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Stable Balanced Expansion in Homogeneous Dynamic Models

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Abstract

This paper establishes conditions for the asymptotic stability of balanced growth paths in dynamic economic models as typical cases of homogeneous dynamical systems. Results for common two-dimensional deterministic and stochastic models are presented and further applications are discussed.
According to Solow & Samuelson (1953) balanced growth paths for deterministic economies are induced by so-called Perron-Frobenius solutions defined by an eigenvalue $\lambda > 0$ (the growth factor) and by an eigenvector $\bar{x}$, a fixed point of the system in intensive form. Contraction Lemma A.1 states for continuous deterministic systems that convergence to a balanced path occurs whenever the product $\lambda \cdot M(\bar{x})$ of the eigenvalue λ multiplied with the contractivity $0 < M(\bar{x}) < 1$ of the stable eigenvector $\bar{x}$ of the intensive form is less than one. For $\lambda \cdot M(\bar{x}) > 1$ all unbalanced orbits in the neighborhood of the balanced path diverge in spite of convergence in intensive form. This confirms that convergence to a stable eigenvector of the intensive form is only a necessary condition for convergence in state space.
In the stochastic case, the condition for asymptotic stability of balanced growth paths (Theorem B.2) uses results from a stochastic analogue of the Perron-Frobenius Theorem on eigenvalues and eigenvectors. Convergence (divergence) occurs if the expectation of the product $\lambda(\omega) \cdot M(\omega)$ is less than (greater than) one, i.e. if the product is mean contractive. This is equivalent to the condition that the sum of the expectations of the logarithmic values of the stochastic growth rate and of the contractivity factor of the intensive form are less than (greater than) zero.

JEL codes: C02, C62, E13, E24, E30, E31, O41, O42

Keywords: balanced growth, stability, stochastic balanced growth, random fixed points, Perron-Frobenius solution

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1 Introduction

In their early contribution Solow & Samuelson (1953), two of the most prominent Nobel Laureates in economics, provided for the first time the formal definition of a balanced growth path as the solution of a nonlinear eigenvector problem of a homogeneous dynamical system, also referred to as the Perron-Frobenius solution of linear systems (see Solow, 1952). It is defined by a growth factor (the eigenvalue) and by a fixed point of the modified time-one map on the unit simplex of the state space (the map in so-called intensive form). Orbits along a balanced growth path of the homogeneous system are defined as a solution exhibiting common constant geometric contraction or expansion of all state variables in the proportions given by the fixed point of the intensive form at a rate equal to the eigenvalue.

Solow and Samuelson discuss in detail existence and uniqueness of balanced paths, and what they call *Relative Stability in the Large* or *Stability in the Small.* By this they refer essentially to convergence in or *stability of proportions* (Solow & Samuelson, 1953, pp. 418) recognizing that there remains an issue of *pathwise* convergence in state space which to them seemed unobtainable at the time. They express their skepticism by writing (p. 419):

> *'The steady growth solution cannot be stable in the absolute sense that changes in initial conditions have effects ultimately damping to zero'.*

In contrast, in his seminal contribution Solow (1956) seems to suggest that under regular conditions a balanced path is attracting in state space if it is unique and if the intensive form is stable while divergence in state space occurs under multiplicity of growth paths if the intensive form assigns instability to one of its paths. In other words, trajectorial convergence holds if the growth path is stable for the intensive form contradictory to the conjecture of instability from the joint publication with Samuelson in *Econometrica* 1953. In almost all publications succeeding the article of 1965, which initiated the era of neoclassical growth theory, economists have been examining stability issues in models of growth primarily for intensive form models supporting the view that stability of the intensive form implies *stable balanced growth paths* in state space as well. The highly regarded survey article by Hahn & Matthews (1964) does not mention the issue. Standard text books of more recent vintage (such as Barro & Sala-I-Martin, 1995; Romer, 1996; Aghion & Howitt, 1998; De La Croix & Michel, 2002) seem to suggest as in Solow (1956) that convergence in per-capita terms or in growth rates implies convergence in state space as well.

With two notable exceptions, discussions of stability in growth models after Solow's 1956 contribution lack an awareness for the need to determine conditions which guarantee convergence to balanced growth paths in state space. Deardorff (1970)[1] shows for the standard Solow model that without depreciation the distance between the unbalanced and the balanced growth path is always exploding for any *positive* growth rate of the population. With such a mathematical condition for divergence economists should have become aware of the fact that convergence to a balanced growth path would also fail for low positive levels of depreciation and some ranges of population growth even when convergence is predicted for intensities. Conversely, convergence to the balanced path might occur for some positive rate of depreciation large enough but less than one which interacts with the rate of population growth. Unfortunately, the stability condition of the intensive form does not reveal any such trade-off between the two rates.

If the instability of balanced growth paths (as conjectured in Solow & Samuelson, 1953) were considered as a structural property of homogeneous dynamic models it implies that it would be

[1] After Deardorff (1970), Jensen (1994) points out that, in general, path-wise convergence in continuous-time growth models cannot be obtained from convergence in per-capita quantities.

impossible to show **convergence** of the capital stock of two identical economies to the same level under growth (with identical consumers, identical labor, and identical initial conditions) when **only** their initial capital equipment differs by $\pm\epsilon$ above/below the growth path. Everything else being identical and in spite of all conditions guaranteeing convergence in intensities, the instability would imply that the difference of their respective capital stocks would **always** grow to infinity, a possibility hard to be conceived of under all general conditions. Solow and Samuelson themselves did not reconsider a discussion of their original conjecture after Deardorff's findings. In the sequel, the discussion of whether additional conditions for convergence exist and could be determined was not pursued by others even in the context of the *convergence debate* of trade theory, of development economics, and of comparative systems (as, for example, in Galor, 1996; Mountford, 1998, 1999).

The opposite conjecture of a general stability of all growth paths under convergence of intensities alone may also be the result of a misinterpretation of the condition for convergence of the usual one-dimensional intensive form using per-capita variables. Formally, this intensive form seems equivalent to assuming that labor supply is constant with $n = 0$. The growth rate of labor enters only in the determination of the *level* of the balanced path and does not seem to matter for the condition of convergence of the intensive form. Under this *'as if'* assumption concavity of the production function is indeed a sufficient condition for convergence to the (trivial) growth path with zero growth of the work force. However, concluding from this fact convergence of orbits in state space to the balanced growth path for *all values* $n > -1$ is not warranted. Nevertheless, this might have lead some researchers to suggest that there is no need for investigating the stability issue further for $n \neq 0$.

For models with labor growing endogenously the growth rate along a balanced path $\bar{k}$ will be a function of the intensity with value $n(\bar{k}) \lesseqgtr 0$. In this case concavity of the production is not a sufficient condition any longer for the stability of the balanced solution $\bar{k}$ in intensive form. In models of dimension higher than two (for example, in a trade model with two or more countries, in a model with two financial assets, etc.) the choice of an intensive form with respect to one particular variable is arbitrary and needs to be taken with care to exhibit convergence of the associated 'real model'. In such cases, it is more appropriate to choose the unit simplex (as suggested originally by Solow & Samuelson, 1953) or the unit sphere as compact domains for the time-one map in intensive form.

Figure 1 portrays geometrically the possible sources for the occurrence of convergence or divergence to the balanced path in the standard two-dimensional growth model of the Solow type, indicated by a decrease or an increase of the distance of an orbit from the balanced path for a given rate of convergence of the intensive form. The difference between the two outcomes lies in the *size* of the expansionary growth rate of labor. This reveals that convergence (divergence) to the balanced path $\bar{k} \subset \mathbb{R}^2_+$ depends on the *relative sizes* of **two** dynamic forces: the contractivity for the intensity and the expansionary one for the growth rate of labor! The examination of the implications of their interaction on the distance from the growth path provides the answer to the question of stability.

The conditions for convergence of orbits to a balanced path in two-dimensional growth models were derived in Böhm, Pampel & Wenzelburger (2005) for models in discrete time and in Pampel (2009) for continuous time. The results for the general n-dimensional situation for continuous homogeneous time-one maps in discrete time are given in Appendix A for deterministic systems and in Appendix B for stochastic ones. For deterministic dynamic economies, as in most models of economic growth, of international trade, or monetary macro, conditions of existence and stability are obtained applying the features of the non-linear generalization of the Perron-Frobenius Theorem. In the stochastic case, the paper introduces the associated notion

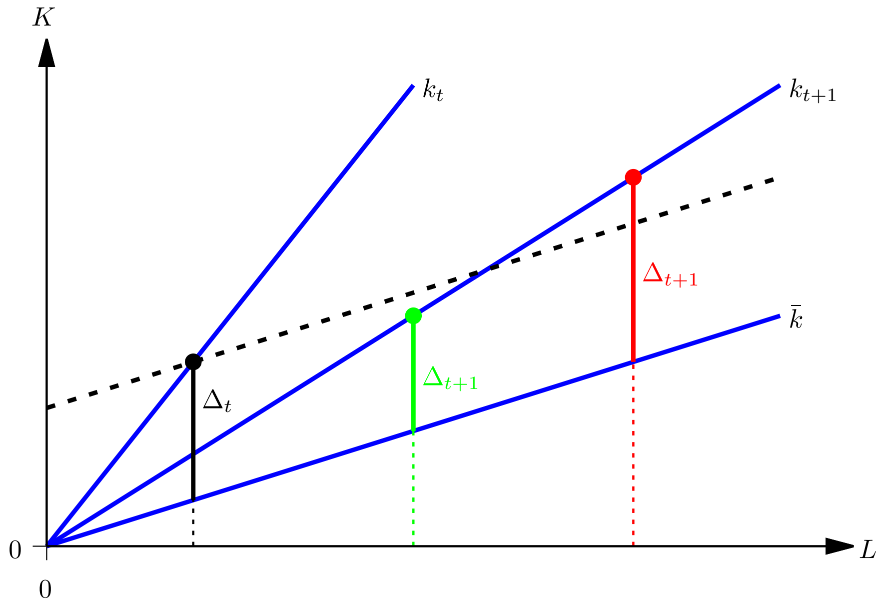


Figure 1: Convergence/divergence in $(L, K)-$space: $\boldsymbol{\Delta_{t+1}} > \boldsymbol{\Delta_t} > \boldsymbol{\Delta_{t+1}}$ for $k_t \to \bar{k}$.

of stochastic balanced paths and derives the conditions for their stability using a recent extension of the Perron-Frobenius Theorem provided by Evstigneev & Pirogov (2010) and Babaei, Evstigneev & Pirogov (2018). The main two sections of this paper present the application of the mathematical results to different economic dynamic models and they discuss further extensions and implications.

2 Stable Expansion in Deterministic Systems

Figure 1 suggests that convergence to a balanced growth path means that the distance of an orbit of the system F to the growth path tends to zero as $t \to +\infty$. Let $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ denote the time-one map of the homogeneous dynamical system and $f : S \to S$ its *intensive form associated with* F being given by $y \mapsto f(y) := F(y)/|F(y)|$ and S (the positive part of) the unit sphere. Let $\bar{x} = f(\bar{x})$ denote an asymptotically stable fixed point of f and $L(\bar{x}) \subset \mathbb{R}^n_+$ the halfline through $\bar{x}$ containing all balanced growth paths.

Define the distance of $x \in \mathbb{R}^n_+$ from $L(\bar{x})$ as

$$\Delta := d(x, L(\bar{x})) = \min_{\alpha \geq 0} |x - \alpha \bar{x}| = |x - \langle x, \bar{x} \rangle \bar{x}| \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product.

Definition 2.1. *An orbit* $\gamma(x)$ *of* F *is said to converge to a balanced growth path (to* $L(\bar{x})$*) if*

$$\Delta_t := d(F^t(x), L(\bar{x})) = |F^t(x) - \langle F^t(x), \bar{x} \rangle \bar{x}| \tag{2.2}$$

converges to zero for $t \to \infty$.

Contraction Lemma A.1
Let $(\lambda, \bar{x})$ denote a Perron-Frobenius solution for F, i.e. $\lambda \bar{x} = F(\bar{x})$ with $|\bar{x}| = 1$ and $\lambda > 0$. Assume that $\bar{x} \in S$ is an asymptotically stable fixed point of f with contractivity $0 < M < 1$.

Then, for all $x_0/|x_0|$ in the basin of attraction of $\bar{x} = f(\bar{x})$:

$$\text{If} \qquad \lambda\, M > 1, \qquad \text{then} \qquad \lim_{t\to\infty} |\Delta_t| = \infty. \tag{2.3}$$

$$\text{If} \qquad \lambda\, M < 1, \qquad \text{then} \qquad \lim_{t\to\infty} |\Delta_t| = 0. \tag{2.4}$$

In other words if the **product** of the two factors of contraction and of expansion is less than one convergence occurs, otherwise divergence from the balanced growth path follows. More details and the proof for Lemma A.1 are given in Appendix A.

2.1 The Solow Growth Model

Let $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ be the concave homogeneous production function inducing the time-one map of the Solow growth model (Solow, 1956, 1988, 1999) $(\mathcal{L}, \mathcal{K}) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, $(L, K) \mapsto (\mathcal{L}(L, K), \mathcal{K}(L, K))$ given by

$$\begin{aligned} L' &= \mathcal{L}(L, K) := (1 + n)L \\ K' &= \mathcal{K}(L, K) := (1 - \delta)K + sA\, F(K, L) \end{aligned} \tag{2.5}$$

with parameters (n, δ, A, s). This implies the common one-dimensional mapping in intensity form

$$k' = G(k) := \frac{1}{1+n}\left((1-\delta)k + sA\, f(k)\right), \qquad k := K/L \quad f(k) := F(K/L, 1). \tag{2.6}$$

A balanced path of the Solow model is induced by a Perron-Frobenius solution of the homogeneous system $(\mathcal{L}, \mathcal{K})$, i.e. by a triple $\lambda > 0$, $(\bar{L}, \bar{K}) \gneqq 0$, $|(\bar{L}, \bar{K})| = 1$, satisfying

$$\lambda \begin{pmatrix} \bar{L} \\ \bar{K} \end{pmatrix} = \begin{pmatrix} (1+n)\bar{L} \\ (1-\delta)\bar{K} + sAF(\bar{K}, \bar{L}) \end{pmatrix} = (1+n)\bar{L} \cdot \begin{pmatrix} 1 \\ G(\bar{k}) \end{pmatrix}, \quad G(\bar{k}) = \bar{k} := \bar{K}/\bar{L}. \tag{2.7}$$

Balanced orbits (or paths) are of the form $\gamma(\alpha(\bar{L}, \bar{K})) = \left\{\lambda^t \cdot \alpha(\bar{L}, \bar{K})\right\}_{t\geq 0}$, $\alpha > 0$. They are all contained in the set $L(\bar{k}) := \left\{(L, K) \in \mathbb{R}^2_+ \,|\, K = \bar{k}L, \bar{k} := \bar{K}/\bar{L} \gneqq 0\right\}$ which is the halfline through $(\bar{L}, \bar{K}) \gneqq 0$, the balanced ray.

If $f(k)$ satisfies the Inada conditions (Inada, 1963), then, for every (n, δ, A, s), there exists a unique Perron-Frobenius solution $\lambda > 0$, $(\bar{L}, \bar{K}) \gg 0$ satisfying

$$\begin{aligned} \frac{f(\bar{k})}{\bar{k}} &:= \frac{n+\delta}{s\,A}, \qquad\qquad \bar{k} := \bar{K}/\bar{L} \\ \lambda &:= 1 + n, \\ M &:= \lim_{k\to\bar{k}} \frac{G(k) - G(\bar{k})}{k - \bar{k}} =: G'(\bar{k}). \end{aligned} \tag{2.8}$$

(1) The steady state $\bar{k} = G(\bar{k})$ of the model in intensive form is asymptotically stable if and only if the elasticity of f, $E_f(\bar{k}) := \bar{k}f'(\bar{k})/f(\bar{k}) < 1$. Since

$$\begin{aligned} G'(\bar{k}) &= \frac{1}{1+n}\left(1 - \delta + Asf'(\bar{k})\right) = \frac{1}{1+n}\left(1 - \delta + (n+\delta)\frac{\bar{k}f'(\bar{k})}{f(\bar{k})})\right) \\ &= 1 + \frac{\delta + n}{1+n}\left(E_f(\bar{k}) - 1\right) < 1 \qquad \iff \qquad E_f(\bar{k}) < 1, \end{aligned} \tag{2.9}$$

this is guaranteed by the concavity of F (respectively of f) for every (n, δ, A, s).

(2) The condition (2.9) is only necessary to guarantee convergence of the orbit in state space to the balanced ray $L(\bar{k})$. According to Lemma A.1 orbits $\gamma(L_0, K_0)$ of the Solow model converge to the halfline through $(\bar{L}, \bar{K})$ if and only if $\lambda M = \lambda G'(\bar{k}) < 1$ which is satisfied if and only if

$$E_f(\bar{k}) < \frac{\delta}{n+\delta}. \tag{2.10}$$

In other words, convergence to the balanced ray (halfline) is guaranteed only for levels of the elasticity E_f smaller than the relative rate of depreciation $\delta/(n+\delta)$. Surely, if $n \leq 0$ stability holds for all pairs $0 \leq (E_f(\bar{k}), \delta) \leq 1$. Conversely, if $n > 0$, all orbits are diverging from the balanced path if $\delta = 0$, which was the result established by Deardorff (1970).

Figure 2 shows the outcome of a numerical example of how a marginal change of the rate of population growth changes the asymptotic behavior near the growth path. Let $E_f(\bar{k}) = .5$ and $0 < \delta = n < 1$. Then, for every small $\epsilon > 0$:

$$\boldsymbol{\delta/(n+\epsilon+\delta)} < E_f(\bar{k}) < \boldsymbol{\delta/(n-\epsilon+\delta)} \tag{2.11}$$

implying that unbalanced growth paths converge for $n-\epsilon$ while they diverge from the balanced path for an increase to $n+\epsilon$.

