Partial differential equations on fractals
Existence, Uniqueness and Approximation results

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Summary

In this thesis, we investigate partial differential equations involving first order terms on fractal spaces, and our main interest is to provide graph approximations for their solutions.

The first part contains a survey of Dirichlet and resistance forms on certain fractal spaces and we also recall basics on metric graphs. Moreover, we provide basic concepts of the analysis of resistance forms. We close this chapter by presenting some examples of spaces that carry a local regular resistance form in the sense of Kigami.

Existence and uniqueness results are presented in the second part. After a brief discussion of fractal analogs of known existence and uniqueness results for linear elliptic and parabolic partial differential equations of second order, we investigate a nonlinear partial differential equation, namely the viscous Burgers equation. We discuss adequate formulations of the viscous Burgers equation and prove existence, uniqueness and continuous dependence on initial conditions for a vector-valued Burgers equation on metric graphs. We also consider the Burgers equation on compact resistance spaces and again we state existence, uniqueness and continuous dependence on initial conditions. The proofs are minor modifications compared to the metric graph case. Furthermore, we show existence of weak solutions to first order equations of continuity type associated to suitably defined vector fields. Our proof is based on a classical vanishing viscosity argument. Up to this point it is not necessary that the form under consideration admits a carré du champ, so the volume measure can be more general. The last part of this chapter concerns p-energies and Sobolev spaces on metric measure spaces that carry a strongly local regular Dirichlet form having a carré du champ. These Sobolev spaces are then used to generalize some basic results from the calculus of variations, such as the existence of minimizers for convex functionals and certain constrained minimization problems. This applies to a number of non-classical situations such as degenerate diffusions, superpositions of diffusions and diffusions on fractals equipped with a Kusuoka type measure or to products of such fractals.

The third part is the heart of the thesis and deals with approximation results. We start again with linear elliptic and parabolic partial differential equations on resistance spaces which involve gradient and divergence terms. For equations on a single resistance space but with varying coefficients we prove that solutions have accumulation points with respect to the uniform convergence in space, provided that the coefficients remain bounded. If the coefficients converge, we can conclude the uniform convergence of the solutions. We then consider equations on a sequence of resistance spaces approximating a target resistance space from within. Under suitable assumptions extensions of linearizations of solutions along this sequence accumulate or even converge uniformly to the solution on the target space. Examples include graph approximations for finitely ramified spaces and metric graph approximations for post-critically finite self-similar spaces. Next, we consider the viscous Burgers equation on a post-critically finite self-similar fractal associated with a regular harmonic structure. Using Post's concept of generalized norm resolvent convergence on varying Hilbert spaces we prove that solutions to the Burgers equation can be approximated in a certain weak sense by solutions to corresponding equations on approximating metric graphs. Finally, we also show that a sequence of solutions to the viscous continuity equation on graphs approximating a finitely ramified fractal converges along a subsequence to a solution to the continuity equation, provided that certain assumptions on the vector fields are satisfied. The proof relies on a diagonal compactness argument combining vanishing diffusion together with a convergence scheme on varying Hilbert spaces in the sense of Kuwae and Shioya.

Contents

Chapter 1

Introduction

By now linear and semilinear elliptic or parabolic partial differential equations on various fractal spaces without first order terms have successfully been studied for quite some time, see for example [Fal99; FH99] and [Str05b]. Less is known about partial differential equations with first order terms.

In this thesis we investigate partial differential equations which involve gradient and divergence terms. We provide abstract formulations of these equations and show existence and uniqueness results for their solutions. Our main interest is to argue that these abstract formulations have a clear physical meaning. To this end, we establish discrete or metric graph approximations for their solutions which indicates that the abstract formulations arise as limits of well known situations.

To describe phenomena in nature it is sometimes better to assume that the underlying space is rough rather than smooth. Metric measure spaces on which neither Poincaré inequalities nor curvature conditions hold provide models of rough spaces that nevertheless possess a very detailed structure. In the following we call these spaces *fractals*. A prominent and by now well known class of examples is Kigami's class of post-critically finite (pcf) self-similar sets having a regular harmonic structure. The simplest nontrivial example in this class is a fractal called *Sierpiński gasket* (Figure 1.1). It is generated by three mappings in the plane, each a similarity with ratio $\frac{1}{2}$ and such that vertices of a triangle are the fixed points of these mappings.

Analysis on fractals is still a relatively young area of research, for example see the works [Bar98; Kig01; Kig03; Kig12; Kus89; Str06]. Highly readable introductions are

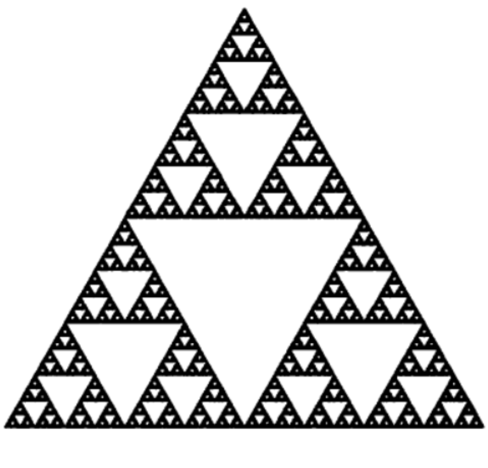

Figure 1.1: Sierpiński gasket,[Kig01, Fig. 0.2 on page 2]

provided in the lecture notes [Bar98] and in the books [Kig01; Str06]. Since fractals like the Sierpiński gasket do not have any smooth structures, to define differential operators like the Laplacian is not possible from the classical viewpoint of analysis. Therefore, the question what is a suitable formulation of the equation is already interesting.

The analysis on fractals is based on energy (Dirichlet) forms and diffusion processes. In the 1980's, Goldstein [Gol87] and Kusuoka [Kus87] proved independently the existence of Brownian motion, and therefore of a Laplacian, on certain fractals. Their proofs rely heavily on the self-similiarity property of the considered fractals. Barlow and Perkins [BP88] followed the probabilistic approach and studied the heat kernel associated with Brownian motion on the Sierpiński gasket.

Using an alternative, more analytic approach, Kigami [Kig89] constructed a Laplacian operator on the Sierpiński gasket as the limit of a sequence of discrete Laplacians on graphs approximating the fractal. Later, Kusuoka and Kigami extended this construction of the Laplacians in the works [Kus89; Kig93a] to the more general class of p.c.f. self-similar fractals. We will follow the analytic approach.

In this thesis, we study equations on a fractal space X that supports a regular, strongly local resistance form $(\mathcal{E},\mathcal{F})$ in the sense of Kigami [Kig12]. Roughly speaking, a non-negative quadratic form $\mathcal{E}$ on a subspace $\mathcal{F}$ of continuous functions is called a *resistance form* if every real valued function on a finite subset $V \subset X$ can be extended to a function $u \in \mathcal{F}$, $(\mathcal{E},\mathcal{F})$ satisfies the Markov property and

$$R(x,y) := \sup\Big\{\frac{(u(x)-u(y))^2}{\mathcal{E}(u,u)} : u \in \mathcal{F}, \mathcal{E}(u,u) > 0\Big\} < \infty$$

exists for any $x, y \in X$.

While linear second order equations without first order terms on fractals are easily tractable whenever we can understand a natural Laplace operator, equations involving first order terms are more difficult and have been studied only recently.

In [CS03] and later in [CS09] and [IRT12], a Hilbert space $\mathcal{H}$ of 1-forms and a related analog ∂ of the exterior derivation (in the L^2-sense) had been introduced by means of tensor products and energy norms. In classical smooth cases this Hilbert space agrees with the Hilbert space of L^2-differential 1-forms. Based on the notion of 1-forms proposed in [CS03; CS09; IRT12], notions of vector fields, gradient and divergence operators are studied in [HRT13]. Moreover, the authors showed that their developed tools can be applied to quasilinear elliptic partial differential equations, in divergence and non-divergence form.

The first order calculus for Dirichlet forms has been studied further in [Hin15; HKT15; HR16; HT15c; HT13]. We would like to emphasize that the implementation of such equations is nontrivial, because the Dirichlet forms involved are not immediately given as integrals involving gradient operators. In fact, the definition of an associated gradient operator is a nontrivial subsequent step. For energy forms on fractals with sufficiently simple structure explicit constructions of gradients had been provided in [Kus89; Kus93], [Kig93b], [Str00] and [Tep00]. In these cases the abstract gradient studied in [CS03; CS09; HRT13; IRT12] extends these constructions.

Mainly following [CS03; CS09; HRT13], we construct on such a space a Hilbert space $\mathcal{H}$ of 1-forms and a derivation operator $\partial : \mathcal{F} \cap C_c(X) \to \mathcal{H}$ that plays the role of a gradient. One can also show that this operator satisfies the identity $\|\partial u\|_{\mathcal{H}}^2 = \mathcal{E}(u,u)$ for any $u \in \mathcal{F} \cap C_c(X)$ and the Leibniz rule. The adjoint operator ∂^* of ∂ plays the role of the divergence. We will use such a derivation operator as our main tool to formulate partial differential equations on fractals.

The content of the present thesis essentially coincides with that of [HM20b; HM20a; HMS20] and with that of a preprint version of [HKM20]. However, the exposition here is more detailed.

1.1 Main results of this thesis

Now we formulate our main results. They are presented in the second and third part of this thesis.

Existence and uniqueness results

We discuss the main results stated in Part II of this thesis.

We start with investigating linear elliptic and parabolic partial differential equations on a separable and locally compact resistance space (X, R) equipped with a regular resistance form $(\mathcal{E}, \mathcal{F})$. Here, we focus on equations involving first order terms $u \mapsto b \cdot \nabla u$ and $u \mapsto \operatorname{div}(u\hat{b})$, where $b \in \mathcal{H}$ and $\hat{b} \in \mathcal{H}$ denote abstract vector fields. In our context these expressions generalize to $u \mapsto b \cdot \partial u$ and $u \mapsto \partial^*(u\hat{b})$, respectively.

Let μ be a finite, positive Borel measure on (X, R). Using [Kig12, Theorem 9.4] one can show that the resistance form $(\mathcal{E}, \mathcal{F} \cap C_c(X))$ induces a regular Dirichlet form on $L^2(X, \mu)$. Suppose that $a : \mathcal{H} \to \mathcal{H}$ is a linear symmetric and bounded operator and c is a bounded function on X. Using the first order calculus for Dirichlet forms, it is not difficult to construct a bilinear form $(\mathcal{Q}, \mathcal{F} \cap C_c(X))$ which involve these coefficients, gradient and divergence terms,

$$\mathcal{Q}(u, v) = \langle a \cdot \partial u, \partial v \rangle_{\mathcal{H}} - \langle v \cdot b, \partial u \rangle_{\mathcal{H}} - \langle u \cdot \hat{b}, \partial v \rangle_{\mathcal{H}} - \langle cu, v \rangle_{L^2(X,\mu)}, \quad u, v \in \mathcal{F} \cap C_c(X). \tag{1.1}$$

Under the assumptions that the coefficient a uniformly elliptic, $c \in L^\infty(X, \mu)$ and that the vector fields b and $\hat{b}$ are 'Hardy' (cf. Section 6.1), we extend the form $(\mathcal{Q}, \mathcal{F} \cap C_c(X))$ to a coercive closed bilinear form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$. Given such a form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ with associated infinitesimal generator $(\mathcal{L}^{\mathcal{Q}}, \mathcal{D}(\mathcal{L}^{\mathcal{Q}}))$, we follow the standard theory for partial differential equations, [GT01, Chapter 8], and Dirichlet forms, [FOT94], and establish existence and uniqueness of weak solutions to elliptic equations of type

$$\mathcal{L}^{\mathcal{Q}} u = f \tag{1.2}$$

for given $f \in L^2(X, \mu)$ and of semigroup solutions to parabolic equations of type

$$\partial_t u(t) = \mathcal{L}^{\mathcal{Q}} u(t), \; t > 0, \quad u(0) = \mathring{u} \tag{1.3}$$

for given $\mathring{u} \in L^2(X, \mu)$.

As a prototype of a nonlinear partial differential equation, we investigate a formulation of the viscous Burgers equation on compact resistance spaces.

Let us put the Burgers equation in a physical context and refer to some selected results in the literature. The viscous Burgers equation, [Bur40; Bur48], is one of the simplest nonlinear partial differential equations. On the real line it reads

$$\partial_t u = \sigma u_{xx} - u_x u, \tag{1.4}$$

see for instance [Eva10; Olv93; Olv14]. The nonlinear term $u_x u = \frac{1}{2}(u^2)_x$ models a convection effect and the viscosity parameter $\sigma > 0$ determines the strength of a competing diffusion. One formulation of (1.4) on higher dimensional Euclidean domains or on manifolds is

$$\partial_t u = \sigma \Delta u - \langle u, \nabla \rangle u, \tag{1.5}$$

it may be seen as a simplification of the incompressible Navier-Stokes equation. Here we denote by Δ the Laplacian acting on vector fields. Depending on the context, also a different formulation of the Burgers equation is studied, then with $\frac{1}{2}\nabla \langle u, u \rangle$ in place of $\langle u, \nabla \rangle u$. However, for gradient field solutions $u = \nabla h$ the terms agree. Equation (1.5) can be solved using the Cole-Hopf transform, [Col51; Flo48; Hop50]: If w is a positive solution to the heat equation $w_t = \sigma\Delta w$, now with the Laplacian Δ acting on scalar valued functions, then the gradient field $u := -2\sigma\nabla \log w$ solves (1.5). See also [Bir03]. This transform is one example of an entire hierarchy of transforms, [KS09; Tas76], and naturally related to integrable systems, [Olv93].

The literature on Burgers equation is extensive. For example, let us mention the paper [KNS08], where the authors study a variant of (1.4) with the Laplacian replaced by a fractional Laplacian. They show finite time blow up of solutions if the fractional power is smaller than 1/2 and global existence (and analyticity) if it is greater than or equal to 1/2.

In [LQ19], a version of (1.4) had been implemented as a semilinear heat equation associated with the Laplacian for scalar functions on the Sierpiński gasket, endowed with the natural self-similar Hausdorff measure and this model is naturally related to control theory and (backward) stochastic differential equations. However, it cannot be solved using the Cole-Hopf transform.

We investigate the Burgers equation as an equation for vector fields and we implement this vector equation using first order calculus, see [CS03; HRT13; IRT12]. On a p.c.f. fractal, the Burgers equation can be formulated as the problem

$$\begin{cases} \partial_t u(t) &= \vec{\mathcal{L}} u(t) - \frac{1}{2}\partial \langle u, u\rangle (t), \quad t > 0, \\ u(0) &= u_0. \end{cases} \tag{1.6}$$

Here $\vec{\mathcal{L}}$ corresponds to the Laplacian acting on vector fields and $\frac{1}{2}\partial \langle u, u\rangle$ can be seen as an abstract version of the convection term $\langle u, \nabla\rangle u$.

One main result in this thesis is the existence and uniqueness of solutions to the viscous Burgers equation for initial conditions that are gradients of energy finite functions. We also show the continuous dependence of the solution on the initial conditions. Our main tool is the Cole-Hopf transform, which also dictates the way we phrase the equation.

Theorem (c.f. Theorem 7.4). *Assume that (X, R) is connected and that μ is such that the semigroup $(e^{t\mathcal{L}})_{t>0}$ is conservative. If we have $u_0 = \partial h_0$ with $h_0 \in \mathcal{D}(\mathcal{E})$ bounded and $w(t)$ denotes the unique solution $e^{t\mathcal{L}} w_0$ to the heat equation (7.4) with initial condition $w_0 := e^{-h_0/2}$, then the function*

$$u(t) := -2\partial \log w(t), \qquad t \geq 0,$$

is the unique solution to (1.6).

We also verify existence and uniqueness of solutions on compact metric graphs, as well as continuous dependence on initial conditions. In the metric graph case the operators involved and their domains admit fairly explicit expressions.

As a prototype example of a first order partial differential equation we investigate the continuity equation on compact resistance spaces.

The continuity equation is a well-known equation with many applications in physics. For example in fluid dynamics, the continuity equation

$$\partial_t u + \operatorname{div}(ub) = 0 \tag{1.7}$$

expresses the condition of mass conservation in the absence of sources or sinks of mass within the fluid, see [Ped87]. In other words, it states that the local increase of density with time must be balanced by a divergence of the mass flux ub.

Under suitable assumptions on the vector field b and its divergence, we establish existence of weak solutions to the continuity equation using the concept of vanishing viscosity.

Following a classical approach already used in [AT14], we approximate the original equation (1.7) by adding a diffusion term $\sigma\Delta u$, $\sigma > 0$. More precisely, in the first step we solve

$$\partial_t u + \operatorname{div}(ub) = \sigma\Delta u \tag{1.8}$$

in the weak sense of duality with some adequate test functions. Then we use Hilbert space techniques to show existence of more regular solutions u_σ to (1.8). After deriving *a priori* estimates we show in the final step that the sequence of solutions u_σ to the modified equation (1.8) converges weakly to a solution u to the first order equation (1.7) if σ tends to 0. We obtain the following result.

Theorem (cf. Theorem 8.3)**.** *Let $b \in L^2(0,T;\mathcal{H})$ be absolutely continuous w.r.t. μ and $\partial^* b \in L^1(0,T;L^\infty(X,\mu))$. Then there exists a weak solution $u \in L^\infty(0,T;L^2(X,\mu))$ to (8.1). Also if $u_0 \geq 0$ then $u \geq 0$.*

Note that the operator ∂^* plays the role of the divergence.

Ambrosio and Trevisan [AT14] already discussed existence and uniqueness of solutions to the continuity equation on quite general metric measure spaces, but their approach is based on the so called carré du champ operator, an operator characterizing the energy density, which many fractals just do not support (unless it is understood in some distributive sense).

At the current state it is difficult to achieve uniqueness statements for solutions to the continuity equation in our setup. Common arguments based on continuity of vector fields as used for example in [BDRS15] do not apply. Basically, the reason is that in our case the 'tangent spaces' can only vary measurably, see Section 4.4 for more details. We hope to find an adaption of other methods used for example for the continuity equation with a nearly incompressible vector field in one dimension, [Gus19].

Approximation results

The main subject we treat in this thesis is the study of approximation results for partial differential equations on fractals.

Energy forms on post-critically finite self-similar sets equipped with a regular harmonic structure can be written as the limit of energy forms on a sequence of discrete graphs approximating the set as proved in [Kig03, Proposition 2.10 and Theorem 2.14]. For metric graph approximations we refer to [Tep08]. To achieve a better understanding of analogs of second order partial differential equations, but also of first order partial differential equations, we investigate whether solutions, in particular on p.c.f. self-similar fractals or on finitely ramified fractals, can be approximated by solutions on the approximating metric or discrete graphs. If so, this might be regarded as a piece of evidence that our proposed formulations of the considered equations are physically meaningful. Moreover, such approximations could serve as a basis for numerical simulations.

We comment now on our main results stated in Part III.

First, we consider linear elliptic and parabolic partial differential equations which involve gradient and divergence terms on a compact resistance space (X,R) such that there exists a sequence of compact resistance spaces $\big((X^{(m)},R^{(m)})\big)_m$ approximating (X,R) from within. Suppose that $(\mathcal{Q},\mathcal{F})$ is a non-symmetric coercive closed form of type (1.1) on the space $L^2(X,\mu)$. The following question arises.

Question: Given certain conditions on vector fields $b,\hat{b} \in \mathcal{H}$ and on coefficients a,c, can we verify the convergence of a sequence $(\mathcal{Q}^{(m)})_m$ of non-symmetric closed forms $\mathcal{Q}^{(m)}$ of a similar type as $\mathcal{Q}$ but defined on the approximating spaces $L^2(X^{(m)},\mu^{(m)})$, respectively, to the form $\mathcal{Q}$ on the target space $L^2(X,\mu)$?

To answer this question, we have to deal with a concept of convergence along a sequence of different Hilbert spaces and it turns out that the *KS-generalized Mosco convergence* for non-symmetric Dirichlet forms based on the works [Hin98; KS03; Töl10] is suitable for our propose.

Hino stated in [Hin98] abstract conditions on generalized (non-symmetric) forms for the strong convergence of the associated resolvents.

In [KS03, Subsections 2.2 - 2.7] Kuwae and Shioya introduced a concept of convergence $H_m \to H$ of Hilbert spaces H_m to a Hilbert space H, including a suitable notion of generalized strong resolvent convergence for self-adjoint operators, cf. [KS03, Definition 2.1]. Their concept is a generalization of the famous Mosco convergence, a variational convergence of symmetric quadratic forms introduced by Mosco [Mos94]. A basic tool for their definitions is a family of identification operators Φ_m defined on a dense subspace $\mathcal{C}$ of the limit space H, each mapping $\mathcal{C}$ into one of the spaces H_m.

In the works [Töl06; Töl10], Tölle examines convergence problems of non-symmetric forms defined on different Hilbert spaces. He generalized Hino's conditions in the Kuwae-Shioya framework to provide necessary and sufficient conditions for the convergence of the associated resolvents and semigroups. We will use his definition of generalized convergence of forms to define KS-generalized Mosco convergence. This concept will entail a suitable convergence of solutions to equations of elliptic type (1.2) and of parabolic type (1.3).

It is not straightforward to provide a correct definition for the restriction of a general vector field b on $\mathcal{H}$ to the approximating space $X^{(m)}$. Therefore, we proceed in the following way: in a first step, we construct a sequence of bilinear closed forms $\left(\mathcal{Q}^{(n)}, \mathcal{F}\right)$ converging in the KS-generalized Mosco sense to $(\mathcal{Q}, \mathcal{F})$. Here we consider convergence of forms on a single compact resistance space (X, R). Since the piecewise harmonic functions are dense in $\mathcal{F}$, we can find sequences $(a_n)_n$, $(b_n)_n$, $(\hat{b}_n)_n$ such that

- $(a_n)_n$ is a sequence of piecewise harmonic functions converging strongly to a,
- $(b_n)_n$, $(\hat{b}_n)_n$ converge to b and $\hat{b}$, respectively, and for each $n \in \mathbb{N}$, b_n and $\hat{b}_n$ are finite sums of the form

  $$b_n = \sum_i g_{n_i} \partial f_{n_i} \qquad \text{and} \qquad \hat{b}_n = \sum_i \hat{g}_{n_i} \partial \hat{f}_{n_i},$$

  where $f_{n_i}, \hat{f}_{n_i}, g_{n_i}, \hat{g}_{n_i}$ are piecewise harmonic functions.

In a second step, we construct a sequence of bilinear closed forms $\left(\mathcal{Q}^{(n,m)}, \mathcal{F}^{(m)}\right)$ converging in the KS-generalized Mosco sense to $(\mathcal{Q}^{(n)}, \mathcal{F})$. Here we consider convergence of forms on varying compact resistance spaces $(X^{(m)}, R^{(m)})$. We use that, under certain assumptions, pointwise restrictions to the approximating space $X^{(m)}$ of piecewise harmonic functions as well as of gradients of piecewise harmonic functions are well defined. In particular, we can define $b_n^{(m)} := b_n|_{X^{(m)}}$ and $\hat{b}_n^{(m)} := \hat{b}_n|_{X^{(m)}}$.

As our answer to the above question we obtain the following uniform approximation result for equations on the target space X, provided that the sequences $\left(b_n^{(m)}\right)_{(n,m)}$, $\left(\hat{b}_n^{(m)}\right)_{(n,m)}$ are bounded. It shows that under suitable assumptions extensions of linearizations converge uniformly to the solution on the target space. To construct these extensions we use harmonic extension operators E_{m_k} and projection operators $H_{m_k}^{m_k}$ that restrict m_k-harmonic functions to the approximating space $X^{(m_k)}$. The operator Φ_m restricts a function $f \in L^2(X, \mu)$ to the space $L^2(X^{(m)}, \mu^{(m)})$.

Theorem (c.f. Theorem 10.4). *Let $a \in \mathcal{F}$ be uniformly elliptic with constants $0 < \lambda < \Lambda$. Let $b, \hat{b} \in \mathcal{H}$ and let $c \in C(X)$. We can find $a_n^{(m)} \in \mathcal{F}^{(m)}$ and $b_n^{(m)}, \hat{b}_n^{(m)} \in \mathcal{H}^{(m)}$ such that* *for any n and m the forms*

$$\mathcal{Q}^{(n,m)}(f,g) := \langle a_n|_{X^{(m)}} \cdot \partial f, \partial g\rangle_{\mathcal{H}^{(m)}} - \left\langle g \cdot b_n^{(m)}, \partial f\right\rangle_{\mathcal{H}^{(m)}} \\ - \left\langle f \cdot \hat{b}_n^{(m)}, \partial g\right\rangle_{\mathcal{H}^{(m)}} - \langle c|_{X^{(m)}} f, g\rangle_{L^2(X^{(m)},\mu^{(m)})}, \quad f,g \in \mathcal{F}^{(m)}$$

are closed in $L^2(X^{(m)}, \mu^{(m)})$, respectively. Writing $(\mathcal{L}^{\mathcal{Q}^{(n,m)}}, \mathcal{D}(\mathcal{L}^{\mathcal{Q}^{(n,m)}}))$ for the generator of $(\mathcal{Q}^{(n,m)}, \mathcal{D}(\mathcal{Q}^{(n,m)}))$, one can observe the following.

(i) Let $f \in L^2(X,\mu)$, u be the unique weak solution to (6.12) on X and $u_n^{(m)}$ be the unique weak solution to (6.12) on $X^{(m)}$ with $\mathcal{L}^{\mathcal{Q}^{(n,m)}}$ and $\Phi_m(f)$ in place of $\mathcal{L}^{\mathcal{Q}}$ and f, respectively. Then there are sequences $(m_k)_k$ and $(n_l)_l$ with $m_k \uparrow +\infty$ and $n_l \uparrow +\infty$ so that

$$\lim_{l\to\infty} \overline{\lim_{k\to\infty}} \left\| E_{m_k} H_{m_k}^{(m_k)} u_{n_l}^{(m_k)} - u\right\|_{\sup} = 0.$$

(ii) Let $\mathring{u} \in L^2(X,\mu)$, u be the unique solution to (6.17) on X and $u_n^{(m)}$ be the unique weak solution to (6.17) on $X^{(m)}$ with $\mathcal{L}^{\mathcal{Q}^{(n,m)}}$ and $\Phi_m(\mathring{u})$ in place of $\mathcal{L}^{\mathcal{Q}}$ and $\mathring{u}$, respectively. Then there are sequences $(m_k)_k$ and $(n_l)_l$ with $m_k \uparrow +\infty$ and $n_l \uparrow +\infty$ so that for any $t > 0$

$$\lim_{l\to\infty} \overline{\lim_{k\to\infty}} \left\| E_{m_k} H_{m_k}^{(m_k)} u_{n_l}^{(m_k)}(t) - u(t)\right\|_{\sup} = 0.$$

Convergence of first order terms associated with strongly local Dirichlet forms and KS-generalized Mosco convergence of forms of type (1.1) have already been discussed in [AST17] and [Suz18] in connection with convergent metric measure spaces, see for instance [AH17; AGS17]. Studies of the first order (and higher order) calculus associated with such Dirichlet forms can be found in [Gig15; Gig18]. The main tool in these papers are Dirichlet forms based on the Cheeger gradient, [Che99]. As a consequence, these Dirichlet forms admit a carré du champ, [BH91], a fact quite fundamental to the methods used there.

We are most interested in equations on fractal spaces, and it is well known that natural strongly local Dirichlet forms on well understood classes of self-similar fractals never admit a carré du champ with respect to the natural self-similar Hausdorff measure, [BST99], [Hin05], [Hin08], [Hin10], so that the methods of the articles mentioned above do not apply.

We continue with the viscous Burgers equation on a self-similar p.c.f. fractal. Again, we use the Cole-Hopf transform and first verify a corresponding statement for solutions of heat equations, in other words, a generalized strong resolvent convergence for the Laplacians for scalar functions on varying L^2-space.

As already mentioned, a suitable concept for convergence on varying Hilbert spaces has been established in [KS03], see for instance [Hin09] for an application to fractals. However, in practice it seems difficult to verify the characterization of such a convergence in terms of Dirichlet forms. It is much easier to verify sufficient conditions for generalized norm resolvent convergence of self-adjoint operators as considered in [Pos12; PS18a; PS18b]. This can be done in a quite straightforward manner if one uses the concept of δ-quasi unitary equivalence introduced in [Pos12, Chapter 4, in particular, Definition 4.4.11, Proposition 4.4.15 and Theorem 4.2.10]. A related concept for sectorial operators was provided in [MNP13]. Mimicking the proof of [PS18a, Theorem 1.1] (where a similar approximation along a sequence of discrete graphs was shown), we verify the norm resolvent convergence of the Laplacians. As a consequence we obtain the convergence of solutions of the heat equations in L^2 in the strong sense and in the Dirichlet form domain in the weak sense. From these convergence results we can deduce the convergence of solutions to the Burgers equation on approximating metric graphs to the solution to the Burgers equation on a connected p.c.f. self-similar structure in a suitable weak sense.

More precisely, we linearize $u_m(t)$ along the edges E_m of metric graphs Γ_m by using the restriction operator H_{Γ_m} and we extend this linearization harmonically by using the extension operator $\mathfrak{E}_m$. To formulate these operators we rely on approximations by piecewise harmonic respectively edge-wise linear functions. Then we compare the resulting function to $u(t)$. Doing so, we discard information, but since we rely on approximation by piecewise harmonic functions anyway, it is natural to proceed in this way.

The next theorem is another main result in this part of the thesis. The identification operator $J^*_{0,m}$ restricts a function $u \in L^2(K, \mu)$ to the space $L^2(X_{\Gamma_m}, \mu_{\Gamma_m})$.

Theorem (c.f. Theorem 11.2)**.** *Assume $u_0 = \partial h_0$ with $h_0 \in \mathcal{D}(\mathcal{E})$. Let $u(t)$ denote the unique solution to (7.12) with initial condition u_0 and for any $m \geq 1$ let $u_m(t)$ denote the unique solution to (7.8) with initial condition $-2d \log J^*_{0,m} e^{-h_0/2}$. Then we have*

$$\lim_{m\to\infty} \langle \mathfrak{E}_m \circ H_{\Gamma_m}(u_m(t)) - u(t), v\rangle_{\mathcal{H}} = 0$$

for any $t \geq 0$ and $v \in \mathcal{H}$.

Let us now turn to an approximation result for the continuity equation. The idea is that we can combine the concept of vanishing viscosity with the convergence scheme for varying Hilbert spaces in the framework of Kuwae and Shioya [KS03]. Under the assumption that the considered vector fields are time-independent and piecewise harmonic 1-forms, i.e. elements of the space $\mathcal{P}^{\perp}\mathcal{H}_k$, see [IRT12] for a definition, we generalize results on *a priori* estimates shown in a previous chapter. For this special class of vector fields we know how to restrict pointwise to $X^{(m)}$. Let $u_n^{(m)}(t)$ be a weak solution to

$$\begin{cases} \partial_t u_n^{(m)}(t) &= -\sigma_n \mathcal{L}^{(m)} u_n^{(m)}(t) + (\partial^*)^{(m)} \left(u_n^{(m)}(t) \cdot b^{(m)} \right), \qquad t > 0, \\ u_n^{(m)}(0) &= u_0^{(m)}. \end{cases} \tag{1.9}$$

Here, (1.9) is the abstract formulation of the Cauchy problem for the viscous continuity equation on the discrete graph $X^{(m)}$. We prove that a sequence of solutions $\left(u_n^{(m)}(t)\right)_{(n,m)}$ converge along a subsequence to a weak solution $u(t)$ to

$$\begin{cases} \partial_t u(t) &= \partial^* \left(u(t) \cdot b \right), \qquad t > 0, \\ u(0) &= u_0, \end{cases} \tag{1.10}$$

the abstract Cauchy problem for the continuity equation on a fractal X, see also the following.

Theorem (c.f. Theorem 12.1)**.** *Let $u_0 \in L^2(X, \mu)$ and let $b \in \mathcal{P}^{\perp}\mathcal{H}_m$ be absolutely continuous w.r.t. μ. For each $m \geq 1$ let $b^{(m)}$ be the pointwise restriction of b to V_m as in* (12.6)*. Moreover, for any $m \geq 1$ let $u_m^{(m)}(t)$ denote the weak solution to* (1.9) *with $\sigma_m = \frac{1}{m}$ and initial condition $\Phi_m u_0$ and let $u(t)$ be the weak solution to* (1.10) *with initial condition u_0. Then there exists a sequence $(m_k)_k$ with $m_k \uparrow \infty$ such that the subsequence $\left(u_{m_k}^{(m_k)}(t)\right)_k$ converges weakly to $u(t)$.*

1.2 Outline

This thesis is organized as follows.

Part I contains three chapters in which we collect all preliminary results and notions that are needed thereafter. Chapter 2 starts with a quick account of Dirichlet forms. In

Chapter 3 we consider resistance forms in the sense of Kigami. In Chapter 4, we develop a first order calculus for resistance forms. Moreover, we present resistance spaces on which we are working on, in Chapter 5.

Part II is devoted to existence and uniqueness results of solutions and is divided into four chapters. Chapter 6 briefly summarizes fractal analogs of standard estimates to obtain existence and uniqueness of solutions to linear elliptic and parabolic partial differential equations on resistance spaces which involve gradient and divergence terms.
In Chapter 7 we study a formulation of the viscous Burgers equation on spaces carrying a local regular resistance form in the sense of Kigami. Here we focus on a formulation which follows from the Cole-Hopf transform and is associated with the Laplacian for vector fields. We show existence and uniqueness of solutions to the Burgers equation and verify the continuous dependence on the initial condition.
Chapter 8 provides an existence result for solutions to the continuity equation on compact resistance spaces.
Chapter 9 contains a review of p-energies and Sobolev spaces on metric measure spaces that carry a strongly local regular Dirichlet form. These Sobolev spaces are then used to generalize some basic results from the calculus of variations. For convenience of the reader, the technical proof of uniform convexity of L^p-spaces is shifted to the appendix of this part. The results of this chapter are based on a preprint version of the published article [HKM20].

In Part III we study two concepts of convergence. We are interested in how one can provide graph approximations on finitely ramified or p.c.f. self-similar spaces for solutions to partial differential equations. This part consists of three chapters. In Chapter 10, we analyze equations on a single resistance space but with varying coefficients and, provided that the coefficients remain bounded, we prove that solutions have accumulation points with respect to the uniform convergence in space. If the coefficients converge, we can conclude the uniform convergence of the solutions. We then consider equations on a sequence of resistance spaces approximating a target resistance space from within. Under suitable assumptions extensions of linearizations of solutions along this sequence accumulate or even converge uniformly to the solution on the target space. Examples include graph approximations for finitely ramified spaces and metric graph approximations for p.c.f. self-similar spaces. We will make results of this chapter and Chapter 6 publicly available in the upcoming article [HM20a].
In Chapter 11, we prove for resistance forms associated with regular harmonic structures on p.c.f. self-similar sets that solutions to the viscous Burgers equation can be approximated in a weak sense by solutions to corresponding equations on approximating metric graphs. Here we use the concept of generalized norm resolvent convergence of self-adjoint operators on varying Hilbert spaces developed by Post. The results of Chapters 7 and 11 are from [HM20b].
Finally in Chapter 12, we provide graph approximations for continuity equations on fractals using the concept of vanishing diffusion and the convergence scheme developed by Kuwae and Shioya. We show that a solution u to the continuity equation can be approximated in a suitable weak sense by a sequence of solutions to the viscous continuity equation on graphs approximating the fractal. Together with the results in Chapter 8 this will be made publicly available in the subsequent article [HMS20].
In the appendix of this part, we provide an auxiliary observation regarding these two concepts of convergence on varying spaces for the interested reader. Further, for the sake of completeness we give a proof of the generalized norm resolvent convergence.

To make the thesis self-contained, we collect some useful results from functional analysis in the global Appendix A.

Part I

Tools and preliminaries

This part contains some notions, results and ways of notation that are used in the main Parts II and III.

Basic notation

Before we start with the actual content of this thesis, we fix some notation that is used throughout this work.

For a topological space X, we denote $C_b(X)$ as the space of all bounded continuous functions on X and $C_c(X)$ as the space of all continuous functions on X with compact support. For quantities $(f, g) \mapsto \mathcal{Q}(f, g)$ depending on two arguments f, g in a symmetric way we use the notation $\mathcal{Q}(f) := \mathcal{Q}(f, f)$. As usual for $f, g : X \to \mathbb{R}$ we set $f \vee g := \sup(f, g)$ and $f \wedge g := \inf(f, g)$.

Chapter 2

Dirichlet forms

Let us summarize the main definitions and properties of Dirichlet forms we use in this thesis. For a more detailed introduction to Dirichlet forms we refer to the book [FOT94] by Fukushima, Oshima and Takeda and the book [MR92] by Ma and Röckner.

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\, ,\, \rangle_{\mathcal{H}}$ and norm $\|\,\|_{\mathcal{H}} := \sqrt{\langle\, ,\, \rangle_{\mathcal{H}}}$.

Definition 2.1. *A pair* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is called a* symmetric closed form *on* $\mathcal{H}$ *if*

- $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ *is a nonnegative definite symmetric bilinear form on a dense linear subspace* $\mathcal{D}(\mathcal{E})$ *of* $\mathcal{H}$*,*
- $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is* closed*, i.e. such that* $\mathcal{D}(\mathcal{E})$ *with the scalar product*

$$\mathcal{E}_1(f,g) := \mathcal{E}(f,g) + \langle f,g\rangle_{\mathcal{H}}, \qquad f,g \in \mathcal{D}(\mathcal{E})$$

  is a Hilbert space.

If in addition $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *satisfies the* Markov property*, which says that*

$$f \in \mathcal{D}(\mathcal{E}) \text{ implies that } g = (0 \vee f) \wedge 1 \in \mathcal{D}(\mathcal{E}) \text{ and } \mathcal{E}(g) \leq \mathcal{E}(f),$$

$(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is called a* Dirichlet form *on* H*.*

For $\alpha \geq 0$ we set

$$\mathcal{E}_\alpha(f,g) = \mathcal{E}(f,g) + \alpha\langle f,g\rangle_{\mathcal{H}} \tag{2.1}$$

for all $f,g \in \mathcal{D}(\mathcal{E})$. Consider the concrete Hilbert space $L^2(X,\mu)$, where X is a locally compact separable metric space and μ is a positive Radon measure such that $\mu(U) > 0$ for any nonempty open set $U \subset X$ and let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form on $L^2(X,\mu)$.

A subset of $\mathcal{C} := C_c(X) \cap \mathcal{D}(\mathcal{E})$ is called a *core* of the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X,\mu)$ if it is both uniformly dense in the space of compactly supported continuous functions $C_c(X)$ and $\mathcal{E}_1$-dense in $\mathcal{D}(\mathcal{E})$. A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X,\mu)$ is called *regular* if it possesses a *core*.

Further, a Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is called *strongly local* if for all $f,g \in \mathcal{D}(\mathcal{E})$ such that $\operatorname{supp} f$ and $\operatorname{supp} g$ are compact and g is constant on a neighbourhood of $\operatorname{supp} f$ it follows that $\mathcal{E}(f,g) = 0$, [FOT94, Section 3.2].

Now let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strongly local regular Dirichlet form on $L^2(X,\mu)$. By the Markov property it holds that

$$\mathcal{E}(f,g)^{\frac{1}{2}} \leq \|f\|_{\sup}\, \mathcal{E}(g)^{\frac{1}{2}} + \|g\|_{\sup}\, \mathcal{E}(f)^{\frac{1}{2}}, \qquad f,g \in \mathcal{C}, \tag{2.2}$$

see [BH91, Cor.I.3.3.2], the space $\mathcal{C}$ is an algebra of bounded functions. Similarly as in [BH91] we say that a regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *admits a carré du champ* if for any $f, g \in \mathcal{C}$ there exists a function $\Gamma(f,g) \in L^1(X,\mu)$ such that for any $h \in \mathcal{C}$ we have

$$\frac{1}{2}\{\mathcal{E}(fh,g) + \mathcal{E}(gh,f) - \mathcal{E}(fg,h)\} = \int_X h\Gamma(f,g)\mu(dx). \tag{2.3}$$

This is the same as to say that *the Dirichlet form admits energy densities with respect to* μ or to say that *the measure* μ *is energy dominant for* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, [Hin10; Hin13a].

Chapter 3

Resistance forms

In much of what follows we will consider resistance forms in the sense of Kigami [Kig01, Definition 2.3.1], see also [Kig03, Definition 2.8]. Kigami developed the theory of resistance forms to study analysis on 'low-dimensional' fractals including for example the Sierpiński gasket or the two dimensional Sierpiński carpet.

By $\ell(X)$ we denote the space of real valued functions on a set X.

Definition 3.1. *A resistance form $(\mathcal{E}, \mathcal{F})$ on a set X is a pair such that*

(i) $\mathcal{F} \subset \ell(X)$ is a linear subspace of $\ell(X)$ containing the constants and $\mathcal{E}$ is a non-negative definite symmetric bilinear form on $\mathcal{F}$ with $\mathcal{E}(u) = 0$ if and only if u is constant.

(ii) Let $\sim$ be the equivalence relation on $\mathcal{F}$ defined by $u \sim v$ if and only if $u - v$ is constant on X. Then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.

(iii) If $V \subset X$ is finite and $v \in \ell(V)$ then there is a function $u \in \mathcal{F}$ so that $u\big|_V = v$.

