

On Quasi-hereditary Structures

Manuel Flores Galicia

Dissertation zur Erlangung des Doktorgrades
der Fakultät für Mathematik
der Universität Bielefeld

März 2020

Abstract

A quasi-hereditary algebra is an Artin algebra A together with the choice of a partial order on the set of isoclasses of simple A -modules which satisfies certain conditions. We refer to this partial order as a *quasi-hereditary structure* on A .

In this thesis we discuss two approaches to investigate all the possible choices that yield quasi-hereditary structures for a given Artin algebra.

The first strategy is to study total orders inducing quasi-hereditary structures via the homological poset, which is a partial order on the set of simple modules reflecting homological properties. The second approach refines the notion of a quasi-hereditary structure considering an appropriate equivalence relation. In particular we exhibit combinatorial characterisations of the homological poset of Auslander algebras arising from truncated polynomial rings and for blocks of Schur algebras of finite representation type. For the case of path algebras of Dynkin type \mathbb{A}_n we find a complete characterisation of all the equivalence classes of quasi-hereditary structures by means of binary trees and certain quiver decompositions.

Acknowledgements

Firstly, I would like to express my genuine gratitude to my supervisor, Prof. Henning Krause, for his advice in preparing this thesis, and for all his support and patience during my doctoral studies as a member of the BIREP research group in Bielefeld.

My sincere thanks also goes to Dr. Baptiste Rognerud for his advice and for providing me with new ideas and suggestions for my research project. Also I would like to thank Dr. Yuta Kimura for his clear advice and interesting discussions. I have to add that doing research with both was a very enriching experience.

I was supported by a joint fellowship between the Consejo Nacional de Ciencia y Tecnología - CONACYT (Mexican National Council for Science and Technology) and the Deutscher Akademischer Austauschdienst - DAAD (German Academic Exchange Service) under the funding programme Länderbezogenes Kooperationsprogramm mit Mexiko: CONACYT Promotion, 2015 (91599352). In the last period of my PhD studies I was supported by the Universität Bielefeld through the Bielefelder Nachwuchsfonds with a Doctorate Completion Scholarship.

Contents

Introduction	vii
1 Preliminaries	1
1.1 General conventions	1
1.2 Representation theory of algebras	1
1.3 Quiver representations	4
1.4 Poset theory	7
1.5 Homological embeddings and homological ring epimorphisms	8
2 Highest weight categories and quasi-hereditary algebras	11
2.1 Highest weight categories	12
2.2 Quasi-hereditary algebras	14
2.2.1 Definition via heredity chains	15
2.2.2 Definition via standard modules	17
2.2.3 The characteristic tilting module	21
2.3 The homological poset	22
2.4 Quasi-hereditary structures	25
3 The Auslander algebra of $\mathbb{K}[x]/(x^n)$	29
3.1 Basic properties of $\text{Aus } T_n$	30
3.2 A basis of $\text{Aus } T_n$	31
3.3 A basis of $\text{Aus } T_n/(e)$	39
3.4 Block decomposition of $\text{Aus } T_n/(e)$	44
3.5 Preprojective algebras of type \mathbb{A}_n	47
3.6 Blocks of $\text{Aus } T_n/(e)$	49
3.7 Homological embeddings in $\text{mod Aus } T_n$	53
3.8 The homological poset of $\text{Aus } T_n$	59
3.9 Homological Hasse quiver of $\text{Aus } T_n$	64
3.10 Homological embeddings over $\text{Aus } T_n$ and compositions	67
3.11 Quasi-hereditary quotients of $\text{Aus } T_n$	76
4 Blocks of Schur algebras of finite representation type	81
4.1 Block decompositions of B_n and \tilde{B}_n	82
4.2 Homological properties of \tilde{B}_n	85
4.3 Homological poset of B_n	89
4.4 Homological Hasse quiver of B_n	94

5	Quasi-hereditary structures of path algebras of type \mathbb{A}_n	99
5.1	Path algebras of type \mathbb{A}_n : equioriented case	99
5.2	Quasi-hereditary structures and deconcatenations	104
	Bibliography	114
	Index	115

Introduction

This thesis is framed within the representation theory of Artin algebras, which roughly speaking is the study of modules over Artin algebras.

The origins of representation theory can be found in the first half of the 19th century with the study of groups by Galois and Lagrange, and Hamilton's work on quaternions, but was DeMorgan who gave the rudiments of the definition of algebra [Gus82]. An *algebra* is vector space equipped with an associative multiplication which is bilinear. The first class of algebras to be completely classified was the one of semisimple algebras, which involves work of Hilbert, Wedderburn and Artin among others. Subsequently, in the first half of the 19th century, the Krull-Remak-Schmidt theorem (cf. Theorem 1.2.4) about indecomposability in group theory is a fundamental tool for future algebraic developments. The concept of a module, coined some years later by E. Noether, provided a new way of investigating algebras, since now their study is translated in terms of modules over an algebra. In the 1950's, homological and categorical methods were introduced, giving rise to new techniques and a robust mathematical language from which representation theory benefited. It is until the 1970's when P. Gabriel introduced quivers as a new way of conceiving algebras: path algebras modulo relations [Gab72], which led him to the classification of path algebras with a finite number of indecomposable modules up to isomorphism. In the same decade, the ideas of M. Auslander and I. Reiten became a cornerstone in modern representation theory, e.g. their AR-quiver permitted the use of new combinatorial techniques that are present in nowadays research in representation theory.

Quasi-hereditary structures

Quasi-hereditary algebras form a distinguished class of Artin algebras introduced in the 1980's by L. Scott [Sco87] in the context of the theory initiated by E. Cline, B. Parshall and L. Scott on highest weight categories arising in the representation theory of semisimple complex Lie algebras and algebraic groups [CPS88a; CPS88b; PS88]. V. Dlab and C. M. Ringel contributed extensively to the study of such algebras from a module theoretical approach, determining important homological and algebraic properties [DR89a; DR89b; DR92]. O. Iyama showed that quasi-hereditary algebras are ubiquitous among Artin algebras [Iya03].

Examples of quasi-hereditary algebras include semisimple algebras, Schur algebras [Par89], path algebras defined by finite acyclic quivers, or by directed quivers modulo relations, algebras of global dimension less or equal than two [DR89b], in particular Auslander algebras, also blocks of the BGG-category \mathcal{O} are quasi-hereditary algebras

[Hum08].

As noted earlier, quasi-hereditary algebras came up in the context of highest weight categories in the sense that a module category over a quasi-hereditary algebra is a highest weight category, and conversely every highest weight category with finitely many simple objects arises in this way [CPS88b].

In a ring theoretical setting, the notion of heredity ideal (cf. Definition 2.2.1) is central to define a quasi-hereditary ring. More precisely, a semiprimary ring Λ is *quasi-hereditary* if there exists a finite chain of ideals

$$0 = \mathfrak{a}_m \subseteq \mathfrak{a}_{m-1} \subseteq \cdots \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_0 = \Lambda$$

such that $\mathfrak{a}_{i-1}/\mathfrak{a}_i$ is a heredity ideal of Λ/\mathfrak{a}_i for $1 \leq i \leq m$. Such a chain of idempotent ideals is called a *heredity chain*. Hidden at first sight, there is an intrinsic partial order on the set of isoclasses of simple Λ -modules defined after every heredity chain. If Λ has n simple modules, then any heredity chain can be refined to a maximal heredity chain of length n .

There is a module theoretical definition of quasi-hereditary algebra equivalent to the one using heredity chains (cf. Definition 2.2.13) in which a partial order is explicitly requested. In this case a quasi-hereditary algebra is a pair (A, \triangleleft) where A is an Artin algebra and \triangleleft is an ordering on the set of simple A -modules, which is used to define a set of standard modules $\Delta = \Delta_{\triangleleft}$, and a set of costandard modules $\nabla = \nabla_{\triangleleft}$. From the notion of highest weight category, the poset \triangleleft must satisfy a basic property in order to have a quasi-hereditary algebra: to be adapted to A (cf. Definition 2.2.15). In particular, an algebra may be quasi-hereditary for one adapted order but not for another. A non-example of the last assertion are semisimple algebras; on the one hand they are quasi-hereditary for any ordering, but on the other hand any order produces the same set of standard modules, i.e. there is only one essential way to turn a semisimple algebra into a quasi-hereditary algebra, in other words, every ordering produces the same set of standard modules. In terms of heredity chains this means that any maximal heredity chain produces a unique set of heredity ideals as factors. In this thesis, the latter is formalised by the notion of quasi-hereditary structure in the following sense.

Let A be an Artin algebra, with $\{S(i)\}_{i \in I}$ a complete set of non-isomorphic simple A -modules. Let \triangleleft_1 and \triangleleft_2 be two partial orders on I , then $\triangleleft_1 \sim \triangleleft_2$ if $\Delta_{\triangleleft_1} = \Delta_{\triangleleft_2}$ and $\nabla_{\triangleleft_1} = \nabla_{\triangleleft_2}$. Then if (A, \triangleleft) is a quasi-hereditary algebra, we say that the class of \triangleleft under \sim is a *quasi-hereditary structure of A* (cf. Definition 2.4.6). Note that the number of quasi-hereditary structures is bounded by $(\text{card } I)!$. Then, the study of quasi-hereditary structures is by definition the study of these equivalence classes.

Many interesting examples of quasi-hereditary algebras come with only one quasi-hereditary structure. For instance, in recent work by K. Coulembier [Cou19] it is provided a sufficient condition for a quasi-hereditary algebra to have a unique quasi-hereditary structure, see Theorem 2.4.10. For example, cellular algebras and Brauer algebras are shown to have at most one quasi-hereditary structure.

In [Rin91] C. M. Ringel proved that if (A, \triangleleft) is quasi-hereditary, then there exists a tilting A -module T_{\triangleleft} , called the characteristic tilting module, such that $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \text{add } T_{\triangleleft}$, where $\mathcal{F}(\Delta)$ denotes the category of all A -modules that admit a Δ -filtration, and similarly for $\mathcal{F}(\nabla)$. Actually, T_{hd} depends only on the class of \triangleleft . Then, we have

that two quasi-hereditary structures are the same if their corresponding characteristic tilting modules are isomorphic.

Contributions

To our knowledge, and after an extensive literature review, no one has tried to do a systematic study of the different quasi-hereditary structures on appropriate families of quasi-hereditary algebras, besides Coulembier's work. This is the main objective of this thesis.

We first present a new approach to study all the possible heredity chains of an algebra A via its *homological poset* $\mathcal{H}(A)$ and its *homological Hasse quiver* $\mathbb{H}(A)$ (cf. Definition 2.3.3). Roughly speaking, $\mathbb{H}(A)$ is a directed graph where the vertices are in bijective correspondence with Serre subcategories of $\text{mod } A$ of the form $\text{mod } A/(e)$, for e an idempotent of A , and the arrows correspond to “minimal” embeddings between Serre subcategories that preserve all extension groups. Then $\mathbb{H}(A)$ determines all the possible heredity chains of A in the following manner.

Theorem 2.3.5. Let A be an Artin algebra, and $\{e_i\}_{i=1}^n$ a complete set of primitive orthogonal idempotents of A . Then we have a bijective correspondence between the set of paths of length n in $\mathbb{H}(A)$ such that each quotient of the corresponding Serre subcategories is semisimple, and the set of maximal heredity chains of A , given by

$$(\emptyset = J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_n = [n]) \mapsto (0 = (e_{J_n^c}) \subset \cdots \subset (e_{J_1^c}) \subset (e_{J_0^c}) = \Lambda).$$

Let Λ_n be the Auslander algebra of the truncated polynomial ring $\mathbb{K}[x]/(x^n)$. We give the following explicit combinatorial characterisation of homological embeddings between Serre subcategories of $\text{mod } \Lambda_n$.

Theorem 3.7.9. Let $J \subseteq I \subseteq \{1, \dots, n\}$ be subsets expressed as disjoint union of intervals $J = \bigsqcup_{j=1}^l J_j$ and $I = \bigsqcup_{i=1}^m I_i$, and $f: \{1, \dots, l\} \rightarrow \{1, \dots, m\}$ a function such that $J_j \subseteq I_{f(j)}$ for all j . Then the embedding $\text{mod } \Lambda_n/(e_{J^c}) \hookrightarrow \text{mod } \Lambda_n/(e_{I^c})$ is homological if and only if

- (i) $1 \in J$ and $J_j = I_{f(j)}$ for all $2 \leq j \leq l$, or
- (ii) $1 \notin J$ and $J_j = I_{f(j)}$ for all $1 \leq j \leq l$.

The key ingredient in the proof of Theorem 3.7.9 is the characterisation of block decompositions of $\Lambda_n/(e)$, for $e \in \Lambda_n$ an idempotent. Moreover, we provide a recursive algorithm to construct the homological Hasse quiver $\mathbb{H}(\Lambda_n)$, confirming that Λ_n admits only one heredity chain. In particular, we find that the cover relations of $\mathcal{H}(\Lambda_n)$ are in bijection with $\mathbb{P}(n)$ the set of parts of n , i.e. the set of all summands in all possible sums of positive integers equal to n .

Theorem 3.10.3. There exists a bijection $\text{Cov } \mathcal{H}(\Lambda_n) \rightarrow \mathbb{P}(n)$.

For a class of algebras Morita equivalent to blocks of Schur algebras of finite representation type, we study their homological poset and homological Hasse quiver, using techniques developed for the case of the Auslander algebras Λ_n , giving some interesting counting formulas.

It is well known that if A is a hereditary algebra, then it is quasi-hereditary for any total order [DR89b], and this assertion can be extended to all adapted orders. In this case, the homological Hasse quiver of A is the n -hypercube, where n is the number of simple A -modules up to isomorphism, and the set of heredity chains of A is in bijective correspondence with the set of all the paths of length n of $\mathbb{H}(A)$, which has cardinality $n!$, but the sets of heredity ideals appearing as factors of two heredity chains may not be the same, i.e. different orderings may produce different sets of standard modules, in other words, the number of quasi-hereditary structures is not $n!$. This led us to a different strategy.

For a second approach, we study quasi-hereditary structures *per se*. First we give the foundations to investigate the set of classes of adapted orders giving quasi-hereditary algebras, denoted by $\text{qh.str}(A)$. A natural way to endow $\text{qh.str}(A)$ with a partial order is to consider the usual partial order for tilting modules, considering that to each quasi-hereditary structure corresponds a characteristic tilting module, defining the *poset of quasi-hereditary structures*. This is a new way to classify quasi-hereditary structures of a given algebra. This is joint work with Yuta Kimura and Baptiste Rognerud [FKR20].

In particular, we classify all quasi-hereditary structures of a path algebra A_n of a Dynkin diagram of type \mathbb{A}_n with linear orientation in terms of binary trees. The nice combinatorial properties of this classification are remarkable and involve Catalan numbers. More precisely, we have the following result.

Theorem 5.1.9. There is a commutative diagram of bijections

$$\begin{array}{ccc} & \left\{ \begin{array}{c} \text{Binary trees} \\ \text{of size } n \end{array} \right\} & \\ & \swarrow \quad \searrow & \\ \text{qh.str}(A_n) & \xrightarrow{\quad} & \left\{ \begin{array}{c} \text{Tilting modules} \\ \text{over } A_n \end{array} \right\} / \cong \end{array}$$

Therefore, the poset of quasi-hereditary structures of A_n is isomorphic to a Tamari lattice of order n . We extend our investigations to the case of a path algebra of type \mathbb{A}_n for any orientation via iterated deconcatenations of quivers at sink or source vertices (cf. Definition 5.2.1).

Theorem 5.2.7. Let $Q^1 \sqcup Q^2 \sqcup \dots \sqcup Q^\ell$ be an iterated deconcatenation of a quiver Q . Let A be a factor algebra of $\mathbb{K}Q$ modulo some admissible ideal and $A^i := A / \langle e_u \mid u \in Q_0^t \setminus \{v\}, t = 1, \dots, \ell, t \neq i \rangle$. Then we have an isomorphism of posets

$$\text{qh.str}(A) \longrightarrow \prod_{i=1}^{\ell} \text{qh.str}(A^i),$$

which is given by $[\triangleleft] \mapsto ([\triangleleft|_{Q_0^i}])_{i=1}^{\ell}$.

Outline

This thesis comprises five chapters organised as follows.

Chapter 1 fixes general conventions and provides the reader the necessary background on representation theory over Artin algebras and finite dimensional algebras. We introduce path algebras and bound quiver algebras and provide central results, fixing important notation for quiver representations. Basic definitions of poset theory are recalled briefly as well. The chapter ends with a brief discussion on homological embeddings and homological epimorphism of rings, proving some important properties used later in the thesis.

In Chapter 2 we start introducing the central concept of highest weight category and give an equivalent definition via filtrations of Serre subcategories. Afterwards we define quasi-hereditary algebras in two equivalent ways, first via heredity chains, then using adapted posets and standard modules, and we present a detailed explanation on the interplay between both definitions and highest weight categories. Then we introduce the homological poset of an algebra and formalise the notion of quasi-hereditary structure, giving some examples.

The Chapter 3 is devoted to studying the homological poset and homological Hasse quiver of the Auslander algebra of the truncated polynomial ring, denoted by Λ . In the first sections we find a good description of a basis of Λ , used later to find block decompositions of the factor algebras $\Lambda/(e)$ for e an idempotent of Λ . The Chinese remainder theorem and a property involving idempotent ideals Lemma 3.4.4 is crucial to find such decompositions. Then we find a characterisation of all homological embeddings over Λ , in particular we find a classification of the cover relations of the homological poset of Λ that yields a recursive method to depict its homological Hasse quiver. In particular we exhibit a bijection between the set of cover relations of $\mathcal{H}(\Lambda)$ and the set of all parts of a positive integer number. At the end of the chapter we characterise the quotients $\Lambda/(e)$ that are quasi-hereditary, and find an induced subposet of $\mathcal{H}(\Lambda)$ with elements corresponding to quasi-hereditary algebras.

In Chapter 4 we recall the notion of a block B of Schur algebras of finite representation type, and define a factor algebra \tilde{B} of B , that is relevant when finding block decompositions of $B/(e)$, for $e \in B$ an idempotent. Then we use several results of Chapter 3 in order to find some homological properties of the algebra \tilde{B} used to classify homological embeddings between Serre subcategories of $\text{mod } B$. We also provide a complete description of the cover relations of the homological poset of B and give a recursive method to depict its homological Hasse quiver.

Chapter 5 begins with the study of quasi-hereditary structures of A_n a path algebra of an equioriented quiver of type \mathbb{A}_n . We construct adapted posets to A_n from binary search trees and show a bijection between binary trees of size n and quasi-hereditary structures of A_n . This bijection is compatible with the one-to-one correspondence between binary trees and tilting modules over A_n (cf. Theorem 5.1.9). This yields a characterisation of characteristic tilting modules over A_n . Next we give a bijection between quasi-hereditary structures of quiver algebras and quasi-hereditary structures of the corresponding quiver algebras given by a deconcatenation of the original quiver. As application we find a classification of quasi-hereditary structures for path algebras of type \mathbb{A}_n in general, via its minimal adapted posets.

Chapter 1

Preliminaries

In this first chapter we recall fundamental definitions and standard results that are used throughout the thesis, it should be noted that this introduction is merely a reminder and is not intended to be exhaustive. We also fix important notation that is constant throughout the text. We assume general knowledge on rings, modules and categories, and only rings with identity are considered. Concerning results of this chapter without proof nor citation, we refer the reader to [ASS06; ARS95; Sch14].

1.1 General conventions

The set of natural numbers is $\mathbb{N} = \{0, 1, \dots\}$, and the set of positive natural numbers is $\mathbb{N}_+ = \{1, 2, \dots\}$. For $n \in \mathbb{N}_+$, we define $[n] := \{1, 2, \dots, n\}$ and $[0] := \emptyset$ the empty set. The set of permutations of $[n]$ is denoted by Σ_n . The cardinality of a set X is denoted by $\text{card } X$.

In this thesis we work mainly with finitely generated left modules. More precisely, let R be a ring. Then $\text{Mod } R$ denotes the category of all left R -modules, and $\text{mod } R$ is the full subcategory of finitely generated left modules over R . For $r \in R$ and $S \subseteq R$, we use the following notations for the two sided ideals generated by r or S : $RrR := (r) = \langle r \rangle_R$ and $RSR := \langle S \rangle_R = \langle S \rangle$.

A path algebra is a bound quiver algebra with zero admissible ideal. For two arrows $\alpha: a \rightarrow b$ and $\beta: b \rightarrow c$ in some quiver, we denote the concatenation of α with β by $\beta\alpha$.

When referring to a poset (S, \triangleleft) , we write just \triangleleft if the underlying set S is known.

1.2 Representation theory of algebras

We start by defining the notion of R -algebra in general, despite the fact that we mainly work with finite dimensional algebras over a field, and more generally over Artin algebras.

Definition 1.2.1. Let A be ring, and R a commutative ring. An R -algebra, or an algebra over R , is a triple (A, R, φ) with $\varphi: R \rightarrow A$ a unit preserving ring homomorphism such that $\varphi(R)$ is contained in the centre of A . In this case we write $ra := \varphi(r)a$ for all $r \in R$ and $a \in A$.

Note that the action of R over A is bilinear, associative and respects the multiplication of A , i.e. $r(ab) = (ra)b = a(rb)$ for all $r \in R$ and $a, b \in A$.

We recall the notion of Artin algebra which is a generalisation of finite dimensional algebras. In this thesis, usually Artin algebras are considered to provide a more general setting for the theory, since the representation theory over Artin algebras is very well studied, see for example [ARS95].

Definition 1.2.2 ([ARS95, II.1]). Let R be a commutative artinian ring. An R -algebra Λ is called *Artin algebra* if Λ is finitely generated as module over R .

From now on, \mathbb{K} denotes an arbitrary field, unless otherwise stated. Thus if A is a \mathbb{K} -algebra, then it has a vector space structure over \mathbb{K} , and A is said to be *finite dimensional* or *infinite dimensional* according to whether A is finite or infinite dimensional as \mathbb{K} -vector space. In particular, every finite dimensional algebra is an Artin algebra. In this thesis we usually denote an Artin algebra with the greek letter Λ and we reserve the letter A for a finite dimensional algebra over a field \mathbb{K} .

The *Jacobson radical* of a ring R is the two-sided ideal of R given by intersection of all its maximal left ideals and is denoted $\text{rad } R$. A two-sided ideal I of R is called *nilpotent* if there exists $m \in \mathbb{N}_+$ such that $I^m = 0$.

We denote by $\text{Mod } \Lambda$ the category of left modules over Λ , and by $\text{mod } \Lambda$ the category of finitely generated left Λ -modules. We denote the opposite algebra of Λ by Λ^{op} , and we identify the category of right Λ -modules with $\text{Mod } \Lambda^{op}$, similarly for $\text{mod } \Lambda^{op}$. If M is a Λ -module, we write ${}_{\Lambda}M$ to specify that it is a left Λ -module. The *regular Λ -module* is Λ with action given by multiplication by the left, denoted ${}_{\Lambda}\Lambda$. In this thesis we work mainly with finitely generated left modules.

Two rings R, S are defined to be *Morita equivalent* provided the categories $\text{Mod } R$ and $\text{Mod } S$ are equivalent. For further details on definitions about module categories we refer the reader to [AF92] and [ARS95].

Let $M \in \text{Mod } \Lambda$. Then M is called *simple* if its unique submodules are 0 and M . M is called *semisimple* if it is sum of simple modules. The algebra Λ is called *semisimple* if Λ is semisimple as left Λ -module. The *radical* $\text{rad } M$ of M is the intersection of all the maximal proper submodules of M , i.e. $\text{rad } M$ is the smallest submodule of M such that $M/\text{rad } M$ is semisimple. Call the quotient $M/\text{rad } M$ the *top* of M , and denote it by $\text{top } M$. The *socle* of M , denoted $\text{soc } M$, is the submodule of M generated by all the simple submodules of M , equivalently $\text{soc } M$ is the largest semisimple submodule of M .

A *composition series* of M is a finite chain of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_t = 0$ such that the factor module M_i/M_{i-1} is simple for all $1 \leq i \leq t$. In this case t is the *length* of the series.

Theorem 1.2.3 (Jordan-Hölder theorem). *Let Λ be an Artin algebra, and $M \in \text{mod } \Lambda$. If $M = M_0 \supset M_1 \supset \cdots \supset M_t = 0$ and $M = N_0 \supset N_1 \supset \cdots \supset N_s = 0$ are two composition series of M , then $s = t$ and there is a bijection between the composition factors of these series such that the corresponding factors are isomorphic.*

Proof. See [ARS95] or [DK94] for the case of finite dimensional. \square

The *length* of M is by definition the length of a composition series of M and is denoted by $\ell(M)$. The last theorem shows that $\ell(M)$ is well defined and that the factors of a composition series of M are unique up to isomorphism and are called the *composition factors* of M . Moreover, if S is a simple Λ -module, the *Jordan-Hölder multiplicity* of S in M , denoted by $[M : S]$, is the number of simple factors of M isomorphic to S .

A module $M \in \text{Mod } \Lambda$ is *indecomposable* provided it is non-zero and cannot be written as a direct sum of non-zero Λ -submodules. If a finitely generated module is not indecomposable, then it can be decomposed into indecomposable direct summands in an essentially unique way as the following theorem asserts.

Theorem 1.2.4 (Krull-Remak-Schmidt theorem). *Let Λ be an Artin algebra. Then the following holds.*

- (a) *Let $M \in \text{mod } \Lambda$. Then M is indecomposable if and only if $\text{End}_\Lambda M$ is local.*
- (b) *For all $M \in \text{mod } \Lambda$, there exists a decomposition $M \cong \bigoplus_{i=1}^s M_i$ such that M_i is an indecomposable Λ -module for all i .*
- (c) *If $M \cong \bigoplus_{i \in I} M_i \cong \bigoplus_{j \in J} N_j$ are two finite decompositions of M into indecomposable modules in $\text{mod } \Lambda$, then there exists a bijection $\sigma: I \rightarrow J$ such that $M_i \cong N_{\sigma(i)}$ for all $i \in I$.*

An algebra Λ is of *finite representation type* if there is a finite number of indecomposable modules up to isomorphism in $\text{mod } \Lambda$.

Idempotent elements play a crucial role in this dissertation. In what follows we recall definitions and some important properties about them.

Let R be a ring. An element $e \in R$ is *idempotent* if $e^2 = e$. A set of idempotents $\{e_1, \dots, e_n\} \subseteq R$ is said to be *complete* provided $e_1 + \dots + e_n = 1_R$. Two idempotents $e, f \in R$ are *orthogonal* if $ef = fe = 0$. Moreover, e is *primitive* if it cannot be expressed as a sum $e = f + g$ with f and g non-zero orthogonal idempotents of R . An idempotent $e \in R$ is called *central* if $er = re$ for all $r \in R$.

A *complete set of primitive orthogonal idempotents* of R is a complete set $E = \{e_1, \dots, e_n\} \subseteq R$ of primitive idempotents which are pairwise orthogonal. In this case, we also say that there exists a decomposition of the identity $1_R = \sum_{i=1}^n e_i$ with primitive orthogonal idempotents.

Proposition 1.2.5. *Let Λ be an Artin algebra. Let $1_\Lambda = e_1 + \dots + e_n = f_1 + \dots + f_m$ be two decompositions of the identity with primitive orthogonal idempotents, then $n = m$ and there exists an invertible element $a \in \Lambda$ and a unique permutation $\sigma \in \Sigma_n$ such that $f_{\sigma(i)} = ae_i a^{-1}$ for all i .*

Proof. For the existence part see [ARS95], and for the second part see [HGK04, Theorem 11.1.7]. \square

Let M and P be modules in $\text{mod } \Lambda$, with P projective. We say that P is a *projective cover* of M if there exists an epimorphism $f: P \rightarrow M$ which induces an isomorphism $P/\text{rad } P \rightarrow M/\text{rad } M$, in this case we write $P = P(M)$. Dually, an injective module $Q \in \text{mod } \Lambda$ is called an *injective envelope* of M if there exists a monomorphism $g: M \rightarrow Q$ such that for any submodule X of Q , $\text{Im } g \cap X = 0$ implies $X = 0$, it is denoted by $Q(M)$. It turns out that both $P(M)$ and $Q(M)$ exist for any $M \in \text{mod } \Lambda$ and are unique up to a unique isomorphism.

Every Artin R -algebra Λ admits a duality $D: \text{mod } \Lambda \rightarrow \text{mod } \Lambda^{op}$ given by $D(-) = \text{Hom}_\Lambda(-, Q(\text{top } R))$, where $Q(\text{top } R)$ is the injective envelope of the top of R . The *standard \mathbb{K} -duality* of a \mathbb{K} -algebra A is the vector space duality $D: \text{mod } A \rightarrow \text{mod } A^{op}$ given by $D(-) = \text{Hom}_{\mathbb{K}}(-, \mathbb{K})$. Then, the simple, projective and injective indecomposable modules are characterised as follows.

Theorem 1.2.6. *Let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of an Artin algebra Λ . Then the following conditions hold.*

- (a) *There exists a decomposition $\Lambda \cong \Lambda e_1 \oplus \dots \oplus \Lambda e_n$ into indecomposable left ideals.*
- (b) *Every simple left Λ -module is isomorphic to a module $S(i) := \text{top } \Lambda e_i$, for some i .*
- (c) *Every indecomposable projective left Λ -module is isomorphic to a module $P(i) := \Lambda e_i$, for some i . Moreover, $P(S(i)) \cong P(i)$, for all i .*
- (d) *Every indecomposable injective left Λ -module is isomorphic to a module $I(i) := D(e_i \Lambda)$, for some i . Moreover, $Q(S(i)) \cong I(i)$, for all i .*

Let $M \in \text{mod } \Lambda$. A *projective resolution* of M is an exact sequence (possibly infinite) $\eta: \dots \rightarrow P_2 \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0 \xrightarrow{f} M \rightarrow 0$ such that P_i is projective for all $i \geq 0$. We say that η is *minimal* if f and $f'_i: P_i \rightarrow \text{Ker } f_{i-1}$ are projective covers for all i . Dually, an *injective coresolution* of M is an exact sequence $\zeta: 0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \dots$ with I_i injective for all $i \geq 0$. Similarly, call ζ *minimal* if the corresponding morphisms are injective hulls. Write $\text{pd } M \leq n$ (pd stands for *projective dimension*) if there exists a projective resolution of length n $0 \rightarrow P_n \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$. If no such a finite resolution exists, we set $\text{pd } M = \infty$, otherwise $\text{pd } M = n$ if n is the length of the shortest projective resolution of M . We define dually the *injective dimension* of M . The *dominant dimension* of M , is the maximum integer n (or ∞) such that if $0 \rightarrow M \rightarrow I_0 \rightarrow \dots \rightarrow I_n \rightarrow \dots$ is a minimal projective resolution of M , then I_j is projective for all $j < n$ (or ∞).

The *global dimension* of Λ , denoted $\text{gl. dim } \Lambda$, is the supremum of the projective dimension of all Λ -modules, or equivalently the supremum of all injective dimensions of Λ -modules.

A ring R is *connected* provided $\{0, 1\}$ are the unique central idempotents of R . This is equivalent to the fact that R cannot be decomposed as a product $S \times T$ of non-trivial rings S and T . Moreover, every finite dimensional algebra A is isomorphic to a direct product $A_1 \times \dots \times A_m$ of finite dimensional connected algebras A_i in an essentially unique way. In this case, the algebras A_i are called the *blocks* of the algebra A .

1.3 Quiver representations

Most of the examples in this thesis are path algebras modulo an admissible ideal. In this section we define all necessary notions to understand basic properties of quiver algebras and quiver representations.

A *quiver* $Q = (Q_0, Q_1, s, t)$ is a finite directed graph, with a set Q_0 of *vertices* and a set Q_1 of *arrows*, equipped with two functions $s, t: Q_1 \rightarrow Q_0$ which associate to each arrow $\alpha \in Q_1$ its source, or starting vertex, $s(\alpha)$ and its target, or terminating vertex, $t(\alpha)$, in this case we write $\alpha: s(\alpha) \rightarrow t(\alpha)$. We say that Q is *finite* provided $\text{card } Q_0 \cup Q_1$ is finite, and Q is *connected* if its underlying graph is connected. The vertices in Q_0 are known as the *paths of length 0* or *trivial paths*, and a *path of length $m \geq 1$* is a sequence of arrows $p = \alpha_m \alpha_{m-1} \dots \alpha_1$ such that $t(\alpha_j) = s(\alpha_{j+1})$ for $j = 1, \dots, m-1$. Set $s(p) := s(\alpha_1)$ and $t(p) := t(\alpha_m)$. Thus the vertices are characterised as the paths ε

of length 0 such that $s(\varepsilon) = t(\varepsilon) = i$, and in this case set $\varepsilon := \varepsilon_i$. For $m \geq 0$, denote by Q_m the paths in Q of length m . If $p \in Q_m$ and $q \in Q_n$ with $t(p) = s(q)$, we denote the *concatenation* of p and q by qp . Note that $qp \in Q_{m+n}$. A path $p \in Q_m$ is called an *oriented cycle* if $m \geq 1$ and $s(p) = t(p)$. A *loop* is a cycle of length 1. A quiver is called *acyclic* if it has no oriented cycles.

A *subquiver* of Q is a quiver $Q' = (Q'_0, Q'_1, s', t')$ such that $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$, $s' = s|_{Q'_1}$ and $t' = t|_{Q'_1}$. Moreover, Q' is called *full* if $Q'_1 = \{\alpha \in Q_1 \mid s(\alpha), t(\alpha) \in Q'_0\}$. The *underlying graph* of Q is the graph obtained from Q by forgetting the directions of the arrows.

The *path algebra* $\mathbb{K}Q$ of a quiver Q is the \mathbb{K} -algebra whose underlying vector space has basis the set of all paths $\bigcup_{m \in \mathbb{N}} Q_m$ with multiplication given by concatenation, that is

$$q \cdot p := \begin{cases} qp & \text{if } t(p) = s(q), \\ 0 & \text{otherwise.} \end{cases}$$

and extended bilinearly to arbitrary elements of $\mathbb{K}Q$.

We resume some general properties of path algebras in the following proposition.

Proposition 1.3.1. *Let Q be a quiver. Then the following conditions hold.*

- (a) $\mathbb{K}Q$ has identity element if and only if Q is finite. In this case, $1_{\mathbb{K}Q} = \sum_{i \in Q_0} \varepsilon_i$ is a decomposition of the identity into primitive orthogonal idempotents.
- (b) $\dim_{\mathbb{K}} \mathbb{K}Q$ is finite if and only if Q is finite and acyclic.
- (c) If Q is finite, then $\mathbb{K}Q$ is connected if and only if Q is connected.
- (d) If Q is finite and acyclic, then $\text{rad } \mathbb{K}Q$ is the two sided ideal generated by all arrows in Q .

From now on, in this dissertation we consider only finite quivers. The last proposition says that if a quiver Q is acyclic, then $\mathbb{K}Q$ is infinite dimensional, thus we want to consider quotients of path algebras by certain ideals that turn out to be finite dimensional.

For the next definitions, we consider a quiver Q and we denote by R_Q the *arrow ideal* of Q , that is, $R_Q = \text{rad } \mathbb{K}Q = \bigoplus_{i \in \mathbb{N}_+} \mathbb{K}Q_i$. The i -th power of R_Q can be decomposed as $R_Q^i = \bigoplus_{j \geq i} \mathbb{K}Q_j$.

An ideal $I \subseteq \mathbb{K}Q$ is said to be *admissible* if there exists $m \in \mathbb{N}$ such that $R_Q^m \subseteq I \subseteq R_Q^2$. Moreover, if $I \subseteq \mathbb{K}Q$ is an admissible ideal, we say that (Q, I) is a *bound quiver* and the quotient algebra $\mathbb{K}Q/I$ is called a *bound quiver algebra*, or simply a *quiver algebra*.

Let A be a finite dimensional algebra with $\{e_1, \dots, e_n\}$ a complete set of primitive orthogonal idempotents. We say that A is *basic* provided $Ae_i \neq Ae_j$, for all $i \neq j$.

A *relation* in a quiver Q (with coefficients in \mathbb{K}) is a linear combination of paths of length at least two $\rho = \sum_{i=1}^m a_i p_i$ with same source and target, that is, $s(p_i) = s(p_j)$ and $t(p_i) = t(p_j)$ for all $i \neq j$. If $m = 1$ we say that ρ is a *monomial relation*, and it is a *commutativity relation* if it is of the form $p_1 - p_2$. Then, we have the following properties of quiver algebras.

Proposition 1.3.2. *Let (Q, I) be a bound quiver, and set $A := \mathbb{K}Q/I$. Then the following conditions hold.*

- (a) A is finite dimensional basic algebra.
- (b) The set $\{e_i := \varepsilon_i + I \mid i \in Q_0\}$ is a complete set of primitive orthogonal idempotents of A .
- (c) A is connected if and only if Q is connected.
- (d) $\text{rad } A = R_Q/I$.
- (e) There exists a finite set of relations $R = \{\rho_1, \dots, \rho_m\}$ such that $I = \langle R \rangle$. In this case we say that the relations ρ_i generate I .

The main connection between finite dimensional algebras and quiver algebras is given by the following result by Gabriel [Gab72].

Theorem 1.3.3. *Let \mathbb{K} be an algebraically closed field, and A a basic and connected finite dimensional algebra. Then there exists a quiver Q_A and admissible ideal I of $\mathbb{K}Q_A$ such that $A \cong \mathbb{K}Q_A/I$.*

The *opposite quiver* Q^{op} of Q is defined as follows. $Q_0^{op} := Q_0$ and $\alpha: i \rightarrow j$ is in Q_1 if and only if $\alpha^{op}: j \rightarrow i$ is in Q^{op} . If (Q, I) is a bound quiver, then there exists an admissible ideal $I^{op} \subseteq \mathbb{K}Q^{op}$ such that $\mathbb{K}Q^{op}/I^{op} \cong (\mathbb{K}Q/I)^{op}$.

A *representation* $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of a quiver Q consists of \mathbb{K} -vector spaces M_i for each vertex $i \in Q_0$, together with linear maps $\varphi_\alpha: M_{s(\alpha)} \rightarrow M_{t(\alpha)}$ for each arrow $\alpha \in Q_1$. A representation M is called *finite dimensional* provided each vector space M_i is finite dimensional. In this case, the *dimension vector* $\underline{\dim} M$ of M is the vector $(\dim_{\mathbb{K}} M_i)_{i \in Q_0}$. Let $M = (M_i, \varphi_\alpha)$ and $N = (N_i, \psi_\alpha)$ be two representations of a quiver Q . A *morphism of representations* $f: M \rightarrow N$ is a collection $f = (f_i)_{i \in Q_0}$ of linear maps $f_i: M_i \rightarrow N_i$ such that for each map $\alpha: i \rightarrow j$ in Q_1 the diagram

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_\alpha} & M_j \\ \downarrow f_i & & \downarrow f_j \\ N_i & \xrightarrow{\psi_\alpha} & N_j \end{array}$$

commutes, i.e. $\psi_\alpha f_i = f_j \varphi_\alpha$. A morphism $f = (f_i): M \rightarrow N$ is a monomorphism (epimorphism, isomorphism, resp.) if each f_i is a monomorphism (epimorphism, isomorphism, resp.). In this way we have the category of representations of Q denoted by $\text{rep } Q$.

Let $M = (M_i, \varphi_\alpha)$ be a representation of Q , and $p = \alpha_m \cdots \alpha_1$ a non-trivial path. We define the linear map $\varphi_p = \varphi_{\alpha_m} \cdots \varphi_{\alpha_1}$. Moreover, if $I \subseteq \mathbb{K}Q$ is an admissible ideal generated by the relations $\{\rho_1, \dots, \rho_m\}$, we say that M is a *representation of the bound quiver* (Q, I) , if $\varphi_\rho = 0$, for each $\rho \in R$, where $\varphi_\rho = \sum_j a_j \varphi_{p_j}$ if $\rho = \sum_j a_j p_j$. The morphisms of representations of bound quivers are defined in the same way as in $\text{rep } Q$. We denote by $\text{rep}(Q, I)$ the category of representations of (Q, I) .

The concepts of module and representation are essentially the same, i.e. if $A = \mathbb{K}Q/I$ is a quiver algebra, then there exists an equivalence of categories $\text{mod } A \cong \text{rep}(Q, I)$, if Q is acyclic, then $\text{mod } \mathbb{K}Q \cong \text{rep } Q$. Thus, in this thesis we always identify both notions.

For example, for $i \in Q_0$ the *simple* module $S(i)$ is given as representation as follows: $S(i)_j = \mathbb{K}$ and $S(i)_i = 0$ for all $j \neq i$, and $\varphi_\alpha = 0$ for all $\alpha \in Q_1$. The representation of the *projective* module $P(i)$ is given by $P(i) = (P(i)_j, \varphi_\alpha)$, where $P(i)_j$ is the vector space with basis the set of all residue classes $p+I$ of paths p from i to j , and if $\alpha : j \rightarrow l$ is in Q_1 , then φ_α is the linear map defined on the basis elements by composing paths from i to j with the arrow α . The injective representations are defined dually. Usually, if no confusion can arise, we write p for a residue class $p+I$ in $\mathbb{K}Q/I$, and set $e_i := \varepsilon_i + I$ for each $i \in Q_0$.

For example, consider the quiver $Q = \begin{matrix} 1 & \xrightleftharpoons[\beta]{\alpha} & 2 \end{matrix}$ with monomial relation $\alpha\beta$, i.e. $I = (\alpha\beta)$. Set $A = \mathbb{K}Q/I$. Then A has dimension 5 with basis $\{e_1, e_2, \alpha, \beta, \beta\alpha\}$. The simple and projective A -modules are given as follows.

$$\begin{aligned} S(1) &= \mathbb{K} \xrightleftharpoons[0]{0} 0 & P(1) &= \mathbb{K}^2 \xrightleftharpoons[\begin{pmatrix} 0 \\ 1 \end{pmatrix}]{\begin{pmatrix} 1 & 0 \end{pmatrix}} \mathbb{K} \\ S(2) &= 0 \xrightleftharpoons[0]{0} \mathbb{K} & P(2) &= \mathbb{K} \xrightleftharpoons[1]{0} \mathbb{K} \end{aligned}$$

In many cases it is usually more efficient to write down a diagram to describe a representation of a quiver. For a more extended explanation we refer to [Bar15, Example 4.21]. For instance, in the above example, the module $P(1)$ can be written as $\begin{matrix} 1 \\ \searrow \\ 2 \end{matrix}$ meaning that $P(1)_1$ is 2-dimensional, $P(1)_2$ is 1-dimensional, i.e. $[P(1) : S(1)] = 2$ and $[P(1) : S(2)] = 1$, $\varphi_\alpha : 1 \mapsto 2$ and $\varphi_\beta : 2 \mapsto 1$ represent the non-zero defining linear functions of $P(1)$ acting on the basis elements. By convention the maps go from upper to lower rows, thus we may avoid writing the edges. Using this new notation we have: $S(1) = 1$, $S(2) = 2$, $P(1) = \begin{matrix} 1 \\ \searrow \\ 2 \end{matrix}$ and $P(2) = \begin{matrix} 1 \\ \swarrow \\ 2 \end{matrix}$.

1.4 Poset theory

A central concept developed in this thesis is the homological poset of an algebra. In what follows we give standard definitions related to posets and Hasse quivers.

Let S be a set. A *binary relation* S is a subset $R \subseteq S \times S$. For simplicity we write xRy for a pair $(x, y) \in R$. A *partially ordered set*, or *poset* or *order* or *ordering* for short, is a pair $P = (S, \triangleleft)$, where \triangleleft is a binary relation on S satisfying three properties: $s \triangleleft s$ for all $s \in S$ (*reflexivity*), if $s \triangleleft t$ and $t \triangleleft s$, then $s = t$ (*antisymmetry*), and if $s \triangleleft t$ and $t \triangleleft u$, then $s \triangleleft u$ (*transitivity*). Two elements $s, t \in S$ are *comparable* if $s \triangleleft t$ or $t \triangleleft s$, otherwise s and t are *incomparable*. We say that t *covers* s provided $s \triangleleft t$ and there is no element $x \in S \setminus \{s, t\}$ such that $s \triangleleft x \triangleleft t$. In this case we call (s, t) a *cover relation* of P . Denote $\text{Cov } P$ the set of all cover relations of P . Sometimes $\text{Cov } P$ is called the *Hasse diagram* of P . An element $m \in P$ is called *maximal* (*minimal*) in P if for all $x \in P$ such that $m \triangleleft x$ ($x \triangleleft m$), then $m = x$. We say that a poset $P' = (S', \triangleleft')$ is a *weak subposet* of P if $S' \subseteq S$, and if $x \triangleleft' y$ then $x \triangleleft y$. Moreover, if $S = S'$ then P is called a *refinement* or *extension* of P' , and we say that P *extends* P' . We say that P' is an *induced subposet* of P if $S' \subseteq S$ and for all $x, y \in S'$, $x \triangleleft' y$ if and only if $x \triangleleft y$.

Let S be a set. The *power set* of S is the set of all subsets of S , and is denoted by 2^S . We equip 2^S with a poset structure given by inclusion, denoted $\mathcal{P}(S) = (2^S, \subseteq)$. In the special case of $[n] = \{1, \dots, n\}$ for $n \in \mathbb{N}$, we write $\mathcal{P}(n) = (2^{[n]}, \subseteq)$. This poset plays a central role in Chapters 3 and 4.

A *totally ordered set*, or *linearly ordered set*, is a poset (S, \triangleleft) in which any two elements are comparable. In this case we say that \triangleleft is a *total order* on S . Note that there are 2^n total orders on the set $\{1, \dots, n\}$.

The *Hasse quiver* of a poset $P = (S, \triangleleft)$ is the quiver with vertices the elements of S , and arrows given by cover relations, i.e. for all $s, t \in S$ there is an arrow $s \rightarrow t$ if and only if (s, t) is a cover relation. Usually we depict the minimal elements at the bottom of the quiver and maximal elements at the top.

1.5 Homological embeddings and homological ring epimorphisms

In this section we introduce the language of homological ring epimorphisms that in the setting of Chapter 2 for instance, coincides with the more general notion of homological embedding. Also we prove some properties of homological ring epimorphisms that are used in Chapter 3 to characterise certain homological embeddings.

Let \mathcal{A}, \mathcal{B} be abelian categories, and $F : \mathcal{A} \rightarrow \mathcal{B}$ an exact functor. For $A, B \in \mathcal{A}$ and $n \geq 0$, denote by

$$F_{A,B}^n : \text{Ext}_{\mathcal{A}}^n(A, B) \rightarrow \text{Ext}_{\mathcal{B}}^n(F(A), F(B))$$

the morphism between n -extension groups in the sense of Yoneda [Wei94; Mac95] induced by F , i.e. if $\xi \in \text{Ext}_{\mathcal{A}}^n(A, B)$ then $F_{A,B}^n(\xi) := F(\xi)$. For $n = 0$, we have

$$F_{A,B}^0 : \text{Hom}_{\mathcal{A}}(A, B) \rightarrow \text{Hom}_{\mathcal{B}}(F(A), F(B)).$$

We say that F is *faithful* if $F_{A,B}^0$ is injective for all $A, B \in \mathcal{A}$. F is *full* if $F_{A,B}^0$ is surjective for all $A, B \in \mathcal{A}$. If F is full and faithful, we say that F is *fully faithful*, or that F is an *embedding*. Occasionally we will omit the subindices, i.e. we write $F^n = F_{A,B}^n$. In the particular case when \mathcal{A} is a subcategory of \mathcal{B} , we denote $\iota : \mathcal{A} \rightarrow \mathcal{B}$ the inclusion functor. A subcategory $\mathcal{A} \subseteq \mathcal{B}$ is called *full* provided $\iota_{A,B}^0$ is invertible for all $A, B \in \mathcal{A}$.

Definition 1.5.1 ([Psa14]). An exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between abelian categories is called a *homological embedding* provided $F_{A,B}^n$ is an isomorphism for all $n \geq 0$ and all $A, B \in \mathcal{A}$.

The following result gives a sufficient condition to have a homological embedding when we consider certain subcategories of $\text{mod } \Lambda$.

Lemma 1.5.2 ([DR89b, Statement 3]). *Let Λ be an Artin algebra, and $e \in \Lambda$ an idempotent such that the ideal (e) is projective as Λ -module. Then $\iota : \text{mod } \Lambda / (e) \rightarrow \text{mod } \Lambda$, given by ${}_{\Lambda/(e)}M \mapsto {}_{\Lambda}M$, is a homological embedding.*

The functors given by a correspondence as in the previous lemma have a standard generic name, see next definition.

Definition 1.5.3. Let R, S be two rings, and $f: R \rightarrow S$ be a ring homomorphism. The functor $\text{Mod } S \rightarrow \text{Mod } R$ induced by f and defined by ${}_S M \mapsto {}_R M$ and the identity on morphisms, is said to be given by *restriction of scalars*, we denote it by f_* .

Functors given by restriction of scalars have the following well known properties.

Proposition 1.5.4. *Let $f: R \rightarrow S$ be a ring homomorphism. Then the following conditions hold.*

- (a) *The functor f_* is faithful.*
- (b) *If f is surjective, then f_* is an embedding, and it restricts to finitely generated modules, i.e. $f_*: \text{mod } S \rightarrow \text{mod } R$.*
- (c) *If $g: S \rightarrow T$ is a ring homomorphism, then $(gf)_* = f_* \circ g_*$.*

The following theorem is crucial to describe homological embeddings.

Theorem 1.5.5 ([GL91, Thm. 4.4]). *Let $f: R \rightarrow S$ be a ring homomorphism. Then the following conditions are equivalent.*

- (a) *f is an epimorphism of rings and $\text{Tor}_i^R(S, S) = 0$ for all $i \geq 1$.*
- (b) *The natural map $f_*^n: \text{Ext}_S^n({}_S X, {}_S Y) \rightarrow \text{Ext}_R^n({}_R X, {}_R Y)$ is an isomorphism for all left modules $X, Y \in \text{Mod } S$ and for all $n \geq 0$.*
- (c) *The natural map $(f^{\text{op}})_*^n: \text{Ext}_{S^{\text{op}}}^n(X_S, Y_S) \rightarrow \text{Ext}_{R^{\text{op}}}^n(X_R, Y_R)$ is an isomorphism for all right modules $X, Y \in \text{Mod } S^{\text{op}}$ and for all $n \geq 0$.*

Following [GL91] we give the next definition.

Definition 1.5.6. A homomorphism of rings $f: R \rightarrow S$ satisfying the equivalent conditions of Theorem 1.5.5 is called a *homological ring epimorphism*.

Corollary 1.5.7. *Let $f: R \rightarrow S$ be a homological ring epimorphism. Then the following conditions hold.*

- (a) *If f is surjective, then $f_*: \text{mod } S \rightarrow \text{mod } T$ is a homological embedding.*
- (b) *$f^{\text{op}}: R^{\text{op}} \rightarrow S^{\text{op}}$ is a homological epimorphism of rings.*

Proof. From Proposition 1.5.4 f_* restricts to finitely generated modules, then (a) is consequence of Theorem 1.5.5. \square

The next results describe some new homological ring epimorphisms from old ones that will permit us to prove important results in Chapter 3.

Lemma 1.5.8. *The following assertions hold.*

- (a) *Let $R \xrightarrow{f} S \xrightarrow{g} T$ be homological ring epimorphisms. Then the composite gf is a homological ring epimorphism.*

(b) Assume that following diagram of ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \alpha \downarrow \wr & & \wr \downarrow \beta \\ R' & \xrightarrow{f'} & S' \end{array}$$

commutes, with α and β isomorphisms. Then f is a homological ring epimorphism if and only if f' is so.

Proof. (a) From Proposition 1.5.4 we have that $(gf)_* = f_* \circ g_*$, thus

$$(gf)_{*TX, TY}^n = f_{*SX, SY}^n \circ g_{*TX, TY}^n$$

for all $X, Y \in \text{Mod } T$, and all $n \geq 0$. Then the conclusion is consequence of Theorem 1.5.5. The assertion (b) follows from (a). \square

Lemma 1.5.9. *Let R, S be two rings. Then the projection $p: R \times S \rightarrow R$, given by $(r, s) \mapsto r$, is a homological ring epimorphism.*

Proof. First note that R is projective as right $R \times S$ -module, since $R \cong R \times 0 = (1_R, 0)R \times S$ as left $R \times S$ -modules, thus $\text{Tor}_n^{R \times S}(R, R) = 0$ for all $n \geq 1$. Then, the result follows from Theorem 1.5.5, since p is surjective. \square

Proposition 1.5.10. *Let $f: R \rightarrow S$ and $g: T \rightarrow U$ be two homological ring epimorphisms. Then $f \times g: R \times T \rightarrow S \times U$, given by $f \times g(r, t) = (f(r), g(t))$, is a homological ring epimorphism.*

Proof. We know that $\text{Mod } S \times U \xrightarrow{\sim} \text{Mod } S \times \text{Mod } U$ via ${}_{S \times U}M \mapsto ((1_S, 0)M, (0, 1_U)M)$, and a quasi-inverse is given by ${}_S M \times {}_U N \mapsto {}_{S \times U}(M \times N)$, thus we have natural isomorphisms

$$\text{Ext}_{S \times U}^i(-, -) \xrightarrow{\sim} \text{Ext}_S^i((1_S, 0)-, (1_S, 0)-) \times \text{Ext}_U^i((0, 1_U)-, (0, 1_U)-)$$

and

$$\text{Ext}_S^i(-, -) \times \text{Ext}_U^i(-, -) \xrightarrow{\sim} \text{Ext}_{S \times U}^i(- \times -, - \times -)$$

for all $i \geq 0$. Now, let X, Y be $S \times U$ -modules and $i \geq 0$. Then,

$$\begin{aligned} \text{Ext}_{S \times U}^i(X, Y) &\cong \text{Ext}_S^i((1_S, 0)X, (1_S, 0)Y) \times \text{Ext}_U^i((0, 1_U)X, (0, 1_U)Y) \\ &\cong \text{Ext}_R^i((1_S, 0)X, (1_S, 0)Y) \times \text{Ext}_T^i((0, 1_U)X, (0, 1_U)Y) \\ &\cong \text{Ext}_{R \times T}^i((1_S, 0)X \times (0, 1_U)X, (1_S, 0)Y \times (0, 1_U)Y) \\ &\cong \text{Ext}_{R \times T}^i(X, Y), \end{aligned}$$

where the last isomorphism holds, since $X \cong (1_S, 0)X \times (0, 1_U)X$ as left $R \times T$ -modules, for any left $S \times U$ -module X . Indeed, $x \mapsto ((1_S, 0)x, (0, 1_U)x)$ is a bijective $R \times T$ -morphism, with inverse given by $((1_S, 0)x, (0, 1_U)x') \mapsto (1_S, 0)x + (0, 1_U)x'$, and the actions are given by $(r, t) \cdot x = ((f \times g)(r, t))x = (f(r), g(t))x$ and $(r, t) \cdot ((1_S, 0)x, (0, 1_U)x') = ((f(r), 0)x, (0, g(t))x')$ \square

Chapter 2

Highest weight categories and quasi-hereditary algebras

The unifying concept of highest weight category was introduced by Cline, Parshall and Scott [CPS88b] as a categorical theoretic generalisation of the BGG category \mathcal{O} of highest weight modules used in the study of the representation theory of semisimple groups or Lie algebras. Such categories arise in many situations, for instance in the theory of quiver algebras or perverse sheaves. The strongest connection with representation theory is given by quasi-hereditary algebras in the sense that every highest weight category with a finite number of isoclasses of simple objects is equivalent to a module category over a quasi-hereditary algebra. Starting from this point of view, it is crucial in this dissertation to define such categories and explain in more detail the connection with quasi-hereditary algebras and its quasi-hereditary structures, via heredity chains and standard modules. More precisely, the definition of a highest weight category depends on the choice of a set of standard modules, occurring as composition factors of a good filtration of the regular module, called a heredity chain, see Definition 2.1.1. Conversely every heredity chain induces a highest weight category structure, i.e. induces a partial order on the set of isoclasses of simples modules, which defines an adequate set of standard modules, see Theorem 2.2.8.