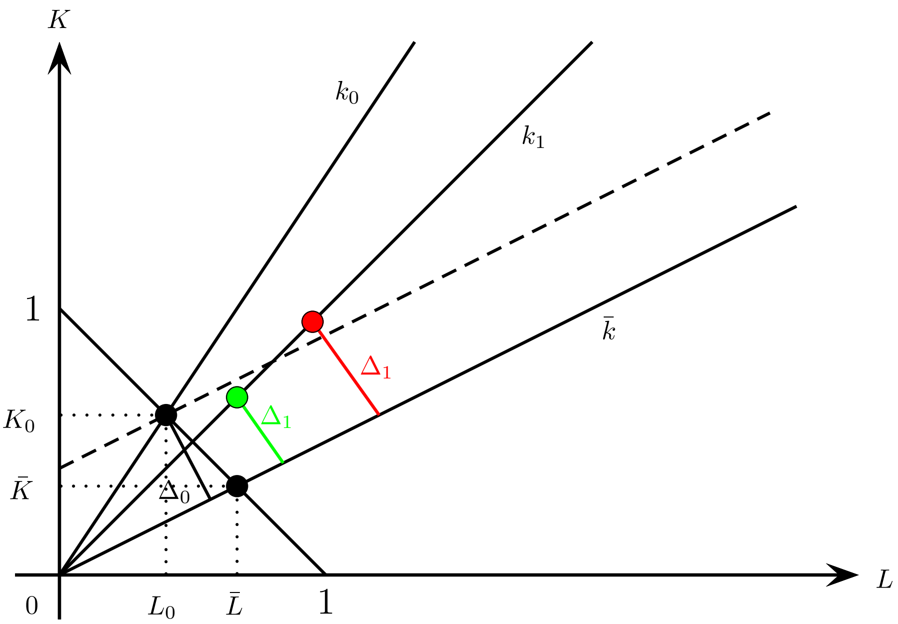


Figure 2: Convergence/divergence in $(L, K)-$space: $\boldsymbol{\Delta_1} > \boldsymbol{\Delta_0} > \boldsymbol{\Delta_1}$ for $k_0 \to \bar{k}$.

In general, for $0 \ll (\delta, n, E_f(\bar{k}))$, equality of condition (2.10) describes the trade-off between (δ, n) and $E_f(\bar{k})$ to maintain stability. Near the boundary of the stability region a decrease in capital depreciation has to be offset by a decrease in elasticity to maintain stability. Figure 3 displays the bifurcation curve and the ranges of $(E_f(\bar{k}), \delta) \in [0,1]^2$ (shaded region) for which *unstable* positive balanced growth occurs for given $n > 0$. In such economies orbits diverge from the balanced path whenever initial conditions are not equal to $\bar{k}$, $0 < E_f(\bar{k}) < 1$ is only a necessary condition for convergence. This occurs in particular for the respective cases with Cobb-Douglas production functions with constant elasticity $E_f(\bar{k})$. Under more general technologies or savings functions monotonic homogeneous systems $(\mathcal{L}, \mathcal{K})$ with multiple balanced paths may exist, all of which may be unstable for large open sets of parameters according to the conditions of Lemma A.1.

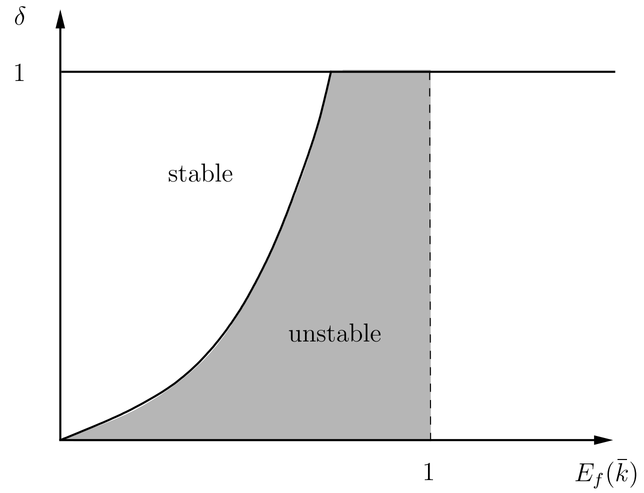


Figure 3: Regions of $(E_f(\bar{k}), \delta) \in [0,1]^2$ with unstable balanced growth; $n > 0$

2.2 Economic Growth with an Aging Workforce and Vintage Capital

Consider a workforce with an overlapping generations structure where the productivity of each generation diminishes with age. Assume that total lifetime of each generation is finite, identical, and equal to some length $N > 2$. Let $L = (L_1, L_2, \ldots, L_N)$ denote the typical vector of the number of workers in an arbitrary period grouped by age, where L_i, $i = 1, \ldots, L_N$ denotes the number of workers with remaining lifetime i.

Assume that the evolution of the workforce follows a linear regeneration process of population dynamics defined by a matrix $L' = \boldsymbol{N}L$ such that

$$L' = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1+n_1 & 1+n_2 & 1+n_3 & & \cdots & 1+n_N \end{pmatrix} L \tag{2.12}$$

where $\boldsymbol{n} := (1{+}n_1, 1{+}n_2, \cdots, 1{+}n_n) \geq 0$ are the fertility rates or growth factors from surviving generations, i.e. the contributions of each generation to the next youngest cohort. The matrix $\boldsymbol{N}$ contains an $N-1$ dimensional unit matrix $\boldsymbol{I}$ in the upper right hand corner while the first column is often assumed to consist of zeroes only. A more elaborate model could include differential death rates $0 \leq \boldsymbol{d} := (0, d_2, d_3, \cdots, d_n) \leq 1$ of generations altering the population matrix to

$$L' = \begin{pmatrix} 0 & 1-d_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1-d_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1-d_4 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1-d_{N-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-d_N \\ 1+n_1 & 1+n_2 & 1+n_3 & & \cdots & 1+n_N \end{pmatrix} L. \tag{2.13}$$

Assume that capital has a fixed finite life time $M > 2$ and that it is non-malleable once produced in time. Let $K = (K_1, K_2, \cdots, K_M)$ denote the vector of the capital equipment in the economy, where K_j is the number of machines with remaining operating life time j, $j = 1, \dots, M$.

Output across time is homogeneous and produced using a homogeneous production function $F : \mathbb{R}^M_+ \times \mathbb{R}^N_+ \to \mathbb{R}_+$, $(K, L) \mapsto F(K, L)$. Then, the formation of new capital under a Solow savings hypothesis implies that

$$K'_M = sF(K, L), \qquad 0 < s < 1. \tag{2.14}$$

The development of the vintage composition follows a linear decay process. Let the list of rates of decay of each vintage machine be given as $\boldsymbol{\delta} := (0, \delta_2, \delta_3, \dots, \delta_M)$ with $0 \leq \delta_j \leq 1$, $j = 2, \dots, M$. Then, the one-step mapping for the change of the vintage capital becomes $K' = \boldsymbol{M}K$ where

$$K' = \begin{pmatrix} 0 & 1-\delta_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1-\delta_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1-\delta_4 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1-\delta_{M-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\delta_M \\ 0 & 0 & & & \cdots & k_M(K,L) \end{pmatrix} K. \tag{2.15}$$

Under the Solow hypothesis with a given propensity to save the entry $k_M(K, L) = sF(K, L)/K_M$ denotes the growth factor of new capital with respect to the previous/latest capital generated from aggregate savings. Thus, one obtains a homogeneous time-one map $(\mathcal{K}, \mathcal{L}) : \mathbb{R}^M_+ \times \mathbb{R}^N_+ \to \mathbb{R}^M_+ \times \mathbb{R}^N_+$ of a dynamical system defined by

$$\begin{aligned} K' &= \mathcal{K}(K, L) := \boldsymbol{M}(K, L)K \\ L' &= \mathcal{L}(K, L) := \boldsymbol{N}L \end{aligned} \tag{2.16}$$

which is linear except for the last component in the vintage capital formation. It describes the joint evolution of capital accumulation and of the demographic development of the work force of a real (non-monetary) economy under a constant aggregate savings propensity $0 < s < 1$ according to a Solow-type savings assumption. The demographic structure of workers as well as the vintage composition of capital is modeled in a linear parametrized form with arbitrary finite lengths of lifetimes of workers and vintage capital which induces a multidimensional Solow model. For $N = M = 1$ and malleability (additivity) of new and old capital the model reduces to the standard one-dimensional model defined by (2.5).

The results for existence, sustainability, and stability from the two-dimensional case can be generalized almost in a one-to-one fashion to the multidimensional model after defining an appropriate intensive form with the positive unit sphere or simplex as state space.

(1) A balanced path of the extended Solow model is defined by a triple $(\lambda, \bar{K}, \bar{L}) \gg 0$, $|(\bar{K}, \bar{L})| = 1$ which solves

$$\lambda \begin{pmatrix} \bar{K} \\ \bar{L} \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}(\bar{K}, \bar{L}) \\ \boldsymbol{N}\bar{L} \end{pmatrix}. \tag{2.17}$$

Thus, balanced orbits of $(\mathcal{K}, \mathcal{L})$ are of the form $\gamma(\alpha(\bar{K}, \bar{L})) = \left\{\lambda^t \alpha(\bar{K}, \bar{L})\right\}_{t \geq 0}$, $\alpha > 0$, which are all contained in the half line $\{(K, L) \,|\, (K, L) = \alpha(\bar{K}, \bar{L}), \alpha > 0\}$.

Since $\boldsymbol{N}$ is a matrix with constant coefficients independent of capital accumulation, $\lambda \bar{L} = \boldsymbol{N} \bar{L}$ must hold. Thus, λ is an eigenvalue and $\bar{L}$ is an eigenvector of the population matrix $\boldsymbol{N}$ defining the stationary distribution of the workforce along balanced orbits. Both are determined parametrically by $(\boldsymbol{d}, \boldsymbol{n})$. Therefore, λ can be greater or less than one and $\bar{L}$ exhibits typically a non-uniform age distribution of workers across generations. The specific structure of the population matrix $\boldsymbol{N}$ of (2.13) implies a simple test for the size of the growth factor λ which turns out to be a leading real eigenvalue of $\boldsymbol{N}$ with multiplicity one[2].

Proposition 2.1. *Let the population matrix* $\boldsymbol{N}$ *be given by* (2.13) *and define* $\bar{p} := \sum_{\ell=1}^{N} \Big((1+n_\ell) \cdot \prod_{k=\ell+1}^{N} (1-d_k) \Big)$. *Then:*

$$\begin{aligned} &\text{if} \quad 0 < \bar{p} < 1, \quad \text{then} \quad \bar{p} < \bar{\lambda} < \bar{p}^{\frac{1}{N}} < 1, \\ &\text{if} \quad\quad 1 < \bar{p}, \quad\;\; \text{then} \quad 1 < \bar{p}^{\frac{1}{N}} < \bar{\lambda} < \bar{p}. \end{aligned} \tag{2.18}$$

(2) Let $S \subset \mathbb{R}_+^{M+N}$ denote the nonnegative subset of the unit sphere and define the 'intensive form' of this Solow model by the mapping $g : S \to S$, $k \mapsto g(k)$ where $k : (K, L)/|(K, L)|$ and

$$g(k) := \frac{(\mathcal{K}(K,L), \mathcal{L}(K,L))}{|(\mathcal{K}(K,L), \mathcal{L}(K,L))|}. \tag{2.19}$$

By construction $\bar{k} := (\bar{K}, \bar{L}) = g(\bar{k}) \in S$ is a fixed point of g and $(\lambda, \bar{K}, \bar{L})$ defines a balanced path with $\lambda = |(\mathcal{K}(\bar{K}, \bar{L}), \mathcal{L}(\bar{K}, \bar{L}))|$.

Let the production function F be strictly monotonically increasing and strictly concave for all x, y, $y \neq \alpha x$ (off rays), and satisfy a generalized weak Inada condition. Then, there exists a unique interior fixed point of $g(\bar{k}) = g(\bar{K}, \bar{L}) = (\bar{K}, \bar{L}) = \bar{k} \in S$ for all $(s, \boldsymbol{d}, \boldsymbol{n}, \boldsymbol{\delta})$. This condition is satisfied in particular when F is isoelastic, i.e. of the Cobb-Douglas type.

(3) If $\bar{k}$ is asymptotically stable under g, for every $k, k' \in \mathcal{B}(\bar{k}) \subset S$, the basin of attraction of $\bar{k}$, one has $|g^m(k) - g^m(k')| < |k - k'|$ for some $m \geq n$, and $\lim k_n = \lim g^n(k) = \lim g^n(k') = \bar{k}$. This implies a contractivity factor $M(\bar{k})$ as

$$M(\bar{k}) := \lim_{k_n \to \bar{k}} \frac{|g(k_n) - \bar{k}|}{|k_n - \bar{k}|} < 1, \tag{2.20}$$

whose size depends jointly on the curvature features of the production function and on $(\boldsymbol{d}, \boldsymbol{n}, \boldsymbol{\delta})$, see Lemma A.1.

(4) Lemma A.1 of the appendix states that orbits $\gamma(K_0, L_0)$ in state space converge to the half line $\{(K, L) \in \mathbb{R}_+^{M+N} \,|\, (K, L) = \alpha(\bar{K}, \bar{L}), \alpha > 0\}$ if $\lambda\, M(\bar{k}) < 1$. They diverge if $\lambda\, M(\bar{k}) > 1$. Therefore, as in the two-dimensional case without demographic or vintage structures, the convergence to balanced growth depends on an interplay between production elasticities embedded in the technology F and the parameters of decay or renewal for capital and for the work force. The *product* of the eigenvalue with the contractivity of the intensive mapping must be less than one, showing again that contractivity of the latter is only a necessary condition for convergence under growth.

There are obvious further applications of Lemma A.1 to examine the conditions for stable balanced growth in models with more general savings behavior than the one of the Solow type:

[2] I am indebted to T. Pampel for pointing out this result.

- all models with endogenous determination of the savings behavior, – as in optimal growth, under differential savings by heterogeneous agents or by income groups (as in Kaldor, 1957; Pasinetti, 1962; Samuelson & Modigliani, 1966), or in OLG models –;
- two-sector growth models (as in Drandakis, 1963; Inada, 1963; Uzawa, 1961, 1963; Galor, 1992);
- models of international trade (Oniki & Uzawa, 1965; Mountford, 1998, 1999);
- models with additional assets other than capital such as public debt (Diamond, 1965);
- models with expanded commodity spaces induced by heterogeneous inputs, natural resources, or public goods;
- convergence to balanced growth in general multisector growth models of the von Neumann type (see von Neumann, 1937; Solow & Samuelson, 1953; Gale, 1956; Kemeny, Morgenstern & Thompson, 1956; Evstigneev & Schenk-Hoppé, 2008) could be examined.

2.3 Examples of Monetary Models

All *consistent* and *complete* intertemporal macroeconomic models which describe time series of monetary data (satisfying the principles of national income accounting) belong to the class of homogeneous systems: the AS-AD macroeconomic model, any complete Keynesian IS-LM model, all complete New-Keynesian models with consistent policies, models of the so-called monetary approach in international trade (for example the Mundell-Fleming Model and others, as in Dornbusch, 1976; Frenkel & Razin, 1987; Gandolfo, 2016, or most models in Krugman, Obstfeld & Melitz, 2015). If their time series are generated by forward recursive time-one maps these will be homogeneous of degree one. The conditions of Lemma A.1 apply and convergence/divergence in state space occurs if the Perron-Frobenius solution and the contractivity of the intensive form satisfy the product rule.

- The two versions of the AS-AD Model with money (Chapters 4.1-4.2 in Böhm, 2017, or with money and sovereign debt in Böhm, 2018) provide explicit applications of these results. They are micro-based completions of the Keynesian IS-LM Model.
- Claas (2019) presents a detailed analysis in a macroeconomic model with efficient bargaining showing how the parameters of taxation, consumption, production, and of union power influence convergence to or divergence from the balanced inflationary path.

3 The Stochastic Solow Growth Model

The stochastic version of the Solow model arises when one or several of the parameters (n, δ, A, s) are subjected to a recurring exogenous random perturbation. Schenk-Hoppé & Schmalfuß (2001) analyze the standard one-dimensional model in its intensive form with general ergodic perturbations of all four parameters. They show existence and convergence to a stationary random orbit defined by a random fixed point of the one-dimensional Solow model. The random fixed point induces a *balanced random growth path* of capital and labor in $\mathbb{R}^2_+$. As in the deterministic case convergence of the intensive form is only a necessary condition for convergence to a balanced growth path in state space[3]. The appendix provides the framework, concepts of

[3]The convergence conditions were presented originally in Böhm, Pampel & Wenzelburger (2005). Here they use the more recent results from Babaei, Evstigneev & Pirogov (2018).

balancedness, and conditions for convergence for finite dimensional stochastic growth models. It derives the conditions for convergence to balanced growth paths in state space (Theorem B.2) from the relationship between the intensive and the state space form of the model.

Let the random variation be a production shock to a standard concave production function given by a bounded positive multiplicative (Hicks neutral) perturbation defined by a random variable $A : \Omega \to [A_{\min}, A_{\max}]$, $0 < A_{\min} < A_{\max} < \infty$, and by a random growth rate of the working population $n : \Omega \to [n_{\min}, n_{\max}]$, $-1 < n_{\min} < n_{\max} < \infty$ for a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\omega = (\dots, -1, 0, 1, \dots) \in \Omega$ is the set of two-sided infinite sequences, $\mathcal{F}$ its Borel sigma-algebra, and $\mathbb{P}$ is a probability measure. The dynamics of the noise process is given by the so-called left shift, $\omega \mapsto \vartheta(\omega)$, an invertible map $\vartheta : \Omega \to \Omega$, defined as $(\vartheta\omega)_s = \omega_{s+1}, s \in \mathbb{Z}$, see Appendix B.

