Value Distribution of Quadratic Forms and Diophantine Inequalities

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Abstract

In this thesis we derive for any $\varepsilon > 0$ effective estimates for the size of a non-zero integral lattice point $m \in \mathbb{Z}^d \setminus \{0\}$ solving the Diophantine inequality $|Q[m]| < \varepsilon$, where Q denotes a non-singular indefinite quadratic form in $d \geq 5$ variables. In order to prove our quantitative variants of the Oppenheim conjecture, we establish effective error bounds on the lattice remainder as well as extend - in the case of diagonal forms - the approach developed by Birch and Davenport [BD58b] non-trivially to higher dimensions than five.

The approximation of the number of lattice points in d-dimensional hyperbolic or elliptic shells $\{m : a < Q[m] < b\}$, which are restricted to rescaled and growing domains $r\,\Omega$, by the volume is a classical question in analytic number theory. Here we prove effective bounds of order $o(r^{d-2})$ for this approximation based on Diophantine approximation properties of the quadratic form Q. This part of the work is a revised variant of the earlier preprint [GM13] of Götze and Margulis with numerous changes and corrections. Using these results together with a Dichotomy argument and Schlickewei's work [Sch85] on small zeros of integral forms, we derive bounds on the size of a non-trivial solution $m \in \mathbb{Z}^d \setminus \{0\}$ of $|Q[m]| < \varepsilon$ in terms of the signature (r, s). For diagonal quadratic forms we can even extend Schlickewei's bounds to the real case in an optimal way up to a negligible growth factor, i.e. our result is already comparable to the integral case if Q is diagonal. The basic strategy in this case is to iterate the Birch-Davenport approach (as introduced in the work [BD58b]) and thereby to prove conditionally improved mean-value estimates for certain products of quadratic Weyl sums.

CONTENT

CHAPTER 1

Introduction

The main objective of this thesis is to develop quantitative versions of the *Oppenheim conjecture* - that is, the study of the size of the least non-trivial integral solution to homogeneous quadratic Diophantine inequalities: We will establish effective norm-bounds for a non-trivial lattice point $m \in \mathbb{Z}^d \setminus \{0\}$ in terms of $\varepsilon > 0$ satisfying

$$|Q[m]| < \varepsilon$$

in at least $d \geq 5$ variables, i.e. the integral point $m \in \mathbb{Z}^d \setminus \{0\}$ should be an approximate zero with 'small' norm. Here we consider non-singular indefinite quadratic forms

$$Q[x] \overset{\text{def}}{=} \langle x, Qx \rangle \quad \text{for} \quad x \in \mathbb{R}^d$$

with signature (r, s), where $Q \in \mathrm{GL}(d, \mathbb{R})$ is the associated symmetric matrix, $\langle \cdot, \cdot \rangle$, resp. $\|\cdot\|$, denotes the standard Euclidean scalar product, resp. norm, on a real d-dimensional Euclidean space $\mathbb{R}^d$ and $d = r + s$.

Our results will depend essentially on both explicit estimates for the lattice remainder and small zeros of integral forms. Based on the preprint [GM13] of Götze and Margulis we shall establish in Chapter 3 effective error bounds for the approximation of the number of lattice points restricted to growing domains for thin and wide shells, where Margulis' averaging method will be presented later in Appendix B. These error bounds will be used in Chapter 4 together with Schlickewei's result [Sch85] on small zeros for integral forms, depending on the signature (r, s) of Q, in order to establish bounds on a least non-trivial solution in the case of general forms, not necessarily diagonal: If Q has 'good' Diophantine properties, we can compare the volume with the number of lattice points. Otherwise Q is near a rational form and here we shall use Schlickewei's bound [Sch85] for small zeros of integral quadratic forms.

In the case of diagonal quadratic forms, we shall prove refined results in Chapter 2 by extending the approach developed by Birch and Davenport [BD58b] to higher dimensions: Compared to the volume argument, Birch and Davenport analyze regular patterns in the frequency picture of the associated counting problem. Starting with the assumption that there are no solutions, they show that specific rational approximations, corresponding to quadratic exponential sums are 'rigid' (we will call this *coupling*). However, their approach is not directly applicable in combination with Schlickewei's bound and will be modified in some parts essentially. We will introduce an iteration of their coupling argument and prove nearly optimal bounds for diagonal forms up to a negligible growth factor.

Both results were published as preprints, see [BGHM19] and [BGH19]. Whereas the latter result was carried out in cooperation with T. Hille (a student of G. Margulis), based on the earlier preprint [BG18], the former result was developed in cooperation with both T. Hille and G. Margulis based on the above-mentioned preprint [GM13].

1.1 The Oppenheim Conjecture: A Short Historical Overview

Before stating our results, we give a short historical overview on the *Oppenheim conjecture* which was first formulated by A. Oppenheim [Opp29] in 1929 and states that $Q[\mathbb{Z}^d]$ is dense in $\mathbb{R}$ if $d \geq 5$ and Q is irrational, i.e. Q is not a multiple of a rational form. This formulation was inspired by the fact that a rational form represents zero non-trivially on $\mathbb{Z}^d$, as proven by Meyer [Mey84] and nowadays deduced from the classical Hasse principle. Extending the Oppenheim conjecture, H. Davenport [DH46] (stated for diagonal forms only) conjectured in 1946 that it is sufficient to have $d \geq 3$ variables.

Actually the density of $Q[\mathbb{Z}^d]$ in $\mathbb{R}$ follows from that Q either represents zero non-trivially or $Q[\mathbb{Z}^d]$ contains non-zero elements with arbitrarily small absolute values, provided that $d \geq 4$[1] and Q is irrational (see [Opp53a; Opp53b; Opp53c] and for instance Section 5 in [Lew73]). Thus, since irrational forms may not represent non-trivially zero in integral points, it is natural to ask for the solvability of the Diophantine inequality $|Q[m]| < \varepsilon$. In particular, our initial problem of finding explicit bounds for 'approximate zeros' is a refinement of the Oppenheim conjecture in the cases considered here, i.e. for forms in $d \geq 5$ variables.

In the later 1950s the validity of the conjecture was confirmed by Birch, Davenport and Ridout in a series of papers [Dav56; BD58a; Dav58; Rid58; DR59] for $d \geq 21$, using mostly analytic number theory methods. In fact, their basic strategy is based on modificated variants of the Hardy-Littlewood circle method, as introduced by Davenport and Heilbronn [DH46], and different diagonalization techniques.[2] In any case, the used diagonalization processes require considerably more variables as the resulting almost diagonal form, leading to the condition $d \geq 21$ on the number of variables. Moreover, since no further progress was achieved by using these methods, the impression arose that the methods of analytic number theory were not sufficient to prove the Oppenheim conjecture for general quadratic forms in a smaller number of variables.

Thirty years later, in 1986, the breakthrough was achieved by the seminal work [Mar89] of Margulis using a connection, noticed by M. S. Raghunathan, between the Oppenheim conjecture and ergodic theory on the homogeneous space G/Γ defined by the Lie group $G = \mathrm{SL}(3, \mathbb{R})$ and the discrete subgroup $\Gamma = \mathrm{SL}(3, \mathbb{Z})$. Considering the orthogonal group

$$H := \mathrm{SO}(Q) = \left\{ U \in \mathrm{GL}(3, \mathbb{R}) \,:\, U^T Q U = Q \right\}$$

of Q, Raghunathan observed[3] that the density of the orbit $H\Gamma$ in G/Γ immediately implies

$$\overline{Q[\mathbb{Z}^3]} = \overline{Q[H\Gamma\mathbb{Z}^3]} = \overline{Q[G\mathbb{Z}^3]} = \mathbb{R}.$$

Since the general problem can be reduced to the case $d = 3$ by restricting the form Q to an appropriate subspace, the Oppenheim conjecture would follow at once. In a first instance, Margulis has proved the weaker statement that $|Q[m]| < \varepsilon$ is non-trivially solvable in integral points $m \in \mathbb{Z}^3$ and, responding to a question by Borel, extended his arguments to prove the solvability of $0 < |Q[m]| < \varepsilon$. The main argument is - similar to Hedlund's theorem for $\mathrm{SL}(2, \mathbb{R})$ - to prove that any non-closed orbit is dense in G/Γ. Since it is well-known that $H\Gamma$ is not closed in G/Γ for irrational Q, the Oppenheim conjecture follows at once. However, Margulis' work is based on the study of minimal invariant sets and the limits of orbits of sequences of points tending to a minimal invariant set. Thus, the available methods at that time were non-effective and not capable to give explicit bounds on the main question of this thesis. Fore more historical information on the Oppenheim conjecture until 1997, we refer the interested reader to [Lew73] and [Mar97].

[1] If $d = 3$ and Q is irrational, then it is only known that the density of $Q[\mathbb{Z}^d]$ in $\mathbb{R}$ and the solvability of $0 < |Q[m]| < \varepsilon$ for any $\varepsilon > 0$ in integral points $m \in \mathbb{Z}^d \setminus \{0\}$ are equivalent.

[2] One of them was introduced by Brauer, see [Bra45].

[3] This connection was already discovered by Cassels and Swinnerton-Dyer implicitly in [CS55], but remained unknown since the language of dynamical systems was not used there.

Remark 1.1.

(a) Applying van der Corput's work [Cor20] on lattice points in the plane, Watson has extended (in 1953) the result [DH46] of Davenport and Heilbronn to forms which include a singlecross-product term, see [Wat53a]. Moreover, using the elementary theory of continued fraction, Watson proved the Oppenheim conjecture for special types of diagonal quadratic forms in three and four variables, see [Wat53b].

(b) In 1975 Iwaniec [Iwa77] proved the Oppenheim conjecture for quaternary quadratic forms of type $x_1^2 + x_2^2 - \theta(x_3^2 + x_4^2)$ with an irrational number $\theta > 0$ using sieve theory.

(c) In 1989 Dani and Margulis have deduced from results on flows on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ that the set of values of Q at primitive integral points is dense as well, see [DM89].

(d) Raghunathan's conjectures are far more profound statements on the distribution of unipotent flows on homogeneous spaces than mentioned here. These generalized conjectures were proved around 1990 by M. Ratner, see e.g. [Rat92].

(e) Baker and Schlickewei [BS87] have already used Schlickewei's work [Sch85] in combination with the methods of Davenport and Ridout [DR59] to prove the Oppenheim conjecture (for non-diagonal forms) in the special cases (i) $d = 18$, $r = 9$, (ii) $n = 20$, $8 \leq 11$, (iii) $d = 20$, $7 \leq r \leq 13$.

Nearly a decade later Eskin, Margulis and Mozes [EMM98; EMM05] gave quantitative versions of these results[4], i.e. counting asymptotically the number of lattice points in fixed hyperbolic shells (with $a, b \in \mathbb{R}$ and $a < b$)

$$E_{a,b} \overset{\text{def}}{=} \{x \in \mathbb{R}^d : \; a < Q[x] < b\} \tag{1.1}$$

which are restricted to growing domains $r\Omega$ with $r \to \infty$. Such results are called *quantitative Oppenheim conjecture* as well, but do not imply in first instance explicit bounds on the size of the least non-trivial integral solution to homogeneous quadratic Diophantine inequalities: To show that the inequality $|Q[m]| < \varepsilon$ admits a non-trivial integer solution, whose size can be bounded, an effective error bound for the lattice remainder is needed. This investigation arguably goes back to the seminal works of Bentkus and Götze [BG97; BG99], establishing effective bounds for the lattice point remainder in the cases $d \geq 9$. However, in these works no explicit connections between theta series and Diophantine approximation of Q were deduced which are needed to obtain explicit bounds on the size of the norm. Later this study was continued by Götze and Margulis [GM13] extending the previous results for $d \geq 5$ and deriving first variants of explicit bounds. Among the extension of the Birch-Davenport approach, we revised the work [GM13] of Götze and Margulis and complete some of their arguments providing improvements on the explicit Diophantine dependency. Moreover, we now use the work [Sch85] of Schlickewei as well. However, we cannot make use of the full strength of Schlickewei's bounds in the general case. Hence, our result for diagonal forms is still considerably sharper.

Remark 1.2.

(a) For completeness, we note that S. Dani and G. Margulis have proved already in 1993 lower asymptotic bounds on the lattice remainder in the case of irrational Q and

[4] One key ingredient in their proof is, in fact, a refined version of Ratner's measure classification theorem.

$d \geq 3$, see [DM93]. More precisely, they showed for appropriate regions Ω that

$$\liminf_{r\to\infty} \frac{\mathrm{vol}_{\mathbb{Z}}\,(E_{a,b} \cap r\Omega)}{\mathrm{vol}\,(E_{a,b} \cap r\Omega)} \geq 1,$$

where vol B denotes the Lebesgue measure of a measurable set $B \subset \mathbb{R}^d$ and $\mathrm{vol}_{\mathbb{Z}} B := \#(B \cap \mathbb{Z}^d)$ denotes the number of integer points in B.

(b) In the case of *positive definite forms* Götze established in 2004 explicit bounds on the lattice remainder. His arguments are based on a direct investigation of the distribution of the first successive minima of a certain symplectic lattice, which is associated to the counting problem via a Weyl-type argument. Showing that there are 'large' gaps in the distribution, he derived bounds for the corresponding averages, see [Göt04]. A variant of this method was applied to *split indefinite forms* in a PhD thesis by G. Elsner [Els09]. In particular, in Chapter 3 important aspects of Götze's approach will be used as well: For example, the rewriting of the lattice remainder in terms of successive minima of a symplectic lattice.

(c) The above-mentioned works [DM93; EMM98; EMM05] were extended by Margulis and Mohammadi to inhomogeneous quadratic forms, see [MM11]. Additionally, they have applied their results on the eigenvalue spacing on flat 2-tori proving a conjecture of Berry and Tabor for these tori under certain Diophantine conditions.

(d) One should also mention related results of Marklof [Mark02; Mark03] investigating the pair correlation densities of inhomogeneous quadratic forms and confirming partly the Berry-Tabor conjecture on the consecutive level spacing distribution of certain quantum systems.

(e) We also note that weaker results, providing upper bounds in terms of the signature for general quadratic forms, were established by Cook [Coo83], [Coo84], and Cook and Raghavan [CR84] using the diagonalization techniques of Birch and Davenport.

1.2 Integer-valued Quadratic Forms

Following the heuristic viewpoint that in the case of irrational forms the number of lattice points should be approximated by the corresponding volume, we expect that the bounds in the real case should be almost as good as in the integral case and, in fact, we will confirm this at least in the case of diagonal forms. Since our argument depends essentially on the solvability of non-degenerate, integral indefinite quadratic forms that are 'close' to scalar multiples of Q, we will need for our purpose explicit bounds on the size of small zeros of integral forms. Such bounds were given by Cassels [Cas55], Birch and Davenport [BD58c] and Schlickewei [Sch85] using techniques from the Geometry of Numbers.

The substantial ideas to establish bounds on the magnitude of a least isotropic lattice point of an integral quadratic form can be already found in Cassels' work [Cas55]. Birch and Davenport modified Cassels' geometric argument in the note [BD58a] and showed that any indefinite quadratic form $F[m] = f_1 m_1^2 + \ldots + f_d m_d^2$ in $d \geq 5$ variables with non-zero integers $f_1, \ldots, f_d$ admits a non-trivial lattice point $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d \setminus \{0\}$ satisfying

$$F[m] = f_1 m_1^2 + \ldots + f_d m_d^2 = 0 \qquad \text{and} \qquad 0 < |f_1| m_1^2 + \ldots + |f_d| m_d^2 \ll_d |f_1 \ldots f_d|, \quad (1.2)$$

where we use Vinogradov's notation $\ll$ as usual. This result of Birch and Davenport has a preparatory role only by providing bounds whose structure is essential for the application of the Birch-Davenport approach [BD58b] which will be discussed in full-detail in Section 2.1.

In 1985 Schlickewei [Sch85] has extended the observation by Birch and Davenport non-trivially by showing that the dimension, say d_0, of a maximal rational isotropic subspace influences the size of possible solutions essentially, rather than the mere indefiniteness, i.e. $d_0 \geq 1$. Using additionally an induction argument combined with Meyer's theorem Schlickewei derived a lower bound for d_0 in terms of the signature (r,s) as well. In Appendix C we will present a complete derivation of these results including an extension of Schlickewei's work by the following theorem. Assuming w.l.o.g. that $r \geq s$ (one can replace A by $-A$), we have

Theorem 1.3 (Schlickewei [Sch85])**.** Let A denote a non-singular quadratic form with signature (r,s) in $r+s=d \geq 5$ variables, which takes integral values on Λ only. Additionally, suppose that $|\det(\Lambda)| \geq 1$ and $\operatorname{Tr} A^2 \geq 1$, then the smallest non-trivial isotropic vector $m \in \Lambda$ of A satisfies the bound

$$0 < \|m\|^2 \ll_d (\operatorname{Tr} A^2)^{\rho} \, |\det \Lambda|^{\frac{4\rho+2}{d}} , \tag{1.3}$$

where

$$\rho := \rho(r,s) := \begin{cases} \frac{1}{2}\frac{r}{s} & \text{for } r \geq s+3 \\ \frac{1}{2}\frac{s+2}{s-1} & \text{for } r = s+2 \text{ or } r = s+1 \, . \\ \frac{1}{2}\frac{s+1}{s-2} & \text{for } r = s \end{cases} \tag{1.4}$$

Remark 1.4. In 1988 Schlickewei and Schmidt [SS88] proved that Schlickewei's bound (in terms of d_0) is qualitatively best possible. Their work is based on a previous counterexample given by Kneser, see [Cas56], and extensions of this example by Watson, see [Wat57], combined with an existence result on linearly independent linear forms with particular geometric properties. Of course, one can also ask if Schlickewei's bound in terms of the signature (r,s) is best possible, as was already conjectured by Schlickewei himself in his first work [Sch85] on small zeros. For the class of integral quadratic forms (not necessarily diagonal) this is known for the cases $r \geq s+3$ and $(3,2)$, see Schmidt [S85].

Remark 1.5.

(a) Applying Theorem 1.3 to diagonal forms, we obtain the following variant of (1.2): For any non-zero integers $f_1, \ldots, f_d$, of which $r \geq 1$ are positive and $s \geq 1$ negative with $d = r+s \geq 5$, there exist integers $m_1, \ldots, m_d$, not all zero, such that

$$f_1 m_1^2 + \ldots + f_d m_d^2 = 0 \qquad \text{and} \qquad 0 < |f_1| m_1^2 + \ldots + |f_d| m_d^2 \ll_d |f_1 \ldots f_d|^{\frac{2\rho+1}{d}}, \tag{1.5}$$

and the implicit constant in (1.5) depends on the dimension d only.

(b) Compared to (1.2), the exponent in (1.5) is smaller for a wide range of signatures (r,s) and in the cases, where the exponent is larger, we can restrict Q by setting some coordinates to zero to arrive at least at the result of the case $d=5$. For example, if one has $r \sim s$, then $2\rho \sim 1$ and therefore the exponent in (1.5) is of order $\sim 2/d$.

1.3 Main Result on Diagonal Indefinite Forms

In this and the following section we present the main theorems of this thesis beginning with the case of diagonal forms. It is worth mentioning that the classical result of Birch and Davenport [BD58b] has provided, until now, the sharpest known bounds within the class of diagonal forms. But, in view of the Schlickewei's work on small zeros of integral forms, it is reasonable to expect that the result of Birch and Davenport can be improved considerably and, in fact, one of our main contributions is to extend their approach to higher dimensions.

The principle strategy of Birch and Davenport is to extend their bound [BD58a] on small zeros of integral forms to the real case: Using a refined variant of the circle method they proved in the case $d = 5$ (assuming that all of the real numbers $q_1, \dots, q_d$ are of absolute value at least one) that for any $\varepsilon > 0$ the Diophantine inequality

$$|Q[m]| = |q_1 m_1^2 + \ldots + q_d m_d^2| < \varepsilon \tag{1.6}$$

is non-trivially solvable in integers and furthermore give an effective estimate on the size of the least solution. More precisely, for any $\delta > 0$ there is a non-trivial integral solution $m = (m_1, \dots, m_d) \in \mathbb{Z}^d \setminus \{0\}$ of (1.6) lying in the elliptic shell defined by

$$|q_1| m_1^2 + \ldots + |q_d| m_d^2 \ll_\delta |q_1 \ldots q_d|^{1+\delta} \varepsilon^{-4-\delta}. \tag{1.7}$$

The reader may note that (1.7) has the same form as the bound (1.2) for integral forms with the choice $\varepsilon = 1$ up to the additional dependency on δ. During the proof we will also see that the weighted norm in (1.7) is an appropriate choice because of the scaling properties with respect to $q_1, \dots, q_d$. Additionally and more importantly, the above result implies for $d \geq 5$ and arbitrarily small $\varepsilon > 0$ that there exists a non-trivial solution of (1.6) with integral $m_1, \dots, m_d$ all of size $\mathcal{O}(\varepsilon^{-2-\delta})$ for any fixed $\delta > 0$.

Guided by Theorem 1.3, we establish improved variants of the bound (1.7) in terms of the signature (r, s): Following the basic idea of Birch and Davenport, we will use Schlickewei's bounds as the main ingredient to bound the size of the least non-trivial solution of (1.6). In doing this, we prove the following bound for the irrational case, which is already comparable to (1.5) up to the determinant ($|f_1 \dots f_d|$ is the determinant of F) being replaced by the d-th power of the largest eigenvalue and an additional growth rate given by (1.9).

Theorem 1.6. Let $q_1, \dots, q_d$ be real numbers, of which $r \geq 1$ are positive and $s \geq 1$ negative, such that $|q_i| \geq e^e$ and $d = r + s \geq 5$. Then there exist integers $m_1, \dots, m_d$, not all zero, satisfying both (2.1) and

$$|q_1| m_1^2 + \ldots + |q_d| m_d^2 \lll_d \big(\max_{i=1,\dots,d} |q_i|\big)^{1+2\rho}, \tag{1.8}$$

where ρ is defined as in (1.4). Here the implicit constant depends on d only and $A \lll B$ stands for

$$A \ll B^{1+\frac{20d^2}{\log\log B}}. \tag{1.9}$$

The reader may note that the growth rate is considerably improved compared with (1.7), since we have

$$B^{1+\frac{20d^2}{\log\log B}} \ll B^{1+\delta}$$

for any $\delta > 0$. This improvement is achieved by replacing the smoothing kernel (in the application of the circle method) by a faster decaying choice. We also note

Corollary 1.7. If $q_1, \dots, q_d$ are fixed, and $\varepsilon > 0$ is arbitrary, then there exists a non-trivial solution $m = (m_1, \dots, m_d) \in \mathbb{Z}^d \setminus \{0\}$ of $|Q[m]| = |q_1 m_1^2 + \ldots q_d m_d^2| < \varepsilon$, whose size is of order $\lll \varepsilon^{-\rho}$.

Obviously, this bound is an improved variant of the above-mentioned bound $\mathcal{O}(\varepsilon^{-2-\delta})$ of Birch and Davenport [BD58b] for higher dimensions in terms of the signature (r, s). Although the general strategy of the proof uses the approach of Birch and Davenport [BD58b] as well, their approach has to be extended non-trivially by counting all intervals, where the peaks of the corresponding Weyl sums occur. To do this, we shall prove (conditionally) improved mean-value estimates for certain products of Weyl sums and iterate the *coupling argument* of Birch and Davenport.

Roughly speaking, their approach is based on an analysis of regular patterns in the frequency picture of the associated counting problem in form of *coupled* Diophantine approximation (see Definition 2.14 for the precise meaning). In the end they deduce a contradiction by counting these points (i.e. establishing an upper and a lower bound for the number of certain Diophantine approximants) under the assumption that there are no solutions of $|Q[m]| < \varepsilon$ in the elliptic shell defined by (1.7). As already mentioned above, our modificated variant of their approach will be described in Section 2.1 in full-detail.

Remark 1.8. The major feature of the Birch-Davenport approach is to avoid involving the explicit size of the Diophantine approximants and the absolute value of certain quadratic Weyl sums, which are related to the Diophantine approximation error via a typical Weyl inequality. In fact, it seems to be impossible to control these parameters sufficiently well via an approach, which aims for an asymptotic approximation of the number of integral solutions of (1.6). In particular, our results for non-diagonal quadratic forms is probably not optimal and one challenging questions is, if it is possible, to extend the Birch-Davenport approach to the non-diagonal case.

1.4 Our Contribution to the Non-Diagonal Case

The proof in the general case is based on effective error bounds for the approximation of the number of lattice points in hyperbolic or elliptic shells $\{m \in \mathbb{Z}^d : a < Q[m] < b\}$ restricted to growing domains, which will be presented in Chapter 3, and explicit estimates of special theta series θ_v, associated to the counting problem, in terms of projective Diophantine approximation, which are proved in Chapter 4. Both chapters are based on the earlier preprint [GM13] with several corrections and improvements included here. In particular, we improve the explicit dependency on the Diophantine properties of Q and make also use of Schlickewei's results [Sch85] on small zeros of integral forms replacing the bound of Birch and Davenport [BD58c].

1.4.1 Value Distribution of Quadratic Forms

Though the explicit results on the lattice remainder will be presented not until Section 3.1 of Chapter 3, we shall outline - as an orientation guide - the basic steps of the Fourier approach used here and mention also the changes added to the new preprint [BGHM19]: Beginning with a smoothing step, we will rewrite the lattice remainder problem in terms of Fourier integrals and split these integrals as usually done in the application of the Davenport-Heilbronn circle method. Guided by the approach introduced by Götze in his work [Göt04] on positive definite forms, the critical part of the Fourier integral will be translated into averages over the α-characteristic of a special $2d$-dimensional symplectic lattice, i.e. the maximum over the reciprocal volume of all d-dimensional sublattices, and then apply Margulis' averaging method giving upper estimates of averages of such functions on the space of lattices. An important role hereby is the choice of the counting region, which will be developed for certain oriented parallelepipeds depending on the width of the shell. Especially, for wide shells we will use modified methods developed by Skriganov [Skr94] on 'admissible lattices'. A more detailed description of these steps will be given in Section 3.2.

The explicit estimates developed here are improved and corrected variants of the announced results of the earlier preprint [GM13] of Götze and Margulis. In particular, the bounds on the lattice remainder stated in [GM13] had to be changed in the case of thin shells and were improved in the case of wide shells. Here the optimization procedure for the smoothing parameters is done differently depending on the width of the shell. In addition, we have changed the splitting of the Fourier integrals as well leading to better dependencies on the smallest and largest eigenvalue of Q and elaborated the discussion on the smoothing of the counting region in full-detail. Now Section 3.5 provides explicit estimates in terms of parameters depending on the parallelepiped region. Moreover, Sections 3.6 and 3.7 are completely revised versions of the corresponding sections in [GM13].

1.4.2 Quantitative Bounds for Diophantine Inequalities

In order to establish quantitative bounds, we will apply our quantitative results on the lattice remainder as follows: Either we have a 'good' approximation of the number of lattice points by the volume or the form is near a rational form and then we can make use of Schlickewei's bound on small zeros of integral forms. In this case, compared to the diagonal case, we use only an l^∞-bound on θ_v to extract the Diophantine behavior of Q (with respect to the scaling size of the region). The indicated dichotomy argument leads finally to the following quantitative bounds in the Oppenheim conjecture.

Theorem 1.9. For all indefinite and non-degenerate quadratic forms Q of dimension $d \geq 5$ and signature (r, s) there exists for any $\delta > 0$ a non-trivial integral solution $m \in \mathbb{Z}^d \setminus \{0\}$ to the Diophantine inequality $|Q[m]| < 1$ satisfying

$$\|Q_+^{1/2} m\| \ll_{\delta,d} (q/q_0)^{(d+1)/(d-2)} q^{\frac{2d}{d-4}\rho+\delta} |\det Q|^{\frac{2\rho+1}{2d}} , \tag{1.10}$$

where ρ is defined as (1.4).

Here Q_+ denotes the unique positive symmetric matrix such that $Q_+^2 = Q^2$ and, if $q_1, \ldots, q_d$ are the eigenvalues of Q, we write

$$q_0 \stackrel{\text{def}}{=} \min_{1\leq j\leq d} |q_j| , \qquad q \stackrel{\text{def}}{=} \max_{1\leq j\leq d} |q_j| , \qquad |Q| \stackrel{\text{def}}{=} |\det Q| . \tag{1.11}$$

Moreover, for technical reasons, we assume additionally that $q_0 \geq 1$. Of course, this can be always achieved by rescaling Q with $1/q_0$.

Corollary 1.10. For indefinite non-degenerate forms in $d \geq 5$ variables of signature (r, s) and eigenvalues in absolute value contained in a compact set $[1, C]$, i.e $1 \leq q_0 \leq q \leq C$, there exist non-trivial solutions $m \in \mathbb{Z}^d$ of $|Q[m]| < \varepsilon$ of size bounded by

$$\|m\| \ll_{C,\delta} \varepsilon^{-\frac{3d-4}{d-4}\rho-\delta}.$$

In particular, we obtain solutions of order $\ll_{C,\delta} \varepsilon^{-\frac{3}{2}-\frac{13d-24}{(d-4)(d-3)}-\delta}$ for the special case $r = s+3$ and for $d = 5$ of order $\ll_{C,\delta} \varepsilon^{-22-\delta}$ for any fixed $\delta > 0$.

The novelty of the revised arguments of [GM13] is, in fact, the derivation of a bound in the non-diagonal case in a small number of variables. Previous works have only established larger bounds for forms with restrictions on the signature and require a large number of variables. However, the proof given in [GM13] was incomplete and here we have remedied one missing argument: The theorem of Meyer requires, of course, that the quadratic form is indefinite. The earlier preprint [GM13] does not address this point, which unfortunately requires additional restrictions leading to worse estimates for the quantitative Oppenheim conjecture. For example, the exponent size of the least solution had to be increased from $\varepsilon^{-12-\delta}$ to $\varepsilon^{-22-\delta}$ in the case $d = 5$. However, at least for higher dimensions and 'good signatures' we have better estimates due to Schlickewei's result and, in addition, the proof was simplified as well.

1.4.3 Diophantine Quadratic Forms

Besides the above-mentioned results, we shall also deduce explicit bounds for another class of forms without the use of small bounds for integral forms. We shall show that in the case of quadratic forms of Diophantine type (κ, A), as will be introduced in Definition 4.1, we can compare the volume with the number of lattice points. Compared to [GM13], we also have added a class of explicit examples: By using Schmidt's Subspace theorem we show that forms with independent algebraic coefficients belong to the class of quadratic forms of Diophantine type.

Theorem 1.11. Let Q be an indefinite quadratic form in at least five variables. Suppose that Q has $k+1$ non-zero entries $y, x_1, \dots, x_k$ such that $x_1/y, \dots, x_k/y$ are algebraic and $1, x_1/y, \dots, x_k/y$ are linearly independent over $\mathbb{Q}$. Then for any $\delta > 0$ there exists a non-trivial solution to the Diophantine inequality $|Q[m]| < \epsilon$ of order

$$\ll_{Q,d,\delta} \epsilon^{-\frac{d(3+2k)-4}{2k(d-4)}-\delta}.$$

For example, for $k = \frac{d(d+1)}{2} - 1$ we can give a bound for the size of the least solution of order $\ll_{Q,d,\delta} \epsilon^{-\frac{d^3+d^2+d-4}{(d^2+d-2)(d-4)}-\delta}$ and in this case for $d = 5$ of order $\ll_{Q,\delta} \epsilon^{-151/28-\delta}$, where $151/28 \approx 5.39$.

Remark 1.12.

(a) Neither in Theorem 1.9 nor in Theorem 1.11 the condition $d \geq 5$ on the dimension can be relaxed. In fact, the proof of Theorem 1.9 relies on results about small zeros of integral-valued quadratic forms and such forms may fail to have non-trivial zeros if $d < 5$. Moreover, the used methods to translate the lattice remainder to averages of certain functions of special symplectic lattices together with Margulis' averaging method require at least $d \geq 5$ variables as well.

(b) If $d = 2$, then the values at integral points may not be dense in $\mathbb{R}$, even when Q is irrational. In fact, since $Q(1, x)$ is a quadratic polynomial which takes positive and negative values, we can write $Q(x, y) = c(x + ay)(x + by)$ with real numbers $a, b, c \in \mathbb{R}$ and therefore zero is an accumulation point of $Q[\mathbb{Z}^2]$ if and only if one of a or b is an irrational number which is not badly approximable.

(c) The remaining cases $d = 3$ and $d = 4$ are expected to be challenging. For quadratic forms in three variables with algebraic coefficients a weak answer is given by Lindenstrauss and Margulis, see [LM14]. As we already indicated, in some special cases (considering diagonal forms of special type) bounds are given in terms of continued fraction expansions by Watson [Wat53b] and also Iwaniec [Iwa77]. In contrast, recent research focus on randomized variants of the above questions as will be discussed in the next section.

1.5 Recent Development on Generic Variants of the Oppenheim Conjecture

Recently, Bourgain [Bou16], Athreya and Margulis [AM18], and Ghosh and Kelmer [GK18] investigated generic variants of the quantitative Oppenheim conjecture. Bourgain [Bou16] proved essentially optimal results for one-parameter families of diagonal ternary indefinite quadratic forms under the Lindelöf hypothesis by using an analytic number theory approach. Compared to [Bou16], Ghosh and Kelmer consider in their work [GK18] the space of all indefinite ternary quadratic forms, equipped with a natural probability measure, and they use an effective mean ergodic theorem for semisimple groups. In contrast, Athreya and Margulis [AM18] applied classical bounds of Rogers for L^2-norm of Siegel transforms to prove that for every $\delta > 0$ and almost every Q (with respect to the Lebesgue measure) with signature (r, s) and $d \geq 3$ variables, there exists a non-trivial integer solution $m \in \mathbb{Z}^d$ of the Diophantine inequality $|Q[m]| < \varepsilon$ whose size is

$$\|m\| \ll_{\delta,Q} \varepsilon^{-\frac{1}{d-2}+\delta}.$$

1.6 Further Research Questions and Open Problems

We have started with the question of the solvability of the Diophantine inequality $|H[m]| < \varepsilon$ in integral points $m \in \mathbb{Z}^d \setminus \{0\}$ for quadratic forms H. Of course, one may ask the same question for homogeneous forms $H\colon \mathbb{R}^d \to \mathbb{R}$ of higher degree k. In this case, relatively little is known and many open questions are expected to be challenging: Though the celebrated theorem of Birch [Bir62] on the Hasse principle, considering integral forms of degree $k \geq 3$, was recently improved by Browning and Prendville [BP17], the same questions for irrational forms remains in most parts unanswered. Additionally, one may ask for further improvements of Birch's work, since for forms of smaller degree particularly rich literature exists requiring a smaller number of variables. See, for example, the work of Heath-Brown [Hea83; Hea07], Hooley [Hoo88], Heath-Brown and Browning [BH09].

Considering irrational forms and following the volume approach used here, one may ask as well if it is possible to establish bounds on the corresponding lattice remainder? Methods based on the study of unipotent subgroups are less likely to be extendable to higher order forms. On the other hand, one may hope to extend the approach of Bentkus-Götze [BG97; BG99] developing a local Weyl-type argument and establishing gaps in the distribution of the corresponding exponential sums. At least for the special class of forms with positive diagonal highest order homogeneous terms there is a quantitative result of Bentkus and Götze [BG01] requiring very large dimensions. Such results on the lattice remainder, corresponding to counting lattice points of irrational forms H, would imply bounds on the size of an integral solution of $|H[m]| < \varepsilon$. In general, no explicit bounds on the size of isotropic vectors for integral forms are known and therefore the dichotomy argument used here is not applicable.

Even in the case of quadratic forms, the range of open questions remains rather broad: In view of the above-mentioned generic results, one can even expect in the case of diagonal forms better results for irrational forms. However, to prove such results one needs to go beyond the standard mean-value estimates. One promising starting point is to use explicit Diophantine properties of Q to get larger gaps in the distribution of the corresponding weighted exponential sum. Moreover, in the non-diagonal case, our Weyl-type argument seems not to be optimal, since the refined Weyl-type argument for diagonal forms gives better dependencies in terms of the Diophantine approximation error. To extend Schlickewei's bounds [Sch85] on small zeros of integral forms to real non-diagonal forms, one additionally needs to establish the rigidity argument of Birch and Davenport [BD58b] in the non-diagonal case. The subtle problem here is that the directions of the successive minima may change and therefore the dichotomy argument in [BD58b] cannot be applied.

So far we have only considered quadratic forms in at least five variables. For quadratic forms Q in dimension $d = 4$ or $d = 3$ there are no appropriate methods to tackle our initial question of the solvability of $|Q[m]| < \varepsilon$. Even the Davenport-Lewis conjecture [DL72] on the density of values at infinity for irrational positive forms remains open.

1.7 Notation and Glossary

Most notations, which are used throughout this thesis, are 'standard' in analytic number theory. We suppose that the reader is familiar with these notations (e.g. Landau symbols). If not otherwise stated, the asymptotics are always considered as $r \to \infty$, respectively $P \to \infty$. Besides the big $\mathcal{O}$ notation, Vinogradov's notation $A \ll_B C$, meaning that $A < c_B C$ with a constant $c_B > 0$ depending on B, will be used as well. We will also write as usual $e(x) = \exp(2\pi i x)$. In addition, $\mathbb{N}$ denotes the set of natural numbers excluding 0 and $\mathbb{N}_0$ with zero element.

General Glossary

Notation	**Description**
q_0	absolute value of the smallest eigenvalue of Q
q	absolute value of the largest eigenvalue of Q
$\lvert Q\rvert$	absolute value of the determinant of Q
ρ	see (1.4)

Glossary on the diagonal case

Notation	**Description**
$S_j(\alpha)$	the quadratic exponential sums corresponding to the eigenvalue q_j as defined in (2.2) on page 13
$K(\alpha)$	smoothing kernel in Chapter 2 given by $K = \widehat{\psi}$ with decay rate given by (2.8) on page 15
P	bound on the size of a non-trivial solution of $\lvert Q[m] < \varepsilon$ in the diagonal case, see (2.7) on page 15
H	see (2.7) on page 15 as well
$и(i)$	is defined by $и(i) := \min\{i, (d-4)\}^{-1}$, see (2.7) on page 15
$\mathcal{F}$	specific subset of $\mathbb{R}$, see (2.26) on page 20
$\mathfrak{D}_j(\alpha)$	see (2.32) on page 22
N_j	number of certain integral pairs, see (2.41) on 23
T_j, U_j	numbers corresponding to a certain dyadic composition, see (2.33) on page 22
$\mathcal{G}$	see (2.40) on page 23
$x_j, y_j, x, y, x'_j, y'_j$	integral numbers corresponding to the coupling argument, see Definition 2.14 on page 24
Q_k	restiction of Q, see (2.58) on page 29
$\mathfrak{p}_i(d)$	see Lemmas 2.25 - 2.27 and Section 7.3 as well

Glossary on the non-diagonal case

Notation	**Description**
$\operatorname{vol} B$	volume of a measurable set B

Notation	**Description**
$\mathrm{vol}_{\mathbb{Z}}\, B$	number of lattice points $m \in \mathbb{Z}^d$ in B
α_l	α_l-characteristic, see (3.1) on page 37 (resp. (3.43) on page 49)
α	maxmium over all α_l-characteristics, see (3.44) on page 49
$\gamma_{[T_-,T],\beta}(r)$	see (3.2) on page 37
$\det \Lambda$	discriminant of a lattice Λ
$M_j(\Lambda)$	j-th successive minima of an n-dimensional lattice Λ
$R(\cdot)$	shortcut for the error term (difference between integral and series), see (3.10) on page 40
$\theta_v(t)$	generalized theta series, see (3.31) on page 46
$\zeta(x)$	defined by $\zeta(x) \coloneqq v(x) \exp\{Q_+[x]\}$, compare (3.3) on page 37
Ω	counting region, see Section 3.5 for a detailed discussion in the case of parallelepiped regions
$\mathrm{Nm}\,\Gamma$	see (3.116) on page 62
Δ_r	lattice point remainder, i.e. $\Delta_r \coloneqq \lvert \mathrm{vol}_{\mathbb{Z}}\, H_r - \mathrm{vol}\, H_r \rvert$
k	fixed smoothing kernel, see Section 3.3.1
g_w	smoothed indicator function, see (3.24) on page 45
$I_\Delta, I_\theta, I_\vartheta$	see (3.33) on page 47
J_0, J_1	intervals defined by $J_0 \coloneqq [-q_0^{-1/2} r^{-1}, q_0^{-1/2} r^{-1}]$ and $J_1 \coloneqq \mathbb{R} \setminus J_0$
Λ_t	special $2d$-dimensional symplectic lattice, see (3.39) on page 47
Λ_Q	see (3.70) on page 54
$\psi(r,t)$	Siegel transform of $\exp\{-x^2\}$ evaluated at the lattice Λ_t, see (6.24) on page 101
D_{rQ}, U_{4tQ}	see (3.40) on page 47
d_r, u_t, k_θ	see (3.67) and (3.68) on page 53
$\delta_{tQ;R}$	approximation error, see (4.4) on page 73

CHAPTER 2

Indefinite Diagonal Quadratic Forms

The main subject of this chapter, which corresponds to the preprint [BGH19], is to prove Theorem 1.6: We shall consider non-singular, indefinite, diagonal quadratic forms

$$Q[m] = q_1 m_1^2 + \ldots + q_d m_d^2$$

of signature (r, s) with $d = r + s \geq 5$ variables only (i.e. $q_1, \ldots, q_d$ are the eigenvalues of Q and $r \geq 1$ of them are positive and $s \geq 1$ negative) and generalize the result of Birch and Davenport [BD58b] to this class. By extending their approach we significantly improve the explicit bounds, established by Birch and Davenport, in terms of the signature (r, s) by means of Schlickewei's work [Sch85] on the size of small zeros of integral quadratic forms. Compared to the earlier preprint [BG18], which already provides optimal results for most forms, we introduce an iteration of the coupling argument of Birch and Davenport to prove conditionally improved mean-value estimates.

To simplify the investigation of the Diophantine inequality $|Q[m]| = |q_1 m_1^2 + \ldots + q_d m_d^2| < \varepsilon$, we may assume that $\varepsilon = 1$. Indeed, replacing all coefficients q_j by q_j/ε it is sufficient to consider the solvability of the inequality

$$|q_1 m_1^2 + \ldots + q_d m_d^2| < 1. \tag{2.1}$$

2.1 Sketch of Proof

First, we shall outline the approach introduced by Birch and Davenport [BD58c], which is a proof by contradiction and consists mainly of two parts: The first step is to pick out all integral solutions to the inequality (2.1) that are contained in a box of a certain size by integrating the product of all associated quadratic exponential sums $S_1(\alpha), \ldots, S_d(\alpha)$, which are defined by

$$S_j(\alpha) \stackrel{\text{def}}{=} \sum_{P < |q_j|^{1/2} m_j < 2dP} e(\alpha q_j m_j^2) \tag{2.2}$$

with a suitable kernel K. Here we write as usual $e(x) = \exp(2\pi i x)$. Assuming that there are no integral solutions contained in the elliptic shell defined by

$$|q_1| m_1^2 + \ldots + |q_d| m_d^2 \leq 4d^3 P^2, \tag{2.3}$$

we deduce (in Lemma 2.2) that the real part of the integral $\int_0^\infty S_1(\alpha) \ldots S_d(\alpha) K(\alpha) \, d\alpha$ vanishes, i.e. there are non-trivial cancellations in the product of the sums $S_1, \ldots, S_d$. To analyze this integral, we will divide the range of integration into four parts, namely

$$0 < \alpha < \frac{1}{(8dP) q^{1/2}}; \qquad \frac{1}{(8dP) q^{1/2}} < \alpha < \frac{1}{(8dP)(q_0)^{1/2}};$$
$$\frac{1}{(8dP)(q_0)^{1/2}} < \alpha < u(P); \qquad u(P) < \alpha, \tag{2.4}$$

where $q_0 = \min_{1 \leq j \leq d} |q_j|$ and $q = \max_{1 \leq j \leq d} |q_j|$, compare (1.11), and $u(P) = \log(P + e)^2$. In the next steps we show that on the first range the mass of the real part is highly concentrated.

In fact, since α is 'very small', van der Corput's lemma can be applied and shows that this part is at least as large as the volume of the restricted hyperbolic shell

$$\{x \in \mathbb{R}^d : |Q[x]| < 1\} \cap \{x \in \mathbb{R}^d : P < |q_j|^{1/2} x_j < 2dP \text{ for all } j = 1, \dots, d\}. \tag{2.5}$$

In comparison, the second and fourth range of the integral is negligible. Consequently the mass contained in the third range - which we will also call $\mathcal{J}$ - has to be of the order of the volume of (2.5) and hence the contribution is 'large' as well when integrating the absolute value of the product $S_1, \dots, S_d$, see Lemma 2.7. Moreover, it remains 'large', even if we restrict ourselves to a subregion (called $\mathcal{F}$) of $\mathcal{J}$, where all factors $S_1, \dots, S_d$ are uniformly 'large' (see Corollary 2.10).

The second step consists in finding an upper and a lower bound for the number N_j of specific rational approximants (x_i, y_i) of $q_i\alpha$ in this subregion of the integral. As in Birch and Davenport [BD58a], it is convenient to consider those parts of this subregion, where for each $i = 1, \dots, d$ both quantities $S_i(\alpha)$ and y_i are all of the same magnitude independent of α. More precisely, we are going to use a dyadic decomposition of $\mathcal{F}$ into $\ll \log(P)^{2d}$ parts and restrict ourselves to one of these sets, say $\mathcal{G}$, where the integral over $\mathcal{G}$ remains 'large', see Lemma 2.12.

The lower bound for N_i will be established by a standard applications of a refined variant of Weyl's inequality, see Corollary 2.13. To derive an upper bound, we shall prove on $\mathcal{G}$ that $d - k$ fractions $x_i y_d / y_i x_d$ are independent of α (see Lemma 2.17), where $k \in \{0, 1, 2, 3\}$ depends on the size of ρ and the order of magnitude of $S_{k+1}, \dots, S_d$ (prior to that, we have already rearranged $S_1, \dots, S_d$ in a certain way, compare (2.34)). Here $S_{k+1}, \dots, S_d$ show a rigid behaviour as in the rational case. Indeed, the previous observation gives rise to a factorization of x_i and y_i as

$$x_i = x x_i' \qquad \text{and} \qquad y_i = y y_i'$$

such that x_i' and y_i' divide a fixed number, which is independent of α. For notational simplicity, we will say that $S_{k+1}, \dots, S_d$ are *coupled* on $\mathcal{G}$ if such a factorization exists, see Definition 2.14 for the precise meaning.

The case $k = 0$ corresponds to Birch and Davenport's paper [BD58c]. However, this setting occurs only if $\rho \geq 2$, i.e. the exponent in the bound (1.7) has to be relatively large. In fact, the main difficulty in the proof of Theorem 1.6 is to overcome this issue: In Section 2.4 this factorization will be used to show that all pairs (x, y) lie in a certain bounded set (see Lemma 2.21). As a consequence, we deduce an upper bound for the number of distinct pairs (x, y), see Corollary 2.22. Based on this, we shall establish an improved mean-value estimate for $S_{k+1} \dots S_d$ on $\mathcal{G}$, which implies better estimates for the order of magnitude of S_k. This improved estimate allows us to conclude that $S_k, \dots, S_d$ are coupled on $\mathcal{G}$ as well. Now, depending on $k \in \{0, 1, 2, 3\}$, we can iterate this argument until $k = 0$ to prove that all remaining coordinates are coupled. In doing this, we are faced with the tedious problem of comparing Schlickewei's exponent (1.4) for Q and all possible restrictions of Q to certain subspaces with k zero coordinates. This results in the number of cases listed in Section 7.3 of Appendix C.

To complete the proof, we deduce an inconsistent inequality (as in Birch and Davenport [BD58c]) by establishing an upper bound for a particular N_i, which contradicts the lower bound found previously.

2.2 Fourier Analysis

Throughout this chapter $q_1, \ldots, q_d$ denote real non-zero numbers, of which $r \geq 1$ are positive and $s \geq 1$ negative and, as usual, the constants throughout the proofs involved in the notation $\ll$ will not be always mentioned explicitly; these will depend on d only unless stated otherwise. We also stress the underlying assumption that $d = r + s \geq 5$, since our argument depends on the solvability of non-degenerate, integral indefinite quadratic forms that are 'close' to scalar multiples of Q. We shall ultimately deduce a contradiction from the following assumption.

Assumption 2.1. Let $q_1, \ldots, q_d$ be as introduced in Theorem 1.6. Suppose that for $C_d > 0$ the inequality

$$\left|q_1 m_1^2 + \ldots + q_d m_d^2\right| < 1$$

has no solutions in integers $m_1, \ldots, m_d$, not all zero, satisfying

$$|q_1|\, m_1^2 + \ldots + |q_d|\, m_d^2 \leq 4d^3 P^2, \tag{2.6}$$

where

$$P = \exp\left\{\left(1 + \tfrac{10d^2}{\log\log H}\right)\log H\right\} \quad \text{and} \quad H = C_d q^{\frac{1}{2}+\rho} \tag{2.7}$$

and ρ is defined as in (1.4).

Until the end of this chapter we shall fix a smoothing kernel $K = \widehat{\psi}$ with decay rate

$$|\widehat{\psi}(t)| \ll \exp(-t/\log(t+\mathrm{e})^2), \tag{2.8}$$

where ψ is a smooth symmetric probability density supported in $[-1, 1]$. Note that the existence of such a function ψ is guaranteed by Lemma 5.11 (of the Appendix A) with the choice

$$u(t) \overset{\text{def}}{=} \log(\mathrm{e} + t)^2. \tag{2.9}$$

Compared to [BD58b] this kernel allows us to reduce the growth rate of the bound (1.8) of Theorem 1.6, since we replace the kernel by a faster decaying one.

2.2.1 Counting via Integration

The starting point of Birch and Davenport's approach is the following observation.

Lemma 2.2. Assumption 2.1 implies

$$\mathrm{Re}\Big(\int_0^\infty S_1(\alpha)\ldots S_d(\alpha)K(\alpha)\,\mathrm{d}\alpha\Big) = 0, \tag{2.10}$$

where the exponential sums S_j are defined as in (2.2).

Proof: Expanding the product shows that

$$\mathrm{Re}\int_0^\infty S_1(\alpha)\ldots S_d(\alpha)K(\alpha)\,\mathrm{d}\alpha = \frac{1}{2}\sum_{P<|q_1|^{1/2}m_1<2dP}\cdots\sum_{P<|q_d|^{1/2}m_d<2dP}\psi(q_1 m_1^2 + \ldots + q_d m_d^2),$$

where we used that ψ and $\mathrm{Re}(S_1 \ldots S_d)$ are symmetric functions. Since the domain of summation is contained in (2.6), we have

$$|q_1 m_1^2 + \ldots + q_d m_d^2| \geq 1$$

by Assumption 2.1. Thus, the sum is zero because ψ is supported in $[-1, 1]$. □

We begin by investigating the first range in (2.4), that is $0 < \alpha < (8dPq^{1/2})^{-1}$, where van der Corput's lemma can be applied in order to relate the integral over the exponential sums $S_1, \dots, S_d$ to the integral over their corresponding exponential integrals as follows.

Lemma 2.3. For

$$0 < \alpha < (8dP)^{-1}|q_j|^{-\frac{1}{2}} \tag{2.11}$$

we have

$$S_j(\alpha) = |q_j|^{-\frac{1}{2}} I(\pm\alpha) + \mathcal{O}(1), \tag{2.12}$$

where the $\pm$ sign is the sign of q_j and

$$I(\alpha) = \int_P^{2dP} \exp(2\pi \mathrm{i}\alpha\xi^2)\,\mathrm{d}\xi. \tag{2.13}$$

Proof: Let $f(x) = \alpha|q_j|x^2$. If $P < |q_j|^{1/2}x < 2dP$, then we have $f''(x) > 0$ and $0 < f'(x) < 1/2$. Hence the conditions of van der Corput's Lemma ([Vin54], Chapter 1, Lemma 13) are fulfilled and therefore we obtain

$$S_j(\alpha) = \int_{P|q_j|^{-\frac{1}{2}}}^{2dP|q_j|^{-\frac{1}{2}}} e(\alpha q_j \zeta^2)\,\mathrm{d}\zeta + \mathcal{O}(1).$$

Using the change of variables $\xi = |q_j|^{1/2}\zeta$ in the last integral proves already (2.12). □

The next lemma will be helpful for estimating the integral $I(\pm\alpha)$ in (2.12).

Lemma 2.4 (Lemma 3 in [BD58b])**.** For $\alpha > 0$ we have

$$|I(\pm\alpha)| \ll \min(P, P^{-1}\alpha^{-1}). \tag{2.14}$$

Proof: Estimating the integral in (2.13) by the length of the integration region shows that $|I(\alpha)| \ll P$. On the other hand, we may change variables via $\xi^2 = \zeta$ to get

$$I(\alpha) = \frac{1}{2}\int_{P^2}^{4d^2P^2} \frac{1}{\sqrt{\zeta}} \exp(2\pi\mathrm{i}\alpha\zeta)\,\mathrm{d}\zeta$$

and after applying partial integration we find

$$I(\alpha) = \frac{1}{4\pi\mathrm{i}\alpha}\frac{\exp(2\pi\mathrm{i}\alpha\zeta)}{\sqrt{\zeta}}\Bigg|_{\zeta=P^2}^{4d^2P^2} + \frac{1}{8\pi\mathrm{i}\alpha}\int_{P^2}^{4d^2P^2} \frac{1}{\zeta^{3/2}}\exp(2\pi\mathrm{i}\alpha\zeta)\,\mathrm{d}\zeta.$$

Here we can bound the integrand (on the right-hand side) by its absolute value in order to get $I(\alpha) \ll (\alpha P)^{-1}$ as well. □

Now we shall give an upper bound for the main integral in a small neighborhood of zero and thus generalize Lemma 4 of [BD58b] to dimensions greater than five.

Lemma 2.5. We have

$$\mathrm{Re}\int_0^{(8dP)^{-1}q^{-\frac{1}{2}}} S_1(\alpha)\dots S_d(\alpha)K(\alpha)\,\mathrm{d}\alpha = M_1 + R_1, \tag{2.15}$$

where the main term satisfies

$$M_1 \gg \delta P^{d-2}|Q|^{-\frac{1}{2}} \tag{2.16}$$

for some $\delta > 0$ depending on the kernel K only and the error term is bounded by

$$|R_1| \ll P^{d-3}q^{\frac{1}{2}}|Q|^{-\frac{1}{2}}. \tag{2.17}$$

Proof: In the domain of integration $0 < \alpha < (8dP)^{-1}q^{-\frac{1}{2}}$ the condition (2.11) of Lemma 2.3 is satisfied for each j=1, ..., d. Thus, we have

$$S_j(\alpha) = |q_j|^{-\frac{1}{2}} I(\pm\alpha) + \mathcal{O}(1)$$

and together with (2.14) of Lemma 2.4 we obtain the bound

$$S_j(\alpha) \ll |q_j|^{-\frac{1}{2}} \min(P, P^{-1}\alpha^{-1}).$$

Combining both relations yields

$$\Big|\prod_{j=1}^{d} S_j(\alpha) - |Q|^{-\frac{1}{2}} \prod_{j=1}^{d} I(\operatorname{sgn}(q_j)\alpha)\Big| \ll \sum_{j=1}^{d-1} \sum_{\{i_1,\ldots,i_j\}\subset\{1,\ldots,d\}} |q_{i_1}\ldots q_{i_j}|^{-\frac{1}{2}} \min(P^j, (P\alpha)^{-j}).$$

Because of $P > q^{1/2}$ and $\alpha^{-1}P^{-1} > q^{1/2}$, we have $\min(P, \alpha^{-1}P^{-1}) > q^{\frac{1}{2}}$ and therefore the right-hand side is bounded by

$$\ll q^{\frac{1}{2}} |Q|^{-\frac{1}{2}} \min\left(P^{d-1}, (P\alpha)^{-(d-1)}\right).$$

Hence, up to a small error, we can replace the sum by an integral and obtain

$$\int_0^{(8dP)^{-1}q^{-\frac{1}{2}}} S_1(\alpha)\ldots S_d(\alpha) K(\alpha)\,\mathrm{d}\alpha = |Q|^{-\frac{1}{2}} \int_0^{(8dP)^{-1}q^{-\frac{1}{2}}} K(\alpha)\Big(\prod_{j=1}^{d} I(\operatorname{sgn}(q_j)\alpha)\Big)\,\mathrm{d}\alpha$$
$$+ \mathcal{O}\bigg(q^{\frac{1}{2}}|Q|^{-\frac{1}{2}} \int_0^{\infty} \min(P^{d-1}, P^{-(d-1)}\alpha^{-(d-1)})\,\mathrm{d}\alpha\bigg).$$

Note that the last error can be absorbed in R_1 by (2.17), because it is bounded by

$$q^{\frac{1}{2}}|Q|^{-\frac{1}{2}}\bigg(\int_0^{P^{-2}} P^{d-1}\,\mathrm{d}\alpha + \int_{P^{-2}}^{\infty} P^{1-d}\alpha^{1-d}\,\mathrm{d}\alpha\bigg) \ll q^{\frac{1}{2}}|Q|^{-\frac{1}{2}} P^{d-3}.$$

We can also extend the integration domain to ∞, since the additional error is given by

$$|Q|^{-\frac{1}{2}} \int_{(8dP)^{-1}q^{-\frac{1}{2}}}^{\infty} I(\pm\alpha)\ldots I(\pm\alpha)K(\alpha)\,\mathrm{d}\alpha \ll |Q|^{-\frac{1}{2}} \int_{(8dP)^{-1}q^{-\frac{1}{2}}}^{\infty} P^{-d}\alpha^{-d}\,\mathrm{d}\alpha$$
$$\ll |Q|^{-\frac{1}{2}} q^{\frac{1}{2}} P^{-1} q^{\frac{d}{2}-1} \ll |Q|^{-\frac{1}{2}} q^{\frac{1}{2}} P^{d-3},$$

where we used that $q^{1/2} < P$. Again, this error can be absorbed in R_1 by (2.17).

Next, we are going to establish a lower bound for the main term

$$M_1 = |q_1\ldots q_d|^{-\frac{1}{2}} \operatorname{Re}\Big(\int_0^{\infty} I(\pm\alpha)\ldots I(\pm\alpha)K(\alpha)\,\mathrm{d}\alpha\Big).$$

Keeping in mind that $\widehat{K} = \psi$, we may rewrite the main term as

$$M_1 = 2^{-1}|q_1\ldots q_d|^{-\frac{1}{2}} \int_P^{2dP} \cdots \int_P^{2dP} \psi(\pm\xi_1^2 \pm \ldots \pm \xi_d^2)\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_d$$
$$= 2^{-d-1}|q_1\ldots q_d|^{-\frac{1}{2}} \int_{P^2}^{4d^2P^2} \cdots \int_{P^2}^{4d^2P^2} (\eta_1\ldots\eta_d)^{-\frac{1}{2}} \psi(\pm\eta_1 \pm \ldots \pm \eta_d)\,\mathrm{d}\eta_1\ldots\mathrm{d}\eta_d.$$

Since $\psi(x)$ is symmetric around $x = 0$, increasing for $x < 0$ and decreasing for $x > 0$ (see Lemma 5.11), we have $\psi(0) \geq 1/2$. In particular, there exists a $\delta \in (0,1)$ such that

$$\psi(\alpha) > 1/4 \quad \text{for all} \quad |\alpha| \leq \delta.$$

Relabeling the variables, if necessary, we may suppose that the sign attached to η_1 is $+$ and that the sign attached to η_2 is $-$. It is easy to see that the region defined by

$$P^2 < \eta_i < 4P^2 \quad \text{for} \quad i = 3, \dots, d \quad \text{and} \quad 4(d-1)P^2 < \eta_2 < (4d(d-1)+7)P^2$$

and

$$|\eta_1 - \eta_2 \pm \eta_3 \pm \ldots \pm \eta_d| < \delta$$

is contained in the region of integration. Therefore, we get the lower bound

$$\begin{aligned} M_1 &> 2^{-d-3}|q_1 \ldots q_d|^{-\frac{1}{2}}(2\delta)(4d^2P^2)^{-\frac{1}{2}} \int_{4(d-1)P^2}^{(4d(d-1)+7)P^2} \eta_2^{-\frac{1}{2}} \mathrm{d}\eta_2 \left(\int_{P^2}^{4P^2} \eta^{-\frac{1}{2}} \,\mathrm{d}\eta \right)^{d-2} \\ &= (2^{-4}\delta)|q_1 \ldots q_d|^{-\frac{1}{2}} \frac{\sqrt{4d(d-1)+7} - \sqrt{4(d-1)}}{d} P^{d-2} \end{aligned}$$

and the latter is at least as large as $(2^{-4}\delta)|q_1 \ldots q_d|^{-\frac{1}{2}} P^{d-2}$. □

In order to guarantee that the (yet to be introduced) Diophantine approximation of $q_j\alpha$ does not vanish, we have to extend the upper integration limit in (2.15) from $(8dP)^{-1}q^{-1/2}$ to $(8dP)^{-1}(q_0)^{-1/2}$. This can be done without changing the lower bound on the main term M_1 in (2.16) and in particular will be important to obtain a rational approximation of the quadratic form αQ with the same signature as Q.

Lemma 2.6. We have

$$R_2 = \int_{(8dP)^{-1}q^{-1/2}}^{(8dP)^{-1}(q_0)^{-1/2}} |S_1(\alpha) \ldots S_d(\alpha)| \,\mathrm{d}\alpha \ll q^{1/2}|Q|^{-1/2}P^{d-3}(\log P). \tag{2.18}$$

A variant of our Lemma 2.6 is also proved in [BD58b] under the stronger assumption $P > |Q|^{1/2}$. Here the situation is even easier, since we have $P > q$. This follows directly from Assumption 2.1 and the fact that $\rho > 1/2$ or more precisely

$$\rho \geq \frac{1}{2}\frac{d+3}{d-3} \quad \text{if } d \text{ is odd} \qquad \text{and} \qquad \rho \geq \frac{1}{2}\frac{d+2}{d-4} \quad \text{if } d \text{ is even,} \tag{2.19}$$

as can be checked easily. Additionally, we will need - apart from Lemma 2.3 - the moment-estimates established in Lemma 5.1, see Section 5.1 in the Appendix A.

Proof of Lemma 2.6: During this proof we will not need that Q is indefinite and therefore we can assume that the eigenvalues are ordered, i.e. $1 \leq |q_1| \leq |q_2| \leq \ldots \leq |q_d|$. In particular, we have $q_0 = |q_1|$ and $q = |q_d|$.

