

A note on stochastic dominance, uniform integrability and lattice properties

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ABSTRACT

In this work, we discuss completeness for the lattice orders of first and second order stochastic dominance. The main results state that both first- and second-order stochastic dominance induce Dedekind super complete lattices, that is, lattices in which every bounded nonempty subset has a countable subset with identical least upper bound and greatest lower bound. Moreover, we show that, if a suitably bounded set of probability measures is directed (for example, a lattice), then the supremum and infimum with respect to first-order or second-order stochastic dominance can be approximated by sequences in the weak topology or in the Wasserstein-1 topology, respectively. As a consequence, we are able to prove that a sublattice of probability measures is complete with respect to first-order stochastic dominance or second-order stochastic dominance and increasing convex order if and only if it is compact in the weak topology or in the Wasserstein-1 topology, respectively. This complements a set of characterizations of tightness and uniform integrability, which are discussed in a preliminary section.

1. Introduction

In this work, we discuss completeness for lattice orders arising from first- and second-order stochastic dominance (see, for instance, Müller and Stoyan [18] or Shaked and Shanthikumar [19]), and their relation to tightness and uniform integrability, respectively. Given a lattice L , it is a well-known result, due to Birkhoff [4, Section X.12, Theorem X.20] and Frink [9], that L is complete, that is, every nonempty subset of L has a least upper bound and a greatest lower bound, if and only if L is compact in the interval topology. The latter is the smallest topology on L such that all closed intervals of the form

$$(-\infty, a] := \{x \in L \mid x \leq a\} \quad \text{and} \quad [a, \infty) := \{x \in L \mid x \geq a\}, \quad \text{for } a \in L,$$

are closed. Due to its definition, the interval topology or, more precisely, open sets in the interval topology are usually not easy to describe. Moreover, in general, the interval topology is not even a Hausdorff topology. We refer to Baer [2] for a characterization of lattices with Hausdorff interval topologies. Stochastic dominance or convex orders are present in many applications in microeconomics and decision theory (cf. Levy [14]), as well as mathematics, for example, martingale optimal transport (cf. Strassen [20]) and, recently, submodular mean field games (cf. Dianetti et al. [7]). It is therefore desirable to obtain a more tractable characterization of complete (with respect to stochastic dominance) lattices of probability measures than the one in terms of compactness in the interval topology. Two of the main results of this paper are a characterization of complete lattices with respect to first- and second-order stochastic dominance in terms of compactness in the weak topology and in the Wasserstein-1 topology, respectively (Theorems 3.5 and 3.12).

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Dianetti et al. [7] propose a lattice-theoretical approach to mean field games using Tarski's fixed point theorem. The crucial step in their analysis is to show that an appropriately chosen lattice L of flows of probability measures is complete. Moreover, the approximation of suprema and infima is of fundamental importance in their analysis. Motivated by this application, in Section 3, for a fixed σ -finite measure space (S, \mathcal{S}, π) , we consider the sets $L^0(S, \mathcal{S}, \pi; \mathcal{P}(\mathbb{R}))$ and $L^0(S, \mathcal{S}, \pi; \mathcal{P}_1(\mathbb{R}))$ of all equivalence classes of \mathcal{S} - $\mathcal{B}(\mathcal{P}(\mathbb{R}))$ and \mathcal{S} - $\mathcal{B}(\mathcal{P}_1(\mathbb{R}))$ measurable flows $(\mu_t)_{t \in S}$ of probability measures, respectively, and address the Dedekind completeness as well as the approximation of suprema and infima for the latter. Here, $\mathcal{P}(\mathbb{R})$ is the set of all probability measures on the Borel σ -algebra of \mathbb{R} and $\mathcal{P}_1(\mathbb{R})$ is the subset of all probability measures with finite first moment, endowed with the lattice orders of first- and second-order stochastic dominance as well as the Borel σ -algebras $\mathcal{B}(\mathcal{P}(\mathbb{R}))$ and $\mathcal{B}(\mathcal{P}_1(\mathbb{R}))$ with respect to the weak topology on $\mathcal{P}(\mathbb{R})$ and the Wasserstein-1 topology on $\mathcal{P}_1(\mathbb{R})$, respectively. Note that, although first- and second-order stochastic dominance induce lattice structures on $\mathcal{P}(\mathbb{R})$ and $\mathcal{P}_1(\mathbb{R})$, they do not induce a lattice order on $\mathcal{P}(\mathbb{R}^d)$ and $\mathcal{P}_1(\mathbb{R}^d)$ for $d > 1$, respectively (cf. Kamae et al. [11] and Müller and Scarsini [17]). Kertz and Rösler [12] proved that the lattices $\mathcal{P}(\mathbb{R})$ and $\mathcal{P}_1(\mathbb{R})$ are Dedekind complete, that is, every bounded nonempty subset has a least upper bound and a greatest lower bound. Müller [16] discusses closure properties of second-order stochastic dominance with respect to the Wasserstein-1 topology, and shows that upper interval bounds can be approximated in the Wasserstein-1 sense by a simple construction of nondecreasing sequences of distributions, see [16, Theorem 4.3]. In the present paper, we show that $L^0(S, \mathcal{S}, \pi; \mathcal{P}(\mathbb{R}))$ and appropriate sublattices of $L^0(S, \mathcal{S}, \pi; \mathcal{P}_1(\mathbb{R}))$ are Dedekind super complete, that is, every bounded nonempty subset has a countable subset with identical least upper bound and greatest lower bound (Theorem 3.2 and Theorem 3.8). We further prove that, for directed and (suitably) bounded sets of probability measures, the supremum and infimum with respect to first-order or second-order stochastic dominance can be approximated by monotone sequences in the weak topology or in the Wasserstein-1 topology, respectively. The proof relies on an abstract lemma (Lemma A.3) in Appendix A, which gives a sufficient and necessary condition for the Dedekind super completeness of lattices. The proof of the latter is an abstract version of the existence proof of the essential supremum for families of random variables (see, for example, Föllmer and Schied [8, Theorem A.32]). Choosing S to be a singleton with $\pi(S) > 0$, we obtain the Dedekind super completeness of $\mathcal{P}(\mathbb{R})$ and $\mathcal{P}_1(\mathbb{R})$, endowed with first- and second-order stochastic dominance, as well as the approximation of suprema/infima in the weak topology and the Wasserstein-1 topology, respectively (see Corollary 3.3 and Proposition 3.9).

In Section 2, we discuss some preliminary results on tightness and uniform integrability. Chandra [6] gives a de La Vallée Poussin-type characterization for tightness and uniform integrability in terms of a function $\psi: [0, \infty) \rightarrow [0, \infty)$ with a certain behavior at infinity, see also Hu and Rosalsky [10]. Inspired by these two works, we relate properties of the function ψ , such as strict monotonicity and (strict) convexity, to integrability conditions on the set of distributions. In particular, we answer two open questions from [6], and derive a perturbation result for families of uniformly integrable (u.i.) random variables in terms of a strictly convex transformation (Lemma 2.4 and Corollary 2.5). More precisely, we show that, for a family H of u.i. random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there exists a strictly convex nondecreasing function $\psi: [0, \infty) \rightarrow [0, \infty)$ with $\frac{\psi(s)}{s} \rightarrow \infty$ as $s \rightarrow \infty$, such that $\{\psi(|X|) \mid X \in H\}$ is again u.i. Leskelä and Vhiola [13] obtained several characterizations of tightness and uniform integrability in terms of first- and second-order stochastic dominance for measures on the positive half line. We extend the results by Leskelä and Vhiola [13] to \mathbb{R} and combine them with the results from Sections 2 and 3 in order to obtain a characterization of tightness and uniform integrability in terms of integrability conditions for a function ψ with certain properties and boundedness conditions with respect to first- and second-order stochastic dominance, respectively (Lemmas 3.4 and 3.11).

Structure of the paper: The paper is organized as follows. In Section 2, we derive some preliminary results on tightness and uniform integrability, which we could not find in the literature in this form. Section 3 is dedicated to the completeness of lattices of (flows of) probability measures ordered by first- and second-order stochastic dominance. The main results are Theorems 3.2, 3.8, 3.5 and 3.12. In Appendix A, we first give some lattice-theoretical definitions (Definition A.1 and Definition A.2), and then derive an abstract auxiliary result (Lemma A.3) that helps to determine, when a lattice is Dedekind super complete, and forms the basis for the proofs in Section 3. The results from Section 2 are proved in Appendix B.

2. *Some preliminary remarks on tightness and uniform integrability*

Let $\mathbb{R}_+ := [0, \infty)$ and $\mathcal{B}(\mathbb{R}_+)$ denote the Borel σ -algebra on \mathbb{R}_+ . We first concentrate on the set $\mathcal{M}(\mathbb{R}_+)$ of all (possibly nonfinite) measures $\mu: \mathcal{B}(\mathbb{R}_+) \rightarrow [0, \infty]$. We start with the following characterization of tightness, which is sort of a folk theorem, see, for example, [6, Lemma 2.1; 13, Remark 7; 15, Lemma D.5.3]. However, in this form, we could not find it in the literature, which is why we provide a proof in Appendix B.