In this chapter we discuss two approaches to study all the possible choices of partial orders inducing quasi-hereditary algebras, or equivalently, highest weight structures. First, the characterisation of highest weight category presented in Theorem 2.1.7 following [Kra17], provides a more conceptual approach via filtrations by Serre subcategories and recollements, useful to find all the possible heredity chains of a given algebra, via the homological poset. For the second strategy we define an equivalence relation on the set of quasi-hereditary structures in order to introduce the more refined notion of poset of quasi-hereditary structures.

The chapter is organised as follows. In the first section we define highest weight categories and recollements, and state Theorem 2.1.7. In Section 2.2 several equivalent definitions of quasi-hereditary algebra are discussed, first by means of heredity chains and then using standard modules where adapted orders are part of the axioms. We open a parenthesis to discuss standardly stratified algebras which are a generalisation of quasi-hereditary algebras. The characteristic tilting module of a quasi-hereditary algebra is defined as well. In Section 2.3 we establish a partial order structure on the

class of Serre subcategories that is used to find heredity chains of a quasi-hereditary algebra, namely the homological poset. In Section 2.4 we introduce the poset of quasi-hereditary structures and state a result about quasi-hereditary algebras admitting only one quasi-hereditary structure, found in [Cou19].

2.1 Highest weight categories

The definition of highest weight category introduced by Cline, Parshall and Scott in [CPS88b, Definition 3.1] considers \mathbb{K} -abelian categories satisfying certain conditions with a non-necessarily finite number of isoclasses of simple objects. In this thesis, we restrict ourselves to the case of module categories with a finite number of simple objects up to isomorphism, particularly we work with module categories over Artin algebras. We start fixing important notation used throughout the thesis.

Let Λ be an Artin algebra. Let (I, \triangleleft) be a finite poset and $\{S(i)\}_{i \in I}$ a complete set of non-isomorphic simple Λ -modules. We denote by $P(i)$ the projective cover of $S(i)$, and by $Q(i)$ the injective envelope of $S(i)$, for $i \in I$. Many times, when referring to a poset (I, \triangleleft) , we write \triangleleft for simplicity, when the underlying set is known. Under this assumptions we define the following central notion.

Definition 2.1.1 ([CPS88b, Definition 3.1]). The pair $(\text{mod } \Lambda, \triangleleft)$ is a *highest weight category* if there exists a collection of finitely generated Λ -modules $\Delta = \{\Delta(i)\}_{i \in I}$ such that the following conditions hold.

- (a) There is a surjective map $\psi_i: \Delta(i) \rightarrow S(i)$, for all $i \in I$.
- (b) If $S(j)$ is a composition factor of $\text{Ker } \psi_i$, then $j \triangleleft i$ and $j \neq i$.
- (c) Every $P(i)$ admits a *good filtration*, i.e. there is a chain of submodules $0 = M_0 \subset M_1 \subset \dots \subset M_t = P(i)$ such that
 - (c1) $P(i)/M_{t-1} \cong \Delta(i)$, and
 - (c2) for $1 \leq s \leq t-1$ there exists $j \in I$ with $M_s/M_{s-1} \cong \Delta(j)$, $i \triangleleft j$ and $i \neq j$.

The elements of (I, \triangleleft) are called the *weights* of $\text{mod } \Lambda$.

Recall that if \mathcal{A} is an abelian category, a full subcategory \mathcal{C} of \mathcal{A} is a Serre subcategory if for every exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in \mathcal{A} , we have $Y \in \mathcal{C}$ if and only if $X, Z \in \mathcal{C}$.

Definition 2.1.2. Let \mathcal{A} be an abelian category and Φ a set of objects in \mathcal{A} . We denote by $\mathcal{F}(\Phi)$ the full subcategory of objects $X \in \mathcal{A}$ that admit a finite filtration $0 = X_0 \subseteq X_1 \subseteq \dots \subseteq X_t = X$ such that each factor X_i/X_{i-1} is isomorphic to an object $X \in \Phi$, in this case we say that X is Φ -*filtered*, and the chain of subobjects is called a Φ -*filtration*.

An ideal I of a ring R is idempotent provided $I^2 = I$. Serre subcategories of a module category over an Artin algebra are characterised as follows.

Proposition 2.1.3. *Let Λ be an Artin algebra, and $\mathcal{A} = \text{mod } \Lambda$. For a full subcategory $\mathcal{C} \subseteq \mathcal{A}$ the following statements are equivalent.*

- (a) \mathcal{C} is a Serre subcategory of \mathcal{A} .
- (b) $\mathcal{C} = \mathcal{F}(\{S_1, \dots, S_n\})$ for a set of simple objects $\{S_1, \dots, S_n\}$ in \mathcal{A} .
- (c) $\mathcal{C} = \text{mod}(\Lambda/\mathfrak{a})$ for some idempotent ideal $\mathfrak{a} \subseteq \Lambda$.
- (d) $\mathcal{C} = \{X \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(P, X) = 0\}$ for some projective object $P \in \mathcal{A}$.

In the case of (b), $\{S_1, \dots, S_n\}$ is a set of representatives of simple objects in $\mathcal{A} \setminus \mathcal{C}$.

Proof. See [Aus74, Section 7] or [GL91, Proposition 5.3]. \square

We will see that every highest weight category induces a sequence of recollements satisfying extra conditions. For, we recall the notion of recollement between abelian categories.

Definition 2.1.4. A *recollement* of abelian categories is a diagram of functors

$$\mathcal{A} \begin{array}{c} \xleftarrow{q} \\ \xleftarrow{i} \longrightarrow \\ \xleftarrow{p} \end{array} \mathcal{B} \begin{array}{c} \xleftarrow{l} \\ \xleftarrow{e} \twoheadrightarrow \\ \xleftarrow{r} \end{array} \mathcal{C}$$

satisfying the following conditions.

- (a) (l, e, r) is an adjoint triple.
- (b) (q, i, p) is an adjoint triple.
- (c) The functors i , l , and r are fully faithful.
- (d) $\text{Im } i = \text{Ker } e$, where $\text{Im } i$ is the essential image of the embedding i .

Following [Psa14], we say that the recollement is *homological* if the embedding i is homological.

Remark 2.1.5. In the situation of a recollement the following properties hold.

- (a) The functors $e: \mathcal{B} \rightarrow \mathcal{C}$ and $i: \mathcal{A} \rightarrow \mathcal{B}$ are exact.
- (b) The composites $ql = pl = 0$.
- (c) The functor i induces an equivalence between \mathcal{A} and the Serre subcategory $\text{Ker } e = \text{Im } i$ of \mathcal{B} .
- (d) There is an equivalence $\mathcal{B}/\mathcal{A} \xrightarrow{\sim} \mathcal{C}$ (cf. [Gab62]).

Example 2.1.6. Let Λ be an Artin algebra. Then every idempotent $e \in \Lambda$ induces a recollement of module categories

$$\text{Mod } \Lambda/(e) \begin{array}{c} \xleftarrow{q} \\ \xleftarrow{i} \longrightarrow \\ \xleftarrow{p} \end{array} \text{Mod } \Lambda \begin{array}{c} \xleftarrow{l} \\ \xleftarrow{e} \twoheadrightarrow \\ \xleftarrow{r} \end{array} \text{Mod } e\Lambda e$$

where

$$\begin{aligned} q &= \Lambda/(e) \otimes_{\Lambda} -, & i &= \text{inc}, & p &= \text{Hom}_{\Lambda}(\Lambda/(e), -), \\ l &= \Lambda e \otimes_{e\Lambda e} -, & e &= \text{Hom}_{\Lambda}(\Lambda e, -) \cong e(-), & r &= \text{Hom}_{e\Lambda e}(e\Lambda, -), \end{aligned}$$

that restricts to finitely generated modules. In this case, the functor $e = \text{Hom}_\Lambda(\Lambda e, -)$ induces an equivalence $\text{Mod } \Lambda / \text{Mod } \Lambda / (e) \cong \text{Mod } e\Lambda e$ that restricts to an equivalence $\text{mod } \Lambda / \text{mod } \Lambda / (e) \cong \text{mod } e\Lambda e$.

Conversely, up to equivalence any recollement with middle term $\text{Mod } \Lambda$ is induced by an idempotent element as before [PV14, Corollary 5.5], and restricts to finitely generated modules [Kra17, Lemma 2.5].

When the weights are totally ordered, the following characterisation of highest weight categories is observed by Krause.

Theorem 2.1.7 ([Kra17, Theorem 3.4]). *Let Λ be an Artin algebra, and \leq the usual order on $\{1, 2, \dots, n\}$. Then the following conditions are equivalent.*

- (a) $(\text{mod } \Lambda, \leq)$ is a highest weight category.
- (b) There is a finite chain of Serre subcategories

$$0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \dots \subseteq \mathcal{A}_n = \text{mod } \Lambda$$

and a sequence of semisimple rings $\Gamma_1, \dots, \Gamma_n$ such that each inclusion $\mathcal{A}_i \hookrightarrow \mathcal{A}_{i+1}$ induces a homological recollement of abelian categories

$$\mathcal{A}_{i-1} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathcal{A}_i \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \text{mod } \Gamma_i.$$

In the latter case, the modules $\Delta(i)$ are obtained by applying the left adjoint $\text{mod } \Gamma_i \hookrightarrow \mathcal{A}_i$ to Γ_i , for $1 \leq i \leq n$. Conversely, the subcategories $\mathcal{A}_i \subseteq \text{mod } \Lambda$ are obtained by setting recursively $\mathcal{A}_{i-1} = \{X \in \mathcal{A}_i \mid \text{Hom}_\Lambda(\Delta_i, X) = 0\}$.

Remark 2.1.8. Note that in presence of Proposition 2.1.3 and Example 2.1.6, condition (b) in Theorem 2.1.7 is equivalent to the existence of a sequence of idempotent elements $\varepsilon_1, \dots, \varepsilon_n$ in Λ inducing homological recollements

$$\text{mod } \Lambda / (\varepsilon_i) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \text{mod } \Lambda \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \text{mod } \varepsilon_i \Lambda \varepsilon_i$$

such that $\varepsilon_i \Lambda \varepsilon_i$ is semisimple, for all i .

2.2 Quasi-hereditary algebras

In this section we present several equivalent definitions of quasi-hereditary algebra. In particular we sketch a proof of the fact that every highest weight category is equivalent to a module category over a quasi-hereditary algebra. We also introduce the notion of standardly stratified algebra, and define the characteristic tilting module of a quasi-hereditary algebra. We start defining quasi-hereditary algebras from a ring theoretical approach.

2.2.1 Definition via heredity chains

A ring Λ is *semiprimary* provided its Jacobson radical $J = \text{rad}(\Lambda)$ is nilpotent and Λ/J is semisimple. For instance, every Artin algebra is a semiprimary ring.

Definition 2.2.1. Let Λ be a semiprimary ring. An ideal $\mathfrak{a} \subseteq \Lambda$ is called a *heredity ideal* if the following three conditions hold.

- (a) \mathfrak{a} is idempotent, i.e. $\mathfrak{a}^2 = \mathfrak{a}$,
- (b) \mathfrak{a} is projective as Λ -module, and
- (c) $\mathfrak{a}J(\Lambda)\mathfrak{a} = 0$.

A *heredity chain*, or *defining sequence*, of Λ is a sequence

$$0 = \mathfrak{a}_n \subseteq \mathfrak{a}_{n-1} \subseteq \cdots \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_0 = \Lambda \quad (2.2.1)$$

of two-sided ideals of Λ such that $\mathfrak{a}_{i-1}/\mathfrak{a}_i$ is a heredity ideal in Λ/\mathfrak{a}_i for all i . In this case we say that the heredity chain has length n .

Definition 2.2.2. A semiprimary ring Λ is called *quasi-hereditary* if it admits a heredity chain.

Lemma 2.2.3. *Let Λ be a semiprimary ring. Then an ideal $\mathfrak{a} \subseteq \Lambda$ is idempotent if and only if there is an idempotent $e \in \Lambda$ such that $\mathfrak{a} = \Lambda e \Lambda$.*

Proof. See [DR89b, Statement 6]. □

Remark 2.2.4. Let $\mathfrak{a} \subseteq A$ be an idempotent ideal. Then Lemma 2.2.3 shows that $\mathfrak{a} = AeA$ for some idempotent $e \in A$. Thus the axiom (c) of Definition 2.2.1 is equivalent to ask that eAe is semisimple.

Note that the length of a heredity chain is arbitrary, but in general we can refine defining sequences in such a way that the length is the number of simples over Λ up to isomorphism. The following definition can be found in [UY90].

Definition 2.2.5. Let Λ be an Artin algebra. A chain of idempotent ideals $0 = \mathfrak{a}_t \subseteq \cdots \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_0 = \Lambda$ is called *maximal* if the length of the chain is the number of simple Λ -modules.

Proposition 2.2.6. *Let Λ be an Artin algebra with N isoclasses of simple modules. Then the every heredity chain (2.2.1) can be refined to a maximal heredity chain $0 = \mathfrak{m}_N \subseteq \cdots \subseteq \mathfrak{m}_1 \subseteq \mathfrak{m}_0 = \Lambda$ such that $\mathfrak{m}_i \neq \mathfrak{m}_{i-1}$ for all $i \in \{1, \dots, N\}$.*

Proof. See [UY90, Proposition 1.3]. □

The next result is an equivalent definition of quasi-hereditary algebra by means of surjective ring homomorphisms.

Lemma 2.2.7. *A semiprimary ring Λ is quasi-hereditary if and only if there exists a finite sequence of surjective ring homomorphisms*

$$\Lambda = \Lambda_n \xrightarrow{f_n} \Lambda_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} \Lambda_1 \xrightarrow{f_1} \Lambda_0 = 0 \quad (2.2.2)$$

such that $\text{Ker}(f_i)$ is an heredity ideal for each $1 \leq i \leq n$.

Proof. Suppose we have a sequence of the form (2.2.2). By Lemma 2.2.3 there are idempotents $e_i \in \Lambda_i$ such that $\text{Ker } f_i = \Lambda_i e_i \Lambda_i$. Define $\mathfrak{a}_n := 0$ and $\mathfrak{a}_i := \text{Ker}(f_{i+1} \cdots f_n)$ for $1 \leq i \leq n-1$. Clearly $\mathfrak{a}_i \subseteq \mathfrak{a}_{i-1}$ for $1 \leq i \leq n$ and $\mathfrak{a}_0 = \text{Ker}(\Lambda \rightarrow 0) = \Lambda$. From the hypothesis $\mathfrak{a}_{n-1} = \text{Ker}(f_n) = (e_n)$ is a heredity ideal in Λ_n . Now fix $i \in \{1, \dots, n-1\}$. We prove that $\mathfrak{a}_{i-1}/\mathfrak{a}_i \subseteq \Lambda/\mathfrak{a}_i$ is heredity. For, set $g := f_{i+1} \cdots f_n$, which is a surjective ring homomorphism, thus we have a commutative diagram

$$\begin{array}{ccccc} \Lambda_n & \xrightarrow{g} & \Lambda_i & \xrightarrow{f_i} & \Lambda_{i-1} \\ & \searrow \pi & & \nearrow \bar{g} & \\ & & \Lambda_n & & \\ & & \mathfrak{a}_i & & \end{array}$$

where π is the canonical projection and \bar{g} is an isomorphism of rings. Then, $\pi(\mathfrak{a}_{i-1}) = \text{Ker}(f_i \bar{g})$, since $g = \bar{g} \pi$. On the other hand, $\text{Ker}(f_i \bar{g}) \xrightarrow{\sim} \text{Ker}(f_i)$ via \bar{g} , as \bar{g} is a bijection, moreover is an isomorphism of Λ/\mathfrak{a}_i -modules: let $\bar{\lambda} \in \Lambda/\mathfrak{a}_i$ and $\bar{x} \in \text{Ker}(f_i \bar{g})$ then $\bar{g}(\bar{\lambda} \cdot \bar{x}) = \bar{g}(\bar{\lambda}) \bar{g}(\bar{x}) = \bar{\lambda} \cdot \bar{g}(\bar{x})$. Let $\bar{e}_i := \bar{g}^{-1}(e_i)$. Then $\text{Ker}(f_i \bar{g}) = \bar{g}^{-1}((e_i)) = (\bar{g}^{-1}(e_i)) = (\bar{e}_i)$. This shows that $\pi(\mathfrak{a}_{i-1}) = \mathfrak{a}_{i-1}/\mathfrak{a}_i$ is an idempotent ideal of Λ/\mathfrak{a}_i and is a projective Λ/\mathfrak{a}_i -module, considering that $\Lambda/\mathfrak{a}_i \cong \Lambda_i$ and $\text{Ker } f_i$ is Λ_i -projective. Finally, let $\bar{x} = \bar{e}_i \bar{x}' \bar{e}_i \in \bar{e}_i \text{rad}(\Lambda/\mathfrak{a}_i) \bar{e}_i$, then $\bar{g}(\bar{x}) = \bar{g}(\bar{e}_i \bar{x}' \bar{e}_i) = e_i \bar{g}(\bar{x}') e_i \in e_i \text{rad}(\Lambda_i) e_i = 0$, therefore $\bar{x} = 0$. This proves that $\mathfrak{a}_{i-1}/\mathfrak{a}_i \subseteq \Lambda/\mathfrak{a}_i$ is heredity.

For the converse, if (2.2.1) is a heredity chain of Λ , then $f_i: \Lambda/\mathfrak{a}_i \rightarrow \Lambda/\mathfrak{a}_{i-1}$ defined by $f_i(\bar{\lambda}) := \lambda + \mathfrak{a}_{i-1}$, for $1 \leq i \leq n$, is a well defined surjective ring homomorphism with kernel $\mathfrak{a}_{i-1}/\mathfrak{a}_i$, thus $\text{Ker } f_i$ is a heredity ideal of Λ/\mathfrak{a}_i , for all $1 \leq i \leq n$. \square

The connection between highest weight categories and quasi-hereditary algebras is given by the following central result due to Cline, Parshall and Scott.

Theorem 2.2.8 ([CPS88b, Theorem 3.6]). *Let Λ be a finite dimensional algebra. Then $\text{mod } \Lambda$ admits the structure of a highest weight category if and only if Λ is a quasi-hereditary algebra.*

Sketch of the proof. Let (I, \triangleleft) be a poset indexing the set of simple Λ -modules $\{S(i)\}_{i \in I}$. Let $\{e_i\}_{i \in I}$ be a complete set of pairwise orthogonal idempotents of Λ .

First assume that $(\text{mod } \Lambda, \triangleleft)$ is a highest weight category, with distinguished objects $\{\Delta(i)\}_{i \in I}$. We construct by induction a heredity chain for Λ as follows. Let $m \in I$ be a maximal element such $P(m) = \Delta(m)$ is the projective cover of $S(m)$. There is an idempotent $e \in \Lambda$ such that $P(m) = \Lambda e$. Moreover, there exists a subset $J \subseteq I$, such that $e = \sum_{j \in J} e_j$. Set $\mathfrak{a} = \Lambda e \Lambda$. Then \mathfrak{a} is a heredity ideal of Λ , and $\text{mod } \Lambda/\mathfrak{a} \cong \mathcal{F}(\{S(i) \mid i \in I \setminus J\})$ is a Serre subcategory of $\text{mod } \Lambda$ which admits the structure of a highest weight category, with poset of weights $(I \setminus J, \triangleleft|_{I \setminus J})$.

Conversely, let $0 = \mathfrak{a}_t \subseteq \mathfrak{a}_{t-1} \subseteq \cdots \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_0 = \Lambda$ be a heredity chain of Λ . This heredity chain induces a partition $I = \bigsqcup_{k=0}^{t-1} I_k$ where

$$I_k = \{i \in I \mid [\text{top}(\mathfrak{a}_k/\mathfrak{a}_{k+1}), S(i)] \neq 0\}.$$

We equip the set I with the following order: $i \triangleleft_H j$ if and only if $i \in I_r$, $j \in I_s$ and $r > s$, where $<$ is the usual order on $\{0, \dots, t-1\}$. For $i \in I_r$, let $\Delta(i)$ be the projective cover of $S(i)$ in $\text{mod } \Lambda/\mathfrak{a}_{r-1}$ viewed as an Λ -module. Note that $\mathfrak{a}_{r-1}/\mathfrak{a}_r$ is a direct sum of copies of $\Delta(i)$, $i \in I_r$. Then $(\text{mod } \Lambda, \triangleleft_H)$ is a highest weight category. \square

Remark 2.2.9. If $(\text{mod } \Lambda, \triangleleft)$ is a highest weight category, we constructed a heredity chain of the form Eq. (2.2.1) for Λ , and then we defined a partial order (I, \triangleleft_H) induced by the last heredity chain. It follows that \triangleleft_H refines \triangleleft .

2.2.2 Definition via standard modules

In this subsection Λ denotes an Artin algebra, $\{S_i\}_{i \in I}$ is a complete set of non-isomorphic simple Λ -modules indexed by a finite poset (I, \triangleleft) . For each $i \in I$, $P(i)$ denotes the projective cover of $S(i)$, and $Q(i)$ its injective envelope.

In what follows we present an alternative definition of quasi-hereditary algebra using the module theoretical approach of Dlab and Ringel [DR92], which simplifies the definition of a highest weight category, considering Theorem 2.2.8. We start defining the following operators.

Definition 2.2.10. Let $M \in \text{Mod } \Lambda$ and \mathcal{U} a class of modules in $\text{Mod } \Lambda$.

- (a) The *trace of Θ in M* is $\text{Tr}_{\mathcal{U}}(M) := \sum_{f: f \in \text{Hom}_{\Lambda}(U, M), U \in \mathcal{U}} \text{Im } f$
- (b) The *reject of Θ in M* is $\text{Rej}_{\mathcal{U}}(M) := \bigcap_{f: f \in \text{Hom}_{\Lambda}(M, U), U \in \mathcal{U}} \text{Ker } f$

Note that $\text{Tr}_{\mathcal{U}}(M)$ is the largest submodule of M generated by \mathcal{U} , and $\text{Rej}_{\mathcal{U}}(M)$ is the submodule N of M such that M/N is the largest factor module of M that is cogenerated by \mathcal{U} .

Lemma 2.2.11 ([DR92]). *Let $(\text{mod } \Lambda, (I, \triangleleft))$ be a highest weight category, with distinguished collection $\Delta = \{\Delta(i)\}_{i \in I}$. Let $\mathcal{U}_i := \{P(j) \mid j \not\triangleleft i\}$. Then*

$$\Delta(i) \cong P(i) / \text{Tr}_{\mathcal{U}_i}(P(i))$$

for all $i \in I$, as Λ -modules.

The last result shows that $\Delta(i)$ is characterised as the largest quotient of $P(i)$ with composition factors $S(j)$, with $j \triangleleft i$. Equivalently, $\Delta(i)$ is the projective cover of $S(i)$ in $\mathcal{F}(\{S(j) \mid j \not\triangleleft i\})$ viewed a Λ -module (compare with the proof of Theorem 2.2.8). This motivates the following definitions in the context of Artin algebras.

Definition 2.2.12. Let (I, \triangleleft) be a poset indexing the simple Λ -modules.

- (a) The *standard module with weight $i \in I$* , denoted by $\Delta(i) = \Delta_{\triangleleft}(i)$, is the maximal factor module of $P(i)$ with composition factors $S(j)$, with $j \triangleleft i$, i.e.

$$\Delta(i) := P(i) / \text{Tr}_{\{P(j) \mid j \not\triangleleft i\}}(P(i)).$$

- (b) The *costandard module with weight $i \in I$* , denoted by $\nabla(i) = \nabla_{\triangleleft}(i)$, is the maximal submodule of $Q(i)$ with composition factors $S(j)$, with $j \triangleleft i$, i.e.

$$\nabla(i) := \text{Rej}_{\{Q(j) \mid j \not\triangleleft i\}}(Q(i)).$$

We set $\Delta = \Delta_{\triangleleft} := \{\Delta(i)\}_{i \in I}$ and $\nabla = \nabla_{\triangleleft} := \{\nabla(i)\}_{i \in I}$.

In the case of a finite dimensional \mathbb{K} -algebra A , the costandard module $\nabla(i)$ is the dual of a standard module: let $D := \text{Hom}_A(-, \mathbb{K})$ be the standard duality and A^{op} the opposite algebra of A , then $\nabla_A(i) = D(\Delta_{A^{op}}(i))$, for all $i \in I$.

Let $M \in \mathcal{F}(\Delta)$ (cf. Definition 2.1.2). The number of times that $\Delta(i)$ appears as quotient in a Δ -filtration of M does not depend on the choice of the filtration (cf. [CPS88b, Theorem 3.11] or [Con16, Remark 1.4.7]), we denote it by $(M : \Delta(i))$. Similarly, for $N \in \mathcal{F}(\nabla)$ the number of times that $\nabla(i)$ appears in a ∇ -filtration is independent of the choice of the filtration, we denote it by $(N : \nabla(i))$.

The next definition is a reinterpretation of Theorem 2.2.8 considering Lemma 2.2.11 and the proof of Theorem 2.2.8.

Definition 2.2.13 ([CPS88b]). The pair $(\Lambda, (I, \triangleleft))$ is a *quasi-hereditary algebra* if the following conditions are satisfied.

- (a) $[\Delta(i) : S(i)] = 1$ for all $i \in I$,
- (b) $P(i) \in \mathcal{F}(\Delta)$ for all $i \in I$, and
- (c) $(P(i) : \Delta(i)) = 1$ for all $i \in I$, and $(P(i) : \Delta(j)) \neq 0$ implies $i \triangleleft j$.

Remark 2.2.14. Note that for each $i \in I$, condition (a) is equivalent to the properties:

- (i) $\text{End}_\Lambda(\Delta(i))$ is a division algebra,
- (ii) $[\nabla(i) : S(i)] = 1$,
- (iii) $\text{End}_\Lambda(\nabla(i))$ is a division algebra;

and condition (b) is equivalent to ${}_\Lambda \Lambda \in \mathcal{F}(\Delta)$.

We can replace condition (c) in Definition 2.2.13 if we consider adapted posets in the following sense.

Definition 2.2.15 ([DR92]). A partial order (I, \triangleleft) is *adapted to Λ* if for every Λ -module M with top $S(i)$ and socle $S(j)$, where i and j are incomparable, there is $k \in I$ with $i \triangleleft k$ and $j \triangleleft k$ and $[M : S(k)] \neq 0$.

Note that if \triangleleft is a total order, then it is adapted to Λ . A very important property of adapted posets is that this property is closed under refinement.

Lemma 2.2.16 ([DR92]). Let (I, \triangleleft_1) be an adapted poset for Λ . Let \triangleleft_2 be a refinement of \triangleleft_1 . For $l = 1, 2$ let $\Delta_l(i)$ be the standard module with weight i for the poset \triangleleft_l . Then

- (a) $\Delta_1(i) = \Delta_2(i)$ for all $i \in I$.
- (b) $\nabla_1(i) = \nabla_2(i)$ for all $i \in I$.
- (c) The poset (I, \triangleleft_2) is adapted.

Lemma 2.2.17 ([Con16, Proposition 1.4.12]). If $(\Lambda, (I, \triangleleft))$ is a quasi-hereditary algebra, then (I, \triangleleft) is adapted to Λ .

Proof. Let M be a module with simple top $S(i)$ and simple socle $S(j)$. Since Λ is Artinian, M is a quotient of $P(i)$. In particular $S(j)$ is a composition factor of $P(i)$. Because the algebra is quasi-hereditary $P(i)$ has a Δ -filtration. So $S(j)$ must appear in a standard module $\Delta(k)$. If $k = i$ then $j \triangleleft i$. If $k \neq i$ and $S(j)$ is at the top of $\Delta(k)$ we have $i \triangleleft j$. And finally if $S(j)$ is not at the top of $\Delta(k)$ we have $i \triangleleft k$ and $j \triangleleft k$. \square

We can now considerably simplify the definition of quasi-hereditary algebra.

Proposition 2.2.18. *The pair $(\Lambda, (I, \triangleleft))$ is a quasi-hereditary algebra, in the sense of Definition 2.2.13, if and only if the following three conditions hold:*

- (a) *The poset (I, \triangleleft) is adapted to Λ .*
- (b) *For all $i \in I$ $[\Delta(i) : S(i)] = 1$.*
- (c) *For all $i \in I$ $P(i) \in \mathcal{F}(\Delta)$.*

Proof. See the proof of [DR92, Theorem 1]. \square

Remark 2.2.19. Many times, when considering adapted posets, one usually restricts to total orders, since Lemma 2.2.16 and Proposition 2.2.18 actually show that this is not an impediment.

The following are equivalent definitions of quasi-hereditary algebras. Condition (e) is due to Soergel [Soe90]. The usual definition of quasi-hereditary algebra is (b).

Proposition 2.2.20 ([DR92, Theorem 1]). *Let (I, \triangleleft) be an adapted poset to Λ , and assume that for all $i \in I$ $[\Delta(i) : S(i)] = 1$. Then the following statements are equivalent.*

- (a) *(Λ, \triangleleft) is a quasi-hereditary algebra.*
- (b) *The module ${}_{\Lambda}\Lambda$ is in $\mathcal{F}(\Delta)$.*
- (c) *$\mathcal{F}(\Delta) = \{X \mid \text{Ext}^1(X, \nabla(i)) = 0 \text{ for all } i \in I\}$.*
- (d) *$\mathcal{F}(\Delta) = \{X \mid \text{Ext}^j(X, \nabla(i)) = 0 \text{ for all } i \in I \text{ and } j \geq 1\}$.*
- (e) *$\text{Ext}^2(\Delta(i), \nabla(j)) = 0$ for all $i, j \in I$.*

We may add the dual conditions:

- (b') *The dual $D(\Lambda_{\Lambda})$ is in $\mathcal{F}(\Delta)$.*
- (c') *$\mathcal{F}(\nabla) = \{Y \mid \text{Ext}^1(\Delta(i), Y) = 0 \text{ for all } i \in I\}$.*
- (d') *$\mathcal{F}(\nabla) = \{Y \mid \text{Ext}^j(\Delta(i), Y) = 0 \text{ for all } i \in I \text{ and } j \geq 1\}$.*

In this case we say that the modules belonging to $\mathcal{F}(\Delta)$ admit a good filtration, and those in $\mathcal{F}(\nabla)$ admit a cogood filtration.

Definition 2.2.21. Let \mathcal{C} be a subcategory of $\text{mod } \Lambda$. We have the following full subcategories of $\text{mod } \Lambda$:

$$\begin{aligned} \mathcal{C}^{\perp} &:= \{M \mid \text{Ext}_{\Lambda}^i(C, M) = 0 \text{ for all } i > 0, \text{ and } C \in \mathcal{C}\} \text{ and} \\ {}^{\perp}\mathcal{C} &:= \{M \mid \text{Ext}_{\Lambda}^i(M, C) = 0 \text{ for all } i > 0, \text{ and } C \in \mathcal{C}\}, \end{aligned}$$

called the *right (resp. left) perpendicular category to \mathcal{C}* .

Ringel showed that the subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are perpendicular to each other, i.e. we have the following result.

Proposition 2.2.22 ([Rin91]). *Let (Λ, \triangleleft) be a quasi-hereditary algebra. Then $\mathcal{F}(\Delta) = {}^\perp\mathcal{F}(\nabla)$ and $\mathcal{F}(\nabla) = \mathcal{F}(\Delta)^\perp$.*

Recall that a ring R is called *right hereditary* if all left ideals are projective. We call an Artin algebra Λ *hereditary* if it is right hereditary, or equivalently if $\text{gl. dim } \Lambda \leq 1$. It is well known that if Q is a finite, connected and acyclic quiver, then $\mathbb{K}Q$ is a hereditary algebra, and all basic hereditary algebras occur in this way [ASS06, Ch. VII, Theorem 1.7].

The next result due to Dlab and Ringel shows that hereditary algebras are quasi-hereditary for any adapted order.

Proposition 2.2.23. *Assume that Λ is hereditary Artin algebra. Then for any adapted order \triangleleft on I , $(\Lambda, (I, \triangleleft))$ is a quasi-hereditary algebra.*

Proof. We may assume that \triangleleft is a total order on I . Then the assertion follows from [DR89b, Theorem 1]. \square

Example 2.2.24. Let \mathbb{K} be a field and Q the quiver $3 \rightarrow 2 \rightarrow 1$. Set $A = \mathbb{K}Q$, and let \triangleleft be given by the cover relations $1 \triangleleft 2$, $1 \triangleleft 3$. Then \triangleleft is not adapted to A , thus (A, \triangleleft) is not a quasi-hereditary algebra. Indeed, $\Delta(1) = 1$, $\Delta(2) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$, $\Delta(3) = 3$. Thus

$P(3) = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$ admits a Δ -filtration $0 \subset \Delta(2) \subset P(3)$, with $(P(3) : \Delta(2)) \neq 0$, but $3 \not\triangleleft 2$,

so the axiom (c) of Definition 2.2.13 fails in this case.

The next result gathers some general properties of quasi-hereditary algebras.

Proposition 2.2.25. *Let Λ be an Artin algebra. Then the following statements hold.*

- (a) *If $(\Lambda, (I, \triangleleft))$ is a quasi-hereditary algebra, then $\text{gl. dim } \Lambda \leq 2n - 2$, where $n = \text{card } I$.*
- (b) *The pair $(\Lambda, (I, \triangleleft))$ is a quasi-hereditary algebra if and only if $(\Lambda, (I, \triangleleft^{op}))$ is a quasi-hereditary algebra.*
- (c) *If $\text{gl. dim } \Lambda \leq 2$, then there exists some adapted poset \triangleleft to Λ , such that (Λ, \triangleleft) is quasi-hereditary.*

Proof. (a) In [PS88, Theorem 4.3 (a)] Parshall and Scoot proved that every quasi-hereditary algebra has finite global dimension. The bound was found later by Dlab and Ringel [DR92, Lemma 2.2]. For (b), see [PS88, Theorem 4.3 (b)]. The assertion (c) when $\text{gl. dim } \Lambda = 2$ was shown by Dlab and Ringel in [DR89b, Theorem 2], for the case $\text{gl. dim } \Lambda \leq 1$ see Proposition 2.2.23. \square

We finish this section recalling a generalisation of the concept of quasi-hereditary algebra, namely standardly stratified algebras.

Definition 2.2.26. Let (I, \triangleleft) be an adapted poset to Λ , and $\Delta = \Delta_{\triangleleft}$. We say that (Λ, \triangleleft) is a *standardly stratified algebra* provided $P(i) \in \mathcal{F}(\Delta)$ for all $i \in I$.

Remark 2.2.27. Note that if Λ has n simples up to isomorphism and $(\{1, \dots, n\}, \triangleleft)$ is a total order, then (Λ, \triangleleft) is standardly stratified if and only if Λ admits a maximal chain of idempotent ideals $0 = (\varepsilon_n) \subset (\varepsilon_{n-1}) \subset (\varepsilon_{n-2}) \subset \dots \subset (\varepsilon_0) = \Lambda$ such that $(\varepsilon_{i-1})/(\varepsilon_i)$ is projective as $\Lambda/(\varepsilon_i)$ -module, for all $1 \leq i \leq n$. Call such a chain *stratifying*.

Proposition 2.2.28. *Let (I, \triangleleft) be an adapted poset to Λ , and $\Delta = \Delta_{\triangleleft}$. Then the following conditions are equivalent.*

- (a) Λ is a quasi-hereditary algebra.
- (b) Λ is a standardly stratified algebra with finite global dimension.
- (c) Λ is a standardly stratified algebra such that $[\Delta(i) : S(i)] = 1$ for all $i \in I$.

Proof. For the equivalence of (a) and (b) see [Wic96, Theorem 1.7] or [Dla96, Corollary 2.6]. The equivalence of (a) and (c) follows from Proposition 2.2.18. \square

2.2.3 The characteristic tilting module

In [Rin91] Ringel studied the full subcategory of modules in $\text{mod } \Lambda$ which have a Δ - and ∇ -filtration, for Λ a quasi-hereditary algebra. In what follows we state precise results that will help us in the classification of quasi-hereditary structures for some quiver algebras (cf. Chapter 5), but first we recall some definitions.

Let Λ be an Artin algebra, and $M \in \text{mod } \Lambda$. The *additive closure* of M , denote by $\text{add } M$, is the full subcategory of $\text{mod } \Lambda$ consisting of all direct summands of any direct sum of finitely many copies of M . We say that M is a *basic module* if it has no direct summand of the form $N \oplus N$, with N a non-zero Λ -module.

Definition 2.2.29. A Λ -module T is called a (generalised) *tilting module* if it satisfies the following three conditions.

- (a) T has finite projective dimension,
- (b) $\text{Ext}_{\Lambda}^i(T, T) = 0$ for all $i > 0$, and
- (c) there exists an exact sequence $0 \rightarrow \Lambda \rightarrow T_0 \rightarrow T_1 \rightarrow \dots \rightarrow T_m \rightarrow 0$, with $T_i \in \text{add } T$.

Moreover, if $\text{pd } T \leq 1$ then T is called a *classical tilting module*.

Proposition 2.2.30 ([Rin91]). *Let $(\Lambda, (I, \triangleleft))$ be a quasi-hereditary algebra. Then there exists a basic tilting module $T \in \text{mod } \Lambda$ such that $\text{add } T = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. Moreover, there exists a decomposition $T = \bigoplus_{i \in I} T(i)$ into indecomposable Λ -modules $T(i)$, such that there are exact sequences*

$$0 \rightarrow \Delta(i) \rightarrow T(i) \rightarrow X(i) \rightarrow 0, \quad 0 \rightarrow Y(i) \rightarrow T(i) \rightarrow \nabla(i) \rightarrow 0$$

where $X(i)$ belongs to $\mathcal{F}(\Delta(j) \mid j \triangleleft i)$ and $Y(i)$ belongs to $\mathcal{F}(\nabla(j) \mid j \triangleleft i)$.

Definition 2.2.31. A module T given as in Proposition 2.2.30 is called a *characteristic tilting module* of $(\Lambda, (I, \triangleleft))$.

Ringel proved that the characteristic tilting module determines both $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ in the following sense.

Proposition 2.2.32 ([Rin91, Corollary 4]). *Let T be the characteristic tilting of a quasi-hereditary algebra (Λ, \triangleleft) . Then $\mathcal{F}(\Delta) = {}^{\perp}T$ and $\mathcal{F}(\nabla) = T^{\perp}$.*

2.3 The homological poset

In this section we present a systematic method to study all the possible heredity chains that an Artin algebra could admit. For, we consider the more general concept of homological embedding between Serre subcategories (cf. Theorem 2.1.7).

In what follows Λ is an Artin algebra, and $\{S_i\}_{i \in I}$ is a finite complete list of non-isomorphic simple Λ -modules. Let $\{e_i\}_{i \in I}$ be a complete set of pairwise orthogonal idempotents of Λ such that $P(i) = \Lambda e_i$ is the projective cover of $S(i)$, for $i \in I$. For $J \subseteq I$ define

$$e_J := \sum_{j \in J} e_j,$$

where $e_\emptyset = 0$. Set $J^c := I \setminus J$, thus $e_{J^c} = 1 - e_J$.

Let $0 = \mathfrak{a}_t \subseteq \mathfrak{a}_{t-1} \subseteq \cdots \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_0 = \Lambda$ be a heredity chain of Λ . Then Lemma 2.2.7 shows that

$$\frac{\Lambda}{\mathfrak{a}_t} \xrightarrow{\pi_t} \frac{\Lambda}{\mathfrak{a}_{t-1}} \xrightarrow{\pi_{t-1}} \cdots \xrightarrow{\pi_2} \frac{\Lambda}{\mathfrak{a}_1} \xrightarrow{\pi_1} \frac{\Lambda}{\mathfrak{a}_0}$$

is a sequence of surjective ring homomorphisms, given by $\pi_i(a + \mathfrak{a}_i) = a + \mathfrak{a}_{i-1}$, with $\text{Ker } \pi_i = \mathfrak{a}_{i-1}/\mathfrak{a}_i$ a heredity ideal of Λ/\mathfrak{a}_i . By Proposition 2.1.3 and Lemma 1.5.2, this sequence induces a chain

$$\text{mod } \Lambda = \text{mod } \Lambda/\mathfrak{a}_t \xleftarrow{\iota_t} \text{mod } \Lambda/\mathfrak{a}_{t-1} \xleftarrow{\iota_{t-1}} \cdots \xleftarrow{\iota_2} \text{mod } \Lambda/\mathfrak{a}_1 \xleftarrow{\iota_1} \text{mod } \Lambda/\mathfrak{a}_0 = 0 \quad (2.3.1)$$

of homological embeddings between Serres subcategories of $\text{mod } \Lambda$, since $\mathfrak{a}_{i-1}/\mathfrak{a}_i$ is projective as Λ/\mathfrak{a}_i -module. The embeddings ι_j are given by restriction of scalars.

Moreover, for each $1 \leq i \leq t$, there is a non-empty subset $J_i \subseteq I$ such that $\text{mod } \Lambda/\mathfrak{a}_i = \mathcal{F}(\{S(j) \mid j \in J_i\})$, more precisely J_i is characterised as the set of indices in I such that $\{S(j)\}_{j \in J_i}$ is a set of representatives of simple objects in $\text{mod } \Lambda \setminus \text{mod } \Lambda/\mathfrak{a}_i$. Thus we have inclusions $\emptyset = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_t = I$. Then it is clear that

$$\text{mod } \Lambda/(e_{J_i^c}) \cong \mathcal{F}(\{S(j) \mid j \in J_i\}) \cong \text{mod } \Lambda/\mathfrak{a}_i,$$

thus $\mathfrak{a}_i = (e_{J_i^c})$, for each $1 \leq i \leq t$. For simplicity, set $\delta_0 := 0$,

$$\varepsilon_i := e_{J_i^c}, \quad \text{and} \quad \delta_i := \varepsilon_{i-1} - \varepsilon_i = e_{J_i \setminus J_{i-1}}$$

for $1 \leq i \leq t$. Therefore, the fact that $\text{Ker } \pi_i = (\varepsilon_{i-1})/(\varepsilon_i) = (\delta_i)$ is a heredity ideal in $\Lambda/(\varepsilon_i)$ means that it is projective as $\Lambda/(\varepsilon_i)$ -module, and $\Gamma_i := \delta_i \Lambda/(\varepsilon_i) \delta_i$ is semisimple. Moreover, for $1 \leq i \leq t$, we have an isomorphism of rings

$$(\Lambda/(\varepsilon_i))/((\varepsilon_{i-1})/(\varepsilon_i)) \cong \Lambda/(\varepsilon_{i-1}),$$

hence we have a homological recollement

$$\text{mod } \Lambda/(\varepsilon_{i-1}) \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longrightarrow \\ \longrightarrow \end{array} \text{mod } \Lambda/(\varepsilon_i) \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longrightarrow \\ \longrightarrow \end{array} \text{mod } \Gamma_i \quad (2.3.2)$$

which induces an equivalence $\text{mod } \Lambda/(\varepsilon_i)/\text{mod } \Lambda/(\varepsilon_{i-1}) \xrightarrow{\sim} \text{mod } \Gamma_i$, for each $1 \leq i \leq t$.

The last discussion motivates a more general construction involving embeddings between Serre subcategories of $\text{mod } \Lambda$ providing a correct language to find all the heredity chains of Λ .

Let $J \subseteq I$. We know that $\mathcal{F}(\{S(j) \mid j \in J\}) \cong \text{mod } \Lambda/(e_{J^c})$ is a Serre subcategory of $\text{mod } \Lambda$, and it is independent of the choice of the set of idempotents E , since it is unique up to conjugacy (cf. Proposition 1.2.5). Thus we have a bijection between the power set of $[n]$ and the Serre subcategories of $\text{mod } A$ of the form $\text{mod } A/(e_{J^c})$, given by $J \mapsto \text{mod } A/(e_{J^c})$.

On the other hand, let $J \subseteq K \subseteq I$. We have a well defined ring surjection

$$\pi_{K,J}: \frac{\Lambda}{(e_{K^c})} \twoheadrightarrow \frac{\Lambda}{(e_{J^c})}$$

given by $\lambda + (e_{K^c}) \mapsto \lambda + (e_{J^c})$, since $(e_{K^c}) \subseteq (e_{J^c})$ by Lemma 3.4.1 (b1). By Proposition 1.5.4 (b) we have an embedding between Serre subcategories of $\text{mod } \Lambda$

$$\iota_{J,K} := (\pi_{K,J})_*: \text{mod } \Lambda/(e_{J^c}) \hookrightarrow \text{mod } \Lambda/(e_{K^c})$$

given by restriction of scalars. Therefore, we regard $\Lambda/(e_{J^c})$ -modules as modules over $\Lambda/(e_{K^c})$, and $\Lambda/(e_{K^c})$ -modules as modules over Λ in the natural way.

Remark 2.3.1. Observe that $\pi_{K,J}$ is a homological ring epimorphism if and only if $\iota_{J,K}$ is a homological embedding, by Theorem 1.5.5.

We will find necessary and sufficient conditions for $\iota_{J,I}$ to be a homological embedding for some classes of algebras, see Chapters 3 and 4. This simple construction defines a poset structure on the power set 2^I .

Definition 2.3.2. Let Λ be an Artin algebra. Let $\{e_i\}_{i \in I}$ be a complete set of primitive orthogonal idempotents of Λ . For $J \subseteq K \subseteq I$, set $J \preceq_{\Lambda} K$ if $\iota_{J,K}: \text{mod } \Lambda/(e_{J^c}) \hookrightarrow \text{mod } \Lambda/(e_{K^c})$ is a homological embedding.

It is clear that \preceq_{Λ} is reflexive and antisymmetric. Moreover, since the composition of homological embeddings is so too, we have that \preceq_{Λ} is transitive, since if $J \preceq_{\Lambda} L \preceq_{\Lambda} K$, then $\iota_{J,K} = \iota_{L,K} \circ \iota_{J,L}$. Thus, \preceq_{Λ} gives a poset structure on 2^I .

Definition 2.3.3. Let Λ be an Artin algebra, and $\{e_i\}_{i \in I}$ a complete set of primitive orthogonal idempotents of Λ . The *homological poset* of Λ is the poset $(2^I, \preceq_{\Lambda})$, denoted by $\mathcal{H}(\Lambda)$. The *homological Hasse quiver* of Λ is the Hasse quiver of $\mathcal{H}(\Lambda)$, and is denoted by $\mathbb{H}(\Lambda)$.

Note that the poset structure \preceq_{Λ} does not depend on the choice of the complete set of primitive pairwise orthogonal idempotents, because if $\{f_i\}_{i \in J}$ is other such a family, by Proposition 1.2.5 there is $a \in \Lambda$ invertible and we can reorder the idempotents f_i in such a way that $f_i = ae_i a^{-1}$ for all $i \in I$. So, for any $J \subseteq I$, we have that $(f_J) = (e_J)$.

In the particular case when $A = \mathbb{K}Q/I$ is a bound quiver algebra, in this thesis we label the vertices of Q as $Q_0 = \{1, \dots, n\}$. We know that the set of classes of paths of length zero $\{e_i := \varepsilon_i + I\}_{i=1}^n$ is a complete set set of primitive orthogonal idempotents of A . We always consider this family when dealing with bound quiver algebras.

Remark 2.3.4. Let $n \geq 0$, and $I = [n] := \{1, \dots, n\}$.

- (a) The poset $\mathcal{P}(n) = (2^{[n]}, \subseteq)$ is a refinement of $\mathcal{H}(\Lambda)$, in other words, $\mathcal{H}(\Lambda)$ is a weak subposet of $\mathcal{P}(n)$.

- (b) $\mathcal{H}(\Lambda)$ is determined by its cover relations since it is finite, i.e. if $J \preceq_{\Lambda} I$, then there exists a chain of cover relations $J \preceq_{\Lambda} X_1 \preceq_{\Lambda} \cdots \preceq_{\Lambda} X_l \preceq_{\Lambda} I$.

Combining Theorem 2.1.7 and Proposition 2.2.6 and our previous discussions, we have the next result.

Theorem 2.3.5. *Let Λ be an Artin algebra, and $\{e_i\}_{i=1}^n$ a complete set of primitive orthogonal idempotents of Λ . Then we have a bijective correspondence between the set of paths of length n in $\mathbb{H}(\Lambda)$ $\eta: \emptyset = J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_n = [n]$ such that the algebra $e_{J_i \setminus J_{i-1}} \Lambda / (e_{J_i^c}) e_{J_i \setminus J_{i-1}}$ is semisimple, and the set of maximal heredity chains of Λ , given by*

$$\eta \mapsto (0 = (e_{J_n^c}) \subset \cdots \subset (e_{J_1^c}) \subset (e_{J_0^c}) = \Lambda).$$

Weakening the hypothesis on the maximal chains of idempotent ideals, we have a similar result in the case of standardly stratified algebras.

Theorem 2.3.6. *Let Λ be an Artin algebra, and $\{e_i\}_{i=1}^n$ a complete set of primitive orthogonal idempotents of Λ . Then we have a bijective correspondence between the set of paths of length n in $\mathbb{H}(\Lambda)$, and the set of maximal stratifying chains of Λ given by*

$$(\emptyset = J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_n = [n]) \mapsto (0 = (e_{J_n^c}) \subset \cdots \subset (e_{J_1^c}) \subset (e_{J_0^c}) = \Lambda).$$

Before exhibiting some examples, we give the following technical observation, which generalises that fact that the embedding $\text{mod } \Lambda / (\varepsilon_{i-1}) \hookrightarrow \text{mod } \Lambda / (\varepsilon_i)$ considered in the recollement (2.3.2) is homological.

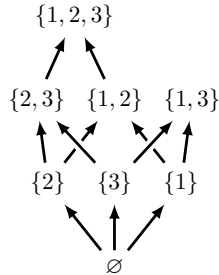
Lemma 2.3.7. *Let Λ be an Artin algebra with $\{e_i\}_{i \in I}$ a complete set of primitive orthogonal idempotents. Let $J \subseteq K \subseteq I$. If $(e_{K \setminus J})$ is projective as $\Lambda / (e_{K^c})$ -module, then $\iota_{J,K}$ is a homological embedding.*

Proof. Since $K^c \subseteq J^c$, we have $J^c = K^c \sqcup J^c \setminus K^c$, thus by Lemma 3.4.1 $(e_{K^c}) \subseteq (e_{J^c}) = (e_{K^c}) + (e_{J^c \setminus K^c}) = (e_{K^c}) + (e_{K \setminus J})$. Set $\Lambda' := \Lambda / (e_{K^c})$, then $\Lambda / (e_{J^c}) \cong \Lambda' / (e_{K \setminus J})$, which shows that we can regard $\iota_{J,K}$ as an embedding $\text{mod } \Lambda' / (e_{K \setminus J}) \hookrightarrow \text{mod } \Lambda'$. The conclusion follows from Lemma 1.5.2. \square

Example 2.3.8. Let \mathbb{K} be a field.

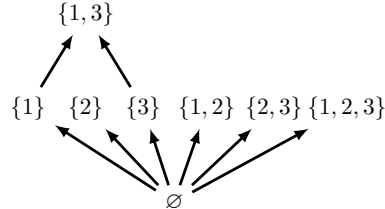
(a) Let A be a hereditary \mathbb{K} -algebra with n simples up to isomorphism. Then every chain of length n of idempotent ideals is a heredity chain by [DR89b, Theorem 1]. Thus $\mathcal{H}(A) = \mathcal{P}(n)$, and $\mathbb{H}(A)$ is the n -hypercube.

(b) Let Q be the quiver $3 \xrightarrow{b} 2 \xrightarrow{a} 1$, and set $A = \mathbb{K}Q / (ab)$. Then $\text{gl. dim } A = 2$, thus A is quasi-hereditary for some adapted poset. $\mathbb{H}(A)$ is the following quiver

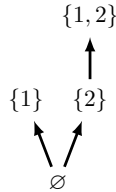


which shows that A admits four different heredity chains.

(c) Let Q be the quiver $1 \begin{matrix} \xrightarrow{a_1} \\ \xleftarrow{b_1} \end{matrix} 2 \begin{matrix} \xrightarrow{a_2} \\ \xleftarrow{b_2} \end{matrix} 3$. Set $A = \mathbb{K}Q/(a_1b_1 - b_2a_2, b_1a_1, a_2b_2)$, thus A is the preprojective algebra of type A_3 , therefore $\text{gl. dim } A = \infty$ and A is not quasi-hereditary for any order. The Hasse quiver of $\mathcal{H}(A)$ is the following.



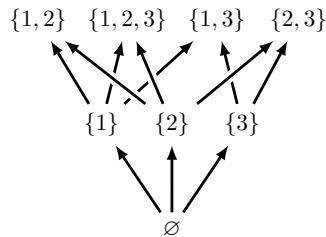
(d) Let Q be the quiver $1 \begin{matrix} \xrightarrow{a} \\ \xleftarrow{b} \end{matrix} 2$. Set $A = \mathbb{K}Q/(baba)$. Then $\text{gl. dim } A = \infty$, thus A is not quasi-hereditary for any ordering, but $\mathbb{H}(A)$ is the following.



Indeed, note that the maximal path in $\mathbb{H}(A)$ corresponds to the chain of Serre subcategories $0 \subset \text{mod } A/(e_1) \subset \text{mod } A$, but $e_1Ae_1 \cong \mathbb{K}(1 \curvearrowright \alpha) / (\alpha^2)$ is clearly not semisimple. Note that A is standardly stratified by Theorem 2.3.6.

(e) Let Q be the quiver $\begin{matrix} & 2 & \\ a \nearrow & & \searrow b \\ 1 & \xleftarrow{c} & 3 \end{matrix}$ and $A = \mathbb{K}Q/(bac, acba)$. Then $\text{gl. dim } A = 4$,

and its homological Hasse quiver is the following.



This shows that A admits no heredity chain. We remark that the algebra A has been already considered in [DR89b].

2.4 Quasi-hereditary structures

In this section we provide the foundations of the notion of quasi-hereditary structures in order to present a new approach to the classification of adapted posets that yield

quasi-hereditary algebras. This is joint work with Yuta Kimura and Baptiste Rognerud [FKR20]. At the end of the section we present a sufficient condition for an algebra to have at most one quasi-heredity structure due to Coulembier.

In what follows Λ denotes an Artin algebra, and (I, \triangleleft) a finite poset indexing a complete set of non-isomorphic simple Λ -modules $\{S(i)\}_{i \in I}$. In this case, $\Delta = \Delta_{\triangleleft}$ denotes the set of standard modules, and $\nabla = \nabla_{\triangleleft}$ the set of costandard modules. Recall that adapted posets (Definition 2.2.15) are closed under refinement (cf. Lemma 2.2.16). We start with the following weaker version of Definition 2.2.15.

Lemma 2.4.1. *A partial order \triangleleft on I is adapted to A if and only if for every A -module M with top $S(i)$ and socle $S(j)$, where i and j are incomparable, there is $k \in I$ with $i \triangleleft k$ or $j \triangleleft k$ and $[M : S(k)] \neq 0$.*

Proof. See [DR92, page 202]. □

Definition 2.4.2. Let \triangleleft_1 and \triangleleft_2 be two partial orders on I . Then \triangleleft_1 is *equivalent to* \triangleleft_2 if $\Delta_{\triangleleft_1} = \Delta_{\triangleleft_2}$ and $\nabla_{\triangleleft_1} = \nabla_{\triangleleft_2}$. In this case we write $\triangleleft_1 \sim \triangleleft_2$.

The relation \sim is an equivalence relation on the set of poset structures of I , and is compatible with the notion of adapted poset in the following sense.

Lemma 2.4.3. *Let \triangleleft_1 be an adapted poset to Λ . If \triangleleft is equivalent to \triangleleft_1 , then \triangleleft is adapted to A .*

Proof. Let M be an indecomposable module with simple top $S(i)$ and simple socle $S(j)$ with i and j are incomparable for \triangleleft . Since M has simple top $S(i)$ it is a quotient of $P(i)$. We denote by $U(i)$ the kernel of the projection from $P(i)$ to $\Delta(i)$. Since i and j are incomparable, the module $S(j)$ is a composition factor of $U(i)$. Then, there is a composition factor $S(k)$ which is at the top of $U(i)$ and which is also a composition factor of M . We denote by N a non-split extension of $S(k)$ and $\Delta(i)$.