In order to apply Lemma 2.3, we split the interval of integration into the $d-1$ intervals

$$I_k = \left\{ \alpha \in (0, \infty) \,:\, (8dP|q_k|^{\frac{1}{2}})^{-1} < \alpha < (8dP|q_{k-1}|^{\frac{1}{2}})^{-1} \right\},$$

where $k = 2, \dots, d$. If $j \leq k-1$, then the condition (2.11) of Lemma 2.3 is satisfied. Therefore, combined with Lemma 2.4, we obtain for $\alpha \in I_k$ the inequality

$$|S_j(\alpha)| \ll |q_j|^{-\frac{1}{2}}P^{-1}\alpha^{-1} + 1 \ll |q_j|^{-\frac{1}{2}}P^{-1}\alpha^{-1}. \tag{2.20}$$

For $j \geq k$ we use the trivial estimate $|S_j(\alpha)| \ll P|q_j|^{-\frac{1}{2}}$ to conclude that

$$|S_1(\alpha)\dots S_d(\alpha)| \ll |Q|^{-\frac{1}{2}}(P\alpha)^{1-k}P^{d-(k-1)}.$$

If $k \geq 3$, then we find the bound

$$\int_{I_k} |S_1(\alpha)\dots S_d(\alpha)| \,\mathrm{d}\alpha \ll |Q|^{-\frac{1}{2}}P^{d-2(k-1)}(P|q_k|^{\frac{1}{2}})^{k-2}$$
$$= |Q|^{-\frac{1}{2}}P^{d-2}(P^{-1}|q_k|^{\frac{1}{2}})^{k-2} \ll |Q|^{-\frac{1}{2}}q^{\frac{1}{2}}P^{d-3}.$$

We are left to treat the case $k = 2$ corresponding to the interval I_2. For $j = 1$ inequality (2.20) still holds and therefore we have

$$|S_1(\alpha)| \ll |q_1|^{-\frac{1}{2}}P^{-1}\alpha^{-1} \ll |q_1|^{-\frac{1}{2}}|q_2|^{\frac{1}{2}}. \tag{2.21}$$

Let $j = 2, \dots, d$. Dividing the interval I_2 into parts of length $|q_j|^{-1}$ (the period of S_j) gives

$$\int_{I_2} |S_j(\alpha)|^{d-1}\,\mathrm{d}\alpha \leq (1 + |q_j|(8dP|q_1|^{\frac{1}{2}})^{-1})\int_0^{|q_j|^{-1}} |S_j(\alpha)|^{d-1}\,\mathrm{d}\alpha \ll \int_0^{|q_j|^{-1}} |S_j(\alpha)|^{d-1}\,\mathrm{d}\alpha,$$

where $P \geq q$ was used. Next we apply Lemma 5.1 to get the estimate

$$\int_{I_2} |S_j(\alpha)|^{d-1}\,\mathrm{d}\alpha \ll |q_j|^{-\frac{d-1}{2}}P^{d-3}(\log P)$$

and use Hölder's inequality to obtain

$$\int_{I_2} |S_2(\alpha)\dots S_d(\alpha)|\,\mathrm{d}\alpha \ll |q_2\dots q_d|^{-\frac{1}{2}}P^{d-3}(\log P).$$

Together with equation (2.21) we find

$$\int_{I_2} |S_1(\alpha)\dots S_d(\alpha)|\,\mathrm{d}\alpha \ll |q_2|^{\frac{1}{2}}|Q|^{-\frac{1}{2}}P^{d-3}(\log P) \ll q^{\frac{1}{2}}|Q|^{-\frac{1}{2}}P^{d-3}(\log P). \qquad \square$$

We end this subsection by combining the previous estimates in order to prove

Lemma 2.7. Under Assumption 2.1, we may choose $C_d \gg 1$, occurring in the definition of P in (2.7), such that

$$\int_{(8dP)^{-1}(q_0)^{-\frac{1}{2}}}^{u(P)} |S_1(\alpha)\dots S_d(\alpha)K(\alpha)|\,\mathrm{d}\alpha \gg |Q|^{-\frac{1}{2}}P^{d-2}. \tag{2.22}$$

Proof: According to Lemmas 2.2, 2.5 and 2.6 we have

$$M_1 + M_2 + R_1 + R_2 + R_3 = 0,$$

where

$$M_1 \gg |Q|^{-\frac{1}{2}}P^{d-2} \quad \text{and} \quad |R_1| + |R_2| \ll q^{\frac{1}{2}}|Q|^{-\frac{1}{2}}P^{d-3}(\log P) \ll |Q|^{-\frac{1}{2}}P^{d-\frac{5}{2}}$$

and

$$M_2 = \mathrm{Re} \int_{(8dP)^{-1}(q_0)^{-\frac{1}{2}}}^{u(P)} S_1(\alpha) \dots S_d(\alpha) K(\alpha) \,\mathrm{d}\alpha,$$

$$R_3 = \mathrm{Re} \int_{u(P)}^{\infty} S_1(\alpha) \dots S_d(\alpha) K(\alpha) \,\mathrm{d}\alpha.$$

We can easily bound R_3: Using the trivial estimate $|S_j(\alpha)| \ll P|q_j|^{-\frac{1}{2}}$ and the decay of K, see (2.8), gives (by applying L'Hôpital's rule)

$$R_3 \ll P^d |Q|^{-\frac{1}{2}} \int_{u(P)}^{\infty} \exp(-\alpha u(\alpha)^{-1}) \,\mathrm{d}\alpha \ll |Q|^{-\frac{1}{2}} P^{d-3}.$$

Combining the previous estimates we end up with

$$|M_1 + M_2| \leq |R_1| + |R_2| + |R_3| \ll |Q|^{-\frac{1}{2}} P^{d-3} \big(1 + P^{\frac{1}{2}}\big).$$

In view of the lower bound for M_1, we may increase $C_d \gg 1$ such that

$$P^{d-2} |Q|^{-\frac{1}{2}} \ll |M_2| \leq \int_{(8dP)^{-1}(q_0)^{-\frac{1}{2}}}^{u(P)} |S_1(\alpha) \dots S_d(\alpha) K(\alpha)| \,\mathrm{d}\alpha. \qquad \square$$

2.2.2 Ordering and Contribution of the Peaks

Next we shall refine the previous bound by showing that the main contribution to the integral (2.22) arises from a certain subregion on which every $S_1, \dots, S_d$ is large. Before doing this, we shall fix an ordering of $S_1, \dots, S_d$ as well, which will be necessary in order to perform the coupling argument and its iteration. To simplify the notation, we define

$$\mathcal{J} := \big\{\, \alpha \in (0, \infty) \,:\, (8dPq_0^{1/2})^{-1} < \alpha < u(P) \,\big\} \tag{2.23}$$

and write

$$\mathcal{J}_\pi := \big\{\, \alpha \in \mathcal{J} \,:\, |q_{\pi(1)}|^{\frac{1}{2}} |S_{\pi(1)}(\alpha)| \leq \dots \leq |q_{\pi(d)}|^{\frac{1}{2}} |S_{\pi(d)}(\alpha)| \,\big\} \tag{2.24}$$

for any permutation π of the set $\{1, \dots, d\}$. Obviously, since all these sets cover $\mathcal{J}$ completely and there are only finitely many permutations of $\{1, \dots, d\}$, Lemma 2.7 implies already

Lemma 2.8. Under Assumption 2.1, there exists a permutation π of $\{1, \dots, d\}$ such that

$$\int_{\mathcal{J}_\pi} |S_1(\alpha) \dots S_d(\alpha) K(\alpha)| \,\mathrm{d}\alpha \gg P^{d-2} |Q|^{-\frac{1}{2}}. \tag{2.25}$$

In this chapter we shall fix a permutation π satisfying the inequality (2.25). As announced, we shall prove next that the integral in (2.25) can be restricted to

$$\mathcal{F} := \big\{\, \alpha \in \mathcal{J}_\pi \,:\, |q_{\pi(i)}|^{\frac{1}{2}} |S_{\pi(i)}(\alpha)| > P(u(P)^2 q)^{-\text{и}(i)} \quad \text{for all} \quad i = 1, \dots, d \,\big\}. \tag{2.26}$$

where $\text{и}(i) := \min\{i, (d-4)\}^{-1}$. Indeed, we have

Lemma 2.9. Independently of Assumption 2.1, the estimate

$$\int_{\mathcal{J}_\pi \setminus \mathcal{F}} |S_1(\alpha) \dots S_d(\alpha)| \,\mathrm{d}\alpha \ll |Q|^{-\frac{1}{2}} P^{d-2} (\log P)^{-1} \tag{2.27}$$

holds, where the error term depends on the dimension d only.

Compared to the original work [BD58b] of Birch and Davenport, the dependency on the maximal eigenvalue in (2.26) can be improved by using the ordering (2.24).

Proof: First we decompose the complement $\mathcal{J}_\pi \setminus \mathcal{F}$ into d many sets given by

$$\mathcal{C}_j \stackrel{\text{def}}{=} \{ \alpha \in \mathcal{J}_\pi \,:\, |q_{\pi(j)}|^{\frac{1}{2}} |S_{\pi(j)}(\alpha)| \leq P(u(P)q)^{-\text{и}(j)} \},$$

where $j = 1, \dots, d$. If $\alpha \in \mathcal{C}_j$, then (2.24) implies that

$$|q_{\pi(1)}|^{1/2} |S_{\pi(1)}(\alpha)| \leq \ldots \leq |q_{\pi(j)}|^{1/2} |S_{\pi(j)}(\alpha)|$$

and therefore the left-hand side of (2.27), restricted to the region $\mathcal{C}_j$, is bounded by

$$\ll |q_{\pi(1)} \dots q_{\pi(k)}|^{-\frac{1}{2}} P^k (u(P)^2 q)^{-1} \int_0^{u(P)} |S_{\pi(k+1)}(\alpha) \dots S_{\pi(d)}(\alpha)| \,\mathrm{d}\alpha, \tag{2.28}$$

where $k = \min(j, d-4)$. Recalling that S_i is a periodic function with period $|q_i|^{-1}$, we find after an application of Lemma 5.1 (of Appendix A) that

$$\int_0^{u(P)} |S_i(\alpha)|^{d-k} \,\mathrm{d}\alpha \ll u(P)|q_i| \int_0^{|q_i|^{-1}} |S_i(\alpha)|^{d-k} \,\mathrm{d}\alpha \ll q P^{d-k-2} |q_i|^{-(d-k)/2} u(P)(\log P).$$

Thus, we can make use of Hölder's inequality to obtain

$$\int_0^{u(P)} |S_{\pi(k+1)}(\alpha) \dots S_{\pi(d)}(\alpha)| \,\mathrm{d}\alpha \ll q \,|q_{\pi(k+1)} \dots q_{\pi(d)}|^{-\frac{1}{2}} P^{d-k-2} u(P) \,(\log P)$$

and combined with (2.28) we conclude that

$$\int_{\mathcal{J}_\pi \setminus \mathcal{F}} |S_1(\alpha) \dots S_d(\alpha)| \,\mathrm{d}\alpha \ll |Q|^{-\frac{1}{2}} P^{d-2} (\log P)^{-1}. \qquad \square$$

Combining both Lemmas 2.8 and 2.9 yields the following corollary.

Corollary 2.10. Under Assumption 2.1, we may increase the constant $C_d \gg 1$, occurring in the definition of P in (2.7), such that

$$\int_{\mathcal{F}} |S_1(\alpha) \dots S_d(\alpha) K(\alpha)| \,\mathrm{d}\alpha \gg P^{d-2} |Q|^{-\frac{1}{2}}. \tag{2.29}$$

Remark 2.11. We note that the usual proof of the Hardy-Littlewood asymptotic formula shows that the mean-value estimates, used here for the products of $S_1, \dots, S_d$, are in general (up to log factors) best possible. In particular, one cannot improve the exponent $\text{и}(i)$ without using additional information on the underlying quadratic form $Q[m] = q_1 m_1^2 + \ldots + q_d m_d^2$. To obtain better moment-estimates (as in Lemma 2.23) we need to iterate the coupling argument of Birch and Davenport and exploit Assumption 2.1: We shall couple certain coordinates (in the sense of Definition 2.14) and establish pointwise bounds for the products of the corresponding exponential sums (see Lemma 2.21).

2.3 First Coupling via Diophantine Approximation

As we have seen in Corollary 2.10, the integral over $\mathcal{F}$ is relatively large. Now we shall split the region $\mathcal{F}$ into parts, where the quantities y_j and S_j have a specified order of magnitude in terms of the following Diophantine approximation: By Dirichlet's approximation theorem there exist for any $\alpha \in \mathcal{J}$ and for each $j = 1, \ldots, d$ a coprime integral pair $(x_j, y_j) \in \mathbb{Z} \times \mathbb{N}$ such that

$$q_j\alpha = \frac{x_j}{y_j} + \beta_j \qquad \text{and} \qquad 0 < y_j \le 8dP|q_j|^{-\frac{1}{2}}, \tag{2.30}$$

where the approximation error is bounded by

$$|\beta_j| < y_j^{-1}(8dP|q_j|^{-\frac{1}{2}})^{-1}. \tag{2.31}$$

For convenience, we introduce the following notations as well: We shall denote by $\mathbb{Z}^2_{\text{prim}}$ the set of coprime integral pairs (x, y) with $y > 0$ and for any $\alpha \in \mathbb{R}$ we define

$$\mathfrak{D}_j(\alpha) \overset{\text{def}}{=} \{(x_j, y_j) \in \mathbb{Z}^2_{\text{prim}} : (x_j, y_j) \text{ are chosen as in (2.30) satisfying (2.31)}\}. \tag{2.32}$$

One important point here is that none of $x_1, \ldots, x_d$ are zero, since

$$|q_j|\alpha > |q_j|(8dP)^{-1}(q_0)^{-\frac{1}{2}} > |\beta_j|$$

holds in the integration region $\mathcal{F}$ of interest. Indeed, we have $|x_j| \ge y_j(|\alpha q_j| - |\beta_j|) > 0$. To localize the peaks relatively to the size of $|S_1(\alpha)|, \ldots, |S_d(\alpha)|$ and $y_1, \ldots, y_d$, we shall decompose the region $\mathcal{F}$ as follows: For each $j = 1, \ldots, d$ let $T_j = 2^{t(j)}$ and $U_j = 2^{u(j)}$ denote dyadic numbers with integer exponents $t(j), u(j) \in \mathbb{Z}$. Corresponding to these numbers we define the sets

$$\begin{aligned}\mathcal{G}(T_1, \ldots, T_d, U_1, \ldots, U_d) = \big\{\alpha \in \mathcal{F} :\ & \exists (x_j, y_j) \in \mathfrak{D}_j(\alpha) \text{ with} \\ & T_jP/2 < |q_j|^{\frac{1}{2}}|S_j(\alpha)| \le T_jP \text{ and} \\ & U_j/2 < y_j \le U_j \text{ for all } j = 1, \ldots, d\big\}.\end{aligned} \tag{2.33}$$

In this chapter we shall assume, for notational simplicity, that the coordinates are relabeled such that (2.24) holds with the trivial permutation and, as a consequence, we can write

$$T_1 \ll \ldots \ll T_d. \tag{2.34}$$

Additionally, we have only to consider those sets $\mathcal{G}(T_1, \ldots, T_d, U_1, \ldots, U_d)$ which are not empty and then for any $\alpha \in \mathcal{G}(T_1, \ldots, T_d, U_1, \ldots, U_d)$ we see that

$$(u(P)^2q)^{-\text{и}(j)} < T_j < 4d, \tag{2.35}$$

where we used, on the one hand, the trivial upper bound $|S_j(\alpha)| \le 2dP|q_j|^{-1/2}$ and, on the other hand, the lower bound in (2.26). Of course, we have

$$U_j \ge y_j \ge 1,$$

i.e. $u(j) \in \mathbb{N}_0$. In order to associate $S_j(\alpha)$ with the Diophantine approximation in (2.30), we shall apply the following refined variant of Weyl's inequality: If (2.31) holds, then Lemma 5.10 (of the Appendix A) states that

$$|S_j(\alpha)| \ll (y_j)^{-\frac{1}{2}}(\log P)\min(P|q_j|^{-\frac{1}{2}}, P^{-1}|q_j|^{\frac{1}{2}}|\beta_j|^{-1}).$$

This can be rewritten by

$$T_j \ll (y_j)^{-\frac{1}{2}}(\log P)\min\left(1, P^{-2}|q_j||\beta_j|^{-1}\right) \ll U_j^{-\frac{1}{2}}(\log P)\min\left(1, P^{-2}|q_j||\beta_j|^{-1}\right). \quad (2.36)$$

In particular, the last inequality yields both

$$U_j \ll (\log P)^2 T_j^{-2} \quad (2.37)$$

and

$$|q_j|^{-1}|\beta_j| \ll P^{-2}(\log P)T_j^{-1}U_j^{-\frac{1}{2}}. \quad (2.38)$$

Lemma 2.12. Under Assumption 2.1, there exist numbers $T_1, \dots, T_d, U_1, \dots, U_d$ such that

$$\int_{\mathcal{G}(T_1,\dots,T_d,U_1,\dots,U_d)} |S_1(\alpha)\dots S_d(\alpha)K(\alpha)|\,\mathrm{d}\alpha \gg |Q|^{-\frac{1}{2}}P^{d-2}(\log P)^{-2d}. \quad (2.39)$$

Proof: On the one hand, we know from Corollary 2.10 that

$$\int_{\mathcal{F}} |S_1(\alpha)\dots S_d(\alpha)K(\alpha)|\,\mathrm{d}\alpha \gg |Q|^{-\frac{1}{2}}P^{d-2}.$$

On the other hand, (2.35) implies

$$1 \gg t(j) \gg -\log\log P - \log q \gg -\log P,$$

and combined with (2.37) we find

$$0 \le u(j) \ll \log\log P + |t(j)| \ll \log P.$$

Hence, the minimal number of choices for $T_1, \dots, T_d, U_1, \dots, U_d$ to cover all $\mathcal{F}$ is at most $\ll (\log P)^{2d}$. Thus, there is at least one choice of $T_1, \dots, T_d, U_1, \dots, U_d$ satisfying

$$\int_{\mathcal{G}(T_1,\dots,T_d,U_1,\dots,U_d)} |S_1(\alpha)\dots S_d(\alpha)K(\alpha)|\,\mathrm{d}\alpha \gg |Q|^{-\frac{1}{2}}P^{d-2}(\log P)^{-2d}. \qquad \square$$

Throughout this chapter we fix a choice $T_1, \dots, T_d, U_1, \dots, U_d$ as in Lemma 2.12, satisfying (2.39), and introduce the abbreviated notation

$$\mathcal{G} = \mathcal{G}(T_1, \dots, T_d, U_1, \dots, U_d). \quad (2.40)$$

Moreover, for each $j = 1, \dots, d$ let

$$N_j \overset{\text{def}}{=} \#\{(x_j, y_j) \in \mathbb{Z}^2_{\text{prim}} : \exists \alpha \in \mathcal{G} \text{ such that } (x_j, y_j) \in \mathfrak{D}_j(\alpha)\} \quad (2.41)$$

denote the number of distinct integer pairs $(x_j, y_j) \in \mathcal{D}_j(\alpha)$ which arise from all $\alpha \in \mathcal{G}$. The previous Lemma 2.12 leads to the next lower bound on N_j.

Corollary 2.13. For the fixed numbers $T_1, \dots, T_d$, $U_1, \dots, U_d$, satisfying (2.39), we have

$$N_j \gg (\log P)^{-2d}(T_1 \dots T_d)^{-1}(T_j U_j^{\frac{1}{2}}). \quad (2.42)$$

Proof: If $\alpha \in \mathcal{G}(T_1, \dots, T_d, U_1, \dots, U_d)$, then we have

$$|S_1(\alpha)\dots S_d(\alpha)| \ll |Q|^{-\frac{1}{2}}P^d(T_1 \dots T_d)$$

and therefore the bound (2.39) implies

$$|\mathcal{G}| \gg P^{-2}(T_1 \dots T_d)^{-1}(\log P)^{-2d}. \quad (2.43)$$

At the same time, the inequality (2.38) shows that for each integer pair $(x_i, y_j) \in \mathfrak{D}_j(\alpha)$, arising from $\alpha \in \mathcal{G}$, α is located in an interval of length bounded by $\ll P^{-2}T_j^{-1}(U_j)^{-1/2}$. Together with (2.43) we get

$$N_j \gg (\log P)^{-2d}(T_1 \dots T_d)^{-1}(T_j U_j^{\frac{1}{2}})$$

as claimed in (2.42). $\square$

2.3.1 Coupling of the Rational Approximants

In the following we shall establish that at least $d-3$ coordinates are coupled and later on iterate this argument to deduce that all coordinates are coupled. To be precise, we define *coupling* as follows.

Definition 2.14. Let $1 \leq j_1 < \ldots < j_k \leq d$, where $k \in \{1, \ldots, d\}$. We say that the coordinates $j_1, \ldots, j_k$ associated to $q_{j_1}, \ldots, q_{j_k}$ (resp. the exponential sums $S_{j_1}, \ldots, S_{j_k}$) can be coupled if for any $\alpha \in \mathcal{G}$ all pairs $(x_j, y_j) \in \mathfrak{D}_j(\alpha)$ are of the form

$$x_j = x x_j' \qquad \text{and} \qquad y_j = y y_j', \tag{2.44}$$

where $x, y > 0$ are coprime integers and x_j', y_j' divide some $L \in \mathbb{N}$ such that L is independent of $\alpha \in \mathcal{G}$.

The following lemma on the number of Diophantine approximations with bounded denominator will be the key tool for the first coupling argument and later on for its iteration as well.

Lemma 2.15. Let $\eta \in (0,1)$ and $X > 0$. Suppose that θ is a real number such that there exist N distinct (non-trivial) integer pairs (x, y) with

$$0 < |x| < X \tag{2.45}$$

and

$$|\theta x - y| < \eta. \tag{2.46}$$

Then either all integer pairs (x, y) have the same ratio y/x or

$$N < 24\eta X. \tag{2.47}$$

We note that this is Lemma 14 in the original work [BD58b] of Birch and Davenport.

Proof: The inequality (2.47) holds trivially whenever $X \leq 1$, because we have $N = 0$. Thus, we may suppose that $X > 1$. Now we distinguish the following two cases. If $\eta \geq 1/2$, then we have for any fixed x at most $2\eta + 1$ possible choices for y, because

$$|y' - y| \leq |\theta x - y| + |\theta x - y'| < 2\eta,$$

and, since there are at most $2X + 1$ integers in the range $0 < |x| < X$, we conclude that

$$N \leq (2X + 1)(2\eta + 1) \leq 12X\eta.$$

In the second case we have $\eta < 1/2$. By Dirichlet's theorem on Diophantine approximation there exist coprime integers a, b with

$$0 < b < 2X \quad \text{and} \quad |b\theta - a| \leq (2X)^{-1}.$$

Now we compare (a, b) with a tuple (x, y) which is restricted to the conditions (2.45) and (2.46) as well. Obviously, such a tuple (x, y) satisfies

$$|xa - yb| \leq |x(a - b\theta)| + |b(x\theta - y)| < X|(a - b\theta)| + b\eta \leq 2(2X)^{-1}q\eta \leq \frac{1}{2} + b\eta. \tag{2.48}$$

If $b\eta \le 1/2$, then the last line implies that $xa = yb$ and this means that the first alternative of the lemma holds. Otherwise we have $b\eta > 1/2$ and then (2.48) shows that the number of possible residue classes for $x \bmod b$ is at most $2(1/2 + b\eta) + 1 < 6b\eta$. If we write $x = bk + r$ with $k \in \mathbb{N}_0$ and $0 \le r < b$, then $|k| \le |x|/b < X/b$. Consequently, we find that the number of possible choices for x is less than

$$6b\eta(2X/b + 1) < 12\eta X + 12X\eta = 24\eta X,$$

because $b < 2X$. In view of (2.46) together with $\eta < 1/2$ we know that x determines y with at most one possibility. This concludes the proof. □

We are going to apply this lemma with the choice $x = x_d y_j$ and $y = y_d x_j$ and show, in view of the upper bound (2.42) for N_j, that the first alternative in the above dichotomy cannot hold. To do so, we need to adapt Lemma 13 of [BD58c] as follows.

Lemma 2.16. Let $j \neq l$ be fixed. For any $\alpha \in \mathcal{G}$ the integral pairs $(x_j, y_j) \in \mathfrak{D}_j(\alpha)$, $(x_l, y_l) \in \mathfrak{D}_l(\alpha)$ satisfy

$$0 < |x_l| y_j \ll |q_l| U_l U_j u(P) \tag{2.49}$$

and also

$$\left| x_l y_j \frac{q_j}{q_l} - x_j y_l \right| \ll |q_j| (U_l U_j)^{\frac{1}{2}} (T_l T_j)^{-1} P^{-2} (\log P)^2. \tag{2.50}$$

Proof: We recall that $x_i \neq 0$ for any $i = 1, \dots, d$ and that the size of $|x_l|$ is of order

$$|q_l|\alpha y_l \ll |x_l| \ll |q_l|\alpha y_l,$$

because the approximation error $|\beta_j|$ is small compared to $|q_l|\alpha y_l$, see (2.31). Thus, we find

$$0 < |x_l| y_j \ll |q_l|\alpha y_l y_j \ll u(P)|q_l| U_l U_j,$$

where we used (2.33), i.e. $y_i \le U_i$, and $\alpha < u(P)$. To prove (2.50), we note first that

$$2\alpha = \frac{1}{q_j}\frac{x_j}{y_j} + \frac{\beta_j}{q_j} = \frac{1}{q_l}\frac{x_l}{y_l} + \frac{\beta_l}{q_l}.$$

Hence after multiplying by $y_l y_j q_j$ and arranging accordingly we see that

$$x_l y_j \frac{q_j}{q_l} - x_j y_l = y_l y_j q_j ({q_j}^{-1}\beta_j - {q_l}^{-1}\beta_l).$$

Consequently, as in the proof of Lemma 13 in [BD58c], we have

$$\left| x_l y_j \frac{q_j}{q_l} - x_j y_l \right| \ll y_l y_j |q_j| (|q_j^{-1}\beta_j| + |q_l^{-1}\beta_l|).$$

The inequality (2.38), that is $|q_i|^{-1}|\beta_i| \ll (\log P) P^{-2} T_i^{-1} U_i^{-1/2}$, combined with the definition (2.33) of U_j shows that the last term can be bounded by

$$\ll |q_j| U_l U_j \big(T_j^{-1} U_j^{-\frac{1}{2}} + T_l^{-1} U_l^{-\frac{1}{2}} \big) P^{-2} (\log P)$$

and, furthermore, this is bounded by

$$\ll |q_j| (U_l U_j)^{\frac{1}{2}} (T_l T_j)^{-1} P^{-2} (\log P)^2$$

because of the lower bound (2.37). □

The next technical lemma is the key step to conclude that at least $d-3$ variables are coupled:

Lemma 2.17. If $d \geq 8$, then for any $j \in \{4, \dots, d-1\}$ and any $\alpha \in \mathcal{G}$ we have

$$\frac{x_j y_d}{y_j x_d} = \frac{A_j}{B_j}, \tag{2.51}$$

where $(x_j, y_j) \in \mathfrak{D}_j(\alpha)$, $(x_d, y_d) \in \mathfrak{D}_d(\alpha)$, A_j, B_j are coprime integers which are independent of α and $B_j > 0$, $A_j \neq 0$. Under the following additional restrictions the same holds also for

(a) $3 \leq j \leq d-1$ if $\rho \geq 2/3$ and $d \geq 7$,

(b) $2 \leq j \leq d-1$ if $\rho \geq 1$ and $d \geq 6$,

(c) $1 \leq j \leq d-1$ if $\rho \geq 2$ and $d \geq 5$.

The second part of Lemma 2.17 will be important for both smaller dimensions and quadratic forms of signature (r, s), where $\rho(r, s)$ is relatively large, see Corollary 2.19.

Proof: The general strategy here is to apply Lemma 2.15 to the integers $x = x_d y_j$ and $y = y_d x_j$, where $(x_j, y_j) \in \mathfrak{D}_j(\alpha)$ and $(x_d, y_d) \in \mathfrak{D}_d(\alpha)$ for some $\alpha \in \mathcal{G}$. We only carry out the proof for $j \in \{4, \dots, d-1\}$ and afterwards outline the required changes for the remaining cases (a)–(c). By Lemma 2.16 we have

$$|xq_j/q_d - y| < \eta \quad \text{and} \quad 0 < |x| < X$$

with

$$X \ll u(P)|q_d|(U_d U_j)u(P) \quad \text{and} \quad \eta \ll |q_j|(U_d U_j)^{\frac{1}{2}}(T_d T_j)^{-1}P^{-2}u(P).$$

According to Lemma 2.15 either $N \leq 24\eta X$, where N denotes the number of distinct integer pairs (x, y) corresponding to any $\alpha \in \mathcal{G}$, or all pairs (x, y) have the same ratio $y\,/\,x$, independent of α, which gives the desired conclusion. We show that the former case is impossible, provided $C_d \gg 1$ is chosen sufficiently large: In this case, we have the upper bound

$$N < 24\eta X \ll |q_d q_j|(U_d U_j)^{\frac{3}{2}}(T_d T_j)^{-1}P^{-2}u(P)^2 \tag{2.52}$$

and, furthermore, the values of x_d, y_d are determined by the divisors of x and y. Since there are $\ll P^{\delta}$ divisors (for any fixed $\delta > 0$) and $x_d \neq 0$, we find

$$N_d \ll P^{\delta} N.$$

Now we may use the lower bound (2.42) from Corollary 2.13 together with the upper bound (2.52) to get

$$(\log P)^{-2d}(T_1 \dots T_d)^{-1}(T_d U_d^{\frac{1}{2}}) \ll |q_1 q_j|(U_d U_j)^{\frac{3}{2}}(T_d T_j)^{-1}P^{-2+\delta}u(P).$$

By (2.37) this can simplified as

$$T_d^4 T_j^4 \ll q^2 P^{-2+\delta}(\log P)^{2d+5}u(P)^2(T_1 \dots T_d). \tag{2.53}$$

Suppose that $j \in \{4, \dots, d-1\}$ and $d \geq 8$. Since $T_1 \ll \dots \ll T_d$, we can cancel T_j^4 and T_d^4 on both sides and obtain together with the bound (2.35) that

$$1 \ll q^2 P^{-2+\delta}(\log P)^{2d}u(P)^2.$$

Since $2\rho \geq (d+3)/(d-3)$, we can choose $\delta > 0$ such that $2 < (2-\delta)(1+2\rho)$ and note that the right-hand side tends to zero. Thus, after increasing $C_d \gg 1$, which occurs in the definition (2.7) of P, we obtain a contradiction.

In the other cases we should use Wigert's divisor bound, i.e. $d(n) \ll_\varepsilon 2^{(1+\varepsilon)\log(n)/\log\log n}$ if $\varepsilon > 0$, regarding that $|x|, |y| \ll P^3$. If $3 \leq j \leq d-1$ and $d \geq 7$, then we can still cancel $T_d^4 T_j^3$ and in addition use the lower bound $T_j \gg q^{-1/3}u(P)^{-2/3}$, compare (2.35). To get a contradiction again, we need at least $1+2\rho \geq 7/3$. If we have $2 \leq j \leq d-1$ and $d \geq 6$, then we cancel $T_d^4 T_j^2$ and use $T_j \geq q^{-1/2}u(P)^{-1}$ to see that at least $1+2\rho \geq 3$ is required. In the last case, i.e. $1 \leq j \leq d-1$ and $d \geq 5$, we need $1+2\rho \geq 5$, since we can cancel $T_d^4 T_j$ only and have $T_j \gg q^{-1}u(P)^{-2}$. □

The above lemma allows us to obtain the factorization of x_j and y_j as formulated in the Definition 2.14, where we have defined the notation of 'coupling'.

Lemma 2.18. In each one of the cases of Lemma 2.17 it holds that all integral pairs $(x_j, y_j) \in \mathfrak{D}_j(\alpha)$, $(x_d, y_d) \in \mathfrak{D}_d(\alpha)$, corresponding to any $\alpha \in \mathcal{G}$, are coupled (in the sense of Definition 2.14) with corresponding dividend $L \in \mathbb{N}$ satisfying the bound

$$0 < L \ll H^{10d}, \tag{2.54}$$

where H is as in (2.7).

Proof: For any $j \neq d$, restricted as in Lemma 2.17, we can rewrite the equation (2.51) as

$$\frac{x_j}{y_j} = \frac{x_d}{y_d}\frac{B_j}{A_j},$$

where $(x_d, y_d) = (x_j, y_j) = (A_j, B_j) = 1$. As a result, we find that

$$x_j = \operatorname{sgn}(x_j)\frac{|x_d|}{(x_d, A_j)}\frac{B_j}{(y_d, B_j)} \quad \text{and} \quad y_j = \frac{y_d}{(y_d, B_j)}\frac{|A_j|}{(x_d, A_j)}.$$

Now let $I \subset \{1, \ldots, d-1\}$ denote the indices of all coordinates for which the factorization (2.51) holds. We define x and y by

$$x = \frac{|x_d|}{(x_d, \prod_{i\in I} A_i)} \quad \text{and} \quad y = \frac{y_d}{(y_d, \prod_{i\in I} B_i)}.$$

Then x and y are non-zero integers and we may factorize x_j and y_j as follows: Define

$$x_d' := \frac{x_d}{x} = \operatorname{sgn}(x_d)(x_d, \textstyle\prod_{i\in I} A_i) \quad \text{and} \quad y_d' := \frac{y_d}{y} = (y_d, \textstyle\prod_{i\in I} B_i)$$

and also

$$x_j' := \frac{x_j}{x} = \operatorname{sgn}(x_j)\frac{|x_d|B_j}{(x_d, A_j)(y_d, B_j)}\frac{(x_d, \prod_{i\in I} A_i)}{|x_d|} = \operatorname{sgn}(x_j)\frac{B_j}{(y_d, B_j)}\frac{(x_d, \prod_{i\in I} A_i)}{(x_d, A_j)}$$

and

$$y_j' := \frac{y_j}{y} = \frac{y_d|A_j|}{(y_d, B_j)(x_d, A_j)}\frac{(y_d, \prod_{i\in I} B_i)}{y_d} = \frac{|A_j|}{(y_d, A_j)}\frac{(y_d, \prod_{i\in I} B_i)}{(y_d, B_j)}.$$

Note that both are non-zero integral numbers. Furthermore, we see that x_j' and y_j' are divisors of

$$L := \textstyle\prod_{i\in I} |A_i B_i|.$$

It remains to find an upper bound for K. By Lemma 2.16 we have

$$|A_j B_j| \leq |x_d| y_j| x_j| y_d \ll u(P)^2 |q_1||q_j| U_d^2 U_j^2.$$

Thus, we find

$$L \ll u(P)^{2(d-1)} q^{2(d-1)} U_d^{2(d-1)} \prod_{i\in I} U_i^2 \ll u(P)^{2(d-1)} (\log P)^{4(d-1)} q^{2(d-1)} T_d^{-4(d-1)} \prod_{i\in I} T_i^{-4},$$

where we used (2.37), and in view of (2.35) this is bounded by

$$\ll u(P)^{2(d-1)} \log(P)^{4(d-1)} q^{2(d-1)} (u(P)^2 q)^{\frac{4(d-1)}{(d-4)} + 4\sum_{i=1}^{d-1} \text{и}(i)} \ll u(P)^{16d+8\log(d)} q^{8d+4\log(d)}.$$

Using the definition of H together with the fact that $\rho \geq 1/2$, we see that the last inequality chain is at most $\ll H^{10d}$. □

Taking into account the definition of $\rho(r,s)$ for a given dimension d and given signature (r,s) we obtain the following

Corollary 2.19. Under Assumption 2.1 the successive minimas $S_4, \ldots, S_d$ are always coupled. Assuming additionally the following conditions imply that $S_{k+1}, \ldots, S_d$ are coupled

(i) $k = 0$ if $d \in \{5, 6\}$ or $r \geq 4s$,

(ii) $k = 1$ if $5 \leq d \leq 10$ or $r \geq 2s$ and $d \geq 11$,

(iii) $k = 2$ if $5 \leq d \leq 22$ or $r \geq 4s/3$ and $d \geq 23$.

2.4 Iteration of the Coupling Argument

In Lemma 2.17 we showed that $S_{k+1}, \ldots, S_d$ are coupled on $\mathcal{G}$ for some $k \in \{0,1,2,3\}$ depending on the size of $\rho(r,s)$. In other words, we know that for any $i = k+1, \ldots, d$ the integer pairs $(x_i, y_i) \in \mathfrak{D}_i(\alpha)$ corresponding to $q_i\alpha$ are of the form

$$x_i = x x_i' \qquad \text{and} \qquad y_i = y y_i', \tag{2.55}$$

with $x > 0$, $y > 0$, $x_i x_i' > 0$, $x_i' \mid L$ and $y_i' \mid L$, where L is independent of $\alpha \in \mathcal{G}$ and $L \ll H^{10d}$. In this section we shall utilize this observation in combination with Schlickewei's bound on small zeros in order to count the number of distinct pairs (x,y). Since the inequality (2.39) depends multiplicatively on $T_1, \ldots, T_d$, we need following multiplicative bound for small zeros of integral quadratic forms.

Corollary 2.20. For any non-zero integers $f_1, \ldots, f_d$, of which $r \geq 1$ are positive and $s \geq 1$ negative with $r \geq s$, $d = r + s \geq 5$, there exist integers $m_1, \ldots, m_d$, not all zero, such that

$$f_1 m_1^2 + \ldots + f_d m_d^2 = 0 \tag{2.56}$$

and

$$0 < |f_1| m_1^2 + \ldots + |f_d| m_d^2 \ll_d |f_1 \ldots f_d|^{\frac{2\rho+1}{d}}, \tag{2.57}$$

where ρ is defined as in (1.4) and the implicit constant depends on the dimension d only.

Here we should emphasize that the exponent in Theorem 1.6 depends essentially on the previous bound on small zeros of diagonal integral forms.

Proof: This is a special case of Corollary 7.4 (of the Appendix C): If we choose there

$$F = \operatorname{diag}(f_1, \ldots, f_d),$$

$A = FF_+^{-1}$, i.e. $A[m] = \sum_{j=1}^d \operatorname{sgn}(f_i) m_i^2$, and $\Lambda = F_+^{1/2}\mathbb{Z}^d$, then we see that there exists a non-trivial solution $m \in \mathbb{Z}^d \setminus \{0\}$ of

$$f_1 m_1^2 + \ldots + f_d m_d^2 = A[F_+^{1/2} m] = 0,$$

whose size can be bounded by

$$|f_1| m_1^2 + \ldots + |f_d| m_2^d = \|F_+^{1/2} m\|^2 \ll (\operatorname{Tr} A^2)^\rho \, |\det \Lambda|^{\frac{4\rho+2}{d}} \ll |f_1 \ldots f_d|^{\frac{2\rho+1}{d}}. \qquad \square$$

In the following we shall always assume that x_i and y_i are factorized as in (2.55) without mentioning this explicitly. For notational simplicity, we also introduce the set $\mathfrak{C}_k(\alpha)$ of all pairs (x, y) corresponding to some fixed $\alpha \in \mathcal{G}$.

Lemma 2.21. Suppose that $S_{k+1}, \ldots, S_d$ are coupled, where $k \in \{0, 1, 2, 3\}$, and that the quadratic form

$$Q_k[m] \overset{\text{def}}{=} q_{k+1} m_{k+1}^2 + \ldots + q_d m_d^2 \tag{2.58}$$

is indefinite of signature (r', s') with $d - k \geq 5$. Then, under Assumption 2.1, the integer pairs $(x, y) \in \mathfrak{C}_k(\alpha)$, corresponding to the factorization (2.55) and any $\alpha \in \mathcal{G}$, satisfy

$$x^{2\rho_k} y^{2\rho_k+2} \ll q^{2\rho_k+1} P^{-2} (\log P) u(P)^{2\rho_k} (U_{k+1} \ldots U_d)^{\frac{4\rho_k+2}{d-k}} \big(\max_{i=k+1,\ldots,d} T_i^{-1} U_i^{-\frac{1}{2}} \big), \tag{2.59}$$

where ρ_k denotes the exponent corresponding to the signature (r', s') of Q_k.

This lemma will be used subsequently to establish improved mean-value estimates and, as a consequence, improved lower bounds for the size of the parameters $T_1, \ldots, T_k$.

Proof: Due to the Diophantine approximation obtained in (2.30), we have for any fixed $\alpha \in \mathcal{G}$ and any integers $m_{k+1}, \ldots, m_d \in \mathbb{Z}$

$$\alpha \sum_{i=k+1}^d q_i m_i^2 = \frac{x}{y} \sum_{i=k+1}^d \frac{x_i'}{y_i'} m_i^2 + \sum_{i=k+1}^d \rho_i m_i^2.$$

Changing variables to $m_i = y_i' n_i$ for $i = k+1, \ldots, d$ yields

$$\alpha \sum_{i=k+1}^d q_i m_i^2 = \frac{x}{y} \sum_{i=k+1}^d x_i' y_i' n_i^2 + \sum_{i=k+1}^d \rho_i y_i'^2 n_i^2, \tag{2.60}$$

and we observe that the first term on the right-hand side, neglecting the factor x/y, is an integral quadratic form whose signature (r', s') coincides with that of Q_k, since the signs of $x_{k+1}' y_{k+1}', \ldots, x_d' y_d'$ are exactly equal to those of $x_{k+1}/y_{k+1}, \ldots, x_d/y_d$ and these have the same signs as $q_{k+1}, \ldots, q_d$. Hence, it follows from Corollary 2.20 that there exist integers $n_{k+1}, \ldots, n_d$, not all zero, such that

$$x_{k+1}' y_{k+1}' n_{k+1}^2 + \ldots + x_d' y_d' n_d^2 = 0$$

and

$$|x'_{k+1}y'_{k+1}|n_{k+1}^2+\ldots+|x'_d y'_d|n_d^2 \ll_d |x'_{k+1}y'_{k+1}\ldots x'_d y'_d|^{(2\rho_k+1)/(d-k)}. \tag{2.61}$$

For the corresponding $m_{k+1},\ldots,m_d$ the first part of the right-hand side in (2.60) vanishes. Thus, we find

$$|q_{k+1}m_{k+1}^2+\ldots+q_d m_d^2| \ll \alpha^{-1}(|\rho_{k+1}|y'^2_{k+1}n_{k+1}^2+\ldots+|\rho_d|y'^2_d n_d^2)$$

and from $\alpha|q_i|\ll|x_i|y_i^{-1}$, (2.61) and $y_i\le U_i$ we deduce that

$$\begin{aligned}|q_{k+1}|m_{k+1}^2+\ldots+|q_d|m_d^2 &\ll \alpha^{-1}xy^{-1}|x'_{k+1}y'_{k+1}\ldots x'_d y'_d|^{(2\rho_k+1)/(d-k)}\\ &\ll \alpha^{-1}x^{-2\rho_k}y^{-2\rho_k-2}|x_{k+1}y_{k+1}\ldots x_d y_d|^{(2\rho_k+1)/(d-k)}\\ &\ll \alpha^{2\rho_k}x^{-2\rho_k}y^{-2\rho_k-2}|q_{k+1}y_{k+1}^2\ldots q_d y_d^2|^{(2\rho_k+1)/(d-k)}\\ &\ll \alpha^{2\rho_k}x^{-2\rho_k}y^{-2\rho_k-2}q^{2\rho_k+1}(U_{k+1}\ldots U_d)^{(4\rho_k+2)/(d-k)}\end{aligned} \tag{2.62}$$

Now we shall apply the Assumption 2.1, made at the beginning: Since Q_k is a restriction of Q, i.e. $Q_k[m]=Q[(0,\ldots,0,m_{k+1},\ldots,m_d)]$, we have either

$$4d^3P^2 < |q_{k+1}|m_{k+1}^2+\ldots+|q_d|m_d^2 \tag{2.63}$$

or

$$1\le|q_{k+1}m_{k+1}^2+\ldots+q_d m_d^2|\le\alpha^{-1}(|\rho_{k+1}|y'^2_{k+1}n_{k+1}^2+\ldots+|\rho_d|y'^2_d n_d^2). \tag{2.64}$$

In the first case we may combine (2.63) together with (2.62) to get

$$P^2\ll\alpha^{2\rho_k}x^{-2\rho_k}y^{-2\rho_k-2}q^{2\rho_k+1}(U_{k+1}\ldots U_d)^{(4\rho_k+2)/(d-k)}$$

and in view of (2.37), that is $T_i^{-1}U_i^{-1/2}\gg\log P$, together with $\alpha<u(P)$ we conclude already that inequality (2.59) holds.

In the second case, (2.64) holds and here we use (2.38), that is

$$|\rho_i|\ll|q_i|P^{-2}(\log P)T_i^{-1}U_i^{-1/2},$$

to obtain

$$1\ll\alpha^{-1}\textstyle\sum_{i=k+1}^d|\rho_i|y'^2_i n_i^2\ll\alpha^{-1}P^{-2}(\log P)\big(\max\limits_{i=k+1,\ldots,d}T_i^{-1}U_i^{-1/2}\big)\big(\sum_{i=k+1}^d|q_i|m_i^2\big),$$

which implies together with (2.62)

$$1\ll\alpha^{2\rho_k-1}x^{-2\rho_k}y^{-2\rho_k-2}q^{2\rho_k+1}P^{-2}(\log P)\big(\max_{i=k+1,\ldots,d}T_i^{-1}U_i^{-1/2}\big)(U_{k+1}\ldots U_d)^{(4\rho_k+2)/(d-k)}.$$

Finally, taking into account that $2\rho_k\ge1$ and $\alpha<u(P)$, we conclude that (2.59) holds. $\square$

All pairs $(x,y)\in\mathfrak{C}_k\overset{\text{def}}{=}\{(x,y)\in\mathbb{Z}^2_{\text{prim}}:(x,y)\in\mathfrak{C}_k(\alpha)\text{ for some }\alpha\in\mathcal{G}\}$ lie in a bounded set determined by condition (2.59). Hence, we can bound the number $\mathfrak{N}_k$ of all these pairs as follows.

Corollary 2.22. Let $\mathfrak{N}_k:=\#\mathfrak{C}_k$. Then, in the situation of Lemma 2.21, we have

$$\mathfrak{N}_k\ll q^{1+\frac{1}{2\rho_k}}P^{-\frac{1}{\rho_k}}(\log P)^{\frac{1}{2\rho_k}}u(P)(U_{k+1}\ldots U_d)^{\frac{4\rho_k+2}{2\rho_k(d-k)}}\big(\max_{i=k+1,\ldots,d}T_i^{-1}U_i^{-\frac12}\big)^{\frac{1}{2\rho_k}}. \tag{2.65}$$

Proof: First note that the expression on the right-hand side of (2.59) must be $\gg 1$, since $\mathcal{G}$ is not empty. Thus, we can apply Dirichlet's hyperbola method to see that the number N of distinct solutions (x,y) of $x^{2\rho_k}y^{2\rho_k+2} \ll Z$ is at most $\ll Z^{1/(2\rho_k)}$. Indeed, we have

$$N = \sum_{x^{2\rho_k}y^{2\rho_k+2} \ll Z} 1 \ll \sum_{y \ll Z^{1/(2\rho_k+2)}} \sum_{x \ll Z^{1/(2\rho_k)}/y^{-1-1/\rho_k}} 1 \ll \sum_{y \ll Z^{1/(2\rho_k+2)}} \frac{Z^{1/(2\rho_k)}}{y^{1+1/\rho_k}} \ll Z^{1/(2\rho_k)}.$$

This already concludes the proof. □

We are in position to establish improved mean-value estimates (conditionally under Assumption 2.1) by controlling the sum over all $(x,y) \in \mathfrak{C}_k$ with the help of the previous corollary.

Lemma 2.23. Suppose that $d \geq 5+k$ and $k \in \{0,1,2,3\}$. Then, in the situation of Lemma 2.21, for any $\delta > 0$ we have

$$\int_{\mathcal{G}} |S_{k+1}(\alpha)\dots S_d(\alpha)K(\alpha)|\,\mathrm{d}\alpha \ll P^{\delta}\frac{P^{d-k-2}}{|q_{k+1}\dots q_d|^{\frac{1}{2}}}\frac{q^{1+\frac{1}{2\rho_k}}}{P^{\frac{1}{\rho_k}}}. \tag{2.66}$$

Proof: We shall decompose the integration domain $\mathcal{G}$ according to the covering induced by the factorization from (2.55), which holds since $S_{k+1},\dots,S_d$ are coupled: For fixed $(x,y) \in \mathfrak{C}_k$ we define

$$\mathfrak{H}_i(x,y) \overset{\text{def}}{=} \{(x_i',y_i') \in \mathbb{Z}^2_{\text{prim}} : x_i = xx_i' \text{ and } y_i = yy_i' \text{ as in (2.55)} \\ \text{with } (x_i,y_i) \in \mathfrak{D}_i(\alpha) \text{ for some } \alpha \in \mathcal{G}\}$$

and

$$\mathcal{J}_i(x_i,y_i) \overset{\text{def}}{=} \{\alpha \in \mathcal{G} : |\alpha q_i y_i - x_i| < |q_i|^{1/2}(8dP)^{-1}\}$$

in order to decompose the integral on the left-hand side of (2.66) as

$$\ll \sum_{(x,y)\in\mathfrak{C}_k} \sum_{(x_{k+1}',y_{k+1}')\in\mathfrak{H}_{k+1}(x,y)} \cdots \sum_{(x_d',y_d')\in\mathfrak{H}_d(x,y)} I(x_{k+1},y_{k+1},\dots,x_d,y_d),$$

where

$$I(x_{k+1},y_{k+1},\dots,x_d,y_d) \overset{\text{def}}{=} \int_{\bigcap_{i=k+1}^d \mathcal{J}_i(x_i,y_i)} |S_{k+1}(\alpha)\dots S_d(\alpha)K(\alpha)|\,\mathrm{d}\alpha.$$

Using the bound $|S_i(\alpha)| \leq |q_i|^{-1/2}PT_i$, compare the definition (2.33) of the set $\mathcal{G}$, yields

$$I(x_{k+1},y_{k+1},\dots,x_d,y_d) \leq \frac{P^{d-k}(\log P)}{|q_{k+1}\dots q_d|^{1/2}}(T_{k+1}\dots T_d)\,\mathrm{mes}\big(\textstyle\bigcap_{i=k+1}^d \mathcal{J}_i(x_i,y_i)\big)$$

and, since the measure of the set $\mathcal{J}_i(x_i,y_i)$ is at most $\ll P^{-2}(\log P)T_i^{-1}U_i^{-1/2}$, Hölder's inequality implies

$$I(x_{k+1},y_{k+1},\dots,x_d,y_d) \ll \frac{P^{d-k-2}(\log P)}{|q_{k+1}\dots q_d|^{1/2}}(T_{k+1}\dots T_d)\textstyle\prod_{i=k+1}^d (T_i^{-1}U_i^{-1/2})^{\frac{1}{d-k}}.$$

Returning to the initial decomposition of the integral, we note that $\#\mathfrak{H}_i(x,y) \ll P^{\delta}$, because x_i', y_i' are divisors of $L \ll H^{10d}$ and there are at most $\ll P^{\delta}$ divisors. Thus, taking all together we find

$$\int_{\mathcal{G}} |S_{k+1}(\alpha)\dots S_d(\alpha)K(\alpha)|\,\mathrm{d}\alpha \ll \frac{P^{d-k-2+\delta}(\log P)}{|q_{k+1}\dots q_d|^{1/2}}\Big(\textstyle\prod_{i=k+1}^d T_i(T_i^{-1}U_i^{-1/2})^{\frac{1}{d-k}}\Big)\mathfrak{N}_k.$$

Next we insert the bound (2.65), established in Corollary 2.22, and get that the last equation is bounded by

$$\ll \frac{P^{d-k-2+\delta}}{|q_{k+1}\dots q_d|^{1/2}} \frac{q^{1+\frac{1}{2\rho_k}}}{P^{\frac{1}{2\rho_k}}} (\log P)^2 u(P) \big(\max_{i=k+1,\dots,d} T_i^{-1} U_i^{-\frac{1}{2}} \big)^{\frac{1}{2\rho_k}} \prod_{i=k+1}^{d} (T_i U_i^{1/2})^{1-\frac{1}{d-k}}$$

where we used that $\frac{4\rho_k+2}{2\rho_k(d-k)} \leq \frac{1}{2}$ holds provided that $d \geq 5+k$. The claim follows now from the fact that

$$\frac{1}{2\rho_k} + \frac{1}{d-k} - 1 \leq -\frac{6}{d-k+3} + \frac{1}{d-k} \leq 0$$

and (2.37), i.e. $T_i U_i^{1/2} \ll \log P$. □

Corollary 2.24. In the situation of Lemma 2.23 we have

$$T_1 \dots T_k \gg P^{-\delta} P^{\frac{1}{\rho_k}} q^{-1-\frac{1}{2\rho_k}}. \tag{2.67}$$

Proof: We recall the lower bound

$$\int_{\mathcal{G}} |S_1(\alpha) \dots S_d(\alpha) K(\alpha)| \, \mathrm{d}\alpha \gg |Q|^{-\frac{1}{2}} P^{d-2} (\log P)^{-2d}$$

obtained in Lemma 2.12 under Assumption 2.1. Combining this inequality together with

$$|S_1(\alpha) \dots S_k(\alpha)| \leq |q_1 \dots q_k|^{-1/2} P^k (T_1 \dots T_k)$$

and the moment estimate derived in Lemma (2.23) shows that

$$|Q|^{-\frac{1}{2}} P^{d-2} (\log P)^{-2d} \ll P^{\delta/2} |Q|^{-\frac{1}{2}} P^{d-2} q^{1+\frac{1}{2\rho_k}} P^{-\frac{1}{\rho_k}} (T_1 \dots T_k).$$ □

2.4.1 Reducing Variables and Corresponding Signatures

Now we shall establish the coupling of the remaining coordinates stepwise beginning with S_k. The basic strategy here is the same as in proofs of Lemma 2.17 and Lemma 2.18, but we additionally make use of the bound (2.67). Compared to the earlier arguments, we need also to consider the ratio between ρ and ρ_k with care, since simple bounds on ρ_k (resp. on ρ) are not sufficient to deduce a contradiction. Thus, we are faced with the problem to specify the possible values of ρ_k depending on the signature (r, s) of Q, which we have moved to Appendix C.

Lemma 2.25. Let $d \geq 8$ and assume that the signature of Q is not of the form $(d-1, 1)$, $(d-2, 2)$ or $(d-3, 3)$. Then, under Assumption 2.1, $S_3, \dots, S_d$ can be coupled on $\mathcal{G}$.

Proof: According to Corollary 2.19 we may assume that $S_4, \dots, S_d$ are coupled. Applying Lemma 2.15 to the integers $x = x_d y_3$ and $y = y_d x_3$ with $(x_d, y_d) \in \mathfrak{D}_d(\alpha)$ and $(x_3, y_3) \in \mathfrak{D}_3(\alpha)$ and assuming that the first alternative of Lemma 2.15 holds, yields inequality (2.53), that is

$$T_d^4 T_3^4 \ll q^2 P^{-2+\rho} (\log P)^{2d+5} u(P)^2 (T_1 \dots T_d).$$

In view of $T_1 \ll \dots \ll T_d$ we can cancel $T_d^4 T_3^3$ and use Corollary 2.24 with $k = 3$ to obtain

$$P^{\frac{1}{3\rho_3} - \frac{\delta}{3}} q^{-\frac{1}{3} - \frac{1}{6\rho_3}} \ll T_3 \ll q^2 P^{-2+\delta} (\log P)^{2d+5}, \tag{2.68}$$

where we used $T_i \ll 1$ as well. Rearranging inequality (2.68) and using that $q \ll P^{\frac{2}{1+2\rho}}$ yields

$$1 \ll P^{2\delta}(\log P)^{2d+5} u(P)^2 P^{\mathfrak{p}_3(d)},$$

where

$$\mathfrak{p}_3(d) \overset{\text{def}}{=} \frac{2}{(1+2\rho)}\left(\frac{7}{3}+\frac{1}{6\rho_3}\right) - \left(2+\frac{1}{3\rho_3}\right).$$

Considering all (bad) cases in the table of Section 7.3 of Appendix C shows that this inequality cannot hold if we increase $C_d > 1$ and choose $\delta > 0$ small enough. To conclude that $S_3, \ldots, S_d$ are coupled, we have to repeat the proof of Lemma 2.18 as well and note that the factorization (2.55) changes if more coordinates are coupled. □

If $d \in \{5, 6\}$ or Q has signature $(d-1, 1)$, then Corollary 2.19 implies that all exponential sums $S_1, \ldots, S_d$ are coupled. Moreover, we also know that $S_3, \ldots S_d$ are coupled if $5 \leq d \leq 22$ or if Q has signature $(d-1, 1)$, $(d-2, 2)$ or $(d-3, 3)$, as can be checked easily. Hence, in view of the previous lemma, we conclude that $S_3, \ldots, S_d$ are always coupled.

Lemma 2.26. Let $d \geq 7$ and assume that the signature of Q is not of the form $(d-1, 1)$ or $(d-2, 2)$. Then, under Assumption 2.1, $S_2, \ldots, S_d$ can be coupled on $\mathcal{G}$.

Proof: In this case, an analogous argument as above yields the inequality

$$P^{\frac{1}{\rho_2}-\rho} q^{-1-\frac{1}{2\rho_2}} \ll T_2^2 \ll q^2 P^{-2+\delta} (\log P)^{2d+5},$$

where we canceled $T_d^4 T_2^2$ and applied Corollary 2.24 with $k = 2$, and after rearranging

$$1 \ll P^{2\delta}(\log P)^{2d+5} u(P)^2 P^{\mathfrak{p}_2(d)},$$

where

$$\mathfrak{p}_2(d) \overset{\text{def}}{=} \frac{2}{(1+2\rho)}\left(3+\frac{1}{2\rho_2}\right) - \left(2+\frac{1}{\rho_2}\right).$$

Considering all (bad) cases in the table of Section 7.3 of Appendix C again shows that this inequality cannot hold if we increase $C_d > 1$ and choose $\delta > 0$ small enough. Finally, we repeat the proof of Lemma 2.18 as well to show the claim. □

By Corollary 2.19 we know that $S_2, \ldots, S_d$ are coupled if $5 \leq d \leq 10$. Hence we may assume that $d \geq 11$ and then $S_2, \ldots, S_d$ are coupled as well if the signature of Q is of the form $(d-1, 1)$ or $(d-2, 2)$. Thus, we have proven that $S_2, \ldots, S_d$ are coupled, regardless of the signature (r, s).

Lemma 2.27. Under Assumption 2.1 all exponential sums $S_1, \ldots, S_d$ can be coupled on $\mathcal{G}$.

Proof: By the previous discussion, we know that $S_2, \ldots, S_d$ are coupled. Moreover, we can assume that $d \geq 7$ and that the signature of Q is not of the form $(d-1, 1)$, since otherwise all coordinates are coupled, see Corollary 2.19. Similar to the previous case, we get

$$P^{\frac{3}{\rho_1}-3\delta} q^{-3-\frac{3}{2\rho_1}} \ll T_1^3 \ll q^2 P^{-2+\delta} (\log P)^{2d+5},$$

where we canceled $T_d^4 T_1^1$ and applied Corollary 2.24 with $k = 1$, and this can be rewritten as

$$1 \ll P^{4\delta}(\log P)^{2d+5} u(P)^2 P^{\mathfrak{p}_1(d)},$$

where
$$\mathfrak{p}_1(d) := \frac{2}{(1+2\rho)}\left(5+\frac{3}{2\rho_1}\right) - \left(2+\frac{3}{\rho_1}\right).$$

For every case, other than $\operatorname{sgn}(Q) = (\frac{d+3}{2}, \frac{d-3}{2})$, we have seen (in Section 7.3 of Appendix C) that $\mathfrak{p}_1(d) < 0$, thus yielding a contradiction. For $\operatorname{sgn}(Q) = (\frac{d+3}{2}, \frac{d-3}{2})$ and $2\rho_1 = \frac{d+1}{d-5}$ we obtain also $\mathfrak{p}_1(d) = -\frac{6(d-5)}{d(d+1)} < 0$. However, if $2\rho_1 = \frac{d+3}{d-5}$, then $\mathfrak{p}_1(d) = 0$. In this case the $(d-1)$-dimensional restriction of the quadratic form is of signature $(\frac{d+1}{2}+1, \frac{d-1}{2}-2)$ and hence we may remove one of the coordinates corresponding to $T_2, \ldots, T_d$ to obtain a $(d-2)$-dimensional restriction of our quadratic form of signature $(\frac{d+1}{2}, \frac{d-1}{2}-2)$. Similarly to Corollary 2.19 we can deduce that

$$T_1 T_l \gg P^{\frac{1}{\rho_2}-\rho} q^{-1-\frac{1}{2\rho_2}},$$

for some $2 \le l \le d$. Arguing again as above, we obtain

$$P^{\frac{3}{\rho_2}-3\delta} q^{-3-\frac{3}{2\rho_2}} \ll T_1^3 \ll q^2 P^{-2+\delta} (\log P)^{2d+5} u(P)^2,$$

which implies
$$1 \ll P^{-2+4\delta} (\log P)^{2d+5} u(P)^2 P^{\mathfrak{p}_1(d)},$$

where
$$\mathfrak{p}_1(d) := \frac{2}{1+2\rho}\left(5+\frac{3}{2\rho_2}\right) - \left(2+\frac{3}{\rho_2}\right) = -\frac{6(d-5)}{d(d+1)} < 0,$$

which yields a contradiction. Since the previous considerations exhaust all cases, we may repeat the proof of Lemma 2.18 again and conclude that all coordinates are coupled. □

Remark 2.28. In the case $\mathfrak{p}_1(d) = 0$ one can use Wigert's divisor bound instead of the above reduction argument. Since the growth rate in (1.8) is limited by the divisor bound, we wanted to emphasize that the last step can be done without using it.

2.5 Proof of Theorem 1.6: Counting Approximants

Finally, we are going to deduce a contradiction in form of an inconsistent inequality consisting of the lower bound for N_j, established in Corollary 2.13, and the upper bound from Corollary 2.22 for the number of distinct pairs (x, y).