(iv) For $x, y \in X$

$$R(x,y) := \sup\Big\{\frac{(u(x)-u(y))^2}{\mathcal{E}(u)} : u \in \mathcal{F}, \mathcal{E}(u) > 0\Big\} < \infty.$$

(v) If $u \in \mathcal{F}$ then $\bar{u} := \max(0, \min(1, u(x))) \in \mathcal{F}$ and $\mathcal{E}(\bar{u}) \leq \mathcal{E}(u)$.

The condition (v) is called the Markov property.

Remark 3.1. Note that the definition of resistance forms does not require any measure on the space X at all.

To R one refers as the *resistance metric*, [Kig03, Definition 2.11] and (X, R) is a metric space, [Kig03, Proposition 2.10], to which we refer as *resistance space*. Metric graphs as in Section 5.1 are resistance spaces, other typical examples are p.c.f. self-similar fractals endowed with limit forms of regular harmonic structures, [Kig89; Kig93a; Kig01], and Sierpiński carpets carrying self-similar resistance forms as in [BB89] (additional information may be found in [BBKT10]).

By Definition 3.1 (iv) we have

$$|u(x) - u(y)|^2 \leq R(x,y)\mathcal{E}(u), \quad u \in \mathcal{F}, \quad x, y \in X. \tag{3.1}$$

Hence every $u \in \mathcal{F}$ is uniformly $\frac{1}{2}$-Hölder continuous with respect to R and in particular, $\mathcal{F} \subset C(X)$ with respect to the topology induced by the resistance metric. For any finite subset $V \subset X$ a resistance form $(\mathcal{E}_V, \ell(V))$ is defined by

$$\mathcal{E}_V(v) = \inf\Big\{\mathcal{E}(u) : u \in \mathcal{F}, u\big|_V = v\Big\} \tag{3.2}$$

where a unique infimum is achieved. The form $\mathcal{E}_V$ is called the *trace* of $\mathcal{E}$ on V, see [Kig12, Def. 8.3]. If $V_1 \subset V_2$ and both are finite, then $(\mathcal{E}_{V_2})_{V_1} = \mathcal{E}_{V_1}$.

We assume X is a nonempty set and $(\mathcal{E}, \mathcal{F})$ is a resistance form on X so that (X, R) is separable. Then we can find a sequence $(V_m)_m$ of finite subsets $V_m \subset X$ with $V_m \subset V_{m+1}$, $m \geq 1$, and $\bigcup_{m\geq 0} V_m$ dense in (X, R). According to [Kig03, Proposition 2.10 and Theorem 2.14] (or [Kig12, Theorem 3.14]), we have

$$\mathcal{E}(u) = \lim_{m\to\infty} \mathcal{E}_{V_m}(u), \quad u \in \mathcal{F}, \tag{3.3}$$

for any such sequence. The limit exists, because for any $u \in \mathcal{F}$ the sequence $(\mathcal{E}_{V_m}(u))_m$ is non-decreasing. Each $\mathcal{E}_{V_m}$ is of the form

$$\mathcal{E}_{V_m}(u) = \frac{1}{2} \sum_{p\in V_m} \sum_{q\in V_m} c(m; p, q)(u(p) - u(q))^2, \quad u \in \mathcal{F}, \tag{3.4}$$

with constants $c(m; p, q) \geq 0$, symmetric in p and q.

Finally, we introduce the definition of energy measures for resistance forms which are well known to exist under the assumptions made above, see [FOT94; Hin05; HN06; Hin10; Kus89; Tep08]. Since we assume that $(\mathcal{E}, \mathcal{F})$ is a regular resistance form on a nonempty, locally compact and separable set X, it follows that for any $f \in \mathcal{F} \cap C_c(X)$ there is a unique finite Radon measure ν_f on X satisfying

$$\mathcal{E}(fg, f) - \frac{1}{2}\mathcal{E}(f^2, g) = \int_X g \, d\nu_f, \quad g \in \mathcal{F} \cap C_c(X), \tag{3.5}$$

the *energy measure of* f. To see this note that obviously $g \mapsto \mathcal{E}(fg, f) - \frac{1}{2}\mathcal{E}(f^2, g)$ defines a linear functional on $\mathcal{F} \cap C_c(X)$. *Mutual energy measures* ν_{f_1,f_2} for $f_1, f_2 \in \mathcal{F} \cap C_c(X)$ are defined using (3.5) and polarization.

Chapter 4

Vector analysis for resistance forms

Basically following [CS03; CS09; IRT12] we can introduce a first order derivation ∂ associated with $(\mathcal{E},\mathcal{F})$.

Throughout this chapter we assume that (X,R) is locally compact and separable and that $(\mathcal{E},\mathcal{F})$ is regular, i.e. such that the algebra $\mathcal{F}\cap C_c(X)$ is uniformly dense in the space $C_c(X)$ of continuous compactly supported functions on (X,R), see [Kig12, Definition 6.2]. We also assume that (X,R) is complete and that closed balls in (X,R) are compact. This is trivially the case if (X,R) is compact, it can also be concluded if the space (X,R) is doubling in the sense of [Kig12, Definition 7.7], see [Kig12, Proposition 7.9].

4.1 Universal derivation

To introduce the first order calculus associated with $(\mathcal{E},\mathcal{F})$, let $l_a(X\times X)$ denote the space of all real valued antisymmetric functions on $X\times X$ and write

$$(g\cdot v)(x,y) := \overline{g}(x,y)v(x,y),\quad x,y\in X, \tag{4.1}$$

for any $v\in l_a(X\times X)$ and $g\in C_c(X)$, where

$$\overline{g}(x,y) := \frac{1}{2}(g(x)+g(y)).$$

Obviously $g\cdot v\in l_a(X\times X)$, and (4.1) defines an action of $C_c(X)$ on $l_a(X\times X)$, turning it into a module. By $d_u:\mathcal{F}\cap C_c(X)\to l_a(X\times X)$ we denote the *universal derivation*,

$$d_uf(x,y) := f(x)-f(y),\quad x,y\in X, \tag{4.2}$$

and by

$$\Omega^1_a(X) := \left\{\sum_i g_i\cdot d_uf_i : g_i\in C_c(X), f_i\in\mathcal{F}\cap C_c(X)\right\}, \tag{4.3}$$

differing slightly from the notation used in [HM20b], the submodule of $l_a(X\times X)$ of finite linear combinations of functions of form $g\cdot d_uf$. A quick calculation shows that for $f,g\in\mathcal{F}\cap C_c(X)$ we have $d_u(fg)=f\cdot d_ug+g\cdot d_uf$.

On $\Omega^1_a(X)$ we can introduce a symmetric nonnegative definite bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ by extending

$$\langle g_1\cdot d_uf_1, g_2\cdot d_uf_2\rangle_{\mathcal{H}} := \lim_{m\to\infty}\frac{1}{2}\sum_{p\in V_m}\sum_{q\in V_m} c(m;p,q)\overline{g_1}(p,q)\overline{g_2}(p,q)d_uf_1(p,q)d_uf_2(p,q) \tag{4.4}$$

linearly in both arguments, respectively, and we write $\|\cdot\|_{\mathcal{H}} = \sqrt{\langle\cdot,\cdot\rangle_{\mathcal{H}}}$ for the associated Hilbert seminorm. In Lemma 4.2 below we will verify that the definition of $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ does not depend on the choice of the sequence $(V_m)_m$.

We factor $\Omega^1_a(X)$ by the elements of zero seminorm and obtain the space $\Omega^1_a(X)/\ker\|\cdot\|_{\mathcal{H}}$. Given an element $\sum_i g_i \cdot d_u f_i$ of $\Omega^1_a(X)$ we write $\left[\sum_i g_i \cdot d_u f_i\right]_{\mathcal{H}}$ to denote its equivalence class. Completing $\Omega^1_a(X)/\ker\|\cdot\|_{\mathcal{H}}$ with respect to $\|\cdot\|_{\mathcal{H}}$ we obtain a Hilbert space $\mathcal{H}$, we refer to it as the *space of generalized L^2-vector fields associated with* $(\mathcal{E},\mathcal{F})$. This is a version of the construction introduced in [CS03; CS09] and studied in [BK19; HR16; HRT13; HT13; HT15c; HT15b; IRT12; LQ19], see also the related sources [Ebe99; Gig15; Gig18; Wea00]. The basic idea is much older, see for instance [BH91, Exercise 5.9], it dates back to ideas of Mokobodzki and LeJan.

4.2 Energy measures and discrete approximations

4.2.1 Energy measures and discrete approximations in the local case

A resistance form $(\mathcal{E},\mathcal{F})$ is called *local* if $\mathcal{E}(u,v)=0$ holds whenever $u,v\in\mathcal{F}$ are such that $R(\mathrm{supp}(u),\mathrm{supp}(v))>0$, see [Kig12, Definition 7.5]. Here $\mathrm{supp}(u)$ is the support of u, and the distance $R(A,B)$ of two sets A and B is defined in the standard way, [Kig12, Definition 5.2].

Lemma 4.1. *Assume that $(\mathcal{E},\mathcal{F})$ is local. Then for any $f_1,f_2,g_1,g_2\in\mathcal{F}\cap C_c(X)$ we have*

$$\langle g_1 d_u f_1, g_2 d_u f_2\rangle_{\mathcal{H}} = \frac{1}{2}\left\{\mathcal{E}(f_1g_1g_2,f_2)+\mathcal{E}(f_1,f_2g_1g_2)-\mathcal{E}(f_1f_2,g_1g_2)\right\}.$$

In particular, the definition of the bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is independent of the choice of the sets V_m.

To prove Lemma 4.1 and to show the independence of $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ of the choice of the sequence $(V_m)_m$ in (4.4) we make use of energy measures and we will also use energy measures to formulate later statements. We wish to briefly point out that their existence can be concluded directly from (3.3).

A standard calculation using (3.3) yields the formula

$$\mathcal{E}(fg,f)-\frac{1}{2}\mathcal{E}(f^2,g)=\frac{1}{2}\lim_{m\to\infty}\sum_{p\in V_m}\sum_{q\in V_m}c(m;p,q)g(p)(f(p)-f(q))^2, \tag{4.5}$$

from which the bound

$$|\mathcal{E}(fg,f)-\frac{1}{2}\mathcal{E}(f^2,g)|\le\|g\|_{\sup}\mathcal{E}(f)$$

and the positivity of the functional are immediate. By the regularity of $(\mathcal{E},\mathcal{F})$ it extends to a positive and bounded linear functional on the space $C_0(X)$ of continuous functions vanishing at infinity, and (3.5) follows from the Riesz representation theorem.

Recall that $B(x,r)$ denotes an open ball in (X,R) centered at x and with radius $r>0$. We prove Lemma 4.1.

Proof. Let $f,g\in\mathcal{F}\cap C_c(X)$, we may assume g is not the zero function and f is not constant. A short calculation shows that

$$\begin{aligned}&\|g d_u f\|^2_{\mathcal{H}}-\mathcal{E}(fg^2,f)+\frac{1}{2}\mathcal{E}(f^2,g^2)\\&\quad=\frac{1}{2}\lim_{m\to\infty}\sum_{p\in V_m}\sum_{q\in V_m}c(m;p,q)\big[\frac{1}{4}(g(p)-g(q))^2\big](f(p)-f(q))^2,\end{aligned}$$

and by polarization and in view of known results, [IRT12], it suffices to show that this equals zero.

Let $\varepsilon > 0$. Since ν_f is Radon, there is a compact set $K_0 \subset X$ such that

$$\nu_f(K_0^c) < \varepsilon/(8\,\|g\|_{\sup}^2).$$

Let $\varphi \in \mathcal{F} \cap C_c(X)$ be such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on K_0. Under our hypothesis the existence of such functions is guaranteed, see [FOT94, Problem 1.4.1]. We write $K := \operatorname{supp}\varphi$. The function $1-\varphi$ is supported in K_0^c and equals one on $K^c \subset K_0^c$, so that

$$\lim_{m\to\infty} \sum_{p\in V_m\cap K^c} \sum_{q\in V_m} c(m;p,q)(f(p)-f(q))^2 \leq \int_X (1-\varphi)d\nu_f < \frac{\varepsilon}{8\,\|g\|_{\sup}^2}. \tag{4.6}$$

Since $c(m;p,q) = c(m;q,p)$ also

$$\lim_{m\to\infty} \sum_{p\in V_m} \sum_{q\in V_m\cap K^c} c(m;p,q)(f(p)-f(q))^2 < \frac{\varepsilon}{8\,\|g\|_{\sup}^2}. \tag{4.7}$$

We next observe that for any $r > 0$ and any $\xi, \eta \in X$ with $R(\xi,\eta) > 6r$ we have

$$\lim_{m\to\infty} \sum_{p\in V_m\cap B(\xi,r)} \sum_{q\in V_m\cap B(\eta,r)} c(m;p,q) = 0. \tag{4.8}$$

To see this, let $\varphi_{\xi,r} \in \mathcal{F}$ be a function such that $0 \leq \varphi_{\xi,r} \leq 1$, $\varphi_{\xi,r} \equiv 1$ on $B(\xi,r)$ and $\operatorname{supp}\varphi_{\xi,r} \subset B(\xi,2r)$, such a function exists by [FOT94, Problem 1.4.1]. Let $\varphi_{\eta,r} \in \mathcal{F}$ be a function with analogous properties. Then

$$\begin{aligned}\lim_{m\to\infty} \sum_{p\in V_m\cap B(\xi,r)} \sum_{q\in V_m\cap B(\eta,r)} c(m;p,q) &\leq \lim_{m\to\infty} \sum_{p\in V_m} \sum_{q\in V_m} c(m;p,q)\varphi_{\xi,r}(p)(\varphi_{\eta,r}(q)-\varphi_{\eta,r}(p)) \\ &= \mathcal{E}(\varphi_{\xi,r},\varphi_{\eta,r}),\end{aligned}$$

and since $R(\operatorname{supp}(\varphi_{\xi,r}),\operatorname{supp}(\varphi_{\eta,r})) \geq 2r$ we have $\mathcal{E}(\varphi_{\xi,r},\varphi_{\eta,r}) = 0$ by the locality of $(\mathcal{E},\mathcal{F})$.

Now let $r' > 0$ be small enough so that by the continuity of q we have

$$\sup_m \sum_{p\in V_m} \sum_{q\in V_m\cap B(p,r')} c(m;p,q)(g(p)-g(q))^2(f(p)-f(q))^2 < \frac{\varepsilon}{4}. \tag{4.9}$$

Let $0 < r < r'/8$ and cover the compact set K by finitely many balls $B(\xi_i, r)$. Then

$$\begin{aligned}&\sum_{p\in V_m\cap K} \sum_{q\in V_m\cap K\cap B(p,r')^c} c(m;p,q)(g(p)-g(q))^2(f(p)-f(q))^2 \\ &\qquad\leq \sum_i \sum_{p\in V_m\cap B(\xi_i,r)} \sum_{q\in V_m\cap K\cap B(\xi_i,r'-r)^c} c(m;p,q)(g(p)-g(q))^2(f(p)-f(q))^2.\end{aligned}$$

The union of the finitely many compact sets $K \cap B(\xi_i, r'-r)^c$ is compact, we can cover it by finitely many balls $B(\eta_j, r)$ and see the above is bounded by

$$\sum_i \sum_j \sum_{p\in V_m\cap B(\xi_i,r)} \sum_{q\in V_m\cap B(\eta_j,r)} c(m;p,q)(g(p)-g(q))^2(f(p)-f(q))^2.$$

Since $R(\xi_i,\eta_j) > 6r$ for all i and j this can be made smaller than $\varepsilon/4$ if m is large enough by (4.8) and the boundedness of f and g. Combined with (4.9), (4.6) and (4.7) this shows that

$$\sum_{p\in V_m} \sum_{q\in V_m} c(m;p,q)\big[(g(p)-g(q))^2\big](f(p)-f(q))^2 < \varepsilon$$

for any large enough m. $\square$

Remark 4.1. Lemma 4.1 implies that the space $\mathcal{H}$ defined above is the same Hilbert space as the one obtained using [IRT12, Definition 2.3], see also [CS03] and [HRT13]. The elements v of $\mathcal{H}$ can no longer be interpreted as a function on $X \times X$, for classical setups such as Euclidean spaces or Riemannian manifolds the space $\mathcal{H}$ is the space of square integrable vector fields, see for instance [HT15b].

4.2.2 Energy measures and discrete approximations in the general case

According to the Beurling-Deny decomposition of $(\mathcal{E}, \mathcal{F})$, see [All75, Théorème 1] (or [FOT94, Section 3.2] for a different context), there exist a uniquely determined symmetric bilinear form $\mathcal{E}^c$ on $\mathcal{F} \cap C_c(X)$ satisfying $\mathcal{E}^c(f, g) = 0$ whenever $f \in \mathcal{F} \cap C_c(X)$ is constant on an open neighborhood of the support of $g \in \mathcal{F} \cap C_c(X)$ and a uniquely determined symmetric nonnegative Radon measure J on $X \times X \setminus \text{diag}$ such

$$\mathcal{E}(f) = \mathcal{E}^c(f) + \int_X \int_X (d_u f(x, y))^2 J(dxdy), \quad f \in \mathcal{F} \cap C_c(X). \tag{4.10}$$

By ν_f^c we denote the local energy measure of a function $f \in \mathcal{F} \cap C_c(X)$, i.e. the energy measures associated with $\mathcal{E}^c$, defined as in (3.5) but with $\mathcal{E}^c$ in place of $\mathcal{E}$.

Lemma 4.2. *For any $f_1, f_2 \in \mathcal{F} \cap C_c(X)$ and $g_1, g_2 \in C_c(X)$ we have*

$$\langle g_1 \cdot \partial f_1, g_2 \cdot \partial f_2 \rangle_{\mathcal{H}} = \int_X g_1 g_2 \, d\nu_{f_1, f_2}^{(c)} + \int_X \int_X \overline{g_1}(x, y) \overline{g_2}(x, y) d_u f_1(x, y) d_u f_2(x, y) \, J(dxdy).$$

In particular, the definition of the bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is independent of the choice of the sets V_m.

Proof. Standard arguments show that for all $v \in C_c(X \times X \setminus \text{diag})$ we have

$$\frac{1}{2} \lim_{\varepsilon \to 0} \lim_{m \to \infty} \sum_{x \in V_m} \sum_{y \in V_m, R(x,y) > \varepsilon} c(m; x, y) v(x, y) = \int_X \int_X v(x, y) J(dxdy), \tag{4.11}$$

see for instance [FOT94, Section 3.2]. The particular case $v = d_u f$, together with (4.10), then implies that

$$\mathcal{E}^c(f) = \frac{1}{2} \lim_{\varepsilon \to 0} \lim_{m \to \infty} \sum_{x \in V_m} \sum_{y \in V_m, R(x,y) \leq \varepsilon} c(m; x, y) (d_u f(x, y))^2 \tag{4.12}$$

for any $f \in \mathcal{F} \cap C_c(X)$. We claim that given such f and $g \in C_c(X)$,

$$\int_X g^2 d\nu_f^c = \frac{1}{2} \lim_{\varepsilon \to 0} \lim_{m \to \infty} \sum_{x \in V_m} \sum_{y \in V_m, R(x,y) \leq \varepsilon} c(m; x, y) \overline{g}(x, y)^2 (d_u f(x, y))^2. \tag{4.13}$$

By (3.5) and (4.12) this follows from the fact that

$$\lim_{\varepsilon} \lim_{m \to \infty} \sum_{x \in V_m} \sum_{y \in V_m, R(x,y) \leq \varepsilon} c(m; x, y) (d_u g(x, y))^2 (d_u f(x, y))^2 = 0,$$

which can be seen following the arguments in the proof of Lemma 4.1. Combining (4.11) with $v = g \cdot d_u f$ and (4.13), we obtain the desired result by polarization. □

4.3 Derivations and generators associated with different energies

The action (4.1) induces an action of $C_c(X)$ on $\mathcal{H}$: Given $v \in \mathcal{H}$ and $g \in C_c(X)$, let $(v_n)_n \subset \Omega_a^1(X)$ be such that $\lim_{n\to\infty}[v_n]_{\mathcal{H}} = v$ in $\mathcal{H}$ and define $g \cdot v \in \mathcal{H}$ by

$$g \cdot v := \lim_{n\to\infty} [g \cdot v_n]_{\mathcal{H}}.$$

Since (4.1) and (4.4) imply

$$\|g \cdot v\|_{\mathcal{H}} \le \|g\|_{\sup} \|v\|_{\mathcal{H}}, \tag{4.14}$$

it follows that the definition of $g \cdot v$ is correct. Given $f \in \mathcal{F} \cap C_c(X)$, we denote the $\mathcal{H}$-equivalence class of the universal derivation $d_u f$ as in (4.2) by ∂f. By the preceding discussion we observe $[g \cdot d_u f]_{\mathcal{H}} = g \cdot \partial f$ for all $f \in \mathcal{F} \cap C_c(X)$ and $g \in C_c(X)$. It also follows that the map $f \mapsto \partial f$ defines a derivation operator

$$\partial : \mathcal{F} \cap C_c(X) \to \mathcal{H}$$

which satisfies the identity $\|\partial f\|_{\mathcal{H}}^2 = \mathcal{E}(f)$ for any $f \in \mathcal{F} \cap C_c(X)$ and the Leibniz rule

$$\partial(fg) = f \cdot \partial g + g \cdot \partial f$$

for any $f, g \in \mathcal{F} \cap C_c(X)$.

Remark 4.2. For Euclidean domains or Riemannian manifolds the operator ∂, defined in an equivalent way, yields the usual gradient operator, see [CS03; HRT13; HT15b].

Let μ be a Borel regular measure on (X, R) so that for any open ball $B(x, r)$ with center $x \in X$ and radius $r > 0$ we have $0 < \mu(B(x, r)) < +\infty$. Under these conditions $\mathcal{F} \cap L^2(X, \mu)$, endowed with the norm

$$\|f\|_{\mathcal{D}(\mathcal{E})} := (\mathcal{E}(f) + \|f\|^2_{L^2(X,\mu)})^{1/2}, \tag{4.15}$$

is a Hilbert space, [Kig12, Lemma 9.2], we write $\langle \cdot, \cdot \rangle_{\mathcal{D}(\mathcal{E})}$ for the corresponding scalar product and $\mathcal{D}(\mathcal{E})$ for the closure of $\mathcal{F} \cap C_c(X)$ in this Hilbert space. If (X, R) is compact, then $\mathcal{D}(\mathcal{E}) = \mathcal{F}$. Under our assumptions the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular Dirichlet form on $L^2(X, \mu)$ in the sense of [FOT94], see [Kig12, Theorem 9.4]. For any $x \in X$ there exists a constant $c_x > 0$ such that

$$|u(x)| \le c_x \|u\|_{\mathcal{D}(\mathcal{E})}, \quad u \in \mathcal{D}(\mathcal{E}), \tag{4.16}$$

this was shown in [Kig12, Lemma 9.6].

The derivation ∂ extends to a closed unbounded operator $\partial : L^2(X, \mu) \to \mathcal{H}$ with domain $\mathcal{D}(\mathcal{E})$. In the case that $(\mathcal{E}, \mathcal{F})$ is local, it satisfies the usual chain rule: If $F \in C^1(\mathbb{R})$ is such that $F(0) = 0$ and $u \in \mathcal{D}(\mathcal{E})$ is bounded, then $\partial F(u) = F(u)\partial u$. The adjoint of ∂ is denoted by ∂^* and its domain by $\mathcal{D}(\partial^*)$. The image $\operatorname{Im} \partial$ of the derivation ∂ is a closed subspace of $\mathcal{H}$, see [HKT15, p.374], and we observe the orthogonal Helmholtz-Hodge type decomposition

$$\mathcal{H} = \operatorname{Im} \partial \oplus \ker \partial^*. \tag{4.17}$$

Remark 4.3. If (X, R) is connected, we have $\ker \partial = \mathbb{R}$, and the spaces $\operatorname{Im} \partial$ and $\mathcal{D}(\mathcal{E})/\mathbb{R}$ are isomorphic as Hilbert spaces.

4.3.1 Scalar Laplacian

Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ denote the generator of the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ i.e. the unique non-positive definite self-adjoint operator such that

$$\mathcal{E}(u, v) = -\langle \mathcal{L}u, v\rangle_{L^2(X,\mu)} \tag{4.18}$$

for all $u \in \mathcal{D}(\mathcal{L})$ and $v \in \mathcal{D}(\mathcal{E})$. A function $u \in \mathcal{D}(\mathcal{E})$ is a member of $\mathcal{D}(\mathcal{L})$ if and only if $\partial u \in \mathcal{D}(\partial^*)$, and in this case we have $\mathcal{L}u = -\partial^*\partial u$.

4.3.2 Vector Laplacian

For the discussion of the viscous Burgers equation (1.4) as equation of vector fields in Chapter 7 we need in addition the following objects.

Viewed as the target space of the derivation ∂, the space $L^2(X, \mu)$ can also be interpreted as the space of L^2-vector fields. Thus, we can introduce a closed quadratic form $(\vec{\mathcal{E}}, \mathcal{D}(\vec{\mathcal{E}}))$ on the Hilbert space $\mathcal{H}$ by setting $\mathcal{D}(\vec{\mathcal{E}}) := \mathcal{D}(\partial^*)$ and

$$\vec{\mathcal{E}}(u, v) := \langle \partial^* u, \partial^* v\rangle_{L^2(X,\mu)}, \quad u, v \in \mathcal{D}(\vec{\mathcal{E}}).$$

The associated generator is $(\vec{\mathcal{L}}, \mathcal{D}(\vec{\mathcal{L}}))$, and $v \in \mathcal{H}$ is in $\mathcal{D}(\vec{\mathcal{L}})$ if and only if $\partial^* v \in \mathcal{D}(\mathcal{E})$. As before we have $(\partial^*)^* = \partial$, because ∂ is densely defined and closed, [RS80, Theorem VIII.1]. For $v \in \mathcal{D}(\vec{\mathcal{L}})$ we have $\vec{\mathcal{L}}v = -\partial\partial^* v$.

4.3.3 Distributional definitions

Let $(\mathcal{D}(\mathcal{E}))^*$ denote the dual space of $(\mathcal{D}(\mathcal{E}))$. We can interpret ∂^* and $\mathcal{L}$ in the distributional sense as bounded linear operators $\partial^* : \mathcal{H} \to (\mathcal{D}(\mathcal{E}))^*$ and $\mathcal{L} : \mathcal{D}(\mathcal{E}) \to (\mathcal{D}(\mathcal{E}))^*$ by

$$\partial^* v(\varphi) := \langle v, \partial\varphi\rangle_{\mathcal{H}} \qquad \text{and} \qquad \mathcal{L}f(\varphi) := -\mathcal{E}(f, \varphi).$$

Using the norm $v \mapsto \|\partial^* v\|_{\mathcal{D}(\mathcal{E})}$ on $\mathcal{D}(\vec{\mathcal{L}})$ we can see that the operator $\vec{\mathcal{L}}$ induces a bounded linear operator $\vec{\mathcal{L}} : L^2(X, \mu) \to (\mathcal{D}(\vec{\mathcal{L}}))^*$, defined by

$$\vec{\mathcal{L}}v(w) := \partial^* v(\partial^* w), \quad w \in \mathcal{D}(\vec{\mathcal{L}}).$$

Finally, we introduce the notion of a generalized convection term by defining $\partial \langle u, u\rangle \in (\mathcal{D}(\vec{\mathcal{L}}))^*$ for any $u \in \mathcal{H}$ via

$$\partial \langle u, u\rangle (v) := \langle (\partial^* v)u, u\rangle_{\mathcal{H}}, \quad v \in \mathcal{D}(\vec{\mathcal{L}}). \tag{4.19}$$

4.4 First order derivatives and measurable bundles

For this section, let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on $L^2(X, \mu)$ and let μ be an energy dominant measure for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, for the definitions we refer to Chapter 2.

In the previous section we have introduced $\mathcal{H}$ as the *space (or rather, module) of generalized L^2-vector fields* associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E})))$. Here, we also provide a fiber-wise interpretation in a measurable sense.

Definition 4.1. *A collection $(\mathcal{H}_x)_{x\in X}$ of Hilbert spaces $(\mathcal{H}_x, \langle\cdot,\cdot\rangle_{\mathcal{H}_x})$ together with a subspace $\mathcal{M}$ of $\prod_{x\in X} \mathcal{H}_x$ is called a* measurable field of Hilbert spaces *if*

(i) an element $\xi \in \prod_{x\in X} \mathcal{H}_x$, $\xi = (\xi_x)_{x\in X}$, is in $\mathcal{M}$ if and only if $x \mapsto \langle \xi_x, \eta_x\rangle_{\mathcal{H}_x}$ is measurable for any $\eta \in \mathcal{M}$,

(ii) there exists a countable set $\{\xi^{(i)} : i \in \mathbb{N}\} \subset \mathcal{M}$ *such that for all* $x \in X$ *the span of* $\{\xi_x^{(i)} : i \in \mathbb{N}\}$ *is dense in* $\mathcal{H}_x$.

The elements $v = (v_x)_{x\in X}$ of $\mathcal{M}$ are usually referred to as *measurable sections.* See for instance [Tak02, Section IV.8].

It was already observed in [Ebe99] that there is a measurable field $(\mathcal{H}_x)_{x\in X}$ of Hilbert spaces (or rather, modules) $\mathcal{H}_x$ on which the action of the core $\mathcal{C}$ is defined by $a(x)\omega_x \in \mathcal{H}_x$, $a \in \mathcal{C}$, $\omega_x \in \mathcal{H}_x$, and such that the direct integral $\int_X^{\oplus} \mathcal{H}_x\,\mu(dx)$ is isometrically isomorphic to $\mathcal{H}$. In particular,

$$\langle u, v\rangle_{\mathcal{H}} = \int_X^{\oplus} \langle u_x, v_x\rangle_{\mathcal{H}_x}\ \mu(dx)$$

for all $u, v \in \mathcal{H}$, where, as above, for any $x \in X$ the symbol v_x denotes the image of the associated projection $v \mapsto v_x$ from $\mathcal{H}$ into $\mathcal{H}_x$. Given $f, g \in \mathcal{D}(\mathcal{E})$, we have

$$\Gamma(f, g)(x) = \langle \partial_x f, \partial_x g\rangle_{\mathcal{H}_x}$$

for μ-a.e. $x \in X$, where $\partial_x f := (\partial f)_x$. See [HRT13, Section 2] for a proof. The spaces $\mathcal{H}_x$ may be viewed as substitutes for tangent spaces, see for instance [HT15b]. The direct integral is also denoted by $L^2(X, \mu, (\mathcal{H}_x)_{x\in X})$, because it is the space of (equivalence classes) of square integrable measurable sections.

Remark 4.4. In contrast to Riemannian manifolds the 'tangent spaces' $\mathcal{H}_x$ do not vary smoothly, but only measurably. Their dimension can change from one base point x to another, see also Example 9.1 (1). Under the additional assumption that μ is minimal in a suitable way, the dimensions of the spaces $\mathcal{H}_x$ are a well-studied and useful quantity referred to as *pointwise index* or *Kusuoka-Hino index* of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, their essential supremum is called the *martingale dimension.* See [Hin08; Hin10; Hin13b] and also [BK19]. For energy forms on self-similar fractals the martingale dimension is known to be bounded (by the spectral dimension) [Hin13b], for p.c.f. self-similar fractals it is known to be one, [Hin08].

As sketched in [HRT13, Section 6] one can also define spaces of p-integrable sections. For a measurable section $v = (v_x)_{x\in X}$ let

$$\|v\|_{L^p(X,\mu,(\mathcal{H}_x)_{x\in X})} := \left(\int_X \|v_x\|_{\mathcal{H}_x}^p \mu(dx)\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$

and define the spaces $L^p(X, \mu, (\mathcal{H}_x)_{x\in X})$ as the collections of the respective equivalence classes of μ-a.e. equal sections having finite norm. By a variant of the classical pointwise Riesz-Fischer argument they are seen to be separable Banach spaces.

Let $\mathcal{B}_b(X)$ denote the space of bounded Borel functions on X. For $f \in \mathcal{B}_b(X)$ and $v = (v_x)_{x\in X} \in L^p(X, \mu, (\mathcal{H}_x)_{x\in X})$ the product fv is defined in the μ-a.e. pointwise sense as the measurable section $x \mapsto f(x)v_x$. Since

$$\|fv\|_{L^p(X,\mu,(\mathcal{H}_x)_{x\in X})} \le \|f(x)\|_{L^\infty(X,\mu)}\|v\|_{L^p(X,\mu,(\mathcal{H}_x)_{x\in X})}$$

the action $v \mapsto fv$ of $\mathcal{B}_b(X)$ on $L^p(X, \mu, (\mathcal{H}_x)_{x\in X})$ is bounded. To the space $L^p(X, \mu, (\mathcal{H}_x)_{x\in X})$ we refer as the *space of generalized* L^p*-vector fields.*

The discussion of first order derivations and concepts of measurable bundles is naturally connected to Sobolev spaces and calculus of variations, see for instance [CG03, Section 4.3]. As a consequence of this connection we construct reflexive $(1, p)$-Sobolev spaces for fractals that carry a local regular Dirichlet form in Chapter 9.

Chapter 5

Examples of resistance spaces

In this chapter we introduce the resistance spaces which we are working on. First, we discuss metric graphs as resistance spaces in Section 5.1. Section 5.2 deals with the class of metric spaces with finitely ramified cell structure as defined in [Tep08]. These spaces are generalizations of p.c.f. self-similar sets introduced by Kigami [Kig89] and of fractafolds introduced by Strichartz [Str03]. Later in Part III we provide graph approximations for finitely ramified spaces for solutions of partial differential equations on resistance spaces. In Section 5.2 we also present a special class of ramified fractals, the so called p.c.f. self-similar fractals endowed with limit forms of regular harmonic structures, [Kig93a; Kig01]. These fractals can be approximated by metric graphs. An example of a infinitely ramified self-similar fractal is presented in Section 5.3.

5.1 Metric graphs

Mainly following [BLS09; Hae11] we provide some basics on metric graphs, related energies and Laplacians. For a reference on the general theory of metric graphs we refer to [Pos12].

A *metric graph* is a quadruple $\Gamma = (E, V, i, j)$ consisting of a countable set E of different copies of open intervals $e = (0, l_e)$ with $l_e \in (0, +\infty]$, a countable set V and maps $i : E \to V$ and $j : \{e \in E \mid l_e < +\infty\} \to V$. To the elements v of V we refer as *vertices*, to the elements e of E as *edges*. Given $e = (0, l_e) \in E$, we call l_e the *length* of e, $i(e)$ its *initial* and $j(e)$ its *terminal vertex*. An edge $e \in E$ and a vertex $p \in V$ are said to be *incident*, $e \sim p$, if p is the initial or the terminal vertex of e. Two distinct vertices $p, q \in V$ are said to be *neighbors*, $p \sim q$, if they are incident to the same edge; two distinct edges $e, e' \in E$ are said to be *neighbors*, $e \sim e'$, if there is some vertex $p \in V$ they are both incident to. A metric graph Γ is called *connected*, if for any distinct $p, q \in V$ there exists $p_0, ..., p_n \in V$ such that $p_0 = p$, $p_n = q$ and $p_i \sim p_{i-1}$ for $i = 1, .., n$. We set $X_e := \{e\} \times (0, l_e)$ and define the disjoint union

$$X_\Gamma := V \cup \bigcup_{e \in E} X_e. \tag{5.1}$$

For any edge e let $\pi_e : X_e \to (0, l_e)$ denote the projection $(e, t) \mapsto t$ onto the second component of X_e. For $e \in E$ with $l_e < +\infty$ we set $\bar{X}_e := X_e \cup \{i(e), j(e)\}$ and for $e \in E$ with $l_e = +\infty$ we set $\bar{X}_e := X_e \cup \{i(e)\}$. Let X_Γ be endowed with the unique topology such that for any $e \in E$ the mapping π_e extends to a homeomorphism $\pi_e : \bar{X}_e \to [0, l(e)]$ that satisfies $\pi_e(i(e)) = 0$ and, in case that $l_e < +\infty$, also $\pi_e(j(e)) = l(e)$. Given a real valued function f on X_Γ we define a function on each edge $e \in E$ by $f_e := f \circ \pi_e^{-1}$. If f is continuous on X_Γ then for each $e \in E$ the function f_e is continuous on e and its value at each vertex is the limit of its values on any adjacent edge. Moreover, the canonical length metric metrizes this topology and makes X_Γ into a locally compact separable metric space. The space X_Γ is compact if and only if E is a finite set and all edges have finite length, and Γ is called *compact* if X_Γ is compact. In what follows we assume that Γ is a compact connected metric graph.

We shall define some notations concerning the function spaces on X_Γ. On each edge $e \in E$ let $\dot{W}^{1,2}(e)$ denote the homogeneous Sobolev space consisting of locally Lebesgue integrable functions g on e such that

$$\mathcal{E}_e(g) := \int_0^{l_e} (g'(s))^2\, ds < +\infty,$$

where the derivative g' of g is understood in the distributional sense.

For a function f on X_Γ such that $f_e \in \dot{W}^{1,2}(e)$ for any $e \in E$ we can define its energy $\mathcal{E}_\Gamma(f)$ on Γ by the sum

$$\mathcal{E}_\Gamma(f) := \sum_{e\in E} \mathcal{E}_e(f_e).$$

We denote the space of continuous functions on X_Γ with finite energy by

$$\dot{W}^{1,2}(X_\Gamma) := \{f \in C(X_\Gamma) : \text{for any } e \in E \text{ we have } f_e \in \dot{W}^{1,2}(e), \text{ and } \mathcal{E}_\Gamma(f) < +\infty\}.$$

By polarization we obtain a nonnegative definite symmetric bilinear form $(\mathcal{E}_\Gamma, \dot{W}^{1,2}(X_\Gamma))$ satisfying the Markov property. Moreover, $(\mathcal{E}_\Gamma, \dot{W}^{1,2}(X_\Gamma))$ is a resistance form on X_Γ in the sense of [Kig03, Definition 2.8], see Chapter 3 above. In particular, on any single edge $e \in E$ the form $\mathcal{E}_e$ satisfies

$$(f_e(s) - f_e(s'))^2 \le l_e \mathcal{E}_e(f_e) \tag{5.2}$$

for any $f \in \dot{W}^{1,2}(X_\Gamma)$ and any $s, s' \in e$.

Now suppose μ_Γ is an atom free nonnegative Radon measure on X_Γ with full support. Then $(\mathcal{E}_\Gamma, \dot{W}^{1,2}(X_\Gamma))$ is a strongly local regular Dirichlet form on $L^2(X_\Gamma, \mu_\Gamma)$ in the sense of [FOT94]. We write $W^{1,2}(X_\Gamma, \mu_\Gamma)$ for the Hilbert space $\dot{W}^{1,2}(X_\Gamma)$ with norm

$$\|f\|_{W^{1,2}(X_\Gamma,\mu_\Gamma)} := \left(\mathcal{E}_\Gamma(f) + \|f\|^2_{L^2(X_\Gamma,\mu_\Gamma)}\right)^{1/2}, \quad f \in W^{1,2}(X_\Gamma, \mu_\Gamma). \tag{5.3}$$

A function $f \in W^{1,2}(X_\Gamma, \mu_\Gamma)$ has zero energy $\mathcal{E}_\Gamma(f) = 0$ if and only if f is constant on X_Γ, and

$$\|f\|_{\sup} \le c\, \|f\|_{W^{1,2}(X_\Gamma,\mu_\Gamma)}, \quad f \in W^{1,2}(X_\Gamma, \mu_\Gamma), \tag{5.4}$$

where $c > 0$ is a constant not depending on f, see [Hae11, Corollary 2.2]. Alternatively, one can follow the arguments of [Kig01, Lemma 5.2.8].

In what follows we assume $(c_e)_{e\in E}$ is a family of real numbers c_e such that $\inf_{e\in E} c_e > 0$ and $\sup_{e\in E} c_e < +\infty$ and that μ_Γ is the measure on X_Γ determined by

$$\mu_\Gamma|_{X_e} \circ \pi_e^{-1} = c_e \lambda^1|_e, \quad e \in E, \tag{5.5}$$

where λ^1 denotes the Lebesgue measure on the real line. This class of measures is sufficiently large for our purposes.

Kirchhoff Laplacian

Under the stated assumption the generator of the Dirichlet form $(\mathcal{E}_\Gamma, W^{1,2}(X_\Gamma, \mu_\Gamma))$ is the nonpositive definite self-adjoint operator $(\mathcal{L}_\Gamma, \mathcal{D}(\mathcal{L}_\Gamma))$ on $L^2(X_\Gamma, \mu_\Gamma)$, where $\mathcal{D}(\mathcal{L}_\Gamma)$

is the collection of all $f \in W^{1,2}(X_\Gamma, \mu_\Gamma)$ such that $f_e \in W^{2,2}(e)$ for all $e \in E$ and $\sum_{e\sim p} U_p(e) f_e'(p) = 0$ for all $p \in V$ and

$$\mathcal{L}_\Gamma f = \sum_{e\in E} c_e^{-1} \mathbf{1}_e\, f_e'' \tag{5.6}$$

for all $f \in \mathcal{D}(\mathcal{L}_\Gamma)$. Here $f_e'(p)$ denotes the trace of $f_e' \in W^{1,2}(e)$ on p, and $U_p(e) = 1$ if $p = j(e)$ and $U_p(e) = -1$ if $p = i(e)$, so that at both points we consider the normals outgoing from the edge e (and ingoing into $i(e)$ and $j(e)$, respectively). To $(\mathcal{L}_\Gamma, \mathcal{D}(\mathcal{L}_\Gamma))$ one refers as *Laplacian with Kirchhoff vertex conditions*, see e.g. [FKW07, Definition 5]. On vertices that are incident to one edge only, this forces zero Neumann boundary conditions.