LEMMA 2.1. *Let $K \subset \mathcal{M}(\mathbb{R}_+)$. Then, the following statements are equivalent:*

- (i) *K is tight, that is, $\sup_{\nu \in K} \nu((s, \infty)) \rightarrow 0$ as $s \rightarrow \infty$;*
- (ii) *there exists a nondecreasing function $\psi: [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$, $\psi(s) \rightarrow \infty$ as $s \rightarrow \infty$ and*

$$\sup_{\nu \in K} \int_0^\infty \psi(s) \, d\nu(s) < \infty.$$

The function ψ in (ii) can be chosen to be continuous. The function ψ can be chosen to be convex if and only if

$$\sup_{\nu \in K} \int_0^\infty (s - M)^+ \, d\nu(s) < \infty \quad \text{or, equivalently, if} \quad \sup_{\nu \in K} \int_M^\infty s \, d\nu(s) < \infty \tag{2.1}$$

for some $M \geq 0$. Additionally, the function ψ can be chosen to be strictly increasing if and only if

$$\sup_{\nu \in K} \nu((s, \infty)) < \infty \quad \text{for all } s > 0. \tag{2.2}$$

Lemma 2.1 answers an open question by an anonymous referee concerning [6, Lemma 2.1], namely if the function ψ in (ii) can be chosen to be convex. In the previous lemma, we have seen that ψ can be chosen to be convex if and only if (2.1) is satisfied for some $M \geq 0$. In the special case, where K is a set of probability measures, (2.1) is equivalent to the uniform boundedness of the first moments of the elements of K , that is,

$$\sup_{\nu \in K} \int_0^\infty s \, d\nu(s) < \infty.$$

DEFINITION 2.2. Let $K \subset \mathcal{M}(\mathbb{R}_+)$. Then, we say that a nondecreasing map $\varphi: [0, \infty) \rightarrow [0, \infty)$ is u.i. for K if

$$\sup_{\nu \in K} \int_M^\infty \varphi(s) \, d\nu(s) \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

We would like to mention at this point that the name ‘uniform integrability’ is actually slightly misleading, since, for sets of nonfinite measures, uniform integrability does not imply integrability, not even for singletons.

REMARK 2.3. Let $\varphi: [0, \infty) \rightarrow [0, \infty)$ be nondecreasing. Then, the uniform integrability of φ for a set $K \subset \mathcal{M}(\mathbb{R}_+)$ can be divided into three cases.

- (i) $\varphi(s) = 0$ for all $s \geq 0$. Then, φ is always u.i. for K .
- (ii) φ is bounded and $\varphi(s) \neq 0$ for some $s \geq 0$. Then, φ is u.i. for K if and only if K is tight.
- (iii) $\varphi(s) \rightarrow \infty$ as $s \rightarrow \infty$. Then, φ is u.i. for K if and only if the identity $[0, \infty) \rightarrow [0, \infty)$, $s \mapsto s$ is u.i. for the set $\{\mu \circ \varphi^{-1} \mid \mu \in K\}$.

Hence, except in the trivial case (i), the question of uniform integrability of φ can be reduced to the question of tightness, which has already been discussed in Lemma 2.1 or uniform integrability of the identity. Therefore, in the next lemma, which is a generalized version of the de La Vallée Poussin Lemma (see also [6]), we will just discuss the uniform integrability of the identity. The proof is a direct consequence of Lemma 2.1, and is also relegated to Appendix B.

In the following, we use the convention $\frac{0}{0} := 0$.

LEMMA 2.4. *Let $K \subset \mathcal{M}(\mathbb{R}_+)$. Then, the following statements are equivalent:*

- (i) *the identity $[0, \infty) \rightarrow [0, \infty)$, $s \mapsto s$ is u.i. for K ;*
- (ii) *there exists a nondecreasing function $\psi: [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$ such that $[0, \infty) \rightarrow [0, \infty)$, $s \mapsto \frac{\psi(s)}{s}$ is nondecreasing, $\frac{\psi(s)}{s} \rightarrow \infty$ as $s \rightarrow \infty$ and*

$$\sup_{\nu \in K} \int_0^\infty \psi(s) \, d\nu(s) < \infty.$$

The function ψ in (ii) can be chosen to be continuously differentiable, convex and u.i. for K . Additionally, the function ψ can be chosen to be strictly convex (and thus strictly increasing) if and only if (2.2) is satisfied.

Note that in the previous lemma, the function ψ in (ii) can be chosen to be strictly convex and u.i. if K is, for example, a set of probability measures. This is actually quite remarkable, since it shows that uniform integrability is preserved under a ‘small’ perturbation in terms of a strictly convex transformation ψ with $\frac{\psi(s)}{s} \rightarrow \infty$ as $s \rightarrow \infty$. It can thus be seen as a perturbation result for families of u.i. functions and answers an open question by an anonymous referee concerning [6, Lemma 1.2]. In view of the standard application of the de La Vallée Poussin Lemma, where the function ψ is chosen to be $\psi(s) = s^p$ for some suitable $p > 1$, this perturbation result is quite natural, since, in this case, any perturbation of the form $s \mapsto s^q$ for $q \in (1, p)$ would remain u.i.

COROLLARY 2.5. *Let H be a set of random variables or random vectors (not necessarily of the same dimension) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $p \in [0, \infty)$. Then, the following statements are equivalent.*

- (i) *H is u.i., that is,*

$$\sup_{X \in H} \int_{\{|X| > s\}} |X| \, d\mathbb{P} \rightarrow 0 \quad \text{as } s \rightarrow \infty.$$

(ii) *There exists a nondecreasing function $\psi: [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$, $\frac{\psi(s)}{s} \rightarrow \infty$ as $s \rightarrow \infty$ and*

$$\sup_{X \in H} \int_{\Omega} \psi(|X|) \, d\mathbb{P} < \infty.$$

The function ψ in (ii) can be chosen to be continuously differentiable, strictly convex and in such a way that $\{\psi(|X|) \mid X \in H\}$ is again u.i.

Proof. This follows from the previous results by considering $K := \{\mathbb{P} \circ |X|^{-1} \mid X \in H\}$. \square

3. Completeness

In this section, we discuss the completeness of lattice orders arising from first- and second-order stochastic dominance. Throughout this section, let $\mathcal{B}(\mathbb{R})$ denote the Borel σ -algebra on \mathbb{R} , $\mathcal{P}(\mathbb{R})$ denote the set of all probability measures on $\mathcal{B}(\mathbb{R})$ and $\mathcal{P}_1(\mathbb{R})$ denote the set of all probability measures on $\mathcal{B}(\mathbb{R})$ with finite first moment, that is, all probability measures μ on $\mathcal{B}(\mathbb{R})$ with

$$\int_{\mathbb{R}} |x| \, d\mu(x) < \infty.$$

3.1. First-order stochastic dominance

Throughout this section, we will identify a distribution $\mu \in \mathcal{P}(\mathbb{R})$ by its survival function μ_0 , that is, we identify

$$\mu(s) = \mu_0(s) := \mu((s, \infty))$$

for all $s \in \mathbb{R}$. On $\mathcal{P}(\mathbb{R})$, we consider the partial-order \leq_{st} arising from first-order stochastic dominance, given by

$$\mu \leq_{\text{st}} \nu \quad \text{if and only if} \quad \mu_0(s) \leq \nu_0(s) \quad \text{for all } s \in \mathbb{R}.$$

Recall that, for $\mu, \nu \in \mathcal{P}(\mathbb{R})$, $\mu \leq_{\text{st}} \nu$ if and only if

$$\int_{\mathbb{R}} h(x) \, d\mu(x) \leq \int_{\mathbb{R}} h(x) \, d\nu(x)$$

for all nondecreasing functions $h: \mathbb{R} \rightarrow \mathbb{R}$. For a detailed discussion on the properties of the partial-order \leq_{st} , we refer to [19, Section 1.A].