Proof of Theorem 1.6: As we have seen in the previous section, all coordinates can be coupled (under the Assumption 2.1) and therefore we can apply Corollary 2.22 with $k = 0$ - in particular, we have $Q_k = Q$ - to find an upper bound for the number N_j of all (x_j, y_j): Since $x_1', y_1', \ldots, x_d', y_d'$ are determined as divisors of an α-independent number $L \ll H^{10d}$, see Lemma 2.18, Wigert's divisor bound implies that

$$N_j^{2\rho} \ll H^{\frac{20d(d-1)}{\log\log H}} \mathfrak{N}_0^{2\rho} \ll H^{\frac{20d(d-1)}{\log\log H}} P^{-2} q^{2\rho+1} u(P)^{2\rho} (U_1 \ldots U_d)^{\rho} \big(\max_{i=1,\ldots,d} T_i^{-1} U_i^{-1/2}\big),$$

where we also used that $(4\rho+2)/d \le \rho$, which can be checked by considering the lower bound (2.19). Next let $j \ne l$, where l is a suffix for which the maximum of $T_i^{-1} U_i^{-1/2}$ is attained. Combined with the lower bound on N_j, obtained in Corollary 2.13, we find

$$(\log P)^{-4d\rho-1} \big(\textstyle\prod_{i=1}^d T_i\big)^{-2\rho} (T_j U_j^{\frac{1}{2}})^{2\rho} \ll H^{\frac{20d(d-1)}{\log\log H}} P^{-2} q^{2\rho+1} u(P)^{2\rho} \big(\textstyle\prod_{i=1}^d U_i\big)^{\rho} (T_l U_l^{\frac{1}{2}})^{-1} \quad (2.69)$$

and this inequality can be simplified by using the notation

$$V_i := U_i^{-\frac{1}{2}} T_i^{-1} (\log P).$$

Indeed, since $V_i \gg 1$ by (2.37), we can rewrite (2.69) as

$$\begin{aligned} 1 \ll (V_1 \dots V_d)^{2\rho} V_j^{-2\rho} V_l^{-1} &\ll H^{-\frac{20d}{\log\log H}} u(P)^{2\rho} (\log P)^{6d\rho+1} \\ &\ll H^{-\frac{1}{\log\log H}} \leq \exp\big(- \tfrac{\log C_d}{\log\log C_d}\big), \end{aligned} \tag{2.70}$$

where $2\rho \geq 1$ was used. If $C_d \gg 1$ is chosen sufficiently large, we get a contradiction. Thus, our initial Assumption 2.1 is false. □

CHAPTER 3

Distribution of Values of Quadratic Forms

In this chapter we shall establish effective estimates on the lattice remainder which, in particular, will be the basis for Chapter 4. This corresponds to the sections 3, 4 and 6 of [GM13] with several corrections and improvements on the parameter dependencies. In order to state the explicit bounds on the lattice point remainder we need to introduce the following notations. Let $\beta > \frac{2}{d}$ such that $0 < \frac{1}{2} - \beta < \frac{1}{2} - \frac{2}{d}$ for $d > 4$. For a lattice $\Lambda \subset \mathbb{R}^{2d}$ with $\dim \Lambda = 2\,d$ and $1 \le l \le d$ we define its α_l-characteristic by

$$\alpha_l(\Lambda) \overset{\text{def}}{=} \sup\Big\{ |\det(\Lambda')|^{-1} : \Lambda' \subset \Lambda,\ \ l\text{-dimensional sublattice of } \Lambda \Big\}, \tag{3.1}$$

which will be the auxiliary tool to transfer the counting problem into the language of dynamical systems. Here $\Lambda' = B\,\mathbb{Z}^d$ is determined by a $(2d) \times l$-matrix B and $\det(\Lambda') = \det(B^T\, B)^{1/2}$ is the volume of a fundamental domain. Introduce

$$\gamma_{[T_-,T],\beta}(r) \overset{\text{def}}{=} \sup\,\Big\{ \big(r^{-d}\alpha_d(\Lambda_t)\big)^{1/2-\beta} :\ T_- \le |t| \le T \Big\}, \tag{3.2}$$

where $\Lambda_t = d_r u_t \Lambda_Q$ denotes a $2d$-dimensional lattice obtained by the diagonal action of $d_r, u_t \in \mathrm{SL}(2,\mathbb{R})$ on $(\mathbb{R}^2)^d$, which will be introduced in (3.67) in full detail, and Λ_Q denotes a fixed $2d$-dimensional lattice depending on Q, which will be introduced later in (3.70) as well. Moreover, we write

$$E_{a,b} = \{x \in \mathbb{R}^d \,:\, a < Q[x] < b\},$$

vol B denotes the Lebesgue measure of a measurable set $B \subset \mathbb{R}^d$ and $\mathrm{vol}_{\mathbb{Z}}\, B := \#(B \cap \mathbb{Z}^d)$ denotes the number of integer points in B. In addition, let $\upsilon(x)$ denote a smooth weight function such that $\zeta(x) := \upsilon(x)\exp\{Q_+[x]\}$ satisfies

$$\sup_{x\in\mathbb{R}^d} \big(|\zeta(x)| + |\widehat{\zeta}(x)|\big)(1 + \|x\|)^{d+1} < \infty. \tag{3.3}$$

Here we should remark that the condition (3.3) will be needed in order to rewrite the remainder problem in terms of special theta series.

Theorem 3.1. Let Q be a non-degenerate quadratic form in $d \ge 5$ variables with $q_0 \ge 1$. Choose $\beta = \frac{2}{d} + \frac{\delta}{d}$ for some arbitrary small $\delta \in (0, \frac{1}{10})$ and let $\varsigma := d(\frac{1}{2} - \beta) = \frac{d}{2} - 2 - \delta$. Write $(b-a)_q := b - a$ if $b - a \le q$ and $(b-a)_q := q^{\beta d - 1/2}$ if $b - a > q$, and $(b-a)^* := (b-a)$ if $b - a \le 1$ and $(b-a)^* := 1$ if $b - a > 1$. Then for any $r \ge q^{1/2}$, $b > a$ and $0 < w < (b-a)/4$ we have

$$\Big|\sum_{m\in E_{a,b}\cap\mathbb{Z}^d} \upsilon(\tfrac{m}{r}) - \int_{E_{a,b}} \upsilon(\tfrac{x}{r})\,\mathrm{d}x\Big| \ll_{\beta,d} \big\{ w\,\|\upsilon\|_Q + \|\widehat{\zeta}\|_1 C_Q\, \rho_{Q,b-a,w}(r)\big\} r^{d-2} \\ + |Q|^{-1/2} r^{d/2} \|\hat{\zeta}\|_{*,r} \log\big(1 + \tfrac{|b-a|}{q_0^{1/2} r}\big), \tag{3.4}$$

where $\|v\|_Q$ is defined in Lemma 3.17, $C_Q := q\,|\det Q|^{-1/4-\beta/2}$, $c_Q := |\det Q|^{1/4-\beta/2}$ and

$$\rho_{Q,b-a,w}(r) \overset{\text{def}}{=} \inf\big\{(b-a)_q\,(c_Q T_-^{\varsigma} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r)\,(1+\log((b-a)^* T_+)) \\ + c_Q^{-1}\,(T_+w)^{-1/2}\mathrm{e}^{-(T_+w)^{1/2}} : T_- \in [q_0^{-1/2}r^{-1},1] \text{ and } T_+ \geq 1\big\}.$$

Furthermore

$$\|\widehat{\zeta}\|_{*,r} \overset{\text{def}}{=} q^{d/4}\Big(\Big(\frac{q}{q_0}\Big)^{d/2}\|\widehat{\zeta}\|_1 + \int_{\|v\|_\infty > r/2} \frac{|\widehat{\zeta}(v)|}{(q^{1/2}r^{-1} + \|vr^{-1}\|_{\mathbb{Z}})^{d/2}}\,\mathrm{d}v\Big) \tag{3.5}$$

and here $\|v\|_{\mathbb{Z}} := \min_{m\in\mathbb{Z}^d}\|v-m\|_\infty$.

Remark 3.2.

a) Note that Theorem 3.1 extends to affine quadratic forms $Q[x+\xi]$ uniformly in $|\xi|_\infty \leq 1$.

b) Depending on the application, the lattice remainder (3.4) will be optimized in the parameters w, ε and T_+ differently: For thin shells the error should also scale with the length $b-a$. This forces T_+ to be large and requires 'strong' Diophantine assumptions. In the case of wide shells it is possible to choose w relatively large.

c) If Q is irrational, then Corollary 4.6 implies that $\rho_{Q,b-a,w}(r) \to 0$ for $r \to \infty$, provided that w and $(b-a)$ are fixed. The first factor in the definition of $\rho_{Q,b-a,w}$ corresponds to small values of t on the Fourier side and the last factor to the decay rate of the w-smoothing of the interval $[a,b]$.

d) The reader may note that the Oppenheim conjecture is equivalent to the statement that if $d \geq 3$ and Q is irrational, then $\mathrm{vol}_{\mathbb{Z}}\,E_{a,b} = \infty$ whenever $a < b$.)

3.1 Effective Estimates

In the following we specialize the choice of the smooth weight function υ in Theorem 3.1 to obtain quantitative bounds for the difference between the volume and the lattice point volume in $E_{a,b}$. Later - in Chapter 4 - the explicit Diophantine dependence will be elaborated as well leading to explicit bounds for a special class of quadratic forms, which will be called of *Diophantine type* (κ, A).

3.1.1 Ellipsoids $E_{0,b}$

Here Q is positive definite and we shall assume that b tends to infinity. Let $r = 2b^{1/2}$ in Theorem 3.1. Then the ellipsoid $E_{0,b} = \{x \in \mathbb{R}^d : Q[x] \leq b\}$ is contained in $r\Omega = Q_+^{-1/2}[-r,r]^d$. Choosing in Theorem 3.1 a smoothing of I_Ω, say υ_ε of width $\varepsilon = \frac{1}{16}$, which equals 1 on $E_{0,b}$, and the smoothing parameter w in terms of T_+, such that the right-hand side in (3.4) is minimal, will lead to

Corollary 3.3. Let Q denote a non-degenerate d-dimensional positive definite form with $d \geq 5$ and $q_0 \geq 1$. For any $r \geq q^{1/2}$ and $r = \sqrt{2b}$ we have with $H_r := E_{0,b}$

$$|\mathrm{vol}_{\mathbb{Z}}\,H_r - \mathrm{vol}\,H_r| \ll_{\beta,d} |Q|^{-1/2}r^{d-2}\big(\rho_Q(r) + q^{d/4}r^{-d/2+2}(q/q_0)^{d/2}\log(r)\big), \tag{3.6}$$

where

$$\rho_Q(r) \overset{\text{def}}{=} \inf\Big\{a_Q\big(q^{\beta d-\frac{1}{2}}(c_Q T_-^{\varsigma} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r)\log(T_+ + 1)\big) + \tfrac{\log(1+qT_+)^2}{T_+}\Big\}$$

and the infimum is taken over $T_- \in [q_0^{-1/2}r^{-1},1]$ and $T_+ \geq 1$, where $a_Q = q\,|\det Q|^{1/4-\beta/2}$, $c_Q = |\det Q|^{1/4-\beta/2}$. Furthermore, $\lim_{r\to\infty}\rho(r) = 0$ as r tends to infinity, provided that Q is irrational.

Compared to the quantitative results in [BG97] and [BG99], this bound holds already for $d \geq 5$. Moreover, Corollary 3.3 refines the estimates obtained in [Göt04].

3.1.2 Hyperboloid Shells $E_{a,b}$

If Q is indefinite, we distinguish, depending on $b-a$, between 'small' and 'wide' shells $E_{a,b}$. Moreover, we restrict ourselves to a special class of rescaled *admissible* parallelepipeds $r\Omega$ for $r>0$: We suppose that $\Omega = A^{-1}[-1,1]^d$ is determined by some $A \in \mathrm{GL}(d,\mathbb{R})$ such that the lattice $\Gamma = A\mathbb{Z}^d$ is admissible in the sense of Subsection 3.5.2, i.e. both (3.116) and (3.89) should be satisfied.

To estimate the lattice point remainder for this restriction of $E_{a,b}$ given by $H_r := E_{a,b} \cap r\Omega$ we smooth the indicator function I_Ω in an ε-neighborhood with an error of order $O(\varepsilon(b-a)r^{d-2})$ using Lemma 3.17. This yields a smooth function υ_ε and a final weight function ζ_ε, according to (3.3) in Theorem 3.1. Since Ω is admissible, both $\|\zeta_\varepsilon\|_1$ and $\|\zeta_\varepsilon\|_{*,r}$ in (3.5) are growing with a power of $|\log\varepsilon|$ only, see Lemmas 3.18 and 3.22.

In the next step we calibrate both smoothing parameters w and ε in order to get Corollary 3.4 below for 'wide' and 'thin' shells. The actual choice of $\varepsilon = \varepsilon(r)$ is then determined by calibrating the main terms εr^{d-2} and $\|\zeta_\varepsilon\|_1 \rho^*_{Q,b-a,w}(r) r^{d-2}$ depending on the speed of convergence of $\lim_{r\to\infty} \rho^*_{Q,b-a,w}(r) = 0$. The resulting error bound for *indefinite forms* will then differ at most by some $|\log\varepsilon|$-factors from the positive definite case, and is thus dominantly influenced by the Diophantine properties reflected in the decay of the $\gamma_{[T_-,T_+],\beta}$ resp. the $\rho^*_{Q,b-a,w}$-characteristic of irrationality. In particular we have uniformly for 'small' and 'wide' shells $E_{a,b}$ and admissible regions Ω the following bound.

Corollary 3.4. Under the assumptions of Theorem 3.1 we get for an admissible region Ω, all $|b|+|a| \leq c_0 r^2$, where $c_0>0$ is chosen as in Lemma 3.17, and $b-a \geq q$

$$\Delta_r \stackrel{\text{def}}{=} |\mathrm{vol}_{\mathbb{Z}} H_r - \mathrm{vol}\, H_r| \ll_{\beta,d} |Q|^{-1/2} r^{d-2}\big(\rho_{Q,b-a}(r) + R_{Q,A}(r)\big), \tag{3.7}$$

where

$$R_{Q,A}(r) \stackrel{\text{def}}{=} q^{\frac{d}{4}} r^{-\frac{d}{2}+2} \log(r+1)^d \Big(\big(\tfrac{q}{q_0}\big)^{\frac{d}{2}} + \frac{c_A^{d/2} q_0^{-\frac{d}{4}}}{\mathrm{Nm}(\Gamma)} \log\big(2+\tfrac{1}{\mathrm{Nm}(\Gamma)}\big)\Big) \log\big(1+\tfrac{b-a}{q_0^{1/2} r}\big), \tag{3.8}$$

$\mathrm{Nm}(\Gamma) := \inf_{\gamma\in\Gamma\setminus\{0\}} |\gamma_1 \dots \gamma_d|$ in standard coordinates $\gamma = (\gamma_1,\dots,\gamma_d)$ and

$$\rho_{Q,b-a}(r) \stackrel{\text{def}}{=} \inf\nolimits^*_{T_+,T_-} \Big\{ \log\big(\tfrac{b-a}{T_-^{\varsigma}}+1\big)^d \Big(a_Q q^{\beta d - \frac{1}{2}} (c_Q T_-^{\varsigma} + \gamma_{[T_-,1],\beta}(r)) + a_Q \gamma_{(1,T_+],\beta}(r) \log(T_+ + 1) + \tfrac{\log(qT_+ + 1)^2}{T_+} \Big) \Big\},$$

where the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$ and $T_+ \geq 1$. If $b-a \leq q$, then (3.7) holds, too, whereby the Diophantine factor $\rho_{Q,b-a}(r)$ have to be replaced by

$$\rho^*_{Q,b-a}(r) \stackrel{\text{def}}{=} \inf\nolimits^*_{T_-,T_+} \Big\{ a_Q \log(1+T_-^{-\varsigma})^d ((b-a)(c_Q T_-^{\varsigma} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r) (\log((b-a)^* T_+) + 1)) \Big\}.$$

In the last equation the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$ and $T_+ \geq 1$ with

$$T_+ \geq 4(b-a)^{-1} T_-^{-\varsigma} \max\{1, \log(c_Q^2 (b-a) T_-^{\varsigma})^2\}. \tag{3.9}$$

These bounds refine the results obtained in [BG99] providing explicit estimates in terms of Q and are valid for $d \geq 5$. Note that, due to the 'uncertainty principle' for the fourier transform, we need to choose T_+ at least as large as in (3.9) if $E_{a,b}$ is 'thin' in order to obtain the small factor $\exp\{-|T_+ w|^{1/2}\}$ which scales with $b-a$. In Section 3.7 we prove a variant of Corollary 3.4 for thin shells and non-admissible regions Ω as well, see Corollary 3.24 on page 70.

3.2 Organization and Sketch of Proof

The proof of the above-mentioned results is divided into three parts: Starting with a smoothing step in Section 3.3, we subsequently transfer the problem to Fourier transforms of the error in terms of special d-dimensional theta sums. One crucial step here is to reformulate the problem via a Weyl-type argument in terms of other theta sums with an underlying symplectic structure on $\mathbb{R}^{2d}$. In fact, in Section 3.4 we prove that the underlying lattice Λ_t is symplectic and provide estimates on the theta sums using basic arguments from the Geometry of Numbers. The crucial estimates for averages of functions on the space of lattices are moved to the Appendix B, where we present Margulis' averaging method in Section 6.2. After proving in Section 3.5 geometric bounds related to parallelepiped regions Ω, which are used here, we combine all these results in Section 3.6 to prove Theorem 3.1 and in Section 3.7 we conclude the results of the previous Section 3.1.

3.2.1 Smooth Weights on $\mathbb{Z}^d$

For the weights $\upsilon_r(x) := \exp\{-Q_+[x]/r^2\}$ our techniques can be used to establish effective bounds for the approximation of a weighted count of lattice points $m \in \mathbb{Z}^d$ with $Q[m] \in [a,b]$ by a corresponding integral with an error

$$R(\upsilon_r I_{E_{a,b}}) \overset{\text{def}}{=} \sum_{m \in E_{a,b} \cap \mathbb{Z}^d} \upsilon_r(m) - \int_{E_{a,b}} \upsilon_r(x)\, \mathrm{d}x. \tag{3.10}$$

The following bounds for $R(\upsilon_r I_{E_{a,b}})$ are *identical* for the case of positive and indefinite d-dimensional forms Q, provided that $d \geq 5$. We have

$$R(\upsilon_r I_{E_{a,b}}) \ll_{Q,d} r^{d-2}\, \bar{\rho}_{Q,b-a}(r) + r^{d/2-1}(b-a), \tag{3.11}$$

provided that $b-a \leq r$. If $r < b-a \ll r^2$ the second term in the bound has to be replaced by $r^{d/2}\log r$. The function $\bar{\rho}_{Q,b-a}(r)$ tends to zero for r tending to infinity if Q is irrational. Moreover, assuming that Q is Diophantine of type (κ, A), as we shall introduce in Definition 4.1, we conclude that $\bar{\rho}_{Q,b-a}(r) \ll_{Q,d,A} r^{-\nu}$, where $\nu \in (0,1)$ depends on d, κ and A (see Corollary 4.4). These results follow from the Theorem 3.1 with parameters chosen for the indefinite, positive and effective Diophantine cases as in the proofs in Section 3.7.

3.2.2 First Steps of the Proof

In order to prove effective bounds as in Theorem 3.1 we need an explicit bound for the error, say $R(I_{r\Omega \cap E_{a,b}})$ with I_A denoting the indicator of a set A, of approximating the number of integral points $m \in E_{a,b}$ in a bounded domain $r\,\Omega$ by the volume.

We start with a simplification of this problem: To bring Fourier analysis into the problem, we replace the weights $I_{r\Omega}(m) = 1$ of integral points $m \in r\,\Omega$ by suitable smoothly changing weights $\upsilon(m/r)$, which tend to zero as m/r tends to infinity. This will be done in Lemma 3.17 for a special class of regions. In fact, we will be forced to restrict the region Ω to parallelepipeds only in order to ensure that the corresponding error has logarithmic growth only. Additionally, we construct a w-smoothing g of the indicator function $[a,b]$ via convolution with an appropriate kernel k whose Fourier transform decays like $|\widehat{k}(t)| \ll \exp\{-\sqrt{|wt|}\}$. This allows us to replace the indicator function of $[a,b]$ in the lattice point counting problem by a smooth function, gaining an additional error in terms of the smoothing parameter w, see Corollary 3.5.

Being in a "smooth setting", we will consider only sufficiently fast decreasing smooth weight functions $\upsilon(x)$, which satisfy additionally the decay condition (3.3). This step is important, because now we can rewrite the lattice remainder in terms of special theta series. Let us sketch this step more detailed: We need to estimate

$$\sum_{m\in\mathbb{Z}^d} \upsilon(\tfrac{m}{r})\, g(Q[m]) - \int_{\mathbb{R}^d} \upsilon(\tfrac{m}{r})\, g(Q[x])\,\mathrm{d}x \overset{\text{def}}{=} V_r - W_r, \tag{3.12}$$

where we write $\upsilon(x) = \zeta(x)\exp\{-Q_+[x]\}$. Using inverse Fourier transforms we may express the weights as

$$g(Q[m]) = \int_{\mathbb{R}} \widehat{g}(t)\exp\{2\pi \mathrm{i} t Q[m]\}\,\mathrm{d}t, \quad \zeta(m) = \int_{\mathbb{R}^d} \widehat{\zeta}(u)\exp\{2\pi\mathrm{i}\langle u,m\rangle\}\,\mathrm{d}u.$$

Combining the resulting factors $\exp\{2\pi\mathrm{i}tQ[m]\}$, $\exp\{2\pi\mathrm{i}\langle v,m\rangle\}$ and $\exp\{-Q_+[\frac{x}{r}]\}$ in (3.12) into terms of the generalized theta series

$$\theta_v(t) \overset{\text{def}}{=} \sum_{m\in\mathbb{Z}^d} \exp\{-2\pi\mathrm{i}\langle v,m\rangle/r - 2\pi\mathrm{i}tQ[m] - Q_+[m]/r^2\}$$

one arrives at an expression for V_r by the following integral (in t and v) over $\theta_v(t)$:

$$V_r = \int_{\mathbb{R}^d} \widehat{\zeta}(v) \int_{\mathbb{R}} \widehat{g}(t)\theta_v(t)\,\mathrm{d}t\,\mathrm{d}v. \tag{3.13}$$

The approximating integral W_r to this sum V_r can be rewritten in exactly the same way by means of the theta integral

$$\vartheta_v(t) \overset{\text{def}}{=} \int_{\mathbb{R}^d} \exp\{-2\pi\mathrm{i}\langle v,x\rangle/r - 2\pi\mathrm{i}tQ[x] - Q_+[x]/r^2\}\,\mathrm{d}x,$$

replacing the theta sum $\theta_v(t)$. Thus, in order to estimate the error $|V_r - W_r|$, the integral over t and v of $|\theta_v(t) - \vartheta_v(t)|\,|\widehat{g}(t)\widehat{\zeta}(v)|$ has to be estimated.

As usually done in such counting problems, we split the integration domain depending on the behavior of the integrands. For $|t| \le q_0^{-1/2} r^{-1}$ and $\|x\| \ll r$ the functions $x \mapsto \exp\{2\pi\mathrm{i}tQ[x]\}$ are sufficiently smooth, so that the sum $\theta_v(t)$ is well approximable by the first term of its Fourier series, that is the corresponding integral $\vartheta_v(t)$, see (6.12) and (6.18). The error of this approximation, after integration over v, yields the second error term in (3.11), which does not depend on the Diophantine properties of Q. Additionally, we may restrict the integration to $|t| \le T_+$ for an appropriate choice of T_+ (depending on the width of the shell) by using the decay rate of the kernel k. So we end up with the remaining error term

$$I = \int_{T_+ > |t| > q_0^{-1/2} r^{-1}} \int_{\mathbb{R}^d} \Big|\theta_v(t)\,\widehat{g}(t)\,\widehat{\zeta}(v)\Big|\,\mathrm{d}v\,\mathrm{d}t, \tag{3.14}$$

which we estimate as follows

$$I \le \|\widehat{\zeta}\|_1 \sup_{v\in\mathbb{R}^d} \int_{T_+\ge|t|>q_0^{-1/2}r^{-1}} |\theta_v(t)|\ |\widehat{g(t)}|\,\mathrm{d}t. \tag{3.15}$$

The second factor in the bound of I in (3.15) encodes both the Diophantine behavior of Q as well as the growth rate with respect to r. Our strategy to extract out of this factor the correct rate of growth will be discussed in more detail later. Let us first state that the resulting bound (the choice of T_+ depending on the width of the shell) is an error bound depending on characteristics of $\widehat{\zeta}(v)$ of the form (see Theorem 3.1)

$$R(I_{E_{a,b}}\upsilon_r) \ll_{\kappa,d,Q} w + \|\widehat{\zeta}\|_1\rho(r,b-a)r^{d-2} + \|\widehat{\zeta}\|_{1,*}r^{d/2}\log\Big(1+\frac{b-a}{r}\Big), \tag{3.16}$$

which has to be optimized in the smoothing size w (compare Theorem 3.3) and $\rho(r,b-a)$ depends on the Diophantine properties of Q and r (see Theorem 3.1).

3.2.3 Mean-Value Estimates

In order to describe the second term in (3.16), we follow [Göt04] and show in Lemma 6.6 by using a Weyl differencing argument that uniformly in v and pointwise in t

$$|\theta_v(t)|^2 \ll r^d\ |\det Q|^{-1/2}\sum_{v\in\Lambda_t}\exp\{-\|v\|^2\}, \tag{3.17}$$

where $(\Lambda_t)_{t\in\mathbb{R}}$ is a family of $2d$-dimensional unimodular lattices generated by orbits of one-parameter subgroups of $\mathrm{SL}(2,\mathbb{R})$ indexed by t and r, see (3.39) for the precise definition. To estimate the right-hand side in (3.17), we will first bound the sum $\psi(t) := \sum_{v\in\Lambda_t}\exp\{-\|v\|^2\}$ by the number of lattice points $v\in\Lambda_t$ with $\|v\|_\infty \ll 1$ and then make use of the symplectic structure of Λ_t, compare Lemma 3.8. Combining these arguments yields the estimate

$$\sum_{v\in\Lambda_t}\exp\{-\|v\|^2\} \ll \frac{1}{M_1(\Lambda_t)\dots M_d(\Lambda_t)} \asymp_d \alpha_d(\Lambda_t),$$

where $M_i(\Lambda_t)$ denotes the i-th successive minima of Λ_t and $\alpha_d(\Lambda_t)$ the d-th α-characteristic of Λ_t, that is

$$\alpha_d(\Lambda_t) = \sup\{|\det(\Lambda')|^{-1} : \Lambda' \text{ is a } d\text{-dimensional sublattice of } \Lambda_t\}.$$

Based on Lemma 6.18 and a local approximation of a certain one-parameter subgroup by the compact group $\mathrm{SO}(2)$ (see Section 3.4.2), the average of $\alpha_d(\Lambda_t)^\beta$ with $0<\beta\le 1/2$ over t is derived in Lemmas 3.15 and 3.16. The proof of Lemma 6.18 uses an involved recursion in the size of r and builds on a method developed in [EMM98] on upper estimates of averages of certain functions on the space of lattices along translates of orbits of compact subgroups. Here we only briefly sketch the main ideas involved in this argument as described in [GM13].

As a first step, we will reduce the problem to the study of the mean-value operator

$$A_g(f)(x) = \frac{1}{2\pi}\int_K f(gkx)\,\mathrm{d}\sigma(k) \quad (x\in \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z}))$$

on $K := \mathrm{SO}(2)$ by finding an appropriate embedding of $\mathrm{SL}(2,\mathbb{R})$ in the standard symplectic group $\mathrm{Sp}(2d,\mathbb{R})$, where the existence of such an embedding will be proven with the help of representation theory of the group $\mathrm{SL}(2,\mathbb{R})$. In fact, taking all mentioned arguments together we will see for thin shells and all intervals I of length at most $\ll 1/q$ that

$$\int_I |\theta_v(t)|\,\mathrm{d}t \ll_\beta r^{d-\beta d}\,|\det Q|^{-1/4}\,\gamma_{I,\beta}(r)\,q^{-1}A_{g(r)}(\alpha^\beta)(\Lambda_I),$$

where $\gamma_{I,\beta}(r)$ contains information on the Diophantine properties of Q and tends to zero for irrational forms as r tends to infinity (see Corollary 4.6) and $g(r)$ and Λ_I are appropriate elements in $\mathrm{SL}(2,\mathbb{R})$, resp. the space of lattices. In the case of wide shells, additionally we need to use the decay rate of the fourier transform $\widehat{g}_w$ of the w-smoothed indicator function of $[a,b]$ to find a similar estimate.

As a special case, we can estimate the growth-rate of A_g for the spherical functions

$$\tau_\lambda(g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|g(\cos(\phi),\sin(\phi))\|^{-\lambda}\,\mathrm{d}\phi$$

as defined in (6.35), since they are the eigenfunctions of A_g. In fact, it is easy to show from this representation that $\tau(g) \ll \|g\|^{\lambda-2}$ if $\lambda > 2$, see (6.48), implying later the effective error bound of order r^{d-2}. Now the main idea is to extend these estimates to a larger class of functions f which do not appear isolated but emerges as the maximum of a family of positive functions $f_1,\dots,f_{2d}$. We require that this family satisfies two properties: First, the value of each f_i on any ball of radius s_0 is bounded (up to a constant depending only on s_0) by its value at the center. Second, the mean value of any f_i satisfies the following functional inequality

$$A_g f_i \ll \tau_{\lambda_i}(g) f_i + \max_{0<j\le \bar{i}} \sqrt{f_{i-j}f_{i+j}},$$

where we set $\bar{i} = \min\{i, 2d-i\}$ and $\lambda_i := \max\{2, \beta\bar{i}\}$. Note that it is possible to reinterpret this function class as a special case of subharmonic functions on Siegel's upper halfspace. We show, in a first instance, that any positive function f satisfying an inequality of the form

$$A_g f \ll \tau_\lambda(g) f + b\tau_\eta, \tag{3.18}$$

for $\lambda > 2$ and $0 < \eta < \lambda$ satisfies already

$$A_g f(1) \ll \tau_\lambda(g) f(1), \tag{3.19}$$

see Corollary 6.10. In other words, the mean value at g of such a function grows as fast as the associated spherical function. In a second instance we apply the already proved estimate (3.19) to the radialized family and obtain a preliminary estimate of the form

$$A_g f(1) \ll_\mu f(1)\tau_\mu(g) \tag{3.20}$$

for any fixed $\mu > \lambda_d$. Then we show inductively, using (3.18), (3.19) and (3.20), that

$$A_g(f_i) \ll f(1)\tau_{\mu_i}(g) \tag{3.21}$$

for all $i \neq d$ and an appropriate sequence $\lambda_d > \mu_i > \lambda_i$. Combining these estimates again with (3.18) yields in the case $i = d$ the inequality $A_g f_d \ll \tau_{\lambda_d}(g) f_d + f(1)\tau_\eta$ for some $\eta < \lambda_d$, which implies together with (3.18) and (3.21) the expected estimate

$$A_g(f)(1) \ll \tau_{\lambda_d}(g) f(1) \asymp f(1)\|g\|^{\lambda_d-2}.$$

To apply this mean-value estimate, it remains to check that the α-characteristics $\alpha_1,\dots,\alpha_{2d}$ satisfy both mentioned properties. This will be done in Lemma 6.17 with the help of the geometric inequality $d(L)d(M) \gg d(L \cap M)d(L+M)$ for any two Δ-rational subspaces L and M, which was established in Lemma 3.6. Finally, we obtain in Theorem 6.18 the bound $A_g \alpha^\beta(\Delta) \ll \alpha(\Delta)^\beta \|g\|^{\beta d-2}$ which implies for any interval I of length $\ll 1/q$

$$\int_I |\theta_v(t)| \, dt \ll q^{-1} r^{d-2} \, |\det Q|^{-1/4} \, \gamma_{I,\beta}(r) \alpha_d(\Lambda_I)^\beta.$$

At this point the current approach is fundamentally different to the approach of previous effective bounds for $R(I_{E_{a,b}} I_{r\Omega})$ by Bentkus and Götze [BG99] (see also [BG97]) valid for $d \geq 9$ and positive as well as indefinite forms. The reduction to (3.17) and $\rho(r, b-a)$ follows the approach used by Götze in [Göt04], where the average on the right-hand side of (3.17) was estimated for $d \geq 5$ by methods from the Geometry of Numbers and essentially required *positive definite forms*. A variant of that method was applied to *split indefinite forms* in a PhD thesis by G. Elsner [Els09].

3.2.4 The Role of the Region Ω

In order to estimate the lattice point deficiency $R(I_{E_{a,b} \cap r\Omega})$ we have to ε-smooth the indicator function of Ω which yields weights $\zeta = \zeta_\varepsilon$ and an additional error of order $\varepsilon(b-a)r^{d-2}$ in case of *indefinite forms* due to the intersection of $E_{a,b}$ with the boundary $\partial r\Omega$. For *positive definite forms* $r\Omega$ contains $E_{a,b}$, that is $\varepsilon > 0$ could be fixed independent of r, since this boundary intersection term is not present here.

In the indefinite case one needs to match the actual size of $r^{d-2}\rho(r, b-a)$ by choosing ε as small as $r^{-d/2+2}$ in (3.16). This leads to a critical dependence on ε through the Fourier transform of ζ_ε and its characteristics. Here $\|\widehat{\zeta_\varepsilon}\|_1$ moderately grows like $(\log 1/\varepsilon)^d$ for arbitrary small ε in the case of polyhedra only, see Lemma 3.18. The dependence of $\|\widehat{\zeta_\varepsilon}\|_{1,*}$, see (3.5), is again critically dependent on Ω and the width $b-a$ of the hyperbolic shell $E_{a,b}$. For $b - a \gg r$ the boundary of $r\Omega \cap E_{a,b}$ will contain a larger segment of $\partial r\Omega$. For a sequence of scalings r theses segments of the $d-1$-polytope potentially contain a large number of lattice points which induce large errors in the lattice point approximation, for which the technical restriction to the region Ω is solely responsible. In order to avoid this artefact which is reflected by a large growth of $\|\widehat{\zeta_\varepsilon}\|_{1,*}$ when ε is small, we restrict ourselves to special admissible regions $r\Omega$, where $\Omega = A^{-1}[-1,1]^d$, and $A \in \mathrm{GL}(d,\mathbb{R})$ is chosen such that the lattice $\Gamma = A\mathbb{Z}^d$ is admissible in the sense of Subsection 3.5.2, i.e. both (3.89) and (3.116) are satisfied. This ensures that the lattice point remainder of $r\Omega$ satisfies $|\mathrm{vol}_\mathbb{Z} r\Omega - \mathrm{vol}\, r\Omega| \ll_\Omega (\log r)^{d-1}$ uniformly which is 'abnormally' small. Likewise $\|\widehat{\zeta_\varepsilon}\|_{1,*}$ grows of order $(\log 1/\varepsilon)^d$ only. The resulting error bounds in Corollary 3.4 for wide shells with $|a| + |b| \ll r^2$ are then comparable up to at most $|\log \varepsilon|^d$ factors to the case of positive forms in Corollary 3.3.

3.3 Lattice Point Remainder via Fourier Representation

We begin by recalling some notations introduced at the beginning: Considering the quadratic form $Q[x] \coloneqq \langle x, Qx \rangle$, $x \in \mathbb{R}^d$, where $\langle \cdot, \cdot \rangle$ resp. $\|\cdot\|$ denote the standard Euclidean scalar product and norm, $Q \colon \mathbb{R}^d \to \mathbb{R}^d$ denotes a symmetric linear operator in $\mathrm{GL}(d,\mathbb{R})$ with eigenvalues $q_1, \ldots, q_d$, we also denote

$$q_0 \overset{\text{def}}{=} \min_{1 \leq j \leq d} |q_j|\,, \qquad q \overset{\text{def}}{=} \max_{1 \leq j \leq d} |q_j|\,, \qquad |Q|^{-1/2} \overset{\text{def}}{=} |\det Q|^{-1/2}\,.$$

Here we always assume that the form is non-degenerate, i.e. $q_0 > 0$.

The first step in the proof of Theorem 3.1 is to rewrite the lattice point counting errors $|\operatorname{vol}_{\mathbb{Z}} H_r - \operatorname{vol} H_r|$ in terms of integrals over appropriate smooth functions. In fact, here we shall consider smooth weights $\upsilon_r(x) := \upsilon(x/r)$ only, which are sufficiently fast decreasing such that the function

$$\zeta(x) \stackrel{\text{def}}{=} \upsilon(x)\exp\{Q_+[x]\} \tag{3.22}$$

satisfies (3.3). Depending on the applications, we will later replace υ by an indicator function of certain parallelepipeds Ω gaining an additional error controlled in Lemma 3.17.

Starting with smooth weights on the lattice points $E_{a,b}$, we shall investigate approximations to the weighted sum

$$\sum_{m\in\mathbb{Z}^d} I_{[a,b]}(Q[m])\upsilon_r(m) = \int_{\mathbb{R}^d} I_{[a,b]}(Q[x])\upsilon_r(x)\,\mathrm{d}x + R(I_{E_{a,b}}\upsilon_r). \tag{3.23}$$

For such weights both sides of (3.23) are well defined and $R(I_{E_{a,b}}\upsilon_r)$ may be estimated by splitting the integration domain appropriately and using Poisson's formula, see [Boc48], §46.

3.3.1 Smooth Approximation of the Indicator Function of $[a, b]$

Before doing this, we shall replace the indicator $I_{[a,b]}$ by a smooth approximation. To this end, we introduce smoothing kernels as follows. By Lemma 5.11 (with $u(t) = \sqrt{t}$) a probability measure $k = k(x)\,\mathrm{d}x$ (symmetric around 0) exists with compact support satisfying $k([-1,1]) = 1$ and $|\widehat{k}(t)| \le C\exp\{-|t|^{1/2}\}$ for all $t\in\mathbb{R}$ and a positive constant $C>0$, where $\widehat{k}(t) := \int g(x)\exp\{-2\pi\mathrm{i}tx\}\,\mathrm{d}x$ denotes the Fourier transform of the measure k. Though Lemma 5.11 provides better kernels, we won't need a better decay rate. For $\tau>0$ let k_τ denote the rescaled measures $k_\tau(A) := k(\tau^{-1}A)$ for any $A\in\mathcal{B}^n$. Using the same notation, let $k_\tau(x) = k_\tau(x_1)\dots k_\tau(x_n)$, $x=(x_1,\dots,x_n)$, denote its multivariate extension on $\mathbb{R}^n$, $n\ge 1$. Furthermore, let $f * k_\tau$ denote the convolution of a function f on $\mathbb{R}^n$ and k_τ.

Lemma 3.5. Let $[a,b]_\tau := [a-\tau, b+\tau]$ and write

$$g_{\pm w} \stackrel{\text{def}}{=} I_{[a,b]_{\pm w}} * k_w \quad \text{and} \quad g^Q_{\pm w}(x) \stackrel{\text{def}}{=} g_{\pm w}(Q[x]),\ x\in\mathbb{R}^d, \tag{3.24}$$

where $0 < w < (b-a)/4$. Then

$$|R(I_{E_{a,b}}\upsilon_r)| \le \max_{\pm} |R(g^Q_{\pm w}\upsilon_r)| + c_d w\|\upsilon\|_Q r^{d-2}, \tag{3.25}$$

where $\|\upsilon\|_Q$ is defined in Lemma 3.17 and c_d is a positive constant depending on d only.

Proof: In Lemma 6.1 we choose the measure μ resp. ν on $\mathbb{R}$ as the induced measure under the map $x\mapsto Q[x]$ of the counting measure with weights $\upsilon_r(m)$ resp. the measure $\upsilon_r(x)\,\mathrm{d}x$. Let $f(z) = I_{[a,b]}(z)$ and $f^\pm_\tau(z) = I_{[a,b]_{\pm\tau}}(z)$. Then (6.1) is satisfied and (6.2) applies with $\tau = w$. In order to bound the remainder term in (6.2) observe that

$$f^+_{2w} - f^-_{2w} \le I\big(\{x\in\mathbb{R}^d : Q[x]\in[a-2w, a+2w]\cup[b-2w, b+2w]\}\big)$$

and apply the geometric estimate of Lemma 3.17; that is (3.95) of the Appendix. □

3.3.2 Rewriting of the Remainder Term

We have reduced the determination of the lattice point remainder $R(I_{E_{a,b}}\upsilon_r)$ to the remainder $R(g^Q_{\pm w}\upsilon_r)$ for smooth weights. Next we rewrite the latter by means of the corresponding Fourier transforms. Rewrite the weight factor υ in (3.23) as $\upsilon(x) = \exp\{-Q_+[x]\}\zeta(x)$. Since by definition

$$|\widehat{g}_{\pm w}(t)| \ll |\widehat{I}_{[a,b]_{\pm w}}(t)\widehat{k}_w(t)| \ll s_{[a,b]_{\pm w}}(t)\exp\{-|tw|^{1/2}\} \quad \text{and} \quad \widehat{\zeta} \in L^1(\mathrm{d}v), \tag{3.26}$$

where

$$s_{[a,b]_{\pm w}}(t) \overset{\text{def}}{=} |(2\pi t)^{-1}\sin(\pi t(b-a\pm 2w))|,$$

we may express the weight functions $g_{\pm w}$ and ζ by their Fourier transforms

$$\widehat{g}_{\pm w}(v) = \int_{\mathbb{R}} g_{\pm w}(x)\exp\{-2\pi\mathrm{i}tx\}\,\mathrm{d}x \quad \text{and} \quad \widehat{\zeta}(v) = \int_{\mathbb{R}^d}\zeta(x)\exp\{-2\pi\mathrm{i}\langle v,x\rangle\}\,\mathrm{d}x.$$

This yields

$$g_{\pm w}(Q[x]) = \int_{\mathbb{R}} \widehat{g}_{\pm w}(t)\exp\{2\pi\mathrm{i}tQ[x]\}\,\mathrm{d}t \tag{3.27}$$

$$\zeta(x) = \int_{\mathbb{R}^d} \widehat{\zeta}(v)\exp\{2\pi\mathrm{i}\langle x,v\rangle\}\,\mathrm{d}v. \tag{3.28}$$

Using (3.27) we obtain by interchanging summation and integration in (3.23)

$$R(g^Q_{\pm w}\upsilon_r) = \int_{\mathbb{R}} R(e_{tQ}\upsilon_r)\widehat{g}_{\pm w}(t)\,\mathrm{d}t \tag{3.29}$$

with $e_{tQ}(x) \coloneqq \exp\{2\pi\mathrm{i}tQ[x]\}$. In the same way, writing

$$\tilde{e}_{v,r}(x) \overset{\text{def}}{=} \exp\{-Q_+[x/r] + 2\pi\mathrm{i}\langle x, vr^{-1}\rangle\},$$

we derive by (3.28) the remainder

$$R(e_{tQ}\upsilon_r) = \int_{\mathbb{R}^d} R(e_{tQ}\tilde{e}_{v,r})\widehat{\zeta}(v)\,\mathrm{d}v. \tag{3.30}$$

The sum $R(e_{tQ}\tilde{e}_{v,r})$ is the remainder between the generalized theta series and its corresponding theta integral, that is $R(e_{tQ}\tilde{e}_{v,r}) = \theta_v(z) - \vartheta_v(t)$, where

$$\theta_v(t) \overset{\text{def}}{=} \sum_{x\in\mathbb{Z}^d}\exp\{Q_{r,v}(t,x)\} \quad \text{and} \quad \vartheta_v(t) \overset{\text{def}}{=} \int_{\mathbb{R}^d}\exp\{Q_{r,v}(t,x)\}\,\mathrm{d}x, \tag{3.31}$$

$$Q_{r,v}(t,x) \overset{\text{def}}{=} 2\pi\mathrm{i}tQ[x] - r^{-2}Q_+[x] + 2\pi\mathrm{i}\langle x, vr^{-1}\rangle. \tag{3.32}$$

3.3.3 Splitting the Fourier Integrals

From here we only consider the weight g_w. The same inequalities hold also for g_w replaced with g_{-w}. Next, we decompose the integral over t in (3.29) into the segments

$J_0 := [-q_0^{-1/2}r^{-1}, q_0^{-1/2}r^{-1}]$ and $J_1 := \mathbb{R} \setminus J_0$, where the choice of J_0 was changed compared to [GM13] leading to improved dependency on Q in the final lattice remainder, and obtain

$$\begin{aligned} \left|R(g_w^Q \upsilon_r)\right| &\ll_d I_\Delta + I_\vartheta + I_\theta, \quad \text{say, where} && (3.33)\\ I_\Delta &\stackrel{\text{def}}{=} \Big|\int_{J_0} R(e_{tQ}\upsilon_r)\,\widehat{g}_w(t)\,\mathrm{d}t\Big|, && (3.34)\\ I_\vartheta &\stackrel{\text{def}}{=} \Big|\int_{J_1} \widehat{g}_w(t)\int_{\mathbb{R}^d} \vartheta_v(t)\,\widehat{\zeta}(v)\,\mathrm{d}v\,\mathrm{d}t\Big|, && (3.35)\\ I_\theta &\stackrel{\text{def}}{=} \Big|\int_{J_1} \widehat{g}_w(t)\int_{\mathbb{R}^d} \theta_v(t)\,\widehat{\zeta}(v)\,\mathrm{d}v\,\mathrm{d}t\Big|. && (3.36) \end{aligned}$$

Most of the technical estimates were moved to the Section (6.1) of Appendix B: In Lemma 6.4 we show that

$$I_\vartheta \ll_d d_Q\,\|\widehat{\zeta}\|_1 \min\{|b-a|\,q_0^{-1/2}r^{-1}, 1\}\,r^{d/2}q_0^{d/4},$$

provided that $d > 2$, and in Lemma 6.5 we prove that

$$I_\Delta \ll_d d_Q\,r^{d/2}\log(1+|b-a|\,q_0^{-1/2}r^{-1})\,\|\widehat{\zeta}\|_{*,r},$$

using the quantity $\|\widehat{\zeta}\|_{*,r}$ as defined in (3.5) for the weights $\zeta(x)$, provided that $d \geq 5$. Thus, applying (3.25) of Lemma 3.5 together with the previous estimates, we may now collect the results obtained so far for the lattice point remainder of (3.23) and obtain

$$\begin{aligned} &\Big|\sum_{m\in\mathbb{Z}^d} I_{[a,b]}(Q[m])\upsilon_r(m) - \int_{\mathbb{R}^d} I_{[a,b]}(Q[x])\upsilon_r(x)\,\mathrm{d}x\Big| \\ &\qquad \ll_d I_\theta + |Q|^{-1/2}r^{d/2}\,\|\widehat{\zeta}\|_{*,r}\,\log(1+|b-a|\,q_0^{-1/2}r^{-1}) + w\,\|v\|_Q\,r^{d-2}. \end{aligned} \tag{3.37}$$

It remains to estimate the term I_θ and this step crucially depends on Margulis' averaging method. We begin to separate the t and v integrals via

$$I_\theta \ll_d \|\widehat{\zeta}\|_1 \sup_{v\in\mathbb{R}^d}\int_{|t|>q_0^{-1/2}r^{-1}} |\widehat{g}_w(t)\theta_v(t)|\,\mathrm{d}t.$$

Applying Lemma 6.6 of the Appendix B, where the bound $|\theta_v(t)| \ll_d |\det Q|^{-1/4}\,r^{d/2}\,\psi(r,t)^{1/2}$ was proven with

$$\psi(r,t) = \sum_{m,n\in\mathbb{Z}^d} \exp\{-H_t(m,n)\},$$

where $H_t(m,n) = r^2 Q_+^{-1}[m - 4tQn] + r^{-2}Q_+[n]$ is a positive quadratic form on $\mathbb{Z}^{2d}$, yields

$$I_\theta \ll_d r^{d/2}\,|\det Q|^{-1/4}\,\|\widehat{\zeta}\|_1 \int_{|t|>q_0^{-1/2}r^{-1}} |\widehat{g}_w(t)|\,\psi(r,t)^{1/2}\,\mathrm{d}t. \tag{3.38}$$

In order to rewrite the Siegel transform $\psi(r,t) = \sum_{v\in\Lambda_t}\exp\{-\|v\|^2\}$ of $\exp\{-\|x\|^2\}$ evaluated at the lattice Λ_t, we introduce the $2d$-dimensional lattice

$$\Lambda_t \stackrel{\text{def}}{=} D_{rQ}U_{4tQ}\mathbb{Z}^{2d}, \tag{3.39}$$

where

$$D_{rQ} = \begin{pmatrix} rQ_+^{-\frac{1}{2}} & \\ & r^{-1}Q_+^{\frac{1}{2}} \end{pmatrix} \quad \text{and} \quad U_{4tQ} = \begin{pmatrix} \mathbb{1}_d & -4tQ \\ & \mathbb{1}_d \end{pmatrix}, \tag{3.40}$$

and note that Lemma 6.6 (with $\varepsilon = 1$) implies the estimate

$$\psi(r,t) \asymp_d \#\{w \in \Lambda_t : \|w\|_\infty \le 1\} \ll_d \#\{w \in \Lambda_t : \|w\| \le d^{1/2}\} \tag{3.41}$$

reducing the problem of estimating the theta series (6.24) to the problem of counting lattice points. In the next section we shall establish a relation between the α_i-characteristics and the successive minima of a lattice. As a consequence, we show that $\psi(r,t) \ll_d \alpha(\Lambda_t)$, where α is the maximum over all α_i-characteristics, compare (3.1).

3.4 Special Symplectic Lattices

Let $n \in \mathbb{N}^+$ be fixed and for every integer l with $1 \le l \le n$ we fix a quasinorm $|\cdot|_l$ on the exterior product $\bigwedge^l \mathbb{R}^n$. Let L be a subspace of $\mathbb{R}^n$ and Δ a lattice in L (i.e. Δ is a free $\mathbb{Z}$-module of full rank $\dim L$), then any two bases of Δ are related by a unimodular transformation, that is, if $u_1, \ldots, u_l$ and $v_1, \ldots, v_l$ are two bases of Δ, where $l = \dim L$, then $v_1 \wedge \cdots \wedge v_l = \pm u_1 \wedge \cdots \wedge u_l$, which implies that the expression $|v_1 \wedge \cdots \wedge v_l|_l$ is independent of the choice of basis.

Let Δ be a lattice in $\mathbb{R}^n$, we say that a subspace L of $\mathbb{R}^n$ is Δ-*rational* if $L \cap \Delta$ is a lattice in L. For any Δ-rational subspace L, we denote by $d_\Delta(L)$, or simply by $d(L)$, the quasinorm $|u_1 \wedge \ldots \wedge u_l|_l$, where $\{u_1, \ldots, u_l\}$, $l = \dim L$, is a basis of $L \cap \Delta$ over $\mathbb{Z}$. For $L = \{0\}$ we write $d(L) \coloneqq 1$. If the quasinorms $|\cdot|_l$ are the norms on $\bigwedge^l \mathbb{R}^n$ induced from the standard Euclidean norm on $\mathbb{R}^n$, then $d(L)$ is equal to the determinant (or discriminant) $\det(L \cap \Delta)$ of the lattice $L \cap \Delta$, that is the volume of $L/(L \cap \Delta)$. In particular, in this case the lattice Δ is said to be unimodular if and only if $d_\Delta(\mathbb{R}^n) = 1$. Additionally, we have the following geometric estimate on the product of the volume of two Δ-rational subspaces.

Lemma 3.6. There is a constant $C \ge 1$ depending only on the quasinorm $|\cdot|_l$ and not on Δ such that

$$C^2 d(L) d(M) \ge d(L \cap M) d(L + M) \tag{3.42}$$

for any two Δ-rational subspaces L and M.

Proof (compare Lemma 5.6 in [EMM98]): Since any two quasinorms on $\bigwedge^l \mathbb{R}^n$ are equivalent, it remains to prove this result when $d(\cdot)$ is equal to the determinant. Additionally, we shall reduce the problem to the case $L \cap M = \{0\}$ as follows.

We denote by $\pi\colon \mathbb{R}^n \to \mathbb{R}^n / L \cap M$ the natural projection and note that $d(L) = d(\pi(M)) d(L \cap M)$, $d(M) = d(\pi(L)) d(L \cap M)$ and $d(L+M) = d(\pi(L+M)) d(L \cap M)$. Since the inequality (3.42) is equivalent to

$$\frac{d(L)}{d(L \cap M)} \frac{d(M)}{d(L \cap M)} \ge \frac{d(L+M)}{d(L \cap M)},$$

we can replace L, M and $L + M$ by $\pi(L)$, $\pi(M)$ and $\pi(L+M)$. Thus, we may assume that $L \cap M = \{0\}$. Now let $e_1, \ldots, e_l$, $l = \dim L$, and $e_{l+1}, \ldots, e_{l+m}$, $m = \dim M$, be bases in L resp. in M. Then, using the Cauchy-Schwarz inequality in the exterior algebra, we get

$$\begin{aligned} d(L)d(M) &= \|e_1 \wedge \ldots \wedge e_l\| \cdot \|e_{l+1} \wedge \ldots \wedge e_{l+m}\| \\ &\ge \|e_1 \wedge \ldots \wedge e_l \wedge e_{l+1} \wedge \ldots \wedge e_{l+m}\| = d(L+M), \end{aligned}$$

where the last equality holds since $e_1, \ldots, e_{l+m}$ is a basis in $L + M$. □

Let us introduce the following notations for $0 \leq l \leq n$,

$$\alpha_l(\Delta) \stackrel{\text{def}}{=} \sup\{d(L)^{-1} : L \text{ is a } \Delta\text{-rational subspace of dimension } l\}, \tag{3.43}$$

$$\alpha(\Delta) \stackrel{\text{def}}{=} \max_{0\leq l\leq n} \alpha_l(\Delta). \tag{3.44}$$

This extends the earlier definition (3.1) of $\alpha_i(\Delta)$ in the introduction of Chapter 3 to the case of general seminorms on $\bigwedge^i \mathbb{R}^n$. In this section the functions α_l and α will be based on standard Euclidean norms, that is, we have $d(L) = \det(L \cap \Delta)$.

In the following we shall use some facts from the geometry of numbers for lattices in $\mathbb{R}^n$, see Davenport [Dav58] and Cassels [Cas97].The successive minima of a lattice Λ are the numbers $M_1(\Lambda) \leq \cdots \leq M_n(\Lambda)$ defined as follows: $M_j(\Lambda)$ is the infimum of $\lambda > 0$ such that the set $\{v \in \Lambda : \|v\| < \lambda\}$ contains j linearly independent vectors, in particular, $M_1(\Lambda)$ is the shortest vector of the lattice Λ. It is easy to see that these infima are attained, that is, there exist linearly independent vectors $v_1, \ldots, v_n \in \Lambda$ such that $\|v_j\| = M_j(\Lambda)$ for all $j = 1, \ldots, n$.

Remark 3.7. In classical reduction theory, one uses a variant of the classical successive minima called primitive vectors: $b_1, \ldots, b_n \in \Lambda$ constitute a basis for Λ with $F(b_j) \asymp_d M_j$. This alternative construction leads to larger constants in Minkowski's second theorem. For details see [Si89], Lecture X, §5–6.

Lemma 3.8. Let F be a norm in $\mathbb{R}^n$ such that $F \asymp_n \|\cdot\|$ and denote by $M_1 \leq \cdots \leq M_n$ the successive minima with respect to F. Let Λ be a lattice in $\Lambda \subset \mathbb{R}^n$, then

$$\alpha_l(\Lambda) \asymp_n (M_1(\Lambda)\cdots M_l(\Lambda))^{-1}, \quad l = 1, \ldots, n. \tag{3.45}$$

Moreover, for any $\mu > 0$, if $1 \leq j \leq n$ is such that $M_j(\Lambda) \leq \mu < M_{j+1}(\Lambda)$, where the right-hand side is omitted if $j = d$, then

$$\#\{v \in \Lambda \,:\, F(v) \leq \mu\} \asymp_n \mu^j\, \alpha_j(\Lambda). \tag{3.46}$$

Proof: In principle, the relation (3.45) is well-known and a proof can be found in Einsiedler-Ward [EW19]. However, for completeness we include the proof here. Let $\Lambda' \subset \Lambda$ be an arbitrary l-dimensional sublattice of Λ and $N_1 \leq \ldots \leq N_l$ the corresponding successive minima of Λ' with respect to F. Moreover, let V' denotes the l-dimensional volume of the convex body $\{F \leq 1\} \cap \mathrm{Span}(\Lambda')$. Then we know by Minkowski's theorem on successive minima that

$$|\det(\Lambda')|\, \frac{2^l}{l!V'} \leq (N_1 \ldots N_l) \leq \frac{2^l}{V'}\, |\det(\Lambda')|\,. \tag{3.47}$$

The last inequality together with $F(\cdot) \asymp_n \|\cdot\|$ shows that $N_1 \cdots N_l \asymp_n |\det(\Lambda')|$ and, because of $M_j \leq N_j$ for all $j = 1, \ldots, l$, we obtain that

$$\alpha_l(\lambda) \ll_n (M_1 \cdots M_l)^{-1}.$$

On the other hand, we have $M_j = F(b_j)$ for some linearly independent lattice points $b_1, \ldots, b_l \in \Lambda$. Therefore, the successive minima of the l-dimensional lattice

$$\Lambda' = \left\{\textstyle\sum_{j=1}^{l} n_j b_j \,:\, n_1, \ldots, n_l \in \mathbb{Z},\right\}$$

are exactly $M_j = N_j$ for all $j = 1, 2, \ldots, l$. This implies $M_1 \cdots M_l \asymp_n |\det(\Lambda')|$ and therefore we find also $\alpha_n(\Lambda) \gg_n M_1 \cdots M_l$. This concludes the proof of (3.45).

Next we shall prove (3.46). Let $\mu > 0$ with $M_j(\Lambda) \le \mu < M_{j+1}(\Lambda)$, where the right-hand side being omitted if $j = m$. Moreover, let $v_1, \dots, v_m$ denote the elements in Λ corresponding to the successive minima $M_i(\Lambda)$, $i = 1, \dots, m$. For $m_1, \dots, m_j \in \mathbb{Z}$ with $|m_i| \le j^{-1}\mu F(v_i)^{-1}$ notice that $v = m_1 v_1 + \dots + m_j v_j$ satisfies $F(v) \le \mu$, thus

$$N(\mu) \overset{\text{def}}{=} \#\{v \in \Lambda \,:\, F(v) \le \mu\} \gg_m \mu^j (M_1(\Lambda) \cdots M_j(\Lambda))^{-1}. \tag{3.48}$$

The upper bound is also proven in Davenport [Dav58], see Lemma 1. Again, we include the argument here for the sake of completeness: First note that any lattice point $v \in \Lambda$ with $F(v) < M_{j+1}$ is linearly dependent on $a_1, \dots, a_l$. Though the points $a_1, \dots, a_n$ in general do not constitute an integral basis for Λ, there exists a basis $b_1, \dots, b_n$ such that a_j is linearly dependent on $b_1, \dots, b_j$, see e.g. Cassels [Cas97], Section I.2 Corollary 2. Hence any element $v \in \Lambda$ with $F(v) \le \mu$ can be written as $v = m_1 b_1 + \dots + m_j b_j$ with $m_i \in \mathbb{Z}$. Suppose $v' \in \Lambda$ is another element with $F(v') \le \mu$. Of course, we can again write $v' = m_1' b_1 + \dots m_j' b_j$ with $m_i' \in \mathbb{Z}$. Now define positive integers $\nu_1, \dots, \nu_j$ by

$$2^{\nu_i - 1} \le \frac{2\mu}{M_i} < 2^{\nu_i}. \tag{3.49}$$

Obviously $\nu_1 \ge \nu_2 \ge \dots \ge \nu_j$. Assuming for the moment that $m_l \equiv m_l' \mod 2^{\nu_l}$ for every $l = 1, \dots, j$ and let i denote the largest index i such that $m_i \ne m_i'$. Then $x := 2^{-\nu_i}(v - v')$ is an element of Λ and linearly independent of $b_1, \dots, b_{i-1}$ and thus also of $a_1, \dots, a_{i-1}$. This implies $F(x) \ge M_i$. On the other hand we have

$$F(x) = 2^{-\nu_i} F(v - v') \le 2^{-\nu_i}(F(v) + F(v')) \le 2^{-\nu_i} 2\mu < M_i$$

by (3.49). This contradiction shows that there is at most one lattice point in Λ such that the coordinates $m_1, \dots, m_j$ lie in the same residue classes to the moduli $2^{\nu_1}, 2^{\nu_2}, \dots, 2^{\nu_j}$ respectively. Hence the number of lattice points $N(\mu)$ in (3.48) is bounded from above by the number of all residue classes, i.e. by $2^{\nu_1} 2^{\nu_2} \dots 2^{\nu_j} \le (4\mu)^j (M_1 \dots M_j)^{-1}$. This shows the upper bound in (3.46). $\square$

Lemma 3.9 (Davenport [Dav58]). Let $\Lambda = g\mathbb{Z}^n$ and $\Lambda' = g^{-T}\mathbb{Z}^n$ denote dual lattices of rank n, then for all $j = 1, \dots, n$ we have

$$1 \le M_j(\Lambda) M_{n+1-j}(\Lambda') \ll_n 1. \tag{3.50}$$

Proof: The proof of (3.50) is given by Davenport in [Dav58], Lemma 2. Once again, we include the argument here for completeness: Let $v_1, \dots, v_n \in \Lambda$, resp. $v_1', \dots, v_n' \in \Lambda'$, be linearly independent such that $\|v_i\| = M_i(\Lambda)$, resp. $\|v_i'\| = M_i(\Lambda')$. Then $v_1, \dots, v_j$ cannot be orthogonal to all lattice points $v_1', \dots, v_{n+1-j}'$, otherwise they would fail to be independent. Thus, we have $\langle v_i, v_k' \rangle \ne 0$ for some $i = 1, \dots, j$ and $k = 1, \dots, n+1-j$, which implies that

$$M_j(\Lambda) M_{n+1-j}(\Lambda') \ge M_i(\Lambda) M_k(\Lambda') = \|v_i\| \|v_k'\| \ge |\langle v_i, v_k' \rangle| \ge 1$$

because of duality. The right-hand side of (3.50) follows from (3.45) in the case $l = d$, which is known as Minkowski's inequality. Indeed, $\det(\Lambda) = \alpha_n(\Lambda) \asymp_n M_1(\Lambda) \dots M_n(\Lambda)$ and since $\det(\Lambda)\det(\Lambda') = 1$ we conclude that

$$M_j(\Lambda) M_{n+1-j}(\Lambda') \ll_n \textstyle\prod_{h=1, h \ne j}^n (M_h(\Lambda) M_{n+1-h}(\Lambda'))^{-1} \ll_n 1. \qquad \square$$

3.4.1 Symplectic Structure of Λ_t

In the following we shall apply the previous results from the Geometry of Numbers to the special $2d$-dimensional lattice Λ_t, introduced in (3.39). Especially, we will make use of the symplectic structure of Λ_t in order to establish a relation between the theta series (6.24) and the α_d-characteristic of Λ_t, see (3.56), and to sharpen Lemma 3.8 and Lemma 3.9 as follows.

Lemma 3.10. Let Λ_t be the lattice defined in (3.39). Then we have for any $t \in \mathbb{R}$

$$M_j(\Lambda_t)\, M_{2d+1-j}(\Lambda_t) \asymp_d 1 \quad (j = 1, \ldots, d), \tag{3.51}$$

$$M_1(\Lambda_t) \le \ldots \le M_d(\Lambda_t) \ll_d 1 \le M_{d+1}(\Lambda_t) \le \ldots \le M_{2d}(\Lambda_t) \tag{3.52}$$

and the lower bound

$$M_1(\Lambda_t) \ge \min\{r^{-1} q_0^{1/2}, r q^{-1/2}\}. \tag{3.53}$$

Corollary 3.11. As a consequence, we find for $\mu \ge 1$

$$\#\{v \in \Lambda_t \,:\, \|v\| \le \mu\} \ll_d \mu^{2d} \alpha_d(\Lambda_t), \tag{3.54}$$

$$\alpha(\Lambda_t) = \max\{\alpha_j(\Lambda_t) \,:\, j = 1, \ldots, 2d\} \asymp_d \alpha_d(\Lambda_t). \tag{3.55}$$

and

$$\psi(r, t) \ll_d \alpha_d(\Lambda_t). \tag{3.56}$$

Proof of Lemma 3.10: First we prove (3.51). Let

$$J \overset{\text{def}}{=} \begin{pmatrix} & \mathbb{1}_d \\ -\mathbb{1}_d & \end{pmatrix},$$

and consider the lattice

$$\Lambda'_t = J D_{rQ} U_{tQ} J^{-1} \mathbb{Z}^{2d},$$

then $J D_{rQ} U_{4tQ} J^{-1} = D_{rQ}^{-1} U_{-4tQ}^T$ and hence Λ'_t is the lattice dual to Λ_t in the sense of Lemma 3.9. We claim that they have identical successive minima. To this end, note that for any $N = (m, \bar{m})^T \in \mathbb{Z}^{2d}$

$$\|D_{rQ} U_{4tQ} N\| = \|J^{-1} J D_{rQ} U_{tQ} J^{-1} J N\| = \|D_{rQ}^{-1} U_{-4tQ}^T J N\|, \tag{3.57}$$

where we use that J is an orthogonal matrix. Since $J\mathbb{Z}^{2d} = \mathbb{Z}^{2d}$, the equation (3.57) implies that the successive minima of Λ_t and Λ'_t are identical and by Lemma 3.9 we conclude $M_j(\Lambda_t) M_{2d+1-j}(\Lambda_t) \asymp_d 1$ for $j = 1, \ldots, d$.