Since \triangleleft_1 is an adapted poset we see that $i \triangleleft_1 k$ and by transitivity if $S(e)$ is a composition factor of $\Delta(i)$ then $e \triangleleft_1 k$. So the largest submodule of N cogenerated by $S(k)$ is a submodule of $\nabla_1(k)$. Since $\nabla(k) = \nabla_1(k)$ and $S(i)$ is a composition factor of this module, we see that $i \triangleleft k$ and the result follows from Lemma 2.4.1. □

Recall that a poset structure on I is a subset $\triangleleft \subseteq I \times I$ satisfying three properties (cf. Section 1.4). Thus we order poset structures on I by inclusion. This gives a poset of posets over I where the minimal element is the equality relation on I , i.e. $\{(i, i) \mid i \in I\}$, and the maximal elements are the total orders on I . Given two poset structures \triangleleft_1 and \triangleleft_2 on I , we can take their intersection $\triangleleft_1 \cap \triangleleft_2$ which is again a poset structure on I , called the *intersection of \triangleleft_1 and \triangleleft_2* .

Lemma 2.4.4. *Let \triangleleft_1 and \triangleleft_2 be two adapted posets to Λ with $\triangleleft_1 \sim \triangleleft_2$.*

- (a) *The intersection $\triangleleft_1 \cap \triangleleft_2$ is an adapted poset to Λ in the same equivalence class.*
- (b) *In each equivalence class of adapted posets to Λ there is a unique minimal poset.*

Proof. It is clear that (a) implies (b). Let \triangleleft_1 and \triangleleft_2 be two posets in the same equivalence class. We denote by $\Delta = \Delta_1 = \Delta_2$ the corresponding set of standard modules. Set \triangleleft_{int} to be the intersection of \triangleleft_1 and \triangleleft_2 , and denote by $\Delta_{int}, \nabla_{int}$ the corresponding sets of standard and costandard modules, respectively.

Let $i \in I$. By definition $\Delta_{int}(i)$ is the largest quotient of $P(i)$ whose composition factors are $S(j)$ such that $j \triangleleft_1 i$ and $j \triangleleft_2 i$. So $\Delta_1(i)$ surjects onto $\Delta_{int}(i)$. If they are not isomorphic, at the top of the kernel there is a simple module $S(j)$ such that $j \triangleleft_1 i$ but j is not smaller than i for \triangleleft_2 . This contradicts $\Delta_1(i) = \Delta_2(i)$. Therefore $\Delta_{int} = \Delta$ and by a dual argument, we see that $\nabla_{int} = \nabla$ and the poset \triangleleft_{int} is equivalent to the posets \triangleleft_1 and \triangleleft_2 . The result follows from Lemma 2.4.3 \square

Let \triangleleft_1 and \triangleleft_2 be two partial orders on I . If $(\Lambda, \triangleleft_1)$ is a quasi-hereditary algebra and $\triangleleft_2 \sim \triangleleft_1$ then $(\Lambda, \triangleleft_2)$ is also a quasi-hereditary algebra since the definition of quasi-hereditary algebra only depends on the set of standard modules. In this case, we can give various characterisations of this equivalence relation. Recall that for a quasi-hereditary algebra (Λ, \triangleleft) there is a tilting module T such that $\text{add } T = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$, called the characteristic tilting module (cf. Proposition 2.2.30).

Lemma 2.4.5. *Let \triangleleft_1 and \triangleleft_2 be two partial orders on I such that $(\Lambda, \triangleleft_1)$ and $(\Lambda, \triangleleft_2)$ are quasi-hereditary algebras. Then the following statements are equivalent.*

- (a) $\triangleleft_1 \sim \triangleleft_2$.
- (b) $\Delta_1 = \Delta_2$.
- (c) $\nabla_1 = \nabla_2$.
- (d) $\mathcal{F}(\Delta_1) = \mathcal{F}(\Delta_2)$.
- (e) $\mathcal{F}(\nabla_1) = \mathcal{F}(\nabla_2)$.
- (f) $T_1 \cong T_2$ where T_i is the characteristic tilting module of $(\Lambda, \triangleleft_i)$ for $i = 1, 2$.

Proof. We show (d) implies (b). For each $i \in I$, let $K(i)$ be the sum of the kernels of non-zero surjective maps $P(i) \rightarrow X$ with $X \in \mathcal{F}(\Delta_1)$. Then $\Delta_1(i) = P(i)/K(i)$ holds by the proof of [Rin91, Corollary 4]. We have $\Delta_1(i) = P(i)/K(i) = \Delta_2(i)$ by the assumption. Dually, (e) implies (c). The remaining equivalences follow from Propositions 2.2.20 and 2.2.30. \square

Definition 2.4.6. Let Λ be an Artin algebra with an adapted poset (I, \triangleleft) indexing the isomorphism classes of simple Λ -modules.

- (a) The equivalence class of \triangleleft under \sim is called a *quasi-hereditary structure* of Λ if $(\Lambda, (I, \triangleleft))$ is a quasi-hereditary algebra. We denote by $[\triangleleft]$ the equivalence class of \triangleleft under \sim .
- (b) The order \triangleleft is called *minimal adapted* if it represents a quasi-hereditary structure and it is minimal among partial orders which represent the same quasi-hereditary structure (cf. Lemma 2.4.4 (b)).

We denote by $\text{qh.str}(\Lambda)$ the set of quasi-hereditary structures on Λ .

Note that the number of quasi-hereditary structures is bounded by $(\text{card } I)!$. Compare with Example 2.4.7 (b).

Example 2.4.7. (a) Let $A = \bigoplus_{i \in I} A_i$ be a semisimple algebra. Then it is clear that any partial order on I is adapted to A , and $\Delta_{\triangleleft}(i) \cong A_i$ for all $i \in I$ and any order (I, \triangleleft) . Thus A has only one quasi-hereditary structure.

(b) Let \mathbb{K} be a field. Let $n \in \mathbb{N}_+$ and Q_{n+1} be a quiver with set of vertices $I = \{1, 2, \dots, n+1\}$ and such that there is a unique arrow from i to j whenever $i > j$. Observe that the underlying graph of Q_{n+1} is a complete graph. Let $K_{n+1} = \mathbb{K}Q_{n+1}$. It follows that any adapted order on I for K_{n+1} is a total order, and two distinct total orders on I induce different quasi-hereditary structures on K_{n+1} . Therefore the number of quasi-hereditary structures of K_{n+1} is $n!$.

There is a poset structure defined on the set of basic tilting modules up to isomorphism given as follows. Let T_1 and T_2 be tilting modules in $\text{mod } \Lambda$, then $T_1 \leq T_2$ if and only if $T_1^\perp \subseteq T_2^\perp$. Happel and Unger [HU05] described the Hasse quiver of this poset in terms of a graph defined previously in work by Riedtmann and Schofield [RS91].

By Lemma 2.4.5, the equivalence class of \triangleleft only depends on its characteristic tilting module. It is then natural to order quasi-hereditary structures of a given quasi-hereditary algebra Λ in the following way.

Definition 2.4.8. Let $[\triangleleft_1]$ and $[\triangleleft_2]$ be two quasi-hereditary structures of Λ with respective sets of standard modules Δ_1 and Δ_2 . We set $[\triangleleft_1] \preceq [\triangleleft_2]$ if $\mathcal{F}(\Delta_2) \subseteq \mathcal{F}(\Delta_1)$. It follows that we have a poset $\text{qh.str}(\Lambda) = (\text{qh.str}(\Lambda), \preceq)$ called the *poset of quasi-hereditary structures of Λ* .

We have the following equivalent definitions of the relation \preceq .

Lemma 2.4.9. For $i = 1, 2$, let $[\triangleleft_i]$ be a quasi-hereditary structure on Λ , with $\nabla_i = \nabla_{\triangleleft_i}$, and T_i a characteristic tilting module of $(\Lambda, \triangleleft_i)$. Then the following statements are equivalent.

- (a) $[\triangleleft_1] \preceq [\triangleleft_2]$.
- (b) $\mathcal{F}(\nabla_1) \subseteq \mathcal{F}(\nabla_2)$.
- (c) $T_1^\perp \subseteq T_2^\perp$.

Proof. Follows from Propositions 2.2.22 and 2.2.32. □

We finish this section with a result by Coulembier which provides a sufficient condition for a finite dimensional algebra A to have a unique quasi-hereditary structure. Let $\{S(i)\}_{i \in I}$ be a complete set of non-isomorphic simple A -modules. In this setting, we say that A has a *duality fixing the simples* if there is an involutive contravariant autoequivalence \mathcal{D} of $\text{mod } A$ that induces the identity on I .

Theorem 2.4.10 ([Cou19, Theorem 2.1.1]). *Let A be a finite dimensional algebra with a duality fixing the simples. Let (I, \triangleleft_1) and (I, \triangleleft_2) be two partial orders indexing the simple modules of A . If (A, \triangleleft_1) and (A, \triangleleft_2) are two quasi-hereditary algebras, then $\triangleleft_1 \sim \triangleleft_2$.*

Chapter 3

The Auslander algebra of $\mathbb{K}[x]/(x^n)$

Let \mathbb{K} be a field. In this chapter we determine the homological poset and homological Hasse quiver of a class of Auslander algebras, more precisely over the Auslander algebra of the truncated polynomial ring $T_n := \mathbb{K}[x]/(x^n)$, for $n \geq 0$. This is done by means of a combinatorial classification of all homological embeddings between Serre subcategories. The crucial point in this characterisation is to determine block decompositions of factor algebras by idempotent ideals, and use the Chinese remainder theorem. Along the process we encounter some interesting integer sequences.

Recall that an Artin algebra Λ is called an *Auslander algebra* if its global dimension is less than or equal to 2 and the dominant dimension of Λ is greater than or equal to 2. Auslander showed that there is a one-to-one correspondence between algebras of finite representation type and Auslander algebras, given by $A \mapsto (\text{End}_A(M))^{op}$, where M is an additive generator of A [Aus74; ARS95]. Usually one takes M to be the direct sum of all indecomposable A -modules up to isomorphism. In this thesis we denote by $\text{Aus } A$ the Auslander algebra of a finite representation type algebra A .

The organisation of the chapter is as follows. We define $\text{Aus } T_n$ in Section 3.1 as a bound quiver algebra and give some properties. In Section 3.2 we find an adequate basis of $\text{Aus } T_n$ that is used in Section 3.3 to define a basis of the indecomposable projective modules over $\text{Aus } T_n/(e)$, for e an idempotent. In Section 3.4 we encounter a sufficient condition concerning idempotent ideals of a \mathbb{K} -algebra A that guarantees block decomposition of the factors $A/(e)$, using a version of the Chinese remainder theorem. We apply it to the case of $\text{Aus } T_n$. In Section 3.5 we introduce the notion of preprojective algebras of type \mathbb{A}_n , and describe an appropriate basis. In Section 3.6 we characterise blocks of the factor algebras $\text{Aus } T_n/(e)$, and find the unique heredity chain of $\text{Aus } T_n$. Section 3.7 is devoted to showing a combinatorial characterisation of homological embeddings between Serre subcategories of $\text{mod } \text{Aus } T_n$, that we use in Section 3.8 to describe explicitly the cover relations of the homological poset of $\text{Aus } T_n$, denoted $\mathcal{H}(\text{Aus } T_n)$, obtaining some counting formulas. In Section 3.9 the homological Hasse quiver of $\text{Aus } T_n$ is described, where some Tribonacci sequences arise in the context of counting methods. In Section 3.10 we define compositions of a positive integer and show a bijection between the cover relations of $\mathcal{H}(\text{Aus } T_n)$ and the set of parts of n via tilings. Finally, Section 3.11 studies the factor algebras $\text{Aus } T_n/(e)$ that are quasi-hereditary algebras,

as consequence some Fibonacci sequences appear in our investigations.

3.1 Basic properties of $\text{Aus } T_n$

Let \mathbb{K} be a field. For $n \geq 1$, set $T_n = \mathbb{K}[x]/(x^n)$ the algebra of truncated polynomials of degree less than n with coefficients over \mathbb{K} . It is clear that T_n is isomorphic to the bound quiver algebra

$$\mathbb{K}(\overset{1}{\bullet} \rightrightarrows \alpha) / (\alpha^n),$$

with isomorphism given by $1 \mapsto \varepsilon_1$ and $x \mapsto \alpha$.

It turns out that $\dim_{\mathbb{K}} T_n = n$, $\text{gl. dim } T_n = \infty$ and T_n is uniserial with unique composition series $T_n \supseteq \text{rad } T_n \supseteq \cdots \supseteq \text{rad}^{n-1} T_n \supseteq \text{rad}^n T_n = 0$, where $\text{rad}^i T_n = \mathbb{K}[x]/(x^{n-i})$, for $i \in \{1, \dots, n\}$. The Auslander-Reiten quiver of T_n is

$$P(1) \begin{array}{c} \overset{\circlearrowleft}{\downarrow} \\ \rightleftarrows \end{array} \text{rad } P(1) \begin{array}{c} \rightleftarrows \\ \overset{\circlearrowright}{\downarrow} \end{array} \cdots \begin{array}{c} \rightleftarrows \\ \overset{\circlearrowright}{\downarrow} \end{array} \text{rad}^{n-1} P(1) = S(1),$$

thus T_n has finite representation type. So, the Auslander algebra of T_n is defined by

$$\Lambda_n = \text{Aus } T_n := \text{End}_{T_n} \left(\bigoplus_{i=0}^{n-1} \text{rad}^i P(1) \right).$$

It turns out that Λ_n is isomorphic to the path algebra $\mathbb{K}Q/\mathbb{I}$, where

$$Q = \overset{1}{\bullet} \begin{array}{c} \xrightarrow{a_1} \\ \xleftarrow{b_1} \end{array} \overset{2}{\bullet} \begin{array}{c} \xrightarrow{a_2} \\ \xleftarrow{b_2} \end{array} \cdots \begin{array}{c} \xrightarrow{a_{n-2}} \\ \xleftarrow{b_{n-2}} \end{array} \overset{n-1}{\bullet} \begin{array}{c} \xrightarrow{a_{n-1}} \\ \xleftarrow{b_{n-1}} \end{array} \overset{n}{\bullet}$$

and \mathbb{I} is the ideal generated by the relations $a_i b_i - b_{i+1} a_{i+1}$ for $1 \leq i \leq n-2$ and $a_{n-1} b_{n-1}$. In this case, $Q_0 = \{1, \dots, n\} =: [n]$. Set $\Lambda_0 := 0$.

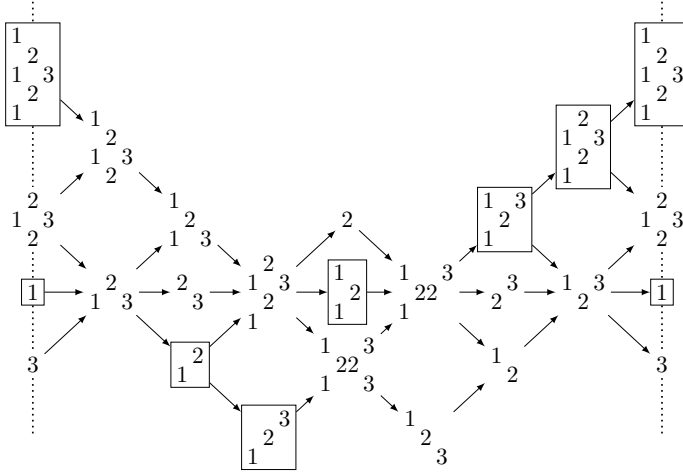
In the next proposition we gather some known properties of the algebras Λ_n .

Proposition 3.1.1. *Let $n \geq 1$. Then the following conditions hold.*

- (a) Λ_n has a unique quasi-hereditary structure given by $(\{1, \dots, n\}, \leq)$, where \leq is the usual order.
- (b) Let $M \in \text{mod } \Lambda_n$, and $\Delta = \Delta_{\leq}$. Then the following conditions are equivalent.
 - (i) $M \in \mathcal{F}(\Delta)$.
 - (ii) $\text{pd } M \leq 1$.
 - (iii) M is torsionless.
 - (iv) $\text{Ext}^1(M, T) = 0$, where T is the characteristic tilting module.
 - (v) The injective envelope of M is projective.
- (c) Λ_n is of finite representation type only if $n \leq 3$.

Proof. For (a) and (c) see [DR92, Sec. 7]. For (b) we refer to [RZ14, Proposition 1]. \square

Example 3.1.2. Consider $n = 3$. The Auslander-Reiten quiver of Λ_3 is the following.



The quiver lies on a cylinder, thus the dotted lines must be identified. The modules belonging to $\mathcal{F}(\Delta)$ are marked with a rectangle. In particular we have the following Λ_3 -modules.

$$\Delta(1) = 1, \quad \Delta(2) = \begin{matrix} 1 & 2 \\ & 1 & 2 \\ & & 1 & 2 & 3 \\ & & & 1 & 2 & 3 \\ & & & & 1 & 2 & 3 \end{matrix}, \quad \Delta(3) = \begin{matrix} 1 & 2 & 3 \\ & 1 & 2 & 3 \\ & & 1 & 2 & 3 \\ & & & 1 & 2 & 3 \\ & & & & 1 & 2 & 3 \end{matrix}, \quad \nabla(1) = 1, \quad \nabla(2) = \begin{matrix} 1 & 2 \\ & 1 & 2 \\ & & 1 & 2 & 3 \\ & & & 1 & 2 & 3 \\ & & & & 1 & 2 & 3 \end{matrix}, \quad \nabla(3) = \begin{matrix} 1 & 2 & 3 \\ & 1 & 2 & 3 \\ & & 1 & 2 & 3 \\ & & & 1 & 2 & 3 \\ & & & & 1 & 2 & 3 \end{matrix}$$

and the characteristic tilting module has the following indecomposable direct summands.

$$T(1) = 1, \quad T(2) = \begin{matrix} 1 & 2 \\ & 1 & 2 \\ & & 1 & 2 & 3 \\ & & & 1 & 2 & 3 \\ & & & & 1 & 2 & 3 \end{matrix}, \quad T(3) = \begin{matrix} 1 & 2 & 3 \\ & 1 & 2 & 3 \\ & & 1 & 2 & 3 \\ & & & 1 & 2 & 3 \\ & & & & 1 & 2 & 3 \end{matrix}$$

3.2 A basis of $\text{Aus } T_n$

The aim of this section is to find a good basis for Λ_n by means of the description of the indecomposable projective Λ_n -modules. The main goal is to find an explicit description of the block of the factor algebras $\Lambda_n/(e)$, for any idempotent e of Λ_n . For example, the projective Λ_4 -modules have the following radical filtrations.

$$P(1) = \begin{matrix} 1 & 2 & 3 & 4 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 2 & 3 & 4 \\ & & & 1 & 2 & 3 & 4 \\ & & & & 1 & 2 & 3 & 4 \\ & & & & & 1 & 2 & 3 & 4 \\ & & & & & & 1 & 2 & 3 & 4 \\ & & & & & & & 1 & 2 & 3 & 4 \end{matrix} \quad P(2) = \begin{matrix} 1 & 2 & 3 & 4 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 2 & 3 & 4 \\ & & & 1 & 2 & 3 & 4 \\ & & & & 1 & 2 & 3 & 4 \\ & & & & & 1 & 2 & 3 & 4 \\ & & & & & & 1 & 2 & 3 & 4 \\ & & & & & & & 1 & 2 & 3 & 4 \end{matrix} \quad P(3) = \begin{matrix} 1 & 2 & 3 & 4 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 2 & 3 & 4 \\ & & & 1 & 2 & 3 & 4 \\ & & & & 1 & 2 & 3 & 4 \\ & & & & & 1 & 2 & 3 & 4 \\ & & & & & & 1 & 2 & 3 & 4 \\ & & & & & & & 1 & 2 & 3 & 4 \end{matrix} \quad P(4) = \begin{matrix} 1 & 2 & 3 & 4 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 2 & 3 & 4 \\ & & & 1 & 2 & 3 & 4 \\ & & & & 1 & 2 & 3 & 4 \\ & & & & & 1 & 2 & 3 & 4 \\ & & & & & & 1 & 2 & 3 & 4 \\ & & & & & & & 1 & 2 & 3 & 4 \end{matrix}$$

Recall that each number j represents a basis element of the \mathbb{K} -vector space $e_j P(i)$, that is a path in Q from i to j . Thus each column in $P(i)$ contains all the paths in Λ_4 starting at i and terminating at j . Therefore, the dimension of $e_j P(i)$ equals the number of elements of the j -column. We will prove that the basis elements of each $P(i)$ only depend on their length and on the target vertex (cf. Theorem 3.2.8). In order to justify that we have the above radical filtrations, we first define the following sets of paths in Q .

Definition 3.2.1. Let $k, j, i \in Q_0$. We denote by (k, j, i) the shortest path in Q from i to k of the form

$$i \longrightarrow \dots \longrightarrow j \longrightarrow \dots \longrightarrow k.$$

For fixed $i \in Q_0$, let $\mathcal{P}_i := \{(k, j, i) \mid k, j \in Q_0\}$.

Notation. If ω is a path in Q , we will denote by $[\omega]$ its corresponding class in Λ , and by $\ell(\omega)$ its length.

Lemma 3.2.2. *The length of (k, j, i) is $|j - k| + |j - i|$.*

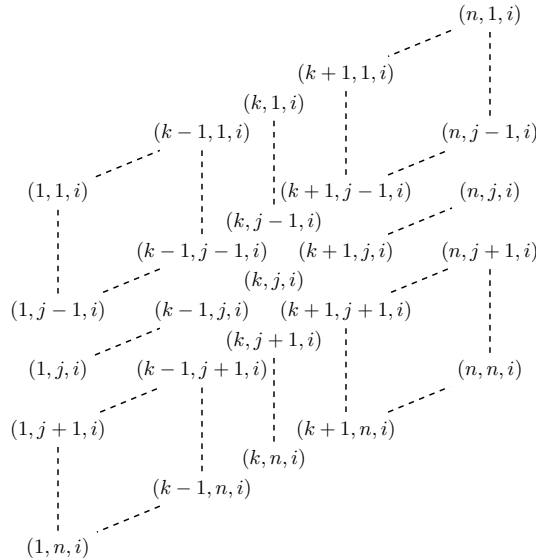
Proof. Straightforward. □

For example,

$$(3, 1, 2) = a_2 a_1 b_1 = 2 \xrightarrow{b_1} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \quad \text{and} \quad \ell(3, 1, 2) = 3.$$

A priori, all the basis elements of Λ lay in $\bigcup_{i \in Q_0} \mathcal{P}_i$, because their length is bounded by $2n - 1$. Indeed, by Corollary 3.2.11 the path $(1, n, 1)$ has maximum length $2n - 2$ in Λ .

Thus, for fixed $i \in [n]$, it is natural to consider the following array of the paths (k, j, i) , where the column k contains the paths starting at i and ending at k . To be more precise, in the row j of the column k lays the element (k, j, i) .



The following result characterizes all the paths in $\bigcup_{i \in Q_0} \mathcal{P}_i$.

Lemma 3.2.3. *Let $k, j, i \in Q_0$.*

- (a) $(i, i, i) = \varepsilon_i$ is the trivial path at vertex i .
- (b) If $k \leq j \leq i$ with $i \neq k$, then $(k, j, i) = b_k \cdots b_{i-1}$.

(c) For $1 \leq k \leq n-1$ and $\max\{k+1, i+1\} \leq j \leq n$, we have

$$(k, j, i) = b_k \cdots b_{j-1} a_{j-1} \cdots a_i = \begin{array}{c} i \\ \bullet \end{array} \xrightarrow{a_i} \cdots \xrightarrow{a_{j-1}} \begin{array}{c} j \\ \bullet \end{array} \xrightarrow{b_{j-1}} \cdots \xrightarrow{b_k} \begin{array}{c} k \\ \bullet \end{array}.$$

(d) If $k \geq j \geq i$ with $i \neq k$, then $(k, j, i) = a_{k-1} \cdots a_i$.

(e) For $2 \leq k, i$ and $1 \leq j \leq \min\{k-1, i-1\}$, we have

$$(k, j, i) = a_{k-1} \cdots a_j b_j \cdots b_{i-1} = \begin{array}{c} i \\ \bullet \end{array} \xrightarrow{b_{i-1}} \cdots \xrightarrow{b_j} \begin{array}{c} j \\ \bullet \end{array} \xrightarrow{a_j} \cdots \xrightarrow{a_{k-1}} \begin{array}{c} k \\ \bullet \end{array}.$$

(f) $(k, k, i) = (k, i, i)$.

Proof. (a) Straightforward.

(b) and (d) follow from the fact that if $1 \leq i < j \leq n$, then the shortest path in Q from i to j is

$$\begin{array}{c} i \\ \bullet \end{array} \xrightarrow{a_i} \begin{array}{c} i+1 \\ \bullet \end{array} \xrightarrow{a_{i+1}} \cdots \xrightarrow{a_{j-1}} \begin{array}{c} j \\ \bullet \end{array},$$

and the shortest path from j to i in Q is

$$\begin{array}{c} i \\ \bullet \end{array} \xleftarrow{b_i} \begin{array}{c} i+1 \\ \bullet \end{array} \xleftarrow{b_{i+1}} \cdots \xleftarrow{b_{j-1}} \begin{array}{c} j \\ \bullet \end{array}.$$

The conditions in (c) imply that $i < j > k$, and the conditions in (e) imply that $1 > j < k$. Then in both cases apply (b) and (d), from where (c) and (e) follow. Finally, (f) shows two different labellings for the same path. \square

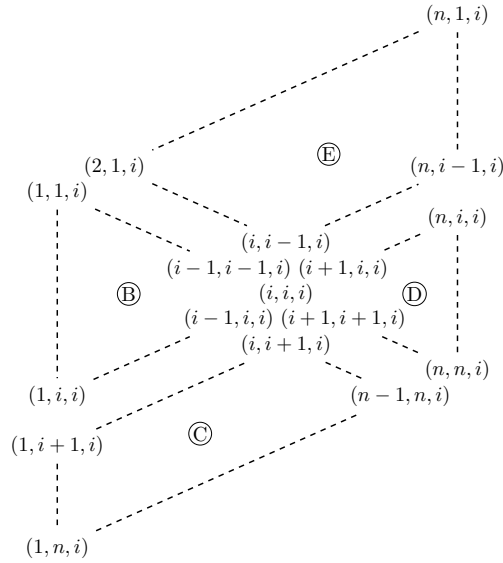
Now we define the following sets.

$$\begin{aligned} \textcircled{\text{A}} &= \{(i, i, i)\} \\ \textcircled{\text{B}} &= \{(k, j, i) \mid k \leq j \leq i, k \neq i\} \\ \textcircled{\text{C}} &= \{(k, j, i) \mid k \neq n, \max\{k+1, i+1\} \leq j \leq n\} \\ \textcircled{\text{D}} &= \{(k, j, i) \mid k \geq j \geq i, k \neq i\} \\ \textcircled{\text{E}} &= \{(k, j, i) \mid k \neq 1, 1 \leq j \leq \min\{k-1, i-1\}\}. \end{aligned}$$

Note that each set corresponds to the elements described in Lemma 3.2.3. It is easy to see that those sets are pairwise disjoint. Thus, we get a partition of \mathcal{P}_i for each $i \in Q_0$

$$\mathcal{P}_i = \textcircled{\text{A}} \sqcup \textcircled{\text{B}} \sqcup \textcircled{\text{C}} \sqcup \textcircled{\text{D}} \sqcup \textcircled{\text{E}} \quad (3.2.1)$$

where $\textcircled{\text{B}} = \textcircled{\text{E}} = \emptyset$ for $i = 1$, and $\textcircled{\text{C}} = \textcircled{\text{D}} = \emptyset$ for $i = n$. The following diagram describes this partition. Recall that all those elements lie in $\mathbb{K}Q$.



Proposition 3.2.4. *Let ω a path from i to j in Q . Then $\ell(\omega) \equiv |j - i| \pmod{2}$.*

Proof. Induction on $\ell(\omega) = l$. If $l = 0$, then $j = i$, i.e. $\omega = \varepsilon_i$, thus $|j - i| = 0 = \ell(\omega)$.

Now let $l > 0$ and $\alpha \in Q_1$ starting at j , so $e(\alpha) = j \pm 1$. Then $\ell(\alpha\omega) = l + 1$, and by induction $\ell(\omega) \equiv |j - i| \pmod{2}$. Therefore $\ell(\alpha\omega) = 1 + \ell(\omega) \equiv 1 + |j - i| \pmod{2}$. Set $m := j - i$, so we have to prove that $1 + |m| \equiv |e(\alpha) - i| \pmod{2}$. If $m \geq 1$ and $e(\alpha) = j + 1$, then $1 + |m| - |m + 1| = 0$. The other cases are similar. This completes the induction. \square

Corollary 3.2.5. *Let $i, j \in Q_0$. Then i and j have the same parity if and only if any path from i to j has even length.*

Proof. Follows directly from Proposition 3.2.4. \square

Remark 3.2.6. Let ω be a path in Q of length $l > 2n - i - j$. By Corollary 3.2.5, if i, j have the same parity, then $2n - i - j$ and l are even. Therefore, the least possible value for l is $2n - i - j + 2$. We conclude the same if i, j have different parity.

For example,

$$\eta_j := \begin{cases} (j, j+1, j)(j, n, i) = b_j a_j b_j \cdots b_{n-2} b_{n-1} a_{n-1} \cdots a_i & \text{if } j \neq n \\ (n, n-1, n)(n, n, i) = a_{n-1} b_{n-1} a_{n-1} \cdots a_i & \text{if } j = n \end{cases}$$

is a path of length $\ell(j, j+1, j) + \ell(j, n, i) = 2n - i - j + 2$.

Lemma 3.2.7. *Let $1 \leq j \leq n - 1$ and $s, t \geq 0$ such that $j + t + s \leq n - 2$. Then*

$$[a_{j+s} \cdots a_j b_j \cdots b_{j+t}] = [b_{j+s+1} \cdots b_{j+t+s+1} a_{j+t+s+1} \cdots a_{j+t+1}],$$

i.e. $[j + s + 1, j, j + t + 1] = [j + s + 1, j + s + t + 2, j + t + 1]$.

Proof. Let j be fixed. We proceed by double induction on (s, t) . The case $(s, t) = (0, 0)$ follows from the definition of Λ . Suppose that the result is true for $t = 0$ and $s \geq 0$, then

$$\begin{aligned} [a_{j+s+1}(a_{j+s} \cdots a_j b_j)] &= [a_{j+s+1}(b_{j+s+1} a_{j+s+1} \cdots a_{j+1})] \\ &= [(b_{j+s+2} a_{j+s+2}) a_{j+s+1} \cdots a_{j+1}], \end{aligned}$$

i.e. the result holds for $(0, s + 1)$. Analogously the result holds for $(0, t + 1)$.

Now suppose that the proposition is true for $(s, t + 1)$, so by induction we have

$$\begin{aligned} [a_{j+s+1}(a_{j+s} \cdots a_j b_j \cdots b_{j+t+1})] &= [a_{j+s+1}(b_{j+s+1} \cdots b_{j+s+t+2} a_{j+s+t+2} \cdots a_{j+t+2})] \\ &= [(b_{j+s+2} \cdots b_{j+s+t+3} a_{j+s+t+3}) a_{j+s+t+2} \cdots a_{j+t+2}] \end{aligned}$$

where the last equality follows by applying the change of variable $j' = j + s + 1$ and considering the base case $(0, t + 1)$, meaning

$$[a_{j'} b_{j'} \cdots b_{j'+t+1}] = [b_{j'+1} \cdots b_{j'+t+2} a_{j'+t+2}],$$

which shows the result in general. \square

Theorem 3.2.8. *Let ω, ω' be two paths in Q from i to j with $\ell(\omega) = \ell(\omega')$. Then $[\omega] = [\omega']$.*

Proof. By Lemma 3.2.7 any expression of ω has the same number of a_j 's and b_j 's. And since ω' is another path from i to j with the same length as ω , it has the same number of a_j 's and b_j 's as ω . Thus $[\omega] = [\omega']$. \square

Theorem 3.2.8 shows a complete classification of paths in Q by their length, that is, any path in Q depends only on the starting and ending vertices and its length. Moreover, for $i, j \in Q_0$ fixed, Proposition 3.2.4 implies that all the possible lengths for a path ω from i to j are of the form $|j - i| + 2l$, for $l \geq 0$.

Corollary 3.2.9. *Let ω be a path from i to j in Q and $l = \ell(\omega)$. Then $l > 2n - i - j$ if and only if $[\omega] = [0]$.*

Proof. (\Rightarrow) Let η_j be the path defined in Remark 3.2.6. It has length $2n - i - j + 2$ and is the least possible value for $l > 2n - i - j$. By Lemma 3.2.7, if $j \neq n$ then

$$[\eta_j] = [b_j(b_{j+1} \cdots b_{n-1} a_{n-1}) b_{n-1} a_{n-1} \cdots a_i] = [0],$$

and clearly $[\eta_n] = [0]$. In general, if ω has length $l = 2n - i - j + 2k$ for $k > 1$, Theorem 3.2.8 shows that for $j \neq n$

$$[\omega] = [(b_j a_j)^{k-1} \eta_j],$$

and $[\omega] = [(a_{n-1} b_{n-1})^{k-1} \eta_n]$ if $j = n$. Thus $[\omega] = [0]$.

(\Leftarrow) Suppose $l \leq 2n - i - j$ and $[\omega] = [0]$. Then $\omega \in \mathbb{I}$, and since ω is a monomial we conclude that

$$\omega = \gamma a_{n-1} b_{n-1} \delta$$

for some paths γ, δ starting at n, i and ending at j, n respectively. Thus $\ell(\gamma) \geq n - j$ and $\ell(\delta) \geq n - i$, so $l = \ell(\gamma) + 2 + \ell(\delta) \geq 2n - j - i + 2$, a contradiction. Therefore, $[\omega] \neq [0]$. \square

Following Corollary 3.2.9 we give the next definition.

Definition 3.2.10. For $i, j \in Q_0$, let $m(i, j) = 2n - i - j$. Thus $m(i, j)$ is the maximal length of a path from i to j in Λ .

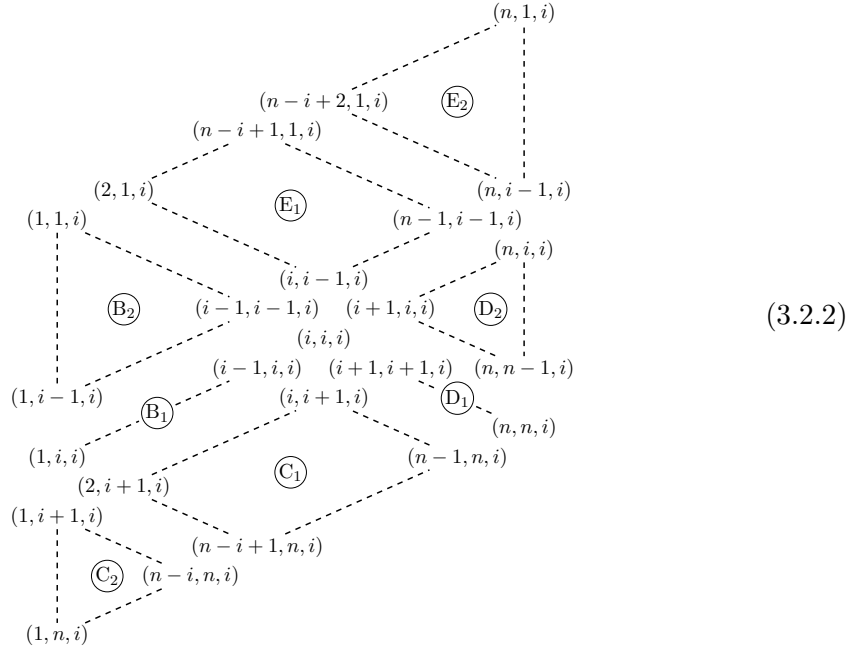
Corollary 3.2.11. $\text{rad}^{2n-1}(\Lambda) = 0$.

Proof. We have that $m(1, 1) = 2n - 2 \geq m(i, j)$ for any $i, j \in Q_0$. Thus by Corollary 3.2.9, any path of length $\geq 2n - 1$ is zero in Λ . \square

Now we describe the radical filtration of the indecomposable Λ -projectives. Let $i \in Q_0$. By Corollary 3.2.9 it suffices to consider only paths (k, j, i) in \mathcal{P}_i because

$$\ell(k, j, i) = |j - k| + |j - i| \leq 2(n - 1) = m(1, 1).$$

Lets analyse the paths described in the partition (3.2.1) of \mathcal{P}_i . By Theorem 3.2.8 we can identify certain labels in Λ , because they have the same length, and same source and target. For that reason we consider the following refinement of the partition of \mathcal{P}_i given in Eq. (3.2.1).



For example, for $n = 7$ and $i = 4$ we have the following diagram.

$$\begin{array}{cccccccc}
& & & & & & & (7, 1, 4) \\
& & & & & & (6, 1, 4) & \vdots \\
& & & & & (5, 1, 4) & (6, 2, 4) & (7, 2, 4) \\
& & & & (4, 1, 4) & (5, 2, 4) & (6, 3, 4) & \vdots \\
& & (3, 1, 4) & & (4, 2, 4) & (5, 3, 4) & (6, 4, 4) & (7, 3, 4) \\
(1, 1, 4) & (2, 1, 4) & (3, 2, 4) & (4, 3, 4) & (5, 4, 4) & (6, 5, 4) & (7, 6, 4) & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(1, 2, 4) & (2, 2, 4) & (3, 3, 4) & (4, 4, 4) & (5, 5, 4) & (6, 6, 4) & (7, 7, 4) & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(1, 3, 4) & (2, 3, 4) & (3, 4, 4) & (4, 5, 4) & (5, 6, 4) & (6, 7, 4) & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
(1, 4, 4) & (2, 4, 4) & (3, 5, 4) & (4, 6, 4) & (5, 7, 4) & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \\
(1, 5, 4) & (2, 5, 4) & (3, 6, 4) & (4, 7, 4) & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
(1, 6, 4) & (2, 6, 4) & (3, 7, 4) & & & & & \\
\vdots & \vdots & \vdots & & & & & \\
(1, 7, 4) & (2, 7, 4) & & & & & &
\end{array} \tag{3.2.3}$$

Now we provide a complete description of this refinement. Compare with (3.2.3) to have an example in mind. First, for $1 \leq k < i$, the k -column of \mathbb{B} is given by $\{(k, k + s, i) \mid s = 0, \dots, i - k\}$, moreover

$$\ell(k, k + s, i) = |k + s - k| + |k + s - i| = i - k,$$

thus elements in the same column in \mathbb{B} represent the same path in Λ . We take the paths of the form (k, i, i) as representatives in Λ , and call this set \mathbb{B}_1 .

For $i < k \leq n$, the k -column in \mathbb{D} is given by $\{(k, k - s, i) \mid s = 0, \dots, k - i\}$, and

$$\ell(k, k - s, i) = |k - s - k| + |k - s - i| = k - i,$$

thus elements in the same column of \mathbb{D} are labels for a single path in Λ . We take the paths of the form (k, k, i) as representatives in Λ , call this set \mathbb{D}_1 .

Let $i > 1$, and $2 \leq k \leq n - 1$, then the k -column of \mathbb{E}_1 is given by

$$(a) \ \{(k, j, i) \mid 1 \leq j \leq \min\{k - 1, i - 1\}\} \text{ if } k < n - i + 2,$$

$$(b) \ \{(k, j, i) \mid k - n + i \leq j \leq \min\{k - 1, i - 1\}\} \text{ if } k \geq n - i + 2,$$

and the k -column of \mathbb{C}_1 is given by

$$(a') \ \{(k, j, i) \mid \max\{i + 1, k + 1\} \leq j \leq k + i - 1\} \text{ if } k < n - i + 2,$$

$$(b') \ \{(k, j, i) \mid \max\{i + 1, k + 1\} \leq j \leq n\} \text{ if } k \geq n - i + 2.$$

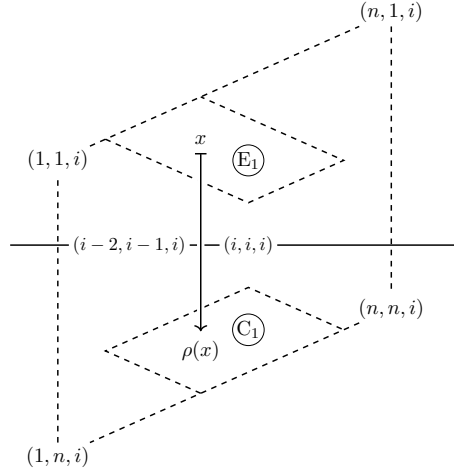
Thus, for (k, j, i) as in (a) or (b), we have that $(k, i + k - j, i)$ is a path as in (a') or (b') respectively, and vice versa. Moreover

$$\ell(k, j, i) = |j - k| + |j - i| = \ell(k, i + k - j, i),$$

therefore $[k, j, i] = [k, i + k - j, i]$. The correspondence

$$x = (k, j, i) \mapsto \rho(x) := (k, i + k - j, i)$$

is just the reflection over the line through $(i, i, i) - (i + 2, i + 1, i)$ or $(i, i, i) - (i - 2, i - 1, i)$ between the k -columns of (\mathbb{E}_1) and (\mathbb{C}_1) . We choose as representatives in Λ the paths in (\mathbb{C}_1) . The following diagram describes the reflection ρ .



Let $i > 1$ and $n - i + 2 \leq k \leq n$, thus $k = n - i + 2 + s$ for $s = 0, \dots, i - 2$. Then the k -column of (\mathbb{E}_2) is given by $\{(n - i + 2 + s, j, i) \mid 1 \leq j \leq 1 + s\}$. Since

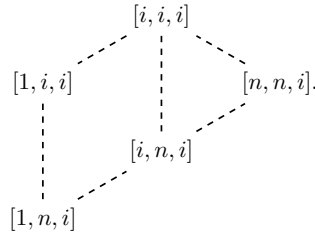
$$\ell(n - i + 2 + s, 1 + s, i) = n - s > n - s - 2 = m(n - i + 2 + s, i)$$

we conclude that the paths in (\mathbb{E}_2) are zero in Λ .

Finally, the paths in (\mathbb{C}_2) are those who have a unique representative of the form (k, j, i) , for $i < n$. Identifying all the paths as above, we get a description of the radical filtration of the indecomposable projectives Λ -modules $P(i)$, for $i \in [n]$.

Notation. Let $i \in Q_0$. Define $e_i = [\varepsilon_i]$. Thus $\{e_1, \dots, e_n\}$ is a complete set of orthogonal primitive pairwise non-isomorphic idempotents of Λ with the natural ordering.

Proposition 3.2.12. *Let $i \in Q_0$. Then a basis for ${}_{\Lambda}P(i) = \Lambda e_i$ is given by the elements*



Therefore, the column k of the radical filtration of $P(i)$ is given by

$$\begin{array}{l} [k, 1, i] \\ [k, 2, i] \\ [k, 3, i] \\ \vdots \\ [k, n, i] \end{array} \quad \text{if } k \leq i, \text{ and by} \quad \begin{array}{l} [k, k, i] \\ [k, k + 1, i] \\ \vdots \\ [k, n, i] \end{array} \quad \text{if } k > i,$$

i.e. $P(i)$ has basis $\mathcal{B}_i := \{[k, j, i] \mid 1 \leq k \leq n, \max\{i, k\} \leq j \leq n\}$. Moreover, $\dim_{\mathbb{K}} P(i) = \frac{(n-i+1)(n+i)}{2}$.

Proof. It follows identifying the paths of diagram (3.2.2) as in the previous discussions. \square

Note that for $2 \leq i \leq n$, we have inclusions

$$P(i) \hookrightarrow P(i-1)$$

given by $\omega = [k, j, i] \mapsto [\omega a_{i-1}] = [k, j, i-1]$. Thus, identifying the images of these inclusions into $P(1)$, we have

$$P(n) \subseteq P(n-1) \subseteq \cdots \subseteq P(1).$$

The following technical result will be useful later.

Lemma 3.2.13. *If $[k, j, i] \in \mathcal{B}_i$ and $0 \leq s \leq n-j$, then $[k, j+s, i]$ factors through $[k, j, i]$.*

Proof. By Theorem 3.2.8 we have

$$[k, j+s, i] = [(b_k a_k)^s] [k, j, i]$$

because both elements have the same length $2s + 2j - k - i$. \square

As a direct consequence of Proposition 3.2.12 we obtain a basis for Λ_n .

Theorem 3.2.14. *Let $n \geq 1$. Then a \mathbb{K} -basis for Λ_n is given by*

$$\mathcal{B}_{\Lambda_n} = \{[k, j, i] \mid 1 \leq k, i \leq n, \max\{i, k\} \leq j \leq n\}.$$

The multiplication between basis elements is given by

$$[k', j', i'] [k, j, i] = \begin{cases} [k', j' + j - k, i] & \text{if } i' = k \text{ and } j' + j - k \leq n \\ 0 & \text{else.} \end{cases}$$

Moreover, $\dim_{\mathbb{K}} \text{Aus } T_n = \frac{n(n+1)(2n+1)}{6}$.

Proof. Suppose that $j' + j - k \leq n$, then $\ell([k', j', k][k, j, i]) = 2(j' + j - k) - k' - i = \ell([k', j' + j - k, i])$, thus $[k', j', k][k, j, i] = [k', j' + j - k, i]$. \square

3.3 A basis of $\text{Aus } T_n/(e)$

Let $\Lambda_n = \text{Aus } T_n$. Recall that we write $(e) := \Lambda_n e \Lambda_n$, for an idempotent e of Λ_n . In this section we describe a basis of $\Lambda_n/(e)$ using the basis of Λ_n constructed in Section 3.2. For the rest of the chapter, let $\{e_i\}_{i=1}^n$ be a complete set of primitive orthogonal idempotents of Λ_n given by the paths of length zero. We start giving the following notation.

Notation. Let $n \geq 1$. We denote the basis elements of Λ_n by $[k, j, i]_n$, and we will omit the subscript if no confusion can arise.

A first and very important instance is the quotient $\Lambda_n/(e_n)$.

Lemma 3.3.1. *Let $n \geq 1$, and $f_n: \Lambda_n \rightarrow \Lambda_{n-1}$ be given by*

$$f_n([k, j, i]_n) := \begin{cases} [k, j, i]_{n-1} & \text{if } 1 \leq k, i, \leq n-1 \text{ and } j \leq n-1 \\ 0 & \text{else.} \end{cases}$$

Then f_n is a well defined \mathbb{K} -algebra surjection, with $\text{Ker}(f_n) = (e_n)$. Thus, f_n induces an algebra isomorphism $\overline{f_n}: \Lambda_n/(e_n) \xrightarrow{\sim} \Lambda_{n-1}$.

Proof. The function f_n is well defined by Theorem 3.2.14, and is a ring homomorphism: let $x = [k', j', k]_n$ and $y = [k, j, i]_n$ be in \mathcal{B}_{Λ_n} , such that $xy \neq 0$, thus $j' + j - k \leq n$ and $xy = [k', j' + j - k, i]_n$. We have two cases, if $j' + j - k \leq n-1$, then necessarily $j, j' \leq n-1$, thus $f_n(xy) = [k', j' + j - k, i]_{n-1} = [k', j', k]_{n-1}[k, j, i]_{n-1} = f_n(x)f_n(y)$. The second case is when $j' + j - k \geq n$, then $f_n(xy) = 0$. If $j = n$ or $j' = n$, the $f_n(x)f_n(y) = 0$, so we can assume that $j, j' \leq n-1$. Then $f_n(x)f_n(y) = [k', j', k]_{n-1}[k, j, i]_{n-1} = 0$, since $j' + j - k \geq n-1$.

Now let's show that $\text{Ker}(f_n) = (e_n)$. For, let $x = [k, j, i] \in \text{Ker}(f_n)$, then necessarily $j = n$, thus $x = (k, n, n)e_n(n, n, i) \in (e_n)$. Finally, $f_n(e_n) = f_n([n, n, n]_n) = 0$, therefore $(e_n) \subseteq \text{Ker}(f_n)$, and the equality holds. \square

We will focus on the quotient modules $P(i)/(e)e_i = \Lambda e_i/\Lambda e \Lambda e_i$ for $i \in Q_0$, because we have an isomorphism of \mathbb{K} -modules

$$\frac{\Lambda}{(e)} = \frac{\Lambda e_1 \oplus \cdots \oplus \Lambda e_n}{(e)} \cong \frac{\Lambda e_1}{(e)e_1} \times \cdots \times \frac{\Lambda e_n}{(e)e_n}.$$

Indeed, let

$$f: \Lambda e_1 \oplus \cdots \oplus \Lambda e_n \rightarrow \frac{\Lambda e_1}{(e)e_1} \times \cdots \times \frac{\Lambda e_n}{(e)e_n},$$

be given by

$$f(\lambda_1 e_1 + \cdots + \lambda_n e_n) = (\lambda_1 e_1 + (e)e_1, \dots, \lambda_n e_n + (e)e_n)$$

with $\lambda_i \in \Lambda$ for all i . Then $\text{Ker } f = (e)$, and the claim follows from the first isomorphism theorem for modules. So, in order to find a basis of $\Lambda/(e)$, it suffices to find bases for the modules $\Lambda e_i/(e)e_i$. The following observation is fundamental in this context.

Lemma 3.3.2. *Let $A = \mathbb{K}Q/I$ be a bound quiver algebra and \mathcal{B} a \mathbb{K} -basis of A . If $e = e_{j_1} + \cdots + e_{j_r}$ is a sum of primitive orthogonal idempotents of A , then*

$$\mathcal{C} = \{b + (e) \mid b \in \mathcal{B}, b \text{ does not factor through } e_{j_k} \text{ for all } k\}$$

is a basis of $A/(e)$. Therefore, a basis of $P(i)/(e)e_i$ is given by

$$\mathcal{C}_i := \{b + (e) \mid b \in \mathcal{B}_i, b \text{ does not factor through } e_{j_k} \text{ for all } k\}.$$

Proof. Follows from the fact that if $b \in \mathcal{B}$, then $b \in (e)$ if and only if b factors through some e_{j_k} . \square

Proposition 3.3.3. *Let $1 \leq k_0 < i \leq n$. Then a basis for $P = P(i)/(e_{k_0})e_i$ is given by*

$$\begin{array}{ccc}
 & [i, i, i] & \\
 [k_0 + 1, i, i] & \text{---} & [n, n, i] \\
 & [n - i + k_0 + 1, n, i] &
 \end{array} \tag{3.3.1}$$

Proof. By Lemma 3.3.2 we have to find the elements of \mathcal{B}_i that factor through e_{k_0} . It is clear that $[k, k_0, i]$ factors through e_{k_0} for all $k \in Q_0$, thus $[k, k_0, i] = [0]$ in P . First consider $1 \leq k \leq k_0$, then $[k, k_0, i] \in (\mathcal{B}_1)$, thus $[k, k_0, i] = [k, i, i] = [0]$ in P , and since

$$x := [k, i + s, i] = [k, i, i][i, i + s, i], \quad 0 \leq s \leq n - i,$$

because they have the same length, we conclude that x is zero in P .

Now consider $k_0 + 1 \leq k \leq n - i + k_0$. We know that $[k, k_0, i] = [k, i + k - k_0, i]$, thus $[k, i + k - k_0, i] = [0]$ in P . But in general, for $k < j > i$ we have that

$$[k, j + s, i] = [k, j, i][i, i + s, i], \quad 0 \leq s \leq n - j,$$

thus for $j = i + k - k_0$ we get that $[k, j + s, i] = [0]$ in P with $0 \leq s \leq n - k - i - k_0$.

It remains to prove that the elements in (3.3.1) do not factor through e_{k_0} . Indeed, those elements can be parameterized as

$$y := [k_0 + 1 + s + t, i + t, i], \quad 0 \leq s \leq i - k_0 - 1, \quad 0 \leq t \leq n - i,$$

then $\ell(y) = |k_0 + 1 + s + t - i - t| + |i + t - i| = |k_0 + 1 + s - i| + t = i - k_0 - 1 - s + t$, but

$$\ell(k_0 + 1 + s + t, k_0, i) = 1 + s + t + i - k_0$$

thus the minimal length path starting at i , ending at $k_0 + 1 + s + t$ and passing by k_0 , has length greater than any y , therefore y does not factor through e_{k_0} . This completes the proof. \square

Corollary 3.3.4. *Let $1 \leq k < i \leq n$. Then a basis for $(e_k)e_i$ is given by*

$$\begin{array}{ccc}
 & [k, i, i] & \\
 [1, i, i] & \text{---} & [n - i + k, n, i] \\
 & [1, n, i] &
 \end{array}$$

Proof. Follows from Propositions 3.2.12 and 3.3.3. \square

Corollary 3.3.5. *Let $1 \leq k_0 < k_1 < \dots < k_r < i \leq n$. Then*

$$\frac{P(i)}{(e_{k_0} + e_{k_1} + \dots + e_{k_r})e_i} = \frac{P(i)}{(e_{k_r})e_i} = P.$$

Proof. It suffices to prove that $(e_{k_0} + e_{k_1} + \cdots + e_{k_r})e_i \subseteq (e_{k_r})e_i$. From the proof of Proposition 3.3.3 we have that all the elements of \mathcal{B}_i that factor through e_{k_0} are zero in P , this is equivalent to say that $(e_{k_0})e_i \subseteq (e_{k_r})e_i$. Then the result follows by induction. \square

Proposition 3.3.6. *Let $1 \leq i < j_0 \leq n$. Then a basis for $P = P(i)/(e_{j_0})e_i$ is given by*

$$\begin{array}{ccc}
 & [i, i, i] & \\
 [1, i, i] & \text{---} & [j_0 - 1, j_0 - 1, i] \\
 | & & / \\
 [1, j_0 - 1, i] & &
 \end{array} \tag{3.3.2}$$

Proof. We proceed as in Proposition 3.3.3: $[k, j_0, i]$ factors through e_{j_0} for all $k \in Q_0$, thus $[k, j_0, i] = [0]$ in P , therefore $[k, j, i] = [0]$ for all $j_0 \leq j \leq n$ and $1 \leq k \leq n$, in particular for the elements in \mathbb{C} . We remain to show that the elements of the four-sided diagram are non-zero in P . Indeed, the largest path in each column has length

$$\ell(k, j_0 - 1, i) = 2j_0 - k - i - 2 < \ell(k, j_0, i), \quad k < j_0,$$

thus all the elements in (3.3.2) do not factor through e_{j_0} . \square

Corollary 3.3.7. *Let $1 \leq i < j \leq n$. Then a basis for $(e_j)e_i$ is given by*

$$\begin{array}{ccc}
 & [j, j, i] & \\
 [1, j, i] & \text{---} & [n, n, i] \\
 | & & / \\
 [1, n, i] & &
 \end{array}$$

Moreover, $(e_j)e_i \cong P(j)$.

Proof. The description of the basis follows from Propositions 3.2.12 and 3.3.6. Recall that $P(i) \hookrightarrow P(j)$, so $(e_j)e_i \subseteq P(i) \subseteq P(j)$, but $\dim_{\mathbb{K}}(e_j)e_i = \dim_{\mathbb{K}} P(j)$, thus the last assertion also follows. \square

A very important case is when we consider the idempotent ideal (e_n) . Next we compute a basis for this ideal using the last result.

Corollary 3.3.8. *Let $1 \leq i \leq n$. Then a basis for $(e_n)e_i$ is given by*

$$\begin{array}{c}
 [n, n, i] \\
 \text{---} \\
 [1, n, i]
 \end{array} \tag{3.3.3}$$

Moreover, $(e_n)e_i$ is isomorphic to $P(n)$.

