Pure point measures with sparse support and sparse Fourier–Bohr support

Michael Baake, Nicolae Strungaru and Venta Terauds

ABSTRACT

Fourier-transformable Radon measures are called doubly sparse when both the measure and its transform are pure point measures with sparse support. Their structure is reasonably well understood in Euclidean space, based on the use of tempered distributions. Here, we extend the theory to second countable, locally compact Abelian groups, where we can employ general cut and project schemes and the structure of weighted model combs, along with the theory of almost periodic measures. In particular, for measures with Meyer set support, we characterise sparseness of the Fourier–Bohr spectrum via conditions of crystallographic type, and derive representations of the measures in terms of trigonometric polynomials. More generally, we analyse positive definite, doubly sparse measures in a natural cut and project setting, which results in a Poisson summation type formula.

1. *Introduction*

The study of translation-bounded, but possibly unbounded, measures on a locally compact Abelian group (LCAG) G, with methods from harmonic analysis, has a long history; compare [1, 7, 12, 13]. Of particular interest are Fourier-transformable measures μ such that both μ and $\widehat{\mu}$ are *sparse*, which means that both are pure point measures and have locally finite support. The best-known example for this type of measure is the uniform Dirac comb [8] of a general lattice $\Gamma \subset \mathbb{R}^d$, which we write as $\delta_\Gamma = \sum_{x \in \Gamma} \delta_x$. This measure is doubly sparse due to the Poisson summation formula (PSF),

$$\widehat{\delta_\Gamma} = \mathrm{dens}(\Gamma)\, \delta_{\Gamma^0}, \tag{1}$$

where Γ^0 denotes the dual lattice of Γ; see [2, Section 9.2] and references therein for background.

The understanding of such measures, and translation-bounded measures and their transforms in general, has reached a reasonably mature state for $G = \mathbb{R}^d$, where they arise in the study of quasicrystals. Here, Meyer's pioneering work on model sets [27, 28] plays a key role; see [5, 31, 32] for a detailed account, and [2, Chapter 9] for an exposition of their appearance in diffraction theory. Although model sets typically lead to diffraction measures with dense support, the methods from this field provide immensely useful tools for the questions at hand. In particular, we will be able to classify, in Theorem 4.10, the few cases of Fourier-transformable

measures that are supported on cut and project sets and have a sparse Fourier transform. While the natural setting of tempered distributions simplifies the harmonic analysis in this case significantly, and powerful complex-analytic techniques may be applied, several interesting open problems remain. We particularly mention those collected and stated by Lagarias [**21**], some of which have recently been answered by Kellendonk and Lenz [**16**], by Favorov [**10**] and by Lev and Olevskii [**25, 26**].

In this paper, we substantially extend the setting and consider *doubly sparse* measures on an LCAG G that is also second countable, hence σ-compact and metrisable. By a doubly sparse measure we mean a Fourier-transformable measure μ such that both $\mathrm{supp}(\mu)$ and $\mathrm{supp}(\widehat{\mu})$ are locally finite point sets (satisfying an upper density condition as detailed in Section 3) in G and $\widehat{G}$, respectively. In particular, both μ and $\widehat{\mu}$ must be pure point measures. Beyond the lattice Dirac comb in (1), other notions and examples of doubly sparse measures have been studied in [**25, 26, 29**] under the name 'crystalline measures'. We do not adopt this term because it has a different meaning elsewhere. Note that some of the measures appearing in these papers are not doubly sparse in our sense, and do not seem to be compatible with the cut and project formalism, which makes them unsuitable for our tools.

In this wider generality, we can no longer work with tempered distributions, but need an extension that is suitable for LCAGs. While one option could be Bruhat–Schwartz theory, compare [**34**] and references therein, it seems more natural to us to employ the general theory of Radon measures on locally compact Abelian groups. A large body of results on such measures has accumulated in recent years, due to the systematic development of the theory of aperiodic order, including the cut and project scheme for measures and their Fourier transforms. We will make extensive use of some of the recent results; these, to our knowledge, have no counterpart yet in Bruhat–Schwartz space. Moreover, we shall employ the connection between Fourier transform and almost periodicity for measures and tempered distributions [**46**]. Since the measures under consideration need not be finite, the notion of transformability is non-trivial; see [**3**, Chapter 4.9] for a detailed exposition and [**7, 12, 13**] for background.

The measures of interest will often display a high degree of long-range translational order. Thus, we may profit from the methods developed in [**6**], which have recently been systematised and extended in [**43**]. In particular, we shall need almost periodic measures of various kinds that emerge from a *cut and project scheme* (CPS) in the sense that they are supported on a projection set with certain properties; see [**2**, Chapter 7] for an introduction, and [**27, 31, 32**] for the general theory and more advanced topics.

In the particular case $G = \mathbb{R}^d$, a natural question is whether one could get more general results via the theory of tempered distributions. It turns out that for a large class of measures, which includes the typical examples we are interested in, the Fourier theory of Radon measures and that of tempered distributions coincide; see Lemma 6.3 for details.

The paper is organised as follows. We recall various concepts and preliminaries in Section 2, followed by Section 3 on the notion and basic properties of sparse point sets in LCAGs. Then, we look more closely at Radon measures with Meyer set support in Section 4, which contains two of our central results, namely Theorems 4.8 and 4.10. They assert that such measures exhibit the following dichotomy: Either μ and $\widehat{\mu}$ are supported on fully periodic sets, or $\widehat{\mu}$ meets the translates of any open set in unboundedly many points.

Then, in Section 5, we consider positive definite measures with uniformly discrete support and sparse Fourier–Bohr spectrum. In particular, we show that any such measure is norm-almost periodic and thus permits a representation in a natural CPS; see Theorem 5.3 and Corollary 5.8. This also allows us to express $\widehat{\mu}$ in terms of a PSF-type formula and to discuss the connection with diffraction theory. Finally, in Section 6, we put our results in perspective with previous results of Lev and Olevskii [**25, 26**] by considering measures supported on $\mathbb{R}^d$, including those arising from fully Euclidean CPSs.

2. *Notation and preliminaries*

Below, we use the general setting of the monograph [**2**], and refer to [**3**, Chapters 4 and 5] for background on the Fourier theory of Radon measures on LCAGs. From now on, unless stated otherwise, the term 'measure' will refer to a (generally complex) Radon measure.

We assume an LCAG G to be equipped with its Haar measure θ_G in a suitable normalisation. This means that we arrange θ_G and $\theta_{\widehat{G}}$, where $\widehat{G}$ is the Pontryagin dual of G, relative to each other in such a way that Parseval's equation holds. In particular, we shall use Lebesgue measure on $\mathbb{R}^d$ and counting measure on $\mathbb{Z}^m$, while the Haar measure will usually be normalised for compact groups. As a consequence, the Haar measure on a finite discrete group will be counting measure divided by the order of the group. For a measurable set $A \subseteq G$, we will often write $\mathrm{vol}(A)$ instead of $\theta_G(A)$ and $\mathrm{d}x$ as a shorthand for $\mathrm{d}\theta_G(x)$, if the reference to G is unambiguous. Below, we will be concerned with certain point sets in G, where the term *point set* refers to an at most countable union of singleton sets.

When G is an LCAG and $g \in L^1(G)$, we write the *Fourier transform* of g as

$$\widehat{g}(\chi) = \int_G \overline{\chi(x)}\, g(x)\, \mathrm{d}x,$$

where $\chi \in \widehat{G}$ is a continuous character, with $\overline{\chi} = \chi^{-1}$. Likewise, the matching inverse transform is given by $\widecheck{g}(\chi) = \int_G \chi(x) g(x)\, \mathrm{d}x$. In this formulation, $\widehat{G}$ is written multiplicatively. This has to be compared with the widely used additive notation for $G = \mathbb{R}^d$, where one writes $\chi(x) = \chi_k(x) = \mathrm{e}^{2\pi\mathrm{i}kx}$ with $k \in \mathbb{R}^d$. Here, and in similar situations such as the d-torus, we then write $\widehat{g}(k) = \int_G \overline{\chi_k(x)} g(x)\, \mathrm{d}x$ with $k \in \widehat{G}$, now written additively. From here, we take the usual route to define the Fourier transform of finite measures, and the notion of Fourier transformability of Radon measures, as in [**43**, Definition 4.9.7].

A *van Hove sequence* $\mathcal{A} = \{A_n\}$ in G is a sequence of compact sets $A_n \subseteq G$ that are nested and exhaustive, meaning $A_n \subseteq A_{n+1}^{\circ}$ together with $\bigcup_n A_n = G$, and also satisfy the asymptotic condition

$$\lim_{n \to \infty} \frac{\theta_G(\partial^K A_n)}{\theta_G(A_n)} = 0$$

for any compact $K \subseteq G$. Here, for compact K and A, the *K-boundary of A* is defined as

$$\partial^K A := \big(\overline{(A + K) \setminus A}\big) \cup \big(A \cap (\overline{G \setminus A} - K)\big), \tag{2}$$

where $A \pm K := \{a \pm k : a \in A, k \in K\}$ denotes the Minkowski sum and difference of the two sets A and K. In particular, for all compact $K \subseteq G$, one has

$$A + K \subseteq A \cup \partial^K A. \tag{3}$$

The nestedness condition implies that $\bigcup_n A_{n+1}^{\circ}$ is an open cover of G, and hence of any compact set $K \subseteq G$. Consequently, $K \subseteq \bigcup_{n \in F} A_{n+1}^{\circ}$ for some finite set $F \subset \mathbb{N}$, which means $K \subseteq A_m$ for all sufficiently large m.

Note that van Hove sequences of the type defined here do exist in all σ-compact LCAGs; see [**40**, p. 145]. In fact, since we included nestedness and exhaustion of G into our definition of a van Hove sequence, the existence of such sequences becomes equivalent to σ-compactness of G. One can go beyond this situation, but we do not attempt that here.

For the induced continuous translation action of G on functions and measures, we start from the relation $(T_t g)(x) = g(x - t)$ for functions. The matching definition for measures is

$$(T_t \mu)(g) = \mu(T_{-t}\, g)$$

for test functions $g \in C_{\mathsf{c}}(G)$. The convolution is defined as usual, and one checks that

$$(T_t \mu) * g = T_t(\mu * g), \tag{4}$$

which makes the notation $T_t \mu * g$ unambiguous. In particular, one finds

$$\big(T_t \mu * g\big)(y) = \big(\mu * g\big)(y - t). \tag{5}$$

Let G be a fixed LCAG. Recall that a measure μ on G is called *translation bounded* if

$$\|\mu\|_E := \sup_{x \in G} |\mu|(x + E) < \infty \tag{6}$$

holds for any compact set E. One can equivalently demand that $\mu * g$ be a bounded function for all $g \in C_{\mathsf{c}}(G)$; see [**40**, Section 1] for the case that G is σ-compact, and [**1**, Theorem 1.1] as well as [**33**, Proposition 4.9.21] for the general case. We denote the set of translation-bounded measures by $\mathcal{M}^\infty(G)$, which will show up many times below.

3.   *Sparse sets*

For the remainder of the paper, unless stated otherwise, G will stand for a second-countable LCAG, and $\widehat{G}$ for its dual group. We generally need second countability of G to define doubly sparse measures on G, and will explicitly mention when our setting can be extended. Recall that a topological group G is *second-countable* if there exists a countable basis for its topology. A second countable group G is both σ-compact and metrisable, which means that $\widehat{G}$ has the same properties [**35**, Theorem 4.2.7].

If μ is a transformable measure on G, we call the measurable support of $\widehat{\mu}$ the *Fourier–Bohr support* of μ, and abbreviate it as FBS from now on. In some papers [**25, 26, 29**], the FBS is also called the *spectrum* or the Fourier–Bohr spectrum of μ. Below, we will not adopt this terminology because the term *spectrum* is already in use in several ways in related questions from dynamical systems and ergodic theory.

3.1.   *General notions and properties*

Given a point set $\Lambda \subseteq G$ and a van Hove sequence $\mathcal{A} = \{A_n\}$ in G, we define the *upper density* and the *uniform upper density* of Λ with respect to $\mathcal{A}$ to be

$$\overline{\mathrm{dens}}_{\mathcal{A}}(\Lambda) := \limsup_{n \to \infty} \frac{\mathrm{card}(\Lambda \cap A_n)}{\mathrm{vol}(A_n)} \qquad \text{and}$$

$$\overline{\mathrm{u\text{-}dens}}_{\mathcal{A}}(\Lambda) := \limsup_{n \to \infty} \sup_{x \in G} \frac{\mathrm{card}\big(\Lambda \cap (x + A_n)\big)}{\mathrm{vol}(A_n)},$$

respectively, and similarly for the lower densities, then denoted as $\underline{\mathrm{dens}}_{\mathcal{A}}(\Lambda)$ and $\underline{\mathrm{u\text{-}dens}}_{\mathcal{A}}(\Lambda)$, with $\limsup$ and $\sup$ replaced by $\liminf$ and $\inf$, respectively. When the lower density of a point set Λ agrees with its upper density, the *density* of Λ with respect to $\mathcal{A}$ exists, and is denoted as $\mathrm{dens}_{\mathcal{A}}(\Lambda)$. The *total uniform upper density* refers to

$$\overline{\mathrm{u\text{-}dens}}(\Lambda) := \sup \big\{ \overline{\mathrm{u\text{-}dens}}_{\mathcal{A}}(\Lambda) : \mathcal{A} \text{ is a van Hove sequence}\big\},$$

again with the matching definition for $\underline{\mathrm{u\text{-}dens}}(\Lambda)$.

Let us add a comment on these notions. When a point set Λ has a finite uniform upper density with respect to *some* van Hove sequence $\mathcal{A}$, it actually has finite uniform upper density with respect to *all* van Hove sequences and, furthermore, the supremum over all of these is finite; see Lemma 3.5 and Remark 3.6. In contrast, a point set may have finite upper density with respect to some van Hove sequence, but infinite upper density with respect to another; see Example 3.4. For this reason, we do not consider the concept of total upper density, and we define sparseness with respect to a particular van Hove sequence in G.

The uniform density is sometimes called *upper Banach density*. When G is a discrete LCAG, this density does not depend on the choice of the Følner sequence [**9**]. One thus has the relation

$\overline{\text{u-dens}}(\varLambda) = \overline{\text{u-dens}}_{\mathcal{A}}(\varLambda) \leqslant 1$ for all $\varLambda$ and every Følner sequence $\mathcal{A}$ in G. The situation seems to be more complicated in non-discrete groups.

DEFINITION 3.1. Given a van Hove sequence $\mathcal{A} = \{A_n\}$ in G, a point set $\varLambda \subseteq G$ is called $\mathcal{A}$-sparse if $\overline{\text{dens}}_{\mathcal{A}}(\varLambda) < \infty$, and strongly $\mathcal{A}$-sparse if $\overline{\text{u-dens}}_{\mathcal{A}}(\varLambda) < \infty$. Moreover, $\varLambda$ is strongly sparse if it is strongly $\mathcal{A}$-sparse for every van Hove sequence $\mathcal{A}$ in G.

REMARK 3.2. If a point set $\varLambda \subseteq G$ is $\mathcal{A}$-sparse for some van Hove sequence $\mathcal{A} = \{A_n\}$ in G, it is automatically locally finite. Indeed, if $K \subseteq G$ is any compact set, there is some A_n in $\mathcal{A}$ with $K \subseteq A_n$, and one has

$$\text{card}(\varLambda \cap K) \leqslant \text{card}(\varLambda \cap A_n) < \infty$$

due to $\mathcal{A}$-sparseness. Local finiteness of $\varLambda$ is then clear, which equivalently means that $\varLambda$ is discrete and closed; compare [**2**, Section 2.1].

Next, we need to recall a notion that is slightly weaker than uniform discreteness, where a point set $\varLambda \in G$ is uniformly discrete if some open neighbourhood U of 0 in G exists such that, for any two distinct points $x, y \in \varLambda$, one has $(x + U) \cap (y + U) = \varnothing$.