To prove (3.52) we note that $M_d \le M_{d+1}$ and $1 \le M_d(\Lambda_t) M_{d+1}(\Lambda_t) \ll_d 1$ implies

$$M_j(\Lambda_t) \le M_d(\Lambda_t) \ll_d 1 \quad \text{and} \quad 1 \le M_{d+1}(\Lambda_t) \le M_{d+j}(\Lambda_t)$$

for all $j = 1, \ldots, d$. Thus, it remains to show the lower bound (3.53) for $M_1(\Lambda_t)$: Take $m, \bar{m} \in \mathbb{Z}^d$ with $M_1(\Lambda_t) = \|D_{rQ} U_{4tQ}(m, \bar{m})\| = H_t[m, \bar{m}]^{1/2}$, where H_t denotes the special norm (6.25) in the theta series (6.24). If $\bar{m} \ne 0$, then we have $M_1(\Lambda_t) \ge r^{-1} \|Q_+^{1/2} \bar{m}\| \ge q_0^{1/2} r^{-1}$, but otherwise $M_1(\Lambda_t) = r\|Q_+^{-1/2} m\| \ge r q^{-1/2}$. □

Proof of Corollary 3.11: We begin with proving (3.54) as follows. Recall that $\mu \geq 1$ and let $2d \geq j \geq 1$ denote the maximal integer with $M_j(\Lambda_t) \leq \mu$. Then Lemma 3.8 implies

$$\#\{v \in \Lambda_t : \|v\| \leq \mu\} \ll_d \mu^j \alpha_j(\Lambda_t) \leq \mu^{2d} \alpha_d(\Lambda_t),$$

since we have $M_j(\Lambda_t) \geq \ldots \geq M_{d+1}(\Lambda_t) \gg 1$ if $j > d$ and $\mu < M_{j+1}(\Lambda_t) \leq \ldots \leq M_d(\Lambda_t) \ll_d 1$ if $j < d$. In the case $\mu < M_1(\Lambda_t)$ the inequality in (3.54) holds trivially. Moreover, this argument also proves (3.55). Finally, the estimate (3.56) follows from the relation (3.41) combined with (3.54) for $\mu = d^{1/2}$. □

For arbitrary t and for small t the following bounds which are independent of the Diophantine properties of Q hold.

Lemma 3.12. Denote by Δ the lattice $Q_+^{1/2}\mathbb{Z}^d$, then

$$\sup_{t\in\mathbb{R}} \alpha_d(d_s u_t \Lambda_Q) \ll_d \varphi_Q(s) \tag{3.58}$$

where

$$\varphi_Q(s) \stackrel{\text{def}}{=} s^d \, |\det Q|^{-1/2} \prod_{j\,:\,M_j(\Delta)>s} (s^{-2}(M_j(\Delta)^2), \;\; s > 0. \tag{3.59}$$

In particular, it follows that

$$\varphi_Q(s) \ll_d s^d \, |\det Q|^{-1/2}, \quad \text{if} \quad |s| \geq q^{1/2}, \tag{3.60}$$

and for small t we get

$$\alpha_d(d_s u_t \Lambda_Q) \ll_d |\det Q|^{1/2} \, (s^{-1} + |t\,s|)^d, \qquad \text{if} \quad q_0^{1/2}\, |t\,s| \geq 1, \tag{3.61}$$

$$\alpha_d(d_s u_t \Lambda_Q) \ll_d |\det Q|^{-1/2} \max\{1, (\sqrt{q}/s)^d\}\, |t\,s|^{-d}, \quad \text{if} \quad q^{1/2}\, |t\,s| \leq 1. \tag{3.62}$$

Proof: If $1/2 < M_1(\Lambda_t)$, then we have obviously

$$\alpha_d(\Lambda_t) \asymp_d (M_1(\Lambda_t) \ldots M_d(\Lambda_t))^{-1} \ll_d \#\{v \in \Lambda_t \, : \, \|v\| \leq 1/2\}. \tag{3.63}$$

Otherwise, there exists an integer $j = 1, \ldots, d$ with $M_j(\Lambda_t) \leq 1/2 < M_{j+1}(\Lambda_t)$, since $1 \leq M_{d+1}(\Lambda_t)$ holds by (3.52). Now, taking $\mu = 1/2$ in (3.46) of Lemma 3.8 shows that

$$\alpha_d(\Lambda_t) \asymp_d (M_1(\Lambda_t) \ldots M_d(\Lambda_t))^{-1} \ll (M_1(\Lambda_t) \ldots M_j(\Lambda_t))^{-1} \asymp_d \#\{v \in \Lambda_t : \|v\| \leq 1/2\},$$

i.e. (3.63) holds also in the second case. Recalling again (6.25), we see that the right-hand side of (3.63) is the same as the number all lattice points $m, \bar{m} \in \mathbb{Z}^d$ satisfying

$$H_t[m, \bar{m}] = s^2 Q_+^{-1}[m - 4tQ\bar{m}] + s^{-2} Q_+[\bar{m}] \leq 1/4. \tag{3.64}$$

Proof of (3.58). If (3.64) holds, then $\|Q_+^{1/2}\bar{m}\| \leq s/2$, which has again by Lemma 3.8 at most $\ll_d \prod_{j\,:\,(M_j(\Delta)\leq s} (s\, M_j(\Delta)^{-1})$ integral solutions. Similarly, for fixed $\bar{m}$ the triangle inequality combined with (3.64) implies

$$\|sQ_+^{-1/2}(m_1 - m_2)\| \leq \sqrt{H_t[m_1, \bar{m}]} + \sqrt{H_t[m_2, \bar{m}]} \leq 1.$$

Thus, for fixed $\bar{m}$, the number of pairs $(m, \bar{m})$ for which (3.64) holds is bounded by the number of elements v in the dual lattice $\Delta' = Q^{-1/2}\mathbb{Z}^d$ to Δ such that $\|v\| \leq s^{-1}$. Since the successive minima for this dual lattice are determined by Lemma 3.9, we may use Lemma 3.8, inequality (3.46), again to determine the upper bound

$$\ll_d \prod_{j\,:\,(M_j(\Delta')\le s^{-1}} (s(M_j(\Delta'))^{-1} \le \prod_{j\,:\,(M_j(\Delta)\ge s} (s^{-1}(M_j(\Delta))$$

for this number as well. The product of both numbers yields the bound

$$\alpha_d(\Lambda_t) \ll_d \#\{v \in \Lambda_t : \|v\| \le 1/2\} \ll_d s^d \big(\prod_{j=1}^d (M_j(\Delta')\big)\big(\prod_{j\,:\,(M_j(\Delta)\ge s} (s^{-2} M_j(\Delta)^2)\big).$$

Finally, using Lemma 3.8 in form of $\prod_{j=1}^d M_j(\Delta') \asymp_d \alpha_d(\Delta')^{-1} = |\det Q|^{-1/2}$ shows the claimed bound in (3.58). Also the inequality (3.60) follows immediately from (3.58).

Proof of (3.61). Assume $q_0^{1/2}\,|ts| \ge 1$ and $q_0 \ge 1$. If $m = 0$ we conclude that $\|\bar m\|^2 \le |4ts|^2\,\|Q_+^{1/2}\bar m\|^2 \le 1/4$. Hence $\bar m = 0$. For any fixed $m \ne 0$ the triangle inequality implies that there is at most one element $\bar m \in \mathbb{Z}^d$ with (3.64). Furthermore, we get $(\|Q_+^{-1/2} m\| - 1/(2s)) \le \|4tQ_+^{1/2}\,\bar m\|$ for that pair $(m, \bar m)$. This implies

$$1/2 \ge \sqrt{H_t(m,\bar m)} \ge s^{-1}\|Q_+^{1/2}\bar m\| \ge \big(\|Q_+^{-1/2} m\| - 1/(2s)\big)/\,|4ts|$$

and hence $\|Q_+^{-1/2} m\| \le (s^{-1} + |4ts|)/2$. Thus

$$\#\{v \in \Lambda_t \;:\; \|v\|^2 \le 1/4\} \ll_d (s^{-1} + |4ts|)^d\;|\det Q|^{1/2}\,.$$

Proof of (3.62). As in the previous case, (3.64) implies by the triangle inequality that

$$\Big|\|Q_+^{-1/2} m\| - \|4tQ_+^{1/2}\,S\,\bar m\|\Big| \le (2s)^{-1} \tag{3.65}$$

and together with $q^{1/2}\,|ts| \le 1$ also $|4ts|\;s^{-1}\|Q_+^{1/2}\bar m\| \le |4ts|\,/2 \le 2q^{-1/2}$. Moreover one of these inequalities is strict and therefore we have

$$q^{-1/2}\|m\| \le \|Q_+^{-1/2} m\| < (2s)^{-1} + (2q^{1/2})^{-1}. \tag{3.66}$$

If $s \ge q^{1/2}$, this leads to a contradiction unless $m = 0$. Hence, the possible solutions for $\bar m$ in (3.65) satisfy $\|Q_+^{1/2}\bar m\| \le |8ts|^{-1}$ which, as in the proof of (3.58), has at most $\ll_d |\det Q|^{-1/2}\,|ts|^{-d}$ solutions. In the second case, i.e. if $s < q^{1/2}$, the inequality (3.66) has at most $\ll_d (q^{1/2}/s)^d$ solutions for m. Now any possible $\bar m$ must satisfy

$$\|Q_+^{1/2}\bar m\| \le |8ts|^{-1} + |4t|^{-1}\,\|Q_+^{-1/2} m\| \le 3\,|4ts|^{-1}$$

again, which completes the proof of (3.62) in view of (3.63). □

3.4.2 Approximation by Compact Subgroups

In the next section we need to average over powers of the α_d-characteristic of the lattice Λ_t introduced in (3.39). In order to use harmonic analysis tools, we shall rewrite the family $\{\Lambda_t\}_{t\in\mathbb{R}}$ as an orbit of a single lattice by means of elements of the one-parameter subgroups $\mathrm{D} := \{d_r : r > 0\}$ and $\mathrm{U} := \{u_t : t \in \mathbb{R}\}$ of $\mathrm{SL}(2,\mathbb{R})$, where

$$d_r \stackrel{\text{def}}{=} \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, \qquad u_t \stackrel{\text{def}}{=} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \tag{3.67}$$

and then approximate the subgroup U locally by the compact subgroup $K = SO(2) = \{k_\theta : \theta \in [0, 2\pi]\}$ parameterized, as usual, by elements

$$k_\theta \stackrel{\text{def}}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.68}$$

Let S be an orthogonal matrix such that $SQQ_+^{-1}S^T = Q_0$, where Q_0 denotes the signature matrix corresponding to Q, that is $Q_0 = \operatorname{diag}(1, \ldots, 1, -1, \ldots, -1)$. A short computation shows that

$$D_{rQ}U_{4tQ} = \begin{pmatrix} S^T & \\ & S^T \end{pmatrix} d_r u_{4t} \begin{pmatrix} SQ_+^{-1/2} & \\ & SQ_+^{1/2} \end{pmatrix},$$

where we embed $SL(2, \mathbb{R})$ into $SL(2d, \mathbb{R})$ according to the following action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a\mathbb{1}_d & b\,Q_0 \\ c\,Q_0 & d\mathbb{1}_d \end{pmatrix}. \tag{3.69}$$

Define the $2d$-dimensional lattice

$$\Lambda_Q \stackrel{\text{def}}{=} \begin{pmatrix} SQ_+^{-1/2} & \\ & SQ_+^{1/2} \end{pmatrix} \mathbb{Z}^{2d}, \tag{3.70}$$

then as claimed

$$\Lambda_t = \begin{pmatrix} S^T & \\ & S^T \end{pmatrix} d_r u_{4t}\, \Lambda_Q. \tag{3.71}$$

Moreover, since S is orthogonal and α_i is invariant under left multiplication by orthogonal matrices we observe that

$$\alpha_i(\Lambda_t) = \alpha_i(d_r u_{4t} \Lambda_Q), \tag{3.72}$$

for all $i = 1, \ldots, 2d$.

Lemma 3.13. With respect to the embedding of $SL(2, \mathbb{R})$ defined in (3.69) we have for $t \in \mathbb{R}$, $s \geq 1$ and any $2d$-dimensional lattice Λ in $\mathbb{R}^{2d}$

$$\alpha_j(d_s\, u_t \Lambda) \ll_d (1 + t^2)^{\frac{j}{2}}\, \alpha_j(d_s\, k_\theta \Lambda), \qquad j = 1, \ldots, 2d, \tag{3.73}$$

where $\theta = \arctan t$.

Proof: Suppose the signature of Q is (p, q) and let $(v, w) \in \mathbb{R}^d \times \mathbb{R}^d$, thought of as a column vector with coordinates $v_1, \ldots, v_d, w_1, \ldots, w_d$, then

$$\|d_s u_t(v, w)\|^2 = \sum_{i=1}^{p} \|d_s u_t(v_i, w_i)\|^2 + \sum_{i=p+1}^{d} \|d_s u_{-t}(v_i, w_i)\|^2. \tag{3.74}$$

Let $x, y \in \mathbb{R}$. Note that $y + t\,x = (1 + t^2)\, y + t\,(x - t\,y)$, which implies that

$$(y + t\,x)^2 \leq 2\,(1 + t^2)^2\,(y)^2 + 2\,t^2\,(x - t\,y)^2,$$

and therefore we find

$$s^2\,(x - t\,y)^2 + s^{-2}\,(y + t\,x)^2 \leq 2(1 + t^2)^2 \left(s^2\,(x - t\,y)^2 + s^{-2} y^2\right), \tag{3.75}$$

provided that $s \geq 1$. Taking $\theta = \arctan t$ and noting that $\cos(\theta) = (t^2+1)^{-1/2}$, resp. $\sin(\theta) = t(t^2+1)^{-1/2}$, we see that (3.75) can be written as

$$\|d_s k_\theta(x,y)\|^2 \leq 2\,(1+t^2)\|d_s u_t(x,y)\|^2, \tag{3.76}$$

and it is easy to see, along the same lines as before, that

$$\|d_s k_\theta^T(x,y)\|^2 \leq 2\,(1+t^2)\|d_s u_{-t}(x,y)\|^2 \tag{3.77}$$

Hence, we obtain in view of (3.74) that

$$\|d_s k_\theta(v,w)\|^2 \leq 2\,(1+t^2)\|d_s u_t(v,w)\|^2, \tag{3.78}$$

from which we deduce that $(1+t^2)^{1/2} M_i(d_s u_t \Lambda) \gg M_i(d_s k_\theta \Lambda)$ for any $i = 1, \ldots, 2d$. The claim follows now from (3.45). $\square$

3.4.3 Application to the Lattice Remainder

In this subsection we shall proceed to estimate the error term I_θ by applying the previous results from the Geometry of Numbers combined with Götze's Fourier-approach and Margulis averaging result. First we recall the bound (3.38) and also the relation $\psi(r,t) \ll_d \alpha_d(\Lambda_t)$, obtained in Corollary 3.11, to conclude that

$$I_\theta \ll_d r^{d/2}\, |\det Q|^{-1/4}\, \|\widehat{\zeta}\|_1 \int_{|t| > q_0^{-1/2} r^{-1}} |\widehat{g}_w(t)|\, \alpha_d(\Lambda_t)^{1/2}\, \mathrm{d}t, \tag{3.79}$$

where Λ_t denotes the lattice defined in (3.39) and g_w the smoothed indicator function of $[a,b]$ with $0 < w < (b-a)/4$, see Corollary 3.5. Since Lemma 3.18 provides estimates for $\|\widehat{\zeta}\|_1$ in the case of both admissible and non-admissible regions Ω, it remains to estimate the integral in (3.79). We shall start by bounding this integral over an interval I of length at most $1/q$ and approximating these integrals by the average over the group SO(2). For this, we introduce the maximum value over I of the α_d-characteristic for the lattice Λ_t via

$$\gamma_{I,\beta}(r) \overset{\text{def}}{=} \sup\big\{ \big(r^{-d}\alpha_d(\Lambda_t)\big)^{\frac{1}{2}-\beta} : \ t \in I \big\} \tag{3.80}$$

and the following family of lattices

$$\Lambda_{Q,t} := d_{q^{1/2}}\, u_{4t}\, \Lambda_Q, \tag{3.81}$$

where Λ_Q is as defined in (3.71). Here $\gamma_{I,\beta}(r)$ depends on the Diophantine properties of Q and tends to zero for growing $r \to \infty$ for irrational Q as we will show in Lemma 4.6.

Lemma 3.14. Let $r \geq q^{1/2}$, $0 < \beta \leq 1/2$ and fix an interval $I = [\tau_1, \tau_2]$ of length at most $1/q$. Then we have

$$\int_I \alpha_d(\Lambda_t)^{1/2}\, |\widehat{g}_w(t)|\, \mathrm{d}t \ll_d \widehat{g}_I\, r^{\frac{d}{2}-\beta d}\, \gamma_{I,\beta}(r) \frac{1}{q} \int_{-\pi}^{\pi} \alpha(d_{r_0} k_\theta \Lambda_{Q,4\tau_1})^\beta \frac{\mathrm{d}\theta}{2\pi}, \tag{3.82}$$

where $r_0 := r\, q^{-1/2}$ and $\widehat{g}_I := \max\{|\widehat{g}_w(t)| : t \in I\}$.

Proof: Using the trivial bound $\alpha_d(\Lambda_t) \le r^{d-2\beta d}\gamma_{I,\beta}(r)^2\,\alpha_d(\Lambda_t)^{2\beta}$ and estimating $|\widehat{g}_w|$ by its maximum $\widehat{g}_I$ on I yields

$$\int_I \alpha_d(\Lambda_t)^{1/2}\,|\widehat{g}_w(t)|\,\mathrm{d}t \le \widehat{g}_I\, r^{\frac{d}{2}-d\beta}\gamma_{I,\beta}(r)\int_I \alpha_d(\Lambda_t)^{\beta}\,\mathrm{d}t. \tag{3.83}$$

Since the group D normalizes U, a computation shows that

$$d_r\, u_{4t} = d_r\, u_{4(t-\tau_1)}\, u_{4a} = d_{r_0}\, u_\tau\, d_{q^{1/2}}\, u_{4\tau_1},$$

where $\tau \coloneqq 4(t-\tau_1)\,q$. Changing variables from t to τ we obtain in terms of the lattices $\Lambda_{Q,s}$, defined in (3.81),

$$\int_I \alpha_d(\Lambda_t)^{\beta}\mathrm{d}t = \int_{\tau_1}^{\tau_2} \alpha_d(d_{r_0}\, u_\tau\, d_{q^{1/2}} u_{4\tau_1}\,\Lambda_Q)^{\beta}\mathrm{d}t \ll \frac{1}{q}\int_0^4 \alpha_d(d_{r_0}\, u_\tau\,\Lambda_{Q,4\tau_1})^{\beta}\mathrm{d}\tau. \tag{3.84}$$

Finally, we estimate the last average with the help of Lemma 3.13 by the average over the group $K = \mathrm{SO}(2)$. Changing variables $\theta(s) = \arctan(\tau)$, $\tau \in [0,4]$, and noting that $|\theta| < \pi$ and $\mathrm{d}\tau = (1+\tau^2)\,\mathrm{d}\theta$, we get by (3.73) of Lemma 3.13 that

$$\int_0^4 \alpha_d(d_{r_0}\, u_\tau\,\Lambda_{Q,4\tau_1})^{\beta}\mathrm{d}\tau \ll \int_0^4 \alpha_d(d_{r_0}\, k_{\theta(\tau)}\,\Lambda_{Q,4\tau_1})^{\beta}\,\mathrm{d}\tau \ll \int_{-\pi}^{\pi} \alpha_d(d_{r_0}\, k_\theta\,\Lambda_{Q,4\tau_1})^{\beta}\,\frac{\mathrm{d}\theta}{2\pi}.$$

Now note that $\alpha_d(\Lambda) \le \alpha(\Lambda)$ holds for any lattice Λ in $\mathbb{R}^{2d}$. Thus, the last inequality together with (3.83) and (3.84) completes the proof. □

Finally, Margulis' averaging results will be applied to prove the following corollary and the lemma thereafter, which will the key-tool to obtain non-trivial estimates of I_θ.

Corollary 3.15. Let $r \ge q^{1/2}$, $I = [t_0, t_0+1]$ with $t_0 \in \mathbb{R}$, $0 < \beta \le 1/2$ with $\beta d > 2$ and $\widehat{g}_I \coloneqq \max\{|\widehat{g}_w(t)| \,:\, t \in I\}$. Using the notation (3.80), we have

$$\int_I \alpha_d(\Lambda_t)^{1/2}\,|\widehat{g}_w(t)|\,\mathrm{d}t \ll_{\beta,d} q\,|\det Q|^{-\beta/2}\;\widehat{g}_I\,\gamma_{I,\beta}(r)\,r^{\frac{d}{2}-2}. \tag{3.85}$$

Note that we need at least $d \ge 5$.

Proof: In order to apply Lemma 3.14, we cover I by intervals $I_j = [s_j, s_{j+1}]$ of length at most $1/q$, where $s_j = t_0 + j/q$ with $j \in J \coloneqq \{0, \dots, \lceil q \rceil\}$. This implies

$$\begin{aligned}\int_I \alpha_d(\Lambda_t)^{1/2}\,|\widehat{g}_w(t)|\,\mathrm{d}t &\le r^{\frac{d}{2}-\beta d}\widehat{g}_I\,\gamma_{I,\beta}\frac{1}{q}\sum_{j\in J}\int_{-\pi}^{\pi}\alpha(d_{r_0}\,k_\theta\,\Lambda_{Q,s_j})^{\beta}\,\frac{\mathrm{d}\theta}{2\pi}\\ &\ll r^{\frac{d}{2}-\beta d}\widehat{g}_I\,\gamma_{I,\beta}\max_{j\in J}\int_{-\pi}^{\pi}\alpha(d_{r_0}\,k_\theta\,\Lambda_{Q,s_j})^{\beta}\,\frac{\mathrm{d}\theta}{2\pi}.\end{aligned} \tag{3.86}$$

Now, we shall apply Theorem 6.18 with $h = d_{r_0}$, $r_0 = r/q^{1/2}$ and the lattices $\Lambda_{Q,s_j} = d_{q^{1/2}}\,u_{s_j}\Lambda_Q$, as defined in (3.81), and obtain

$$\max_{j\in J}\int_{-\pi}^{\pi}\alpha_d(d_{r_0}\,k_\theta\,\Lambda_{Q,s_j})^{\beta}\,\frac{d\theta}{2\pi} \ll_{\beta,d} \max_{j\in J}\alpha(\Lambda_{Q,s_j})^{\beta}\|d_{r_0}\|^{\beta d-2} \ll_d r^{\beta d-2}\big(q|Q|^{-\beta/2}\big),$$

where we have used $\|d_{r_0}\| = r_0 = r/q^{1/2}$ and (3.60) in form of

$$\alpha(\Lambda_{Q,s_j}) \ll_d \alpha_d(\Lambda_{Q,s_j}) \ll_d |\det Q|^{-1/2}\,q^{d/2}.$$

Note that we have applied Corollary 3.11 with $r = q^{1/2}$ and $t = s_j$ in order to get $\alpha(\Lambda_{Q,s_j}) \asymp_d \alpha_d(\Lambda_{Q,s_j})$. Finally, in view of (3.86), this concludes the proof of (3.85). □

In order to bound the lattice point remainder for 'wide shells', that is $b-a>q^{1/2}$, we need to extend the averaging result, established in Corollary 3.15, for small values of t_0. To do this, we recall the bound

$$|\widehat{g}_w(t)| \ll \min\{|b-a|\,,|t|^{-1}\}\exp\{-|tw|^{1/2}\} \tag{3.87}$$

for the integrand $\widehat{g}_w(t)$ in (3.82), provided that $0<w<(b-a)/4$. Note that it is of size $b-a$ for $|t|\le 1/(b-a)$ and changes rapidly if $|b-a|>1$ grows with r.

Lemma 3.16. If $r\ge q^{1/2}$, $\beta d>2$ and $0<w<|b-a|/4$, then

$$\int_{q_0^{-1/2}r^{-1}}^{q^{-1/2}} \alpha_d(\Lambda_t)^{1/2}|\widehat{g}_w(t)|\,\mathrm{d}t \ll_{\beta,d} q^{\beta d+1/2}\,|\det Q|^{-\beta/2}\,\gamma_{I,\beta}(r)\,r^{d-2}, \tag{3.88}$$

where $I=[q_0^{-1/2}r^{-1},q^{-1/2}]$.

Proof: Starting as in the proof of Lemma 3.14 and changing variables to $s=t^{-1}$ yields

$$\int_{q_0^{-1/2}r^{-1}}^{q^{-1/2}} \alpha_d(\Lambda_t)^{1/2}|\widehat{g}_w(t)|\,\mathrm{d}t \ll_d \gamma_{I,\beta}(r)\,r^{d/2-\beta d}\int_{q^{1/2}}^{rq_0^{1/2}} \alpha_d(d_r\,u_{4s^{-1}}\Lambda_Q)^\beta\,|\widehat{g}_w(s^{-1})|\,\frac{\mathrm{d}s}{s^2}.$$

Let $N=\lceil r(q_0/q)^{1/2}\rceil$, then the integral on the right-hand side is bounded by $\sum_{j=2}^N I_j$, where

$$I_j \overset{\text{def}}{=} \int_{q^{1/2}(j-1)}^{q^{1/2}j} \alpha_d(d_r u_{4s^{-1}}\Lambda_Q)^\beta\,|\widehat{g}_w(s^{-1})|\,\frac{\mathrm{d}s}{s^2}.$$

For $2\le j\le N$ write $t_j=q^{-1/2}j^{-1}$, then using that

$$d_r u_{4s^{-1}} = d_r\,u_{4(s^{-1}-t_j)}\,u_{4t_j} = d_{4rj^{-1}}\,u_{4^{-1}j^2(s^{-1}-t_j)}\,d_{4^{-1}j}\,u_{4t_j}$$

together with the change of variables $v=4^{-1}j^2(s^{-1}-t_j)$ yields

$$\begin{aligned} I_j &\le \frac{4}{j^2}\int_0^1 \alpha_d(d_{4rj^{-1}}\,u_v\,d_{4^{-1}j}\,u_{4t_j}\Lambda_Q)^\beta\,|\widehat{g}_w(4vj^{-2}+t_j)|\,\mathrm{d}v \\ &\ll_d \frac{q^{1/2}}{j}\int_0^1 \alpha_d(d_{4rj^{-1}}\,u_v\,d_{4^{-1}j}\,u_{4t_j}\Lambda_Q)^\beta\,\mathrm{d}v, \end{aligned}$$

where the last inequality is a consequence of $|\widehat{g}_w(t)|\ll|t|^{-1}$. Hence, since $4rj^{-1}\ge 1$ and $q^{1/2}jt_j=1$, we deduce from Lemma 3.13, Theorem 6.18 and (3.62) of Lemma 3.12 that

$$\begin{aligned} I_j &\ll_d \frac{q^{1/2}}{j}\int_{\mathrm{K}} \alpha_d(d_{4rj^{-1}}\,k\,d_{4^{-1}j}\,u_{4t_j}\Lambda_Q)^\beta\,\mathrm{d}\sigma(k) \\ &\ll_d r^{\beta d-2}\,|\det Q|^{-\beta/2}\,q^{\beta d/2+1/2}j^{1-\beta d}\max\{1,(4q^{1/2}j^{-1})^{\beta d}\}. \end{aligned}$$

Summing the last inequality over $2\le j\le N$, we observe that it suffices to show that the following estimate holds

$$\textstyle\sum_{j=2}^N j^{1-\beta d}\max\{1,(4q^{1/2}j^{-1})^{\beta d}\} \ll_{\beta,d} r^{\beta d-2}|\det Q|^{-\beta/2}q^{\beta d+1/2}.$$

Indeed, split the previous sum according to whether $j\le 4q^{1/2}$ or $j>4q^{1/2}$. The sum over $j>4q^{1/2}$ can be bounded by

$$r^{\beta d-2}|\det Q|^{-\beta/2}q^{\beta d/2+1/2}\textstyle\sum_{j=\lceil 4q^{1/2}\rceil}^N j^{1-\beta d} \ll_{\beta,d} r^{\beta d-2}|\det Q|^{-\beta/2}\,q^{3/2}\,r^{\beta d-2},$$

and the sum over $2\le j\le 4q^{1/2}$ by

$$r^{\beta d-2}|\det Q|^{-\beta/2}q^{\beta d+1/2}\textstyle\sum_{j=2}^{\lfloor 4q^{1/2}\rfloor} j^{1-2\beta d} \ll_{\beta,d} r^{\beta d-2}|\det Q|^{-\beta/2}q^{\beta d+1/2}. \qquad \square$$

3.5 Smoothing of Special Parallelepiped Regions

This section corresponds to the appendix of [GM13] with major changes on the explicit dependency on the parameters depending on the region Ω. In the following Lemma 3.17 we shall bound the volume of ε-boundaries of $r\Omega \cap E_{a,b}$ and in Lemma 3.18 we estimate integrals of the Fourier transform of the region Ω. For wide shells the lattice point counting remainders will reflect the Diophantine properties of Q more directly when using counting regions Ω which are 'admissible' convex polyhedra. Here we confine ourselves to study a specially oriented parallelepiped $\Omega = A^{-1}[-1,1]^d$ with

$$Q_+ \le A^T A \le c_A Q_+ \tag{3.89}$$

for a suitable $A \in \mathrm{GL}(d,\mathbb{R})$ and a positive constant $c_A \ge 1$ depending on A. In this case, the Minkowski functional of Ω is given by $M(x) = \max(\langle g_{i,\pm}, x\rangle \,:\, i = 1,\dots,d)$, where $g_{i,\pm} = \pm A^T e_i$ are $2d$ outward normal vectors of the faces of Ω. Note that the inequalities in (3.89) imply the norm equivalence

$$d^{-1/2}\|Q_+^{1/2}x\| \le M(x) \le (c_A)^{1/2}\|Q_+^{1/2}x\|. \tag{3.90}$$

We now approximate I_Ω by smooth weight functions. For this, introduce

$$\Omega_{\pm\varepsilon} \overset{\text{def}}{=} (1\pm\varepsilon)\Omega, \quad (\partial\Omega)_\varepsilon \overset{\text{def}}{=} \Omega_\varepsilon \setminus \Omega_{-\varepsilon} \quad \text{and} \quad \upsilon_{\pm\varepsilon} \overset{\text{def}}{=} I_{\Omega_{\pm\varepsilon}} * k_{A,\varepsilon}, \tag{3.91}$$

where $k_{A,\varepsilon}(B) = k_\varepsilon(AB)$ for any $B \in \mathcal{B}^n$ and k_ε denotes the rescaled measure on $\mathbb{R}^d$ introduced in the beginning of Subsection 3.3.1. Moreover, we also will need the technical restriction $\varepsilon \in (0,\varepsilon_0)$ with $\varepsilon_0 := 1/100$. Since Lemma 6.1 can be adapted to this situation, taking $\upsilon_{\pm\varepsilon,r}(x) := \upsilon_{\pm\varepsilon}(x/r)$, we get for the lattice point remainder (3.23)

$$\big|R(I_{E_{a,b}} I_{r\Omega})\big| \le \max_{\pm} \big|R(I_{E_{a,b}}\upsilon_{\pm\varepsilon,r})\big| + R_{\varepsilon,r}, \tag{3.92}$$

where, in view of (6.2), the remainder term is given by

$$R_{\varepsilon,r} \overset{\text{def}}{=} \int_{\mathbb{R}^d} I_{(\partial\Omega)_{2\varepsilon}}(x/r)\, I_{[a,b]}(Q[x])\,\mathrm{d}x. \tag{3.93}$$

Using the bound (3.96) from the following Lemma 3.17 yields

$$\big|R(I_{E_{a,b}} I_{r\Omega})\big| \le \max_{\pm} \big|R(I_{E_{a,b}}\upsilon_{\pm\varepsilon,r})\big| + |Q|^{-1/2}(b-a)\,\varepsilon\, r^{d-2}. \tag{3.94}$$

Lemma 3.17. Let $\partial_w[a,b] := [a-2w, a+2w] \cup [b-2w, b+2w]$ for $0 < w < (b-a)/4$. Consider a weight function υ such that the integral in (3.101) exists, resp. (3.102) is bounded. Then

$$\int I_{\partial_w[a,b]}(Q[x])\,\upsilon(x/r)\,\mathrm{d}x \ll_d w\,\|\upsilon\|_Q\, r^{d-2}, \tag{3.95}$$

where $\|\upsilon\|_Q$ is defined in (3.101), resp. (3.102). Assuming additionally $|a| + |b| < c_0 r^2$ with $c_0 = (c_A)^{-1}/5$, following estimates hold for indefinite forms Q.

$$R_{\varepsilon,r} \ll_d |Q|^{-1/2}(b-a)\,\varepsilon\, r^{d-2} \tag{3.96}$$

$$\operatorname{vol} H_r \gg_d |Q|^{-1/2}(\sqrt{c_A})^{-(d-2)}(b-a)\, r^{d-2} \tag{3.97}$$

Moreover, for the special choice $\upsilon = \upsilon_{\pm\varepsilon}$, as defined in (3.91), we have

$$\|\upsilon_{\pm\varepsilon}\|_Q \ll_d |Q|^{-1/2}, \tag{3.98}$$

whereby the condition $|a| + |b| < c_0 r^2$ can be dropped if Q is positive definite.

The lower bound (3.97) can be also found in [BG99], see Lemma 8.2. Moreover, Lemma 3.8 in [EMM98] provides an asymptotic formula for the volume of H_r.

Proof: For a bounded measurable function g on $\mathbb{R}$ with compact support we introduce

$$R_g \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} g(Q[x])\,\upsilon(x/r)\,\mathrm{d}x.$$

Let $S_Q = QQ_+^{-1}, L_Q = Q_+^{1/2}$ and let U denote a rotation in $\mathbb{R}^d$ such that UQU^{-1} and hence UL_QU^{-1} are diagonal. Write $\upsilon_Q(y) := \upsilon(L_Q^{-1}U^{-1}y)$ with integrable weight function υ. Changing variables via $x = rL_Q^{-1}U^{-1}y$ in $\mathbb{R}^d$ with $y \in \mathbb{R}^p \times \mathbb{R}^q$, $d = p + q$ and using polar coordinates, $y = (r_1\eta_1, r_2\eta_2)$, where $r_1, r_2 > 0$ and $\eta_1 \in S^{p-1}$, $\eta_2 \in S^{q-1}$, that is $\|\eta_1\| = \|\eta_2\| = 1$, we may write $Q[x] = r^2(r_1^2 - r_2^2)$. Thus we obtain by Fubini's theorem

$$R_g = r^d|Q|^{-1/2}\int_0^\infty\int_0^\infty r_1^{p-1}r_2^{q-1}g(r^2(r_1^2 - r_2^2))\,\varphi_\upsilon(r_1, r_2)\,\mathrm{d}r_1\,\mathrm{d}r_2, \tag{3.99}$$

where

$$\varphi_\upsilon(r_1, r_2) \stackrel{\text{def}}{=} \int_{S^{p-1}\times S^{q-1}} \upsilon_Q(r_1\eta_1, r_2\eta_2)\,\mathrm{d}\sigma(\eta_1)\,\mathrm{d}\sigma(\eta_2).$$

Note that in the case of positive definite forms Q (i.e. $p = 0$ or $q = 0$), the double integral (3.99) must be replaced by a single one. Next, we change variables via $v := r_1^2 - r_2^2$ and $u := r_1$, so that $r_1^2 + r_2^2 = 2u^2 - v$ and $r_2 = \sqrt{u^2 - v}$. Thus, we get

$$R_g = r^d\frac{|Q|^{-1/2}}{2}\int_{\mathbb{R}} g(r^2v)\int_0^\infty I(u^2 \ge v)\,u^{p-1}\varphi_\upsilon(u, \sqrt{u^2 - v})(u^2 - v)^{(q-2)/2}\mathrm{d}u\,\mathrm{d}v. \tag{3.100}$$

In order to prove (3.95), we choose $g = I_{\partial_w[a,b]}$ in (3.100). Since the length of $r^{-2}\operatorname{supp} g$ is at most $\ll |w|\,r^{-2}$, we get $R_g \ll_d |w|\,r^{d-2}\|\upsilon\|_Q$, where

$$\|\upsilon\|_Q \stackrel{\text{def}}{=} |Q|^{-1/2}\sup_{v\in r^{-2}\partial_w[a,b]}\left|\int_0^\infty I(u^2 \ge v)\,u^{p-1}\varphi_\upsilon(u, \sqrt{u^2 - v})(u^2 - v)^{(q-2)/2}\mathrm{d}u\right|. \tag{3.101}$$

If Q is positive definite, $\|\upsilon\|_Q$ has to be replaced by

$$\|\upsilon\|_Q \stackrel{\text{def}}{=} |Q|^{-1/2}\sup_{v\in r^{-2}\partial_w[a,b]}\left|v^{d-1}\varphi_\upsilon(v)\right|. \tag{3.102}$$

Next we prove (3.97): Taking $g = I_{[a,b]}$, $\upsilon(x) = I_\Omega(x) = I(M(x) \le 1)$ and using

$$\|y\|d^{-1/2} \le M(L_Q^{-1}U^{-1}y) \le \|y\|(c_A)^{1/2} \tag{3.103}$$

gives the lower bound

$$\begin{aligned}\varphi_\upsilon(r_1, r_2) &\ge \int_{S^{p-1}\times S^{q-1}} I(\|(r_1\eta_1, r_2\eta_2)\| \le (c_A)^{-1/2})\,\mathrm{d}\sigma(\eta_1)\,\mathrm{d}\sigma(\eta_2)\\ &\gg_d I(2u^2 + |v| \le (c_A)^{-1}).\end{aligned}$$

Thus, we find

$$\begin{aligned}\operatorname{vol} H_r &\gg_d r^d|Q|^{-1/2}\int_{r^{-2}a}^{r^{-2}b}\int_0^\infty I(u^2 \ge v)\,I(2u^2 + |v| \le (c_A)^{-1})\,u^{p-1}(u^2 - v)^{(q-2)/2}\,\mathrm{d}u\,\mathrm{d}v\\ &\gg_d r^d|Q|^{-1/2}\int_{r^{-2}a}^{r^{-2}b} I(|v| \le c_0)\int_0^\infty I(\tfrac{5}{4}c_0 \le u^2 \le 2c_0)u^{p-1}(u^2 - v)^{(q-2)/2}\mathrm{d}u\,\mathrm{d}v\\ &\gg_d r^{d-2}(b - a)|Q|^{-1/2}(\sqrt{c_0})^{d-2}.\end{aligned}$$

Proof of (3.96). In (3.100) we choose $g = I_{[a,b]}$ and $v = I_{(\partial\Omega)_{2\varepsilon}}$ with $0 < \varepsilon < \varepsilon_0$. By the properties of the polyhedron Ω, see (3.90), we have $I_{(\partial\Omega)_{2\varepsilon}}(x) \le I(M(x) \in J_{1,2\varepsilon})$, where $J_{1,2\varepsilon} \coloneqq [1-2\varepsilon, 1+2\varepsilon]$. Let $g_1, \dots, g_{2d}$ denote the $2d$-tuple of normal vectors defining Ω and let $f_m = UL_Q^{-1} g_m$, $m = 1, \dots, 2d$, be the transformed vectors. Since

$$I(M(L_Q^{-1}U^{-1}y) \in J_{1,2\varepsilon}) \le \textstyle\sum_{m=1}^{2d} I(\langle y, f_m\rangle \in J_{1,2\varepsilon})$$

we may bound $\varphi_v(r_1, r_2)$ in (3.100) as follows

$$\varphi(r_1, r_2) \le \textstyle\sum_{m=1}^{2d} \varphi_m(r_1, r_2),$$

where

$$\varphi_m(r_1, r_2) \overset{\text{def}}{=} \int_{S^{p-1}\times S^{q-1}} I\big[\langle (r_1\eta_1, r_2\eta_2), f_m\rangle \in J_{1,2\varepsilon}\big]\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2.$$

Recall $|v| \le c_0$, $v = r_1^2 - r_2^2$, $u = r_1$ and $r_2 = \sqrt{u^2 - v}$. The inequality (3.103) implies

$$(1+2\varepsilon)^2 d \ge r_1^2 + r_2^2 = 2u^2 - v \ge (1-2\varepsilon)^2 (c_A)^{-1}.$$

Therefore $\varphi_v(u, \sqrt{u^2 - v}) = 0$ if

$$0 \le u \le 2^{-\frac{1}{2}} \sqrt{5c_0(1-2\varepsilon)^2 - c_0} \quad \text{or} \quad u > C_\Omega \overset{\text{def}}{=} \frac{(1+2\varepsilon)}{\sqrt{2}} \sqrt{d + c_0}.$$

Because of

$$2^{-\frac{1}{2}} \sqrt{5c_0(1-2\varepsilon)^2 - c_0} > c_\Omega \overset{\text{def}}{=} \sqrt{\frac{19c_0}{10}}$$

and $u^2 - v \ge 9c_0/10$, we get

$$\begin{aligned} R_g &\ll r^d |Q|^{-1/2} \int_{r^{-2}a}^{r^{-2}b} \left(\int_{c_\Omega}^{C_\Omega} u^{p-1}(u^2 - v)^{\frac{q-2}{2}} \varphi(u, \sqrt{u^2 - v})\, \mathrm{d}u \right) \mathrm{d}v \\ &\le r^d |Q|^{-1/2} \sum_{m=1}^{2d} \int_{r^{-2}a}^{r^{-2}b} \left(\int_{c_\Omega}^{C_\Omega} u^{p-1}(u^2 - v)^{\frac{q-2}{2}} \varphi_m(u, \sqrt{u^2 - v})\, \mathrm{d}u \right) \mathrm{d}v. \end{aligned} \tag{3.104}$$

By interchanging the variables r_1 and r_2 we can suppose that $q \ge 2$. Thus, since $u \ll_d 1$ and $\sqrt{u^2 - v} \ll_d 1$, we see that

$$\int_{c_\Omega}^{C_\Omega} u^{p-1}(u^2 - v)^{\frac{q-2}{2}} \varphi_m(u, \sqrt{u^2 - v})\, \mathrm{d}u \ll_d \int_{c_\Omega}^{C_\Omega} \varphi_m(u, \sqrt{u^2 - v})\, \mathrm{d}u. \tag{3.105}$$

We claim that

$$R_g \ll_d |Q|^{-1/2} \varepsilon (b - a) r^{d-2} \tag{3.106}$$

holds. In view of (3.104) and (3.105), the estimates

$$R_m \overset{\text{def}}{=} \int_{c_\Omega}^{C_\Omega} \varphi_m(u, \sqrt{u^2 - v})\, \mathrm{d}u \ll_d \varepsilon c_\Omega$$

for all $m = 1, \dots, 2d$ will prove the bound (3.106).

Thus let $F_m(u) \coloneqq \langle (u\eta_1, (u^2 - v)^{1/2}\eta_2), f_m\rangle$ for fixed $|v| \le c_0$ and (η_1, η_2). If

$$\Big|\frac{\partial}{\partial u} F_m(u)\Big| \ge c_1 > 0 \tag{3.107}$$

for all $c_\Omega \le u \le C_\Omega$ with $F_m(u) \in [1-2\varepsilon, 1+2\varepsilon]$ uniformly in (η_1, η_2) and v, then

$$\int_{c_\Omega}^{C_\Omega} I(F_m(u) \in [1-2\varepsilon, 1+2\varepsilon])\, \mathrm{d}u \ll \frac{\varepsilon}{c_1}$$

and hence $R_m \ll_d c_1^{-1}\varepsilon$ for all $m = 1, \dots, 2d$. Note that

$$\frac{\partial}{\partial u} F_m(u) = \frac{1}{u}\left(F_m(u) + \frac{v}{\sqrt{u^2 - v}} \langle (0, \eta_2), f_m \rangle \right)$$

and because of $\|L_Q^{-1}A^T\| = \|A L_Q^{-1}\| \le \sqrt{c_A}$ we see that

$$\Big| \frac{\partial}{\partial u} F_m(u) \Big| \ge \frac{1}{u}\Big(|F_m(u)| - \frac{c_0}{\sqrt{9c_0/10}} \|f_m\| \Big) \ge \frac{1}{u}\Big(1 - 2\varepsilon - \frac{1}{2} \Big) \gg c_\Omega^{-1}.$$

Thus, (3.107) holds and the assertion (3.106) is proved. This yields the claimed bound for $R_{\varepsilon,r}$, compare (3.93). Finally, we prove (3.98). Here we have $v = v_{\pm\varepsilon}$ and $v_{\pm\varepsilon}(x) \le I(M(x) \le 1 + 2\epsilon)$. In view of (3.103), we find that the u-integral in (3.101) can be restricted to $2u^2 \le 2d + v$. Hence

$$\|v_{\pm\varepsilon}\|_Q \ll_d |Q|^{-1/2} \sup_{v \in r^{-2}\partial_w[a,b]} (1 + |v|)^{(d-3)/2} \int_0^\infty I(v \le u^2 \le d + v/2)\, \mathrm{d}u \ll |Q|^{-1/2},$$

because $|v| \le r^{-2}(|a| + |b|) \le c_0 \le 1$. Since φ_v is supported in $\|\cdot\|$-ball of radius $2d^{1/2}$, we get also in the case of positive definite forms that (3.102) is bounded by $\ll_d |Q|^{-1/2}$. □

3.5.1 Fourier Transform of Weights for Polyhedra

Here we continue to estimate the remainder terms in (3.94). Since the bounds for $R(g_w^Q v_{-\varepsilon,r})$ are exactly the same as for $R(g_w^Q v_{+\varepsilon,r})$ we shall consider the latter only. We shall now modify the weight v_ε, defined in (3.91), as follows. Define $\varphi = I_{[-2,2]} * k$, where k is again the probability measure from Subsection 3.3.1. Of course, φ is smooth and $\varphi(u) = 1$ if $\|u\|_\infty \le 1$ and $\varphi(u) = 0$ if $\|u\|_\infty \ge 3$. Let $s_d := d(1 + 2\varepsilon_0)^2$. Now, by construction $\varphi(Q_+[x]s_d^{-1})$ is identical to 1 on the support of the ε-smoothed indicator of $\Omega_\varepsilon = A^{-1}[-(1+\varepsilon), (1+\varepsilon)]^d$, that is $v_\varepsilon(x)$. Hence we may rewrite the weights ζ of (3.22) via

$$\zeta_\varepsilon(x) = v_\varepsilon(x) \exp\{Q_+[x]\} = v_\varepsilon(x)\psi(x) \tag{3.108}$$

using the C^∞ function of bounded support $\psi(x) := \exp\{Q_+[x]\}\varphi(Q_+[x]s_d^{-1})$, whose Fourier transform is easy to handle.

Lemma 3.18. The following estimate holds

$$\int_{\mathbb{R}^d} |\widehat{\zeta_\varepsilon}(v)|\, \mathrm{d}v \ll_d \int |\widehat{I}_{[-1,1]^d}|(v) \textstyle\prod_{j=1}^d \exp\{-|\varepsilon v_j|^{1/2}\}\, \mathrm{d}v \ll_d (\log \varepsilon^{-1})^d. \tag{3.109}$$

Remark 3.19. In the general case, when Ω has finite Minkowski surface measure c_Ω only, defined via $\mathrm{meas}(\partial_\varepsilon \Omega) \le c_\Omega \varepsilon$, we have

$$\|\widehat{I_\Omega}\|_{1,\varepsilon} \overset{\text{def}}{=} \int_{\mathbb{R}^d} |\widehat{I_\Omega}(v)| \exp\{-\|\varepsilon v\|^{1/2}\}\, \mathrm{d}v \ll_\Omega \varepsilon^{-(d+1)/2}.$$

These bounds are best possible. For the case of general Ω, we may use the bound in Theorem 2.9 of [BCT97] for the average $\eta \mapsto |\widehat{I_\Omega}(s\eta)|$ over the unit sphere $\mathbb{S}^{d-1}$ for polyhedra. That paper contains examples showing that these averages are sharp. In our setting, i.e. in the case of specially oriented parallelepipeds Ω, more elementary arguments can be used.

Proof: Note that by definition

$$\int_{\mathbb{R}^d} |\widehat{\zeta_\varepsilon}(v)|\,\mathrm{d}v = \int_{\mathbb{R}^d} |\widehat{\upsilon_\varepsilon \psi}(v)|\,\mathrm{d}v = \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \widehat{\upsilon_\varepsilon}(v-x)\widehat{\psi}(x)\,\mathrm{d}x \Big| \mathrm{d}v \le \|\widehat{\upsilon_\varepsilon}\|_1 \|\widehat{\psi}\|_1. \quad (3.110)$$

Since

$$\widehat{\psi}(x) = |\det Q|^{-1/2} \int_{\mathbb{R}^d} \exp[v^2]\varphi(v^2 s_d^{-1})\,\mathrm{e}^{-2\pi\mathrm{i}\langle v, Q_+^{-1/2}x\rangle}\,\mathrm{d}v$$

we easily conclude that

$$|\widehat{\psi}(x)| \le |\det Q|^{-1/2}\, c(d,k)(1+Q_+^{-1}[x])^{-k},\ x\in\mathbb{R}^d, \quad \text{and thus} \quad \|\widehat{\psi}\|_1 \le c(d). \quad (3.111)$$

Defining $B \coloneqq (A^{-1})^T$ and changing variables shows also that

$$\widehat{I}_{\Omega_\varepsilon}(v) = (1+\varepsilon)^d \widehat{I}_{\Omega}((1+\varepsilon)v) = (1+\varepsilon)^d\,|\det A|^{-1}\,\widehat{I}_{[-1,1]^d}((1+\varepsilon)Bv) \quad (3.112)$$

and

$$|\widehat{k}_{A,\varepsilon}(v)| \le \exp\{-\varepsilon^{1/2}\textstyle\sum_{j=1}^d |(Bv)_j|^{1/2}\}. \quad (3.113)$$

Thus we get for $\upsilon_\varepsilon = I_{\Omega_\varepsilon} * k_{A,\varepsilon}$

$$\|\widehat{\upsilon_\varepsilon}\|_1 = \|\widehat{I}_{\Omega_\varepsilon}\widehat{k}_{A,\varepsilon}\|_1 \ll_d \int_{\mathbb{R}^d} |\widehat{I}_{[-1,1]^d}((1+\varepsilon)v)|\textstyle\prod_{j=1}^d \exp\{-|\varepsilon v_j|^{1/2}\}\,\mathrm{d}v. \quad (3.114)$$

Finally, using $\widehat{I}_{[-1,1]^d}(v) = \prod_{j=1}^d \sin(2\pi v_j)/(\pi v_j)$ together with (3.114) gives the estimate

$$\|\widehat{\upsilon_\varepsilon}\|_1 \ll_d \Big(\int_0^\infty \frac{1}{u+\varepsilon}\mathrm{e}^{-\sqrt{u}}\,\mathrm{d}u\Big)^d \ll_d \Big(1+\int_0^1 \frac{1}{u+\varepsilon}\,\mathrm{d}u\Big)^d \ll_d \log(\varepsilon^{-1})^d. \quad (3.115)$$

We now obtain the estimate (3.109) from (3.110) combined with (3.111) and (3.115). □

3.5.2 Lattice Point Remainders for Admissible Parallelepipeds

Now we restrict the parallelepiped $\Omega = A^{-1}[-1,1]^d$, as defined in (3.89), such that its faces are in a general position relative to the standard lattice $\mathbb{Z}^d$. This ensures that the lattice point remainder for $r\Omega$ is of 'abnormally' small error *uniformly* in r. To construct it, we may alternatively construct lattices $A\mathbb{Z}^d$ such that the faces of $[-1,1]^d$ have this property. Following Skriganov [Skr94], we call a lattice $\Gamma \subset \mathbb{R}^d$ of full rank, and likewise Ω, 'admissible' if

$$\operatorname{Nm}\Gamma \overset{\text{def}}{=} \inf_{\gamma\in\Gamma\setminus\{0\}} |\operatorname{Nm}\gamma| > 0, \quad (3.116)$$

where $\operatorname{Nm}\gamma = |\gamma_1\cdots\gamma_d|$ in standard coordinates $\gamma = (\gamma_1,\dots,\gamma_d)$.

Remark 3.20. This definition is a special case of 'admissible lattices' for *star-bodies*, see Chapter IV.4 in [Cas97]. Here, the star-body is given by $\{F<1\}$ with the *distance function*

$$F(x) = |x_1\cdots x_d|^{1/d}.$$

As shown in Lemma 3.1 of [Skr94], the dual lattice $\Gamma^* = B\mathbb{Z}^d$ of Γ, where $B^TA = \mathrm{Id}$, is admissible as well. Another property of admissible lattices is that there exists a cube $[-r_0,r_0]^d$ containing a fundamental domain F of Γ such that $r_0>0$ depends only by means of the invariants $\det\Gamma$ and $\operatorname{Nm}\Gamma$.

Example 3.21. Well known examples are provided by the *Minkowski embedding* of a *totally real* algebraic number field $\mathbb{F}$ of degree d into $\mathbb{R}^d$. Given all embeddings $\sigma_1, \ldots, \sigma_d$ of $\mathbb{F}$, the Minkowski embedding $\sigma \colon \mathbb{F} \to \mathbb{R}^d$ is defined by $\sigma = (\sigma_1, \ldots, \sigma_d)$. In this case $\operatorname{Nm} \sigma(\alpha) = \left|N_{\mathbb{F}/\mathbb{Q}}(\alpha)\right|$ is the field norm of any $\alpha \in \mathbb{F}$, where we interpret multiplication by α as a $\mathbb{Q}$-linear map. Thus, the image of the ring of integers $\mathcal{O}_{\mathbb{F}}$ is an admissible lattice Γ with $\operatorname{Nm} \Gamma \geq 1$. For more information, see Chapter 2.3 in [BS66].

We also note that for any natural number $n \in \mathbb{N}$ we may choose a real number field of degree n which is normal over the rational numbers. In fact, let $m \in \mathbb{N}$ be chosen such that $2n \mid \varphi(m)$ and let ξ_m be a primitive m-th root of unity. Then $\mathbb{Q}(\xi_m + \xi_m^{-1})$ is a real number field of degree $\varphi(m)/2$, which is also normal and its Galois group G is abelian. Since G contains a subgroup H of order $\varphi(m)/(2n)$, the fixed field of H is real, normal and of degree n. Thus, there exists an admissible region Ω satisfying (3.89) with $c_A \asymp_d q/q_0$ and $\operatorname{Nm}(A) \asymp_d q^{d/2}$.

Lemma 3.22. Assume that the lattice $\Gamma = A\mathbb{Z}^d$ is admissible and A satisfies (3.89). For $0 < \varepsilon < \varepsilon_0$ and $r \geq 1$ we get for the parallelepiped $\Omega = A^{-1}[-1,1]^d$ and the corresponding weights $\zeta_\varepsilon(x) = \upsilon_\varepsilon(x)\psi(x)$ introduced in Subsection 3.5.1

$$I_\zeta \overset{\text{def}}{=} \int_{\|v\|_\infty > r/2} \frac{|\widehat{\zeta_\varepsilon}(v)|}{(q^{1/2}r^{-1} + \|r^{-1}v\|_{\mathbb{Z}^d})^{d/2}} \,\mathrm{d}v \ll_d q_0^{-d/4} |Q|^{-1/2} \, |\det A| \, \lambda_{r,\varepsilon}^{d-1} \frac{\bar{\lambda}_{r,\varepsilon,\Gamma}}{\operatorname{Nm}(\Gamma)}, \quad (3.117)$$

where $\lambda_{r,\varepsilon} := \min\{\log(r+1), \log(\varepsilon^{-1})\}$ and $\bar{\lambda}_{r,\varepsilon,\Gamma} := \max\{\lambda_{r,\varepsilon}, \log(2 + \frac{1}{\operatorname{Nm}(\Gamma) r\varepsilon})\}$. For any inadmissible parallelepiped Ω only the estimate

$$I_\zeta \ll_d |Q|^{-1/2} q^{d/2} c_A^{(d+1)/2} \varepsilon^{-d} \quad (3.118)$$

holds. Additionally, we also have $|Q|^{-1/2} \, |\det A| \leq (c_A)^{d/2}$.

Proof: We start by making the change of variables $w = r^{-1}Bv$ in (3.117) and then splitting I_ζ into integrals over cells $C^* := B[-\frac{1}{2}, \frac{1}{2})^d$, where $\Gamma^* := B\mathbb{Z}^d$ denotes the dual lattice to Γ, that is $B = (A^T)^{-1}$, in order to get

$$I_\zeta = \sum_{\gamma^* \in \Gamma^* \setminus \{0\}} I_\zeta(\gamma^*), \quad \text{where} \quad I_\zeta(m) \overset{\text{def}}{=} r^d \, |\det A| \int_{C^*} \frac{|\widehat{\zeta_\varepsilon}(B^{-1} r(\gamma^* + v))|}{(q^{\frac{1}{2}} r^{-1} + \|B^{-1}v\|_\infty)^{\frac{d}{2}}} \,\mathrm{d}v. \quad (3.119)$$

Note that Γ^* satisfies $\|B\| \leq \|Q_+^{-1/2}\| \leq q_0^{-1/2}$, since the first inequality in (3.89) implies

$$1 \geq \|Q_+^{1/2} A^{-1}\| = \|((A^T)^{-1} Q_+^{1/2})^T\| = \|(A^T)^{-1} Q_+^{1/2}\| = \|B Q_+^{1/2}\|. \quad (3.120)$$

In particular, the fundamental domain C^* is contained in $q_0^{-1/2}\sqrt{d}[-\frac{1}{2}, \frac{1}{2}]^d$. Next, we shall bound the Fourier transform of ζ_ε. Recall that by definition

$$\widehat{\zeta_\varepsilon}(u) = ((\widehat{I_{\Omega_\varepsilon}} \cdot \widehat{k}_{A,\varepsilon}) * \widehat{\psi})(u). \quad (3.121)$$

As verified in (3.112), we have in coordinates $u = (u_1, \ldots, u_d)$

$$|\widehat{I}_{\Omega_\varepsilon}(B^{-1}u)| \ll_d |\det A|^{-1} \prod_{j=1}^{d} \left| \frac{\sin[2\pi(1+\varepsilon)u_j]}{(1+\varepsilon)u_j} \right| \ll_d |\det A|^{-1} \prod_{j=1}^{d} (1 + |u_j|)^{-1}. \quad (3.122)$$

Since (3.120) also implies $\|Q_+^{-1/2}(B^{-1}u)\| \geq \|u\|$, we can rewrite (3.111) by

$$|\psi(B^{-1}u)| \ll_{d,k} |\det Q|^{-1/2} (1+\|u\|^2)^{-k} \ll_{d,k} |\det Q|^{-1/2} \prod_{j=1}^d (1+u_j^2)^{-k/d}, \tag{3.123}$$

where we applied the AM-GM inequality. In view of (3.113) we have the bound

$$|\widehat{k}_{A,\varepsilon}(B^{-1}u)| \leq \exp\{-\textstyle\sum_{j=1}^d |\varepsilon u_j|^{1/2}\} \tag{3.124}$$

as well. Combining these estimates yields

$$|\widehat{\zeta}_\varepsilon(B^{-1}rw)| \ll_{d,k} |Q|^{-1/2} \int_{\mathbb{R}^d} \prod_{j=1}^d \frac{1}{(1+u_j^2)^{k/d}} \frac{\exp\{-\varepsilon^{1/2} |rw_j - u_j|^{1/2}\}}{1+|rw_j - u_j|} \,\mathrm{d}u.$$

Thus, we get for a fixed lattice point $\gamma^* = (\gamma_1^*, \ldots, \gamma_d^*) \in \Gamma^*$

$$I_\zeta(\gamma^*) \ll_{d,k} \int_{C^*} \frac{|\det Q|^{-1/2} \, |\det A|}{(qr^{-1} + \|B^{-1}v\|_\infty)^{d/2}} \int_{\mathbb{R}^d} \prod_{j=1}^d \bar{\omega}(u_j) \frac{\omega(\varepsilon r(\gamma_j^* + v_j - \frac{u_j}{r}))}{r^{-1} + \left|\gamma_j^* + v_j - \frac{u_j}{r}\right|} \,\mathrm{d}u \,\mathrm{d}v,$$

where $\bar{\omega}(x) \coloneqq (1+x^2)^{-k/d}$ and $\omega(x) \coloneqq \exp\{-|x|^{1/2}\}$. We now estimate the last double integral coordinatewise: Note that we have $|v_i| \leq \bar{v} \coloneqq \sqrt{d}/2$ and

$$(q^{1/2}r^{-1} + \|B^{-1}v\|_\infty)^{d/2} \gg_d q_0^{d/4}(r^{-1} + \|v\|_\infty)^{d/2} \geq q_0^{d/4} \textstyle\prod_{j=1}^d (r^{-1} + |v_i|)^{1/2},$$

since $\|B^{-1}v\|_\infty \gg_d \|B\|^{-1}\|v\|_\infty \geq q_0^{1/2}\|v\|_\infty$. Hence, we find

$$I_\zeta(\gamma^*) \ll_{d,k} q_0^{-d/4} |Q|^{-1/2} \, |\det A| \textstyle\prod_{j=1}^d J_\zeta(\gamma_j^*; \mathbb{R}),$$

where

$$J_\zeta(\gamma_j^*; D) \stackrel{\text{def}}{=} \int_{-\bar{v}}^{\bar{v}} \frac{1}{(r^{-1} + |v|)^{1/2}} \int_D \bar{\omega}(u) \frac{\omega(\varepsilon r(\gamma_j^* + v - \frac{u}{r}))}{r^{-1} + \left|\gamma_j^* + v - \frac{u}{r}\right|} \,\mathrm{d}u \,\mathrm{d}v.$$

In order to estimate $J_\zeta(\gamma_j^*; \mathbb{R})$, we decompose the integral into parts corresponding to the extremal points of the integrands. Defining $D_j \coloneqq \{|u| \geq r \left|\gamma_j^* + v\right|/2\}$, we get

$$J_\zeta(\gamma_j^*; D_j) \leq \int_{-\bar{v}}^{\bar{v}} \frac{r}{|v|^{1/2}} \int_{D_j} \bar{\omega}(u) \,\mathrm{d}u \,\mathrm{d}v \ll_{k,d} \int_{-\bar{v}}^{\bar{v}} \frac{1}{|v|^{1/2}} \frac{r}{(1 + r\left|\gamma_j^* + v\right|)^{\frac{k}{d}-1}} \,\mathrm{d}v.$$

In the case $\left|\gamma_j^*\right| \geq \sqrt{d}$, we have $\left|\gamma_j^* + v\right| \geq \left|\gamma_j^*\right|/2$ and hence

$$J_\zeta(\gamma_j^*; D_j) \ll_d \frac{r}{(1+\left|r\gamma_j^*\right|)^{d+2}} \int_{-\bar{v}}^{\bar{v}} \frac{1}{|v|^{1/2}} \,\mathrm{d}v \ll_d \frac{1}{(1+\left|r\gamma_j^*\right|)^{d+1}}$$

if we take $k = d(d+3)$. In the other case $\left|\gamma_j^*\right| < \sqrt{d}/2$, we split the v-integral into two parts as follows in order to find the estimate

$$\begin{aligned} J_\zeta(\gamma_j^*; D_j) &\ll_d \int_{-\bar{v}}^{\bar{v}} \frac{\left|\gamma_j^*\right|^{-\frac{1}{2}} r\, I(|v| \geq \left|\gamma_j^*\right|/2)}{(1+r\left|\gamma_j^* + v\right|)^{d+2}} \,\mathrm{d}v + \int_0^{\left|\gamma_j^*\right|/2} \frac{r}{v^{\frac{1}{2}}(1+r(\left|\gamma_j^*\right| - v))^{d+2}} \,\mathrm{d}v \\ &\ll_d \left|\gamma_j^*\right|^{-\frac{1}{2}} + \frac{\left|\gamma_j^*\right|^{\frac{1}{2}} r}{(r\left|\gamma_j^*\right| + 1)^{d+2}} \int_0^{1/2} \frac{1}{v^{\frac{1}{2}}(1-v)^{d+2}} \,\mathrm{d}v \ll_d \left|\gamma_j^*\right|^{-\frac{1}{2}}. \end{aligned}$$

In the complement $u \in D_j^c$ we have $\left|\gamma_j^* + v - \frac{u}{r}\right| \geq \left|\gamma_j^* + v\right|/2$ and thus

$$J_\zeta(\gamma_j^*; D_j^c) \ll_d \int_{-\bar{v}}^{\bar{v}} |v|^{-\frac{1}{2}} \frac{\omega(\varepsilon r(\gamma_j^* + v)/2)}{r^{-1} + \left|\gamma_j^* + v\right|} \, \mathrm{d}v.$$

If $\left|\gamma_j^*\right| \geq \sqrt{d}$, then we easily conclude that $J_\zeta(\gamma_j^*; D_j^c) \ll_d \omega(\varepsilon r \gamma_j^*/4) \left|\gamma_j^*\right|^{-1}$. At last, we consider the case $\left|\gamma_j^*\right| < \sqrt{d}$. The v-integral over the region $\{\bar{v} \geq |v| \geq \left|\gamma_j^*\right|/2\}$ can be bounded by

$$\begin{aligned}
&\ll_d \left|\gamma_j^*\right|^{-1/2} \int_{-\bar{v}}^{\bar{v}} \frac{I(|v| \geq \left|\gamma_j^*\right|/2)}{(r^{-1} + \left|\gamma_j^* + v\right|)(1 + \varepsilon r \left|\gamma_j^* + v\right|)} \, \mathrm{d}v \\
&\ll_d \left|\gamma_j^*\right|^{-1/2} \int_0^{3\sqrt{d}/2} \frac{1}{r^{-1} + v} \frac{1}{1 + \epsilon r v} \, \mathrm{d}v \ll_d \left|\gamma_j^*\right|^{-1/2} \min\{\log(\varepsilon^{-1}), \log(r+1)\}
\end{aligned}$$

and similar over the complement by

$$\ll_d \int_0^{\left|\gamma_j^*\right|/2} \frac{v^{-1/2}}{r^{-1} + \left|\gamma_j^*\right| - v} \, \mathrm{d}v \ll_d \left|\gamma_j^*\right|^{-1/2}.$$

Hence we conclude that

$$I_\zeta \ll_d q_0^{-d/4} |Q|^{-1/2} \, |\det A| \sum_{(\gamma_1^*, \dots, \gamma_d^*) \in \Gamma^* \setminus \{0\}} \prod_{j=1}^d \frac{H_{r,\varepsilon}(\gamma_j^*)}{\left|\gamma_j^*\right|}, \tag{3.125}$$

where

$$H_{r,\varepsilon}(x) := \lambda_{r,\varepsilon} \, |x|^{1/2} \, I(|x| < \sqrt{d}) + (1 + \varepsilon r \, |x|)^{-d} I(|x| \geq \sqrt{d}). \tag{3.126}$$

In view of the following Lemma 3.23 this concludes the proof of the bound (3.117).