REMARK 3.1. (a) By identifying μ with its survival function μ_0 , the set $\mathcal{P}(\mathbb{R})$ coincides with the set of all nonincreasing right-continuous functions $F: \mathbb{R} \rightarrow [0, 1]$ with $\lim_{s \rightarrow -\infty} F(s) = 1$ and $\lim_{s \rightarrow \infty} F(s) = 0$. In particular, the partial-order \leq_{st} induces a lattice structure on $\mathcal{P}(\mathbb{R})$ via

$$(\mu \vee_{\text{st}} \nu)(s) := \mu_0(s) \vee \nu_0(s) \quad \text{and} \quad (\mu \wedge_{\text{st}} \nu)(s) := \mu_0(s) \wedge \nu_0(s) \quad \text{for all } s \in \mathbb{R}.$$

Further, we would like to recall that the weak topology on $\mathcal{P}(\mathbb{R})$ is metrizable and that the weak convergence coincides with the pointwise convergence of survival functions at every continuity point, that is, $\mu^n \rightarrow \mu$ weakly as $n \rightarrow \infty$ if and only if

$$\mu_0^n(s) \rightarrow \mu_0(s) \quad \text{as } n \rightarrow \infty \quad \text{for every continuity point } s \in \mathbb{R} \text{ of } \mu_0.$$

Therefore, the weak convergence behaves well with the pointwise lattice operations \vee_{st} and \wedge_{st} . More precisely, the maps $(\mu, \nu) \mapsto \mu \vee_{\text{st}} \nu$ and $(\mu, \nu) \mapsto \mu \wedge_{\text{st}} \nu$ are continuous

$\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R})$. Moreover, the weak topology is finer than the interval topology, since, by the previous argumentation, every closed interval is weakly closed.

(b) Recall that a nonincreasing function $\mathbb{R} \rightarrow \mathbb{R}$ is right-continuous if and only if it is lower semicontinuous (lsc). Hence, for a sequence $(\mu^n)_{n \in \mathbb{N}} \in \mathcal{P}(\mathbb{R})$, which is bounded above, the supremum $\sup_{n \in \mathbb{N}} \mu^n$ with respect to \leq_{st} exists, and is exactly the pointwise supremum of the survival functions $(\mu_0^n)_{n \in \mathbb{N}}$.

(c) For a nonincreasing function $F: \mathbb{R} \rightarrow \mathbb{R}$, we define its lsc-envelope $F_*: \mathbb{R} \rightarrow \mathbb{R}$ by

$$F_*(s) := \sup_{\delta > 0} F(s + \delta) \quad \text{for } s \in \mathbb{R}.$$

Note that $F(s) \geq F_*(s) \geq F(s + \varepsilon)$ for all $s \in \mathbb{R}$ and $\varepsilon > 0$. That is, F_* differs from F only at discontinuity points of F . Intuitively speaking, F_* is the right-continuous version of F . For a sequence $(\mu^n)_{n \in \mathbb{N}} \in \mathcal{P}(\mathbb{R})$, which is bounded below, the infimum $\inf_{n \in \mathbb{N}} \mu^n$ with respect to \leq_{st} exists, and is given by the lsc-envelope of the pointwise infimum of the survival functions $(\mu_0^n)_{n \in \mathbb{N}}$. That is, one has to modify the pointwise infimum at all its discontinuity points in order to be right-continuous.

(d) A combination of the previous remarks, leads to the following insight: Every bounded and nondecreasing or nonincreasing sequence $(\mu^n)_{n \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R})$ converges weakly to its supremum or infimum with respect to \leq_{st} , respectively.

Let (S, \mathcal{S}, π) be a σ -finite measure space. We denote the Borel σ -algebra of the weak topology by $\mathcal{B}(\mathcal{P}(\mathbb{R}))$ and the lattice of all equivalence classes of \mathcal{S} - $\mathcal{B}(\mathcal{P}(\mathbb{R}))$ -measurable functions $S \rightarrow \mathcal{P}(\mathbb{R})$ by $L_{\text{st}}^0 = L^0(\mathcal{P}(\mathbb{R})) = L^0(S, \mathcal{S}, \pi; \mathcal{P}(\mathbb{R}))$. An arbitrary element μ of L_{st}^0 will be denoted in the form $\mu = (\mu_t)_{t \in S}$. On L_{st}^0 we consider the order relation $\leq_{L_{\text{st}}^0}$, given by $\mu \leq_{L_{\text{st}}^0} \nu$ if and only if $\mu_t \leq_{\text{st}} \nu_t$ for π -almost all $t \in S$.

THEOREM 3.2. (a) *The lattice L_{st}^0 is Dedekind super complete (cf. Definition A.1).*

(b) *If $M \subset L_{\text{st}}^0$ is a nonempty set, which is bounded above or below and directed upward or downward (cf. Definition A.1), then there exists a sequence $(\mu^n)_{n \in \mathbb{N}} \subset M$ with $\mu^n \leq_{L_{\text{st}}^0} \mu^{n+1}$ for all $n \in \mathbb{N}$ and $\mu^n \rightarrow \sup M$ weakly π -almost everywhere as $n \rightarrow \infty$ or $\mu^n \geq_{L_{\text{st}}^0} \mu^{n+1}$ for all $n \in \mathbb{N}$ and $\mu^n \rightarrow \inf M$ weakly π -almost everywhere as $n \rightarrow \infty$, respectively.*

Proof. Since every σ -finite measure can be transformed into a finite measure without changing the null-sets, we may without loss of generality assume that π is finite. By Remark 3.1, L_{st}^0 is Dedekind σ -complete. Let Φ be the cumulative distribution function of the standard normal distribution, that is,

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy \quad \text{for all } x \in \mathbb{R}.$$

Alternatively, one could consider, for example, $\Phi = \arctan$. The map $S \rightarrow \mathbb{R}, t \mapsto \int_{\mathbb{R}} \Phi(x) d\mu_t(x)$ is \mathcal{S} - $\mathcal{B}(\mathbb{R})$ -measurable for every $\mu \in L_{\text{st}}^0$, since $\Phi \in C_b(\mathbb{R})$ induces a continuous (with respect to the weak topology) linear functional $\mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$. Hence,

$$F: L_{\text{st}}^0 \rightarrow \mathbb{R}, \quad \mu \mapsto \int_S \int_{\mathbb{R}} \Phi(x) d\mu_t(x) d\pi(t)$$

is well defined and strictly increasing (cf. Definition A.2), since Φ is strictly increasing, see, for example, [19, Theorem 1.A.8]. The assertions now follow from Lemma A.3 with $R = \mathbb{R}$; Remarks A.4 and 3.1(d). □

In the case, where S is a singleton and $\pi(S) > 0$, we obtain the following corollary.

COROLLARY 3.3. (a) The lattice $\mathcal{P}(\mathbb{R})$, endowed with the lattice-order \leq_{st} , is Dedekind super complete.

(b) If $M \subset \mathcal{P}(\mathbb{R})$ is a nonempty set, which is bounded above or below and directed upward or downward, then there exists a sequence $(\mu^n)_{n \in \mathbb{N}} \subset M$ with $\mu^n \leq_{st} \mu^{n+1}$ for all $n \in \mathbb{N}$ and $\mu^n \rightarrow \sup M$ weakly as $n \rightarrow \infty$ or $\mu^n \geq_{st} \mu^{n+1}$ for all $n \in \mathbb{N}$ and $\mu^n \rightarrow \inf M$ weakly as $n \rightarrow \infty$, respectively.

We now turn our focus on characterizing the complete sublattices of $\mathcal{P}(\mathbb{R})$. We start with the following lemma, which gives two more characterizations of tightness. It is basically a combination and generalization of [13, Proposition 1] and Lemma 2.1.

LEMMA 3.4. Let $K \subset \mathcal{P}(\mathbb{R})$. Then, the following statements are equivalent.

- (i) K is tight, that is, $\sup_{\mu \in K} \mu([-s, s]^c) \rightarrow 0$ as $s \rightarrow \infty$.
- (ii) K is \leq_{st} -bounded, that is, there exist $\underline{\mu}, \bar{\mu} \in \mathcal{P}(\mathbb{R})$ with $\underline{\mu} \leq_{st} \mu \leq_{st} \bar{\mu}$ for all $\mu \in K$.
- (iii) Every nonempty subset of K has a least upper bound and a greatest lower bound.
- (iv) There exists a nondecreasing function $\psi: [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$, $\psi(s) \rightarrow \infty$ as $s \rightarrow \infty$ and

$$\sup_{\mu \in K} \int_{\mathbb{R}} \psi(|x|) d\mu(x) < \infty.$$

The function ψ in (iv) can be chosen to be continuous and strictly increasing. Moreover, the function ψ can be chosen to be convex if and only if

$$\sup_{\mu \in K} \int_{\mathbb{R}} |x| d\mu(x) < \infty.$$

Proof. The equivalence of (i) and (iv) has been discussed in the previous section. By Corollary 3.3, (ii) and (iii) are equivalent. If K is \leq_{st} -bounded, then

$$\sup_{\mu \in K} \mu((s, \infty)) \leq \bar{\mu}_0(s) \rightarrow 0 \quad \text{as } s \rightarrow \infty$$

and

$$\sup_{\mu \in K} \mu((-\infty, s)) = 1 - \inf_{\mu \in K} \mu([s, \infty)) \leq 1 - \underline{\mu}_0(s) \rightarrow 0 \quad \text{as } s \rightarrow -\infty,$$

which shows that K is tight. It remains to show that (i) implies (ii). We assume that K is tight. Then,

$$\inf_{\mu \in K} \mu_0(s) \rightarrow 1 \quad \text{as } s \rightarrow -\infty \quad \text{and} \quad \sup_{\mu \in K} \mu_0(s) \rightarrow 0 \quad \text{as } s \rightarrow \infty. \tag{3.1}$$