A function $f \in L^2(X_\Gamma, \mu_\Gamma)$ is already uniquely determined by the functions f_e, and we may write $f = (f_e)_{e\in E}$. Given a function $f \in W^{1,2}(X_\Gamma, \mu_\Gamma)$ we can define a function $df = ((df)_e)_{e\in E}$ in $L^2(X_\Gamma, \mu_\Gamma)$ by $(df)_e = c_e^{-1/2} f_e'$ for any $e \in E$ where each f_e' is understood in distributional sense. This yields a bounded linear operator $d : W^{1,2}(X_\Gamma, \mu_\Gamma) \to L^2(X_\Gamma, \mu_\Gamma)$, note that for any $f, g \in W^{1,2}(X_\Gamma, \mu_\Gamma)$ we have

$$\langle df, dg \rangle_{L^2(X_\Gamma,\mu_\Gamma)} = \mathcal{E}(f, g). \tag{5.7}$$

Remark 5.1. Since X_Γ is connected, the kernel of d consists exactly of the constants, $\ker d = \mathbb{R}$, so that $W^{1,2}(X_\Gamma, \mu_\Gamma)/\mathbb{R}$ and the image $\operatorname{Im} d$ of d in $L^2(X_\Gamma, \mu_\Gamma)$ are isomorphic as vector spaces and by (5.7) even isomorphic as Hilbert spaces.

Since due to (5.4) the space $W^{1,2}(X_\Gamma, \mu_\Gamma)$ is an algebra with pointwise multiplication, we can observe the Leibniz rule $d(fg) = (df)g + fdg$, for any f, g from this space. The operator d may also be seen as a densely defined closed linear operator on $L^2(X_\Gamma, \mu_\Gamma)$ with domain $W^{1,2}(X_\Gamma, \mu_\Gamma)$, and using integration by parts on the individual edges and Fubini's theorem, the adjoint d^* of d is seen to be $d^* f = ((d^* f)_e)_{e\in E}$ with $(d^* f)_e = -c_e^{-1/2} f_e'$ for f from its domain $\mathcal{D}(d^*)$ consisting of all $f \in L^2(X_\Gamma, \mu_\Gamma)$ such that $f_e \in \dot{W}^{1,2}(e)$ for all $e \in E$ and

$$\sum_{e\sim p} c_e^{1/2} U_p(e) f_e(p) = 0 \tag{5.8}$$

for all $p \in V$. Similarly as before $f_e(p)$ is understood in the sense of traces. By general theory d^* is closed in $L^2(X_\Gamma, \mu_\Gamma)$ and its domain $\mathcal{D}(d^*)$ is dense. A function $f \in W^{1,2}(X_\Gamma, \mu_\Gamma)$ is in $\mathcal{D}(\mathcal{L}_\Gamma)$ if and only if df is in $\mathcal{D}(d^*)$, and in this case we have $\mathcal{L}_\Gamma f = -d^* df$.

Vector Laplacian

Viewed as the target space of the derivation d, the space $L^2(X_\Gamma, \mu_\Gamma)$ can also be interpreted as the space of L^2-vector fields. Its subspace $\ker d^*$ is trivial if and only if Γ has no cycles (i.e. is a tree), see [IRT12, Proposition 5.1]. We follow [BK19] and define a natural nonnegative definite closed quadratic form on the space $L^2(X_\Gamma, \mu_\Gamma)$ of L^2-vector fields by setting $\mathcal{D}(\vec{\mathcal{E}}_\Gamma) := \mathcal{D}(d^*)$ and

$$\vec{\mathcal{E}}_\Gamma(u, v) := \langle d^* u, d^* v \rangle_{L^2(X_\Gamma,\mu_\Gamma)}, \quad u, v \in \mathcal{D}(\vec{\mathcal{E}}_\Gamma). \tag{5.9}$$

Remark 5.2.

(i) If Γ has only one single edge e, then $(\vec{\mathcal{E}}_\Gamma, \mathcal{D}(\vec{\mathcal{E}}_\Gamma))$ is the Dirichlet form associated with the Laplacian on e with Dirichlet boundary conditions, [BK19, Example 4.1].

(ii) In general $(\vec{\mathcal{E}}_\Gamma, \mathcal{D}(\vec{\mathcal{E}}_\Gamma))$ is not a Dirichlet form. Suppose Γ has a vertex $p \in V$ with at least three incident edges e_1, e_2, e_3, and $c_{e_i} = 1$, $i = 1, 2, 3$. If e_1 and e_2 have p as terminal and e_3 has it as initial vertex, consider a function $v \in \mathcal{D}(\vec{\mathcal{E}}_\Gamma) \cap L^\infty(X_\Gamma, \mu_\Gamma)$ that satisfies $v_{e_1} = 1$, $v_{e_2} = 1$ and $v_{e_3} = 2$. Then the square v^2 of v violates (5.8) at p. Consequently, the Markov property cannot hold.

The generator of $(\vec{\mathcal{E}}_\Gamma, \mathcal{D}(\vec{\mathcal{E}}_\Gamma))$ is the nonnegative definite self-adjoint operator $(\vec{\mathcal{L}}_\Gamma, \mathcal{D}(\vec{\mathcal{L}}_\Gamma))$, given by $\vec{\mathcal{L}}_\Gamma v := -dd^*v$ for all functions v from its domain $\mathcal{D}(\vec{\mathcal{L}}_\Gamma)$. This domain $\mathcal{D}(\vec{\mathcal{L}}_\Gamma)$ is the space of all $v \in \mathcal{D}(d^*)$ such that $d^*v = (-v_e')_{e\in E}$ is in $W^{1,2}(X_\Gamma, \mu_\Gamma)$, as follows from the identity $(d^*)^* = d$, valid because d is a densely defined and closed operator, see e.g. [RS80, Theorem VIII.1].

Remark 5.3. To the vertex conditions associated with $\vec{\mathcal{L}}_\Gamma$ the authors of [BK19] referred to as anti-Kirchhoff conditions, they slightly differ from those specified in [FKW07, Definition 6].

Since X_Γ is compact, a function $f \in W^{1,2}(X_\Gamma, \mu_\Gamma)$ satisfies $\langle d^*v, f\rangle_{L^2(X_\Gamma,\mu_\Gamma)} = 0$ for all d^*v with $v \in \mathcal{D}(\vec{\mathcal{L}}_\Gamma)$ if and only if f is constant on X_Γ: In fact, this is equivalent to requiring $\langle v, df\rangle_{L^2(X_\Gamma,\mu_\Gamma)} = 0$ for such v, and since $\mathcal{D}(\vec{\mathcal{L}}_\Gamma)$ is dense in $L^2(X_\Gamma, \mu_\Gamma)$ this is equivalent to $f \in \ker d$. Moreover, because the constants form a closed subspace of $L^2(X_\Gamma, \mu_\Gamma)$ it follows that each function $\varphi \in W^{1,2}(X_\Gamma, \mu_\Gamma)$ can uniquely be written as a sum

$$\varphi = d^*v + c \tag{5.10}$$

for some $v \in \mathcal{D}(\vec{\mathcal{L}}_\Gamma)$ and $c \in \mathbb{R}$.

Distributional definitions

Let $(W^{1,2}(X_\Gamma, \mu_\Gamma))^*$ denote the dual space of $W^{1,2}(X_\Gamma, \mu_\Gamma)$. We can interpret d^* and $\mathcal{L}_\Gamma$ in the distributional sense as bounded linear operators $d^* : L^2(X_\Gamma, \mu_\Gamma) \to (W^{1,2}(X_\Gamma, \mu_\Gamma))^*$, defined by

$$d^*v\,(\varphi) := \langle v, d\varphi\rangle_{L^2(X_\Gamma,\mu_\Gamma)}, \quad \varphi \in W^{1,2}(X_\Gamma, \mu_\Gamma), \tag{5.11}$$

and $\mathcal{L}_\Gamma : W^{1,2}(X_\Gamma, \mu_\Gamma) \to (W^{1,2}(X_\Gamma, \mu_\Gamma))^*$, defined by

$$\mathcal{L}_\Gamma f(\varphi) := -\mathcal{E}_\Gamma(f, \varphi), \quad \varphi \in W^{1,2}(X_\Gamma, \mu_\Gamma). \tag{5.12}$$

The operator $\vec{\mathcal{L}}_\Gamma$ may also be interpreted in the distributional sense as a bounded linear operator $\vec{\mathcal{L}}_\Gamma : L^2(X_\Gamma, \mu_\Gamma) \to (\mathcal{D}(\vec{\mathcal{L}}_\Gamma))^*$ defined by

$$\vec{\mathcal{L}}_\Gamma v(w) := -d^*v(d^*w), \quad w \in \mathcal{D}(\vec{\mathcal{L}}_\Gamma), \tag{5.13}$$

where (5.11) is used. Finally, we also define the operator d on $L^1(X_\Gamma, \mu_\Gamma)$ in a suitable distributional sense: Let $\mathcal{D}(\vec{\mathcal{L}}_\Gamma)$ be endowed with the norm $v \mapsto \|d^*v\|_{W^{1,2}(X_\Gamma,\mu_\Gamma)}$ and let $(\mathcal{D}(\vec{\mathcal{L}}_\Gamma))^*$ denote its topological dual. We define $d : L^1(X_\Gamma, \mu_\Gamma) \to (\mathcal{D}(\vec{\mathcal{L}}_\Gamma))^*$ by

$$df(v) := \int_{X_\Gamma} d^*v\, f\, d\mu_\Gamma, \quad v \in \mathcal{D}(\vec{\mathcal{L}}_\Gamma). \tag{5.14}$$

Then $|df(v)| \le c\,\|d^*v\|_{W^{1,2}(X_\Gamma,\mu_\Gamma)}\,\|f\|_{L^1(X_\Gamma,\mu_\Gamma)}$ for any $f \in L^1(X_\Gamma, \mu_\Gamma)$ by (5.4), and for $f \in W^{1,2}(X_\Gamma, \mu_\Gamma)$ we have $df(v) = \langle v, df\rangle_{L^2(X_\Gamma,\mu_\Gamma)}$. If Γ has a single edge e only and f_e' denotes the distributional derivative of f_e on e, then $df(v) = c_e^{1/2} f_e'(v)$.

5.2 Finitely ramified fractals with regular resistance forms

We consider fractals which have finitely ramified cell structures as introduced in [Tep08, Definition 2.1] and used e.g. in [IRT12].

Definition 5.1. *A* finitely ramified set *X is a compact metric space which has a* cell structure *$\{X_\alpha\}_{\alpha\in\mathcal{A}}$ and a* boundary (vertex) structure *$\{V_\alpha\}_{\alpha\in\mathcal{A}}$ such that the following hold:*

(i) *$\mathcal{A}$ is a countable index set;*

(ii) *each X_α is a distinct compact connected subset of X;*

(iii) *each V_α is a finite subset of X_α;*

(iv) *if $X_\alpha = \bigcup_{j=1}^k X_{\alpha_j}$ then $V_\alpha \subset \bigcup_{j=1}^k X_{\alpha_j}$;*

(v) *there is a filtration $\{\mathcal{A}_n\}_n$ such that*

(v.a) *each $\mathcal{A}_n$ is a finite subset of $\mathcal{A}$, $\mathcal{A}_0 = \{0\}$, and $X_0 = X$;*

(v.b) *$\mathcal{A}_n \cap \mathcal{A}_m = \emptyset$ if $n \neq m$;*

(v.c) *for any $\alpha \in \mathcal{A}_n$ there are $\alpha_1, ..., \alpha_k \in \mathcal{A}_{n+1}$ such that $X_\alpha = \bigcup_{j=1}^k X_{\alpha_j}$;*

(vi) *$X_{\alpha'} \cap X_\alpha = V_{\alpha'} \cap V_\alpha$ for any two distinct $\alpha, \alpha' \in \mathcal{A}_n$;*

(vii) *for any strictly decreasing infinite sequence $X_{\alpha_1} \supsetneq X_{\alpha_2} \supsetneq ...$ there exists $x \in X$ such that $\bigcap_{n\geq 1} X_{\alpha_n} = \{x\}$.*

Under these conditions the triple $(X, \{X_\alpha\}_{\alpha\in\mathcal{A}}, \{V_\alpha\}_{\alpha\in\mathcal{A}})$ is called a finitely ramified cell structure*.*

We denote $V_n = \bigcup_{\alpha\in\mathcal{A}_n} V_\alpha$. Note that $V_n \subset V_{n+1}$ for all $n > 0$. From now on, we say that X_α is an n-cell if $\alpha \in \mathcal{A}_n$.

Remark 5.4.

1. Roughly speaking, we consider fractals which are 'barely connected', i.e. the cells X_α intersect each other in only finitely many points. Thus, by removing these finite number of points, the fractals become disconnected.

2. We emphasize that the considered spaces may have no self-similarity in any sense and may have infinitely many cells connected at every junction point.

3. Note that in this definition the vertex boundary V_0 of $X_0 = X$ can be arbitrary, and in general may have no relation with the topological structure of X.

In the sequel we assume that $(\mathcal{E}, \mathcal{F})$ is a resistance form on $V_* := \bigcup_{n\geq 0} V_n$ satisfying the following.

Assumption 5.1.

(i) Each $\mathcal{E}_n$ is irreducible on each V_α;

(ii) all n-harmonic functions are continuous in the topology of X.

The energy measure ν_u of an n-harmonic function $u \in H_n(X)$ satisfies

$$\nu_u(X_\alpha) = \mathcal{E}_\alpha(u|_{V_\alpha}, u|_{V_\alpha}) \tag{5.15}$$

for all $\alpha \in \mathcal{A}_m$, $m \geq n$. See [IRT12, Proposition 2.16], [Kig03; Tep08]. Assumption 5.1 (i) implies that X is locally connected in the resistance metric.

Special case: post critically finite self-similar fractals equipped with regular harmonic structures

Here, we consider a more structured class of fractals as resistance spaces.

Definition 5.2. *[Kig01, Definition 1.3.1] Let K be a compact metrizable topological space and let S be a finite set, where we may assume that $S = \{1, ..., N\}$. Further, let $F_1, ..., F_N$ be continuous injections from K into itself. Then $(K, S, \{F_j\}_{j\in S})$ is called a* self-similar structure *if there exists a continuous surjection $\pi : S^{\mathbb{N}} \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$, $i = 1, ..., N$, where $\sigma_i(w_1w_2...) = iw_1w_2...$ for all $w_1w_2... \in S^{\mathbb{N}}$.*

If K is a self-similar set with respect to injective contractions $\{F_1, \dots, F_N\}$, then $(K, S, \{F_j\}_{j\in S})$ is a self-similar structure. We recall a Theorem that ensures existence and uniqueness of self-similar sets from [Kig01].

Theorem 5.1. *[Kig01, Theorem 1.1.4] Let (X, d) be a complete metric space. If $F_j : X \to X$ is a contraction with respect to the metric d for $j = 1, 2, \dots, N$, then there exits a unique non-empty compact subset of X that satisfies*

$$K = \bigcup_{j=1}^{N} F_j(K).$$

K is called the self-similar set with respect to $\{F_1, \dots, F_N\}$.

Definition 5.3. *[Kig01, Definitions 1.3.4 and 1.3.13] Let $(K, S, \{F_j\}_{j\in S})$ be a self-similar structure. $(K, S, \{F_j\}_{j\in S})$ is said to be* post critically finite (p.c.f.) *if and only if the set $\mathcal{P} = \bigcup_{n\geq 1} \sigma^n(\mathcal{C})$ is finite, where $\mathcal{C}_K = \bigcup_{i,j\in S, i\neq j}(F_i(K) \cap F_j(K))$ and $\mathcal{C} := \pi^{-1}(\mathcal{C}_K)$. Here we use the notation $\sigma(w_1w_2w_3...) = w_2w_3...$ for all $w_1w_2... \in S^{\mathbb{N}}$. We write $V_0 := \pi(\mathcal{P})$, note that for $N_0 = |V_0|$ we have $N_0 \leq N$.*

Examples 5.1. The **Sierpiński gasket** (see Figure 1.1) is a well-known example of p.c.f. self-similar sets, see [Kig01] or [Str06] for more details.
Let $X = \mathbb{R}^2$ and $V_0 = p_1, p_2, p_3$ be a set of vertices of an equilateral triangle and consider a set of three mappings $F_j : \mathbb{R}^2 \to \mathbb{R}^2$, $j = 1, 2, 3$, defined by

$$F_j(z) = (z - p_j)/2 + p_j.$$

Then the self-similar set K with respect to $\{F_1, F_2, F_3\}$ is called the Sierpiński gasket, i.e. it is the unique non-empty compact subset K of $\mathbb{R}^2$ that satisfies the self-similar identity

$$K = F_1(K) \cup F_2(K) \cup F_3(K).$$

Roughly speaking, the Sierpiński gasket is a union of three smaller copies of itself and these copies intersect each other at a finite set of points.

Throughout the following $(K, S, \{F_j\}_{j\in S})$ is a post critically finite self-similar structure and in addition we assume throughout that K is connected.

As usual we write $W_m := S^m$ for the space of finite words $w = w_1w_2...w_m$ of length $|w| = m$ over the alphabet S. Given a word $w \in W_m$ we write $F_w = F_{w_1} \circ F_{w_2} \circ ... \circ F_{w_m}$ and the abbreviations $K_w := F_w(K)$ and $V_w := F_w(V_0)$. For two different words w, w' of the same length we have $K_w \cap K_{w'} = V_w \cap V_{w'}$, see for instance [Kig01, Proposition 1.3.5 (2)]. We write $V_m := \cup_{|w|=m} V_w$ for $m \geq 1$ and note that $V_m \subset V_{m+1}$, $m \geq 0$, and we use the notation $V_* := \cup_{m\geq 0} V_m$. See [Kig01, Lemma 1.3.11].

We assume that $((\mathcal{E}_m, \ell(V_m))_m$ is a sequence of Dirichlet forms associated with a regular harmonic structure on K, i.e. there are $0 < r_i < 1$, $i = 1, ..., N$, and a Dirichlet form $\mathcal{E}_0(u) = \frac{1}{2}\sum_{p\in V_0}\sum_{q\in V_0} c(0; p, q)(u(p) - u(q))^2$ on $\ell(V_0)$ so that for all $m \geq 1$ we have

$$\mathcal{E}_m(u, v) = \sum_{w\in W_m} r_w^{-1}\mathcal{E}_0(u \circ F_w, v \circ F_w), \quad u, v \in \ell(V_m), \tag{5.16}$$

where $r_w := r_{w_1} \dots r_{w_m}$ for $w = w_1...w_m$, and $(\mathcal{E}_{m+1})_{V_m} = \mathcal{E}_m$ for all $m \geq 0$ (here we use notation (3.2)). See [Kig01, Definitions 3.1.1 and 3.1.2]. In this case the limit form

$$\mathcal{E}(u) := \lim_{m\to\infty} \mathcal{E}_m(u) \tag{5.17}$$

with domain $\{u \in \ell(V_*) : \lim_{m\to\infty} \mathcal{E}_m(u) < +\infty\}$ is a resistance form on V_*, the completion of V_* with respect to the associated resistance metric R is (K, R), and this space is compact, [Kig01, Theorem 3.3.4]. Each function from this domain extends uniquely to a continuous function on K, and writing $\mathcal{F}$ for the space of these continuous extensions to K, we obtain a local regular resistance form $(\mathcal{E}, \mathcal{F})$ on K, see the proof of [Kig01, Theorem 3.4.6]. The form $(\mathcal{E}, \mathcal{F})$ is self-similar in the sense that for any fixed m we have

$$\mathcal{E}(u) = \sum_{w\in W_m} \mathcal{E}_{K_w}(u), \quad u \in \mathcal{F},$$

where $\mathcal{E}_{K_w}(u) := r_w^{-1}\mathcal{E}(u\circ F_w)$, see [Kig01, Proposition 3.3.1] and equation (3.3.1) following it. For a fixed word $w \in W_m$ of length $|w| = m$ the form $\mathcal{E}_{K_w}$ satisfies

$$(u(x) - u(y))^2 \leq r_w\, \mathcal{E}_{K_w}(u) \tag{5.18}$$

for any $u \in \mathcal{F}$ and any $x, y \in K_w$.

5.3 An example of a non-finitely ramified fractal

In this section, we discuss briefly the Sierpiński carpet (see Figure 5.1) as a simple example of an infinitely ramified fractal.

The Sierpiński carpet is defined in the following way: Let $X = \mathbb{C}$ and let $p_1 = 0$, $p_2 = \frac{1}{2}$, $p_3 = 1$, $p_4 = 1 + \frac{\sqrt{-1}}{2}$, $p_5 = 1 + \sqrt{-1}$, $p_6 = \frac{1}{2} + \sqrt{-1}$ $p_7 = \sqrt{-1}$ and $p_8 = \frac{\sqrt{-1}}{2}$. We set

$$F_i(z) = \frac{(z - p_i)}{3} + p_i \qquad \text{for} \qquad p_i = 1, 2, \dots, 8.$$

The self-similar set K_{SC} with respect to $\{F_i\}_{i=1,2,...,8}$ is called the Sierpiński carpet.

So, the Sierpiński carpet consists of eight smaller copies of itself and each pair of adjacent smaller copies intersects on a one-dimensional interval. Note that the corresponding self-similar structure $(K_{SC}, \{1, 2, 3, 4, 5, 6, 7, 8, \}, \{F_i\}_{i=1,2,...,8})$ is *not* post critically finite [Kig01, Example 1.3.17]. In fact, $\mathcal{C}_{K_{SC}}$, $\mathcal{C}$ and $\mathcal{P}$ are infinte sets. In particular, V_0 equals the boundary of the unit square $[0, 1] \times [0, 1]$.

Remark 5.5. The Sierpiński carpet is much harder to study than the more familiar Sierpiński gasket. This is because the carpet is not finitely ramified, while the gasket is finitely ramified. The difficulty of working on the Sierpiński carpet, as well its beauty, attracts several mathematicians to attack various problems on such spaces. In their breakthrough paper [BB89], Barlow and Bass have constructed a Brownian motion on the Sierpiński carpet as the scaling limit of a sequence of reflected Brownian motions on Euclidean domains approximating the carpet. Subsequently, Kusuoka and Yin [KY92] were able to give a different construction of a Brownian motion. They used graph approximations of

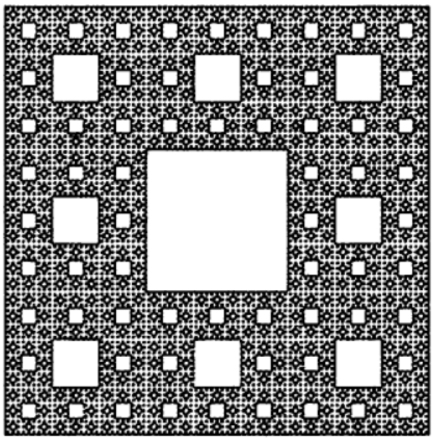

Figure 5.1: Sierpiński carpet, [Kig01, Fig. 0.4 on page 6]

the carpet and Dirichlet form techniques. In particular, the two Brownian motions given by [BB89] and [KY92] are the same. Uniqueness of such a Brownian motion was an open question for more than two decades and was finally proven by Barlow, Bass, Kumagai and Teplyaev [BBKT10].

Part II

Existence and uniqueness results

The purpose of this part is the discussion of existence and uniqueness results for equations on fractal spaces. Our results concern equations involving first order differential operators and also vector valued equations and are based on the first order calculus for Dirichlet forms proposed by Cipriani and Sauvageot [CS03; CS09].

Part II is organized as follows.
Chapter 6 is devoted to the discussion of Dirichlet problems for linear equations of elliptic and parabolic type where the linear operators have principal part in divergence form. Chapter 7 deals with two possible formulations of the viscous Burgers equation on compact resistance spaces. Here, we focus on well-posedness of Cole-Hopf solutions which solve vector-valued Burgers equations. In Chapter 8 we present an existence result for a first order equation of continuity type. We would like to point out that for the results in this part so far, it is not necessary that the considered fractals are post critically finite. Finally, we discuss existence of minimizers for convex functionals on metric measure spaces that carry a strongly local regular Dirichlet form in Chapter 9.

Chapter 6

Linear equations of elliptic and parabolic type on resistance spaces

The considerations in this chapter are straightforward generalizations of the standard theory for partial differential equations, [GT01, Chapter 8], and Dirichlet forms, [FOT94], see for instance [FK04].

Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a nonempty set X so that (X, R) is separable and locally compact and assume that $(\mathcal{E}, \mathcal{F})$ is regular. Let μ be a Borel measure on (X, R) such that for any $x \in X$ and $R > 0$ we have $0 < \mu(B(x, R)) < +\infty$. Then by [Kig12, Theorem 9.4] the form $(\mathcal{E}, \mathcal{F} \cap C_c(X))$ is closable on $L^2(X, \mu)$ and its closure, which we denote by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, is a regular Dirichlet form. Recall that if (X, R) is compact then $\mathcal{D}(\mathcal{E}) = \mathcal{F}$.

By the closedness of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ the derivation ∂, defined as in the preceding chapter, extends to a closed unbounded linear operator $\partial : L^2(X, \mu) \to \mathcal{H}$ with domain $\mathcal{D}(\mathcal{E})$, we write $\operatorname{Im} \partial$ for the image of $\mathcal{D}(\mathcal{E})$ under ∂. The adjoint operator $(\partial^*, \mathcal{D}(\partial^*))$ of $(\partial, \mathcal{D}(\mathcal{E}))$ can be interpreted as minus the divergence operator, and for the generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ we have $\partial f \in \mathcal{D}(\partial^*)$ whenever $f \in \mathcal{D}(\mathcal{L})$. Remember in this case we have $\mathcal{L}f = -\partial^* \partial f$.

6.1 Coercive closed forms

We call a symmetric bounded linear operator $a : \mathcal{H} \to \mathcal{H}$ a *uniformly elliptic symmetric coefficient* if there are universal constants $0 < \lambda < \Lambda$ such that

$$\lambda \|v\|_{\mathcal{H}}^2 \le \langle a\, v, v \rangle_{\mathcal{H}} \le \Lambda \|v\|_{\mathcal{H}}^2, \quad v \in \mathcal{H}. \tag{6.1}$$

We follow [FK04] and say that an element $b \in \mathcal{H}$ is in the *Hardy class* if there are constants $\delta(b) \in (0, \infty)$ and $\gamma(b) \in [0, \infty)$ such that

$$\|g \cdot b\|_{\mathcal{H}}^2 \le \delta(b)\mathcal{E}(g) + \gamma(b) \|g\|_{L^2(X,\mu)}^2, \quad g \in \mathcal{F} \cap C_c(X). \tag{6.2}$$

Given uniformly elliptic a as in (6.1), b, $\hat{b} \in \mathcal{H}$ in the Hardy class and $c \in L^\infty(X, \mu)$ we consider the bilinear form on $\mathcal{F} \cap C_c(X)$ defined by

$$\mathcal{Q}(f, g) = \langle a \cdot \partial f, \partial g \rangle_{\mathcal{H}} - \langle g \cdot b, \partial f \rangle_{\mathcal{H}} - \left\langle f \cdot \hat{b}, \partial g \right\rangle_{\mathcal{H}} - \langle cf, g \rangle_{L^2(X,\mu)}, \quad f, g \in \mathcal{F} \cap C_c(X). \tag{6.3}$$

Definition 6.1. *(see also [MR92, Def. 2.4]) We say that a densely defined bilinear form $(Q, \mathcal{D}(Q))$ on $L^2(X, \mu)$ is a* coercive closed form *if* *(i)* $(\mathcal{D}(Q), \tilde{Q}_1)$ *is a Hilbert space, where* $\tilde{Q}$ *denotes the symmetric part of* Q*, defined by*

$$\tilde{Q}(f,g) = \frac{1}{2}\left(Q(f,g) + Q(g,f)\right), \quad f,g \in \mathcal{D}(Q).$$

(ii) $(Q, \mathcal{D}(Q))$ *satisfies the weak sector condition, i.e. there exists a constant* $K > 0$ *such that*

$$|\mathcal{Q}_1(f,g)| \leq K \mathcal{Q}_1(f)^{\frac{1}{2}} \mathcal{Q}_1(g)^{\frac{1}{2}}, \qquad f,g \in \mathcal{D}(Q)).$$

Proposition 6.1. *Assume that* $a : \mathcal{H} \to \mathcal{H}$ *is a uniformly elliptic symmetric coefficient satisfying (6.1),* $b, \hat{b} \in \mathcal{H}$ *are in the Hardy class and such that*

$$\lambda_0 := \frac{1}{2}\left(\lambda - \sqrt{\delta(b)} - \sqrt{\delta(\hat{b})}\right) > 0 \tag{6.4}$$

and $c \in L^\infty(X,\mu)$ *is such that*

$$c_0 := \operatorname*{ess\,inf}_{x \in X}\left(-c(x)\right) - \frac{\gamma(b) + \gamma(\hat{b})}{2\lambda_0} > 0. \tag{6.5}$$

Then $(\mathcal{Q}, \mathcal{F} \cap C_c(X))$ *extends to a coercive closed form* $(\mathcal{Q}, \mathcal{D}(\mathcal{E}))$ *on* $L^2(X,\mu)$ *such that*

$$\lambda_0\, \mathcal{E}(f) + c_0 \|f\|^2_{L^2(X,\mu)} \leq \mathcal{Q}(f) \leq \Lambda_\infty\, \mathcal{E}(f) + c_\infty \|f\|^2_{L^2(X,\mu)}, \quad f \in \mathcal{D}(\mathcal{E}), \tag{6.6}$$

where

$$\Lambda_\infty := \Lambda + \sqrt{\delta(b)} + \sqrt{\delta(\hat{b})} + 1 \quad \text{and} \quad c_\infty := \frac{\gamma(b) + \gamma(\hat{b})}{2} + \|c\|_{L^\infty(X,\mu)}. \tag{6.7}$$

Proof. Let $f \in \mathcal{F} \cap C_c(X)$. We first observe that from Cauchy Schwarz, (6.1) and (6.2) it follows that

$$\begin{aligned}
\mathcal{Q}(f) &\leq \Lambda\, \mathcal{E}(f) + \left(\|f \cdot b\|_{\mathcal{H}} + \|f \cdot \hat{b}\|_{\mathcal{H}}\right) \mathcal{E}(f)^{\frac{1}{2}} + \|c\|_{L^\infty(X,\mu)} \|f\|^2_{L^2(X,\mu)} \\
&\leq \Lambda\, \mathcal{E}(f) + \left(\sqrt{\delta(b)\, \mathcal{E}(f) + \gamma(b)\|f\|^2_{L^2(X,\mu)}} + \sqrt{\delta(\hat{b})\, \mathcal{E}(f) + \gamma(\hat{b})\|f\|^2_{L^2(X,\mu)}}\right) \mathcal{E}(f)^{\frac{1}{2}} \\
&\quad + \|c\|_{L^\infty(X,\mu)} \|f\|^2_{L^2(X,\mu)} \\
&\leq \left(\Lambda + \sqrt{\delta(b)} + \sqrt{\delta(\hat{b})}\right) \mathcal{E}(f) + \left(\sqrt{\gamma(b)} + \sqrt{\gamma(\hat{b})}\right) \mathcal{E}(f)^{\frac{1}{2}} \|f\|_{L^2(X,\mu)} \\
&\quad + \|c\|_{L^\infty(X,\mu)} \|f\|^2_{L^2(X,\mu)}.
\end{aligned}$$

Using Young's inequality we obtain

$$\mathcal{Q}(f) \leq \Lambda_\infty\, \mathcal{E}(f) + c_\infty \|f\|^2_{L^2(X,\mu)}, \qquad f \in \mathcal{F} \cap C_c(X), \tag{6.8}$$

with Λ_∞ and c_∞ defined as in (6.7). Similarly, we have

$$\begin{aligned}
\mathcal{Q}(f) &\geq \lambda\, \mathcal{E}(f) - \left(\|f \cdot b\|_{\mathcal{H}} + \|f \cdot \hat{b}\|_{\mathcal{H}}\right) \mathcal{E}(f)^{\frac{1}{2}} - \langle cf, f \rangle_{L^2(X,\mu)} \\
&\geq 2\lambda_0\, \mathcal{E}(f) - \left(\sqrt{\gamma(b)} + \sqrt{\gamma(\hat{b})}\right) \mathcal{E}(f)^{\frac{1}{2}} \|f\|_{L^2(X,\mu)} - \langle cf, f \rangle_{L^2(X,\mu)}
\end{aligned}$$

with λ_0 defined as in (6.4). Now using again Young's inequality we obtain the lower bound

$$\begin{aligned}
\mathcal{Q}(f) &\geq \left(2\lambda_0 - \frac{\lambda_0}{2}\right) \mathcal{E}(f) + \left(\operatorname*{ess\,inf}_{x \in X}\left(-c(x)\right) - \frac{\gamma(b) + \gamma(\hat{b})}{2\lambda_0}\right) \|f\|^2_{L^2(X,\mu)} \\
&\geq \lambda_0\, \mathcal{E}(f) + c_0 \|f\|^2_{L^2(X,\mu)}, \qquad f \in \mathcal{F} \cap C_c(X),
\end{aligned} \tag{6.9}$$

where c_0 is defined as in (6.5).

Now let $(u_n)_{n\in\mathbb{N}}$, $u_n \in \mathcal{F}\cap C_c(X)$, $n\in\mathbb{N}$, be a sequence such that $\lim_{n\to\infty} u_n = 0$ in $L^2(X,\mu)$ and $\lim_{n,m\to\infty} \mathcal{Q}(u_n-u_m) = 0$ in $\mathcal{F}\cap C_c(X)$. By (6.9), $(u_n)_n$ also satisfies $\lim_{n,m\to\infty} \mathcal{E}(u_n-u_m) = 0$ in $\mathcal{F}\cap C_c(X)$. Recall that $(\mathcal{E},\mathcal{F}\cap C_c(X))$ is a symmetric closable form and extends to the closed coercive form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on $L^2(X,\mu)$. Hence, we have $\lim_{n\to\infty}\mathcal{E}(u_n) = 0$ in $\mathcal{F}\cap C_c(X)$ and by (6.8), $\lim_{n\to\infty}\mathcal{Q}(u_n) = 0$ in $\mathcal{F}\cap C_c(X)$. Consequently, $(\mathcal{Q},\mathcal{F}\cap C_c(X))$ is closable. Moreover, application of (6.1) and (6.2) yield

$$\begin{aligned}
|\mathcal{Q}_1(f,g)| &\le \Lambda\,\mathcal{E}(f)^{\frac12}\,\mathcal{E}(g)^{\frac12} + \|g\cdot b\|_{\mathcal{H}}\,\mathcal{E}(f)^{\frac12} + \|f\cdot\hat{b}\|_{\mathcal{H}}\,\mathcal{E}(g)^{\frac12}\\
&\quad + \left(1+\|c\|_{L^\infty(X,\mu)}\right)\|f\|_{L^2(X,\mu)}\|g\|_{L^2(X,\mu)}\\
&\le \left(\Lambda + \sqrt{\delta(b)} + \sqrt{\delta(\hat{b})}\right)\mathcal{E}(f)^{\frac12}\,\mathcal{E}(g)^{\frac12} + \sqrt{\gamma(b)}\,\mathcal{E}(f)^{\frac12}\|g\|_{L^2(X,\mu)}\\
&\quad + \sqrt{\gamma(\hat{b})}\,\mathcal{E}(g)^{\frac12}\|f\|_{L^2(X,\mu)} + \left(1+\|c\|_{L^\infty(X,\mu)}\right)\|f\|_{L^2(X,\mu)}\|g\|_{L^2(X,\mu)}.
\end{aligned}$$

Using (6.9), we obtain

$$|\mathcal{Q}_1(f,g)| \le K\,\mathcal{Q}_1(f)^{\frac12}\mathcal{Q}_1(g)^{\frac12}, \qquad f,g\in\mathcal{F}\cap C_c(X), \tag{6.10}$$

where $K := \frac{1}{\lambda_0}\left(\Lambda + \sqrt{\delta(b)} + \sqrt{\delta(\hat{b})} + \sqrt{\gamma(b)} + \sqrt{\gamma(\hat{b})}\right) + 1 + \|c\|_{L^\infty(X,\mu)}$. Thus, we have proved that $(\mathcal{Q},\mathcal{F}\cap C_c(X))$ satisfies the weak sector condition.

Note that $\mathcal{F}\cap C_c(X)$ is dense in $L^2(X,\mu)$. By continuity inequalities (6.8), (6.9) and (6.10) extend to all $f,g\in\mathcal{D}(\mathcal{E})$, so we finished to prove that $(\mathcal{Q},\mathcal{D}(\mathcal{E}))$ is a closed coercive form on $L^2(X,\mu)$ in the sense of Definition 6.1 such that (6.6) is satisfied. □

Remark 6.1. These conditions are chosen for convenience, we do not claim their optimality. We would like to point out that certain standard estimates, as for instance used in [Suz18], do not apply unless one assumes that energy measures are absolutely continuous with respect to μ, an assumption we wish to avoid.

Suppose that the hypotheses of Proposition 6.1 are satisfied. Let $(\mathcal{L}^{(\mathcal{Q})},\mathcal{D}(\mathcal{L}^{(\mathcal{Q})}))$ denote the infinitesimal generator of $(\mathcal{Q},\mathcal{D}(\mathcal{E}))$, that is, the unique closed operator on $L^2(X,\mu)$ associated with $(\mathcal{Q},\mathcal{D}(\mathcal{E}))$ by the identity

$$\mathcal{Q}(f,g) = -\left\langle \mathcal{L}^{\mathcal{Q}} f, g\right\rangle_{L^2(X,\mu)}, \quad f\in\mathcal{D}(\mathcal{L}^{\mathcal{Q}}),\ g\in\mathcal{D}(\mathcal{E}).$$

A direct calculation shows the following.

Corollary 6.1. *Let the hypotheses of Proposition 6.1 be satisfied and let notation be as there. The generator* $(\mathcal{L}^{(\mathcal{Q})},\mathcal{D}(\mathcal{L}^{(\mathcal{Q})}))$ *satisfies the sector condition*

$$|\left\langle(-\mathcal{L}^{\mathcal{Q}}-\varepsilon)f,g\right\rangle_{L^2(X,\mu)}| \le K\left\langle(-\mathcal{L}^{\mathcal{Q}}-\varepsilon)f,f\right\rangle_{L^2(X,\mu)}^{1/2}\left\langle(-\mathcal{L}^{\mathcal{Q}}-\varepsilon)g,g\right\rangle_{L^2(X,\mu)}^{1/2}, \tag{6.11}$$

$f,g\in\mathcal{D}(\mathcal{L}^{\mathcal{Q}})$, *with same sector constant* K *as in* (6.10) *and uniformly for all* $0\le\varepsilon\le c_0/2$.

6.2 Linear elliptic and parabolic problems

Suppose throughout this section that a, b, $\hat{b}$ and c satisfy the hypotheses of Proposition 6.1. It is straightforward to formulate equations of elliptic type. Given $f\in L^2(X,\mu)$, we say that $u\in L^2(X,\mu)$ is a *weak solution* to

$$\mathcal{L}^{\mathcal{Q}}u = f \tag{6.12}$$

if $u\in\mathcal{D}(\mathcal{E})$ and $\mathcal{Q}(u,g) = -\langle f,g\rangle_{L^2(X,\mu)}$ for all $g\in\mathcal{D}(\mathcal{E})$.

Remark 6.2. Formally, the generator $(\mathcal{L}^{\mathcal{Q}}, \mathcal{D}(\mathcal{L}^{\mathcal{Q}}))$ of $(\mathcal{Q}, \mathcal{D}(\mathcal{E}))$ has the structure

$$\mathcal{L}^{\mathcal{Q}}u = -\partial^*(a\,\partial u) + b \cdot \partial u + \partial^*(u \cdot \hat{b}) + cu,$$

so that equation (6.12) is an abstract version of the elliptic equation

$$\operatorname{div}(a\nabla u) + b \cdot \nabla u - \operatorname{div}(ub) + cu = f.$$

It follows from the lower estimate in (6.6) that the Green operator $G^{\mathcal{Q}} = (-\mathcal{L}^{\mathcal{Q}})^{-1}$ of $\mathcal{L}^{\mathcal{Q}}$ exists as a bounded linear operator $G^{\mathcal{Q}} : L^2(X, \mu) \to L^2(X, \mu)$ and satisfies

$$\mathcal{Q}(G^{\mathcal{Q}}f, g) = \langle f, g \rangle_{L^2(X,\mu)}, \quad f \in L^2(X, \mu),\ g \in \mathcal{D}(\mathcal{E}). \tag{6.13}$$

Remark 6.3. Actually, we can weaken our assumption on the function c. In the case that $c \in L^\infty(X, \mu)$ does not satisfy (6.5), we just consider the modified equation

$$\mathcal{L}^{\mathcal{Q}}u - \check{c}u = f, \tag{6.14}$$

where $\check{c} \in \mathbb{R}$ is an arbitrary, positive fixed constant such that $c_0 + \check{c} > 0$ holds. Then the form $\left(\check{\mathcal{Q}}, \mathcal{F} \cap C_c(X)\right)$,

$$\check{\mathcal{Q}}(f, g) = \mathcal{Q}(f, g) + \check{c}\langle f, g \rangle_{L^2(X,\mu)}, \quad f, g \in \mathcal{F} \cap C_c(X), \tag{6.15}$$

can be extended to a closed form $\left(\check{\mathcal{Q}}, \mathcal{D}(\mathcal{Q})\right)$ by Proposition 6.1 and in particular, (6.6) holds with $c_0 + \check{c}$ instead of c_0 and (6.6), (6.7) with $\|c\|_{L^\infty(X,\mu)} + \check{c}$ instead of $\|c\|_{L^\infty(X,\mu)}$.