If the region Ω is not admissible, then we change variables to $w = r^{-1}v$ and split the left-hand side of (3.117) into integrals over unit cells $E := [-\frac{1}{2}, \frac{1}{2})^d$ in order to find

$$I_\zeta = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} I_\zeta(m), \quad \text{where} \quad I_\zeta(m) \stackrel{\text{def}}{=} r^d \int_E \frac{|\widehat{\zeta}_\varepsilon(r(m + w)|}{(q^{1/2} r^{-1} + \|w\|_\infty)^{d/2}} \, \mathrm{d}w.$$

Because of $\sum_{j=1}^d |u_j|^{1/2} \geq \|u\|^{1/2}$ we can further estimate (3.124) by

$$\widehat{k}_{A,\varepsilon}(B^{-1}u) \leq \exp\{-\|\varepsilon u\|^{1/2}\}.$$

Recalling the definition (3.121) and the estimates (3.122)–(3.123) for $u = Bw$ shows that

$$\widehat{\zeta}_\varepsilon(rw) \ll_k |Q|^{-1/2} \varepsilon^{-k+1} (r\|Bw\| + 1)^{-k} \ll |Q|^{-1/2} \varepsilon^{-k+1} (q c_A)^{k/2} (r\|w\| + 1)^{-k}.$$

Thus, taking $k = d + 1$ we find

$$I_\zeta \ll_d |Q|^{-1/2} q^{d/2} c_A^{(d+1)/2} \varepsilon^{-d}.$$

The last remark easily follows by comparing the volume of the bodies $\{\|Ax\| \leq 1\}$ and $\{\|Q_+^{1/2} x\| \leq 1\}$: Using (3.89) leads to $|\det Q|^{1/2} \leq |\det A| \leq (c_A)^{d/2} \, |\det Q|^{1/2}$. □

Lemma 3.23. For an admissible lattice Γ we have for any weight function $\omega(x) > 0$ on $\mathbb{R}$, such that $\omega_\infty := 1 + \max_x \omega(x)(1+|x|)^p < \infty$, where $p \in \mathbb{N}$ and $\varepsilon > 0$, the bound

$$S_{\Gamma,\varepsilon} \stackrel{\text{def}}{=} \sum_{(\gamma_1,\dots,\gamma_d)\in\Gamma\setminus\{0\}} \Big|\frac{\omega_{r,\varepsilon}(\gamma_1)\dots\omega_{r,\varepsilon}(\gamma_d)}{\gamma_1\dots\gamma_d}\Big| \ll_d \omega_\infty \lambda_{r,\varepsilon}^{d-1}\frac{\bar\lambda_{r,\varepsilon,\Gamma}}{\mathrm{Nm}(\Gamma)}, \tag{3.127}$$

where $\omega_{r,\varepsilon}(x) := \lambda_{r,\varepsilon}\,|x|^{\frac12}\, I(|x| < \sqrt{d}) + \omega(\varepsilon r x) I(|x| \ge \sqrt{d})$.

Proof: First, we make a decomposition of Γ as follows. For any $(x_1,\dots,x_d) \in \mathbb{R}^d$ with $|x_1\cdots x_d| \ge \mathrm{Nm}(\Gamma)$ let $m_j \in \mathbb{Z}$ be the unique integers satisfying $2 > |2^{m_j}x_j|\,d^{-1/2} \ge 1$ for $j = 2,\dots,d$. We have $|x_1| \ge \mathrm{Nm}(x)\,|x_2\dots x_d|^{-1} \ge \mathrm{Nm}(\Gamma)d^{(1-d)/2}\prod_{j=2}^d 2^{m_j-1}$ and this implies that $|2^{m_1}x_1| \in [k c_\Gamma,(k+1)c_\Gamma)$ for a unique integer $k \ge 1$, where $m_1 \in \mathbb{Z}$ is determined by $m_1 + m_2 + \ldots + m_d = 0$ and $c_\Gamma = d^{(1-d)/2}2^{-d+1}\,\mathrm{Nm}(\Gamma)$. Introducing the lattice

$$E_d := \{m = (m_1,\dots,m_d) \in \mathbb{Z}^d \,:\, m_1 + \ldots + m_d = 0\} \subset \mathbb{Z}^d$$

and the interval $B_k := [k c_\Gamma,(k+1)c_\Gamma)$, we can write

$$I(|x_1\dots x_d| \ge \mathrm{Nm}(\Gamma)) = \sum_{m\in E_d}\sum_{k\in\mathbb{N}} I_{B_k}(|2^{m_1}x_1|)\prod_{j=2}^d I_{[\sqrt{d},2\sqrt{d})}(|2^{m_j}x_j|),$$

and hence

$$S_{\Gamma,\varepsilon} = \sum_{m\in E_d}\sum_{k\in\mathbb{N}}\sum_{\gamma\in\Gamma} I_{B_k}(|2^{m_1}\gamma_1|)\prod_{j=2}^d I_{[\sqrt{d},2\sqrt{d})}(|2^{m_j}\gamma_j|)\Big|\frac{\omega_{r,\varepsilon}(\gamma_1)\dots\omega_{r,\varepsilon}(\gamma_d)}{\gamma_1\dots\gamma_d}\Big|. \tag{3.128}$$

We also introduce the obvious notations $\mathrm{Nm}(x) := |x_1\cdots x_d|$, $2^m x = (2^{m_1}x_1,\dots 2^{m_d}x_d)$, $m \in E_d$ and $2^m\Gamma$ for the rescaled lattice $\{2^m\gamma : \gamma\in\Gamma\}$. Note that $\mathrm{Nm}(2^m\gamma) = \mathrm{Nm}(\gamma)$ and hence $\mathrm{Nm}(\Gamma) = \mathrm{Nm}(2^m\Gamma)$. Defining $C_k := B_k \times [\sqrt{d},2\sqrt{d})^{d-1}$ and $h(x) := (1+|x|)^{-p}$, we may rewrite and bound (3.128) by

$$\begin{aligned} S_{\Gamma,\varepsilon} &= \sum_{m\in E_d}\Big(\sum_{k\in\mathbb{N}}\sum_{\eta\in 2^m\Gamma} I_{C_k}(\eta)\prod_{j=1}^d\frac{\omega_{r,\varepsilon}(2^{-m_j}\eta_j)}{|\eta_j|}\Big) \\ &\ll_d \omega_\infty \sum_{m\in E_d}\sum_{k\in\mathbb{N}}\Big(\Big(\sum_{\eta\in 2^m\Gamma} I_{C_k}(\eta)\Big)\frac{h_{r,\varepsilon}(c_\Gamma 2^{-m_1}k)}{c_\Gamma k}\Big)\prod_{j=2}^d h_{r,\varepsilon}(2^{-m_j}), \end{aligned} \tag{3.129}$$

where $h_{r,\varepsilon}(x) := \lambda_{r,\varepsilon}\,|x|^{\frac12}\, I(|x| < 1) + h(\varepsilon r x) I(|x| \ge 1)$. In order to perform the summation in k and η in (3.129) we first observe that

$$\sum_{\eta\in 2^m\Gamma} I_{C_k}(\eta) \le 1. \tag{3.130}$$

Proof of (3.130): Assume that two different lattice points $\eta,\eta' \in 2^m\Gamma$ lie in C_k. Then we have $|\eta_1 - \eta_1'| < c_\Gamma$ and $\max_{2\le j\le d}|\eta_j - \eta_j'| < \sqrt{d}$. Since $\eta - \eta' \in 2^m\Gamma\setminus\{0\}$ implies $|\eta_2 - \eta_2'|\cdots|\eta_d - \eta_d'| \ge (\mathrm{Nm}\,\Gamma)/c_\Gamma = d^{(d-1)/2}2^{(d-1)}$ and hence $|(\eta_2 - \eta_2')| \ge 2\sqrt{d}$ for some $j \ge 2$, we get at a contradiction which proves (3.130).

Estimating the sum following in k by an integral, we obtain

$$\sum_{k=1}^{\infty} \frac{h_{r,\varepsilon}(\alpha k)}{k} \ll \lambda_{r,\varepsilon} I(\alpha < 1) + \log\left(1 + \frac{2}{\alpha r\varepsilon}\right) \overset{\text{def}}{=} \bar{h}(\alpha). \tag{3.131}$$

Hence, making use of (3.130) and (3.131) in (3.129), shows that

$$S_{\Gamma,\varepsilon} \ll_d \omega_\infty(c_\Gamma)^{-1} \textstyle\sum_{m\in E_d} H(2^{-m}), \tag{3.132}$$

where $2^m := (2^{m_1}, \ldots, 2^{m_d})$ and $H(x) := \bar{h}(c_\Gamma x_1) h_{r,\varepsilon}(x_2) \cdots h_{r,\varepsilon}(x_d)$.

Let E_d' denote the subset of E_d consisting of all lattice points $(m_1, \ldots, m_d) \in E_d$ with $m_1 \le 0$. We claim that

$$\sum_{m\in E_d'} H(2^{-m}) \ll_d \big(\lambda_{r,\varepsilon} + \log(1 + \tfrac{1}{\mathrm{Nm}(\Gamma) r\varepsilon})\big) \lambda_{r,\varepsilon}^{d-1}. \tag{3.133}$$

Proof of (3.133): Let $m \in E_d' \setminus \{0\}$. Assume for definiteness that $m_1, \ldots, m_{l-1} \le 0$ and $m_l, \ldots, m_d > 0$. By definition of E_d we get $2\sum_{j=l}^{m} m_j = \sum_{j=1}^{d} |m_j| \ge \|m\|_2$. Since $h_{r,\varepsilon}(2^{-k}) \le 1$ for $k \le 0$ and otherwise $h_{r,\varepsilon}(2^{-k}) = \lambda_{r,\varepsilon} 2^{-k/2}$, we obtain

$$\begin{aligned} H(2^{-m}) &\ll_d \big(\lambda_{r,\varepsilon} + \log(1 + \tfrac{1}{\mathrm{Nm}(\Gamma) r\varepsilon})\big) \lambda_{r,\varepsilon}^{d-l} \textstyle\prod_{j=l}^{d} 2^{-m_j/2} \\ &\ll_d \big(\lambda_{r,\varepsilon} + \log(1 + \tfrac{1}{\mathrm{Nm}(\Gamma) r\varepsilon})\big) \lambda_{r,\varepsilon}^{d-l} 2^{-\|m\|/4}. \end{aligned}$$

Thus, splitting the sum according to the number of positive coordinates and then summing over the $d-1$-dimensional lattice E_d yields (3.133).

In order to bound the sum over the complement of E_d', we again split the sum according to the number of positive coordinates. For simplicity, we may assume that $m_1, m_2, \ldots, m_l > 0$ and $m_{l+1}, \ldots, m_d \le 0$. Similar to the previous case, we find that

$$H(2^{-m}) \ll_d \big(\|m\| + \lambda_{r,\varepsilon} + \log(1 + \tfrac{1}{\mathrm{Nm}(\Gamma) r\varepsilon})\big) \lambda_{r,\varepsilon}^{l-1} \Big(\prod_{j=2}^{l} 2^{-\frac{m_j}{2}}\Big) \min(1, (r\varepsilon)^{-dp} 2^{-p\|m\|/2}).$$

If we parameterize the $d-1$-dimensional lattice E_d by $(m_1, \bar{m})$, where $m_1 = -(m_2 + \ldots + m_d)$ and $\bar{m} = (m_2, \ldots, m_d) \in \mathbb{Z}^{d-1}$, and split the summation into a ball of radius $\|\bar{m}\|_2 \le R_\varepsilon := 3d \log(2 + (r\varepsilon)^{-1})$ and its complement, where $(r\varepsilon)^{-dp} 2^{-p\|m\|_2/2} \le (r\varepsilon)^{-dp} 2^{-p\|\bar{m}\|_2/2} \le 1$, we can bound the sum corresponding to a fixed l by

$$\begin{aligned} &\ll_d \lambda_{r,\varepsilon}^{l-1} \Big(\sum_{\|\bar{m}\|_2 \le R_\varepsilon} (\bar{\lambda}_{r,\varepsilon,\Gamma} + \|\bar{m}\|) \prod_{j=2}^{l} 2^{-m_j/2} + \sum_{\|\bar{m}\|_2 > R_\epsilon} (\bar{\lambda}_{r,\varepsilon,\Gamma} + \|\bar{m}\|)(r\varepsilon)^{-dp} 2^{-p\|\bar{m}\|_2/2} \Big) \\ &\ll_d \lambda_{r,\varepsilon}^{l-1} \Big(\bar{\lambda}_{r,\varepsilon,\Gamma} \log(2 + \tfrac{1}{r\varepsilon})^{d-1-(l-1)} + \bar{\lambda}_{r,\varepsilon,\Gamma} \Big) \ll_d \lambda_{r,\varepsilon}^{d-1} \bar{\lambda}_{r,\varepsilon,\Gamma}, \end{aligned}$$

where we have estimated the sums by comparison with the corresponding integrals. Using this estimate for each $l = 1, \ldots, d-1$ together with (3.133) in (3.132) yields the bound (3.127). □

3.6 Proof of Theorem 3.1

We are left to collect all error bounds and to adjust the choice of parameters in order to prove Theorem 3.1. In view of (3.37), it remains to estimate I_θ. By (3.79), with $K_0 := [q_0^{-1/2} r^{-1}, 1]$ and $K_j := (j, j+1]$, $j \geq 1$, we have

$$I_\theta \ll_d |\det Q|^{-\frac{1}{4}} \|\widehat{\zeta}\|_1 \Big(I_{\theta,0} + \sum_{j=1}^{\infty} I_{\theta,j} \Big), \quad \text{where} \quad I_{\theta,j} \overset{\text{def}}{=} \int_{K_j} |\widehat{g}_w(t)| \alpha_d(\Lambda_t)^{\frac{1}{2}} \,\mathrm{d}t \tag{3.134}$$

with $j \geq 0$. For fixed $r \geq q^{1/2}$ we may choose

$$0 < w < (b-a)/4, \quad 1 \geq T_- \geq q_0^{-1/2} r^{-1}, \quad T_+ \geq 1 \quad \text{and} \quad \frac{d}{2} > \beta d > 2. \tag{3.135}$$

Also recall that $C_Q = q\, |\det Q|^{-1/4-\beta/2}$ as introduced in Corollary 3.15.

Step 1: Estimate of $I_{\theta,0}$. We consider the case $b-a \leq q$ first. Here we apply Corollary 3.15 to bound the integral over K_0 combined with $\widehat{g}_{K_0} \ll b-a$. Note that we didn't use the restriction $b-a \leq q$ at all. For wide shells, i.e. in the case $b-a > q$, we use Lemma 3.16 for $t \in K_0$, $q_0^{-1/2} r^{-1} \leq |t| \leq q^{-1/2}$ and Corollary 3.15 for the other t in K_0 together with $\widehat{g}_{[q^{-1/2},1]} \ll q^{1/2}$. Furthermore, for both cases of $b-a$, split $K_0 = K_{00} \cup K_{01}$, where $K_{00} := [q_0^{-1/2} r^{-1}, T_-]$ and $K_{01} := (T_-, 1]$. Then (3.61) yields

$$\gamma_{K_{00},\beta}(r) \ll_d \big(|\det Q|^{\frac{1}{2}}\, T_-^d \big)^{\frac{1}{2}-\beta} = T_-^{\varsigma}\, |\det Q|^{\frac{1}{4}-\frac{\beta}{2}}, \quad \varsigma \overset{\text{def}}{=} d(\tfrac{1}{2} - \beta) \tag{3.136}$$

with the notation (3.80). Using $C_Q q^{(2\beta d-1)/2} = \bar{C}_Q$, we may bound $I_{\theta,0}$ as

$$I_{\theta,0} \ll_d C_Q\, (b-a)_q \big(|\det Q|^{\frac{1}{4}-\frac{\beta}{2}}\, T_-^{\varsigma} + \gamma_{K_{01},\beta}(r) \big)\, r^{d-2}, \quad \text{where} \tag{3.137}$$

$$(b-a)_q \overset{\text{def}}{=} (b-a) I(b-a \leq q) + q^{(2\beta d-1)/2} I(b-a > q). \tag{3.138}$$

Step 2: Estimate of $I_{\theta,j}$ for $j \geq 1$. Similar as before, applying Corollary 3.15 (with $\beta = 1/2$) combined with the estimate (3.60) of Lemma 3.12 shows that

$$I_{\theta,j} \ll_d \widehat{g}_{K_j}\, C_Q\, \gamma_{K_j,\beta}(r)\, r^{d-2} \ll_d \widehat{g}_{K_j}\, q\, |\det Q|^{-1/2}\, r^{d-2}. \tag{3.139}$$

We recall the bound (3.87) for $\widehat{g}_w$ and the choices of T_+ and w in (3.135) in order to get

$$\sum_{j=T_+}^{\infty} \widehat{g}_{K_j} \ll \int_{T_+}^{\infty} \frac{\exp\{-|sw|^{1/2}\}}{s}\,\mathrm{d}s \ll \frac{1}{\sqrt{T_+ w}} \exp\{-|T_+ w|^{1/2}\}.$$

Thus, we obtain

$$\textstyle\sum_{j=T_+}^{\infty} I_{\theta,j} \ll_d r^{d-2} q\, |\det Q|^{-1/2}\, (T_+ w)^{-1/2} \exp\{-|T_+ w|^{1/2}\}. \tag{3.140}$$

Furthermore, for $b-a > 1$ we can use $|\widehat{g}_{K_j}| \ll j^{-1}$ to bound the remaining sum. Whereas for $b-a \leq 1$ we use $|\widehat{g}_{K_j}| \ll b-a$ for $1 \leq j \leq S-1$ and $|\widehat{g}_{K_j}| \ll j^{-1}$ for $S \leq j \leq T_+ - 1$ and minimize the resulting expression in S. In both cases this leads to

$$\textstyle\sum_{j=1}^{T_+-1} \widehat{g}_{K_j} \ll 1 + \log((b-a)^* T_+), \tag{3.141}$$

where

$$(b-a)^* \overset{\text{def}}{=} (b-a) I(b-a \leq 1) + I(b-a > 1).$$

Hence, using (3.134) combined with (3.137), (3.140) and (3.141) with (3.139), we get

$$I_\theta \ll_d \|\widehat{\zeta}\|_1 r^{d-2} C_Q \Big((b-a)_q (c_Q T_-^{\varsigma} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r)(1+\log((b-a)^* T_+)) + c_Q^{-1} \tfrac{\exp(-(T_+ w)^{1/2})}{(T_+ w)^{1/2}}\Big), \tag{3.142}$$

where $c_Q = |\det Q|^{\frac{1}{4}-\frac{\beta}{2}}$. Together with the inequality (3.37) we obtain

$$\begin{aligned}\Delta_r(\upsilon) &\overset{\text{def}}{=} \Big|\sum_{m\in\mathbb{Z}^d} I_{[a,b]}(Q[m])\upsilon_r(m) - \int_{\mathbb{R}^d} I_{[a,b]}(Q[x])\upsilon_r(x)\,\mathrm{d}x\Big| \\ &\ll_{\beta,d} r^{d-2}\big(\|\widehat{\zeta}\|_1 C_Q \rho_{Q,b-a,w}(r) + w\|\upsilon\|_Q\big) + |Q|^{-\frac12} r^{\frac d2} \|\widehat{\zeta}\|_{*,r} \log\Big(1+\tfrac{|b-a|}{q_0^{1/2} r}\Big),\end{aligned} \tag{3.143}$$

where

$$\rho_{Q,b-a,w}(r) \overset{\text{def}}{=} \inf\big\{(b-a)_q(c_Q T_-^{\varsigma} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r)(1+\log((b-a)^* T_+)) + c_Q^{-1}(T_+ w)^{-1/2} \mathrm{e}^{-(T_+ w)^{1/2}} : T_- \in [q_0^{-1/2} r^{-1}, 1] \text{ and } T_+ \geq 1\big\}$$

under the condition $0 < w < (b-a)/4$. This completes the proof of Theorem 3.1. □

3.7 Applications of Theorem 3.1

We start by smoothing the indicator function of the region Ω. We choose weights $\upsilon = \upsilon_{\pm\varepsilon}$ as defined in (3.91) and the related $\zeta = \zeta_\varepsilon$, see Section 3.5.1, corresponding to parallelepipeds $\Omega = A^{-1}[-1,1]^d$ satisfying $Q_+ \leq A^T A \leq c_A Q_+$, compare (3.89). Recalling (3.94), where we have used Lemma 3.17 to estimate the ε-smoothing error, yields a total error

$$\Delta_r \ll_d |Q|^{-\frac12}(b-a)\varepsilon r^{d-2} + \max_{\pm} \big|R(I_{E_{a,b}} \upsilon_{\pm\varepsilon,r})\big| \tag{3.144}$$

for the lattice remainder

$$\Delta_r \overset{\text{def}}{=} |\mathrm{vol}_{\mathbb{Z}}(E_{a,b} \cap r\Omega) - \mathrm{vol}(E_{a,b}\cap r\Omega)|\,.$$

Now we can apply Theorem 3.1 in order to bound the latter remainder $\big|R(I_{E_{a,b}}\upsilon_{\pm\varepsilon,r})\big|$ as follows. In (3.143) we shall estimate $\|\widehat{\zeta_\varepsilon}\|_{*,r}$ by using $\|\upsilon_\varepsilon\|_Q \ll_d |Q|^{-1/2}$ of Lemma 3.17, $\|\widehat{\zeta_\varepsilon}\|_1 \ll_d (\log\varepsilon^{-1})^d$ of Lemma 3.18 and

$$\|\widehat{\zeta_\varepsilon}\|_{*,r} \ll_d q^{d/4}\Big(\big(\tfrac{q}{q_0}\big)^{d/2}\log(\varepsilon^{-1})^d + q_0^{-d/4} c_A^{d/2} \lambda_{r,\varepsilon}^{d-1} \tfrac{\bar\lambda_{r,\varepsilon,\Gamma}}{\mathrm{Nm}(\Gamma)}\Big) \tag{3.145}$$

of Lemma 3.22 for admissible regions Ω, i.e. (3.116) holds, to get

$$\begin{aligned}\Delta_r \ll_{\beta,d}\, & |Q|^{-\frac12} r^{d-2}\Big(\varepsilon(b-a) + w + a_Q(\log\tfrac1\varepsilon)^d \rho_{Q,b-a,w}(r)\Big) \\ &+ |Q|^{-\frac12} q^{d/4} r^{d/2}\Big(\big(\tfrac{q}{q_0}\big)^{d/2}\log(\varepsilon^{-1})^d + q_0^{-d/4} c_A^{d/2}\lambda_{r,\varepsilon}^{d-1}\tfrac{\bar\lambda_{r,\varepsilon,\Gamma}}{\mathrm{Nm}(\Gamma)}\Big)\log\Big(1+\tfrac{b-a}{q_0^{1/2} r}\Big),\end{aligned} \tag{3.146}$$

where $a_Q \coloneqq q c_Q = q\,|\det Q|^{1/4-\beta/2} = C_Q |Q|^{1/2}$, provided that $0 < w < (b-a)/4$. This bound holds for admissible parallelepipeds Ω only. If Ω is not admissible, then we have to replace the smoothing error (3.145) by

$$\|\widehat{\zeta_\varepsilon}\|_{*,r} \ll_d q^{d/4}\big((q/q_0)^{d/2}\log(\varepsilon^{-1})^d + |Q|^{-1/2} q^{d/2} (c_A)^{(d+1)/2} \varepsilon^{-d}\big), \tag{3.147}$$

that is (3.118) of Lemma 3.22. With these bounds we are ready to prove the main statements on the lattice point remainder for hyperbolic shells.

Proof of Corollary 3.4: For wide shells, i.e. $b-a > q$, we optimize (3.146) in the smoothing parameter w first by choosing $w = \mathrm{W}(qT_+/2)^2/T_+$, where W denotes the principal branch of the Lambert-W-function. Note that we have $w \le q/(2\mathrm{e}) < (b-a)/4$ as required in the restrictions (3.135). This leads to the partial bound

$$|Q|^{-1/2}w + C_Q c_Q^{-1}(T_+w)^{-1/2}\mathrm{e}^{-(T_+w)^{1/2}} \ll |Q|^{-1/2}\tfrac{\mathrm{W}(qT_+/2)^2}{T_+} \ll |Q|^{-1/2}\tfrac{\log(qT_++1)^2}{T_+}.$$

Next, we calibrate the ε-dependent terms in (3.146) by choosing $\varepsilon = T_-^{\varsigma}(b-a)^{-1}/9$. Again, this choice satisfies the required restrictions, i.e. $\varepsilon < \varepsilon_0$. Because of

$$\varepsilon(b-a) \le a_Q(b-a)_q c_Q T_-^{\varsigma}, \quad \log\varepsilon^{-1} \ll \log(r+1) \quad \text{and}$$
$$\frac{\bar\lambda_{r,\varepsilon,\Gamma}}{\log(r+1)} \ll \max\Big\{1, \frac{\log(2+\frac{r^{d+1}}{\mathrm{Nm}(\Gamma)})}{\log(r+1)}\Big\} \ll_d \log(2+\tfrac{1}{\mathrm{Nm}(\Gamma)}),$$

compare the definition in Lemma 3.22, we can simplify (3.146) to

$$\begin{aligned}\Delta_r \ll_{\beta,d} & |Q|^{-\frac12} r^{d-2}\rho_{Q,b-a}(r) \\ & + |Q|^{-\frac12} q^{\frac d4} r^{\frac d2} \log(r+1)^d\big((\tfrac{q}{q_0})^{\frac d2} + \tfrac{c_A^{d/2} q_0^{-d/4}}{\mathrm{Nm}(\Gamma)}\log(2+\tfrac{1}{\mathrm{Nm}(\Gamma)})\big)\log\Big(1+\tfrac{b-a}{q_0^{1/2}r}\Big),\end{aligned} \tag{3.148}$$

where

$$\begin{aligned}\rho_{Q,b-a}(r) \overset{\mathrm{def}}{=} \inf{}^*_{T_+,T_-}\Big\{ & \log\big(\tfrac{b-a}{T_-^{\varsigma}}+1\big)^d\Big(a_Q q^{(2\beta d-1)/2}(c_Q T_-^{\varsigma} + \gamma_{[T_-,1],\beta}(r)) \\ & + a_Q\gamma_{(1,T_+],\beta}(r)\log(T_++1) + \tfrac{\log(qT_++1)^2}{T_+}\Big)\Big\}\end{aligned}$$

and the infimum is taken over all $T_- \in [q_0^{-1/2}r^{-1}, 1]$, $T_+ \ge 1$. This proves the first part of Corollary 3.4. Next, we consider the case of thin shells, i.e. $b-a \le q$. Here we take $\varepsilon = T_-^{\varsigma}/9$ and $w = T_-^{\varsigma}(b-a)/4$ in (3.146), noting that $|Q|^{-1/2}(w+\varepsilon(b-a)) \le a_Q(b-a)c_Q T_-^{\varsigma}$, in order to get the bound (3.148), whereby the factor $\rho_{Q,b-a}(r)$, depending on the Diophantine properties of Q, has to be replaced by

$$\begin{aligned}\rho^*_{Q,b-a}(r) \overset{\mathrm{def}}{=} \inf{}^*_{T_-,T_+}\Big\{ & a_Q\log(1+T_-^{-\varsigma})^d((b-a)(c_Q T_-^{\varsigma} + \gamma_{[T_-,1],\beta}(r)) \\ & + \gamma_{(1,T_+],\beta}(r)(\log((b-a)^*T_+)+1))\Big\}.\end{aligned}$$

In the last equation the infimum is taken over all $T_- \in [q_0^{-1/2}r^{-1}, 1]$ and $T_+ \ge 1$ with

$$T_+ \ge 4(b-a)^{-1}T_-^{-\varsigma}\max\{1, \log(c_Q^2(b-a)T_-^{\varsigma})^2\},$$

where the last condition ensures that

$$c_Q^{-1}(T_+w)^{-1/2}\mathrm{e}^{-(T_+w)^{1/2}} \le c_Q(b-a)T_-^{\varsigma}.$$

Finally, we note that Corollary 4.6 implies that $\gamma_{[T_-,1],\beta}(r) \to 0$ and also $\gamma_{[1,T_+],\beta}(r) \to 0$ for $r \to \infty$ and any fixed $T_- \in [q_0^{-1/2}r^{-1}, 1]$, $T_+ \ge 1$, when Q is irrational. Thus, we conclude that $\rho_{Q,b-a}(r) \to 0$, resp. $\rho^*_{Q,b-a}(r) \to 0$, for $r \to \infty$ and fixed $b-a$. □

Corollary 3.24. Consider a (not necessary admissible) parallelepiped Ω satisfying (3.89) and $|a|+|b| \le c_0 r^2$, where $c_0 > 0$ is chosen as in Lemma 3.17. Then for all $b-a \le 1$

$$\Delta_r \ll_{\beta,d} |Q|^{-1/2}r^{d-2}\big(\rho_{Q,b-a}(r) + (b-a)r^{1-d/2}q^{(d-2)/4}\log(1+r)^d(q/q_0)^{(d+1)/2}(c_A)^{(d+1)/2}\big),$$

where $\rho_{Q,b-a}$ is defined in (3.149). In particular, for irrational Q we have $\rho_{Q,b-a}(r) \to 0$ for $r \to \infty$, provided that $b-a$ is fixed.

Proof: Since Ω is not necessarily admissible, we can only use (3.147) to bound $\|\widehat{\zeta_\varepsilon}\|_{*,r}$. Additionally we consider here thin shells only, i.e. $b-a \le 1$. Taking

$$\varepsilon = (9\log(1+T_-^{-\varsigma}))^{-1} \quad \text{and} \quad w = T_-^{\varsigma}(b-a)/4,$$

leads to the bound

$$\Delta_r \ll_{\beta,d} |Q|^{-1/2} r^{d-2} \rho_{Q,b-a}(r) + |Q|^{-1/2} q^{d/4} r^{d/2}\Big(\log(1+r)^d (q/q_0)^{d/2} + |Q|^{-1/2} q^{d/2} (c_A)^{(d+1)/2} \log(1+r)^d\Big) \log\Big(1 + \tfrac{|b-a|}{q_0^{1/2} r}\Big),$$

where

$$\rho_{Q,b-a}(r) \stackrel{\text{def}}{=} \inf\Big\{ a_Q \log(1+T_-^{-\varsigma})^d \Big((b-a)(c_Q T_-^{\varsigma} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r)\log((b-a)T_+)\Big) + \tfrac{b-a}{\log(1+T_-^{-\varsigma})}\Big\} \tag{3.149}$$

and the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$ and

$$T_+ \ge \frac{4}{(b-a)T_-^{\varsigma}} \max\{1, \log(c_Q^2 (b-a) T_-^{\varsigma})^2\}. \qquad \square$$

The next corollary provides a lower bound for the number of lattice points and will be useful for proving quantitative bounds in the Oppenheim conjecture.

Corollary 3.25. For the special choice $A = Q_+^{1/2}$, i.e. $\Omega = Q_+^{-1/2}[-1,1]^d$ and $c_A = 1$, and all $|a| + |b| \le r^2/5$ and $b - a \le 1$ there exists a constant $b_{\beta,d} > 0$, depending on β and d only, such that

$$\Delta_r \le \frac{\operatorname{vol} H_r}{10} + b_{\beta,d} |Q|^{-1/2} r^{d-2} \big(\rho_{Q,b-a}(r) + q^{(d-2)/4} (b-a) r^{-d/2+1} (q/q_0)^{(d+1)/2}\big), \tag{3.150}$$

where $c_Q = |\det Q|^{1/4-\beta/2}$, $a_Q = q\, c_Q$ and

$$\rho_{Q,b-a}(r) \stackrel{\text{def}}{=} \inf\nolimits^*_{T_-,T_+} \{ a_Q \big((b-a)(c_Q T_-^{\varsigma} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r) \log((b-a)T_+)\big)\} \tag{3.151}$$

and the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$ and $T_+ \ge 1$ with

$$T_+ \ge C_{\beta,d} \frac{1}{(b-a)} \max\Big\{ \log\Big(\frac{b-a}{q\, c_{\beta,d}}\Big)^2, 1\Big\}$$

and $C_{\beta,d}, c_{\beta,d} \ge 1$ are constants depending on d and β only.

Proof: In view of (3.97), established in Lemma 3.17, we can take $\varepsilon = (30\, a_{\beta,d}\, b_d)^{-1}$ and $w = (b-a)\varepsilon$ in the optimization of (3.146), where the error bound (3.147) is used instead of (3.145), since Ω is not necessarily admissible, and $b_d \ge 1$, resp. $a_{\beta,d} \ge 1$, denotes the implicit constant in (3.97), resp. (3.146). If we choose $T_+ \ge w^{-1} \max\{\log(q^{-1}w)^2, 1\}$ additionally, then we also have $a_{\beta,d}\, q\, |Q|^{-1/2} (T_+ w)^{-1/2} \exp(-|T_+ w|^{1/2}) \le \operatorname{vol} H_r/30$. $\square$

Now we consider elliptic shells as well and optimize the lattice remainder as in the case of 'wide shells'. In contrast to the previous cases, the error caused by the smoothing of the region Ω is not present here.

Proof of Corollary 3.3: In the case of ellipsoids, i.e. Q is a positive definite form, we choose the (not necessary admissible) parallelepiped $\Omega := A^{-1}[-1,1]^d$ with $A = Q_+^{1/2}$ and $r = \sqrt{2b} \geq q^{1/2}$, resp. $2b = r^2$, $a = 0$ and $\varepsilon = 1/4$. Then (3.89) is satisfied with $c_A = 1$ and $E_{0,b} \subset r\Omega$, i.e. $H_r := E_{a,b} \cap r\Omega = E_{a,b}$. Moreover, since $E_{0,b}$ does not intersect $r(\partial\Omega)_{2\epsilon}$ (the $2\varepsilon r$-boundary of $r\Omega$ as defined in (3.91)), we get an error $R_{\varepsilon,r} = 0$ for smoothing the indicator function of $r\Omega$. Hence, we may remove the term proportional to $(b-a)\varepsilon$ in (3.144). Note that apart from Lemma 3.17 of the appendix the *indefiniteness* of Q has not been used in all arguments so far. In contrast to the case of hyperbolic shells, we optimize (3.143) in w first. Again including the bound $\|\upsilon_\varepsilon\|_Q \ll_d |Q|^{-1/2}$ of Lemma 3.17 and here taking $w = \mathrm{W}(qT_+/4)^2/T_+$, where W denotes the principal branch of the Lambert-W-function, and noting that $w \leq q/(4e) < (b-a)/4$, leads (as in the proof of Corollary 3.4) to the bound

$$\begin{aligned}\Delta_r \ll_{\beta,d} r^{d-2}\Big(C_Q\big(q^{(2\beta d-1)/2}(c_Q T_-^{\varsigma} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r)\log(T_+ + 1)\big)\\ +|Q|^{-\frac{1}{2}}\tfrac{\log(1+qT_+)^2}{T_+}\Big) + |Q|^{-\frac{1}{2}}q^{\frac{d}{4}}r^{\frac{d}{2}}\big((q/q_0)^{\frac{d}{2}} + |Q|^{-\frac{1}{2}}q^{\frac{d}{2}}\big)\log\Big(1+\tfrac{r}{q_0^{1/2}}\Big),\end{aligned} \tag{3.152}$$

where $T_- \in [q_0^{-1/2}r^{-1}, 1]$ and $T_+ \geq 1$. This can be rewritten as

$$\Delta_r \ll_{\beta,d} |Q|^{-1/2}r^{d-2}\rho_Q(r) + |Q|^{-1/2}q^{d/4}r^{d/2}(q/q_0)^{d/2}\log(1 + r/q_0^{1/2})$$

with

$$\rho_Q(r) \overset{\text{def}}{=} \inf\Big\{a_Q\big(q^{\beta d-\frac{1}{2}}(c_Q T_-^{\varsigma} + \gamma_{[T_-,1],\beta}(r)) + \gamma_{(1,T_+],\beta}(r)\log(T_+ + 1)\big) + \tfrac{\log(1+qT_+)^2}{T_+}\Big\},$$

where the infimum is taken over all $T_- \in [q_0^{-1/2}r^{-1}, 1]$ and $T_+ \geq 1$. Note that as in the indefinite case $\lim_{r\to\infty}\rho(r) = 0$ if Q is irrational by Corollary 4.6. This proves Corollary 3.3. Furthermore, we remark that $\operatorname{vol} H_r = \operatorname{vol}(r\Omega \cap E_{0,b}) = |Q|^{-1/2}\omega_d r^d$, where ω_d denotes the volume of the unit d-ball. □

To establish explicit bounds, it remains to bound the Diophantine factors (in terms of a certain Diophantine approximation error). This will be done in the next chapter leading to quantitative variants of the Oppenheim conjecture.

CHAPTER 4

General Indefinite Quadratic Forms

In the following we shall prove the introduced quantitative variants of the Oppenheim conjecture for non-diagonal forms, i.e. we prove Theorems 1.9 and 1.11. To do this, we shall finally combine the results established in Chapter 3 on thin shells together with explicit estimates on the α_d-characteristic in terms of a projective Diophantine approximation of Q which will be introduced in the next section. Compared to [GM13] the explicit dependency on the largest eigenvalue of Q has been improved in Lemma 4.5 and the dependency on the determinant was removed. Additionally, we show that forms with algebraic coefficients are Diophantine forms leading to explict bounds on the size of an integral solution of $|Q[m]| < \varepsilon$.

4.1 Quadratic Forms of Diophantine Type (κ, A)

For any fixed $T > 1 > T_-$ and irrational Q we will prove in Corollary 4.6 that

$$\lim_{r\to\infty} \gamma_{[T_-,T],\beta}(r) = 0, \tag{4.1}$$

with a speed depending on the Diophantine properties of Q. For fixed $b - a > 0$ we get

$$\lim_{r\to\infty} \rho_{Q,b-a}(r) = 0 \tag{4.2}$$

and hence $\Delta_r = o(r^{d-2})$ as $r \to \infty$, where Δ_r denotes the lattice remainder (introduced in the previous chapter). This holds *uniformly* for all intervals $[a, b]$ with $0 < u_r \le b-a \le v_r \le c_0 r^2$ and sequences $\lim_r u_r = 0$, $\lim_r v_r = \infty$, $r \to \infty$ depending on Q. However, in order to get effective bounds, we need effective estimates on the rate of convergence as well. Hence, one may introduce the following class of Diophantine matrices.

Definition 4.1. We call Q Diophantine of type (κ,A), where $\kappa, A > 0$, if for any $m \in \mathbb{Z}\setminus\{0\}$ and $M \in M(d, \mathbb{Z})$ we have

$$\inf_{t\in[1,2]} \|M - mtQ\| \ge A\ |m|^{-\kappa}, \tag{4.3}$$

where $\|\cdot\|$ denotes the operator norm induced by the Euclidean norm on $\mathbb{R}^n$.

Equivalently we may require that Q satisfy

$$\inf_{t\in[1,2]} \delta_{tQ;R} > AR^{-\kappa} \quad \text{for all } R \ge 1,$$

where $\delta_{tQ;R}$ is the truncated rational approximation error defined by

$$\delta_{tQ;R} \overset{\text{def}}{=} \min\Big\{\|M - mtQ\| \,:\, m \in \mathbb{Z}, 0 < |m| \le R,\, M \in \operatorname{Sym}(d, \mathbb{Z})\Big\}. \tag{4.4}$$

Remark 4.2. As an aside, we remark that the property of Q being Diophantine in the above sense is easily seen to be equivalent to the requirement that

$$\|M - tQ\| > t^{-\tilde{\kappa}}, \quad \text{for all } t \ge 2 \text{ and } M \in \operatorname{Sym}(d, \mathbb{Z}), \text{ for some } \tilde{\kappa} > 0,$$

which was introduced in [EMM98] in the context of forms that are (EWAS). However, this formulation ignores the constant A which is of major importance in Diophantine approximation.

Applying Corollary 3.25 we will prove the following corollary establishing a quantitative variant of the Oppenheim conjecture for quadratic forms Q of Diophantine types (A, κ).

Corollary 4.3. Let Q be an indefinite quadratic form of Diophantine type (κ, A) and $\delta > 0$. Then for any $\varepsilon > 0$ there exists a non-trivial lattice point $m \in \mathbb{Z}^d \setminus 0$ satisfying

$$|Q[m]| < \varepsilon \quad \text{and} \quad \|m\| \ll_{Q,d,\delta} \varepsilon^{-\frac{2d+3\kappa d-4\kappa}{2d-8}-\delta}.$$

Moreover, using Corollary 4.6, we may estimate the Diophantine factors of Corollary 3.4 for quadratic forms Q of Diophantine types (A, κ) as follows.

Corollary 4.4. Consider an indefinite quadratic form $Q[x]$ with matrix Q which is Diophantine of type (κ, A). Moreover, let $\beta = 2/d + \delta$ for some sufficiently small $0 < \delta < \frac{1}{10}$. Then for the case of wide shells $b - a \geq q$ in Corollary 3.4 we have

$$\rho_{Q,b-a}(r) \ll_{\beta,d} \log(r+1)^d h_Q q^{\beta d - \frac{1}{2}} (1 + A^{-\nu})(r^{-\frac{2\nu\sigma}{\nu+\sigma}} + r^{-\frac{2\nu}{\kappa\nu+1}} \log(qr+1)), \tag{4.5}$$

where $h_Q = q\,|\det Q|^{1/2-\beta}$ and $\nu = (1-2\beta)/(2\kappa+2)$. Thus for an admissible region Ω satisfying (3.89) we have for all $r \geq q^{1/2}$ and $b - a \leq c_0 r^2$

$$\left|\frac{\operatorname{vol}_{\mathbb{Z}} H_r}{\operatorname{vol} H_r} - 1\right| \ll_{Q,\Omega,\beta,d} \frac{\log(r+1)^d}{b-a}\Big(r^{-\frac{(1-2\beta)d}{1+(\kappa+1)d}} + r^{-\frac{2-4\beta}{2+(3-2\beta)\kappa}} + r^{-\frac{d}{2}+2}\log\big(1+\tfrac{b-a}{r}\big)\Big), \tag{4.6}$$

where the implied constant in (4.6) can be explicitly determined. For thin shells, i.e. $b-a \leq q$, we have

$$\begin{aligned}\rho^*_{Q,b-a}(r) \ll_{\beta,d} \inf\nolimits^*_{T_-,T_+}\Big\{ \log\big(1+T_-^{-\varsigma}\big)^d h_Q\big((b-a)(T_-^{\varsigma} + A^{-\nu}T_-^{-\nu}r^{-2\nu}) \\ + A^{-\nu}T_+^{\kappa\nu}r^{-2\nu}(\log((b-a)^*T_+)) + 1)\big)\Big\},\end{aligned}$$

where the infimum is taken over all $T_- \in [q_0^{-1/2}r^{-1}, 1]$ and $T_+ \geq 1$ restricted to

$$T_+ \geq 4(b-a)^{-1}T_-^{-\varsigma}\max\{1, \log(c_Q^2(b-a)T_-^{-\varsigma})^2\}.$$

4.2 Irrational and Diophantine Lattices

In this section we shall establish a connection between the α_d-characteristic $\alpha_d(\Lambda_t)$ and Diophantine approximations of tQ by symmetric integral matrices with approximation error $\delta_{tQ;R}$, introduced in (4.4), in order to pave the way for applying the results on small zeros of integral forms and the approximation property (4.3) of quadratic forms of Diophantine type (κ, A) as introduced in Definition 4.1. To do this, we introduce the rescaled α_d-characteristic

$$\beta_{t;r} := \alpha_d(\Lambda_t) r^{-d} |\det Q|^{1/2} \tag{4.7}$$

and note that by Lemma 3.12 we have the uniform bound $\beta_{t;r} \ll_d 1$ for $r \geq q^{1/2}$. This bound will be refined in the following Lemma 4.5, showing that larger values of $\beta_{t;r}$ enforce smaller values of the rational approximation error $\delta_{4tQ;R}$.

Lemma 4.5. Assume that $q_0 \geq 1$. Then we have for all $t \in \mathbb{R}$ and $r \geq q^{1/2}$

$$\delta_{4tQ;\beta_{t;r}^{-1}} \ll_d q r^{-2} \beta_{t;r}^{-1}. \tag{4.8}$$

Note that this bound is non-trivial for $\beta_{t;r} > qr^{-2}$ only.

As a remark, we note that the restriction in (4.4) to symmetric matrices can be dropped, since Q is a symmetric matrix. Before proving (4.8), we shall state the following important consequences.

Corollary 4.6. Consider any interval $[T_-, T_+]$ with $T_- \in (0,1]$ and $T_+ \geq 1$.

i) If Q is *irrational*, then

$$\lim_{r\to\infty} \Big(\sup_{T_-\leq t\leq T_+} \alpha_d(\Lambda_t) r^{-d} \Big) = 0. \tag{4.9}$$

ii) If Q is *Diophantine* of type (κ, A), where $\kappa > 0$ and $A > 0$, that is,

$$\inf_{t\in[1,2]} \delta_{tQ,R} > A\, R^{-\kappa} \quad \text{for all} \quad R \geq 1, \tag{4.10}$$

then

$$\sup_{T_-\leq t\leq T_+} \alpha_d(\Lambda_t) r^{-d} \ll_d (q\,|\det Q|^{\frac{1}{2}}\, A^{-1} r^{-2})^{\frac{1}{\kappa+1}} \max\big\{(T_-)^{-\frac{1}{\kappa+1}}, (T_+)^{\frac{\kappa}{\kappa+1}}\big\}. \tag{4.11}$$

A variant of (i) in terms of the successive minima of Λ_t can also be found in [Göt04], see Lemma 3.11, yielding an alternative proof of (4.9) when combined with (3.45).

Proof: *i)* Assume that there is an $\varepsilon > 0$ and sequences r_j, t_j such that $\lim_j r_j = \infty$ and $\beta_{t_j;r_j} > \varepsilon$. Passing to a subsequence we may assume that $\lim_j t_j = t$ for some $t \in [T_-, T_+]$. Thus (4.8) yields $\lim_j \delta_{4t_jQ,R_j^*} = 0$ with $R_j^* := \beta_{t_j;r_j}^{-1} < \varepsilon^{-1}$. By definition, this means that $\lim_j \|M_j - 4t_j m_j Q\| = 0$ for some $M_j \in \mathrm{Sym}(d,\mathbb{Z})$ and $m_j \in \mathbb{Z}$ with $|m_j| \leq \varepsilon^{-1}$. Obviously both, $\|M_j\|$ and $|m_j|$, are bounded. Hence there exist integral elements M, m and an infinite subsequence j' of j with $M_{j'} = M$, $m_{j'} = m$ and by construction $\lim_{j'} t_{j'} = t$. These limit values satisfy $\|M - 4mtQ\| = 0$, i.e. Q is a multiple of a rational form.

ii) First we note that for any $t \in [1, T_+]$ we have by (4.10)

$$(\delta_{tQ,R})^{-1} \leq \sup\nolimits_{t'\in[1,2]} (\delta_{t'Q,4tR})^{-1} < A^{-1}(4tR)^{\kappa} \leq A^{-1}(T_+)^{\kappa}(4R)^{\kappa}$$

and similarly for $t \in [T_-, 1]$

$$(T_-)^{-1}\delta_{4tQ,R} \gg \lceil t^{-1}\rceil \delta_{tQ,4R} \geq \delta_{(\lceil t^{-1}\rceil t)Q,4R} > A(4R)^{-\kappa}.$$

Thus, we obtain for every $t \in [T_-, T_+]$ that

$$\beta_{t;r} \ll_d q\,|\det Q|^{\frac{1}{2}}\, r^{-2} (\delta_{4tQ,\beta_{t;r}^{-1}})^{-1} \ll_d 4^{\kappa} q\,|\det Q|^{\frac{1}{2}}\, r^{-2} A^{-1} \max\{(T_-)^{-1}, (T_+)^{\kappa}\} (\beta_{t;r})^{-\kappa},$$

where we used (4.8). Therefore we conclude (4.11) as claimed. □

As a preparation for the proof below, we recall that the standard Euclidean inner product and the corresponding norm on the exterior product $\bigwedge^m \mathbb{R}^n$ (with $1 \leq m \leq n$) can be introduced as follows. Using the universal property of the exterior product twice, we see that the alternating multilinear form $\langle\cdot,\cdot\rangle\colon (\mathbb{R}^n)^m \times (\mathbb{R}^n)^m \to \mathbb{R}$ defined by

$$\langle v_1 \wedge \ldots \wedge v_m, w_1 \wedge \ldots \wedge w_m \rangle \overset{\text{def}}{=} \det(\langle v_i, w_j\rangle,\, 1 \leq i,j \leq m) \tag{4.12}$$

can be extended to $(\bigwedge^m \mathbb{R}^n) \times (\bigwedge^m \mathbb{R}^n)$, which we also call $\langle\cdot,\cdot\rangle$. The uniqueness of this extension shows that this map is symmetric as well. Additionally, we see that $e_I := e_{i_1} \wedge$

$\ldots \wedge e_{i_m}$, passing all subsets $I = \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ with m elements, constitute an orthogonal basis. Thus, writing $v = \sum_I a_i e_I$ we find that

$$\langle v, w \rangle = \sum_{I,J} a_I a_J \langle e_I, e_J \rangle = \sum_I a_I^2$$

showing the positive definiteness of $\langle \cdot, \cdot \rangle$. The reader may note that the definition (4.12) is directly related to the volume of a lattice Λ with basis $b_1, \ldots, b_m$, because we have

$$\det(\Lambda) = \sqrt{\det(\langle b_i, b_j \rangle,\ 1 \leq i, j \leq m)} = \|b_1 \wedge \ldots \wedge b_m\|.$$

This relation will be used in the following proof to rewrite the α_d-characteristic of Λ_t.

Proof of Lemma 4.5: We begin by recalling that $\Lambda_t = D_{rQ} U_{4tQ} \mathbb{Z}^{2d}$ (see (3.39)), where

$$D_{rQ} = \begin{pmatrix} rQ_+^{-1/2} & 0 \\ 0 & r^{-1} Q_+^{1/2} \end{pmatrix} \qquad \text{and} \qquad U_{4tQ} = \begin{pmatrix} \mathrm{I}_d & -4tQ \\ 0 & \mathrm{I}_d \end{pmatrix}.$$

Since the d-th exterior powers of D_{rQ} and U_{4tQ} are invertible, we see that

$$\|D_{rQ} U_{4tQ}(v_1 \wedge \ldots \wedge v_d) - D_{rQ} U_{4tQ}(w_1 \wedge \ldots \wedge w_d)\| \gg \|(v_1 \wedge \ldots \wedge v_d) - (w_1 \wedge \ldots \wedge w_d)\|,$$

where the implicit constant depends on Q, t, r. Now the right-hand side takes positive integer values only and therefore we find that the α_d-characteristic of Λ_t is attained at some sublattice. In other words, we can write $\alpha_d(\Lambda_t) = \|w_1 \wedge \ldots \wedge w_d\|^{-1}$ by means of vectors $w_j := D_{rQ} U_{4tQ} l_j$ with linear independent points $l_1, \ldots, l_d \in \mathbb{Z}^{2d}$ depending on t. Moreover, we write $l_j = (m_j, n_j)$, where $m_j, n_j \in \mathbb{Z}^d$ and the coordinates of (m_j, n_j) are the coordinates of the vectors m_j and n_j in the corresponding order. Additionally, we introduce the $d \times d$ integer matrices N and M with columns $n_1, \ldots, n_d$ and $m_1, \ldots, m_d$ as well. Using this notation, we may write

$$w_1 \wedge \ldots \wedge w_d = (D_{rQ} U_{4tQ}) \begin{pmatrix} M \\ N \end{pmatrix} e_1 \wedge \ldots \wedge e_d. \tag{4.13}$$

First, we shall prove that

$$\alpha_d(\Lambda_t) > q\, d_Q r^{d-2} \qquad \text{implies} \qquad \beta_{t;r}^{-1} > |\det(N)| > 0. \tag{4.14}$$

Note that the left-hand side of (4.14) can be rewritten as $\beta_{t;r} > q r^{-2}$ and we may assume that this inequality holds, since otherwise the bound (4.8) is trivial.

Let us assume that $\operatorname{rank}(N) = d - k$. According to elementary divisor theory (for matrices with entries in a principal ideal domain) there exist $P, P' \in \mathrm{GL}(d, \mathbb{Z})$ such that $P'NP$ is a diagonal matrix with positive entries of the form $\operatorname{diag}(0, \ldots, 0, a_{k+1}, \ldots, a_d)$ with $a_i \mid a_{i+1}$, $a_i \in \mathbb{N}$. In particular NP is a matrix whose first k columns are zero. Moreover, since $\det P = \pm 1$, it is obvious that

$$\begin{pmatrix} MP \\ NP \end{pmatrix} e_1 \wedge \ldots \wedge e_d = \pm \begin{pmatrix} M \\ N \end{pmatrix} e_1 \wedge \ldots \wedge e_d$$

and hence we can assume from now on that $N = (0, \ldots, 0, n_{k+1}, \ldots, n_d)$ with linearly independent vectors $n_{k+1}, \ldots, n_d \in \mathbb{Z}^d$. Since $l_1, \ldots, l_d$ constitute a basis for a d-dimensional lattice, we note that $m_1, \ldots, m_k$ are necessarily linearly independent. Now we shall express $w_1 \wedge \ldots \wedge w_d$ in terms of the standard basis $e_I \wedge e_J$ indexed by pairs of subsets $I \subset \{1, \ldots, d\}$ and $J \subset \{d+1, \ldots, 2d\}$ with $|I| + |J| = d$, i.e. we write

$$w_1 \wedge \ldots \wedge w_d = \sum_{I,J} \beta_{I,J} e_I \wedge e_J.$$

Let $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_{d-m}\}$, then the coefficients $\beta_{I,J}$ are given by

$$\beta_{I,J} \overset{\text{def}}{=} \det \begin{pmatrix} A_I & * \\ 0 & B_J \end{pmatrix}, \tag{4.15}$$

where

$$A_I \overset{\text{def}}{=} \begin{pmatrix} \langle rQ_+^{-\frac{1}{2}} m_1, e_{i_1} \rangle & \ldots & \langle rQ_+^{-\frac{1}{2}} m_k, e_{i_1} \rangle \\ \vdots & & \vdots \\ \langle rQ_+^{-\frac{1}{2}} m_1, e_{i_m} \rangle & \ldots & \langle rQ_+^{-\frac{1}{2}} m_k, e_{i_m} \rangle \end{pmatrix}$$

$$B_J \overset{\text{def}}{=} \begin{pmatrix} \langle r^{-1} Q_+^{\frac{1}{2}} n_{k+1}, e_{j_1} \rangle & \ldots & \langle r^{-1} Q_+^{\frac{1}{2}} n_d, e_{j_1} \rangle \\ \vdots & & \vdots \\ \langle r^{-1} Q_+^{\frac{1}{2}} n_{k+1}, e_{j_{d-m}} \rangle & \ldots & \langle r^{-1} Q_+^{\frac{1}{2}} n_d, e_{j_{d-m}} \rangle \end{pmatrix}.$$

Since the matrix in (4.15) is of block-type, we find

$$\begin{aligned} \alpha_d(\Lambda_t)^{-2} &= \|w_1 \wedge \ldots \wedge w_d\|^2 \\ &\geq \sum_{|I|=k} \sum_{|J|=d-k} \beta_{I,J}^2 = \Big(\sum_{|I|=k} (\det A_I)^2 \Big) \Big(\sum_{|J|=d-k} (\det B_J)^2 \Big) \\ &= r^{4k-2d} \|Q_+^{-\frac{1}{2}} (m_1 \wedge \ldots \wedge m_k)\|^2 \, \|Q_+^{\frac{1}{2}} (n_{k+1} \wedge \ldots \wedge n_d)\|^2. \end{aligned} \tag{4.16}$$

Without loss of generality assume that the eigenvalues of Q are indexed such that $|q_1| \leq \cdots \leq |q_d|$. Since $q_0 \geq 1$, note that the minimal eigenvalue of the k-th exterior power of $Q_+^{-1/2}$ is given by $|q_{d-k+1} \cdots q_d|^{-1/2}$ and that of the $(d-k)$-th exterior power of $Q_+^{1/2}$ is precisely $|q_1 \ldots q_{d-k}|^{1/2}$. Hence, since $m_1, \ldots, m_k$ and $n_{k+1}, \ldots, n_d$ are linearly independent and integral, we obtain the following lower bound

$$\alpha_d(\Lambda_t)^{-1} \geq r^{2k-d} \left(\frac{|q_1 \ldots q_{d-k}|}{|q_{d-k+1} \cdots q_d|} \right)^{1/2} \geq q^{-1} \, |\det Q|^{1/2} \, r^{2-d}.$$

where we used that $r \geq q^{1/2}$. In view of (4.14), this strict inequality yields a contradiction unless $k = 0$. Thus, we proved that $\det N \neq 0$ and $k = 0$. Now (4.16) also implies $\beta_{t;r}^{-1} \geq |\det N|$. Hence, the upper bound for $|\det N|$ in (4.14) holds as well.

Finally, we shall prove (4.8). Since N is invertible, we can rewrite $w_1 \wedge \ldots w_d$ by

$$(D_{rQ} U_{4tQ}) \begin{pmatrix} MN^{-1} \\ \mathbb{1}_d \end{pmatrix} N e_1 \wedge \ldots \wedge e_d = (\det N)(D_{rQ} U_{4tQ}) \begin{pmatrix} MN^{-1} \\ \mathbb{1}_d \end{pmatrix} e_1 \wedge \ldots \wedge e_d, \tag{4.17}$$

i.e. we parametrized the subspace spanned by $l_1, \ldots, l_d$. Introduce also the $2d \times d$ matrix

$$W \overset{\text{def}}{=} (D_{rQ} U_{4tQ}) \begin{pmatrix} MN^{-1} \\ \mathbb{1}_d \end{pmatrix} = \begin{pmatrix} rQ_+^{-\frac{1}{2}} (MN^{-1} - 4tQ) \\ r^{-1} Q_+^{\frac{1}{2}} \end{pmatrix}$$

and note that W^TW is a positive definite symmetric $d \times d$ matrix. Thus, there exists an orthogonal matrix $V \in O(d)$ such that $D := V^TW^TWV$ is diagonal with positive entries. Since $(\det V)(e_1 \wedge \ldots \wedge e_d) = V(e_1 \wedge \ldots \wedge e_d)$ it follows that

$$\begin{aligned} \|W(e_1 \wedge \ldots \wedge e_d)\|^2 &= \|WV(e_1 \wedge \ldots \wedge e_d)\|^2 \\ &= \langle D(e_1 \wedge \ldots \wedge e_d), (e_1 \wedge \ldots \wedge e_d)\rangle = \prod_{i=1}^d \|De_i\| = \prod_{i=1}^d \|Wv_i\|^2, \end{aligned} \tag{4.18}$$

where $v_1, \ldots, v_d$ denote the columns of V. Next observe that

$$\max_{1 \le i \le d} \|Wv_i\| \ge \max_{1 \le i \le d} \|rQ_+^{-\frac{1}{2}}(MN^{-1} - 4tQ)v_i\| \gg_d rq^{-\frac{1}{2}}\|MN^{-1} - 4tQ\|. \tag{4.19}$$

Now let i_0 be a subscript for which $\|Wv_i\|$ is maximal. Similar to the proof of (4.16) we may write $W(\wedge_{i \ne i_0} v_i) = \sum \beta_{I,J} e_I \wedge e_J$, where the sum is taken over subsets $I \subset \{1, \ldots, d\}$ and $J \subset \{d+1, \ldots, 2d\}$ with $|I| + |J| = d - 1$, and find that

$$\|W(\wedge_{i \ne i_0} v_i)\|^2 \ge \sum_{|I|=0, |J|=d-1} \beta_{I,J}^2 = \|r^{-1}Q_+^{\frac{1}{2}}(\wedge_{i \ne i_0} v_i)\|^2 \ge r^{-2(d-1)}q^{-1} |\det Q| . \tag{4.20}$$

Combining (4.17) together with (4.18)–(4.20) yields

$$\begin{aligned} \alpha_d(\Lambda_t)^{-1} &= |\det(N)| \; \|Wv_{i_0}\| \textstyle\prod_{i \ne i_0} \|Wv_i\| = |\det(N)| \; \|Wv_{i_0}\| \, \|W(\wedge_{i \ne i_0} v_i)\| \\ &\gg_d r^{-(d-2)}q^{-1} |\det Q|^{\frac{1}{2}} \; |\det N| \; \|MN^{-1} - 4tQ\|. \end{aligned}$$

Since $(\det N)N^{-1}$ is an integral matrix, the last line together with (4.14) implies

$$\min\{\|\bar{M} - 4mtQ\| \,:\, 0 < |m| \le \beta_{t;r}^{-1}, \, m, \bar{M} \text{ integral}\} \ll_d qr^{-2}\beta_{t;r}^{-1},$$

and, since Q is symmetric, we may take $\bar{M}$ symmetric as well, which proves (4.8). □

4.3 Proofs of Theorems 1.9, 1.11 and Corollaries 4.3, 4.4

Now we are in position to prove the Theorems 1.9 and 1.11. In the case of Theorem 1.9, we consider the solubility of the Diophantine inequality $|Q[m]| < 1$. In order to get solutions of $|Q[m]| < \varepsilon$ with explicit bounds for the norm of $m \in \mathbb{Z}^d \setminus \{0\}$ (in terms of $\varepsilon > 0$), one can replace Q by Q/ε. The general approach here is to compare the volume with the number of lattice points if Q has 'good' Diophantine properties. If Q has not 'good' Diophantine properties, we will see that Q is near a rational form and here we shall use Schlickewei's bound [Sch85] for small zeros of integral quadratic forms.