Proof. Fix $i \in [n]$. A basis for $(e_n)e_i$ is just the complement of Eq. (3.3.2), for $j_0 = n$, in the basis \mathcal{B}_i of $P(i)$ (cf. Proposition 3.2.12), and this is given by Eq. (3.3.3). For the last assertion, the isomorphism is given by $[j, n, i] \mapsto [j, n, i][i, i, n] = [j, n, n]$, for any $j \in [n]$. \square

Corollary 3.3.9. *Let $1 \leq i < j_0 < j_1 < \cdots < j_r \leq n$. Then*

$$\frac{P(i)}{(e_{j_0} + e_{j_1} + \cdots + e_{j_r})e_i} = \frac{P(i)}{(e_{j_0})e_i} = P.$$

Proof. It suffices to prove that $(e_{j_0} + e_{j_1} + \cdots + e_{j_r})e_i \subseteq (e_{j_0})e_i$. In the proof of Proposition 3.3.6 we showed that all the elements of \mathcal{B}_i that factor through e_{j_1} are zero in P , which is equivalent to the fact that $(e_{j_1})e_i \subseteq (e_{j_0})e_i$. Then the result follows by induction. \square

Proposition 3.3.10. *Let $1 \leq k_0 < i < j_0 \leq n$. Then a basis for $P = P(i)/(e_{k_0} + e_{j_0})e_i$ is given by*

$$\begin{array}{c} [i, i, i] \\ \swarrow \quad \searrow \\ [k_0 + 1, i, i] \quad [j_0 - 1, j_0 - 1, i] \\ \swarrow \quad \searrow \\ [k_0 + j_0 - i, j_0 - 1, i] \end{array} \quad (3.3.4)$$

Proof. By Lemma 3.3.2, a basis of P is formed by those elements of \mathcal{B}_i that do not factor through e_{k_0} nor through e_{j_0} . By Propositions 3.3.3 and 3.3.6 those elements are the intersection of (3.3.1) and (3.3.2), and this is just (3.3.4). \square

Corollary 3.3.11. *Let $1 \leq k < i < j \leq n$. Then a basis for $(e_k + e_j)e_i$ is given by*

$$\begin{array}{c} [k, i, i] \quad [j, j, i] \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ [1, i, i] \quad [k + j - i, j, i] \quad [n, n, i] \\ \swarrow \quad \searrow \\ [1, n, i] \end{array}$$

Proof. Follows from Propositions 3.2.12 and 3.3.10. \square

The previous results give us a simple description of bases for the modules

$$\frac{P(i)}{(e)e_i},$$

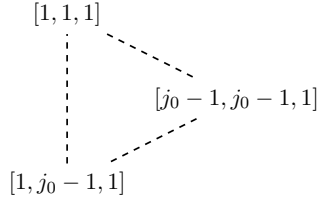
for e an idempotent in Λ .

Theorem 3.3.12. *Let $i \in Q_0$.*

(a) *If $i = 1 < j_0 < j_1 < \cdots < j_s \leq n$, and $e = e_{j_0} + \cdots + e_{j_s}$, then*

$$\frac{P(1)}{(e)e_1} = \frac{P(1)}{(e_{j_0})e_1},$$

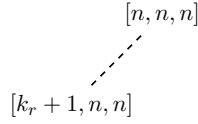
and a basis is given by



(b) If $1 \leq k_0 < k_1 < \cdots < k_r < n = i$, and $e = e_{k_0} + \cdots + e_{k_r}$, then

$$\frac{P(n)}{(e)e_n} = \frac{P(n)}{(e_{k_r})e_n},$$

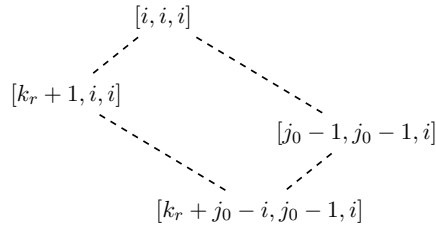
and a basis is given by



(c) Let $1 \leq k_0 < \cdots < k_r < i < j_0 < \cdots < j_s \leq n$, thus $i \neq 1, n$. If $e := e_{k_0} + \cdots + e_{k_r} + e_{j_0} + \cdots + e_{j_s}$, then

$$\frac{P(i)}{(e)e_i} = \frac{P(i)}{(e_{k_r} + e_{j_0})e_i},$$

and a basis is given by



In other words, we have that the shape of $P(i)/(e)e_i$, in either case, depends only on the neighbour indices of i , i.e. on k_r and j_0 .

Proof. (c) is consequence of Corollaries 3.3.5 and 3.3.9 and Proposition 3.3.10. \square

3.4 Block decomposition of $\text{Aus } T_n/(e)$

In this section we find block decompositions of the factor algebras $\text{Aus } T_n/(e)$ for idempotent elements e in Λ_n and characterise its blocks. The main tool used to find such decompositions is a particular version of the Chinese remainder theorem for non-commutative rings. In general we find a sufficient condition on the set of idempotent ideals of an algebra that guaranties the existence of a block decomposition as the one that we obtain for the case of $\text{Aus } T_n$. We start discussing some properties of idempotent ideals.

Let R a ring and $F \subseteq R$. Denote by $\langle F \rangle_+$ the set of all finite sums of elements of F union with $\{0_R\}$.

In what follows, $E = \{e_1, \dots, e_n\}$ denotes a complete set of primitive orthogonal idempotents of a ring R or \mathbb{K} -algebra A . It follows for any element $e \in \langle E \rangle_+$ there exists a unique $J \subseteq [n] = \{1, \dots, n\}$ such that

$$e = e_J := \sum_{j \in J} e_j,$$

where $e_\emptyset := 0$, i.e. there is a bijection between the power set of $[n]$ and $\langle E \rangle_+$ given by $J \mapsto e_J$.

For natural numbers $s \leq t$, the set $[s, t] := \{x \in \mathbb{N} \mid s \leq x \leq t\}$ is called a *discrete interval*. Therefore, if $J \subseteq [n]$ is non-empty, then J can be expressed as disjoint union of discrete intervals in a unique way:

$$J = [s_1, t_1] \sqcup \dots \sqcup [s_r, t_r], \quad (3.4.1)$$

where $1 \leq r \leq \lfloor \frac{n+1}{2} \rfloor$, $s_i \leq t_i$ for all $1 \leq i \leq r$, and $s_{i+1} - t_i \geq 2$ for $1 \leq i \leq r-1$. In this case, define $J_i := [s_i, t_i]$, for $1 \leq i \leq r$, and we use this notation for the rest of this section. Note that $e_{[n]} = 1_R$. Moreover, we set $J^c := [n] \setminus J$. Thus, $e_{J^c} = 1_R - e_J$ and

$$J^c = \bigcap_{i=1}^r J_i^c.$$

The next result gathers some technical properties of two-sided ideals generated by idempotent elements.

Lemma 3.4.1. *Let R be a ring. Then the following conditions hold.*

- (a) *If $e, f \in R$ are orthogonal idempotents, then $e + f$ is an idempotent and $(e) \subseteq (e + f) = (e) + (f)$. In general this is not a direct sum.*
- (b) *Let $E = \{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of R , and J and I subsets of $[n]$.*
 - (b0) *The product of any two idempotents in $\langle E \rangle_+$ is idempotent, more precisely, $e_J e_I = e_{J \cap I}$.*
 - (b1) *$(e_{J \cap I}) \subseteq (e_J) \cap (e_I) \subseteq (e_I)$. In particular, if $J \subseteq I$ then $(e_J) \subseteq (e_I)$.*
 - (b2) *$J \cap I = \emptyset$ if and only if e_J, e_I are orthogonal. In this case, $e_{J \cup I} = e_J + e_I$ is an idempotent and $(e_{J \cup I}) = (e_J) + (e_I)$.*

Proof. (a) It is clear that $e + f$ is idempotent, and since $e = (e + f)e \in (e + f)$, we have $(e) \subseteq (e + f)$. On the other hand, we have $(e + f) \subseteq (e) + (f)$. For the reverse inclusion let $xey + w fz$ be in $(e) + (f)$, thus $xey + w fz = x(e + f)ey + w f(e + f)z \in (e + f)$, and the assertion holds. As counterexample for the direct sum consider $R = \text{Aus } T_2$, then $[1, 2, 1] \in (e_1) \cap (e_2)$.

(b0) Since $e_j e_i = 0$ for all $j \neq i$ and $e_i^2 = e_i$, then

$$\begin{aligned} e_J e_I &= e_J \sum_{i \in I} e_i = \sum_{i \in I} e_J e_i = \sum_{i \in I} \left(\sum_{j \in J} e_j \right) e_i \\ &= \sum_{(i,j) \in I \times J} e_j e_i = \sum_{k \in J \cap I} e_k = e_{J \cap I}. \end{aligned}$$

(b1) follows from (b0). The first part of (b2) follows from (b0), because $e_\emptyset = 0$, and the last assertion of (b2) is due to (a), considering $J \cap I = \emptyset$. \square

Next we recall the Chinese remainder theorem for non-commutative rings.

Proposition 3.4.2. *Let I_1, \dots, I_r be two-sided ideals of a ring R such that $I_j + I_k = R$ for all $j \neq k$. Then the map*

$$\varphi: R \rightarrow \frac{R}{I_1} \times \cdots \times \frac{R}{I_r}, \quad r \mapsto (r + I_1, \dots, r + I_r)$$

is a ring surjection such that $\text{Ker } \varphi = \bigcap_{i=1}^r I_i$.

The following result is a less general version of the Chinese remainder theorem, adapted to our purposes when we consider ideals generated by idempotent elements.

Corollary 3.4.3. *Let $J = \bigsqcup_{i=1}^r J_i \subseteq [n]$. Then the correspondence*

$$\varphi_J: R \rightarrow \frac{R}{(e_{J_1^c})} \times \cdots \times \frac{R}{(e_{J_r^c})}, \quad r \mapsto (r + (e_{J_1^c}), \dots, r + (e_{J_r^c}))$$

is a surjective ring homomorphism with $\text{Ker } \varphi_J = \bigcap_{i=1}^r (e_{J_i^c})$.

Proof. Follows from Proposition 3.4.2, since $1_R \in (e_{J_i^c}) + (e_{J_j^c})$ for all $i \neq j$. \square

We will find a description of $\text{Ker } \varphi_J$ for algebras satisfying certain conditions. Some results that we have proven for the algebra $\text{Aus } T_n$ motivate the following lemma.

Lemma 3.4.4. *Let A be an algebra with n simples, and $E = \{e_1, \dots, e_n\}$ a complete set of primitive orthogonal idempotents of A satisfying the following property: for $i \in [n]$, and any subsets $K = \{k_0, \dots, k_l\}$ and $L = \{l_0, \dots, l_m\}$ of $[n]$ with $1 \leq k_0 < \dots < k_l < i < l_0 < \dots < l_m \leq n$, the following conditions (I) and (II) hold:*

- (I) *If $i \leq n$ then $(e_K)e_i = (e_{k_l})e_i$,*
- (II) *If $1 \leq i$ then $(e_L)e_i = (e_{l_0})e_i$.*

Then, if $J = \bigsqcup_{i=1}^r J_i \subseteq [n]$, we have that $(e_{J^c}) = \bigcap_{i=1}^r (e_{J_i^c})$.

Proof. First note that $(e_K + e_L)e_i = (e_{k_l} + e_{l_0})e_i$. Indeed, clearly $I \cap J = \emptyset$, thus by Lemma 3.4.1 (a) $(e_K + e_L) = (e_K) + (e_L)$, thus by (I) and (II) $(e_K + e_L)e_i = (e_K)e_i + (e_L)e_i = (e_{k_l})e_i + (e_{l_0})e_i = ((e_{k_l}) + (e_{l_0}))e_i = (e_{k_l} + e_{l_0})e_i$.

By Lemma 3.4.1 (b1), we have $(e_{J^c}) = (e_{\bigcap_{i=1}^r J_i^c}) \subseteq \bigcap_{i=1}^r (e_{J_i^c})$. Now we show the reverse inclusion. For, let $x \in \bigcap_{i=1}^r (e_{J_i^c})$, and $a \in [n]$. If $a \in \bigcap_{i=1}^r J_i^c = J^c$, then $xe_a \in (e_{J^c})$. Now suppose $a \notin \bigcap_{i=1}^r J_i^c = J^c \Leftrightarrow a \in J$, thus there exists $i_0 \in [r]$ such that $a \in J_{i_0}$. We have three cases: $i_0 \in \{1, r\}$ or $1 < i_0 < r$.

First suppose $i_0 = 1$. Thus $a \in J_1 = [s_1, t_1]$. We have three subcases. If $s_1 = 1$ and $t_1 < n$, then $t_1 + 1 \in J^c$, otherwise $t_1 + 1 \in J$ which is a contradiction. Thus $J_1^c = [t_1 + 1, n]$ and $a < t_1 + 1$. In particular $xe_a \in (e_{J_1^c})e_a \subseteq (e_{t_1+1})e_a \subseteq (e_{J^c})e_a \subseteq (e_{J^c})$, where the first and second inclusions follow from (II) and Lemma 3.4.1 (b1) respectively.

If $1 < s_1 \leq a \leq t_1 < n$, then $s_1 - 1, t_1 + 1 \in J^c$. Thus $J_1^c = [1, s_1 - 1] \sqcup [t_1 + 1, n]$, and as before we have $xe_a \in (e_{J_1^c})e_a \subseteq (e_{s_1-1} + e_{t_1+1})e_a \subseteq (e_{J^c})e_a \subseteq (e_{J^c})$, but now using the first claim in this proof. Finally, if $t_1 = n$, then $J = J_1$ and the claim is trivial. Similar arguments, using also the property (I), show the other two remaining cases, which completes the proof. \square

Note that Lemma 3.4.4 depends on the ordering of the set E of idempotents, i.e. it could happen that if we reorder the elements of E , then the requested property could not hold. Thus, we are interested in some order of the set of idempotents where the requested property actually holds, if it exists.

Theorem 3.4.5. *Let A be a \mathbb{K} -algebra satisfying the property in Lemma 3.4.4 for some complete set of pairwise orthogonal idempotents $E = \{e_1, \dots, e_n\}$, $J = \bigsqcup_{i=1}^r J_i \subseteq [n]$, and $\pi_J : A \rightarrow A/(e_{J^c})$ the canonical projection. Then the map*

$$\bar{\varphi}_J : \frac{A}{(e_{J^c})} \xrightarrow{\sim} \frac{A}{(e_{J_1^c})} \times \cdots \times \frac{A}{(e_{J_r^c})},$$

given by $\bar{\varphi}(\pi_J(a)) := \varphi_J \pi_J(a)$ is a \mathbb{K} -algebra isomorphism.

Proof. The ring surjection φ_J is \mathbb{K} -linear by componentwise scalar multiplication. Then the theorem is consequence of Corollary 3.4.3 and Lemma 3.4.4. \square

Corollary 3.4.6. *Let $\Lambda_n = \text{Aus } T_n$, $E = \{[i, i, i] \mid i \in [n]\}$ with the canonical order, and $J = \bigsqcup_{i=1}^r J_i \subseteq [n]$. Then*

$$\bar{\varphi}_J : \frac{\Lambda_n}{(e_{J^c})} \xrightarrow{\sim} \frac{\Lambda_n}{(e_{J_1^c})} \times \cdots \times \frac{\Lambda_n}{(e_{J_r^c})},$$

given as in Theorem 3.4.5, is a \mathbb{K} -algebra isomorphism.

Proof. By Theorem 3.4.5, it is enough to show that Λ_n satisfies the property given in Lemma 3.4.4 for E . Indeed, the inclusions \subseteq of (I) and (II) were shown in the proofs of Corollaries 3.3.5 and 3.3.9 respectively, the reversed inclusions follow from Lemma 3.4.1 (b1). \square

Let $\Lambda_n = \text{Aus } T_n$. We finish this section with the following result that plays an important role when computing homological embeddings for Λ_n . To avoid confusions, if $e \in \Lambda_n$ is an idempotent, we use the notation $\langle e \rangle = \Lambda_n e \Lambda_n$.

Proposition 3.4.7. *Let $1 \leq i \leq n$. Then $\langle e_i + \cdots + e_n \rangle$ is a projective left Λ_n -module.*

Proof. We have

$$\begin{aligned} \langle e_i + \cdots + e_n \rangle &= \langle e_i + \cdots + e_n \rangle (e_1 + \cdots + e_n) \cong \bigoplus_{i=1}^n \Lambda_n (e_i + \cdots + e_n) \Lambda_n e_i \\ &= \langle e_i \rangle e_1 \oplus \cdots \oplus \langle e_i \rangle e_{i-1} \oplus P(i) \oplus \cdots \oplus P(n), \end{aligned}$$

where the last equality holds since Λ_n satisfies the property in Lemma 3.4.4. From Corollaries 3.3.7 and 3.3.8 we have that this is a direct sum of projective Λ_n -modules. \square

3.5 Preprojective algebras of type \mathbb{A}_n

We will prove that some blocks of $\text{Aus } T_n/(e)$ are isomorphic to preprojective algebras of type \mathbb{A}_n . In this section we define such algebras and find a basis of its indecomposable projective modules.

Definition 3.5.1. For $n \geq 2$, let A_n be the quiver

$$\begin{array}{ccccccc} 1 & \xleftarrow{b_1} & 2 & \xleftarrow{b_2} & \dots & \xleftarrow{b_{n-1}} & n \\ \bullet & & \bullet & & & & \bullet \end{array}$$

of Dynkin type \mathbb{A}_n . Let $\Pi = \Pi_n = \mathcal{P}(A_n)$ be the *preprojective algebra of type \mathbb{A}_n* . Thus $\Pi = \mathbb{K}\overline{A_n}/\mathbb{J}$, where the double quiver $\overline{A_n}$ of A_n is the following:

$$\begin{array}{ccccccc} 1 & \xrightarrow{a_1} & 2 & \xrightarrow{a_2} & \dots & \xrightarrow{a_{n-1}} & n \\ \bullet & & \bullet & & & & \bullet \\ & \xleftarrow{b_1} & & \xleftarrow{b_2} & & \xleftarrow{b_{n-1}} & \\ & & & & & & \end{array}$$

and the ideal \mathbb{J} is generated by b_1a_1 , $a_{n-1}b_{n-1}$, and $a_i b_i - b_{i+1}a_{i+1}$ for $1 \leq i \leq n-2$.

We will describe a basis for the preprojective algebra Π using what we have done so far, but first note that Π is quotient of $\Lambda = \text{Aus } T_n$. Indeed, recall that $\Lambda = \mathbb{K}Q/\mathbb{I}$, and denote by $\pi : \mathbb{K}Q \rightarrow \Lambda$ the canonical projection. Consider the ideal $\mathfrak{b} := ([b_1a_1]) + \mathbb{I}$ in Λ . Thus by the (Noether) correspondence theorem we have that

$$\frac{\Lambda}{\mathfrak{b}} \cong \frac{\mathbb{K}Q}{\pi^{-1}(\mathfrak{b})},$$

but $\pi^{-1}(\mathfrak{b}) = \mathbb{J}$. Therefore $\Lambda/\mathfrak{b} \cong \Pi$. So, in order to find a basis of Π , it is enough to get a basis of Λ/\mathfrak{b} , and this is equivalent to find bases for the projectives $\Lambda e_i/\mathfrak{b}e_i = P(i)/\mathfrak{b}e_i$, with $1 \leq i \leq n$.

Proposition 3.5.2. *Let $1 \leq i \leq n$, then a basis for $P(i)/\mathfrak{b}e_i = \Pi e_i$ is given by*

$$\begin{array}{c} [i, i, i] \\ \diagdown \quad \diagup \\ [1, i, i] \quad [n, n, i] \\ \diagup \quad \diagdown \\ [n-i+1, n, i] \end{array} \quad (3.5.1)$$

Proof. Fix $i \in \{1, \dots, n\}$. As before, we have to show that the elements in (3.5.1) are those elements in the basis \mathcal{B}_i of $P(i)$ that do not factor through b_1a_1 . In what follows we will use Lemma 3.2.7 without mentioning it.

First consider the elements below the diagonal $[1, i, i] - [n-i+1, n, i]$ (see diagram in Proposition 3.2.12). Those elements are parameterized by $[k, i+k, i]$ with $1 \leq k \leq n-i$. It is clear that $[1, 2, 1]$ factors through b_1a_1 , so consider $i = 1$ and $1 < k \leq n-1$. Then,

$$[k, k+1, 1] = [(b_k a_k \cdots a_2)a_1] = [(a_{k-1} \cdots a_1 b_1)a_1].$$

Now let $i > 1$. If $k = 1$, then

$$[1, i+1, i] = [b_1(b_2 \cdots b_i a_i)] = [b_1(a_1 b_1 \cdots b_{i-1})].$$

So, if $k > 1$ we get

$$\begin{aligned} [k, i+k, i] &= [b_k(b_{k-1} \cdots b_{i+k-1} a_{i+k-1} \cdots a_i)] = [b_k(a_k \cdots a_1 b_1 \cdots b_{i-1})] \\ &= [(b_k a_k \cdots a_2)a_1 b_1 \cdots b_{i-1}] = [(a_{k-1} \cdots a_1 b_1)a_1 b_1 \cdots b_{i-1}]. \end{aligned}$$

Thus by Lemma 3.2.13 all the elements below this diagonal factor through b_1a_1 . So, let's show that the diagonal elements do not factor through b_1a_1 . Indeed, these elements are parameterized by $[k, i + k - 1, i]$ for $1 \leq k \leq n - i + 1$. We proceed by induction on k . For $[1, 1, 1]$ the assertion is trivial, so consider $i > 1$ and $k = 1$. Then the claim is also clear for $[1, i, i] = [b_1 \cdots b_{i-1}]$. By induction, suppose that for $k > 1$ the element $\eta = [k - 1, i + k - 2, i]$ does not factor through b_1a_1 . If

$$[k, i + k - 1, i] = [a_{k-1}\eta] = [a_{k-1}b_{k-1} \cdots b_{i+k-3}a_{i+k-3} \cdots a_i]$$

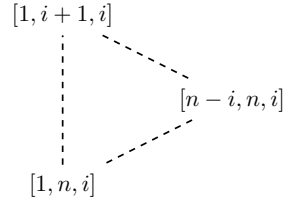
factors through b_1a_1 , then the only possibility to find such a factorization is when

$$[a_{k-1}b_{k-1}] = [b_k a_k] = [b_1 a_1]$$

because η does not admit the sequence b_1a_1 in any expression. Thus $k = 1$, a contradiction. Therefore $[k, i + k - 1, i]$ does not factor through b_1a_1 .

In the proof of Proposition 3.3.3 we showed that the remaining elements of (3.5.1) do not factor through e_1 , thus they cannot factor through b_1a_1 . Thus the result follows in general. \square

Corollary 3.5.3. *Let $1 \leq i \leq n - 1$, then a basis for $\mathfrak{b}e_i \subseteq \Lambda_n$ is given by*



and $\mathfrak{b}e_n = 0$. Thus the elements $[k, j, i] \in \mathfrak{b}$ are characterised by $i \in [1, n - 1]$, $k \in [1, n - i]$ and $k + i \leq j \leq n$.

In order to distinguish the basis elements of Λ and Π , we will use the notation $\llbracket k, j, i \rrbracket$ for basis elements of Π . Thus, we can state the following result.

Theorem 3.5.4. *Let $n \geq 1$. Then a \mathbb{K} -basis for Π_n is given by*

$$\mathcal{B}_{\Pi_n} = \{ \llbracket k, j, i \rrbracket \mid 1 \leq k, i \leq n, \max\{i, k\} \leq j \leq \min\{k + i - 1, n\} \}.$$

The multiplication between basis elements is given by

$$\llbracket k', j', i' \rrbracket \llbracket k, j, i \rrbracket = \begin{cases} \llbracket k', j' + j - k, i \rrbracket & \text{if } i' = k \text{ and } j' + j - k \leq \min\{k' + i - 1, n\} \\ 0 & \text{else.} \end{cases}$$

Proof. Follows from Theorem 3.2.14 and Proposition 3.5.2 \square

3.6 Blocks of $\text{Aus } T_n/(e)$

Let $\Lambda_n = \text{Aus } T_n$. In what follows we characterise the blocks of the block decomposition of $\Lambda_n/(e)$ given in Section 3.4. It turns out that those blocks are isomorphic to $\text{Aus } T_m$ or Π_m , for some $m \in \mathbb{N}_+$. We start observing an important property of the morphisms $f_i: \Lambda_i \rightarrow \Lambda_{i-1}$ defined at the beginning of Section 3.3.

Lemma 3.6.1. *Let $1 \leq m \leq n$ and $f_i: \Lambda_i \rightarrow \Lambda_{i-1}$ the function defined in Lemma 3.3.1. Then the composite $f_m \cdots f_n: \Lambda_n \rightarrow \Lambda_{m-1}$ has kernel $\langle e_m + \cdots + e_n \rangle$.*

Proof. Let $x = [k, j, i]_n \in \text{Ker}(f_m \cdots f_n)$, then there exists $m_0 \in \mathbb{N}$, with $m \leq m_0 \leq n$, such that $j \geq m_0$, otherwise $j \leq m - 1$, so $f_m \cdots f_n(x) = [k, j, i]_{m-1} \neq 0$, which is a contradiction. Thus, by Lemma 3.4.1 $x \in \langle e_j \rangle \subseteq \langle e_m + \cdots + e_n \rangle$, since $m \leq m_0 \leq j \leq n$. For the converse, set $[j]_i := [j, j, j]_i = e_j \in \Lambda_i$, for $1 \leq j \leq i \leq n$, then, since $f_i([i]_i) = 0$ for all i , we have

$$\begin{aligned} f_m \cdots f_n([m]_n + \cdots + [n]_n) &= f_m \cdots f_{n-1}([m]_{n-1} + \cdots + [n-1]_{n-1}) \\ &= \cdots = f_m([m]_m) = 0. \end{aligned}$$

Therefore $\langle e_m + \cdots + e_n \rangle \subseteq \text{Ker}(f_m \cdots f_n)$. \square

The following technical result is used in the proof of Theorem 3.6.3.

Lemma 3.6.2. *Let $1 < s \leq t < n$ and $J = [s, t]$. If $x := [k, j, i] \in \Lambda_n$, then the following conditions hold.*

- (a) $x \in (e_{J^c})$ if and only if
 - (i) $i \in J^c$, or
 - (ii) $i \in J$ and $k \in J^c$, or
 - (iii) $i, k \in J$ and $\min\{i + k - s + 1, t + 1\} \leq j$.
- (b) $x \notin (e_{J^c})$ if and only if $i, k \in J$ and $j \leq \min\{i + k - s, t\}$.

Proof. (b) Let $i, k \in J$, then $x \notin (e_{J^c})$ if and only if x is a basis element of $P(i)/(e_{J^c})e_i$, but by Theorem 3.3.12 (c) those elements are characterised by $j \leq \min\{i + k - s, t\}$.

(a) Follows from (b), since the conditions are dual. \square

Theorem 3.6.3. *Let $J = [s, t] \subseteq \{1, \dots, n\}$, and $m := t - s + 1$. Then*

$$\frac{\Lambda_n}{(e_{J^c})} \cong \begin{cases} 0 & \text{if } J = \emptyset, \\ \Lambda_{m=t} & \text{if } s = 1, \\ \Pi_m & \text{if } s > 1. \end{cases}$$

as \mathbb{K} -algebras. If $m = 1$ we identify $\Lambda_1 \cong S(t) \cong \Pi_1 \cong \mathbb{K}$.

Proof. Let $A := \Lambda_n/(e_{J^c})$. The case $J = \emptyset$ is trivial, so we can assume $J \neq \emptyset$. If $m = 1$, we have that $J = \{t\}$, thus the algebra A has only one primitive idempotent, namely e_t . Therefore $A \cong \mathbb{K} \cdot e_t \cong \mathbb{K}$.

Now let $1 < m = n$. Then $J = [1, n]$, thus $(e_{J^c}) = 0$, and $A = \Lambda_n$. So we can assume $1 < m < n$. If $s = 1$ then $m = t$ and $e_{J^c} = e_{t+1} + \cdots + e_n$. Therefore, by Lemma 3.6.1

$$\psi := \overline{f_{t+1} \cdots f_n}: \frac{\Lambda_n}{(e_{J^c})} \rightarrow \Lambda_t$$

is an isomorphism of rings, where $f_i: \Lambda_i \rightarrow \Lambda_{i-1}$ is defined in Lemma 3.3.1.

Now let $s > 1$, then $t \leq n$. If $t < n$, then $J^c = [1, s-1] \sqcup [t+1, n]$, and we define a function

$$\psi = \psi_J: \frac{\Lambda_n}{(e_{J^c})} \rightarrow \frac{\Lambda_m}{(b_1 a_1)} = \Pi_m$$

given, for $[k, j, i]_n \in \Lambda_n$, by

$$\psi([k, j, i]_n + (e_{J^c})) := \begin{cases} [k-s+1, j-s+1, i-s+1]_m + (b_1 a_1) & \text{if } s \leq k, i \leq t, j \leq t \\ \bar{0} & \text{else.} \end{cases}$$

It is clear that ψ is well defined, since $[k, j, i]_n + (e_{J^c}) = [k', j', i']_n + (e_{J^c}) \Leftrightarrow [k, j, i]_n = [k', j', i']_n$ or $[k, j, i]_n, [k', j', i']_n \in (e_{J^c})$.

Now, we prove that ψ is a ring homomorphism. For, let $y = [k', j', j]_n, x = [k, j, i]_n$ be in Λ_n . Write $\bar{x} := x + (e_{J^c})$. Then $yx = [k', j' + j - k, i]_n$, considering $yx = 0$ if $j' + j - k > n$. Thus,

$$\psi(\overline{yx}) = [k' - s + 1, j' + j - k - s + 1, i - s + 1]_m + (b_1 a_1)$$

if $s \leq k', i \leq t$ and $j' + j - k \leq t$, and $\psi(\overline{yx}) = \bar{0}$ else,

$$\psi(\overline{y}) := \begin{cases} [k' - s + 1, j' - s + 1, k - s + 1]_m + (b_1 a_1) & \text{if } s \leq k', k \leq t \text{ and } j' \leq t, \\ \bar{0} & \text{else,} \end{cases}$$

$$\psi(\overline{x}) := \begin{cases} [k - s + 1, j - s + 1, i - s + 1]_m + (b_1 a_1) & \text{if } s \leq k, i \leq t \text{ and } j \leq t, \\ \bar{0} & \text{else.} \end{cases}$$

We can distinguish two cases (a) and (b).

(a) $\overline{yx} = \bar{0} \Leftrightarrow j' + j - k > n$ or $yx \in (e_{J^c})$. In the first instance, if $s \leq k', k, i \leq t$ and $j', j \leq j$, then $\psi(\overline{y})\psi(\overline{x}) = [k' - s + 1, j' + j - k - s + 1, i - s + 1]_m = \bar{0}$, since $j' + j - k - s + 1 > n - s + 1 > t - s + 1 = m$; if the latter conditions do not hold, then $\psi(\overline{y}) = \bar{0}$ or $\psi(\overline{x}) = \bar{0}$, thus their product is zero. For the second instance, by Lemma 3.6.2, $yx \in (e_{J^c})$ if and only if one of the following three conditions hold:

- (i) $i \in J^c$, or
- (ii) $i \in J$ and $k' \in J^c$, or
- (iii) $i, k' \in J$ and $\min\{i + k' - s + 1, t + 1\} \leq j' + j - k$.

Then, (i) implies $\psi(\overline{x}) = \bar{0}$, (ii) implies $\psi(\overline{y}) = \bar{0}$, and (iii) implies in particular that $j' + j - k - s + 1 > t - s + 1 = m$, thus $\psi(\overline{y}) = \bar{0}$ or $\psi(\overline{x}) = \bar{0}$ or $\psi(\overline{y})\psi(\overline{x}) = \bar{0}$. In either case (i), (ii) or (iii), we have $\psi(\overline{y})\psi(\overline{x}) = \bar{0}$.

(b) By Lemma 3.6.2, $\overline{yx} \neq \bar{0} \Leftrightarrow yx \notin (e_{J^c}) \Leftrightarrow i, k' \in J$ and $j' + j - k \leq \min\{i + k' - s, t\}$. In particular, these conditions imply $j', j \leq t$ and

$$\psi(\overline{yx}) = [k' - s + 1, j' + j - k - s + 1, i - s + 1]_m + (b_1 a_1).$$

Thus, if $k \in J$, then $\psi(\overline{y})\psi(\overline{x}) = \psi(\overline{yx})$. On the other hand, $k \in J^c$ cannot occur, otherwise $x \in (e_{J^c})$, which implies $\overline{yx} = \bar{0}$, a contradiction. This shows that ψ is a ring homomorphism.

Moreover, we have decompositions

$$\frac{\Lambda_n}{(e_{J^c})} \cong \frac{\Lambda_n e_s}{(e_{J^c})e_s} \times \cdots \times \frac{\Lambda_n e_t}{(e_{J^c})e_t}, \quad \text{and} \quad \frac{\Lambda_m}{(b_1 a_1)} \cong \frac{\Lambda_m e_1}{(b_1 a_1)e_1} \times \cdots \times \frac{\Lambda_m e_m}{(b_1 a_1)e_m}$$

as Λ_n -modules. But Theorem 3.3.12 and Propositions 3.3.10 and 3.5.2 show that the correspondence

$$\frac{\Lambda_n e_i}{(e_{J^c})e_i} = \frac{\Lambda_n e_i}{(e_{s-1} + e_{t+1})e_i} \longrightarrow \frac{\Lambda_m e_{i-s+1}}{(b_1 a_1)e_{i-s+1}}$$

given by

$$[k, j, i]_n + (e_{J^c})e_i \mapsto [k - s + 1, j - s + 1, i - s + 1]_m + (b_1 a_1)e_{i-s+1},$$

is bijective for all $i \in J = [s, t]$. This proves that ψ is a bijection, thus a ring isomorphism.

Finally, if $t = n$, then $J^c = [1, s - 1]$, and similar arguments as before show that the function

$$\psi = \psi_J: \frac{\Lambda_n}{(e_{J^c})} \longrightarrow \frac{\Lambda_m}{(b_1 a_1)} = \Pi_m$$

given, for $[k, j, i]_n \in \Lambda_n$, by

$$\psi([k, j, i]_n + (e_{J^c})) := \begin{cases} [k - s + 1, j - s + 1, i - s + 1]_m + (b_1 a_1) & \text{if } s \leq k, i, \\ \bar{0} & \text{else,} \end{cases}$$

is a ring isomorphism. This completes the proof of the theorem. \square

Corollary 3.6.4. *Let $J \subseteq \{1, \dots, n\}$. Then the blocks of the factor algebra $\Lambda_n/(e_{J^c})$ are isomorphic to Λ_r or Π_t , for some $1 \leq r, t \leq n$.*

Proof. Follows from Corollary 3.4.6 and Theorem 3.6.3. \square

Now we are able to prove that Λ_n admits a unique heredity chain using some results of the previous sections.

Lemma 3.6.5. *The ideal (e_n) is a heredity ideal of Λ_n .*

Proof. By Lemma 2.2.3 (e_n) is an idempotent ideal, and Proposition 3.4.7 shows that it is a projective Λ_n -module. Finally, $e_n \Lambda_n e_n \cong e_n P(n) \cong \mathbb{K} \cdot e_n \cong S(n)$ because the only element in the basis of $P(n)$ with target n is $[n, n, n] = e_n$. Thus, $e_n \Lambda_n e_n$ is semisimple. \square

Proposition 3.6.6. *Let $n \geq 0$. Then Λ_n is quasi-hereditary, with unique heredity chain*

$$0 \subseteq (e_n) \subseteq (e_{n-1} + e_n) \subseteq \cdots \subseteq (e_1 + \cdots + e_n) = \Lambda_n. \quad (3.6.1)$$

Proof. By induction on n , we construct a sequence of surjective ring morphisms $f_i: \Lambda_i \rightarrow \Lambda_{i-1}$, such that $\text{Ker}(f_i)$ is a heredity ideal. The cases $n = 0, 1$ are trivial, so consider $n > 1$. By induction there is a sequence of surjective ring homomorphisms

$$\Lambda_{n-1} \xrightarrow{f_{n-1}} \Lambda_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_2} \Lambda_1 \xrightarrow{f_1} \Lambda_0 = 0$$

such that $\text{Ker}(f_i)$ is a heredity ideal, for $1 \leq i \leq n-1$. By Lemma 3.3.1, there exists a surjective ring morphism $f_n: \Lambda_n \rightarrow \Lambda_{n-1}$, with $\text{Ker}(f_n) = (e_n)$, which is a heredity ideal by Lemma 3.6.5. This proves that Λ_n is quasi-hereditary for all $n \geq 0$.

In order to prove that (3.6.1) is a heredity chain, it is enough to show that for $1 \leq m \leq n$, $\text{Ker}(f_m \cdots f_n) = \langle e_m + \cdots + e_n \rangle$, but this follows from Lemma 3.6.1. Finally, if we show that (e_n) is the unique heredity ideal of the form $\langle e \rangle$, for some primitive idempotent $e \in \Lambda_n$, then it follows by induction that (3.6.1) is unique. Indeed, this holds, since for $1 \leq i \leq n-1$, we have that $[i, i+1, i] \in e_i \text{rad}(\Lambda_n)e_i \subseteq \langle e_i \rangle \text{rad}(\Lambda_n)\langle e_i \rangle \neq 0$. \square

3.7 Homological embeddings in $\text{mod Aus } T_n$

In this section we assume that \mathbb{K} is an algebraically closed field. Set $\Lambda_n = \text{Aus } T_n$. The aim of this section is to compute homological embeddings between Serre subcategories of $\text{mod } \Lambda_n$, using the block decompositions obtained in Section 3.4 and the classification of homological embeddings over preprojective algebras of type \mathbb{A}_n found in [Mar17]. Our proof uses the language of homological ring epimorphisms.

Recall that if R is a ring, we fix $E = \{e_1, \dots, e_n\}$ a complete set of primitive orthogonal idempotents of R . If $J \subseteq I \subseteq [n]$, then the embedding

$$\iota_{J,I}: \text{mod } R/(e_{J^c}) \hookrightarrow \text{mod } R/(e_{I^c})$$

is given by restriction of scalars, i.e. $\iota_{J,I} := (\pi_{I,J})_*$, see Section 2.3.

Proposition 3.7.1. *The embedding $\iota_{[n-1],[n]}: \text{mod } \Lambda_n/(e_n) \hookrightarrow \text{mod } \Lambda_n$ is homological.*

Proof. From Proposition 3.4.7 we know that $(e_n) \subseteq \Lambda_n$ is projective. Then the result follows by Lemma 1.5.2. \square

Corollary 3.7.2. *Let $n \geq 1$. The morphism $f_n: \Lambda_n \rightarrow \Lambda_{n-1}$ is a homological ring epimorphism. Thus, for any $1 \leq m \leq n$, the composite $f_m \cdots f_n: \Lambda_n \rightarrow \Lambda_{m-1}$ is a homological ring epimorphism as well.*

Proof. By Proposition 3.7.1 $\iota_{[n-1],[n]} = (\pi_{[n],[n-1]})_*$ is a homological embedding, i.e. $\pi_{[n],[n-1]}$ is a homological ring epimorphism (cf. Remark 2.3.1). From Lemma 3.3.1 we have that f_n induces an isomorphism $\bar{f}_n: \Lambda_n/(e_n) \xrightarrow{\sim} \Lambda_{n-1}$, and it is easy to see that $f_n = \bar{f}_n \pi_{[n],[n-1]}$, thus f_n is a homological ring epimorphism by Lemma 1.5.8 (a). The last assertion follows by induction. \square

The next theorem due to Marks will help us to prove that there are no non-trivial homological embeddings into $\text{mod } \Pi_n$, with Π_n the preprojective algebra of type \mathbb{A}_n .

Theorem 3.7.3 ([Mar17, Thm. 6.2]). *Suppose \mathbb{K} is an algebraically closed field. Let A be a preprojective algebra of Dynkin type and $F: \text{mod } B \rightarrow \text{mod } A$ be a homological embedding that is neither zero nor an equivalence. Then A must be of type \mathbb{A}_n*

1 — 2 — \dots — n for $n \geq 2$ and the algebra B is Morita equivalent to \mathbb{K} . In fact, for each $n \geq 2$ there are precisely two such choices for F , up to equivalence, which correspond to the Weyl group elements $s_{n-1}(s_{n-2}s_{n-1}) \cdots (s_1s_2 \cdots s_{n-1})$ and $s_n(s_{n-1}s_n) \cdots (s_2s_3 \cdots s_n)$, respectively.

Corollary 3.7.4. *Let Π_n be the preprojective algebra of type \mathbb{A}_n . If $\iota: \text{mod } B \rightarrow \text{mod } \Pi_n$ is a homological embedding given by restriction of scalars, then ι is zero or an equivalence.*

Proof. Suppose that ι is neither zero nor an equivalence. Then Theorem 3.7.3 implies that B necessarily is Morita equivalent to \mathbb{K} , and the essential image of ι is $\text{add}(\Pi_n P(1))$ or $\text{add}(\Pi_n P(n))$, in either case ι is not given by restriction of scalars, a contradiction. Hence, ι is zero or an equivalence. \square

Corollary 3.7.5. *Let $J = [s, t] \subsetneq I = [r, u] \subseteq [n]$, such that $1 \notin J$. Then the embedding $\iota_{J,I}: \text{mod } \frac{\Lambda_n}{(e_{J^c})} \rightarrow \text{mod } \frac{\Lambda_n}{(e_{I^c})}$ is not homological.*

Proof. If $1 \in I$, then by Theorem 3.6.3 $\Lambda_n/(e_{J^c}) \cong \Pi_{t-s+1}$ and $\Lambda_n/(e_{I^c}) \cong \Lambda_{u-r+1}$, thus $\text{gl. dim } \Lambda_n/(e_{J^c}) = \infty$ and $\text{gl. dim } \Lambda_n/(e_{I^c}) = 2$. Therefore, $\iota_{J,I}$ is not a homological embedding.

If $1 \notin I$, Theorem 3.6.3 implies $\Lambda_n/(e_{J^c}) \cong \Pi_{t-s+1}$ and $\Lambda_n/(e_{I^c}) \cong \Pi_{u-r+1}$. Thus, since $J \neq I$ the embedding $\iota_{I,J}$, induced by restriction of scalars by $\pi_{I,J}$, is not zero nor an equivalence. Hence, Corollary 3.7.4 implies that $\iota_{I,J}$ is not a homological embedding. \square

The next technical results will be used to characterise the homological embeddings between Serre subcategories of $\text{mod } \Lambda_n$.

Lemma 3.7.6. *Let $J = [s, t] \subseteq I = [r, u] \subseteq [n]$ such that $1 \notin I$. Set $p := u - r + 1$, $m := t - s + 1$ and $q := s - r$. Then the correspondence*

$$\gamma = \gamma_{I,J}: \frac{\Lambda_{u-r+1}}{(b_1 a_1)} \longrightarrow \frac{\Lambda_{t-s+1}}{(b_1 a_1)}$$

given by

$$\gamma([k, j, i]_p + (b_1 a_1)) = \begin{cases} [k - q, j - q, i - q]_m + (b_1 a_1) & \text{if } k, i \in [q + 1, q + m], j \leq q + m, \\ \bar{0} & \text{else,} \end{cases}$$

is the composite $\psi_J \circ \pi_{I,J} \circ \psi_I^{-1}$. Therefore, γ is a \mathbb{K} -algebra surjection.

Proof. Let J and I be as in the hypothesis. In the proof of Theorem 3.6.3 we defined an isomorphism $\psi_I: \Lambda_n/(e_{I^c}) \rightarrow \Lambda_{u-r+1}/(b_1 a_1)$, with inverse $\psi_I^{-1}: \Lambda_{u-r+1}/(b_1 a_1) \rightarrow \Lambda_n/(e_{I^c})$ given by $[k, j, i]_p + (b_1 a_1) \mapsto [k + r - 1, j + r - 1, i + r - 1]_n + (e_{I^c})$. Therefore,

$$\begin{aligned} \psi_J \circ \pi_{I,J} \circ \psi_I^{-1}([k, j, i]_p + (b_1 a_1)) &= \psi_J \circ \pi_{I,J}([k + r - 1, j + r - 1, i + r - 1]_n + (e_{I^c})) \\ &= \psi_J([k + r - 1, j + r - 1, i + r - 1]_n + (e_{J^c})) \\ &= [k + r - s, j + r - s, i + r - s]_{t-s+1} + (b_1 a_1) \\ &= \gamma([k, j, i]_p + (b_1 a_1)). \end{aligned}$$

Thus, the diagram

$$\begin{array}{ccc} \frac{\Lambda_n}{(e_{I^c})} & \xrightarrow{\pi_{I,J}} & \frac{\Lambda_n}{(e_{J^c})} \\ \psi_I^{-1} \uparrow \wr & & \wr \downarrow \psi_J \\ \frac{\Lambda_{u-r+1}}{(b_1 a_1)} & \xrightarrow{\gamma} & \frac{\Lambda_{t-s+1}}{(b_1 a_1)} \end{array}$$

commutes, proving the lemma. \square

Lemma 3.7.7. *Let $J \subsetneq I \subseteq [n]$, such that $J = [s, t]$ and $I = [r, u]$. Then $I^c \subsetneq J^c$, and the following holds.*

(a) *If $1 \in J$ (thus $1 = s = r$) there is a commutative diagram*

$$\begin{array}{ccc} \frac{\Lambda_n}{(e_{I^c})} & \xrightarrow{\pi_{I,J}} & \frac{\Lambda_n}{(e_{J^c})} \\ \overline{f_{u+1} \cdots f_n} \downarrow & & \downarrow \overline{f_{t+1} \cdots f_n} \\ \Lambda_u & \xrightarrow{f_{t+1} \cdots f_u} & \Lambda_t \end{array} \quad (3.7.1)$$

(b) *If $1 \notin I$ (thus $1 \notin J$) there is a commutative diagram*

$$\begin{array}{ccc} \frac{\Lambda_n}{(e_{I^c})} & \xrightarrow{\pi_{I,J}} & \frac{\Lambda_n}{(e_{J^c})} \\ \psi_I \downarrow & & \downarrow \psi_J \\ \Pi_{u-r+1} & \xrightarrow{\gamma_{I,J}} & \Pi_{t-s+1} \end{array} \quad (3.7.2)$$

Proof. (a) Let $\lambda \in \Lambda_n$. Then

$$\begin{aligned} f_{t+1} \cdots f_u \circ \overline{f_{u+1} \cdots f_n}(\lambda + (e_{I^c})) &= f_{t+1} \cdots f_u(\overline{f_{u+1} \cdots f_n}(\lambda + (e_{I^c}))) \\ &= f_{t+1} \cdots f_u(f_{u+1} \cdots f_n(\lambda)) = f_{t+1} \cdots f_n(\lambda) \\ &= \overline{f_{t+1} \cdots f_n}(\lambda + (e_{J^c})) = \overline{f_{t+1} \cdots f_n}(\pi_{I,J}(\lambda + (e_{I^c}))) \\ &= \overline{f_{t+1} \cdots f_n} \circ (\pi_{I,J})(\lambda + (e_{I^c})). \end{aligned}$$

(b) Follows from Lemma 3.7.6, since $\gamma_{I,J} \circ \psi_I = \psi_J \circ \pi_{I,J} \circ \psi_I^{-1} \circ \psi_I = \psi_J \circ \pi_{I,J}$. \square

Let $J \subsetneq I \subseteq \{1, \dots, n\}$. Thus we can write them uniquely, up to permutation, as disjoint union of discrete intervals $J = \bigsqcup_{j=1}^l J_j$ and $I = \bigsqcup_{i=1}^m I_i$, with $J_j = [s_j, t_j]$ and $I_i = [r_i, u_i]$ for all $j \in [l]$ and $i \in [m]$. Then, it is clear that there exists a unique function $f = f_{I,J}: [l] \rightarrow [m]$ such that $J_j \subseteq I_{f(j)}$. Moreover, if $1 \in J$, we always choose an ordering of the intervals such that $1 \in J_1$. Under this conditions, the following holds.

Lemma 3.7.8. *Let $J \subsetneq I \subseteq [n]$ as above, and suppose that $l \leq m$. Then we can reorder the intervals I_i in such a way that $f(j) = j$ for all $1 \leq j \leq l$, i.e. $J_j \subseteq I_j$ for $1 \leq j \leq l$, and the following diagram of ring homomorphisms*

$$\begin{array}{ccc} \frac{\Lambda_n}{(e_{I^c})} & \xrightarrow{\pi_{I,J}} & \frac{\Lambda_n}{(e_{J^c})} \\ \overline{\varphi}_I \downarrow \wr & & \wr \downarrow \overline{\varphi}_J \\ \times_{i=1}^m \frac{\Lambda_n}{(e_{I_i^c})} & \xrightarrow{p_{I,J}} & \times_{j=1}^l \frac{\Lambda_n}{(e_{J_j^c})} \end{array}$$

commutes, where $p_{I,J}(\lambda_1 + (e_{I_1^c}), \dots, \lambda_m + (e_{I_m^c})) := (\lambda_1 + (e_{J_1^c}), \dots, \lambda_l + (e_{J_l^c}))$

Proof. The first assertion is clear. Also it is clear that $p_{I,J}$ is a well defined ring surjection. Now we prove that the diagram commutes. For, let $\lambda \in \Lambda_n$, then

$$\begin{aligned} p_{I,J} \circ \bar{\varphi}_I(\lambda + (e_{I^c})) &= p_{I,J}(\lambda + (e_{I_1^c}), \dots, \lambda + (e_{I_m^c})) \\ &= (\lambda + (e_{J_1^c}), \dots, \lambda + (e_{J_l^c})) \\ &= \bar{\varphi}_J(\lambda + (e_{J^c})) = \bar{\varphi}_J \circ \pi_{I,J}(\lambda + (e_{I^c})), \end{aligned}$$

proving the lemma. \square

Now we are able to present a complete characterisation of homological embeddings between Serre subcategories of $\text{mod } \Lambda_n$.

Theorem 3.7.9. *Let $J \subseteq I \subseteq [n]$. The embedding $\iota_{J,I}: \text{mod } \Lambda_n/(e_{J^c}) \rightarrow \text{mod } \Lambda_n/(e_{I^c})$ is homological if and only if*

- (i) $1 \in J$ and $J_j = I_{f(j)}$ for all $2 \leq j \leq l$, or
- (ii) $1 \notin J$ and $J_j = I_{f(j)}$ for all $1 \leq j \leq l$.

Proof. (\Leftarrow) Suppose first that (i) holds. Then $1 \in J_1 \subseteq I_1$, and necessarily $l \leq m$. Reordering the intervals I_i we can assume $f(j) = j$ for all $2 \leq j \leq l$, therefore $I = I_1 \sqcup \bigsqcup_{j=2}^l J_j \sqcup \bigsqcup_{i=l+1}^m I_i$, where $\bigsqcup_{i=l+1}^m I_i$ is empty whenever $l = m$. Then we have the following diagram of ring morphisms:

$$\begin{array}{ccc} \frac{\Lambda_n}{(e_{I^c})} & \xrightarrow{\pi_{I,J}} & \frac{\Lambda_n}{(e_{J^c})} \\ \bar{\varphi}_I \downarrow \wr & & \wr \downarrow \bar{\varphi}_J \\ \frac{\Lambda_n}{(e_{I_1^c})} \times \times_{j=2}^l \frac{\Lambda_n}{(e_{J_j^c})} \times \times_{i=l+1}^m \frac{\Lambda_n}{(e_{I_i^c})} & \xrightarrow{p_{I,J}} & \frac{\Lambda_n}{(e_{J_1^c})} \times \times_{j=2}^l \frac{\Lambda_n}{(e_{J_j^c})} \\ \overline{f_{u_1+1} \cdots f_n} \times \times_{j=2}^l \psi_{J_j} \times \times_{i=l+1}^m \psi_{I_i} \downarrow \wr & & \wr \downarrow \overline{f_{t_1+1} \cdots f_n} \times \times_{j=2}^l \psi_{J_j} \\ \Lambda_{u_1} \times \times_{j=2}^l \Pi_{t_j-s_j+1} \times \times_{i=l+1}^m \Pi_{u_i-r_i+1} & \xrightarrow{\eta} & \Lambda_{t_1} \times \times_{j=2}^l \Pi_{t_j-s_j+1} \end{array}$$

where $\eta := f_{t_1+1} \cdots f_{u_1} \times p$, and $p: R \times S \rightarrow R$ is just the projection $(r, s) \mapsto r$, and R and S are the corresponding direct products of preprojective algebras in the diagram, if $l < m$. When $l = m$, set $p = 1_R$. Moreover, if $t_1 = u_1$, we replace $f_{t_1+1} \cdots f_{u_1}$ by $1_{\Lambda_{u_1}}$.

By Theorem 3.6.3 and Corollary 3.4.6 the vertical arrows are isomorphisms, and by Lemma 3.7.8 the upper square commutes. Now we show the commutativity of the lower

square. For, let $\lambda_i \in \Lambda_n$ for $1 \leq i \leq m$, then

$$\begin{aligned}
\eta \circ \overline{f_{u_1+1} \cdots f_n} &\times \prod_{j=2}^l \psi_{J_j} \times \prod_{i=l+1}^m \psi_{I_i}(\lambda_1 + (e_{I_1^c}), \dots, \lambda_m + (e_{I_m^c})) = \\
&= \eta(\overline{f_{u_1+1} \cdots f_n}(\lambda_1 + (e_{I_1^c}), (\psi_{J_j}(\lambda_j + (e_{J_j^c})))_j, (\psi_{I_i}(\lambda_i + (e_{I_i^c})))_i)) \\
&= (\overline{f_{t_1+1} \cdots f_{u_1} f_{u_1+1} \cdots f_n}(\lambda_1 + (e_{I_1^c}), (\psi_{J_j}(\lambda_j + (e_{J_j^c})))_j)) \\
&= (\overline{f_{t_1+1} \cdots f_n} \circ \pi_{I, J_1}(\lambda_1 + (e_{I_1^c}), (\psi_{J_j}(\lambda_j + (e_{J_j^c})))_j)) \\
&= (\overline{f_{t_1+1} \cdots f_n}(\lambda_1 + (e_{J_1^c}), (\psi_{J_j}(\lambda_j + (e_{J_j^c})))_j)) \\
&= (\overline{f_{t_1+1} \cdots f_n} \times \prod_{j=2}^l \psi_{J_j})(\lambda_1 + (e_{J_1^c}), (\lambda_j + (e_{J_j^c}))_j) \\
&= (\overline{f_{t_1+1} \cdots f_n} \times \prod_{j=2}^l \psi_{J_j}) \circ p_{I, J}(\lambda_1 + (e_{I_1^c}), \dots, \lambda_m + (e_{I_m^c}))
\end{aligned}$$

using Lemma 3.7.7 (a) in the third equality. Hence the lower square commutes.

Now, p and $f_{t_1+1} \cdots f_{u_1}$ (or $1_{\Lambda_{u_1}}$ if $t_1 = u_1$) are homological ring epimorphisms by Lemma 1.5.9 and Corollary 3.7.2 respectively, therefore η is so considering Proposition 1.5.10. Thus $\pi_{I, J}$ is a homological ring epimorphism, so Corollary 1.5.7 (a) implies that $\iota_{J, I}$ is a homological embedding.

Now suppose (ii) holds. As before, we can reorder the intervals I_i in such a way that $f(j) = j$ for $1 \leq j \leq l$. Moreover, $l < m$, since $J \neq I$. Thus, $I = \bigsqcup_{j=1}^l J_j \sqcup \bigsqcup_{i=l+1}^m I_i$, and we have the following diagram.

$$\begin{array}{ccc}
\frac{\Lambda_n}{(e_{I^c})} & \xrightarrow{\pi_{I, J}} & \frac{\Lambda_n}{(e_{J^c})} \\
\downarrow \bar{\varphi}_I \wr & & \downarrow \wr \bar{\varphi}_J \\
\prod_{j=1}^l \frac{\Lambda_n}{(e_{J_j^c})} \times \prod_{i=l+1}^m \frac{\Lambda_n}{(e_{I_i^c})} & \xrightarrow{p_{I, J}} & \prod_{j=1}^l \frac{\Lambda_n}{(e_{J_j^c})} \\
\downarrow \wr \prod_{j=1}^l \psi_{J_j} \times \prod_{i=l+1}^m \psi_{I_i} & & \downarrow \wr \prod_{j=1}^l \psi_{J_j} \\
\prod_{j=1}^l \Pi_{t_j-s_j+1} \times \prod_{i=l+1}^m \Pi_{u_i-r_i+1} & \xrightarrow{p} & \prod_{j=1}^l \Pi_{t_j-s_j+1}
\end{array}$$

where p is the projection onto the first component. Note that the vertical arrows are isomorphisms. Thus similar arguments as in case (i) show that the diagram is commutative, and therefore $\pi_{I, J}$ is a homological ring epimorphism. Thus, by Corollary 1.5.7 (a), $\iota_{J, I}$ is a homological embedding.