DEFINITION 3.3. A point set $\varLambda \subseteq G$ is called weakly uniformly discrete if, for each compact $K \subseteq G$ and all $x \in G$, $\text{card}\big(\varLambda \cap (x + K)\big)$ is bounded by a constant that depends only on K.

Weak uniform discreteness of $\varLambda$ is equivalent to $\delta_{\varLambda}$ being a translation-bounded measure; compare [**43**, p. 288] as well as [**40**, Section 1]. Note also that strong $\mathcal{A}$-sparseness clearly implies $\mathcal{A}$-sparseness, but not vice versa. Let us illustrate these connections as follows.

EXAMPLE 3.4. Consider the point set $\varLambda \subset \mathbb{R}$ defined as

$$\varLambda = \bigcup_{n \in \mathbb{N}} \left\{ n + \frac{k}{n} : 0 \leqslant k < n \right\}.$$

The set $\varLambda$ fails to be weakly uniformly discrete because $\text{card}\big(\varLambda \cap (n + [0, 1])\big) = n$ is unbounded. For the same reason, $\varLambda$ cannot be strongly $\mathcal{A}$-sparse, as any van Hove sequence $\mathcal{A} = \{A_n\}$ in $\mathbb{R}$ has the property that the compact sets A_n contain a translate of $[0,1]$ for all sufficiently large n, so $\overline{\text{u-dens}}_{\mathcal{A}}(\varLambda) = \infty$, and thus also $\overline{\text{u-dens}}(\varLambda) = \infty$.

However, $\varLambda$ can still be $\mathcal{A}$-sparse for certain van Hove sequences. In general, the density with respect to a given van Hove sequence need not be zero, but can take any value $\geqslant 0$, even including ∞. Indeed, choosing A_n as $[-n^3, n]$, $[-\alpha n^2, n]$ with $\alpha > 0$ or $[-n, n^2]$, one gets $\mathcal{A}$-density 0, $\frac{1}{2\alpha}$ or ∞, respectively.

LEMMA 3.5. If $\varLambda \subseteq G$ is weakly uniformly discrete, one has

$$\sup \big\{ \overline{\text{dens}}_{\mathcal{A}}(\varLambda) : \mathcal{A} \text{ is van Hove in } G \big\} \leqslant \overline{\text{u-dens}}(\varLambda) < \infty.$$

In [**22**, Lemma 9.2], the authors prove this result for the larger class of translation-bounded measures (compare also with [**40**, Lemma 1.1]). Here, we prefer to give an independent argument as follows.

Proof. Observe first that $\overline{\text{dens}}_{\mathcal{A}}(\varLambda) \leqslant \overline{\text{u-dens}}_{\mathcal{A}}(\varLambda)$ obviously holds for any van Hove sequence $\mathcal{A}$ in G, hence also $\overline{\text{dens}}_{\mathcal{A}}(\varLambda) \leqslant \overline{\text{u-dens}}(\varLambda)$ for all $\mathcal{A}$, and the first inequality is clear. It remains to show that there is a constant $C < \infty$ with $\overline{\text{u-dens}}(\varLambda) \leqslant C$.

Select some non-negative $f \in C_{\mathrm{c}}(G)$ with $\theta_G(f) = \int_G f(x)\,\mathrm{d}x = 1$, and set $K = \mathrm{supp}(f)$. Since Λ is weakly uniformly discrete, the Dirac comb δ_Λ is translation bounded, and $f * \delta_\Lambda$ is a non-negative continuous function that is bounded. We thus have $C := \|f * \delta_\Lambda\|_\infty < \infty$ and $0 \leqslant \bigl(f * \delta_\Lambda\bigr)(x) \leqslant C$ for all $x \in G$.

Let $\mathcal{A}$ be any van Hove sequence in G. Then, using Fubini, we can estimate

$$\mathrm{card}\bigl(\Lambda \cap (x + A_n)\bigr) = \int_G \int_G f(t)\,\mathrm{d}t\, 1_{x+A_n}(s)\,\mathrm{d}\delta_\Lambda(s)$$

$$= \int_G \int_G f(t - s)\,\mathrm{d}t\, 1_{x+A_n}(s)\,\mathrm{d}\delta_\Lambda(s) = \int_G \int_G f(t - s)\, 1_{x+A_n}(s)\,\mathrm{d}\delta_\Lambda(s)\,\mathrm{d}t.$$

Now, observe that $f(t - s)\, 1_{x+A_n}(s) = 0$ unless $t \in x + A_n + K$, hence

$$0 \leqslant f(t - s)\, 1_{x+A_n}(s) = f(t - s)\, 1_{x+A_n}(s)\, 1_{x+A_n+K}(t) \leqslant f(t - s)\, 1_{x+A_n+K}(t),$$

and we get

$$\mathrm{card}\bigl(\Lambda \cap (x + A_n)\bigr) \leqslant \int_G \int_G f(t - s)\, 1_{x+A_n+K}(t)\,\mathrm{d}\delta_\Lambda(s)\,\mathrm{d}t \tag{7}$$

$$= \int_G 1_{x+A_n+K}(t) \int_G f(t - s)\,\mathrm{d}\delta_\Lambda(s)\,\mathrm{d}t = \int_G 1_{x+A_n+K}(t) \bigl(f * \delta_\Lambda\bigr)(t)\,\mathrm{d}t$$

$$\leqslant C\, \mathrm{vol}(x + A_n + K) = C\, \mathrm{vol}(A_n + K) \leqslant C\bigl(\mathrm{vol}(A_n) + \mathrm{vol}(\partial^K A_n)\bigr),$$

independently of x, with the last step following from equation (3). Consequently, we have

$$\sup_{x \in G} \frac{\mathrm{card}\bigl(\Lambda \cap (x + A_n)\bigr)}{\mathrm{vol}(A_n)} \leqslant C\left(1 + \frac{\mathrm{vol}(\partial^K A_n)}{\mathrm{vol}(A_n)}\right),$$

where n is arbitrary. Hence, by the van Hove property, $\overline{\mathrm{u\text{-}dens}}_{\mathcal{A}}(\Lambda) \leqslant C$. Since this bound does not depend on $\mathcal{A}$, our claim follows. $\qquad \square$

REMARK 3.6. When $\Lambda \subseteq G$ is a point set that violates weak uniform discreteness, one gets $\overline{\mathrm{u\text{-}dens}}_{\mathcal{A}}(\Lambda) = \infty$, for any van Hove sequence $\mathcal{A}$. Indeed, the sets A_n are compact, and we may, without loss of generality, assume that all of them have non-empty interior. For any $n \in \mathbb{N}$, this implies

$$\|\delta_\Lambda\|_{A_n} = \sup_{x \in G} |\delta_\Lambda|(x + A_n) = \infty,$$

which really is a statement in the norm topology [6]; compare [43, equation (5.3.1)]. This property means that

$$\sup_{x \in G} \frac{\mathrm{card}\bigl(\Lambda \cap (x + A_n)\bigr)}{\mathrm{vol}(A_n)} = \infty$$

and hence $\overline{\mathrm{u\text{-}dens}}_{\mathcal{A}}(\Lambda) = \infty$.

Under the conditions of Remark 3.6, for any van Hove sequence $\mathcal{A}$, there is a sequence $\{t_n\}$ of translations such that $\mathrm{card}\bigl(\Lambda \cap (t_n + A_n)\bigr)/\mathrm{vol}(A_n) > n$, which is unbounded. However, this does not imply $\overline{\mathrm{dens}}_{\mathcal{A}}(\Lambda) = \infty$, as Example 3.4 shows. Also, Lemma 3.5 and Remark 3.6 imply the following: If $\overline{\mathrm{u\text{-}dens}}_{\mathcal{A}}(\Lambda) < \infty$ holds for *some* van Hove sequence $\mathcal{A}$, the same estimate holds for *all* van Hove sequences. We can now strengthen the relations as follows.

THEOREM 3.7. *For a point set $\Lambda \subseteq G$, the following properties are equivalent.*

(1) Λ is weakly uniformly discrete.
(2) Λ is strongly sparse.
(3) One has $\overline{\text{u-dens}}(\Lambda) < \infty$.
(4) One has $\overline{\text{u-dens}}_{\mathcal{A}}(\Lambda) < \infty$ for some van Hove sequence $\mathcal{A}$.

Proof. (1) $\Rightarrow$ (3) follows from Lemma 3.5, while (3) $\Rightarrow$ (2) $\Rightarrow$ (4) is an immediate consequence of Definition 3.1. Finally, (4) $\Rightarrow$ (1) follows from Remark 3.6. $\square$

3.2. *Sparse cut and project sets*

Let us begin by briefly recalling the setting of a CPS, which is based on [**27, 31, 32**]. A CPS consists of two LCAGs, G and H, together with a lattice[†] $\mathcal{L} \subseteq G \times H$ and several mappings with some specific conditions. This is denoted by the triple $(G, H, \mathcal{L})$ and usually summarised in a diagram as follows.

$$
\begin{array}{ccccc}
G & \xleftarrow{\pi_G} & G \times H & \xrightarrow{\pi_H} & H \\
\cup & & \cup & & \cup \;\; \text{dense} \\
\pi_G(\mathcal{L}) & \xleftarrow{1-1} & \mathcal{L} & \longrightarrow & \pi_H(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\;\;\star\;\;} & & L^{\star}
\end{array}
\tag{8}
$$

Here, the mapping $(\cdot)^{\star}\colon L \longrightarrow H$ is well defined; see [**31, 32**] for a general exposition and [**2**] for further details, in particular for the case of $G = \mathbb{R}^d$, which we call a *Euclidean* CPS. When also $H = \mathbb{R}^n$, it is called *fully Euclidean*.

For some arguments, we also need the *dual CPS*, denoted by $(\widehat{G}, \widehat{H}, \mathcal{L}^0)$ and nicely explained in [**31**]; see also [41]. Here, $\widehat{G}$ and $\widehat{H}$ are the dual groups, while $\mathcal{L}^0$ is the annihilator of $\mathcal{L}$ from (8), and a lattice in $\widehat{G \times H} \simeq \widehat{G} \times \widehat{H}$. Diagrammatically, we get the following.

$$
\begin{array}{ccccc}
\widehat{G} & \xleftarrow{\pi_{\widehat{G}}} & \widehat{G} \times \widehat{H} & \xrightarrow{\pi_{\widehat{H}}} & \widehat{H} \\
\cup & & \cup & & \cup \;\; \text{dense} \\
\pi_{\widehat{G}}(\mathcal{L}^0) & \xleftarrow{1-1} & \mathcal{L}^0 & \longrightarrow & \pi_{\widehat{H}}(\mathcal{L}^0) \\
\| & & & & \| \\
L^0 & & \xrightarrow{\;\;\star\;\;} & & (L^0)^{\star}
\end{array}
\tag{9}
$$

Note that the existence of a $\star$-map in the dual CPS follows from that in the original one, whence we use the same symbol for it, though the mappings are, of course, different.

Recall that, once a CPS $(G, H, \mathcal{L})$ with its natural projections and its $\star$-map is given, a *cut and project set*, or CP set for short, is a set of the form

$$
\curlywedge(U) = \{x \in \pi_G(\mathcal{L}) : x^{\star} \in U\} = \{x \in L : x^{\star} \in U\}
\tag{10}
$$

for some coding set or *window* $U \subseteq H$. When U is relatively compact with non-empty interior, $\curlywedge(U)$ is called a *model set*. Note that model sets are Meyer sets[‡], and that any Meyer set is a subset of a model set; see [**33**, Theorem 5.7.8]. For a function g on H such that

$$
\omega_g := \sum_{x \in \pi_G(\mathcal{L})} g(x^{\star}) \, \delta_x
$$

[†]In an LCAG G, a lattice simply is a discrete, co-compact subgroup.

[‡]Recall that $\Lambda \subseteq G$ is a Meyer set if it is relatively dense and if $\Lambda - \Lambda \subseteq \Lambda + F$ holds for some finite set $F \subseteq G$. Another characterisation together with further aspects will be discussed in Remark 4.4.

is a measure on G, we call ω_g a *weighted Dirac comb* for $(G, H, \mathcal{L})$; see [**37**, Section 4.1] for details. When the support of ω_g is a model set, we call it a *weighted model comb*.

Recall that the density of a lattice, such as $\mathcal{L}$ in $G \times H$, exists uniformly, so does not depend on the choice of a van Hove sequence. We thus write $\mathrm{dens}(\mathcal{L})$ in this situation. Let us begin by proving a density formula for CP sets with open sets as windows, which will be a key input for many of our later computations. Here, we invoke and extend [**15**, Proposition 3.4], which is a density formula for relatively compact sets as windows that is substantially based on [**39**, Theorem 1]. We note that, while the point sets we employ in our results often fail to be model sets themselves, projection sets with unbounded windows of finite measure will play an important role in our arguments. Also, it is essential for the proofs to come that the windows need not be regular. There is quite some recent interest in the corresponding theory of weak model sets; compare [**4, 17, 18, 41**].

PROPOSITION 3.8. *Let $(G, H, \mathcal{L})$ be a CPS, let $\mathcal{A}$ be a van Hove sequence in G and let $U \subseteq H$ be an open set. Then,*

$$\theta_H(U) \leqslant \frac{\overline{\mathrm{dens}}_{\mathcal{A}}\big(\curlywedge(U)\big)}{\mathrm{dens}(\mathcal{L})}.$$

In particular, if $\curlywedge(U)$ is $\mathcal{A}$-sparse, one has $\theta_H(U) < \infty$.

Proof. Let $K \subseteq U$ be any compact set. Then, by [**15**, Proposition 3.4], we have

$$\theta_H(K) \leqslant \frac{\overline{\mathrm{dens}}_{\mathcal{A}}\big(\curlywedge(K)\big)}{\mathrm{dens}(\mathcal{L})}.$$

Next, as $K \subseteq U$, we have $\curlywedge(K) \subseteq \curlywedge(U)$ and hence $\overline{\mathrm{dens}}_{\mathcal{A}}\big(\curlywedge(K)\big) \leqslant \overline{\mathrm{dens}}_{\mathcal{A}}\big(\curlywedge(U)\big)$. This shows that, for all $K \subseteq U$ compact, we have

$$\theta_H(K) \leqslant \frac{\overline{\mathrm{dens}}_{\mathcal{A}}\big(\curlywedge(U)\big)}{\mathrm{dens}(\mathcal{L})}.$$

Finally, by the inner regularity of θ_H, we have

$$\theta_H(U) = \sup_{\substack{K \subseteq U \\ \text{compact}}} \theta_H(K) \leqslant \frac{\overline{\mathrm{dens}}_{\mathcal{A}}\big(\curlywedge(U)\big)}{\mathrm{dens}(\mathcal{L})},$$

which completes the argument. $\square$

REMARK 3.9. It is worth mentioning that, given a relatively compact window $W \subseteq H$ and an arbitrary van Hove sequence $\mathcal{A}$ in G, one has the following chain of estimates,

$$\mathrm{dens}(\mathcal{L})\, \theta_H\big(W^{\circ}\big) \leqslant \underline{\mathrm{u\text{-}dens}}\big(\curlywedge(W)\big) \leqslant \underline{\mathrm{u\text{-}dens}}_{\mathcal{A}}\big(\curlywedge(W)\big)$$

$$\leqslant \underline{\mathrm{dens}}_{\mathcal{A}}\big(\curlywedge(W)\big) \leqslant \overline{\mathrm{dens}}_{\mathcal{A}}\big(\curlywedge(W)\big)$$

$$\leqslant \overline{\mathrm{u\text{-}dens}}_{\mathcal{A}}\big(\curlywedge(W)\big) \leqslant \overline{\mathrm{u\text{-}dens}}\big(\curlywedge(W)\big) \leqslant \mathrm{dens}(\mathcal{L})\, \theta_H\big(\overline{W}\big),$$

which puts Proposition 3.8 in a more general perspective.

An immediate consequence of Proposition 3.8 is the following.