Proof of Theorem 1.9: Let $d \ge 5$, $q_0 \ge 1$ and $\beta = 2/d + \delta'/d$ with appropriate $\delta' > 0$ depending on $\delta > 0$. Applying Corollary 3.25 with $b = -a = 1/10$ (note that both conditions $|a| + |b| \le r^2/5$ and $b - a \le 1$ are always satisfied) gives the bound

$$\Delta_r \le \frac{\operatorname{vol} H_r}{10} + b_{\beta,d}\, d_Q\, r^{d-2}\big(\rho_Q(r) + q^{(d-2)/4} r^{1-d/2} (q/q_0)^{(d+1)/2}\big),$$

where

$$\rho_Q(r) = q \, |\det Q|^{\frac{1-2\beta}{4}} \inf\nolimits^*_{T_-,T_+} \big\{ \, |\det Q|^{\frac{1-2\beta}{4}} \; T_-^{d(\frac{1}{2}-\beta)} + \gamma_{[T_-,1],\beta}(r) + \gamma_{(1,T_+],\beta}(r) \log(T_+) \big\}$$

and the infimum is taken over all $T_- \in [q_0^{-1/2} r^{-1}, 1]$ and

$$T_+ \gg_{\beta,d} \max\{1, \log(10qc_{\beta,d}))^2\}.$$

Thus, we can take $T_+ \asymp_{\beta,d} \log(q+1)^2$. In view of the lower bound $\mathrm{vol}_{\mathbb{Z}} H_r \gg_d d_Q r^{d-2}$, established in Lemma 3.17 (see (3.97)), we may also take at least

$$r \gg_{\beta,d} q^{1/2}(q/q_0)^{(d+1)/(d-2)} \tag{4.21}$$

in order to get

$$b_{\beta,d} d_Q r^{d/2-1} q^{(d-2)/4} (q/q_0)^{(d+1)/2} \leq \frac{\mathrm{vol}\, H_r}{10},$$

where we used $r \geq q^{1/2}$ in the form $q^{(d-2)/4} \leq r^{d/2-1}$. Likewise, we may take

$$T_- \asymp_{\beta,d} (q^{-1} |\det Q|^{\beta-1/2})^{\frac{2}{d(1-2\beta)}} = q^{-\frac{2}{d(1-2\beta)}} |\det Q|^{-1/d}$$

to obtain

$$b_{\beta,d} d_Q r^{d-2} q |\det Q|^{1/2-\beta} T_-^{\varsigma} \leq \frac{\mathrm{vol}\, H_r}{10}.$$

In order to guarantee that $T_- \in [q_0^{-1/2} r^{-1}, 1]$ is satisfied, we have to choose

$$r \gg_{\beta,d} q_0^{-1/2} q^{2/(d-4)+\delta} |\det Q|^{1/d}.$$

First Case: We consider first classes of (irrational) quadratic forms Q for which we can approximate the number of lattice points by the volume: Corresponding to Diophantine properties of Q, we assume that

$$b_{\beta,d} q |\det Q|^{1/4-\beta/2} \left(\gamma_{[T_-,1],\beta}(r) + \gamma_{[1,T_+],\beta}(r)\log(T_+)\right) \leq h_{\beta,d} \tag{4.22}$$

with some constant $h_{\beta,d} > 0$ depending on d and β only such that

$$5\mathrm{vol}_{\mathbb{Z}} H_r \geq \mathrm{vol}\, H_r,$$

compare with (3.97) of Lemma 3.17. Note that $r \geq q^{1/2}$ is fixed here. Taking a priori

$$r \asymp_{\beta,d} q^{1/2}(q/q_0)^{(d+1)/(d-2)} q^{2/(d-4)+\delta} \tag{4.23}$$

guarantees that $\mathrm{vol}_{\mathbb{Z}} H_r \geq 2$, i.e. there exists at least one non-zero lattice point $m \in \mathbb{Z}^d \setminus \{0\}$ satisfying both $|Q[m]| \leq \varepsilon$ and $\|Q_+^{1/2} m\| \leq r$. Note that the choice (4.23) ensures also that both conditions (4.21) and (4.3) are satisfied, whereby we may increase the implicit constant if necessary.

Second Case: Now we assume that the inequality in (4.22) does not hold. Then there exists a $t_0 \in [T_-, T_+]$ such that the reciprocal α_d-characteristic satisfies at least

$$\alpha_d(\lambda_{t_0}) r^{-d} \geq E(t_0) \overset{\mathrm{def}}{=} (h_{\beta,d}/b_{\beta,d})^{\frac{2}{1-2\beta}} q^{-\frac{2}{1-2\beta}} |\det Q|^{-1/2} \log(T_+)^{-1} \tag{4.24}$$

By Lemma 4.5 we have a 'good' rational approximation of $t_0 Q$: There exists a symmetric integral-valued matrix $M \in \mathrm{Sym}(d, \mathbb{Z})$ and a positive integer $k \in \mathbb{N}$ such that

$$\|M - k t_0 Q\| \ll_d q d_Q (\alpha_d(\Lambda_{t_0}) r^{-d})^{-1} r^{-2} \leq q d_Q E(t_0)^{-1} r^{-2}, \tag{4.25}$$

where $1 \le k \le \beta_{t;r} \le d_Q E(t_0)^{-1}$. In view of (4.25) and

$$\log(q+1)^2 \gg_{\beta,d} T_+ \ge t_0 \ge T_- \gg_{\beta,d} q^{-1-\frac{2}{d-4}-\delta/2},$$

we need that

$$r \gg_d E(t_0)^{-1/2}(q/q_0)^{1/2}q^{1/(d-4)+\delta/4}d_Q^{1/2} \gg_{\beta,d} (q/q_0)^{1/2}q^{2/(d-4)+\delta},$$

where we used $q^\delta \gg_\delta \log(\log(q+e))$. In fact, increasing the implicit constant in (4.23) ensures that the last condition is satisfied. This choice together with the Courant-Fischer theorem guarantees that M and kt_0Q have the same signature. In particular, M is invertible. Now we shall apply Theorem 7.1 with $\Lambda = M_+^{1/2}$ and $A[x] := M[M_+^{-1/2}x]$. Hence there exists a non-trivial lattice point $m \in \mathbb{Z}^d \setminus \{0\}$, which is an isotropic point of M, i.e. $M[m] = 0$, and is bounded as follows:

$$\|M_+^{1/2}m\| \ll_d |\det M|^{1/(2d_0)} \ll |\det Q|^{1/(2d_0)}\,(kt_0)^{d/(2d_0)}. \tag{4.26}$$

Although the dimension d_0 of a maximal isotropic $\mathbb{Q}$-subspace depends on M, the bound (7.3) depends on (r,s) only; that is $d/d_0 \le 2\rho+1$, where ρ is defined as in (1.4). Because of $\|M_+^{1/2}m\| \gg (q_0kt_0)^{1/2}\|m\|$ we conclude in combination with (4.25) and (4.26) that

$$\begin{aligned} |Q[m]| &= \big|(t_0k)^{-1}M[m] - Q[m]\big| \\ &\le (t_0k)^{-1}\|M - t_0kQ\| \cdot \|m\|^2 \\ &\le (q/q_0)(kt_0)^{2\rho-1}\,|\det Q|^{\frac{2\rho+1}{d}}\, d_Q E(t_0)^{-1}r^{-2} \\ &\ll_{\beta,d} (q/q_0)(\log(\varepsilon)^2 + \log(q)^2)\,|\det Q|^{\frac{2\rho+1}{d}-\rho}\,\varepsilon^{1-2\rho}E(t_0)^{-2\rho}r^{-2}. \end{aligned} \tag{4.27}$$

In view of (4.24) we need to take

$$r \asymp_{\beta,d} (q/q_0)^{1/2}\,q^{\frac{2d}{d-4}\rho+\delta}\,|\det Q|^{\frac{2\rho+1}{2d}}\,\varepsilon^{-\delta-\frac{3d-4}{d-4}\rho}$$

in order to guarantee that $|Q[m]| < \varepsilon$. For this choice we also find

$$\begin{aligned} \|Q_+^{1/2}m\| \ll (q/q_0)^{1/2}(kt_0)^{-1/2}\|M_+^{1/2}m\| &\ll_{\beta,d} (q/q_0)^{1/2}\,|\det Q|^{\frac{2\rho+1}{2d}}\,(kt_0)^\rho \\ &\ll_{\beta,d} (q/q_0)^{1/2}\,q^{\frac{2d}{d-4}\rho+\delta}\,|\det Q|^{\frac{2\rho+1}{2d}}\,\varepsilon^{-\delta-\frac{3d-4}{d-4}\rho}. \end{aligned}$$

This concludes the proof of Theorem 1.9. □

Given a quadratic form Q of Diophantine type (κ, A), i.e. Q satisfies (4.3), we shall apply Corollary 4.6 in order to estimate the Diophantine factors explicitly. Hereby, we prove quantitative bounds in the Oppenheim conjecture by comparing the weighted volume with the corresponding lattice sum.

Proof of Corollary 4.3: We begin by applying Corollary 3.25: Taking $b = -a = \varepsilon$ and $T_- = (10\,a_d b_{\beta,d} q)^{-1/\varsigma}\,|\det Q|^{-1/d}$, where $a_d > 0$ denotes the implicit constant from (3.97), yields the lattice remainder bound

$$\begin{aligned} \Delta_r \le \frac{\operatorname{vol} H_r}{5} &+ r^{d-2}C_Q\,b_{\beta,d}\,(2\varepsilon\,\gamma_{[T_-,1],\beta}(r) + \gamma_{(1,T_+],\beta}(r)\log(2\varepsilon T_+)) \\ &+ 2\varepsilon\, b_{\beta,d}\,|Q|^{-1/2}\,q^{(d-2)/4}\,r^{d/2-1}\,(q/q_0)^{(d+1)/2}, \end{aligned}$$

provided that $r \geq q^{1/2}$, $r\,q_0^{1/2} \geq (10\,a_d b_{\beta,d}\,q)^{1/\varsigma}\,|\det Q|^{1/d}$ and

$$T_+ = C_{\beta,d}(2\varepsilon)^{-1}\max\{1, \log(2\varepsilon(qc_{\beta,d})^{-1})^2\}.$$

Since Q is of Diophantine type (κ, A), we can use Corollary 4.6 in order to find that

$$\gamma_{[T_-,1],\beta}(r) \ll_d A^{-\frac{1-2\beta}{2(\kappa+1)}}\,|\det Q|^{\frac{1}{4}-\frac{\beta}{2}+\frac{1-2\beta}{2d(\kappa+1)}}\;q^{\frac{1}{d(\kappa+1)}}\,r^{-\frac{1-2\beta}{\kappa+1}}$$

and also that

$$\gamma_{(1,T_+],\beta}(r) \ll_{\beta,d} A^{-\frac{1-2\beta}{2(\kappa+1)}}\,|\det Q|^{\frac{1}{4}-\frac{\beta}{2}}\,\varepsilon^{-\frac{\kappa}{\kappa+1}(\frac{1}{2}-\beta)}\,r^{-\frac{1-2\beta}{\kappa+1}}\max\{1, \log(\tfrac{2\varepsilon}{q\,c_{\beta,d}})^2\}^{\frac{\kappa}{\kappa+1}(\frac{1}{2}-\beta)}.$$

In view of (3.97), we may increase $r \gg_{Q,\beta,d} 1$ to get

$$2\varepsilon\, b_{\beta,d}\, r^{d-2}\big(C_Q\gamma_{[T_-,1],\beta}(r) + r^{-d/2+1}|Q|^{-1/2}q^{(d-2)/4}(q/q_0)^{(d+1)/2}\big) \leq \frac{\operatorname{vol} H_r}{5}.$$

Because of the condition $1/2 > \beta > 2/d$, we shall take $\beta = 2/d + \delta'$ for a suitable δ' depending on δ. Now, we choose $r \asymp_{Q,\delta,d} \varepsilon^{-(2d+3\kappa d-4\kappa)/(2d-8)-\delta}$ in order to obtain

$$b_{\beta,d}\,C_Q\,r^{d-2}\log(2\varepsilon T_+)\,\gamma_{(1,T_+],\beta}(r) \leq \frac{\operatorname{vol} H_r}{5}.$$

All in all, we have

$$5\operatorname{vol}_{\mathbb{Z}} H_r \geq \operatorname{vol} H_r \gg_d |Q|^{-1/2}\,\varepsilon\, r^{d-2}.$$

Since $(2d + 3\kappa d - 4\kappa)/(2d - 8) \geq 1/(d - 2)$ holds if $d \geq 5$, we find that $\operatorname{vol}_{\mathbb{Z}} H_r > 1$. This means that there exists at least one non-zero lattice point $m \in \mathbb{Z}^d$ satisfying both $|Q[m]| < \varepsilon$ and also $\|Q_+^{1/2}m\| \ll_d r$. □

Using the Diophantine estimates for quadratic forms Q of Diophantine type (κ, A) again, we can estimate $\rho_{Q,b-a}$ and $\rho^*_{Q,b-a}$ in Corollary 3.4 explicitly as follows.

Proof of Corollary 4.4: First, we consider 'wide shells', i.e. $b - a \geq q$. By applying Corollary 4.6, we can bound the Diophantine factor from Corollary 3.4 by

$$\rho_{Q,b-a}(r) \ll_d \inf\nolimits^*_{T_-,T_+}\Big\{\log\big(\tfrac{b-a}{T_-^{\varsigma}}+1\big)^d\Big(h_Q\big(q^{\beta d-\frac{1}{2}}(T_-^{\varsigma} + A^{-\nu}T_-^{-\nu}r^{-2\nu}) + A^{-\nu}T_+^{\kappa\nu}r^{-2\nu}\log(T_+ + 1)\big) + \tfrac{\log(qT_+ + 1)}{T_+}\Big)\Big\},$$

where $h_Q := q\,|\det Q|^{\frac{1}{2}-\beta}$, $\nu := (1 - 2\beta)/(2\kappa + 2)$ and the infimum is taken over all $T_- \in [q_0^{-1/2}r^{-1}, 1]$ and $T_+ \geq 1$. Optimizing this expression by taking $T_- = r^{-2\nu/(\nu+\sigma)}$, note that $T_- \in [q_0^{-1/2}r^{-1}, 1]$ because of $\sigma \geq \nu$, and $T_+ = r^{(2\nu)/(\kappa\nu+1)}$ leads to

$$\rho_{Q,b-a}(r) \ll_{\beta,d} \log(r+1)^d\,h_Q\,q^{\beta d-\frac{1}{2}}(1 + A^{-\nu})(r^{-\frac{2\nu\sigma}{\nu+\sigma}} + r^{-\frac{2\nu}{\kappa\nu+1}}\log(qr + 1)).$$

In view of the bound from Corollary 3.4 and (3.97) we get the relative lattice error

$$\Big|\frac{\operatorname{vol}_{\mathbb{Z}} H_r}{\operatorname{vol} H_r} - 1\Big| \ll_{Q,\Omega,\beta,d} (b-a)^{-1}\log(r+1)^d\Big(r^{-\frac{2\nu\sigma}{\nu+\sigma}} + r^{-\frac{2\nu}{\kappa\nu+1}}\log(r+1) + r^{-\frac{d}{2}+2}\log\big(1+\tfrac{b-a}{r}\big)\Big).$$

For 'thin shells', i.e. $b-a \le q$, we have

$$\rho^*_{Q,b-a}(r) \ll_{\beta,d} \inf^*_{T_-,T_+}\big\{ \log\big(1+T_-^{-\varsigma}\big)^d h_Q\big((b-a)(T_-^{\varsigma} + A^{-\nu}T_-^{-\nu}r^{-2\nu}) + A^{-\nu}T_+^{\kappa\nu}r^{-2\nu}(\log((b-a)^*T_+)\big)+1)\big)\big\},$$

where the infimum is taken over all $T_- \in [r^{-1},1]$ and $T_+ \ge 1$ satisfying

$$T_+ \ge 4(b-a)^{-1}T_-^{-\varsigma}\max\{1, \log(c_Q^2(b-a)T_-^{-\varsigma})^2\}. \qquad \square$$

In order to prove Theorem 1.11, we need to show that quadratic forms with $\mathbb{Q}$-independent algebraic coefficients are Diophantine forms in the sense of Definition 4.1. In fact, we have

Lemma 4.7. Suppose Q is a form such that $k+1$ non-zero entries $y, x_1, \dots, x_k$ satisfy the property that

$$\max_{i=1,\dots,k} |q\, x_i/y + p_i| > Aq^{-\kappa}$$

for all k-tuples $(p_1/q, \dots, p_k/q)$ of rationals. Then Q is Diophantine of type (κ, A'), where A' depends on $A, y, x_1/y, \dots, x_k/y$ only (see (4.28)).

Proof: Let $M \in \mathrm{Sym}(d,\mathbb{Z})$, $m \in \mathbb{Z}\setminus\{0\}$ and $t \in [1,2]$. Denoting the entries in M corresponding to the coordinates of Q in which $y, x_1, \dots, x_k$ appear by $q, p_1, \dots, p_k$, we find the inequality

$$\|M - m\,tQ\| \ge \max\big\{ \max_{1\le i\le k} |p_i - m\,tx_i|, |q - m\,ty|\big\}.$$

Suppose that the expression on the right-hand side is strictly less than $A'm^{-\kappa}$, where

$$A' = \min\{A\,(5y\,(1+\max_{1\le i\le k}|x_i/y|))^{-1}, 1/2\}. \tag{4.28}$$

Note first that $|m| \ge |mty|/(2y) > q/(4y)$ and hence

$$\left|\frac{x_i}{y}q - p_i\right| \le \left|\frac{x_i}{y}\right| |q - mty| + |mtx_i - p_i| < A'm^{-\kappa}(1+|x_i/y|) < Aq^{-\kappa}$$

for all $i = 1, \dots, k$, which yields a contradiction. $\square$

In particular any form Q for which one ratio of two of its entries is a Diophantine number, is Diophantine in the sense of Definition 4.1 and hence almost all forms are Diophantine in this sense. An example of Diophantine forms for which we can control the exponent κ is the following: Suppose Q is a form with $k+1$ entries $y, x_1, \dots, x_k$ such that $x_1/y, \dots, x_k/y$ are algebraic and $1, x_1/y, \dots, x_k/y$ are linearly independent over $\mathbb{Q}$, then Schmidt's Subspace Theorem together with Lemma 4.7 implies that for any $\eta > 0$ the form Q is Diophantine of type $(1/k + \eta, A')$, where A' is a constant depending only on $\eta, A, y, x_1/y, \dots, x_k/y$, proving Theorem 1.11. However, as is usually the case in Diophantine approximation, the constant A and hence A' is ineffective in the sense that these constants cannot be determined explicitly.

4.4 Davenport-Lewis Conjecture

As a side remark, we illustrate that the techniques developed here are capable to prove the Davenport-Lewis Conjecture for indefinite and positive definite forms as well. Davenport and Lewis [DL72] have conjectured for *positive definite* quadratic forms that the distance between successive values v_n of the quadratic form $Q[x]$ on $\mathbb{Z}^d$ converges to zero as $n \to \infty$, provided that the dimension d is at least five and Q is irrational. This conjecture was proved by Götze in [Göt04]. Using the explicit error bounds for the lattice point counting problem we shall investigate the density of values of the quadratic form as well:

Corollary 4.8. Let Q denote a non-degenerate indefinite form in $d \geq 5$ Diophantine of type (κ, A), for some $\kappa > 0$ and $A > 0$. For $\delta > 0$ and a fixed, sufficiently small constant $c_1 > 0$, depending on Ω, we obtain for the maximal gap $d(r)$ between successive values of the quadratic form in the set $V(r)$

$$d(r) \leq c_{\delta,d,\Omega,\kappa,A,Q}\, r^{-\nu_0+\delta}, \tag{4.29}$$

for sufficiently large r, where $\nu_0 := \frac{2d-8}{2d+3\kappa d-4\kappa}$ and $c_{\delta,d,\Omega,\kappa,A,Q} > 0$ denotes a constant depending on κ, A, Q, Ω and d and $0 < \delta < 1/10$. For a more detailed description see Corollaries 3.4 and 4.4 below.

Proof: It is sufficient to prove that $\mathrm{vol}_{\mathbb{Z}^d}(r\Omega \cap E_{a,b}) > 0$ for any $a, b \in [-c_1 r^2, c_1 r^2]$ with $c_{\beta,d,\Omega,Q}\, r^{-\nu_0} = b - a$ and a sufficiently large constant $c_{\beta,d,\Omega,Q} > 0$. Moreover, we have $b - a \leq 1$ for sufficiently large $r \geq q^{1/2}$. Since Corollary 3.25 can be also extended to general (not necessarily admissible) parallelepipeds $\Omega = A^{-1}[-1,1]$, which satisfies (3.89), we can argue as in the last proof: Taking $r = (c_{\beta,d,\Omega,Q})^{-1/\nu_0}(b-a)^{-1/\nu_0}$ in Corollary 3.25, where $\nu_0 := \frac{2d-8}{2d+3\kappa d-4\kappa} - \delta$, leads to $\mathrm{vol}_{\mathbb{Z}^d}(r\Omega \cap E_{a,b}) > 0$. □

CHAPTER 5

Appendix A

The Appendix A constitutes sufficient preparation for the diagonal case. We revisit well-known moment estimates for quadratic exponential sums and prove a refined variant of Weyl's inequality as well. Additionally, we discuss the existence of compactly-supported smoothing kernels with fast-decaying Fourier transforms.

5.1 Mean-Value Estimates for Quadratic Exponential Sums

For the special case of diagonal quadratic forms, i.e. Q is of the form $Q[m] = \sum_{k=1}^{d} q_k m_k^2$ with eigenvalues $q_1, \dots, q_d$ of absolute values at least one, we will need the following well-known (and simple) moment estimates for the associated quadratic exponential sums

$$S_j(\alpha) = \sum_{P<|q_j|^{1/2}m<2dP} \exp(2\pi \mathrm{i} \alpha q_j m^2),$$

where $j = 1, \dots, d$ and $P \gg |q_j|^{1/2}$.

Lemma 5.1. For any $n \geq 4$ we have

$$\int_0^{|q_j|^{-1}} |S_j(\alpha)|^n \,\mathrm{d}\alpha \ll |q_j|^{-\frac{n}{2}} P^{n-2} (\log P). \tag{5.1}$$

Remark 5.2. Lemma 5.1 will be used in the proof of Lemma 2.6 in order to bound integrals of the type $\int_0^X |S_1(\alpha) \dots S_{d-1}(\alpha)| \,\mathrm{d}\alpha$. It is possible to deduce these estimates by applying a special form of the large sieve combined with the effective error bounds from Chapter 3: In 1974 A. Selberg used Beurling's function in order to obtain

$$\int_\alpha^\beta \left| \sum_{k=1}^{N} a(k) \mathrm{e}^{2\pi \mathrm{i} \nu_k t} \right|^2 \mathrm{d}t \ll (\beta - \alpha + \delta^{-1}) \sum_{k=1}^{N} |a(k)|^2,$$

where $\nu_1, \dots, \nu_N$ are well-spaced real numbers in the sense that $|\nu_n - \nu_m| \geq \delta$ whenever $m \neq n$. See Vaaler's survey article [Vaa85] for a detailed discussion on Beurling's function and applications.

Proof: Using the trivial estimate $|S_j(\alpha)| \ll |q_j|^{-\frac{1}{2}} P$ we shall reduce the problem to the case $n = 4$ as follows:

$$\int_0^{|q_j|^{-1}} |S_j(\alpha)|^n \,\mathrm{d}\alpha \ll |q_j|^{-\frac{n-4}{2}} P^{n-4} \int_0^{|q_j|^{-1}} |S_j(\alpha)|^4 \,\mathrm{d}\alpha.$$

Next we make the change of variables $\alpha = |q_j|^{-1}\theta$ and get

$$\int_0^{|q_j|^{-1}} |S_j(\alpha)|^n \,\mathrm{d}\alpha \ll |q_j|^{-\frac{n-2}{2}} P^{n-4} \int_0^1 \left| \sum_{P<|q_j|^{1/2}m<2dP} \mathrm{e}^{2\pi \mathrm{i} \theta m^2} \right|^4 \mathrm{d}\theta.$$

Obviously, the integral on the right-hand side represents the number of solutions of

$$v_1^2 + v_2^2 = w_1^2 + w_2^2,$$

where v_i, w_i range over the interval of summation. This number can be bounded by

$$\sum_{n \le N} r^2(n), \tag{5.2}$$

where $r(n)$ denotes the number of representations of n as a sum of two squares and $N = 8d^2P^2|q_j|^{-1}$. In view of the next Lemma 5.3 we see that this sum is $\ll N \log N$. □

By applying Dirichlet's hyperbola method we will prove the following folklore estimate for the second moment of $r(n)$.

Lemma 5.3. Let $r(n) := \#\{(x, y) \in Z^2 : x^2 + y^2 = n\}$ denote the number of representations of n as a sum of two squares (with multiplicity). Then we have

$$\sum_{1 \le n \le N} r(n)^2 \ll N \log N. \tag{5.3}$$

Remark 5.4. In the case $n \ge 10$ one might appeal to the Hardy-Littlewood asymptotic formula (see e.g. [Nat96], Theorem 5.7) and for $n \ge 6$ we could use the results in [CKO05] to drop the term $\log N$ as well, but this wouldn't have any effect on Theorem 1.6. For completeness, we also note that the best known asymptotic formula for (5.2) can be found in [Küh93].

Proof of Lemma 5.3: Because of $r(n)^2 = \#\{(x_1, y_1, x_2, y_2) \in \mathbb{Z}^4 : x_1^2 + y_1^2 = x_2^2 + y_2^2 = n\}$ we see that the sum in (5.3) is bounded by

$$\#\{(x_1, y_1), (x_2, y_2) \in (\mathbb{Z} \cap [-\sqrt{N}, \sqrt{N}])^2 : x_1^2 + y_1^2 = x_2^2 + y_2^2\}.$$

We shall transform this problem into a multiplicative one by introducing the new variables

$$X_1 = x_1 + x_2, \qquad X_2 = x_1 - x_2, \qquad Y_1 = y_2 + y_1, \qquad Y_2 = y_2 - y_1.$$

In fact, we can rewrite the equation $x_1^2 + y_1^2 = x_2^2 + y_2^2$ by

$$X_1X_2 = (x_1 + x_2)(x_1 - x_2) = x_1^2 - x_2^2 = y_2^2 - y_1^2 = (y_2 + y_1)(y_2 - y_2) = Y_1Y_2$$

and each solution (x_1, y_2, x_2, y_2) of the initial Diophantine equation gives rise to only one integer solution (X_1, X_2, Y_1, Y_2) with $|X_1|, |X_2|, |Y_1|, |Y_2| \le 2\sqrt{N}$. Thus, we have

$$\textstyle\sum_{1 \le n \le N} r(n)^2 \le 2^4 \#\{(X_1, X_2, Y_1, Y_2) \in \mathbb{Z}^4 : 0 \le X_1, X_2, Y_1, Y_2 \le 2\sqrt{N},\ X_1X_2 = Y_1Y_2\},$$

where the factor 2^4 comes from all possible sign patterns. Now we count the number of integral points $0 \le X_1, X_2, Y_1, Y_2 \le 2\sqrt{N}$ satisfying $X_1X_2 = Y_1Y_2$. If one of the variables is zero, then necessarily another variable is also zero, and therefore there are at most $4 \cdot (2\sqrt{N})^2 \ll N$ solutions. Hence, we can suppose that all coordinates are none zero. In this case, we can factorize any solution (X_1, X_2, Y_1, Y_2) in coprime factors and get

$$a_1 := \frac{X_1}{(X_1, Y_1)} = \frac{Y_2}{(Y_2, X_2)} \qquad \text{and} \qquad a_2 := \frac{Y_1}{(X_1, Y_1)} = \frac{X_2}{(Y_2, X_2)}.$$

Thus, writing $b_1 = (X_1, Y_1)$ and $b_2 = (X_2, Y_2)$, we find that all solutions with non-zero coordinates can be bounded by

$$\sum_{\substack{1\leq a_1,a_2\leq 2\sqrt{N}\\(a_1,a_2)=1}} \sum_{1\leq b_1,b_2\leq 2\sqrt{N}\min(a_1^{-1},a_2^{-1})} 1 \leq 2 \sum_{1\leq a_1\leq a_2\leq 2\sqrt{N}} \sum_{1\leq b_1,b_2\leq 2\sqrt{N}a_2^{-1}} 1.$$

Here, the last inequality follows by symmetry. Furthermore, since there are at most $4Na_2^{-2}$ integers b_1, b_2, we find the bound

$$\ll N \sum_{1\leq a_1\leq a_2\leq 2\sqrt{N}} \frac{1}{a_2^2} \leq N \sum_{1\leq a_2\leq 2\sqrt{N}} \frac{1}{a_2} \ll N \log N,$$

where we used again that there are at most a_2^{-1} integers a_1. This concludes the proof of the second moment bound (5.3). □

5.2 A Refined Variant of Weyl's Inequality

The purpose of this section is to prove Lemma 5.10 - a refined variant of Weyl's inequality for the exponential sums $S_j(\alpha) = \sum_{P<|q_j|^{1/2}m<2dP} \exp(2\pi \mathrm{i}\alpha q_j m^2)$, where $q_1, \ldots, q_d$ denote the eigenvalues of the diagonal form $Q[m] = \sum_{k=1}^{d} q_k m_k^2$. Both results, Lemmata 5.5 and 5.10, are already proven in [BD58b] (see Lemma 9 and the subsequent Corollary) for the case $d = 5$. In fact, the same proofs apply here if the endpoints of summation and integration are adjusted; the main idea is to split the sum on the left-hand side of (5.6) according to the residue classes $\bmod\, q$ and then apply Poisson's summation formula to each of these sums.

Lemma 5.5. Suppose that $A \gg_k 1$ and that α is a real number with approximation

$$\alpha = \frac{x}{y} + \beta, \tag{5.4}$$

where $x, y \in \mathbb{Z}$ are coprime integers satisfying

$$0 < y \ll_k A, \qquad 4k|\beta| < y^{-1}A^{-1}. \tag{5.5}$$

and $k > 1$ is a fixed integer. Then

$$\sum_{A<m<kA} \exp(2\pi \mathrm{i}\alpha m^2) = y^{-1} S_{x,y} \int_A^{kA} \exp(2\pi \mathrm{i}\beta\xi^2)\, \mathrm{d}\xi + \mathcal{O}(y^{1/2}\log 2y), \tag{5.6}$$

where the $\mathcal{O}$-term dependent on k only and

$$S_{x,y} \overset{\mathrm{def}}{=} \sum_{m=1}^{y} \exp(2\pi \mathrm{i} x m^2/y). \tag{5.7}$$

Remark 5.6. This lemma can be generalized to higher powers m^k, $k > 2$, as already done by Davenport in his investigations on Waring's problem for cubes and quartics, see [Dav39a] and [Dav39b] for more details.

Before proving this lemma, we introduce for any integer $z \in \mathbb{Z}$ the generalized Gauss sum

$$S_{x,y,z} \overset{\mathrm{def}}{=} \sum_{m=1}^{y} \exp(2\pi \mathrm{i}(xm^2 + zm)/y)$$

as well. Following the arguments in [Vin54] (Chapter 2),we shall first establish the bound

$$|S_{x,y,z}| \ll y^{1/2}, \tag{5.8}$$

which will be used in the proof of Lemmata 5.5 and 5.10. We begin by identifying $\mathbb{Z}/y\mathbb{Z}$ with $\mathbb{Z}/y_1\mathbb{Z} \times \ldots \mathbb{Z}/y_m\mathbb{Z}$ by the Chinese remainder theorem, where $y = y_1 \ldots y_m$ is the canonical decomposition into coprime numbers $y_1, \ldots, y_m$: Let $m_j = \prod_{i \neq j} y_i$ and choose $a_j \in \mathbb{Z}$ with $a_j m_j \equiv 1 \bmod y_j$, then the map $\mathbb{Z}/y_1\mathbb{Z} \times \ldots \mathbb{Z}/y_m\mathbb{Z} \to \mathbb{Z}/y\mathbb{Z}$ via $(r_1, \ldots, r_m) \mapsto \sum_{j=1}^m a_j m_j r_j$ is bijective. This implies

$$S_{x,y,z} = \sum_{r_1=1}^{y_1} \ldots \sum_{r_m=1}^{y_m} \exp(2\pi \mathrm{i}(x(\textstyle\sum_{j=1}^m a_j m_j r_j)^2 + z\sum_{j=1}^m a_j m_j r_j)/y) = \prod_{j=1}^m S_{xa_j^2 m_j, y_j, z a_j}.$$

Thus, the bound (5.8) follows at once from the case, where $y = p^k$ is a power of a prime number p. Note that we have here $(xa_j^2 m_j, q_j) = 1$, too. Moreover, if p is an odd prime number, there exists an integer $x^* \in \mathbb{Z}$ with $2xx^* \equiv 1 \bmod p^k$ and then we have $xm^2 + zm \equiv x(m + zx^*)^2 - xz^2(x^*)^2 \bmod p^k$ and consequently $|S_{x,p^k,z}| = |S_{x,p^k}|$. In other words, we can assume that $z = 0$ if p is odd. In the case $p = 2$ this argument cannot be applied and, in particular, it is possible that $|S_{x,2^k,z}| \neq |S_{x,2^k}|$. We begin by considering the case of odd prime numbers.

Lemma 5.7. For any odd prime number $p \in \mathbb{P}$ and coprime integer $a \in \mathbb{Z}$ we have

$$|S_{a,p}| \leq \sqrt{p}. \tag{5.9}$$

Proof: Since $\mathbb{Z}/p\mathbb{Z}$ is a finite field, the group $(\mathbb{Z}/p\mathbb{Z})^*$ is generated by one element g. Obviously, $m^2 \equiv w \bmod p$, $w \in (\mathbb{Z}/p\mathbb{Z})^*$, is solvable if and only if the index of w is a multiple of $h := (2, p-1)$ and then there are h solutions. Thus, using

$$\sum_{m=0}^{h-1} \exp\left(2\pi\mathrm{i}\frac{m \operatorname{ind} w}{h}\right) = \begin{cases} h & \text{if } h \mid \operatorname{ind} w \\ 0 & \text{otherwise} \end{cases},$$

we may write

$$S_{a,p} = 1 + \sum_{m=0}^{h-1}\sum_{w=1}^{p-1} \exp\left(2\pi\mathrm{i}\frac{m \operatorname{ind} w}{h} + 2\pi\mathrm{i}\frac{aw}{p}\right) = \sum_{m=1}^{h-1}\sum_{w=1}^{p-1} \exp\left(2\pi\mathrm{i}\frac{m \operatorname{ind} w}{h}\right) \exp\left(2\pi\mathrm{i}\frac{aw}{p}\right).$$

Now we use the basic technique of Weyl differencing: The Cauchy-Schwarz inequality implies

$$|S_{a,p}|^2 \leq (h-1) \sum_{m=1}^{h-1} \sum_{v,w=1}^{p-1} \exp\left(2\pi\mathrm{i}m\frac{\operatorname{ind} v - \operatorname{ind} w}{h}\right) \exp\left(2\pi\mathrm{i}\frac{v-w}{p}\right)$$

and, because the map $t \mapsto wt$ is a bijection of $(\mathbb{Z}/p\mathbb{Z})^*$, this can be rewritten as

$$|S_{a,p}|^2 \leq (h-1) \sum_{m=1}^{h-1} \sum_{t,w=1}^{p-1} \exp\left(2\pi\mathrm{i}m\frac{\operatorname{ind} t}{h}\right) \exp\left(2\pi\mathrm{i}w\frac{t-1}{p}\right).$$

Since the sum over $w = 1, \ldots, p-1$ for $t = 1$ is exactly $p - 1$ and (by adding all terms with $z = 0$, i.e. $\sum_{t=2}^{p-1} \exp(2\pi\mathrm{i}m \operatorname{ind} t/h)$) is zero otherwise, we conclude further that

$$|S_{a,p}|^2 \leq (h-1) \sum_{m=1}^{h-1} \Big(p - 1 - \sum_{t=2}^{p-1} \exp(2\pi\mathrm{i}m \operatorname{ind} t/h)\Big).$$

If t runs from 1 to $p-1$, the index of t takes all values $\bmod h$ equally often and therefore the last sum is $(h-1)^2 p$ and thus (5.9) holds as claimed. □

Remark 5.8.

(1) The above argument can be used in order to prove the more general estimate

$$\left|\sum_{m=1}^{p} \exp(2\pi \mathrm{i} a m^k/p)\right| \leq [(k, p-1) - 1]p^{1/2}$$

as well, where p is an odd prime number, a is an integer not divisible by p and $k \in \mathbb{N}$.

(2) One can even prove that $|S_{a,p}| = p^{1/2}$ for odd prime numbers p. To do this, one verify that $S_{a,q}$ is the *Gauss sum* associated with the character $\chi(n) = (n \mid p)$ induced by the Legendre symbol and, since χ is a primitive character modulo p, that $|S_{a,p}|^2 = p$. For details see e.g. Theorem 8.15 in [Apo76].

(3) As the reader probably already knows, Gauss has proved the explicit formula

$$S_{1,m} = \frac{1}{m}\sqrt{m}(1+\mathrm{i})(1+\mathrm{e}^{-\pi \mathrm{i} m/2}) \quad (m \in \mathbb{N})$$

in his famous Disquisitiones Arithmeticae (published 1801). This formula can be utilized in order to prove the quadratic reprocity law (see e.g. [Apo76], Sections 9.9–9.11). In view of the separability property $S_{a,p} = (a \mid p)S_{1,p}$, which holds for any odd prime number p, we can even determine the complex sign of $S_{a,q}$.

The previous Lemma 5.7 together with the following Lemma 5.9 already implies $|S_{x,p^k,z}| \leq p^{k/2}$ for any odd prime number $p \in \mathbb{P}$: We have $|S_{x,p^k,z}| = |S_{x,p^k}| = p^j |S_{x,p^{k-2j}}|$ with some $j \in \mathbb{N}_0$ such that $k - 2j \in \{0, 1\}$ and also $|S_{x,p^{k-2j}}| \leq p^{k/2-j}$. Moreover, we see also that the inequality (5.8) cannot be improved, since for example $|S_{x,p^2}| = \sqrt{p}$.

Lemma 5.9. If $k \geq 2$, $p \in \mathbb{P}$ is an odd prime number and $(a, p) = 1$, then we have

$$S_{a,p^k} = pS_{a,p^{k-2}}.$$

On the other hand, we have for $k \geq 4$

$$S_{a,2^k} = 2S_{a,2^{k-2}}.$$

Proof: If p is an odd prime number, we transform the sum S_{a,p^k} by changing variables via $m = p^{k-1}t + r$, where $t = 0, \ldots, p-1$ and $r = 0, \ldots, p^{k-1} - 1$ are taken independently. This gives the identity

$$S_{a,p^k} = \sum_{t=0}^{p-1} \sum_{r=0}^{p^{k-1}-1} \exp\left(2\pi \mathrm{i}\left(\frac{ar^2}{p^k} + \frac{2atr}{p}\right)\right),$$

where we have used that $\exp(2\pi \mathrm{i} a p^{k-2} t^2) = 1$ provided that $k \geq 2$. For any fixed r, which is not divisible by p, the sum over t runs through a complete residue system (because $2ar$ is invertible in $\mathbb{F}_p$) and therefore vanishes. Hence we obtain

$$S_{a,p^k} = p \sum_{m=0}^{p^{k-2}-1} \exp\left(2\pi \mathrm{i}\frac{am^2}{p^{k-2}}\right) = pS_{a,p^{k-2}}.$$

In the second case we introduce the more general sums

$$\widetilde{S}_{x,y,z} \stackrel{\text{def}}{=} \sum_{m=0}^{q-1} \exp(\pi \mathrm{i}(xm^2 + zm)/y)$$

and also $\widetilde{S}_{x,y} := \widetilde{S}_{x,y,0}$. Similarly, changing variables as above shows that

$$S_{a,2^k,b} = \sum_{r=0}^{2^{k-1}-1} \mathrm{e}^{2\pi\mathrm{i}(ar^2+br)/2^k} + \sum_{r=0}^{2^{k-1}-1} \mathrm{e}^{2\pi\mathrm{i}[a(2^{k-1}+r)^2+b(2^{k-1}+r)]/2^k} = (1+\mathrm{e}^{\pi\mathrm{i}b})\widetilde{S}_{a,2^{k-1},b} \quad (5.10)$$

provided that $k \geq 2$. In particular $S_{a,2^k} = 2\widetilde{S}_{a,2^{k-1}}$. On the other hand, we may change variables via $m = 2^{k-2}t + r$, where $y = 0, \ldots, 3$ and $z = 0, \ldots, 2^{k-2} - 1$, in order to get

$$S_{a,2^k} = \sum_{r=0}^{2^{k-2}-1} \exp\left(2\pi\mathrm{i}\frac{ar^2}{2^k}\right) \sum_{t=0}^{3} \exp(\pi\mathrm{i}tar) = 2\sum_{r=0}^{2^{k-2}-1} \exp\left(2\pi\mathrm{i}\frac{ar^2}{2^k}\right)(1+(-1)^r),$$

since $k \geq 4$ and a is odd. Moreover, this can be rewritten as

$$S_{a,2^k} = 4\sum_{r=0}^{2^{k-3}-1} \exp\left(\pi\mathrm{i}\frac{ar^2}{2^{k-3}}\right) = 4\widetilde{S}_{a,2^{k-3}} = 2S_{a,2^{k-2}}. \qquad \square$$

We are left to prove (5.8) for the case $y = 2^k$. By (5.10) we have $S_{x,2^k,z} = 0$ if z is odd. Otherwise, we can argue as in the case of odd prime numbers: Write $z = 2z'$ with $z' \in \mathbb{Z}$ and take $x^* \in \mathbb{Z}$ with $xx^* \equiv 1 \bmod 2^k$ in order to get $xm^2 + zm = x(m + z'x^*)^2 - x(z'x^*)^2 \bmod 2^k$. Thus, we have $|S_{x,2^k,z}| = |S_{x,2^k}|$ and now (5.8) follows by the same arguments as before.

Proof of Lemma 5.5: By varying A we may suppose that A and kA are irrational and not near an integer (without contributing $A \gg_k 1$). Note that the sum and the integral in (5.6) change by an amount of $\mathcal{O}(1)$ only. Then we split the sum in (5.6) according to the residue classes modulo q to get

$$\begin{aligned}\sum_{A<m<kA} \exp(2\pi\mathrm{i}\alpha m^2) &= \sum_{r=1}^{y} \exp(2\pi\mathrm{i}xr^2/y) \sum_{\substack{A<m<kA \\ m\equiv r \bmod y}} \exp(2\pi\mathrm{i}\beta m^2) \\ &= \sum_{r=1}^{y} \exp(2\pi\mathrm{i}xr^2/y) \sum_{(A-r)/y<m<(kA-r)/y} \exp(2\pi\mathrm{i}\beta(ym+r)^2).\end{aligned} \quad (5.11)$$

Here we need a generalized variant of Poisson's summations formula and therefore we repeat the proof in a special case: Let $f(s) := 1_{((A-r)/y,(kA-r)/y)}(s)\exp(2\pi\mathrm{i}\alpha(ys+r)^2)$ and

$$F(s) \overset{\text{def}}{=} \sum_{l\in\mathbb{Z}} f(s+l)$$

the periodic extension of f. F is locally a finite sum, 1-periodic and has jumping discontinuities at $(A-r)/y + \mathbb{Z}$ and $(kA-r)/y + \mathbb{Z}$. Moreover, we can write $F = P + \mathrm{i}Q$ with piecewise continuous and monotonic function P, Q on $[0,1]$. Hence, the Fourier series of F converges towards the mean of the left- and right-hand limits of F (see e.g. [Vin54], Chapter 1, Lemma 11). Since A and kA are irrational, $s = 0$ is a point of continuity for F. Therefore, we may find that the inner sum in (5.11) is equal to

$$\int_{(A-r)/y}^{(kA-r)/y} \exp(2\pi\mathrm{i}\beta(y\eta+r)^2)\,\mathrm{d}\eta + \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \int_{(A-r)/y}^{(kA-r)/y} \exp(2\pi\mathrm{i}[\beta(y\eta+r)^2+\nu\eta])\,\mathrm{d}\eta,$$

where the convergence is understood in the sense that the summation shall be taken symmetric over negative and positive integers. Next, we make the substitution $y\eta + r = \xi$ in order to obtain

$$\sum_{A<m<kA} \exp(2\pi \mathrm{i} \alpha m^2) = \frac{1}{y} \sum_{r=1}^{y} \mathrm{e}^{2\pi \mathrm{i} x r^2/y} \left(\int_A^{kA} \mathrm{e}^{2\pi \mathrm{i} \beta \xi^2} \,\mathrm{d}\xi + \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \int_A^{kA} \mathrm{e}^{2\pi \mathrm{i}[\beta\xi^2 + \nu(\xi - r)/y]} \,\mathrm{d}\xi \right)$$

and this can be rewritten as

$$\sum_{A<m<kA} \exp(2\pi \mathrm{i} \alpha m^2) = \frac{S_{x,y}}{y} \int_A^{kA} \exp(2\pi \mathrm{i} \beta \xi^2) \,\mathrm{d}\xi + R$$

with an error term R, which is given by

$$R \overset{\text{def}}{=} \frac{1}{y} \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} S_{x,y,-\nu} \int_A^{kA} \exp(2\pi \mathrm{i}[\beta\xi^2 + \nu\xi/y]) \,\mathrm{d}\xi. \tag{5.12}$$

In order to bound (5.12) we would like to make a change of variables in the last integral as well: For this, we write

$$\beta\xi^2 + \frac{\nu\xi}{y} = \beta \left(\xi + \frac{\nu}{2y\beta} \right)^2 - \frac{\nu^2}{4\beta y^2}$$

and note that the last condition in (5.5) implies

$$\frac{|\nu|}{2y|\beta|} > 2kA.$$

Therefore the new variable $|\beta|^{-1}\zeta = \{\xi + \nu/(2y\beta)\}^2$ is not vanishing (neither for positive nor negative ν) and the integral in (5.12) becomes

$$\exp\left(-\pi \mathrm{i} \frac{\nu^2}{2\beta y^2} \right) \frac{|\beta|^{-1/2}}{2} \operatorname{sgn}(\nu\beta) \int_{\zeta_1}^{\zeta_2} \zeta^{-1/2} \exp(\operatorname{sgn}(\beta) 2\pi \mathrm{i} \zeta) \,\mathrm{d}\zeta,$$

where

$$\zeta_1 \overset{\text{def}}{=} |\beta| \left(A + \frac{\nu}{2y\beta} \right)^2 \qquad \text{and} \qquad \xi_2 = |\beta| \left(kA + \frac{\nu}{2y\beta} \right)^2.$$

For notational simplicity we may suppose that $\beta > 0$. Because of $\zeta_1 > 0$ and $\zeta_2 > 0$ we obtain by integrating by parts

$$\int_{\zeta_1}^{\zeta_2} \zeta^{-1/2} \mathrm{e}^{2\pi \mathrm{i} \zeta} \,\mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \left(\zeta_2^{-1/2} \mathrm{e}^{2\pi \mathrm{i} \zeta_2} - \zeta_1^{-1/2} \mathrm{e}^{2\pi \mathrm{i} \zeta_1} \right) - \frac{1}{4\pi \mathrm{i}} \int_{\zeta_1}^{\zeta_2} \zeta^{-3/2} \mathrm{e}^{2\pi \mathrm{i} \zeta} \,\mathrm{d}\zeta. \tag{5.13}$$

Again integrating by parts shows that the last integral is of order $\mathcal{O}(\zeta_1^{-3/2} + \zeta_2^{-3/2}) = \mathcal{O}(y^3|\beta|^{3/2}|\nu|^{-3})$. In view of (5.8) the contribution of these terms to (5.12) is at most

$$\ll y^2|\beta| \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} |S_{x,y,-\nu}| \cdot |\nu|^{-3} \ll y^{5/2}|\beta| \ll_k y^{3/2} A^{-1} \ll_k y^{1/2},$$

where (5.5) was used in the last two steps. It remains to bound the sum over the first terms on the right-hand side of (5.13). Since both terms can be treated similarly, we consider only of one of them. For example write

$$\frac{1}{2\pi \mathrm{i}}\zeta_2^{-1/2}\exp(2\pi \mathrm{i}\zeta_2) = \frac{y|\beta|^{1/2}}{\pi \mathrm{i}}\operatorname{sgn}(\nu)\frac{\exp(2\pi \mathrm{i}[k^2A^2\beta + kA\nu/y])}{\nu + 2kAy\beta}\exp\left(\pi \mathrm{i}\frac{\nu^2}{2\beta y^2}\right)$$

and note that the sum over $|\nu| \le y^2$ can be bounded by

$$\ll \sum_{0<|\nu|\le y^2} |S_{x,y,-\nu}||\nu + 2kAy\beta|^{-1} \ll y^{1/2}\sum_{0<|\nu|\le y^2} |\nu|^{-1} \ll y^{1/2}\log 2y,$$

where we used the typical integral comparison argument in the last inequality. The remaining sum over $|\nu| > y^2$ can be written as

$$\sum_{r=1}^{y}\exp(2\pi \mathrm{i}xr^2/y)\sum_{|\nu|>y^2}\frac{1}{v + 2kAy\beta}\exp(2\pi \mathrm{i}\nu(kA - r)/y),$$

except for a factor of absolute value one. Splitting the summation (of the inner sum) into positive and negative ν, we shall apply summation by parts (Abel's lemma) to each part separately. Again, both cases can be treated similarly. For example we define

$$A(t) := \sum_{0<\nu\le t}\exp(2\pi \mathrm{i}\nu(kA - r)/y), \quad \varphi(x) = \frac{1}{v + 2kAy\beta}$$

and note that $|A(t)| \le |\sin(\pi(kA - r)/y)| \ll \{(kA - r)/y\}^{-1}$, where $\{\theta\} = \min_{m\in\mathbb{Z}}|m - \theta|$ denotes the distance to the nearest integer. Recall that we have supposed that A is not near an integer, and therefore $\{(kA - r)/y)\} \gg_k y^{-1}$. In particular $\varphi(s)A(s) \to 0$ for $s \to \infty$. All in all, summation by parts shows that

$$\begin{aligned}\sum_{\nu>q^2}\frac{1}{v + 2kAy\beta}\exp(2\pi \mathrm{i}\nu(kA - r)/y) &= -A(y^2)\varphi(y^2) - \int_{y^2}^{\infty} A(u)\varphi'(u)\,\mathrm{d}u \\ &\ll y^{-2}\{(kA - r)/y\}^{-1} \ll_k y^{-1}\end{aligned}$$

and hence the overall contribution is $\mathcal{O}(1)$. This completes the proof of (5.6). □

Finally we shall apply Lemma 5.5 to the quadratic exponential sums S_j as follows.

Lemma 5.10. Suppose that $x_j, y_j \in \mathbb{Z}$ are coprime integers with $0 < y_j \le 8dP|q_j|^{-\frac{1}{2}}$ and

$$q_j\alpha = \frac{x_j}{y_j} + \beta_j,$$

where $|\beta_j| < y_j^{-1}(8dP|q_j|^{-\frac{1}{2}})^{-1}$ and $P \gg |q_j|^{1/2}$. Then we have

$$|S_j(\alpha)| \ll y_j^{-\frac{1}{2}}(\log P)\min(P|q_j|^{-\frac{1}{2}}, P^{-1}|q_j|^{\frac{1}{2}}|\beta_j|^{-1}). \tag{5.14}$$

Proof: Applying Lemma 5.5 with $A = P|q_j|^{-1/2}$, $k = 2d$, $x = x_j$, $y = y_j$ and α replaced by $q_j\alpha$ shows that

$$S_j(\alpha) = y_j^{-1}S_{x_j,y_j}\int_{P|q_j|^{-1/2}}^{2dP|q_j|^{-1/2}}\exp(2\pi \mathrm{i}\beta_j\xi^2)\,\mathrm{d}\xi + \mathcal{O}(y_j^{1/2}\log 2y_j).$$

Because of $y_j^{1/2} \ll y_j^{-1/2} P|q_j|^{-1/2}$, $\log(2y_j) \ll \log(P)$ and $y_j^{1/2} \ll y_j^{-1/2} P^{-1}|q_j|^{1/2}|\beta_j|^{-1}$, we are left to estimate the integral. Using $S_{x_j,y_j} \ll y_j^{1/2}$, see (5.8), and bounding the integral by the length of the integration interval yield $\ll y_j^{-1/2} P|q_j|^{-1/2}$. On the other hand, we can use the alternative representation

$$\int_{P|q_j|^{-1/2}}^{2dP|q_j|^{-1/2}} \exp(2\pi \mathrm{i} \beta_j \xi^2)\, \mathrm{d}\xi = \frac{1}{2} \int_{P^2|q_j|^{-1}}^{4d^2P^2|q_j|^{-1}} |\zeta|^{-1/2} \exp(2\pi \mathrm{i} \beta_j \zeta)\, \mathrm{d}\zeta$$

and apply partial integration in order to conclude that the integral is $\ll |\beta_j|^{-1} P^{-1} |q_j|^{1/2}$. Together with $S_{x_j,y_j} \ll y_j^{1/2}$ this proves already (5.14). □

5.3 Smoothing Kernels

In the next lemma we shall construct compactly-supported smoothing kernels with fast-decaying Fourier transforms. This construction extends the commonly used kernels (in the context of the circle method - see e.g. Lemma 1 in [Dav56] or [BK01]) by using convergent infinite convolution products (instead of convolving finitely many times). Since our kernel is used only rarely, we decided to include the proof (following the arguments in [BR86]).

Lemma 5.11 (Theorem 10.2 in [BR86])**.** Let $u\colon [1,\infty) \to [0,\infty)$ be a positive, continuous, strictly increasing function satisfying the decay condition

$$\int_1^\infty \frac{1}{tu(t)}\, \mathrm{d}t < \infty. \tag{5.15}$$

Then there exists a smooth symmetric probability density ψ on $\mathbb{R}$ such that

(1) ψ is supported in $[-1,1]$,

(2) ψ is increasing for $x < 0$ and ψ decreasing for $x > 0$,

(3) $|\widehat{\psi}(t)| \ll \exp(-|t|u(|t|)^{-1})$ and $\widehat{\psi}$ is real-valued and symmetric.

Remark 5.12. It is worth mentioning that this kernel will be used in both cases: Only after approximating the indicator function of $[a,b]$ and of the region Ω we can rewrite the lattice point counting error in terms of Fourier integrals. But also the circle method of Birch and Davenport, as used in the case of diagonal quadratic forms, relies on Fourier methods and requires a suitable kernel.

Proof: Let us first introduce the notation $U([-a,a]) = (2a)^{-1} 1_{[-a,a]}$, i.e. $U([-a,a])$ denotes the density of the uniform distribution on the interval $[-a,a]$, $a > 0$. As simple to check, the Fourier transform is given by

$$\widehat{U}([-a,a])(t) = \frac{1}{2a} \int_{-a}^{a} \cos(2\pi t x)\, \mathrm{d}x = \frac{\sin(2\pi a t)}{2\pi a t}. \tag{5.16}$$

We shall use this simple kernel as a basis for the infinite convolution product. From the condition (5.15) we see that there exists a positive integer $n_0 \in \mathbb{N}$ and a non-decreasing sequence of non-negative numbers $(a_n)_{n\in\mathbb{N}}$, given by

$$a_n = \begin{cases} \frac{\mathrm{e}}{n_0 u(n_0)} & \text{if } 1 \le n \le n_0 \\ \frac{\mathrm{e}}{n u(n)} & \text{if } n > n_0 \end{cases},$$

which satisfies

$$\sum_{n=1}^{\infty} a_r = \frac{\mathrm{e}}{u(n_0)} + \mathrm{e} \sum_{n=n_0+1}^{\infty} \frac{1}{nu(n)} \leq 1. \tag{5.17}$$

We shall check that

$$\psi := \lim_{n\to\infty} U([-a_1, a_1]) * \ldots * U([-a_n, a_n])$$

is uniformly convergent and has the claimed properties. To do this, we write $\psi_n := U([-a_1, a_1]) * \ldots * U([-a_n, a_n])$ and claim that ψ_n, $n > 1$, is Lipschitz-continuous with Lipschitz constant $1/(4a_0a_1)$. In fact, if $0 < b \leq a$, then we find

$$U([-a,a]) * U([-b,b])(t) = \begin{cases} 0 & \text{if } |t| \geq a+b \\ \frac{1}{2a} & \text{if } |t| \leq a-b\,. \\ \frac{a+b-|t|}{4ab} & \text{else} \end{cases} \tag{5.18}$$

Hence the above remark is true for $n = 2$. The general case follows by induction:

$$|u_{n+1}(s) - u_{n+1}(t)| \leq \frac{1}{2a_{n+1}} \int_{-a_{n+1}}^{a_{n+1}} |u_n(s-h) - u_n(t-h)| \,\mathrm{d}h \leq \frac{1}{4a_0a_1}|t-s|.$$

Proceeding in the same manner, we see that

$$\begin{aligned} |u_{n+1}(t) - u_n(t)| &\leq \frac{1}{2a_{n+1}} \int_{-a_{n+1}}^{a_{n+1}} |u_n(t-h) - u_n(t)| \,\mathrm{d}h \\ &\leq \frac{1}{4a_0a_1} \frac{1}{2a_{n+1}} \int_{-a_{n+1}}^{a_{n+1}} |h| \,\mathrm{d}h = \frac{a_{n+1}}{8a_0a_1}, \end{aligned}$$

if $n > 1$. In view of (5.17) this shows that $(\psi_n)_{n\in\mathbb{N}}$ is uniformly convergent, say to ψ. Obviously, ψ is non-negative and continuous. Moreover, since ψ_n has compact support lying in $[-\sum_{k=1}^n a_k, \sum_{k=1}^n a_k]$, we find that $\operatorname{supp}\psi \subset [-1,1]$. By the fundamental theorem of calculus we get that ψ_n is $(n-1)$-times continuous differentiable, where $n > 1$, and together with the uniform convergence we conclude that ψ is smooth with $\lim_{n\to\infty, n>k} \psi_n^{(k)} = \psi^{(k)}$ uniformly on $\mathbb{R}$. In particular, we have

$$\int \psi(x) \,\mathrm{d}x = \lim_{n\to\infty} \int \psi_n(x) \,\mathrm{d}x = 1$$

and hence ψ is a probability density. Additionally, we see by induction that every ψ_n is symmetric and thus also ψ. Similarly, we shall prove (2) by induction: For $n = 2$ this follows at once from (5.18). If $n \geq 2$ we have

$$\psi'_{n+1}(t) = \frac{1}{2a_{n+1}} \{\psi_n(t + a_{n+1}) - \psi_n(t - a_{n+1})\}.$$

At this point we may use the symmetry of ψ_n in order to conclude that both $\psi'_{n+1}(t) \geq 0$ if $t \leq 0$ and $\psi'_{n+1}(t) \leq 0$ if $t \geq 0$ hold, as claimed. Letting $n \to \infty$ yields (2) for ψ. Finally, it remains to prove (3). The uniform convergence combined with the explicit formula (5.16) implies the identity

$$\widehat{\psi}(t) = \prod_{n=1}^{\infty} \frac{\sin(2\pi a_n t)}{2\pi a_n t},$$

where the convergence is uniform on compact sets. Note that (5.15) necessarily implies $u(t) \to \infty$ if $t \to \infty$ and therefore there exists a $t_0 > 0$ such that $u(t) \geq 1$ for all $t \geq t_0$. Let $|t| \geq t_0$. In view of the bound

$$|\widehat{\psi}(t)| \leq \prod_{k=1}^{n} \left(\frac{1}{2\pi |a_k t|} \right) \leq \frac{1}{|a_n t|^n} = \left(\frac{n u(n)}{\mathrm{e}|t|} \right)^n$$

we may take $n = \lfloor |t| u(|t|)^{-1} \rfloor$, i.e. the integer part of $|t|u(|t|)^{-1}$, to obtain

$$|\widehat{\psi}(t)| \leq \left(\frac{u(n)}{\mathrm{e} u(|t|)} \right)^n \leq \mathrm{e}^{-n} \ll \exp\{-|t| u(|t|)^{-1}\}.$$

In the last line we used that u is non-decreasing and that $|t| \geq n$, since $|t| \geq t_0$. This completes the proof of Lemma 5.11. □

Remark 5.13. We note that Lemma 5.11 is due to Ingham [Ing34]. In general, there are many known variants of the *uncertainty principle*, i.e. that "a pair of transforms f and $\widehat{f}$ cannot both be very small"[1] as remarked by N. Wiener. As examples, one can mention Hardy's clarification of this remark [Har33] or a uniqueness theorem of Beurling, see [Hör91]. Also the identity theorem for analytic (resp. quasianalytic) functions is directly related to this question: If we suppose that $\widehat{f}$ has decay rate $\mathcal{O}(\exp\{-\varepsilon|x|\})$ for $|x| \to \infty$ and some $\varepsilon > 0$, then f can be extended analytically to the ε-stripe $\{z \in \mathbb{C} : -\varepsilon < \mathrm{Im}(z) < \varepsilon\}$ and therefore f cannot have compact support (unless f is zero everywhere). Ingham has extended this observation by proving that compactly supported functions f exist with $\widehat{f}(t) = \mathcal{O}(\exp\{-|t|u(t)^{-1}\})$, provided that (5.15) holds, and also that this condition is necessary. That means if $\int_1^\infty (tu(t))^{-1}\,\mathrm{d}t = \infty$, then there does not exist any finite measure with compact support and Fourier transform of growth $\mathcal{O}(\exp\{-|t|u(t)^{-1}\})$. A similar result can be found in [PW87] as well, see Theorem XII.