3.1 State Space vs. Intensive Form

If $(A_{t-1}, n_{t-1}) = (A(\omega_{t-1}), n(\omega_{t-1})) = (A(\vartheta^{t-1}\omega), n(\vartheta^{t-1}\omega))$ is a pair of realizations of the noise process at time $t-1$ within the above frame work for an arbitrary $\omega \in \Omega$, the standard formula (2.5) of the Solow model defines a pair of homogeneous random difference equations

$$\begin{aligned} L_t &= (1 + n_{t-1})L_{t-1} \\ K_t &= (1-\delta)K_{t-1} + sA_{t-1}\,F(K_{t-1}, L_{t-1}) \end{aligned} \tag{3.1}$$

determining the one-step realization of capital and labor. They induce a random family of homogeneous mappings[4] $G(\omega) := (\mathcal{L}(\omega), \mathcal{K}(\omega)) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, $(L, K) \mapsto G(\omega)(L, K)$,

$$G(\omega)(L,K) := \begin{pmatrix} \mathcal{L}(\omega)(L,K) \\ \mathcal{K}(\omega)(L,K) \end{pmatrix} := \begin{pmatrix} (1+n(\omega))L \\ (1-\delta)K + sA(\omega)\,F(K,L) \end{pmatrix} \qquad \mathbb{P}\text{-a.s.}. \tag{3.2}$$

The standard intensive form of the stochastic Solow model is given by the maps $g(\omega) : \mathbb{R}_+ \to \mathbb{R}_+$

$$g(\omega)k := \frac{1}{1+n(\omega)}\left((1-\delta)k + sA(\omega)\,f(k)\right), \qquad k := K/L \quad f(k) := F(K/L, 1). \tag{3.3}$$

Homogeneity implies the relation between the two mappings $G(\omega) = (1+n(\omega))\begin{pmatrix} 1 \\ g(\omega) \end{pmatrix}$, $\mathbb{P}$-a.s., since

$$G(\omega)(L,K) = (1+n(\omega))L \begin{pmatrix} 1 \\ \dfrac{(1-\delta)k + sA(\omega)\,f(k)}{1+n(\omega)} \end{pmatrix} = (1+n(\omega))L \begin{pmatrix} 1 \\ g(\omega)\left(\frac{K}{L}\right) \end{pmatrix}. \tag{3.4}$$

It provides a convenient way to compare orbits in state space with those of the intensive form. With the definition of the two random families of maps G (respectively g), the list $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ forms a *random dynamical system* in the sense of Arnold (1998) allowing the usage of the methods of the associated theory.

Given ω and any initial condition (L_0, K_0), the state (L_t, K_t) of the system G after $t > 0$ periods is generated by the mapping

$$C(t,\omega)(K_0, L_0) := \begin{cases} G(\omega)(\vartheta^{t-1}\omega) \circ \dots \circ G(\omega)(K_0, L_0) & t > 0 \\ id_{\mathbb{R}^2_+} & t = 0 \end{cases} \tag{3.5}$$

[4]The notational convention for the result of the application of the function $F(\omega)$ to the point x will be $F(\omega)x$ instead of $F(\omega)(x)$.

which satisfies

$$C(t+s,\omega) = C(t,\vartheta^s\omega) \circ C(s,\omega) \qquad \text{for all } t,s. \tag{3.6}$$

Therefore, for any $\omega \in \Omega$, the orbit of capital and labor in state space with initial condition $(L_0, K_0) \in \mathbb{R}^2_+$ is given by $\gamma(\omega,(L_0,K_0)) := \{C(t,\omega)(L_0,K_0)\}_t^\infty$. Similarly, for any initial condition $k_0 \in \mathbb{R}_+$, orbits of the intensive form $g(\omega)$ are given by $\gamma(\omega,k_0) := \{c(t,\omega)k_0\}_t^\infty$ since the state k_t of the intensive form system is generated by the one-dimensional mapping $c(t,\omega) : \mathbb{R}_+ \to \mathbb{R}_+$

$$c(t,\omega)k_0 := \begin{cases} g(\vartheta^{t-1}\omega) \circ \cdots \circ g(\omega)k_0 & t > 0 \\ id_{\mathbb{R}_+} & t = 0 \end{cases} \tag{3.7}$$

which also satisfies $c(t+s,\omega) = c(t,\vartheta^s\omega) \circ c(s,\omega)$, for all t,s. For notational consistency the normalized mapping $\tilde{c}(t,\omega) := (1, c(t,\omega)) : S \to S$, $S := \left\{(x_1,x_2) \in \mathbb{R}^2_+ \,|\, x_1 = 1\right\}$, will also be used.

$$\tilde{c}(t,\omega)\begin{pmatrix}1\\k_0\end{pmatrix} := \begin{cases} \begin{pmatrix}1\\c(t,\omega)k_0\end{pmatrix} & t > 0 \\ id_S & t = 0 \end{cases} \tag{3.8}$$

satisfying $\tilde{c}(t+s,\omega) = \tilde{c}(t,\vartheta^s\omega) \circ \tilde{c}(s,\omega)$, for all t,s.

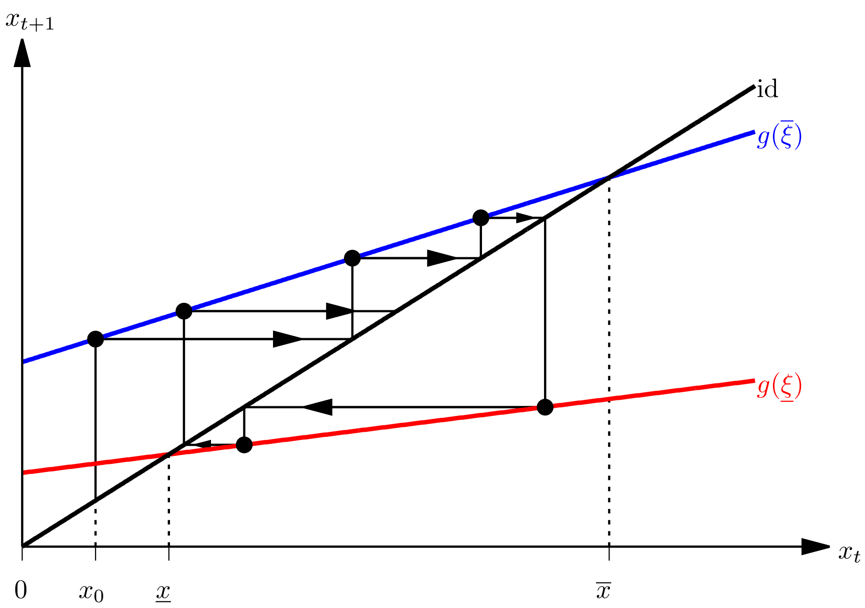


Figure 4: A random orbit $\gamma(\omega,x_0)$ for $\omega = (\ldots,\overline{\xi},\overline{\xi},\overline{\xi},\underline{\xi},\underline{\xi},\overline{\xi},\ldots)$

Figure 4 portrays the evolution of parts of an orbit of $g(\omega) \equiv (g(\underline{\xi}), g(\overline{\xi}))$, with a discrete two-point perturbation $\Omega = \cdots \times \left\{\underline{\xi},\overline{\xi}\right\} \times \left\{\underline{\xi},\overline{\xi}\right\} \times \cdots$. Observe that an orbit is a sequence of successive points on the two graphs of the maps $(g(\underline{\xi}), g(\overline{\xi}))$. If both are contractions with fixed points $(\underline{x},\overline{x})$, the interval $[\underline{x},\overline{x}]$ is a forward invariant set of the random dynamical system.

Stationary solutions of intensive form growth models are given by *random fixed points* which are the stochastic analogue for stochastic difference equations of the concept of a deterministic fixed point (see Schenk-Hoppé & Schmalfuß, 2001, or Definition B.2).

Definition 3.1. *A random fixed point of the Solow growth model* (3.3) *in intensive form is a measurable mapping (i.e. a random variable)* $k^* : \Omega \to \mathbb{R}_+$ *solving*

$$k^\star(\vartheta\omega) = g(\omega)k^\star(\omega) := \frac{(1-\delta)k^\star(\omega) + sA(\omega)\, f(k^\star(\omega))}{1+n(\omega)}, \quad \mathbb{P}\text{-}a.s. \tag{3.9}$$

Under the usual assumptions (Inada conditions and concavity of the production function, and boundedness of stationary perturbations) a unique fixed point $k^\star$ exists (see Schenk-Hoppé & Schmalfuß, 2001).

3.2 Balanced Random Growth Paths

Definition 3.2. *A pair of measurable mappings* $\lambda : \Omega \to \mathbb{R}_{++}$ *and* $\xi : \Omega \to S$ *is called a Perron-Frobenius solution for the Solow model* $G(\omega)$ *if* $\mathbb{P}$*-a.s.:*

$$G(\omega)\xi(\omega) = \lambda(\omega) \cdot \xi(\vartheta(\omega)), \tag{3.10}$$

$$\begin{pmatrix} 1 \\ g(\omega) \end{pmatrix} \xi(\omega) = \xi(\vartheta(\omega)), \qquad \xi(\omega), \xi(\vartheta\omega) \in S. \tag{3.11}$$

Condition (3.11) imposes that the random variable $\xi : \Omega \to \mathbb{R}^2_+$ is a random fixed point of the normalized map $(1, g(\omega)) : \Omega \to S$, which is the intensive form of $G(\omega)$, while (3.10) states that the two random forces of expansion/contraction and of deviation of intensity act in a separable way on scale and on intensity. They are factorized in a multiplicative way, where the randomness of intensity is governed by the intensive form map alone while the random expansionary force, the growth factor $\lambda(\omega)$, is independent of the state and of the intensity, i.e. randomness of intensity and randomness of scale are processes depending only on the perturbation.

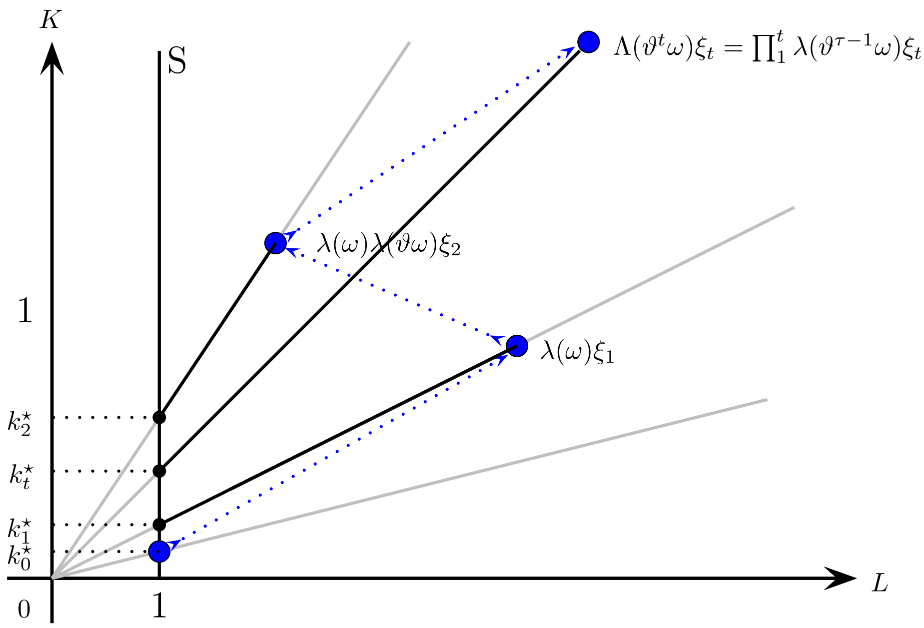


Figure 5: A balanced growth path $\{C(t,\omega)\xi(\omega)\}$ of the Solow model for the random fixed point $\xi(\omega) \equiv (1, k^\star(\omega))$ and Perron-Frobenius solution $\lambda(\omega)\xi(\vartheta\omega) = G(\omega)\xi(\omega)$.

Definition 3.3. *An orbit* $\gamma(\omega,(L_0,K_0)) := \{C(t,\omega)(L_0,K_0)\}_t^\infty$ *of* $G(\omega)$ *is called balanced if there exists a Perron-Frobenius solution* (λ,ξ), $\lambda:\Omega\to\mathbb{R}_+$, $\xi:\Omega\to\mathbb{R}^2_+$, *such that*

$$C(t,\omega)\xi(\omega) = \left(\prod_{\tau=1}^{t}\lambda(\vartheta^{\tau-1}\omega)\right)\cdot\tilde{c}(t,\omega)\xi(\omega) =: \Lambda(\vartheta^t\omega)\cdot\tilde{c}(t,\omega)\xi(\omega). \tag{3.12}$$

In other words, each state along a balanced growth path is given by the state of a random eigenvector **multiplied** by the product of the cumulative growth factors of the preceding states. Thus, the balanced growth factors depend on the previous growth factors but not on the current or previous states along the path. By construction, one obtains the following lemma.

Lemma 3.1. *A random fixed point* $k^\star$ *of the intensive form of the Solow model induces a balanced growth path with growth factor* $\lambda(\omega) = (1+n(\omega))$.

Proof. The equality $G(\omega) = (1+n(\omega))\begin{pmatrix}1\\ g(\omega)\end{pmatrix}$ from (3.4) and $\xi(\omega) = \begin{pmatrix}1\\ k^\star(\omega)\end{pmatrix}$ imply

$$\begin{aligned}\lambda(\omega)\xi(\vartheta(\omega)) = G(\omega)\xi(\omega) &= (1+n(\omega))\begin{pmatrix}1\\ g(\omega)\end{pmatrix}\xi(\omega)\\ &= (1+n(\omega))\begin{pmatrix}1\\ g(\omega)\end{pmatrix}\begin{pmatrix}1\\ k^\star(\omega)\end{pmatrix} = (1+n(\omega))\begin{pmatrix}1\\ k^\star(\vartheta\omega)\end{pmatrix}\\ &= (1+n(\omega))\xi(\vartheta\omega),\end{aligned} \tag{3.13}$$

so that $\lambda(\omega) = 1+n(\omega)$ follows. □

Figure 5 shows the relationship between the orbit $\{k_t^\star\} \equiv \{k^\star(\vartheta^t\omega)\} = \{c(t,\omega)k^\star(\omega)\}$ of the random fix point $k^\star$ and the balanced growth path $\gamma(\omega,\xi(\omega)) = \{C(t,\omega)\xi(\omega)\}$.

3.3 Stable Balanced Growth Paths

In order to discuss convergence and stability of random orbits in the Solow model the notion of the stability of a random fixed point is used (for more details see Appendix B).

Definition 3.4. *A random fixed point* $k^\star:\Omega\to\mathbb{R}_+$ *of the Solow model in intensive form is called asymptotically stable if*

$$\lim_{t\to\infty}|c(t,\omega)k_0 - k^\star(\vartheta^t\omega)| = 0 \quad \mathbb{P}\text{-a.s.} \tag{3.14}$$

for all $k_0\in\mathcal{B}(k^\star(\omega))$, *the basin of attraction of* $k^\star$, *where* $c(t,\omega)$ *is the mapping* (3.7) *associated with* $g(\omega)$.

Let $\{(L_t,K_t) = C(t,\omega)(L_0,K_0)\}$ denote an orbit in state space and $\{k_t = c(t,\omega)k_0\}$ one of the intensive form with $K_0 = k_0L_0$ and $S := \{(x_1,x_2)\in\mathbb{R}^2_+ \mid x_1 = 1\}$.

Definition 3.5. *The distance of* $\{C(t,\omega)(L_0,K_0)\}$ *of* G *to the balanced one* $\{C(t,\omega)\xi(\omega)\}$ *associated with the random fixed point* $\xi(\omega)\equiv(1,k^\star(\omega))$, $\xi:\Omega\to S$ *is given by*

$$\begin{aligned}\Delta_t = \Delta(t,\omega)(L_0,K_0) :=& |C(t,\omega)(L_0,K_0) - C(t,\omega)\xi(\omega))|\\ =& |C(t,\omega)(L_0,K_0) - \Lambda(\vartheta^t\omega)\xi(\vartheta^t\omega))|.\end{aligned} \tag{3.15}$$

An orbit $\{C(t,\omega)(L_0,K_0)\}$ *is said to* ***converge*** *to the balanced orbit associated with* k^* *if for all* $k_0\in\mathcal{B}(\xi(\omega))\subset S$ *and for all* $(L_0,K_0) = (1,k_0)\neq\xi(\omega)$:

$$\lim_{t\to\infty}|c(t,\omega)k_0 - \xi(\vartheta^t\omega)| = 0 \quad \text{and} \quad \lim_{t\to\infty}|\Delta(t,\omega)(L_0,K_0)| = 0, \quad \mathbb{P}\text{-a.s.}. \tag{3.16}$$

Figure 6 displays the implications of a large versus a small expansionary growth factor of labor for a given contractionary effect of intensity for the Solow model implying an increase or a decrease of the induced distance.