Corollary 6.2. *For any $f \in L^2(X, \mu)$ the function $u = -G^{\mathcal{Q}}f \in \mathcal{D}(\mathcal{L}^{\mathcal{Q}})$ is the unique weak solution to (6.12). It satisfies*

$$\mathcal{Q}_1(u) \leq \left(\frac{2}{c_0} + \frac{4}{c_0^2}\right) \|f\|^2_{L^2(X,\mu)}. \tag{6.16}$$

Remark 6.4. The constant in (6.16) is chosen just for convenience. The only fact that matters is that it may be chosen in a way that depends monotonically on c_0.

Proof. The first part is clear, the second follows from (6.13), Cauchy-Schwarz and because for any $0 < \varepsilon \leq c_0/2$ with c_0 as in (6.5) the operator $\mathcal{L}^{\mathcal{Q}} + \varepsilon$ generates a strongly continuous contraction semigroup, so that

$$\left\|G^{\mathcal{Q}}f\right\|_{L^2(X,\mu)} = \left\|\left(\varepsilon + (-\varepsilon - \mathcal{L}^{\mathcal{Q}})\right)^{-1} f\right\|_{L^2(X,\mu)} \leq \frac{1}{\varepsilon}\|f\|_{L^2(X,\mu)}.$$

□

Related parabolic problems can be discussed in a similar manner. Given $\mathring{u} \in L^2(X, \mu)$ we say that a function $u : (0, +\infty) \to L^2(X, \mu)$ is a *solution* to the Cauchy problem

$$\partial_t u(t) = \mathcal{L}^{\mathcal{Q}}u(t),\ t > 0, \quad u(0) = \mathring{u}, \tag{6.17}$$

if u is an element of $C^1((0, +\infty), L^2(X, \mu)) \cap C([0, +\infty), L^2(X, \mu))$, we have $u(t) \in \mathcal{D}(\mathcal{L}^{\mathcal{Q}})$ for any $t > 0$ and (6.17) holds. See [Paz83, Chapter 4, Section 1].

Remark 6.5. Problem (6.17) is an abstract version of the parabolic problem

$$\partial_t u(t) = \operatorname{div}(a\nabla u(t)) + b \cdot \nabla u(t) - \operatorname{div}(u(t)b) + cu(t),\ t > 0, \quad u(0) = \mathring{u}.$$

Let $(T_t^{\mathcal{Q}})_{t>0}$ denote the strongly continuous contraction semigroup on $L^2(X, \mu)$ generated by the operator $\mathcal{L}^{\mathcal{Q}}$. The following is standard.

Corollary 6.3. *For any* $\mathring{u} \in L^2(X,\mu)$ *the Cauchy problem (6.17) has the unique solution* $u(t) = T_t^{\mathcal{Q}}\mathring{u}$, $t > 0$. *For any* $t > 0$ *it satisfies* $u(t) \in \mathcal{D}(\mathcal{L}^{\mathcal{Q}})$ *and*

$$\mathcal{Q}_1(u(t)) \leq \left(\frac{C_K}{t} + 1\right) \|\mathring{u}\|^2_{L^2(X,\mu)}, \tag{6.18}$$

where $C_K > 0$ *is a constant depending only on the sector constant* K *in (6.11).*

Proof. Again the first part of the Corollary is standard. To see estimate (6.18) recall that the operator $(\mathcal{L}^{\mathcal{Q}}, \mathcal{D}(\mathcal{L}^{\mathcal{Q}}))$ satisfies the sector condition (6.11). Consequently the semigroup $(T_t^{\mathcal{Q}})_{t>0}$ generated by $(\mathcal{L}^{\mathcal{Q}} + \varepsilon, \mathcal{D}(\mathcal{L}^{\mathcal{Q}}))$ extends to a holomorphic contraction semigroup on the sector $\{z \in \mathcal{C} : |\operatorname{Im} z| \leq K^{-1}\operatorname{Re} z\}$, see for instance [Kat95, Chapter XI, Theorem 1.24], or [MR92, Theorem 2.20 and Corollary 2.21]. By (6.6) zero is contained in the resolvent set of $\mathcal{L}^{\mathcal{Q}}$. This implies that for any $t > 0$ we have

$$\|\mathcal{L}^{\mathcal{Q}} T_t^{\mathcal{Q}} f\|_{L^2(X,\mu)} \leq \frac{C_K}{t} \|f\|_{L^2(X,\mu)}, \quad f \in L^2(X,\mu), \tag{6.19}$$

for some $C_K \in (0,\infty)$ depending only on the sector constant K, as an inspection of the classical proofs of (6.19) shows, see for instance [EN00, Theorem 4.6] or [Paz83, Section 2.5, Theorem 5.2]. Now (6.18) follows using (6.19), Cauchy-Schwarz and contractivity. □

6.3 Comments on the coefficients

It is a trivial observation that if $a \in C(X)$ satisfies

$$\lambda < a(x) < \Lambda, \quad x \in X, \tag{6.20}$$

then a, interpreted as a bounded linear map $v \mapsto a \cdot v$ from $\mathcal{H}$ into itself, satisfies the uniform ellipticity condition (6.1).

Remark 6.6. Our main interest is to understand the drift terms and therefore we restrict attention to coefficients a of form (6.20). A discussion of more general a should involve suitable coordinates, see [Hin08; HT15b; Tep08]. It is well known that natural local energy forms on p.c.f. self-similar sets have pointwise index one, [BK19; Kus93; Hin10], which means that for such examples basically all continuous diffusion coefficients a will be functions $a \in C(X)$.

Under certain geometric assumptions on (X,R) and μ we can observe that any vector field $b \in \mathcal{H}$ satisfies the Hardy condition. Recall that (X,R) is called *metrically doubling with doubling constant* $K_R > 1$ if for any $x \in X$ and any $r > 0$ the ball $B(x,r)$ can be covered by K_R balls of radius r. We say that the measure μ *has a uniform lower bound* V if V is an increasing function $V : (0,+\infty) \to (0,+\infty)$ so that

$$\mu(B(x,r)) \geq V(r), \quad x \in X, \quad r > 0. \tag{6.21}$$

The key part of the following proposition is a partial refinement of [HR16, Lemma 4.2].

Proposition 6.2. *Suppose that the space* (X,R) *is metrically doubling with doubling constant* $K_R > 1$ *and that* μ *has the uniform lower bound* V. *Then for any* $g \in \mathcal{F} \cap C_c(X)$, *any* $b \in \mathcal{H}$ *and any* $M > 0$ *we have*

$$\|g \cdot b\|^2_{\mathcal{H}} \leq \frac{1}{M}\mathcal{E}(g) + \mathcal{V}(M\|b\|^2_{\mathcal{H}})\|b\|^2_{\mathcal{H}}\|g\|^2_{L^2(K,\mu)}, \tag{6.22}$$

where $\mathcal{V}$ *is the increasing function defined by*

$$\mathcal{V}(s) = \frac{2K_R}{V\left(\frac{1}{2K_R s}\right)}, \quad s > 0.$$

In particular, any $b \in \mathcal{H}$ is in the Hardy class, and for any $M > 0$ it satisfies the estimate (6.2) with

$$\delta(b) = \frac{1}{M} \quad \text{and} \quad \gamma(b) = \mathcal{V}(M\,\|b\|_{\mathcal{H}}^2)\,\|b\|_{\mathcal{H}}^2\,.$$

Moreover, for any $\lambda > 0$ condition (6.4) holds if we choose $M > 2/\lambda$ for both b and $\hat{b}$.

A proof of an inequality of type (6.22) was already given in [HR16, Lemma 4.2], but the function $\mathcal{V}$ had not been specified there. We include the proof for completeness.

Proof. We may assume $\|b\|_{\mathcal{H}} > 0$.

In the first step we use the metric doubling property to conclude that X can be covered by finitely many balls $B_j = B(x_j, 2r)$ with $r \in \left(0, (4MK_R\,\|b\|_{\mathcal{H}}^2)^{-1}\right]$ such that

$$\sum_j \mathbf{1}_{B_j} \le K_R.$$

Fix $0 < r \le (4MK_R\,\|b\|_{\mathcal{H}}^2)^{-1}$. We note that a maximal set $\{B(x_j, r)\}$ of disjoint balls has the property that $\cup_j B_j \supset X$. Consider $x_{j_1}, \dots, x_{j_n} \in B(x, 2r)$. Thus, covering by K_R balls $B(y_k, r)$, we see each x_{ji} is in some $B(y_k, r)$ but no two can be in the same ball $B(y_k, r)$ else $y_k \in B(x_{j_l}, r) \cap B(x_{j_{l'}}, r)$ contradicts disjointness, so $n \le K_R$.

In the next step we use the cover to estimate $g(x)^2$. Suppose that $g \in \mathcal{F} \cap C_c(X)$. Given the case that $x \in B_j$ Jensen's inequality together with the resistance estimate (3.1) imply that

$$\begin{aligned}
|g(x)|^2 &\le 2\left|\frac{1}{\mu(B_j)}\int_{B_j} g(x) - g(y)d\mu(y)\right|^2 + 2\left|\frac{1}{\mu(B_j)}\int_{B_j} g(y)d\mu(y)\right|^2 \\
&\le 2\left(\frac{1}{\mu(B_j)}\int_{B_j} |g(x) - g(y)|d\mu(y)\right)^2 + 2\frac{1}{\mu(B_j)}\int_{B_j} g^2(y)d\mu(y) \\
&\le 2\left(\frac{1}{\mu(B_j)}\,\mathcal{E}(g)^{\frac{1}{2}}\int_{B_j} R(x,y)^{\frac{1}{2}}d\mu(y)\right)^2 + 2\frac{1}{\mu(B_j)}\int_{B_j} g^2(y)d\mu(y) \\
&\le 4\,\mathcal{E}(g)r + 2\frac{1}{\mu(B_j)}\int_{B_j} g^2(y)d\mu(y).
\end{aligned}$$

For any $x \in X$ we have

$$\begin{aligned}
|g(x)|^2 &\le \sum_j |g(x)|^2 \mathbf{1}_{B_j}(x) \\
&\le \sum_j \left(4\,\mathcal{E}(g)r + 2\frac{1}{\mu(B_j)}\int_{B_j} g^2(y)d\mu(y)\right)\mathbf{1}_{B_j}(x) \\
&\le 4K_R\,\mathcal{E}(g)r + 2\sum_j \frac{\mathbf{1}_{B_j}(x)}{\mu(B_j)}\int_{B_j} g^2(y)d\mu(y).
\end{aligned}$$

The lower uniformity property (6.21) yields a bound for the last term,

$$2K_R\|g\|_{L^2(K,\mu)}^2 (V(2r))^{-1}.$$

Our choice of r leads to the simplification

$$|g(x)|^2 \le \frac{\mathcal{E}(g)}{M\|b\|_{\mathcal{H}}} + 2K_R\|g\|_{L^2(K,\mu)}^2 \left(V\left(\frac{1}{2MK_R\|b\|_{\mathcal{H}}^2}\right)\right)^{-1}.$$

Finally, it suffices to use (4.14) to obtain the desired inequality. □

Chapter 7

The viscous Burgers equation

Here our main aim is to propose a formulation of Burgers equation (1.4) on compact resistance spaces that can be solved using the Cole-Hopf transform. More precisely, we use the Cole-Hopf transform to verify existence and uniqueness of solutions in the case the initial condition is a gradient of an energy finite function. Also their continuous dependence on the initial condition is addressed. For notational simplicity we consider the viscosity $\sigma = 1$ only.

The results of this chapter are based on joint work with Michael Hinz [HM20b].

7.1 Different formulations of the formal problem

Two conceptually different generalizations of the Burgers equation emerge. The interpretation of (1.4) (with $\sigma = 1$) as a semilinear heat equation for scalar functions motivates a formulation of Burgers equation as the formal problem

$$\begin{cases} g_t(t) &= -d^* dg(t) - \frac{1}{2} d(g^2)(t), \\ g(0) &= g_0, \end{cases} \tag{7.1}$$

where we symbolically write d for the gradient operator taking a function into a vector field and d^* for its adjoint (such that $-d^*$ is the divergence operator). This is a semilinear heat equation for the Laplacian $-d^*d$ acting on functions. In [LQ19] it had been implemented as an L^2-Cauchy problem on the Sierpiński gasket, endowed with the natural self-similar Hausdorff measure, and the authors showed existence, uniqueness and regularity of weak solutions for (7.1) with Dirichlet boundary conditions. As discussed in [LQ18; LQ19], this model is naturally related to control theory and (backward) stochastic differential equations. However, it cannot be solved using the Cole-Hopf transform.

An alternative viewpoint upon (1.4) is to interpret it as an equation for vector fields, similar to (1.5). This suggests to formulate Burgers equation as the formal problem

$$\begin{cases} u_t(t) &= -d\, d^* u(t) - \frac{1}{2} d(u^2)(t), \\ u(0) &= u_0. \end{cases} \tag{7.2}$$

Here $-d\,d^*$ is the Laplacian acting on vector fields, so that (7.2) has to be seen as a vector equation and it can be implemented using first order calculus, [CS03; HRT13; IRT12].

The difference between (7.1) and (7.2) admits a very natural interpretation if one considers these equations on metric graphs, [BK19; BLS09; EP07; FKW07; Hae11; KS99; KS00; Kuc04; Kuc05; Mug14]. In this case (7.1) is a semilinear heat equation for the Laplacian $d^*\, d$ with Kirchhoff vertex conditions, while (7.2) employs the Laplacian $d\, d^*$ with another different type of vertex conditions.

In the following, we study (7.2) on sets X endowed with a local regular resistance form and a fairly general volume measure.

7.2 Heat and Burgers equation on metric graphs

In this section we provide adequate formulations of (7.1) and (7.2) on metric graphs defined as in Section 5.1 and prove existence, uniqueness and continuous dependence on initial conditions for (7.2).

7.2.1 Kirchhoff Burgers equation

On the unit interval the viscous Burgers equation is given by (1.4). If we now consider Burgers equation with respect to Dirichlet boundary conditions, existence and uniqueness for arbitrary finite time horizons can for instance be obtained in a monotone operator setup, [Liu11, Theorem 1.1 and Example 3.2]. If endowed with Neumann boundary conditions, the unit interval $[0,1]$ can be seen as the metric graph having only the single edge $e=(0,1)$ and vertex set $V=\{i(e),j(e)\}$, and this suggests to generalize the Cauchy problem for (1.4) to a compact connected metric graph Γ by considering the formal problem (7.1). There are various ways to formulate (7.1) rigorously as a Cauchy problem

$$\begin{cases} g_t(t) &= \mathcal{L}_\Gamma g(t) - \frac{1}{2} d(g^2)(t), \\ g(0) &= g_0 \end{cases} \tag{7.3}$$

with initial condition $g_0 \in L^2(X_\Gamma, \mu_\Gamma)$. Imposing additional Dirichlet boundary conditions on a finite subset of Γ and assuming $g_0 \in W^{1,2}(X_\Gamma, \mu_\Gamma)$, one can invoke well known semigroup methods to obtain solutions to (7.3) on Γ for sufficiently small time T, [Paz83, Section 6.3, Theorem 3.1].

We strongly believe that the arguments of [LQ19], which make heavy use of (5.4), can be combined with known heat kernel estimates, see [Hae11] and the references cited there, to obtain global weak solutions under Dirichlet boundary conditions, [LQ19, Definition 4.13].

7.2.2 Burgers equation via Cole-Hopf

As before we assume that Γ is a compact connected metric graph. An alternative generalization of (1.4) to Γ can be obtained applying the Cole-Hopf transform to solutions of the *heat equation*

$$\begin{cases} w_t(t) &= \mathcal{L}_\Gamma w(t), \quad t>0, \\ w(0) &= w_0, \end{cases} \tag{7.4}$$

for the Kirchhoff Laplacian $\mathcal{L}_\Gamma$ as defined in (5.6). This leads to the formal problem (7.2) which in general is different from (7.1).

Assume that $w_0 \in L^2(X_\Gamma, \mu_\Gamma)$ is strictly positive μ_Γ-a.e. The unique solution to (7.4), seen as a Cauchy problem in $L^2(X_\Gamma, \mu_\Gamma)$, is $w(t) = e^{t\mathcal{L}_\Gamma} w_0$. For any $t>0$ the function $w(t)$ is in $W^{1,2}(X_\Gamma, \mu_\Gamma))$, it is bounded, continuous and also strictly positive on X_Γ, because under the stated hypotheses $(e^{t\mathcal{L}_\Gamma})_{t>0}$ is conservative. Therefore, by the chain rule (with respect to t),

$$h := -2 \log w \tag{7.5}$$

defines a differentiable function $h : (0,\infty) \to W^{1,2}(X_\Gamma, \mu_\Gamma)$, and it satisfies the *potential Burgers equation*

$$\langle h_t(t), \varphi \rangle_{L^2(X_\Gamma, \mu_\Gamma)} = \mathcal{L}_\Gamma h(t)(\varphi) - \frac{1}{2} \langle dh(t), dh(t) \rangle (\varphi) \tag{7.6}$$

for any $\varphi \in W^{1,2}(X_\Gamma, \mu_\Gamma)$, where we write $\langle dh(t), dh(t)\rangle(\varphi) := \langle \varphi\, dh(t), dh(t)\rangle_{L^2(X_\Gamma,\mu_\Gamma)}$. See for instance [Olv14, Section 8.4]. Its derivative

$$u(t) := dh(t), \tag{7.7}$$

is a function $u : (0, \infty) \to L^2(X_\Gamma, \mu_\Gamma)$, and writing $u_t(t)(v) := \langle u_t(t), v\rangle_{L^2(X_\Gamma,\mu_\Gamma)}$, $v \in \mathcal{D}(\vec{\mathcal{L}}_\Gamma)$, we can formulate (7.2) rigorously as the Cauchy problem

$$\begin{cases} u_t(t) &= \vec{\mathcal{L}}_\Gamma u(t) - \frac{1}{2} d(u^2)(t), \quad t > 0, \\ u(0) &= u_0, \end{cases} \tag{7.8}$$

where $\vec{\mathcal{L}}_\Gamma u(t) := \vec{\mathcal{L}}_\Gamma(u(t))$ and $d(u^2)(t) := d(u^2(t))$ are understood in terms of the distributional definitions (5.13) and (5.14). We use the following notion of solution.

Definition 7.1. *A function* $u \in C([0, +\infty), L^2(X_\Gamma, \mu_\Gamma)) \cap C^1((0, +\infty), L^2(X_\Gamma, \mu_\Gamma))$ *is called a* solution *to (7.8)* with initial condition $u_0 \in L^2(X_\Gamma, \mu_\Gamma)$ *if u satisfies the first identity in (7.8) in* $(\mathcal{D}(\vec{\mathcal{L}}_\Gamma))^*$ *and the second in* $L^2(X_\Gamma, \mu_\Gamma)$.

7.2.3 Existence and uniqueness results

We first observe the structure of solutions. The space $\operatorname{Im} d$ is a closed subspace of $L^2(X_\Gamma, \mu_\Gamma)$ and $L^2(X_\Gamma, \mu_\Gamma)$ admits the orthogonal decomposition

$$L^2(X_\Gamma, \mu_\Gamma) = \operatorname{Im} d \oplus \ker d^*.$$

Theorem 7.1. *Suppose u is a solution to (7.8) with initial condition u_0. Let $\eta_0 \in \ker d^*$ and $h_0 \in W^{1,2}(X_\Gamma, \mu_\Gamma)$ be such that $u_0 = dh_0 + \eta_0$. Then u is of form*

$$u(t) = dh(t) + \eta_0, \qquad t \geq 0,$$

with a function $h : [0, +\infty) \to W^{1,2}(X_\Gamma, \mu_\Gamma)$ satisfying $dh(0) = dh_0$.

Proof. For any $t \geq 0$ there exist $\eta(t) \in \ker d^*$, uniquely determined, and $h(t) \in W^{1,2}(X_\Gamma, \mu_\Gamma)$, unique up to an additive constant, such that $u(t) = dh(t) + \eta(t)$. Since by definition $t \mapsto u(t)$ is differentiable on $(0, +\infty)$ and continuous on $[0, +\infty)$, so is its orthogonal projection η to $\ker d^*$ and therefore also dh, and by Remark 5.1 even h, seen as a function with values in $W^{1,2}(X_\Gamma, \mu_\Gamma)/\mathbb{R}$. For any $v \in \ker d^* \subset \mathcal{D}(\vec{\mathcal{L}}_\Gamma)$ and any fixed t we have

$$\langle \eta_t(t), v\rangle_{L^2(X_\Gamma,\mu_\Gamma)} = -\langle h_t(t), d^*v\rangle_{L^2(X_\Gamma,\mu_\Gamma)} - d^*u(t)(d^*v) - \frac{1}{2}\int_{X_\Gamma} u^2(t)\, d^*v\, d\mu_\Gamma = 0,$$

so that also

$$\langle \eta(t) - \eta_0, v\rangle_{L^2(X_\Gamma,\mu_\Gamma)} = \int_0^t \langle \eta_\tau(\tau), v\rangle_{L^2(X_\Gamma,\mu_\Gamma)}\, d\tau = 0.$$

However, this implies that $\eta(t) - \eta_0 \perp \ker d^*$, which means this difference must be zero in $L^2(X_\Gamma, \mu_\Gamma)$. □

The Cole-Hopf transform (7.5) and (7.7) guarantees the existence and uniqueness of solution fields for initial conditions of gradient type.

Theorem 7.2. *Assume that $u_0 = dh_0$ with $h_0 \in W^{1,2}(X_\Gamma, \mu_\Gamma)$. Let $w(t)$ denote the unique solution $e^{t\mathcal{L}_\Gamma}w_0$ to (7.4) with initial condition $w_0 := e^{-h_0/2}$. Then the function*

$$u(t) := -2d \log w(t), \qquad t \geq 0,$$

is the unique solution to (7.8).

Both the existence and the uniqueness part follow well-known standard arguments, see for instance [Bir03] or [Olv14, Section 8.4]. We adapt them to our setup.

Proof. To verify that u is a solution, let h be as in (7.5). The stated hypotheses imply $u_t(t) = dh_t(t)$ in $L^2(X_\Gamma, \mu_\Gamma)$ for any $t > 0$. We have $\langle d^*u(t), d^*v\rangle = \langle \mathcal{L}_\Gamma h(t), d^*v\rangle$ for test functions $\varphi = d^*v$ with $v \in \mathcal{D}(\vec{\mathcal{L}}_\Gamma)$. From (7.6) it follows that u satisfies the first identity in (7.8).

To verify the continuity of u at zero, note that by nonnegativity and conservativity of the semigroup we have

$$\inf_{s\in X_\Gamma} w(t,s) \geq e^{-\|h_0\|_{\sup}/2}$$

for any $t \geq 0$. Since the function $w : [0,+\infty) \to W^{1,2}(X_\Gamma, \mu_\Gamma)$ is continuous, also its reciprocal $w(\cdot)^{-1} : [0,+\infty) \to W^{1,2}(X_\Gamma, \mu_\Gamma)$ is continuous. Therefore

$$\begin{aligned}
&\|u(t) - u_0\|_{L^2(X_\Gamma,\mu_\Gamma)} \\
&\leq 2\left\|(dw(t) - dw_0)w(t)^{-1}\right\|_{L^2(X_\Gamma,\mu_\Gamma)} + 2\left\|(w(t)^{-1} - w_0^{-1})dw_0\right\|_{L^2(X_\Gamma,\mu_\Gamma)} \\
&\leq 2e^{\|h_0\|_{\sup}/2}\left\|w(t) - w_0\right\|_{W^{1,2}(X_\Gamma,\mu_\Gamma)} + 2e^{\|h_0\|_{\sup}/2}\left\|w(t)^{-1} - w_0^{-1}\right\|_{L^\infty(X_\Gamma,\mu_\Gamma)},
\end{aligned} \tag{7.9}$$

what by (5.4) converges to zero as t goes to zero.

To see uniqueness we may, by Theorem 7.1, assume that $u(t) \in \operatorname{Im} d$ for any $t \geq 0$. In this case there is a potential $\tilde{h} : [0,+\infty) \to W^{1,2}(X_\Gamma, \mu_\Gamma)$ such that $u(t) = d\tilde{h}(t)$ for all $t \geq 0$. According to Definition 7.1 $\tilde{h}$ is continuous on $[0,+\infty)$ and differentiable on $(0,+\infty)$, and we have (7.6) for $\tilde{h}$ in place of h and all test functions φ of type $\varphi = d^*v$, $v \in \mathcal{D}(\vec{\mathcal{L}}_\Gamma)$. In order to have (7.6) for all test functions from $W^{1,2}(X_\Gamma, \mu_\Gamma)$, which also detect additive constants, we need to readjust the choice of the potential. For each $t \geq 0$ set now

$$g(t) := \frac{1}{\mu_\Gamma(X_\Gamma)}\left\{-\left\langle \tilde{h}_t(t), \mathbf{1}\right\rangle_{L^2(X_\Gamma,\mu_\Gamma)} - \frac{1}{2}\left\langle d\tilde{h}(t), d\tilde{h}(t)\right\rangle_{L^2(X_\Gamma,\mu_\Gamma)}\right\}$$

and let $G : [0,+\infty) \to \mathbb{R}$ be a differentiable function satisfying $G_t = g$. Then the readjusted potential $h(t) := \tilde{h}(t) + G(t)$ satisfies (7.6) for all $\varphi \in W^{1,2}(X_\Gamma, \mu_\Gamma)$: Suppose the decomposition (5.10) of φ reads $\varphi = d^*v + c$ with $v \in \mathcal{D}(\vec{\mathcal{L}}_\Gamma)$ and $c \in \mathbb{R}$, then

$$\begin{aligned}
\langle h_t(t), d^*v + c\rangle_{L^2(X_\Gamma,\mu_\Gamma)} &= \left\langle \tilde{h}_t(t), d^*v\right\rangle_{L^2(X_\Gamma,\mu_\Gamma)} + \left\langle \tilde{h}_t(t), c\right\rangle_{L^2(X_\Gamma,\mu_\Gamma)} + cg(t)\mu_\Gamma(X_\Gamma) \\
&= \mathcal{L}_\Gamma \tilde{h}(t)(d^*v) - \frac{1}{2}\left\langle (d^*v + c)d\tilde{h}(t), d\tilde{h}(t)\right\rangle_{L^2(X_\Gamma,\mu_\Gamma)} \\
&= \mathcal{L}_\Gamma h(t)(d^*v) - \frac{1}{2}\left\langle (d^*v + c)dh(t), dh(t)\right\rangle_{L^2(X_\Gamma,\mu_\Gamma)},
\end{aligned}$$

where we have used (5.12) and $\ker d = \mathbb{R}$. As a consequence, the continuous $W^{1,2}(X_\Gamma, \mu_\Gamma)$-valued function

$$w(t) := e^{-h(t)/2}, \quad t \geq 0,$$

is the unique solution to the Cauchy problem for the heat equation (7.4) in $L^2(X_\Gamma, \mu_\Gamma)$ with initial condition w_0. To see this, note that

$$\begin{aligned}
\mathcal{L}_\Gamma w(t) &= d^*\left(-\frac{1}{2}\, e^{-h(t)/2} dh(t)\right) \\
&= -\frac{1}{2}\, e^{-h(t)/2}\left(\mathcal{L}_\Gamma h(t) - \frac{1}{2}\left\langle dh(t), dh(t)\right\rangle\right) \\
&= -\frac{1}{2}\, e^{-h(t)/2} h_t(t) = w_t(t)
\end{aligned}$$

in $(W^{1,2}(X_\Gamma,\mu_\Gamma))^*$, which follows from [HRT13, Lemma 3.2]. However, since $w_t(t)$ is in $L^2(X_\Gamma,\mu_\Gamma)$, also $\mathcal{L}_\Gamma w(t)$ must be in $L^2(X_\Gamma,\mu_\Gamma)$, and since $W^{1,2}(X_\Gamma,\mu_\Gamma)$ is dense in $L^2(X_\Gamma,\mu_\Gamma)$ the equality must hold in $L^2(X_\Gamma,\mu_\Gamma)$. If now $\overline{u}$ was another solution of (7.8) with initial condition u_0 different from u and having a potential $\overline{h}$ satisfying (7.6) for all $\varphi\in W^{1,2}(X_\Gamma,\mu_\Gamma)$ then h and $\overline{h}$ would have to differ on $(0,+\infty)$ by a nonconstant function. However, this would lead to two different solutions w and $\overline{w}$ of the Cauchy problem (7.4), a contradiction. □

The following is immediate from [IRT12, Proposition 5.1].

Corollary 7.1. *If Γ has no cycles, i.e. is a tree, then for any initial condition $u_0\in L^2(X_\Gamma,\mu_\Gamma)$ the problem (7.8) has a unique solution.*

We provide some rudimentary estimates.

Corollary 7.2. *Let u_0, h_0 and u be as in Theorem 7.2.*

(i) We have

$$\sup_{t>0}\|u(t)\|_{L^2(X_\Gamma,\mu_\Gamma)}\le c_1\|u_0\|_{L^2(X_\Gamma,\mu_\Gamma)}\,e^{c_2\|u_0\|_{L^2(X_\Gamma,\mu_\Gamma)}}$$

with positive constants c_1 and c_2 independent of u_0.

(ii) Assume in addition that $h_0(s_0)=0$ for some $s_0\in X_\Gamma$. If $\widetilde{u}_0=d\widetilde{h}_0$ is another initial condition with $\widetilde{h}_0\in W^{1,2}(X_\Gamma,\mu_\Gamma)$ such that $\widetilde{h}_0(s_0)=0$, and $\widetilde{u}$ the corresponding solution, then

$$\sup_{t>0}\|u(t)-\widetilde{u}(t)\|_{L^2(X_\Gamma,\mu_\Gamma)}$$
$$\le c_3(\|u_0\|_{L^2(X_\Gamma,\mu_\Gamma)}+1)^2e^{c_4(\|u_0\|_{L^2(X_\Gamma,\mu_\Gamma)}+\|\widetilde{u}_0\|_{L^2(X_\Gamma,\mu_\Gamma)})}\|u_0-\widetilde{u}_0\|_{L^2(X_\Gamma,\mu_\Gamma)}$$

with positive constants c_3 and c_4 independent of u_0 and $\widetilde{u}_0$.

The proof relies on standard arguments.

Proof. From spectral theory it is easy to see that $(e^{t\mathcal{L}_\Gamma})_{t>0}$ is contractive also on $\dot{W}^{1,2}(X_\Gamma,\mu_\Gamma)$ with respect to the seminorm $\mathcal{E}_\Gamma(\cdot)^{1/2}$. Therefore

$$\begin{aligned}\|u(t)\|_{L^2(X_\Gamma,\mu_\Gamma)}&=2\mathcal{E}_\Gamma(\log w(t))^{1/2}\\&\le 2e^{\|h_0\|_{\sup}/2}\mathcal{E}_\Gamma(w(t))^{1/2}\\&\le 2e^{\|h_0\|_{\sup}/2}\|w_0\|_{W^{1,2}(X_\Gamma,\mu_\Gamma)}\le c\,e^{\|h_0\|_{\sup}}\mathcal{E}_\Gamma(h_0)^{1/2},\end{aligned}$$

what shows (i).

To see (ii) note that since $(\mathcal{E}_\Gamma,\dot{W}^{1,2}(X_\Gamma))$ is a resistance form and X_Γ is bounded in resistance metric, there is a constant $c>0$ such that for any $f\in\dot{W}^{1,2}(X_\Gamma)$ with $f(s_0)=0$ for some $s_0\in X_\Gamma$ we have $\|f\|_{\sup}\le c\,\mathcal{E}_\Gamma(f)^{1/2}$. A second fact we use is that there is a constant $c>0$ such that for any C^2-function $F:\mathbb{R}\to\mathbb{R}$ with bounded derivatives and any $f,g\in W^{1,2}(X_\Gamma,\mu_\Gamma)$ we have

$$\|F(f)-F(g)\|_{W^{1,2}(X_\Gamma,\mu_\Gamma)}\le c(\|F'\|_{\sup}+\|F''\|_{\sup})(\|f\|_{W^{1,2}(X_\Gamma,\mu_\Gamma)}+1)\|f-g\|_{W^{1,2}(X_\Gamma,\mu_\Gamma)},\tag{7.10}$$

as follows for instance from [HZ12, Proposition 3.1] and its proof, combined with the above estimate. Let us write $M_0:=\max(\|h_0\|_{\sup},\|\widetilde{h}_0\|_{\sup})$. Allowing constants to vary and using (7.10),

$$\begin{aligned}\|u(t)-\widetilde{u}(t)\|_{L^2(X_\Gamma,\mu_\Gamma)} &= 2\mathcal{E}_\Gamma(\log w(t)-\log\widetilde{w}(t))^{1/2}\\ &\le c\,e^{\check{M}_0}(\mathcal{E}(w(t))^{1/2}+1)\,\mathcal{E}(w(t)-\widetilde{w}(t))^{1/2}\\ &\le c\,e^{\check{M}_0}(\mathcal{E}(w_0)^{1/2}+1)\,\mathcal{E}(w_0-\widetilde{w}_0)^{1/2}\\ &\le c\,e^{c\check{M}_0}(\mathcal{E}(h_0)^{1/2}+1)^2\,\mathcal{E}(h_0-\widetilde{h}_0)^{1/2},\end{aligned}$$

what entails (ii), note that $\check{M}_0 \le c\,(\|u_0\|_{L^2(X_\Gamma,\mu_\Gamma)} + \|\widetilde{u}_0\|_{L^2(X_\Gamma,\mu_\Gamma)})$. □

7.3 Heat and Burgers equations on resistance spaces

In this section we formulate (7.1) and (7.2) on resistance spaces, these formulations are analogs of (7.3) and (7.8). We then analyze the formulation of (7.2) in more detail.

7.3.1 Hodge star operators and scalar Burgers equation

The formulation of a counterpart of (7.1) and (7.3) on resistance spaces is non-trivial, note that a priori $\partial(g^2)$ is not a scalar function. However, if the space is one-dimensional in a certain way, [Kus89; Hin10], the gradient field $\partial(g^2)$ can be interpreted as a function.

To make this precise, we assume that ν is a minimal energy dominant measure on X, i.e. a measure with respect to which all energy measures ν_f, $f \in \mathcal{F}\cap C_c(X)$, are absolutely continuous (see formula (3.5) in Section 4.2 for a definition of energy measures), and that ν is minimal in the sense that it is absolutely continuous with respect to any other measure having this property, see [Hin10; HRT13]. As discussed in Section 4.4 and shown in [HRT13, Section 2] there exists a measurable field $(\mathcal{H}_x)_{x\in X}$ of Hilbert spaces $(\mathcal{H}_x, \|\cdot,\cdot\|_{\mathcal{H}_x})$ such that the space $\mathcal{H}$ is isometrically isomorphic to the direct integral $\int_X^\oplus \mathcal{H}_x\,\nu(dx)$ with respect to ν.

Remark 7.1. To have this direct integral representation the given volume measure μ itself does not have to be energy dominant. For example, for the standard energy form on the Sierpiński gasket one can take μ to be the natural (normalized) self-similar Hausdorff measure of dimension $\frac{\log 3}{\log 2}$ and ν to be the Kusuoka measure. These two measures are mutually singular, [BST99]. This singularity is typical for local Dirichlet forms on self-similar fractals, [Hin05; HN06].

To the ν-essential supremum of the dimensions $\dim \mathcal{H}_x$ one refers as the *index (or martingale dimension)* of the Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$, see [Hin10].

For the rest of this subsection suppose the index equals one and let $\omega \in \mathcal{H}$ be such that satisfies $\|\omega\|_{\mathcal{H}_x} = 1$ for ν-a.e. $x \in X$. Such ω can always be found, see [BK19, Lemma 4.3]. Consequently for any $v \in \mathcal{H}$ there is a uniquely defined function $g \in L^2(X,\nu)$ such that $v = g\omega$. It was shown in [BK19, Section 4] that the map $\star_\omega : \mathcal{H} \to L^2(X,\mu)$, defined by $\star_\omega v := g$, provides an isometric isomorphism from $\mathcal{H}$ onto $L^2(X,\nu)$, [BK19, Proposition 4.5]. To $\star_\omega$ we refer as the *Hodge star operator associated with* ω, [BK19, Definition 4.4]. In the following, let $\omega \in \mathcal{H}$ with $\|\omega\|_{\mathcal{H}_x} = 1$ ν-a.e. be fixed.

Remark 7.2. A very well known observation, conjectured and partially proved by Kusuoka, [Kus89], and finally established in [Hin08; Hin10], is that for local Dirichlet forms on a large class of self-similar sets the index is one.

Mathematically it seems reasonable to formulate (7.1) as the Cauchy problem

$$\begin{cases} g_t(t) &= \mathcal{L}g(t) - \frac{1}{2}\star_\omega \partial(g^2)(t),\\ g(0) &= g_0. \end{cases} \tag{7.11}$$

In [LQ19] the authors consider (7.11) on the Sierpiński gasket endowed with its standard energy form. Making heavy use of resistance estimate (3.1) they skillfully establish Sobolev inequalities on the Sierpiński gasket for mutually singular measures. They combine them with known results on heat kernels to obtain the existence and uniqueness of weak solutions, [LQ19, Definition 4.13], to a counterpart of (7.11) subject to Dirichlet boundary conditions and in the case that μ is the natural Hausdorff measure, [LQ19, Theorem 4.16]. Their results work for arbitrary finite time intervals $[0, T]$. Without mentioning it explicitely, they make use of a Hodge star operator $\star_\omega$. In fact, in a probabilistic form it already appeared in [Kus89, Theorem 5.4 (ii)] (for $p = 1$), as can be seen using Nakao's theorem, see for example [HRT13, Theorem 9.1]. Under additional conditions also well known semigroup methods may be applied to obtain existence and uniqueness of solutions to (7.11), [Paz83, Section 6.3, Theorem 3.1], at least for small time intervals.

7.3.2 Vector Burgers equation

Here we focus on (7.2) under the assumptions made in Chapter 4.

In what follows let μ be an atom free Radon measure on X with full support. Then, as discussed in the preceding section, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with $\mathcal{D}(\mathcal{E}) = \mathcal{F}$ is a local regular Dirichlet form on $L^2(X, \mu)$, see [Kig01, Theorem 3.4.6]. We assume μ is such that the associated Markov semigroup $(e^{t\mathcal{L}})_{t>0}$ is conservative, i.e. such that $e^{t\mathcal{L}}\mathbf{1} = \mathbf{1}$, $t > 0$.

Suppose that $w(t) = e^{t\mathcal{L}}w_0$ is the unique solution to the heat equation (7.4) for $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$. Again we assume the initial condition $w_0 \in L^2(X, \mu)$ to be strictly positive μ-a.e. The $\mathcal{D}(\mathcal{E})$-valued function $h := -2 \log w$ satisfies (7.6) for any $\varphi \in \mathcal{D}(\mathcal{E})$. We write $\partial \langle u, u \rangle (t) := \partial \langle u(t), u(t) \rangle$, the latter defined as in (4.19), and consider the Cauchy problem

$$\begin{cases} u_t(t) &= \vec{\mathcal{L}}u(t) - \frac{1}{2}\partial \langle u, u \rangle (t), \quad t > 0, \\ u(0) &= u_0. \end{cases} \tag{7.12}$$

The definition of solution is similar to the metric graph case.

Definition 7.2. *A function $u \in C([0, +\infty), \mathcal{H}) \cap C^1((0, +\infty), \mathcal{H})$ is called a solution to (7.12) with initial condition $u_0 \in \mathcal{H}$ if u satisfies the first identity in (7.12) in $(\mathcal{D}(\vec{\mathcal{L}}))^*$ and the second in $\mathcal{H}$.*

7.3.3 Existence and uniqueness results

The structure of solutions is as in the metric graph case, provided the space is connected.

Theorem 7.3. *Assume that (X, R) is connected. Suppose u is a solution to (7.12) with initial condition $u_0 \in \mathcal{H}$. Let $\eta_0 \in \ker \partial^*$ and $h_0 \in \mathcal{D}(\mathcal{E})$ be such that $u_0 = \partial h_0 + \eta_0$. Then u is of form*

$$u(t) = \partial h(t) + \eta_0, \qquad t \geq 0,$$

with a function $h : [0, +\infty) \to \mathcal{D}(\mathcal{E})$ satisfying $dh(0) = dh_0$.

Proof. Considering ∂ and ∂^* in place of d and d^*, respectively and using (4.17) we can follow the proof of Theorem 7.1. □

Again we can conclude an existence and uniqueness statement for solutions.