COROLLARY 3.10. *Let $(G, H, \mathcal{L})$ be a CPS, let $h \in C_0(H)$ and set $U := \{z \in H : h(z) \neq 0\}$. If the weighted Dirac comb ω_h has $\mathcal{A}$-sparse support for some van Hove sequence $\mathcal{A}$ in G, one has $\theta_H(U) < \infty$. In particular, $\theta_H(U) < \infty$ whenever $\mathrm{supp}(\omega_h)$ is weakly uniformly discrete.*

Proof. Observe that

$$\mathrm{supp}(\omega_h) = \{x \in L : h(x^\star) \neq 0\} = \{x \in L : x^\star \in U\} = \lambda(U).$$

Then, $\mathcal{A}$-sparseness of the support means that $\overline{\mathrm{dens}}_{\mathcal{A}}\big(\lambda(U)\big)$ is finite, and the result follows from Proposition 3.8.

The last claim now follows via the implication $(1) \Rightarrow (4)$ from Theorem 3.7. $\square$

To continue, we will have to consider a group G and its dual, $\widehat{G}$. Unless stated otherwise, we assume that we have selected a van Hove sequence in each of these two groups, namely $\mathcal{A}$ for G and $\mathcal{B}$ for $\widehat{G}$. In the case of a self-dual group, such as $\mathbb{R}^d$, we might think of taking the same sequence for both. In contrast, for $G = \mathbb{Z}^m$ hence $\widehat{G} = \mathbb{T}^m$, we fix some $\mathcal{A}$ for $\mathbb{Z}^m$, say a sequence of centred cubes or balls, while it would be natural to take $\mathcal{B} = \{B_n\}$ as the constant sequence, so $B_n = \mathbb{T}^m$ for all n, and similarly for other LCAGs H that are compact. Note that this is consistent with our nestedness condition because H is both open and closed.

DEFINITION 3.11. Let G be an LCAG with σ-compact dual group, $\widehat{G}$. Assume that a van Hove sequence $\mathcal{B}$ for $\widehat{G}$ has been selected. Then, we say that a measure $\mu \in \mathcal{M}^\infty(G)$ has *sparse Fourier–Bohr support* (sparse FBS) with respect to $\mathcal{B}$ if

(1) μ is Fourier transformable, with transform $\widehat{\mu}$;
(2) the support, $\mathrm{supp}(\widehat{\mu})$, is a $\mathcal{B}$-sparse point set in $\widehat{G}$.

Moreover, if also G is σ-compact and a van Hove sequence $\mathcal{A}$ for G is given, a measure μ with $\mathcal{A}$-sparse support and $\mathcal{B}$-sparse FBS is called *doubly sparse* with respect to $(\mathcal{A}, \mathcal{B})$, or $(\mathcal{A}, \mathcal{B})$-sparse for short. If μ is $(\mathcal{A}, \mathcal{B})$-sparse for any pair of van Hove sequences, we simply call μ *doubly sparse*.

REMARK 3.12. Note that the notion of a $\mathcal{B}$-sparse FBS does not require the existence of a van Hove sequence in G. In fact, μ has $\mathcal{B}$-sparse FBS if and only if μ is Fourier transformable, $\widehat{\mu}$ is a pure point measure and the point set $\{\chi \in \widehat{G} : \widehat{\mu}(\{\chi\}) \neq 0\}$ is $\mathcal{B}$-sparse.

Moreover, if a measure $\mu \in \mathcal{M}^\infty(G)$ has sparse FBS, $\widehat{\mu}$ is pure point and, consequently, μ must be strongly almost periodic [**33**, Corollary 4.10.13]. Here, strongly almost periodic for a measure μ means that $\mu * g$ is uniformly (or Bohr) almost periodic for every $g \in C_c(G)$, and any such measure μ must be translation bounded. In fact, for $\mu \neq 0$, $\mathrm{supp}(\mu) \subseteq G$ is relatively dense by [**43**, Lemma 5.9.1]. In particular, it then follows that a measure μ with sparse FBS has Meyer set support if and only if $\mu \neq 0$ and $\mathrm{supp}(\mu)$ is a subset of a Meyer set.

EXAMPLE 3.13. All crystallographic measures on $\mathbb{R}^d$ have a strongly sparse FBS. Indeed, $\omega \in \mathcal{M}^\infty(\mathbb{R}^d)$ is *crystallographic* if it is of the form $\omega = \mu * \delta_\Gamma$ with μ a *finite* measure and $\Gamma \subset \mathbb{R}^d$ a lattice; compare [**2**, Section 9.2.3]. Any such measure is Fourier transformable, with

$$\widehat{\omega} = \widehat{\mu} \cdot \widehat{\delta_\Gamma} = \mathrm{dens}(\Gamma)\, \widehat{\mu}\, \delta_{\Gamma^0} = \mathrm{dens}(\Gamma) \sum_{k \in \Gamma^0} \widehat{\mu}(k)\, \delta_k$$

by an application of the convolution theorem in conjunction with the PSF from equation (1). Here, $\widehat{\mu}$ is a continuous function on $\mathbb{R}^d$, and the dual lattice, Γ^0, is a uniformly discrete point set; see [**2**, Example 9.2]. This means that $\mathrm{supp}(\widehat{\omega}) \subseteq \Gamma^0$ is a strongly sparse point set in $\mathbb{R}^d$, and ω is *doubly sparse* when μ has finite support, which is to say that it is of the form $\mu = \sum_{x \in F} \mu(\{x\})\, \delta_x$ for some finite set $F \subset \mathbb{R}^d$.

In what follows, we shall consider a slight generalisation of this idea, namely, measures that are supported within finitely many translates of a lattice, but with coefficients that are

not necessarily lattice-periodic. Such measures thus have a support with a crystallographic structure, without actually being crystallographic in the above sense.

To continue, we need the following notion for continuous functions.

DEFINITION 3.14. We say that a continuous function $h\colon H \longrightarrow \mathbb{C}$ has *finite-measure support* if $\theta_H\big(\{x \in H : h(x) \neq 0\}\big) < \infty$.

FACT 3.15. Let H be an arbitrary LCAG, and consider $h \in C_0(H)$. If h has finite-measure support, then $h \in L^1(H)$.

Proof. Any $h \in C_0(H)$ is bounded. With $U := \{x \in H : h(x) \neq 0\}$, we thus get

$$\int_H |h(z)|\,\mathrm{d}z = \int_U |h(z)|\,\mathrm{d}z \leqslant \|h\|_\infty \,\theta_H(U) \,<\, \infty,$$

which implies the claim. $\qquad\square$

4. Measures with Meyer set support and sparse FBS

In this section, we characterise translation-bounded measures, so $\mu \in \mathcal{M}^\infty(G)$, with the additional properties that $\mathrm{supp}(\mu) - \mathrm{supp}(\mu)$ is uniformly discrete and that μ has a sparse FBS. Simple examples are Dirac combs of lattices in $\mathbb{R}^d$, as mentioned in equation (1) and in Example 3.13. An important tool will be the structure of compactly generated LCAGs, which we recall for convenience from [**13**, Theorem 9.8]; see also [**35**, Theorem 4.2.29].

FACT 4.1. If the LCAG H is compactly generated, there are non-negative integers d and m such that H, as a topological group, is isomorphic with $\mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{K}$, where the Abelian group $\mathbb{K}$ is compact. $\qquad\square$

Recall that the m-torus, $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$, is the dual group of $\mathbb{Z}^m$. For convenience in explicit calculations, we represent it as $[0,1)^m$ with addition modulo 1, which is fully compatible with writing the elements of $\mathbb{T}^m$ as $x + \mathbb{Z}^m$ with $x \in [0,1)^m$. Before we continue, we need the following simple variant of the classic Paley–Wiener theorem [**47**, Section VI.4].

LEMMA 4.2. Let $d, m \in \mathbb{N}$ be fixed. Then, for any fixed $f \in C_{\mathsf{c}}(\mathbb{R}^d \times \mathbb{Z}^m)$, there exists an analytic function $F\colon \mathbb{R}^d \times \mathbb{R}^m \longrightarrow \mathbb{C}$ such that F is $\mathbb{Z}^m$-periodic in the second argument and that $\widehat{f}(x, y + \mathbb{Z}^m) = F(x,y)$ holds for all $x \in \mathbb{R}^d$ and $y \in [0,1)^m$.

Proof. Define $\mu\colon C_{\mathsf{c}}(\mathbb{R}^d \times \mathbb{R}^m) \longrightarrow \mathbb{C}$ by

$$\mu(g) = \sum_{v \in \mathbb{Z}^m} \int_{\mathbb{R}^d} g(u,v) f(u,v)\,\mathrm{d}u,$$

which is a finite measure of compact support because $\mathrm{supp}(f)$ is compact by assumption, and thus also a tempered distribution. Its (distributional) Fourier transform is an analytic function on $\mathbb{R}^{d+m}$, by the easy direction of the Paley–Wiener–Schwartz theorem for distributions, see [**11**, Theorems III.2.2 and III.4.5], and reads

$$F(x,y) = \widehat{\mu}(x,y) = \sum_{v \in \mathbb{Z}^m} \mathrm{e}^{-2\pi\mathrm{i}vy} \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi\mathrm{i}ux} f(u,v)\,\mathrm{d}u,$$

with a unique continuation to an entire function on $\mathbb{C}^{d+m}$. The relation $F(x, y+k) = F(x,y)$ for arbitrary $k \in \mathbb{Z}^m$ and all $x \in \mathbb{R}^d$, $y \in \mathbb{R}^m$ is clear.

On the other hand, when $x \in \mathbb{R}^d$ and $y + \mathbb{Z}^m \in \mathbb{T}^m$, we get

$$\widehat{f}(x, y + \mathbb{Z}^m) = \int_{\mathbb{R}^d} \sum_{v \in \mathbb{Z}^m} \mathrm{e}^{-2\pi \mathrm{i} v y} \, \mathrm{e}^{-2\pi \mathrm{i} u x} f(u, v) \, \mathrm{d}u$$

$$= \int_{\mathbb{R}^d \times \mathbb{Z}^m} \mathrm{e}^{-2\pi \mathrm{i}(ux + vy)} \, \mathrm{d}\mu(u, v) = F(x, y),$$

which completes the argument. $\qquad\square$

Here, we are interested in the following consequence.

COROLLARY 4.3. *Let $d, m \in \mathbb{N}$ be fixed and consider a function $f \in C_{\mathsf{c}}(\mathbb{R}^d \times \mathbb{Z}^m)$ with $f \not\equiv 0$. Then, the set*

$$U := \{(x, y + \mathbb{Z}^m) \in \mathbb{R}^d \times \mathbb{T}^m : \widehat{f}(x, y + \mathbb{Z}^m) = 0\}$$

has measure 0 in $\mathbb{R}^d \times \mathbb{T}^m$.

Proof. Defining the function F as in Lemma 4.2, $V := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^m : F(x, y) = 0\}$ is a null set in $\mathbb{R}^d \times \mathbb{R}^m$ because F is analytic (see also [**30**] for this point). Since F is $\mathbb{Z}^m$-periodic in its second variable, we have a canonical projection $\pi \colon \mathbb{R}^d \times \mathbb{R}^m \longrightarrow \mathbb{R}^d \times \mathbb{T}^m$ such that $U = \pi(V)$ and $V = \pi^{-1}(U)$, which implies the claim. $\qquad\square$

At this point, we can harvest the constructive approach to the CPS of a given Meyer set with methods from [**6, 43**].

REMARK 4.4. Recall that a subset Λ of a locally compact Abelian group G is a Meyer set if it is relatively dense and $\Lambda - \Lambda - \Lambda$ is uniformly discrete; see [**20**] for background. If G is compactly generated, then the second condition is equivalent to the uniform discreteness of $\Lambda - \Lambda$; see [**19**, Theorem 1.1] for $G = \mathbb{R}^d$ and [**5**, Appendix] for the general case, as well as [**2**, Remark 2.1]. We note further that the following results do not require the stronger Meyer set condition; that is, in this case, it is sufficient to require that $\mathrm{supp}(\mu) \subseteq \Lambda$, where $\Lambda - \Lambda$ is uniformly discrete; compare [**43**, Theorem 5.5.2].

PROPOSITION 4.5. *Let $\mu \neq 0$ be a translation-bounded measure on G. If $\mathrm{supp}(\mu)$ is a subset of a Meyer set and if μ has $\mathcal{B}$-sparse FBS for some van Hove sequence $\mathcal{B}$ in $\widehat{G}$, then μ is $(\mathcal{A}, \mathcal{B})$-sparse for every van Hove sequence $\mathcal{A}$ in G.*

Moreover, there is a CPS $(G, \mathbb{Z}^m \times \mathbb{K}, \mathcal{L})$, with $m \in \mathbb{N}_0$ and $\mathbb{K}$ a compact Abelian group, and some $h \in C_{\mathsf{c}}(\mathbb{Z}^m \times \mathbb{K})$ with $\widehat{h} \in C_0\big(\mathbb{T}^m \times \widehat{\mathbb{K}}\big) \cap L^1\big(\mathbb{T}^m \times \widehat{\mathbb{K}}\big)$ such that

$$\mu = \omega_h \quad \text{and} \quad \widehat{\mu} = \mathrm{dens}(\mathcal{L}) \, \omega_{\widecheck{h}}.$$

Proof. Since $\mathrm{supp}(\mu)$ is contained in a Meyer set, it is uniformly discrete, hence strongly sparse, and thus $\mathcal{A}$-sparse for every van Hove sequence $\mathcal{A}$ in G.

By definition, compare Remark 3.12, $\mu \in \mathcal{M}^\infty(G)$ having $\mathcal{B}$-sparse FBS means that μ is Fourier transformable and $\widehat{\mu}$ is a pure point measure. Consequently, by [**33**, Corollary 4.10.13], μ is a strongly almost periodic measure. By Remark 3.12, we know that $\mathrm{supp}(\mu)$ must actually be a Meyer set, so [**43**, Theorem 5.5.2] implies that there exists a CPS $(G, H, \mathcal{L})$, with H compactly generated, and some function $h \in C_{\mathsf{c}}(H)$ such that $\mu = \omega_h$. In other words, μ is a weighted model comb.

Since μ is Fourier transformable, [**37**, Theorem 5.3] implies that we have $\check{h} \in L^1(\widehat{H})$ and $\widehat{\omega_h} = \mathrm{dens}(\mathcal{L})\omega_{\check{h}}$, where $\check{h} \in C_0(\widehat{H})$ is clear from [**38**, Theorem 1.2.4] (or from the Riemann–Lebesgue lemma). By assumption, $\mathrm{supp}(\omega_{\check{h}}) = \mathrm{supp}(\widehat{\mu})$ is $\mathcal{B}$-sparse in $\widehat{G}$. Via the dual CPS $(\widehat{G}, \widehat{H}, \mathcal{L}^0)$, and applying Corollary 3.10 to the set $U = \{z \in \widehat{H} : \check{h}(z) \neq 0\}$, we see that

$$\theta_{\widehat{H}}(U) < \infty, \tag{11}$$

and that the FBS of μ is the CP set $\lambda(U)$ in the dual CPS.