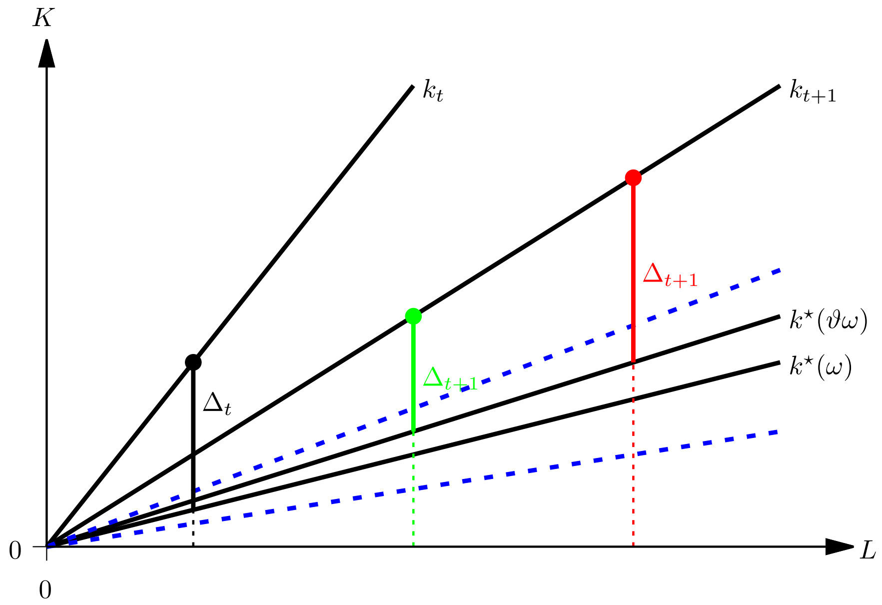


Figure 6: Convergence/divergence to balanced path in (K,L)-space: $\boldsymbol{\Delta_{t+1}} > \boldsymbol{\Delta_t} > \boldsymbol{\Delta_{t+1}}$

Theorem 3.1.
Let $k^\star : \Omega \to \mathbb{R}_+$ be an asymptotically stable random fixed point of $g(\omega)$ inducing the rate of contraction

$$M(\omega, \xi^\star(\omega)) := \lim_{k_0 \to k^\star(\omega)} \left| \frac{|g(\omega)k_0 - g(\omega)k^\star(\omega)|}{|k_0 - k^\star(\omega)|} \right| < 1, \quad \mathbb{P}\text{-a.s.} \tag{3.17}$$

of $g(\omega)$ at $\xi^\star(\omega) = (1, k^\star(\omega))$. For almost all $\omega \in \Omega$ and any $(L_0, k_0 L_0)$, $k_0 \in \mathcal{B}(g(\omega))$, $k_0 \neq k^\star(\omega)$ with $\lim_{t\to\infty} |c(t,\omega)k_0 - k^\star(\vartheta^t\omega)| = 0$, the distance $\Delta_t := |C(t,\omega)(L_0, K_0) - \Lambda(t,\omega) \cdot \xi^\star(\vartheta^t\omega))|$ satisfies $\mathbb{P}$-a.s.:

$$\lim_{t\to\infty} |\Delta_t| = 0 \qquad \text{if} \qquad \mathbb{E}\log(\lambda(\omega, \xi^\star(\omega)) + \mathbb{E}\log M(\omega, \xi^\star(\omega)) < 0 \tag{3.18}$$

$$\lim_{t\to\infty} |\Delta_t| = \infty \qquad \text{if} \qquad \mathbb{E}\log(\lambda(\omega, \xi^\star(\omega)) + \mathbb{E}\log M(\omega, \xi^\star(\omega)) > 0. \tag{3.19}$$

The proof is identical to the one given in the appendix for Theorem B.2 by replacing the unit simplex S as the domain for the intensive form in Theorem B.2 by the set $S = \{(x_1, x_2) \in \mathbb{R}^2_+ \,|\, x_1 = 1\}$ in Theorem 3.1.

Corollary 3.1. *Let $g(\omega) : \mathbb{R}_+ \to \mathbb{R}_+$ be monotonically increasing and differentiable and assume that $k^\star : \Omega \to \mathbb{R}_+$ is an asymptotically stable random fixed point of $g(\omega)$. The derivative of $g(\omega)$ at $k^\star(\omega)$ is given by*

$$g'(\omega)k^\star(\omega) = \lim_{k_0 \to k^\star(\omega)} \left| \frac{|g(\omega)k_0 - g(\omega)k^\star(\omega)|}{|k_0 - k^\star(\omega)|} \right| =: M(\omega, \xi^\star(\omega)), \quad \mathbb{P}\text{-a.s.}. \tag{3.20}$$

For almost all $\omega \in \Omega$ and any $(L_0, k_0 L_0)$, $k_0 \in \mathcal{B}(g(\omega))$, $k_0 \neq \xi^\star(\omega)$ with $\lim_{t\to\infty} |c(t,\omega)k_0 - k^\star(\vartheta^t\omega)| = 0$, the distance $\Delta_t := |C(t,\omega)(L_0, K_0) - \Lambda(t,\omega) \cdot \xi^\star(\vartheta^t\omega))|$ satisfies $\mathbb{P}$-a.s.:

$$\lim_{t\to\infty} |\Delta_t| = 0 \qquad \text{if} \qquad \mathbb{E}\left[(1 + n(\omega)) \cdot g'(\omega)\xi^\star(\omega)\right] < 1 \tag{3.21}$$

$$\lim_{t\to\infty} |\Delta_t| = \infty \qquad \text{if} \qquad \mathbb{E}\left[(1 + n(\omega)) \cdot g'(\omega)\xi^\star(\omega)\right] > 1 \tag{3.22}$$

Proof. Conditions (B.24) and (B.25) in the proof of Theorem B.2 imply convergence or divergence for the linear maps $\overline{\Delta}_t$ and $\underline{\Delta}_t$ if the expectation of the product $(1+n(\omega)) \cdot M(\omega, \xi(\omega))$ is less than one or bigger than one. Since, $\underline{\Delta}_t \leq \Delta(t, \omega) \leq \overline{\Delta}_t$, (3.21) and (3.22) hold. □

The corollary reveals that the random eigenvalue of the Perron-Frobenius solution has the product form $\lambda(\omega) = (1+n(\omega)) \cdot M(\omega, \xi(\omega))$. Its expectation plays a critical role for upper and lower bounds of the growth factor $\lambda(\omega)$ near balanced paths.

As in the deterministic case, pointwise convergence to the random fixed point $k^\star$ is only a necessary condition for convergence of an orbit in state space to the balanced growth path, the structural reason being the same:

> ***Convergence*** *in state space depends on the interplay of the* ***contracting*** *forces in intensity and the* ***expanding*** *forces of the growing labor supply. Convergence of the intensive form evaluates the contractionary forces of intensities only and disregards the size of the expansionary forces of labor supply.*

Formally, *mean-contractivity* does not require that either of the two interacting variables must be contractive almost surely (see Arnold & Crauel, 1992). Nevertheless, a sufficient degree of contractivity of the intensive form is needed to assure its convergence. It is a challenge to investigate the consequences of both of these observation for other models of economic growth.

There are direct further applications of these results to more general growth models.

- The multidimensional stochastic version of the extended Solow model with an aging workforce, i.e. with OLG consumers, and vintage capital (as introduced in Section 2.2) arises when the coefficients $(\boldsymbol{d}, \boldsymbol{n}, \boldsymbol{\delta})$ of the two matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ are random. Their properties together with those of the production function determine the conditions for convergence to their random balanced growth paths in state space according to Theorem B.2.
- Questions of viability or sustainability in models with public debt, financial assets, with insurance, or pension systems are connected to the convergence issue under expansion in homogeneous models. Böhm & Hillebrand (2007) presents an application of an intensive form model examining the efficiency of Pay-As-You-Go pension systems in a stochastic economy with multiperiod overlapping generations of consumers where compulsory public retirement savings coexists with private savings and assets. The convergence issue is not treated.
- Multisector models with heterogeneous resources, of countries, industries, or of the environment are further examples for applications of the features of the theorem.
- Last but not least, Theorem B.2 and 3.1 could be used to determine how the mean contractivity rule translates into specific conditions for stable random paths in optimal growth and in models with overlapping generations.

4 A Stochastic AS-AD Model with Money

Monetary macroeconomic models make up another important area where methods of an orbit-oriented approach are useful when stability or convergence of time series are to be investigated. This seems to be particularly desirable for stochastic models with rational expectations. One such model was presented in Böhm (2017) which is a particular tractable version of a closed demand consistent temporary equilibrium model with a stochastic aggregate supply function of the Lucas-type. The introduction of a multiplicative random production shock in the deterministic model with perfect foresight turns the parametrized deterministic AS-AD-system (see Chapter 4 in Böhm, 2017) into a two-dimensional homogeneous random dynamical system under rational expectations. Its dynamic properties are reanalyzed here applying the properties of random Perron-Frobenius solutions from Babaei, Evstigneev & Pirogov (2018).

Let the random perturbation be given by a bounded positive multiplicative (Hicks neutral) production shock defined by a random variable $Z : \Omega \to [Z_{\min}, Z_{\max}]$, $0 < Z_{\min} < Z_{\max} < \infty$ for a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with time-shift $\vartheta : \Omega \to \Omega$, see Appendix B. This implies two homogeneous stochastic difference equations under rational expectations in the state variables money balances M_t and the mean of expected future prices p^e_t given by

$$\begin{pmatrix} M_{t+1} \\ p^e_{t+1} \end{pmatrix} = \begin{pmatrix} \mathcal{M}(M_t, p^e_t, Z_t) \\ \Psi(M_t, p^e_t, Z_t) \end{pmatrix} := \begin{pmatrix} M_t \left(\dfrac{\tilde{c} - \tau^\star}{\tilde{c}}\right) (1 + \bar{g}\mathcal{P}(1, \psi^*(1, p^e_t/M_t), Z_t)) \\ M_t\, \psi^*(1, p^e_t/M_t) \end{pmatrix} \tag{4.1}$$

inducing a random family of mappings $G(\omega) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, $(M, p^e) \mapsto G(\omega)(M, p^e)$,

$$G(\omega)(M, p^e) = \begin{pmatrix} M \left(\dfrac{\tilde{c} - \tau^\star}{\tilde{c}}\right) (1 + \bar{g}\mathcal{P}(1, \psi^*(1, p^e/M), Z(\omega)) \\ M\, \psi^*(1, p^e/M) \end{pmatrix}, \quad \mathbb{P}\text{-a.s..} \tag{4.2}$$

Here, $\mathcal{P}$ is the random equilibrium-price law, a mapping which is homogeneous of degree one in (M_t, p^e_t) for each level of Z, with $\bar{g} > 0$ being the level of government real demand. $(\tilde{c} - \tau^*)(\tilde{c}$ denotes the net consumption multiplier of aggregate demand arising from isoelastic consumption characteristics. Therefore, the first equation of (4.1) describes the evolution of monetary growth.

The variable $p^e_t \equiv p^e_{t-1,t}$ denotes the prediction made in $t-1$ for the mean price in period t, and ψ^* is the unbiased predictor (a homogeneous forecasting rule depending on money balances and the previous prediction) making the mean prediction unbiased along orbits[5]. Thus, the second equation guarantees rational expectations along orbits in the usual sense. For simplicity it is assumed that the noise is an i.i.d. process. This makes the unbiased predictor ψ^* a deterministic function independent of $Z(\omega)$.

4.1 State Space vs. Intensive Form

As for the two-dimensional Solow growth model the following concepts and definitions are to be used. The one-dimensional intensive form $g(\omega) : \mathbb{R}_+ \to \mathbb{R}_+$, of the AS-AD model with $q^e := p^e/M$ is defined as

$$g(\omega)q^e := \frac{\Psi(M, p^e, Z(\omega))}{\mathcal{M}(M, p^e, Z(\omega))} = \left(\frac{\tilde{c}}{\tilde{c} - \tau^\star}\right) \frac{\psi^*(1, q^e)}{1 + \bar{g}\mathcal{P}(1, \psi^*(1, q^e), Z(\omega))}, \quad \mathbb{P}\text{-a.s.} \tag{4.3}$$

which implies the pointwise relationship between the two mappings $\mathbb{P}$-a.s.

$$G(\omega)(M, p^e) = M \left(\frac{\tilde{c} - \tau^\star}{\tilde{c}}\right) (1 + \bar{g}\mathcal{P}(1, \psi^*(1, q^e), Z(\omega)) \begin{pmatrix} 1 \\ g(\omega)q^e \end{pmatrix}, \quad p^e = q^e M. \tag{4.4}$$

[5]For an i.i.d. perturbation $Z(\omega)$ with measure $\mu \in \text{Prob}\,[Z_{\min}, Z_{\max}]$, an *unbiased predictor* is a recursive mapping $\psi^* : \mathbb{R}^2_+ \to \mathbb{R}_+$, $(M, p^e) \mapsto \psi^*(M, p^e) = p^e_1$ which solves $(\mathbb{E}_\mu \mathcal{P})(M, \psi^*(M, p^e)) = p^e$ for every (M, p^e). Thus, ψ^* is an inverse of the mean-price law $(\mathbb{E}_\mu \mathcal{P})$ with respect to p^e.

Given ω and any initial condition (M_0, p_0^e), the state (M_t, p_t^e) of the system G at date $t > 0$ periods is generated by the mapping

$$C(t,\omega)(M_0,p_0^e) := \begin{cases} G(\omega)(\vartheta^{t-1}\omega) \circ \cdots \circ G(\omega)(M_0,p_0^e) & t > 0 \\ id_{\mathbb{R}^2_+} & t = 0 \end{cases} \tag{4.5}$$

which satisfies

$$C(t+s,\omega) = C(t,\vartheta^s\omega) \circ C(s,\omega) \qquad \text{for all } t, s. \tag{4.6}$$

Therefore, for any $\omega \in \Omega$, the orbit of money balances and price expectations in state space with initial condition $(M_0, p_0^e) \in \mathbb{R}^2_+$ is given by $\gamma(\omega, (M_0, p_0^e)) := \{C(t,\omega)(M_0, p_0^e)\}_t^\infty$. Similarly, for any initial condition $q_0^e \in \mathbb{R}_+$, orbits of the intensive form $g(\omega)$ are given by $\gamma(\omega, q_0^e) := \{c(t,\omega)q_0^e\}_t^\infty$ since the state q_t^e of the intensive form system is generated by the one-dimensional mapping $c(t,\omega) : \mathbb{R}_+ \to \mathbb{R}_+$

$$c(t,\omega)q_0^e := \begin{cases} g(\vartheta^{t-1}\omega) \circ \cdots \circ g(\omega)q_0^e & t > 0 \\ id_{\mathbb{R}_+} & t = 0 \end{cases} \tag{4.7}$$

which also satisfies $c(t+s,\omega) = c(t,\vartheta^s\omega) \circ c(s,\omega)$, for all t, s. For notational consistency the two-dimensional mapping $\tilde{c}(t,\omega) := (1, c(t,\omega)) : S \to S$, $S := \{1\} \times \mathbb{R}_+$

$$\tilde{c}(t,\omega)\begin{pmatrix} 1 \\ k_0 \end{pmatrix} := \begin{cases} \begin{pmatrix} 1 \\ c(t,\omega)k_0 \end{pmatrix} & t > 0 \\ id_S & t = 0 \end{cases} \tag{4.8}$$

will also be used satisfying $\tilde{c}(t+s,\omega) = \tilde{c}(t,\vartheta^s\omega) \circ \tilde{c}(s,\omega)$, for all t, s as well.

4.2 Balanced Monetary Growth

As for the growth model in the previous section, stationary solutions of the real part of the AS-AD model are generated by *random fixed points* of the intensive form (4.3), see Definition B.2 in Appendix B.

Definition 4.1. *A random fixed point of the intensive form* (4.3) *of the* AS-AD *model is a random variable* $q^* : \Omega \to \mathbb{R}_+$ *solving*

$$q^\star(\vartheta\omega) = g(\omega)q^\star(\omega) := \left(\frac{\tilde{c}}{\tilde{c} - \tau^\star}\right) \frac{\psi^*(1, q^\star(\omega))}{1 + \bar{g}\mathcal{P}(1, \psi^*(1, q^\star(\omega)), Z(\omega))}, \quad \mathbb{P}\text{-}a.s. \tag{4.9}$$

$$= \psi^*(1, q^\star(\omega)) \frac{\left(\frac{\tilde{c}}{\tilde{c} - \tau^\star}\right)}{1 + \bar{g}\mathcal{P}(1, \psi^*(1, q^\star(\omega)), Z(\omega))}, \quad \mathbb{P}\text{-}a.s.. \tag{4.10}$$

In other words, the fixed point is the mean prediction deflated by the money growth rate. This implies the following definition of a Perron-Frobenius solution of $G(\omega)$.

Definition 4.2. *A pair of random variables* $\lambda : \Omega \to \mathbb{R}_{++}$ *and* $\xi : \Omega \to S$ *is a Perron-Frobenius solution for the* AS-AD *model* $G(\omega)$ *if* $\mathbb{P}$*-a.s.:*

$$G(\omega)\xi(\omega) = \lambda(\omega)\xi(\vartheta(\omega)), \tag{4.11}$$

$$\begin{pmatrix} 1 \\ g(\omega) \end{pmatrix} \xi(\omega) = \xi(\vartheta(\omega)), \qquad \xi(\omega), \xi(\vartheta\omega) \in S. \tag{4.12}$$

This provides the prerequisites for the definition of a balanced monetary path.

Definition 4.3. *An orbit* $\gamma(\omega, (M_0, p_0^e)) := \{C(t,\omega)(M_0, p_0^e)\}_t^\infty$ *of* $G(\omega)$ *is called balanced if there exists a Perron-Frobenius solution* (λ, ξ)*,* $\lambda : \Omega \to \mathbb{R}_+$*,* $\xi : \Omega \to \mathbb{R}_+^2$*, such that* $\mathbb{P}$*-a.s. :*

$$C(t,\omega)\xi(\omega) = \left(\prod_{\tau=1}^{t} \lambda(\vartheta^{\tau-1}\omega)\right) \cdot \tilde{c}(t,\omega)\xi(\omega) =: \Lambda(\vartheta^t\omega) \cdot \tilde{c}(t,\omega)\xi(\omega). \tag{4.13}$$

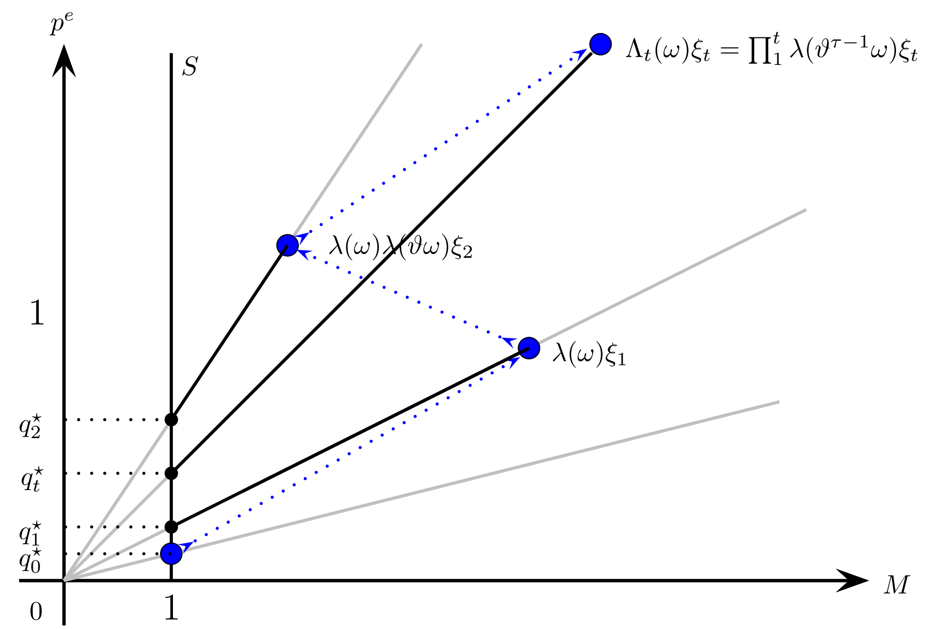


Figure 7: A balanced monetary path $\{C(t,\omega)\xi(\omega)\}$ of the AS-AD model for the random fixed point $\xi(\omega) \equiv (1, q^\star(\omega))$ and Perron-Frobenius solution $\lambda(\omega)\xi(\vartheta\omega) = G(\omega)\xi(\omega)$.