Theorem 7.4. *Assume that (X, R) is connected and that μ is such that $(e^{t\mathcal{L}})_{t>0}$ is conservative. If we have $u_0 = \partial h_0$ with $h_0 \in \mathcal{D}(\mathcal{E})$ bounded and $w(t)$ denotes the unique solution $e^{t\mathcal{L}}w_0$ to (7.4) with initial condition $w_0 := e^{-h_0/2}$ then the function*

$$u(t) := -2\partial \log w(t), \qquad t \geq 0,$$

is the unique solution to (7.12).

Proof. The proof follows the same arguments as that of Theorem 7.2. To see the continuity of the Cole-Hopf solution at zero note that by the chain rule and (4.14) we have

$$\|u(t)-u_0\|_{\mathcal{H}} \leq 2e^{\|h\|_{\sup}/2}\,\|w(t)-w_0\|_{\mathcal{D}(\mathcal{E})} + 2\left(\int_X (w^{-1}(t)-w_0^{-1})\,d\nu_{w_0}\right)^{1/2},$$

where ν_{w_0} denotes the energy measure of w_0, see Section 4.2. Clearly the first summand goes to zero for $t\to 0$. Because the semigroup is conservative, we have

$$\inf_{x,\in X} w(t,x) \geq e^{\|h_0\|_{\sup}/2}$$

for all $t\geq 0$ (here $w(t,x) := w(t)(x)$) so that also $w(\cdot)^{-1} : [0,+\infty) \to \mathcal{D}(\mathcal{E})$ is continuous and uniformly bounded. Since the measure ν_{w_0} is finite, (4.16) and bounded convergence imply that also the second summand goes to zero for $t\to 0$. The uniqueness follows from (4.17) together with Theorem 7.3. □

Also the following estimates are as before, see Corollary 7.2.

Corollary 7.3. *Let u_0, h_0 and u be as in Theorem 7.4.*

(i) We have

$$\sup_{t>0} \|u(t)\|_{\mathcal{H}} \leq c_1\,\|u_0\|_{\mathcal{H}}\,e^{c_2\|u_0\|_{\mathcal{H}}}$$

with positive constants c_1 and c_2 independent of u_0.

(ii) Assume in addition that (X,R) is compact and that $h_0(x_0)=0$ for some $x_0\in X$. If $\widetilde{u}_0 = d\widetilde{h}_0$ is another initial condition with $\widetilde{h}_0\in\mathcal{D}(\mathcal{E})$ such that $\widetilde{h}_0(x_0)=0$, and $\widetilde{u}$ the corresponding solution, then

$$\sup_{t>0} \|u(t)-\widetilde{u}(t)\|_{\mathcal{H}} \leq c_3(\|u_0\|_{\mathcal{H}}+1)^2 e^{c_4(\|u_0\|_{\mathcal{H}}+\|\widetilde{u}_0\|_{\mathcal{H}})}\,\|u_0-\widetilde{u}_0\|_{\mathcal{H}}$$

with positive constants c_3 and c_4 independent of u_0 and $\widetilde{u}_0$.

Remark 7.3. It is well known that in the context of classical partial differential equations the Cole-Hopf transform connects an entire hierarchy of equations and allows to obtain exact solutions to non-linear equations from solutions to linear equations on each particular level, [KS09; Tas76]. On fractals linear second order ('heat') equations (7.4) are tractable whenever we can understand a natural Laplace operator. In comparison, linear first order ('transport') equations of type $g_t = g_x$ are more difficult to analyze, and due to possible energy singularity the existing methods, such as [AT14], may work for some volume measures, but certainly not for all. Linear equations of higher order, for instance $g_t = g_{xxx}$, have not yet been studied on fractals, and it is an interesting open question how to formulate them in a meaningful way.

Chapter 8

Existence of solutions to the continuity equation

In the joint work [HMS20], based on the master thesis [Sch19], we provided an existence result for weak solutions of continuity equations on resistance spaces by adapting the arguments of Ambrosio and Trevisan in [AT14]. We present this result in the remainder of this chapter.

8.1 Weak solutions to continuity equations

Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance form on a nonempty set X so that (X, R) is compact and metrically doubling with doubling constant $K_R > 1$, and let μ be a finite Borel measure on (X, R) with a uniform lower bound V. According to [Kig12, Theorem 9.4], $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with $\mathcal{D}(\mathcal{E}) = \mathcal{F}$ is a regular Dirichlet form on $L^2(X, \mu)$.

Our aim is to study the continuity equation

$$\begin{cases} \partial_t u + \operatorname{div}(ub) = 0 & \text{in } (0,T) \times X, \\ u(0) = u_0 & \text{on } X. \end{cases} \tag{8.1}$$

The desired weak formulation of a solution $u : [0,T] \times X \to \mathbb{R}$ for some vector field $b : [0,T] \to \mathcal{H}$ would be

$$-\int_0^T \psi'(t) \int_X u(t)\phi d\mu dt = \int_0^T \psi(t) \langle u(t)b(t), \partial\phi \rangle_{\mathcal{H}} dt + \psi(0) \int_X u_0 \phi d\mu, \tag{8.2}$$

for every $\phi \in \mathcal{D}(\mathcal{E})$, $\psi \in C^1([0,T])$ with $\psi(T) = 0$.

8.2 Existence for time-dependent vector fields

8.2.1 Variational solutions to viscous continuity equations

Following [AT14], we use the approach of vanishing viscosity. This means that we need to study a modified continuity equation to derive an existence result for (8.1).

Let us consider the continuity equation modified by adding a second order term $\sigma \Delta u$, also called diffusion term, for some $\sigma > 0$.

$$\begin{cases} \partial_t u + \operatorname{div}(ub) = \sigma \Delta u & \text{in } (0,T) \times X, \\ u(0) = u_0 & \text{on } X. \end{cases} \tag{8.3}$$

From now on, we call this modified equation the *viscous continuity equation*. Thus, by proving extra regularity, we will show that the sequence of solutions $(u_\sigma)_\sigma$, $\sigma > 0$, to (8.3) converges to a solution u to (8.1) if σ tends to 0. We use the following notion of solution.

Definition 8.1. *Let $b \in L^2(0,T;\mathcal{H})$ and $u_0 \in L^2(X,\mu)$. A function $u \in L^2(0,T;\mathcal{D}(\mathcal{E}))$ is called a* weak solution *to (8.3) if for every $\psi \in C^1([0,T])$ with $\psi(T) = 0$ and every $\phi \in \mathcal{D}(\mathcal{E})$ the following equation holds:*

$$-\int_0^T \psi'(t) \int_X u(t)\phi d\mu dt = \int_0^T \psi(t) \left[\langle u(t)b(t), \partial\phi\rangle_{\mathcal{H}} - \sigma\mathcal{E}(u(t),\phi)\right] dt + \psi(0) \int_X u_0 \phi d\mu. \tag{8.4}$$

In the following proofs we will frequently use an inequality already mentioned in (4.16). Thanks to [Kig12, Lemma 9.2], there exists a constant $C_R > 0$ such that for every $u \in \mathcal{D}(\mathcal{E})$ we have

$$\|u\|_{\sup} \le C_R \mathcal{E}(u)^{\frac{1}{2}}. \tag{8.5}$$

To prove existence we will use the following extension of the well known Lax-Milgram Lemma.

Theorem 8.1. *Let $b \in L^\infty(0,T;\mathcal{H})$ with $\partial^* b \in L^\infty(0,T;L^\infty(X,\mu))$ and $u_0 \in L^2(X,\mu)$. Then, for every $\sigma \in (0, \frac{1}{2}]$, there exists a weak solution $u \in L^2(0,T;\mathcal{D}(\mathcal{E}))$ to (8.3) with*

$$\left\| e^{-\lambda t} u \right\|_{L^2(0,T;\mathcal{D}(\mathcal{E}))} \le \frac{1}{\sigma} \|u_0\|_{L^2(X,\mu)}, \tag{8.6}$$

where

$$\lambda = \frac{1}{2} \left\| (\partial^* b)^+ \right\|_{L^\infty(0,T;L^\infty(X,\mu))} + \sigma. \tag{8.7}$$

Proof. Let $H = L^2(0,T;\mathcal{D}(\mathcal{E}))$ and $V = \operatorname{span}\{\psi \cdot \phi \mid \psi \in C^1([0,T]), \psi(T) = 0 \text{ and } \phi \in \mathcal{D}(\mathcal{E})\}$ with

$$\|v\|_V^2 := \|v\|^2_{L^2(0,T;\mathcal{D}(\mathcal{E}))} + \|v(0)\|^2_{L^2(X,\mu)}. \tag{8.8}$$

Clearly, H is a Hilbert space and $V \subset H$ a linear, normed space with $V \hookrightarrow H$. We fix the functional $\ell \in V'$ by

$$\ell(v) = \int_X u_0 v(0) d\mu.$$

Now we define the bilinear form $B : H \times V \to \mathbb{R}$ by

$$B(h,v) = -\int_0^T \int_X h(t)\partial_t v(t) - \lambda h(t)v(t) d\mu + \langle h(t)b(t), \partial v(t)\rangle_{\mathcal{H}} - \sigma\mathcal{E}(h(t),v(t)) dt. \tag{8.9}$$

We claim that B is a coercive and continuous bilinear form in the sense of the Lions-Lax-Milgram theorem. First the continuity, i.e. for fixed $v \in V$ we show $B(\cdot, v) \in H'$. We estimate every term on its own:

$$\left| \int_0^T \int_X h(t)\partial_t v(t) d\mu dt \right| \le \|h\|_{L^2(0,T;L^2(X,\mu))} \|\partial_t v\|_{L^2(0,T;L^2(X,\mu))},$$

$$\left| \int_0^T \int_X \lambda h(t)v(t) d\mu dt \right| \le \lambda \|h\|_{L^2(0,T;L^2(X,\mu))} \|v\|_{L^2(0,T;L^2(X,\mu))},$$

$$\left| \int_0^T \langle h(t)b(t), \partial v(t)\rangle_{\mathcal{H}} dt \right| \le C \|h\|_{L^2(0,T;\mathcal{D}(\mathcal{E}))} \|b\|_{L^\infty(0,T;\mathcal{H})} \|v\|_{L^2(0,T;\mathcal{D}(\mathcal{E}))},$$

$$\left| \int_0^T \sigma\mathcal{E}(h(t),v(t)) dt \right| \le \sigma \|h\|_{L^2(0,T;\mathcal{D}(\mathcal{E}))} \|v\|_{L^2(0,T;\mathcal{D}(\mathcal{E}))}.$$

Note that we used (8.5) in the third term. Thus $\|B(\cdot,v)\|_{H'}$ is bounded by

$$\|\partial_t v\|_{L^2(0,T;L^2(X,\mu))}+\lambda \|v\|_{L^2(0,T;L^2(X,\mu))}+C\|b\|_{L^\infty(0,T;\mathcal{H})}\|v\|_{L^2(0,T;\mathcal{D}(\mathcal{E}))}+\sigma\|v\|_{L^2(0,T;\mathcal{D}(\mathcal{E}))}. \tag{8.10}$$

Now the coercivity, i.e. for every $v \in V$ we have $B(v,v) \geq \sigma \|v\|_V^2$. For the first term we use that $v\partial_t v^2 = \frac{1}{2}\partial_t v$. Hence by Fubini

$$-\int_0^T \int_X v(t)\partial_t v(t) d\mu dt = -\frac{1}{2}\int_X [v(T)^2 - v(0)^2] d\mu = \frac{1}{2}\|v(0)\|^2_{L^2(X,\mu)}. \tag{8.11}$$

The second and third term will be estimated together. Here we also use $\partial v \cdot v = \frac{1}{2}\partial v^2$ to get $\langle v(t)b(t), \partial v(t)\rangle_{\mathcal{H}} = \frac{1}{2}\langle \partial^* b(t), v(t)^2\rangle_{\mathcal{H}}$. Hence

$$\int_0^T \int_X \lambda v(t)^2 d\mu - \langle v(t)b(t), \partial v(t)\rangle_{\mathcal{H}} dt \geq \left(\lambda - \frac{1}{2}\left\|(\partial^* b)^+\right\|_{L^\infty(0,T;L^\infty(X,\mu))}\right)\|v\|^2_{L^2(0,T;L^2(X,\mu))}. \tag{8.12}$$

We use the definition of λ and $\frac{1}{2} \geq \sigma$ and conclude $B(v,v) \geq \sigma\|v\|_V^2$. Thus Lions-Lax-Milgram implies the existence of $h \in H$ such that

$$B(h,v) = \ell(v) \quad \text{for every } v \in V \quad \text{and} \quad \|h\|_H \leq \frac{1}{\sigma}\|\ell\|_{V'}. \tag{8.13}$$

Now let us set $u(t) = e^{\lambda t}h(t)$. Clearly $u \in H$. For any $v \in V$ define the function $t \mapsto \widetilde{v}(t) = e^{\lambda t}v(t) \in V$. Then by linearity, since $e^{\lambda t}$ only depends on time, and

$$\partial_t \widetilde{v}(t) = \lambda \widetilde{v}(t) + e^{\lambda t}\partial_t v(t) \tag{8.14}$$

we verify

$$-\int_0^T \int_X u(t)\partial_t v(t) d\mu + \langle u(t)b(t), \partial v(t)\rangle_{\mathcal{H}} - \sigma\mathcal{E}(u(t),v(t))dt = B(h,\widetilde{v}). \tag{8.15}$$

Thus (8.13) and (8.15) together with $\|\ell\|_{V'} \leq \|u_0\|_{L^2(X,\mu)}$ implies that u is a weak solution to (8.3) with $\left\|e^{-\lambda t}u\right\|_{L^2(0,T;\mathcal{D}(\mathcal{E}))} \leq \frac{1}{\sigma}\|u_0\|_{L^2(X,\mu)}$. □

8.2.2 *A priori* estimates

This section is dedicated to *a priori* estimates.

Theorem 8.2. *Let $b \in L^\infty(0,T;\mathcal{H})$ with $\partial^* b \in L^\infty(0,T;L^\infty(X,\mu))$ and $u_0 \in L^2(X,\mu)$. Then there exists a weak solution*

$$u \in L^2(0,T;\mathcal{D}(\mathcal{E})) \cap L^\infty(0,T;L^2(X,\mu)) \tag{8.16}$$

to (8.3) *with*

$$\sup_t \left\|u(t)^\pm\right\|_{L^2(X,\mu)} \leq \left\|u_0^\pm\right\|_{L^2(X,\mu)}. \tag{8.17}$$

In particular if $u_0 \geq 0$ then $u \geq 0$.

The proof of Theorem 8.2 follows the arguments of [AT14, Theorem 4.6], but uses the duality between $\mathcal{D}(\mathcal{E})$ and $(\mathcal{D}(\mathcal{E}))^*$.

Proof. Let $u \in L^2(0,T;\mathcal{D}(\mathcal{E}))$ be a weak solution to (8.3). For any $s > 0$ we introduce the notation $f_s(t) = P_s f(t)$ for appropriate functions f. Using the formulation of (8.3) we deduce for every $\psi \in C_c^1(0,T)$ and $\phi \in \mathcal{D}(\mathcal{E})$:

$$
\begin{aligned}
-\int_0^T \psi'(t) \int_X u_s(t)\phi d\mu dt &= -\int_0^T \psi'(t) \int_X u(t)\phi_s(t) d\mu dt \\
&= \int_0^T \psi(t) \left[\langle u(t)b(t), \partial\phi_s\rangle_{\mathcal{H}} - \sigma\mathcal{E}(u(t),\phi_s)\right] dt \\
&= \int_0^T \psi(t) \left[\langle P_s\partial^*(u(t)b(t)), \phi(t)\rangle + \sigma\langle \Delta u_s(t), \phi\rangle\right] dt.
\end{aligned}
$$

Hence distributionally in $(\mathcal{D}(\mathcal{E}))^*$

$$\frac{d}{dt}u_s = P_s\partial^*(ub) + \sigma\Delta u_s = \partial^*(u_s b) + \sigma\Delta u_s + \mathcal{C}_s \tag{8.18}$$

with commutator

$$\mathcal{C}_s = P_s\partial^*(ub) - \partial^*(u_s b). \tag{8.19}$$

Lemma 8.1. *u_s has an absolutely continuous representative $\bar{u}_s \in AC([0,T];L^2(X,\mu))$ with $\bar{u}_s(0) = P_s u_0$.*

Proof of Lemma 8.1. It is enough to show that $P_s\partial^*(ub) + \sigma\Delta u_s \in L^1(0,T;L^2(X,\mu))$. Since $\Delta u_s \in L^2(0,T;L^2(X,\mu))$ follows immediately from the regularization estimate

$$\|\Delta P_s f\|_{L^2(X,\mu)} \le \frac{1}{s}\|f\|_{L^2(X,\mu)} \quad \text{for every } f \in L^2(X,\mu). \tag{8.20}$$

And $P_s\partial^*(ub) \in L^2(0,T;L^2(X,\mu))$ follows from the regularization estimate

$$\mathcal{E}(f_s) \le \frac{1}{2s}\|f\|^2_{L^2(X,\mu)} \quad \text{for every } f \in L^2(X,\mu) \tag{8.21}$$

because for every $\phi \in \mathcal{D}(\mathcal{E})$ we have

$$
\begin{aligned}
|\langle u(t)b(t), \partial\phi_s\rangle_{\mathcal{H}}| &\le \|u(t)b(t)\|_{\mathcal{H}} \|\partial\phi_s\|_{\mathcal{H}} \le \|u(t)\|_{\sup} \|b(t)\|_{\mathcal{H}} \mathcal{E}(\phi_s)^{\frac{1}{2}} \\
&\le \frac{C}{\sqrt{2s}} \|u(t)\|_{\mathcal{D}(\mathcal{E})} \|b(t)\|_{\mathcal{H}} \|\phi\|_{L^2(X,\mu)}.
\end{aligned}
$$

By using (8.3), i.e. test with functions $\psi \in C^1([0,T])$ with $\psi(T) = 0$, one concludes $\bar{u}_s(0) = P_s u_0$. ☐

Now define $\beta, \beta_n : \mathbb{R} \to [0,\infty)$ by $\beta(z) = (z^+)^2$ and

$$\beta_n(z) = \begin{cases} (z^+)^2 & , z \le n \\ 2nz^+ - n^2 & , z > n \end{cases}. \tag{8.22}$$

It is easy to see that β_n is C^1 and convex with derivative

$$\beta_n'(z) = \begin{cases} 2z^+ & , z \le n \\ 2n & , z > n \end{cases}. \tag{8.23}$$

Thus β_n' is Lipschitz with $\beta_n'(0) = 0$ and β_n' and $\beta_n(z)/z$ are bounded on $\mathbb{R}$. We claim that $t \mapsto \int_X \beta_n(u_s(t))d\mu$ is almost everywhere differentiable with

$$\frac{d}{dt}\int_X \beta_n(u_s(t))d\mu = \langle \frac{d}{dt}u_s(t), \beta_n'(u_s(t))\rangle_{\langle(\mathcal{D}(\mathcal{E}))^*,\mathcal{D}(\mathcal{E})\rangle}, \tag{8.24}$$

where u_s denotes its absolute continuous representative from the lemma above.

We also need the following lemma.

Lemma 8.2. *[AGS14, Lemma 2.9] Let $L \in L^1(0,1)$ nonnegative and $g : [0,1] \to [0,\infty]$ measurable with $\int_0^1 L(t)dt > 0$ and $\int_0^1 g(t)L(t)dt < \infty$. Further let $w : [0,1] \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous with $w < +\infty$ a.e. on $\{L \neq 0\}$ and*

$$w(t) - w(s) \leq g(t) \int_s^t L(r)dr \quad \text{for all } t, s \in \{w < +\infty\} \tag{8.25}$$

and, for arbitrary $0 \leq a < b \leq 1$:

$$\int_a^b L(t)dt = 0 \quad \Rightarrow \quad w \text{ constant on } [a,b]. \tag{8.26}$$

Then $\{w = +\infty\}$ is empty and $w \in AC([0,1])$.

First note that $\beta_n'(u_s(t)) \in \mathcal{D}(\mathcal{E})$, see [BH91, Prop. 3.3.1] with

$$\begin{aligned}
\left\|\beta_n'(u_s(t))\right\|_{\mathcal{D}(\mathcal{E})}^2 &\leq \sup_z \left|\frac{\beta_n'(z)}{z}\right|^2 \|u_s(t)\|_{L^2(x,\mu)}^2 + \mathrm{Lip}(\beta_n')^2 \mathcal{E}(u_s(t)) \\
&\leq \left(\sup_z \left|\frac{\beta_n'(z)}{z}\right|^2 + \mathrm{Lip}(\beta_n')^2\right) \|u(t)\|_{\mathcal{D}(\mathcal{E})}^2 .
\end{aligned}$$

The convexity of β_n implies for every $0 \leq t' < t \leq T$:

$$\begin{aligned}
\int_X \beta_n(u_s(t))d\mu - \int_X \beta_n(u_s(t'))d\mu &\leq \int_X \beta_n'(u_s(t))(u_s(t) - u_s(t'))d\mu \\
&= \int_{t'}^t \langle \frac{d}{dr} u_s(r), \beta_n'(u_s(t))\rangle_{\langle(\mathcal{D}(\mathcal{E}))^*, \mathcal{D}(\mathcal{E})\rangle} dr \\
&\leq \left\|\beta_n'(u_s(t))\right\|_{\mathcal{D}(\mathcal{E})} \int_{t'}^t \left\|\frac{d}{dr} u_s(r)\right\|_{(\mathcal{D}(\mathcal{E}))^*} dr.
\end{aligned}$$

Note that $\left\|\frac{d}{dt} u_s(t)\right\|_{\mathcal{D}(\mathcal{E})^*} \leq (C_R \|b(t)\|_{\mathcal{H}} + \sigma)\, \mathcal{E}(u(t))^{\frac{1}{2}} \in L^1(0,T)$ since

$$|\langle \frac{d}{dt} u_s(t), \phi\rangle| \leq |\langle u(t)b(t), \partial\phi_s\rangle| + \sigma|\mathcal{E}(u_s(t),\phi)| \leq (C_R \|b(t)\|_{\mathcal{H}} + \sigma)\, \mathcal{E}(u(t))^{\frac{1}{2}} \mathcal{E}(\phi)^{\frac{1}{2}}.$$

The lower semicontinuity follows again from the convexity of β and the continuity of u_s. Hence $t \mapsto \int_X \beta_n(u_s(t))d\mu$ is absolutely continuous. Using the chain rule one concludes the claimed derivative. Using the identity of the distributional derivative of u_s we get

$$\begin{aligned}
\frac{d}{dt} \int_X \beta_n(u_s(t))d\mu &= \langle \frac{d}{dt} u_s(t), \beta_n'(u_s(t))\rangle \\
&= \langle u_s(t)b(t), \partial\beta_n'(u_s(t))\rangle_{\mathcal{H}} - \sigma\mathcal{E}(u_s(t), \beta_n'(u_s(t)) + \langle \mathcal{C}_s(t), \beta_n'(u_s(t))\rangle.
\end{aligned}$$

For the first term we can use the integration by parts formula and chain rule of ∂ to deduce

$$\begin{aligned}
\langle u_s(t)b(t), \partial\beta_n'(u_s(t))\rangle_{\mathcal{H}} &= \int_X \partial^* b(t)\beta_n'(u_s(t))u_s(t)d\mu - \langle b(t), \partial\beta_n(u_s(t))\rangle_{\mathcal{H}} \\
&= \int_X \partial^* b(t) \left[\beta_n'(u_s(t))u_s(t) - \beta_n(u_s(t))\right] d\mu \\
&\leq \left\|\partial^* b(t)^+\right\|_{L^\infty(X,\mu)} \int_X \beta_n(u_s(t))d\mu,
\end{aligned}$$

where we used $0 \leq \beta_n'(z)z - \beta_n(z) \leq \beta_n(z)$ in the last line. For the second term, since β_n' is Lipschitz with $\beta_n'' \geq 0$, we get

$$\mathcal{E}(u_s(t), \beta_n'(u_s(t))) = \int_X d\Gamma(u_s(t), \beta_n'(u_s(t)) = \int_X \beta_n''(u_s(t))d\Gamma(u_s(t)) \geq 0.$$

Thus

$$\frac{d}{dt}\int_X \beta_n(u_s(t))d\mu \leq \left\|\partial^* b(t)^+\right\|_{L^\infty(X,\mu)} \int_X \beta_n(u_s(t))d\mu + \langle \mathcal{C}_s(t), \beta_n'(u_s(t))\rangle. \tag{8.27}$$

Now Gronwall's inequality implies

$$\int_X \beta_n(u_s(t))d\mu \leq e^{\left\|\partial^* b^+\right\|_{L^1(0,T;L^\infty(X,\mu))}} \left(\int_X \beta_n(u_s(0))d\mu + \int_0^t \langle \mathcal{C}_s(r), \beta_n'(u_s(r))\rangle dr\right). \tag{8.28}$$

We study the convergence $s \to 0$. The convergence of $\int_X \beta_n(u_s(t))d\mu$ and $\int_X \beta_n(u_s(0))d\mu$ follows from the convexity of β_n and the uniform bound on the L^2-norm of $\beta_n'(u_s(t))$. Now we show that $\langle \mathcal{C}_s(t), \beta_n'(u_s(t))\rangle \to 0$ and $\|\langle \mathcal{C}_s, \beta_n'(u_s)\rangle\|_{L^1(0,T)}$ is uniformly bounded in s. For arbitrary $\phi \in \mathcal{D}(\mathcal{E})$ we have

$$\begin{aligned}
|\langle \mathcal{C}_s(t), \phi\rangle| &= |\langle u(t)b(t), \partial\phi_s\rangle_{\mathcal{H}} - \langle u_s(t)b(t), \partial\phi\rangle_{\mathcal{H}}| \\
&\leq |\langle (u(t) - u_s(t))b(t), \partial\phi_s\rangle_{\mathcal{H}}| + |\langle u_s(t)b(t), \partial\phi - \partial\phi_s\rangle_{\mathcal{H}}| \\
&\leq C_R \mathcal{E}(u(t) - u_s(t))^{\frac{1}{2}} \|b(t)\|_{\mathcal{H}} \mathcal{E}(\phi)^{\frac{1}{2}} + C_R \mathcal{E}(u(t))^{\frac{1}{2}} \|b(t)\|_{\mathcal{H}} \mathcal{E}(\phi - \phi_s)^{\frac{1}{2}}.
\end{aligned}$$

Now set $\phi = \beta_n'(u_s(t))$. Since $\beta_n'(u_s(t)) \in \mathcal{D}(\mathcal{E})$ uniformly in s, the first term converges to 0. For the second term

$$\begin{aligned}
\mathcal{E}(P_s\beta_n'(u_s(t)) - \beta_n'(u_s(t)))^{\frac{1}{2}} &\leq \mathcal{E}(P_s\left[\beta_n'(u_s(t)) - \beta_n'(u(t))\right])^{\frac{1}{2}} + \mathcal{E}(P_s\beta_n'(u(t)) - \beta_n'(u(t)))^{\frac{1}{2}} \\
&\quad + \mathcal{E}(\beta_n'(u(t)) - \beta_n'(u_s(t)))^{\frac{1}{2}} \\
&\leq 2\mathcal{E}(\beta_n'(u(t)) - \beta_n'(u_s(t)))^{\frac{1}{2}} + \mathcal{E}(P_s\beta_n'(u(t)) - \beta_n'(u(t)))^{\frac{1}{2}}.
\end{aligned}$$

The convergence of the second term is clear since $\beta_n'(u(t)) \in \mathcal{D}(\mathcal{E})$. The convergence $\beta_n'(u_s(t)) \to \beta_n'(u(t))$ in $\mathcal{D}(\mathcal{E})$ follows from the fact that $u_s(t) \to u(t)$ in $\mathcal{D}(\mathcal{E})$ and β' being Lipschitz continuous, see for instance [BH91, Theorem 3.3.3]. Hence $\langle \mathcal{C}_s(t), \beta_n'(u_s(t))\rangle \to 0$.

For the uniform bound we use

$$|\langle \mathcal{C}_s(t), \beta_n'(u_s(t))\rangle| \leq 2C_R \|b(t)\|_{\mathcal{H}} \mathcal{E}(u(t))^{\frac{1}{2}} \mathcal{E}(\beta_n'(u_s(t)))^{\frac{1}{2}}$$

together with the uniform bound on $\beta_n'(u_s(t)) \in \mathcal{D}(\mathcal{E})$. Thus the convergence $s \to 0$ implies for every n

$$\int_X \beta_n(u(t))d\mu \leq e^{\left\|\partial^* b^+\right\|_{L^1(0,T;L^\infty(X,\mu))}} \int_X \beta_n(u_0)d\mu. \tag{8.29}$$

Now let $n \to \infty$ and use the monotone convergence $\beta_n \to \beta$ to conclude the proof in the case $u(t)^+$. To prove the inequality for $u(t)^-$ note that $v(t) = -u(t)$ solves (8.3) with initial condition $-u_0$. Since $v(t)^+ = u(t)^-$ the proof is done. □

8.2.3 Vanishing viscosity and existence of solutions

Definition 8.2. *Let $b \in L^2(0,T;\mathcal{H})$. We call b absolutely continuous w.r.t. μ, if there exists a linear Operator $\Gamma_b : \mathcal{D}(\mathcal{E}) \to L^1(0,T;L^2(X,\mu)), \phi \mapsto (t \mapsto \Gamma_{b(t)}(\phi))$ such that for every $v \in L^\infty(0,T;\mathcal{D}(\mathcal{E}))$ and $\phi \in \mathcal{D}(\mathcal{E})$ the following equation holds*

$$\langle v(t)b(t), \partial\phi\rangle_{\mathcal{H}} = \int_X v(t)\Gamma_{b(t)}(\phi)d\mu \quad \textit{for a.e. } t. \tag{8.30}$$

In particular we can now define for $u \in L^\infty(0,T;L^2(X,\mu))$ and $\phi \in \mathcal{D}(\mathcal{E})$:

$$\langle u(t)b(t), \partial\phi\rangle_{\mathcal{H}} := \int_X u(t)\Gamma_{b(t)}(\phi)d\mu \quad \textit{for a.e. } t.$$

Now we prove the main result of this chapter.

Theorem 8.3. *Let $b \in L^2(0,T;\mathcal{H})$ be absolutely continuous w.r.t. μ and $\partial^* b \in L^1(0,T;L^\infty(X,\mu))$. Then there exists a weak solution $u \in L^\infty(0,T;L^2(X,\mu))$ to (8.1). Also if $u_0 \geq 0$ then $u \geq 0$.*

One main ingredient in the proof is the well-known Banach-Alaoglu theorem. Since it will be used many times throughout this thesis we include the statement of the theorem in the Appendix (cf. Appendix A).

Proof. Let $\eta \in C_c^\infty(\mathbb{R})$ be the standard mollifier on $(-1,1)$. For $\delta > 0$ we set

$$b_\delta(t) = (\eta_\delta * b)(t) = \int_0^T \eta\left(\frac{t-r}{\delta}\right) b(r)dr = \int_{t-T}^t \eta\left(\frac{r}{\delta}\right) b(t-r)dr. \tag{8.31}$$

Thus $b^\delta \in L^\infty(0,T;\mathcal{H})$. Further we see that

$$\partial^*(b_\delta)(t) = (\partial^* b)_\delta(t) = \int_0^T \eta\left(\frac{t-r}{\delta}\right) \partial^* b(r)dr. \tag{8.32}$$

So $\partial^* b_\delta := \partial^*(b_\delta) \in L^\infty(0,T;L^\infty(X,\mu))$. Let $n \geq 2$. By Theorem 8.2 there exists a function

$$u^{n,\delta} \in L^\infty(0,T;L^2(X,\mu)) \cap L^2(0,T;\mathcal{D}(\mathcal{E}))$$

solving (8.3) with $\sigma = \frac{1}{n}$ and b_δ instead of b. Since

$$\left\|(\partial^* b_\delta)^+\right\|_{L^\infty(0,T;L^\infty(X,\mu))} \leq \|\partial^* b\|_{L^1(0,T;L^\infty(X,\mu))} \tag{8.33}$$

the Lions-Lax-Milgram estimate implies for every $\delta > 0$ and $n \geq 2$

$$\frac{1}{n}\left\|u^{n,\delta}\right\|_{L^2(0,T;\mathcal{D}(\mathcal{E}))} \leq e^{\|\partial^* b\|_{L^1(0,T;L^\infty(X,\mu))}T+\frac{1}{2}} \|u_0\|_{L^2(X,\mu)}. \tag{8.34}$$

The *a priori* estimate implies for every $\delta > 0$ and $n \geq 2$

$$\left\|(u^{n,\delta})^\pm\right\|_{L^\infty(0,T;L^2(X,\mu))} \leq e^{\frac{T}{2}\|\partial^* b\|_{L^1(0,T;L^\infty(X,\mu))}} \left\|u_0^\pm\right\|_{L^2(X,\mu)}. \tag{8.35}$$

We see that $(u^{n,\delta})_\delta$ is bounded in $L^2(0,T;\mathcal{D}(\mathcal{E})) \cap L^\infty(0,T;L^2(X,\mu))$. Thus Banach-Alaoglu implies the existence of a function $u^n \in L^2(0,T;\mathcal{D}(\mathcal{E})) \cap L^\infty(0,T;L^2(X,\mu))$ and a subsequence $(u^{n,\delta_k})_k$, w.r.t. $\delta \to 0$, such that

$$u^{n,\delta_k} \overset{*}{\rightharpoonup} u^n \quad \text{in } L^2(0,T;\mathcal{D}(\mathcal{E})) \cap L^\infty(0,T;L^2(X,\mu)).$$

Note that we additionally have

$$\left|\int_0^T \psi(t)\langle u^{n,\delta}(t)\left(b_\delta(t) - b(t)\right), \partial\phi\rangle_{\mathcal{H}} dt\right| \leq C\|\psi\|_{\sup}\left\|u^{n,\delta}\right\|_{L^2(0,T;\mathcal{D}(\mathcal{E}))} \|b_\delta - b\|_{L^2(0,T;\mathcal{H})} \mathcal{E}(\phi)^{\frac{1}{2}},$$

the uniform bound (8.34) of $\left\|u^{n,\delta}\right\|_{L^2(0,T;\mathcal{D}(\mathcal{E}))}$ in δ and $b_\delta \to b$ in $L^2(0,T;\mathcal{H})$. Hence by the formulation of a weak solution to (8.3) we conclude

$$-\int_0^T \psi'(t)\int_X u^n(t)\phi d\mu dt = \int_0^T \psi(t)\left[\langle u^n(t)b(t), \partial\phi\rangle_{\mathcal{H}} - \frac{1}{n}\mathcal{E}(u^n(t),\phi)\right] dt + \psi(0)\int_X u_0\phi d\mu$$

for every $\psi \in C^1([0,T])$ with $\psi(T) = 0$ and $\phi \in \mathcal{D}(\mathcal{E})$.

We know that u^n inherits the bounds (8.34) and (8.35). So $(u^n/n)_n$ is bounded in $L^2(0,T;\mathcal{D}(\mathcal{E}))$ and $(u^n)_n$ is bounded in $L^\infty(0,T;L^2(X,\mu))$. Hence we can extract a subsequence $(u^{n_k})_{n_k}$ converging $*$-weakly to some $u \in L^\infty(0,T;L^2(X,\mu))$ and $u^{n_k}/n_k \rightharpoonup 0$ in $L^2(0,T;\mathcal{D}(\mathcal{E}))$. Thus we easily conclude

$$\int_0^T \psi(t)\langle u(t)b(t), \partial\phi\rangle_{\mathcal{H}} dt = -\int_0^T \psi'(t)\int_X u(t)\phi d\mu dt - \psi(0)\int_X u_0\phi d\mu.$$

for every $\psi \in C^1([0,T])$ with $\psi(T) = 0$ and $\phi \in \mathcal{D}(\mathcal{E})$, because of

$$\lim_{k\to\infty}\int_0^T \psi(t)\frac{1}{n_k}\mathcal{E}(u^{n_k}(t),\phi)dt = 0.$$

So u is a weak solution to (8.1). Note that u inherits (8.35), which implies the positivity if $u_0 \geq 0$. □

Chapter 9

Calculus of variations on fractals

In this chapter we generalize some basic facts from the calculus of variations on metric measure spaces that carry a strongly local regular Dirichlet form.

Section 9.1 contains a short discussion about p-energies and $(1,p)$-Sobolev spaces for fractals that carry a local regular Dirichlet form having a carré du champ. This partially generalizes former definitions in [HRT13, Section 6], which covered the cases $2 \leq p < +\infty$. Under certain assumptions and in connection with the concept of measurable bundles as introduced in Section 4.4 we show the *reflexivity* of these Sobolev spaces. This fact is then used in Section 9.2 to show existence of minimizers for convex functionals in the present setup. Some constrained models are discussed in Section 9.3.

The results of this chapter are based on a preprint version of the published article [HKM20].

9.1 p-Energies and reflexive Sobolev spaces

Throughout this section (X, d) is assumed to be a locally compact separable metric space and μ a nonnegative Radon measure on X such that $\mu(U) > 0$ for any nonempty open set $U \subset X$. Further, let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a fixed, regular Dirichlet form on $L^2(X, \mu)$.

We would like to give quick account on p-energies and $(1,p)$-Sobolev spaces for fractals that carry a local regular Dirichlet form. Our starting point is the linear theory for $p = 2$, and we would like to take this as a base to define p-energies and Sobolev spaces. The naive idea is to mimick the classical definitions, and clearly this can work only under certain assumptions.

Assumption 9.1. The Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, \mu)$ is strongly local and admits a carré du champ, $\mu(X) < +\infty$, and $\mathcal{A}$ is a subspace of $\mathcal{D}(\mathcal{E}) \cap L^\infty(X, \mu)$, dense in $\mathcal{D}(\mathcal{E})$ and dense in all $L^p(X, \mu)$-spaces, $1 \leq p < +\infty$, such that

$$\Gamma(f) \in L^\infty(X, \mu) \quad \text{for all } f \in \mathcal{A}. \tag{9.1}$$

In many typical examples $\mathcal{A}$ will be a suitable core of $\mathcal{E}$ so that its density in $C_c(X)$ implies its density in the $L^p(X, \mu)$-spaces. We emphasize that the considered Dirichlet form does not have to be induced by a resistance form.

Under Assumption 9.1 we can define associated *p-energies*, $1 \leq p < +\infty$, by

$$\mathcal{E}^{(p)}(f) := \int_X \Gamma(f)^{p/2} d\mu, \quad f \in \mathcal{A}.$$

We wish to extend $\mathcal{E}^{(p)}$ to a subspace of $L^p(X, \mu)$ consisting of all functions f for which $\mathcal{E}^{(p)}(f)$ can be defined as a finite quantity, and we wish this pool of functions to become a

Banach space. The functional $(\mathcal{E}^{(p)}, \mathcal{A})$ is said to be *closable in* $L^p(X,\mu)$ if for any sequence $(f_n)_n \subset \mathcal{A}$ that is Cauchy in the seminorm $\mathcal{E}^{(p)}(\cdot)^{1/p}$ and such that $\lim_{n\to\infty} f_n = 0$ in $L^p(X,\mu)$ we have $\lim_{n\to\infty} \mathcal{E}^{(p)}(f_n) = 0$.

For $p \geq 2$ closability is easily seen, for $1 < p < +\infty$ we make an additional assumption, it implies a 'distributional' integration by parts identity, see (9.3). Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ denote the generator of the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ as defined in (4.18).

Assumption 9.2. There is a space of functions $\mathcal{A}_{\mathcal{L}} \subset \mathcal{D}(\mathcal{L})$ so that Assumption 9.1 holds with $\mathcal{A}_{\mathcal{L}}$ in place of $\mathcal{A}$ and such that we have

$$\mathcal{L}f \in L^\infty(X,\mu) \quad \text{for all } f \in \mathcal{A}_{\mathcal{L}}.$$

Remark 9.1.

(i) For the definitions of p-energies and Sobolev spaces as above the algebra $\mathcal{A}$ and the measure μ are part of the setup. Therefore the definitions depend on the choice of these items, at least a priori. It would be interesting to find out more about the possible equivalence of definitions for different $\mathcal{A}$ and μ. Valuable hints might be found in [Sch17].

(ii) Strong locality of the form and finiteness of the measure in Assumption 9.1 may not be too restrictive, but the assumption of a carré du champ excludes many interesting examples, such as for instance the standard self-similar Dirichlet forms on the classical Sierpiński gasket and carpet, considered with the standard normalized self-similar Hausdorff measure, respectively. See for instance [BST99; Hin05; HN06]. This is a clear disadvantage. However, for a given strongly local regular Dirichlet form one can always find a finite measure μ and an algebra $\mathcal{A}$, [HKT15, Lemma 2.1], [HT18, Remark 4.1], such that after a standard change of measure procedure we arrive at a Dirichlet form satisfying Assumption 9.1, see [HRT13, Lemma 7.1 and Theorem 7.1]. For change of measure results see [CF12, Corollary 5.2.10], [FOT94, Section 6.2, p. 275], [FL91; FST91; KN91], or [Hin16; HRT13]. This clearly alters the metric measure space under consideration, but it provides a rich class of examples of fractal spaces that can be analyzed. The study of fractals with energy dominant measures,[Kus89; Tep08], is also referred to as *measurable Riemannian geometries*, see [Kaj12; Kig08], and also the related studies [Hin13a; KZ12]. The prototype of these examples is the classical Sierpiński gasket with standard energy form and Kusuoka measure. We would finally like to point out that in the case of resistance forms a considerable amount of theory can be developed in a measure-free context, this has been done in [Kig03].