Now, by Fact 4.1, we have $H \cong \mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{K}$ for some $d, m \in \mathbb{N}_0$ and $\mathbb{K}$ a compact Abelian group, and we identify H with this group. We shall now show that, in fact, $d = 0$. Since $\mu \neq 0$ by assumption, we have $h \not\equiv 0$ and thus $\check{h}(x_0, y_0, z_0) \neq 0$ for some $(x_0, y_0, z_0) \in \mathbb{R}^d \times \mathbb{T}^m \times \widehat{\mathbb{K}}$. From (11), we get

$$\theta_{\widehat{H}}\big(U \cap (\mathbb{R}^d \times \mathbb{T}^m \times \{z_0\})\big) < \infty. \tag{12}$$

Now, for each $x \in \mathbb{R}^d, y \in \mathbb{T}^m$, we have

$$\check{h}(x, y, z_0) = \int_{\mathbb{R}^d} \int_{\mathbb{Z}^m} \int_{\mathbb{K}} \chi_x(s)\, \chi_y(t)\, \chi_{z_0}(u)\, h(s, t, u)\, \mathrm{d}\theta_{\mathbb{K}}(u)\, \mathrm{d}\theta_{\mathbb{Z}^m}(t)\, \mathrm{d}\theta_{\mathbb{R}^d}(s) = \check{f}(x, y), \tag{13}$$

where $f \colon \mathbb{R}^d \times \mathbb{Z}^m \longrightarrow \mathbb{C}$ is defined by

$$(s, t) \mapsto f(s, t) := \int_{\mathbb{K}} \chi_{z_0}(u)\, h(s, t, u)\, \mathrm{d}\theta_{\mathbb{K}}(u),$$

which satisfies $\check{f}(x_0, y_0) = \check{h}(x_0, y_0, z_0) \neq 0$. By Lemma 4.2, the function $\check{f}$ is analytic and satisfies $\check{f} \not\equiv 0$. Thus, by Corollary 4.3, the set

$$Z := \{(x, y) \in \mathbb{R}^d \times \mathbb{T}^m : \check{f}(x, y) = 0\}$$

has measure 0 in $\mathbb{R}^d \times \mathbb{T}^m$. Let

$$V = \{(x, y) \in \mathbb{R}^d \times \mathbb{T}^m : \check{h}(x, y, z_0) \neq 0\},$$

so that, by (13), we have $\mathbb{R}^d \times \mathbb{T}^m = Z \dot\cup V$. Since $\widehat{\mathbb{K}}$ is discrete, $\theta_{\widehat{\mathbb{K}}}$ is proportional to counting measure, which means

$$\theta_{\mathbb{R}^d \times \mathbb{T}^m}(V) = c\, \theta_{\mathbb{R}^d \times \mathbb{T}^m \times \widehat{\mathbb{K}}}(V \times \{z_0\})$$

for some $c > 0$, where the right-hand side is finite as a consequence of equation (12). We thus get

$$\theta_{\mathbb{R}^d \times \mathbb{T}^m}(\mathbb{R}^d \times \mathbb{T}^m) = \theta_{\mathbb{R}^d \times \mathbb{T}^m}(V \dot\cup Z) = \theta_{\mathbb{R}^d \times \mathbb{T}^m}(V) < \infty,$$

which is only possible if $d = 0$. Consequently, $H = \mathbb{Z}^m \times \mathbb{K}$ together with $\widehat{H} = \mathbb{T}^m \times \widehat{\mathbb{K}}$, and our claims follow. $\qquad\square$

REMARK 4.6. The last part of the above proof may alternatively be shown by invoking the *qualitative uncertainty principle* (QUP) for LCAGs, as nicely summarised in [**14**]. Let $\mathbb{K}_0$ be the (connected) identity component of $\mathbb{K}$. Then, $\mathbb{R}^d \times \{0\} \times \mathbb{K}_0$ is the identity component of H. Since f and $\widehat{f}$ have finite-measure support by assumption, with $f \not\equiv 0$, the QUP fails in H. By [**14**, Theorem 1], the identity component of H must then be compact, thus $d = 0$.

Recall that the Eberlein (or volume-averaged) convolution of two measures $\mu, \nu \in \mathcal{M}^\infty(G)$, relative to a given van Hove sequence $\mathcal{A}$, is defined by

$$\mu \circledast \nu = \lim_{n \to \infty} \frac{\mu_n * \nu_n}{\mathrm{vol}(A_n)},$$

where μ_n and ν_n are the restrictions of μ and ν to the set A_n. Here, the existence of the vague limit is assumed, which is always the case in our setting. An explicit proof of the following result is given in [**37**, Proposition 5.1], and need not be repeated here; see also [**2**, Section 9.4] as well as [**6**].

COROLLARY 4.7. *Under the conditions of Proposition* 4.5, *which comprise the transformability of* μ, *the autocorrelation* $\gamma := \mu \circledast \widetilde{\mu}$ *is well defined, and one has the relations*

$$\gamma = \mathrm{dens}(\mathcal{L})\,\omega_{h*\widetilde{h}} \quad and \quad \widehat{\gamma} = \mathrm{dens}(\mathcal{L})^2\omega_{|\check{h}|^2}.$$

Moreover, setting $S := \mathrm{supp}(\widehat{\mu})$, *we also have the representation* $\widehat{\mu} = \sum_{y\in S}\widehat{\mu}(\{y\})\,\delta_y$ *together with* $\widehat{\gamma} = \sum_{y\in S}|\widehat{\mu}(\{y\})|^2\delta_y$. □

We are now ready to formulate our first main result.

THEOREM 4.8. *Let* $\mu \in \mathcal{M}^\infty(G)$ *with* $\mu \neq 0$ *be such that* $\mathrm{supp}(\mu)$ *is contained in a Meyer set and that the FBS of* μ *is* $\mathcal{B}$-*sparse for some van Hove sequence* $\mathcal{B}$ *in* $\widehat{G}$. *Then, there is a lattice* Γ *in* G *together with finite sets* $F \subseteq G$ *and* $F' \subseteq \widehat{G}$ *such that*

$$\mathrm{supp}(\mu) \subseteq \Gamma + F \quad and \quad \mathrm{supp}(\widehat{\mu}) \subseteq \Gamma^0 + F'.$$

Proof. By assumption, we have $\mu \neq 0$. By Proposition 4.5, there exists a CPS $(G, H, \mathcal{L})$, with $H := \mathbb{Z}^m \times \mathbb{K}$, and some $h \in C_{\mathsf{c}}(\mathbb{Z}^m \times \mathbb{K})$ with $\widehat{h} \in L^1(\mathbb{T}^m \times \widehat{\mathbb{K}})$ continuous such that $\mu = \omega_h$. Now, consider

$$\Gamma := \curlywedge\bigl(\{0\}\times\mathbb{K}\bigr).$$

Since $H_0 := \{0\}\times\mathbb{K}$ is a subgroup of H, and the $\star$-map is a group homomorphism, Γ is a subgroup of G. Moreover, since H_0 is both compact and open, Γ is a Delone set. This shows that Γ is a lattice in G.

Next, since $\mathrm{supp}(h)$ is compact, it can be covered by finitely many translates of the open set H_0. More precisely, there is a finite set $S \subset \mathbb{Z}^m$ such that $\mathrm{supp}(h) \subseteq \bigcup_{t\in S}\bigl((t,0) + H_0\bigr)$. If we set $F := \curlywedge(S\times\{0\})$, we see that F is finite and

$$\mathrm{supp}(\mu) = \curlywedge\bigl(\mathrm{supp}(h)\bigr) \subseteq \curlywedge(H_0) + \curlywedge(S\times\{0\}) = \Gamma + F.$$

To gain the corresponding result for $\widehat{\mu}$, we need to show that $\mathrm{supp}\bigl(\check{h}\bigr)$ is compact. From the above, we know that $\mathrm{supp}(h) \subseteq S\times\mathbb{K}$. For $t \in S$, set

$$h_t(\xi) := h(t, \xi),$$

so that $h_t \in C(\mathbb{K})$ and $h = \sum_{t\in S} 1_{\{t\}} \otimes h_t$. For any $t \in S$ and $y \in \widehat{\mathbb{K}}$, we have

$$\check{h}_t(y) = \int_{\mathbb{K}} \chi_y(u)\,h_t(u)\,\mathrm{d}\theta_{\mathbb{K}}(u) \in \mathbb{C}.$$

Then, for arbitrary $(x, y) \in \widehat{H} = \mathbb{T}^m \times \widehat{\mathbb{K}}$, a simple calculation shows that

$$\check{h}(x, y) = \sum_{t\in S} \chi_x(t)\,\check{h}_t(y). \tag{14}$$

Fix $y \in \widehat{\mathbb{K}}$ and define $g_y \colon \mathbb{T}^m \longrightarrow \mathbb{C}$ by $x \mapsto g_y(x) = \check{h}(x, y)$. Next, for each $t \in S$, define $\chi_t \colon \mathbb{T}^m \longrightarrow \mathbb{C}$ by $\chi_t(x) = \chi_x(t)$. Note that χ_t simply is $t \in S \subset \mathbb{Z}^m$ viewed as a character on $\mathbb{T}^m = \widehat{\mathbb{Z}^m}$. Then, by equation (14), we have

$$g_y(x) = \sum_{t\in S} \check{h}_t(y)\,\chi_t(x)$$

for each $x \in \mathbb{T}^m$, so that g_y, for any fixed y, is a trigonometric polynomial on $\mathbb{T}^m$. Applying Lemma 4.2 in conjunction with Corollary 4.3, we see that either $g_y \equiv 0$ or the set of zeros of g_y is a null set in $\mathbb{T}^m$. Now, consider

$$U_y := \{x \in \mathbb{T}^m : \check{h}(x,y) \neq 0\} = \{x \in \mathbb{T}^m : g_y(x) \neq 0\}.$$

By the above, we see that either $U_y = \varnothing$ or $\theta_{\mathbb{T}^m}(U_y) = 1$.

Since $\widehat{\mathbb{K}}$ is discrete, we may repeat this process to obtain such a set U_y for each $y \in \widehat{\mathbb{K}}$. Then, for every $y \in \widehat{\mathbb{K}}$, we have either $U_y \times \{y\} = \varnothing$ or $\theta_{\mathbb{T}^m \times \mathbb{K}}(U_y \times \{y\}) = 1$. Next, consider

$$J := \{y \in \widehat{\mathbb{K}} : U_y \neq \varnothing\}.$$

Recall from equation (11) that the set $U = \{z \in \mathbb{T}^m \times \widehat{\mathbb{K}} : \check{h}(z) \neq 0\}$ has finite measure. We have

$$U = \bigcup_{y \in \widehat{\mathbb{K}}} U_y \times \{y\} = \bigcup_{y \in J} U_y \times \{y\}$$

and thus, since $\theta_{\mathbb{T}^m}(U_y) = 1$ for all $y \in J$ but $\theta_{\widehat{H}}(U) < \infty$, we conclude that J is a finite set. Noting that $U_y = \varnothing$ for $y \notin J$, we see that, for any $y \notin J$, we have

$$\check{h}(x,y) = 0 \quad \text{for all } x \in \mathbb{T}^m.$$

This implies $\operatorname{supp}(\check{h}) \subseteq \mathbb{T}^m \times J$ and, reasoning as we previously did for μ, we find that

$$\operatorname{supp}(\widehat{\mu}) \subseteq \Gamma' + F',$$

where $\Gamma' = \curlywedge(\mathbb{T}^m \times \{0\})$ is a lattice in $\widehat{G}$ and $F' = \curlywedge(\{0\} \times J)$ is a finite set, this time referring to the dual CPS, $(\widehat{G}, \widehat{H}, \mathcal{L}^0)$.

To finish the proof, we need to show that $\Gamma' = \Gamma^0$, where the lattice Γ^0 is the annihilator of Γ. Recall that we had $\Gamma = \curlywedge(\{0\} \times \mathbb{K})$ and $\Gamma' = \curlywedge(\mathbb{T}^m \times \{0\})$. Let $y \in \Gamma'$, which means $(y, y^\star) \in \mathcal{L}^0$ together with $y^\star \in \mathbb{T}^m \times \{0\}$. Similarly, $x \in \Gamma$ means $(x, x^\star) \in \mathcal{L}$ with $x^\star \in \{0\} \times \mathbb{K}$. But $(x, x^\star) \in \mathcal{L}$ and $(y, y^\star) \in \mathcal{L}^0$ implies $\chi_y(x) \chi_{y^\star}(x^\star) = 1$.

Now, $x^\star \in \{0\} \times \mathbb{K}$ gives us $x^\star = (0, \xi)$ with $\xi \in \mathbb{K}$, while $y^\star \in \mathbb{T}^m \times \{0\}$ implies the form $y^\star = (\eta, 0)$ with $\eta \in \mathbb{T}^m$. Then, $\chi_{y^\star}(x^\star) = \chi_\eta(0) \chi_0(\xi) = 1$. Employing the previous relation, we are thus left with $\chi_y(x) = 1$, which implies $y \in \Gamma^0$ because $x \in \Gamma$ was arbitrary. Since this works for any $y \in \Gamma'$, we have $\Gamma' \subseteq \Gamma^0$.

To establish the converse inclusion, let $k \in \Gamma^0$ be arbitrary but fixed, so $\chi_k(x) = 1$ for all $x \in \Gamma = \curlywedge(\{0\} \times \mathbb{K})$. We work with the CPS $(G, H, \mathcal{L})$ from above, and write elements of $H = \mathbb{Z}^m \times \mathbb{K}$ as (t, κ). Since $\pi_H(\mathcal{L})$ is dense in H and $\{t\} \times \mathbb{K}$ is open in H for any $t \in \mathbb{Z}^m$, we may conclude

$$\pi_{\mathbb{Z}^m}(\mathcal{L}) = \mathbb{Z}^m,$$

which implies that, for any $t \in \mathbb{Z}^m$, there are elements $x \in G$ and $\kappa \in \mathbb{K}$ such that $(x, t, \kappa) \in \mathcal{L}$.

Define the mapping $\psi \colon \mathbb{Z}^m \longrightarrow \mathbb{S}^1$ by $\psi(t) = \chi_k(x)$, which turns out to be well defined. Indeed, if (x_1, t, κ_1) and (x_2, t, κ_2) are both elements of $\mathcal{L}$, then so is their difference, where we have $(x_1 - x_2)^\star = (0, \kappa_1 - \kappa_2) \in \{0\} \times \mathbb{K}$. But this implies $x_1 - x_2 \in \curlywedge(\{0\} \times \mathbb{K}) = \Gamma$, so $\chi_k(x_1 - x_2) = 1$ due to $k \in \Gamma^0$, and hence $\chi_k(x_1) = \chi_k(x_2)$.

Next, we show that ψ defines a character on $\mathbb{Z}^m$. Since $\mathbb{Z}^m$ carries the discrete topology, ψ is continuous. Now, for any $t_1, t_2 \in \mathbb{Z}^m$, there are $x_1, x_2 \in G$ and $\kappa_1, \kappa_2 \in \mathbb{K}$ so that $(x_i, t_i, \kappa_i) \in \mathcal{L}$, and we get $\psi(t_i) = \chi_k(x_i)$ by definition. Since the sum of two lattice points is again a lattice point, we also get $\psi(t_1 + t_2) = \chi_k(t_1 + t_2) = \chi_k(t_1) \chi_k(t_2) = \psi_k(x_1) \psi_k(x_2)$ as required.

Finally, since ψ is a character on $\mathbb{Z}^m$, there exists an element $\ell \in \mathbb{T}^m$ such that $\psi = \chi_\ell$. We now claim that $(k, -\ell, 0) \in \mathcal{L}^0$. Indeed, for all $(x, t, \kappa) \in \mathcal{L}$, we have

$$\chi_k(x) \chi_{-\ell}(t) \chi_0(\kappa) = \chi_k(x) \overline{\chi_\ell(t)} = \chi_k(x) \overline{\psi(t)} = \chi_k(x) \overline{\chi_k(x)} = 1.$$

This also means that $k \in \lambda\left(\mathbb{T}^m \times \{0\}\right) = \Gamma'$, which completes the argument. $\qquad\square$

For any fixed $y \in J$, with the set J from the proof of Theorem 4.8, the function defined by $x \mapsto \check{h}(x, y)$ is a trigonometric polynomial on $\mathbb{T}^m$. In fact, we can say more.

LEMMA 4.9. *Let $\Gamma \subseteq G$ be a lattice and μ a Fourier-transformable measure on G such that $\operatorname{supp}(\mu) \subseteq \Gamma + F$ and $\operatorname{supp}(\widehat{\mu}) \subseteq \Gamma^0 + F'$ for finite sets $F \subseteq G$ and $F' \subseteq \widehat{G}$. Then, there is a set $\{\tau_1, \ldots, \tau_N\} \subseteq F$ such that one can represent μ as*

$$\mu = \sum_{j=1}^{N} \sum_{x \in \Gamma + \tau_j} P_j(x)\, \delta_x,$$

where each P_j is a trigonometric polynomial on G.