By definition, each state along a balanced monetary path is given by the state of the random eigenvector **multiplied** by the product of the cumulative growth factors of money (the eigenvalues at the preceding dates). The eigenvalue $\lambda(\omega)$ depends on the noise process alone so do their cumulative products of the preceding dates. Therefore, balanced factors of monetary expansion depend on the previous growth factors but not on the current nor on previous states along the path. Thus, along the balanced path the expansionary forces of scale and the contractionary forces governing stationary intensities are uncoupled stochastically. By construction their interaction follows from the associated random fixed point of the intensive form, as stated in the next lemma (which is identical to Lemma 3.1 of the Solow growth case).

Lemma 4.1. *A random fixed point* $q^\star : \Omega \to \mathbb{R}_{++}$ *of the intensive form of the* AS-AD *model induces a balanced monetary path with growth factor*

$$\lambda(\omega) = \left(\frac{\tilde{c} - \tau^\star}{\tilde{c}}\right)(1 + \bar{g}\mathcal{P}(1, \psi^*(1, q^\star(\omega)), Z(\omega))\,. \tag{4.14}$$

.

Proof. According to Lemma B.2, a random fixed point induces a Perron-Frobenius solution with eigenvalue equal to the growth factor of the fixed point. □

Lemma 4.1 together with Definition 4.3 imply geometrically that each state of the fixed point in S is the radial projection of an orbital state onto S, as shown in Figure 7. Loosely speaking, the orbit of the fixed point is the radial projection of the balanced orbit in state space. Figure 7 and Figure 5 portray identical features for two very similar two-dimensional models.

To understand the properties of balanced monetary expansion, it is useful to reveal some further properties of the random fixed point. For all $t \geq 0$, one has

$$\frac{p_t^e}{M_t} = q_t^e = q^\star(\vartheta^t\omega). \tag{4.15}$$

Equations (4.2) and (4.14) imply that along the balanced path money holdings and rational mean price predictions are generated by two *linear* random difference equations in diagonal form

$$\begin{pmatrix} p_{t+1}^e \\ M_{t+1} \end{pmatrix} = \begin{pmatrix} \dfrac{\psi^*(1, q^\star(\vartheta^t\omega))}{q^\star(\vartheta^t\omega)} & 0 \\ 0 & \left(\dfrac{\tilde{c}-\tau^\star}{\tilde{c}}\right)(1+\bar{g}\mathcal{P}(1, \psi^*(1, q^\star(\vartheta^t\omega)), Z(\vartheta^t\omega))) \end{pmatrix} \begin{pmatrix} p_t^e \\ M_t \end{pmatrix}$$
$$=: \begin{pmatrix} \pi(\omega) & 0 \\ 0 & \mu(\omega) \end{pmatrix} \begin{pmatrix} p_t^e \\ M_t \end{pmatrix} \tag{4.16}$$

with growth rates $p_{t+1}^e/p_t^e \equiv \pi : \Omega \to \mathbb{R}_+$, $M_{t+1}/M_t \equiv \mu : \Omega \to \mathbb{R}_+$, and $\mu(\omega) \equiv \lambda(\omega)$. In other words, the eigenvalue of the Perron-Frobenius solution, induced by the random fixed point (4.14), is the monetary growth factor which is perfectly correlated with the mean forecast of the price level. In addition, the two growth factors satisfy

$$\pi(\omega)q^*(\omega) = \psi^*(1, q^\star(\omega)) = \frac{1+\bar{g}\mathcal{P}(1, \psi^*(1, q^\star(\omega)), Z(\omega))}{\left(\frac{\tilde{c}}{\tilde{c}-\tau^\star}\right)} q^*(\vartheta\omega) = \mu(\omega)q^*(\vartheta\omega), \ \mathbb{P}\text{-a.s.}. \tag{4.17}$$

This also implies

$$g(\omega)q^*(\omega) = q^*(\vartheta\omega) = \frac{\pi(\omega)}{\mu(\omega)} q^*(\omega), \quad \mathbb{P}\text{-a.s.}, \tag{4.18}$$

i.e. orbits of the random fixed point $q^* : \Omega \to \mathbb{R}_+$ satisfy a one-dimensional linear difference equation with random coefficient $\pi(\omega)/\mu(\omega)$,

$$q_{t+1}^\star = \frac{p_{t+1}^e}{M_{t+1}} = \frac{\pi(\vartheta^t\omega)}{\mu(\vartheta^t\omega)} \frac{p_t^e}{M_t} = \frac{\pi(\vartheta^t\omega)}{\mu(\vartheta^t\omega)} q_t^\star \tag{4.19}$$

In other words, in this highly nonlinear AS-AD model, rational mean predictions deflated by money growth are generated by a *linear* random difference equation if the underlying process of productivity shocks is multiplicative and i.i.d.. The formulas (4.15) - (4.19) reflect the typical serial correlation of expected inflation rates and money balances for balanced monetary expansion under rational expectations due to the expectational lead in consumption demand of the AS-AD model.

4.3 Stable Monetary Growth

Orbits in state space of the AS-AD system (4.2) are defined as sequences generated by the mapping (4.5). Let $C(t,\omega) = (M(t,\omega), p^e(t,\omega))$, $C(t,\omega) : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ denote its two component maps for money balances and price predictions, i.e.

$$C(t,\omega)(M_0, p_0^e) \equiv \begin{pmatrix} M(t,\omega) \\ p^e(t,\omega) \end{pmatrix} (M_0, p_0^e). \tag{4.20}$$

Then, given ω and any initial condition (M_0, p_0^e), an orbit in state space can be written as $\gamma(\omega)(M_0, p_0^e) = \{M(t,\omega)(M_0, p_0^e), p^e(t,\omega)(M_0, p_0^e)\}_0^\infty$. Define the distance of an orbit of (4.2) to the balanced path associated with $q^\star$ as

$$\Delta_t = \Delta(t,\omega)(M_0, p_0^e) := p^e(t,\omega)(M_0, p_0^e) - q^\star(\vartheta^t\omega) \cdot M(t,\omega)(M_0, p_0^e). \tag{4.21}$$

Definition 4.4. *A balanced orbit associated with the random fixed point* $q^\star : \Omega \to \mathbb{R}_+$ *of* $g(\omega)$ *is called* ***asymptotically stable*** *if, for all* $q_0^e \in \mathcal{B}(q^\star(\omega))$, (the basin of attraction of $q^\star$), *and* (M_0, p_0^e) *in a neighborhood* $\mathcal{U}(\bar{M}_0, \bar{p}_0^e, \omega)$, $\bar{p}_0^e = \bar{M}_0\, q^\star(\omega)$, $p_0^e = q_0^e M_0$*:*

$$\lim_{t\to\infty} |c(t,\omega)q_0^e - q^\star(\vartheta^t\omega)| = 0 \quad \textit{and} \quad \lim_{t\to\infty} |\Delta(t,\omega)(M_0, p_0^e)| = 0, \quad \mathbb{P}\text{-}a.s.. \tag{4.22}$$

The first condition imposes that all orbits from the basin of attraction of the random fixed point in intensive form converge to the orbit of the fixed point. The second one requires in addition that the associated orbit in state space $\{C(t,\omega)(M_0, p_0^e)\}_0^\infty$ converges pointwise to the balanced path.

Figure 8 displays the convergence issue in state space. The two rays $q^\star(\omega)$ and $q^\star(\vartheta\omega)$ describe the one-step movement of the stochastic fixed point moving within the cone of the two blue dashed lines (an attracting set of the fixed point). The one-step move of the orbit in intensive form converging to the random fixed point is indicated by the pair q_t^e and q_{t+1}^e. For (M_t, p_t^e) with distance $\Delta_t \equiv \Delta(t,\omega)$ the two possible cases of convergence Δ_{t+1} or of divergence with $\Delta_{t+1} > \Delta_t > \Delta_{t+1}$ are indicated for the same $\omega \in \Omega$. The diagram visualizes the possibility of convergence or divergence depending on the rate of monetary expansion (given the rate of contraction of the fixed point) which is shown to be larger for Δ_{t+1} than for Δ_{t+1} in the one-step description for an arbitrary point (M_t, p_t^e) along the orbit. Theorem 4.1 states that

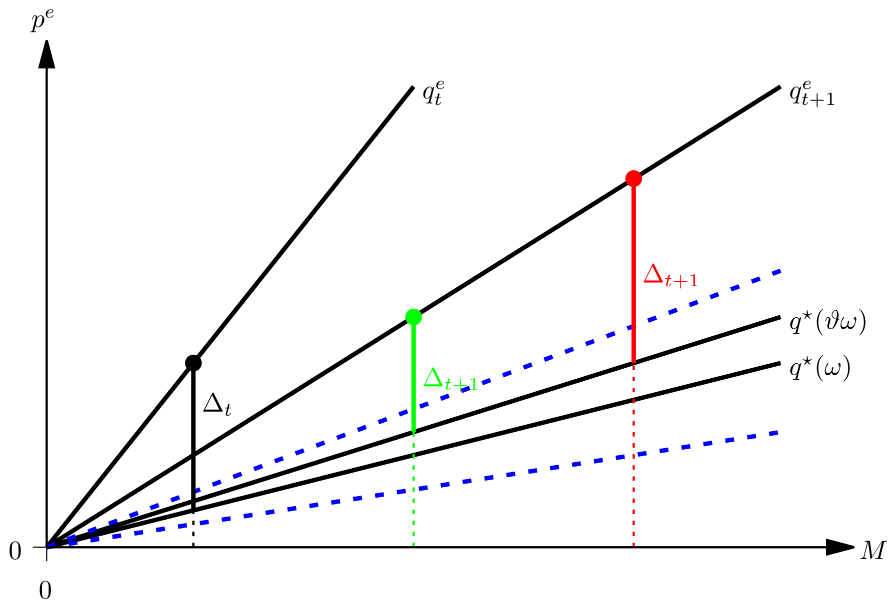


Figure 8: Convergence/divergence to balanced path in (M, p^e)-space: $\boldsymbol{\Delta_{t+1}} > \boldsymbol{\Delta_t} > \boldsymbol{\Delta_{t+1}}$

convergence (divergence) occurs when the expansionary forces are appropriately dominated (or not dominated) *on average* by the rate of contraction of the random fixed point, i.e. keeping the random intensity sufficiently bounded relative to the rate of expansion along the random fixed point.

Theorem 4.1.
Let g be differentiable and increasing with respect to q^e and let $q^\star : \Omega \to \mathbb{R}_+$ be an asymptotically stable random fixed point of (4.3)

$$q^e_{t+1} = g(\vartheta^t\omega)q^e_t := \left(\frac{\tilde{c}}{\tilde{c} - \tau^\star}\right)\frac{\psi^*(q^e_t)}{1 + \bar{g}\mathcal{P}(1, \psi^*(q^e_t), Z(\vartheta^t\omega))}.$$

Then, for almost all $\omega \in \Omega$ and any $q^e_0 \in \mathcal{B}(q^\star(\omega))$, $q^e_0 \neq q^\star(\omega)$ with $\lim_{t\to\infty}|c(t,\omega)q^e_0 - q^\star(\vartheta^t\omega)| = 0$ the distance $\Delta_t := p^e(t,\omega)(M_0, p^e_0) - q^\star(\vartheta^t\omega)\, M(t,\omega)(M_0, p^e_0)$ satisfies $\mathbb{P}$-a.s.:

$\lim_{t\to\infty}|\Delta_t| = 0$ *if*

$$\mathbb{E}\log(g'(\omega)q^\star(\omega) + \mathbb{E}\log\left(\frac{\tilde{c} - \tau^\star}{\tilde{c}}\right)(1 + \bar{g}\mathcal{P}(1, \psi^*(q^\star(\omega)), Z(\omega))) < 0 \tag{4.23}$$

$\lim_{t\to\infty}|\Delta_t| = \infty$ *if*

$$\mathbb{E}\log(g'(\omega)q^\star(\omega) + \mathbb{E}\log\left(\frac{\tilde{c} - \tau^\star}{\tilde{c}}\right)(1 + \bar{g}\mathcal{P}(1, \psi^*(q^\star(\omega)), Z(\omega))) > 0. \tag{4.24}$$

The distance function Δ defines a second random difference equation $\Delta(\omega) : S \times \mathbb{R} \to \mathbb{R}$, making the pair (g, Δ) a two dimensional random dynamical system $(g, \Delta)(\omega) : S \times \mathbb{R} \to S \times \mathbb{R}$ with random fixed point $(q^\star, 0) : \Omega \to S \times \mathbb{R}$.

Proof. Let $q^e_0 \in \mathcal{B}(q^\star(\omega))$ with $\lim_{t\to\infty}|c(t,\omega)q^e_0 - q^\star(\vartheta^t\omega)| = 0$ and $p^e_0 = q^e_0 M_0$ such that $(M_0, p^e_0) \in \mathcal{U}(\bar{M}_0, \bar{p}^e_0, \omega)$, $\bar{p}^e_0 = \bar{M}_0\, q^*(\omega)$. From the definition $\Delta_t = M_t(q^e_t - q^\star(\vartheta^t\omega))$ and (4.3) one has for $q^e_t = c(t,\omega)q^e_0$:

$$\begin{aligned}
\Delta_{t+1} &= M_{t+1}(q^e_{t+1} - q^\star(\vartheta^{t+1}\omega)) = M_{t+1}(g(\vartheta^t\omega)q^e_t - g(\vartheta^t\omega)q^\star(\vartheta^t\omega))) \\
&= \frac{M_{t+1}}{M_t}\,\frac{g(\vartheta^t\omega)q^e_t - g(\vartheta^t\omega)q^\star(\vartheta^t\omega))}{q^e_t - q^\star(\vartheta^t\omega)}\Delta_t \\
&= \frac{\tilde{c} - \tau^*}{\tilde{c}}\left(1 + g\mathcal{P}(1, \psi^*(q^e_t), Z(\vartheta^t\omega))\right)\frac{g(\vartheta^t\omega)q^e_t - g(\vartheta^t\omega)q^\star(\vartheta^t\omega))}{q^e_t - q^\star(\vartheta^t\omega)}\Delta_t
\end{aligned}$$

implying

$$\frac{\Delta_{t+1}}{\Delta_t} = \frac{\tilde{c} - \tau^*}{\tilde{c}}\left(1 + \bar{g}\mathcal{P}(1, \psi^*(q^e_t), Z(\vartheta^t\omega))\right)\frac{g(\vartheta^t\omega)q^e_t - g(\vartheta^t\omega)q^\star(\vartheta^t\omega))}{q^e_t - q^\star(\vartheta^t\omega)}. \tag{4.25}$$

Since $\lim_{t\to\infty}|q^e_t - q^\star(\vartheta^t\omega)| = \lim_{t\to\infty}|c(t,\omega) - q^\star(\vartheta^t\omega)| = 0$, $\mathbb{P}$-a.s., there exists an $\varepsilon > 0$ sufficiently small and $t_0 = t_0(\varepsilon, \omega) > 0$ sufficiently large such that

$$\left|\mathcal{P}(1, \psi^*(q^e_t), Z(\vartheta^t\omega)) - \mathcal{P}(1, \psi^*(q^\star(\vartheta^t\omega), Z(\vartheta^t\omega))\right| < \varepsilon \tag{4.26}$$

$$\left|\frac{g(\vartheta^t\omega)q^e_t - g(\vartheta^t\omega)q^\star(\vartheta^t\omega)}{q^e_t - q^\star(\vartheta^t\omega)} - g'(\vartheta^t\omega)q^\star(\vartheta^t\omega)\right| < \varepsilon. \tag{4.27}$$

By induction one has $\underline{\Delta}_t \leq |\Delta_t| \leq \overline{\Delta}_t$ for all $t \geq t_0$, for the two linear random dynamical systems

$$\overline{\Delta}_{t+1} = \left(\frac{\tilde{c} - \tau^*}{\tilde{c}}\right)\left[\left(1 + \bar{g}\mathcal{P}(1, \psi^*(q^\star(\vartheta^t\omega)), Z(\vartheta^t\omega))\right)\, g'(\vartheta^t\omega)q^\star(\vartheta^t\omega) + \varepsilon\right]\overline{\Delta}_t \tag{4.28}$$

$$\underline{\Delta}_{t+1} = \left(\frac{\tilde{c} - \tau^*}{\tilde{c}}\right)\left[\left(1 + \bar{g}\mathcal{P}(1, \psi^*(q^\star(\vartheta^t\omega)), Z(\vartheta^t\omega))\right)\, g'(\vartheta^t\omega)q^\star(\vartheta^t\omega) - \varepsilon\right]\underline{\Delta}_t \tag{4.29}$$

with $\overline{\Delta}_{t_0} = |\Delta_{t_0}|$ and $\underline{\underline{\Delta}}_{t_0} = |\Delta_{t_0}|$. Therefore, assumption (4.23) implies that the upper bound (4.28) and Δ_t converge to zero $\mathbb{P}$-a.s.. Conversely, condition (4.24) implies that the lower bound grows to infinity eventually. In this case the distance of an orbit to the balanced path diverges under assumption (4.24). □

The findings of Theorem 4.1 are completely parallel to Theorem 3.1 of the random growth model, both of which are an application of Theorem B.2 of the appendix. Stable balanced monetary expansion depends on the size of the *product* of the eigenvalue of a stochastic Perron-Frobenius solution with the contractivity of the random fixed point of the underlying intensive form. The central equation (4.25)

$$\Delta_{t+1} = \frac{\tilde{c} - \tau^*}{\tilde{c}} \left(1 + \bar{g}\mathcal{P}(1, \psi^*(q_t^e), Z(\vartheta^t\omega))\right) \cdot \frac{g(\vartheta^t\omega, q_t^e) - g(\vartheta^t\omega, q^\star(\vartheta^t\omega))}{q_t^e - q^\star(\vartheta^t\omega)} \cdot \Delta_t$$

reveals that, for $t \to \infty$, the distance function Δ is given by a linear random dynamical system whose coefficient consists of the product of the eigenvalue with the contractivity of the intensive form model. If the expectation of the product $\lambda(\omega) \cdot g'(q^\star(\omega))$ is less than one, i.e. if the product is *mean contractive*, convergence occurs $\mathbb{P}$-a.s. which is equivalent to condition (4.23). As for the growth model, this means formally that neither of the two interacting variables need to be contractive almost surely. However, the asymptotic stability of the random fixed point requires that $g'(\omega)q^\star(\omega)$ must be less than one sufficiently often. Figure 13 in the appendix visualizes geometrically the opposing effects of the tradeoff between the expansionary and the contractionary forces acting along the balanced path.