(\Rightarrow) We proceed by contraposition. So, we have three cases:

- (1) $1 \in J$ and there exists $j_0 \in \{2, \dots, l\}$ such that $J_{j_0} \subsetneq I_{f(j_0)}$,
- (2) $1 \notin J$ and there exists $j_0 \in \{1, \dots, l\}$ such that $J_{j_0} \subsetneq I_{f(j_0)}$, and
- (3) there exists $j_0 \in \{2, \dots, l\}$ such that $J_{j_0} \subsetneq I_{f(j_0)}$.

We need to show that in either case, the embedding $\iota_{J,I}$ is not homological. For, first consider (1). Then, $1 \notin J_{j_0}$, otherwise $1 \in J_1 \cap J_{j_0}$, and since the intervals are disjoint, we have $J_1 = J_{j_0}$, contradiction. Then we have the following commutative diagram

$$\begin{array}{ccc}
\frac{\Lambda_n}{(e_{I^c})} & \xrightarrow{\pi_{I,J}} & \frac{\Lambda_n}{(e_{J^c})} \\
\downarrow \overline{\varphi}_I \wr & & \downarrow \wr \overline{\varphi}_J \\
\prod_{i=1}^m \frac{\Lambda_n}{(e_{I_i^c})} & & \prod_{j=1}^l \frac{\Lambda_n}{(e_{J_j^c})} \\
\downarrow p & & \downarrow p' \\
\frac{\Lambda_n}{(e_{I_{f(j_0)}^c})} & \xrightarrow{\pi_{I_{f(j_0)}, J_{j_0}}} & \frac{\Lambda_n}{(e_{J_{j_0}^c})}
\end{array}$$

where p and p' are the projections onto the components $f(j_0)$ and j_0 respectively. It induces the following commutative diagram of functors given by restriction of scalars

$$\begin{array}{ccc}
\text{mod } \Lambda_n/(e_{I^c}) & \xleftarrow{\iota_{J,I}} & \text{mod } \Lambda_n/(e_{J^c}) \\
(\overline{\varphi}_I)_* \wr \uparrow & & \wr \uparrow (\overline{\varphi}_J)_* \\
\text{mod } \prod_{i=1}^m \Lambda_n/(e_{I_i^c}) & & \text{mod } \prod_{j=1}^l \Lambda_n/(e_{J_j^c}) \\
p_* \wr \uparrow & & \wr \uparrow p'_* \\
\text{mod } \Lambda_n/(e_{I_{f(j_0)}^c}) & \xleftarrow{\iota_{J_{j_0}, I_{f(j_0)}}} & \text{mod } \Lambda_n/(e_{J_{j_0}^c})
\end{array}$$

Then, from Corollary 3.7.5 we have that $\iota_{J_{j_0}, I_{f(j_0)}}$ is not a homological embedding, since $1 \notin J_{j_0}$, thus there exist left modules $X, Y \in \text{mod } \Lambda_n/(e_{J_{j_0}^c})$ and $q > 0$ such that

$$\text{Ext}_{\frac{\Lambda_n}{(e_{J_{j_0}^c})}}^q(X, Y) \not\cong \text{Ext}_{\frac{\Lambda_n}{(e_{I_{f(j_0)}^c})}}^q(X, Y).$$

On the other hand, p_* and p'_* are homological embeddings (cf. Lemma 1.5.9), and since $(\overline{\varphi}_I)_*$ and $(\overline{\varphi}_J)_*$ are equivalences, we get

$$\begin{aligned}
\text{Ext}_{\frac{\Lambda_n}{(e_{J^c})}}^q(X, Y) &\cong \text{Ext}_{\prod_{j=1}^l \frac{\Lambda_n}{(e_{J_j^c})}}^q(X, Y) \cong \text{Ext}_{\frac{\Lambda_n}{(e_{J_{j_0}^c})}}^q(X, Y) \\
&\not\cong \text{Ext}_{\frac{\Lambda_n}{(e_{I_{f(j_0)}^c})}}^q(X, Y) \cong \text{Ext}_{\prod_{i=1}^m \frac{\Lambda_n}{(e_{I_i^c})}}^q(X, Y) \cong \text{Ext}_{\frac{\Lambda_n}{(e_{I^c})}}^q(X, Y),
\end{aligned}$$

thus $\text{Ext}_{\frac{\Lambda_n}{(e_{J^c})}}^q(X, Y) \not\cong \text{Ext}_{\frac{\Lambda_n}{(e_{I^c})}}^q(X, Y)$, proving that the embedding $\iota_{J,I}$ is not homological.

Now assume that (2) holds. Since $1 \notin J$, we have $1 \notin J_{j_0}$, and Corollary 3.7.5 shows that $\iota_{J_{j_0}, I_{f(j_0)}}$ is not a homological embedding, and the same calculations as before show that $\iota_{J,I}$ is not homological as well. Finally (3) follows from (1) and (2). This completes the proof. \square

3.8 The homological poset of $\text{Aus } T_n$

Let $\Lambda_n = \text{Aus } T_n$. In last section we characterised homological embeddings between Serre subcategories of $\text{mod } \Lambda_n$. In what follows we describe the set $\text{Cov } \mathcal{H}(\Lambda_n)$ of cover relations of the homological poset $\mathcal{H}(\Lambda_n)$. Recall that this poset is defined on the power set $2^{[n]}$, and for two subsets $J \subseteq I \subseteq [n]$ the relation is given by $J \preceq_{\Lambda_n} I$ if and only if the embedding $\iota_{J,I}: \text{mod } \Lambda_n/(e_{J^c}) \hookrightarrow \text{mod } \Lambda_n/(e_{I^c})$ is homological.

Notation. If $J = \bigsqcup_{j=1}^l J_j \subseteq [n]$, set $\ell(J) := l$. Thus, $\ell(\emptyset) = 0$.

For simplicity, we write \preceq instead of \preceq_{Λ_n} . The next proposition characterises the elements of $\text{Cov } \mathcal{H}(\Lambda_n)$.

Proposition 3.8.1. *Let $J \prec I \subseteq [n]$.*

(a) *If $1 \in J$, then I covers J if and only if*

- (a1) $I_1 = J_1$ and $\ell(I) = \ell(J) + 1$, or
- (a2) $u_1 = t_1 + 1$ and $\ell(I) = \ell(J)$.

(b) *If $1 \notin J$, then I covers J if and only if*

- (b1) $1 \notin I$ and $\ell(I) = \ell(J) + 1$, or
- (b2) $I \setminus J = \{1\}$.

Moreover, all these conditions are pairwise disjoint and characterise all the cover relations of $\mathcal{H}(\Lambda_n)$.

Proof. Assume that $J \prec I$. In particular $J \subseteq I$ and $\ell(J) \leq \ell(I)$, thus there is a function $f = f_{J,I}: \{1, \dots, \ell(J)\} \rightarrow \{1, \dots, \ell(I)\}$, such that if $1 \leq j \leq j' \leq \ell(J)$, then $f(j) \leq f(j')$.

(a) Suppose $1 \in J$. Then $J_1 \subseteq I_1$ and $J_j = I_{f(j)}$ for all $2 \leq j \leq \ell(J)$. In particular, $J_1 = [1, t_1]$ and $I_1 = [1, u_1]$ with $t_1 \leq u_1$.

(\Rightarrow) We proceed by contraposition. Thus we have four cases.

Case 1. $J_1 \subsetneq I_1$ and $u_1 \neq t_1 + 1$. Then $u_1 < t_1 + 1 \Leftrightarrow u_1 \leq t_1$, and since $t_1 \leq u_1$, we have $t_1 = u_1$, a contradiction, since $J_1 \neq I_1$. Thus we can assume $u_1 > t_1 + 1$. Then $t_1 + 2 \leq u_1 = u_{f(1)} < r_{f(2)} = s_2$. Set $J' := J \cup \{t_1 + 1\}$, then $J' = [1, t_1 + 1] \sqcup J_2 \sqcup \dots \sqcup J_{\ell(J)}$, since $t_1 + 2 < s_2$. Thus $J \prec J' \prec I$, given $J' \neq I$.

Case 2. $J_1 \subsetneq I_1$ and $\ell(J) < \ell(I)$. Then f is not surjective, otherwise $\ell(J) \geq \ell(I)$, a contradiction. Thus there exists $i_0 \in [\ell(I)]$, such that $i_0 \notin \text{Im } f$, i.e. $J_j \not\subseteq I_{i_0}$ for all $j \in [\ell(J)]$, moreover $2 \leq i_0$ because $J_1 \subset I_1$, thus $J_j = I_{f(j)} \not\subseteq I_{i_0}$ for all $2 \leq j \leq \ell(J)$. Set $J' := J \cup I_{i_0}$. Thus $J' = J \sqcup I_{i_0}$. Therefore, $J \prec J' \prec I$, since $J_1 \subset I_1$.

Case 3. $\ell(I) \neq \ell(J) + 1$ and $u_1 \neq t_1 + 1$. We have four subcases.

(i) $\ell(I) > \ell(J) + 1$ and $u_1 > t_1 + 1$. Then $t_1 + 2 \leq u_1 < r_{f(2)} = s_2$. Set $J' := J \cup \{t_1 + 1\}$. Then $J' = [1, t_1 + 1] \sqcup J_2 \sqcup \dots \sqcup J_{\ell(J)}$, given that $t_1 + 2 < s_2$. Therefore $J \prec J' \prec I$, since $\ell(J') = \ell(J) < \ell(I)$.

(ii) $\ell(I) > \ell(J) + 1$ and $u_1 < t_1 + 1$. Then $u_1 = t_1$, i.e. $J_1 = I_1$, and since $\ell(I) > \ell(J)$, as in Case 2, there exists $2 \leq i_0 \leq \ell(I)$, such that $i_0 \notin \text{Im } f$. Defining $J' := J \cup I_{i_0}$, we get $J' = J \sqcup I_{i_0}$. Thus $J \prec J' \prec I$, since $\ell(J') = \ell(J) + 1 < \ell(I)$.

(iii) $\ell(I) < \ell(J) + 1$ and $u_1 > t_1 + 1$. Then $\ell(I) = \ell(J)$, thus $f = 1_{[\ell(J)]}$, i.e. $J_j \subseteq I_j$ for all $j \in [\ell(J)]$. Therefore $t_1 + 2 \leq u_1 < r_{f(2)} = r_2 = s_2$, considering $I_2 = J_2$. Setting $J' := J \cup \{t_1 + 1\}$, we get $J' = [1, t_1 + 1] \sqcup J_2 \sqcup \cdots \sqcup J_{\ell(J)}$, since $t_1 + 2 < s_2$. Thus $J \prec J' \prec I$, since $t_1 + 1 < u_1$.

(iv) $\ell(I) < \ell(J) + 1$ and $u_1 < t_1 + 1$. Then $\ell(I) = \ell(J)$ and $u_1 = t_1$. Thus $J_j = I_j$ for all $1 \leq j \leq \ell(J)$, so $J = I$, contradiction. Then this subcase does not hold.

Case 4. $\ell(I) \neq \ell(J) + 1$ and $\ell(I) \neq \ell(J)$. Then we have two subcases, since $\ell(J) \leq \ell(I)$.

(i) $\ell(I) > \ell(J) + 1$ and $\ell(I) > \ell(J)$. This is equivalent to $\ell(I) > \ell(J) + 1$. So, as in Case 2, there exists $2 \leq i_0 \leq \ell(I)$, such that $i_0 \notin \text{Im } f$. Set $J' := J \cup I_{i_0}$, thus $J' = J \sqcup I_{i_0}$. Then $J \prec J' \prec I$, since $\ell(J') = \ell(J) + 1 < \ell(I)$.

(ii) $\ell(I) < \ell(J) + 1$ and $\ell(I) > \ell(J)$. Then $\ell(I) = \ell(J)$ and $\ell(I) > \ell(J)$, a contradiction. Thus, this subcase does not occur.

Therefore in each case, we conclude that I does not cover J . Completing the proof for the necessity of (a).

(\Leftarrow) Set $l := \ell(J)$. Suppose (a1) holds. Since $\ell(I) = l + 1$ and $I_1 = J_1$, there exists a unique $2 \leq i_0 \leq l + 1$ such that $i_0 \notin \text{Im } f$, and since $J \prec I$, we have $I \setminus J = I_{i_0}$. Suppose that there exists $J' \subseteq [n]$, with $J \prec J' \prec I$, then necessarily there exists non-empty $L \subsetneq I_{i_0}$ such that $J' = J \sqcup L$. If $\ell(L) > 1$, then $\ell(J') = l + \ell(L) > l + 1 = \ell(I)$, a contradiction, since $\ell(J') \leq \ell(I)$, considering that $J' \prec I$. Thus $L = [p, q] \subset I_{i_0}$ for some integers $p \leq q$. So, since $J' \prec I$ and $i_0 \geq 2$, Theorem 3.7.9 implies that $L = I_{i_0}$, a contradiction. Thus such a J' does not exist, proving that I covers J .

Now suppose (a2) holds. Since $\ell(I) = \ell(J) = l$, we can choose an ordering of the intervals of I such that $J_j \subseteq I_j$ for all $1 \leq j \leq l$. In particular, $J_j = I_j$ for all $2 \leq j \leq l$, considering $J \prec J$. Moreover, $J_1 = [1, t_1] \subset I_1 = [1, t_1 + 1]$, thus $J \setminus I = \{t_1 + 1\}$, i.e. I covers J in $(2^{[n]}, \subseteq)$, hence the same holds in $(2^{[n]}, \preceq)$. This completes the proof of (a).

(b) Suppose $1 \notin J$. Then $J_j = I_{f(j)}$ for all $1 \leq j \leq \ell(J)$.

(\Rightarrow) We proceed by contraposition. Thus we have two cases.

Case 1. $1 \in I$ and $I \setminus J \neq \{1\}$. We can distinguish two subcases.

(i) $\ell(I) = \ell(J)$. Then the conditions in (b) show that $f = 1_{[\ell(J)]}$, and in particular $J_1 = I_1$, thus $1 \in J_1 \subseteq J$, a contradiction. Thus this subcase does not hold.

(ii) $\ell(I) > \ell(J) \Leftrightarrow \ell(I) \geq \ell(J) + 1$. If $\ell(I) > \ell(J) + 1$, then as before f is not surjective, thus there exists $1 \leq i_0 \leq \ell(I)$ such that $i_0 \notin \text{Im } f$. Then, for $J' := J \cup I_{i_0}$, we get $J' = J \sqcup I_{i_0}$, and $J \prec J' \prec I$.

If $\ell(I) = \ell(J) + 1$, since $J_j = I_{f(j)}$ for all $1 \leq j \leq \ell(J) = \ell(I) - 1$ and $1 \in I \setminus J$, we have $I = I_1 \sqcup J_1 \sqcup \cdots \sqcup J_{\ell(J)}$. Then $I \setminus J = I_1$, thus $I_1 \neq \{1\}$ by assumption, so $u_1 \geq 2$. On the other hand, $1 < f(1)$, otherwise $1 = f(1)$, so $J_1 = I_1$, thus $1 \in I_1 = J_1 \subseteq J$, a contradiction. Then $2 \leq u_1 < r_{f(1)} = s_1$, thus $3 \leq s_1$. Therefore, setting $J' := J \cup \{1\}$, we get $J' = [1, 1] \sqcup J$. Then $I \neq J'$, given $I \setminus J' = I_1 \setminus [1, 1] \neq \emptyset$. Thus $J \prec J' \prec I$.

Case 2. $\ell(I) \neq \ell(J) + 1$ and $I \setminus J \neq \{1\}$. As before, we have two subcases.

(i) $\ell(I) < \ell(J) + 1$. Then $\ell(I) = \ell(J)$. Thus $f = 1_{[\ell(J)]}$, considering (b), so $I = J$, a contradiction.

(ii) $\ell(I) > \ell(J) + 1$. Similar as in Case 1 (ii).

Therefore in each case, we conclude that I does not cover J . This completes the proof for the necessity of (b).

(\Leftarrow) Consider that (b1) holds. Similar arguments as in case (a1) show that I covers J .

Note that condition $1 \notin I$ is necessary to have J covered by I . Indeed, as counterexample consider $J = [4, 4]$ and $I = [1, 2] \sqcup [4, 4]$, then $J \prec I$, $1 \notin J$ and $\ell(I) = 2 = \ell(J) + 1$. But $J' = [1, 1] \sqcup [4, 4]$ satisfies $J \prec J' \prec I$, showing that I does not cover J .

Finally suppose that (b2) holds. Then I covers J in $(2^{[n]}, \subseteq)$, thus the same holds in $(2^{[n]}, \preceq)$. The last assertion is clear. This completes the proof of the proposition. \square

In order to find some combinatorial properties of $\mathcal{H}(\Lambda_n)$, we define the following sets which contain the cover relations given by Proposition 3.8.1.

Definition 3.8.2. Let $n \geq 0$. Define the following subsets of $2^{[n]} \times 2^{[n]}$.

$$A(n) := \{(J, I) \in 2^{[n]} \times 2^{[n]} \mid J \prec I, 1 \in J, I_1 = J_1 \text{ and } \ell(I) = \ell(J) + 1\}.$$

$$B(n) := \{(J, I) \in 2^{[n]} \times 2^{[n]} \mid J \prec I, 1 \in J, u_1 = t_1 + 1 \text{ and } \ell(I) = \ell(J)\}.$$

$$C(n) := \{(J, I) \in 2^{[n]} \times 2^{[n]} \mid J \prec I, 1 \notin I, \text{ and } \ell(I) = \ell(J) + 1\}.$$

$$D(n) := \{(J, I) \in 2^{[n]} \times 2^{[n]} \mid J \prec I, I \setminus J = \{1\}\}.$$

We denote the cardinality of each set using the corresponding lower case letter, e.g. $a(n) = \text{card } A(n)$.

Therefore,

$$\text{Cov } \mathcal{H}(\Lambda_n) = A(n) \sqcup B(n) \sqcup C(n) \sqcup D(n)$$

and $h(n) := \text{card Cov } \mathcal{H}(\Lambda_n) = a(n) + b(n) + c(n) + d(n)$ is the number of cover relations in $\mathcal{H}(\Lambda_n)$, by Proposition 3.8.1. Now, we determine $h(n)$ as follows.

Proposition 3.8.3. For $n \geq 0$, the following equalities hold.

$$(a) \quad a(n) = 0 \text{ for } n = 0, 1, \text{ and } a(n) = (n - 2) \cdot 2^{n-3} \text{ for } n \geq 2.$$

$$(b) \quad b(n) = 0 \text{ for } n = 0, 1, \text{ and } b(n) = 2^{n-2} \text{ for } n \geq 2.$$

$$(c) \quad c(n) = 0 \text{ for } n = 0, 1, \text{ and } c(n) = n \cdot 2^{n-3} \text{ for } n \geq 2.$$

$$(d) \quad d(0) = 0, d(1) = 1, \text{ and } d(n) = 2^{n-2} \text{ for } n \geq 2.$$

Proof. (a) It is clear that $A(n) = \emptyset$ for $n = 0, 1, 2$, and $A(3) = \{(\{1\}, \{1, 3\})\}$, thus $a(3) = 1 = (3 - 2) \cdot 2^{3-3}$. Now, let $n \geq 4$. Therefore, by induction we get that $a(n - 1) = (n - 3) \cdot 2^{n-4}$. We define the following three functions.

$$\begin{aligned} \alpha: A(n - 1) &\rightarrow A(n) & (J, I) &\mapsto (J, I) \\ \beta: A(n - 1) &\rightarrow A(n) & (J, I) &\mapsto \begin{cases} (J, I \cup \{n\}) & \text{if } n - 1 \in I \setminus J \\ (J \cup \{n\}, I \cup \{n\}) & \text{if } n - 1 \notin I \setminus J \end{cases} \\ \gamma: \{J \mid J \subseteq \{2, \dots, n - 2\}\} &\rightarrow A(n) & J &\mapsto (\{1\} \cup J, \{1, n\} \cup J) \end{aligned}$$

Note that α is well defined. Now we show that β is well defined. For, let $(J, I) \in A(n - 1)$, and write $J = [1, t_1] \sqcup \dots \sqcup [s_l, t_l]$. Set $I' := I \cup \{n\}$. First, suppose that $n - 1 \in I \setminus J$. Thus $I = J \sqcup [r_{l+1}, n - 1]$ with $t_l + 2 \leq r_{l+1} \leq n - 1$, so $I' = J \sqcup [r_{l+1}, n]$. Then it is easy to verify that $(J, I') \in A(n)$. Secondly, suppose that $n - 1 \notin I \setminus J$. We have two cases.

(i) $n-1 \notin I$. Thus, $n-1 \notin J$, so $(J, I) \in A(n-2)$, thus $J' := J \cup \{n\} = J \sqcup [n, n]$ and $I' = I \sqcup [n, n]$. Therefore, $(J', I') \in A(n)$, since $\ell(I') = \ell(I) + 1 = \ell(J) + 1 + 1 = \ell(J') + 1$.

(ii) $n-1 \in J$. Thus $n-1 \in I$. In this case, we can write $J = [1, t_1] \sqcup \cdots \sqcup [s_l, n-1]$, $I = I_1 \sqcup \cdots \sqcup I_{l+1}$, and without loss of generality we can assume $I_1 = J_1$ and $I_{l+1} = J_l$, considering that $\{1, n-1\} \subseteq J \subseteq I$. Thus, there exists a unique $i_0 \in \{2, \dots, l\}$ such that $I \setminus J = I_{i_0}$, so $I = J \sqcup I_{i_0}$. Therefore, $J' := J \cup \{n\} = [1, t_1] \sqcup \cdots \sqcup [s_l, n]$ and $I' = J' \sqcup I_{i_0}$, thus $(J', I') \in A(n)$. This proves that β is well defined.

To prove that γ is well defined, let $J \subseteq \{2, \dots, n-2\}$. Write $J = [s_1, t_1] \sqcup \cdots \sqcup [s_l, t_l]$, and set $J' := \{1\} \cup J$ and $I' := \{1, n\} \cup I$. If $2 \in J$, then $J' = [1, t_1] \sqcup \cdots \sqcup [s_l, t_l]$ and $I' = [1, t_1] \sqcup \cdots \sqcup [s_l, t_l] \sqcup [n, n]$, so clearly $(J', I') \in A(n)$. On the other hand, if $2 \notin J$, then $J' = [1, 1] \sqcup J$ and $I' = J' \cup [n, n]$, thus $(J', I') \in A(n)$, proving that γ is well defined. Thus, this shows that $\text{Im } \alpha \cup \text{Im } \beta \cup \text{Im } \gamma \subseteq A(n)$.

Next we prove that $A(n) \subseteq \text{Im } \alpha \cup \text{Im } \beta \cup \text{Im } \gamma$. For, let $(J', I') \in A(n)$. If $n \notin I'$, then $(J', I') \in A(n-1) = \text{Im } \alpha$. So, we can assume that $n \in I'$. We have two cases.

Case 1. $n \notin J'$. Then $J' = [1, t_1] \sqcup \cdots \sqcup [s_l, t_l]$ and $I' = J' \sqcup [r_{l+1}, n]$, with $t_l + 2 \leq r_{l+1} \leq n$. If $n-1 \in I'$, then $r_{l+1} \leq n-1$, thus $I := I' \setminus \{n\}$ satisfies that $(I, J') \in A(n-1)$, and $\beta(J', I) = (J', I')$, since $n-1 \in I \setminus J'$. If $n-1 \notin I'$, then $r_{l+1} = n$, and $J := I' \setminus \{1, n\} \subseteq \{2, \dots, n-2\}$ satisfies $\gamma(J) = (\{1\} \cup J, \{1, n\} \cup J) = (J', I')$, since $J' = I' \setminus n = 1 \cup (I' \setminus \{1, n\}) = 1 \cup J$.

Case 2. $n \in J'$. Since $(J', I') \in A(n)$, we can write $J' = [1, t_1] \sqcup \cdots \sqcup [s_l, n]$ and $I' = J' \sqcup I'_{i_0}$ for some unique interval $I'_{i_0} \subseteq \{t_1 + 2, \dots, s_l - 2\}$. Set $I := I' \setminus \{n\}$ and $J := J' \setminus \{n\}$, hence $(J, I) \in A(n-1)$ and $\beta(J, I) = (J', I')$, since $n-1 \notin I_{i_0} = I \setminus J$. This proves that $A(n) = \text{Im } \alpha \cup \text{Im } \beta \cup \text{Im } \gamma$.

Now we show that the images are pairwise disjoint. Indeed, if $(J', I') \in \text{Im } \alpha \cap \text{Im } \beta$, then $I' \subseteq [n-1]$ and $n \in I'$, a contradiction, thus $\text{Im } \alpha \cap \text{Im } \beta = \emptyset$. Similarly $\text{Im } \alpha \cap \text{Im } \gamma = \emptyset$. Now, let $(J', I') \in \text{Im } \beta \cap \text{Im } \gamma$. In particular $n-1 \notin I'$ and there exists $I \subseteq [n-1]$ such that $I' = I \cup \{n\}$, thus there exists $J \subseteq I$ such that $J' = J \cup \{n\}$. Thus $I' \setminus J' = \{n\}$ and $I' \setminus J' \not\subseteq n$, a contradiction. Thus $\text{Im } \beta \cap \text{Im } \gamma = \emptyset$.

Note that α, β and γ are injective functions, hence the previous proved conditions imply that

$$\begin{aligned} a(n) &= \text{card } A(n) = \text{card}(\text{Im } \alpha) + \text{card}(\text{Im } \beta) + \text{card}(\text{Im } \gamma) \\ &= \text{card } A(n-1) + \text{card } A(n-1) + 2^{n-3} = 2a(n-1) + 2^{n-3} \\ &= (n-3) \cdot 2^{n-3} + 2^{n-3} = (n-2) \cdot 2^{n-3}. \end{aligned}$$

(b) It is clear that $B(n) = \emptyset$ for $n = 0, 1$, and $B(2) = \{(\{1\}, \{1, 2\})\}$, thus $a(2) = 1 = 2^{2-2}$. Now, suppose by induction that $b(n-1) = 2^{n-3}$ for $n > 2$. We define the correspondences

$$\begin{aligned} \delta: B(n-1) &\rightarrow B(n) & (J, I) &\mapsto (J, I) \\ \varepsilon: B(n-1) &\rightarrow B(n) & (J, I) &\mapsto \begin{cases} (J \cup \{n\}, I \cup \{n\}) & \text{if } I \neq [1, n-1] \\ (J \cup \{n-1\}, I \cup \{n\}) & \text{if } I = [1, n-1] \end{cases} \end{aligned}$$

It is clear that δ is well defined. Next we show that ε is a well defined function. For, let $(J, I) \in B(n-1)$. If $\ell(J) = 1$, then necessarily $\ell(I) = 1$, and $J = [1, t_1]$, $I = [1, t_1 + 1]$ with $t_1 + 1 \leq n-1$. If $t_1 + 1 = n-1$, then $\varepsilon(J, I) = ([1, n-1], [1, n]) \in B(n)$, and

if $t_1 + 1 < n - 1$, then $\varepsilon(J, I) = (J \cup \{n\}, I \cup \{n\}) = (J \sqcup [n, n], I \sqcup [n, n]) \in B(n)$. If $\ell(J) = \ell(I) = l > 1$, then $J = [1, t_1] \sqcup \cdots \sqcup [s_l, t_l]$ and $I = [1, t_1 + 1] \sqcup \cdots \sqcup [s_l, t_l]$, and it follows easily that $\varepsilon(J, I) \in B(n)$. This proves that ε is well defined, and that $\text{Im } \delta \cup \text{Im } \varepsilon \subseteq B(n)$.

Next we prove that $B(n) \subseteq \text{Im } \delta \cup \text{Im } \varepsilon$. For, let $(J', I') \in B(n)$. As before, if $n \notin J'$, then $(J', I') \in B(n - 1)$ and $(J', I') = \delta(J', I')$. Thus, we can assume that $n \in I'$. Set $J := J' \setminus \{n\}$ and $I := I' \setminus \{n\}$. If $I' = [1, n]$, then necessarily $J' = [1, n - 1]$. Thus, $I = [1, n - 1]$, and $\varepsilon([1, n - 2], I) = (J', I')$. So, let's assume that $I' \neq [1, n]$. If $\ell(J') = \ell(I') = 1$, then $J' = [1, t_1]$ and $I' = [1, t_1 + 1]$ with $t_1 + 1 \leq n - 1$, and $n \in I'$, a contradiction. Thus, if $I' \neq [1, n]$, and $n \in I'$, then $\ell(I') = \ell(J') = l > 1$. So, $J' = [1, t_1] \sqcup [s_2, t_2] \sqcup \cdots \sqcup [s_l, n]$ and $I' = [1, t_1 + 1] \sqcup [s_2, t_2] \cdots \sqcup [s_l, n]$, where $t_1 + 1 \leq s_2 - 2$. If $s_l = n$, then $J = [1, t_1] \sqcup [s_2, t_2] \sqcup \cdots \sqcup [s_{l-1}, t_{l-1}]$ and $I = [1, t_1 + 1] \sqcup [s_2, t_2] \sqcup \cdots \sqcup [s_{l-1}, t_{l-1}]$ with $t_{l-1} \leq n - 2$, thus $(J, I) \in B(n - 2) \subseteq B(n - 1)$, and $\varepsilon(J, I) = (J', I')$. On the other hand, if $s_l \leq n - 1$, then $J = [1, t_1] \sqcup [s_2, t_2] \sqcup \cdots \sqcup [s_l, n - 1]$ and $I = [1, t_1 + 1] \sqcup [s_2, t_2] \sqcup \cdots \sqcup [s_l, n - 1]$, thus $(J, I) \in B(n - 1)$, and $\varepsilon(J, I) = (J', I')$. This shows that $B(n) = \text{Im } \delta \cup \text{Im } \varepsilon$.

Moreover, $\text{Im } \delta \cap \text{Im } \varepsilon = \emptyset$ and δ, ε are injective functions, hence

$$\begin{aligned} b(n) &= \text{card } B(n) = \text{card}(\text{Im } \delta) + \text{card}(\text{Im } \varepsilon) \\ &= \text{card } B(n - 1) + \text{card } B(n - 1) = 2 \cdot 2^{n-3} = 2^{n-2}. \end{aligned}$$

(c) We have that $C(0) = C(1) = \emptyset$, and $C(2) = \{(\emptyset, \{2\})\}$, thus $c(2) = 1 = 2 \cdot 2^{2-3}$. So, by induction suppose that $n \geq 4$ and $c(m) = m \cdot 2^{m-3}$ for $2 \leq m \leq n - 1$. Hence, $\sum_{i=2}^{n-1} c(i) = (n - 2) \cdot 2^{n-3}$ by induction. Now we define the following correspondences.

$$\begin{aligned} \eta: \{2, 3\} &\rightarrow C(n) & s &\mapsto (\emptyset, [s, n]) \\ \theta: C(n - 1) &\rightarrow C(n) & (J, I) &\mapsto (J, I), \end{aligned}$$

and for $2 \leq i \leq n - 2$, let

$$\begin{aligned} \mu_i: C(i) &\rightarrow C(n) & (J, I) &\mapsto (J \sqcup [i + 2, n], I \sqcup [i + 2, n]) \\ \nu_i: \{J \mid J \subseteq [2, i]\} &\rightarrow C(n) & J &\mapsto (J, J \sqcup [i + 2, n]). \end{aligned}$$

Note that these functions are well defined. Thus $\text{Im } \eta \cup \text{Im } \theta \cup \bigcup_{i=2}^{n-2} (\text{Im } \mu_i \cup \text{Im } \nu_i) \subseteq C(n)$. Now we show the reverse inclusion. For, let $(J', I') \in C(n)$. If $n \notin I'$, then $(J', I') \in C(n - 1)$, thus $(J', I') = \theta(J', I')$.

Now, suppose that $n \in I'$. If $\ell(I') = 1$, then $I' = [r, n]$ for some $2 \leq r \leq n$ and $\ell(J') = 0$, thus $J' = \emptyset$. Then, if $r \in \{2, 3\}$ we have $(J', I') = \eta(r)$, and if $r \geq 4$, then $\nu_{r-2}(\emptyset) = (\emptyset, [r, n]) = (J', I')$. Then we can assume that $\ell(I') = \ell(J') + 1 \geq 2$, i.e. $l = \ell(J') \geq 1$. If $n \in J'$, then there exists $4 \leq r \leq n$ such that $J'_l = [r, n]$, since $\ell(I') \geq 2$. Set $J := J' \setminus [r, n]$ and $I := I' \setminus [r, n]$, then it follows that $(J, I) \in C(r - 2)$, hence $\mu_{r-2}(J, I) = (J', I')$. On the other hand, if $n \notin J'$, then there exists $4 \leq r \leq n$ such that $I'_{l+1} = [r, n]$ and $I' = J' \sqcup [r, n]$, otherwise $\ell(I') = 1$, a contradiction. Thus $J' \subseteq [2, r - 2]$, therefore $\nu_{r-2}(J') = (J', J' \sqcup [r, n]) = (J', I')$ as needed. This shows that $C(n) = \text{Im } \eta \cup \text{Im } \theta \cup \bigcup_{i=2}^{n-2} (\text{Im } \mu_i \cup \text{Im } \nu_i)$.

Note that the images of $\eta, \theta, \mu_i, \nu_j$ are pairwise disjoint for all $2 \leq i, j \leq n-2$, and since they are injective functions, we get

$$\begin{aligned} c(n) &= \text{card } C(n) = \text{card}(\text{Im } \eta) + \text{card}(\text{Im } \theta) + \sum_{i=2}^{n-2} \text{card}(\text{Im } \mu_i) + \sum_{i=2}^{n-2} \text{card}(\text{Im } \nu_i) \\ &= 2 + \text{card } C(n-1) + \sum_{i=2}^{n-2} \text{card } C(i) + \sum_{i=2}^{n-2} \text{card}\{J \mid J \subseteq [2, i]\} \\ &= 2 + \sum_{i=2}^{n-1} c(i) + \sum_{i=1}^{n-3} 2^i = 2 + (n-2) \cdot 2^{n-3} + 2^{n-2} - 2 = n \cdot 2^{n-3}. \end{aligned}$$

(d) It is clear that $d(0) = 0$, $D(1) = \{(\emptyset, \{1\})\} = D(2)$, thus we can assume that $n \geq 3$, and define the correspondence

$$\sigma: \{J \mid J \subseteq \{3, \dots, n\}\} \rightarrow D(n) \quad J \mapsto (J, \{1\} \cup J).$$

It follows that σ is a well defined injection, since $J \prec \{1\} \cup J$. Next we show that σ is surjective. For, let $(J', I') \in D(n)$. If $1 \in J'$, then $I' \setminus J' = \emptyset$, a contradiction. Thus, $1 \notin J'$. If $2 \in J'$, then $J' = [2, t_1] \sqcup \dots \sqcup [s_l, t_l]$, and since $I' \setminus J' = \{1\}$, we conclude that $I' = \{1\} \cup J' = [1, t_1] \sqcup \dots \sqcup [s_l, t_l]$, thus $J' \not\prec I'$, contradiction. Therefore $J' \subseteq \{3, \dots, n\}$, thus $I' = \{1\} \cup J'$, i.e. $\sigma(J') = (J', I')$, proving that σ is surjective, thus bijective. So, $d(n) = \text{card } D(n) = 2^{n-2}$. \square

Corollary 3.8.4. $h(0) = 0$ and $h(n) = 2^{n-2}(n+1)$ for $n \geq 1$.

Proof. Follows from Proposition 3.8.3. \square

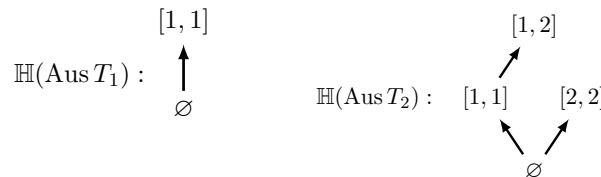
The numbers $h(n)$ form the sequence [OEIS:A001792](#) shifted by -1 . Moreover, they satisfy the recursive formula $h(n) = 2 \cdot h(n-1) + 2^{n-2}$, for $n \geq 2$. The first values of the sequence $h(n)$ appear below.

n	1	2	3	4	5	6	7	8	9	10
$h(n)$	1	3	8	20	48	112	256	576	1280	2816

3.9 Homological Hasse quiver of $\text{Aus } T_n$

In this section we provide an explicit description of the Hasse diagram of the homological poset of $\Lambda_n = \text{Aus } T_n$ using the classification of the cover relations of the homological poset of Λ_n given in Section 3.8, and construct its Hasse quiver $\mathbb{H}(\text{Aus } T_n)$.

From the proof of Proposition 3.8.3, we have the following homological Hasse quivers. $\mathbb{H}(\text{Aus } T_0): \emptyset$



Note that $h(n-1) = c(n)$, thus the proof of Proposition 3.8.3 (c) motivates the following construction of $\mathbb{H}(\text{Aus } T_n)$ using recursion methods. More specifically, we will prove that $\mathbb{H}(\text{Aus } T_n)$ is obtained from $\mathbb{H}(\text{Aus } T_{n-1})$ by attaching to it exactly one copy of $\mathbb{H}(\text{Aus } T_i)$, for $0 \leq i \leq n-2$, and connecting the vertices in certain way, and finally adding the arrow $([1, n-1], [1, n])$, which corresponds to the embedding $\text{mod } \Lambda_n / (e_n) \hookrightarrow \text{mod } \Lambda_n$.

For the rest of the section, set $H(n) := \mathcal{H}(\text{Aus } T_n)$. For $n \geq 2$ and $0 \leq i \leq n-2$ define

$$\begin{aligned} H'(i) &:= \{(J \sqcup [i+2, n], I \sqcup [i+2, n]) \mid (J, I) \in \text{Cov } H(i)\}, \text{ and} \\ U(i) &:= \{(I, I \sqcup [i+2, n]) \mid I \subseteq [i]\}, \text{ where } [0] := \emptyset. \end{aligned}$$

Note that these sets are contained in $\text{Cov } H(n)$ and are pairwise disjoint. We regard $H'(i)$ as the copy of $\text{Cov } H(i)$ inside $\text{Cov } H(n)$ that we mentioned above. The elements of $U(i)$ correspond to arrows that connect the vertices of $\mathbb{H}(\text{Aus } T_{n-1})$ with the corresponding ones in the copies of $\mathbb{H}(\text{Aus } T_i)$, for $0 \leq i \leq n-2$.

The following proposition shows that this construction actually give us the complete Hasse diagram of $\mathcal{H}(\text{Aus } T_n)$.

Proposition 3.9.1. *Let $n \geq 2$. Then*

$$h(n) = \sum_{i=0}^{n-1} h(i) + \sum_{i=0}^{n-2} 2^i + 1 = \sum_{i=0}^{n-1} h(i) + 2^{n-1}.$$

Thus, $\text{Cov } H(n) = \text{Cov } H(n-1) \sqcup \bigsqcup_{i=0}^{n-2} (H'(i) \sqcup U(i)) \sqcup \{([1, n-1], [1, n])\}$.

Proof. For $n=2$ we have $h(2) = 3 = 1 + 1 + 1 = h(1) + 2^0 + 1$, thus we can assume by induction that the result holds for $n-1$, for $n > 2$. Then,

$$\sum_{i=0}^{n-1} h(i) + 2^{n-1} = \sum_{i=0}^{n-1} h(i) + \sum_{i=0}^{n-2} 2^i + 1 = h(n-1) + h(n-1) + 2^{n-2} = h(n). \quad (3.9.1)$$

Moreover, since the subsets $H'(i)$ and $U(j)$ are contained in $\text{Cov } H(n)$ and are pairwise disjoint, for $0 \leq i, j \leq n-2$, Eq. (3.9.1) shows the last claim of the proposition. \square

Fig. 3.1 illustrates the construction of $\mathbb{H}(\text{Aus } T_3)$ and $\mathbb{H}(\text{Aus } T_4)$ using the previous recursive method.

Remark 3.9.2. By Proposition 3.6.6, the unique maximal length path in $\mathbb{H}(\text{Aus } T_n)$ corresponds to the unique heredity chain of $\text{Aus } T_n$.

Now we count the number of maximal elements of $H(n)$. For, let $M(n)$ be the set of maximal elements of $H(n)$, and set $m(n) := \text{card } M(n)$. Thus, $m(0) = 0$, $m(1) = 1$, $m(2) = 2$, $m(3) = 4$ and $m(4) = 7$ (cf. Fig. 3.1). Note that, for $0 \leq i \leq n-2$, the elements $I \sqcup [i+2, n]$ lie in $M(n)$ if and only if $I \in M(i)$, by Proposition 3.9.1. Set $M'(i) := \{I \sqcup [i+2, n] \mid I \in M(i)\}$, for $0 \leq i \leq n-2$, thus we can say that the copies $H'(i)$ preserve the maximality of its elements, if $i \in [n-2]$.

On the other hand, for $n \geq 5$, we have that $M(n-1) \not\subseteq M(n)$, for, consider $J = \{1\} \sqcup [3, n-2] \in M(n-1)$, and observe that it is not a maximal element of $H(n)$, since

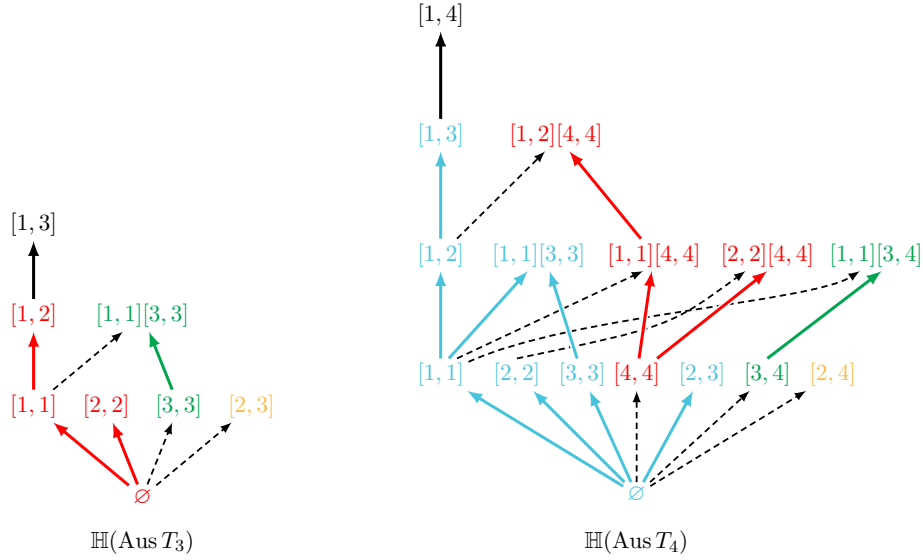


Figure 3.1: The homological Hasse quivers of $\text{Aus } T_3$ and $\text{Aus } T_4$. We omit the symbols \sqcup in the expression of $J \subseteq [n]$ as disjoint union of intervals, e.g. $[1, 1][3, 3] = [1, 1] \sqcup [3, 3]$. The copies of $\mathcal{H}(\text{Aus } T_0)$ are indicated in **orange**, $\mathcal{H}(\text{Aus } T_1)$ in **green**, $\mathcal{H}(\text{Aus } T_2)$ in **red** and $\mathcal{H}(\text{Aus } T_3)$ in **blue**, respectively. Dashed arrows indicate elements of the sets $U(i)$, and the unique solid black arrow corresponds to the embedding $\iota_{[n-1],[n]}$, for $n = 3, 4$.

$J \prec_{\text{Aus } T_n} J \sqcup \{n\}$. Actually, all the maximal elements of $H(n-1)$, not considering $[n-1]$, that are not maximal in $H(n)$ are those with maximal element $n-2$, denote this set by $N(n)$. Now define $N'(i) := \{J \sqcup [i+2, n-2] \mid J \in M(i)\}$, for $i \in [n-4]$. It is clear that $N'(i) \subseteq M(n-1)$, but $N'(i) \not\subseteq M(n)$. Hence, by induction we have that $N(n) = \bigcup_{i=1}^{n-4} N'(i) \cup \{[2, n-2]\}$, thus $M(n-1) \setminus (N(n) \cup \{[n-1]\}) \subseteq M(n)$. For simplicity, set

$$M'(n-1) := M(n-1) \setminus (N(n) \cup \{[n-1]\}).$$

We have the following result.

Proposition 3.9.3. *Let $n \geq 4$. Then $m(n) = m(n-1) + m(n-2) + m(n-3)$.*

Proof. We have already shown that $\bigcup_{i=0}^{n-1} M'(i) \sqcup \{[n]\} \subseteq M(n)$, moreover the reverse inclusion also holds, considering Proposition 3.9.1 and the fact that the subsets of $M'(n-1)$ are maximal elements of $H(n-1)$ with maximal element $n-1$, thus the equality holds. Since all the involved subsets are pairwise disjoint we have the following equalities.

$$\begin{aligned} \text{card } M(n) &= \sum_{i=0}^{n-1} \text{card } M'(i) + 1 = \sum_{i=0}^{n-2} M'(i) + \text{card } M(n-1) - \left(\sum_{i=1}^{n-4} \text{card } N'(i) + 1 \right) \\ &= 1 + \sum_{i=1}^{n-1} m(i) - \sum_{i=1}^{n-4} m(i) - 1 = m(n-1) + m(n-2) + m(n-3). \quad \square \end{aligned}$$

The sequence $\{m(n)\}_{n \in \mathbb{N}}$ is a Tribonacci sequence with initial conditions $m(1) = 1$, $m(2) = 2$ and $m(3) = 4$, and this coincides, up to a shift, with the sequence $a(n)$ given

by [OEIS:A000073](#), more precisely $m(n) = a(n+2)$, for $n \geq 1$. The following table exhibits the first values of $m(n)$.

n	1	2	3	4	5	6	7	8	9	10
$m(n)$	1	2	4	7	13	24	44	81	149	274

Now we show that $H(n)$ admits a rank function in the following sense.

Definition 3.9.4. A *rank function* of a poset P is function $\rho: P \rightarrow \mathbb{N}$ satisfying the following properties:

- (a) If $x \in P$ is minimal, then $\rho(x) = 0$.
- (b) If y covers x , then $\rho(y) = \rho(x) + 1$.

Lemma 3.9.5. Let $n \geq 0$. Then the function $\rho_n: 2^{[n]} \rightarrow \mathbb{N}$ given by

$$\rho_n(J) := \begin{cases} \ell(J) & \text{if } 1 \notin J \\ t_1 + \ell(J) - 1 & \text{if } 1 \in J. \end{cases}$$

where $J = [s_1, t_1] \sqcup \cdots \sqcup [s_l, t_l] \subseteq [n]$, is a rank function of $H(n)$.

Proof. Set $\rho = \rho_n$. Since $\mathcal{P}(n) = (2^{[n]}, \subseteq)$ is a refinement of $H(n)$, we have that $H(n)$ has only one minimal element, namely \emptyset . Thus, $\rho(\emptyset) = \ell(\emptyset) = 0$. Now, suppose that (J, I) is a cover relation in $H(n)$, with $J = \bigsqcup_{j=1}^l [s_j, t_j]$ and $I = \bigsqcup_{i=1}^m [r_i, u_i]$. From Proposition 3.8.1 we can distinguish four cases.

(i) $1 \in J$, $I_1 = J_1$ and $m = l+1$. Then $\rho(J) = t_1 + l - 1$ and $\rho(I) = u_1 + m - 1 = t_1 + l$. Thus $\rho(I) = \rho(J) + 1$.

(ii) $1 \in J$, $u_1 = t_1 + 1$ and $m = l$. The same calculations as in (i) show that $\rho(I) = \rho(J) + 1$.

(iii) $1 \notin J$, $1 \notin I$ and $m = l + 1$. Then $\rho(J) = l = m - 1 = \rho(I) - 1$.

(iv) $I \setminus J = \{1\}$. The cases $n = 1, 2$ are trivial. So, suppose that $n \geq 3$. In the proof of Proposition 3.8.3 (d) we showed that if (J, I) satisfies (iv), then $(J, I) = (J, \{1\} \sqcup J)$ for some $J \subseteq \{3, \dots, n\}$, thus $u_1 = 1$. So, $\rho(I) = u_1 + m - 1 = m = l + 1 = \ell(J) + 1 = \rho(J) + 1$. This shows that ρ is a rank function of $H(n)$. \square

3.10 Homological embeddings over $\text{Aus}T_n$ and compositions

In this section we study some interesting combinatorial properties of cover relations of the poset $H(n) = \mathcal{H}(\text{Aus}T_n)$ involving compositions of an integer. We start with some definitions.

Let n be a positive integer. A *composition* of n is a sequence of positive integers $\sigma = (\sigma_1, \dots, \sigma_k)$ such that $\sum_{i=1}^k \sigma_i = n$. We write often $\sigma = \sigma_1 + \cdots + \sigma_k$, and we say that σ is a k -composition of n , the numbers σ_i are called the *parts* of σ . We denote $\text{Comp}(n)$ the set of all compositions of n , and $\mathbb{P}(n)$ denotes the set of parts of all compositions of the positive integer n .

The main motivation for considering compositions of positive integers comes from the description of [OEIS:A001792](#), where we can see that $h(n)$ coincides with the number of parts in all compositions of n . For example, $h(3) = 8$, and all the compositions of 3 are $1 + 1 + 1$, $2 + 1$, $1 + 2$, 3 . In what follows we will exhibit a bijection between $\text{Cov } H(n)$ and $\mathbb{P}(n)$, but first we provide a new description of the cover relations of $H(n)$ that is crucial to find such a correspondence. Recall that $\text{Cov } H(n) = A(n) \sqcup B(n) \sqcup C(n) \sqcup D(n)$ (cf. Proposition 3.8.1 and Definition 3.8.2).

Lemma 3.10.1. *Let $(J, I) \in \text{Cov } H(n)$, then $D := I \setminus J$ has one of the following forms.*

- (a) $D = [d_1, d_2]$ for some $3 \leq d_1 \leq d_2 \leq n$, if $(J, I) \in A(n)$.
- (b) $D = \{d\}$ for some $2 \leq d \leq n$, if $(J, I) \in B(n)$.
- (c) $D = [d_1, d_2]$ for some $2 \leq d_1 \leq d_2 \leq n$, if $(J, I) \in C(n)$.
- (d) $D = \{1\}$ if $(J, I) \in D(n)$.

Moreover, the conditions (a)-(d) determine the cover relation (J, I) , in the sense that if $J \prec I$ and their set difference D satisfies (a), (b), (c) or (d), then (J, I) lies in $A(n)$, $B(n)$, $C(n)$ or $D(n)$, respectively. Thus, any cover relation (J, I) is of the form $(J, J \sqcup D)$, for some set $D \subset [n]$ as in (a)-(d).

Proof. Follows from Proposition 3.8.1 and Definition 3.8.2. □

For simplicity, in the presence of Lemma 3.10.1, we use the following notation.

Definition 3.10.2. Let $(J, J \sqcup D)$ be a cover relation of $H(n)$. Then $\langle\langle J, D \rangle\rangle := (J, J \sqcup D)$.

Now we define a correspondence $\Psi = \Psi_n: \mathbb{P}(n) \rightarrow \text{Cov } H(n)$. Let $\sigma = \sigma_1 + \cdots + \sigma_k$ be a k -composition of n . For $1 \leq j \leq j' \leq k$ set $\sigma_j^{j'} = \sigma_j + \cdots + \sigma_{j'}$.

Case 1. If $\sigma_1 \geq 2$, Ψ is given by:

$$\sigma_1 \mapsto \left\langle\left\langle [1, \sigma_1 - 1] \sqcup \bigsqcup_{\substack{j \geq 2, \\ \sigma_j \geq 2}} [\sigma_1^{j-1} + 2, \sigma_1^j], [\sigma_1, \sigma_1] \right\rangle\right\rangle, \quad (3.10.1)$$

and for $i \geq 2$,

$$\sigma_i \mapsto \left\langle\left\langle [1, \sigma_1 - 1] \sqcup \bigsqcup_{\substack{1 < j < i, \\ \sigma_j \geq 2}} [\sigma_1^{j-1} + 1, \sigma_1^j - 1] \sqcup \bigsqcup_{\substack{j > i, \\ \sigma_j \geq 2}} [\sigma_1^{j-1} + 2, \sigma_1^j], [\sigma_1^{i-1} + 1, \sigma_1^i] \right\rangle\right\rangle. \quad (3.10.2)$$

For (3.10.1) we have that $1 \leq \sigma_1 - 1 < \sigma_1 + 2 \leq \sigma_1^{j-1} + 2$ for any $j \geq 2$ such that $\sigma_j \geq 2$; and for $2 \leq j < j'$ such that $\sigma_j, \sigma_{j'} \geq 2$, we get $\sigma_1^j < \sigma_1^j + 2 \leq \sigma_1^{j'-1} + 2$, meaning that the intervals involved in the right hand side of (3.10.1) are disjoint and determine the correct expression of their underlying set as union of them. Hence the first assignment is well defined. Similar arguments show that the assignment (3.10.2) is well defined.

Case 2. If $\sigma_1 = 1$, then Ψ is given by

$$\sigma_1 \mapsto \left\langle\left\langle \bigsqcup_{\substack{j \geq 2, \\ \sigma_j \geq 2}} [\sigma_1^{j-1} + 2, \sigma_1^j], [1, 1] \right\rangle\right\rangle, \quad (3.10.3)$$

and for $i \geq 2$

$$\sigma_i \mapsto \left\langle \bigsqcup_{\substack{1 < j < i, \\ \sigma_j \geq 2}} [\sigma_1^{j-1} + 1, \sigma_1^j - 1] \sqcup \bigsqcup_{\substack{j > i, \\ \sigma_j \geq 2}} [\sigma_1^{j-1} + 2, \sigma_1^j], [\sigma_1^{i-1} + 1, \sigma_1^i] \right\rangle. \quad (3.10.4)$$

Similar arguments as before, show that the assignments (3.10.3) and (3.10.4) are well defined. Moreover, from Lemma 3.10.1 we get

$$\Psi(\sigma_i) \in \begin{cases} A(n) & \text{if } \sigma_1 \geq 2 \text{ and } i \geq 2, \\ B(n) & \text{if } \sigma_1 \geq 2 \text{ and } i = 1, \\ C(n) & \text{if } \sigma_1 = 1 \text{ and } i \geq 2, \\ D(n) & \text{if } \sigma_1 = 1 \text{ and } i \geq 1. \end{cases} \quad (3.10.5)$$

As an example, we write explicitly the correspondence $\Psi_2: \mathbb{P}(2) \rightarrow \text{Cov } H(2)$, since the case $n = 1$ is trivial. Indeed, we know that there are only two compositions of 2: $\sigma = 1 + 1$ and $\sigma' = 2$, thus $\text{card } \mathbb{P}(2) = 3$. Hence, Ψ_2 is given as follows:

$$\sigma_1 = 1 \mapsto \langle \emptyset, \{1\} \rangle, \quad \sigma_2 = 1 \mapsto \langle \emptyset, \{2\} \rangle, \quad \sigma'_1 = 2 \mapsto \langle \{1\}, \{2\} \rangle,$$

and the inverse $\Phi_2: \text{Cov } H(2) \rightarrow \mathbb{P}(2)$ is given by

$$(\emptyset, \{1\}) \mapsto \sigma_1 = 1, \quad (\emptyset, \{2\}) \mapsto \sigma_2 = 1, \quad (\{1\}, \{1, 2\}) \mapsto \sigma'_1 = 2.$$

In Theorem 3.10.3 we define the correspondence Φ_n for any $n \geq 2$ and prove that Φ_n and Ψ_n are inverse from each other. For, we use the following notation. Set $\mathbb{1}(m) := 1, \dots, 1$ m -times, for $m \geq 0$. Thus, $\mathbb{1}(0)$ is the empty word. Also note that if σ is a k -composition of n , then k can be recovered from the parts of σ that are larger than 1, i.e.

$$\begin{aligned} k &= n - (\sigma_1 + \dots + \sigma_k) + k = n - [(\sigma_1 - 1) + \dots + (\sigma_k - 1)] \\ &= n - \sum_{\substack{j, \\ \sigma_j \geq 2}} (\sigma_j - 1), \end{aligned} \quad (3.10.6)$$

similarly, if $2 \leq i \leq k$ and $\tau := \sigma_1 + \dots + \sigma_{i-1} + 1 = n'$, then we can recover the number i from the parts of τ that are larger than 1, i.e.

$$\begin{aligned} i &= n' - (\sigma_1 + \dots + \sigma_{i-1} + 1) + i = n' - [(\sigma_1 - 1) + \dots + (\sigma_{i-1} - 1)] \\ &= n' - \sum_{\substack{1 \leq j \leq i-1, \\ \sigma_j \geq 2}} (\sigma_j - 1). \end{aligned} \quad (3.10.7)$$

Theorem 3.10.3. *Let $n \geq 1$. There exists a function $\Phi = \Phi_n: \text{Cov } \mathcal{H}(\Lambda_n) \rightarrow \mathbb{P}(n)$ such that $\Psi \circ \Phi = 1_{\text{Cov } H(n)}$ and $\Phi \circ \Psi = 1_{\mathbb{P}(n)}$.*

Proof. We have already shown the cases $n \in \{1, 2\}$, so consider $n \geq 3$. We define Φ explicitly by cases. For, let $x = (J, I) = \langle J, D \rangle \in \text{Cov } H(n)$.