[1] This quote can be found in [Har33].

CHAPTER 6

Appendix B

The Appendix B is a revised and corrected version of certain technical parts of the preprint [GM13] of Götze and Margulis on approximating the number of lattice points in d-dimensional hyperbolic or elliptic shells $\{m \in \mathbb{Z}^d : a < Q[m] < b\}$, which are restricted to rescaled and growing domains $r\Omega$, by the volume.

6.1 Fourier Analysis, Smoothing and Theta-Series

In order to rewrite the lattice remainder, we will need certain well-known estimates for smooth approximations, estimates on the Theta-series $\vartheta_v(t)$ corresponding to the lattice counting problem and bounds on $\vartheta_v(t)$ in terms of the α-characteristic of a special symplectic lattice Λ_t as well, which was already introduced in (3.39). First, we prove the following smoothing estimate.

Lemma 6.1. Suppose that μ and ν are positive finite measures on $(\mathbb{R}^n, \mathcal{B}^n)$. Moreover, let f and $f_\tau^\pm, \tau > 0$, denote bounded real-valued Borel-measurable functions on $\mathbb{R}^n$ satisfying for any $\tau > 0$ the inequalities

$$\begin{aligned} f_\tau^-(x) &\le \inf\{f(y) : \|y-x\|_\infty < \tau\} \quad \text{and} \quad f_\tau^+(x) \ge \sup\{f(y) : \|y-x\|_\infty < \tau\}, \\ f_{2\tau}^-(x) &\le \inf\{f_\tau^-(y) : \|y-x\|_\infty < \tau\} \quad \text{and} \quad f_{2\tau}^+(x) \ge \sup\{f_\tau^+(y) : \|y-x\|_\infty < \tau\}. \end{aligned} \tag{6.1}$$

Then we have

$$\Big| \int f \, \mathrm{d}(\mu - \nu) \Big| \le \max_{\pm} \Big| \int f_\tau^\pm \, \mathrm{d}(\mu-\nu) * k_\tau \Big| + \int (f_{2\tau}^+ - f_{2\tau}^-) \, \mathrm{d}\nu. \tag{6.2}$$

Proof: Since k_τ is a probability measure with support contained in a $\|\cdot\|_\infty$-ball of radius τ, we conclude in view of (6.1) that the chain of inequalities

$$f_{2\tau}^- \le f_\tau^- * k_\tau \le f \le f_\tau^+ * k_\tau \le f_{2\tau}^+$$

holds. The inequality $f \le f_\tau^+ * k_\tau$ implies, for instance, the upper bound

$$\int f \, \mathrm{d}(\mu-\nu) \le \int f_\tau^+ * k_\tau \, \mathrm{d}(\mu-\nu) + \int (f_\tau^+ * k_\tau - f) \, \mathrm{d}\nu.$$

Now we may use $f \ge f_\tau^- * k_\tau \ge f_{2\tau}^-$ in the last integral in order to obtain

$$\int f \, \mathrm{d}(\mu-\nu) \le \int f_\tau^+ * k_\tau \, \mathrm{d}(\mu-\nu) + \int (f_\tau^+ * k_\tau - f_\tau^- * k_\tau) \, \mathrm{d}\nu. \tag{6.3}$$

In the same manner we get also the lower bound

$$\int f \, \mathrm{d}(\mu-\nu) \ge \int f_\tau^- * k_\tau \, \mathrm{d}(\mu-\nu) + \int (f_\tau^- * k_\tau - f) \, \mathrm{d}\nu$$

and together with $f \le f_\tau^+ * k_\tau \le f_{2\tau}^+$ also

$$\int f \, \mathrm{d}(\mu-\nu) \ge \int f_\tau^- * k_\tau \, \mathrm{d}(\mu-\nu) - \int (f_\tau^+ * k_\tau - f_{2\tau}^-) \, \mathrm{d}\nu. \tag{6.4}$$

Both estimates (6.3) and (6.4) together yields the claimed inequality (6.2). □

6.1.1 Estimates for $\vartheta_v(t)$

In order to estimate the terms I_Δ and I_ϑ in (3.33) we need to estimate $|\vartheta_v(t)|$ first. To do this, we need the following lemma on the multidimensional Gaussian integral.

Lemma 6.2. For any symmetric complex $d \times d$-matrix Ω, whose imaginary part is positive definite, we have

$$\int_{\mathbb{R}^d} \exp\{\pi \mathrm{i}\Omega[x] + 2\pi\mathrm{i}\langle x, v\rangle\}\,\mathrm{d}x = (\det(\Omega/\mathrm{i}))^{-1/2} \exp\left\{-\pi\mathrm{i}\Omega^{-1}[v]\right\}, \tag{6.5}$$

where we choose the branch of the square root which takes positive values on purely imaginary Ω, $v \in \mathbb{R}^d$ and $\Omega^{-1}\big[x\big]$ denotes the quadratic form $\langle \Omega^{-1}x, x\rangle$, defined by the inverse operator $\Omega^{-1}\colon \mathbb{C}^d \to \mathbb{C}^d$ whose imaginary part is negative definite.

Proof (see also [Mum83], p. 195, Lemma 5.8 and (5.6)): If Ω is purely imaginary, this formula follows at once by applying the change of variables $y = -\mathrm{i}\Omega$ and taking into account that $x \mapsto \exp(-\pi x^2)$ is invariant under Fourier transformation. The general case follows by splitting $\Omega = \Omega_1 - \mathrm{i}\Omega_2$ into the real and imaginary part and using successively that both sides in (6.5) are analytic as one-variable functions in the entries of Ω_1. □

Corollary 6.3 (Simple Bound for $\vartheta_v(t)$)**.** We have

$$|\vartheta_{ur}(t)| \ll_d d_Q r^{d/2} r_t^{d/2} \exp\Big\{ -\pi^2 r_t^2 Q_+^{-1}[u]\Big\}, \tag{6.6}$$

where $r_t := r(4\pi^2 t^2 r^4 + 1)^{-1/2}$ and $d_Q := |\det Q|^{-1/2}$ as already defined in (1.11).

Proof: Here we shall apply (6.5) to ϑ_v (as introduced (3.31) and (6.13)) by taking

$$\Omega_t \overset{\text{def}}{=} \mathrm{i}\pi^{-1}\tilde{Q}_t = 2tQ + \mathrm{i}\pi^{-1}r^{-2}Q_+.$$

This yields

$$\vartheta_v(t) = \int_{\mathbb{R}^d} \exp\left\{\pi\mathrm{i}\Omega_t[x] + 2\pi\mathrm{i}\big\langle x, v/r\big\rangle\right\}\mathrm{d}x = \big(\det(\Omega_t/\mathrm{i})\big)^{-1/2} \exp\{-\pi\mathrm{i}\Omega_t^{-1}[v/r]\} \tag{6.7}$$

and therefore the Fourier transform of $x \mapsto \exp\{Q_{r,v}(t,x)]\}$ is given by

$$\det\big(\pi^{-1}\tilde{Q}_t\big)^{-1/2} \exp\Big\{-\pi^2\tilde{Q}_t^{-1}[u - v/r]\Big\} = \vartheta_{v-ru}(t) = \vartheta_{ru-v}(t). \tag{6.8}$$

Since Q and Q_+ are simultaneously diagonalizable, a short calculation shows that

$$\tilde{Q}_t^{-1} = (4\pi^2 t^2 + r^{-4})^{-1}(2\pi\mathrm{i}tQ^{-1} + r^{-2}Q_+^{-1})$$

and likewise

$$\det\tilde{Q}_t^{-1} = (4\pi^2t^2 + r^{-4})^{-d} \prod_{i=1}^{d}(2\pi\mathrm{i}tq_i^{-1} + r^{-2}|q_i|^{-1}). \tag{6.9}$$

By taking the absolute value of (6.7) and (6.9) we find the bound (6.6). □

6.1.2 Estimation of I_ϑ and I_Δ

By means of the previous Lemma 6.1 we can already estimate I_ϑ as follows.

Lemma 6.4 (Estimation of I_ϑ)**.** If $d > 2$, then we have

$$I_\vartheta \ll_d d_Q \|\widehat{\zeta}\|_1 \min\{|b-a|\, q_0^{-1/2} r^{-1}, 1\}\, r^{d/2} q_0^{d/4}. \tag{6.10}$$

Proof: As a starting point, we note that that we have $|\vartheta_v(t)| \ll_d d_Q r^{d/2} r_t^{d/2}$ and $\widehat{g}_w(t) \ll s_{[a,b]_\omega}(t)$. The first bound follows from (6.6) with $v = ur$ and the second one is a direct consequence of (3.26). Using these bounds we may estimate the integral in (3.35) by

$$I_\vartheta = \left| \int_{J_1} \widehat{g}_w(t) \int_{\mathbb{R}^d} \vartheta_v(t)\, \widehat{\zeta}(v)\, \mathrm{d}v\, \mathrm{d}t \right| \ll_d d_Q r^{d/2} \|\widehat{\zeta}\|_1 \int_{|t| > q_0^{-1/2} r^{-1}} s_{[a,b]_\omega}(t)\, r_t^{d/2} \mathrm{d}t. \tag{6.11}$$

Depending on the length of the interval $[a, b]$ we distinguish following two cases. If $|b-a|^{-1} \leq q_0^{-1/2} r^{-1}$, then we can use $s_{[a,b]_w}(t) \leq |t|^{-1}$ in order to get the bound

$$\int_{q_0^{-1/2} r^{-1}}^{\infty} s_{[a,b]_w}(t)\, r_t^{d/2}\, \mathrm{d}t \leq r^{-d/2} \int_{q_0^{-1/2} r^{-1}}^{\infty} t^{-d/2-1}\, \mathrm{d}t \ll_d q_0^{d/4}.$$

In the case $|b-a|^{-1} > q_0^{-1/2} r^{-1}$, we have the bound $s_{[a,b]_w}(t) \leq |b-a+2w|\,/2$ and thus

$$\int_{|t| > q_0^{-1/2} r^{-1}} s_{[a,b]_\omega}(t)\, r_t^{d/2} \mathrm{d}t \ll r^{-d/2}\, |b-a+2w| \int_{q_0^{-1/2} r^{-1}}^{\infty} t^{-d/2}\, \mathrm{d}t \ll_d \frac{|b-a|}{q_0^{1/2}\, r}\, q_0^{d/4},$$

where $|w| < (b-a)/4$ and $d > 2$ was used as well. The last two estimates together with (6.11) yield the claimed bound (6.10). □

The error term I_Δ, corresponding to small values of the t-integration, will be estimated by using the following representations of $R(e_{tQ}\, \tilde{e}_{v,r}) = \theta_v(z) - \vartheta_v(t)$ in (3.29) by means of Poisson's formula (see [Boc48], §46). In fact, we can apply the Poisson summation formula and obtain

$$\theta_v(t) - \vartheta_v(t) = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \vartheta_{v-r\,m}(t). \tag{6.12}$$

Note that by definition (3.31) the Fourier transform of $x \mapsto \exp\{Q_{r,v}(t,x)\}$ at $u \in \mathbb{R}^d$ is given by $\vartheta_{v-r\,u}(t)$, where

$$\exp\{Q_{r,v}(t,x)\} = \exp\{-\tilde{Q}_t[x] + 2\pi \mathrm{i} \langle x, v\, r^{-1} \rangle\} \quad \text{and} \quad \tilde{Q}_t \stackrel{\text{def}}{=} r^{-2} Q_+ - 2\pi \mathrm{i}\, t\, Q. \tag{6.13}$$

In view of (3.30) and (6.12) we have

$$R(e_{tQ}\, v_r) = \int_{\mathbb{R}^d} \Big(\sum_{m \in \mathbb{Z}^d \setminus \{0\}} \vartheta_{v-r\,m}(t) \Big) \widehat{\zeta}(v)\, \mathrm{d}v. \tag{6.14}$$

Lemma 6.5 (Estimation of I_Δ)**.** Using the quantity $\|\widehat{\zeta}\|_{*,r}$ as defined in (3.5) for the weights $\zeta(x)$, we have the estimate

$$I_\Delta \ll_d d_Q r^{d/2} \log(1 + |b-a|\, q_0^{-1/2} r^{-1})\, \|\widehat{\zeta}\|_{*,r}. \tag{6.15}$$

Proof: According to (3.34), (3.30) and (6.12) we may write

$$I_\Delta = \left| \int_{J_0} \widehat{g}_w(t)\, R(e_{tQ}\, \upsilon_r)\, \mathrm{d}t \right|, \qquad \text{where}$$
$$R(e_{tQ}\,\upsilon_r) = \int_{\mathbb{R}^d} S_{t,v}\,\widehat{\zeta}(v)\,\mathrm{d}v, \qquad S_{t,v} \overset{\text{def}}{=} \sum_{m\in\mathbb{Z}^d\setminus\{0\}} \vartheta_{v-rm}(t). \tag{6.16}$$

In order to use the estimate (6.6) let $v \in \mathbb{R}^d$ and write $v = ru$ with $u = u_0 + m_u$, where $u_0 \in [-1/2, 1/2]^d$ and $m_u \in \mathbb{Z}^d$, then

$$|S_{t,v}| \le \sum_{m\neq m_u} |\vartheta_{r(u_0+m)}(t)| \ll d_Q r^{d/2} r_t^{d/2} \sum_{m\neq m_u} \exp\{-\pi^2 r_t^2 Q_+^{-1}[u_0+m]\}. \tag{6.17}$$

Note that $\|m+u_0\| \ge \|m+u_0\|_\infty \ge \frac{1}{2}$ for any $m \in \mathbb{Z}^d \setminus \{0\}$ and therefore $\frac{\pi^2}{2} Q_+^{-1}[u_0+m] \ge \frac{\pi^2}{8} q^{-1} \ge q^{-1}$ which yields the bound

$$|S_{t,v}| \ll d_Q r^{d/2} |r_t|^{d/2} \Big(\mathrm{e}^{-\pi^2 r_t^2 Q_+^{-1}[u_0]} I_r(v) + \mathrm{e}^{-\frac{r_t^2}{q}} K_{u_0} \Big) \tag{6.18}$$

where $I_r(v) := I(\|v\|_\infty \ge r/2)$ and $K_{u_0} := \sum_{m\in\mathbb{Z}^d} \exp\{-\frac{\pi^2}{2} r_t^2 Q_+^{-1}[m+u_0]\}$. The sum K_{u_0} may be estimated by an integral as follows: Since $r_t^2 \ge q_0/(4\pi^2+1)$ for $|t| \le q_0^{-1/2} r^{-1}$ as $r \ge q^{\frac{1}{2}}$, we have $\exp\{-\pi^2 r_t^2 Q_+^{-1}[u]\} \le \exp\{-\frac{q_0}{5} Q_+^{-1}[u]\}$. Let $I := [-\frac{1}{2}, \frac{1}{2}]^d$ and note that $Q_+^{-1}[x] \le \frac{1}{4q_0}$ for $x \in I$, from which we deduce that

$$k_u \overset{\text{def}}{=} \int_I \exp\{-\tfrac{q_0}{5} Q_+^{-1}[u+x]\}\,\mathrm{d}x \gg_d \exp\{-\tfrac{q_0}{5} Q_+^{-1}[u]\} \int_I \exp\{-\tfrac{2q_0}{5} \langle Q_+^{-1}u, x\rangle\}\,\mathrm{d}x,$$

where the integral on the right-hand side is at least 1 by Jensen's inequality. Hence

$$K_{u_0} \le \sum_{m\in\mathbb{Z}^d} \mathrm{e}^{-\frac{q_0}{5} Q_+^{-1}[m+u_0]} \ll_d \sum_{m\in\mathbb{Z}^d} k_{m+u_0} = \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{q_0}{5} Q_+^{-1}[x]}\,\mathrm{d}x \ll_d \Big(\frac{q}{q_0}\Big)^{\frac{d}{2}}. \tag{6.19}$$

Using (6.16) together with (6.18) and (6.19), we may now estimate I_Δ by the following integrals. Writing $v_0 = v - rm$, $\|v_0\|_\infty \le \frac{r}{2}$, $m \in \mathbb{Z}^d$, we have

$$I_\Delta \ll_d d_Q \int_{J_0} |\widehat{g}_w(t)| \big(\Theta_{t,1} + \Theta_{t,2}\big)\,\mathrm{d}t, \tag{6.20}$$

where

$$\Theta_{t,1} \overset{\text{def}}{=} \Big(\frac{q}{q_0}\Big)^{d/2} r^{d/2} r_t^{d/2} \mathrm{e}^{-\frac{r_t^2}{q}} \int_{\mathbb{R}^d} |\widehat{\zeta}(v)|\,\mathrm{d}v$$
$$\Theta_{t,2} \overset{\text{def}}{=} r^{d/2} r_t^{d/2} \int_{\|v\|_\infty > r/2} \exp\{-\pi^2 r_t^2 Q_+^{-1}[v_0 r^{-1}]\}\, |\widehat{\zeta}(v)|\,\mathrm{d}v.$$

Note that $t \mapsto r_t^2$ is strictly monotonically decreasing on J_0 from $r_0^2 = r^2$ to $q_0/50 \le r_t^2 \le q_0$ for $t = \pm q_0^{-1/2} r^{-1}$. If we write $h(s;x) := s^{d/4}\mathrm{e}^{-sx}$ with $s, x > 0$, then the maximum of $h(s;x)$ is attained at $s_0 = d/(4x)$. Hence, $\max_{t\in J_0} h(r_t^2; x) \ll_d \min(x^{-d/4}, r^{d/2}) \ll_d (x + \frac{1}{r^2})^{-d/4}$. Thus, we obtain with $x = 1/q$

$$\max_{t\in J_0} \Theta_{t,1} \ll_d (q/q_0)^{d/2} r^{d/2} q^{d/4} \|\widehat{\zeta}\|_1. \tag{6.21}$$

In order to estimate $\Theta_{t,2}$, we choose $x = Q_+^{-1}[v_0/r]/4$ and get

$$\sup_{t\in J_0} \Theta_{t,2} \ll_d r^{d/2} \int_{\|v\|_\infty > r/2} \frac{|\widehat{\zeta}(v)|}{(r^{-2} + Q_+^{-1}[v_0/r])^{d/4}} \,\mathrm{d}v. \tag{6.22}$$

Now we integrate the bounds (6.21) and (6.22) in $t \in J_0$ weighted with $|\widehat{g}_w(t)|$: In view of (3.26) we have $\int_{J_0} |\widehat{g}_w(t)| \,\mathrm{d}t \ll \log(1 + |b-a|\, q_0^{-1/2} r^{-1})$ and thus we finally get, using the quantity $\|\widehat{\zeta}\|_{*,r}$ as defined in (3.5) for the weights $\zeta(x)$, the estimate (6.15). □

6.1.3 Rewriting of I_θ

In this section we proceed to estimate (3.38), where we have already started to treat the term I_θ, see (3.36). As already announced, we shall bound the theta series $\theta_v(t)$ uniformly in v by another theta series in dimension $2d$ with symplectic structure. This step is crucial in order to transform the problem to averages over functions on the space of lattices subjected to actions of $\mathrm{SL}(2,\mathbb{R})$. We have

Lemma 6.6. Let $\theta_v(t)$ denote the theta function in (3.31) depending on Q and $v \in \mathbb{R}^d$. For $r \ge 1$, $t \in \mathbb{R}$ the following bound holds uniformly in $v \in \mathbb{R}^d$

$$\big|\theta_v(t)\big| \ll_d (\det Q_+)^{-1/4} r^{d/2} \psi(r,t)^{1/2}, \qquad \text{where} \tag{6.23}$$

$$\psi(r,t) \stackrel{\text{def}}{=} \sum_{m,n\in\mathbb{Z}^d} \exp\{-H_t(m,n)\}, \qquad \text{and} \tag{6.24}$$

$$H_t(m,n) \stackrel{\text{def}}{=} r^2 Q_+^{-1}[m - 4tQn] + r^{-2} Q_+[n], \tag{6.25}$$

and $H_t(m,n)$ is a positive quadratic form on $\mathbb{Z}^{2d}$.

Proof: For any $x, y \in \mathbb{R}^d$ the equalities

$$\begin{aligned} 2\,(Q_+[x] + Q_+[y]) &= Q_+[x+y] + Q_+[x-y], \\ \langle Q\,(x+y),\, x-y\rangle &= Q[x] - Q[y] \end{aligned} \tag{6.26}$$

hold. Rearranging $\theta_v(z)\,\overline{\theta_v(z)}$ and using (6.26), we would like to use $m+n$ and $m-n$ as new summation variables on a lattice. But both vectors have the same parity, i.e. $m+n \equiv m-n \mod 2$. Since they are dependent one has to consider the 2^d affine sublattices indexed by $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_j \in \{0,1\}$ for $1 \le j \le d$:

$$\mathbb{Z}^d_\alpha \stackrel{\text{def}}{=} \{m \in \mathbb{Z}^d \,:\, m \equiv \alpha \mod 2\},$$

where, for $m = (m_1, \dots, m_d)$, $m \equiv \alpha \mod 2$ means $m_j \equiv \alpha_j \mod 2$ for all $1 \le j \le d$. Thus writing

$$\theta_{v,\alpha}(t) \stackrel{\text{def}}{=} \sum_{m\in\mathbb{Z}^d_\alpha} \exp\left[-\frac{1}{r^2} Q_+[m] - 2\pi \mathrm{i} t Q[m] + 2\pi\mathrm{i}\,\langle m, \frac{v}{r}\rangle\right],$$

we obtain $\theta_v(t) = \sum_\alpha \theta_{v,\alpha}(t)$ and hence by the Cauchy-Schwarz inequality

$$\big|\theta_v(t)\big|^2 \le 2^d \textstyle\sum_{\alpha\in\{0,1\}^d} \big|\theta_{v,\alpha}(t)\big|^2. \tag{6.27}$$

Using (6.26) and the absolute convergence of $\theta_\alpha(t)$, we can write

$$\begin{aligned}|\theta_{v,\alpha}(t)|^2 &= \sum_{m,n\in\mathbb{Z}^d_\alpha} \exp\Big[-\frac{1}{r^2}\big(Q_+[m]+Q_+[n]\big) - 2\pi\mathrm{i}t\big(Q[m]-Q[n]\big) - 2\pi\mathrm{i}\langle m-n, \frac{v}{r}\rangle\Big] \\ &= \sum_{m,n\in\mathbb{Z}^d_\alpha} \exp\Big[-\frac{2}{r^2}\big(Q_+[\bar m]+Q_+[\bar n]\big) - 4\pi\mathrm{i}\big\langle 2tQ\bar m + \frac{v}{r}, \bar n\big\rangle\Big] \end{aligned} \tag{6.28}$$

where $\bar m = \frac{m+n}{2}$, $\bar n = \frac{m-n}{2}$. Note that the map

$$H\colon \bigcup\nolimits_{\alpha\in\{0,1\}^d} \mathbb{Z}^d_\alpha \times \mathbb{Z}^d_\alpha \longrightarrow \mathbb{Z}^d\times\mathbb{Z}^d, \quad (m,n)\longmapsto \Big(\frac{m+n}{2}, \frac{m-n}{2}\Big)$$

is a bijection. Therefore we get by (6.27)

$$\begin{aligned}\big|\theta_v(t)\big|^2 &\ll_d \sum_{\alpha\in\{0,1\}^d}\sum_{m,n\in\mathbb{Z}^d_\alpha} \exp\Big[-\frac{2}{r^2}\big(Q_+[\bar m]+Q_+[\bar n]\big) - 2\mathrm{i}\big\langle 2tQ\bar m+\frac{v}{r},\bar n\big\rangle\Big] \\ &= \sum_{\bar m,\bar n\in\mathbb{Z}^d} \exp\Big[-\frac{2}{r^2}\big(Q_+[\bar m]+Q_+[\bar n]\big) - 2\mathrm{i}\big\langle 2tQ\bar m+\frac{v}{r},\bar n\big\rangle\Big].\end{aligned} \tag{6.29}$$

In this double sum fix $\bar n$ and sum over $\bar m\in\mathbb{Z}^d$ first, and call the inner sum $\theta_v(t,\bar n)$. Using (6.5) with $\Omega = 2\mathrm{i}Q_+r^{-2}/\pi$ and $v = -4tQ\bar n + m$, we get for $\delta := \big(\det\big(\frac{2}{\pi r^2}Q_+\big)\big)^{-\frac12}$ by the symmetry of Q and Poisson's formula (see [Boc48], §46)

$$\begin{aligned}\theta_v(t,\bar n) &\overset{\text{def}}{=} \sum_{\bar m\in\mathbb{Z}^d} \exp\Big[-\frac{2}{r^2}\big(Q_+[\bar m]+Q_+[\bar n]\big) - 4\pi\mathrm{i}\big\langle 2tQ\bar m+\frac{v}{r},\bar n\big\rangle\Big] \\ &= \delta\sum_{m\in\mathbb{Z}^d}\exp\Big[-\frac{\pi^2r^2}{2}Q_+^{-1}[m-4t\,Q\bar n] - \frac{2}{r^2}Q_+[\bar n] - 4\pi\mathrm{i}\langle\frac{v}{r},\bar n\rangle\Big].\end{aligned}$$

Thus, we have uniformly in $v\in\mathbb{R}^d$

$$\big|\theta_v(t,\bar n)\big| \le \delta\sum_{m\in\mathbb{Z}^d}\exp\Big\{-\frac{\pi^2r^2}{2}Q_+^{-1}[m-4t\,Q\bar n] - \frac{2}{r^2}Q_+[\bar n]\Big\}. \tag{6.30}$$

Hence we obtain by (6.29) and (6.30)

$$\big|\theta_v(t)\big|^2 \ll_d (\det Q_+)^{-1/2} r^d \sum_{m,n\in\mathbb{Z}^d}\exp\{-G_t[m,n]\},$$

where $G_t[m,n] := \frac{\pi^2r^2}{2}Q_+^{-1}[m-4tQn] + \frac{2}{r^2}Q_+[n]$. Since $\pi^2/2 > 1$ we may bound $G_t[m,n]$ from below as follows:

$$G_t[m,n] \ge r^2Q_+^{-1}[m-4tQn] + r^{-2}Q_+[n] = H_t[m,n]$$

which proves the claimed estimate (6.23). □

We end this section by establishing the following lower and upper bound on the Siegel transform of a lattice.

Lemma 6.7. Let Λ be a lattice in $\mathbb{R}^d$. Assume that $0 < \varepsilon \leq 1$, then

$$\exp\{-d\varepsilon\}\#H \leq \sum_{v\in\Lambda} \exp\{-\varepsilon\,\|v^2\|\} \ll_d \varepsilon^{-d/2}\#H, \tag{6.31}$$

where $H \coloneqq \{v \in \Lambda : \|v\|_\infty < 1\}$.

We may apply Lemma 6.7 to find that $\psi(r,t) = \sum_{v\in\Lambda_t} \exp\{-\|v\|^2\} \asymp_d \#\{w \in \Lambda_t : \|w\|_\infty \leq 1\} \ll_d \#\{w \in \Lambda_t : \|w\| \leq d^{1/2}\}$, where Λ_t is defined as in (3.39). This step will be further carried out in Section 3.3.3.

Proof: The lower bound for the sum is obvious by restricting summation to the set of elements in H. As for the upper bound introduce for $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{Z}^d$ the sets

$$B_\mu \stackrel{\text{def}}{=} \left[\mu_1 - \frac{1}{2}, \mu_1 + \frac{1}{2}\right) \times \cdots \times \left[\mu_d - \frac{1}{2}, \mu_d + \frac{1}{2}\right)$$

such that $\mathbb{R}^d = \bigcup_{\mu\in\mathbb{Z}^d} B_\mu$. For any fixed $w^* \in H_\mu \coloneqq \Lambda \cap B_\mu$ we have $w - w^* \in H$ for all $w \in H_\mu$. Hence we conclude for any $\mu \in \mathbb{Z}^d$

$$\#H_\mu \leq \#H.$$

Since $x \in B_\mu$ implies $\|x\|_\infty \geq \|\mu\|_\infty/2$, we obtain

$$\sum_{v\in\Lambda} \mathrm{e}^{-\varepsilon\,\|v\|^2} \leq \sum_{v\in\Lambda} \mathrm{e}^{-\varepsilon\,\|v\|_\infty^2} \leq \sum_{\mu\in\mathbb{Z}^d}\ \sum_{v\in\Lambda\cap B_\mu} \mathrm{e}^{-\frac{\varepsilon}{4}\,\|\mu\|_\infty^2} \leq \#H \sum_{\mu\in\mathbb{Z}^d} \mathrm{e}^{-\frac{\varepsilon}{4}\,\|\mu\|^2} \ll_d \varepsilon^{-d/2}\#H.$$

This concludes the proof of Lemma 6.7. □

6.2 Margulis' Averaging Result

In the following paragraphs we present Margulis' averaging method which will be used in Section 3.4.3 to prove explicit bounds for averages over the group K of type $\int_{\mathrm{K}} \alpha_d(d_r k\Lambda)^\beta \,\mathrm{d}k$.

6.2.1 Operators A_g and Functions τ_λ on $\mathrm{SL}(2,\mathbb{R})$

Let $\mathrm{G} = \mathrm{SL}(2,\mathbb{R})$. We consider the following two subgroups of G:

$$\mathrm{K} = \mathrm{SO}(2) = \{k_\theta : 0 \leq \theta < 2\pi\} \quad \text{and} \quad \mathrm{T} = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} : a > 0,\, b \in \mathbb{R}\right\},$$

where k_θ is defined as in (3.68) by

$$k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

According to the Iwasawa decomposition, any $g \in G$ can be uniquely represented as a product of elements from K and T, that is

$$g = k(g)t(g), \quad k(g) \in \mathrm{K},\, t(g) \in \mathrm{T}.$$

Now let

$$d_a \stackrel{\text{def}}{=} \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \text{ for } a > 0 \text{ and } \mathrm{D}^+ = \{d_a : a \geq 1\}.$$

According to the Cartan decomposition, we have

$$\mathrm{G} = \mathrm{KD^+K},\ g = k_1(g)d(g)k_2(g),\ g \in \mathrm{G}, k_1(g), k_2(g) \in \mathrm{K},\ d(g) \in \mathrm{D}^+.$$

In this decomposition $d(g)$ is determined by g, and if $g \notin \mathrm{K}$ then $k_1(g)$ and $k_2(g)$ are also determined by g up to a factor of ± 1 on k_1 and k_2. It is clear that $\|g\| = \|d(g)\|$, where $\|\cdot\|$ denotes the operator norm on $\mathrm{GL}(n, \mathbb{R})$ induced by the standard Euclidean norm on $\mathbb{R}^n$. Note that, in the simple case $g = d_a$, this norm is given by $\|d_a\| = a$. Since d_a is the conjugate of $d_{a^{-1}}$ by $k_{\pi/2}$, we see that $g^{-1} \in \mathrm{K}g\mathrm{K}$ or equivalently, $d(g) = d(g^{-1})$ for any $g \in \mathrm{G}$. Therefore, $\|g\| = \|g^{-1}\|$, $g \in \mathrm{G}$.

We say that a function f on G is *left* K-*invariant* (resp. *right* K-*invariant*, resp. *bi*-K-*invariant*) if $f(\mathrm{K}g) = f(g)$ (resp. $f(g\mathrm{K}) = f(g)$, resp. $f(\mathrm{K}g\mathrm{K}) = f(g)$). Any bi-K-invariant function on G is completely determined by its restriction to D^+. Hence for any bi-K-invariant function f on G, there is a function $\hat{f}$ on $[1, \infty)$ such that $f(g) = \hat{f}(\|g\|)$, $g \in \mathrm{G}$.

For any $\lambda \in \mathbb{R}$ we define a character x_λ of T by

$$\chi_\lambda \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^{-\lambda}$$

and the function $\varphi_\lambda \colon \mathrm{G} \to \mathbb{R}^+$ by

$$\varphi_\lambda(g) = \chi_\lambda(t(g)), \quad g \in G. \tag{6.32}$$

The function φ_λ has the property

$$\varphi_\lambda(kgt) = \chi_\lambda(t)\varphi_\lambda(g), \quad g \in \mathrm{G},\ k \in \mathrm{K},\ t \in \mathrm{T}, \tag{6.33}$$

and it is completely determined by this property and the condition $\varphi_\lambda(1) = 1$.

Definition 6.8. For $g \in \mathrm{G}$ and a continuous action of G on a topological space X, we define the operator A_g on the space of continuous functions on X by

$$(A_g f)(x) = \int_{\mathrm{K}} f(gkx)\, \mathrm{d}\sigma(k), \quad x \in X, \tag{6.34}$$

where σ is the normalized Haar measure on K, or, using the parametrization of K, by

$$(A_g f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(gk(\theta)x)\, \mathrm{d}\theta, \quad x \in X.$$

The operator A_g is a linear map into the space of left K-invariant functions on X. If $X = \mathrm{G}$ and G acts on itself by left translations, then A_g commutes with right translations. From these two remark we get that

$$A_g\varphi_\lambda(k\tilde{g}t) = A_g\varphi_\lambda(\tilde{g}t) = A_g(\varphi_\lambda(\cdot t))(\tilde{g}) = \chi_\lambda(t)A_g\varphi_\lambda(\tilde{g}) \quad (\tilde{g} \in \mathrm{G},\ k \in \mathrm{K},\ t \in \mathrm{T}),$$

i.e. $A_g\varphi_\lambda$ has the property (6.33). Hence φ_λ is an eigenfunction for A_g with the eigenvalue

$$\tau_\lambda(g) \overset{\text{def}}{=} (A_g\varphi_\lambda)(1) = \int_{\mathrm{K}} \varphi_\lambda(gk)\, \mathrm{d}\sigma(k) = \int_{\mathrm{K}} \chi_\lambda(t(gk))\, \mathrm{d}\sigma(k). \tag{6.35}$$

We see from (6.35) that τ_λ is obtained from φ_λ by averaging over right translations by elements of K. But φ_λ is left K-invariant and A_g commutes with right translations. Hence the function τ_λ is bi-K-invariant and it is an eigenfunction for A_g with the eigenvalue $\tau_\lambda(g)$, that is

$$(A_g\tau_\lambda)(h) = \tau_\lambda(g)\tau_\lambda(h) \quad \text{for all} \quad h \in \mathrm{G}. \tag{6.36}$$

We have that

$$\varphi_\lambda(g) = \|ge_1\|^{-\lambda}, \quad g \in G,\, e_1 = (1,0), \tag{6.37}$$

where $\|\cdot\|$ denotes the usual Euclidean norm on $\mathbb{R}^2$. Indeed

$$\varphi_\lambda(g) = \chi_\lambda(t(g)) = \|t(g)e_1\|^{-\lambda} = \|k(g)t(g)e_1\|^{-\lambda} = \|ge_1\|^{-\lambda}.$$

From (6.35) and (6.37) we get

$$\begin{aligned}\tau_\lambda(g) &= \int_K \|gke_1\|^{-\lambda}\,\mathrm{d}\sigma(k) = \frac{1}{2\pi}\int_0^{2\pi} \|gk(\theta)e_1\|^{-\lambda}\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \|g(\cos\theta,\sin\theta)\|^{-\lambda}\,\mathrm{d}\theta = \int_{S^1} \|gu\|^{-\lambda}\,\mathrm{d}\ell(u),\end{aligned} \tag{6.38}$$

where S^1 is the unit circle in $\mathbb{R}^2$ and ℓ denotes the normalized rotation invariant measure on S^1. One can easily see that $\|gu\|^{-2}$, $g \in G$, $u \in S^1$, is equal to the Jacobian at u of the diffeomorphism $v \mapsto gv/\|gv\|$ of S^1 onto S^1. On the other hand, it follows from the change of variables formula that

$$\int_M J_f^\lambda = \int_M J_{f^{-1}}^{1-\lambda}, \quad \lambda \in \mathbb{R},$$

where $f\colon M \to M$ is a diffeomorphism of a compact differentiable manifold M and J_f (resp. $J_{f^{-1}}$) denotes the Jacobian of f (resp. f^{-1}). Now using (6.38) we get

$$\tau_\lambda(g) = \tau_{2-\lambda}(g^{-1}) = \tau_{2-\lambda}(g),\ g \in G,\ \lambda \in \mathbb{R}. \tag{6.39}$$

The second equality in (6.39) is true because τ_λ is bi-K-invariant and $g^{-1} \in \mathrm{K}g\mathrm{K}$. Since, obviously, $\tau_0(g) = 1$, it follows that

$$\tau_2(g) = \tau_0(g) = 1. \tag{6.40}$$

Since $t^{-\lambda}$ is a strictly convex function of λ for any $t > 0, t \neq 1$, it follows from (6.38) that $\tau_\lambda(g)$ is a strictly convex function of λ for any $g \in G$. From this, (6.39) and (6.40) we deduce that

$$\begin{aligned}&\tau_\eta(g) < \tau_\lambda(g) && \text{for any } g \notin \mathrm{K} \text{ and } 1 \le \eta < \lambda \le 2, \\ &\tau_\eta(g) < 1 \text{ and } \tau_\lambda(g) > 1 && \text{for any } g \notin \mathrm{K},\ 0 < \eta < 2, \lambda > 2, \text{ and} && (6.41) \\ &\tau_\eta(g) < \tau_\lambda(g) && \text{for any } g \notin \mathrm{K},\ \lambda \ge 2,\ 0 < \eta < \lambda. && (6.42)\end{aligned}$$

Since the function $\tau_\lambda(g)$ is bi-K-invariant, it depends only on the norm $\|g\|$ of g. Thus, we can write

$$\tau_\lambda(g) = \hat{\tau}_\lambda(\|g\|), \quad g \in \mathrm{G}, \tag{6.43}$$

where for $a \ge 1$

$$\hat{\tau}_\lambda(a) = \tau_\lambda(d_a) = \int_{\mathrm{K}} \|d_a ke_1\|^{-\lambda}\,\mathrm{d}\sigma(k) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\theta}{(a^2\cos^2\theta + a^{-2}\sin^2\theta)^{\lambda/2}}. \tag{6.44}$$

In view of (6.36) and the definition of A_g, we get

$$\int_K \hat{\tau}_\lambda(\|gkd_a\|)\,\mathrm{d}\sigma(k) = \tau_\lambda(g)\hat{\tau}_\lambda(a), \quad g \in \mathrm{G}, a \geq 1. \tag{6.45}$$

Since $\|g\| = \|g^{-1}\|$ for all $g \in \mathrm{G}$,

$$\frac{a}{\|g\|} \leq \|gkd_a\| \leq a\|g\|$$

for all $k \in \mathrm{K}$ and $g \in \mathrm{G}$. From this, (6.41) and (6.45) we deduce that, for any $\lambda > 2$, the continuous function $\hat{\tau}_\lambda(a), a \geq 1$, does not have a local maximum. Hence $\hat{\tau}_\lambda$ is strictly increasing for all $\lambda > 2$ or, equivalently,

$$\tau_\lambda(g) < \tau_\lambda(h) \quad \text{if } \|g\| < \|h\|,\ g, h \in G,\ \lambda > 2. \tag{6.46}$$

Using (6.39) and (6.44) yields

$$\hat{\tau}_\lambda(a) = \hat{\tau}_{2-\lambda}(a) = \frac{1}{2\pi}\int_0^{2\pi} (a^2\cos^2\theta + a^{-2}\sin^2\theta)^{\frac{\lambda}{2}-1}\,\mathrm{d}\theta. \tag{6.47}$$

Since $a^2\cos^2\theta \leq a^2\cos^2\theta + a^{-2}\sin^2\theta \leq a^2$, we deduce from (6.47) the estimates

$$c(\lambda)a^{\lambda-2} \leq \hat{\tau}_\lambda(a) \leq a^{\lambda-2},\ a \geq 1,\ \lambda \geq 2, \tag{6.48}$$

where

$$c(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} |\cos\theta|^{\lambda-2}\,\mathrm{d}\theta = \frac{2}{\pi}\int_0^{\pi/2} \cos(\theta)^{\lambda-2}\,\mathrm{d}\theta = \frac{\mathrm{B}\left(\frac{\lambda-1}{2}, \frac{1}{2}\right)}{\pi} = \frac{\Gamma(\frac{\lambda-1}{2})}{\Gamma(\frac{\lambda}{2})\sqrt{\pi}}, \tag{6.49}$$

B denotes the beta function. Here we have used the identity $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ as well as $\Gamma(1/2) = \sqrt{\pi}$. From (6.47) we also conclude that for any $\lambda > 2$ the ratio $\frac{\hat{\tau}_\lambda(a)}{a^{\lambda-2}}$ is a strictly decreasing function of $a \geq 1$ and

$$\lim_{a\to\infty} \frac{\hat{\tau}_\lambda(a)}{a^{\lambda-2}} = c(\lambda). \tag{6.50}$$

We note that the constant $c(\lambda)$ is usually referred to as Harish-Chandra's c-function and it is well-known that its value is given by (6.49), see [Hel00], Introduction Theorem 4.5, or [Lan85], Chapter V §5.

Lemma 6.9. Let $g \in \mathrm{G}, g \notin \mathrm{K}$, $\lambda > 2$, $0 < \eta < \lambda$, $b \geq 0$, $B > 1$, and let f be a left K-invariant positive continuous function on G. Assume that

$$A_g f \leq \tau_\lambda(g) f + b\tau_\eta \tag{6.51}$$

and that

$$f(yh) \leq Bf(h) \quad \text{if} \quad h, y \in \mathrm{G} \text{ and } \|y\| \leq \|g\|. \tag{6.52}$$

Then for all $h \in G$

$$(A_h f)(1) = \int_{\mathrm{K}} f(hk)\,\mathrm{d}\sigma(k) \leq s\tau_\lambda(h),$$

where

$$s = B\left(f(1) + \frac{b}{\tau_\lambda(g) - \tau_\eta(g)}\right). \tag{6.53}$$

Proof: We define

$$f_{\mathrm{K}}(h) \stackrel{\mathrm{def}}{=} \int_{\mathrm{K}} f(hk)\,\mathrm{d}\sigma(k), \quad h \in \mathrm{G}.$$

Since A_g commutes with right translations, and τ_η is right K-invariant, it follows from (6.51) that $A_g f_{\mathrm{K}} \le \tau_\lambda(g) f_{\mathrm{K}} + b\tau_\eta$. If h and y are as in (6.52), then $f(yhk) \le Bf(hk)$ for every $k \in \mathrm{K}$ and therefore $f_{\mathrm{K}}(yh) \le Bf_{\mathrm{K}}(h)$. On the other hand, it is clear that

$$f_{\mathrm{K}}(h) = (A_h f_{\mathrm{K}})(1) = (A_h f)(1).$$

Thus we can replace f by f_{K} and assume that f is bi-K-invariant. Then we have to prove that $f \le s\tau_\lambda$. Assume the contrary, then $f(h) > s'\tau_\lambda(h)$ for some $h \in G$ and $s' > s$. In view of (6.42) and (6.53), $s' > s \ge Bf(1)$. From this, (6.46) and (6.52) we get that $\|h\| > \|g\|$ and

$$f(yh) > \frac{s'}{B}\,\tau_\lambda(yh) \quad \text{if} \quad \|y\| \le \|g\| \;\text{ and }\; \|yh\| \le \|h\|. \tag{6.54}$$

Using the Cartan decomposition, we see that any $x \in \mathrm{G}$ with $\frac{\|h\|}{\|g\|} \le \|x\| \le \|h\|$ can be written as $x = k_1 y h k_2$, where $k_1, k_2 \in \mathrm{K}$, $\|y\| \le \|g\|$ and $\|yh\| \le \|h\|$. (In fact, if $x = k_3 d_a k_4$ and $h = k_5 d_b k_6$ with $a, b \ge 1$, $k_3, \ldots, k_6 \in K$, then we can take $k_1 = k_3$, $k_2 = k_6^{-1} k_4$ and $y = d_{a/c} k_5^{-1}$, where $\|y\| = c/a$.) But the functions f and τ_λ are bi-K-invariant. Therefore it follows from (6.54) that

$$f(x) > \frac{s'}{B}\,\tau_\lambda(x) \quad \text{if} \quad \frac{\|h\|}{\|g\|} \le \|x\| \le \|h\|. \tag{6.55}$$

Let

$$\begin{aligned} a_\eta &\stackrel{\mathrm{def}}{=} \frac{b}{\tau_\lambda(g) - \tau_\eta(g)}, \quad a_\lambda \stackrel{\mathrm{def}}{=} \frac{s'}{B} > f(1) + \frac{b}{\tau_\lambda(g) - \tau_\eta(g)}, \text{ and} \\ \omega &\stackrel{\mathrm{def}}{=} f - a_\lambda \tau_\lambda + a_\eta \tau_\eta. \end{aligned}$$

In view of (6.36) and (6.51), we see that

$$\begin{aligned} A_g\omega - \tau_\lambda(g)\omega &= A_g(f - a_\lambda\tau_\lambda + a_\eta\tau_\eta) - \tau_\lambda(g)(f - a_\lambda\tau_\lambda + a_\eta\tau_\eta) \\ &= [A_g f - \tau_\lambda(g) f] - a_\lambda[A_g\tau_\lambda - \tau_\lambda(g)\tau_\lambda] + a_\eta\,[A_g\tau_\eta - \tau_\lambda(g)\tau_\eta] \\ &\le b\tau_\eta + a_\eta\,[\tau_\eta(g)\tau_\eta - \tau_\lambda(g)\tau_\eta] = 0. \end{aligned} \tag{6.56}$$

Since $\tau_\lambda(1) = \tau_\eta(1) = 1$, we have

$$\omega(1) = f(1) - a_\lambda + a_\eta < 0. \tag{6.57}$$

It follows from (6.42) that $a_\eta \ge 0$. Using additionally (6.53) and (6.55), we get that

$$\begin{aligned} \omega(x) &= f(x) - a_\lambda\tau_\lambda(x) + a_\eta\tau_\eta(x) \ge f(x) - a_\lambda\tau_\lambda(x) \\ &> \left(\frac{s'}{B} - a_\lambda\right)\tau_\lambda(x) = 0 \quad \text{if} \quad \frac{\|h\|}{\|g\|} \le \|x\| \le \|h\|. \end{aligned} \tag{6.58}$$

Let $v \in \mathrm{G}$, satisfying $\|v\| \le \|h\|$, be a point where the continuous function ω attains its minimum on the set $\{x \in \mathrm{G} : \|x\| \le \|h\|\}$. It follows from (6.57) and (6.58) that

$$\omega(v) < 0 \quad \text{and} \quad \|v\| \le \frac{\|h\|}{\|g\|}.$$

Because of $\tau_\lambda(g) > 1$ and $\|gkv\| \le \|g\|\|v\|$ for all $k \in \mathrm{K}$ we conclude

$$(A_g\omega)(v) = \int_{\mathrm{K}} \omega(gkv)\,\mathrm{d}\sigma(k) \ge \omega(v) > \tau_\lambda(g)\omega(v).$$

Thus, we get a contradiction with (6.56). □

As a special case ($\eta = 2$ and $b = 0$) of Lemma 6.9, we have the following

Corollary 6.10. Let $g \in \mathrm{G}$, $g \notin \mathrm{K}$, $\lambda > 2$, $B > 1$, and let f be a left K-invariant positive continuous function on G satisfying the inequality (6.52). Assume that

$$A_g f \le \tau_\lambda(g) f.$$

Then for all $h \in G$

$$(A_h f)(1) = \int_{\mathrm{K}} f(hk)\,\mathrm{d}\sigma(k) \le B f(1)\tau_\lambda(h).$$

Lemma 6.11. Let $g \in \mathrm{G}$, $g \notin \mathrm{K}$, $2 < \lambda < \mu$, $B > 1$, $M > 1$, $n \in \mathbb{N}^+$ and let f_i, $0 \le i \le n$, be left K-invariant positive continuous functions on G. We denote $\min\{i, n-i\}$ by $\bar{i}$ and $\sum_{0\le i\le n} f_i$ by f. Assume that

$$\begin{aligned} f_i(yh) &\le B f_i(h) \quad \text{if } \ 0 \le i \le n, \ \ h, y \in \mathrm{G} \ \text{ and } \ \|y\| \le \|g\|, \\ A_g f_i &\le \tau_\lambda(g) f_i + M \max_{0<j\le \bar{i}} \sqrt{f_{i-j} f_{i+j}}, \quad 0 \le i \le n, \end{aligned} \tag{6.59}$$

so in particular $A_g f_0 \le \tau_\lambda(g) f_0$ and $A_g f_n \le \tau_\lambda(g) f_n$. Then there is a constant $C = C(g, \lambda, \mu, B, M, n)$ such that for all $h \in \mathrm{G}$,

$$(A_h f)(1) = \int_{\mathrm{K}} f(hk)\,\mathrm{d}\sigma(k) \le C f(1)\tau_\mu(h). \tag{6.60}$$

Proof: For any $0 < \varepsilon \le 1$ and $0 \le i \le n$ we define

$$f_{i,\varepsilon} = \varepsilon^{q(i)} f_i \quad \text{where } q(i) \overset{\text{def}}{=} i(n-i).$$

Using the inequality (6.59) for all i, $0 \le i \le n$, we see that

$$\begin{aligned} A_g f_{i,\varepsilon} = \varepsilon^{q(i)} A_g f_i &\le \varepsilon^{q(i)} \tau_\lambda(g) f_i + \varepsilon^{q(i)} M \max_{0<j\le\bar{i}} \sqrt{\varepsilon^{-q(i-j)} f_{i-j,\varepsilon} \varepsilon^{-q(i+j)} f_{i+j,\varepsilon}} \\ &= \tau_\lambda(g) f_{i,\varepsilon} + M \max_{0<j\le\bar{i}} \varepsilon^{q(i)-\frac{1}{2}[q(i-j)+q(i+j)]} \sqrt{f_{i-j,\varepsilon} f_{i+j,\varepsilon}}. \end{aligned}$$

Direct computation shows that

$$q(i) - \frac{1}{2}[q(i-j) + q(i+j)] = j^2.$$

Hence for all i, $0 \le i \le n$,

$$A_g f_{i,\varepsilon} \le \tau_\lambda(g) f_{i,\varepsilon} + \varepsilon M \max_{0<j\le\bar{i}} \sqrt{f_{i-j,\varepsilon} f_{i+j,\varepsilon}}. \tag{6.61}$$

Let $f_\varepsilon := \sum_{0\le i\le n} f_{i,\varepsilon}$. Summing (6.61) over all i, $0 \le i \le n$, and using the inequalities $f_\varepsilon > \sqrt{f_{i-j,\varepsilon}\, f_{i+j,\varepsilon}}$, which are satisfied for any $1 \le i \le n-1$, $0 < j \le \bar{i}$, we get

$$A_g f_\varepsilon = \sum_{0\le i\le n} A_g f_{i,\varepsilon} \le \tau_\lambda(g) f_\varepsilon + \varepsilon M(n-1) f_\varepsilon = (\tau_\lambda(g) + \varepsilon M(n-1))\, f_\varepsilon. \tag{6.62}$$

Write

$$\varepsilon_0 = \min\left\{1, \frac{\tau_\mu(g) - \tau_\lambda(g)}{M(n-1)}\right\}$$

in order to get from (6.62) that

$$A_g f_{\varepsilon_0} \le \tau_\mu(g) f_{\varepsilon_0}.$$

Since f_ε also satisfies (6.52), we can apply Corollary 6.10 to f_{ε_0} and get that

$$(A_h f)(1) < \varepsilon_0^{-n^2}(A_h f_{\varepsilon_0})(1) \le \varepsilon_0^{-n^2} f_{\varepsilon_0}(1)\tau_\mu(h) \le \varepsilon_0^{-n^2} B f(1)\tau_\mu(h)$$

for all $h \in \mathrm{G}$. Hence (6.60) is true with $C = \varepsilon_0^{-n^2} B$. □

Proposition 6.12. Let $g \in \mathrm{G}$, $g \notin \mathrm{K}$, $d \in \mathbb{N}^+$, $B > 1$, $M > 1$. For every $0 \le i \le 2d$, let $\lambda_i \ge 2$ and let f_i be a left K-invariant positive continuous function on G. We denote $\min\{i, 2d-i\}$ by $\bar{i}$ and $\sum_{0\le i\le 2d} f_i$ by f. Assume that

$$\lambda_d > \lambda_i \quad \text{for any } i \ne d,$$

$$f_i(yh) \le B f_i(h) \quad \text{if} \quad 0 \le i \le 2d,\ h, y \in \mathrm{G} \text{ and } \|y\| \le \|g\|, \tag{6.63}$$

$$A_g f_i \le \tau_{\lambda_i}(g) f_i + M \max_{0<j\le \bar{i}} \sqrt{f_{i-j} f_{i+j}}, \quad 0 \le i \le 2d, \tag{6.64}$$

in particular,

$$A_g f_0 \le \tau_{\lambda_0}(g) f_0 \quad \text{and} \quad A_g f_{2d} \le \tau_{\lambda_{2d}}(g) f_{2d}.$$

Then, we have that

(a) For all $h \in \mathrm{G}$ and $0 \le i \le 2d,\ i \ne d$,

$$(A_h f_i)(1) = \int f_i(hk)\,\mathrm{d}\sigma(k) \ll f(1)\tau_\eta(h),$$

where

$$\eta = \lambda_d - 3^{-(d+1)}(\lambda_d - \eta') < \lambda_d, \quad \eta' = \max\{\lambda_i : 0 \le i \le 2d, i \ne d\}. \tag{6.65}$$

(b) For all $h \in \mathrm{G}$

$$(A_h f_d)(1) = \int_{\mathrm{K}} f_d(hk)\,\mathrm{d}\sigma(k) \ll f(1)\tau_{\lambda_d}(h).$$

(c) For all $h \in \mathrm{G}$

$$(A_h f)(1) = \int_{\mathrm{K}} f(hk)\,\mathrm{d}\sigma(k) \ll f(1)\|h\|^{\lambda_d - 2}.$$

Here the notation $\ll$ means (until the end of the proof of this proposition) that the left hand side is bounded from above by the right-hand side multiplied by a constant which depends on $g, \lambda_0, \ldots, \lambda_{2d}, B$ and M, and does not depend on $f_0, \ldots, f_{2d}$.

Proof: (a) Let

$$f_{i,\mathrm{K}}(h) \stackrel{\text{def}}{=} \int_{\mathrm{K}} f_i(hk)\,\mathrm{d}\sigma(k), \quad h \in \mathrm{G}.$$

The Cauchy-Schwarz inequality implies

$$\int_{\mathrm{K}} \sqrt{f_{i-j}(hk) f_{i+j}(hk)}\, \mathrm{d}\sigma(k) \le \sqrt{\int_{\mathrm{K}} f_{i-j}(hk)\, \mathrm{d}\sigma(k)} \sqrt{\int_{\mathrm{K}} f_{i+j}(hk)\, \mathrm{d}\sigma(k)}$$
$$= \sqrt{f_{i-j,\mathrm{K}}(h) f_{i+j,\mathrm{K}}(h)}.$$

Hence

$$\begin{aligned}\int_{\mathrm{K}} \max_{0<j\le\bar{i}} \sqrt{f_{i-j}(hk) f_{i+j}(hk)}\, \mathrm{d}\sigma(k) &\le \sum_{0<j\le\bar{i}} \int_{\mathrm{K}} \sqrt{f_{i-j}(hk) f_{i+j}(hk)}\, \mathrm{d}\sigma(k) \\ &\le \sum_{0<j\le\bar{i}} \sqrt{f_{i-j,\mathrm{K}}(h) f_{i+j,\mathrm{K}}(h)} \\ &\le d \max_{0<j\le\bar{i}} \sqrt{f_{i-j,\mathrm{K}}(h) f_{i+j,\mathrm{K}}(h)}.\end{aligned}$$

On the other hand, we have

$$(A_g f_{i,\mathrm{K}})(h) = \int_{\mathrm{K}} (A_g f_i)(hk)\, \mathrm{d}\sigma(k)$$

and according to (6.64)

$$(A_g f_i)(hk) \le \tau_{\lambda_i}(g) f_i(hk) + M \max_{0<j\le\bar{i}} \sqrt{f_{i-j}(hk) f_{i+j}(hk)}.$$

Therefore

$$A_g f_{i,\mathrm{K}} \le \tau_{\lambda_i}(g) f_{i,\mathrm{K}} + dM \max_{0<j\le\bar{i}} \sqrt{f_{i-j,\mathrm{K}} f_{i+j,\mathrm{K}}}.$$

But $f_{\mathrm{K}}(1) = f(1)$,

$$f_{i,\mathrm{K}}(h) = (A_h f_{i,\mathrm{K}})(1) = (A_h f_i)(1)$$

and, as easily follows from (6.63), we have

$$f_{i,\mathrm{K}}(yh) \le B f_{i,\mathrm{K}}(h)$$

if $h, y \in \mathrm{G}$, and $\|y\| \le \|g\|$. Thus, replacing f_i by $f_{i,\mathrm{K}}$ and M by dM, we can assume that the functions f_i are bi-K-invariant. Then we have to prove that

$$f_i \ll f(1)\tau_\eta \quad \text{for all } 0 \le i \le 2d,\ i \ne d. \tag{6.66}$$

Let $\eta' = \max\{\lambda_i : 0 \le i \le 2d, i \ne d\}$, as in (6.65). We define μ_i, $0 \le i \le 2d$, by

$$\mu_d = \lambda_d + 3^{-(d+1)}(\lambda_d - \eta') \quad \text{and} \tag{6.67}$$
$$\mu_i = \mu_d - 3^{-\bar{i}}(\lambda_d - \eta'),\ 0 \le i \le 2d,\ i \ne d. \tag{6.68}$$

Since (6.42) together with $\mu_d \ge \lambda_d \ge \lambda_i \ge 2$ implies $\tau_{\lambda_i}(g) \le \tau_{\mu_d}(g)$, it follows from Lemma 6.11 that

$$f_i \ll f(1)\tau_{\mu_d}, \quad 0 \le i \le 2d. \tag{6.69}$$

One can easily check that $\eta > \mu_i > \lambda_i \ge 2$ and therefore $\tau_\eta \ge \tau_{\mu_i}$ for all $0 \le i \le 2d, i \ne d$. Thus, to prove (6.66), it is enough to show that

$$f_i \ll f(1)\tau_{\mu_i} \quad \text{for all } 0 \le i \le 2d,\ i \ne d. \tag{6.70}$$

We will prove (6.70) for $i \le d-1$ by using induction in i; the proof in the case $i \ge d+1$ is similar. For $i = 0$ we have $\tau_{\mu_0}(g) > \tau_{\lambda_0}(g)$ because of (6.42) and thus it is enough to use Corollary 6.10. Let $1 \le m \le d-1$ and assume that (6.70) is proved for all $i < m$. Using (6.69) for all $0 < j \le m$ we find that

$$\sqrt{f_{m-j} f_{m+j}} \ll f(1)\sqrt{\tau_{\mu_{m-j}} \tau_{\mu_d}} \le f(1)\sqrt{\tau_{\mu_{m-1}} \tau_{\mu_d}} \ll f(1)\tau_{(\mu_{m-1}+\mu_d)/2}. \tag{6.71}$$

Note that the second inequality in (6.71) follows from (6.42) and (6.68), and the third one follows from (6.43) and (6.48).

Combining (6.64) and (6.66) we get

$$A_g f_m \le \tau_{\lambda_m}(g) f_m + C f(1) \tau_{(\mu_{m-1}+\mu_d)/2},$$

where $C \ll 1$. On the other hand, we have $\lambda_m < \mu_m$ and

$$(\mu_{m-1} + \mu_d)/2 < \mu_m$$

by (6.67) and (6.68). Now, to prove that $f_m \ll f(1)\tau_{\mu_m}$, it remains to apply Lemma 6.9 combined with (6.42).

(b) As in the proof of (a), we can assume that the functions f_i are bi-K-invariant. Then we get from (6.64) and (6.66) that

$$A_g f_d \le \tau_{\lambda_d} f_d + D f(1) \tau_\eta,$$

where $D \ll 1$. Since $\eta < \lambda_d$, Lemma 6.9 implies that $f_d \ll f(1)\tau_{\lambda_d}$ which proves (b).

(c) Follows from (a), (b), (6.42), (6.43) and (6.48). □

6.2.2 Quasinorms and Representations of $\mathrm{SL}(2,\mathbb{R})$

We say that a continuous function $v \mapsto |v|$ on a real topological vector space V is a *quasinorm* if it satisfies the following properties

(i) $|v| \ge 0$ and $|v| = 0$ if and only if $v = 0$,

(ii) $|\lambda v| = |\lambda| \cdot |v|$ for all $\lambda \in \mathbb{R}$ and $v \in V$.

If V is finite dimensional, then any two quasinorms on V are equivalent in the sense that their ratio lies between two positive constants.