We define $\underline{\mu}_0, \bar{\mu}_0: \mathbb{R} \rightarrow [0, 1]$ by

$$\underline{\mu}_0(s) := \sup_{\delta > 0} \inf_{\mu \in K} \mu_0(s + \delta) \quad \text{and} \quad \bar{\mu}_0(s) := \sup_{\mu \in K} \mu_0(s)$$

for $s \in \mathbb{R}$. Then, $\underline{\mu}_0$ and $\bar{\mu}_0$ are nonincreasing and right-continuous functions, cf. Remark 3.1, and, by (3.1), $\lim_{s \rightarrow -\infty} \underline{\mu}_0(s) = 1$ and $\lim_{s \rightarrow \infty} \bar{\mu}_0(s) = 0$ for $F = \underline{\mu}_0, \bar{\mu}_0$. This shows that $\underline{\mu}_0$ and $\bar{\mu}_0$ give rise to two probability measures $\underline{\mu}, \bar{\mu} \in \mathcal{P}(\mathbb{R})$ with $\underline{\mu} \leq_{st} \mu \leq_{st} \bar{\mu}$ for all $\mu \in K$. \square

THEOREM 3.5. Let K be a sublattice of $\mathcal{P}(\mathbb{R})$. Then, the following statements are equivalent:

- (ii) K is weakly compact.
- (i) K is complete with respect to the lattice-order \leq_{st} .

Proof. Since, by Remark 3.1(a), the weak topology is finer than the interval topology, it follows that every weakly compact subset of $\mathcal{P}(\mathbb{R})$ is compact in the interval topology, and therefore complete. Now, assume that K is complete. We first show that K is weakly closed. Let $(\mu^n)_{n \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R})$ with $\mu^n \rightarrow \mu \in \mathcal{P}(\mathbb{R})$ weakly as $n \rightarrow \infty$. Then,

$$\mu = \inf_{n \in \mathbb{N}} \sup_{k \geq n} \mu^k \in K,$$

since K is complete. Here, we used the fact that the weak convergence is equivalent to the pointwise convergence at every continuity point. Since the weak topology is metrizable, it follows that K is weakly closed. By the previous lemma, K is tight, since it is complete and thus \leq_{st} -bounded. Since K is weakly closed, it is therefore weakly compact by Prokhorov’s theorem (see, for example, [3, Theorems 6.1 and 6.2; 5, Theorem 8.6.2]). \square

3.2. Second-order stochastic dominance and increasing convex order

In this section, we identify a probability measure $\mu \in \mathcal{P}_1(\mathbb{R})$ via its (negative) integrated distribution function $\mu_{1,-}$, which is given by

$$\mu_{1,-}(s) = \int_{\mathbb{R}} (x - s)^- d\mu(x) = - \int_{-\infty}^s \mu_0(u) du \quad \text{for all } s \in \mathbb{R},$$

and via its integrated survival function $\mu_{1,+}$, which is given by

$$\mu_{1,+}(s) = \int_{\mathbb{R}} (x - s)^+ d\mu(x) = \int_s^{\infty} \mu_0(u) du \quad \text{for all } s \in \mathbb{R}.$$

On $\mathcal{P}_1(\mathbb{R})$, we then consider the partial order arising from second -order stochastic dominance (also called increasing concave order) \leq_{icv} , given by

$$\mu \leq_{icv} \nu \quad \text{if and only if} \quad \mu_{1,-}(s) \leq \nu_{1,-}(s) \quad \text{for all } s \in \mathbb{R},$$

and the increasing convex -order \leq_{icx} , given by

$$\mu \leq_{icx} \nu \quad \text{if and only if} \quad \mu_{1,+}(s) \leq \nu_{1,+}(s) \quad \text{for all } s \in \mathbb{R}.$$

Recall that, for $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$, $\mu \leq_{icv} \nu$ ($\mu \leq_{icx} \nu$) if and only if

$$\int_{\mathbb{R}} h(x) d\mu(x) \leq \int_{\mathbb{R}} h(x) d\nu(x) \tag{3.2}$$

for all nondecreasing concave (convex) functions $h: \mathbb{R} \rightarrow \mathbb{R}$. In particular, $\mu \leq_{st} \nu$ implies $\mu \leq_{icv} \nu$ and $\mu \leq_{icx} \nu$. If (3.2) holds for all convex functions $h: \mathbb{R} \rightarrow \mathbb{R}$, we write $\mu \leq_{cx} \nu$ (convex order). Note that $\mu \leq_{cx} \nu$ if and only if $\nu \leq_{icv} \mu$ and $\mu \leq_{icx} \nu$. For $\mu \in \mathcal{P}_1(\mathbb{R})$, let $\mu' \in \mathcal{P}_1(\mathbb{R})$ denote the probability measure, given by

$$\mu'((-\infty, s]) := \mu([-s, \infty)) \quad \text{for all } s \in \mathbb{R}.$$

That is, for a random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution μ , μ' is the distribution of $-X$. Then, for all $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$,

$$\mu \leq_{icx} \nu \quad \text{if and only if} \quad \mu' \geq_{icv} \nu'. \tag{3.3}$$

For a detailed discussion on the properties of the partial orders \leq_{icv} and \leq_{icx} , we refer to [19, Section 4.A]. In this subsection, we investigate the interplay between the partial orders \leq_{icv} and \leq_{icx} and the Wasserstein-1 topology on the Wasserstein-1 space $\mathcal{P}_1(\mathbb{R})$. For a definition of the Wasserstein-1 metric and a detailed discussion on its properties, we refer to [1, 21].

REMARK 3.6. (a) Note that by identifying μ via its integrated distribution (survival) function $\mu_{1,-}$ ($\mu_{1,+}$), $\mathcal{P}_1(\mathbb{R})$ coincides with the set of all nonincreasing concave (convex)

functions $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ with $\varphi(s) + s - m \rightarrow 0$ as $s \rightarrow \infty$ ($s \rightarrow -\infty$) for some constant $m \in \mathbb{R}$ and $\varphi(s) \rightarrow 0$ as $s \rightarrow -\infty$ ($s \rightarrow \infty$). This follows from the observation that, for a concave (convex) function ψ , $\psi(s) \rightarrow 0$ as $s \rightarrow \pm\infty$ implies that $\psi'(s) \rightarrow 0$ as $s \rightarrow \pm\infty$, where ψ' denotes the right-continuous version of the derivative of ψ . Therefore, $-\varphi'$ is the distribution (survival) function μ_0 of a probability measure $\mu \in \mathcal{P}_1(\mathbb{R})$ with $\int_{\mathbb{R}} x \, d\mu(x) = m$. In particular, the partial orders \leq_{icv} and \leq_{icx} induce lattice structures on $\mathcal{P}_1(\mathbb{R})$ via

$$(\mu \wedge_{icv} \nu)(s) := \mu_{1,-}(s) \wedge \nu_{1,-}(s), \quad (\mu \vee_{icx} \nu)(s) := \mu_{1,+}(s) \vee \nu_{1,+}(s)$$

and

$$(\mu \vee_{icv} \nu)(s) := \inf\{\lambda s + a \mid \lambda, a \in \mathbb{R} \text{ with } \lambda t + a \geq \mu_{1,-}(t) \vee \nu_{1,-}(t) \text{ for all } t \in \mathbb{R}\},$$

$$(\mu \wedge_{icx} \nu)(s) := \sup\{\lambda s + a \mid \lambda, a \in \mathbb{R} \text{ with } \lambda t + a \leq \mu_{1,+}(t) \wedge \nu_{1,+}(t) \text{ for all } t \in \mathbb{R}\}$$

for all $s \in \mathbb{R}$. That is, the minimum (maximum) of two distributions is the pointwise minimum (maximum) of their integrated distribution (survival) functions and the maximum (minimum) of two distributions is the upper concave (lower convex) envelope of the pointwise maximum (minimum) of their integrated survival functions. Recall that, for a function $F: \mathbb{R} \rightarrow [0, \infty)$, its upper concave envelope F^* and its lower convex envelope F_* are given by

$$F^*(s) := \inf\{\lambda s + a \mid \lambda, a \in \mathbb{R} \text{ with } \lambda t + a \geq F(t) \text{ for all } t \in \mathbb{R}\} \quad \text{and}$$

$$F_*(s) := \sup\{\lambda s + a \mid \lambda, a \in \mathbb{R} \text{ with } \lambda t + a \leq F(t) \text{ for all } t \in \mathbb{R}\} \quad \text{for } s \in \mathbb{R}.$$

Since the convergence in the Wasserstein-1 topology implies the pointwise convergence of the integrated distribution (survival) functions (see, for example, [21, Theorem 6.9]), the lattice operations $(\mu, \nu) \mapsto \mu \vee_{icx} \nu$ and $(\mu, \nu) \mapsto \mu \wedge_{icx} \nu$ are continuous $\mathcal{P}_1(\mathbb{R}) \times \mathcal{P}_1(\mathbb{R}) \rightarrow \mathcal{P}_1(\mathbb{R})$. Moreover, this implies that the Wasserstein-1 topology is finer than both interval topologies, since closed intervals are closed with respect to the Wasserstein-1 metric.