(a) If $x \in A(n)$, then $D = [d_1, d_2]$ with $3 \leq d_1 \leq d_2 \leq n$, and we can write

$$I = J \sqcup D = [s_1 = 1, t_1] \cdots [s_l, t_l][d_1, d_2][r_1, u_1] \cdots [r_m, u_m],$$

where $1 \leq l$, and $1 \leq m$, if $J \cap [d_2 + 1, n] \neq \emptyset$, otherwise the intervals $[r_j, u_j]$ do not appear. Set

$$\begin{aligned} \alpha_j &:= t_j - s_j + 2 & \beta_j &:= u_j - r_j + 2 \\ \alpha^j &:= s_{j+1} - t_j - 2 & \beta^j &:= r_j - u_{j-1} - 2, \end{aligned} \quad (3.10.8)$$

and define a map $\Phi_A : A(n) \rightarrow \text{Comp}(n)$ by

$$\Phi_A(x) = \sigma := (\alpha_1, \mathbb{1}(\alpha^1), \dots, \alpha_l, \mathbb{1}(\alpha^l), d_2 - d_1 + 1, \mathbb{1}(\beta^1), \beta_1, \dots, \mathbb{1}(\beta^m), \beta_m, \mathbb{1}(n - u_m))$$

where, by convention, we set $s_{l+1} = d_1$ and $u_0 = d_2$. If $J \cap [d_2 + 1, n] = \emptyset$ we identify $u_m = d_2$ and we omit to write all the coordinates involving β^j or β_j , for example if $d_2 = n$, then the last coordinate of σ is $d_2 - d_1 + 1$. Note that Φ_A is well defined, since the sum of its parts is

$$\begin{aligned} & \sum_{j=1}^l (\alpha_j + \alpha^j) + d_2 - d_1 + 1 + \sum_{j=1}^m (\beta^j + \beta_j) + n - u_m \\ &= -1 + d_1 + d_2 - d_1 + 1 - d_2 + u_m + n - u_m = n. \end{aligned}$$

Now, for $1 \leq j \leq j'$, let $s_j^{j'} := s_j + \cdots + s_{j'}$. Similarly for $t_j^{j'}$, $r_j^{j'}$ and $u_j^{j'}$. Note that the part α_1 occurs in the position 1, and if it is the case, for $2 \leq j \leq l$, α_j occurs in position $(j) := s_1^j - t_1^{j-1} - j + 1$; the part $d_2 - d_1 + 1$ occurs in position $(d) := (l) + d_1 - t_l - 1$; and for $1 \leq j \leq m$ the part β_j occurs in position $(j^*) := (d) + r_1^j - u_0^{j-1} - j$. Thus, we define for $x \in A(n)$

$$\Phi(x) := (\Phi_A(x))_{(d)} = \sigma_{(d)} = d_2 - d_1 + 1.$$

It is easy to prove by induction that $\sigma_1^{(j)} = t_j + 1$, $\sigma_1^{(j)-1} = s_j - 1$ for $2 \leq j \leq l$, $\sigma_1^{(d)} = d_2$, $\sigma_1^{(d)-1} = d_1$, and for $1 \leq j \leq m$, we have $\sigma_1^{(j^*)} = u_j$ and $\sigma_1^{(j^*)-1} = r_j - 2$. Moreover, σ is a k -composition of n , where

$$\begin{aligned} k &= (1 + \alpha^1) + \cdots + (1 + \alpha^l) + 1 + (\beta^1 + 1) + \cdots + (\beta^m + 1) + n - u_m \\ &= n - l - m + s_1^{l+1} - t_1^l + r_1^m - u_0^m. \end{aligned}$$

Thus, (3.10.2) applies, since $\sigma_1 \geq 2$ and $(d) > 1$, and we get

$$\Psi(\Phi(x)) = \Psi(\sigma_{(d)}) = \langle J, D \rangle = x.$$

(b) If $x \in B(n)$, then $D = \{d\}$ for $2 \leq d \leq n$, and we can write

$$I = [1, d][r_1, u_1] \cdots [r_m, u_m]$$

if $J \cap [d + 1, n] \neq \emptyset$, otherwise the intervals $[r_j, u_j]$ do not appear. Then, we define a correspondence $\Phi_B : B(n) \rightarrow \text{Comp}(n)$ by

$$\Phi_B(x) = \sigma := (d, \mathbb{1}(\beta^1), \beta_1, \dots, \mathbb{1}(\beta^m), \beta_m, \mathbb{1}(n - u_m))$$

where we use the same notations and conventions as in (a), with the conditions $u_0 := d$ and $(d) = 1$. It is easy to show that σ is a k -composition of n , where $k = n - m + 1 + r_1^m - u_0^m$. Thus, we define for $x \in B(n)$

$$\Phi(x) := (\Phi_B(x))_1 = \sigma_1 = d,$$

and similar calculations as in (a) show that $\Psi(\Phi(x)) = \Psi(\sigma_1) = x$.

(c) If $x \in C(n)$, then $D = [d_1, d_2]$ with $2 \leq d_1 \leq d_2 \leq n$, and we can write

$$I = J \sqcup D = [s_1, t_1] \cdots [s_l, t_l][d_1, d_2][r_1, u_1] \cdots [r_m, u_m],$$

for some $l, m \geq 1$ if $J \cap [2, d_1 - 1] \neq \emptyset$ and $J \cap [d_2 + 1, n] \neq \emptyset$, otherwise the intervals $[s_j, t_j]$ or $[r_j, u_j]$ do not appear respectively. Using the same conventions and notation of (a), we define a correspondence $\Phi_C : C(n) \rightarrow \text{Comp}(n)$ by

$$\begin{aligned} \Phi_C(x) = \sigma := & (\mathbb{1}(s_1 - 1), \alpha_1, \mathbb{1}(\alpha^1), \dots, \alpha_l, \mathbb{1}(\alpha^l), d_2 - d_1 + 1, \\ & \mathbb{1}(\beta^1), \beta_1, \dots, \mathbb{1}(\beta^m), \beta_m, \mathbb{1}(n - u_m)). \end{aligned}$$

In this case σ is a k -composition of n , where $k = n - l - m + s_1^{l+1} - t_1^l + r_1^m - u_0^m$. Thus, we define for $x \in C(n)$

$$\Phi(x) := (\Phi_C(x))_{(d)} = \sigma_{(d)} = d_2 - d_1 + 1,$$

and similar calculations as in (a) show that $\Psi(\Phi(x)) = \Psi(\sigma_{(d)}) = x$.

(d) If $x \in D(n)$, then $D = \{1\}$, and we can write

$$I = [1, 1][r_1, u_1] \cdots [r_m, u_m],$$

for some $m \geq 1$ if $J \cap [2, n] \neq \emptyset$, otherwise $I = [1, 1]$. As before, we use the same conventions and notations as in (a) to define a correspondence $\Phi_D : D(n) \rightarrow \text{Comp}(n)$ by

$$\Phi_D(x) = \sigma := (1, \mathbb{1}(\beta^1), \beta_1, \dots, \mathbb{1}(\beta^m), \beta_m, \mathbb{1}(n - u_m))$$

where $u_0 := 1$ and $(d) := 1$. Note that σ is a k -composition of n , with $k = n - m + 1 + r_1^m - u_0^m$. Thus, we define for $x \in D(n)$

$$\Phi(x) := (\Phi_D(x))_1 = \sigma_1 = 1.$$

Similarly as in (a), we get that $\Psi(\Phi(x)) = \Psi(\sigma_1) = x$. This shows that $\Psi \circ \Phi = 1_{\text{Cov } H(n)}$.

For the converse, let $\sigma = \sigma_1 + \cdots + \sigma_k$ be a k -composition of n , with $\sigma_1 \geq 2$, and take $2 \leq i \leq k$. Then, from Eq. (3.10.2) we get

$$\Psi(\sigma_i) = \left\langle \left\langle [1, \sigma_1 - 1] \sqcup \bigsqcup_{j=2}^l [\sigma_1^{(j)-1} + 1, \sigma_1^{(j)} - 1] \sqcup \bigsqcup_{j=1}^m [\sigma_1^{(j^*)-1} + 2, \sigma_1^{j^*}], [\sigma_1^{i-1} + 1, \sigma_1^i] \right\rangle \right\rangle,$$

where (j) are those indices $j' \in \{2, \dots, i-1\}$ such that $\sigma_{j'} \geq 2$, and by (j^*) we denote the indices $j' \in \{i+1, \dots, k\}$ such that $\sigma_{j'} \geq 2$, with the convention $1 < (2) < \cdots <$

$(l) < i < (1^*) < \cdots < (m^*)$. Define $(1) := 1$, $G := \{(1), (2), \dots, (l), i, (1^*), \dots, (m^*)\}$, and

$$\begin{array}{llll} & & \text{for } 2 \leq j \leq l : & \text{for } 1 \leq j \leq m : \\ s_1 = 1 & d_1 = \sigma_1^{i-1} + 1 & s_j = \sigma_1^{(j)-1} + 1 & r_j = \sigma_1^{(j^*)-1} + 2 \\ t_1 = \sigma_1 - 1 & d_2 = \sigma_1^i & t_j = \sigma_1^{(j)} - 1 & u_j = \sigma_1^{(j^*)}. \end{array}$$

From (3.10.5) we have $\Psi(\sigma_i) \in A(n)$, hence we can set $\sigma' := \Phi_A(\Psi(\sigma_i))$. Therefore, using the equalities (3.10.8), we have that

$$\begin{aligned} \sigma'_1 &= \alpha_1 = t_1 - s_1 + 2 = t_1 + 1 = \sigma_1, \\ \sigma'_{(j)} &= \alpha_j = t_j - s_j + 2 = \sigma_1^{(j)} - 1 - (\sigma_1^{(j)-1} + 1) + 2 = \sigma_{(j)} \quad \text{for } 2 \leq j \leq l, \\ \sigma'_{(j^*)} &= \beta_j = u_j - r_j + 2 = \sigma_1^{(j^*)} - (\sigma_1^{(j^*)-1} + 2) + 2 = \sigma_{(j^*)} \quad \text{for } 1 \leq j \leq m, \\ \sigma'_i &= d_2 - d_1 + 1 = \sigma_i, \quad \text{and} \\ \sigma'_j &= 1 = \sigma_j \quad \text{for any } j \in [k] \setminus G, \end{aligned}$$

moreover, σ' is a k' -composition of n by construction, where $k' = n - l - m + s_1^{l+1} - t_1^l + r_1^m - u_0^m$. Hence,

$$\begin{aligned} k' &= n - l - m + 1 + (\sigma_1^{(2)-1} + 1) + \cdots + (\sigma_1^{(l)-1} + 1) + (\sigma_1^{i-1} + 1) \\ &\quad - [(\sigma_1 - 1) + (\sigma_1^{(2)} - 1) + \cdots + (\sigma_1^{(l)} - 1)] + (\sigma_1^{(1^*)-1} + 2) + \cdots + (\sigma_1^{(m^*)-1} + 2) \\ &\quad - [\sigma_1^i + \sigma_1^{(1^*)} + \cdots + \sigma_1^{(m^*)}] \\ &= n - [(\sigma_{(1)} - 1) + \cdots + (\sigma_{(l)} - 1) + (\sigma_i - 1) + (\sigma_{(1^*)} - 1) + \cdots + (\sigma_{(m^*)} - 1)] \\ &= n - \sum_{j \in G} (\sigma_j - 1) = n - \sum_{\substack{j, \\ \sigma_j \geq 2}} (\sigma_j - 1) = k \end{aligned}$$

where the last equality follows from Eq. (3.10.6). Therefore, $\sigma = \sigma'$ as k -compositions of n . Also, we have that

$$\begin{aligned} (d) &= (l) + d_1 - t_l - 1 = s_1^l - t_1^l - l + d_1 \\ &= 1 + (\sigma_1^{(2)-1} + 1) + \cdots + (\sigma_1^{(l)-1} + 1) \\ &\quad - [(\sigma_1 - 1) + (\sigma_1^{(2)} - 1) + \cdots + (\sigma_1^{(l)} - 1)] - l + \sigma_1^{i-1} + 1 \\ &= \sigma_1^{i-1} + 1 - [(\sigma_1 - 1) + (\sigma_{(2)} - 1) + \cdots + (\sigma_{(l)} - 1)] = i \end{aligned}$$

where the last equality follows from Eq. (3.10.7). Therefore,

$$\Phi(\Psi(\sigma_i)) = (\Phi_A(\Psi(\sigma_i)))_{(d)} = \sigma'_{(d)} = \sigma_{(d)} = \sigma_i.$$

Now for the case $i = 1$, we have that

$$\Psi(\sigma_1) = \left\langle [1, \sigma_1 - 1] \sqcup \bigsqcup_{j=1}^m [\sigma_1^{(j^*)-1} + 2, \sigma_1^{j^*}], [\sigma_1, \sigma_1] \right\rangle,$$

where we use the same notation as in the previous case. From Eq. (3.10.5) we have $\Psi(\sigma_1) \in B(n)$, hence we can set $\sigma' := \Phi_B(\Psi(\sigma_1))$. Similar arguments as before show that $\sigma = \sigma'$ as k -compositions of n . Therefore,

$$\Phi(\Psi(\sigma_1)) = (\Phi_B(\Psi(\sigma_1)))_1 = \sigma'_1 = \sigma_1.$$

The case when $\sigma = 1 + \sigma_2 + \dots + \sigma_k$ is proven using the same techniques as in the case $\sigma_1 \geq 2$. Thus $\Phi \circ \Psi = 1_{\mathbb{P}(n)}$. This completes the proof of the theorem. \square

Concerning the proof of Theorem 3.10.3, note that the maps Φ_B and Φ_D are injective for all $n \geq 1$; Φ_C is not injective for $n \geq 3$, since $\Phi_C(\langle \emptyset, \{2\} \rangle) = 1+1+1 = \Phi_C(\langle \emptyset, \{3\} \rangle)$; and Φ_A is not injective for all $n \geq 4$, since $\Phi_A(\langle \{1\}, \{3\} \rangle) = 2+1+1 = \Phi_A(\langle \{1\}, \{4\} \rangle)$.

In the following paragraphs, we describe the maps Φ_A, Φ_B, Φ_C , and Φ_D in terms of other combinatorial objects. For, we first introduce the binary notation of subsets of $[n]$ as follows. For this purpose we always consider the elements of $J \subseteq [n]$ ordered by the usual ordering of \mathbb{N} .

Let $\text{Bin}(n)$ be the set of binary words of length n , i.e. words $w_1 w_2 \dots w_n$ such that $w_i \in \{0, 1\}$ for all i . We may also depict a binary word w of length n as a n -tuple (w_1, \dots, w_n) .

Recall that $\text{card } 2^{[n]} = \text{card } \text{Bin}(n) = 2^n$, via the bijective map $\text{Bin}_n: 2^{[n]} \rightarrow \text{Bin}(n)$ given by $J \mapsto \text{Bin}_n(J)$, where $\text{Bin}_n(J)$ is the word $w_1 \dots w_n$ with $w_j = 1$ if $j \in J$, and $w_j = 0$ otherwise. We call $\text{Bin}_n(J)$ the *binary notation* of J , and for seek of simplicity we identify those notations, when we fix n . For example, if $J = \{1, 3, 4, 5\} \subseteq [6]$, we have $J = \text{Bin}_6(J) = 101110 = (1, 0, 1, 1, 1, 0)$.

It is well known that we can visualise a composition $\sigma = \sigma_1 + \dots + \sigma_k$ (of n) as a *tiling* of a $1 \times n$ board with tiles of size $1 \times \sigma_i$ corresponding to each part of σ in the natural way. For example, $1 + 2 + 3 + 1 = 7$ is represented by the tiling

$$\boxed{1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7} \tag{3.10.9}$$

which has 4 tiles of lengths 1, 2, 3 and 1, resp. The numbers in the tiling (3.10.9) represent the positions of the underlying tiles of size 1×1 when considering the composition $\sigma = (1, 1, 1, 1, 1, 1, 1)$, counting from left to right. In what follows, we stick to this convention. Moreover, we assign $\sigma_i - 1$ dots to each tile of size $1 \times \sigma_i$, as in the following picture:

$$\boxed{\quad \mid \bullet \quad \mid \bullet \quad \mid \bullet \quad \mid \quad} \tag{3.10.10}$$

1 2 3 4 5 6

and we identify the tilings (3.10.9) and (3.10.10). In the latter case, the dots are drawn over all the internal edges of each tile of length $\sigma_i \geq 2$. Actually, this representation gives a bijection φ between $\text{Comp}(n)$ (viewed as tilings) and the set $\text{Bin}(n - 1)$ of all binary sequences of length $n - 1$, e.g. (3.10.10) corresponds to the binary sequence $(0, 1, 0, 1, 1, 0)$: the ones correspond to the positions of the dots over the $n - 1$ internal edges of the board labelled with the natural order. The inverse correspondence is given by first assigning to the binary sequence $s = (s_1, \dots, s_{n-1})$ its conjugate s' , where $s'_i := s_i + 1 \pmod 2$, and then marking the internal edges of the board corresponding to the positions of the ones in s' to get a tiling, and therefore a composition of n . For example,

$$\varphi: (1, 0, 1, 1, 0, 1) \mapsto (0, 1, 0, 0, 1, 0) \mapsto \begin{array}{|c|c|c|c|c|c|} \hline & \blacksquare & & & \blacksquare & \\ \hline 1 & 2 & 3 & 4 & 5 & 6 \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline \bullet & \blacksquare & \bullet & \bullet & \blacksquare & \bullet \\ \hline 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

which corresponds to the composition $2 + 3 + 2 = 7$. Thus, there are 2^{n-1} compositions of n . So, it is natural to equip $\text{Comp}(n)$ with the following order: let T and T' be in $\text{Comp}(n)$, then $T \leq_{\text{til}} T'$ if and only if $\varphi(T)_i \leq \varphi(T')_i$, for all $i \in [n - 1]$. Hence, $(2^{[n-1]}, \subseteq)$ and $(\text{Comp}(n), \leq_{\text{til}})$ are isomorphic posets, thus they have isomorphic Hasse quivers. Fig. 3.2 shows the Hasse quiver of $(\text{Comp}(3), \leq_{\text{til}})$.

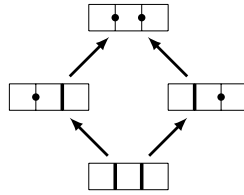
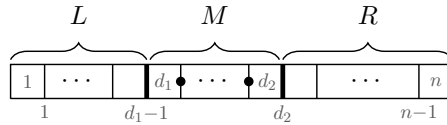


Figure 3.2: Hasse quiver of $(\text{Comp}(3), \leq_{\text{til}})$. The vertices are the compositions of 3 depicted as tilings of a 1×3 board. Note that all the elements of $\mathbb{P}(3)$ appear in this diagram as tiles.

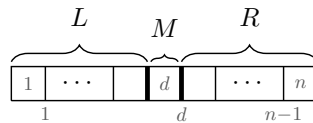
Now, we exhibit the corresponding tiling of the composition $\Phi_A(x)$ ($\Phi_B(x)$, $\Phi_C(x)$, $\Phi_D(x)$) for x in $A(n)$ ($B(n)$, $C(n)$, $D(n)$, respectively). Recall that for $x = (J, I) \in \text{Cov} H(n)$, we set $D = I \setminus J$.

- (a) If $x \in A(n)$, then $D = [d_1, d_2]$ for $3 \leq d_1 \leq d_2 \leq n$. If $d_1 < d_2$, the tiling corresponding to $\Phi_A(x)$ is the following:



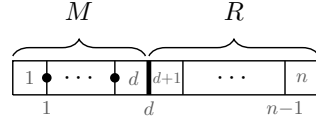
where L denotes the parts $\alpha_1, \mathbb{1}(\alpha^1), \dots, \alpha_l, \mathbb{1}(\alpha^l)$ of $\Phi_A(x)$ and it is obtained by drawing dots over the internal edges with positions j , for any $j \in J \cap [1, d_1 - 2]$; note that it has at least one dot, since $1 \in J \cap [1, d_1 - 2]$. M indicates the part $d_2 - d_1 + 1$ that occurs in position (d) . If $d_2 = n$, then R is not defined; but if $d_2 < n$, R is obtained by drawing dots over the internal edges with positions $j - 1$, for every $j \in J \cap [d_2 + 1, n]$ provided the last intersection is non-empty, otherwise R is given by $n - d_2$ tiles of length 1, thus R represents the parts $\mathbb{1}(\beta^1), \beta_1, \dots, \mathbb{1}(\beta^m), \beta_m, \mathbb{1}(n - u_m)$. Therefore, $\Phi(x) = (\Phi_A(x))_{(d)} = d_2 - d_1 + 1$ is represented by the tile M .

If $d_1 = d_2 = d$, we just consider the tiling



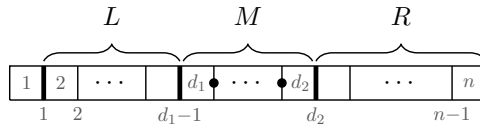
and proceed as before.

- (b) If $x \in B(n)$, then $D = \{d\}$ for some $2 \leq d \leq n$. Thus, the tiling representation of $\Phi_B(x)$ is the following:



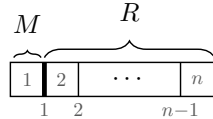
where R is given and represents the same parts as in (a). Therefore, $\Phi(x) = (\Phi_B(x))_1 = d$ is represented by M .

- (c) If $x \in C(n)$, then $D = [d_1, d_2]$ for some $2 \leq d_1 \leq d_2 \leq n$. If $d_1 < d_2$, the tiling representing $\Phi_C(x)$ is the following:



where R is defined and represents the same parts as in (a). Note that in this case $1 \notin J$, hence the first tile has length 1. If $d_1 = 2$, then L is not defined; if $d_1 = 3$, then L consists of the single tile in position 2; if $d_1 \geq 4$ L is given by drawing dots over the internal edges with positions j , for every $j \in J \cap [2, d_1 - 2]$ provided the last intersection is non-empty, otherwise L consists of $d_1 - 2$ tiles of length 1 in positions $2, \dots, d_1 - 1$. In this way L represents the parts $(\Phi_C(x))_j$ for $2 \leq j \leq (d) - 1$. Therefore, $\Phi(x) = (\Phi_C(x))_{(d)} = d_2 - d_1 + 1$ is represented by M . If $d_1 = d_2$ then M is represented by a tile of length 1 as in the last part of (a).

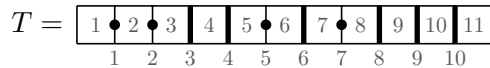
- (d) If $x \in D(n)$, then $D = \{1\}$. Hence, the tiling representing $\Phi_D(x)$ is the following:



where R is given and represents the same parts as in (a). Since $1 \notin J$, the first tile has length 1. Therefore, $\Phi(x) = (\Phi_D(x))_1 = 1$ is represented by M .

Observe that the construction given above may give the same tiling for different cover relations of $H(n)$, actually if σ is a k -composition of n , then there are k elements of $\text{Cov } H(n)$ with the same tiling T , but each one corresponding to different tiles of T . In particular if (J, I) and (J', I') are elements of $\text{Cov } H(n)$, with $1 \in J$ and $1 \notin J'$, then their corresponding parts will never belong to the same composition, see Fig. 3.3.

The inverse correspondence is given by Ψ . We explain this map using tilings by an example as follows. The special case occurs when $\sigma_1 \geq 2$. For, consider the tiling



For convenience, we keep the numbering of the edges and tiles as explained before. Thus, T corresponds to the composition $\sigma = (3, 1, 2, 2, 1, 1, 1) \in \text{Comp}(11)$. In this case $\sigma_1 = 3 \geq 2$, hence, Eq. (3.10.1) implies that it corresponds to the cover relation

$x = \langle\langle [1, \sigma_1 - 1] \sqcup J, \{\sigma_1\} \rangle\rangle$, where J consists of the numbers $j + 1$ with j a label of an edge marked with a dot appearing to the right of the edge σ_1 , i.e. $J = \{6, 8\}$, hence $x = \langle\langle \{1, 2, 6, 8\}, \{3\} \rangle\rangle$. For $i > 1$, Eq. (3.10.2) indicates that σ_i corresponds to the cover relation $\langle\langle J \sqcup J', D \rangle\rangle$, where J consists of the numbers j that label an edge marked with a dot that appear to the left of the tile T' corresponding to the part σ_i ; J' consists of the numbers $j + 1$ with j a label of a dotted edge to the right of T' , and D consists of the positions of the underlying 1×1 tiles of T' . For instance, σ_3 corresponds to $\langle\langle \{1, 2, 8\}, \{5, 6\} \rangle\rangle$, and σ_6 to $\langle\langle \{1, 2, 5, 7\}, \{9\} \rangle\rangle$. The cases when $\sigma_1 = 1$ are analogous to the the last two examples.

Fig. 3.3 displays the Hasse quiver of $(\text{Comp}(4), \leq_{\text{til}})$ with the cover relations of $\mathcal{H}(\text{Aus } T_4)$ corresponding to the parts of each composition of 4.

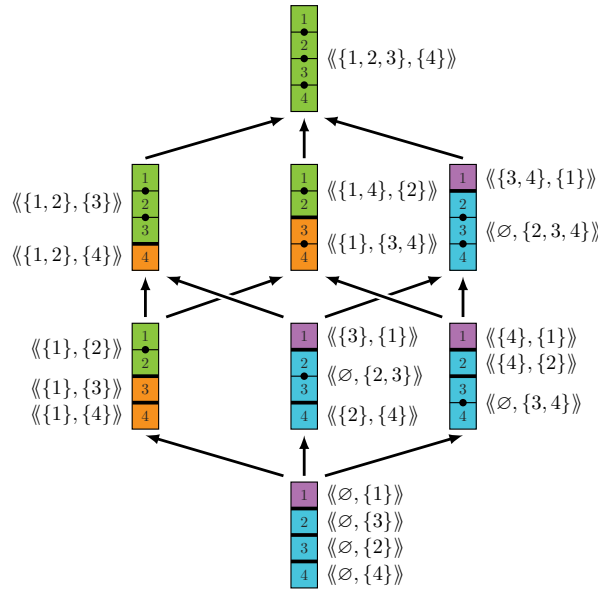


Figure 3.3: Hasse quiver of $(\text{Comp}(4), \leq_{\text{til}})$ with tilings depicted vertically for a better display, with the 1×1 tile labelled by 1 on the top. Next to each tile appears the corresponding cover relation of $\mathcal{H}(\text{Aus } T_4)$ using the notation $\langle\langle J, D \rangle\rangle$. The green tiles correspond to elements of $A(4)$, orange tiles to elements of $B(4)$, blue tiles to elements of $C(4)$, and purple tiles to elements of $D(4)$. Note that if we erase the tiles involving a 4, then we get the Hasse quiver of $(\text{Comp}(3), \leq_{\text{til}})$ with the corresponding elements of $\text{Cov } \mathcal{H}(\text{Aus } T_3)$.

3.11 Quasi-hereditary quotients of $\text{Aus } T_n$

Observe that if $J \subseteq [n]$ and $\text{Bin}_n(J)$ has a subword of the form 011, then $A_J := \text{Aus } T_n / (e_{J^c})$ has a block isomorphic to Π_m , for some $m \geq 2$, by Theorem 3.6.3 and Corollary 3.6.4, thus A_J is not quasi-hereditary since $\text{gl. dim } \Pi_m = \infty$. The converse is also true. We denote by $<$ the usual order on J .

Proposition 3.11.1. *$(A_J, (J, <))$ is a quasi-hereditary algebra if and only if 011 is not a subword of $\text{Bin}_n(J)$.*

Proof. (\Leftarrow) Let $J = \bigsqcup_{i=1}^r J_i$, with $J_i = [s_i, t_i]$, for $i \in [r]$. We proceed by induction on r . The case $r = 1$ means that $J = [s_1, t_1]$, so we have two cases: if $s_1 = 1$, then $A_J \cong \text{Aus } T_{t_1}$, and A_J is quasi-hereditary; if $s_1 > 1$, then necessarily $s_1 = t_1$, and A_J is simple, thus quasi-hereditary. Now let $r > 1$, and set $J' = \bigsqcup_{j=1}^{r-1} [s_j, t_j]$, thus $J = J' \sqcup [s_r, t_r]$. Without loss of generality we can assume $s_r > 1$. Since 011 is not a subword of $\text{Bin}_n(J)$, we have that $s_r = t_r$, thus $A_J \cong A_{J'} \times \mathbb{K}e_{s_r}$, hence $(e_{s_r}) = \mathbb{K}e_{s_r}$ is clearly a heredity ideal of A_J , and $f: A_J \rightarrow A_{J'}$ given by the projection onto the first coordinate, is a ring surjection with kernel (e_{s_r}) . Then, the result follows by induction. This completes the proof. \square

Recall that an induced subposet of a poset (P, \leq) is a poset (P', \leq') , where $P' \subseteq P$, and for all $x, y \in P'$, we have $x \leq' y$ if and only if $x \leq y$. In this case, we write $\leq|_{P'} = \leq'$.

Using the characterisation of the quasi-hereditary quotients A_J , we can restrict ourselves to homological embeddings in $\mathcal{H}(\text{Aus } T_n)$ between quasi-hereditary quotients, i.e. we have an induced subposet of $\mathcal{H}(\text{Aus } T_n)$ given as follows.

Set $P(n) := \{J \subseteq [n] \mid 011 \text{ is not subword of } \text{Bin}_n(J)\}$. Then, it is clear that we have an induced subposet

$$\mathcal{QH}(\text{Aus } T_n) := (P(n), \leq_{\text{Aus } T_n} |_{P(n)}).$$

Set $p(n) := \text{card } P(n)$. Note that $P(1) = 2^{[1]}$ and $P(2) = 2^{[2]}$, thus $p(1) = 2$ and $p(2) = 4$, hence $\mathcal{QH}(\text{Aus } T_n) = \mathcal{H}(\text{Aus } T_n)$, for $n = 1, 2$. The following lemma provides a recursive method to construct $P(n)$.

Lemma 3.11.2. *Let $n \geq 3$. Then $p(n) = p(n-1) + p(n-2) + 1$.*

Proof. Set $P'(k-2) = \{J \sqcup \{k\} \mid J \in P(k-2)\}$, for $k \geq 3$. It is enough to prove that $P(n) = P(n-1) \sqcup P'(n-2) \sqcup \{[n]\}$. For, we use binary notation. Indeed, it is clear that $P(n-1) \sqcup P'(n-2) \sqcup \{[n]\} \subseteq P(n)$. For the converse inclusion, let $w \in P(n)$. If $w_n = 0$, then $w \in P(n-1)$. On the other hand, if $w_n = 1$, we have two cases: $w_{n-1} = 1$, thus $w_i = 1$ for all $i \in [n-2]$, otherwise 011 is a subword of w , a contradiction, therefore, $w = [n]$. Second, if $w_{n-1} = 0$, then $w \in P'(n-2)$. This completes the proof. \square

The sequence $\{p(n)\}_{n \geq 1}$ satisfies the recursive defining formula of the Fibonacci numbers $\text{Fib}(n)$ minus one, but with initial values 2, 4, i.e. $p(n) = \text{Fib}(n+3) - 1$, for $n \geq 1$, where $\text{Fib}(n) = \text{A000045}(n)$. We observe also that the numbers $\text{Fib}(n) - 1$ appear as the sequence [OEIS:A000071](#).

In what follows we will see that the number $q(n)$ of cover relations of $\mathcal{QH}(\text{Aus } T_n)$ is given by the sequence [OEIS:A023610](#) (cf. Proposition 3.11.4), which counts the number of parts of all compositions of n that use only 1 and 2, i.e. setting $\text{Comp}_{\leq 2}(n) := \{\sigma \in \text{Comp}(n) \mid \sigma_i \leq 2 \text{ for all } i\}$ and $\mathbb{P}(\text{Comp}_{\leq 2}(n)) := \{\sigma_i \in \mathbb{P}(n) \mid \sigma \in \text{Comp}_{\leq 2}(n)\}$, then $\text{card } \mathbb{P}(\text{Comp}_{\leq 2}(n)) = \text{A023610}(n-1)$.

Note that the image of the restriction of the map $\Phi: \text{Cov } H(n) \xrightarrow{\sim} \mathbb{P}(n)$ to the subset of cover relations $(J, I) \in \text{Cov } H(n)$ with $J, I \in P(n)$ is not completely contained in $\mathbb{P}(\text{Comp}_{\leq 2}(n))$, e.g. $\Phi([2], [3]) = 3$, we need to characterise those elements $\Phi((J, I))$ that

do not lie in $\mathbb{P}(\text{Comp}_{\leq 2}(n))$. For, we define the following subsets of $\text{Comp}(n)$:

$$F(n) := \{\sigma \in \text{Comp}_{\leq 2}(n) \mid \sigma_i = 2 \text{ for some } i \geq 2\}, \text{ and}$$

$$G(n) := \{\sigma \in \text{Comp}(n) \mid \sigma_1 \geq 3, \sigma_i \leq 2 \text{ for all } i > 1\}.$$

For simplicity, we write a k -composition σ of n as a word w of length k , and we identify both notations, e.g. if $\sigma = (1, 2, 4) \in \text{Comp}(7)$, then $\sigma = 124$. Using this identification, the following table shows the elements of $F(n)$ and $G(n)$, for $n = 3, 4, 5$.

n	$F(n)$	$G(n)$
3	12	3
4	121, 112, 22	31, 4
5	1211, 1121, 1112, 221, 212, 122	311, 41, 32, 5

Additionally we set,

$$\mathbb{F}(n) := \{\sigma_i \in \mathbb{P}(n) \mid \sigma \in F(n), i \geq 2, \sigma_i = 2\}, \text{ and}$$

$$\mathbb{G}(n) := \{\sigma_i \in \mathbb{P}(n) \mid \sigma \in G(n), \sigma_i \neq 2\}.$$

Note that $\text{card } \mathbb{F}(n) = \text{card } \mathbb{G}(n)$ for $1 \leq n \leq 5$. We show that this is not a coincidence.

Lemma 3.11.3. $\text{card } \mathbb{F}(n) = \text{card } \mathbb{G}(n)$, for $n \geq 1$.

Proof. Let $n \geq 5$. For $X_n \in \{F(n), G(n)\}$ and $m \in \mathbb{N}$, set $X_n[m] := \{wm \mid w \in X_n\}$. First we show that

$$F(n) = F(n-1)[1] \sqcup F(n-2)[2] \sqcup \{x_n, y_n\}, \quad (3.11.1)$$

where $x_n := 1 \cdots 12$ is a $n-1$ -composition of n , and $y_n := 21 \cdots 12$ is a $n-2$ -composition of n . It is clear that the right hand side of Eq. (3.11.1) is contained in $F(n)$. Conversely, let $\sigma \in F(n)$ be a k -composition. If $\sigma_k = 1$, then $\sigma \in F(n-1)[1]$. If $\sigma_k = 2$, and there exists $1 < i < k$, then $\sigma \in F(n-2)[2]$, otherwise $\sigma \in \{x_n, y_n\}$. A similar proof shows that equality

$$G(n) = G(n-1)[1] \sqcup G(n-2)[2] \sqcup \{n\} \quad (3.11.2)$$

also holds. Hence,

$$\text{card } \mathbb{F}(n) = \text{card } \mathbb{F}(n-1) + \text{card } \mathbb{F}(n-2) + \text{card } F(n-2) + 2, \text{ and}$$

$$\text{card } \mathbb{G}(n) = \text{card } \mathbb{G}(n-1) + \text{card } G(n-1) + \text{card } \mathbb{G}(n-2) + 1.$$

Moreover, from (3.11.1) we have that $\text{card } F(n) = \text{card } F(n-1) + \text{card } F(n-2) + 2$, thus, (3.11.2) and an easy induction show that $\text{card } G(n) = \text{card } F(n-1) + 1$. The result follows by induction. \square

Proposition 3.11.4. Let $n \geq 1$. Then $q(n) = \text{A023610}(n-1)$. Moreover, $q(n) = q(n-1) + q(n-2) + \text{Fib}(n+1)$, for $n \geq 3$.

Proof. Define

$$\begin{aligned} R &= \{\sigma_i \in \mathbb{P}(n) \mid \sigma \in \text{Comp}_{\leq 2}(n), \sigma_i = 1\}, \text{ and} \\ T &= \{\sigma_1 \in \mathbb{P}(n) \mid \sigma \in \text{Comp}_{\leq 2}(n), \sigma_1 = 2\}. \end{aligned}$$

Hence, $\mathbb{P}(\text{Comp}_{\leq 2}(n)) = \mathbb{F}(n) \sqcup R \sqcup T$, and $\text{card } \mathbb{F}(n) \sqcup R \sqcup T = \mathbf{A023610}(n-1)$.

Let $x = \langle\langle J, \bar{D} \rangle\rangle \in \text{Cov } H(n)$. If we prove that J and $J \sqcup D$ are in $P(n)$ if and only if $\Phi(x) \in R \sqcup T \sqcup \mathbb{G}(n)$, then the result follows from Lemma 3.11.3, since $\Phi: \text{Cov } H(n) \rightarrow \mathbb{P}(n)$ is a bijection. So, we prove the last claim.

Set $I := J \sqcup D$, and suppose that $J, J \sqcup D \in P(n)$. From the proof of Theorem 3.10.3, we have four cases.

Case 1. $x \in A(n)$. Then $I = \{1, 2, \dots, t_1, s_2, \dots, s_l, d, r_1, \dots, r_m\}$, with no consecutive integers after t_1 , for some $d \geq 3$. Thus

$$\Phi_A(x) \in \begin{cases} G(n) & \text{if } t_1 \geq 2, \\ F(n) & \text{if } t_1 = 1 \text{ and } l \geq 2 \text{ or } m \geq 1, \\ \{21 \cdots 1\} & \text{if } I = \{1, d\}. \end{cases}$$

In the first case we get $\Phi(x) = 1 \in \mathbb{G}(n)$, and in the last two cases $\Phi(x) = 1 \in R$, since $I \setminus [1, t_1]$ is union of singletons.

Case 2. $x \in B(n)$. Then $I = \{1, 2, \dots, d, r_1, \dots, r_m\}$, with no consecutive integers after d , for some $d \geq 2$. Thus,

$$\Phi_B(x) \in \begin{cases} G(n) & \text{if } d \geq 3, \\ F(n) & \text{if } d = 2. \end{cases}$$

In the first case we get $\Phi(x) = d \in \mathbb{G}(n)$, and in the second $\Phi(x) = 2 \in T$.

Case 3. $x \in C(n)$. Then $I = \{s_1, \dots, s_l, d, r_1, \dots, r_m\}$, with no consecutive integers and $d \geq 2$. Thus,

$$\Phi_C(x) \in \begin{cases} F(n) & \text{if } l \geq 1 \text{ or } m \geq 1, \\ \{1 \cdots 1\} & \text{if } I = \{d\}. \end{cases}$$

In both instances we find that $\Phi(x) = 1 \in R$.

Case 4. $x \in D(n)$. Then $I = \{1, r_1, \dots, r_m\}$, with no consecutive integers. Thus,

$$\Phi_D(x) \in \begin{cases} F(n) & \text{if } m \geq 1, \\ \{1 \cdots 1\} & \text{if } I = \{1\}. \end{cases}$$

In both instances we find that $\Phi(x) = 1 \in R$.

For the converse, let $\Phi(x) = \sigma$ be a k -composition of n such that $\sigma_i \leq 2$, for $2 \leq i \leq k$. Set $J_{\sigma_i}^- := \{j \mid 1 < j < i, \sigma_j = 2\}$ and $J_{\sigma_i}^+ := \{j \mid i < j < k, \sigma_j = 2\}$. Recall that if $1 \leq j \leq j' \leq k$, then $\sigma_j^{j'} = \sigma_j + \cdots + \sigma_{j'}$. Hence, if $j \in J_{\sigma_i}^-$, then $\sigma_1^j - 1 = \sigma_1^{j-1} + 1$, and if $j \in J_{\sigma_i}^+$, then $\sigma_1^j = \sigma_1^{j-1} + 2$. Set $\Sigma_{\sigma_i}^- = \{\sigma_1^j - 1 \mid j \in J_{\sigma_i}^-\}$, $\Sigma_{\sigma_i}^+ = \{\sigma_1^j \mid j \in J_{\sigma_i}^+\}$, and $\Sigma_{\sigma_i} = \Sigma_{\sigma_i}^- \sqcup \Sigma_{\sigma_i}^+$. From the definition of Ψ (cf. page 68) we have the following.

If $\sigma_i = 1 \in R$, then

$$\Psi(\sigma_i) = \begin{cases} \langle\langle \{1\} \sqcup \Sigma_{\sigma_i}, \{\sigma_1^i\} \rangle\rangle & \text{if } \sigma_1 = 2, \\ \langle\langle \Sigma_{\sigma_i}^+, \{1\} \rangle\rangle & \text{if } \sigma_1 = 1 \text{ and } i = 1, \\ \langle\langle \Sigma_{\sigma_i}, \{\sigma_1^i\} \rangle\rangle & \text{if } \sigma_1 = 1 \text{ and } i > 1. \end{cases}$$

If $\sigma_1 = 2 \in T$, then $\Psi(\sigma_1) = \langle\langle \{1\} \sqcup \Sigma_{\sigma_1}^+, \{2\} \rangle\rangle$.

Finally, if $\sigma_i \in \mathbb{G}(n)$, then

$$\Psi(\sigma_i) = \begin{cases} \langle\langle [1, \sigma_1 - 1] \sqcup \Sigma_{\sigma_i}^+, \{\sigma_1\} \rangle\rangle & \text{if } i = 1. \\ \langle\langle [1, \sigma_1 - 1] \sqcup \Sigma_{\sigma_i}^+, \{\sigma_1^i\} \rangle\rangle & \text{if } i > 1. \end{cases}$$

In all cases we can see that $J, D \in P(n)$, thus $J, I \in P(n)$. This shows the first part of the proposition; the second assertion is a well known property of the sequence [OEIS:A023610](#). \square

The following table shows the values of $q(n)$ for small n .

n	1	2	3	4	5	6	7	8	9	10
$q(n)$	1	3	7	15	30	58	109	201	365	655

Now it is easy to construct recursively the Hasse diagram of $\mathcal{QH}(\text{Aus } T_n)$. For, set $Q(n) := \text{Cov } \mathcal{QH}(\text{Aus } T_n)$,

$$Q'(n-2) := \{(J \sqcup \{n\}, I \sqcup \{n\}) \mid (J, I) \in Q(n-2)\},$$

$$U(n-2) := \{(J, J \sqcup \{n\}) \mid J \in P(n-2)\}.$$

Corollary 3.11.5. $Q(n) = Q(n-1) \sqcup Q'(n-2) \sqcup U(n-2) \sqcup \{([n-1], [n])\}$.

Proof. Write $X = Q(n-1) \sqcup Q'(n-2) \sqcup U(n-2) \sqcup \{([n-1], [n])\}$. Then it is clear that $X \subseteq Q(n)$, and $\text{card } X = q(n-1) + q(n-2) + p(n-2) + 1$. Since $p(n-2) = \text{Fib}(n+1) - 1$, then $\text{card } X = q(n-1) + q(n-1) + \text{Fib}(n-1)$ and the result follows from Proposition [3.11.4](#). \square

Chapter 4

Blocks of Schur algebras of finite representation type

Classical Schur algebras are defined as the endomorphism ring of some permutation modules over symmetric groups. It is well known that Schur algebras are quasi-hereditary for some suitable order of the simples [Don81; Par89; Gre90], whose highest weight theory corresponds to those of general linear groups. In [Erd93] Erdmann classified Schur algebras of finite type, and following this work, Donkin and Erdmann classified blocks of finite representation type of Schur algebras, up to Morita equivalence in [DR94]. In this chapter we describe the homological poset of blocks of Schur algebras of finite representation type, using a similar procedure as in the case of the Auslander algebra of the truncated polynomial ring in Chapter 3.

Let \mathbb{K} be a fixed algebraically closed field of characteristic p . For a vector space V of dimension n over \mathbb{K} , denote by $V^{\otimes r}$ the r -th tensor power of V , with $r \geq 0$. Then the symmetric group Σ_n acts on $V^{\otimes r}$ by place permutations. The *Schur algebra* $S_{\mathbb{K}}(n, r)$ is the endomorphism ring $\text{End}_{\mathbb{K}\Sigma_n}(V^{\otimes r})$. For more details on Schur algebras and representation theory we refer the reader to [Mar08].

An interesting fact about blocks of Schur algebras of finite representation type is that they are Morita equivalent to quiver algebras of the form $\mathbb{K}Q/I$, where Q is the quiver

$$1 \begin{array}{c} \cdot \\ \leftarrow \\ \cdot \end{array} \xrightleftharpoons[b_1]{a_1} 2 \begin{array}{c} \cdot \\ \leftarrow \\ \cdot \end{array} \xrightleftharpoons[b_2]{a_2} \dots \xrightleftharpoons[b_{n-2}]{a_{n-2}} n-1 \begin{array}{c} \cdot \\ \leftarrow \\ \cdot \end{array} \xrightleftharpoons[b_{n-1}]{a_{n-1}} n \begin{array}{c} \cdot \\ \leftarrow \\ \cdot \end{array}$$

and I is the ideal generated by $a_{i+1}a_i$, $b_i b_{i+1}$, $a_i b_i - b_{i+1} a_{i+1}$, for $1 \leq i \leq n-2$, and $a_{n-1} b_{n-1}$, for some $n \geq 1$ [DR94, Theorem 2.1]. We denote this algebra by B_n , setting $B_i = \text{Aus } T_i$, for $i \in \{0, 1, 2\}$.

It is also well known that the algebras B_n are quasi-hereditary for a unique order of the simples, since they have a duality fixing the simples [DR94; Cou19]. In this case the only admissible order is the usual one: $1 < 2 < \dots < n$. In this chapter we describe the homological poset of the algebras B_n . Firstly we describe the factors of some block decomposition of B_n , to do so we define the following class of algebras: for $n \geq 1$, set $\tilde{B}_n := B_n/(b_1 a_1)$. Note that $\tilde{B}_2 = \Pi_2$ (cf. Definition 3.5.1). Recall that for $n \in \mathbb{N}_+$, we set $[n] = \{1, \dots, n\}$.

The chapter is organised as follows. In Section 4.1 we find block decompositions of the factor algebras $B_n/(e)$ and $\tilde{B}_n/(e)$, for e an idempotent in B_n or \tilde{B}_n , respectively, and we

also characterise its blocks. Section 4.2 is devoted to explaining and prove homological properties of the algebra \tilde{B}_n that we utilise later to show that certain embeddings are not homological. In Section 4.3 we describe the set of cover relations of the homological poset of B_n using properties of the algebras $\text{Aus } T_n$ discussed in Chapter 3. Finally, in Section 4.4 we construct the homological Hasse quiver of B_n using similar methods as in Section 3.9.

4.1 Block decompositions of B_n and \tilde{B}_n

In this section we find block decompositions and characterise its blocks in the case of the algebras B_n and \tilde{B}_n . We proceed in a similar way as in the case of $\text{Aus } T_n$ in Chapter 3.

Let $E = \{e_i\}_{i \in [n]}$ be a complete set of primitive orthogonal idempotents of B_n corresponding to the paths of length zero. The next result characterises the indecomposable projective B_n -modules and the idempotent ideals $(e) = B_n e B_n$. For the convenience of the reader, we use two notations for the basis elements.

Proposition 4.1.1. *Let $n \geq 1$, and $P(i) := B_n e_i$, for $1 \leq i \leq n$. Then the following conditions hold.*

$$(a) \ P(1) = \begin{matrix} e_1 \\ b_1 a_1 \end{matrix} a_1 = \begin{matrix} 1 \\ 1 \end{matrix} 2, \ P(n) = \begin{matrix} e_n \\ b_{n-1} \end{matrix} e_n = \begin{matrix} n \\ n-1 \end{matrix}, \text{ and}$$

$$(b) \ P(i) = \begin{matrix} e_i \\ b_{i-1} \\ b_i a_i \end{matrix} a_i = \begin{matrix} i-1 \\ i \\ i+1 \end{matrix}, \text{ for } 2 \leq i \leq n-1.$$

$$(c) \ \text{If } 2 \leq i \leq n-1 \text{ then } (e_{i-1})e_i = \begin{matrix} i-1 \\ i \end{matrix}, \text{ and } (e_{n-1})e_n \cong \mathbb{K} b_{n-1}.$$

$$(d) \ \text{If } 1 \leq i \leq n-1 \text{ then } (e_{i+1})e_i = \begin{matrix} i+1 \\ i \end{matrix}.$$

$$(e) \ \text{If } |i-j| \geq 2 \text{ then } (e_j)e_i = 0.$$

$$(f) \ \text{Let } I = \{i_j \mid 1 \leq j \leq k\} \subseteq [n] \text{ with } i_1 < \dots < i_k. \text{ If } i_k < i \text{ then } (e_I)e_i = (e_{i_k})e_i, \text{ and if } i < i_1 \text{ then } (e_I)e_i = (e_{i_1})e_i.$$

$$(g) \ \text{The ideal } (e_n) \text{ is projective, more precisely } (e_n) \cong P(n) \oplus P(n).$$

Proof. Since there are no paths of length greater than 2, (a) and (b) follow directly from the defining relations of B_n .

(c), (d) and (e) follow easily from (a) and (b). For (f), first assume $i_k < i$. If $i - i_k \geq 2$, then $(e_I)e_i = 0$ since there are no paths starting in i and factoring through some i_j by (c), thus $(e_{i_k})e_i = 0 = (e_{i_k})e_i$. If $i - i_k \leq 1$, then necessarily $i_k = i - 1$. It is clear that $(e_{i_k})e_i \subseteq (e_I)e_i$. Let $x \in (e_I)e_i$, then $x = p_1 e_I p_2 e_i = \sum_{j=1}^k p_1 e_{i_j} p_2 e_i = p_1 e_{i-1} p_2 e_i \in (e_{i-1})e_i$, by (e), from which the first part of (f) follows. The second part of (f) is similar. Finally, (g) follows from (a), (d) and (e). \square

Corollary 4.1.2. *Let $J = \bigsqcup_{i=1}^r J_i \subseteq [n]$. Then the map*

$$\bar{\varphi}_J : \frac{B_n}{(e_{J^c})} \xrightarrow{\sim} \frac{B_n}{(e_{J_1^c})} \times \cdots \times \frac{B_n}{(e_{J_r^c}),}$$

given by $b + (e_{J^c}) \mapsto (b + (e_{J_1^c}), \dots, b + (e_{J_r^c}))$, is a \mathbb{K} -algebra isomorphism.

Proof. By Proposition 4.1.1 (f) the algebra B_n satisfies the property in Lemma 3.4.4. Thus, the result follows from Theorem 3.4.5. \square

Lemma 4.1.3. *The embedding $\iota_{[n-1],[n]} : \text{mod } B_n/(e_n) \hookrightarrow \text{mod } B_n$ is homological.*

Proof. By Proposition 4.1.1 (g), the ideal (e_n) is projective as B_n -module, thus the result follows from Lemma 1.5.2. \square

For simplicity, we use the same notation $E = \{e_i\}_{i \in [n]}$ for classes of paths of length zero of \tilde{B}_n , if no confusion can arise. Then we have the following descriptions of the indecomposable projective modules and idempotent ideals of \tilde{B}_n . This follows directly from Proposition 4.1.1.

Proposition 4.1.4. *Let $n \geq 2$, and $P(i) := \tilde{B}_n e_i$, for $1 \leq i \leq n$. Then*

$$(a) \ P(1) = \begin{matrix} e_1 \\ a_1 \end{matrix} = \begin{matrix} 1 \\ 2 \end{matrix}, \ P(n) = \begin{matrix} e_n \\ b_{n-1} \end{matrix} = \begin{matrix} n \\ n-1 \end{matrix}, \text{ and}$$

$$(b) \ P(i) = \begin{matrix} e_i \\ b_{i-1} a_i \end{matrix} = \begin{matrix} i \\ i-1 \end{matrix} \begin{matrix} i \\ i+1 \end{matrix}, \text{ for } 2 \leq i \leq n-1.$$

$$(c) \ \text{If } 2 \leq i \leq n-1 \text{ then } (e_{i-1})e_i = \begin{matrix} i-1 \\ i \end{matrix}, \text{ and } (e_{n-1})e_n \cong \mathbb{K}b_{n-1}.$$

$$(d) \ \text{If } 2 \leq i \leq n-1 \text{ then } (e_{i+1})e_i = \begin{matrix} i+1 \\ i \end{matrix}, \text{ and } (e_2)e_1 \cong \mathbb{K}a_1.$$

$$(e) \ \text{If } |i-j| \geq 2 \text{ then } (e_j)e_i = 0.$$

$$(f) \ \text{Let } I = \{i_j \mid 1 \leq j \leq k\} \subseteq [n] \text{ with } i_1 < \dots < i_k. \text{ If } i_k < i \text{ then } (e_I)e_i = (e_{i_k})e_i, \\ \text{and if } i < i_1 \text{ then } (e_I)e_i = (e_{i_1})e_i.$$

$$(g) \ \text{If } n \geq 3 \text{ then } (e_i) \cong P(i) \oplus P(i), \text{ for } i = 1, n.$$

Proof. To avoid confusion, we use the notation $\langle e \rangle := \tilde{B}_n e \tilde{B}_n$.

(g) $\langle e_1 \rangle = \langle e_1 \rangle (e_1 + \dots + e_n) \cong \langle e_1 \rangle e_1 \oplus \langle e_1 \rangle e_2 \cong P(1) \oplus P(1)$, by (a), (c) and (e). The case $i = n$ is analogous. \square

The assertion (f) of Proposition 4.1.4 implies that \tilde{B}_n has the property stated in Lemma 3.4.4. Thus, by Theorem 3.4.5 we can describe the quotients $\tilde{B}_n/(e)$, for $e \in \tilde{B}_n$ a sum of elements of E .

Corollary 4.1.5. *Let $J = \bigsqcup_{i=1}^r J_i \subseteq [n]$. Then*

$$\overline{\varphi}_J : \frac{\tilde{B}_n}{\langle e_{J^c} \rangle} \xrightarrow{\sim} \frac{\tilde{B}_n}{\langle e_{J_1^c} \rangle} \times \cdots \times \frac{\tilde{B}_n}{\langle e_{J_r^c} \rangle},$$

given as in Theorem 3.4.5, is a \mathbb{K} -algebra isomorphism, with the usual order of the idempotents.

Now we are able to characterise the blocks of $B_n/(e)$ as algebras of the form B_m and \tilde{B}_n . The crucial step for this characterisation is to consider the case of $\text{Aus } T_n$ (cf. Theorem 3.6.3) and the following observation.

Proposition 4.1.6. *Let A and A' be \mathbb{K} -algebras. Consider two-sided ideals I, J of A , and denote by $\pi_X : A \rightarrow A/X$ the canonical projections for $X \in \{I, J\}$. If there exists a \mathbb{K} -algebra isomorphism $\varphi : \pi_J(A) \rightarrow A'$, then the map*

$$\theta : \frac{\pi_I(A)}{\pi_I(J)} \rightarrow \frac{A'}{\varphi\pi_J(I)}, \quad \pi_I(a) + \pi_I(J) \mapsto \varphi\pi_J(a) + \varphi\pi_J(I)$$

is a \mathbb{K} -algebra isomorphism.