Some additional qualitative features of the AS-AD economy can be described in a numerical analysis of a parametric example[6]. Let this be given by isoelastic production and consumption characteristics and a production shock with an i.i.d. discrete two point perturbation $Z \sim \{Z_{\min}, Z_{\max}\}$ with equal probability. A standard parametrization implies an isoelastic random

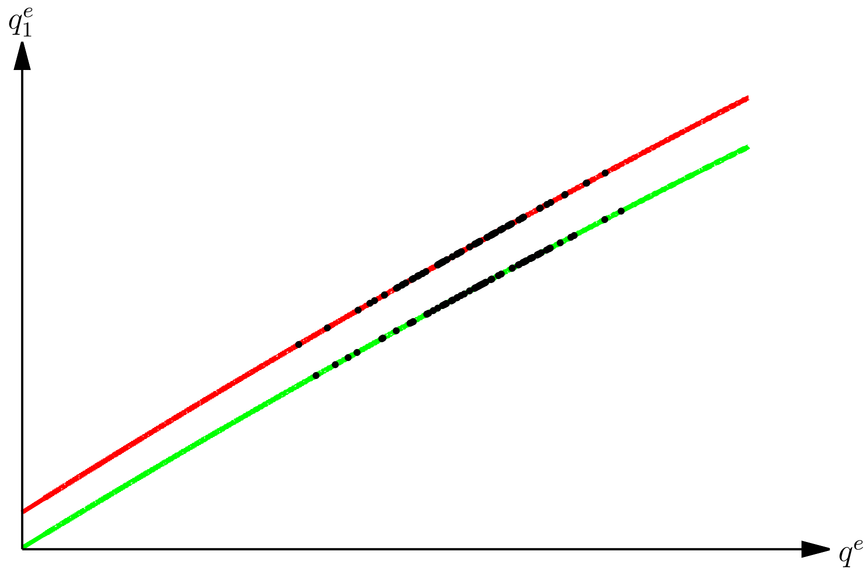


Figure 9: Graph of $g(\omega)q_3^\star(\omega)$ and part of an orbit $\{c(t,\omega)q_3^\star(\omega)\}$

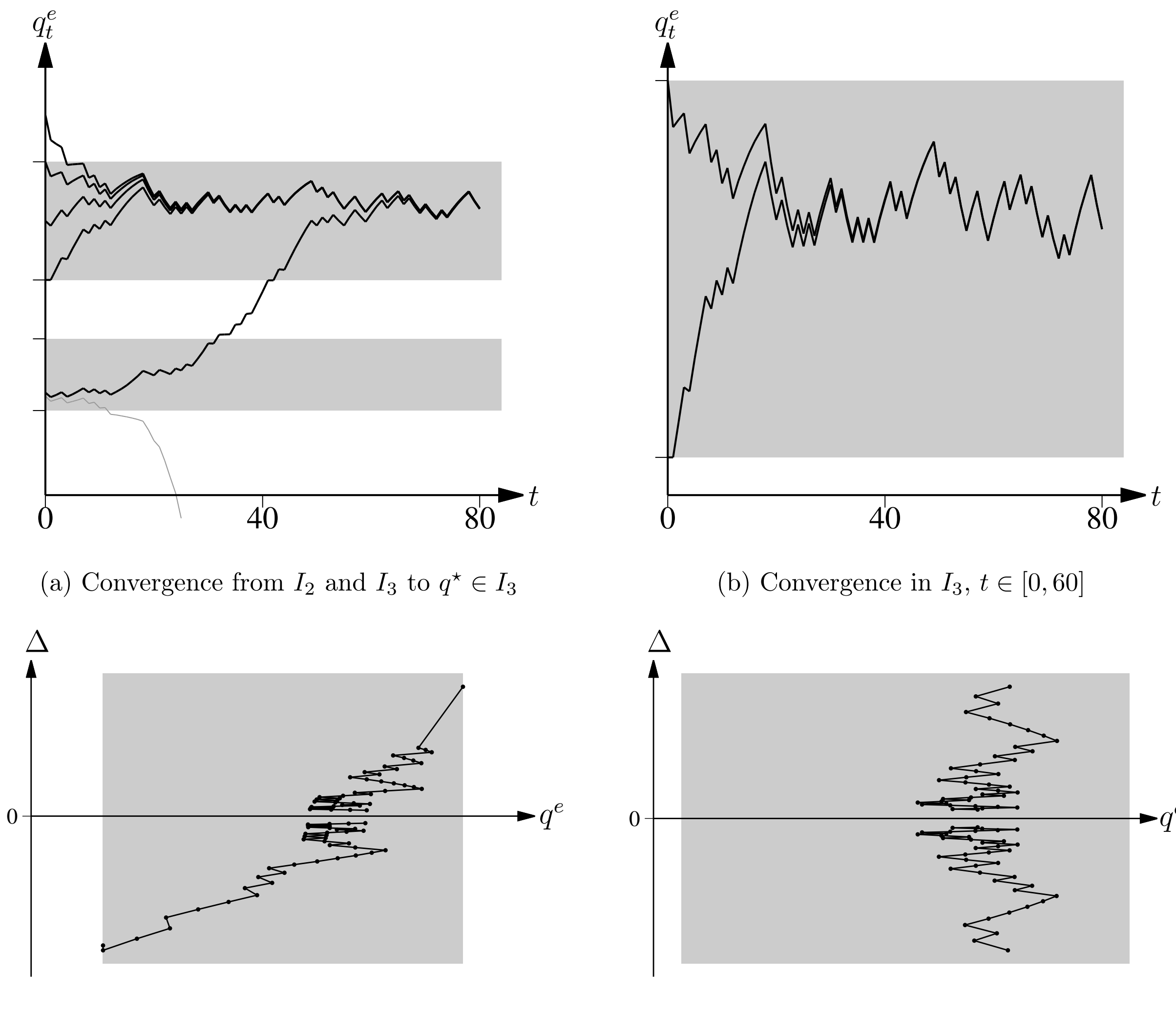
aggregate supply function of the Lucas type and a deterministic aggregate demand function

[6]see Chapter 4 of Böhm (2017) for details.

with a constant multiplier $(\tilde{c} - \tau^*)/\tilde{c}$. The associated random system $G(\omega)$ exhibits three random fixed points $q_i^\star$, $i = 1, 2, 3$ of the intensive form $g(\omega)$: a stable trivial one $q_1^\star(\omega) \equiv 0$ with range $I_1 \equiv 0$ and two strictly positive ones on disjoint intervals $I_2 \cap I_3 = \emptyset$, satisfying $0 < I_2 < I_3$, where $q_2^\star : \Omega \to I_2$ with $g^{-1}(\omega)I_2 \subset I_2$ is unstable while $q_3^\star : \Omega \to I_3$ with $g(\omega)I_3 \subset I_3$ is stable. Figure 9 shows the two-piece graph of the intensive form map $g(\omega)$ with part of an orbit $\{c(t,\omega)q_3^\star(\omega)\}$ in (q_t^e, q_{t+1}^e)-space. For the small shock chosen both graphs are almost linear maps with slope less than one within the range I_3.

Figure 10 displays the convergence features of $q_3^\star$ for six different initial conditions, five converging to I_3 and one diverging from I_2. The two invariant sets $I_2 < I_3$ are marked as gray bands in panel (a). Panel (b) indicates that convergence in I_3 is relatively fast. Panels (c) and (d) show the convergence in (q^e, Δ)-space for the two initial conditions plotted in panel (b) indicating that the distance Δ decreases monotonically to zero for the chosen ω. This translates into monotonic pointwise convergence to the balanced path in state space (M, p^e).

For different values of the fiscal parameters with lower government taxes τ^* and higher government demand $\bar{g}$ deficits and inflation along orbits increase causing a higher rate of monetary growth $\mu^\star(\omega)$ while the contractivity $g'(\omega)q_3^\star(\omega)$ is not strengthened substantially. This causes the balanced monetary path to become unstable since $\mathbb{E}\mu^\star(\omega) \cdot g'(\omega)q_3^\star(\omega) > 1$. Figure 11



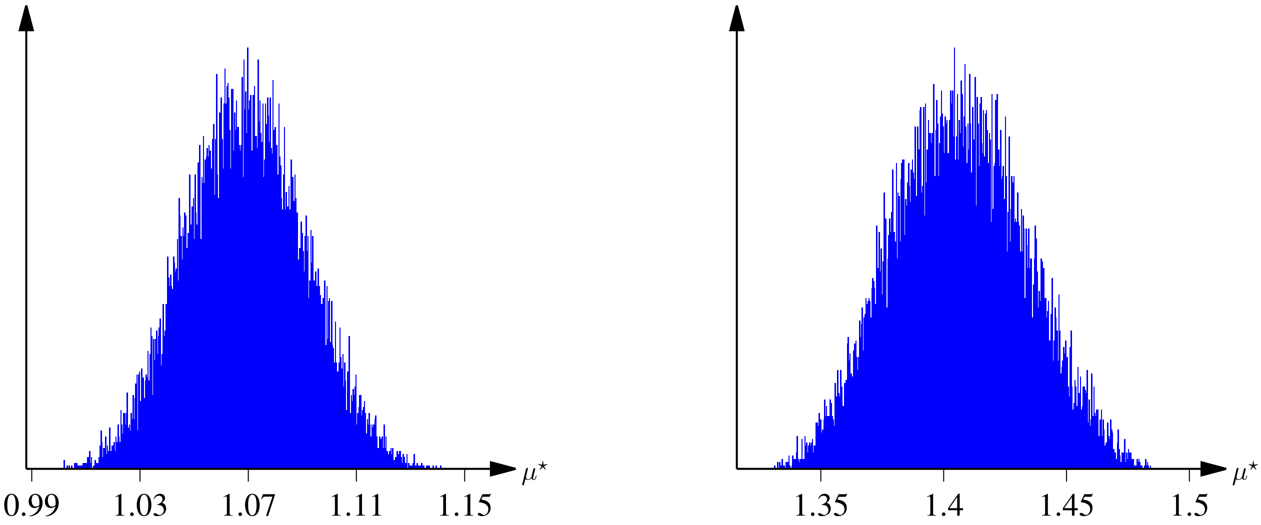
(a) Convergence from I_2 and I_3 to $q^\star \in I_3$

(b) Convergence in I_3, $t \in [0, 60]$

(c) Convergence in I_3: $t \in [0, 30]$

(d) Symmetry of orbits for $t > 30$

Figure 10: Convergence in (q^e, Δ)-space



(a) Stable balanced growth: $\mathbb{E}\mu^\star = 1.0690$ (b) Unstable balanced growth: $\mathbb{E}\mu^\star = 1.4063$

Figure 11: Impact of fiscal policy on monetary growth: $T = 2 \cdot 10^4$

displays the sizable mean shift of the distribution of the growth rate of money (the eigenvalue of the Perron-Frobenius solution) initiating the switch from convergence to divergence caused by the different values of the policy parameters. If $g'(\omega)q_3^\star(\omega) < 1, \mathbb{P}$-a.s., as in the numerical example, the stability requirement stipulates that the rate of monetary expansion may well be larger than one $\mathbb{P}$-a.s. along the whole orbit of $q_3^\star$, as long as it makes the product with $g'(\omega)q_3^\star(\omega)$ less than one on average. While money grows at a rate of less than 7 percent in the stable case, subfigure (a), it is about 40 percent in (b) indicating the reason for the balanced orbit to become unstable.

A Expansion of Deterministic Homogeneous Systems

Let $(F_i)_{i=1}^n$ denote a list of continuous mappings $F_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ and define $(F_i)_{i=1}^n \equiv F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $x \mapsto F(x)$. $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is the time-one map of a dynamical system. F is called *homogeneous* of degree one if all functions $F_i, i = 1, \dots, n$, are homogeneous of degree one, i.e. if for all $\lambda \geq 0$ and all $x \in \mathbb{R}^n_+$: $F(\lambda x) = \lambda F(x)$. Depending on applications other properties of the mapping F like concavity or monotonicity may be defined componentwise.

The mapping in so-called *intensive form associated with the homogeneous map* F from (the positive part of) the unit sphere S into itself is defined by $f : S \to S$, $y \mapsto f(y) := F(y)/|F(y)|$. For most applications the choice of the domain of the time-one map in intensive form is of secondary importance. Solow & Samuelson (1953) use the unit simplex. In most two-dimensional homogeneous systems, the ratio of the two variables is convenient and customary. The stability results presented here are independent of the choice made for the intensive form.

Definition A.1. *A pair* $(\lambda, \bar{x})$ *solving* $\lambda\bar{x} = F(\bar{x})$ *with* $\lambda > 0$ *and* $\bar{x} \in S$ *is called a Perron-Frobenius solution of* F.

λ is called an eigenvalue of F which satisfies $\lambda = |F(\bar{x})|$. $\bar{x}$ is a fixed point of f and is called an eigenvector of F.

Definition A.2. *An orbit* $\gamma(x) := \{F^t(x)\}_{t\geq 0}$ *of* F *is called balanced if for all* $t \geq 0$: $F^t(x) = \lambda^t x$, *for some* $\lambda > 0$. *Then,* $\gamma(\alpha\bar{x}) = \{F^t(\alpha\bar{x}\}_{t=0}^{\infty} = \{\lambda^t(\alpha\bar{x})\}_{t=0}^{\infty}$ *is a balanced orbit for all* $\alpha > 0$ *if and only if* $(\lambda, \bar{x})$ *is a Perron-Frobenius solution for* F.

One has the immediate result:
An orbit $\gamma(\alpha\bar{x}) = \{F^t(\alpha\bar{x}\}_{t=0}^{\infty} = \{\lambda^t(\alpha\bar{x})\}_{t=0}^{\infty}$ is a balanced orbit for all $\alpha > 0$ if and only if $(\lambda, \bar{x})$ is a Perron-Frobenius solution for F.

Often, the set $\left\{x^n \in \mathbb{R}^n_+ \,|\, x^n = \lambda^n \alpha\bar{x}, n = 0, 1, \ldots\right\}$ is referred to as a balanced path. The union of all balanced paths $\cup_{\alpha\geq 0}\left\{x^n \in \mathbb{R}^n_+ \,|\, x^n = \lambda^n\alpha\bar{x}\right\}$ is a subset of the ray or halfline $L(\bar{x}) := \left\{x \in \mathbb{R}^n_+ \,|\, x = \alpha\bar{x}, \alpha \geq 0\right\}$ associated with $\bar{x}$, referred to as the balanced ray or halfline.

Define the distance of $x \in \mathbb{R}^n_+$ from $L(\bar{x})$ as

$$\Delta := d(x, L(\bar{x})) = \min_{\alpha\geq 0} |x - \alpha\bar{x}| = |x - \langle x, \bar{x}\rangle\bar{x}| \tag{A.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product.

Definition A.3. *An orbit* $\gamma(x)$ *of* F *is said to converge to a balanced path (to* $L(\bar{x})$*) if*

$$\Delta_t := d(F^t(x), L(\bar{x})) = |F^t(x) - \langle F^t(x), \bar{x}\rangle\bar{x}| \tag{A.2}$$

converges to zero for $t \to \infty$.

A.1 A Contraction Lemma for Deterministic Systems

Lemma A.1.
Consider a continuous and homogeneous time-one map $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *and its associated mapping in intensive form* $f : S \to S$ *where* $S := \left\{x \in \mathbb{R}^n_+ \,|\, |x| = 1\right\}$.

Let $(\lambda, \bar{x})$ *denote a Perron-Frobenius solution for* F, *i.e.* $\lambda\bar{x} = F(\bar{x})$ *with* $|\bar{x}| = 1$ *and* $\lambda > 0$. *Assume that* $\bar{x} \in S$ *is an asymptotically stable fixed point of* f *with contractivity* $0 < M < 1$, *i.e.* $|f^m(y) - f^m(x)| \leq M|y - x|$ *for* $m \geq m_0$.