Proof. Given a lattice Γ and a finite set F, there exists a minimal finite set, $F_0 \subseteq F$ say, such that $\Gamma + F_0 = \Gamma + F$. Without loss of generality, we may assume that F and F' are minimal in this sense. Then, applying [45, Remark 5] to the measure $\gamma = \widehat{\mu}$, we gain the existence of a finite measure ν on G such that

$$\mu^\dagger = \widehat{\widehat{\mu}} = \left(\sum_{x \in \Gamma} \sum_{\chi \in F'} \chi(x)\, \delta_x \right) * \nu,$$

where $\mu^\dagger(g) := \mu(g \circ I)$ with $I(x) = -x$, so $(\mu^\dagger)^\dagger = \mu$ and $(\mu * \nu)^\dagger = \mu^\dagger * \nu^\dagger$. Consequently, with $(\delta_x)^\dagger = \delta_{-x}$ and $\chi(-x) = \overline{\chi(x)}$, one gets

$$\mu = \left(\sum_{x \in \Gamma} \sum_{\chi \in F'} \overline{\chi(x)}\, \delta_x \right) * \nu^\dagger.$$

Define the measures $\nu_1 := \sum_{x \in \Gamma + F} \nu^\dagger(\{x\})\delta_x$ and $\nu_2 := \nu^\dagger - \nu_1$. Then,

$$\mu = \left(\sum_{x \in \Gamma} \sum_{\chi \in F'} \overline{\chi(x)}\, \delta_x \right) * \nu_1 + \left(\sum_{x \in \Gamma} \sum_{\chi \in F'} \overline{\chi(x)}\, \delta_x \right) * \nu_2 \; =: \; \mu_1 + \mu_2.$$

Since μ and μ_1 are supported in $\Gamma + F$, we have $\operatorname{supp}(\mu_2) \subseteq \Gamma + F$. Observe that $\nu_2(\{x\}) = 0$ for all $x \in \Gamma + F$ by construction, which implies $\mu_2(\{x\}) = 0$ for all $x \in \Gamma + F$ by a simple calculation. Consequently, $\mu_2 = 0$ as a measure, and we have

$$\mu = \left(\sum_{x \in \Gamma} \sum_{\chi \in F'} \overline{\chi(x)}\, \delta_x \right) * \nu_1,$$

where ν_1 is a finite pure point measure with $\operatorname{supp}(\nu_1) \subseteq \Gamma + F$.

Now, let $F = \{\tau_1, \ldots, \tau_N\}$. Then, recalling $\delta_x * \delta_y = \delta_{x+y}$, we can explicitly write

$$\nu_1 = \sum_{j=1}^{N} \left(\sum_{x \in \Gamma} \nu_1\left(\{x + \tau_j\}\right) \delta_x \right) * \delta_{\tau_j} = \sum_{j=1}^{N} \eta_j * \delta_{\tau_j}$$

with $\eta_j := \sum_{x \in \Gamma} \nu_1(\{x + \tau_j\})\delta_x$. Note that each η_j is supported inside Γ and is finite, meaning that $\sum_{x \in \Gamma} |\eta_j(\{x\})| < \infty$. This finiteness allows us to choose (and change) the order of summation in what follows. Now, decompose μ as $\mu = \sum_{j=1}^{N} \vartheta_j * \delta_{\tau_j}$ with

$$\vartheta_j = \left(\sum_{x \in \Gamma} \sum_{\chi \in F'} \overline{\chi(x)}\, \delta_x \right) * \eta_j.$$

Then, with $\eta_j(\{x\}) = \nu_1(\{x + \tau_j\})$ for $x \in \Gamma$, we obtain

$$\vartheta_j = \sum_{y \in \Gamma} \eta_j(\{y\}) \sum_{x \in \Gamma} \sum_{\chi \in F'} \overline{\chi(x)}\, \delta_{x+y} = \sum_{z \in \Gamma} \sum_{\chi \in F'} \sum_{y \in \Gamma} \eta_j(\{y\})\chi(y)\, \overline{\chi(z)}\, \delta_z,$$

where we have used a change of variable transformation in the lattice Γ together with the relation $\Gamma - \Gamma = \Gamma$. With $a_\chi := \sum_{y \in \Gamma} \eta_j(\{y\})\chi(y)$ for $\chi \in F'$, we get

$$\vartheta_j = \sum_{z \in \Gamma} \sum_{\chi \in F'} a_\chi\, \overline{\chi}(z)\, \delta_z = \sum_{z \in \Gamma} P_j(z + \tau_j)\, \delta_z,$$

where $P_j := \sum_{\chi \in F'} a_\chi\, \chi(\tau_j)\, \overline{\chi}$ is a finite linear combination of characters from $\widehat{G}$, and thus a trigonometric polynomial on G. Consequently, we also have

$$\vartheta_j * \delta_{\tau_j} = \sum_{z \in \Gamma + \tau_j} P_j(z)\, \delta_z.$$

Repeating this construction for each j, we find

$$\mu = \sum_{j=1}^{N} \vartheta_j * \delta_{\tau_j} = \sum_{j=1}^{N} \sum_{z \in \Gamma + \tau_j} P_j(z)\, \delta_z,$$

which was our original claim. $\square$

Now, consider a measure $\mu \in \mathcal{M}^\infty(G)$ with Meyer set support that is Fourier transformable. By [**37**, Theorem 5.9] or [**33**, Theorem 4.9.32], we know that $\widehat{\mu}$ is transformable as well. Then, interchanging the roles of G and $\widehat{G}$ as well as those of μ and $\widehat{\mu}$, the previous result can be derived for $\widehat{\mu}$, this time with $\{\sigma_1, \ldots, \sigma_M\} \subseteq F'$. This leads to the following general statement.

THEOREM 4.10.   *Under the conditions of Theorem* 4.8, *we can write*

$$\mu = \sum_{i=1}^{N} \sum_{x \in \Gamma + \tau_i} P_i(x)\, \delta_x \qquad \text{and} \qquad \widehat{\mu} = \sum_{j=1}^{M} \sum_{y \in \Gamma^0 + \sigma_j} Q_j(y)\, \delta_y,$$

where each P_i, respectively Q_j, is a trigonometric polynomial on the group G, respectively $\widehat{G}$, while N and M are the cardinalities of the minimally chosen finite sets F and F'. $\square$

REMARK 4.11.   Note that the polynomials P_i and Q_j in Theorem 4.10 are not unique. Indeed, if χ is any character that is constant on $\Gamma + \tau_i$ (meaning $\chi \in \Gamma^0$) and if c is the corresponding constant, then $(\overline{c}\chi)P_j$ is another polynomial that agrees with P_j on $\Gamma + \tau_i$. A (somewhat) canonical choice for the polynomials can now be made as follows.

Given polynomials P_i such that the first relation in Theorem 4.10 holds, there exist characters $\chi_1, \ldots, \chi_M$ and coefficients c_{ij} with $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant M$ such that $P_i = \sum_{j=1}^{M} c_{ij}\chi_j$. This is possible because the χ_j can comprise *all* characters which appear in the polynomials P_i, then with the possibility that some of the coefficients c_{ij} vanish.

Now, using $\psi_{\tau_i}(\chi) := \chi(\tau_i)$, a simple computation gives

$$\widehat{\mu} = \left(\sum_{i=1}^{N} \left(\sum_{j=1}^{M} c_{ij}\, \chi_j \right) \left(\delta_\Gamma * \delta_{\tau_i} \right) \right)^{\widehat{}} = \operatorname{dens}(\Gamma) \sum_{i=1}^{N} \left(\sum_{j=1}^{M} c_{ij}\, \delta_{\chi_j} \right) * \left(\psi_{\tau_i}\, \delta_{\Gamma^0} \right)$$

$$= \operatorname{dens}(\Gamma) \sum_{i=1}^{N} \sum_{j=1}^{M} c_{ij}\, \chi_j(\tau_i)\, \psi_{\tau_i} \delta_{\Gamma^0 + \chi_j} = \sum_{j=1}^{M} \left(\sum_{i=1}^{N} \operatorname{dens}(\Gamma)\, c_{ij}\, \chi_j(\tau_i)\, \psi_{\tau_i} \right) \delta_{\Gamma^0 + \chi_j}.$$

We thus see that there are coefficients c_{ij}, translations $\tau_1, \ldots, \tau_N$ and characters $\chi_1, \ldots, \chi_M$ such that we get

$$\mu = \sum_{i=1}^{N} P_i\, \delta_{\Gamma + \tau_i} \quad \text{and} \quad \widehat{\mu} = \sum_{j=1}^{M} Q_j\, \delta_{\Gamma^0 + \chi_j}$$

for the choice

$$P_i = \sum_{j=1}^{M} c_{ij} \chi_j \quad \text{and} \quad Q_j = \operatorname{dens}(\Gamma) \sum_{i=1}^{N} c_{ij}\, \chi_j(\tau_i)\, \psi_{\tau_i},$$

which seems to be a reasonable standardisation.

The results of this section may be applied more generally. Firstly, we consider a transformable measure μ whose Fourier transform may have a continuous part. We refer to the pure point part of a measure ν by ν_{pp}, and use μ_{s} and μ_0 for the strongly almost periodic and null-weakly almost periodic parts of a weakly almost periodic measure μ; see [**3**, Section 4.10] for definitions and properties around the underlying Eberlein decomposition, $\mu = \mu_{\mathsf{s}} + \mu_0$.

COROLLARY 4.12. *Let μ be a translation-bounded, Fourier-transformable measure on G such that $\operatorname{supp}(\mu)$ is a subset of a Meyer set and that $\operatorname{supp}(\widehat{\mu}_{\mathsf{pp}}) \neq \varnothing$ is $\mathcal{B}$-sparse for some van Hove sequence $\mathcal{B}$ in $\widehat{G}$. Then, there exists a lattice Γ together with finite sets $F = \{\tau_1, \ldots, \tau_N\}$ and F_0 in G such that*

$$\mu_{\mathsf{s}} = \sum_{i=1}^{N} \sum_{x \in \Gamma + \tau_i} P_i(x)\, \delta_x \quad \text{and} \quad \operatorname{supp}(\mu_0) \subseteq \Gamma + F_0.$$

Moreover, one has $\operatorname{supp}(\widehat{\mu}_{\mathsf{pp}}) \subseteq \Gamma^0 + F'$ for some finite set $F' \subseteq \widehat{G}$.

Proof. Since μ has Meyer set support, then so does μ_{s}, as follows from [**42, 45**]. Now, the first part of the claim becomes a consequence of Theorem 4.10. For the second part, we know that $\operatorname{supp}(\mu)$ can be embedded inside a model set $\Lambda \subseteq G$ with compact window. Then, $\operatorname{supp}(\mu_0), \operatorname{supp}(\mu_{\mathsf{s}}) \subseteq \Lambda$. Since μ_{s} is strongly almost periodic and non-trivial, $\operatorname{supp}(\mu_{\mathsf{s}})$ is relatively dense. So, applying [**43**, Lemma 5.5.1], we find a finite set, $F_1 \subseteq G$, such that

$$\Lambda \subseteq \operatorname{supp}(\mu_{\mathsf{s}}) + F_1.$$

From the first part, $\operatorname{supp}(\mu_{\mathsf{s}}) \subseteq \Gamma + F$, so we have

$$\operatorname{supp}(\mu_0) \subseteq \Lambda \subseteq \operatorname{supp}(\mu_{\mathsf{s}}) + F_1 \subseteq (\Gamma + F) + F_1.$$

The last claim follows from Theorem 4.8. □

In general, a measure need not (and generally will not) be Fourier transformable in order to possess an autocorrelation and a diffraction. We can extend these results to the class of weakly

almost periodic measures; see [**3**, Section 4.10] for definitions and [**24**, **33**] for details. This class contains all translation-bounded, Fourier-transformable measures [**33**]. Next, we require the concept of Fourier–Bohr coefficients; compare [**24**, Definition 2.18].

DEFINITION 4.13.   Let μ be a weakly almost periodic measure on a group G. The *Fourier–Bohr coefficients* of μ are defined, for each $\chi \in \widehat{G}$, by

$$c_\chi(\mu) \,:=\, \mathbb{M}\big(\overline{\chi}\mu\big) = \lim_{n \to \infty} \frac{\big(\overline{\chi}\mu\big)(A_n)}{\mathrm{vol}(A_n)},$$

where $\mathcal{A} = \{A_n\}$ is any van Hove sequence in G.

Note that the existence of the Fourier–Bohr coefficients, in the above form, follows from [**33**, Lemma 4.10.7], and their values do not depend on the van Hove sequence chosen; see also [**12**].

COROLLARY 4.14.   *Let μ be a weakly almost periodic measure on G such that $\mathrm{supp}(\mu)$ is contained in a Meyer set. If $\mu_\mathsf{s} \neq 0$ and the set $S := \{\chi \in \widehat{G} : c_\chi(\mu) \neq 0\}$ is $\mathcal{B}$-sparse in $\widehat{G}$ for some van Hove sequence $\mathcal{B}$ in $\widehat{G}$, there exists a lattice Γ in G together with a finite set $F \subseteq G$ such that*

$$\mathrm{supp}(\mu_\mathsf{s}), \, \mathrm{supp}(\mu_0) \,\subseteq\, \Gamma + F,$$

together with $S \subseteq \Gamma^0 + F'$ for some finite set $F' \subseteq \widehat{G}$.

Proof.   By [**24**, Theorem 7.6], μ has the unique autocorrelation γ_μ and pure point diffraction with support S. Since γ_μ is translation bounded, transformable and supported in the Meyer set $\mathrm{supp}(\mu) - \mathrm{supp}(\mu)$, we may apply Corollary 4.12 to γ_μ to obtain

$$\mathrm{supp}(\gamma_\mu) \,\subseteq\, \Gamma + F.$$

Next, let Λ be any Meyer set in G such that $\mathrm{supp}(\mu) \subseteq \Lambda$. Then, $\mathrm{supp}(\gamma_\mu) \subseteq \Lambda - \Lambda$, and [**43**, Lemma 5.5.1] guarantees the existence of a finite set F_1 such that

$$\Lambda - \Lambda \,\subseteq\, \mathrm{supp}(\gamma_\mu) + F_1.$$

Fix some $s \in \Lambda$. Then, we get

$$\mathrm{supp}(\mu) \,\subseteq\, \Lambda \,\subseteq\, (\Lambda - \Lambda) + s \,\subseteq\, \mathrm{supp}(\gamma_\mu) + F_1 + s \,\subseteq\, \Gamma + F + F_1 + s.$$

The claim now follows from Corollary 4.12. $\square$

5.   *Sparseness of positive definite measures*

This is the moment where we need to recall further concepts of almost periodicity and study their consequences and relations in the context of measures with sparse supports. Since we are interested in measures with pure point Fourier transform, it is natural to begin with the class of strongly almost periodic measures, or $\mathcal{SAP}$-measures for short, which were defined in Remark 3.12 and have been used several times already.

5.1.   *Positive definite measures with uniformly discrete support*

Here, we consider positive definite, pure point $\mathcal{SAP}$-measures, where we are able to tighten the type of almost periodicity when the support is uniformly discrete. In particular, we will show that any such measure is also sup-almost periodic.

For any pure point measure ω on G, one can consider

$$\|\omega\|_\infty := \sup_{x \in G} |\omega|\big(\{x\}\big),$$

which defines a norm on $\mathcal{M}_{\mathsf{pp}}^\infty$, the space of translation-bounded pure point measures. Now, recall from [**43**, Definition 5.3.4] that a measure μ from this class is called *sup-almost periodic* if the set $P_\varepsilon := \{t \in G : \|\mu - T_t\mu\|_\infty < \varepsilon\}$ is relatively dense for every $\varepsilon > 0$.

THEOREM 5.1. *Let $0 \neq \mu = \sum_{x \in \Lambda} a(x)\,\delta_x$ be a positive definite, strongly almost periodic (hence also translation-bounded) measure on G. If Λ is uniformly discrete, the following statements hold.*

(1) *The set $B_\varepsilon := \{x \in \Lambda : \mathrm{Re}\big(a(x)\big) \geqslant a(0) - \varepsilon\}$ is relatively dense for every $\varepsilon > 0$.*
(2) *The measure μ is sup-almost periodic.*
(3) *The measure μ is norm-almost periodic.*
(4) *There is a CPS $(G, H, \mathcal{L})$ and some $h \in C_0(H)$ such that $\mu = \omega_h$.*

Proof. First, let us note that, since μ is a positive definite, pure point measure, the function a on G given by $a(x) = \mu(\{x\})$ is positive definite and supported inside Λ by [**23**, Proposition 2.4]. Therefore, we have $a(0) \in \mathbb{R}$, and $|a(x)| \leqslant a(0)$ holds for all $x \in \Lambda$. Consequently[†], we also have $0 \in \Lambda$ (since $\mu \neq 0$) and $|\mathrm{Re}\big(a(x)\big)| \leqslant a(0)$ for all $x \in \Lambda$.