(iii) Assumption 9.2 is also satisfied for many interesting examples, for instance to the Sierpiński gasket endowed with the standard energy form and the Kusuoka measure, see Example 9.1(3). This example generalizes to more general finitely ramified fractals, [Tep08, Section 8]. Secondly, if (in addition to Assumption 9.1) the Markovian semigroup uniquely on $L^2(X,\mu)$ associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Feller semigroup (i.e. a positivity preserving and strongly continuous contraction semigroup on the space $C_0(X)$ of continuous functions vanishing at infinity) and $\mathcal{A}_{\mathcal{L}}$ is a dense subalgebra of the domain of the Feller generator, then Assumption 9.2 is satisfied. This can be seen as in [BBKT10, Lemma 2.8]. Finally, Assumption 9.2 also holds if (in addition to the other assumptions) (X, μ, Γ) is a Diffusion Markov triple in the sense of [BGL14, Definition 3.1.8].

Theorem 9.1. *The functional* $(\mathcal{E}^{(p)}, \mathcal{A})$ *is closable in* $L^p(X,\mu)$, $2 \leq p < +\infty$. *If Assumption 9.2 is satisfied, then it is also closable in* $L^p(X,\mu)$, $1 < p < 2$.

In order to prove Theorem 9.1 we need some preparations. We start with a version of [HRT13, Lemma 7.2]. Because the proof is an inessential modification of the one given there, we omit it.

Lemma 9.1. *The space $\mathcal{A}\otimes\mathcal{A}$ is dense in all $L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$, $1<p<+\infty$. Under Assumption 9.2 also the space $\mathcal{A}_{\mathcal{L}}\otimes\mathcal{A}$ is dense in all $L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$, $1<p<+\infty$.*

We rely on an integration by parts formula which involves the divergence. Recall that the adjoint operator $\partial^*:\mathcal{H}\to L^2(X,\mu)$ of ∂ is defined by saying that $v\in\mathcal{H}$ is a member of $\mathcal{D}(\partial^*)$ if there exists $v^*\in L^2(X,\mu)$ such that $\langle f,v\rangle_{L^2(X,m)}=\langle\partial f,v\rangle_{\mathcal{H}}$ for all $f\in\mathcal{D}(\mathcal{E})$. In this case $\partial^*v:=v^*$ and

$$\langle f,\partial^*v\rangle_{L^2(X,\mu)}=\langle\partial f,v\rangle_{\mathcal{H}},\quad f\in\mathcal{D}(\mathcal{E}).$$

The operator $-\partial^*$ is a *generalized divergence*. Alternatively ∂^*v can be defined for any $v\in\mathcal{H}$ in a distributional sense by setting

$$\partial^*v(\varphi):=\langle\partial\varphi,v\rangle_{\mathcal{H}},\quad\varphi\in\mathcal{A}.$$

For $v=g\partial f$ with $f,g\in\mathcal{A}$ we then have $\partial^*(g\partial f)(\varphi)=\int_X g\Gamma(f,\varphi)d\mu$, $\varphi\in\mathcal{A}$, and therefore

$$|\partial^*(g\partial f)(\varphi)|\leq\|g\|_{\sup}\mathcal{E}(f)^{1/2}\mathcal{E}(\varphi)^{1/2},\tag{9.2}$$

as pointed out in [HRT13, Section 3]. This is sufficient to prove closability for $p\geq 2$, [HRT13, Theorem 6.1].

Now suppose Assumption 9.2 is in force. For $f\in\mathcal{A}_{\mathcal{L}}$ and $g\in\mathcal{A}$ we have, similarly as in [HRT13, Lemma 3.2], the identity

$$\partial^*(g\partial f)(\varphi)=\mathcal{E}(g\varphi,f)-\int_X\varphi d\Gamma(f,g)d\mu,\quad\varphi\in\mathcal{A}.\tag{9.3}$$

By (4.18) this implies

$$|\partial^*(g\partial f)(\varphi)|\leq(\|g\|_{L^\infty(X,\mu)}\|\mathcal{L}f\|_{L^\infty(X,\mu)}+\|\Gamma(f)\|^{1/2}_{L^\infty(X,\mu)}\|\Gamma(g)\|^{1/2}_{L^\infty(X,\mu)})\|\varphi\|_{L^1(X,\mu)}\tag{9.4}$$

for all $\varphi\in\mathcal{A}$, and since $\mathcal{A}$ is dense in $L^1(X,\mu)$, the functional $\partial^*(g\partial f)$ and the estimate (9.4) extend to all $\varphi\in L^1(X,\mu)$. By linear extension we can therefore define ∂^*v for any $v\in\mathcal{A}_{\mathcal{L}}\otimes\mathcal{A}$ as an element of $L^\infty(X,\mu)$. In particular, $\mathcal{A}_{\mathcal{L}}\otimes\mathcal{A}\subset\mathcal{D}(\partial^*)$.

This allows to define gradients ∂f in a *distributional sense* for all $f\in L^1(X,\mu)$. The space $\mathcal{A}_{\mathcal{L}}\otimes\mathcal{A}$ can be endowed with the norm $v\mapsto\|\partial^*v\|_{L^\infty(X,\mu)}+\|v\|_{\mathcal{H}}$. Now suppose $f\in L^1(X,\mu)$. Setting

$$\partial f(v):=\int_X f\,\partial^*v\,d\mu,\quad v\in\mathcal{A}_{\mathcal{L}}\otimes\mathcal{A},\tag{9.5}$$

we observe

$$|\partial f(v)|\leq\|f\|_{L^1(X,\mu)}(\|\partial^*v\|_{L^\infty(X,\mu)}+\|v\|_{\mathcal{H}}),$$

so that ∂f is is seen to be an element of the dual space $(\mathcal{A}_{\mathcal{L}}\otimes\mathcal{A})'$, and its norm in that space is bounded by $\|f\|_{L^1(X,\mu)}$. Note that since $\mathcal{A}\subset L^1(X,\mu)$ and, by the definition of ∂^*,

$$\langle v,\partial f\rangle_{\mathcal{H}}=\int_X f\,\partial^*v\,d\mu,\quad v\in\mathcal{A}_{\mathcal{L}}\otimes\mathcal{A},\tag{9.6}$$

for any $f\in\mathcal{A}$, we see that for such functions f the gradient ∂f, defined as in Section 4.4, and the gradient defined in (9.5) coincide in $(\mathcal{A}_{\mathcal{L}}\otimes\mathcal{A})'$.

We prove Theorem 9.1.

Proof. Suppose $(f_n)_n \subset \mathcal{A}$ is $\mathcal{E}^{(p)}(\cdot)^{1/p}$-Cauchy and $\lim_{n\to\infty} f_n = 0$ in $L^p(X,\mu)$. Then $(\partial f_n)_n$ is Cauchy in $L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$ and therefore converges to some limit ξ in this space. It suffices to show $\xi = 0$.

If $2 \le p < +\infty$ we can proceed as in [HRT13, Theorem 6.1]: In this case the finiteness of μ implies that $(f_n)_n$ is $\mathcal{E}$-Cauchy, what by (9.2) shows that for any $f, g \in \mathcal{A}$ we have

$$\int_X \langle g\partial f, \xi\rangle_{\mathcal{H}_x}\, \mu(dx) = \lim_{n\to\infty} \langle g\partial f, \partial f_n\rangle_{\mathcal{H}} = \lim_{n\to\infty} \partial^*(g\partial f)(f_n) = 0.$$

Taking linear combinations and applying Lemma 9.1 we conclude that $\xi = 0$.

If $1 < p < 2$ and Assumption 9.2 holds, then for any $f \in \mathcal{A}_{\mathcal{L}}$ and $g \in \mathcal{A}$ we obtain

$$\int_X \langle g\partial f, \xi\rangle_{\mathcal{H}_x}\, \mu(dx) = \lim_{n\to\infty} \langle g\partial f, \partial f_n\rangle = \lim_{n\to\infty} \int_X f_n\, \partial^*(g\partial f)\, \mu(dx) = 0$$

by (9.6) and because $\partial^*(g\partial f) \in L^\infty(X,\mu) \subset L^q(X,\mu)$. Here $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $L^q(X,\mu,(\mathcal{H}_x)_{x\in X})$ and $L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$. Again linear combinations and Lemma 9.1 show $\xi = 0$. □

The closability can also be stated in terms of the operator ∂.

Corollary 9.1. *The linear operator $(\partial, \mathcal{A})$ is closable in $L^p(X,\mu)$, $2 \le p < +\infty$. If Assumption 9.2 is satisfied, then it is also closable in $L^p(X,\mu)$, $1 < p < 2$.*

Now suppose that $2 \le p < +\infty$ or that Assumption 9.2 is satisfied. Then, if $f \in L^p(X,\mu)$ is such that there exists a sequence $(f_n)_n \subset \mathcal{A}$, Cauchy in the seminorm $\mathcal{E}^{(p)}(\cdot)^{1/p}$ and convergent to f in $L^p(X,\mu)$, we define

$$\mathcal{E}^{(p)}(f) := \lim_{n\to\infty} \mathcal{E}^{(p)}(f_n),$$

which by Theorem 9.1 is a correct definition. We denote the vector space of all such $f \in L^p(X,\mu)$ by $H_0^{1,p}(X,\mu)$. The spaces $H_0^{1,p}(X,\mu)$ are Banach with norms

$$\|f\|_{H_0^{1,p}(X,\mu)} = \|f\|_{L^p(X,\mu)} + \mathcal{E}^{(p)}(f)^{1/p}, \quad f \in H_0^{1,p}(X,\mu).$$

The functional $(\mathcal{E}^{(p)}, H_0^{1,p}(X,\mu))$ is called the *closure* (or *smallest closed extension*) of $(\mathcal{E}^{(p)}, \mathcal{A})$ in $L^p(X,\mu)$. Formula (9.1) remains valid for all $f \in H_0^{1,p}(X,\mu)$, their energy densities $\Gamma(f)$ can be defined by approximation.

Definition 9.1. *To the spaces $H_0^{1,p}(X,\mu)$, $1 \le p < +\infty$, we refer as* Sobolev spaces. *Given an open set $\Omega \subset X$ we define the* Sobolev spaces $H_0^{1,p}(\Omega,\mu)$ on Ω *as the completion in $H_0^{1,p}(X,\mu)$ of all elements of $\mathcal{A}$ supported in Ω, respectively.*

Corollary 9.1 implies that under the respective hypoheses ∂ extends to a closed unbounded linear operator $\partial : L^p(X,\mu) \to L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$ with domain $H_0^{1,p}(X,\mu)$. In this case we can upgrade (9.6) to

$$\langle v, \partial f\rangle = \int_X f\, \partial^* v\, d\mu, \quad v \in \mathcal{A}_{\mathcal{L}} \otimes \mathcal{A},$$

for any $f \in H_0^{1,p}(X,\mu)$ and with the left hand side interpreted as dual pairing. This shows that for $f \in H_0^{1,p}(X,\mu)$ the gradient ∂f in the sense of Corollary 9.1 coincides in $(\mathcal{A}_{\mathcal{L}} \otimes \mathcal{A})'$ with the distributional gradient of f as defined in (9.5).

Remark 9.2.

(i) We wish to point out that, as in the case $p = 2$, the closability of $(\mathcal{E}^{(p)}, \mathcal{A})$ is equivalent to the lower semicontinuity of $\mathcal{E}^{(p)}$, seen as a functional on $L^p(X, \mu)$ taking values in $[0, +\infty]$. The proof that the existence of a closed extension implies lower semicontinuity uses Banach-Alaoglu (together with Eberlein-Šmulian and Mazur's lemma) and the reflexivity of $H_0^{1,p}(X, \mu)$.

(ii) Similarly as in the case $p = 2$ closability and lower semicontinuity of $\mathcal{E}^{(p)}$ with respect to the norm in $L^p(X, \mu)$ are equivalent to the closability of $\mathcal{E}^{(p)}$ with respect to the supremum norm and also equivalent to the lower semicontinuity of $\mathcal{E}^{(p)}$ with respect to the supremum norm. This can be seen similarly as in [Hin16, Sections 6, 8 and 10], see also [HT15a]. This use of the supremum norm goes back to [Mok95]. A detailed and very general discussion of closability and lower semicontinuity of energy forms in L^p-spaces can be found in [Sch17].

These spaces are obvious generalizations of the classical Sobolev $H_0^{1,p}(\Omega)$ spaces over bounded domains Ω in Euclidean spaces, defined by completion, [AF03; Maz11]. For $p = 2$ the Sobolev spaces are Hilbert, and $H_0^{1,2}(X, \mu) = \mathcal{D}(\mathcal{E})$ in the sense of equivalently normed Hilbert spaces.

Remark 9.3. The definition of p-energies and $(1, p)$-Sobolev spaces based on a given Dirichlet form differs from other well established approaches:

(i) Sobolev spaces on metric measure spaces via mean-value type inequalities have been proposed in [Haj96], see also [Hei01, Section 5.4]. For Sobolev spaces on metric measure spaces via rectifiable curves and upper gradients see [BB11; BBS03; HKST15; KM02; Sha00; Sha03] or [Hei01, Section 7], an equivalent approach is provided in [Che99]. By a lack of rectifiable curves this upper gradient approach does not apply to fractals.

(ii) Originating in Dirichlet form theory for homogeneous spaces [BM95; MM99], a sort of axiomatic approach to nonlinear energy forms was suggested in [Cap03a; Cap03b; Mos05], it is related to certain metrics. For fractal curves such nonlinear energy forms can be obtained from a simple bare hands definition, see e.g. [CL02].

(iii) Yet a different approach was taken in [HPS04], where the authors considered the Sierpiński gasket and, mimicking the construction of energy forms on post-critically finite self-similar sets, [Kig01], constructed p-energy forms on the gasket solving a renormalization problem. Related p-Laplacians were defined in [SW04]. These p-energies are quite different from those we defined above if μ is taken to be the Kusuoka measure, [Kus89; Kaj12; Kig08; Str06], and $\mathcal{E}$ is the standard energy form on the gasket.

Remark 9.4. As a by-product of the above proof of closability for $1 < p < 2$ one can provide an analog of the most classical definition of Sobolev spaces $W^{1,p}(\Omega)$. Let Assumption 9.2 be in force.

For any $1 < p < +\infty$ set

$$W^{1,p}(X, \mu) := \{f \in L^p(X, \mu) : \partial f \in L^p(X, \mu)\}$$

and endow this vector space with the norm

$$f \mapsto (\|f\|_{L^p(X,\mu)} + \|\partial f\|_{L^p(X,\mu,(\mathcal{H}_x)_{x\in X})})^{1/p}.$$

Now the classical proof shows that they are Banach spaces: If $(f_n)_n$ is Cauchy in $W^{1,p}(X,\mu)$ then there exist some $f \in L^p(X,\mu)$ such that $f = \lim_{n\to\infty} f_n$ in $L^p(X,\mu)$ and some $\xi \in L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$ such that $\lim_{n\to\infty} \partial f_n = \xi$ in $L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$. Since

$$\xi(v) = \langle v, \xi\rangle = \lim_{n\to\infty} \langle v, \partial f_n\rangle = \lim_{n\to\infty} \partial f_n(v) = \lim_{n\to\infty} \int_X \partial^* v\, f_n\, d\mu = \int \partial^* v\, f\, d\mu = \partial f(v)$$

for all $v \in \mathcal{A}_{\mathcal{L}} \otimes \mathcal{A}$, we have $\xi = \partial f \in L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$, what shows that $\lim_{n\to\infty} f_n = f$ in $W^{1,p}(X,\mu)$, as desired.

Obviously

$$H_0^{1,p}(X,m) \subset W^{1,p}(X,\mu).$$

Looking at the classical p-energies on bounded Euclidean domains shows that in general the converse inclusion will not hold: In this case ∂f for $f \in L^1(\Omega) \subset L^1_{\text{loc}}(\Omega)$ coincides with ∇f, seen as a regular distribution on Ω, and we have $W^{1,p}(X,\mu) = W^{1,p}(\Omega)$, which is strictly larger than $H_0^{1,p}(X,m) = W_0^{1,p}(\Omega)$.

Recall that we refer to the space $L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$ as the *space of generalized L^p-vector fields*. Using the fiber-wise interpretation in a measurable sense we can also write for any $1 \le p < +\infty$

$$\mathcal{E}^{(p)}(f) = \int_X \|\partial_x f\|_{\mathcal{H}_x}^p\, \mu(dx), \quad f \in \mathcal{A}.$$

The next fact was noted in [CG03, Lemma 4.3] for continuous fields of Hilbert spaces.

Proposition 9.1. *The spaces $L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$, $1 < p < +\infty$, are uniformly convex and in particular, reflexive. For each $1 < p < +\infty$ the spaces $L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$ and $L^q(X,\mu,(\mathcal{H}_x)_{x\in X})$ with $1 = 1/p + 1/q$ are the dual of each other.*

A proof is provided in the appendix to this chapter. Proposition 9.1 implies the following useful fact.

Corollary 9.2. *The Sobolev spaces $H_0^{1,p}(X,\mu)$ are separable for $1 \le p < +\infty$ and reflexive for $1 < p < +\infty$.*

Corollary 9.2 can be seen using the following well-known standard trick, see for instance [Bre11, Proposition 8.1 and 9.1].

Proof. Since Cartesian products of reflexive spaces are reflexive, $L^p(X,\mu)\times L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$ is reflexive for $1 < p < +\infty$. The operator $T : H_0^{1,p}(X,\mu) \to L^p(X,\mu)\times L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$, $Tf := (f, \partial f)$ is an isometry from $H_0^{1,p}(X,\mu)$ onto the closed subspace $T(H_0^{1,p}(X,\mu))$ of $L^p(X,\mu) \times L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$, and closed subspaces of reflexive spaces are reflexive. Therefore $T(H_0^{1,p}(X,\mu))$ is reflexive and consequently also $H_0^{1,p}(X,\mu)$. For $1 \le p < +\infty$ separability follows similarly, because it is stable under products and inherited to subsets. □

Remark 9.5. Although in the present setup the reflexivity of the spaces $H_0^{1,p}(X,\mu)$ may seem rather trivial to the reader, we would like to point out that in other approaches to Sobolev spaces on metric measure spaces it is a serious issue and may fail to hold, for some comments see [HKST15, p. 204].

We provide some examples for which our Assumptions 9.1 and 9.2 are satisfied.

Examples 9.1.

(1) **Degenerate forms**
Let $\Omega = (-1,1)^2 \subset \mathbb{R}^2$ and consider the quadratic form

$$\mathcal{E}(f) = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial f}{\partial x_1}\right)^2 dx_1dx_2 + \int_{-1}^{1}\int_{0}^{1} x_2\left(\frac{\partial f}{\partial x_2}\right)^2 dx_1dx_2, \quad f \in C_c^\infty((-1,1)^2).$$

Since obviously $\frac{\partial}{\partial x_i}(x_2 \vee 0) \in L^2((-1,1)^2)$, $i = 1,2$, the form is closable in $L^2((-1,1)^2)$, [FOT94, Section 3.1, (1°.a)], and its closure satisfies Assumptions 9.1 and 9.2 with μ being the two-dimensional Lebesgue measure, $d\mu = dx_1dx_2$ and $\mathcal{A}_{\mathcal{L}} = \mathcal{A} = C_c^\infty(\Omega)$. We have

$$\Gamma(f)(x_1,x_2) = \left(\frac{\partial f}{\partial x_1}(x_1,x_2)\right)^2 + (x_2 \vee 0)^2\left(\frac{\partial f}{\partial x_2}(x_1,x_2)\right)^2.$$

For a.e. $x = (x_1,x_2) \in (-1,1)\times(-1,0)$ the spaces $\mathcal{H}_x$ are one-dimensional and for a.e. $x \in (-1,1)\times(0,1)$ two-dimensional. Roughly speaking, this means that in the lower half of the square the diffusion can move only in x_1-direction, while in the upper half it can also move in x_2-direction. The associated Sobolev spaces $H_0^{1,p}(X,\mu)$ inherit this degeneracy.

(2) **Superpositions**
We revisit a special case of [Hin13b, Example 2.3]. Again let $X = (-1,1)^2 \subset \mathbb{R}^2$, we write $x = (x_1,x_2)$ for its elements. Now consider

$$\mathcal{E}(f) = \int_{-1}^{1}\int_{-1}^{1}|\nabla f(x_1,x_2)|^2 dx_1dx_2 + \int_{-1}^{1}\left(\frac{\partial f}{\partial x_1}(x_1,0)\right)^2 dx_1, \quad f \in C_c^\infty((-1,1)^2).$$

This form is closable in $L^2((-1,1)^2,\mu)$ with $d\mu = dx_1dx_2 + dx_1 \times \delta_0(dx_2)$, where δ_0 is the Dirac measure at $0 \in (-1,1)$, [FOT94, Section 3.1 (2°), p.103], and clearly μ is energy dominant. Now

$$\Gamma(f) = |\nabla f|^2 + \left(\frac{\partial f}{\partial x_1}\right)^2.$$

There is an μ-null set outside of which we have $\dim \mathcal{H}_x = 2$ if $x_2 \neq 0$ and $\dim \mathcal{H}_x = 1$ if $x_2 = 0$. Again both Assumptions 9.1 and 9.2 are satisfied with $\mathcal{A}_{\mathcal{L}} = \mathcal{A} = C_c^\infty((-1,1)^2)$.

(3) **Sierpiński gasket**
Let X be the classical Sierpiński gasket K and $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ its standard energy form, see for instance [Str06]. We consider this form on $L^2(K,\nu)$ where $\mu = \nu$ is the Kusuoka measure. The latter is defined as the sum $\nu = \nu_{h_1} + \nu_{h_2}$ of the energy measures of h_1 and h_2, where $\{h_1,h_2\}$ is an energy orthonormal system of non-constant harmonic functions on K. See for instance [Kaj12; Kig08; Kus89; Tep08]. Assumptions 9.1 and 9.2 are satisfied: The algebra $C^1(K)$ of functions of type $f = F(h_1,h_2)$ with $F \in C^1(\mathbb{R}^2)$ is dense in $\mathcal{D}(\mathcal{E})$ and by the chain rule for energy measures, can be taken as the space $\mathcal{A}$. In fact, we have

$$\Gamma(f)(x) = \langle \nabla F(y), Z(x)\nabla F(y)\rangle_{\mathbb{R}^2} \tag{9.7}$$

for ν-a.e. $x \in K$, where ∇F denotes the usual gradient of F in $\mathbb{R}^2$, $y(x) := (h_1(x),h_2(x))$, $x \in K$, and $Z = (Z(x))_{x\in X}$ is a measurable (2×2)-matrix valued function on K such that $\operatorname{rank} Z(x) = 1$ for ν-a.e. $x \in K$. The map y is a homeomorphism $y : K \to y(K)$ of the compact (in Euclidean or resistance topology) space K onto its image $y(K)$ in $\mathbb{R}^2$. Consequently $\Gamma(f) \in L^\infty(K,\nu)$. The density of $\mathcal{A}$ in the continuous functions follows from Stone-Weierstrass. Similarly we can use the space $C^2(K)$ of functions $f = F(h_1, h_2)$ with $F \in C^2(\mathbb{R}^2)$ as $\mathcal{A}_{\mathcal{L}}$, note that $\mathcal{L}f(x) = \mathrm{Tr}(Z_x D^2 F(y))$, where D^2F denotes the Hessian of F, and clearly this is in $L^\infty(K,\nu)$. This space is also $\mathcal{E}$-dense in $\mathcal{D}(\mathcal{E})$. For details see [Tep08, Theorem 8]. Note that the result on the rank of Z dictates that for ν-a.e. $x \in K$ the dimension of $\mathcal{H}_x$ is one.

(4) **Products of fractals**
For simplicity consider $X = K \times J$, where K is the classical Sierpiński gasket and $J = [0,1]$. We endow X with the product measure $d\mu := d\nu \times dx$, where ν is the Kusuoka measure on K and on J we use the one-dimensional Lebesgue measure dx. Let $\mathcal{E}_K$ be the standard energy form on K with domain $\mathcal{D}(\mathcal{E}_K)$ and let $\mathcal{E}_J(f) := \int_0^1 (f')^2 dx$ be the Dirichlet integral on $(0,1)$ with domain $W_0^{1,2}(0,1)$. On $L^2(K\times J,\mu)$ one can consider the product Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ defined by

$$\mathcal{D}(\mathcal{E}) := \big\{ f \in L^2(K \times J, \mu) : \text{for a.e. } x_2 \in J \text{ we have } f(\cdot, x_2) \in \mathcal{D}(\mathcal{E}_K) \\ \text{and for } \nu\text{-a.e. } x_1 \in K \text{ we have } f(x_1, \cdot) \in W_0^{1,2}(0,1) \big\}$$

and

$$\mathcal{E}(f) = \int_J \mathcal{E}_K(f(\cdot, x_2))dx_2 + \int_K \int_J \left(\frac{\partial f}{\partial x_2}(x_1, x_2) \right)^2 dx_2 \nu(dx_1),$$

see [BH91, Chapter V], [Str05a] or [Str06, Section 5.6]. We have

$$\Gamma(f)(x_1, x_2)) = \Gamma_K(f(\cdot, x_2))(x_1) + \left(\frac{\partial f}{\partial x_2}(x_1, x_2) \right)^2,$$

and it is not difficult to see that for μ-a.e. $x = (x_1, x_2) \in K \times J$ the spaces $\mathcal{H}_x$ equal (up to isometry) the products $\mathcal{H}_{K,x_1} \times \mathcal{H}_{J,x_2}$ of the individual fibers. In particular, they are two-dimensional μ-a.e. The Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular and local and satisfies Assumption 9.1 with $\mathcal{A} = C^1(K) \otimes C_c^\infty((0,1))$ (with obvious multiplication).

9.2 Existence of minimizers for convex functionals

In this section we formulate the direct method for an abstract setup. To do so we follow classical presentations as can for instance be found in [Dac08, Sections 3.2 and 3.4] or in [JL98, Chapter 4]. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form satisfying Assumption 9.1, and whenever $1 < p < 2$ also Assumption 9.2.

We start by observing lower semicontinuity for integral functionals on $L^p(X, \mu, (\mathcal{H}_x)_{x\in X})$, [JL98, Lemma 4.3.1.]

Lemma 9.2. *Let $1 \le p < +\infty$ and let $f = (f_x)_{x\in X}$ be a family of mappings $f_x : \mathcal{H}_x \to \mathbb{R}$, $x \in X$, such that*

(i) for every $v \in L^p(X, \mu, (\mathcal{H}_x)_{x\in X})$ the function $x \mapsto f_x(v_x)$ is Borel measurable,

(ii) f_x is lower semicontinuous for every $x \in X$,

(iii) there are a function $a \in L^1(X,\mu)$ and constant $b > 0$ such that

$$f_x(v_x) \ge -a(x) + b\|v_x\|^p_{\mathcal{H}_x} \tag{9.8}$$

for μ-a.e. $x \in X$ and all $v \in L^p(X, \mu, (\mathcal{H}_x)_{x\in X})$.

Then

$$I[v] := \int_X f_x(v_x)\mu(dx)$$

is a lower semicontinuous functional on $L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$.

Proof. For any $v \in L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$ the integral $I(v)$ is well-defined as an element of the extended real axis because of (i) and (9.8). Suppose $(v_n)_n$ converges to v in $L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$. Then we can find a subsequence, for convenience again denoted by $(v_n)_n$, such that its norms $\|(v_n)_x\|_{\mathcal{H}_x}$ converge pointwise μ-a.e. to $\|v_x\|_{\mathcal{H}_x}$. Since f_x is lower semicontinuous, we have

$$f_x(v_x) - b\|v_x\|^p_{\mathcal{H}_x} \leq \liminf_{n\to\infty} \left(f_x((v_n)_x) - b\|(v_n)_x\|^p_{\mathcal{H}_x}\right)$$

μ-a.e. and using (9.8) and Fatou's lemma,

$$\int_X \left(f_x(v_x) - b\|v_x\|^p_{\mathcal{H}_x}\right)\mu(dx) \leq \liminf_{n\to\infty}\int_X \left(f_x((v_n)_x) - b\|(v_n)_x\|^p_{\mathcal{H}_x}\right)\mu(dx).$$

Because $(v_n)_n$ converges to v in $L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$, we have

$$\int_X b\|v_x\|^p_{\mathcal{H}_x}\mu(dx) = \lim_{n\to\infty}\int_X b\|(v_n)_x\|^p_{\mathcal{H}_x}\mu(dx),$$

so that by the superadditivity of $\liminf$,

$$\int_X f_x(v_x)\mu(dx) \leq \liminf_{n\to\infty}\int_X f_x((v_n)_x)\mu(dx).$$

□

Convexity is inherited from the integrand to the functional.

Lemma 9.3. *Suppose in addition to the assumptions in Lemma 9.2 that f_x is convex for every $x \in X$. Then the functional I is also convex.*

Proof. Let $v, w \in L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$ and $t \in [0,1]$. The convexity of each f_x implies

$$\int_X f_x\left(tv_x + (1-t)w_x\right)\mu(dx) \leq \int_X \left(tf_x(v_x) + (1-t)f_x w_x\right)\mu(dx).$$

□

It is a well known general fact that by convexity we can pass from the strong to the weak topology. For a proof see for instance [JL98, Lemma 4.2.2.]

Lemma 9.4. *Let V be a convex subset of a separable reflexive Banach space, $F: V \to \bar{\mathbb{R}}$ convex and lower semicontinuous. Then F is also lower semicontinuous w.r.t. weak convergence.*

We are interested in minimizing the convex functional

$$I[u] := \int_X f_x(\partial_x u)\mu(dx).$$

The following is a version of a well known existence result, see e.g. [JL98, Theorem 4.3.1], adapted to our situation. Given an open set $\Omega \subset X$ and a function $g \in H_0^{1,p}(X,\mu)$ we write $g + H_0^{1,p}(\Omega,\mu)$ for the collection of all elements of $H_0^{1,p}(X,\mu)$ of form $g + \varphi$ with $\varphi \in H_0^{1,p}(\Omega,\mu)$. This encodes a *generalized Dirichlet boundary condition.*

Theorem 9.2. *Let $1 < p < +\infty$ let $\Omega \subset X$ be an open set and assume that the (global) Poincaré inequality*

$$\|u\|^p_{L^p(\Omega,\mu)} \leq c\,\mathcal{E}^{(p)}(u), \quad u \in H^{1,p}_0(\Omega,\mu),$$

holds, where $c > 0$ is constant depending only on Ω and p. Let $f = (f_x)_{x\in X}$ be a family of mappings $f_x : \mathcal{H}_x \to \mathbb{R}$, $x \in X$ such that

(i) for every $v \in L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$ the function $x \mapsto f_x(v_x)$ is Borel measurable,

(ii) the function f_x is lower semicontinuous and convex for all $x \in X$,

(iii) there are a function $a \in L^1(X,\mu)$ and a constant $b > 0$ such that

$$f_x(v_x) \geq -a(x) + b\|v_x\|^p_{\mathcal{H}_x} \tag{9.9}$$

is satisfied for almost all $x \in X$ and all $v \in L^p(X,\mu,(\mathcal{H}_x)_{x\in X})$.

Then for any $g \in H^{1,p}_0(X,\mu)$ the functional

$$I\,[u] = \int_X f_x(\partial_x u)\mu(dx)$$

admits its infimum on $g + H^{1,p}_0(\Omega,\mu)$, i.e. there exists $u_0 \in g + H^{1,p}_0(\Omega,\mu)$ with

$$I\,[u_0] = \inf_{u\in g+H^{1,p}_0(\Omega,\mu)} I\,[u]\,.$$

Proof. By Lemmas 9.2 and 9.3 the functional I is lower semicontinuous and convex on $H^{1,p}_0(X,\mu)$. Since $H^{1,p}_0(X,\mu)$ is separable and reflexive (Corollary 9.2), I is weakly lower semicontinuous on $H^{1,p}_0(X,\mu)$ by Lemma 9.4.

Let $(u_n)_n$ be a minimizing sequence in $g + H^{1,p}_0(\Omega,\mu)$, i.e. such that

$$\lim_{n\to\infty} I[u_n] = \inf_{u\in g+H^{1,p}_0(\Omega,\mu)} I[u].$$

From (9.9) we obtain

$$\int_X \|\partial_x u_n\|^p_{\mathcal{H}_x}\,\mu(dx) \leq \frac{1}{b} I\,[u_n] + \frac{1}{b}\int_X a(x)\mu(dx).$$

This implies that $(u_n)_n$ is bounded in $H^{1,p}_0(X,\mu)$. By Corollary 9.2 together with the theorems of Banach-Alaoglu and Eberlein-Šmulian we can find a subsequence, which we will again denote by $(u_n)_n$, that converges weakly in $H^{1,p}_0(X,\mu)$ to some limit u_0. Since $g + H^{1,p}_0(\Omega,\mu)$ is convex and closed, it is weakly closed, so that $u_0 \in H^{1,p}_0(\Omega,\mu)$.

Combined with the weakly lower semicontinuity of I, this implies

$$I\,[u_0] \leq \liminf_{n\to\infty} I\,[u_n] = \lim_{n\to\infty} I\,[u_n] = \inf_{u\in g+H^{1,p}_0(\Omega,\mu)} I\,[u] \leq I\,[u_0]\,.$$

and since $u_0 \in g + H^{1,p}_0(\Omega,\mu)$, we must have equality. □

Remark 9.6. Under a strict convexity assumption one can also obtain uniqueness, see [Dac08, Theorem 3.30].

Remark 9.7. If $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a resistance form (or induced by a resistance form), such as in Chapter 3, the minimizer is trivially Hölder continuous. In the general case related Hölder regularity results can be found in [BM06; KS01]. They follow classical methods of DeGiorgi and Moser, respectively, and therefore need volume doubling and the validity of suitable (localized) Poincaré inequalities.

The considered functionals do not have to be isotropic anymore.

Examples 9.2 (Anisotropic functionals). Let $1 < p < +\infty$, if $1 < p < 2$ let Assumption 9.2 be in force. Suppose that for μ-a.e. $x \in X$ the space $\mathcal{H}_x$ is two-dimensional, as for instance in Example 9.1(4). Let $\eta^{(1)}, \eta^{(2)} \in \mathcal{H}$ be such that for any $x \in X$ with $\dim \mathcal{H}_x = 2$, $\left\{\eta_x^{(1)}, \eta_x^{(2)}\right\}$ is an orthonormal basis in $\mathcal{H}_x$, see for instance [Tak02, Lemma 8.12]. By Theorem 9.2 we can find a minimizer in $g + H_0^{1,p}(\Omega, \mu)$ for the functional I with integrand defined by

$$f_x(v) = \|v\|_{\mathcal{H}_x}^p + \left|\left\langle v, \eta_x^{(1)}\right\rangle_{\mathcal{H}_x}\right|^p, \quad v \in \mathcal{H}_x,$$

if $\mathcal{H}_x$ is two-dimensional and by $f_x \equiv 0$ otherwise. This anisotropic functional could not be expressed in terms of the carré operator $u \mapsto \Gamma(u)$ only.

9.3 Constrained minimization problems

We translate some problems with integral constraints, [Eva10, Section 8], to our setup.

9.3.1 Nonlinear Poisson equation

Let $1 < p < \infty$, if $1 < p < 2$ let Assumption 9.2 be satisfied. Suppose that $\Omega \subset X$ is open and $g \in H_0^{1,p}(X, \mu)$. We wish to minimize the energy functional

$$I[w] := \int_X \|\partial_x w\|_{\mathcal{H}_x}^p \mu(dx)$$

in the class $g + H_0^{1,p}(\Omega, \mu)$, but now subject to the additional condition that

$$J[w] := \int_X G(w(x))\mu(dx) = 0,$$

where $G : \mathbb{R} \to \mathbb{R}$ is a given smooth function such that $|G'(z)| \leq C\left(|z|^{p-1} + 1\right)$ for some constant C. We introduce the admissible class

$$\mathfrak{A} := \{w \in g + H_0^{1,p}(\Omega, \mu) \mid J[w] = 0\}.$$

The following is a version of [Eva10, Section 8.4.1, Theorem 1].

Theorem 9.3. *Assume that the embedding of $H_0^{1,p}(X, \mu) \subset L^p(X, \mu)$ is compact. Then, if the admissible class $\mathfrak{A}$ is nonempty, there exists $u \in \mathfrak{A}$ satisfying $I[u] = \min_{w \in \mathfrak{A}} I[w]$.*

Proof. Let $(u_n)_n \subset \mathfrak{A}$ be such that $\lim_{n\to\infty} I[u_n] = \inf_{w\in\mathfrak{A}} I[w]$, we may assume it converges weakly to some u in $g + H_0^{1,p}(\Omega, \mu)$ with $I[u] \leq \inf_{w\in\mathfrak{A}} I[w]$. Using the boundedness of this sequence and the compact embedding, we may (by switching to a subsequence) assume it also converges to some limit in $L^p(X, \mu)$, and by Mazur's lemma this limit must be u. Now

$$\begin{aligned}|J(u)| = |J(u) - J(u_n)| &\leq \int_X |G(u(x)) - G((u_n(x))|\mu(dx) \\ &\leq C \int_X |u(x) - u_n(x)| \left(1 + |u(x)|^{p-1} + |u_n(x)|^{p-1}\right) \mu(dx),\end{aligned}$$

and by Hölder's inequality the right hand side converges to zero, proving $J(u) = 0$ and therefore $u \in \mathfrak{A}$. ∎

We turn to the corresponding Euler-Lagrange equation, [Eva10, Section 8.4.1, Theorem 2].

Theorem 9.4. *Assume that Ω is connected and let $\mathfrak{A}_0 := \{w \in H_0^{1,p}(\Omega,\mu) \mid J[w] = 0\}$. Suppose there exists $u \in \mathfrak{A}_0$ such that $I[u] = \min_{w\in\mathfrak{A}_0} I[w]$. Then we can find a real number λ such that*

$$\int_X \|\partial_x u\|_{\mathcal{H}_x}^{p-2} \langle \partial_x u, \partial_x v\rangle_{\mathcal{H}_x} \mu(dx) = \lambda \int_X G'(u(x))v(x)\mu(dx) \tag{9.10}$$

for all $v \in H_0^{1,p}(\Omega,\mu)$.

The number λ is the *Lagrange multiplier* corresponding to the integral constraint $J[u] = 0$. The function u as in the Theorem 9.4 is a *weak solution of the nonlinear Poisson equation* $-\Delta_p u = \lambda G'(u)$ in Ω with zero Dirichlet boundary condition on $X \setminus \Omega$ for the p-Laplacian Δ_p. In the case $p = 2$ this is a nonlinear eigenvalue problem, see Section 8.4.1 in [Eva10].

To see the last theorem one can follow the proof of Theorem 2 in Section 8.4.1 of [Eva10], as Lagrange multiplier λ one has to choose

$$\lambda := \frac{\int_X \|\partial_x u\|_{\mathcal{H}_x}^{p-2} \langle \partial_x u, \partial_x w\rangle_{\mathcal{H}_x} \mu(dx)}{\int_X G'(u(x))w(x)\mu(dx)}.$$

9.3.2 Variational inequality

From now on we assume $p = 2$ and discuss variational problems with one-sided constraints. Let $\Omega \subset X$ be open. We are interested in minimizing the energy functional

$$I[w] := \int_X \|\partial_x w\|_{\mathcal{H}_x}^2 - f(x)w(x)\mu(dx)$$

among all functions w belonging to the admissible class

$$\mathfrak{A} := \{w \in g + H_0^{1,2}(\Omega,\mu) \mid w \geq h \text{ a.e. in } \Omega\},$$

where $f \in L^1(X,\mu)$, $f \not\equiv 0$ and $h \in H_0^{1,2}(X,\mu)$. The function h is called the *obstacle*. We revisit a well known existence and uniqueness result, see [Eva10, Section 8.4.2].

Theorem 9.5. *Assume the admissible set $\mathfrak{A}$ is nonempty. Then there exists a unique function $u \in \mathfrak{A}$ satisfying $I[u] = \min_{w\in\mathfrak{A}} I[w]$.*

Proof. The existence of a minimizer follows as before. To see uniqueness, we assume $u, \tilde{u} \in \mathfrak{A}$ are minimizers and $u \neq \tilde{u}$. Then $w := \frac{1}{2}(u + \tilde{u}) \in \mathfrak{A}$, and

$$\begin{aligned}
I[w] &= \int_X \frac{1}{4} \|\partial_x u + \partial_x \tilde{u}\|_{\mathcal{H}_x}^2 - \frac{1}{2} f(x)(u(x) + \tilde{u}(x))\mu(dx) \\
&= \int_X \frac{1}{4} \left(2\|\partial_x u\|_{\mathcal{H}_x}^2 + 2\|\partial_x \tilde{u}\|_{\mathcal{H}_x}^2 - \|\partial_x u - \partial_x \tilde{u}\|_{\mathcal{H}_x}^2\right) - \frac{1}{2} f(x)(u(x) + \tilde{u}(x))\mu(dx) \\
&< \frac{1}{2} \int_X \|\partial_x u\|_{\mathcal{H}_x}^2 - f(x)u(x)\mu(dx) + \frac{1}{2} \int_X \|\partial_x \tilde{u}\|_{\mathcal{H}_x}^2 - f(x)\tilde{u}(x)\mu(dx) \\
&= \frac{1}{2} I[u] + \frac{1}{2} I[\tilde{u}].
\end{aligned}$$

The strict inequality follows because $u \neq \tilde{u}$ and because $I[\mathbf{1}] = \int_X f(x)\mu(dx)$, so that the functional cannot produce the same value for two functions that differ by a constant. The above inequality contradicts the minimality of u and $\tilde{u}$. ☐

For the present problem the Euler-Lagrange equation is replaced by an inequality.