Let $\gamma(x_0)$ *be an orbit of* F *and define* $y_0 := x_0/|x_0| \neq \bar{x}$, *and* $\Delta_0 := |x_0 - \langle x_0, \bar{x}\rangle\bar{x}| \neq 0$. *Let* $0 < \bar{x} \in S$ *be an asymptotically stable fixed point of* f *and* $y_0 \in \mathcal{B}(\bar{x}) \subset S$, *its basin of attraction. Then, for all* $x_0/|x_0| \in \mathcal{B}(\bar{x})$:

$$\textit{If} \quad \lambda M > 1, \quad \textit{then} \quad \lim_{t\to\infty} |\Delta_t| = \infty. \tag{A.3}$$

$$\textit{If} \quad \lambda M < 1, \quad \textit{then} \quad \lim_{t\to\infty} |\Delta_t| = 0. \tag{A.4}$$

Proof. Since

$$\begin{aligned}
\Delta_1 &= |F(x) - \langle F(x), \bar{x}\rangle\bar{x}| \\
&= \frac{|F(x)|}{|x|} \cdot \frac{|F(x)/|F(x)| - \langle F(x)/|F(x)|, \bar{x}\rangle\bar{x}|}{|x/|x| - \langle x/|x|, \bar{x}\rangle\bar{x}|}\Delta \\
&= |F(x/|x|)| \cdot \frac{|F(x)/|F(x)| - \langle F(x)/|F(x)|, \bar{x}\rangle\bar{x}|}{|x/|x| - \langle x/|x|, \bar{x}\rangle\bar{x}|}\Delta \\
&= |F(y)| \cdot \frac{|f(y) - \langle f(y), \bar{x}\rangle\bar{x}|}{|y - \langle y, \bar{x}\rangle\bar{x}|}\Delta =: \mathcal{D}(y, \Delta), \qquad y \in S
\end{aligned} \tag{A.5}$$

the last equation defines a time-one map $\mathcal{D}$ for Δ as a function of (y, Δ). In other words, the mapping F induces a time-one map $(f, \mathcal{D}) : S \times \mathbb{R}_+ \to S \times \mathbb{R}_+$ of an auxiliary system whose fixed points are $(\bar{x}, 0)$. Thus, asymptotic convergence of $\{y_t, \Delta_t\}$ to $(\bar{x}, 0)$ holds if and only if the orbit $\gamma(x_0)$ converges to $L(\bar{x})$ in state space of the original dynamical system. Therefore,

$$\lim |F(y_n)| = |F(\bar{x})| = |\lambda \bar{x}| = \lambda > 0 \tag{A.6}$$

and

$$\begin{aligned}\lim \frac{|f(y_n) - \langle f(y_n), \bar{x}\rangle \bar{x}|}{|y_n - \langle y_n, \bar{x}\rangle \bar{x}|} &= \lim \frac{|f(y_n) - \langle \bar{x}, \bar{x}\rangle \bar{x}|}{|y_n - \langle \bar{x}, \bar{x}\rangle \bar{x}|} = \lim \frac{|f(y_n) - |\bar{x}|^2 \bar{x}|}{|y_n - |\bar{x}|^2 \bar{x}|} \\ &= \lim \frac{|f(y_n) - \bar{x}|}{|y_n - \bar{x}|} = M\end{aligned} \tag{A.7}$$

imply

$$\lim_{n\to\infty} \frac{\Delta_{n+1}}{\Delta_n} = \lambda\, M. \tag{A.8}$$

This means $|\frac{\Delta_{t+1}}{\Delta_t} - \lambda\, M| < \epsilon$ for t larger than some t_0. Thus,

$$[\lambda\, M - \epsilon]|\Delta_t| < |\Delta_{t+1}| < [\lambda\, M + \epsilon]|\Delta_t|, \quad t \geq t_0,$$

and by induction

$$[\lambda\, M - \epsilon]^\tau |\Delta_{\tau+t_0}| < |\Delta_{t+t_0}| < [\lambda\, M + \epsilon]^\tau |\Delta_{t_0}|, \quad \tau > 0.$$

Therefore, for ϵ sufficiently small,

$$\lambda\, M < 1 \quad \Rightarrow \quad \lambda\, M + \epsilon < 1$$

so that $\lim_{t\to\infty} \Delta_t = 0$. Conversely,

$$\lambda\, M > 1 \quad \Rightarrow \quad \lambda\, M - \epsilon > 1$$

so that $\lim_{t\to\infty} |\Delta_t| = \infty$. □

A.2 Stable Balanced Growth in Two-Dimensional Models

The Solow growth model serves as the work horse model for the examination of the stability of two-dimensional balanced growth paths. Since the induced models in intensive form with their associated distance functions are equivalent, the statements of Lemma A.1, of Theorem A.1, and of Corollary A.1 are directly comparable and describe asymptotic convergence/divergence of hyperbolic fixed points $(\Delta, k) = (0, \bar{k}) \in \mathbb{R} \times S$ for the respective norm $|\cdot|$.

Let

$$\begin{aligned} L' &= \mathcal{L}(L, K) := (1+n)L \\ K' &= \mathcal{G}_s(L, K) := (1-\delta)K + S(L, K) \end{aligned} \tag{A.9}$$

denote a Solow growth model with a general monotonic homogeneous savings function $S(L, K)$ and parameters (n, δ). Such savings functions appear in many models of optimal growth or with two-period overlapping generations of consumers with perfect foresight. The system (A.9) induces the one-dimensional mapping in intensive form

$$k' = G_s(k) := \frac{1}{1+n}\left((1-\delta)k + s(k)\right), \qquad k := K/L \quad s(k) := S(1, k). \tag{A.10}$$

If the labor force grows at a constant rate $n > -1$ so that $L_{t+1} = (1+n)L_t$, one obtains a dynamical system for L and K with state space $\mathbb{R}^2_+$ governed by the two maps

$$(\mathcal{L}, \mathcal{G}_s) : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+, \quad (L, K) \mapsto \big(\mathcal{L}(L,K), \mathcal{G}_s(L,K)\big), \tag{A.11}$$

where $\mathcal{L}(L,K) := (1+n)L$. Let $(\mathcal{L}, \mathcal{G}_s)^t$ denote the t-th iterate of the map $(\mathcal{L}, \mathcal{G}_s)$, i.e.,

$$(L_t, K_t) = (\mathcal{L}, \mathcal{G}_s)^t(L_0, K_0) := \underbrace{(\mathcal{L}, \mathcal{G}_s) \circ \cdots \circ (\mathcal{L}, \mathcal{G}_s)}_{t-\text{times}}(L_0, K_0).$$

Thus, an orbit $\gamma(L_0, K_0)$ of the *time-one map* (A.11) is given by

$$\gamma(L_0, K_0) := \{(\mathcal{L}, \mathcal{G}_s)^t(L_0, K_0)\}_{t \in \mathbb{N}},$$

where the sequence $\{(L_t, K_t)\}_{t=0}^{\infty}$ is also called a *growth path*. The theorem and its proof are taken from Böhm, Pampel & Wenzelburger (2005).

Theorem A.1. *Let s be differentiable and let $\bar{k}$ be an asymptotically stable fixed point of G_s. Let*[7] *$\mathcal{B}(\bar{k}) \cap (G_s)^{-1}(\{\bar{k}\}) = \{\bar{k}\}$, where $\mathcal{B}(\bar{k})$ is the basin of attraction of $\bar{k}$ and $(G_s)^{-1}(\{\bar{k}\})$ is the preimage of $\bar{k}$. Consider the time-one map $(\mathcal{L}, \mathcal{G}_s)$ as given in (A.11). Let $\gamma(L_0, K_0)$ be an arbitrary orbit of $(\mathcal{L}, \mathcal{G}_s)$ with $K_0/L_0 \in \mathcal{B}(\bar{k})$, $K_0/L_0 \neq \bar{k}$, implying $\triangle_0 = K_0 - \bar{k}L_0 \neq 0$. Then the following holds:*

$$\text{If} \qquad E_s(\bar{k}) < \frac{\delta}{n+\delta}, \quad \text{then} \quad \lim_{t\to\infty} \triangle_t = 0; \tag{A.12}$$

$$\text{If} \qquad E_s(\bar{k}) > \frac{\delta}{n+\delta}, \quad \text{then} \quad \lim_{t\to\infty} |\triangle_t| = \infty, \tag{A.13}$$

where $E_s(\bar{k}) = \dfrac{\bar{k}s'(\bar{k})}{s(\bar{k})}$ *denotes the elasticity of the function s at the steady state $\bar{k}$.*

Proof. Let $k_0 > 0$ and $k_0 \neq \bar{k}$ arbitrary but fixed. Under the hypotheses of this Theorem, one has

$$\triangle_{t+1} = K_{t+1} - \bar{k}L_{t+1} = L_{t+1}[G_s(k_t) - \bar{k}], \quad t \in \mathbb{N},$$

where $\bar{k}$ denotes the steady state of G_s. Then,

$$\frac{\triangle_{t+1}}{\triangle_t} = (1+n)\frac{G_s(k_t) - \bar{k}}{k_t - \bar{k}}.$$

k_t converges to $\bar{k}$ since $k_0 \in \mathcal{B}(\bar{k})$. Therefore,

$$\lim_{t\to\infty} \frac{\triangle_{t+1}}{\triangle_t} = (1+n)G_s'(\bar{k}).$$

This implies that $|\frac{\triangle_{t+1}}{\triangle_t} - (1+n)G_s'(\bar{k})| < \epsilon$ for all t larger than some t_0. It follows that

$$[(1+n)G_s'(\bar{k}) - \epsilon]|\triangle_t| < |\triangle_{t+1}| < [(1+n)G_s'(\bar{k}) + \epsilon]|\triangle_t|, \quad t \geq t_0,$$

and by induction

$$[(1+n)G_s'(\bar{k}) - \epsilon]^\tau|\triangle_{\tau+t_0}| < |\triangle_{t+t_0}| < [(1+n)G_s'(\bar{k}) + \epsilon]^\tau|\triangle_{t_0}|, \quad \tau > 0.$$

[7] This assumption is always satisfied, if s is strictly increasing.

Now, if $(1+n)G_s'(\bar{k}) < 1$, then $(1+n)G_s'(\bar{k})+\epsilon < 1$ for sufficiently small ϵ such that $\lim_{t\to\infty} \triangle_t = 0$. On the other hand, if $(1+n)G_s'(\bar{k}) > 1$, then $(1+n)G_s'(\bar{k}) - \epsilon > 1$ for sufficiently small ϵ and hence $\lim_{t\to\infty} |\triangle_t| = \infty$. The rest of the statement follows from

$$(1+n)G_s'(\bar{k}) < 1 \iff E_s(\bar{k}) < \frac{\delta}{n+\delta}$$

and the fact that $k_0 \in \mathcal{B}(\bar{k})$, $k_0 \neq \bar{k}$ was arbitrary. □

Theorem A.1 applies to functions G_s which are strictly increasing or strictly decreasing in a neighborhood of an asymptotically stable fixed point. As an immediate implication one obtains the following corollary.
Corollary A.1. *A balanced growth path of the standard Solow model* (2.5) *is stable if*

$$E_f(\bar{k}) < \frac{\delta}{n+\delta}, \tag{A.14}$$

and unstable if

$$E_f(\bar{k}) > \frac{\delta}{n+\delta}. \tag{A.15}$$

For the standard Solow model the elasticity of the savings function always coincides with the elasticity of the production function. Since the production function f is strictly concave, its elasticity $E_f(k)$ is always less than one.

B Balanced Expansion of Random Homogeneous Systems

Following Evstigneev & Pirogov (2007, 2010), let $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ denote an ergodic dynamical system[8]with probability space $(\Omega, \mathcal{F}, \mathbb{P})$, its automorphism $\vartheta : \Omega \to \Omega$, (i.e. a one-to-one mapping such that ϑ and ϑ^{-1} are measurable and preserve the measure $\mathbb{P}$), and $F(\omega) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ an associated random family of continuous, homogeneous time-one maps which are $\mathcal{F}$-measurable in ω. $F(\omega)$ induces the random difference equation

$$x_t = F(\vartheta^{t-1}\omega)x_{t-1} \qquad \text{for all } t. \tag{B.1}$$

Random orbits $\gamma(\omega, x_0) := \{C(t,\omega)x_0\}_t^\infty$ of $F(\omega)$ are generated by the mapping

$$x_t = C(t,\omega)x_0 := \begin{cases} F(\vartheta^{t-1}\omega) \circ \cdots \circ F(\omega)x_0 & t > 0 \\ id_X & t = 0 \end{cases} \tag{B.2}$$

which satisfies

$$C(t+s,\omega) = C(t,\vartheta^s\omega) \circ C(s,\omega) \qquad \text{for all } t, s. \tag{B.3}$$

The mapping $C(t,\omega)$ is a cocyle over the dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ (see Arnold, 1998). In the following, the notational convention $C(t,\omega)x$ and $F(\omega)x$ will be used for the result of the application of the corresponding function to the point x (as in Evstigneev & Pirogov, 2010).

[8]For concreteness, let Ω be the space of two-sided infinite sequences of the perturbation, $\mathcal{F}$ the Borel-σ-algebra, and $\mathbb{P}$ the probability measure generated by the marginal distributions. $\vartheta : \Omega \to \Omega$ is the so-called left-shift on Ω, i.e. $(\vartheta\omega)_s = (\omega_{s+1}), s \in \mathbb{Z}$.

Definition B.1. *If $F(\omega) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is homogeneous of degree one, its associated* mapping in intensive form $f(\omega) \equiv F(\omega)/|F(\omega)| : S \to S$, $y \mapsto f(\omega)y$ *is defined as*

$$f(\omega)y \equiv (F(\omega)/|F(\omega)|)y := \frac{1}{|F(\omega)y|} \cdot F(\omega)y = F(\omega)\frac{y}{|F(\omega)y|} \tag{B.4}$$

where $S := \{y \in \mathbb{R}^n_+ \mid |y| = 1\}$.

Definition B.2. *A random fixed point of $f(\omega)$ is a measurable mapping $\xi : \Omega \to S$ such that*

$$\xi(\vartheta\omega) = f(\omega)\xi(\omega) \quad \mathbb{P}\text{-}a.s. \tag{B.5}$$

Definition B.3. *Let $F(\omega)$ be homogeneous and $\xi : \Omega \to S$ be a random fixed point of $f(\omega)$. An orbit $\gamma(\omega, \xi(\omega)) = \{(C(t, \omega)\xi(\omega))\}$ of F is called balanced if there exists a random variable $\lambda : \Omega \to \mathbb{R}_{++}$ such that*

$$C(t, \omega)\xi(\omega) = F(\vartheta^{t-1}\omega) \circ \cdots \circ \; F(\omega)\xi(\omega) = \left(\prod_{\tau=1}^{t} \lambda(\vartheta^{\tau-1}\omega)\right) \cdot c(t, \omega)\xi(\omega) \tag{B.6}$$

where $c(t, \omega) := f(\vartheta^{t-1}\omega) \circ \cdots \circ f(\omega)$ is the cocycle associated with the mapping $f(\omega)$, see (B.2)*.*

Definition B.4 (Perron-Frobenius)**.** *A pair of measurable mappings $(\lambda(\omega), \xi(\omega))$, $\lambda : \Omega \to \mathbb{R}_+$, $\xi : \Omega \to \mathbb{R}^N_+$ is called a Perron-Frobenius solution of $F(\omega)$ if $\mathbb{P}$-a.s. :*

$$\lambda(\omega)\xi(\vartheta\omega) = F(\omega)\xi(\omega) \quad \textit{with} \quad |\xi(\omega)| = 1 \quad \textit{and} \quad \lambda(\omega) > 0 \tag{B.7}$$

(Evstigneev & Pirogov, 2010).

Theorem B.1 (Evstigneev & Pirogov (2010))**.**
Let $F(\omega)$ be homogeneous and strictly monotone[9]*. There exists a unique Perron-Frobenius solution $(\lambda, \xi) : \Omega \to \mathbb{R}_+ \times \mathbb{R}^N_+$*

$$\lambda(\omega)\xi(\vartheta\omega) = F(\omega)\xi(\omega) \quad \textit{with} \quad |\xi(\omega)| = 1 \quad \textit{and} \quad \lambda(\omega) > 0 \tag{B.8}$$

Lemma B.1.
Every Perron-Frobenius solution (λ, ξ) induces a balanced orbit $\{C(t, \omega)\xi(\omega)\}_0^\infty$ of $F(\omega)$.

Proof. The homogeneity of F implies that $F(\omega)\alpha x = \alpha \cdot F(\omega)x$ for all $\alpha > 0$ and $x \gneqq 0$.

[9] F is called strictly monotone if $x \gneqq y$ implies $F(\omega)x \gg F(\omega)y$, for all ω, see Evstigneev & Pirogov (2010), Theorem 1.