Proof. First note that θ is well defined: suppose $\pi_I(a) + \pi_I(J) = \pi_I(b) + \pi_I(J)$, then $\pi_I(a - b) \in \pi_I(J)$, thus $a - b \in J$, then $\varphi\pi_J(a - b) = 0$, i.e. $\varphi\pi_J(a) = \varphi\pi_J(b)$, from where the claim follows. Now, let $\pi_I(a) + \pi_I(J) \in \text{Ker } \theta$, thus $\varphi\pi_J(a) \in \varphi\pi_J(I)$, since φ is invertible, we get $\pi_J(a) \in \pi_J(I)$, thus $a \in I$ which shows that $\text{Ker } \theta$ is trivial. Finally, since φ is surjective, we conclude that θ is an isomorphism. \square

For the rest of the section, if S is a subset of a ring R , denote $\langle S \rangle_R$ the two-sided ideal of R generated by the elements of S .

For the following result, set $\Lambda_n = \text{Aus } T_n$ and denote Π_n the preprojective algebra of type \mathbb{A}_n . Consider the set $I_n := \{a_{i+1}a_i \mid 1 \leq i \leq n-2\} \cup \{b_i b_{i+1} \mid 1 \leq i \leq n-2\}$, as subset either of Λ_n or Π_n . Then we have the two-sided ideals $\mathbb{I}_n := \langle I_n \rangle_{\Lambda_n}$ and $\tilde{\mathbb{I}}_n := \langle I_n \rangle_{\Pi_n}$. With this notation, observe that $B_n \cong \Lambda_n/\mathbb{I}_n$ and $\tilde{B}_n \cong \Pi_n/\tilde{\mathbb{I}}_n$ as algebras, and we have the canonical projections $\pi_n : \Lambda_n \rightarrow \Lambda_n/\mathbb{I}_n = B_n$ and $\tilde{\pi}_n : \Pi_n \rightarrow \Pi_n/\tilde{\mathbb{I}}_n = \tilde{B}_n$. Note that if $e \in \Lambda_n$ is a primitive idempotent associated to a path of length 0, then $\pi_n(\langle e \rangle_{\Lambda_n}) = \langle e \rangle_{B_n}$, since $\pi_n(e) = e$.

Theorem 4.1.7. *Let $n \geq 3$, $J = [s, t] \subseteq \{1, \dots, n\}$, and $m := t - s + 1$. Then*

$$\frac{B_n}{\langle e_{J^c} \rangle_{B_n}} \cong \begin{cases} 0 & \text{if } J = \emptyset, \\ B_{m=t} & \text{if } s = 1, \\ \tilde{B}_m & \text{if } s > 1. \end{cases}$$

as \mathbb{K} -algebras. If $m = 1$ we identify $B_1 \cong \mathbb{K}e_t \cong \tilde{B}_1$.

Proof. The cases $J = \emptyset$ and $m \leq 2$ have been already considered in Theorem 3.6.3, thus we can assume $J \neq \emptyset$ and $m \geq 3$.

Let $\pi_J : \Lambda_n \rightarrow \Lambda_n/\langle e_{J^c} \rangle_{\Lambda_n}$ and $\tilde{\pi}_J : \Pi_n \rightarrow \Pi_n/\langle e_{J^c} \rangle_{\Pi_n}$ be the canonical projections. Then, by Theorem 3.6.3 we have isomorphisms $\psi : \pi_J(\Lambda_n) \xrightarrow{\sim} \Lambda_t$ if $s = 1$,

and $\psi: \pi_J(\Lambda_n) \xrightarrow{\sim} \Pi_m$, if $s > 1$. Recall the notation $[i+1, i+1, i]_n = a_{i+1}a_i$ and $[i, i+1, i+1]_n = b_i b_{i+1}$ as elements of Λ_n or Π_n . Thus, from the proof of Theorem 3.6.3 we get $\psi([i+2, i+1, i]_n + \langle e_{J^c} \rangle_{\Lambda_n}) = [i+2, i+1, i]_t = a_{i+1}a_i$ and $\psi([i, i+1, i+2]_n + \langle e_{J^c} \rangle_{\Lambda_n}) = [i+2, i+1, i]_t = b_i b_{i+1}$ only if $1 \leq i \leq t-2$, and 0 otherwise, therefore $\psi\pi_J(\mathbb{I}_n) = \mathbb{I}_t$ if $s = 1$, and $\psi\tilde{\pi}_J(\tilde{\mathbb{I}}_n) = \tilde{\mathbb{I}}_m$ if $s > 1$. Hence, if $s = 1$, by Proposition 4.1.6 we have

$$\frac{B_n}{\langle e_{J^c} \rangle_{B_n}} = \frac{\pi_n(\Lambda_n)}{\pi_n(\langle e_{J^c} \rangle_{\Lambda_n})} \cong \frac{\Lambda_t}{\psi\pi_J(\mathbb{I}_n)} = \frac{\Lambda_t}{\mathbb{I}_t} = B_t,$$

and if $s > 1$ we get

$$\frac{B_n}{\langle e_{J^c} \rangle_{B_n}} = \frac{\pi_n(\Lambda_n)}{\pi_n(\langle e_{J^c} \rangle_{\Lambda_n})} \cong \frac{\Pi_m}{\psi\tilde{\pi}_J(\tilde{\mathbb{I}}_n)} = \frac{\Pi_m}{\tilde{\mathbb{I}}_m} = \tilde{B}_m.$$

□

Corollary 4.1.8. *Let $n \geq 3$ and $J = [s, t] \subseteq [n]$, $J \neq \emptyset$. Then $\frac{\tilde{B}_n}{\langle e_{J^c} \rangle} \cong \tilde{B}_{t-s+1}$.*

Proof. We consider B_{n+1} with primitive orthogonal idempotents labelled e_0, e_1, \dots, e_n . So, by Theorem 4.1.7 we have that $B_{n+1}/\langle e_0 \rangle \cong \tilde{B}_n$. Let $\{\tilde{e}_i\}_{i=1}^n$ be the usual set of primitive idempotents of \tilde{B}_n . Then, motivated by the latter isomorphism, we identify $e_i = \tilde{e}_i$, for $1 \leq i \leq n$. Set $J^{c'} := \{0, 1, \dots, n\} \setminus J$, since $J^c = \{1, \dots, n\} \setminus J$, and $0 \notin J$, we get $J^{c'} = J^c \cup \{0\}$. Hence

$$\frac{\tilde{B}_n}{\langle e_{J^c} \rangle} \cong \frac{B_{n+1}}{\langle e_{J^c} + e_0 \rangle} = \frac{B_{n+1}}{\langle e_{J^c \cup \{0\}} \rangle} = \frac{B_{n+1}}{\langle e_{J^{c'}} \rangle} \cong \tilde{B}_{t-s+1}$$

by Theorem 4.1.7. □

Corollary 4.1.9. *Let $J \subseteq [n]$. Then the factor algebra $B_n/\langle e_{J^c} \rangle$ has blocks isomorphic to B_r or \tilde{B}_t , for some $1 \leq r, t \leq n$.*

Proof. Follows from Corollary 4.1.2 and Theorem 4.1.7. □

4.2 Homological properties of \tilde{B}_n

In this section we study some homological properties involving the syzygies of the simple \tilde{B}_n -modules which will permit us characterise homological embeddings between Serre subcategories of $\text{mod } B_n$. More precisely, we show that if $n \geq 5$ and $1 < i < j < n$ then $\text{Ext}_{\tilde{B}_n}^{j-i}(S(i), S(j)) \cong \mathbb{K}$, to do so we need explicit descriptions of the k -th syzygies of the \tilde{B}_n -simples S_1, \dots, S_n for some k 's. The next results will explain this in more detail. We start recalling the definition of syzygy of a module.

Definition 4.2.1. Let Λ be an Artin algebra. Let M be a Λ -module, and $P_\bullet: \dots \rightarrow P_k \xrightarrow{f_k} \dots \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0$ a projective resolution of M , with M in degree -1 . Then f_i is the i -th differential of P_\bullet , for $i \geq 0$, and $\Omega_\Lambda^k M := \text{Ker } f_{k-1}$ is the k -th syzygy of M , for all $k \geq 1$.

From now on, in this section we work over the algebra \tilde{B}_n , so we write $\Omega^k M = \Omega_{\tilde{B}_n}^k M$, for any module $M \in \text{mod } \tilde{B}_n$. Note that by dimension shifting we have that $\text{Ext}_{\tilde{B}_n}^k(M, N) \cong \text{Hom}_{\tilde{B}_n}(\Omega^k M, N)$, for any $N \in \text{mod } \tilde{B}_n$ and $k \leq 1$.

For $1 \leq i \leq n-2$ and $0 \leq t \leq \frac{n-i}{2}$, $t \in \mathbb{Z}$, denote $M(i, i+2t)$ the string \tilde{B}_n -module

$$\begin{array}{ccccccc} i & & i+2 & & \cdots & & i+2t \\ & \searrow & / & \searrow & \cdots & & / \\ & i+1 & & \cdots & & & i+2t-1 \end{array}$$

and the projective \tilde{B}_n -module $P(i, i+2t) := \bigoplus_{j=0}^t P(i+2j)$. By Proposition 4.1.4, there are inclusions $\iota_{i,t}: M(i, i+2t) \hookrightarrow P(i+1, i+2t-1)$. Observe that there are projective covers of the form $h_{i,t}: P(i, i+2t) \rightarrow M(i, i+2t)$.

From Proposition 4.1.4, direct and easy computations show that

$$\Omega^1 S(i) \cong \begin{cases} S(2) & \text{if } i = 1, \\ M(i-1, i+1) & \text{if } 2 \leq i \leq n-2, \\ S(n-1) & \text{if } i = n. \end{cases} \quad (4.2.1)$$

For the next proposition consider the functions $h, g: \mathbb{N} \rightarrow \mathbb{Z}$ given by

$$h(n) := \begin{cases} m-2 & \text{if } n = 2m+1, \\ m-2 & \text{if } n = 2m, \end{cases} \quad g(n) := \begin{cases} 2 & \text{if } n = 3, \\ m+2 & \text{if } n = 2m+1 \neq 3, \\ m+1 & \text{if } n = 2m. \end{cases}$$

Proposition 4.2.2. *Let $n \geq 5$.*

(a) *We have the following exact sequences in $\text{mod } \tilde{B}_n$.*

$$\begin{aligned} 0 &\rightarrow M(1, 3) \rightarrow P(2) \rightarrow P(1) \rightarrow S(1) \rightarrow 0, \\ 0 &\rightarrow M(2, 4) \rightarrow P(1, 3) \rightarrow P(2) \rightarrow S(2) \rightarrow 0, \\ 0 &\rightarrow M(i-2, i+2) \rightarrow P(i-1, i+1) \rightarrow P(i) \rightarrow S(i) \rightarrow 0, \text{ for } 2 \leq i \leq n-1, \\ 0 &\rightarrow M(n-2, n) \rightarrow P(n-1) \rightarrow P(n) \rightarrow S(n) \rightarrow 0. \end{aligned}$$

(b) *If $2 \leq k \leq n-2$ we have $\Omega^k S(1) \cong \begin{cases} M(1, k+1) & \text{if } k \text{ is even,} \\ M(2, k+1) & \text{if } k \text{ is odd.} \end{cases}$*

(c) *If $2 \leq k \leq h(n)$ and $k+3 \leq i \leq n-1-k$, then $\Omega^k S(i) \cong M(i-k, i+k)$.*

(d) *If $3 \leq k \leq n-2$, then $\Omega^k S(1) \cong \Omega^{k-1} S(2) \cong \dots \cong \Omega^{k+1-g(k)} S(g(k))$.*

The diagram in Fig. 4.1 depicts the syzygies considered in Proposition 4.2.2, in the sense that the point in the row label by k and column i represents $\Omega^k S(i)$. Note that the syzygies appearing on the left hand side of the picture satisfy $\Omega^k S(i) \cong \Omega^{k+i-1} S(1)$, for $2 \leq k \leq n-3$ and $2 \leq i \leq g(k)$, by Proposition 4.2.2 (d), thus they are characterised by Proposition 4.2.2 (b). The remaining cases, i.e. the syzygies represented by points in the triangle to the right are completely described by Proposition 4.2.2 (c).

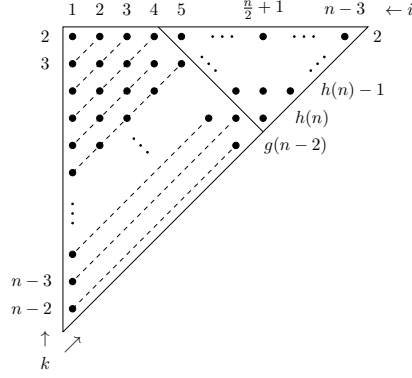


Figure 4.1: For n even, the points in diagram represent $\Omega^k S(i)$. The triangle to the right correspond to syzygies characterised in Proposition 4.2.2 (c). The dashed lines denote the isomorphic syzygies described in Proposition 4.2.2 (d). The diagram is similar when $n \geq 9$ is odd, in that case the row $k = h(n)$ in the triangle has 2 points corresponding to $\Omega^{h(n)} S(\frac{n+1}{2})$ and $\Omega^{h(n)} S(\frac{n+3}{2})$. If $n = 5, 6, 7$ the triangle to the right does not appear.

Proof of Proposition 4.2.2. (a) is consequence of (4.2.1).

(b) From (a), we construct a projective resolution of $S(1)$ by induction. If $n = 2m + 1$, with $m \geq 2$, then $n - 2 = 2m - 1$. Thus, for $2 \leq k \leq m$ we have an exact sequence

$$P(2, 2k) \xrightarrow{h_{2k-1}} P(1, 2k - 1) \xrightarrow{h_{2k-2}} P(2, 2k - 2) \rightarrow \cdots \\ \xrightarrow{h_4} P(2, 4) \xrightarrow{h_3} P(1, 3) \xrightarrow{h_2} P(2) \xrightarrow{h_1} P(1) \xrightarrow{h_0} S(1) \rightarrow 0$$

where $P(1)$ and $P(2, 2k)$ are in degrees 0 and $2k - 1$ respectively. By induction, $\Omega^{2k-1} S(1) = \text{Ker } h_{2k-2} = \text{Im } h_{2k-1} \cong M(2, 2k)$ and $\Omega^{2k-2} S(1) \text{ Ker } h_{2k-3} = \text{Im } h_{2k-2} \cong M(1, 2k - 1)$.

Similarly, if $n = 2m$, with $m \geq 3$, then $n - 2 = 2m - 2$. Thus, for $2 \leq k \leq m$ we construct an exact sequence

$$P(1, 2k - 1) \xrightarrow{h_{2k-2}} P(2, 2k - 2) \xrightarrow{h_{2k-3}} P(1, 2k - 3) \rightarrow \cdots \\ \xrightarrow{h_4} P(2, 4) \xrightarrow{h_3} P(1, 3) \xrightarrow{h_2} P(2) \xrightarrow{h_1} P(1) \xrightarrow{h_0} S(1) \rightarrow 0$$

where $P(1)$ and $P(1, 2k - 1)$ are in degrees 0 and $2k - 2$ respectively. By induction, $\Omega^{2k-2} S(1) = \text{Ker } h_{2k-3} = \text{Im } h_{2k-2} \cong M(1, 2k - 1)$ and $\Omega^{2k-3} S(1) = \text{Ker } h_{2k-4} = \text{Im } h_{2k-3} \cong M(2, 2k - 2)$.

(c) In this case, consider $n \geq 8$, otherwise the conditions are empty and there is nothing to prove. Note that if $k = 2$, the result follows from (a). Then, if $k + 3 \leq i \leq n - 1 - k$, usig induction on k we find an exact sequence

$$P(i - (k - 2), i + k - 2) \xrightarrow{h_{k-2}} P(i - (k - 3), i + k - 3) \xrightarrow{h_{k-3}} \cdots \\ \rightarrow P(i - 2, i + 2) \xrightarrow{h_2} P(i - 1, i + 1) \xrightarrow{h_1} P(i) \xrightarrow{h_0} S(i) \rightarrow 0,$$

then $\text{Ker } h_{k-2} = \Omega^{k-1} S(i) \cong M(i - k + 1, i + k - 1)$, which implies that the following term in the sequence is $P = P(i - k + 1, i + k - 1)$ and $\Omega^k S(i) \cong M(i - k, i + k)$.

(d) We proceed by induction on k . For $k = 3, 4, 5, 6$ the result follows by direct inspection. Thus, we can assume $k \geq 6$. First suppose that $k = 2m + 1$, with $m \geq 3$. Then $g(k - 1) = m + 1$ and $g(k) = m + 2$. Thus, by induction hypothesis we have $\Omega^{2m}S(1) \cong \Omega^{2m-1}S(2) \cong \dots \cong \Omega^m S(m + 1)$. Using these isomorphisms, by induction we continue the construction of projective resolutions of $S(1), \dots, S(m + 1)$ such that $\Omega^{2m+1}S(1) \cong \Omega^{2m}S(2) \cong \dots \cong \Omega^{m+1}S(m + 1)$, but by (b) these syzygies are isomorphic to $M(2, 2m + 2) = M(2, k + 1)$.

On the other hand, note that the numbers $k' := m - 1$ and $i' := m + 2$ satisfy $2 \leq k' \leq h(n)$ and $k' + 3 = i'$, thus by (c) we have $\Omega^{m-1}S(m + 2) \cong M(3, 2m + 1) = M(3, k)$, hence, by induction, the $m - 1$ -st differential of a projective cover of $S(m + 2)$ has the form

$$\begin{array}{ccc} P(3, k) & \xrightarrow{h_{m-1}} & P(4, k - 1) \\ & \searrow & \nearrow \\ & & M(3, k) \end{array}$$

thus $\Omega^m S(m + 2) \cong M(2, k + 1)$, showing the claim for $k = 2m + 1$, since $k + 1 - g(k) = m$ and $g(k) = m + 2$. The case when $k = 2m$ is similar. This completes the proof. \square

Corollary 4.2.3. *Let $n \geq 2$, and $i \in [n]$. Then $\text{pd } S(i) = \infty$ as \tilde{B}_n -module. Thus, $\text{gl. dim } \tilde{B}_n = \infty$.*

Proof. From Proposition 4.2.2 we can construct for each $S(i)$, $i = 1, \dots, n$, an infinite projective resolution of period 2, with repeating differentials $P(1, n) \rightarrow P(2, n - 1)$ if n is odd, and $P(1, n - 1) \rightarrow P(2, n)$ if n is even. \square

We have two crucial implications of Proposition 4.2.2 that will be used in the proof of Theorem 4.3.3. Namely we have the following observation.

Remark 4.2.4. If $2 \leq k \leq n - 2$ and $1 \leq i \leq n - 1 - k$, then $\Omega_{\tilde{B}_n}^k S(i) \cong M(x, i + k)$ for some integer $1 \leq x \leq n - 1$, and for $i \in [n]$, $\text{Ext}_{\tilde{B}_n}^2(S(i), S(i)) \cong \text{Hom}_{\tilde{B}_n}(\Omega^2 S(i), S(i)) \cong \mathbb{K}$, since $S(i)$ is direct summand of $\text{top } \Omega^2 S(i)$.

The following is a key lemma used in the characterisation of homological embeddings into $\text{mod } B_n$.

Lemma 4.2.5. *Let $n \geq 2$. Then $\text{Ext}_{B_n}^2(S(i), S(i)) \cong \mathbb{K}$ for $2 \leq i \leq n$.*

Proof. Induction on n . For $n = 2$, we have that $0 \rightarrow P(2) \rightarrow P(1) \rightarrow P(2) \rightarrow S(2) \rightarrow 0$ is a projective resolution of $S(2)$ in $\text{mod } B_2$, thus $\Omega^2 S(2) \cong P(2)$. By dimension shifting we get $\text{Ext}_{B_2}^2(S(2), S(2)) \cong \text{Hom}_{B_2}(\Omega^2 S(2), S(2)) \cong \mathbb{K}$, since $S(2) = \text{top } P(2)$.

Now let $n > 2$. By Theorem 4.1.7 $B_n/(e_n)$ is isomorphic to B_{n-1} , thus Lemma 4.1.3 implies that the composition $\text{mod } B_{n-1} \xrightarrow{\sim} \text{mod } B_n/(e_n) \hookrightarrow \text{mod } B_n$ is a homological embedding, so $\text{Ext}_{B_n}(S(i), S(i)) \cong \mathbb{K}$ for all $i \in \{2, \dots, n - 1\}$ by induction. It is left to prove the case $i = n$. Indeed, the following is an exact sequence in $\text{mod } B_n$:

$$0 \rightarrow K \rightarrow P(n - 1) \rightarrow P(n) \rightarrow S(n) \rightarrow 0,$$

where $K = \begin{smallmatrix} n-2 \\ \searrow \\ n-1 \end{smallmatrix} \nearrow^n$, hence $K = \Omega^2 S(n)$. So, $\text{Ext}_{B_n}^2(S(n), S(n)) \cong \text{Hom}_{B_n}(K, S(n)) \cong \mathbb{K}$, because $S(n)$ is direct summand of $\text{top } K$, finishing the proof. \square

4.3 Homological poset of B_n

Now we are prepare to give a characterisation of the cover relations of the homological poset of B_n . We start describing the homological embeddings into $\text{mod } B_n$.

Theorem 4.3.1. *Let $J = \bigsqcup_{j=1}^r J_j \subsetneq [n]$, J non-empty. Then the embedding $\iota_J := \iota_{J,[n]}: \text{mod } B_n/(e_{J^c}) \hookrightarrow \text{mod } \tilde{B}_n$ is homological if and only if $r = 1$ and $1 \in J$.*

Proof. (\Leftarrow) Follows by induction using Lemma 4.1.3 and Theorem 4.1.7.

(\Rightarrow) We proceed by contraposition. First assume, without loss of generality, that $r = 2$. Write $J_i = [s_i, t_i]$ for $i = 1, 2$, with $t_1 + 2 \leq s_2$, and set $B_J := B_n/(e_{J^c})$. Then

$$B_J \cong \frac{B_n}{(e_{J_1^c})} \times \frac{B_n}{(e_{J_2^c})},$$

where the second block is isomorphic to $\tilde{B}_{t_2-s_2+1}$, since $1 \notin J_2$. We distinguish two cases. If $\text{card } J_2 = 1$, i.e. $s_2 = t_2$, then $\tilde{B}_{t_2-s_2+1} \cong \mathbb{K}e_{t_2} \cong S(t_2)$ as B_J -modules, which shows that $S(t_2)$ is B_J -projective, so $\text{Ext}_{B_J}^i(S(t_2), S(t_2)) = 0$ for all $i \geq 1$. On the other hand, by Lemma 4.2.5 $\text{Ext}_{B_n}^2(S(t_2), S(t_2)) \cong \mathbb{K}$, given $t_2 > 2$. Therefore ι_J is not homological.

Now, if $\text{card } J_2 > 1$, then $\tilde{B}_{t_2-s_2+1}$ has infinite global dimension by Corollary 4.2.3, but B_n has finite global dimension, considering it is quasi-hereditary, hence ι_J cannot be homological.

Finally, if $1 \notin J$, B_J is isomorphic to a product of algebras of the form \tilde{B}_m for some $m \in \mathbb{N}_+$, so similar arguments as before show that ι_J is not a homological embedding. This completes the proof. \square

Lemma 4.3.2. *Let $n \geq 3$. Then, the embedding $\iota_i: \text{mod } \tilde{B}_n/(e_i) \hookrightarrow \text{mod } \tilde{B}_n$ given by restriction of scalars is homological for $i = 1, n$.*

Proof. It is enough to observe that (e_1) and (e_n) are projective \tilde{B}_n -modules by Proposition 4.1.4 (g). \square

Recall that we identify $\tilde{B}_2 = \Pi_2$, thus in either case, the only homological embeddings of the form $\text{mod } A \hookrightarrow \text{mod } \tilde{B}_2$ are the trivial ones, i.e. when $A = \tilde{B}_2$ or $A = 0$ (cf. Corollary 3.7.4).

Theorem 4.3.3. *Let $n \geq 3$, $J = \bigsqcup_{i=1}^r J_i \subsetneq [n]$, J non-empty. Then, the embedding $\iota_J := \iota_{J,[n]}: \text{mod } \tilde{B}_n/(e_{J^c}) \hookrightarrow \text{mod } \tilde{B}_n$ is homological if and only if $r = 1$ and $\text{card } J \geq 2$.*

Proof. Set $B := \tilde{B}_n/(e_{J^c})$.

(\Rightarrow) We proceed by contraposition. For, suppose $r \geq 2$. Without loss of generality we can consider $r = 2$. Write $J_i = [u_i, t_i]$, for $i = 1, 2$. Then, by Corollary 4.1.5 we have

$$B \cong \frac{\tilde{B}_n}{(e_{J_1^c})} \times \frac{\tilde{B}_n}{(e_{J_2^c})}.$$

If $u_i = t_i$, for some $i = 1, 2$, then Corollary 4.1.8 implies that $\tilde{B}_n/(e_{J_i^c}) \cong \tilde{B}_1 \cong \mathbb{K}e_{u_i}$, thus $\text{pd}_B S(u_i) = \text{pd}_{\tilde{B}_n/(e_{J_i^c})} S(u_i) = \text{pd}_{\mathbb{K}} \mathbb{K} = 0$. But in Remark 4.2.4 we observed that $\text{Ext}_{\tilde{B}_n}^2(S(u_i), S(u_i)) \cong \mathbb{K}$. Hence, ι_J is not homological in this case.

Now assume $u_i < t_i$, for $i = 1, 2$. Thus $1 < t_1 < u_2 < n$. Since $S(t_1)$ and $S(u_2)$ are supported in different blocks as B -modules, $\text{Ext}_B^*(S(t_1), S(u_2)) = 0$. On the other hand, there exists $2 \leq k \leq n - 2$ such that $u_2 = t_1 + k$. By dimension shifting we have

$$\text{Ext}_{\tilde{B}_n}^k(S(t_1), S(t_1 + k)) \cong \text{Hom}_{\tilde{B}_n}(\Omega^k S(t_1), S(t_1 + k)). \quad (4.3.1)$$

From Proposition 4.2.2 we know that $\Omega^k S(t_1) \cong M(p, t_1 + k)$ as \tilde{B}_n -modules, for some $1 \leq p \leq n - 1$. Then, recalling the shape of $M(p, t_1 + k)$, it is easy to see that the right hand side of (4.3.1) is isomorphic to \mathbb{K} , since $S(t_1 + k)$ is direct summand of $\text{top } \Omega^k S(t_1)$, showing that ι_J is not homological.

(\Leftarrow) We proceed by induction on n . For $n = 3$, the only possibilities for J are $[1, 2]$ and $[2, 3]$. Since J^c is equal to $\{3\}$ and $\{1\}$ respectively, the claim follows from Lemma 4.3.2. Now let $n > 3$ and $J \subseteq [n]$ as in the hypothesis. We distinguish two cases.

First, if $n \notin J$, then $J \subseteq [n - 1]$. Write $J^c = [n - 1] \setminus J$, and recall that $J^c = [n] \setminus J$. Then,

$$\frac{\tilde{B}_n}{(e_{J^c})} \cong \tilde{B}_{t-s+1} \cong \frac{\tilde{B}_{n-1}}{(e_{J^c})}.$$

By induction $\iota'_j: \text{mod } \tilde{B}_{n-1}/(e_{J^c}) \rightarrow \text{mod } \tilde{B}_{n-1}$ is a homological embedding. Also note that $\tilde{B}_{n-1} \cong \tilde{B}_n/(e_n)$, and $\iota_n: \text{mod } \tilde{B}_n/(e_n) \rightarrow \text{mod } \tilde{B}_n$ is homological. Hence, the composition

$$\text{mod } \frac{\tilde{B}_n}{(e_{J^c})} \xrightarrow{\sim} \text{mod } \frac{\tilde{B}_{n-1}}{(e_{J^c})} \xrightarrow{\iota'_j} \text{mod } \tilde{B}_{n-1} \xrightarrow{\sim} \text{mod } \frac{\tilde{B}_n}{(e_n)} \xrightarrow{\iota_n} \text{mod } \tilde{B}_n$$

equals ι_J , since all functors are given by restriction of scalars, thus ι_J is homological.

Second, if $n \in J$, then necessarily $1 \notin J$, and the proof is done similarly as in the previous case, but now considering that $\tilde{B}_{n-1} \cong \tilde{B}_n/(e_1)$ and that the embedding $\iota_1: \text{mod } \tilde{B}_n/(e_1) \rightarrow \text{mod } \tilde{B}_n$ is homological. This completes the proof. \square

Recall that $\mathcal{H}(A)$ is a weak subposet of the power set of $[n]$ ordered by inclusion $\mathcal{P}(n)$. Thus, if $(J, I) \in \mathcal{H}(A)$ and $\text{card } I \setminus J = 1$, then $(J, I) \in \text{Cov } \mathcal{H}(A)$.

Corollary 4.3.4. *Let $n \geq 3$, and $J \subseteq [n]$. Then,*

- (a) $(J, [n]) \in \text{Cov } \mathcal{H}(B_n)$ if and only if $J = [1, n - 1]$.
- (b) $(J, [n]) \in \text{Cov } \mathcal{H}(\tilde{B}_n)$ if and only if $J = [1, n - 1]$ or $J = [2, n]$.

Proof. (a) From Theorem 4.3.1 we have $r := (J, [n]) \in \mathcal{H}(B_n)$ if and only if $J = [1, t]$ for some $1 \leq t \leq n - 1$. Thus, if $t = n - 1$, then clearly r is a cover relation. Conversely, if $t \leq n - 2$, then $\iota_{[t], [n]} = \iota_{[n-1], [n]} \circ \iota_{[t], [n-1]}$ is a non-trivial factorisation of $\iota_{[t], [n]}$ into homological embeddings, thus $[n]$ does not cover $[t]$ in $\mathcal{H}(B_n)$.

(b) From Theorem 4.3.3 we have $r := (J, [n]) \in \mathcal{H}(\tilde{B}_n)$ if and only if $J = [s, t]$ for some $1 \leq s < t \leq n$. Thus, if $(s, t) = (1, n - 1)$ or $(s, t) = (2, n)$ then r is a cover relation in $\mathcal{H}(\tilde{B}_n)$. Conversely, if $2 \leq \text{card } J \leq n - 2$, then $J \subsetneq [1, n - 1]$ or $J \subsetneq [2, n]$. In the first case, $\iota_{J, [n]} = \iota_{[n-1], [n]} \circ \iota_{J, [n-1]}$ is a non-trivial factorisation of $\iota_{J, [n]}$ into homological embeddings, thus r is not a cover relation in $\mathcal{H}(\tilde{B}_n)$. The other case is similar. \square

In what follows we use the following assumptions. Let $J \subseteq I \subseteq [n]$. As before, we split them into intervals, i.e. we find $1 \leq r \leq t$ and closed intervals $J_j = [s_j, t_j]$, $I_i = [r_i, u_i]$, for $1 \leq j \leq r$ and $1 \leq i \leq t$, such that $J = \bigsqcup_{j=1}^r J_j$ and $I = \bigsqcup_{i=1}^t I_i$. We also assume that $t_j + 2 \leq s_{j+1}$ and $u_i + 2 \leq r_{i+1}$, for $j \in [r-1]$ and $i \in [t-1]$, unless otherwise stated. Moreover, we have a unique function $\sigma: [r] \rightarrow [t]$ such that $J_j \subseteq I_{\sigma(j)}$ for all $j \in [r]$. In the case when $t = r$, we assume that $\sigma = 1_{[r]}$, thus, if $1 \in J$, then $1 \in J_1 \subseteq I_1$, and in these cases, $\ell(J) = r$ and $\ell(I) = t$. Also, for simplicity, we denote $B_J := B_n/(e_{J^c})$ for any $J \subseteq [n]$.

Next we show that the cover relations of $\mathcal{H}(\text{Aus}T_n)$ (cf. Proposition 3.8.1) are actually relations in $\mathcal{H}(B_n)$, but not necessarily cover relations.

Proposition 4.3.5. *Let $(J, I) \in \mathcal{H}(\text{Aus}T_n)$, then $(J, I) \in \mathcal{H}(B_n)$.*

Proof. It is enough to prove the assertion for the cover relations of $\mathcal{H}(\text{Aus}T_n)$. So, let $(J, I) \in \text{Cov} \mathcal{H}(\text{Aus}T_n)$, and set $B_I := B_n/(e_{I^c})$ for simplicity. Then, by Lemma 2.3.7, it is sufficient to show that the ideal $(e_{I \setminus J})$ is projective as B_I -module.

From Proposition 3.8.1 we have four cases.

Case 1. $t = r + 1$, $1 \in J$, $J_1 = I_1$ and $J_j = I_{\sigma(j)}$ for $2 \leq j \leq r$. By Lemma 3.10.1, we have $I \setminus J = I_{i_0}$ for some $i_0 \in \{2, \dots, t\}$, i.e. $I = J \sqcup I_{i_0}$. Then

$$B_I \cong \prod_{j=1}^r \frac{B_n}{(e_{J_j^c})} \times \frac{B_n}{(e_{I_{i_0}^c})},$$

note that the last block is isomorphic to $(e_{I \setminus J}) = (e_{I_{i_0}})$ as B_I -module, hence B_I -projective.

Case 2. $t = r$, $1 \in J$, $u_1 = t_1 + 1$, and $J_j = I_j$ for $2 \leq j \leq r$. Thus, $I_1 = [1, u_1]$ and $I \setminus J = \{u_1\}$, so

$$B_I \cong \frac{B_n}{(e_{I_1^c})} \times \prod_{j=2}^r \frac{B_n}{(e_{J_j^c})} \cong B_{u_1} \times \prod_{j=2}^r \frac{B_n}{(e_{J_j^c})},$$

then $(e_{I \setminus J}) = B_I e_{u_1} B_I \cong \frac{B_n}{(e_{I_1^c})} e_{u_1} \frac{B_n}{(e_{I_1^c})} \cong B_{u_1} e_{u_1} B_{u_1} =: M$ as B_I -modules. Note that M is a projective B_{u_1} -module (Proposition 4.1.1 (g)). Set $B' = \prod_{j=2}^r B_n/(e_{J_j^c})$, then B' is an ideal of B_I and $B' \subseteq \text{ann}_{B_I} M$, hence $M \cong (e_{I \setminus J})$ is a projective B_I -module.

Case 3. $t = r + 1$, $1 \notin I$ and $J_j = I_{\sigma(j)}$ for all $j \in [r]$. Then, there exists $i_0 \in [t]$ such that $I \setminus J = I_{i_0}$, i.e. $I = J \sqcup I_{i_0}$, thus

$$B_I \cong \prod_{j=1}^r \frac{B_n}{(e_{J_j^c})} \times \frac{B_n}{(e_{I_{i_0}^c})}$$

and similarly as in Case 1, we conclude that $(e_{I \setminus J})$ is B_I -projective.

Case 4. $J \subseteq \{3, \dots, n\}$ and $I = [1, 1] \sqcup J$. Thus, $I \setminus J = \{1\}$, so

$$B_I \cong \frac{B_n}{(e_{[2, n]})} \times \prod_{j=1}^r \frac{B_n}{(e_{J_j^c})} \cong \mathbb{K}e_1 \times \prod_{j=1}^r \frac{B_n}{(e_{J_j^c})},$$

thus $(e_{I \setminus J}) = (e_1) \cong \mathbb{K}e_1$ as B_I -modules, showing that $(e_{I \setminus J})$ is projective. \square

Note that the relations (J, I) described in Cases 2 and 4 in the last proof are cover relations in $\mathcal{H}(B_n)$. We keep the same notation for the intervals J_j and I_i as before.

Lemma 4.3.6. *Let $n \geq 3$, $J \subseteq I \subseteq [n]$, $r = \ell(J)$ and $t = \ell(I)$. Then the following conditions hold.*

- (a) *If $(J, I) \in \text{Cov } \mathcal{H}(B_n)$, then $r \leq t \leq r + 1$.*
- (b) *Let $t = r$. Then $(J, I) \in \text{Cov } \mathcal{H}(B_n)$ if and only if there exists $i_0 \in [r]$ such that $(J_{i_0}, I_{i_0}) \in \text{Cov } \mathcal{H}(B_{I_{i_0}})$ and $J_i = I_i$ for all $i \neq i_0$.*
- (c) *Let $t = r + 1$. Then $(J, I) \in \text{Cov } \mathcal{H}(B_n)$ if and only if there exists $i_0 \in [t]$ such that $i_0 \notin \text{Im } \sigma$, $(\emptyset, I_{i_0}) \in \text{Cov } \mathcal{H}(B_{I_{i_0}})$ and $J_j = I_{\sigma(j)}$ for all $j \in [r]$.*

Proof. (a) First suppose by contradiction that $r + 1 < t$. Without loss of generality we can assume $t = r + 2$. Then $I = J \sqcup I_{i_0} \sqcup I_{i_1}$, thus similar arguments to those used in the proof of Proposition 4.3.5 Case 1 show that $J \prec_{B_n} J \sqcup I_{i_0} \prec_{B_n} I$, a contradiction. Moreover, $r \leq t$, otherwise $t < r$, thus there exist $i_0 \in [t]$ and $j_0 \in [r]$ such that $J_{j_0} \sqcup J_{j_0+1} \subseteq I_{i_0}$. If $1 \in I_{i_0}$, then B_I has a block of the form $B_n/(e_{I_{i_0}^c}) \cong B_m$, for some m , hence, after relabelling the idempotents if necessary, Theorem 4.3.1 shows that $(J_{j_0} \sqcup J_{j_0+1}, I_{i_0})$ is not a homological embedding in $\mathcal{H}(B_m)$, thus $(J, I) \notin \mathcal{H}(B_n)$, a contradiction. A similar contradiction holds if $1 \notin I_{i_0}$. Therefore, $r \leq t \leq r + 1$.

(b) It is clear that the implication (\Leftarrow) is consequence of Corollary 4.1.2. For the converse, we proceed by contradiction. In particular we have $(J_i, I_i) \in \mathcal{H}(B_{I_i})$ holds for all $i \in [r]$. First, suppose that there is no $i_0 \in [r]$ such that $(J_{i_0}, I_{i_0}) \in \text{Cov } \mathcal{H}(B_{I_{i_0}})$, i.e. $(J_i, I_i) \notin \text{Cov } \mathcal{H}(B_{I_i})$ for all $i \in [r]$, then there exists $K_i \subseteq [n]$ such that $J_i \prec K_i \prec I_i$ in B_{I_i} , thus $J \prec_{B_n} J \sqcup K_i \prec_{B_n} I$ contradicts the fact that (J, I) is a cover relation. Then, we can assume that such an i_0 exists. By contradiction, we assume also that there exists $i_1 \in [j]$, $i_0 \neq i_1$, such that $J_{i_1} \subsetneq I_{i_1}$. Then, $J_{i_0} \prec I_{i_0}$ in $\mathcal{H}(B_{I_{i_0}})$ and $J_{i_1} \prec I_{i_1}$ in $\mathcal{H}(B_{I_{i_1}})$, hence $J \prec (J \setminus J_{i_0}) \sqcup I_{i_0} \prec I$ is a non-trivial factorisation of the relation (J, I) in $\mathcal{H}(B_n)$, since J_{i_1} is contained in the middle set, a contradiction. Thus, the converse holds as well.

Note that if $(J, I) \in \text{Cov } \mathcal{H}(B_n)$, then $t = r$ if and only if $J_i \neq \emptyset$ for all $i \in [r]$, thus (c) is consequence of (b) when we let some J_i to be empty. This completes the proof. \square

The last lemma shows that the cover relations (J, I) in $\mathcal{H}(B_n)$ are determined by the cover relations of any block B_I , and these cover relations are completely characterised in Corollary 4.3.4, since each block is of the form B_m or \tilde{B}_m , for some m .

Proposition 4.3.7. *Let $n \geq 3$, and $J \subseteq I \subseteq [n]$ as before. Consider the following conditions.*

- (C1) *$t = r + 1$, $1 \in J$, $J_j = I_{\sigma(j)}$ for all $j \in [r]$, and $\text{card } I_{i_0} \in \{1, 2\}$ for the unique $i_0 \in [t] \setminus \text{Im } \sigma$.*
- (C2) *$t = r$, $1 \in J_1 \subsetneq I_1$, $J_j = I_j$ for $2 \leq j \leq r$, and $u_1 = t_1 + 1$.*
- (C3) *$t = r + 1$, $1 \notin I$, $J_j = I_{\sigma(j)}$ for all $j \in [r]$, and $\text{card } I_{i_0} \in \{1, 2\}$ for the unique $i_0 \in [t] \setminus \text{Im } \sigma$.*

(C4) $I = [1, 1] \sqcup J$, with $J \subseteq \{3, \dots, n\}$.

(C5) $1 \in J$, there exists $2 \leq j_0 \leq r$ such that $s_{j_0} < t_{j_0}$ and $I \setminus J = \{s_{j_0} - 1\}$ or $I \setminus J = \{t_{j_0} + 1\}$.

(C6) $1 \notin I$, there exists $1 \leq j_0 \leq r$ such that $s_{j_0} < t_{j_0}$ and $I \setminus J = \{s_{j_0} - 1\}$ or $I \setminus J = \{t_{j_0} + 1\}$.

If the pair (I, J) satisfies one of the properties (C1)-(C6), then $(I, J) \in \text{Cov } \mathcal{H}(B_n)$. Conversely, all the cover relations of $\mathcal{H}(B_n)$ are given by the conditions (C1)-(C6).

Proof. First, note that all the conditions are pairwise disjoint. Moreover, by Propositions 3.8.1 and 4.3.5 the pairs (J, I) in (C1) to (C4) are relations in $\mathcal{H}(B_n)$. For the rest of the proof, set $B_I := B_n/(e_I^c)$.

Next, we show that (C5) and (C6) give relations in $\mathcal{H}(B_n)$. For, suppose that (I, J) satisfies (C5) with $I \setminus J = \{s_{j_0} - 1\}$. Then, $I = I_{j_0} \sqcup \bigsqcup_{j \neq j_0} J_j$, where $I_{j_0} = [s_{j_0} - 1, t_{j_0}]$. Since $1 \in J$, we have that $1 \in I_1$, and $1 \notin I_{j_0}$, given $2 \leq j_0$. Thus,

$$B_I \cong \frac{B_n}{(e_{I_{j_0}}^c)} \times \prod_{j \neq j_0} \frac{B_n}{(e_{J_j}^c)} \cong \tilde{B}_m \times \prod_{j \neq j_0} \frac{B_n}{(e_{J_j}^c)}, \quad (4.3.2)$$

where $m := t_{j_0} - s_{j_0} + 2 \geq 3$. Then, $(e_{I \setminus J}) = B_I e_{s_{j_0} - 1} B_I \cong \tilde{B}_m e_{s_{j_0} - 1} \tilde{B}_m =: M$ as B_I -modules. By Proposition 4.1.4 (g), and after relabelling the idempotents of \tilde{B}_m as $\{e_{j_0-1}, e_{j_0}, \dots, e_{t_{j_0}}\}$, we conclude that M is a projective \tilde{B}_m -module, thus a projective B_I -module, considering the isomorphism (4.3.2). Hence, Lemma 2.3.7 implies that $\iota_{J,I}$ is homological in this case.

Now, if (J, I) satisfies (C5) with $I \setminus J = \{t_{j_0} + 1\}$, or (C6), we find an isomorphism (4.3.2), thus similar arguments as before show that $(J, I) \in \mathcal{H}(B_n)$.

On the other hand, the pairs (J, I) with properties (C2), (C4), (C5) or (C6) satisfy that $\text{card } I - \text{card } J = 1$, thus they are cover relations in $\mathcal{H}(B_n)$. If (J, I) has the conditions in (C1) or (C3), then $I = J \sqcup I_{i_0}$ for some $i_0 \in [t]$, and $\text{card } I_{i_0} \leq 2$. If $\text{card } I_{i_0} = 1$, then $\text{card } I - \text{card } J = 1$ and (J, I) is a cover relation. If $\text{card } I_{i_0} = 2$, then $B_n/(e_{I_{i_0}}^c) \cong \tilde{B}_2 = \Pi_2$ would be a block of B_I by Corollary 4.1.2 and Theorem 4.1.7, but we know that there are no non-trivial homological embeddings $\text{mod } A \hookrightarrow \text{mod } \Pi_2$ given by restriction of scalars, so there is no $K \subseteq [n]$ such that $J \prec_{B_n} K \prec_{B_n} I$, i.e. $(J, I) \in \text{Cov } \mathcal{H}(B_n)$.

Now, we prove that these are all the cover relations of $\mathcal{H}(B_n)$. Indeed, let $(J, I) \in \text{Cov } \mathcal{H}(B_n)$, then by Lemma 4.3.6 (a) we know that $r \leq t \leq r + 1$.

First, if $t = r$, Lemma 4.3.6 (b) implies that there exists $i_0 \in [r]$ such that $(J_{i_0}, I_{i_0}) \in \text{Cov } \mathcal{H}(B_{I_{i_0}})$ and $J_i = I_i$ for all $i \neq i_0$. Since $I_{i_0} \neq \emptyset$, we have that

$$B_{I_{i_0}} \cong \begin{cases} B_m & \text{if } 1 \in I_{i_0}, \\ \tilde{B}_m & \text{if } 1 \notin I_{i_0}, \end{cases}$$

for some $m \geq 2$. In the first instance we have that $i_0 = 1$, thus Corollary 4.3.4 (a) shows that the only possibility for J_1 , and therefore for J , is the one given in (C2). For the second case, if $m \geq 3$, then Corollary 4.3.4 (b) shows that the unique possibilities for J_{i_0} are given by (C5) if $1 \in J$, or by (C6) if $1 \notin I$. If $m = 2$, the unique cover relation

in $\mathcal{H}(B_{I_{i_0}})$ is (\emptyset, I_{i_0}) , i.e. $J_{i_0} = \emptyset$, thus $t = r + 1$, a contradiction. Thus, this case does not hold.

If $t = r + 1$, Lemma 4.3.6 (c) shows that there exists i_0 such that $I = J \sqcup I_{i_0}$ and $(\emptyset, I_{i_0}) \in \text{Cov } \mathcal{H}(B_{I_{i_0}})$. Then necessarily $\text{card } I_{i_0} \in \{1, 2\}$ by Corollary 4.3.4, thus

$$B_{I_{i_0}} \cong \begin{cases} B_1 & \text{if } \text{card } I_{i_0} = 1, \\ \Pi_2 & \text{if } \text{card } I_{i_0} = 2. \end{cases}$$

Hence, the unique possibilities for (J, I) are given by (C1), (C3) and (C4), depending on whether $1 \in J$, $1 \notin I$ or $1 \in I \setminus J$, respectively. This shows that the conditions (C1)-(C6) describe all the cover relations in $\mathcal{H}(B_n)$. \square

4.4 Homological Hasse quiver of B_n

Next we count the number of cover relations given in Proposition 4.3.7. For, we use the binary notation of subsets $J \subseteq [n]$ introduced in Section 3.10. Observe that with this notation, the conditions (C1)-(C6) in Proposition 4.3.7 are equivalent to statements using binary notation and subwords, for instance (C1) is equivalent to ask $\text{Bin}_n(I)_1 = 1$ and 010 or 0110 are subwords of $\text{Bin}_n(I)$; or $\text{Bin}_n(I) = (1, *, \dots, *, 0, 1)$ or $\text{Bin}_n(I) = (1, *, \dots, *, 0, 1, 1)$, where $*$ $\in \{0, 1\}$.

For $i \in \{1, \dots, 6\}$, we denote by $C(i)$ the set of pairs (J, I) given by the condition (Ci) in Proposition 4.3.7.

Lemma 4.4.1. *Let $n \geq 4$. Then the following equalities hold.*

- (a) $\text{card } C(1) = 2^{n-5}(3n - 4)$.
- (b) $\text{card } C(3) = 2^{n-5}(3n + 2)$.
- (c) $\text{card } C(5) = 2^{n-4}(n - 4)$.
- (d) $\text{card } C(6) = 2^{n-4}(n - 2)$.

Proof. (a) Let $(J, I) \in C(1)$. From the previous discussion, we have four general shapes for the binary notation of I :

- (i) $I = (1, *, \dots, *, 0, 1, 0, *, \dots, *)$,
- (ii) $I = (1, *, \dots, *, 0, 1)$,
- (iii) $I = (1, *, \dots, *, 0, 1, 1, 0, *, \dots, *)$,
- (iv) $I = (1, *, \dots, *, 0, 1, 1)$.

Note that the first two correspond to the property that for some i_0 $\text{card } I_{i_0} = 1$, and the last two when $\text{card } I_{i_0} = 2$. Hence, for (i) the first 0 of the subword 010 cannot be placed at entries $1, n - 1$ or n of I , and there are $n - 4$ free slots marked with $*$, thus there are $2^{n-4}(n - 3)$ choices for I as in (i). It is clear that for (ii) there are 2^{n-3} choices. Similar arguments show that there are $2^{n-5}(n - 4)$ choices for the case (iii) and 2^{n-4} for (iv).

Hence, $\text{card } C(1) = 2^{n-5}(3n-4)$, since once I is determined, J is uniquely defined in these cases.

(b) Let $(J, I) \in C(3)$. Then I is determined by the conditions: $\text{Bin}_n(I)_1 = 0$ and there exists i_0 such that $\text{card } I_{i_0} \in \{1, 2\}$. If

$$(i) \quad I = (0, *, \dots, *, 0, 1, 0, *, \dots, *),$$

$$(ii) \quad I = (0, 1, 0, *, \dots, *),$$

$$(iii) \quad I = (0, *, \dots, *, 0, 1).$$

Using the same counting methods as in (a), we have $2^{n-4}(n-3)$ choices for (i), and 2^{n-3} for (ii) and (iii). On the other hand, if $\text{card } I_{i_0} = 2$, we have also three shapes for $\text{Bin}_n(I)$:

$$(i) \quad I = (0, *, \dots, *, 0, 1, 1, 0, *, \dots, *),$$

$$(ii) \quad I = (0, 1, 1, 0, *, \dots, *),$$

$$(iii) \quad I = (0, *, \dots, *, 0, 1, 1).$$

For (i) there are $2^{n-5}(n-4)$ choices, and 2^{n-4} for (ii) and (iii). Since J is uniquely determined, once I is given, we get that $\text{card } C(3) = 2^{n-5}(3n+2)$.

(c) Let $(J, I) \in C(5)$. Then, it is clear that I is characterised by: $\text{Bin}_n(I) = 1$ and 0111 is a subword of $\text{Bin}_n(I)$. Thus $I = (1, *, \dots, *, 0, 1, 1, 1, *, \dots, *)$, so there are $2^{n-5}(n-4)$ choice for such an I . In this case, J is not uniquely determined by I , there are exactly two choices for it, considering (C5). Therefore, $\text{card } C(5) = 2^{n-4}(n-4)$.

(d) Let $(J, I) \in C(6)$. Then, I is characterised by: $\text{Bin}_n(I) = 0$ and 0111 is a subword of $\text{Bin}_n(I)$. Thus $I = (0, *, \dots, *, 0, 1, 1, 1, *, \dots, *)$, for which there are $2^{n-5}(n-4)$ choices, or $I = (0, 1, 1, 1, *, \dots, *)$, for which there are 2^{n-4} choices. Thus, there are $2^{n-5}(n-2)$ choices for I satisfying the conditions in (C6). As before, there are two choices for J , thus, $\text{card } C(6) = 2^{n-4}(n-2)$. \square

Corollary 4.4.2. *Let $n \geq 3$. Then, $\text{card Cov } \mathcal{H}(B_n) = 2^{n-4}(5n+1)$.*

Proof. We know that $\text{card } C(2) = 2^{n-2} = \text{card } C(4)$ (cf. Proposition 3.8.3), then the result follows from Lemma 4.4.1, when $n \geq 4$. Direct computations show that $\mathcal{H}(B_3) = \mathcal{H}(\text{Aus } T_3)$, thus $\text{Cov } \mathcal{H}(B_3) = 8$. \square

In what follows we will describe the Hasse quiver $\mathbb{H}(B_n)$ of B_n recursively, but first we need the next arithmetic formula of $\text{card Cov } \mathcal{H}(B_n)$. Recall that $B_n = \text{Aus } T_n$ for $n \in \{1, 2\}$, thus we define $b(1) := 1$, $b(2) := 3$ and $b(n) := 2^{(n-4)}(5n+1)$, for $n \geq 3$. Hence, $b(n) = \text{card Cov } \mathcal{H}(B_n)$, for $n \geq 1$. We remark that the sequence $\{b(n)\}_{n \in \mathbb{N}_+}$ is not listed in the on-line encyclopedia of integer sequences [OEIS \[OEI20\]](#) until the publication date of this thesis.

Lemma 4.4.3. *Let $n \geq 3$. Then $b(n) = \sum_{i=1}^{n-1} b(i) + 5 \cdot 2^{n-3} - 1$.*

Proof. Follows easily by induction. \square

Note that for $n \geq 3$,

$$b(n) = \sum_{i=1}^{n-1} b(i) + 2^{n-3} + 2^{n-2} + 2(2^{n-3} - 1) + 1. \quad (4.4.1)$$

This motivates a recursive construction of $\mathbb{H}(B_n)$. For, we define the following sets.

$$\begin{aligned} H(i) &:= \{(J \sqcup [i+2, n], I \sqcup [i+2, n]) \mid (J, I) \in \text{Cov } \mathcal{H}(B_i)\}, \text{ for } i \in [n-2], \\ U(i) &:= \{(J, J \sqcup [i+2, n]) \mid J \subseteq [i]\}, \text{ for } i \in \{n-3, n-2\}, \\ T(n) &:= \{(J \sqcup [n-2, n-1], J \sqcup [n-2, n]) \mid J \subsetneq [n-3]\}, \\ T'(n) &:= \{(J \sqcup [n-1, n], J \sqcup [n-2, n]) \mid J \subsetneq [n-3]\}. \end{aligned}$$

Proposition 4.4.4. *Let $n \geq 3$. Then*

$$\text{Cov } \mathcal{H}(B_n) = \bigsqcup_{i=1}^{n-1} H(i) \sqcup U(n-3) \sqcup U(n-2) \sqcup T(n) \sqcup T'(n) \sqcup \{([n-1], [n])\},$$

where $H(n-1) := \text{Cov } \mathcal{H}(B_{n-1})$.

Proof. First, observe that the sets involved in the equality are pairwise disjoint. Moreover, Lemma 4.3.6 implies that $H(i) \subseteq \text{Cov } \mathcal{H}(B_n)$ for all $i \in [n-1]$, in particular Lemma 4.3.6 (c) shows that $U(i) \subseteq \text{Cov } \mathcal{H}(B_n)$ for $i \in \{n-3, n-2\}$, since $(\emptyset, [2]) \in \text{Cov } \mathcal{H}(\Pi_2)$ and $(\emptyset, [1]) \in \text{Cov } \mathcal{H}(B_1)$. On the other hand, $T(n) \sqcup T'(n)$ is the subset of relations (J, I) in $C(5) \sqcup C(6)$ such that $\text{Bin}_n(I) = (*, \dots, *, 0, 1, 1, 1)$, thus $T(n) \sqcup T'(n) \subseteq \text{Cov } \mathcal{H}(B_n)$. It is straightforward to see that $\text{card } H(i) = b(i)$, for $i \in [n-1]$, $\text{card } U(i) = 2^i$, for $i \in \{n-3, n-2\}$, and $\text{card } T(n) = \text{card } T'(n) = 2^{n-3} - 1$. Finally, note that $([n-1], [n]) \in \text{Cov } \mathcal{H}(B_n)$ by Corollary 4.3.4 (a), hence the result follows from (4.4.1). \square

The last result says that, for $i \in [n-1]$, then $H(i)$ can be considered as a copy of $\text{Cov } \mathcal{H}(B_i)$ inside $\text{Cov } \mathcal{H}(B_n)$. As an example, in Fig. 4.2 we depict the Hasse quivers of $\mathcal{H}(B_n)$, for $n \in \{4, 5\}$. The cases $n = 1, 2$ or 3 are shown in Fig. 3.1 on page 66.