(b) Let $(\mu^n)_{n \in \mathbb{N}} \in \mathcal{P}_1(\mathbb{R})$ be a sequence, which is bounded above with respect to \leq_{icv} (\leq_{icx}), and let

$$\nu^n := \mu^1 \vee_{icv} \cdots \vee_{icv} \mu^n \quad (\nu^n := \mu^1 \vee_{icx} \cdots \vee_{icx} \mu^n) \quad \text{for all } n \in \mathbb{N}.$$

Then, the supremum of $(\mu^n)_{n \in \mathbb{N}}$ with respect to \leq_{icv} (\leq_{icx}) exists, and its integrated distribution (survival) function is exactly the pointwise limit, as $n \rightarrow \infty$, of the integrated distribution (survival) functions of $(\nu^n)_{n \in \mathbb{N}}$.

(c) Let $(\mu^n)_{n \in \mathbb{N}} \in \mathcal{P}_1(\mathbb{R})$ be a sequence, which is bounded below with respect to \leq_{icv} (\leq_{icx}), and let

$$\nu^n := \mu^1 \wedge_{icv} \cdots \wedge_{icv} \mu^n \quad (\nu^n := \mu^1 \wedge_{icx} \cdots \wedge_{icx} \mu^n) \quad \text{for all } n \in \mathbb{N}.$$

Then, the infimum of $(\mu^n)_{n \in \mathbb{N}}$ with respect to \leq_{icv} (\leq_{icx}) exists, and its integrated distribution (survival) function is exactly the pointwise limit, as $n \rightarrow \infty$, of the integrated distribution (survival) functions of $(\nu^n)_{n \in \mathbb{N}}$.

Before we state the first main result of this subsection, we briefly discuss some connections between uniform integrability, and boundedness with respect to \leq_{icv} and \leq_{icx} .

REMARK 3.7. (a) Let $K \subset \mathcal{P}_1(\mathbb{R})$ and $\underline{\mu}, \bar{\mu} \in \mathcal{P}_1(\mathbb{R})$ with $\underline{\mu} \leq_{icv} \mu \leq_{icx} \bar{\mu}$ for all $\mu \in K$. Then, the identity is u.i. for K . In fact,

$$\sup_{\mu \in K} \int_{2s}^{\infty} x \, d\mu(x) \leq \sup_{\mu \in K} 2 \int_s^{\infty} x - s \, d\mu(x) \leq 2 \int_s^{\infty} x - s \, d\bar{\mu}(x) \rightarrow 0 \quad \text{as } s \rightarrow \infty$$

and

$$\sup_{\mu \in K} \int_{-\infty}^{2s} -x \, d\mu(x) \leq \sup_{\mu \in K} 2 \int_{-\infty}^s s - x \, d\mu(x) \leq 2 \int_{-\infty}^s s - x \, d\mu(x) \rightarrow 0 \quad \text{as } s \rightarrow -\infty.$$

(b) Let $(\mu^n)_{n \in \mathbb{N}} \subset \mathcal{P}_1(\mathbb{R})$ with $\underline{\mu} \leq_{\text{icv}} \mu^n \leq_{\text{icx}} \bar{\mu}$ and $\mu^n \leq_{\text{icv}} \mu^{n+1}$ ($\mu^n \leq_{\text{icx}} \mu^{n+1}$) for all $n \in \mathbb{N}$. Then, by part a), $(\mu^n)_{n \in \mathbb{N}}$ is u.i. Therefore, every subsequence of $(\mu^n)_{n \in \mathbb{N}}$ has a Wasserstein-1 convergent subsequence (cf. [1, Proposition 7.1.5; 21, Theorem 6.9]). Let $\mu \in \mathcal{P}_1(\mathbb{R})$ denote the least upper bound of $(\mu^n)_{n \in \mathbb{N}}$ with respect to \leq_{icv} (\leq_{icx}), $(\nu^n)_{n \in \mathbb{N}}$ be a convergent subsequence of $(\mu^n)_{n \in \mathbb{N}}$ and $\nu := \lim_{n \rightarrow \infty} \nu^n$. Then, $\nu^n \leq_{\text{icv}} \mu$ ($\nu^n \leq_{\text{icx}} \mu$) and therefore, $\nu \leq_{\text{icv}} \mu$ ($\nu \leq_{\text{icx}} \mu$) by Remark 3.6 a). On the other hand, for all $n \in \mathbb{N}$, there exists some $N \in \mathbb{N}$ with $\nu^k \geq_{\text{icv}} \mu^n$ ($\nu^k \geq_{\text{icx}} \mu^n$) for all $k \geq N$. Therefore, by Remark 3.6(a), $\nu \geq_{\text{icv}} \mu^n$ ($\nu \geq_{\text{icx}} \mu^n$) for all $n \in \mathbb{N}$, that is, $\nu \geq_{\text{icv}} \mu$ ($\nu \geq_{\text{icx}} \mu$). This shows that $\nu = \mu$. Recall that a sequence converges if and only if every subsequence has a convergent subsequence with the same limit. We have therefore shown that $(\mu^n)_{n \in \mathbb{N}}$ converges to its supremum with respect to \leq_{icv} (\leq_{icx}) in the Wasserstein-1 topology.

(c) Note that the \leq_{icv} -boundedness from below (\leq_{icx} -boundedness from above) in parts (a) and (b) cannot be replaced by \leq_{icx} -boundedness from below (\leq_{icv} -boundedness from above). Recall that every set $K \subset \mathcal{P}_1(\mathbb{R})$ with $\sup_{\mu \in K} \int_{\mathbb{R}} |x| \, d\mu(x) < \infty$ is \leq_{icv} -bounded from above (\leq_{icx} -bounded from below). This follows from the fact that, by Jensen's inequality, $\mu \leq_{\text{icv}} \delta_a$ ($\delta_a \leq_{\text{icx}} \mu$) for all $a \in \mathbb{R}$ and $\mu \in \mathcal{P}_1(\mathbb{R})$ with $a \geq \int_{\mathbb{R}} x \, d\mu(x)$ ($a \leq \int_{\mathbb{R}} x \, d\mu(x)$). However, the uniform boundedness of first moments is not sufficient in order to achieve uniform integrability. We would further like to point out that $\underline{\mu} \leq_{\text{icv}} \mu$ ($\mu \leq_{\text{icx}} \bar{\mu}$) for all $\mu \in K$ and $\int_{\mathbb{R}} x \, d\mu(x) = \int_{\mathbb{R}} x \, d\nu(x)$ for all $\mu, \nu \in K$ implies $\mu \leq_{\text{icx}} \underline{\mu}$ ($\bar{\mu} \leq_{\text{icv}} \mu$) for $\mu \in K$ and thus the uniform integrability of the identity for K .

We consider a σ -finite measure space (S, \mathcal{S}, π) . We denote the Borel σ -algebra with respect to the Wasserstein-1 topology by $\mathcal{B}(\mathcal{P}_1(\mathbb{R}))$ and the lattice of all equivalence classes of \mathcal{S} - $\mathcal{B}(\mathcal{P}_1(\mathbb{R}))$ -measurable functions $S \rightarrow \mathcal{P}_1(\mathbb{R})$ by $L^0(\mathcal{P}_1(\mathbb{R})) = L^0(S, \mathcal{S}, \pi; \mathcal{P}_1(\mathbb{R}))$. An arbitrary element μ of $L^0(\mathcal{P}_1(\mathbb{R}))$ will again be denoted in the form $\mu = (\mu_t)_{t \in S}$. On $L^0(\mathcal{P}_1(\mathbb{R}))$ we consider the order relations $\leq_{L^0_{\text{icv}}}$ and $\leq_{L^0_{\text{icx}}}$ given by $\mu \leq_{L^0_{\text{icv}}} \nu$ if and only if $\mu_t \leq_{\text{icv}} \nu_t$ for π -almost all $t \in S$ and $\mu \leq_{L^0_{\text{icx}}} \nu$ if and only if $\mu_t \leq_{\text{icx}} \nu_t$ for π -almost all $t \in S$, respectively.

THEOREM 3.8. (a) For every fixed $\nu \in L^0(\mathcal{P}_1(\mathbb{R}))$, the lattice $\{\mu \in L^0(\mathcal{P}_1(\mathbb{R})) \mid \mu \leq_{\text{icx}} \nu \text{ (} \mu \geq_{\text{icv}} \nu)\}$, together with the order $\leq_{L^0_{\text{icv}}}$ ($\leq_{L^0_{\text{icx}}}$), is Dedekind super complete (cf. Definition A.1).