Therefore,

$$\begin{aligned}
C(t,\omega)\xi(\omega) &= F(\vartheta^{t-1}\omega)\circ\cdots\circ F(\omega)\xi(\omega)\\
&= F(\vartheta^{t-1}\omega)\circ\cdots\circ F(\vartheta\omega)\circ\lambda(\omega)\cdot f(\omega)\xi(\omega)\\
&= \lambda(\omega)\cdot F(\vartheta^{t-1}\omega)\circ\cdots\circ F(\vartheta\omega)\circ f(\omega)\xi(\omega)\\
&= \lambda(\omega)\cdot F(\vartheta^{t-1}\omega)\circ\cdots\circ F(\vartheta\omega)\circ f(\omega)\xi(\omega)\\
&= \lambda(\omega)\cdot F(\vartheta^{t-1}\omega)\circ\cdots\circ F(\vartheta\omega)\xi(\vartheta\omega)\\
&= \lambda(\omega)\cdot F(\vartheta^{t-1}\omega)\circ\cdots\circ F(\vartheta^2\omega)\circ F(\vartheta\omega)\xi(\vartheta\omega)\\
&= \lambda(\omega)\cdot F(\vartheta^{t-1}\omega)\circ\cdots\circ F(\vartheta^2\omega)\circ\lambda(\vartheta\omega)\cdot f(\vartheta\omega)\xi(\vartheta\omega) \qquad \text{(B.9)}\\
&= \lambda(\omega)\cdot\lambda(\vartheta\omega)\cdot F(\vartheta^{t-1}\omega)\circ\cdots\circ F(\vartheta^2\omega)\circ f(\vartheta\omega)\xi(\vartheta\omega)\\
&= \lambda(\omega)\cdot\lambda(\vartheta\omega)\cdot F(\vartheta^{t-1}\omega)\circ\cdots\circ F(\vartheta^2\omega)\circ f(\vartheta\omega)\circ f(\omega)\xi(\omega)
\end{aligned}$$

and by induction

$$\begin{aligned}
&= \lambda(\omega)\cdot\lambda(\vartheta\omega)\cdots\lambda(\vartheta^{t-1}\omega)\cdot f(\vartheta^{t-1}\omega)\circ\cdots\circ f(\omega)\xi(\omega)\\
&= \left(\prod_{\tau=1}^{t}\lambda(\vartheta^{\tau-1}\omega)\right)\cdot c(t,\omega)\xi(\omega) =: \Lambda(\vartheta^t\omega)\cdot c(t,\omega)\xi(\omega)
\end{aligned}$$

□

Lemma B.2 (Stochastic Case).
Assume that $\xi^\star:\Omega\to S$ *is a random fixed point of* $f(\omega)$. *There exists* $\lambda:\Omega\to\mathbb{R}_+$, $\lambda(\omega)>0$ *such that* $\mathbb{P}$*-a.s.*

$$F(\omega)\xi^\star(\omega)=\lambda(\omega)\cdot\xi^\star(\vartheta\omega)=\lambda(\omega)\cdot f(\omega)\xi^\star(\omega)=\lambda(\omega)\cdot\frac{F(\omega)}{|F(\omega)|}\xi^\star(\omega) \qquad \text{(B.10)}$$

Proof. Define the random variable

$$\lambda^\star(\omega):=\frac{|F(\omega)\xi^\star(\omega)|}{|(F(\omega)/|F(\omega)|)\xi^\star(\omega)|} \qquad \text{(B.11)}$$

Then,

$$\begin{aligned}
\lambda^\star(\omega)\cdot\xi^\star(\vartheta\omega) &= \frac{|F(\omega)\xi^\star(\omega)|}{|(F(\omega)/|F(\omega)|)\xi^\star(\omega)|}\cdot f(\omega)\xi^\star(\omega)\\
&= \frac{|F(\omega)\xi^\star(\omega)|}{|(F(\omega)/|F(\omega)|)\xi^\star(\omega)|}\cdot\frac{F(\omega)\xi^\star(\omega)}{|F(\omega)\xi^\star(\omega)|} \qquad \text{(B.12)}\\
&= \frac{|F(\omega)\xi^\star(\omega)|}{|f(\omega)\xi^\star(\omega)|}\cdot\frac{F(\omega)\xi^\star(\omega)}{|F(\omega)\xi^\star(\omega)|}=F(\omega)\xi^\star(\omega)
\end{aligned}$$

Thus, the pair $(\lambda^\star(\omega),\xi^\star(\omega))$ is a Perron-Frobenius solution inducing random balanced orbits given by

$$\begin{aligned}
\gamma(\omega,\xi^\star(\omega)) &= \{C(t,\omega)\xi^\star(\omega)\}_{t=0}^{\infty}\\
C(t,\omega)\xi^\star(\omega) &= \left(\prod_{\tau=1}^{t}\lambda^\star(\vartheta^{\tau-1}\omega)\right)\cdot c(t,\omega)\xi^\star(\omega).
\end{aligned} \qquad \text{(B.13)}$$

. □

Figure 12 portrays the relationship between the Perron-Frobenius solution $\lambda(\omega)\xi(\vartheta\omega)=F(\omega)\xi(\omega)$, the orbit of the fixed point $\{(x_t\equiv c(t,\omega)\xi(\omega))\}_0^\infty$, and the associated balanced random orbit.

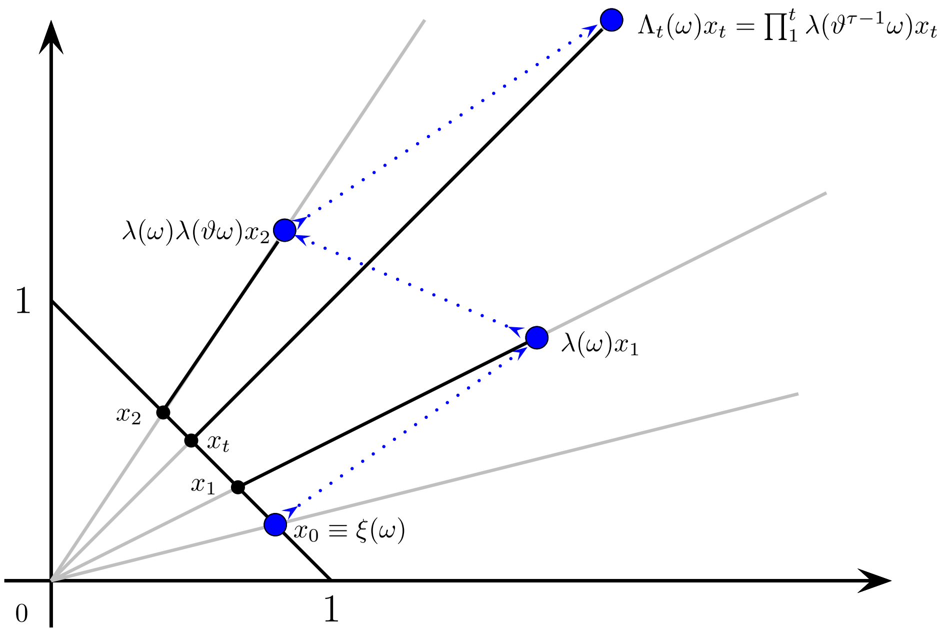


Figure 12: A balanced random orbit for the Perron-Frobenius solution $\lambda(\omega)\xi(\vartheta\omega) = F(\omega)\xi(\omega)$.

B.1 A Contraction Theorem for Random Systems

Definition B.5. *A random fixed point $\xi : \Omega \to S$ is called asymptotically stable if*

$$\lim_{t\to\infty} |c(t,\omega)x_0 - \xi(\vartheta^t\omega)| = 0 \quad \mathbb{P}\text{-}a.s. \tag{B.14}$$

for all $x_0 \in \mathcal{B}(\xi(\omega))$, the basin of attraction of ξ, where $c(t,\omega)$ is the cocycle associated with $f(\omega)$, see equation (B.2).

Definition B.6. *Define the distance of an orbit $\{C(t,\omega)X_0\}$ of F with $|X_0 = 1|$ to the balanced one $\{C(t,\omega)\xi(\omega)\}$ associated with the random fixed point $\xi : \Omega \to S$ as*

$$\Delta_t = \Delta(t,\omega)X_0 := |C(t,\omega)X_0 - C(t,\omega)\xi(\omega))| = |C(t,\omega)X_0 - \Lambda(\vartheta^t\omega)\xi(\vartheta^t\omega))|. \tag{B.15}$$

An orbit $\{C(t,\omega)X_0\}$ is said to ***converge*** *to a balanced orbit if for all $x_0 \in \mathcal{B}(\xi(\omega)) \subset S$ and for all $X_0 = x_0 \neq \xi(\omega)$:*

$$\lim_{t\to\infty} |c(t,\omega)x_0 - \xi(\vartheta^t\omega)| = 0 \quad \text{and} \quad \lim_{t\to\infty} |\Delta(t,\omega)X_0| = 0, \quad \mathbb{P}\text{-}a.s.. \tag{B.16}$$

Theorem B.2.
Let $\xi^\star : \Omega \to \mathbb{R}^n_+$ be an asymptotically stable random fixed point of $f(\omega)$ inducing the rate of contraction

$$M(\omega, \xi^\star(\omega)) := \lim_{x_0\to\xi^\star(\omega)} \left| \frac{|f(\omega)x_0 - f(\omega)\xi^\star(\omega)|}{|x_0 - \xi^\star(\omega)|} \right| < 1, \quad \mathbb{P}\text{-}a.s. \tag{B.17}$$

of f at $\xi^\star(\omega)$.

Then, for almost all $\omega \in \Omega$ and any $x_0 \in \mathcal{B}(\xi^\star(\omega))$, $x_0 \neq \xi^\star(\omega)$ with $\lim_{t\to\infty} |c(t,\omega)x_0 - \xi^\star(\vartheta^t\omega)| = 0$ the distance $\Delta_t := |C(t,\omega)X_0 - \Lambda(t,\omega)\cdot\xi^\star(\vartheta^t\omega))|$ satisfies $\mathbb{P}$-a.s.:

$$\lim_{t\to\infty} |\Delta_t| = 0 \qquad \text{if} \qquad \mathbb{E}\log(\lambda(\omega,\xi^\star(\omega)) + \mathbb{E}\log M(\omega,\xi^\star(\omega)) < 0 \tag{B.18}$$

$$\lim_{t\to\infty} |\Delta_t| = \infty \qquad \text{if} \qquad \mathbb{E}\log(\lambda(\omega,\xi^\star(\omega)) + \mathbb{E}\log M(\omega,\xi^\star(\omega)) > 0 \tag{B.19}$$

Proof. Let $\gamma(\omega, \xi^\star(\omega)) = \{C(t,\omega)\xi^\star(\omega)\}_{t=0}^{\infty}$ denote the balanced orbit associated with the random fixed point $\xi^\star$ of f given by the two associated cocycles and $C(t,\omega)$ resp. $c(t,\omega)$

$$C(t,\omega)\xi^\star(\omega) = \left(\prod_{\tau=1}^{t} \lambda^\star(\vartheta^{\tau-1}\omega)\right) \cdot c(t,\omega)\xi^\star(\omega) \equiv \Lambda(\vartheta^t\omega) \cdot c(t,\omega)\xi^\star(\omega). \tag{B.20}$$

From definition (B.15) one has

$$\begin{aligned}
\Delta_t = \Delta(t,\omega)X_0 =& |C(t,\omega)X_0 - \Lambda(\vartheta^t\omega)\cdot\xi^\star(\vartheta^t\omega))| \\
=& \Lambda(\vartheta^t\omega)\cdot|c(t,\omega)x_0 - \xi^\star(\vartheta^t\omega)| \\
& \text{and} \\
\Delta_{t+1} = \Delta(t+1,\omega)X_0 =& |C(t+1,\omega)X_0 - \Lambda(\vartheta^{t+1}\omega)\cdot\xi^\star(\vartheta^{t+1}\omega))| \\
=& \lambda(\vartheta^t\omega)\cdot\Lambda(\vartheta^t\omega)\cdot|c(t+1,\omega)x_0 - \xi^\star(\vartheta^{t+1}\omega)| \\
=& \lambda(\vartheta^t\omega)\cdot\Lambda(\vartheta^t\omega)\cdot|f(\vartheta^{t+1}\omega)\circ c(t,\omega)x_0 - f(\vartheta^{t+1}\omega)\xi^\star(\vartheta^t\omega)|
\end{aligned} \tag{B.21}$$

implying

$$\begin{aligned}
\frac{\Delta_{t+1}}{\Delta_t} =& \frac{\lambda(\vartheta^t\omega)\cdot\Lambda(\vartheta^t\omega)\cdot|f(\vartheta^{t+1}\omega)\circ c(t,\omega)x_0 - f(\vartheta^{t+1}\omega)\xi^\star(\vartheta^t\omega)|}{\Lambda(\vartheta^t\omega)\cdot|c(t,\omega)x_0 - \xi^\star(\vartheta^t\omega)|} \\
=& \frac{\lambda(\vartheta^t\omega)\cdot|f(\vartheta^{t+1}\omega)\circ c(t,\omega)x_0 - f(\vartheta^{t+1}\omega)\xi^\star(\vartheta^t\omega)|}{|c(t,\omega)x_0 - \xi^\star(\vartheta^t\omega)|}.
\end{aligned} \tag{B.22}$$

Since $\lim_{t\to\infty}|c(t,\omega)x_0 - \xi^\star(\vartheta^t\omega)| = 0, \mathbb{P}$-a.s., there exists an $\varepsilon > 0$ sufficiently small and $t_0 = t_0(\varepsilon,\omega) > 0$ sufficiently large such that

$$\left|\frac{|f(\vartheta^{t+1}\omega)\circ c(t,\omega)x_0 - f(\vartheta^{t+1}\omega)\xi^\star(\vartheta^t\omega)|}{|c(t,\omega)x_0 - \xi^\star(\vartheta^t\omega)|} - M(\vartheta^t\omega, \xi^\star(\omega))\right| < \varepsilon \tag{B.23}$$

for $t \geq t_0$. By induction, for all $t \geq t_0$, there are upper and lower bounds satisfying $\underline{\Delta}_t \leq |\Delta_t| \leq \overline{\Delta}_t$ for the two linear random dynamical systems

$$\overline{\Delta}_{t+1} = \left[\lambda(\vartheta^t\omega)\cdot\ M(\vartheta^t\omega, \xi^\star(\omega)) + \varepsilon\right]\overline{\Delta}_t \tag{B.24}$$

$$\underline{\Delta}_{t+1} = \left[\lambda(\vartheta^t\omega)\cdot\ M(\vartheta^t\omega, \xi^\star(\omega)) - \varepsilon\right]\underline{\Delta}_t \tag{B.25}$$

with $\overline{\Delta}_{t_0} = |\Delta_{t_0}|$ and $\underline{\Delta}_{t_0} = |\Delta_{t_0}|$. Therefore,

$$\mathbb{E}\log(\lambda(\omega, \xi^\star(\omega)) + \mathbb{E}\log M(\omega, \xi^\star(\omega)) < 0$$

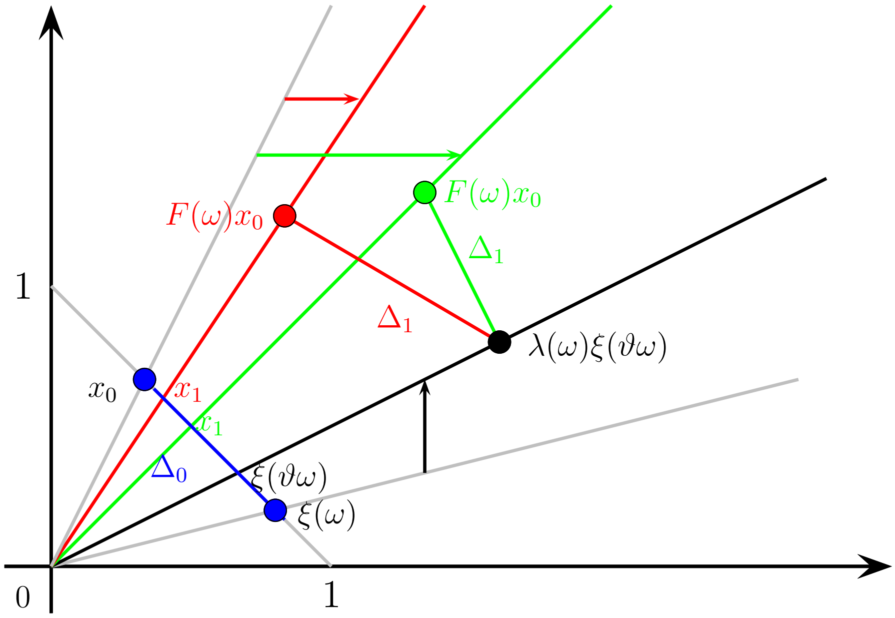
implies that the upper bound (B.24) converges to zero $\mathbb{P}$-a.s. so that $\lim_{t\to\infty}|\Delta_t| = 0$.

Conversely,

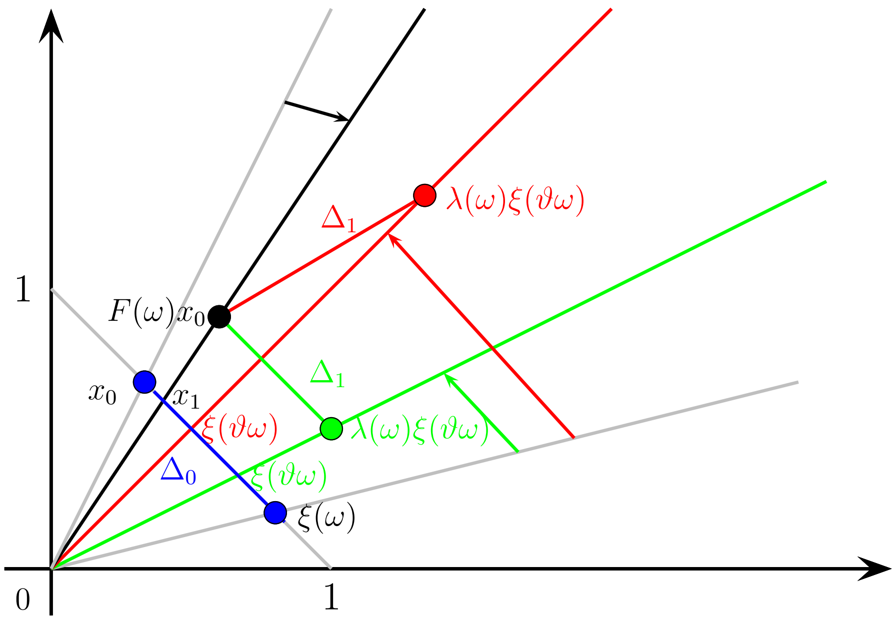
$$\mathbb{E}\log(\lambda(\omega, \xi^\star(\omega)) + \mathbb{E}\log M(\omega, \xi^\star(\omega)) > 0$$

implies that the lower bound grows to infinity and $\lim_{t\to\infty}|\Delta_t| = \infty$. □

Figure 13 displays the sources of convergence or divergence for two alternative scenarios: one originating from weak versus strong contractivity of the intensive form map for a *given* rate of expansion of the Perron-Frobenius solution (subfigure (a)) or one originating from a weakly expanding versus a strongly expanding Perron-Frobenius solutions with a *given* contractivity factor (subfigure (b)).



(a) **Weak**/**Strong** contractivity for a given Perron-Frobenius solution $(\boldsymbol{\lambda}, \boldsymbol{\xi})$



(b) **Weakly**/**strongly** expanding Perron-Frobenius solutions for given contractivity $\boldsymbol{M(\xi^\star)}$

Figure 13: **Convergence**/**Divergence** of stochastic balanced paths: $\boldsymbol{\Delta_1} < \boldsymbol{\Delta_0} < \boldsymbol{\Delta_1}$

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