On the other hand, in Section 3.11 we characterised the quasi-hereditary algebras of the form $\text{Aus } T_n/(e_{J^c})$ for subsets $J \subseteq [n]$. In the case of the quotients $B_n/(e_{J^c})$ we have the same conclusion.

Proposition 4.4.5. *Let $J \subseteq [n]$, and $<$ the usual order on J . Then $(B_n/(e_{J^c}), (J, <))$ is a quasi-hereditary algebra if and only if 011 is not a subword of $\text{Bin}_n(J)$.*

Proof. Follows similarly as in the proof of Proposition 3.11.1 considering Corollary 4.1.9 and that \tilde{B}_m has infinite global dimension for $m \geq 2$. \square

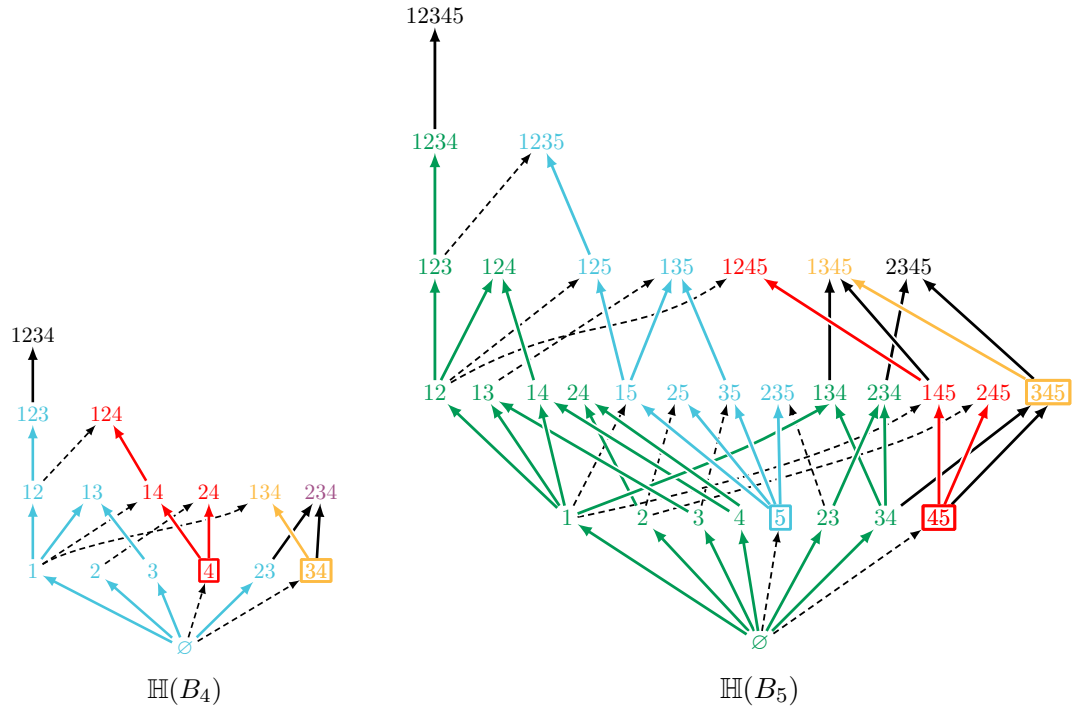


Figure 4.2: Homological Hasse quivers of B_4 and B_5 . For simplicity, we write the subsets $J \subseteq [n]$ as words over the ordered alphabet $\{1, \dots, n\}$, in the natural way, for $n = 4, 5$. The copies of $\mathbb{H}(B_1)$ are indicated in **yellow**, $\mathbb{H}(B_2)$ in **red**, $\mathbb{H}(B_3)$ in **blue** and $\mathbb{H}(B_4)$ in **green**, respectively; in each case, the empty set corresponds to the elements marked with a rectangle. For $n \in \{4, 5\}$, **black** solid arrows terminate at vertices I , with $\{n - 2, n - 1, n\} \subseteq I$, and they correspond to elements of $T(n) \sqcup T'(n)$ and the embedding $\iota_{[n-1], [n]}$. Dashed arrows indicate elements of the sets $U(i)$.

Chapter 5

Quasi-hereditary structures of path algebras of type \mathbb{A}_n

Let \mathbb{K} be a field. The Dynkin diagram of type \mathbb{A}_n is the graph $\overset{1}{\bullet} - \overset{2}{\bullet} - \dots - \overset{n-1}{\bullet} - \overset{n}{\bullet}$. In this chapter we classify all the quasi-hereditary structures of path algebras of type \mathbb{A}_n , i.e. path algebras $\mathbb{K}Q$, where Q has underlying graph \mathbb{A}_n . First for an equioriented quiver, and then for any orientation. This is joint work with Yuta Kimura and Baptiste Rognerud [FKR20].

The chapter is organised as follows. In Section 5.1 we define binary search trees which we use to construct minimal adapted posets to $\Lambda_n := \mathbb{K}A_n$ in the sense of Definition 2.2.15, where A_n is an equioriented quiver of type \mathbb{A}_n . Then we show a bijection between binary trees and quasi-hereditary structures of Λ_n that is compatible with the classification of Λ_n -tilting modules presented in [Hil06], determining that $\text{qh.str}(\Lambda_n)$ and the Tamari lattice of size n are isomorphic posets. Section 5.2 is devoted to studying quasi-hereditary structures of quiver algebras under deconcatenations of the original quiver. As application we find all the quasi-hereditary structures of path algebras of type \mathbb{A}_n in the general case.

5.1 Path algebras of type \mathbb{A}_n : equioriented case

Let $A_n = \overset{1}{\bullet} \leftarrow \overset{2}{\bullet} \leftarrow \dots \leftarrow \overset{n}{\bullet}$ be an equioriented quiver of type \mathbb{A}_n . In the setting of quiver representations, it was noticed by Gabriel in [Gab81] that tilting modules over $\Lambda_n := \mathbb{K}A_n$ are counted by the n -th Catalan number

$$c_n = \frac{1}{n+1} \binom{2n}{n}.$$

Moreover, Buan and Krause studied the poset structure of tilting modules in $\text{mod } \Lambda_n$, determining that the latter is isomorphic to the Tamari lattice of size n as posets [BK04]. The Tamari lattice of size n , denoted by $\mathcal{T}(n)$, is defined on the set of bracketings of a string of $n+1$ -letters, and relations given by the rule $(xy)z \rightarrow x(yz)$ [Sta99], it was introduced by Dov Tamari [Tam62]. There are several ways to describe the Tamari lattice, all of them involve bijections between families of objects counted by Catalan numbers [Sta99, Corollary 6.2.3]; among them we find the set of binary trees.

In this section we show that the number of different quasi-hereditary structures of the path algebra Λ_n also coincides with c_n , and prove that this bijection is compatible with the poset structure on the set of quasi-hereditary structures of Λ_n . The classification of quasi-hereditary structures of Λ_n is given by binary trees with a specific labelling. We also describe all the characteristic tilting modules over Λ_n using minimal adapted posets. We start defining binary trees.

Definition 5.1.1. A *binary tree* T is either the empty set or a tuple (r, L, R) where r is a singleton set, called the *root of T* , and L and R are two binary trees, called *left* and *right subtree*, respectively. The root of L (R , resp.) is called the *left* (*right*, resp.) *child of r* . The empty set has no vertex but has one leaf. The set of *leaves* of $T = (r, L, R)$ is the disjoint union of the set of leaves of L and R . The *size* of the tree is the number of leaves minus 1. We depict binary trees as in Fig. 5.1.

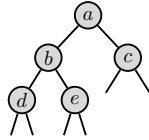


Figure 5.1: Graphic representation of the binary tree $T = (a, (b, (d, \emptyset, \emptyset), (e, \emptyset, \emptyset)), (c, \emptyset, \emptyset))$. The vertices correspond to singletons, which are connected by an edge to their left and right subtrees. The leaves are marked by edges connected to only one vertex. In this case T has 6 leaves, thus it has size 5. The top-most vertex corresponds to the root of T .

From now on we identify binary trees with their graphic representation. Therefore, the size of a tree T is the number of vertices appearing in T . It is well known that the number of binary trees is given by the Catalan numbers.

Definition 5.1.2. A *binary search tree* is a binary tree labelled by integers such that if a vertex x is labelled by k , then the vertices of the left subtree (resp. right subtree) of x are labelled by integers less than or equal (resp. superior) to k .

If T is a binary tree with n vertices, there is a unique labelling of the vertices by each of the integers $1, 2, \dots, n$ that makes it a binary search tree. This procedure is sometimes called the *in-order traversal* of the tree or simply as the *in-order algorithm* (recursively visit left subtree, root and right subtree). The first vertex visited by the algorithm is labelled by 1, the second by 2 and so on, see Fig. 5.2 for an example.

From now on, every binary search tree will be labelled by the in-order traversal method.

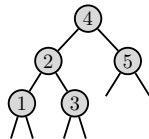
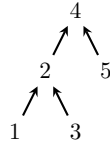


Figure 5.2: Binary tree of Fig. 5.1 viewed as a binary search tree using the in-order algorithm. The vertices of T are visited in the following order: d, b, e, a, c , thus d is labelled by 1, b by 2, and so on.

Let T be a binary tree of size n viewed as a binary search tree. Then T induces a poset \triangleleft_T on $\{1, 2, \dots, n\}$ by setting $i \triangleleft_T j$ if i labels a vertex in some subtree of the vertex labelled by j . In other words, erasing the leaves of T , and converting each edge into an arrow pointing from a lower level to a higher level, we get the corresponding Hasse quiver of \triangleleft_T .

Example 5.1.3. Consider the binary search tree T of Fig. 5.2. Then \triangleleft_T is the transitive closure of the relations: $1 \triangleleft_T 2$, $3 \triangleleft_T 2$, $2 \triangleleft_T 4$ and $5 \triangleleft_T 4$, i.e. the Hasse quiver of \triangleleft_T is the following.



The next proposition is a key result for proving a bijection between binary search trees and adapted posets to Λ_n . From now on, we denote the usual order on $\{1, \dots, n\}$ by $<$.

Proposition 5.1.4. *Let \triangleleft be a partial order on $\{1, 2, \dots, n\}$. Then there is a binary tree T such that $\triangleleft = \triangleleft_T$ if and only if*

- (a) *For every $i < j$ incomparable for \triangleleft , there exists k such that $i < k < j$ and $i \triangleleft k$ and $j \triangleleft k$.*
- (b) *For every $i < j < k$, if $i \triangleleft k$ then $j \triangleleft k$ and if $k \triangleleft i$ then $j \triangleleft i$.*

Proof. See [CPP19, Proposition 2.21]. □

Remark 5.1.5. Condition (a) is equivalent to the following weaker condition: for every $i < j$ incomparable there exists k such that $i < k < j$ and $i \triangleleft k$ or $j \triangleleft k$.

Lemma 5.1.6. *Let $n \geq 1$. Then the following conditions hold.*

- (a) *Let T be a binary tree of size n . Then \triangleleft_T is an adapted poset for Λ_n .*
- (b) *If \triangleleft is an adapted poset for Λ_n , then there is a binary tree T such that $\triangleleft \sim \triangleleft_T$.*

Proof. The indecomposable Λ_n -modules can be identified with intervals in $\{1, 2, \dots, n\}$. Then, (a) follows from Proposition 5.1.4.

(b) Let \triangleleft be an adapted poset for Λ_n . Let R be the set consisting of all the relations of \triangleleft that satisfy the condition (b) of Proposition 5.1.4. This set is non-empty since it contains all the trivial relations $i \triangleleft i$ and all the relations of length 2. Because the condition (b) is stable under transitivity, it is easy to see that R is a partial order on $\{1, 2, \dots, n\}$.

Let us look more carefully at the failure of condition (b). Let $i < j$ such that $i \triangleleft j$ and there is $i < k < j$ with $k \not\triangleleft j$. We choose k maximal for this property. There are two possibilities: either $j \triangleleft k$ or j and k are incomparable. Assume the second possibility. Since the poset is adapted, there is $k < x < j$ with $k \triangleleft x$ and $j \triangleleft x$. By the maximality of k , there is a relation $x \triangleleft j$ which contradicts the anti-symmetry of \triangleleft . Thus, only the first possibility occurs, and by transitivity we have $i \triangleleft k$. Note that the relation $j \triangleleft k$

lies in R because if $k < x < j$, the maximality of k implies that $x \triangleleft j$, and by transitivity we have $x \triangleleft k$. The relation $i \triangleleft k$ not necessarily belongs to R , but we can use the same argument with i and k and by induction we prove the following. If a relation $i \triangleleft j$ with $i < j$ is not in R , there is $i < k < j$ such that $i \triangleleft j$, $j \triangleleft k \in R$ and $i \triangleleft k \in R$. By symmetry we have the same result for the decreasing relations.

Now we prove that R satisfies the condition (a) of Proposition 5.1.4. Let $i < j$ such that i and j are incomparable in R . There are two possibilities: either i and j are comparable for \triangleleft or not. In the first case, the discussion above implies the existence of k such that $i < k < j$ and $i \triangleleft k \in R$ and $j \triangleleft k \in R$. Let us assume that i and j are incomparable for \triangleleft , then there is $i < k < j$ such that $i \triangleleft k$ and $j \triangleleft k$. If the last two relations are in R , we are done. Otherwise, we use the discussion above and we see that there is $i < t < j$ such that $i \triangleleft t \in R$ and $j \triangleleft t \in R$.

In conclusion, the poset R satisfies the two conditions of Proposition 5.1.4, so there is a binary tree T such that $R = \triangleleft_T$. Moreover the poset \triangleleft is an extension of R , so by Lemma 2.2.16, we have that $R \sim \triangleleft$. \square

Now we are prepared to show that the quasi-hereditary structures of Λ_n are counted by the Catalan numbers.

Proposition 5.1.7. *Let $n \geq 1$. The map sending a binary tree T to the equivalence class of the adapted poset \triangleleft_T is a bijection between the set of binary trees of size n and the set of quasi-hereditary structures of Λ_n . Therefore, $\text{card}(\text{qh. str } \Lambda_n) = c_n$.*

Proof. We already know that this map is surjective, since by Proposition 2.2.23 every adapted poset induces a quasi-hereditary algebra. We need to see that it is injective. For that we explain how we can recover the tree for the set of standard and costandard modules.

Let T be a binary tree. Then \triangleleft_T is an adapted poset for Λ_n and $(\Lambda_n, \triangleleft_T)$ is a quasi-hereditary algebra. The standard module $\Delta(i)$ is the largest quotient of $P(i)$ having composition factors $S(j)$ such that $j \triangleleft_T i$. By construction, this implies that j labels a vertex in the left subtree of the vertex labelled by i . Conversely, the label of the left subtree of i is of the form $[j, i]$, so we see that $\Delta(i)$ is the indecomposable module with composition factor the interval consisting of i the labels of its left subtree. Similarly, $\nabla(i)$ is the indecomposable module with composition factors indexed by the interval consisting of i and the labels of its right subtree. It follows that two different trees induce two non-equivalent posets. \square

Lemma 5.1.8. *Let T be a binary tree of size n . Then \triangleleft_T is a minimal adapted order to Λ_n .*

Proof. Let \triangleleft' be an adapted poset to Λ_n such that $\triangleleft_T \sim \triangleleft'$. Then there exists a binary tree T' such that $\triangleleft' \sim \triangleleft_{T'}$, by Lemma 5.1.6. Thus $T = T'$ by Proposition 5.1.7, and the proof of Lemma 5.1.6 (b) shows that \triangleleft_T is extended by \triangleleft' , which shows the claim. \square

Lemma 5.1.8 proves that the Hasse quiver in Example 5.1.3 corresponds to a minimal adapted order to Λ_5 . In general, all minimal adapted orders to Λ_n are obtained in this way (cf. Theorem 5.1.9).

Let $(\Lambda_n, \triangleleft)$ be a quasi-hereditary algebra, and T the associated characteristic tilting module, which is characterised by $\text{add}(T) = \mathcal{F}(\Delta_{\triangleleft}) \cap \mathcal{F}(\nabla_{\triangleleft})$ (cf. Proposition 2.2.30).

Observe that the module T is a classical tilting module in this case, since Λ_n is hereditary. Moreover, the tilting module T only depends on the equivalence class of the partial order \triangleleft by Lemma 2.4.5. So we have a well defined map char from the set quasi-hereditary structures of Λ_n to the set of isomorphism classes of tilting modules for Λ_n which sends the equivalence class of \triangleleft to the characteristic tilting module of $(\Lambda_n, \triangleleft)$. Moreover, by Proposition 5.1.7 and [Gab81], the correspondence char is one-to-one.

On the other hand, Hille exhibited a bijective correspondence φ between binary trees of size n and tilting Λ_n -modules up to isomorphism [Hil06, Sec. 9]. This bijection is compatible with our findings in the following sense.

Theorem 5.1.9. *We have a commutative diagram of bijections*

$$\begin{array}{ccc}
 & \left\{ \begin{array}{c} \text{Binary trees} \\ \text{of size } n \end{array} \right\} & \\
 \psi \swarrow & & \searrow \varphi \\
 \text{qh. str}(\Lambda_n) & \xrightarrow{\text{char}} & \left\{ \begin{array}{c} \text{Tilting modules} \\ \text{over } \Lambda_n \end{array} \right\} / \cong
 \end{array}$$

where ψ is given by $T \mapsto [\triangleleft_T]$.

Proof. In the proof of Proposition 5.1.7 we determined the set of standard modules and costandard modules from a binary tree T' . We claim that the indecomposable direct summand $T(i)$ of the characteristic tilting module T is the indecomposable module with composition factors indexed by the interval consisting of i and the labels of its subtrees (left and right). Since the map φ sends T' to the module constructed in this way, the proof follows from this claim.

By induction on the size of the subtrees we show that the module $T(i)$ belongs to $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. This is clear for the subtrees of size one since in this case $T(i) = S(i) = \Delta(i) = \nabla(i)$.

In the general case, if i_l (resp. i_r) denotes the left (resp. right) child of i we have two exact sequences

$$0 \rightarrow \Delta(i) \rightarrow T(i) \rightarrow T(i_r) \rightarrow 0$$

and

$$0 \rightarrow T(i_l) \rightarrow T(i) \rightarrow \nabla(i) \rightarrow 0.$$

If i has not left (resp.) right child then we let $T(i_l) = 0$ (resp. $T(i_r) = 0$) and we still have the two exact sequences. By induction $T(i_r) \in \mathcal{F}(\Delta)$ and $T(i_l) \in \mathcal{F}(\nabla)$, so $T(i) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. The results follows. \square

Remark 5.1.10. Theorem 5.1.9 does not hold for other orientations of \mathbb{A}_n , for instance in Section 5.2 we show that if $Q = 3 \rightarrow 2 \leftarrow 1$, then $\text{card}(\text{qh. str}(\mathbb{K}Q)) = 4$, but $\mathbb{K}Q$ has 5 tilting modules, i.e. not all tilting modules are characteristic tilting modules.

The next result is intrinsic in the proof of Theorem 5.1.9.

Corollary 5.1.11. *Let \triangleleft be a minimal adapted poset to Λ_n and $\text{char}([\triangleleft]) = \bigoplus_{i \in I} T(i)$ the associated characteristic tilting module. Then for any weights $i, j \in I$, we have that $[T(i) : S(j)] \neq 0$ if and only if $j \triangleleft i$.*

Proof. Let T be a binary tree such that $\triangleleft_T = \triangleleft$. Thus $\bigoplus_{i \in I} T(i) = \text{char}([\triangleleft]) = \text{char}([\triangleleft_T]) \cong \varphi(T)$, by Theorem 5.1.9. Then, from the definition of φ , we have that $T(i)$ has composition factors $S(j)$, with $j \in I$, if and only if $j \triangleleft_T i$. \square

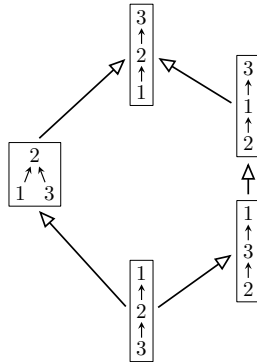
Example 5.1.12. Let T be the binary tree of Fig. 5.2, and $\text{char}([\triangleleft_T]) = \bigoplus_{i=1}^5 T(i)$. Then

$$T(1) = 1, \quad T(2) = \begin{matrix} 3 \\ 2 \\ 1 \end{matrix}, \quad T(3) = 3, \quad T(4) = \begin{matrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{matrix}, \quad T(5) = 5.$$

Corollary 5.1.13. Let $n \geq 1$. Then $\text{qh.str}(\Lambda_n)$ and the Tamari lattice $\mathcal{T}(n)$ are isomorphic as partially ordered sets.

Proof. From Lemma 2.4.9 and Theorem 5.1.9 we have that $\text{qh.str}(\Lambda_n)$ and the set of tilting modules over Λ_n are isomorphic as posets. Then the assertion is consequence of [BK04, Theorem 5.2]. \square

Example 5.1.14. The Hasse quiver of $\text{qh.str}(\Lambda_3)$ is depicted below. The vertices correspond to the minimal adapted orders to Λ_3 displayed as Hasse diagrams. Note that for the bottom-most order the standard modules are $S(1), S(2), S(3)$. On the other hand, for the top-most order we have that $\Delta = \{P(1), P(2), P(3)\}$.



5.2 Quasi-hereditary structures and deconcatenations

In this section we define certain kind of quiver decomposition, namely deconcatenation of a quiver, and study the relation between quasi-hereditary structures of a quiver algebra and quasi-hereditary structures of the corresponding quiver algebras given by a deconcatenation of the original quiver. As application of this results, we generalise the classification of quasi-hereditary structures of Section 5.1 to path algebras of type \mathbb{A}_n for any orientation.

For the rest of the section, let Q be a finite connected quiver. Recall that a vertex $v \in Q_0$ is a *sink* if all arrows connected to v point towards v , and v is a *source* if all arrows connected to v point away from v .

Definition 5.2.1. Let $v \in Q_0$ be a sink or a source. A *deconcatenation* of Q at v is a disjoint union $Q^1 \sqcup Q^2 \sqcup \cdots \sqcup Q^\ell$ of full subquivers Q^i of Q such that each Q^i is a connected full subquiver of Q having a vertex v , $Q_0 = (Q_0^1 \setminus \{v\}) \sqcup \cdots \sqcup (Q_0^\ell \setminus \{v\}) \sqcup \{v\}$, $Q_0^i \cap Q_0^j = \{v\}$ and there are no arrows between the elements of $Q_0^i \setminus \{v\}$ and $Q_0^j \setminus \{v\}$, for $1 \leq i \neq j \leq \ell$.

Note that a deconcatenation of Q at a sink or source is not a unique. Furthermore, it is easy to see that if $Q^1 \sqcup Q^2$ is a deconcatenation of Q at a vertex v , and $Q^3 \sqcup Q^4$ is a deconcatenation of Q^2 at the vertex w , then $Q^1 \sqcup Q^3 \sqcup Q^4$ is a deconcatenation of Q . For instance, consider $Q = 1 \rightarrow 2 \leftarrow 3 \leftarrow 4 \rightarrow 5$. Then we have two deconcatenations of Q :

$$(1 \rightarrow 2) \sqcup (2 \leftarrow 3 \leftarrow 4 \rightarrow 5), \quad (1 \rightarrow 2 \leftarrow 3 \leftarrow 4) \sqcup (4 \rightarrow 5).$$

Moreover, the first has a deconcatenation at 4, the second has a deconcatenation at 2, and the resulting deconcatenation is the same:

$$(1 \rightarrow 2) \sqcup (2 \leftarrow 3 \leftarrow 4) \sqcup (4 \rightarrow 5).$$

Therefore, we consider deconcatenations which are the disjoint union of two full subquivers.

Let $Q^1 \sqcup Q^2$ be a deconcatenation of Q at a sink or a source v . Let I be an admissible ideal of $\mathbb{K}Q$ and $A := \mathbb{K}Q/I$. Set $\bar{1} = 2$ and $\bar{2} = 1$. For each $\ell = 1, 2$, let

$$A^\ell := \frac{A}{\langle e_u \mid u \in Q_0^\ell \setminus \{v\} \rangle}. \quad (5.2.1)$$

Thus we have a surjective morphism of algebras $A \rightarrow A^\ell$, and there exists an embedding $\text{mod } A^\ell \rightarrow \text{mod } A$ given by restriction of scalars. Using this functor, we regard $\text{mod } A^\ell$ as a full subcategory of $\text{mod } A$. Therefore, an A -module M is an A^ℓ -module if and only if $e_u M = 0$ for any $u \in Q_0^\ell \setminus \{v\}$. For a vertex $i \in Q_0^\ell$, let $P^\ell(i)$, $I^\ell(i)$, $S^\ell(i)$ be the indecomposable projective, indecomposable injective and the simple A^ℓ -module associated to the vertex i , respectively. The next lemma is proven easily.

Lemma 5.2.2. *Let $Q^1 \sqcup Q^2$ be a deconcatenation of Q at a sink or a source v . Fix $\ell = 1, 2$.*

- (a) *For any vertex $i \in Q_0^\ell$, we have $S(i) \cong S^\ell(i)$ in $\text{mod } A$.*
- (b) *If v is a sink, then for any vertex $i \in Q_0^\ell$, we have $P(i) \cong P^\ell(i)$ in $\text{mod } A$.*
- (c) *If v is a source, then for any vertex $i \in Q_0^\ell$, we have $I(i) \cong I^\ell(i)$ in $\text{mod } A$.*
- (d) *Let M be a non-zero A -module. If both the top and the socle of M are simple, then one of $M \in \text{mod } A^1$ or $M \in \text{mod } A^2$ holds.*
- (e) *Let $M \in \text{mod } A^\ell$ and $i \in Q_0$. If $[M : S(i)] \neq 0$, then $i \in Q_0^\ell$ holds.*

Let \triangleleft be a partial order on Q_0 . By restricting this order, we have a partial order $\triangleleft|_{Q_0^\ell}$ on Q_0^ℓ , for $\ell = 1, 2$. We first compare standard and costandard modules associated to these orders.

Lemma 5.2.3. *Let $Q^1 \sqcup Q^2$ be a deconcatenation of Q at a sink or a source v . Let \triangleleft be a partial order on Q_0 and Δ (∇ , respectively) the standard (costandard, respectively) A -modules associated to \triangleleft . We denote by Δ_ℓ (∇_ℓ , respectively) the standard (costandard, respectively) A^ℓ -modules associated to $\triangleleft|_{Q_0^\ell}$. Then we have the following statements.*

- (a) *If v is a sink, then we have $\Delta_\ell(i) \cong \Delta(i)$ for any $\ell = 1, 2$ and any $i \in Q_0^\ell$.*
- (b) *If v is a source, then we have $\nabla_\ell(i) \cong \nabla(i)$ for any $\ell = 1, 2$ and any $i \in Q_0^\ell$.*
- (c) *If \triangleleft defines a quasi-hereditary structure on A , then $\triangleleft|_{Q_0^\ell}$ defines a quasi-hereditary structure on A^ℓ for each $\ell = 1, 2$.*

Proof. (a) Let $i \in Q_0^\ell$. By Lemma 5.2.2, we have $P(i) \cong P^\ell(i)$. If $j \in Q_0$ satisfies $[P(i) : S(j)] \neq 0$, then $j \in Q_0^\ell$. Thus, for a composition factor $S(j)$ of $P(i) \cong P^\ell(i)$, we get that $j \triangleleft i$ if and only if $j \triangleleft|_{Q_0^\ell} i$. This means that $\Delta(i) \cong \Delta_\ell(i)$. By a similar argument, (b) follows.

(c) Assume that v is a sink. Let $i \in Q_0^\ell$. By (1), $\Delta(i) \cong \Delta_\ell(i)$. Thus we have $[\Delta_\ell(i) : S^\ell(i)] = 1$ by Lemma 5.2.2. Since any composition factor of $P(i)$ is a simple A^ℓ -module, a filtration of $P(i)$ by Δ in $\text{mod } A$ gives a filtration of $P^\ell(i)$ by Δ_ℓ in $\text{mod } A^\ell$. Clearly, this filtration satisfies the axiom (c) of Definition 2.2.13. Thus $\triangleleft|_{Q_0^\ell}$ defines a quasi-hereditary structure on A^ℓ . The assertion also holds in the case when v is a source by a similar argument. \square

Next we construct a partial order on Q_0 from partial orders on Q_0^ℓ . Recall that we write $\bar{1} = 2$ and $\bar{2} = 1$.

Definition 5.2.4. Let \triangleleft_ℓ be partial orders on Q_0^ℓ for $\ell = 1, 2$. We define a partial order $\triangleleft = \triangleleft(\triangleleft_1, \triangleleft_2)$ on Q_0 as follows: for $i, j \in Q_0$, set $i \triangleleft j$ if one of the following two statements holds:

- (a) $i, j \in Q_0^\ell$ and $i \triangleleft_\ell j$ holds for some ℓ , or
- (b) $i \in Q_0^\ell, j \in Q_0^{\bar{\ell}}, i \triangleleft_\ell v$ and $v \triangleleft_{\bar{\ell}} j$ hold.

Lemma 5.2.5. *Let $Q^1 \sqcup Q^2$ be a deconcatenation of Q at a sink or source v . Let \triangleleft_ℓ be a partial order on Q_0^ℓ and Δ_ℓ (∇_ℓ , respectively) the standard (costandard) A^ℓ -modules associated to \triangleleft_ℓ for $\ell = 1, 2$. We denote by Δ (∇ , respectively) the standard (costandard, respectively) A -modules associated to $\triangleleft = \triangleleft(\triangleleft_1, \triangleleft_2)$. Then we have the following statements.*

- (a) *If v is a sink, then $\Delta(i) \cong \Delta_\ell(i)$ for any $\ell = 1, 2$ and any $i \in Q_0^\ell$.*
- (b) *If v is a source, then $\nabla(i) \cong \nabla_\ell(i)$ for any $\ell = 1, 2$ and any $i \in Q_0^\ell$.*
- (c) *If \triangleleft_ℓ defines a quasi-hereditary structure on A^ℓ for $\ell = 1, 2$, then $\triangleleft = \triangleleft(\triangleleft_1, \triangleleft_2)$ defines a quasi-hereditary structure on A .*

Proof. (a) Let $i \in Q_0^\ell$. By Lemma 5.2.2, $P(i) \cong P^\ell(i)$ holds. By the definition of \triangleleft and a similar argument as the proof of Lemma 5.2.3 (a), we have that, for a composition factor $S(j)$ of $P(i) \cong P^\ell(i)$, $j \triangleleft i$ if and only if $j \triangleleft_\ell i$. This implies the assertion. By a similar argument, (b) holds.

(c) Assume that v is a sink. Let $i \in Q_0^\ell$. From (a) we have $\Delta(i) \cong \Delta_\ell(i)$. In particular $[\Delta(i) : S(i)] = 1$. By Lemma 5.2.2, $P(i) \cong P^\ell(i)$, for all $i \in Q_0^\ell$. Since \triangleleft_ℓ defines a quasi-hereditary structure on A^ℓ , $P(i)$ is filtered by $\Delta(i)$ for $i \in Q_0^\ell$. Since the number $(P(i) : \Delta(j))$ does not depend on the choice of the filtration, the axiom (c) of Definition 2.2.13 is satisfied. If v is a source, then the dual argument show the claim. \square

Let $Q^1 \sqcup Q^2$ be a deconcatenation of Q at a sink or a source v . Let I be an admissible ideal of $\mathbb{K}Q$ and $A := \mathbb{K}Q/I$. By Lemmas 5.2.3 and 5.2.5, we have the following map:

$$\Phi: \text{qh.str}(A) \longrightarrow \text{qh.str}(A^1) \times \text{qh.str}(A^2), \quad [\triangleleft] \mapsto ([\triangleleft|_{Q_0^1}], [\triangleleft|_{Q_0^2}]).$$

We also have an inverse map

$$\Psi: \text{qh.str}(A^1) \times \text{qh.str}(A^2) \longrightarrow \text{qh.str}(A), \quad ([\triangleleft_1], [\triangleleft_2]) \mapsto [\triangleleft(\triangleleft_1, \triangleleft_2)].$$

Let (A, \leq_A) and (B, \leq_B) be two posets. For $(a_1, b_1), (a_2, b_2) \in A \times B$, we set $(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq_A a_2$ and $b_1 \leq_B b_2$. Then $(A \times B, \leq)$ is a poset, called the *product poset*.

Proposition 5.2.6. *The map Φ is an isomorphism of posets, with inverse Ψ .*

Proof. Consider the product poset on $\text{qh.str}(A^1) \times \text{qh.str}(A^2)$. Then the assertion follows directly from Lemmas 2.4.5, 5.2.3 and 5.2.5. \square

Let $Q^1 \sqcup Q^2 \sqcup \dots \sqcup Q^\ell$ be a deconcatenation of Q at a vertex v . If $Q^{\ell+1} \sqcup \dots \sqcup Q^m$ is a deconcatenation of Q^ℓ at a vertex u , then we have a disjoint union $Q^1 \sqcup Q^2 \sqcup \dots \sqcup Q^{\ell-1} \sqcup Q^{\ell+1} \sqcup \dots \sqcup Q^m$ of full subquivers of Q , and so on for each connected quiver Q^i . We call a disjoint union $Q^1 \sqcup Q^2 \sqcup \dots \sqcup Q^\ell$ of full subquivers of Q obtained by iterated operations as above an *iterated deconcatenation* of Q .

Then we have the following result, which is a generalisation of Proposition 5.2.6.

Theorem 5.2.7. *Let $Q^1 \sqcup Q^2 \sqcup \dots \sqcup Q^\ell$ be an iterated deconcatenation of Q . Let A be a factor algebra of $\mathbb{K}Q$ modulo some admissible ideal and $A^i := A/\langle e_u \mid u \in Q_0^i \setminus \{v\}, t = 1, \dots, \ell, t \neq i \rangle$. Then we have an isomorphism of posets*

$$\text{qh.str}(A) \longrightarrow \prod_{i=1}^{\ell} \text{qh.str}(A^i),$$

which is given by $[\triangleleft] \mapsto ([\triangleleft|_{Q_0^i}]_{i=1}^{\ell})$.

Proof. By applying Proposition 5.2.6 iteratively, we have the assertion. \square

Example 5.2.8. Let $Q = 1 \rightarrow 2 \leftarrow 3 \leftarrow 4 \rightarrow 5$. We have an isomorphism of posets

$$\text{qh.str}(\mathbb{K}Q) \longrightarrow \text{qh.str}(\Lambda_2) \times \text{qh.str}(\Lambda_3) \times \text{qh.str}(\Lambda_2),$$

where Λ_n is the path algebra of an equioriented quiver of type \mathbb{A}_n , for $n = 2, 3$. Thus $\text{card}(\text{qh.str } \mathbb{K}Q) = 2 \times 5 \times 2 = 20$.

The following result determines how minimal adapted orders behave under deconcatenations.

Proposition 5.2.9. *Let $Q^1 \sqcup Q^2$ be a deconcatenation of Q at a sink or a source v . Let I be an admissible ideal of $\mathbb{K}Q$ and $A = \mathbb{K}Q/I$. Let \triangleleft ($\triangleleft_1, \triangleleft_2$ respectively) be a partial order on Q_0 (Q_0^1, Q_0^2 , respectively) defining a quasi-hereditary structure on A (A^1, A^2 , respectively). Then the following statements hold.*

- (a) *If \triangleleft is a minimal adapted order, then both $\triangleleft|_{Q_0^1}$ and $\triangleleft|_{Q_0^2}$ are minimal adapted orders.*
- (b) *If \triangleleft_1 and \triangleleft_2 are minimal adapted orders, then $\triangleleft(\triangleleft_1, \triangleleft_2)$ is a minimal adapted order.*

In particular, if \triangleleft is minimal, then $\triangleleft = \triangleleft(\triangleleft|_{Q_0^1}, \triangleleft|_{Q_0^2})$ holds.

Proof. We show only (a) and the last assertion. The assertion (b) is shown similarly. We show that $\triangleleft_1 := \triangleleft|_{Q_0^1}$ is a minimal adapted order. Let \triangleleft'_1 be a partial order on Q_0^1 such that $\triangleleft'_1 \sim \triangleleft_1$. Let $i, j \in Q_0^1$ and assume that $i \triangleleft_1 j$. This implies that $i \triangleleft j$. By Proposition 5.2.6, we have $\triangleleft' := \triangleleft(\triangleleft'_1, \triangleleft|_{Q_0^2}) \sim \triangleleft$. Since \triangleleft is minimal, then $i \triangleleft' j$. By the definition of \triangleleft' , we have that $i \triangleleft'_1 j$. The last assertion follows from the uniqueness of a minimal adapted order, stated in Lemma 2.4.4 (b). \square

Theorem 5.2.10. *Let $\mathbb{K}Q$ be a path algebra of type \mathbb{A}_n , and $Q^1 \sqcup Q^2 \sqcup \dots \sqcup Q^\ell$ an iterated deconcatenation of Q such that each Q^i is an equioriented quiver of type \mathbb{A}_{n_i} , for some $n_i \in \mathbb{N}_+$. Then there is a bijection*

$$\text{qh. str}(\mathbb{K}Q) \longrightarrow \prod_{i=1}^{\ell} \text{qh. str}(\Lambda_{n_i})$$

given by $[\triangleleft] \mapsto ([\triangleleft|_{Q_0^i}]_{i=1}^{\ell})$. Moreover, if \triangleleft is a minimal adapted order, then there exists a binary tree T_i of size n_i such that $\triangleleft|_{Q_0^i} = \triangleleft_{T_i}$, for each $1 \leq i \leq \ell$.

Proof. The bijection follows from Theorem 5.2.7. The second assertion is consequence of Propositions 5.1.7 and 5.2.9 and Lemma 5.1.8. \square

Example 5.2.11. Let $A = \mathbb{K}Q$, where $Q = 1 \rightarrow 2 \leftarrow 3 \leftarrow 4 \rightarrow 5$ as before. Then we have that

$$\begin{array}{ccccc} & 1 & & 3 & & 5 \\ & \uparrow & & \nearrow \searrow & & \uparrow \\ 2 & & & 2 & 4 & 4 \end{array}$$

are the Hasse quivers of some minimal adapted posets to Λ_2, Λ_3 and Λ_2 , respectively. Then, by Proposition 5.2.9 we have that the concatenation of the last Hasse quivers

$$\begin{array}{ccccc} & 1 & & 3 & & 5 \\ & \searrow \nearrow & & \searrow \nearrow & & \\ & 2 & & 4 & & \end{array}$$

is the corresponding minimal adapted poset to A , which represents a quasi-hereditary structure of A . Moreover, we know that $\text{card}(\text{qh. str } A) = 20$. In Fig. 5.3 we depict the Hasse quiver of $\text{qh. str}(A)$. The vertices correspond to minimal adapted orders to A which represent all the quasi-hereditary structures to A . Note that if a total order is a minimal adapted order, then it is the unique element in its equivalence class.

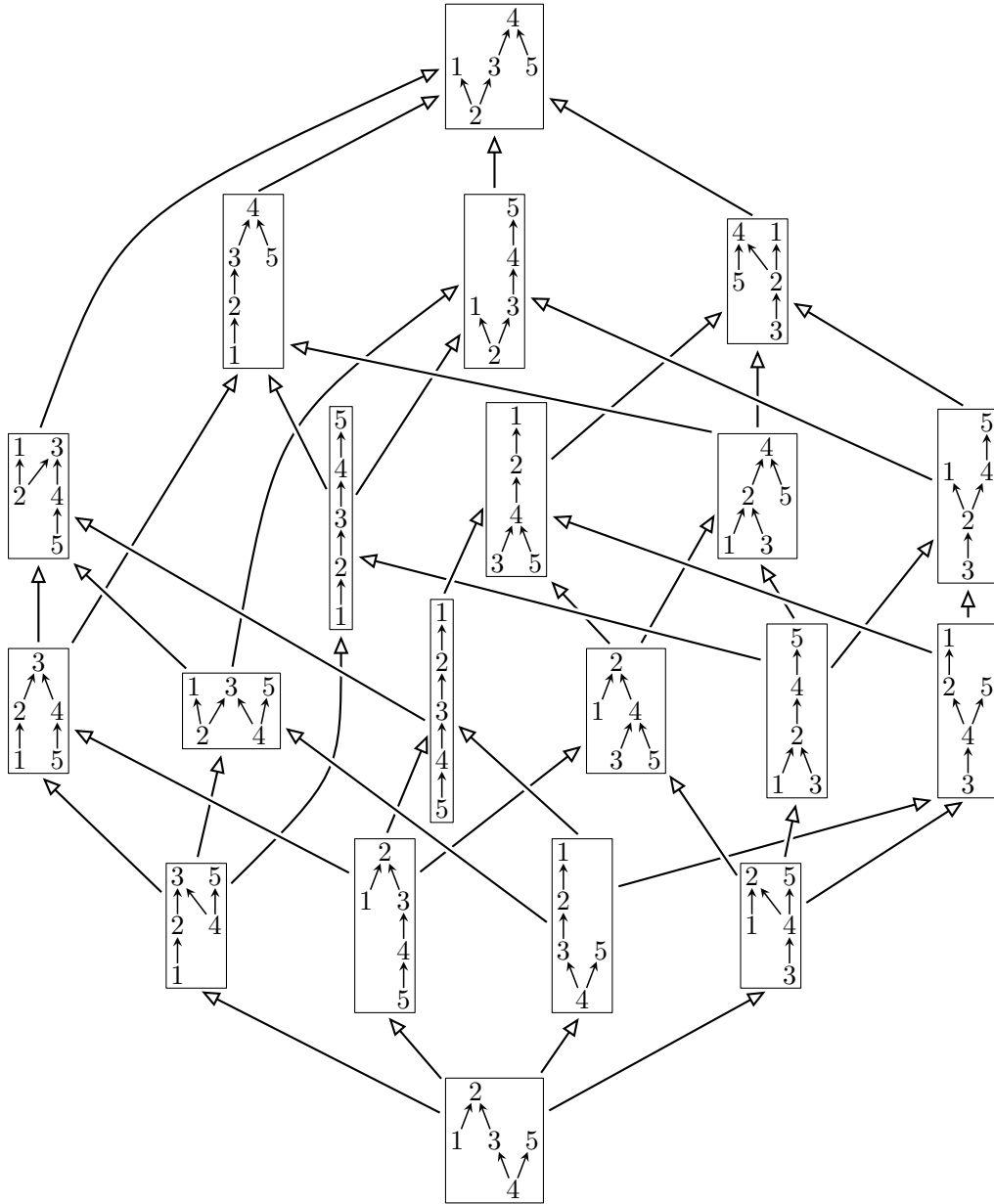


Figure 5.3: Poset of quasi-hereditary structures of $A = \mathbb{K}(1 \rightarrow 2 \leftarrow 3 \leftarrow 4 \rightarrow 5)$.

Bibliography

- [AF92] F. W. Anderson and K. R. Fuller. *Rings and categories of modules*. Second. Vol. 13. Graduate Texts in Mathematics. Springer-Verlag, New York, 1992, pp. x+376.
- [ASS06] I. Assem, D. Simson, and A. Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*. Vol. 65. London Mathematical Society Student Texts. Techniques of representation theory. Cambridge University Press, Cambridge, 2006, pp. x+458.
- [Aus74] M. Auslander. “Representation theory of Artin algebras. I”. In: *Comm. Algebra* 1 (1974), pp. 177–268.
- [ARS95] M. Auslander, I. Reiten, and S. O. Smalø. *Representation theory of Artin algebras*. Vol. 36. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995, pp. xiv+423.
- [Bar15] M. Barot. *Introduction to the representation theory of algebras*. Springer, Cham, 2015, pp. x+179.
- [BK04] A. B. Buan and H. Krause. “Tilting and cotilting for quivers of type \tilde{A}_n ”. In: *J. Pure Appl. Algebra* 190.1-3 (2004), pp. 1–21.
- [CPP19] G. Chatel, V. Pilaud, and V. Pons. “The weak order on integer posets”. In: *Algebr. Comb.* 2.1 (2019), pp. 1–48.
- [CPS88a] E. Cline, B. Parshall, and L. Scott. “Algebraic stratification in representation categories”. In: *J. Algebra* 117.2 (1988), pp. 504–521.
- [CPS88b] E. Cline, B. Parshall, and L. Scott. “Finite-dimensional algebras and highest weight categories”. In: *J. Reine Angew. Math.* 391 (1988), pp. 85–99.
- [Con16] T. Conde. “On certain strongly quasihereditary algebras”. PhD thesis. University of Oxford, 2016.
- [Cou19] K. Coulembier. “The classification of blocks in BGG category \mathcal{O} ”. In: *arXiv e-prints* (Mar. 2019). arXiv: [1903.02717](https://arxiv.org/abs/1903.02717) [[math.RT](#)].
- [Dla96] V. Dlab. “Quasi-hereditary algebras revisited”. In: vol. 4. 2. Representation theory of groups, algebras, and orders (Constanța, 1995). 1996, pp. 43–54.
- [DR89a] V. Dlab and C. M. Ringel. “Auslander algebras as quasi-hereditary algebras”. In: *J. London Math. Soc. (2)* 39.3 (1989), pp. 457–466.
- [DR89b] V. Dlab and C. M. Ringel. “Quasi-hereditary algebras”. In: *Illinois J. Math.* 33.2 (1989), pp. 280–291.

- [DR92] V. Dlab and C. M. Ringel. “The module theoretical approach to quasi-hereditary algebras”. In: *Representations of algebras and related topics (Kyoto, 1990)*. Vol. 168. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1992, pp. 200–224.
- [Don81] S. Donkin. “A filtration for rational modules”. In: *Math. Z.* 177.1 (1981), pp. 1–8.
- [DR94] S. Donkin and I. Reiten. “On Schur algebras and related algebras. V. Some quasi-hereditary algebras of finite type”. In: *J. Pure Appl. Algebra* 97.2 (1994), pp. 117–134.
- [DK94] Y. A. Drozd and V. V. Kirichenko. *Finite-dimensional algebras*. Translated from the 1980 Russian original and with an appendix by Vlastimil Dlab. Springer-Verlag, Berlin, 1994, pp. xiv+249.
- [Erd93] K. Erdmann. “Schur algebras of finite type”. In: *Quart. J. Math. Oxford Ser. (2)* 44.173 (1993), pp. 17–41.
- [FKR20] M. Flores, Y. Kimura, and B. Rognerud. “Combinatorics of quasi-hereditary structures”. In: *arXiv e-prints* (Apr. 2020). arXiv: [2004.04726](https://arxiv.org/abs/2004.04726) [[math.RT](#)].
- [Gab62] P. Gabriel. “Des catégories abéliennes”. fr. In: *Bulletin de la Société Mathématique de France* 90 (1962), pp. 323–448.
- [Gab81] P. Gabriel. “Un jeu? Les nombres de Catalan”. In: *Uni Zürich, Mitteilungsblatt des Rektorats* 6 (1981), pp. 4–5.
- [Gab72] P. Gabriel. “Unzerlegbare Darstellungen. I”. In: *Manuscripta Math.* 6 (1972), 71–103, correction, *ibid.* 6 (1972), 309.
- [GL91] W. Geigle and H. Lenzing. “Perpendicular categories with applications to representations and sheaves”. In: *J. Algebra* 144.2 (1991), pp. 273–343.
- [Gre90] J. A. Green. “Schur algebras”. Unpublished lectures given at Bielefeld. 1990.
- [Gus82] W. H. Gustafson. “The history of algebras and their representations”. In: *Representations of algebras (Puebla, 1980)*. Vol. 944. Lecture Notes in Math. Springer, Berlin-New York, 1982, pp. 1–28.
- [HU05] D. Happel and L. Unger. “On a partial order of tilting modules”. In: *Algebr. Represent. Theory* 8.2 (2005), pp. 147–156.
- [HGK04] M. Hazewinkel, N. Gubareni, and V. V. Kirichenko. *Algebras, rings and modules. Vol. 1*. Vol. 575. Mathematics and its Applications. Kluwer Academic Publishers, Dordrecht, 2004, pp. xii+380.
- [Hil06] L. Hille. “On the volume of a tilting module”. In: *Abh. Math. Sem. Univ. Hamburg* 76 (2006), pp. 261–277.
- [Hum08] J. E. Humphreys. *Representations of semisimple Lie algebras in the BGG category \mathcal{O}* . Vol. 94. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008, pp. xvi+289.
- [Iya03] O. Iyama. “Finiteness of representation dimension”. In: *Proc. Amer. Math. Soc.* 131.4 (2003), pp. 1011–1014.

- [Kra17] H. Krause. “Highest weight categories and recollements”. In: *Ann. Inst. Fourier (Grenoble)* 67.6 (2017), pp. 2679–2701.
- [Mac95] S. Mac Lane. *Homology*. Classics in Mathematics. Reprint of the 1975 edition. Springer-Verlag, Berlin, 1995, pp. x+422.
- [Mar17] F. Marks. “Homological embeddings for preprojective algebras”. In: *Math. Z.* 285.3-4 (2017), pp. 1091–1106.
- [Mar08] S. Martin. *Schur algebras and representation theory*. Vol. 112. Cambridge Tracts in Mathematics. Reprint of the 1993 original. Cambridge University Press, Cambridge, 2008, pp. xvi+232.
- [OEI20] OEIS Foundation Inc. (2020). *The On-Line Encyclopedia of Integer Sequences*. URL: <https://oeis.org/>.
- [Par89] B. Parshall. “Finite-dimensional algebras and algebraic groups”. In: *Classical groups and related topics (Beijing, 1987)*. Vol. 82. Contemp. Math. Amer. Math. Soc., Providence, RI, 1989, pp. 97–114.
- [PS88] B. Parshall and L. Scott. “Derived Categories, quasi-hereditary algebras, and algebraic groups”. In: *Proc. Ottawa–Moosonee Workshop*. Carleton–Ottawa Math. Lecture Note Series 3. 1988, pp. 1–105.
- [Psa14] C. Psaroudakis. “Homological theory of recollements of abelian categories”. In: *Journal of Algebra* 398 (2014), pp. 63–110.
- [PV14] C. Psaroudakis and J. Vitória. “Recollements of module categories”. In: *Appl. Categ. Structures* 22.4 (2014), pp. 579–593.
- [RS91] C. Riedtmann and A. Schofield. “On a simplicial complex associated with tilting modules”. In: *Comment. Math. Helv.* 66.1 (1991), pp. 70–78.
- [Rin91] C. M. Ringel. “The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences”. In: *Math. Z.* 208.2 (1991), pp. 209–223.
- [RZ14] C. M. Ringel and P. Zhang. “From submodule categories to preprojective algebras”. In: *Math. Z.* 278.1-2 (2014), pp. 55–73.
- [Sch14] R. Schiffler. *Quiver representations*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, Cham, 2014, pp. xii+230.
- [Sco87] L. Scott. “Simulating algebraic geometry with algebra. I. The algebraic theory of derived categories”. In: *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*. Vol. 47. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1987, pp. 271–281.
- [Soe90] W. Soergel. “Construction of projectives and reciprocity in an abstract setting”. Preprint. 1990.
- [Sta99] R. P. Stanley. *Enumerative combinatorics. Vol. 2*. Vol. 62. Cambridge Studies in Advanced Mathematics. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. Cambridge University Press, Cambridge, 1999, pp. xii+581.
- [Tam62] D. Tamari. “The algebra of bracketings and their enumeration”. In: *Nieuw Arch. Wisk. (3)* 10 (1962), pp. 131–146.

- [UY90] M. Uematsu and K. Yamagata. “On serial quasi-hereditary rings”. In: *Hokkaido Math. J.* 19.1 (1990), pp. 165–174.
- [Wei94] C. A. Weibel. *An introduction to homological algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450.
- [Wic96] D. D. Wick. “A generalization of quasi-hereditary rings”. In: *Comm. Algebra* 24.4 (1996), pp. 1217–1227.

Index

- Δ -filtered module, 12, 18
- Δ -filtration, 12, 18
- adapted order, 18
- additive closure, 21
- algebra, 1
 - Artin, 2
 - basic, 5
 - finite dimensional, 2
- arrow ideal, 5
- Artin algebra, *see* algebra, Artin
- Auslander algebra, 29
- basic
 - algebra, *see* algebra, basic module, 21
- binary notation, 73
- binary relation, 7
 - antisymmetry, 7
 - reflexive, 7
 - transitivity, 7
- binary search tree, 98
- binary tree, 98
 - leaves, 98
 - root, 98
 - size, 98
- block, 4
- bound quiver algebra, 5
- characteristic tilting module, 21
- child of a root, 98
- comparable elements of a poset, 7
- complete
 - set of idempotents, 3
 - set of primitive orthogonal idempotents, 3
- composition
 - factors, 2
 - series, 2
- composition of an integer, 67
- concatenation, 5
- connected
 - quiver, *see* quiver, connected
 - ring, 4
- costandard module, 17
- cover relation, 7
- deconcatenation, *see* quiver, deconcatenation
 - iterated, 105
- defining sequence, 15
- differential, 85
- dimension vector, 6
- discrete interval, 45
- dominant dimension, 4
- duality
 - fixing the simples, 28
 - standard, 3
- embedding, 8
- equivalent posets, 26
- finite representation type, 3
- functor
 - faithful, 8
 - full, 8
 - fully faithful, 8
- global dimension, 4
- good filtration, 12, 19
- Hasse
 - diagram of a poset, 7
 - quiver of a poset, 8
- hereditary algebra, 20
- heredity

- chain, 15
- ideal, 15
- highest weight category, 12
- homological
 - embedding, 8
 - Hasse quiver of an algebra, 23
 - poset of an algebra, 23
 - ring epimorphism, 9
- ideal
 - admissible, 5
 - nilpotent, 2
- idempotent, 3
 - central, 3
 - orthogonal, 3
 - primitive, 3
- in-order algorithm, 98
- in-order traversal, 98
- incomparable elements of a poset, 7
- indecomposable module, 3
- injective
 - coresolution, 4
 - dimension, 4
 - envelope, 3
- Jacobson radical, 2
- Jordan-Hölder
 - multiplicity, 2
- length
 - of a composition series, 2
 - of a module, 2
- loop, 5
- maximal
 - element of a poset, 7
 - heredity chain, 15
- minimal
 - adapted order, 27
 - element of a poset, 7
 - injective coresolution, 4
 - projective resolution, 4
- Morita equivalence, 2
- order, *see* partially ordered set
- oriented cycle, 5
- part of a composition, 67
- partially ordered set, 7
- path, 4
 - trivial, 4
- path algebra, 5
- perpendicular category, 19
- poset, *see* partially ordered set
 - extension, 7
 - intersection, 26
 - of quasi-hereditary structures, 28
 - product, 105
 - refinement, 7
- power set, 8
- preprojective algebra of type A_n , 48
- projective
 - cover, 3
 - dimension, 4
 - representation, 7
 - resolution, 4
- quasi-hereditary
 - algebra, 18, 19
 - ring, 15
 - structure, 27
- quiver, 4
 - acyclic, 5
 - bound, 5
 - commutativity relation, 5
 - connected, 4
 - deconcatenation, 102
 - finite, 4
 - monomial relation, 5
 - opposite, 6
 - relation, 5
 - representation, 6
- quiver algebra, *see* bound quiver algebra
- radical of a module, 2
- rank function, 67
- recollement, 13
 - homological, 13
- regular module, 2
- reject of a module, 17
- representation, *see* quiver, representation
 - finite dimensional, 6
 - of a bound quiver, 6
 - morphism, 6
- restriction of scalars, 9

- right hereditary ring, 20
- semiprimary ring, 14
- semisimple
 - algebra, 2
 - module, 2
- simple
 - module, 2
 - representation, 7
- sink, 102
- socle, 2
- source, 102
- standard module, 17
- standardly stratified algebra, 20
- stratifying chain, 21
- subcategory
 - full, 8
- subposet
 - induced, 7
 - weak, 7
- subquiver, 5
 - full, 5
- subtree, 98
- syzygy of a module, 85

- tiling, 73
- tilting module, 21
 - classical, 21
- top, 2
- total order, 8
- totally ordered set, 8
- trace of a module, 17

- underlying graph, 5

- weight, 12