(b) Let $M \subset L^0(\mathcal{P}_1(\mathbb{R}))$ be nonempty, $\leq_{L^0_{\text{icv}}}$ -bounded below and $\leq_{L^0_{\text{icx}}}$ -bounded above. If M is directed upward/downward (cf. Definition A.1) with respect to $\leq_{L^0_{\text{icv}}}$ ($\leq_{L^0_{\text{icx}}}$), then there exists a nondecreasing/nonincreasing sequence $(\mu^n)_{n \in \mathbb{N}} \subset M$ with $\mu^n \rightarrow \mu$ π -almost everywhere in the Wasserstein-1 topology as $n \rightarrow \infty$, where $\mu \in L^0(\mathcal{P}_1(\mathbb{R}))$ is the supremum/infimum of M with respect to $\leq_{L^0_{\text{icv}}}$ ($\leq_{L^0_{\text{icx}}}$).

Proof. Since every σ -finite measure can be transformed into a finite measure without changing the null sets, we may without loss of generality assume that π is finite. By Remarks 3.6 and 3.7, the set $\{\mu \in L^0(\mathcal{P}_1(\mathbb{R})) \mid \mu \leq_{L^0_{\text{icx}}} \nu \text{ (} \mu \geq_{L^0_{\text{icv}}} \nu)\}$ together with $\leq_{L^0_{\text{icv}}}$ ($\leq_{L^0_{\text{icx}}}$) is Dedekind σ -complete. Let $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$\Phi^\pm(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (x - s)^\pm e^{-s^2/2} \, ds \quad \text{for all } x \in \mathbb{R}.$$

The map $S \rightarrow \mathbb{R}$, $t \mapsto \int_{\mathbb{R}} \Phi^{\pm}(x) d\mu_t(x)$ is $\mathcal{S}\text{-}\mathcal{B}(\mathbb{R})$ -measurable for every $\mu \in L^0(\mathcal{P}_1(\mathbb{R}))$, since Φ^{\pm} is 1-Lipschitz and thus induces a continuous (with respect to the Wasserstein-1 topology) linear functional $\mathcal{P}_1(\mathbb{R}) \rightarrow \mathbb{R}$. Hence,

$$F^{\pm}: L^0(\mathcal{P}_1(\mathbb{R})) \rightarrow \mathbb{R}, \quad \mu \mapsto \int_{\mathbb{R}} \Phi^{\pm}(x) d\mu_t(x) d\pi(t)$$

is well defined. Moreover, F^- (F^+) is strictly increasing with respect to \leq_{icv} (\leq_{icx}), cf. Definition A.2, since Φ is nondecreasing and strictly concave (convex), see, for example, [19, Theorem 4.A.49]. The assertions now follow from Lemma A.3, and Remarks A.4 and 3.7(b). \square

Again, in the case, where S is a singleton and $\pi(S) > 0$, we obtain the following proposition.

PROPOSITION 3.9. (a) *The lattice $\mathcal{P}_1(\mathbb{R})$, endowed with the lattice-order \leq_{icv} (\leq_{icx}), is Dedekind super complete.*

(b) *Let $M \subset \mathcal{P}_1(\mathbb{R})$ be nonempty, \leq_{icv} -bounded below and \leq_{icx} -bounded above. If M is directed upward/downward (cf. Definition A.1) with respect to \leq_{icv} (\leq_{icx}), then there exists a nondecreasing/nonincreasing sequence $(\mu^n)_{n \in \mathbb{N}} \subset M$ with $\mu^n \rightarrow \mu$ in the Wasserstein-1 topology as $n \rightarrow \infty$, where $\mu \in \mathcal{P}_1(\mathbb{R})$ is the supremum/infimum of M with respect to \leq_{icv} (\leq_{icx}).*

Proof. By Remark 3.6(b) and (c), $\mathcal{P}_1(\mathbb{R})$ together with \leq_{icv} (\leq_{icx}) is Dedekind σ -complete. The assertions now follow from the same arguments that were employed in the proof of Theorem 3.8. \square

REMARK 3.10. For fixed $m \in \mathbb{R}$, let $\mathcal{P}_{1,m}(\mathbb{R})$ denote the set of all $\mu \in \mathcal{P}_1(\mathbb{R})$ with $\int_{\mathbb{R}} x d\mu(x) = m$. Then, $\nu \leq_{icv} \mu$ if and only if $\mu \leq_{icx} \nu$ if and only if $\mu \leq_{cx} \nu$ for all $\mu, \nu \in \mathcal{P}_{1,m}(\mathbb{R})$. By Theorem 3.8, the lattices $L^0(\mathcal{P}_{1,m}(\mathbb{R}))$, defined in a similar way as $L^0(\mathcal{P}_1(\mathbb{R}))$ and equipped with the order $\leq_{L_{icx}^0} := \leq_{L_{icx}^0}$, as well as $\mathcal{P}_{1,m}(\mathbb{R})$ together with the order relation \leq_{cx} are then Dedekind super complete. Moreover, the supremum/infimum of any bounded set M , which is directed upward/downward can be approximated in the (π -almost everywhere) Wasserstein-1 sense by a nondecreasing/nonincreasing sequence in M .

We now turn our focus on characterizing the complete sublattices of $\mathcal{P}_1(\mathbb{R})$. We start with the following lemma, which is a combination and generalization of [13, Theorem 1] and Lemma 2.4. It complements the characterizations of uniform integrability from Section 2 by two more equivalences.

LEMMA 3.11. *Let $K \subset \mathcal{P}_1(\mathbb{R})$. Then, the following statements are equivalent.*

- (i) *The identity is u.i. for K , that is, $\sup_{\mu \in K} \int_{\mathbb{R}} 1_{[-s,s]^c}(x) |x| d\mu(x) \rightarrow 0$ as $s \rightarrow \infty$.*
- (ii) *There exist $\underline{\mu}, \bar{\mu} \in \mathcal{P}_1(\mathbb{R})$ with $\underline{\mu} \leq_{icv} \mu \leq_{icx} \bar{\mu}$ for all $\mu \in K$.*
- (iii) *Every nonempty subset of K has a least upper bound and a greatest lower bound with respect to \leq_{icv} and \leq_{icx} .*
- (iv) *There exists a nondecreasing function $\psi: [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$, $\frac{\psi(s)}{s} \rightarrow \infty$ as $s \rightarrow \infty$ and*

$$\sup_{\mu \in K} \int_{\mathbb{R}} \psi(|x|) d\mu(x) < \infty.$$

In this case, for $\underline{\mu} = \inf_{\text{icv}} K$ and $\bar{\mu} = \sup_{\text{icx}} K$,

$$\int_{\mathbb{R}} x \, d\underline{\mu}(x) = \inf_{\mu \in K} \int_{\mathbb{R}} x \, d\mu(x) \quad \text{and} \quad \int_{\mathbb{R}} x \, d\bar{\mu}(x) = \sup_{\mu \in K} \int_{\mathbb{R}} x \, d\mu(x). \tag{3.4}$$

Moreover, the function ψ in (iv) can be chosen to be continuously differentiable, strictly convex and u.i. for K .

Proof. The equivalence of (i) and (iv) has been discussed in Section 2. By Proposition 3.9, (ii) and (iii) are equivalent. Moreover, (ii) implies (i) by Remark 3.7. It remains to show that (i) implies (ii). Assume that K is u.i., and define $\underline{\mu}_{1,-}, \bar{\mu}_{1,+} : \mathbb{R} \rightarrow [0, \infty)$ by

$$\underline{\mu}_{1,-}(s) := \inf_{\mu \in K} \mu_{1,-}(s) \quad \text{and} \quad \bar{\mu}_{1,+}(s) := \sup_{\mu \in K} \mu_{1,+}(s)$$

for $s \in \mathbb{R}$. Then, $\underline{\mu}_{1,-}$ is nonincreasing and concave, and $\bar{\mu}_{1,+}$ is nonincreasing and convex. Since the identity is u.i. for K , it follows that

$$\lim_{s \rightarrow \infty} \underline{\mu}_{1,-}(s) + s = \inf_{\mu \in K} \int_{\mathbb{R}} x \, d\mu(x) \quad \text{and} \quad \lim_{s \rightarrow -\infty} \bar{\mu}_{1,+}(s) + s = \sup_{\mu \in K} \int_{\mathbb{R}} x \, d\mu(x).$$

Moreover, $\lim_{s \rightarrow -\infty} \underline{\mu}_{1,-}(s) = 0$ and $\lim_{s \rightarrow \infty} \bar{\mu}_{1,+}(s) = 0$, which shows that $\underline{\mu}_{1,-}$ and $\bar{\mu}_{1,+}$ give rise to two probability measures $\underline{\mu}, \bar{\mu} \in \mathcal{P}_1(\mathbb{R})$ with (3.4). By definition of $\underline{\mu}$ and $\bar{\mu}$, it follows that $\underline{\mu} = \inf_{\text{icv}} K$ and $\bar{\mu} = \sup_{\text{icx}} \bar{K}$. \square

Note that the previous lemma is quite interesting in view of Lemma 3.4. A combination of these two results yields that uniform integrability of the identity implies tightness and thus \leq_{st} -boundedness, which in turn implies \leq_{icv} -boundedness and \leq_{icx} -boundedness and thus uniform integrability. At first glance, this gives the impression that tightness is equivalent to uniform integrability of the identity and thus seems to be a contradiction. However, the \leq_{st} -bounds of tight sets, for which the identity is not u.i., do not lie in $\mathcal{P}_1(\mathbb{R})$, and thus the two lemmas are not contradictory. We are now ready to state the second main result of this subsection.