Next, as Λ is uniformly discrete, we can find a relatively compact open neighbourhood U of 0 such that $(x + U) \cap (y + U) \neq \varnothing$ for any $x, y \in \Lambda$ implies $x = y$.

Let $g \in C_{\mathsf{c}}(G)$ be a function with values in $[0,1]$ such that $g(0) = 1$, $g(-x) = g(x)$ and $\mathrm{supp}(g) \subseteq U$. The previous property implies that any translate of $\mathrm{supp}(g)$ meets Λ in at most one point. Therefore, if $(\delta_\Lambda * g)(z) \neq 0$ for some $z \in G$, there exists a point $y \in \Lambda$ so that

$$\big(\mu * g\big)(z) = g(z - y)\, a(y). \tag{15}$$

As μ is strongly almost periodic, $\mu * g$ is uniformly (or Bohr) almost periodic, and the set

$$V_\varepsilon := \big\{t \in G : \|T_t\mu * g - \mu * g\|_\infty < \varepsilon\big\}$$

is relatively dense for each $\varepsilon > 0$. We claim that

$$V_\varepsilon \subseteq B_\varepsilon + U,$$

with B_ε as defined in statement (1) of the theorem. As U is a relatively compact subset of G, once we prove this claim, the relative denseness of V_ε implies the relative denseness of B_ε.

To proceed, let $t \in V_\varepsilon$. Then, as $\|T_t\mu * g - \mu * g\|_\infty < \varepsilon$, we certainly have the inequality $|\big(T_t\mu * g\big)(t) - \big(\mu * g\big)(t)| < \varepsilon$. By equation (5), this is equivalent to

$$\big|\big(\mu * g\big)(0) - \big(\mu * g\big)(t)\big| < \varepsilon.$$

Since $B_\varepsilon \subseteq B_{\varepsilon'}$ for $\varepsilon \leqslant \varepsilon'$, it suffices to show relative denseness of the sets B_ε for all sufficiently small $\varepsilon > 0$. Thus, we now assume

$$\big(\mu * g\big)(0) = a(0) > \varepsilon > 0 \tag{16}$$

and hence $\big(\mu * g\big)(t) \neq 0$. Therefore, by equation (15), there exists a unique $y \in \Lambda$ so that

$$\big(\mu * g\big)(t) = g(t - y)\, a(y). \tag{17}$$

[†]Note that, since Λ is uniformly discrete and the coefficients $a(x)$ are bounded by $a(0)$, we also obtain the translation-boundedness of μ directly from positive definiteness, without reference to strong almost periodicity.

With the relations from equations (16) and (17), we obtain the following chain of implications,

$$\big|(\mu * g)(0) - (\mu * g)(t)\big| < \varepsilon \quad \Rightarrow \quad \big|\mathrm{Re}\big(a(0) - g(t-y)a(y)\big)\big| < \varepsilon$$

$$\Rightarrow \quad \big|a(0) - \mathrm{Re}\big(g(t-y)a(y)\big)\big| < \varepsilon \quad \Rightarrow \quad \mathrm{Re}\big(g(t-y)a(y)\big) \geqslant a(0) - \varepsilon > 0$$

$$\Rightarrow \quad \mathrm{Re}\big(a(y)\big) \geqslant g(t-y)\,\mathrm{Re}\big(a(y)\big) \geqslant a(0) - \varepsilon,$$

where the last step follows from $0 \leqslant g(t-y) \leqslant 1$ in conjunction with the observation that $g(t-y)a(y) = (\mu * g)(t) \neq 0$. This implies $y \in B_\varepsilon$ together with $t - y \in \mathrm{supp}(g) \subseteq U$. Thus

$$t = y + (t - y) \in B_\varepsilon + U,$$

and we are done with the first claim.

Next, consider claim (2). As a is positive definite, Krein's inequality [**23**, Corollary 2.5] implies

$$|a(x+t) - a(x)|^2 \leqslant 2a(0)[a(0) - \mathrm{Re}(a(t))].$$

Therefore, one has

$$\big\|\mu - T_t\mu\big\|_\infty^2 \leqslant 2a(0)[a(0) - \mathrm{Re}(a(t))].$$

This shows that

$$B_{\frac{\varepsilon^2}{2a(0)}} \subseteq P_\varepsilon,$$

where the P_ε are the sets of ε-almost periods which define sup-almost periodicity.

By [**43**, Lemma 5.3.6], sup-almost periodicity implies norm-almost periodicity in this case, as Λ is uniformly discrete. Finally, claim (4) is [**43**, Theorem 5.4.2]; a stronger version will be given in Theorem 5.3. $\qquad\square$

Recall that positive definite measures are Fourier transformable [**7**, Theorem 4.5], and that strong almost periodicity then implies the Fourier transform to be a pure point measure; see [**46**, Corollary 4.10.13]. Thus, under the conditions of Theorem 5.1, μ and $\widehat{\mu}$ are both pure point. However, for μ to be doubly sparse, we need to add a condition on the support of $\widehat{\mu}$.

Indeed, the autocorrelation measure of the Fibonacci chain, see [**2**] for details, provides an example of a positive definite $\mathcal{SAP}$-measure, $\mu \in \mathcal{M}^\infty(\mathbb{R})$, with Meyer set support such that $\widehat{\mu}$ is a positive, pure point measure on $\mathbb{R}$ with *dense* support, and the same situation applies to the autocorrelation measures of aperiodic regular model sets in general; see also [**36**] for some interesting extensions beyond bounded windows.

At this point, it seems worthwhile to state the following improvement of Theorem 5.1 for the case that the support of μ is FLC, where we do not need second countability of G. We refer to [**10, 16**] for Dirac combs with Delone set support.

THEOREM 5.2. *Let μ be a positive definite, pure point measure on the metrisable LCAG G, and assume that μ has FLC support and sparse FBS. Then, one has*

$$\mu = \sum_{i=1}^{N} \sum_{x \in \Gamma + \tau_i} P_i(x)\delta_x \quad \text{and} \quad \widehat{\mu} = \sum_{j=1}^{M} \sum_{y \in \Gamma^0 + \sigma_j} Q_j(y)\delta_y$$

for some lattice $\Gamma \subseteq G$ and some trigonometric polynomials P_i on G and Q_j on $\widehat{G}$.

Proof. Recall first that $\mathrm{supp}(\mu)$ being FLC means that $\mathrm{supp}(\mu) - \mathrm{supp}(\mu)$ is locally finite. Then, since μ is positive definite, we have $|\mu(\{x\})| \leqslant \mu(\{0\})$ for all $x \in G$ as in the proof of Theorem 5.1, and μ is thus translation bounded. The assumption that μ has a sparse FBS

implies that μ is transformable as a measure and that $\widehat{\mu}$ is pure point, hence μ is strongly almost periodic.

Now, by Theorem 5.1, μ is norm-almost periodic. Invoking the implication (i) $\Rightarrow$ (vi) from [**43**, Theorem 5.5.2], we see that there exists a CPS $(G, H, \mathcal{L})$ and a function $h \in C_{\mathsf{c}}(H)$ such that $\mu = \omega_h$. In particular, $\mathrm{supp}(\mu) \subseteq \curlywedge\big(\mathrm{supp}(h)\big)$, so $\mathrm{supp}(\mu)$ is a subset of a Meyer set.

The claim now follows from Theorem 4.10. $\qquad\square$

5.2. *Doubly sparse sup-almost periodic measures*

Our aim here is to characterise positive definite measures μ with uniformly discrete support and sparse FBS. The key to this characterisation is the sup-almost periodicity of such a measure as obtained above.

In fact, given a sup-almost periodic measure μ, our results require only weak uniform discreteness of its support. In line with Theorem 5.1, note that, when the support of μ is weakly uniformly discrete, μ is sup-almost periodic if and only if it is norm-almost periodic [**43**, Lemma 5.3.6]. This means that all measures we consider in this section are actually norm-almost periodic.

THEOREM 5.3. *Let* $0 \neq \mu = \sum_{x \in \Lambda} a(x)\, \delta_x$ *be a translation-bounded, sup-almost periodic measure on* G. *If* μ *has weakly uniformly discrete support,* Λ, *and sparse FBS, there is a CPS* $(G, H, \mathcal{L})$ *and some* $h \in C_0(H)$ *such that*

(1) $\mu = \omega_h$;
(2) $h \in L^1(H)$ *with support of finite measure;*
(3) $\widehat{\mu} = \mathrm{dens}(\mathcal{L})\, \omega_{\widecheck{h}}$;
(4) $\widecheck{h} \in L^1\big(\widehat{H}\big) \cap C_0\big(\widehat{H}\big)$, *with support of finite measure;*
(5) H *has an open and closed compact subgroup,* $\mathbb{K}$;
(6) $\widehat{H}$ *has an open and closed compact subgroup.*

Proof. Since μ is sup-almost periodic, [**43**, Theorem 5.4.2] implies the existence of a CPS $(G, H, \mathcal{L})$, with the group H metrisable, and that of a function $h \in C_0(H)$ such that $\mu = \omega_h$. Let $U = \{y \in H : h(y) \neq 0\}$ as before. Since $\mathrm{supp}(\mu)$ is weakly uniformly discrete, $\theta_H(U)$ is finite, by Corollary 3.10. Then, $h \in L^1(H)$ by Fact 3.15, and claims (1) and (2) are verified.

Let $(\widehat{G}, \widehat{H}, \mathcal{L}^0)$ be the dual CPS. To show claim (3), we use a function of compact support to construct a measure whose Fourier–Bohr coefficients are 'close' to those of μ. Fix a van Hove sequence $\mathcal{A}$ in G, set $d = \overline{\mathrm{dens}}_{\mathcal{A}}(\Lambda)$, which is finite because Λ is weakly uniformly discrete, and fix an arbitrary $\varepsilon > 0$. Since $h \in C_0(H)$, there exists a compact set $K_0 \subseteq H$ such that

$$|h(y)| \;<\; \frac{\varepsilon}{d + \mathrm{dens}(\mathcal{L})\theta_H(U)} \;=:\; \varepsilon_1$$

holds for every $y \notin K_0$. We may choose a relatively compact open set $V \supseteq K_0$ and an $f \in C_{\mathsf{c}}(H)$ such that $1_{K_0} \leqslant f \leqslant 1_V$. Then, setting $g := fh$, we have

$$|g(y)| \;\leqslant\; |h(y)| \quad \text{for all } y \in H.$$

Further, for $y \in K_0$, we have $h(y) = g(y)$ and, for $y \notin K_0$,

$$|h(y) - g(y)| = |h(y)(1 - f(y))| \;\leqslant\; |h(y)| \;<\; \varepsilon_1.$$

Consequently, we get

$$\|h - g\|_\infty \;<\; \varepsilon_1.$$

Now, consider the measure ω_g. In general, we may not assume that ω_g is transformable but, since $g \in C_c(H)$, ω_g is strongly almost periodic by [**22**, Theorem 3.1]. Then, by Definition 4.13, we may consider the Fourier–Bohr coefficients of ω_g,

$$c_\chi(\omega_g) \,=\, \lim_{n\to\infty} \frac{(\overline{\chi}\omega_g)(A_n)}{\mathrm{vol}(A_n)},$$

for each $\chi \in \widehat{G}$. Now, [**22**, Theorem 3.3] implies

$$c_\chi(\omega_g) = \mathrm{dens}(\mathcal{L}) \int_H \chi^\star(t) g(t)\,\mathrm{d}t = \mathrm{dens}(\mathcal{L})\, \breve{g}(\chi^\star)$$

for all $\chi \in \pi_{\widehat{G}}(\mathcal{L}^0)$, and $c_\chi(\omega_g) = 0$ otherwise.

Since $|g| \leqslant |h|$, we have $\mathrm{supp}(\omega_g) \subseteq \Lambda$, while $\|h - g\|_\infty < \varepsilon_1$ implies $|\omega_h - \omega_g|(t) < \varepsilon_1$ for all $t \in \Lambda$. Finally, since μ is Fourier transformable, we have $\widehat{\mu}(\{\chi\}) = \mathbb{M}(\overline{\chi}\mu)$ for all $\chi \in \widehat{G}$, by an application of [**37**, Proposition 3.14]. For all $\chi \in \widehat{G}$, this gives us

$$\begin{aligned}
\left|\widehat{\mu}(\{\chi\}) - c_\chi(\omega_g)\right| \,&=\, \lim_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \left| \int_{A_n} \chi(t)\,\mathrm{d}(\mu - \omega_g) \right| \\[2mm]
&\leqslant\, \limsup_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \sum_{t\in\Lambda\cap A_n} \left| \chi(t)\big(\omega_h(\{t\}) - \omega_g(\{t\})\big) \right| \\[2mm]
&<\, \limsup_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \sum_{t\in\Lambda\cap A_n} \varepsilon_1 \,=\, d\varepsilon_1.
\end{aligned}$$

For $\chi \notin \pi_{\widehat{G}}(\mathcal{L}^0)$, since $c_\chi(w_g) = 0$ (and $d\varepsilon_1$ is a product of fixed constants with an arbitrary $\varepsilon > 0$, so that $d\varepsilon_1 = \mathcal{O}(\varepsilon)$), we thus have

$$\widehat{\mu}(\{\chi\}) = 0, \quad \text{for all } \chi \notin \pi_{\widehat{G}}(\mathcal{L}^0). \tag{18}$$

Let $\chi \in \pi_{\widehat{G}}(\mathcal{L}^0)$. Then, from above,

$$\left|\widehat{\mu}(\{\chi\}) - \mathrm{dens}(\mathcal{L})\breve{g}(\chi^\star)\right| \,<\, d\varepsilon_1.$$

Now, $|g| \leqslant |h|$ implies $0 = h(y) = g(y)$ for all $y \notin U$, and thus

$$\begin{aligned}
\left|\breve{g}(\chi^\star) - \breve{h}(\chi^\star)\right| \,&=\, \left| \int_H \chi^\star(y)(g(y) - h(y))\,\mathrm{d}y \right| = \left| \int_U \chi^\star(y)(g(y) - h(y))\,\mathrm{d}y \right| \\[2mm]
&\leqslant\, \int_U |g(y) - h(y)|\,\mathrm{d}y \,<\, \theta_H(U)\,\varepsilon_1.
\end{aligned}$$

Multiplying this with $\mathrm{dens}(\mathcal{L})$ and combining it with the previous inequality, we obtain

$$\left|\widehat{\mu}(\{\chi\}) - \mathrm{dens}(\mathcal{L})\,\breve{h}(\chi^\star)\right| \,<\, \big(d + \mathrm{dens}(\mathcal{L})\,\theta_H(U)\big)\varepsilon_1 \,=\, \varepsilon,$$

and since $\varepsilon > 0$ was arbitrary, we have

$$\widehat{\mu}(\{\chi\}) = \mathrm{dens}(\mathcal{L})\,\breve{h}(\chi^\star) \quad \text{for all } \chi \in \pi_{\widehat{G}}(\mathcal{L}^0). \tag{19}$$

Combining (18) and (19) gives claim (3), and as μ has sparse FBS, claim (4) is direct from Corollary 3.10 (and [**38**, Theorem 1.2.4]).