Theorem 9.6. *Let* $u \in \mathfrak{A}$ *be the unique solution of* $I[u] = \min_{w\in\mathfrak{A}} I[w]$*. Then*

$$\int_X \langle \partial_x u, \partial_x (w-u)\rangle_{\mathcal{H}_x} \mu(dx) \geq \int_X f(x)(w(x)-u(x))\mu(dx)$$

for all $w \in \mathfrak{A}$*.*

Proof. Fix any element $w \in \mathfrak{A}$. The convexity of $\mathfrak{A}$ implies that for any $\tau \in [0,1]$ the function $u + \tau(w-u) = (1-\tau)u + \tau w$ is an element of $\mathfrak{A}$. Consequently, if we set $i(\tau) := I[u+\tau(w-u)]$, we see that $i(0) \leq i(\tau)$ for all $\tau \in [0,1]$. Hence $i'(0) \geq 0$. Now if $\tau \in (0,1]$, then

$$\frac{i(\tau)-i(0)}{\tau} = \int_X \langle \partial_x u, \partial_x (w-u)\rangle_{\mathcal{H}_x} + \frac{\tau}{2} \|\partial_x (w-u)\|^2 - f(x)(w(x)-u(x))\mu(dx),$$

and taking the limit $\tau \to 0$, we obtain the result. □

Problems of this type occur for instance in elastic plastic torsion problems, [Tin66; Tin67]. It would be interesting to see whether there are meaningful fractal analogs of such models.

Chapter A2

Appendix to Part II

Proof of Proposition 9.1

We follow [Köt69, Chapter Five, §26, Section 7] for a proof. We wish to point out that if one is interested in reflexivity only, one could give a slightly shorter proof (as discussed in the cited reference).

Recall first that a normed space $(V, \|\cdot\|)$ is called *uniformly convex* if for any $0 < \varepsilon \leq 2$ there exists some $\delta > 0$ such that for all $u, v \in V$ with $\|u\| \leq 1$, $\|v\| \leq 1$ and $\|u - v\| > \varepsilon$ we have $\left\|\frac{1}{2}(u+v)\right\| \leq 1 - \delta$. The condition for uniform convexity may be seen as the generalization of the parallelogram identity in (pre-) Hilbert spaces, from which it is immediate that any (pre-) Hilbert space is uniformly convex. By Milman's theorem, [Köt69, Chapter Five, §26, Section 6, (4)] every uniformly convex Banach space is reflexive. We also need the following inequalities due to Clarkson.

Proposition A2.1. *[Köt69, Chapter Five, §26, Section 7, p. 357]*
Suppose $(V, \|\cdot\|)$ is a uniformly convex normed space and $1 < p < \infty$. Given $\varepsilon > 0$ there exists some $\delta > 0$ such that for any $u, v \in V$ with $\|u\| \leq 1$, $\|v\| \leq 1$ and $\|u - v\| \geq \varepsilon$ we have

$$\left\|\frac{1}{2}(u+v)\right\|^p \leq (1-\delta)\left(\frac{\|u\|^p + \|v\|^p}{2}\right).$$

For any $u, v \in V$ therefore

$$\left\|\frac{1}{2}(u+v)\right\|^p \leq \left(1 - \delta\left(\frac{\|u-v\|}{\max(\|u\|, \|v\|)}\right)\right)\left(\frac{\|u\|^p + \|v\|^p}{2}\right). \tag{A2.1}$$

We prove Proposition 9.1. To shorten notation we will use the abbreviation L^p for $L^p(X, \mu, (\mathcal{H}_x)_{x \in X})$. For simplicity we assume that μ is a probability measure.

Proof. Let $1 < p < +\infty$. We verify the uniform convexity of L^p, the reflexivity follows. Suppose $0 < \varepsilon \leq 2$, $u, v \in L^p$, $\|u\|_{L^p} \leq 1$, $\|v\|_{L^p} \leq 1$ and $\|u - v\|_{L^p} \geq \varepsilon$. In the following we work with fixed μ-versions of u and v, denoted by the same symbols; the result does not depend on their choice. Let $M \subset X$ be the set of all $x \in X$ such that

$$\|u_x - v_x\|_{\mathcal{H}_x}^p \geq \frac{\varepsilon^p}{4}(\|u_x\|_{\mathcal{H}_x}^p + \|v_x\|_{\mathcal{H}_x}^p) \geq \frac{\varepsilon^p}{4}\max(\|u_x\|_{\mathcal{H}_x}^p, \|v_x\|_{\mathcal{H}_x}^p). \tag{A2.2}$$

Applying (A2.1) to the Hilbert space $\mathcal{H}_x$, we obtain

$$\left\|\frac{1}{2}(u_x + v_x)\right\|_{\mathcal{H}_x}^p \leq \left(1 - \delta\frac{\varepsilon}{4^{1/p}}\right)\left(\frac{1}{2}(\|u_x\|_{\mathcal{H}_x}^p + \|v_x\|_{\mathcal{H}_x}^p)\right) \tag{A2.3}$$

for all $x \in M$. For $X \setminus M$ we have

$$\int_{X\setminus M} \|u_x - v_x\|_{\mathcal{H}_x} \, \mu(dx) \le \frac{\varepsilon^p}{4} \int_X (\|u_x\|^p_{\mathcal{H}_x} + \|v_x\|^p_{\mathcal{H}_x})\mu(dx) \le \frac{\varepsilon^p}{2},$$

so that, using the above assumptions on u and v,

$$\int_M \|u_x - v_x\|_{\mathcal{H}_x} \, \mu(dx) \ge \frac{\varepsilon^p}{2}.$$

Consequently $2\max(\|u|_M\|_{L^p}, \|v|_M\|_{L^p}) \ge \|u|_M - v|_M\|_{L^p} \ge \varepsilon/2^{1/p}$, i.e.

$$\max(\|u|_M\|^p_{L^p}, \|v|_M\|^p_{L^p}) \ge \frac{\varepsilon^p}{2^{p+1}}. \tag{A2.4}$$

Since by the elementary inequality $a^p + b^p \ge 2^{1-p}(a+b)^p$ for $a, b \ge 0$ the integrand is nonnegative, we have

$$\int_X \left(\frac{1}{2}(\|u_x\|^p_{\mathcal{H}_x} + \|v_x\|^p_{\mathcal{H}_x}) - (\frac{1}{2}\|u_x + v_x\|_{\mathcal{H}_x})^p\right)\mu(dx)$$
$$\ge \int_M \left(\frac{1}{2}(\|u_x\|^p_{\mathcal{H}_x} + \|v_x\|^p_{\mathcal{H}_x}) - (\frac{1}{2}\|u_x + v_x\|_{\mathcal{H}_x})^p\right)\mu(dx).$$

By (A2.3) this is greater or equal to

$$\delta\frac{\varepsilon}{4^{1/p}}\frac{1}{2}\int_M (\|u_x\|^p_{\mathcal{H}_x} + \|v_x\|^p_{\mathcal{H}_x})\mu(dx) \ge \delta\frac{\varepsilon}{4^{1/p}}\frac{\varepsilon^p}{2^{p+2}}.$$

where we have used (A2.4). This implies

$$\left\|\frac{1}{2}(u+v)\right\|_{L^p} \le \left(1 - \delta\frac{\varepsilon}{4^{1/p}}\frac{\varepsilon^p}{2^{p+2}}\right).$$

It remains to show that for any $1 < p < +\infty$ the space L^q, $\frac{1}{p} + \frac{1}{q} = 1$, is the dual of L^p. We repeat the classical arguments to point out that a Radon-Nikodym theorem is not needed. Given $v \in L^q$ consider the linear functional

$$v \mapsto \langle u, v\rangle := \int_X \langle u_x, v_x\rangle_{\mathcal{H}_x} \, \mu(dx), \qquad u \in L^p.$$

By Hölder's inequality this is a member of $(L^p)'$, hence L^q can be identified with a closed subspace of $(L^p)'$. We claim that $\sup_{\|u\|_{L^p}\le 1} |\langle u, v\rangle| = \|v\|_{L^q}$. The inequality $\le$ is clear from Hölder. Now define a measurable section $u = (u_x)_{x\in X}$ by $u_x := \|v_x\|^{q-2}_{\mathcal{H}_x} v_x$ for $x \in \{v \ne 0\}$ and $u_x := 0$ for $x \in \{v = 0\}$. Then $\|u\|_{L^p} = \|v\|^q_{L^q}$, so that $u \in L^p$. Moreover, $\left\langle v, \frac{u}{\|u\|_{L^p}}\right\rangle = \|v\|_{L^q}$, proving the claim. Consequently on L^q the norm of $(L^p)'$ coincides with the norm in L^q, and since L^q is complete, it is a closed subspace of $(L^p)'$. If L^q were a proper subspace we could find a nontrivial bounded linear functional on $(L^p)'$ vanishing on L^q. Since L^p is reflexive, this functional must be given by some $u \in L^p$. But then $\langle u, v\rangle = 0$ for all $v \in L^q$, and for $v \in L^q$ defined by $v = (v_x)_{x\in X}$ by $v_x := \|u_x\|^{q-2}_{\mathcal{H}_x} u_x$ for $x \in \{u \ne 0\}$ and $v_x := 0$ for $x \in \{u = 0\}$ we obtain $u = 0$ in L^p, a contradiction. □

Part III

Approximation results

The purpose of this part is to provide graph approximations for finitely ramified spaces and metric graph approximations for p.c.f. self-similar spaces for solutions of partial differential equations on resistance spaces.

We apply two abstract convergence schemes of forms defined on different Hilbert spaces to the case of certain fractals. The first concept we use is a generalization of the convergence scheme for varying separable Hilbert spaces developed by Kuwae and Shioya in [KS03]. The second concept we introduce is the concept of generalized norm resolvent convergence of self-adjoint operators on varying Hilbert spaces developed by Post in [Pos12; Pos06]. It relies on the concept of quasi-unitary equivalence of forms, see Definition 11.1.

Both concepts are based on the idea of using identification operators Φ_m respectively J_m (and J'_m) that are 'close to being unitary operators', see [KS03, Section 2.2, p. 611] and [Pos12, Section 4.1, in particular Lemma 4.1.4].

This part is organized as follows.
Chapter 10 is devoted to convergence results for linear elliptic and parabolic partial differential equations on resistance spaces which involve gradient and divergence terms. Here we use the generalized strong resolvent convergence in the sense of Kuwae and Shioya. Chapter 11 deals with a generalized norm resolvent convergence result in the sense of Post for the viscous Burgers equation on a post-critically finite self-similar fractal associated with a regular harmonic structure. Finally, we discuss a approximation result for the continuity equation on a finitely ramified fractal in Chapter 12. The proof relies on a diagonal compactness argument combining vanishing diffusion together with a convergence scheme on varying Hilbert spaces in the sense of Kuwae and Shioya.

Chapter 10

Generalized strong resolvent convergence for linear PDEs on compact resistance spaces

The aim of this chapter is to provide a suitable setup and sufficient conditions that allow to conclude that the solutions of (6.12) and (6.17) (for fixed $t > 0$) associated with the forms $\mathcal{Q}^{(m)}$ converge to those associated with the form $\mathcal{Q}$, uniformly on X. If X is a finitely ramified space one can, given coefficients a, b, $\hat{b}$ and c for $\mathcal{Q}$, define coefficients of approximating forms $\mathcal{Q}^{(m)}$ by restriction operations, defined in a rather straightforward way, see Section 10.3. Our main result is the KS-generalized Mosco convergence of coercive closed forms in the sense of Definition 10.2. This convergence is a generalization of the famous Mosco convergence of symmetric forms in the Kuwae and Shioya framework. We give a quick account on this convergence scheme in the next section.

The results of this chapter are based on joint work in progress with Michael Hinz [HM20a].

10.1 KS-generalized Mosco convergence for non-symmetric Dirichlet forms

We review the notion of KS-generalized Mosco convergence for non-symmetric Dirichlet forms as studied by Tölle [Töl06; Töl10] and we also give necessary and sufficient conditions for strong convergence of the associated resolvents. Tölle's notion of generalized convergence is a generalization of Hino's conditions, see [Hin98, Section 3] in the Kuwae-Shioya framework, see [KS03], which we will briefly describe. In this section we omit proofs and give detailed references to the literature.

In [KS03, Subsections 2.2 - 2.7] Kuwae and Shioya introduced a concept of convergence $H_m \to H$ of Hilbert spaces H_m to a Hilbert space H, including a suitable notion of generalized strong resolvent convergence for self-ajoint operators, cf. [KS03, Definition 2.1]. A basic tool for their definitions is a family of identification operators Φ_m defined on a dense subspace $\mathcal{C}$ of the limit space H, each mapping $\mathcal{C}$ into one of the spaces H_m.

Let H, H_1, H_2, ... be separable Hilbert spaces. The sequence $(H_m)_m$ is said to *converge to H in KS-sense*, $\lim_{m\to\infty} H_m = H$, if there are a dense subspace $\mathcal{C}$ of H and operators

$$\Phi_m : \mathcal{C} \to H_m \tag{10.1}$$

such that

$$\lim_{m\to\infty} \|\Phi_m w\|_{H_m} = \|w\|_H , \quad w \in \mathcal{C}, \tag{10.2}$$

[KS03, Subsection 2.2].

We recall suitable simplifications of [KS03, Definition 2.4, 2.5 and 2.6] and formulate [KS03, Theorem 2.1 (i)] as a definition.

Definition 10.1.

(i) A sequence $(u_m)_m$ with $u_m \in H_m$ is said to converge KS-strongly to $u \in H$ *if there is a sequence $(\widetilde{u}_m)_m \subset \mathcal{C}$ such that*

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}} \|\Phi_m \widetilde{u}_n - u_m\|_{H_m} = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|\widetilde{u}_n - u\|_H = 0. \tag{10.3}$$

(ii) A sequence $(u_m)_m$ with $u_m \in H_m$ is said to converge KS-weakly to $u \in H$ *if*

$$\lim_{m\to\infty} (u_m, v_m)_{H_m} = (u, v)_H$$

for every sequence $(v_m)_m$ KS-strongly convergent to v.

(iii) A sequence $(B_m)_m$ of bounded linear operators $B_m : H_m \to H_m$ is said to converge KS-strongly *to a bounded linear operator $B : H \to H$ if for any sequence $(u_m)_m$ with $u_m \in H_m$ converging KS-strongly to $u \in H$ the sequence $(B_m u_m)_m$ converges KS-strongly to Bu.*

(iv) A sequence $(A_m)_m$ of nonnegative definite self-adjoint operators $A_m : H_m \to H_m$ is said to converge in KS-generalized strong resolvent sense *to a nonnegative definite self-adjoint operator $A : H \to H$ if for some (hence all) $\lambda > 0$ the λ-resolvent operators $G_\lambda^{A_m}$ of the A_m converge KS-strongly to the λ-resolvent operator G_λ^A of A.*

Remark 10.1.

(i) In the classical case where $H_m \equiv H$ and $\Phi_m \equiv \mathrm{id}_H$ for all m the strong convergence of bounded linear operators B_m defined in (iii) differs from the classical definition of strong convergence of bounded linear operators on Hilbert spaces, as pointed out in [KS03, Section 2.3]. However, a sequence $(B_m)_m$ of bounded linear operators $B_m : H \to H$ admitting a uniform bound in operator norm $\sup_m \|B_m\| < +\infty$ converges KS-strongly to a bounded linear operator $B : H \to H$ if and only if it converges strongly to B in the usual sense, [KS03, Lemma 2.8 (1)].

(ii) For any $\lambda > 0$ the sequence $(G_\lambda^{A_m})_m$ of λ-resolvent operators in (iv) satisfies

$$\sup_m \left\| G_\lambda^{A_m} \right\| < 1/\lambda.$$

In the classical case where $H_m \equiv H$ and $\Phi_m \equiv \mathrm{id}_H$ for all m we therefore observe that the sequence of operators $(A_m)_m$ as in (iv) converges to A as in (iv) in the KS-generalized strong resolvent sense if and only if it converges to A in the usual strong resolvent sense, [RS80, Section VIII.7].

For each $m \in \mathbb{N}$, let $(\mathcal{Q}^{(m)}, \mathcal{D}(\mathcal{E}^{(m)}))$ be a (not necessarily symmetric) coercive closed form with sector constant K_m.We recall a shorted version of [Töl10, Def.7.14] and [Töl06, Def. 2.43].

Definition 10.2. *Assume that the sector constants K_m of the $\mathcal{Q}^{(m)}$'s are uniformly bounded. We say that a sequence $(\mathcal{Q}^{(m)}, \mathcal{D}(\mathcal{E}^{(m)}))_m$* converges in the KS-generalized Mosco sense *to a form $(\mathcal{Q}, \mathcal{D}(\mathcal{E}))$, if there exists a subset $\mathcal{C} \subset \mathcal{D}(\mathcal{E})$ densely and if the following two conditions hold:*

(1) *If a sequence* $(u_m)_m$ *KS-weakly converges to* u *in* $L^2(X,\mu)$ *satisfies* $\liminf_m \left(\tilde{\mathcal{Q}}_1^{(m)}(u_m)\right)^{\frac{1}{2}} < \infty$, *then* $u \in \mathcal{D}(\mathcal{E})$.

(2) *For any sequence* $m_k \uparrow \infty$ *and every* $w \in \mathcal{C}$, $u \in \mathcal{D}(\mathcal{E})$ *and any sequence* $(u_k)_k$, $u_k \in L^2(X^{(m_k)},\mu^{(m_k)})$, $k \in \mathbb{N}$, *converging KS-weakly to* u *and satisfying*

$$\sup_k \left(\tilde{\mathcal{Q}}_1^{(m_k)}(u_k)\right)^{\frac{1}{2}} < \infty$$

one has a sequence $(w_k)_k$, $w_k \in L^2(X^{(m_k)},\mu^{(m_k)})$, $k \in \mathbb{N}$ *converging KS-strongly in* $L^2(X,\mu)$ *to* w *with*

$$\liminf_k \mathcal{Q}^{(m_k)}(w_k,u_k) \leq \mathcal{Q}(w,u).$$

Remark 10.2. We emphasize that Hino and Tölle have introduced further criteria of convergence which we will not mentioned here to keep the presentation simple, see [Töl10, Section 7.3.2] and [Hin98, Section 3]. Using these stronger conditions the assumption of uniformly bounded sector constants K_m can be omitted.

The next Theorem is a special case of [Töl10, Theorem 7.15, Corollary 7.16 and Remark 7.17] (see also [Töl06, Theorem 2.4.1 and Corollary 2.4.1]), which generalize [Hin98, Theorem 3.1].

Theorem 10.1. *For each* m *let* $(\mathcal{Q}^{(m)},\mathcal{D}(\mathcal{E}^{(m)}))$ *be a coercive closed form on* H_m *and assume that the corresponding sector constants are uniformly bounded,* $\sup_m K_m < +\infty$. *Let* $\left(G_\alpha^{\mathcal{Q}^{(m)}}\right)_{\alpha>0}$, $\left(T_t^{\mathcal{Q}^{(m)}}\right)_{t>0}$ *and* $(\mathcal{L}^{\mathcal{Q}^{(m)}},\mathcal{D}(\mathcal{L}^{\mathcal{Q}^{(m)}}))$ *be the associated resolvent, semigroup and generator on* H_m. *Suppose that* $(\mathcal{Q},\mathcal{D}(\mathcal{E}))$ *is a coercive closed form on* H *with resolvent* $\left(G_\alpha^{\mathcal{Q}}\right)_{\alpha>0}$, *semigroup* $\left(T_t^{\mathcal{Q}}\right)_{t>0}$ *and generator* $(\mathcal{L}^{\mathcal{Q}},\mathcal{D}(\mathcal{L}^{\mathcal{Q}}))$. *Then the following are equivalent:*

(1) *The sequence of forms* $(\mathcal{Q}^{(m)},\mathcal{D}(\mathcal{E}^{(m)}))_m$ *converges to* $(\mathcal{Q},\mathcal{D}(\mathcal{E}))$ *in the KS-generalized Mosco sense.*

(2) *The sequence of operators* $\left(G_\alpha^{\mathcal{Q}^{(m)}}\right)_m$ *converges to* $G_\alpha^{\mathcal{Q}}$ *KS-strongly for any* $\alpha > 0$.

(3) *The sequence of operators* $\left(T_t^{\mathcal{Q}^{(m)}}\right)_m$ *converges to* $T_t^{\mathcal{Q}}$ *KS-strongly for any* $t > 0$.

(4) *The sequence of operators* $(\mathcal{L}^{\mathcal{Q}^{(m)}},\mathcal{D}(\mathcal{L}^{\mathcal{Q}^{(m)}}))$ *converges to* $(\mathcal{L}^{\mathcal{Q}},\mathcal{D}(\mathcal{L}^{\mathcal{Q}}))$ *in the KS-generalized strong resolvent sense.*

Remark 10.3.

(i) Theorem 10.1 and Definition 10.2 provide a characterization of convergence in the (KS-generalized) strong resolvent sense in terms of the associated bilinear forms.

(ii) In case of symmetric forms, Theorem 10.1 yields that conditions in Definition 10.2 are just another characterization of Mosco convergence as first introduced by Mosco in [Mos94] for the special case $H = H_m$, $m \in \mathbb{N}$, and later generalized by Kuwae and Shioya in [KS03] for varying spaces.

(iii) If we do not assume that the sector constants are uniformly bounded, we only have that (1) implies (2), (3) and (4). The equivalence between (2) and (3) still holds as shown in Theorem [Töl06, Thm. 2.21], a generalization of Kato's Theorem [Kat95, Thm. IX.2.16] for strong convergence of semigroups.

10.2 Convergence of solutions on a single space

Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance form on a nonempty set X so that (X, R) is compact and metrically doubling with doubling constant $K_R > 1$, and let μ be a finite Borel measure on (X, R) with a uniform lower bound V.

Similar to the Chapter 6 we consider bilinear forms $\mathcal{Q}_{(m)}$ on $L^2(X, \mu)$ defined by

$$\mathcal{Q}_{(m)}(f, g) := \langle a_m \cdot \partial f, \partial g \rangle_{\mathcal{H}} - \langle g \cdot b_m, \partial f \rangle_{\mathcal{H}} - \left\langle f \cdot \hat{b}_m, \partial g \right\rangle_{\mathcal{H}} - \langle cf, g \rangle_{L^2(X,\mu)}, \quad f, g \in \mathcal{F}. \tag{10.4}$$

where we suppose that a_m uniformly elliptic as in (6.1) and b_m, $\hat{b}_m \in \mathcal{H}$ in the Hardy class and $c \in L^\infty(X, \mu)$ are given. Note that the coefficients a_m b_m and $\hat{b}_m$ may vary with m. To keep the exposition more transparent and since it is rather trivial to vary it, we keep c fixed.

We consider the unique weak solutions to elliptic problems (6.12) and unique solutions at fixed positive times of parabolic problems (6.17) with these coefficients. In Corollary 10.3 we show that we can find accumulation points with respect to the uniform convergence on X of these solutions, and these accumulation points are elements of $\mathcal{F}$ if there exists a sequence $(a_m)_m$ satisfying (6.1) uniformly in m, the sequences $(b_m)_m$ and $(\hat{b}_m)_m$ are bounded and c is small enough. If coefficients a, b, $\hat{b}$ and c are given and the sequences $(a_m)_m$, $(b_m)_m$ and $(\hat{b}_m)_m$ converge to a b and $\hat{b}$, respectively, then we can conclude the uniform convergence of the solutions to the respective solutions of the target problem, see Theorem 10.2.

10.2.1 Boundedness and convergence of vector fields

We record two useful consequences of Proposition 6.2. The first states that if the norms of vector fields in a sequence are uniformly bounded we may choose uniform constants in the Hardy condition (6.2).

Corollary 10.1. *Let the hypotheses of Proposition 6.2 be satisfied. If $(b_m)_m \subset \mathcal{H}$ satisfies $\sup_m \|b_m\|_{\mathcal{H}} < +\infty$ then for any $M > 0$ there is a constant $\gamma_M > 0$ independent of m such that (6.2) holds for each b_m with constants $\delta(b_m) = \frac{1}{M}$ and $\gamma(b_m) = \gamma_M$.*

Proof. Since $\mathcal{V}$ is increasing we can take

$$\gamma_M := \mathcal{V}(M \sup_m \|b_m\|^2_{\mathcal{H}}) \sup_m \|b_m\|^2_{\mathcal{H}}. \tag{10.5}$$

□

The second is a continuity statement.

Corollary 10.2. *Let the hypotheses of Proposition 6.2 be satisfied. If $b \in \mathcal{H}$ and $(b_m)_m \subset \mathcal{H}$ is a sequence with $\lim_{m\to\infty} b_m = b$ in $\mathcal{H}$ then for any $g \in C_c(X)$ we have*

$$\lim_{m\to\infty} \|g \cdot b_m - g \cdot b\|^2_{\mathcal{H}} = 0.$$

Proof. This is immediate from the definition of $\mathcal{V}$ and the fact that the uniform lower bound V is strictly positive and increasing. □

10.2.2 Accumulation points

For each m let $a_m \in C(X)$ satisfy (6.20) with the same constants $0 < \lambda < \Lambda$. Suppose $M > 0$ is large enough so that $\lambda_0 := \lambda/2 - 1/M > 0$ and that $b_m, \hat{b}_m \in \mathcal{H}$ satisfy

$$\sup_m \|b_m\|_{\mathcal{H}}^2 < +\infty \quad \text{and} \quad \sup_m \|\hat{b}_m\|_{\mathcal{H}}^2 < +\infty. \tag{10.6}$$

Let γ_M be as in (10.5) and $\hat{\gamma}_M$ defined in the same way with the $\hat{b}_m$ replacing the b_m and suppose $c \in L^\infty(X, \mu)$ is such that

$$c_0 := \operatorname*{ess\,inf}_{x \in X} (-c(x)) - \frac{\gamma_M + \hat{\gamma}_M}{\lambda - 2/M} > 0. \tag{10.7}$$

Then by Proposition 6.1 and Corollary 10.1 the forms $\mathcal{Q}_{(m)}$ as stated in (10.4) are closed forms on $L^2(X, \mu)$ for each m. They satisfy (6.6) with $\delta(b_m) = \delta(\hat{b}_m) = 1/M$ and γ_M, $\hat{\gamma}_M$ replacing $\gamma(b)$, $\gamma(\hat{b})$ in (6.7). Their generators $(\mathcal{L}^{\mathcal{Q}_{(m)}}, \mathcal{D}(\mathcal{L}^{\mathcal{Q}_{(m)}}))$ satisfy the sector conditions (6.11) with the same sector constant K. As a consequence we observe uniform energy bounds for the solutions of (6.12) and (6.17).

We write $\mathcal{Q}_{(m),\alpha}$ for the form defined like $\mathcal{E}_\alpha$ in (2.1) but with $\mathcal{Q}_{(m)}$ in place of $\mathcal{E}$.

Proposition 10.1. *Let a_m, b_m, $\hat{b}_m$ and c be as above such that (10.6) and (10.7) hold.*

(i) If $f \in L^2(X, \mu)$, and u_m is the unique weak solution to (6.12) with $\mathcal{L}^{\mathcal{Q}_{(m)}}$ in place of $\mathcal{L}$, then we have $\sup_m \mathcal{Q}_{(m),1}(u_m) < +\infty$.

(ii) If $\mathring{u} \in L^2(X, \mu)$, and u_m is the unique solution to (6.17) with $\mathcal{L}^{\mathcal{Q}_{(m)}}$ in place of $\mathcal{L}$, then for any $t > 0$ we have $\sup_m \mathcal{Q}_{(m),1}(u_m(t)) < +\infty$.

Proof. Since (10.7) and (6.11) hold with the same constants c_0 and K for all m, the statements follow from Corollaries 6.2 and 6.3. □

The compactness of X implies the existence of accummulation points in $C(X)$.

Corollary 10.3. *Let a_m, b_m, $\hat{b}_m$ and c be as above such that (10.6) and (10.7) are satisfied.*

(i) If $f \in L^2(X, \mu)$ and u_m is the unique weak solution to (6.12) with $\mathcal{L}^{\mathcal{Q}_{(m)}}$ in place of $\mathcal{L}^{\mathcal{Q}}$, then each subsequence of $(u_m)_m$ has a subsequence converging to a limit $\widetilde{u} \in \mathcal{F}$ uniformly on X.

(ii) If $\mathring{u} \in L^2(X, \mu)$ and u_m is the unique weak solution to (6.12) with $\mathcal{L}^{\mathcal{Q}_{(m)}}$ in place of $\mathcal{L}^{\mathcal{Q}}$, then for each $t > 0$ each subsequence of $(u_m(t))_m$ has a further subsequence converging to a limit $\widetilde{u}_t \in \mathcal{F}$ uniformly on X.

At this point we can of course not claim that the $C(X)$-valued function $t \mapsto \widetilde{u}_t$ has any good properties or significance.

Proof. Since all forms $\mathcal{Q}_{(m)}$ satisfy (6.6) with the same constants, Proposition 10.1 implies that $\sup_m \mathcal{E}_1(u_m) < +\infty$. By [Kig12, Lemma 9.7] the embedding $\mathcal{F} \subset C(X)$ is compact, hence $(u_m)_m$ has a subsequence that converges uniformly on X to a limit $\widetilde{u}$. Since also this subsequence is bounded in $\mathcal{F}$, it has a further subsequence that converges to a limit $\widetilde{w} \in \mathcal{F}$ (respectively $\widetilde{w} \in \mathcal{F}_0$) weakly in $L^2(X, \mu)$, as follows from a Banach-Saks type argument, for the concrete statement see Appendix A. This forces $\widetilde{w} = \widetilde{u}$.

Statement (ii) is proved in the same manner. □

10.2.3 Strong resolvent convergence

Let $a \in \mathcal{F}$ be such that (6.20) holds with constants $0 < \lambda < \Lambda$ and let $(a_m)_m \subset C(X)$ be such that

$$\lim_{m\to\infty} \|a_m - a\|_{\sup} = 0. \tag{10.8}$$

Without loss of generality we may then assume that also the functions a_m satisfy (6.20) with the very same constants $0 < \lambda < \Lambda$. Suppose $M > 0$ is large enough so that $\lambda_0 := \lambda/2 - 1/M > 0$. Let $b, \hat{b} \in \mathcal{H}$ and let $(b_m)_m \subset \mathcal{H}$ and $(\hat{b}_m)_m \subset \mathcal{H}$ be sequences such that

$$\lim_{m\to\infty} \|b_m - b\|_{\mathcal{H}} = 0 \quad \text{and} \quad \lim_{m\to\infty} \left\|\hat{b}_m - \hat{b}\right\|_{\mathcal{H}} = 0. \tag{10.9}$$

Note that this implies (10.6). Let γ_M be as in (10.5) and $\hat{\gamma}_M$ similarly but with the $\hat{b}_m$. Let $\mathcal{Q}$ be as in (6.3) and $\mathcal{Q}_{(m)}$ as in (10.4).

The next theorem states that the solutions to (6.12) and (6.17) depend continuously on the coefficients a, b and $\hat{b}$.

Theorem 10.2. *Let a, a_m, b, b_m, $\hat{b}$ and $\hat{b}_m$ be as above such that (10.8) and (10.9) hold. Then*

$$\lim_{m\to\infty} \mathcal{L}^{\mathcal{Q}_{(m)}} = \mathcal{L}^{\mathcal{Q}}$$

in the strong resolvent sense, and the following hold.

(i) Let $f \in L^2(X,\mu)$, u be the unique weak solution to (6.12) and u_m be the unique weak solution to (6.12) with $\mathcal{L}^{\mathcal{Q}_{(m)}}$ in place of $\mathcal{L}^{\mathcal{Q}}$, respectively. Then

$$\lim_{m\to\infty} u_m = u$$

in $L^2(X,\mu)$. Moreover, there is a sequence $(m_k)_k$ with $m_k \uparrow +\infty$ such that

$$\lim_{k\to\infty} u_{m_k} = u \qquad \text{uniformly on } X.$$

(ii) Let $\mathring{u} \in L^2(X,\mu)$, u be the unique solution to (6.17) and u_m be the unique solution to (6.17) with $\mathcal{L}^{\mathcal{Q}_{(m)}}$ in place of $\mathcal{L}$, respectively. Then for any $t > 0$ we have

$$\lim_{m\to\infty} u_m(t) = u(t)$$

in $L^2(X,\mu)$. Moreover, there is a sequence $(m_k)_k$ with $m_k \uparrow +\infty$ such that for any $t > 0$

$$\lim_{k\to\infty} u_{m_k}(t) = u(t) \qquad \text{uniformly on } X.$$

Proof. By [Hin98, Theorem 3.1], the claimed strong resolvent convergence and the stated convergences in $L^2(X,\mu)$ follow once we have verified the conditions in Definition 10.2, see Theorem 10.1 and Remark 10.3 in the previous section. The statements on uniform convergence then also follow using Corollary 10.3.

If $c \in L^\infty(X,\mu)$ does *not* satisfy (10.7), we use an easy shift argument. More precisely, we take an arbitrary $\check{c} \in \mathbb{R}$ such that the inequality

$$\operatorname*{ess\,inf}_{x\in X}(-c(x)) + \check{c} - \frac{\gamma_M + \hat{\gamma}_M}{\lambda - 2/M} > 0$$

holds. Thus, instead of $(\mathcal{Q}_{(m)}, \mathcal{F})$ with associated generator $(\mathcal{L}^{\mathcal{Q}_{(m)}}, \mathcal{D}(\mathcal{L}^{\mathcal{Q}_{(m)}}))$ we prove the statemets of the theorem for new bilinear forms $(\check{\mathcal{Q}}_{(m)}, \mathcal{F})$ with associated generators $(\mathcal{L}^{\check{\mathcal{Q}}_{(m)}}, \mathcal{D}(\mathcal{L}^{\mathcal{Q}_{(m)}}))$,

$$\check{\mathcal{Q}}_{(m)}(f,g) := \mathcal{Q}_{(m)}(f,g) + \check{c}\langle f,g\rangle_{L^2(X,\mu)}, \ f,g \in \mathcal{F} \qquad \text{and} \qquad \mathcal{L}^{\check{\mathcal{Q}}_{(m)}} := \mathcal{L}^{\mathcal{Q}_{(m)}} - \check{c}.$$

If $\lim_{m\to\infty}\mathcal{L}^{\check{\mathcal{Q}}_{(m)}}=\mathcal{L}^{\check{\mathcal{Q}}}$ is satisfied, the KS-strong convergence of the associated resolvents $\left(G_\alpha^{\check{\mathcal{Q}}_{(m)}}\right)_{\alpha>0}$ to $G_\alpha^{\check{\mathcal{Q}}}$ implies the KS-strong convergence of the resolvents $\left(G_{\alpha+\check{c}}^{\mathcal{Q}_{(m)}}\right)_{\alpha+\check{c}>0}$, $G_{\alpha+\check{c}}^{\mathcal{Q}_{(m)}}=G_\alpha^{\check{\mathcal{Q}}_{(m)}}$ associated to $(\mathcal{Q}_{(m)},\mathcal{F})$ to $G_{\alpha+\check{c}}^{\mathcal{Q}}$, so we have $\lim_{m\to\infty}\mathcal{L}^{\mathcal{Q}_{(m)}}=\mathcal{L}^{\mathcal{Q}}$ in the KS-generalized resolvent sense and (i) holds. In addition, for fixed $t>0$ the semigroup $\left(T_t^{\mathcal{Q}_{(m)}}\right)_{t>0}$ associated to $(\mathcal{Q}_{(m)},\mathcal{F})$ converges KS-strongly to $e^{\check{c}t}T_t^{\check{\mathcal{Q}}}$, since the rescaled semigroup is given by $T_t^{\mathcal{Q}_{(m)}}=e^{\check{c}t}T_t^{\check{\mathcal{Q}}_{(m)}}$, so (ii) follows. Therefore, we also obtain the claimed strong resolvent convergence and the stated convergences in $L^2(X,\mu)$ for the original equations. In this case, the statements on uniform convergence follow from Corollary 10.3, since all $\check{\mathcal{Q}}_{(m)}$ satisfy by Remark 6.3 the inequalities in (6.6) with modified constants $c_0\mapsto c_0+\check{c}$ and $c_\infty\mapsto c_\infty+\check{c}$ and for all m, for all $u\in\mathcal{F}$ we have $\mathcal{Q}_{(m)}(u)\le\check{\mathcal{Q}}_{(m)}(u)$.

Thanks to (6.4), (6.5), (6.6) and (6.7) together with Proposition 6.2 and Corollaries 10.1 and 10.2 we can find a constant $C>0$ such that for every sufficiently large m we have

$$C\,\mathcal{E}_1(f)\le\mathcal{Q}_{(m),1}(f)\le C^{-1}\,\mathcal{E}_1(f),\quad f\in\mathcal{F}. \tag{10.10}$$

Suppose that $\lim_{m\to\infty}u_m=u$ weakly in $L^2(X,\mu)$ with $\liminf_m Q_{(m),1}(u_m,u_m)<\infty$. Then there is a subsequence $(u_{m_k})_k$ such that $\sup_k\mathcal{Q}_{(m_k),1}(u_{m_k})<+\infty$, and by the preceding we have $\sup_k\mathcal{E}_1(u_{m_k},u_{m_k})<+\infty$. A standard Banach-Saks type argument shows that a subsequence of $(u_{m_k})_k$ converges to a limit $u_{\mathcal{E}}\in\mathcal{F}$ weakly in $(\mathcal{F},\mathcal{E})$ and that $u_{\mathcal{E}}=u$, what proves condition (1) in Definition 10.2.

To verify condition (2) in Definition 10.2 suppose that $(m_k)_k$ be a sequence of natural numbers with $m_k\uparrow\infty$, $\lim_{k\to\infty}u_k=u$ weakly in $L^2(X,\mu)$ with $\sup_k Q_{(m_k),1}(u_k,u_k)<\infty$ and $u\in\mathcal{F}$. By (10.10) we have $\sup_k\mathcal{E}_1(u_k)<\infty$, what implies that $(u_k)_k$ has a subsequence $(u_{k_j})_j$ converging to $u\in\mathcal{F}$ weakly in $\mathcal{F}$ and uniformly on X, and such that its averages $N^{-1}\sum_{j=1}^N u_{k_j}$ converge to u in $\mathcal{F}$. Here the statement on uniform convergence is again a consequence of the compact embedding $\mathcal{F}\subset C(X)$, [Kig12, Lemma 9.7]. Combined with the weak convergence in $L^2(X,\mu)$ it follows that $(u_{k_j})_j$ converges weakly in $(\mathcal{F}/\sim,\mathcal{E})$. Moreover, using (4.10), the convergence of averages and the linearity of d we may assume that $(du_{k_j})_j$ converges to du weakly in $L^2(X\times X\setminus\operatorname{diag},J)$. As a consequence, we also have

$$\begin{aligned}\lim_{j\to\infty}\mathcal{E}^c(u_{k_j},v)&=\lim_{j\to\infty}\mathcal{E}(u_{k_j},v)-\lim_{j\to\infty}\int_X\int_X du_{k_j}(x,y)dv(x,y)J(dxdy)\\&=\mathcal{E}(u,v)-\int_X\int_X du(x,y)dv(x,y)J(dxdy)\\&=\mathcal{E}^c(u,v)\end{aligned}$$

for all $v\in\mathcal{F}$.