THEOREM 3.12. *Let K be a sublattice of $\mathcal{P}_1(\mathbb{R})$. Then, the following statements are equivalent:*

- (i) K is compact in the Wasserstein-1 topology;
- (ii) K and $K' := \{\mu' \mid \mu \in K\}$ are complete with respect to the lattice-order \leq_{icv} (\leq_{icx});
- (iii) K is complete with respect to the lattice orders \leq_{icv} and \leq_{icx} .

Proof. If $K = \emptyset$, the statement is trivial. Therefore, we assume that K is nonempty. Since the Wasserstein-1 topology is finer than the interval topologies of \leq_{icv} and \leq_{icx} , it follows that every Wasserstein-1 compact subset of $\mathcal{P}_1(\mathbb{R})$ is compact in both interval topologies and thus complete with respect to \leq_{icv} and \leq_{icx} . On the other hand, if K is compact in the interval topologies of \leq_{icv} (\leq_{icx}), it is closed in the Wasserstein-1 topology. In fact, let $(\mu^n)_{n \in \mathbb{N}} \subset \mathcal{P}_1(\mathbb{R})$ with $\mu^n \rightarrow \mu \in \mathcal{P}_1(\mathbb{R})$ as $n \rightarrow \infty$ with respect to the Wasserstein-1 metric. Then,

$$\mu_{1,-} = \sup_{n \in \mathbb{N}} \inf_{k \geq n} \mu_{1,-}^k \in K \quad \left(\mu_{1,+} = \inf_{n \in \mathbb{N}} \sup_{k \geq n} \mu_{1,+}^k \in K \right),$$

which shows that $\mu \in K$, since K is complete. Here, we used the fact that the Wasserstein-1 convergence implies the pointwise convergence of the integrated distribution (survival) functions. This shows that K is Wasserstein-1 closed. Moreover, K is \leq_{icv} -bounded and \leq_{icx} -bounded as it is complete with respect to both partial orders. By Lemma 3.11 (or Remark 3.7(a)), it follows that K is u.i., and therefore Wasserstein-1 compact, as it is closed in

the Wasserstein-1 topology (cf. [1, Proposition 7.1.5; 21, Theorem 6.9]). We have thus shown that (i) and (iii) are equivalent. The equivalence of (ii) and (iii) follows from (3.3). \square

The following corollary is a direct consequence of Theorem 3.12 and Remark 3.7(c).

COROLLARY 3.13. *Let K be a sublattice of $\mathcal{P}_1(\mathbb{R})$ with $\int_{\mathbb{R}} x d\mu(x) = \int_{\mathbb{R}} x d\nu(x)$ for all $\mu, \nu \in K$. Then, K is complete with respect to \leq_{cx} if and only if K is compact in the Wasserstein-1 topology.*

Appendix A. An auxiliary result

In this section, we prove an auxiliary result that helps to determine when a Dedekind σ -complete lattice is Dedekind super complete. We start with the following definitions:

DEFINITION A.1. Let L be a lattice, that is, a partially ordered set (poset) in which every finite nonempty subset has a least upper bound and a greatest lower bound.

(a) We say that L is Dedekind σ -complete if every countable nonempty subset, that is bounded above or below, has a least upper bound or a greatest lower bound, respectively. We say that L is Dedekind complete if every nonempty subset, that is bounded above or below, has a least upper bound or a greatest lower bound, respectively. We say that L is Dedekind super complete if every nonempty subset, that is bounded above or below, has a countable subset with the same least upper bound or greatest lower bound, respectively. We say that L is complete if every nonempty subset of L has the least upper bound and the greatest lower bound.

(b) We say that a set $M \subset L$ is directed upward or directed downward if, for all $x, y \in M$, there exists some $z \in M$ with $x \vee y \leq z$ or $x \wedge y \geq z$, respectively.

DEFINITION A.2. Let L and R be two posets. We say that a map $F: L \rightarrow R$ is strictly increasing if

- (i) $F(x) \leq F(y)$ for all $x, y \in L$ with $x \leq y$;
- (ii) for all $x, y \in L$ with $x \leq y$ and $F(x) = F(y)$, it follows that $x = y$;

The following lemma gives a sufficient and necessary condition for a Dedekind σ -complete lattice to be Dedekind super complete. The proof is a generalized version of the existence proof of the essential supremum for families of real-valued random variables (see, for example, Föllmer-Schied [8, Theorem A.32]).

LEMMA A.3. *Let L be a Dedekind σ -complete lattice. Then, L is Dedekind super complete if and only if there exists a strictly increasing map $F: L \rightarrow R$ for some Dedekind super complete lattice R .*

Proof. If L is Dedekind super complete, we may choose $R = L$ and F as the identity. In order to prove the converse implication, let $M \subset L$ be a nonempty subset of L , which is bounded above. Then, for every countable nonempty set $\Psi \subset M$, we denote by $x_\Psi := \sup \Psi \in L$. Let

$$M_0 := \{F(x_\Psi) \mid \Psi \subset M \text{ nonempty and countable}\} \quad \text{and} \quad c := \sup M_0 \in R.$$

Note that M_0 is nonempty since M is nonempty, and bounded above since F is nondecreasing (property (i) in Definition A.2) and M is bounded above. Therefore, $F(a) \in R$ is an upper

bound of M_0 for every upper bound $a \in L$ of M . Since R is Dedekind super complete, there exists a sequence $(\Psi^n)_{n \in \mathbb{N}}$ of countable nonempty subsets of M with

$$\sup_{n \in \mathbb{N}} F(x_{\Psi^n}) = c.$$

Then, $\Psi^* := \bigcup_{n \in \mathbb{N}} \Psi^n$ is a countable nonempty subset of M since it is a countable union of countable nonempty subsets of M , and we set $x^* := x_{\Psi^*}$. Note that $x_{\Psi^n} \leq x^*$ for all $n \in \mathbb{N}$, and therefore,

$$F(x_{\Psi^n}) \leq F(x^*) \leq c \quad \text{for all } n \in \mathbb{N},$$

where the last inequality follows from the fact that Ψ^* is a countable nonempty subset of M . Taking the supremum over all $n \in \mathbb{N}$, it follows that $F(x^*) = c$. We now show that $x^* \geq x$ for all $x \in M$. In order to see this, fix some arbitrary $x \in M$, and let $\Psi' := \Psi^* \cup \{x\}$. Then, Ψ' is again a countable nonempty subset of M , and we obtain that

$$c = F(x^*) \leq F(x_{\Psi'}) \leq c.$$

Since $x^* \leq x_{\Psi'}$ and F is strictly increasing (property (ii) in Definition A.2), it follows that $x^* = x_{\Psi'}$, which implies that $x \leq x^*$. We have thus shown that $x^* \in L$ is an upper bound of M . Now, let $a \in L$ be an upper bound of M . Then, a is also an upper bound of $\Psi^* \subset M$, which shows that $a \geq \sup \Psi^* = x^*$. Therefore, x^* is the least upper bound of M and of the countable subset Ψ^* of M . Analogously, one shows that M has a countable subset with the same infimum if M is bounded below. \square

REMARK A.4. Let M be a nonempty subset of a Dedekind super complete lattice L . If M is bounded above or below and directed upward or downward, then there exists a nondecreasing or nonincreasing sequence $(x^n)_{n \in \mathbb{N}} \subset M$ with $\sup M = \sup_{n \in \mathbb{N}} x^n$ or $\inf M = \inf_{n \in \mathbb{N}} x^n$, respectively. In fact, let $(y^n)_{n \in \mathbb{N}} \subset M$ with $\sup M = \sup_{n \in \mathbb{N}} y^n$. Since M is directed upward, there exists a sequence $(x^n)_{n \in \mathbb{N}} \subset M$ with $x^{n+1} \geq x^n \geq y^1 \vee \dots \vee y^n$ for all $n \in \mathbb{N}$. The sequence $(x^n)_{n \in \mathbb{N}}$ can be constructed recursively by defining $x^1 := y^1$, and by choosing $x^{n+1} \in M$ with $x^{n+1} \geq x^n \vee y^{n+1}$ for all $n \in \mathbb{N}$. Since $(x^n)_{n \in \mathbb{N}} \subset M$, it follows that $\sup_{n \in \mathbb{N}} x^n \leq \sup M$. On the other hand, $y^n \leq x^n$ for all $n \in \mathbb{N}$, and consequently,

$$\sup M = \sup_{n \in \mathbb{N}} y^n \leq \sup_{n \in \mathbb{N}} x^n \leq \sup M.$$

The statement for the infimum follows in an analogous way.