To see claim (5), we reason as in Remark 4.6. Since both h and $\breve{h}$ have finite-measure support, the QUP fails for H. Then, by [**14**, Theorem 1], the identity component of H must be compact. Recall that H, as an LCAG, has an open and closed subgroup of the form $\mathbb{R}^d \times \mathbb{K}$. Since the identity component of H is compact, we have $d = 0$, and hence H has an open and closed subgroup $\mathbb{K}$ which is compact. The same argument applied to the dual group $\widehat{H}$ verifies claim (6), and we are done. $\qquad\square$

REMARK 5.4. Note that we have $\mu = \omega_h$ together with $\widehat{\mu} = \operatorname{dens}(\mathcal{L})\,\omega_{\breve{h}}$ in Theorem 5.3, from claims (1) and (3). Let us emphasise that this actually is a PSF for the lattice $\mathcal{L}^0$. Indeed, with

$$K_2(G) := \operatorname{span}_{\mathbb{C}}\{f * g : f, g \in C_{\mathsf{c}}(G)\},$$

the second relation means $\langle \omega_h, g \rangle = \operatorname{dens}(\mathcal{L})\langle \omega_{\breve{h}}, \breve{g} \rangle$ for all $g \in K_2(G)$; see [33, Section 4.9] for background. By definition, one has

$$\langle \omega_h, g \rangle = \sum_{x \in L} h(x^\star)\,\delta_x(g) = \sum_{(x,x^\star) \in \mathcal{L}} g(x)\,h(x^\star) = \delta_{\mathcal{L}}(g \otimes h),$$

while the other side contains

$$\big\langle \omega_{\breve{h}}, \breve{g} \big\rangle = \sum_{(\chi, \chi^\star) \in \mathcal{L}^0} \breve{h}(\chi^\star)\,\breve{g}(\chi) = \delta_{\mathcal{L}^0}\big(\breve{g} \otimes \breve{h}\big).$$

Consequently, $\widehat{\omega_h} = \operatorname{dens}(\mathcal{L})\,\omega_{\breve{h}}$ simply means that, for all $g \in K_2(G)$, we have

$$\langle \delta_{\mathcal{L}}, g \otimes h \rangle = \operatorname{dens}(\mathcal{L})\,\big\langle \delta_{\mathcal{L}^0}, \widetilde{g \otimes h} \big\rangle,$$

which justifies the interpretation; see [36] for related results.

The last proof, in conjunction with [24, Theorem 7.6], has a direct consequence as follows.

COROLLARY 5.5. *Under the conditions of Theorem 5.3, the measure $\mu = \omega_h$ has a unique autocorrelation measure, namely $\gamma = \omega_h \circledast \widetilde{\omega_h} = \operatorname{dens}(\mathcal{L})\omega_{h*\widetilde{h}}$, and the corresponding diffraction measure is $\widehat{\gamma} = \operatorname{dens}(\mathcal{L})^2 \omega_{|\breve{h}|^2}$.* □

REMARK 5.6. Since $\mathbb{K}$ is open and closed in H, the factor group $H/\mathbb{K}$ is discrete. Therefore,

$$\mathbb{K}^0 := \{\chi \in \widehat{H} : \chi \equiv 1 \text{ on } \mathbb{K}\} \simeq \widehat{H/\mathbb{K}}$$

is closed and compact.

PROPOSITION 5.7. *Under the conditions of Theorem 5.3, the measure μ may be approximated in any K-norm $\|.\|_K$ for measures by strongly almost periodic measures μ_n that are supported inside sets $\Gamma + F_n$, where Γ is a lattice in G and the $F_n \subseteq G$ are finite. Moreover, the Fourier–Bohr coefficients of the measures μ_n converge to those of μ.*

Proof. We employ the setting of the proof of Theorem 5.3. Fix a compact set $K \subseteq G$ and set $\Gamma := \lambda(\mathbb{K})$. We will now construct an increasing sequence of finite sets $F_n \subseteq G$ such that $\mu_n = \mu|_{\Gamma + F_n} \in \mathcal{SAP}(G)$ and $\|\mu_n - \mu\|_K \leqslant \frac{1}{n}$, where $\|.\|_K$ is defined as in (6).

Since Λ is weakly uniformly discrete, there exists an $N \in \mathbb{N}$ such that, for all $t \in G$,

$$\operatorname{card}((t + \Lambda) \cap K) < N. \tag{20}$$

As usual, let $U = \{y \in H : h(y) \neq 0\}$, where $h \in C_0(H)$. For each $n \in \mathbb{N}$, there exists a compact set $W_n \subseteq U$ such that

$$|h(x)| < \frac{1}{nN} \quad \text{for all } x \notin W_n. \tag{21}$$

Since $\pi_H(\mathcal{L})$ is dense in H and $\mathbb{K}$ is open in H, we have $\pi_H(\mathcal{L}) + \mathbb{K} = H$. By the compactness of W_n, we can find a finite set $F_n^\star \subseteq \pi_H(\mathcal{L})$ such that $W_n \subseteq F_n^\star + \mathbb{K}$. Then, let

$$F_n := \lambda(F_n^\star) \quad \text{and} \quad h_n := h 1_{F_n^\star + \mathbb{K}}.$$

Since $\mathbb{K}$ is open and closed in H and $F_n^\star$ is finite, $F_n^\star + \mathbb{K}$ is also open and closed in H. Consequently, h_n is continuous. Moreover, as $F_n^\star + \mathbb{K}$ is compact, we have $h_n \in C_{\mathsf{c}}(H)$. Setting

$$\mu_n := \omega_{h_n},$$

we have $\mu_n \in \mathcal{SAP}(G)$ by [**22**, Theorem 3.1] and $\mu_n = \mu|_{\Gamma + F_n}$ by construction. Now, (21) ensures that $|h(y) - h_n(y)| < \frac{1}{nN}$ for all $y \in H$ and thus that $\big| \mu(\{x\}) - \mu_n(\{x\}) \big| < \frac{1}{nN}$ for all $x \in G$. It is clear that $\mathrm{supp}(\mu - \mu_n) \subseteq \Lambda$. Consequently, via (20), we see that

$$\| \mu - \mu_n \|_K \, < \, \frac{1}{n}.$$

Note that the sets W_n, and thus $F_n^\star$ and F_n, may be chosen to be increasing, as claimed.

Finally, for $\chi \in \widehat{G}$, observe that

$$\big| \widehat{\mu}(\{\chi\}) - c_\chi(\mu_n) \big| = \mathbb{M}\big(\overline{\chi}(\mu - \mu_n) \big) \leqslant C \| \mu - \mu_n \|_K,$$

for some $C > 0$, which verifies the convergence of the Fourier–Bohr coefficients. $\qquad\square$

Comparing the results of this section with those of Section 4, we see that, while sup-almost periodicity of a pure point measure μ enables its representation as a model comb, the weight function h has compact support if and only if $\mathrm{supp}(\mu)$ is FLC (or Meyer); see [**43**, Theorem 5.5.2]. This makes the calculations for doubly sparse sup-almost periodic measures with only uniformly discrete support a little more delicate. Nevertheless, we obtain almost everything that we did in Section 4, apart from the support of μ being crystallographic, and even this we 'almost' get.

Now, we combine these results with Theorem 5.1 to obtain the characterisation for positive definite measures with uniformly discrete support.

COROLLARY 5.8. *Let $0 \neq \mu = \sum_{x \in \Lambda} a(x)\, \delta_x$ be a positive definite measure with uniformly discrete support, Λ, and sparse FBS. Then, the conclusions of Theorem 5.3, Corollary 5.5 and Proposition 5.7 hold.*

Proof. By the argument used in the proof of Theorem 5.2, μ is translation bounded. From Remark 3.12, we see that μ is an $\mathcal{SAP}$-measure. Now, Theorem 5.1 implies that μ is sup-almost periodic, and the rest is clear. $\qquad\square$

There is also a Fourier-dual version, which we can formulate as follows.

COROLLARY 5.9. *Let $0 \neq \mu = \sum_{x \in \Lambda} a(x)\, \delta_x$ be a positive measure with sparse support, Λ. Further, assume that μ is Fourier transformable and that $\mathrm{supp}(\widehat{\mu})$ is uniformly discrete. Then, the conclusions of Theorem 5.3, Corollary 5.5 and Proposition 5.7 hold.* $\qquad\square$

6. Specific results for $G = \mathbb{R}^d$

For arguments in $G = \mathbb{R}^d$, the usual framework is that of tempered distributions. We use $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ to denote the spaces of Schwartz functions and tempered distributions on $\mathbb{R}^d$, respectively, and $\langle T, \varphi \rangle := T(\varphi)$ for the pairing of a distribution and a test function. The distributional Fourier transform of $\mu \in \mathcal{S}'(\mathbb{R}^d)$ is the distribution $\nu \in \mathcal{S}'(\mathbb{R}^d)$ such that $\langle \nu, \varphi \rangle = \langle \mu, \widehat{\varphi} \rangle$ for all test functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Recall that a translation-bounded measure on $\mathbb{R}^d$ is always a tempered distribution, referred to as a tempered measure; see [**47**] for general background, and [**46**] for further notions, such as translation-boundedness for tempered distributions.

In previous sections, we have assumed our measures to be translation bounded and Fourier transformable. The connection between the distributional Fourier transform and Fourier transformability as an unbounded Radon measure, as we have considered, was clarified in [**44**], where it was shown that a measure μ on $\mathbb{R}^d$ is Fourier transformable as a measure if and only if it is tempered and its distributional Fourier transform is a translation-bounded measure. Thus, in the Euclidean setting, a measure μ is translation bounded and Fourier transformable if and only if its distributional Fourier transform ν is a translation-bounded and Fourier-transformable measure. We begin this section by establishing some sufficient conditions for transformability and translation-boundedness.

In [**46**], the notions of weak and strong almost periodicity for tempered distributions were defined, and it was shown that these definitions coincide with the classical ones for the class of translation-bounded measures on $\mathbb{R}^d$.

LEMMA 6.1. *Let $\mu \in \mathcal{S}'(\mathbb{R}^d)$ be a measure with uniformly discrete support that is weakly almost periodic as a tempered distribution. Then, μ is a translation-bounded measure and thus $\mu \in \mathcal{WAP}(\mathbb{R}^d)$.*

Proof. By [**46**, Remark 5.1], $\mu = \sum_{x \in \Lambda} a(x)\delta_x$ is translation bounded as a tempered distribution, meaning that $\mu * f \in C_{\mathsf{u}}(\mathbb{R}^d)$ for all $f \in \mathcal{S}(\mathbb{R}^d)$. Now, since $\operatorname{supp}(\mu)$ is uniformly discrete, we may choose an open neighbourhood U of 0 such that $(x + U) \cap (y + U) = \varnothing$ for all $x, y \in \operatorname{supp}(\mu)$ with $x \neq y$. Select a function $f \in C_{\mathsf{c}}^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}(f) \subset U$ and $f(0) = 1$. Via a simple calculation, one can verify that

$$\big|\mu * f\big|(x) = \big(|\mu| * |f|\big)(x) = \big|a(x)\big|$$

holds for all $x \in \operatorname{supp}(\mu)$; compare [**43**, Lemma 5.8.3]. Then, there exists a $C > 0$ such that

$$|a(x)| \leqslant C \quad \text{for all } x \in \operatorname{supp}(\mu)$$

and thus, since $\operatorname{supp}(\mu)$ is uniformly discrete, μ is a translation-bounded measure. Hence, by [**46**, Theorem 5.3], μ is also weakly almost periodic as a measure. □

REMARK 6.2. In particular, any tempered measure, μ, whose distributional Fourier transform is a measure, is a weakly almost periodic tempered distribution [**46**, Theorem 5.1]. Thus, if μ also has uniformly discrete support, the conclusions of Lemma 6.1 hold.

The following generalises [**25**, Lemma 2], which assumes that both $\Lambda = \operatorname{supp}(\mu)$ and $S = \operatorname{supp}(\nu)$ are uniformly discrete, and [**26**, Lemma 3.1], which shows only the translation-boundedness of μ.

LEMMA 6.3. *Let $\Lambda \subset \mathbb{R}^d$ be uniformly discrete, let $S \subset \mathbb{R}^d$ be weakly uniformly discrete, and consider $\mu, \nu \in \mathcal{S}'(\mathbb{R}^d)$ as given by*

$$\mu = \sum_{x \in \Lambda} a(x)\,\delta_x \quad \text{and} \quad \nu = \sum_{y \in S} b(y)\,\delta_y. \tag{22}$$

Now, if ν is the distributional Fourier transform of μ, then μ and ν are translation-bounded measures that are Fourier transformable as measures and, as such, satisfy $\widehat{\mu} = \nu$.

Proof. As noted above, the translation-boundedness of μ is a consequence of Lemma 6.1. To see the translation-boundedness of ν, it suffices to show that the set of coefficients, namely $\{b(y) : y \in S\}$, is bounded.

Let $y \in S$ be arbitrary but fixed, and select $c \in C_{\mathsf{c}}^\infty(\mathbb{R}^d)$ with $\widehat{c}(y) = 1$ and $\int_{\mathbb{R}^d} |c(x)|\,\mathrm{d}x \leqslant 2$, which is clearly possible. From [**46**, Proposition 4.1], we know that the function $g = \mu * c$

is bounded and uniformly continuous, and thus defines a regular tempered distribution. Its distributional Fourier transform, $\nu\widehat{c} =: \rho$, is a finite measure. Now, by [**46**, Theorem 7.2], we have

$$\rho(\{y\}) = \mathbb{M}\big(e^{-2\pi i y(.)}g\big).$$

But this gives us

$$|b(y)| = |\widehat{c}(y)b(y)| = |\rho(\{y\})| = \big|\mathbb{M}\big(e^{-2\pi i y(\cdot)}g\big)\big| \leqslant \mathbb{M}\big(|e^{-2\pi i y(\cdot)}g|\big) = \mathbb{M}\big(|\mu * c|\big)$$

$$= \lim_{n\to\infty} \frac{1}{(2n)^d} \int_{[-n,n]^d} |\mu * c|(y)\,\mathrm{d}y \leqslant \limsup_{n\to\infty} \frac{1}{(2n)^d} \int_{[-n,n]^d} \big(|\mu| * |c|\big)(y)\,\mathrm{d}y$$

$$= \limsup_{n\to\infty} \frac{1}{(2n)^d} \int_{[-n,n]^d} \int_{\mathbb{R}^d} |c(y-t)|\,\mathrm{d}|\mu|(t)\,\mathrm{d}y.$$

Since c has compact support, there exists some m such that $\mathrm{supp}(c) \subseteq [-m,m]^d$. Then, for all $y \in [-n,n]^d$, we have $c(y-t) = 0$ outside of $[-n-m,n+m]^d$. Via Fubini, we thus get

$$|b(y)| \leqslant \limsup_n \frac{1}{(2n)^d} \int_{[-n,n]^d} \int_{[-n-m,n+m]^d} |c(y-t)|\,\mathrm{d}|\mu|(t)\,\mathrm{d}y$$

$$= \limsup_n \frac{1}{(2n)^d} \int_{[-n-m,n+m]^d} \int_{[-n,n]^d} |c(y-t)|\,\mathrm{d}y\,\mathrm{d}|\mu|(t)$$

$$\leqslant \limsup_n \frac{1}{(2n)^d} \int_{[-n-m,n+m]^d} \int_{\mathbb{R}^d} |c(y-t)|\,\mathrm{d}y\,\mathrm{d}|\mu|(t)$$

$$\leqslant \limsup_n \frac{2}{(2n)^d} |\mu|\big([-n-m,n+m]^d\big) =: C.$$

Since μ is translation bounded, we have $C < \infty$ and hence $|b(y)| \leqslant C$ for all $y \in S$, which proves that the set of coefficients is indeed bounded. Finally, since S is weakly uniformly discrete, it follows that ν is translation bounded.

Now, since the measure μ is tempered and its Fourier transform as a tempered distribution is a translation-bounded measure, μ is Fourier transformable as a measure, by [**44**, Theorem 5.2]. The same statements hold for the measure ν, and we have $\widehat{\mu} = \nu$ as claimed. □

REMARK 6.4. An interesting question in the context of Lemma 6.3 is whether one could relax the condition of uniform discreteness of Λ to weak uniform discreteness, which would strengthen some results in this section. At present, we do not have an answer to this question. Also, if true, a proof would need other methods. Indeed, already Lemma 6.1 fails for measures with weakly uniformly discrete support, as

$$\mu := \sum_{k=1}^{\infty} k \left(\delta_{k+\frac{1}{k}} + \delta_{k-\frac{1}{k}} - 2\delta_k \right)$$

clearly demonstrates.