Lemma 6.13. Let ρ be a (continuous) representation of $\mathrm{G} = \mathrm{SL}(2,\mathbb{R})$ in a real topological vector space V, let $|\cdot|$ be a $\rho(\mathrm{K})$-invariant quasinorm on V and let $v \in V, v \ne 0$, be an eigenvector for ρ corresponding to the character $\chi_{-r}, r \in \mathbb{R}$, that is

$$\rho \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} v = a^r v.$$

Then for any $g \in \mathrm{G}$ and $\beta \in \mathbb{R}$

$$|\rho(g)v|^{-\beta} = \varphi_{\beta r}(g)\, |v|^{-\beta}, \tag{6.72}$$

where $\varphi_{\beta r}$ is defined as in (6.32), and

$$\int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{|\rho(gk)v|^\beta} = \tau_{\beta r}(g)\, |v|^{-\beta}. \tag{6.73}$$

Proof: Using the $\rho(\mathrm{K})$-invariance of $|\cdot|$ we get that

$$\begin{aligned}|\rho(g)v|^{-\beta} &= |\rho(k(g))\rho(t(g))v|^{-\beta} = |\rho(t(g))v|^{-\beta} = |\chi_{-r}(t(g))v|^{-\beta} = \chi_{\beta r}(t(g))\,|v|^{-\beta}\\ &= \varphi_{\beta r}(g)\,|v|^{-\beta}\,.\end{aligned}$$

The equality (6.73) follows from (6.72) and from the definition of $\tau_{\beta r}(g)$, see (6.35). □

Let $\|z\|$ denote the norm of $z \in \mathbb{C}^2$ corresponding to the standard Hermitian inner product on $\mathbb{C}^2$, that is

$$\|z\|^2 = \|x\|^2 + \|y\|^2 \quad \text{where } z = x + iy,\ x, y \in \mathbb{R}^2.$$

Lemma 6.14. For any $z \in \mathbb{C}^2$, $z \neq 0$, $g \in G$ and $\beta > 0$, we have

$$F(z) = F_{g,\beta}(z) \stackrel{\text{def}}{=} \|z\|^\beta \int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{\|gkz\|^\beta} \leq \tau_\beta(g). \tag{6.74}$$

Proof: Since the measure σ on K is translation invariant, we have

$$F(kz) = F(z) \text{ for any } k \in \mathrm{K}. \tag{6.75}$$

Also for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $z \in \mathbb{C}^2, z \neq 0$,

$$F(\lambda z) = F(z), \tag{6.76}$$

because $\|\lambda v\| = |\lambda| \cdot \|v\|$, $v \in \mathbb{C}^2$, and because $\mathrm{G} = \mathrm{SL}(2,\mathbb{R})$ acts $\mathbb{C}$-linearly on $\mathbb{C}^2$. Any nonzero vector $x \in \mathbb{R}^2$ can be represented as $x = \lambda k e_1$ with $\lambda \in \mathbb{R}$, $k \in \mathrm{K}$, $e_1 = (1,0)$. Then, using (6.38) from Section 6.2.1, we get from (6.75) and (6.76) that

$$F(x) = F(e_1) = \tau_\beta(g) \text{ for all } x \in \mathbb{R}^2, x \neq 0. \tag{6.77}$$

Let now $z = x + iy$, $x, y \in \mathbb{R}^2$, $z \neq 0$. We write $e^{i\theta}z = x_\theta + iy_\theta$, $x_\theta, y_\theta \in \mathbb{R}^2$. Then $\frac{\|x_\theta\|}{\|y_\theta\|}$ is a continuous function of θ with values in $\mathbb{R}_{\geq 0} \cup \{\infty\}$. But $e^{i\pi/2}z = iz = -y + ix$ and therefore $\frac{\|x_{\pi/2}\|}{\|y_{\pi/2}\|} = \left(\frac{\|x_0\|}{\|y_0\|}\right)^{-1}$. Hence there exists θ such that $\|x_\theta\| = \|y_\theta\|$. Replacing then z by $e^{i\theta}z$ and using (6.76) we can assume that $\|x_\theta\| = \|y_\theta\|$. Now using the convexity of the function $t \to t^{-\beta/2}$, $t > 0$, and the identity (6.77) we get that

$$\begin{aligned}\int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{\|gkz\|^\beta} &= \int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{(\|gkx\|^2 + \|gky\|^2)^{\beta/2}}\\ &\leq \frac{2^{-\beta/2}}{2}\left[\int_K \frac{\mathrm{d}\sigma(k)}{\|gkx\|^\beta} + \int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{\|gky\|^\beta}\right] = \frac{2^{-\beta/2}}{2}\left[\frac{\tau_\beta(g)}{\|x\|^\beta} + \frac{\tau_\beta(g)}{\|y\|^\beta}\right]\\ &= 2^{-\beta/2}\tau_\beta(g)\frac{1}{\|x\|^\beta} = 2^{-\beta/2}\tau_\beta(g)\cdot\frac{1}{\|z\|^\beta \cdot 2^{-\beta/2}} = \frac{\tau_\beta(g)}{\|z\|^\beta}.\end{aligned} \tag{6.78}$$

Clearly the last inequality (6.78) implies (6.74). □

Let us recall some basic facts of the finite-dimensional representation theory of $\mathrm{G} = \mathrm{SL}(2,\mathbb{R})$. Let W be a finite-dimensional complex vector space, there is a correspondence between complex-linear representations of $\mathfrak{sl}(2,\mathbb{C})$ on W and representations of G on W, under which invariant subspaces and equivalences are preserved (see [Kna01] Proposition 2.1). It is well-known that any finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ is fully reducible, that is, it can be decomposed into the direct sum of irreducible representations (see [Kna02] Corollary

1.70). Moreover, for each $m \geq 1$ there exists up to equivalence a unique irreducible complex-linear representation of $\mathfrak{sl}(2,\mathbb{C})$ on a complex vector space of dimension m (see [Kna02] Corollary 1.63). Hence, any finite-dimensional representation of G is fully reducible and any two irreducible finite-dimensional representations of the same degree must be isomorphic. Let $\mathcal{P}_m$ denote the $(m+1)$-dimensional complex vector space of complex polynomials in two variables homogeneous of degree m, and let ψ_m denote the regular representation of $\mathrm{G} = \mathrm{SL}(2,\mathbb{R})$ on $\mathcal{P}_m$ defined by $(\psi_m(g)P)(z) = P(g^{-1}z)$, for $g \in \mathrm{G}$, $z \in \mathbb{C}^2$ and $P \in \mathcal{P}_m$. It is well-known that the representation ψ_m is irreducible for any m (see [Kow14] Example 2.7.11) and hence it is, up to isomorphism, the unique irreducible finite-dimensional representation of G of degree m. We define

$$I(\rho) = \{\, m \in \mathbb{N}^+ \,:\, \psi_m \text{ is isomorphic to a subrepresentation of } \rho \,\}.$$

Proposition 6.15. Let ρ be a representation of $\mathrm{G} = \mathrm{SL}(2,\mathbb{R})$ on a finite-dimensional space W. Then there exists a $\rho(\mathrm{K})$-invariant quasinorm $|\cdot| = |\cdot|_\rho$ on W such that for any $w \in W, w \neq 0$, $g \in \mathrm{G}$ and $\beta > 0$,

$$\int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{|\rho(gk)w|^\beta} \leq \max_{m \in I(\rho)} \{\tau_{\beta m}(g)\} \frac{1}{|w|^\beta}.$$

Proof: Let $W = \bigoplus_{i=1}^n W_i$ be the decomposition of W into the direct sum of $\rho(\mathrm{G})$-irreducible subspaces, and let $\pi_i \colon W \to W_i$ denote the natural projection. Suppose that we constructed for each $i = 1, \ldots, n$ a $\rho(\mathrm{K})$-invariant quasinorm $|\cdot|_i = |\cdot|_{\rho_i}$ on W_i such that for any $w \in W_i, w \neq 0, g \in \mathrm{G}$, and $\beta > 0$,

$$\int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{|\rho_i(gk)w|_i^\beta} \leq \tau_{\beta m(i)}(g)\, \frac{1}{|w|_i^\beta}, \tag{6.79}$$

where ρ_i denotes the restriction of ρ to W_i and $m(i) \in I(\rho)$ is defined by the condition that $\psi_{m(i)}$ is isomorphic to ρ_i. Then we define $|w| = |w|_\rho$ by

$$|w| = \max_{1 \leq i \leq n} |\pi_i(w)|_i, \;\; w \in W. \tag{6.80}$$

Clearly $|\cdot|_\rho$ is a $\rho(\mathrm{K})$-invariant quasinorm. Let us fix now $w \in W, w \neq 0$. Then

$$\begin{aligned}\int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{|\rho(gk)w|^\beta} &\leq \min_{1 \leq i \leq n} \int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{|\pi_i(\rho(gk)w)|_i^\beta} = \min_{1 \leq i \leq n} \int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{|\rho_i(gk)\pi_i(w)|_i^\beta} \\ &\leq \min_{1 \leq i \leq n} \tau_{\beta m(i)}(g) \frac{1}{|\pi_i(w)|_i^\beta} \leq \max_{m \in I(\rho)} \{\tau_{\beta m}(g)\} \frac{1}{|w|^\beta}.\end{aligned}$$

Thus, it is enough to prove the proposition for representations ψ_m. For this, let $P \in \mathcal{P}_m, P \neq 0$. We consider P as a polynomial on $\mathbb{C}^2$ and decompose P, using the fundamental theorem of algebra, into the product of m linear forms

$$P = \ell_1 \cdot \ldots \cdot \ell_m, \quad \text{where} \quad \ell_i(z_1, z_2) = a_i z_1 + b_i z_2, \quad a_i, b_i z_1, z_2 \in \mathbb{C}.$$

There is a natural K-invariant norm on the space of linear forms on $\mathbb{C}^2$:

$$\|\ell\|^2 = |a|^2 + |b|^2, \;\; \ell(z_1, z_2) = a z_1 + b z_2.$$

Now we define a quasinorm on $\mathcal{P}_m$ by the equation

$$|P| = \|\ell_1\| \cdot \ldots \cdot \|\ell_m\|. \tag{6.81}$$

This definition is correct because the factorization (6.81) is unique up to the order of factors and the multiplication of ℓ_i, $1 \le i \le n$, by constants. We denote by $\tilde{\psi_1}$ the extension of ψ_1 to the space of linear forms on G. It is isomorphic to the standard representation of G on $\mathbb{C}^2$. Then using Lemma 6.14 and the generalized Hölder inequality, we get that

$$\begin{aligned}\int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{|\psi_m(gk)P|^\beta} = \int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{\prod_{i=1}^m \|\tilde{\psi_1}(gk)\ell_i\|^\beta} &\le \prod_{i=1}^m \left(\int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{\|\tilde{\psi_1}(gk)\ell_i\|^{\beta m}}\right)^{1/m} \\ &\le \prod_{i=1}^m \left(\frac{\tau_{\beta m}(g)}{\|\ell_i\|^{\beta m}}\right)^{1/m} = \frac{\tau_{\beta m}(g)}{|P|^\beta}.\end{aligned} \tag{6.82}$$

Since $I(\psi_m) = \{m\}$, (6.82) implies (6.79) for $\rho = \psi_m$. □

We recall from Section 6.2.1, see (6.41) and (6.42), that $\tau_\mu(g) < 1$ and $\tau_\eta(g) < \tau_\lambda(g)$ for any $g \notin \mathrm{K}$, $0 < \mu < 2$, $\lambda \ge 2$ and $0 < \eta < \lambda$. Using this, we deduce from the previous Proposition 6.15 the following corollary.

Corollary 6.16. Let ρ be a representation of $\mathrm{G} = \mathrm{SL}(2,\mathbb{R})$ in a finite dimensional space W, and let m be the largest number in $I(\rho)$. Then there exists a $\rho(\mathrm{K})$-invariant quasinorm $|\cdot| = |\cdot|_\rho$ on W such that

(i) if $\beta > 0$ and $\beta m \ge 2$ then for any $w \in W$, $w \neq 0$, and $g \in \mathrm{G}$

$$\int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{|\rho(gk)w|^\beta} \le \tau_{\beta m}(g) \frac{1}{|w|^\beta},$$

(ii) if $\beta > 0$ and $\beta m < 2$ then for any $w \in W$, $w \neq 0$, and $g \in \mathrm{G}$, $g \notin \mathrm{K}$,

$$\int_{\mathrm{K}} \frac{\mathrm{d}\sigma(k)}{|\rho(gk)w|^\beta} < \frac{1}{|w|^\beta}.$$

6.2.3 Estimates of Special Functions on the Space of Lattices

Let ρ be a representation of $\mathrm{G} = \mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^n$ and for each $1 \le l \le n$ let $|\cdot|_l$ be a $(\wedge^i\rho)(\mathrm{K})$-invariant quasinorm on the exterior product $\bigwedge^l \mathbb{R}^n$. Throughout this section the underlying quasinorms in the definition of the lattice functions α_l and α are taken to be with respect to this particular choice of quasinorms, see (3.43) and (3.44). For every compact subset $A \subset \mathrm{G}$ note that

$$\begin{aligned}&\sup\left\{\frac{|(\wedge^i\rho)(h)v|_i}{|v|_i} : h \in A, v \in \wedge^i\mathbb{R}^n, v \neq 0\right\} \\ &= \sup\{|(\wedge^i\rho)(h)v|_i : h \in A, v \in \wedge^i\mathbb{R}^n, |v|_i = 1\}\end{aligned}$$

is finite for every $i, 1 \le i \le n$. Hence, if we fix $g \in \mathrm{G}, g \notin \mathrm{K}$, then there exists some $B > 1$ such that for any $i, 1 \le i \le n$, and $v \in \wedge^i\mathbb{R}^n$, $v \neq 0$,

$$B^{-1} < \frac{|(\wedge^i\rho)(y)v|_i}{|v|_i} < B \quad \text{if } y \in \mathrm{G} \text{ and } \|y\| \le \|g\|, \tag{6.83}$$

where $\|h\| = \|h^{-1}\|$ denotes the norm of $h \in \mathrm{G} = \mathrm{SL}(2,\mathbb{R})$ with respect to the standard Euclidean norm on $\mathbb{R}^2$. Now, let Δ be a lattice in $\mathbb{R}^n$ and L a Δ-rational subspace, i.e. $L \cap \Delta$ is a full-rank lattice in L. For any $h \in \mathrm{SL}(2,\mathbb{R})$ observe that hL is an $h\Delta$-rational subspace and if $v_1, \ldots, v_i$ is a basis of $\Delta \cap L$ then $hv_1, \ldots, hv_i$ is a basis of $h\Delta \cap hL$. This observation together with (6.83) implies that

$$B^{-1} < \frac{d_{y\Delta}(yL)}{d_\Delta(L)} < B \quad \text{if } y \in \mathrm{G} \text{ and } \|y\| \leq \|g\|. \tag{6.84}$$

Hence, for any $i \in \{0, \ldots, n\}$ it follows that

$$\alpha_i(y\Delta) < B\alpha_i(\Delta) \quad \text{if } y \in \mathrm{G} \text{ and } \|y\| \leq \|g\|. \tag{6.85}$$

For any $\beta > 0$ and $1 \leq i \leq n$ we define the functions $F_{i,\beta}$ on $\wedge^i\mathbb{R}^n \setminus \{0\}$ by

$$F_{i,\beta}(w) \stackrel{\text{def}}{=} \int_K \frac{|w|_i^\beta}{|(\wedge^i\rho)(gk)w|_i^\beta} \,\mathrm{d}\sigma(k), \quad w \in \wedge^i\mathbb{R}^n, w \neq 0.$$

It is clear that the functions $F_{i,\beta}$ are continuous and that $F_{i,\beta}(\lambda w) = F_{i,\beta}(w)$ for any $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Let $c_{0,\beta} := 1$ and for $1 \leq i \leq n$

$$c_{i,\beta} \stackrel{\text{def}}{=} \sup\{F_{i,\beta}(w) : w \in \wedge^i\mathbb{R}^n, w \neq 0\} = \sup\{F_{i,\beta}(w) : w \in \wedge^i\mathbb{R}^n, |w|_i = 1\}. \tag{6.86}$$

We note that $c_{n,\beta} = 1$, since the image of any continuous homomorphism $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ is contained in $\mathrm{SL}(n,\mathbb{R})$. (In fact, composing any continuous homomorphism $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ with the determinant map $\mathrm{GL}(n,R) \to \mathbb{R}^*$ gives a continuous homomorphism $f\colon \mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}^*$. Since $\mathrm{SL}(2,\mathbb{R})$ is connected, the image $f(\mathrm{SL}(2,\mathbb{R}))$ lies in $\mathbb{R}_{>0}$. As $\mathbb{R}_{>0} \sim \mathbb{R}$ topologically and algebraically, the map f is trivial, because any continuous homomorphism $\mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}$ is already the trivial homomorphism. The last statement can be easily checked by using the Iwasawa decomposition.)

Lemma 6.17. For any i, $0 \leq i \leq n$,

$$A_g\alpha_i^\beta \leq c_{i,\beta}\alpha_i^\beta + C^\beta B^{2\beta} \max_{0<j\leq\bar{i}} \sqrt{\alpha_{i-j}^\beta \alpha_{i+j}^\beta}, \tag{6.87}$$

where $\bar{i} = \min\{i, n-i\}$, the constant $C \geq 1$ is from Lemma 3.6 and the operator A_g is defined by (6.34) from Section 6.2.1.

Proof: Let Δ be a lattice in $\mathbb{R}^n$. We have to prove that

$$\int_{\mathrm{K}} \alpha_i(gk\Delta)^\beta \,\mathrm{d}\sigma(k) \leq c_{i,\beta}\alpha_i(\Delta)^\beta + C^\beta B^{2\beta} \max_{0<j\leq\bar{i}} \sqrt{\alpha_{i-j}(\Delta)^\beta \alpha_{i+j}(\Delta)^\beta}. \tag{6.88}$$

There exists a Δ-rational subspace L of dimension i such that

$$\frac{1}{d_\Delta(L)} = \alpha_i(\Delta). \tag{6.89}$$

Let us denote the set of Δ-rational subspaces M of dimension i with $d_\Delta(M) < B^2 d_\Delta(L)$ by Ψ_i. For a Δ-rational i-dimensional subspace $M \notin \Psi_i$ we get from (6.84) that

$$d_{gk\Delta}(gkM) > d_{gk\Delta}(gkL).$$

If $\Psi_i = \{L\}$, then it follows from this and the definitions of α_i and $c_{i,\beta}$ that

$$\int_{\mathrm{K}} \alpha_i(gk\Delta)^\beta \, \mathrm{d}\sigma(k) \leq c_{i,\beta}\alpha_i(\Delta)^\beta. \tag{6.90}$$

Assume now that $\Psi_i \neq \{L\}$. Let $M \in \Psi_i$, $M \neq L$. Then $\dim(M + L) = i + j$, $0 < j \leq \bar{i}$. Now we obtain by (6.84), (6.89) and Lemma 3.6 for any $k \in K$ that

$$\begin{aligned} \alpha_i(gk\Delta) < B\alpha_i(\Delta) = \frac{B}{d_\Delta(L)} \leq \frac{B^2}{\sqrt{d_\Delta(L)d_\Delta(M)}} &\leq \frac{CB^2}{\sqrt{d_\Delta(L \cap M)d_\Delta(L + M)}} \\ &\leq CB^2\sqrt{\alpha_{i-j}(\Delta)\alpha_{i+j}(\Delta)}. \end{aligned}$$

Hence, if $\Psi_i \neq \{L\}$,

$$\int_{\mathrm{K}} \alpha_i(gk\Delta)^\beta \, \mathrm{d}\sigma(k) \leq C^\beta B^{2\beta} \max_{0<j\leq\bar{i}} \sqrt{\alpha_{i-j}(\Delta)^\beta\alpha_{i+j}(\Delta)^\beta}. \tag{6.91}$$

Combining (6.90) and (6.91), we get (6.88). □

Theorem 6.18. Let $d \in \mathbb{N}^+$ and let ρ_d be a representation of $\mathrm{G} = \mathrm{SL}(2, \mathbb{R})$ isomorphic to the direct sum of d copies of the standard 2-dimensional representation. Let β be a positive number such that $\beta d > 2$. Then there is a constant R, depending only on β and the choice of the K-invariant quasinorms $|\cdot|_i$ involved in the definition of α_i, such that for any $h \in \mathrm{G}$ and any lattice Δ in $\mathbb{R}^{2d}$

$$(A_h\alpha^\beta)(\Delta) = \int_{\mathrm{K}} \alpha(hk\Delta)^\beta \, \mathrm{d}\sigma(k) \leq R\alpha(\Delta)^\beta \|h\|^{\beta d - 2}.$$

Proof: As in Section 6.2.2, we define for a finite dimensional representation ρ of G

$$I(\rho) = \{m \in \mathbb{N}^+ : \psi_m \text{ is isomorphic to a subrepresentation of } \rho\},$$

where ψ_m denotes the regular representation of G in the space of complex homogeneous polynomials in two variables homogeneous of degree m. Let m_i be the largest number in $I(\wedge^i\rho_d)$, $1 \leq i \leq 2d$. It is well-known that

$$m_i = \bar{i} \overset{\text{def}}{=} \min\{i, 2d - i\}. \tag{6.92}$$

We fix $g \in \mathrm{G}, g \notin \mathrm{K}$. It follows from (6.92) and from Corollary 6.16 that we can choose quasi-norms $|\cdot|_i$ on $\wedge^i\mathbb{R}^{2d}$ in such a way that for $w \in \wedge^i\mathbb{R}^{2d}$, $w \neq 0$,

$$\int_{\mathrm{K}} \frac{|w|_i^\beta}{|(\wedge^i\rho_d)(g)w|_i^\beta} \, \mathrm{d}\sigma(k) \leq \begin{cases} \tau_{\beta\bar{i}}(g) & \text{if } \beta\bar{i} \geq 2 \\ 1 & \text{if } \beta\bar{i} < 2. \end{cases}$$

Hence

$$c_{i,\beta} \leq \tau_{\beta\bar{i}}(g) \quad \text{if} \quad \beta\bar{i} \geq 2 \quad \text{and} \quad c_{i,\beta} \leq 1 \quad \text{if} \quad \beta\bar{i} < 2. \tag{6.93}$$

where $c_{i,\beta}$, $1 \leq i \leq 2d$, is defined by (6.86) and $c_{0,\beta} = 1$. As a remark, we notice that $c_{i,\beta} = \tau_{\beta\bar{i}}(g)$ if $\beta\bar{i} \geq 2$.

According to Lemma 6.17, the functions α_i^β, $0 \leq i \leq 2d$, satisfy the following system of inequalities

$$A_g\alpha_i^\beta \leq c_{i,\beta}\alpha_i^\beta + C^\beta B^{2\beta} \max_{0<j\leq\bar{i}} \sqrt{\alpha_{i-j}^\beta\alpha_{i+j}^\beta}. \tag{6.94}$$

Let

$$\lambda_i \stackrel{\text{def}}{=} \max\{2, \beta\bar{i}\}, \quad 0 \le i \le 2d. \tag{6.95}$$

Since $\tau_2(g) = 1$, see (6.40) in Section 6.2.1, it follows from (6.93)-(6.95) that

$$A_g \alpha_i^\beta \le \tau_{\lambda_i}(g)\alpha_i^\beta + C^\beta B^{2\beta} \max_{0<j\le\bar{i}} \sqrt{\alpha_{i-j}^\beta \alpha_{i+j}^\beta}, \quad 0 \le i \le 2d. \tag{6.96}$$

Now we fix a lattice Δ in $\mathbb{R}^{2d}$ and define functions f_i, $0 \le i \le 2d$, on G by

$$f_i(h) = \alpha_i(h\Delta)^\beta, \quad h \in \mathrm{G}.$$

Then it follows from (6.96) that

$$A_g f_i \le \tau_{\lambda_i}(g) f_i + C^\beta B^{2\beta} \max_{0<j\le\bar{i}} \sqrt{f_{i-j} f_{i+j}}, \quad 0 \le i \le 2d.$$

On the other hand, in view of (6.85),

$$f_i(yh) \le B^\beta f_i(h), \quad \text{if} \quad 0 \le i \le 2d,\ h, y \in \mathrm{G} \quad \text{and} \quad \|y\| \le \|g\|.$$

Since $\beta d > 2$, we have that $\beta d = \lambda_d > \lambda_i$ for any $i \ne d$. Now we can apply Proposition 6.12 (c) in order to get that

$$\begin{aligned}(A_h\alpha^\beta)(\Delta) < (A_h \sum_{0\le i\le 2d} \alpha_i^\beta)(\Delta) = (A_h \sum_{0\le i\le 2d} f_i)(1) \ll (\sum_{0\le i\le 2d} f_i(1))\,\|h\|^{\lambda_d-2} \\ = (\sum_{0\le i\le 2d} \alpha_i(\Delta)^\beta)\|h\|^{\lambda_d-2} \le 2d\alpha(\Delta)^\beta\,\|h\|^{\beta d-2}.\end{aligned} \tag{6.97}$$

The inequality (6.97) proves the theorem for our specific choice of the quasinorms $|\cdot|_i$. Now it remains to notice that any two quasinorms on $\bigwedge^i\mathbb{R}^n$ are equivalent. □

CHAPTER 7

Appendix C

7.1 Integer-valued Quadratic Forms

In the following we summarize some essential results on small zeros of integer-valued quadratic forms and revisit Schlickewei's work [Sch85], including a complete derivation. These norm-bounds (depending on the signature of Q) for isotropic vectors will be used together with our effective equidistribution results, resp. our extension of the Birch-Davenport approach, in order to obtain the quantitative versions of the Oppenheim conjecture stated in Theorem 1.9 and Theorem 1.6.

Here we shall suppose that

$$A[x] = \sum_{i,j=1}^{d} a_{i,j} x_i x_j$$

is an indefinite quadratic form in d variables and Λ is a full-rank lattice in $\mathbb{R}^d$ such that $A[m]$ takes integral values on Λ. It is well-known that such form represents non-trivially zero on Λ if the rank of the associated matrix A is at least 5. Meyer [Mey84] was the first who proved this (reformulated for $\Lambda = \mathbb{Z}^d$) using elementary arguments. Nowadays this result is usually deduced from the Hasse-Minkowski theorem, which is a *local-global principle*: A rational form A represents zero non-trivially over $\mathbb{Q}$ if and only if it represents zero over any completion of $\mathbb{Q}$, i.e. over the field $\mathbb{Q}_p$ of p-adic numbers for all $p \in \mathbb{P}$ and over $\mathbb{R}$ (see [Ger08], Theorem 5.7). Since A is isotropic over all $\mathbb{Q}_p$ for p finite, provided that A is regular and $d \geq 5$, Meyer's Theorem follows immediately (see [Ger08], Corollary 5.10). In contrast, it is possible that an indefinite integral form in four variables does not represent zero.

7.2 Schlickewei's Work on Small Zeros of Integral Quadratic Forms

Similarly to the result of Birch and Davenport [BD58b] on diagonal forms in five variables, our quantitative bounds depend essentially on explicit bounds for small zeros of integral forms as well, since our argument depends on rational approximations that are 'close' to scalar multiples of Q. First bounds of this kind were proved by Cassels [Cas55] based on a geometric argument using Minkowski's theorem on successive minima. Birch and Davenport [BD58c] improved Cassels' result as follows: If $d \geq 3$ and $A[m]$ admits a non-trivial zero on the lattice Λ, then there exists an isotropic $m \in \Lambda \setminus \{0\}$ with Euclidean norm

$$0 < \|m\|^2 \leq \gamma_{d-1}^{d-1} (2 \operatorname{Tr} A^2)^{(d-1)/2} (\det \Lambda)^2, \tag{7.1}$$

where γ_d denotes Hermite's constant in dimension d. This bound is essentially best possible in view of an example by M. Kneser if A has signature $(n-1, 1)$, see [Cas56]. On the other hand, a result of Schmidt [S79a] may lead us to expect that (7.1) can be improved by considering rational isotropic subspaces. In fact, Schlickewei [Sch85] proved that the dimension, say d_0, of a maximal rational isotropic subspace influences the size of possible solutions essentially as follows.

Theorem 7.1 (Schlickewei [Sch85]). Let Λ be a d-dimensional lattice and A a non-trivial quadratic form in d variables taking integral values on Λ. Also let $d_0 \geq 1$ be maximal such that there exist linearly independent lattice points $m_1, \dots, m_{d_0} \in \Lambda$ with the property that A vanishes on the linear subspace generated by $m_1, \dots, m_{d_0}$. Then there exist such points $m_1, \dots, m_{d_0} \in \Lambda$ satisfying

$$(\|m_1\| \dots \|m_{d_0}\|)^2 \ll_d (\operatorname{Tr} A^2)^{(d-d_0)/2} (\det \Lambda)^2. \tag{7.2}$$

In the same way as Birch and Davenport [BD58c] deduce their Theorem B from their Theorem A, we may conclude

Theorem 7.2 (Schlickewei [Sch85]). Let $F, G \neq 0$ be quadratic forms in d variables and suppose in addition that G is positive definite. Let d_0 be maximal such that F vanishes on a rational subspace of dimension d_0. Then there exist d_0 linearly independent lattice points $m_1, \dots, m_{d_0} \in \mathbb{Z}^d$ such that F vanishes on the corresponding subspace and

$$G[m_1] \cdots G[m_{d_0}] \ll_d (\operatorname{Tr}(FG^{-1})^2)^{(d-d_0)/2} \det G,$$

where the implicit constant depends on d only.

Additionally, Schlickewei derived also the following lower bound (7.3) for the dimension of a maximal rational isotropic subspace in terms of the signature (r, s), compare with Hilfssatz of Section 4 in [Sch85]. We shall reproduce his proof of (7.3), which relies on an induction argument combined with Meyer's theorem, as well.

Theorem 7.3. Suppose that A takes integral values on Λ and that A has signature $(r, s; t)$ with $r + s + t = d$. Moreover, let $r \geq s$. Then A vanishes on a subspace of dimension at least d_0, generated by linearly independent lattice points $m_1, \dots, m_{d_0} \in \Lambda$, where

$$d_0 \geq \begin{cases} s+t & \text{if } r \geq s+3 \\ s+t-1 & \text{if } r = s+2 \text{ or } r = s+1 \\ s+t-2 & \text{if } r = s \end{cases}. \tag{7.3}$$

Obviously, a straightforward combination of the upper bound (7.3) together with Theorem 7.1 yields explicit bounds on the smallest non-trivial isotropic vector. However this application can be improved in the cases $r = s + 2$ and $r = s$ by reducing the problem to dimension $d - 1$ as done by Schlickewei in *Folgerung 3* of [Sch85]: He proved that for any integral quadratic form A with signature (r, s) there exists an isotropic lattice point $m \in \mathbb{Z}^d \setminus \{0\}$ such that $\|m\|^2 \ll_d (\operatorname{Tr} A^2)^\rho$, where ρ is defined as in (1.4) by

$$\rho := \rho(r, s) := \begin{cases} \frac{1}{2}\frac{r}{s} & \text{for } r \geq s+3 \\ \frac{1}{2}\frac{s+2}{s-1} & \text{for } r = s+2 \text{ or } r = s+1 \\ \frac{1}{2}\frac{s+1}{s-2} & \text{for } r = s \end{cases}.$$

We will extend this result to general lattices leading to the following strengthening of (7.1).

Corollary 7.4. Let A denote a non-singular quadratic form with signature (r, s) in $r + s = d \geq 5$ variables, which takes integral values on Λ only. Additionally suppose that $|\det(\Lambda)| \geq 1$ and $\operatorname{Tr} A^2 \geq 1$, then the smallest non-trivial isotropic vector $m \in \Lambda$ of A satisfies the bound

$$0 < \|m\|^2 \ll_d (\operatorname{Tr} A^2)^\rho \, |\det \Lambda|^{\frac{4\rho+2}{d}}, \tag{7.4}$$

where ρ is defined as in (1.4).

Remark 7.5.

(a) We recall the second part of Remark 1.5: Compared to (7.1), the exponent in (7.4) is smaller for a wide range of signatures (r, s) and in the cases, where the exponent is larger, we can restrict A by setting certain coordinates to zero to arrive at least at the result of the case $d = 5$. For example, if one has $r \sim s$, then $2\rho \sim 1$ and therefore $(2\rho + 1)/d \sim 2/d$.

(b) In a series of papers [S85; Sch85; SS88] Schlickewei and Schmidt have shown that the above-mentioned bounds are - in most cases - best possible. More details on the optimality of these bounds were already mentioned in Remark 1.4.

Proof of Theorem 7.1: Let $m_1, \ldots, m_{d_0} \in \Lambda$ be a basis of a d_0-dimensional subspace on which A vanishes. Additionally, we suppose that this basis is chosen such that

$$\|m_1 \wedge \ldots \wedge m_{d_0}\|^2 = \det(\langle m_i, m_j \rangle : i, j = 1, \ldots, d_0)$$

is minimal. Here, $\langle \cdot, \cdot \rangle$ denotes the standard euclidean inner product on $\mathbb{R}^d$. (Note that the minimum is attained, because the above norm takes values in a discrete set. In fact, if we write $\Lambda = G\mathbb{Z}^d$ with $G \in \mathrm{GL}(\mathbb{R}, d)$, then the d_0-th exterior power of G is invertible with inverse $\wedge_{i=1}^{d_0}(G^{-1})$ and thus $\|(\wedge_{i=1}^{d_0} G)v\| \geq \|\wedge_{i=1}^{d_0}(G^{-1})\|^{-1}\|v\|$ for any $v \in \wedge_{i=1}^{d_0}\mathbb{R}^d$.) Moreover, for notational simplicity we write

$$\Delta \stackrel{\text{def}}{=} \det(\langle m_i, m_j \rangle : i, j = 1, \ldots, d_0)^{1/2}. \tag{7.5}$$

Let M denote the subspace, respectively Λ_{d_0} the lattice, generated by $m_1, \ldots, m_{d_0}$, and $M^\perp = \{x \in \mathbb{R}^d : \langle m_i, x \rangle = 0 \ \forall i = 1, \ldots, d_0\}$ the orthogonal complement of M. By choice, the volume of a fundamental domain of Λ_{d_0} is the determinant Δ. Furthermore, we denote by $\Lambda^\perp$ the $(d - d_0)$-dimensional lattice arising as the projection of the lattice Λ onto $M^\perp$. According to (7.5) we have

$$\det(\Lambda) = \det(\Lambda_{d_0}) \det(\Lambda^\perp) = \Delta \det(\Lambda^\perp). \tag{7.6}$$

Now we may use Minkowski's convex body theorem (see [Cas97], Section III.2.2) and see that there exists a non-trivial lattice point $v \in \Lambda^\perp$ satisfying

$$\|v\| \ll_d \det(\Lambda^\perp)^{1/(d-d_0)} = (\det(\Lambda)/\Delta)^{1/(d-d_0)}. \tag{7.7}$$

Therefore, there exist $u \in \Lambda$ and $\lambda_1, \ldots, \lambda_{d_0} \in \mathbb{R}$ such that

$$u = \lambda_1 m_1 + \ldots + \lambda_{d_0} m_{d_0} + v. \tag{7.8}$$

Since we have $\langle x, Ay \rangle = 0$ for all $x, y \in M$, the maximality of d_0 implies the existence of a non-trivial point $x \in M_u := M \oplus \mathbb{R}u$ with

$$\langle x, Au \rangle \neq 0. \tag{7.9}$$

The points $m_1, \ldots, m_{d_0}, u$ generate a lattice $\Lambda_{d_0+1} \subset M_u$ of dimension $d_0 + 1$. Since $v \in M^\perp$, we obtain from (7.5), (7.7) and (7.8) that

$$\det(\Lambda_{d_0+1}) = \Delta\|v\| \leq \Delta^{1-1/(d-d_0)} \det(\Lambda)^{1/(d-d_0)}.$$

The next step is to construct a d_0-dimensional sublattice Λ' of Λ_{d_0+1} such that A vanishes on Λ' and the determinant of Λ is bounded by

$$\det(\Lambda') \ll_d \Delta^{1-2/(d-d_0)} \det(\Lambda)^{2/(d-d_0)} (\operatorname{Tr} A^2)^{1/2}. \tag{7.10}$$

At this point we should note that this idea is essentially due to Cassels [Cas55]. Assuming (7.10) we get in view of the minimality of Δ that

$$\Delta \ll_d \Delta^{1-2/(d-d_0)} \det(\Lambda)^{2/(d-d_0)} (\operatorname{Tr} A^2)^{1/2}.$$

In other words, we would obtain that

$$\Delta \ll_d (\operatorname{Tr} A^2)^{(d-d_0)/4} \det(\Lambda). \tag{7.11}$$

This would already complete the proof, since Minkowski's theorem on successive minima implies that the successive minima $n_1, \ldots, n_{d_0}$ of Λ_{d_0} satisfy

$$\|n_1\| \ldots \|n_{d_0}\| \asymp_d \Delta.$$

Thus, we are left to construct a lattice with (7.10): We recall that A takes integral values on Λ and hence $2\langle Ax, y\rangle \in \mathbb{Z}$ for all $x, y \in \Lambda$, where we used the decomposition $A[x+y] = A[x] + 2\langle x, Ay\rangle + A[y]$. Especially, we may take the points

$$x = 2x_1 m_1 + \ldots + 2x_{d_0} m_{d_0} + \nu u = (2x_1 + \nu\lambda_1)m_1 + \ldots + (2x_{d_0} + \nu\lambda_{d_0})m_{d_0} + \nu v$$

with $x_1, \ldots, x_{d_0}, \nu \in \mathbb{Z}$ and $y = u$ to get $2\langle x, Au\rangle \in \mathbb{Z}$. According to (7.8) we also have

$$\langle x, Au\rangle = \nu\langle v, Av\rangle + \sum_{i=1}^{d_0} (2x_i + 2\lambda_i)\langle v, Am_i\rangle.$$

Now let $\mathfrak{L}$ denote the (d_0+1)-dimensional lattice spanned by

$$m_1' = 4m_1, \ldots, m_{d_0}' = 4m_{d_0}, u' = 4\lambda_1 m_1 + \ldots + 4\lambda_{d_0} m_{d_0} + 2v.$$

As we have seen, any point $x = x_1 m_1' + \ldots + x_{d_0} m_{d_0}' + \nu u'$ of $\mathfrak{L}$ satisfies

$$\langle x, Av\rangle = 2\left(\textstyle\sum_{i=1}^{d_0} (2x_i + 2\lambda_i)\langle v, Am_i\rangle + \nu\langle v, Av\rangle\right) = 2\langle \textstyle\sum_{i=1}^{d_0} 2x_i m_i + \nu v, Au\rangle \in \mathbb{Z}. \tag{7.12}$$

Since we assumed that d_0 is maximal, there exists an $x \in \mathfrak{L}$ with $\langle x, Av\rangle \neq 0$. Let us fix a point $\mathfrak{U} \in \mathfrak{L}$ such that the map $x \mapsto \langle x, Av\rangle$ is minimal and positive, say with value $a \in \mathbb{N}$. Obviously, this lattice point $\mathfrak{U}$ has the property that

$$\langle \mathfrak{U}, a^{-1}Av\rangle = 1. \tag{7.13}$$

Now if $x \in \mathfrak{L}$ and $\langle x, Av\rangle = qa + r$ with $0 \le r < a$, then $x - q\mathfrak{U} \in \mathfrak{L}$ and thus $\langle x - q\mathfrak{U}, Av\rangle = r$. Since $a > 0$ was minimal, we see that $r = 0$. This argument shows that $\langle x, a^{-1}Av\rangle \in \mathbb{Z}$ for all $x \in \mathfrak{L}$. Similarly, we see also that $\mathfrak{U}$ must be a primitive lattice point in $\mathfrak{L}$. Hence, we can extended $\mathfrak{U}$ to a basis $\mathfrak{U}, b_1, \ldots, b_{d_0}$ of $\mathfrak{L}$. If we replace b_i by $b_i - \langle b_i, a^{-1}Av\rangle\mathfrak{U}$, we get $\langle b_i, Av\rangle = 0$ for all $i = 1, \ldots, d_0$ as well. Here we used that $\langle b_i, a^{-1}Av\rangle \in \mathbb{Z}$. In particular, the lattice $\mathfrak{L}'$ generated by $b_1, \ldots, b_{d_0}$ lies in the subspace W determined by the condition

$$\langle x, a^{-1}Av\rangle = 0, \ x \in \operatorname{span}(\mathfrak{L}) \tag{7.14}$$

and $b_1, \ldots, b_{d_0}$ is a basis of W. We also see that the point $\mathfrak{U}$ has distance $\det(\mathfrak{L})/\det(\mathfrak{L}')$ to W. Since $a^{-1}Av$ is orthogonal to W, we conclude together with (7.13) that

$$\det(\mathfrak{L}') = \det(\mathfrak{L})a^{-1}\|Av\|. \tag{7.15}$$

Recall that $\mathfrak{L}$ is generated by the basis $4m_1, \ldots, 4m_{d_0}, 4\lambda_1 m_1 + \ldots + 4\lambda_{d_0} m_{d_0} + 2v$ and $v \in M^\perp$. Hence, using (7.5) and (7.7), we conclude that

$$\det(\mathfrak{L}) \ll_d \|v\|\Delta \ll_d \Delta^{1-1/(d-d_0)}(\det(\Lambda))^{1/(d-d_0)}$$

and in view of (7.15) together with (7.7) also that

$$\det(\mathfrak{L}') \leq \det(\mathfrak{L})(\operatorname{Tr} A^2)^{1/2}\|v\| \ll_d \Delta^{1-2/(d-d_0)}(\det(\Lambda))^{2/(d-d_0)}(\operatorname{Tr} A^2)^{1/2}, \tag{7.16}$$

where we used that $\sqrt{\operatorname{Tr} A^2}$ is the Hilbert-Schmidt norm of A. Denote by $\mathfrak{X}_i' = \sum_{j=1}^{d_0} x_{j,i} m_j' + y_i u'$ the successive minima of $\mathfrak{L}'$. Again, by Minkowski's theorem we know that

$$\|\mathfrak{X}_1'\| \ldots \|\mathfrak{X}_{d_0}'\| \asymp_d \det(\mathfrak{L}'). \tag{7.17}$$

The linearly independence of $\mathfrak{X}_1', \ldots, \mathfrak{X}_{d_0}'$ implies also that the points

$$\mathfrak{X}_i = x_{1,i} m_1 + \ldots + x_{d_0,i} m_{d_0} + y_i u, \quad i = 1, \ldots, d_0$$

are linearly independent. In fact, suppose that $\sum_{i=1}^{d_0} n_i \mathfrak{X}_i = 0$. Then we have

$$\sum_{i=1}^{d_0} n_i \mathfrak{X}_i' = \sum_{i=1}^{d_0} 4n_i \mathfrak{X}_i - 2\sum_{i=1}^{d_0} n_i y_i v = -2\sum_{i=1}^{d_0} n_i y_i v.$$

Since $m_1, \ldots, m_{d_0}, v$ are linearly independent, comparing the coefficients yields that

$$\sum_{i=1}^{d_0} n_i y_i = 0$$

and thus $n_i = 0$ for all $i = 1, \ldots, d_0$. Note that by choice we have $\mathfrak{X}_i \in \Lambda$. In the same way, using that $v \in M^\perp$, we find

$$\|\mathfrak{X}_i\| \asymp \|\mathfrak{X}_i'\|. \tag{7.18}$$

Combining (7.16) together with (7.17) and (7.18) yields

$$\|\mathfrak{X}_1\| \ldots \|\mathfrak{X}_{d_0}\| \ll \Delta^{1-2/(d-d_0)}(\det(\Lambda))^{2/(d-d_0)}(\operatorname{Tr} A^2)^{1/2}.$$

If we verify that $A[x]$ vanishes on the subspace spanned by $\mathfrak{X}_1, \ldots, \mathfrak{X}_{d_0}$, then the claimed assertion follows: Let $\mathfrak{L}''$ denote the lattice generated by $\mathfrak{X}_1, \ldots, \mathfrak{X}_{d_0}$. Recalling that Δ was chosen minimal, we get

$$\begin{aligned}\|\mathfrak{X}_1\| \ldots \|\mathfrak{X}_{d_0}\| &\ll \Delta^{1-2/(d-d_0)}(\det(\Lambda))^{2/(d-d_0)}(\operatorname{Tr} A^2)^{1/2} \\ &\ll \det(\mathfrak{L}'')^{1-2/(d-d_0)}(\det(\Lambda))^{2/(d-d_0)}(\operatorname{Tr} A^2)^{1/2} \\ &\ll (\|\mathfrak{X}_1\| \ldots \|\mathfrak{X}_{d_0}\|)^{1-2/(d-d_0)}(\det(\Lambda))^{2/(d-d_0)}(\operatorname{Tr} A^2)^{1/2}\end{aligned}$$

and thus (7.2) holds. In order to show that $A[x]$ vanishes on the subspace spanned by

$\mathfrak{X}_1, \ldots, \mathfrak{X}_{d_0}$ it is sufficient to prove that

$$\langle \mathfrak{X}_i, A\mathfrak{X}_j \rangle = 0 \quad \forall i, j = 1, \ldots, d_0. \tag{7.19}$$

To do this, note that $\mathfrak{X}_i$ can be rewritten as

$$\mathfrak{X}_i = (x_{1,i} + \lambda_1 y_i)m_1 + \ldots (x_{d_0,i} + \lambda_{d_0} y_i)m_{d_0} + y_i v.$$

Using this representation we find

$$\begin{aligned}\langle \mathfrak{X}_i, A\mathfrak{X}_k \rangle = y_i &\left(\sum_{k=1}^{d_0} (x_{k,j} + \lambda_k y_j) \langle m_k, Av \rangle + \frac{1}{2} y_j \langle v, Av \rangle \right) \\ &+ y_j \left(\sum_{k=1}^{d_0} (x_{k,i} + \lambda_k y_i) \langle m_k, Av \rangle + \frac{1}{2} y_i \langle v, Av \rangle \right) = \frac{y_i}{4} \langle \mathfrak{X}'_j, Av \rangle + \frac{y_j}{4} \langle \mathfrak{X}'_i, Av \rangle.\end{aligned}$$

Since $\mathfrak{X}'_1, \ldots, \mathfrak{X}'_{d_0} \in \mathfrak{L}'$ and (7.14) holds on $\mathfrak{L}'$, both terms in the last line are zero. This concludes the proof of (7.19). □

Remark 7.6. The above arguments show the existence of an isotropic subspace of dimension d_0 with small determinant, provided that there exists a d_0-dimensional isotropic subspace. In the Geometry of Numbers it is often the case that one can use the existence of a lattice points satisfying some inequality in order to get several independent points satisfying a joint inequality: Schlickewei and Schmidt [SS87; SS89] proved the existence of $d - d_0 + 1$ many isotropic subspaces $\Gamma_0, \ldots, \Gamma_{d-d_0}$ with the properties

(1) $\Gamma_0 \cap \Gamma_j$ has dimension $d_0 - 1$ for each $j = 1, \ldots, d - d_0$,

(2) the union of $\Gamma_0, \ldots, \Gamma_{d-d_0}$ spans $\mathbb{R}^n$ and

(3) $\det \Gamma_0 \det \Gamma_j \ll (\operatorname{Tr} A^2)^{(d-d_0)/2} (\det \Lambda)^2$ for each $j = 1, \ldots, d - d_0$.

Clearly, the last inequality immediately implies Schlickewei's result on small zeros of integral forms. In addition, this extends (with $d_0 = 1$) Davenport's work [Dav71] and generalizes a result of Schulze-Pillot [Sch83], which states that there exist d linearly independent isotropic lattice points $x_0, \ldots, x_{d-1}$ with

$$\|x_0\|^{d_0 - 1} \|x_1\| \ldots \|x_{d-1}\| \ll (\operatorname{Tr} A^2)^{(d-1)^2/2} (\det \Lambda)^{2(d-1)}.$$

In fact, one corollary of (3) is that the lattices $\Gamma_0, \ldots, \Gamma_{d-d_0}$ satisfy

$$(\det \Gamma_0)^{d-d_0} \det \Gamma_1 \ldots \det \Gamma_{d-d_0} \ll (\operatorname{Tr} A^2)^{(d-d_0)^2/2} (\det \Lambda)^{2(d-d_0)}.$$

Proof of Theorem 7.3: Here, as usual, A denotes the symmetric matrix corresponding to the quadratic form $A[x]$. According to the rank assumptions, we have $\dim(\ker A) = t$. We may factorize A via $\pi \colon \mathbb{R}^d \to \mathbb{R}^d / \ker A$. Note that any maximal isotropic subspace in $\mathbb{R}^d / \ker A$ with dimension d_0 corresponds to a maximal isotropic subspace in $\mathbb{R}^d$ with dimension $d_0 + t$. Moreover, the image of the lattice Λ is also a full-rank lattice in $\mathbb{R}^d / \ker A$ and A takes integral-values on this lattice. Thus we can assume w.l.o.g. that $t = 0$ and A is not singular. Additionally, we may suppose that $\Lambda = \mathbb{Z}^d$: If $\Lambda = B\mathbb{Z}^d$ with $B \in \mathrm{GL}(d)$, then the quadratic form $M[m] = 2A[Bm]$ has integral coefficients and by Sylvester's law of inertia also the same signature as A.

Overall we are reduced to the case when A is a non-singular integral matrix with signature (r,s) and w.l.o.g. $r \geq s$. Now we shall prove that $A[x]$ vanishes on a rational subspace of dimension at least

$$d' \stackrel{\text{def}}{=} \begin{cases} s & \text{if } r \geq s+3, \\ s-1 & \text{if } r = s+2 \text{ or } r = s+1, \\ s-2 & \text{if } r = s. \end{cases}$$

Recall that by Meyer's theorem any indefinite non-singular form with integral coefficients in at least five variables represents zero non-trivially on $\mathbb{Z}^d$. Here, we have $r + s = d \geq 5$ with $r \geq s > 0$. Let $\mathfrak{U}_1 \in \mathbb{Z}^d \setminus \{0\}$ be an isotropic vector of A. Because A is non-singular, there exists $\mathfrak{B} \in \mathbb{Z}^d \setminus \{0\}$ with

$$\langle \mathfrak{U}_1, A\mathfrak{B} \rangle \neq 0.$$

Note that the subspace spanned by $\mathfrak{U}_1$ and $\mathfrak{B}$ is a hyperbolic plane. Next we restrict A on the subspace M_1 determined by

$$\langle \mathfrak{X}, A\mathfrak{B} \rangle = 0 \quad \text{and} \quad \langle \mathfrak{X}, A\mathfrak{U}_1 \rangle = 0,$$

which is $(d-2)$-dimensional. Obviously, A is not singular on M_1, has signature $(r-1, s-1)$ and $A[x]$ takes integral values on $M_1 \cap \mathbb{Z}^d$. Thus, if $s-1 > 0$ and $r+s-2 = d-2 \geq 5$, we can proceed by using Meyer's theorem again to get a second isotropic lattice point $\mathfrak{U}_2 \in M_1 \cap \mathbb{Z}^d$. In view of (7.2) we see that $A[x]$ vanishes on $\operatorname{span}(\mathfrak{U}_1, \mathfrak{U}_2)$. Repeated application of this argument leads to the following cases:

(a) If $r - s \geq 3$, then we get in the $n-1$-th application of our argument that the quadratic form, restricted on the corresponding subspace, has signature $(r-s+1, 1)$. Because of $r - s + 1 \geq 4$ we can apply Meyer's theorem in order to get another isotropic lattice point. Thus $A[x]$ vanishes on a rational subspace of dimension s.

(b) If $r - s = 2$, then we have in the $n-2$-th step a quadratic form with signature $(r-s+2, 2) = (4,2)$ and again we may apply Meyer's theorem in the same way as before. Thus, we see that $A[x]$ vanishes on a rational space of dimension s.

(c) In the case $r - s = 1$ we get, as before, a $(s-1)$-dimensional rational subspace on which $A[x]$ vanishes.

(d) If $r = s$, then the rational subspace, on which $A[x]$ vanishes, has dimension $s - 2$. □

Proof of Corollary 7.4: As can be checked easily, in the cases $r \geq s+3$ and $r = s+1$ the bound (7.4) follows immediately by Theorem 7.1 together with (7.3), since we have $d/d_0 \leq 2\rho + 1$ in these cases. If $r = s$ or $r = s+2$, then this relation does not hold. Here we fix a reduced basis $v_1, \ldots, v_d$ of Λ with $\|v_1\| \leq \ldots \leq \|v_d\|$ and

$$|\det(\Lambda)| \asymp_d \|v_1\| \ldots \|v_d\|.$$

Let $\Lambda_0 := \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_{d-1}$, which is a $d-1$ dimensional sublattice of Λ, and note that Hadamard's inequality (for positive definite matrices), applied on $B^T B = (b_{i,j})_{1 \leq i,j \leq d}$ with $B := (v_1, \ldots, v_{d-1})$, shows that

$$\det(\Lambda_0) = (\det(B^T B))^{1/2} \leq \prod_{k=1}^{d-1} \sqrt{b_{k,k}} = \prod_{k=1}^{d-1} \|v_k\|.$$

In other words, we showed that $\|v_1 \wedge \ldots \wedge v_{d-1}\| \leq \|v_1\| \ldots \|v_{d-1}\|$ and therefore

$$\det(\Lambda_0) \ll_d \det(\Lambda)^{(d-1)/d}. \tag{7.20}$$

Now denote by A_0 the restriction of A to the subspace generated by $v_1, \ldots, v_{d-1}$. It follows that A_0 has signature either $(r, s-1)$ or $(r-1, s)$ and, since $(\operatorname{Tr} A^2)^{1/2} = \|A\|_{\mathrm{HS}}$, also that $\operatorname{Tr} A_0^2 \leq \operatorname{Tr} A^2$. Applying Theorem 7.1 to A_0 and Λ_0 shows that there exists an isotropic lattice point $m \in \Lambda_0 \setminus \{0\}$ such that

$$\|m\|^2 \ll_d (\operatorname{Tr} A_0^2)^{\frac{d-1-d_0}{2d_0}} |\det \Lambda_0|^{\frac{2}{d_0}} \ll_d (\operatorname{Tr} A^2)^{\frac{d-1-d_0}{2d_0}} |\det \Lambda|^{\frac{d-1}{d}\frac{2}{d_0}},$$

where we used (7.20) in the last step (in doing so, we need $\operatorname{Tr} A^2 \geq 1$ and $|\det \Lambda| \geq 1$). Here d_0 denotes the dimension of a maximal isotropic subspace of A_0 (instead of A). Completing the proof, we note that in both cases $r = s+2$ and $r = s$ one has

$$\frac{d-1}{d_0} \leq 2\rho + 1,$$

as can be readily seen. □

7.3 Discrete Optimization: Possible Signatures and Exponents

In this section we treat the discrete optimization problem with which we are faced in the diagonal case: We have to determine all possible values of ρ_k (defined as in Lemma 2.21) depending on the signature (r, s) of Q and then find an upper bound for $\mathfrak{p}_1(d), \ldots, \mathfrak{p}_3(d)$, which appear in the iteration of the coupling argument (see Lemmas 2.25 - 2.27).

Even d							
Sign(Q)	2ρ	Sign(Q_3)	$2\rho_3$	Sign(Q_2)	$2\rho_2$	Sign(Q_1)	$2\rho_1$
$(\frac{d}{2}, \frac{d}{2})$	$\frac{d+2}{d-4}$	$(\frac{d-6}{2}, \frac{d}{2})$ $(\frac{d-4}{2}, \frac{d-2}{2})$ $(\frac{d-2}{2}, \frac{d-4}{2})$ $(\frac{d}{2}, \frac{d-6}{2})$	$\frac{d}{d-6}$	$(\frac{d-4}{2}, \frac{d}{2})$ $(\frac{d-2}{2}, \frac{d-2}{2})$ $(\frac{d}{2}, \frac{d-4}{2})$	$\frac{d}{d-6}$	$(\frac{d-2}{2}, \frac{d}{2})$ $(\frac{d}{2}, \frac{d-2}{2})$	$\frac{d+2}{d-4}$
$(\frac{d+2}{2}, \frac{d-2}{2})$	$\frac{d+2}{d-4}$	$(\frac{d-4}{2}, \frac{d-2}{2})$ $(\frac{d-2}{2}, \frac{d-4}{2})$ $(\frac{d}{2}, \frac{d-6}{2})$ $(\frac{d+2}{2}, \frac{d-8}{2})$	$\frac{d}{d-6}$ $\frac{d}{d-6}$ $\frac{d}{d-6}$ $\frac{d+2}{d-8}$	$(\frac{d-2}{2}, \frac{d-2}{2})$ $(\frac{d}{2}, \frac{d-4}{2})$ $(\frac{d+2}{2}, \frac{d-6}{2})$	$\frac{d}{d-6}$ $\frac{d}{d-6}$ $\frac{d+2}{d-6}$	$(\frac{d}{2}, \frac{d-2}{2})$ $(\frac{d+2}{2}, \frac{d-4}{2})$	$\frac{d+2}{d-4}$
$(\frac{d+4}{2}, \frac{d-4}{2})$	$\frac{d+4}{d-4}$	$(\frac{d-2}{2}, \frac{d-4}{2})$ $(\frac{d}{2}, \frac{d-6}{2})$ $(\frac{d+2}{2}, \frac{d-8}{2})$ $(\frac{d+4}{2}, \frac{d-10}{2})$	$\frac{d}{d-6}$ $\frac{d}{d-6}$ $\frac{d+2}{d-8}$ $\frac{d+4}{d-10}$	$(\frac{d}{2}, \frac{d-4}{2})$ $(\frac{d+2}{2}, \frac{d-6}{2})$ $(\frac{d+4}{2}, \frac{d-8}{2})$	$\frac{d}{d-6}$ $\frac{d+2}{d-6}$ $\frac{d+4}{d-8}$	$(\frac{d+2}{2}, \frac{d-4}{2})$ $(\frac{d+4}{2}, \frac{d-6}{2})$	$\frac{d+2}{d-4}$ $\frac{d+4}{d-6}$
$(\frac{d+2l}{2}, \frac{d-2l}{2})$ $l \geq 3$	$\frac{d+2l}{d-2l}$	$(\frac{d+2l-6}{2}, \frac{d-2l}{2})$ $(\frac{d+2l-4}{2}, \frac{d-2l-2}{2})$ $(\frac{d+2l-2}{2}, \frac{d-2l-4}{2})$ $(\frac{d+2l}{2}, \frac{d-2l-6}{2})$	$\frac{d+2l-6}{d-2l}$ $\frac{d+2l-4}{d-2l-2}$ $\frac{d+2l-2}{d-2l-4}$ $\frac{d+2l}{d-2l-6}$	$(\frac{d+2l-4}{2}, \frac{d-2l}{2})$ $(\frac{d+2l-2}{2}, \frac{d-2l-2}{2})$ $(\frac{d+2l}{2}, \frac{d-2l-4}{2})$	$\frac{d+2l-4}{d-2l}$ $\frac{d+2l-2}{d-2l-2}$ $\frac{d+2l}{d-2l-4}$	$(\frac{d+2l-2}{2}, \frac{d-2l}{2})$ $(\frac{d+2l}{2}, \frac{d-2l-2}{2})$	$\frac{d+2l-2}{d-2l}$ $\frac{d+2l}{d-2l-2}$

Odd d							
Sign(Q)	2ρ	Sign(Q_3)	$2\rho_3$	Sign(Q_2)	$2\rho_2$	Sign(Q_1)	$2\rho_1$
$\left(\frac{d+1}{2}, \frac{d-1}{2}\right)$	$\frac{d+3}{d-3}$	$\left(\frac{d-5}{2}, \frac{d-1}{2}\right)$ $\left(\frac{d-3}{2}, \frac{d-3}{2}\right)$ $\left(\frac{d-1}{2}, \frac{d-5}{2}\right)$ $\left(\frac{d+1}{2}, \frac{d-7}{2}\right)$	$\frac{d-1}{d-7}$ $\frac{d-1}{d-7}$ $\frac{d-1}{d-7}$ $\frac{d+1}{d-7}$	$\left(\frac{d-3}{2}, \frac{d-1}{2}\right)$ $\left(\frac{d-1}{2}, \frac{d-3}{2}\right)$ $\left(\frac{d+1}{2}, \frac{d-5}{2}\right)$	$\frac{d+1}{d-5}$	$\left(\frac{d-1}{2}, \frac{d-1}{2}\right)$ $\left(\frac{d+1}{2}, \frac{d-3}{2}\right)$	$\frac{d+1}{d-5}$
$\left(\frac{d+3}{2}, \frac{d-3}{2}\right)$	$\frac{d+3}{d-3}$	$\left(\frac{d-3}{2}, \frac{d-3}{2}\right)$ $\left(\frac{d-1}{2}, \frac{d-5}{2}\right)$ $\left(\frac{d+1}{2}, \frac{d-7}{2}\right)$ $\left(\frac{d+3}{2}, \frac{d-9}{2}\right)$	$\frac{d-1}{d-7}$ $\frac{d-1}{d-7}$ $\frac{d+1}{d-7}$ $\frac{d+3}{d-9}$	$\left(\frac{d-1}{2}, \frac{d-3}{2}\right)$ $\left(\frac{d+1}{2}, \frac{d-5}{2}\right)$ $\left(\frac{d+3}{2}, \frac{d-7}{2}\right)$	$\frac{d+1}{d-5}$ $\frac{d+1}{d-5}$ $\frac{d+3}{d-7}$	$\left(\frac{d+1}{2}, \frac{d-3}{2}\right)$ $\left(\frac{d+3}{2}, \frac{d-5}{2}\right)$	$\frac{d+1}{d-5}$ $\frac{d+3}{d-5}$
$\left(\frac{d+5}{2}, \frac{d-5}{2}\right)$	$\frac{d+5}{d-5}$	$\left(\frac{d-1}{2}, \frac{d-5}{2}\right)$ $\left(\frac{d+1}{2}, \frac{d-7}{2}\right)$ $\left(\frac{d+3}{2}, \frac{d-9}{2}\right)$ $\left(\frac{d+5}{2}, \frac{d-11}{2}\right)$	$\frac{d-1}{d-7}$ $\frac{d+1}{d-7}$ $\frac{d+3}{d-9}$ $\frac{d+5}{d-11}$	$\left(\frac{d+1}{2}, \frac{d-5}{2}\right)$ $\left(\frac{d+3}{2}, \frac{d-7}{2}\right)$ $\left(\frac{d+5}{2}, \frac{d-9}{2}\right)$	$\frac{d+1}{d-5}$ $\frac{d+3}{d-7}$ $\frac{d+5}{d-9}$	$\left(\frac{d+3}{2}, \frac{d-5}{2}\right)$ $\left(\frac{d+5}{2}, \frac{d-7}{2}\right)$	$\frac{d+3}{d-5}$ $\frac{d+5}{d-7}$
$\left(\frac{d+2l+1}{2}, \frac{d-2l-1}{2}\right)$ $l \geq 3$	$\frac{d+2l+1}{d-2l-1}$	$\left(\frac{d+2l-5}{2}, \frac{d-2l-1}{2}\right)$ $\left(\frac{d+2l-3}{2}, \frac{d-2l-3}{2}\right)$ $\left(\frac{d+2l-1}{2}, \frac{d-2l-5}{2}\right)$ $\left(\frac{d+2l+1}{2}, \frac{d-2l-7}{2}\right)$	$\frac{d+2l-5}{d-2l-1}$ $\frac{d+2l-3}{d-2l-3}$ $\frac{d+2l-1}{d-2l-5}$ $\frac{d+2l+1}{d-2l-7}$	$\left(\frac{d+2l-3}{2}, \frac{d-2l-1}{2}\right)$ $\left(\frac{d+2l-1}{2}, \frac{d-2l-3}{2}\right)$ $\left(\frac{d+2l+1}{2}, \frac{d-2l-5}{2}\right)$	$\frac{d+2l-3}{d-2l-1}$ $\frac{d+2l-1}{d-2l-3}$ $\frac{d+2l+1}{d-2l-5}$	$\left(\frac{d+2l-1}{2}, \frac{d-2l-1}{2}\right)$ $\left(\frac{d+2l+1}{2}, \frac{d-2l-3}{2}\right)$	$\frac{d+2l-1}{d-2l-1}$ $\frac{d+2l+1}{d-2l-3}$

Note that in both tables the last case in every row is the worst when compared to ρ. Thus, considering all theses cases, one can derive the following bound on the exponent $\mathfrak{p}_i(d)$.

Sign(Q)	$\mathfrak{p}_3(d) \leq$	$\mathfrak{p}_2(d) \leq$	$\mathfrak{p}_1(d) \leq$
$\left(\frac{d}{2}, \frac{d}{2}\right)$	$-\frac{6d-4}{d(d-1)}$	$-\frac{6(d-2)}{d(d-1)}$	$-\frac{6}{d-1}$
$\left(\frac{d+2}{2}, \frac{d-2}{2}\right)$	$-\frac{14}{3(d-1)}$	$-\frac{4}{d-1}$	$-\frac{6}{d-1}$
$\left(\frac{d+2l}{2}, \frac{d-2l}{2}\right)$, $l \geq 2$	$-\frac{2(2l-1)}{d}$	$-\frac{4(l-1)}{d}$	$-\frac{2(2l-3)}{d}$
$\left(\frac{d+1}{2}, \frac{d-1}{2}\right)$	$-\frac{16}{3(d+1)}$	$-\frac{6(d-1)}{d(d+1)}$	$-\frac{6(d-5)}{d(d+1)}$
$\left(\frac{d+3}{2}, \frac{d-3}{2}\right)$	$-\frac{4}{d}$	$-\frac{2}{d}$	$*$
$\left(\frac{d+2l+1}{2}, \frac{d-2l-1}{2}\right)$, $l \geq 2$	$-\frac{4l}{d}$	$-\frac{2(2l-1)}{d}$	$-\frac{4(l-1)}{d}$

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