Appendix B. Proofs of Lemmas 2.1 and 2.4

Proof of Lemma 2.1. First, assume that there exists a nondecreasing function $\psi: [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$, $\psi(s) \rightarrow \infty$ as $s \rightarrow \infty$ and

$$C := \sup_{\nu \in K} \int_0^\infty \psi(s) \, d\nu(s) < \infty.$$

Then, by Markov's inequality,

$$\sup_{\nu \in K} \nu((s, \infty)) \leq \frac{1}{\psi(s)} \sup_{\nu \in K} \int_0^\infty \psi(u) \, d\nu(u) \leq \frac{C}{\psi(s)} \rightarrow 0 \quad \text{as } s \rightarrow \infty.$$

Now, assume that K is tight, and let $(M^n)_{n \in \mathbb{N}} \subset [0, \infty)$ with $1 < M^n \leq M^{n+1}$ and

$$\sup_{\nu \in K} \nu((M^n, \infty)) \leq 2^{-n}$$

for all $n \in \mathbb{N}$. Define $\psi: [0, \infty) \rightarrow [0, \infty)$ by

$$\psi(s) := \sum_{n \in \mathbb{N}} 1_{(M^n, \infty)}(s) \quad \text{for all } s \geq 0.$$

Then, ψ is nondecreasing with $\psi(0) = 0$ and $\psi(s) \rightarrow \infty$ as $s \rightarrow \infty$. Moreover,

$$\sup_{\nu \in K} \int_0^\infty \psi(s) \, d\nu(s) \leq \sum_{n \in \mathbb{N}} \nu((M^n, \infty)) \leq \sum_{n \in \mathbb{N}} 2^{-n} = 1.$$

Choosing ψ , instead, as the linear interpolation of the points $(0,0)$ and $(M^n, n-1)$ for all $n \in \mathbb{N}$, we obtain that ψ is continuous with $\psi(0) = 0$ and

$$\sup_{\nu \in K} \int_0^\infty \psi(s) \, d\nu(s) \leq 1,$$

since $\psi \leq \sum_{n \in \mathbb{N}} 1_{(M^n, \infty)}$. If (2.1) is satisfied for some $M \geq 0$, then ψ can, for example, be chosen as $\psi(s) := (s - M)^+$ for all $s \geq 0$. On the other hand, if ψ is convex, there exist $\alpha > 0$ and $\beta \geq 0$ such that $\psi(s) \geq \alpha s - \beta$ for all $s \geq 0$. Then, for $M := \frac{\beta}{\alpha}$,

$$\sup_{\nu \in K} \int_0^\infty (s - M)^+ \, d\nu(s) = \frac{1}{\alpha} \int_0^\infty (\alpha s - \beta)^+ \, d\nu(s) \leq \frac{1}{\alpha} \sup_{\nu \in K} \int_0^\infty \psi(s) \, d\nu(s) < \infty.$$

Next, assume that (2.2) is satisfied. Up to now, we saw that the function ψ can be chosen to be strictly increasing on $[M, \infty)$ for $M \geq 0$ sufficiently large. It remains to show that ψ can be chosen to be strictly increasing on $[0, M]$ for all $M > 0$. Let $M > 0$ and

$$\alpha^0 := \sup_{\nu \in K} \nu((M, \infty)) < \infty \quad \text{and} \quad \alpha^n := \sup_{\nu \in K} \nu\left(\left(\frac{M}{n+1}, \frac{M}{n}\right]\right) < \infty \quad \text{for all } n \in \mathbb{N}.$$

Next, we choose a sequence $(c^n)_{n \in \mathbb{N}_0} \subset (0, \infty)$ with $c^{n+1} < c^n$ for all $n \in \mathbb{N}_0$, $c^n \rightarrow 0$ as $n \rightarrow \infty$ and $\sum_{n \in \mathbb{N}_0} c^n \alpha^n < \infty$. Then,

$$\sup_{\nu \in K} \int_0^\infty c^0 1_{(M, \infty)}(s) + \sum_{n \in \mathbb{N}} c^n 1_{\left(\frac{M}{n+1}, \frac{M}{n}\right]}(s) \, d\nu(s) \leq \sum_{n \in \mathbb{N}_0} c^n \alpha^n < \infty.$$

Let ψ_M denote the linear interpolation of the points $(\frac{M}{n+1}, c^n)$ for all $n \in \mathbb{N}_0$ with $\psi_0(0) := 0$ and $\psi_M(s) := c_0$ for all $s > M$. Then, ψ_M is strictly increasing on $[0, M]$. Choosing either $\psi(s) = \psi_M(s) + (s - M)^+$ or $\psi = \psi_{M^1}(s) + \eta(s)$ for $s \geq 0$, where η is the linear interpolation of the points $(M^n, n-1)$, we see that ψ is strictly increasing. On the other hand, if ψ is strictly increasing, then $\psi(s) > 0$ for all $s > 0$, and therefore, by Markov's inequality,

$$\sup_{\nu \in K} \nu((s, \infty)) \leq \frac{1}{\psi(s)} \sup_{\nu \in K} \int_0^\infty \psi(u) \, d\nu(u) < \infty \quad \text{for all } s > 0. \quad \square$$

Proof of Lemma 2.4. Note that $[0, \infty) \rightarrow [0, \infty)$, $s \mapsto s$ is u.i. for K if and only if the set $L = \{\nu_1 \mid \nu \in K\}$ is tight, where, for $\nu \in K$, the measure $\nu_1: \mathcal{B}(\mathbb{R}_+) \rightarrow [0, \infty]$ is given by

$$\nu_1(B) := \int_B u \, d\nu(u) \quad \text{for all } B \in \mathcal{B}(\mathbb{R}_+).$$

By the previous lemma, the set L is tight if and only if there exists a nondecreasing function $\eta: [0, \infty) \rightarrow [0, \infty)$ with $\eta(0) = 0$, $\eta(s) \rightarrow \infty$ as $s \rightarrow \infty$ and

$$\sup_{\sigma \in L} \int_0^\infty \eta(s) \, d\sigma(s) < \infty.$$

Defining $\psi(s) := \eta(s)s$ or $\eta(s) := \frac{\psi(s)}{s}$ for all $s \geq 0$, respectively, the equivalence of (i) and (ii) follows. By Lemma 2.1, we may assume that η is continuous. Let $\alpha \in (0, 1]$ and

$\psi_\alpha(s) := \int_0^s \eta(u)^\alpha du$ for all $s \geq 0$. Since η is continuous and nondecreasing, ψ_α is continuously differentiable, convex and nondecreasing with $\psi_\alpha(0) = 0$. Since $\psi_\alpha(s) \leq \eta(s)s$ for all $s \geq 0$, it follows that

$$\sup_{\nu \in K} \int_0^\infty \psi_\alpha(s) d\nu(s) < \infty.$$

By the transformation theorem, $\frac{\psi_\alpha(s)}{s} = \int_0^1 \eta(su)^\alpha du$ for all $s \geq 0$, which shows that the map $[0, \infty) \rightarrow [0, \infty)$, $s \mapsto \frac{\psi_\alpha(s)}{s}$ is nondecreasing with $\frac{\psi_\alpha(s)}{s} \rightarrow \infty$ as $s \rightarrow \infty$. Moreover, by Hölder's inequality or Jensen's inequality,

$$\frac{\psi_\alpha(s)}{\psi(s)} = \eta(s)^{-1} \frac{\psi_\alpha(s)}{s} = \eta(s)^{-1} \int_0^1 \eta(su)^\alpha du \leq \eta(s)^{\alpha-1}.$$

Hence,

$$\sup_{\nu \in K} \int_s^\infty \psi_\alpha(u) d\nu(u) \leq \eta(s)^{\alpha-1} \sup_{\nu \in K} \int_0^\infty \psi(u) d\nu(u),$$

which shows that ψ_α is u.i. for K if $\alpha \in (0, 1)$. By Lemma 2.1, the function η can, additionally, be chosen to be strictly increasing if and only if (2.2) is satisfied. In this case, ψ_α can be chosen to be strictly convex and thus strictly increasing. On the other hand, if the function ψ in (ii) is strictly convex, it is strictly increasing, and therefore (2.2) has to be satisfied by Lemma 2.1. \square

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