Recalling from Section 5 that positive definite Radon measures are Fourier transformable, and again using [**44**, Theorem 5.2], we summarise some useful sufficient conditions as follows.

COROLLARY 6.5. *Let μ be a tempered measure on $\mathbb{R}^d$ such that its distributional Fourier transform, ν, is a measure. Under any of the following conditions, μ is translation bounded and Fourier transformable (and thus so is $\nu = \widehat{\mu}$) :*

(1) $\mathrm{supp}(\mu)$ is uniformly discrete and $\mathrm{supp}(\nu)$ is weakly uniformly discrete;
(2) $\mathrm{supp}(\mu)$ is uniformly discrete and μ is positive definite;
(3) $\mathrm{supp}(\mu)$ is uniformly discrete and ν is translation bounded;
(4) μ is translation bounded and positive definite;
(5) μ and ν are both translation bounded. $\hfill\square$

Note that a pure point measure may be translation bounded and still have dense support, such as the diffraction measure of the Fibonacci chain [**2**, Section 9.4.1]. Consequently, sparseness conditions on the support of a measure can be sufficient but are not necessary for translation-boundedness.

At this point, we can harvest Lemma 6.3 to state the following slightly stronger version of Theorem 4.10 for the specific case $G = \mathbb{R}^d$.

THEOREM 6.6. *Let* $0 \neq \mu \in \mathcal{M}(\mathbb{R}^d)$ *be a tempered measure such that* $\mathrm{supp}(\mu)$ *is contained in a Meyer set, and assume that the distributional Fourier transform* ν *of* μ *is a measure whose support is* $\mathcal{B}$*-sparse for some van Hove sequence* $\mathcal{B}$ *in* $\mathbb{R}^d$*. Then, there is a lattice* $\Gamma \subset \mathbb{R}^d$ *together with elements* $\tau_1, \ldots, \tau_N, \sigma_1, \ldots, \sigma_M \in \mathbb{R}^d$ *and trigonometric polynomials* P_i *and* Q_j *on* $\mathbb{R}^d$ *such that*

$$\mu = \sum_{i=1}^{N} \sum_{x \in \Gamma + \tau_i} P_i(x)\, \delta_x \qquad and \qquad \widehat{\mu} = \sum_{j=1}^{M} \sum_{y \in \Gamma^0 + \sigma_j} Q_j(y)\, \delta_y. \qquad \square$$

The above results allow us to use the results of Section 4 in considering a question posed by Meyer [**29**], namely whether there exists a pair of tempered measures μ, ν on $\mathbb{R}^d$, defined as in (22), such that ν is the distributional Fourier transform of μ, Λ is a fully Euclidean model set and S is locally finite.

REMARK 6.7. Recall that Meyer's definition of a model set in the context of this question requires that the internal space be $H = \mathbb{R}^n$. As stated in Section 3.2, we always refer to a CPS of the form $(\mathbb{R}^d, \mathbb{R}^n, \mathcal{L})$ as a *fully Euclidean CPS*.

We require one further result as follows.

LEMMA 6.8. *Let* $(\mathbb{R}^d, \mathbb{R}^n, \mathcal{L})$ *be a fully Euclidean CPS, and let* $\curlywedge(W)$ *be a model set in this CPS. If there exists a lattice* $\Gamma \subset \mathbb{R}^d$ *and a finite set* $F \subset \mathbb{R}^d$ *such that*

$$\curlywedge(W) \subseteq \Gamma + F,$$

then $n = 0$ *and* $\curlywedge(W)$ *is a lattice in* $\mathbb{R}^d$*.*

Proof. Suppose that such sets $\Gamma, F \subset \mathbb{R}^d$ exist. We first show that we can choose them such that $\Gamma, F \subseteq L = \pi_G(\mathcal{L})$. Note that [**43**, Lemma 5.5.1] implies $\Gamma + F \subseteq \curlywedge(W) + F_0$ for some finite set F_0. Then, we get

$$\Gamma \subseteq \Gamma + F - F = \curlywedge(W) + F'$$

with $F' = F_0 - F$, which is a finite set. As Γ is a lattice in $\mathbb{R}^d$, there are vectors $v_1, \ldots, v_d \in \mathbb{R}^d$ such that $\Gamma = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_d$. For any fixed $j \in \{1, \ldots, d\}$, we have $mv_j \in \Gamma \subseteq \curlywedge(W) + F'$ for every $m \in \mathbb{N}$. As F' is finite, there exist positive integers $n_1 \neq n_2$ and a $t \in F'$ such that $n_1 v_j$ and $n_2 v_j$ lie in $\curlywedge(W) + t$. Thus, we have

$$(n_1 - n_2)v_j \in \curlywedge(W) - \curlywedge(W) \subseteq L.$$

In this way, for each $j \in \{1, \ldots, d\}$, we find some $m_j \in \mathbb{N}_0$ such that $m_j v_j \in L$. Now, setting $\ell = \mathrm{lcm}(m_1, \ldots, m_d)$, we have

$$\ell \Gamma \subseteq L.$$

Now, as $\ell \Gamma \subseteq \Gamma$ has finite index, there exists a finite set $J \subset \mathbb{R}^d$ such that $\Gamma \subseteq \ell \Gamma + J$, and we get

$$\curlywedge(W) \subseteq \ell \Gamma + F + J.$$

Define $F'' = (F + J) \cap L$. For $x \in \curlywedge(W)$, there exist elements $y \in \ell \Gamma$ and $z \in (F + J)$ such that $x = y + z$. But as $x \in L$ and $y \in L$, we must have $z \in L \cap (F + J)$, Consequently,

$$\curlywedge(W) \subseteq \ell \Gamma + F'',$$

where the lattice $\ell \Gamma$ and the finite set F'' are both contained in L.

To continue, we relabel so that, without loss of generality, $\curlywedge(W) \subseteq \Gamma + F$ with $\Gamma, F \subseteq L$. Now, invoking [**43**, Lemma 5.5.1], there exists a finite set, $F_1 \subset \mathbb{R}^d$, such that

$$\Gamma + F \subseteq \curlywedge(W) + F_1,$$

and since $\Gamma, F, \curlywedge(W) \subseteq L$, we may as above choose F_1 such that $F_1 \subset L$. Then,

$$\Gamma \subseteq \curlywedge(W) + F_1 - F = \curlywedge(W) - F_2,$$

with a finite set $F_2 \subseteq L$, and thus

$$\Gamma \subseteq \curlywedge(W + F_2^\star).$$

Define $Z := \{x^\star : x \in \Gamma\} = \Gamma^\star$. Then, Z is a subgroup of $\mathbb{R}^n$, and so is its closure, $\overline{Z}$. Since $\overline{Z} \subseteq \overline{W} + F_2^\star$, we see that $\overline{Z}$ is a compact subgroup of $\mathbb{R}^n$, so we must have $\overline{Z} = \{0\}$.

Now, recalling that $\curlywedge(W) \subseteq \Gamma + F$, we have $W \subseteq \overline{Z} + F = F$, so W is finite. But W has non-empty interior, so we must have $n = 0$. To conclude, we note that, since $W \subseteq \{0\} = \mathbb{R}^0$, we have $W = \{0\}$. Hence, $\curlywedge(W)$ is a subgroup of $\mathbb{R}^d$ and thus is a lattice. $\qquad \square$

By combining Theorem 4.8 with Lemma 6.8 and Corollary 6.5, we can answer a weaker version of Meyer's question. Recall that a sparse point set (precisely, $\mathcal{B}$-sparse for some van Hove sequence $\mathcal{B}$) is necessarily locally finite.

COROLLARY 6.9. *There is no tempered measure $0 \neq \mu = \sum_{\lambda \in \Lambda} a(\lambda)\, \delta_\lambda$ supported inside a model set $\Lambda \subset \mathbb{R}^d$ in a non-trivial, fully Euclidean CPS $(\mathbb{R}^d, \mathbb{R}^n, \mathcal{L})$ such that the distributional Fourier transform ν is a translation-bounded measure with sparse support.*

Proof. Suppose to the contrary that such a measure, $\mu \neq 0$, exists. By Corollary 6.5, μ is translation bounded and Fourier transformable as a measure, with measure Fourier transform ν. Then, by Theorem 4.8, $\mathrm{supp}(\mu)$ is a subset of $\Gamma + F$, for some lattice Γ and $F \subset \mathbb{R}^d$ finite.

Next, as $\mu \in \mathcal{SAP}(\mathbb{R}^d)$, $\mathrm{supp}(\mu)$ is relatively dense. Therefore, by [**43**, Lemma 5.5.1], there exists a finite set $F' \subset \mathbb{R}^d$ such that $\Lambda \subseteq \mathrm{supp}(\mu) + F'$. This implies

$$\Lambda \subseteq L + (F + F'),$$

so, by Lemma 6.8, the CPS has internal space $\mathbb{R}^0 = \{0\}$. $\qquad \square$

Note that translation-boundedness of ν in the above result may be replaced by any of the sufficient conditions in Corollary 6.5. In fact, a result of Lev and Olevskii allows us to answer Meyer's question in a little more generality, namely for the case that ν is a slowly increasing measure. Recall that a tempered measure ν is *slowly increasing* when $|\nu|(B_r) = \mathcal{O}(r^n)$ as $r \to \infty$

for some $n \in \mathbb{N}$, where B_r denotes the ball of radius r around 0, which is a mild restriction when ν is a signed or complex measure.

For slowly increasing measures μ and ν, defined as in (22), with ν the distributional Fourier transform of μ and $\mathrm{supp}(\mu) = \Lambda$ inside a Meyer set, [26, Theorem 7.1] states that $S = \mathrm{supp}(\nu)$ is either uniformly discrete or has a relatively dense set of accumulation points. This means that local finiteness of S forces S to be uniformly discrete in this case (and Corollary 6.5 then implies that μ is translation bounded and transformable, so we may proceed as above). The result is also implied by [26, Theorem 2.3], which is an $\mathbb{R}^d$-version of our Theorem 4.10.

COROLLARY 6.10. *Let $\mu = \sum_{\lambda \in \Lambda} a(\lambda)\, \delta_\lambda$ be tempered and supported inside a model set Λ in a non-trivial, fully Euclidean CPS and let ν, the distributional Fourier transform of μ, be a slowly increasing measure. Then, if ν_{pp} has locally finite support, it must be trivial, $\nu_{\mathsf{pp}} = 0$.*

Proof. If μ is supported inside a fully Euclidean model set, then so is μ_{s}. By [46, Theorem 6.1], ν_{pp} is the distributional Fourier transform of μ_{s}, so applying Corollary 6.9 and the comments following it to μ_{s} gives the result. $\square$

Lemma 6.1 allows us to use some properties of weakly almost periodic measures, compare [24], to make some general statements about the diffraction of measures on $\mathbb{R}^d$ that have uniformly discrete support. The following generalises [26, Lemma 10.5], where we employ the Fourier–Bohr coefficients of a measure μ, denoted by $c_\chi(\mu)$, from Definition 4.13.

PROPOSITION 6.11. *Let μ be a translation-bounded measure on $\mathbb{R}^d$ such that its distributional Fourier transform, denoted by ν, is also a measure, and let $S := \{\chi \in \mathbb{R}^d : c_\chi(\mu) \neq 0\}$. Then, one has the following properties.*

(1) *The autocorrelation γ of μ is unique.*
(2) *μ possesses the pure point diffraction measure $\widehat{\gamma} = \sum_{\chi \in S} |c_\chi(\mu)|^2\, \delta_\chi$.*
(3) *$\mathrm{supp}(\nu_{\mathsf{pp}}) = S$.*

Proof. Via Lemma 6.1 and Remark 6.2, we see that $\mu \in \mathcal{WAP}(\mathbb{R}^d)$. Then, claims (1) and (2) are clear from [24, Theorem 7.6], while claim (3) now follows from [46, Theorem 7.2]; compare Definition 4.13. $\square$

The following result generalises [26, Theorem 10.4], since we do not require the measure μ to be translation bounded.

THEOREM 6.12. *Let μ be a tempered measure that is supported in a Meyer set and has an autocorrelation, γ. The support of the pure point part of the diffraction, $S := \mathrm{supp}(\widehat{\gamma}_{\mathsf{pp}})$, is either uniformly discrete and contained in finitely many translates of a lattice, or is not locally finite and has a relatively dense set of accumulation points.*

Proof. From [46, Theorem 5.1], γ is a weakly almost periodic, tempered distribution, so by Lemma 6.1, γ is a weakly almost periodic, translation-bounded measure. Since γ is supported inside a Meyer set, γ_{s} is supported inside a Meyer set as well [42 45].

Noting that γ_{s} and its Fourier transform, $\widehat{\gamma_{\mathsf{s}}} = \widehat{\gamma}_{\mathsf{pp}}$, are both translation-bounded and hence slowly increasing measures, we may apply [26, Theorem 7.1] to the measure γ_{s} to see that either S has a relatively dense set of accumulation points or is uniformly discrete.

The latter case is non-trivial only when $\gamma \neq 0$. Then, observing that γ is positive definite and supported inside a Meyer set, we see that γ is translation bounded and transformable by Corollary 6.5, so we may apply Theorem 4.8 to γ to obtain the result. $\square$

We combine the results of this section as follows.

COROLLARY 6.13.   *Let μ be a tempered measure on $\mathbb{R}^d$ such that its distributional Fourier transform ν is also a measure. If μ is supported inside a Meyer set, Λ say, and if we set $S := \operatorname{supp}(\nu_{\mathsf{pp}}) \neq \varnothing$, precisely one of the following situations applies.*

(1) *S contains a relatively dense set of accumulation points.*
(2) *There exists a lattice Γ in $\mathbb{R}^d$ together with finite sets $F, F' \subset \mathbb{R}^d$ such that $\Lambda \subseteq \Gamma + F$ and $S \subseteq \Gamma^0 + F'$.*

Proof.   By Lemma 6.1, the measure μ is translation bounded. Now, due to Proposition 6.11, μ has unique autocorrelation γ and diffraction

$$\widehat{\gamma} = \sum_{\chi \in S} |c_\chi(\mu)|^2 \, \delta_\chi,$$

where $S = \operatorname{supp}(\widehat{\gamma}) = \operatorname{supp}(\nu_{\mathsf{pp}})$.

Then, by Theorem 6.12, either claim (1) holds, or S is uniformly discrete. In the latter case, S is $\mathcal{B}$-sparse for all van Hove sequences, and hence, by Corollary 4.14, claim (2) holds.   □

The explicit structure can then be summarised as follows.

COROLLARY 6.14.   *Let μ and ν be as in Corollary 6.13. Then, there exists a CPS $(\mathbb{R}^d, H, \mathcal{L})$ and some $h \in C_{\mathsf{c}}(H)$ such that the autocorrelation and diffraction of μ are*

$$\gamma = \operatorname{dens}(\mathcal{L}) \, \omega_{h * \widetilde{h}} \quad and \quad \widehat{\gamma} = \operatorname{dens}(\mathcal{L})^2 \omega_{|\check{h}|^2},$$

and that the two cases are then as follows.

(1) *$\mu_{\mathsf{s}} = \omega_h$ and $\nu_{\mathsf{pp}} = \operatorname{dens}(\mathcal{L}) \, \omega_{\check{h}}$.*
(2) *$\mu = \omega_h$ and $\nu = \operatorname{dens}(\mathcal{L}) \, \omega_{\check{h}}$.*

Further, in the second case, μ and $\nu = \widehat{\mu}$ have the form given in Theorem 4.10, with an internal space of the form $H = \mathbb{Z}^m \times \mathbb{K}$.   □

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Michael Baake
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld
Germany

mbaake@math.uni-bielefeld.de

Nicolae Strungaru
Department of Mathematical Sciences
MacEwan University
10700 104 Avenue
Edmonton Alberta
Canada T5J 4S2

strungarun@macewan.ca

Venta Terauds
Discipline of Mathematics
University of Tasmania
Private Bag 37
Hobart, TAS 7001
Australia

venta.terauds@utas.edu.au