Now let $w\in\mathcal{F}$. Then we have

$$|Q_{(m_{k_j})}(w,u_{k_j})-Q(w,u)|\le|Q_{(m_{k_j})}(w,u_{k_j})-Q(w,u_{k_j})|+|Q(w,u_{k_j}-u)|. \tag{10.11}$$

Since c is kept fixed, the first summand on the right hand side of the inequality (10.11) is bounded by

$$\begin{aligned}&\left|\left\langle(a_{m_{k_j}}-a)\cdot\partial w,\partial u_{k_j}\right\rangle_{\mathcal{H}}\right|+\left|\langle u_{k_j}\cdot(b_{m_{k_j}}-b),\partial w\rangle_{\mathcal{H}}\right|+\left|\left\langle w\cdot(\hat{b}_{m_{k_j}}-\hat{b}),\partial u_{k_j}\right\rangle_{\mathcal{H}}\right|\\&\le\|a_{m_{k_j}}-a\|_{\sup}\mathcal{E}(w)^{1/2}\mathcal{E}(u_{k_j})^{1/2}+\|u_{k_j}\|_{\sup}\|b_{m_{k_j}}-b\|_{\mathcal{H}}\,\mathcal{E}(w)^{1/2}+\|w\|_{\sup}\|\hat{b}_{m_{k_j}}-\hat{b}\|_{\mathcal{H}}\,\mathcal{E}(u_{k_j})^{1/2},\end{aligned}$$

where we have used Cauchy-Schwarz and (4.14). By the hypotheses on the coefficients and the boundedness of $(u_{k_j})_j$ in energy and in uniform norm this converges to zero. The second summand on the right hand side of (10.11) is bounded by

$$|\left\langle \partial w, a\cdot\partial(u_{k_j}-u)\right\rangle_{\mathcal{H}}|+|\langle (u_{k_j}-u)\cdot b, \partial w\rangle_{\mathcal{H}}|+|\langle w\cdot\hat{b}, \partial(u_{k_j}-u)\rangle_{\mathcal{H}}|+|\langle cw, u_{k_j}-u\rangle_{L^2(X,\mu)}|.$$

The last summand in this line obviously converges to zero, and also the second does, note that

$$|\langle (u_{k_j}-u)\cdot b, \partial w\rangle_{\mathcal{H}}| \leq \|u_{k_j}-u\|_{\sup}\|b\|_{\mathcal{H}}\,\mathcal{E}(w)^{1/2}$$

by Cauchy-Schwarz and (4.14). By Lemma 4.2 we have

$$\left\langle \partial w, a\cdot\partial(u_{k_j}-u)\right\rangle_{\mathcal{H}} = \int_X a\, d\nu^c_{w,u_{k_j}-u} + \int_X\int_X \overline{a}(x,y)dw(x,y)d(u_{k_j}-u)(x,y)\; J(dxdy).$$

Since

$$\|\overline{a}dw\|_{L^2(X\times X\setminus \mathrm{diag},J)} \leq \|a\|_{\sup}\,\mathcal{E}(w)^{1/2},$$

the double integral converges to zero by the weak convergence of $(du_{k_j})_j$ to du in $L^2(X\times X\setminus\mathrm{diag},J)$. By (2.2) we have $\sup_j \mathcal{E}_1(au_{k_j})^{1/2}<+\infty$ and $\mathcal{E}_1(wu_{k_j})^{1/2}<+\infty$. Thinning out the sequence $(u_{k_j})_j$ once more we may, using the arguments above, assume that $\lim_{j\to\infty}\mathcal{E}^c(au_{k_j},v)=\mathcal{E}^c(au,v)$ and $\lim_{j\to\infty}\mathcal{E}^c(wu_{k_j},v)=\mathcal{E}^c(wu,v)$ for all $v\in\mathcal{F}$. Then

$$\int_X a\, d\nu^c_{w,u_{k_j}-u} = \frac{1}{2}\left\{\mathcal{E}^c(aw,u_{k_j}-u)+\mathcal{E}^c(a(u_{k_j}-u),w)-\mathcal{E}^c(w(u_{k_j}-u),a)\right\}$$

also converges to zero, what implies that $\lim_{j\to\infty}\left\langle \partial w, a\cdot\partial(u_{k_j}-u)\right\rangle_{\mathcal{H}}=0$. Finally, note that by the Leibniz rule for ∂,

$$\left\langle \hat{b}, w\cdot\partial(u_{k_j}-u)\right\rangle_{\mathcal{H}} = \left\langle \hat{b}, \partial(w(u_{k_j}-u))\right\rangle_{\mathcal{H}} - \left\langle \hat{b}, (u_{k_j}-u)\cdot\partial w\right\rangle_{\mathcal{H}}.$$

As before we see easily that the second summand on the right hand side goes to zero. For the first, let $\hat{b}=\partial f+\eta$ be the unique decomposition of $\hat{b}$ into a gradient ∂f of a function $f\in\mathcal{F}$ and a vector field $\eta\in\ker\partial^*$. Then

$$\left\langle \hat{b}, \partial(w(u_{k_j}-u))\right\rangle_{\mathcal{H}} = \left\langle \partial f, \partial(w(u_{k_j}-u))\right\rangle_{\mathcal{H}} = \mathcal{E}(f, w(u_{k_j}-u)),$$

which converges to zero by the preceding arguments. Combining, we see that

$$\lim_{j\to\infty}\mathcal{Q}_{n_{k_j}}(w,u_{k_j})=\mathcal{Q}(w,u),$$

and since $w\in\mathcal{F}$ was arbitrary, this implies condition (2) in Definition 10.2. □

10.3 Convergence of solutions on varying spaces

In this section we will basically repeat the above program, but now on varying resistance spaces.

10.3.1 Setup and basic assumptions

Let $(\mathcal{E},\mathcal{F})$ be a resistance form on a nonempty set X and suppose that (X,R) is compact and metrically doubling with doubling constant $K_R>1$. By compactness $(\mathcal{E},\mathcal{F})$ is regular. We also assume (X,R) is complete and connected and that $(\mathcal{E},\mathcal{F})$ is local, that is, $\mathcal{E}(u,v)=0$ whenever $u,v\in\mathcal{F}$ are such that $R(\mathrm{supp}(u),\mathrm{supp}(v))>0$, see [Kig12, Definition 7.5].

As in (3.3) let $(V_m)_m$ be an increasing sequence of finite subsets $V_m \subset X$ such that $\bigcup_{m\geq 0} V_m$ is dense in (X, R). Given $m \geq 0$ and a function $v \in \ell(V_m)$ there exists a unique function $h_m(v) \in \mathcal{F}$ such that $h_m(v)|_{V_m} = v$ in $\ell(V_m)$ and

$$\mathcal{E}(h_m(v)) = \mathcal{E}_{V_m}(v) = \min\{\mathcal{E}(u) : \ u \in \mathcal{F}, u|_{V_m} = v\},$$

see [Kig03, Proposition 2.15]. To this function $h_m(v)$ we refer as the *harmonic extension* of v, and as usual we say that a function $u \in \mathcal{F}$ is *m-harmonic* if $u = h_m(u|_{V_m})$. For any $u \in \mathcal{F}$ we have $\lim_m \mathcal{E}(u - h_m(u|_{V_m})) = 0$, see for instance [Str06, Lemma 1.4.1], and using (3.1) it follows that also $\lim_m \|u - h_m(u|_{V_m})\|_{\sup,X} = 0$, where $\|\cdot\|_{\sup,X}$ denotes the supremum norm on X. We write $H_m(X)$ to denote the space of m-harmonic functions on X and $H_m(X)/\sim$ for the space of m-harmonic functions on X modulo constants. For each m the space $H_m(X)$ is a closed subspace of $\mathcal{F}$ (and the space $H_m(X)/\sim$ is a closed subspace of $\mathcal{F}/\sim$). By H_m we denote the projection from $\mathcal{F}$ onto $H_m(X)$. Since $H_m\mathbf{1} = \mathbf{1}$ it naturally induces an orthogonal projection in $(\mathcal{F}/\sim, \mathcal{E})$ onto $H_m(X)/\sim$, which we denote by the same symbol.

Suppose that $(X^{(m)})_m$ is a sequence of subsets $X^{(m)} \subset X$ such that for each $m \geq 0$ we have $X^{(m)} \subset X^{(m+1)}$ and $V_m \subset X^{(m)}$. Suppose that for each $m \geq 0$, $(\mathcal{E}^{(m)}, \mathcal{F}^{(m)})$ is a resistance form on $X^{(m)}$ so that $(X^{(m)}, R^{(m)})$ is metrically doubling with the same doubling constant $K_R > 1$ and compact. Then each $(\mathcal{E}^{(m)}, \mathcal{F}^{(m)})$ is regular. We assume that each space $(X^{(m)}, R^{(m)})$ is continuously embedded in (X, R). In particular, for any m and any $f \in \mathcal{F}^{(m)}$ the energy measure the $\nu_f^{(m)}$ of f may be interpreted as a Borel measure on X.

We make some further assumptions. The first expresses a connection between the resistance forms in terms of m-harmonic functions.

Assumption 10.1.

(i) For each m the pointwise restriction $u \mapsto u|_{X^{(m)}}$ defines a linear map from $H_m(X)$ into $\mathcal{F}^{(m)}$ which is injective and satisfies

$$\mathcal{E}^{(m)}(u|_{X^{(m)}}) = \mathcal{E}(u), \quad u \in H_m(X). \tag{10.12}$$

(ii) We have

$$\nu_u = \lim_{m\to\infty} \nu^{(m)}_{H_m(u)|_{X^{(m)}}}, \quad u \in \mathcal{F}, \tag{10.13}$$

in the sense of weak convergence of measures on X.

(iii) The forms $\mathcal{E}^{(m)}$ are purely local or $\bigcup_{m\geq 0} H_n(X)$ is a special standard core for $(\mathcal{E}, \mathcal{F})$.

Note that as a particular consequence of (10.13) we have

$$\mathcal{E}(u) = \lim_{m\to\infty} \mathcal{E}^{(m)}(H_m(u)|_{X^{(m)}}), \quad u \in \mathcal{F}, \tag{10.14}$$

and together with (10.12) it can easily be concluded that

$$\lim_{m\to\infty} \mathcal{E}(u - H_m u) = 0, \quad u \in \mathcal{F}. \tag{10.15}$$

Remark 10.4. For approximations by discrete graphs (10.13) follows from (10.14) and (4.5). For metric graph approximations (10.13) is verified in Lemma 10.8, the use of products in (3.5) hinders a direct conclusion of (10.13) from (10.14). By the locality of $(\mathcal{E}, \mathcal{F})$ together with Assumption 5.1 (iii) explicit assumptions on the jump measures of the forms $\mathcal{E}^{(m)}$ are not needed.

$$\begin{array}{ccc} \mathcal{F}/\sim & & \mathcal{F}^{(m)}/\sim \\ \Big\downarrow{\scriptstyle H_m} & & \Big\downarrow{\scriptstyle H_m^{(m)}} \\ H_m(X)/\sim & \underset{E_m}{\overset{E_m^{-1}=\cdot|_{X^{(m)}}}{\rightleftarrows}} & H_m(X^{(m)})/\sim \end{array}$$

Figure 10.1: Connection between the operators H_m, $H_m^{(m)}$ and E_m

Conjecture 1. The convergence of energy measures as in (10.13) may not be needed and Assumption 10.1(i) may be sufficient.

Now let

$$H_m(X^{(m)}) := \{u|_{X^{(m)}} : u \in H_m(X)\}$$

denote the image of $H_m(X)$ under the pointwise restriction $u \mapsto u|_{X^{(m)}}$, which by (10.12) induces an isometry from $(H_m(X)/\sim, \mathcal{E})$ onto the Hilbert space $(H_m(X^{(m)})/\sim, \mathcal{E}^{(m)})$. The space $H_m(X^{(m)})$ is a closed linear subspace of $\mathcal{F}^{(m)}$ and the space $H_m(X^{(m)})/\sim$ is a closed linear subspace of $\mathcal{F}^{(m)}/\sim$. Let $H_m^{(m)}$ denote the projection from $\mathcal{F}^{(m)}$ onto $H_m(X^{(m)})$. It satisfies $H_m^{(m)}\mathbf{1} = \mathbf{1}$ and induces an orthogonal projection from $(\mathcal{F}^{(m)}/\sim, \mathcal{E}^{(m)})$ onto $H_m(X^{(m)})/\sim$ so that in particular,

$$\mathcal{E}^{(m)}(H_m^{(m)}v) \leq \mathcal{E}^{(m)}(v), \quad v \in \mathcal{F}^{(m)}. \tag{10.16}$$

Let $id_{\mathcal{F}^{(m)}}$ denote the identity operator in $\mathcal{F}^{(m)}$.

We need an assumption on the decay of the operators $id_{\mathcal{F}^{(m)}} - H_m^{(m)}$ as m goes to infinity.

Assumption 10.2.

(i) For any sequence $(u_m)_m$ with $u_m \in \mathcal{F}^{(m)}$ such that $\sup_m \mathcal{E}^{(m)}(u_m) < +\infty$ we have

$$\lim_{m\to\infty} \left\| u_m - H_m^{(m)}u_m \right\|_{\sup, X_m} = 0. \tag{10.17}$$

(ii) For $u, w \in H_n(X)$ we have

$$\lim_{m\to\infty} \mathcal{E}^{(m)}(u|_{X^{(m)}}w|_{X^{(m)}} - H_m^m(u|_{X^{(m)}}w|_{X^{(m)}})) = 0. \tag{10.18}$$

Remark 10.5. For discrete graph approximations we have $H_m^{(m)}v = v$, $v \in \mathcal{F}^{(m)}$, so that Assumption 10.2 is trivially satisfied. For metric graph approximations this operator equals to the operator H_{Γ_m} as introduced in Section 10.4.2.

Now let μ be a finite Borel measure so that $\mu(B(x,r)) > 0$ for any $x \in X$ and $r > 0$, and each m let $\mu^{(m)}$ be a finite Borel measure so that $\mu^{(m)}(B^{(m)}(x,r)) > 0$ for any $x \in X^{(m)}$ and $r > 0$. Then $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}^{(m)}, \mathcal{F}^{(m)})$ are regular Dirichlet forms on $L^2(X,\mu)$ and $L^2(X^{(m)}, \mu^{(m)})$, respectively, [Kig12, Theorem 9.4], and the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is strongly local.

We make an assumption on the connection between the spaces $L^2(X,\mu)$ and $L^2(X^{(m)}, \mu^{(m)})$ and the consistency of this connection with the projections and pointwise restrictions. By $E_m : H_m(X^{(m)}) \to H_m(X)$ we denote the inverse of the bijection $u \mapsto u|_{X^{(m)}}$ from $H_m(X)$ onto $H_m(X^{(m)})$, see also Figure 10.1.

Assumption 10.3.

(i) Each measure $\mu^{(m)}$ admits a uniform lower bound $V^{(m)}$, and for any $r > 0$ we have $\inf_m V^{(m)}(r) > 0$.

(ii) There are linear operators $\Phi_m : L^2(X,\mu) \to L^2(X^{(m)},\mu^{(m)})$ such that

$$\sup_m \|\Phi_m\|_{L^2(X,\mu)\to L^2(X^{(m)},\mu^{(m)})} < +\infty, \tag{10.19}$$

$$\lim_{m\to\infty} \left\|\Phi_m u\right\|_{L^2(X^{(m)},\mu^{(m)})} = \|u\|_{L^2(X,\mu)}, \quad u \in L^2(X,\mu), \tag{10.20}$$

and for any n and $u \in H_n(X)$ we have

$$\lim_{m\to\infty} \|\Phi_m^*\Phi_m u - u\|_{L^2(X,\mu)} = 0, \tag{10.21}$$

where for any m the symbol Φ_m^* denotes the adjoint of Φ_m.

(iii) For any sequence $(u_m)_m \subset \mathcal{F}$ with $\sup_m \mathcal{E}(u_m) < +\infty$ we have

$$\lim_{m\to\infty} \left\|\Phi_m u_m - u_m|_{X^{(m)}}\right\|_{L^2(X^{(m)},\mu^{(m)})} = 0. \tag{10.22}$$

(iv) For any sequence $(u_m)_m$ with $u_m \in \mathcal{F}^{(m)}$ such that $\sup_m \mathcal{E}_1^{(m)}(u_m) < +\infty$ we have

$$\sup_m \left\|E_m \circ H_m^{(m)} u_m\right\|_{L^2(X,\mu)} < +\infty. \tag{10.23}$$

In Section 10.4 we verify that graph approximations for finitely ramified spaces and metric graph approximations for p.c.f. self-similar spaces satisfy all of our assumptions.

Let $\mathcal{H}$ and $\mathcal{H}^{(m)}$ denote the spaces of generalized L^2-vector fields on X and $X^{(m)}$ associated with $(\mathcal{E},\mathcal{F})$ and $(\mathcal{E}^{(m)},\mathcal{F}^{(m)})$, respectively. The corresponding gradient operators we denote by ∂ and $\partial^{(m)}$. If a, b, $\hat{b}$ and c satisfy the hypotheses of Proposition 6.1 then

$$\mathcal{Q}(f,g) := \langle a\,\partial f, \partial g\rangle_{\mathcal{H}} - \langle g\cdot b, \partial f\rangle_{\mathcal{H}} - \left\langle f\cdot \hat{b}, \partial g\right\rangle_{\mathcal{H}} - \langle cf, g\rangle_{L^2(X,\mu)}, \quad f,g\in\mathcal{F},$$

defines a coercive closed form $(\mathcal{Q},\mathcal{F})$ on $L^2(X,\mu)$. If a and c are suitable continuous functions on X and b, $\hat{b}$, $b^{(m)}$ and $\hat{b}^{(m)}$ are vector fields of a suitable form, then we can define coercive closed forms $(\mathcal{Q}^{(m)},\mathcal{F}^{(m)})$ on the spaces $L^2(X^{(m)},\mu^{(m)})$, respectively, by

$$\begin{aligned}\mathcal{Q}^{(m)}(f,g) :=& \langle a|_{X^{(m)}}\cdot\partial f, \partial g\rangle_{\mathcal{H}^{(m)}} - \left\langle g\cdot b^{(m)}, \partial f\right\rangle_{\mathcal{H}^{(m)}} \\ &- \left\langle f\cdot \hat{b}^{(m)}, \partial g\right\rangle_{\mathcal{H}^{(m)}} - \langle c|_{X^{(m)}} f, g\rangle_{L^2(X^{(m)},\mu^{(m)})}, \quad f,g\in\mathcal{F}^{(m)}.\end{aligned} \tag{10.24}$$

Below we observe that under simple boundedness assumptions the solutions of (6.12) and (6.17) (for fixed $t>0$) associated with the forms $\mathcal{Q}^{(m)}$ on the spaces $X^{(m)}$ accumulate in a suitable sense, see Proposition 10.2. In Theorem 10.3 and Corollary 10.8 we then conclude that they actually converge to the solutions to the respective equation associated with the form $\mathcal{Q}$ on X, as announced before.

We first record some consequences of the assumptions and discuss possible choices for b, $\hat{b}$, $b^{(m)}$ and $\hat{b}^{(m)}$.

10.3.2 Some consequences of the assumptions

The following is due to Assumption 5.1.

Corollary 10.4. *For any $f_1, f_2 \in H_n(X)$ and $g_1, g_2 \in C(X)$ we have*

$$\begin{aligned}\langle g_1\cdot\partial f_1, g_2\cdot\partial f_2\rangle_{\mathcal{H}} &= \lim_{m\to\infty}\left\langle g_1|_{X^{(m)}}\cdot\partial^{(m)}(f_1|_{X^{(m)}}), g_2|_{X^{(m)}}\cdot\partial^{(m)}(f_2|_{X^{(m)}})\right\rangle_{\mathcal{H}^{(m)}} \\ &= \lim_{m\to\infty}\int_{X^{(m)}} g_1|_{X^{(m)}} g_2|_{X^{(m)}}\, d\nu^{(m)}_{f_1|_{X^{(m)}},f_2|_{X^{(m)}}}.\end{aligned}$$

Proof. If all $\mathcal{E}^{(m)}$'s are local then by Lemma 4.2 we have

$$\left\|g|_{X^{(m)}} \cdot \partial^{(m)}(f|_{X^{(m)}})\right\|^2_{\mathcal{H}^{(m)}} = \int_{X^{(m)}} (g|_{X^{(m)}})^2 d\nu^{(m),c}_{f|_{X^{(m)}}} = \int_X g^2 d\nu^{(m)}_{f|^{(m)}_X}$$

for all $f \in H_n(X)$ and $g \in C(X)$, where $\nu_f^{(m),c}$ denotes the local energy measure of f with respect to $(\mathcal{E}^{(m)}, \mathcal{F}^{(m)})$. By (10.13) this converges to

$$\int_X g^2 d\nu_f = \|g \cdot \partial f\|^2_{\mathcal{H}}.$$

Suppose the $\mathcal{E}^{(m)}$'s have nontrivial jump measures $J^{(m)}$. If $f \in H_n(X)$ and $g \in H_{n'}(X)$ have disjoint supports then by Lemma 4.2, the locality of $\mathcal{E}^{(m)}$, (10.12) and the locality of $\mathcal{E}$ we have

$$\begin{aligned}
-2\lim_{m\to\infty} \int_X \int_X f(x)g(y)\, J^{(m)}(dxdy) &= \lim_{m\to\infty} \int_{X^{(m)}} \int_{X^{(m)}} (f(x)-f(y))(g(x)-g(y)) J^{(m)}(dxdy) \\
&= \lim_{m\to\infty} \mathcal{E}^{(m)}(f,g) \\
&= \mathcal{E}(f,g) \\
&= 0. \qquad (10.25)
\end{aligned}$$

Since $\bigcup_{n\geq 0} H_n(X)$ is a core this shows that $\lim_{m\to\infty} J^{(m)} = 0$ vaguely on $X \times X \setminus \text{diag}$, for details see [FOT94, Section 3.2]. Now suppose $f \in H_n(X)$ and $g \in C(X)$. Then, by Lemma 4.2 and (10.13) we have

$$\|g \cdot d_u f\|^2_{\mathcal{H}} = \lim_{m\to\infty} \left\{ \int_{X^{(m)}} g^2 d\nu_f^{(m),c} + \frac{1}{2} \int_{X^{(m)}} \int_{X^{(m)}} (g^2(x) + g^2(y))(d_u f(x,y))^2 J^{(m)}(dxdy) \right\}.$$

Using (10.25) together with the arguments in the proof of Lemma 4.1 we can conclude

$$\lim_{m\to\infty} \int_{X^{(m)}} \int_{X^{(m)}} (d_u g(x,y))^2 (d_u f(x,y))^2 J^{(m)}(dxdy) = 0,$$

so that the above limit is seen to equal

$$\begin{aligned}
&\lim_{m\to\infty} \left\{ \int_{X^{(m)}} g^2 d\nu_f^{(m),c} + \frac{1}{2} \int_{X^{(m)}} \int_{X^{(m)}} (\overline{g}(x,y))^2 (d_u f(x,y))^2 J^{(m)}(dxdy) \right\} \\
&= \lim_{m\to\infty} \int_{X^{(m)}} (g|_{X^{(m)}})^2\, d\nu^{(m)}_{f|_{X^{(m)}}},
\end{aligned}$$

and polarization yields the result. □

Another consequence of the assumptions is the convergence of the L^2-spaces and the energy domains in the sense of Definition 10.1.

Corollary 10.5.

(i) We have

$$\lim_{m\to\infty} L^2(X^{(m)}, \mu^{(m)}) = L^2(X, \mu) \qquad (10.26)$$

in the KS-sense with identification operators Φ_m as above.

(ii) We have

$$\lim_{m\to\infty} \mathcal{F}^{(m)} = \mathcal{F} \qquad (10.27)$$

in the KS-sense with identification operators $u \mapsto (H_m u)|_{X^{(m)}}$ mapping from $\mathcal{F}$ into $\mathcal{F}^{(m)}$ respectively.

(iii) If $f \in \mathcal{F}$ and $(f_m)_m$ is a sequence of functions $f_m \in \mathcal{F}^{(m)}$ such that $\lim_{m\to\infty} f_m = f$ KS-strongly w.r.t. (10.27) then we also have $\lim_{m\to\infty} f_m = f$ KS-strongly w.r.t. (10.26).

Proof. Statement (i) is immediate from (10.20).

To see statement (ii) let $u \in \mathcal{F}$. If $x_0 \in V_0$ is fixed, we have $H_m u(x_0) = u(x_0)$ for any m and therefore, by (3.1) and (10.15),

$$\lim_{m\to\infty} \|u - H_m u\|^2_{L^2(X,\mu)} \le \mu(X) \lim_{m\to\infty} \|u - H_m u\|^2_{\sup} \le \mu(X)\,\mathrm{diam}(X) \lim_{m\to\infty} \mathcal{E}(u - H_m u) = 0.$$

Using (10.19), we obtain $\lim_{m\to\infty} \|\Phi_m(H_m u) - \Phi_m(u)\|_{L^2(X,\mu)} = 0$, and combining with (10.22) and (10.20),

$$\begin{aligned}
&\lim_{m\to\infty} \|(H_m u)|_{X^{(m)}}\|_{L^2(X^{(m)},\mu^{(m)})} \\
&\quad = \lim_{m\to\infty} \|(H_m u)|_{X^{(m)}} - \Phi_m(H_m u)\|_{L^2(X^{(m)},\mu^{(m)})} + \lim_{m\to\infty} \|\Phi_m(H_m u)\|_{L^2(X^{(m)},\mu^{(m)})} \\
&\quad = \lim_{m\to\infty} \|\Phi_m(u)\|_{L^2(X^{(m)},\mu^{(m)})} \\
&\quad = \|u\|_{L^2(X,\mu)}\,.
\end{aligned}$$

Together with (10.14) this shows that $\lim_{m\to\infty} \mathcal{E}_1^{(m)}((H_m u)|_{X^{(m)}}) = \mathcal{E}_1(u)$ for all $u \in \mathcal{F}$.

To see (iii) note that according to the hypothesis, there exists some $\varphi_n \in \mathcal{F}$ such that

$$\lim_{n\to\infty} \mathcal{E}_1(\varphi_n - f) = 0 \quad \text{and} \quad \lim_{n\to\infty} \overline{\lim_{m\to\infty}}\, \mathcal{E}_1^{(m)}((H_m\varphi_n)|_{X^{(m)}} - f_m) = 0.$$

This implies

$$\lim_{n\to\infty} \|\varphi_n - f\|_{L^2(X,\mu)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \overline{\lim_{m\to\infty}}\, \|(H_m\varphi_n)|_{X^{(m)}} - f_m\|_{L^2(X^{(m)},\mu^{(m)})} = 0.$$

Conditions (10.20) and (10.22), applied to the constant function $\mathbf{1}$, yield $\lim_{m\to\infty} \mu^{(m)}(X^{(m)}) = \mu(X)$, and in particular,

$$\sup_m \mu(X^{(m)}) < +\infty. \tag{10.28}$$

We may therefore use (10.17) to conclude

$$\lim_{m\to\infty} \|(H_m\varphi_n)|_{X^{(m)}} - \Phi_m(H_m\varphi_n)\|_{L^2(X^{(m)},\mu^{(m)})} = 0$$

for any n, so that

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}}\, \|\Phi_m(H_m\varphi_n) - f_m\|_{L^2(X^{(m)},\mu^{(m)})} = 0.$$

Let $x_0 \in V_0$. Then, since $H_m\varphi_n(x_0) = \varphi(x_0)$ for all m and n, the resistance estimate (3.1) implies $\lim_{m\to\infty} \|H_m\varphi_n - \varphi_n\|_{L^2(X,\mu)} = 0$ for all n. Together with (10.19) it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \overline{\lim_{m\to\infty}}\, \|\Phi_m(H_m\varphi_n) - \Phi_m(\varphi_n)\|_{L^2(X^{(m)},\mu^{(m)})} \\
&\qquad \le \left(\sup_m \|\Phi_m\|_{L^2(X,\mu)\to L^2(X^{(m)},\mu^{(m)})}\right) \lim_{n\to\infty} \overline{\lim_{m\to\infty}}\, \|H_m\varphi_n - \varphi_n\|_{L^2(X,\mu)} \\
&\qquad = 0,
\end{aligned}$$

what entails $\lim_{n\to\infty} \overline{\lim}_{m\to\infty} \|\Phi_m(\varphi_n) - f_m\|_{L^2(X^{(m)},\mu^{(m)})} = 0$. □

We finally record a property of weak convergence w.r.t. (10.20) that will be useful later on.

Lemma 10.1. *If* $\lim_{m\to\infty} f_m = f$ *KS-weakly and* $w \in \mathcal{F}$ *then there is a sequence* $(m_k)_k$ *with* $\lim_{k\to\infty} m_k = +\infty$ *such that*

$$\lim_{k\to\infty} w|_{X^{(m_k)}} f_{m_k} = wf \quad \textit{KS-weakly.}$$

Proof. By (10.22) we have

$$\lim_{m\to\infty} w|_{X^{(m)}} = w \quad \text{KS-strongly}$$

for all $w \in \mathcal{F}$. Fix $w \in \mathcal{F}$. Clearly $\sup_m \|w|_{X^{(m)}} f_m\|_{L^2(X^{(m)},\mu^{(m)})} < +\infty$ by the boundedness of w, hence $\lim_{k\to\infty} w|_{X^{(m_k)}} f_{m_k} = \widetilde{w}$ KS-weakly for some $\widetilde{w} \in L^2(X,\mu)$ and some sequence $(m_k)_k$. For any $v \in \mathcal{F}$ we have $vw \in \mathcal{F}$ and trivially $(vw)|_{X^{(m)}} = v|_{X^{(m)}} w|_{X^{(m)}}$, hence

$$\begin{aligned}\langle \widetilde{w}, v\rangle_{L^2(X,\mu)} &= \lim_{k\to\infty} \langle w|_{X^{m_k}} f_{m_k}, v|_{X^{m_k}}\rangle_{L^2(X^{(m_k)},\mu^{(m_k)})} \\ &= \lim_{k\to\infty} \langle f_{m_k}, (vw)|_{X^{m_k}}\rangle_{L^2(X^{(m_k)},\mu^{(m_k)})} \\ &= \langle f, wv\rangle_{L^2(X,\mu)} = \langle wf, v\rangle_{L^2(X,\mu)},\end{aligned}$$

what by the density of $\mathcal{F}$ implies $\widetilde{w} = wf$ and therefore the lemma. □

10.3.3 Boundedness and compatibility of vector fields

Similarly as in Corollary 10.1, uniform norm bounds on the vector fields allow to choose uniform constants in the Hardy condition (6.2).

Corollary 10.6. *Suppose* $b^{(m)} \in \mathcal{H}^{(m)}$ *are such that* $\sup_m \left\|b^{(m)}\right\|_{\mathcal{H}^{(m)}} < +\infty$*. Then for any* $M > 0$ *there is a constant* $\gamma_M > 0$ *independent of* m *so that each* $b^{(m)}$ *satisfies (6.2) with*

$$\delta(b^{(m)}) = \frac{1}{M} \quad \textit{and} \quad \gamma(b^{(m)}) = \gamma_M.$$

Proof. Since each $(X^{(m)}, R^{(m)})$ is metrically doubling and each measure $\mu^{(m)}$ has a uniform lower bound $V^{(m)}$, Proposition 6.2 implies that

$$\|g_m \cdot b_m\|_{\mathcal{H}}^2 \leq \frac{1}{M}\,\mathcal{E}^{(m)}(g_m) + \widetilde{V}^{(m)}(M\,\|b_m\|_{\mathcal{H}^{(m)}}^2)\,\|b_m\|_{\mathcal{H}^{(m)}}^2\,\|g_m\|_{L^2(X^{(m)},\mu^{(m)})}^2$$

for any m and any $g_m \in \mathcal{F}^{(m)}$. Writing $s := \sup_n \|b_n\|_{\mathcal{H}^{(n)}}^2$ we have

$$\widetilde{V}^{(m)}(M\,\|b_m\|_{\mathcal{H}^{(m)}}^2) \leq \widetilde{V}^{(m)}(Ms) \leq \frac{2K_R}{\inf_m V^{(m)}\left(\frac{1}{2K_R s}\right)} < +\infty$$

for any m, and the statement follows with

$$\gamma_M := \sup_m \widetilde{V}^{(m)}(M\,\|b_m\|_{\mathcal{H}^{(m)}}^2)\,\sup_m \|b_m\|_{\mathcal{H}^{(m)}}^2. \tag{10.29}$$

□

An analog of Corollary 10.2 for vector fields on varying spaces is less straightforward, see Remark 10.6 below. Since our main interest is the approximation of equations on X, it seems convenient to restrict attention to certain elements of the module $\Omega_a^1(X)$ and their equivalence classes in $\mathcal{H}$ and $\mathcal{H}^{(m)}$ which then define vector fields b on X and $b^{(m)}$ on $X^{(m)}$ suitable to allow an approximation procedure. Given an element of $\Omega_a^1(X)$ of the special form $\sum_i g_i \cdot d_u f_i$ with $g_i \in C(X)$ and $f_i \in H_n(X)$, let b defined as its $\mathcal{H}$-equivalence class $\big[\sum_i g_i \cdot d_u f_i\big]_{\mathcal{H}}$ as in Section 4.1,

$$b := \sum_i g_i \cdot \partial f_i. \tag{10.30}$$

By Assumption 5.1 we have $f_i|_{X^{(m)}} \in \mathcal{F}^{(m)}$ for all i and m, so that $\sum_i g_i|_{X^{(m)}} \cdot d_u(f_i|_{X^{(m)}})$ is an element of $\Omega^1_a(X^{(m)})$. We define $b^{(m)}$ to be its $\mathcal{H}^{(m)}$-equivalence class $\big[\sum_i g_i|_{X^{(m)}} \cdot d_u(f_i|_{X^{(m)}})\big]_{\mathcal{H}^{(m)}}$, i.e.

$$b^{(m)} := \sum_i g_i|_{X^{(m)}} \cdot \partial^{(m)}(f_i|_{X^{(m)}}). \tag{10.31}$$

The following convergence result may be seen as a partial generalization of (10.13). It is an immediate consequence of (10.13), Corollary 10.4 and bilinear extension.

Corollary 10.7. *Suppose b and $b^{(m)}$ are as in (10.30) and (10.31) and $g \in C(X)$. Then we have*

$$\lim_{m\to\infty} \left\| g|_{X^{(m)}} \cdot b^{(m)} \right\|_{\mathcal{H}^{(m)}} = \|g \cdot b\|_{\mathcal{H}}. \tag{10.32}$$

If the space X has a certain cell structure, we can also fix a suitable vector field b on X and obtain approximating vector fields $b^{(m)}$ on $X^{(m)}$ by a well defined restriction operation. Following [IRT12] we define subspaces $\mathcal{H}_m$ of $\mathcal{H}$ by

$$\mathcal{H}_m := \Big\{ \sum_{\alpha\in\mathcal{A}_m} \mathbf{1}_{X_\alpha} \partial h_\alpha : \quad h_\alpha \in H_m(X) \text{ for all } \alpha \in \mathcal{A}_m \Big\}.$$

From Definition 5.1 it follows that $\mathcal{H}_m \subset \mathcal{H}_{m+1}$ for all m, see [IRT12, Lemma 5.3] for a proof. For a particular element $\sum_{\alpha\in\mathcal{A}_m} \mathbf{1}_{X_\alpha}\partial h_\alpha$ of $\mathcal{H}_m$ we have

$$\Big\| \sum_{\alpha\in\mathcal{A}_m} \mathbf{1}_{X_\alpha} \partial h_\alpha \Big\|^2_{\mathcal{H}} = \sum_{\alpha\in\mathcal{A}_m} \mathcal{E}_\alpha(h_\alpha, h_\alpha), \tag{10.33}$$

[IRT12, Theorem 5.4]. Moreover, $\bigcup_{m\geq 0} \mathcal{H}_m$ is dense in $\mathcal{H}$.

To generalize this we define a pointwise restriction of elements of $\mathcal{H}_m$ to $X^{(m)}$ by

$$\Big(\sum_{\alpha\in\mathcal{A}_m} \mathbf{1}_{X_\alpha} \partial h_\alpha \Big)|_{X^{(m)}} := \sum_{\alpha\in\mathcal{A}_m} \mathbf{1}_{X_\alpha \cap X^{(m)}} \partial^{(m)}(h_\alpha|_{X^{(m)}}), \tag{10.34}$$

and clearly this restriction operation maps $\mathcal{H}_m$ into $\mathcal{H}^{(m)}$. Thanks to the finitely ramified cell structure of X it is straightforward to see that this definition is correct.

Lemma 10.2. *For any $b \in \mathcal{H}_n$ and any $g \in C(X)$ we have*

$$\lim_{m\to\infty} \|g|_{X^{(m)}} \cdot b|_{X^{(m)}}\|_{\mathcal{H}^{(m)}} = \|g \cdot b\|_{\mathcal{H}}. \tag{10.35}$$

Proof. Let $\varepsilon > 0$. Choose $n_g \geq n$ sufficiently large such that

$$\sup_{\beta\in\mathcal{A}_{n_g}} \sup_{x,y\in X_\beta} |g(x)^2 - g(y)^2| < \frac{\varepsilon}{5\sum_{\alpha\in\mathcal{A}_n} \mathcal{E}(h_\alpha)}.$$

For all $\beta \in \mathcal{A}_{n_g}$ choose $x_\beta \in X_\beta \setminus V_{n_g}$ and define $\widetilde{g}(x) := g(x_\beta)$ if $x \in X_\beta \setminus V_{n_g}$ and $\widetilde{g}(x) := 0$ if $x \in V_{n_g}$. Then we we have

$$\sup_{\beta\in\mathcal{A}_{n_g}} \sup_{x\in X_\beta\setminus V_{n_g}} |g(x)^2 - \widetilde{g}(x)^2| < \frac{\varepsilon}{5\sum_{\alpha\in\mathcal{A}_n} \mathcal{E}(h_\alpha)}$$

and therefore

$$\left|\sum_{\alpha\in\mathcal{A}_n}\int_{X_\alpha\setminus V_{n_g}} g|^2_{X^{(m)}}d\nu^{(m)}_{h^\alpha|_{X^{(m)}}}-\int_{X_\alpha\setminus V_{n_g}}\widetilde{g}|^2_{X^{(m)}}d\nu^{(m)}_{h^\alpha|_{X^{(m)}}}\right|<\frac{\varepsilon}{5} \tag{10.36}$$

for all m and also

$$\left|\sum_{\alpha\in\mathcal{A}_n}\int_{X_\alpha\setminus V_{n_g}} g^2 d\nu_{h^\alpha}-\int_{X_\alpha\setminus V_{n_g}}\widetilde{g}^2 d\nu_{h^\alpha}\right|<\frac{\varepsilon}{5}. \tag{10.37}$$

The energy measures ν_{h_α} are nonatomic, hence by (10.13) and the Portmanteau lemma we can find a positive integer $m_\varepsilon \geq n_g$ so that for all $m\geq m_\varepsilon$ and all $\alpha\in\mathcal{A}_n$ we have

$$\nu^{(m)}_{h_\alpha|_{X^{(m)}}}(V_{n_g})<\frac{\varepsilon}{2|\mathcal{A}_n|^2\,\|g\|^2_{\sup}} \tag{10.38}$$

and

$$\left|\nu^{(m)}_{h_\alpha|_{X^{(m)}}}(X_\beta\setminus V_{n_g})-\nu_{h_\alpha}(X_\beta\setminus V_{n_g})\right|<\frac{\varepsilon}{2|\mathcal{A}_n|\,\|g\|^2_{\sup}}. \tag{10.39}$$

Since (10.39) implies

$$\begin{aligned}\left|\sum_{\alpha\in\mathcal{A}_n}\sum_{\beta\in\mathcal{A}_{n_g}} g(x_\beta)^2\nu^{(m)}_{h_\alpha|_{X^{(m)}}}(X_\alpha\cap X_\beta\cap V^c_{n_g})-\sum_{\alpha\in\mathcal{A}_n}\sum_{\beta\in\mathcal{A}_{n_g}} g(x_\beta)^2\nu_{h_\alpha}(X_\alpha\cap X_\beta\cap V^c_{n_g})\right|\\ \leq \|g\|^2_{\sup}\sum_{\beta\in\mathcal{A}_{n_g}}\left|\nu^{(m)}_{h_\alpha|_{X^{(m)}}}(X_\beta\setminus V_{n_g})-\nu_{h_\alpha}(X_\beta\setminus V_{n_g})\right|\\ \leq\varepsilon,\end{aligned}$$

we can use (10.36) and (10.37) to obtain

$$\left|\sum_{\alpha\in\mathcal{A}_n}\int_{X_\alpha\setminus V_{n_g}} g|^2_{X^{(m)}}d\nu^{(m)}_{h_\alpha|_{X^{(m)}}}-\sum_{\alpha\in\mathcal{A}_n}\int_{X_\alpha\setminus V_{n_g}} g^2 d\nu_{h_\alpha}\right|<\frac{3\varepsilon}{5}. \tag{10.40}$$

On the other hand, we have

$$\begin{aligned}\|g|_{X^{(m)}}\cdot b|_{X^{(m)}}\|^2_{\mathcal{H}^{(m)}}=&\sum_{\alpha\in\mathcal{A}_n}\int_{X_\alpha} g|^2_{X^{(m)}}d\nu^{(m)}_{h_\alpha|_{X^{(m)}}}\\ &+\sum_{\alpha,\alpha'\in\mathcal{A}_n,\alpha'\neq\alpha}\int_{X_\alpha\cap X_{\alpha'}} g|^2_{X^{(m)}}d\nu^{(m)}_{h_\alpha|_{X^{(m)}},h_{\alpha'}|_{X^{(m)}}}.\end{aligned}$$

By (10.38), Cauchy-Schwarz for energy measures and Definition 5.1 (vi) we see that the second summand on the right hand side is bounded by

$$\left(\sum_{\alpha\in\mathcal{A}_n}\nu^{(m)}_{h_\alpha|_{X^{(m)}}}(V_{n_g})^{1/2}\right)^2<\frac{\varepsilon}{5},$$

and using (10.38) once more, we obtain

$$\left|\|g|_{X^{(m)}}\cdot b|_{X^{(m)}}\|^2_{\mathcal{H}^{(m)}}-\sum_{\alpha\in\mathcal{A}_n}\int_{X_\alpha\setminus V_{n_g}} g|^2_{X^{(m)}}d\nu^{(m)}_{h_\alpha|_{X^{(m)}}}\right|<\frac{2\varepsilon}{5}. \tag{10.41}$$

Combining (10.40), (10.41) and the fact that $\|g\cdot b\|^2_{\mathcal{H}}=\sum_{\alpha\in\mathcal{A}_n}\int_{X_\alpha} g^2 d\nu_{h_\alpha}$, we arrive at (10.35). □

Remark 10.6. As discussed in Remarks 4.3 and 5.1, the spaces $\operatorname{Im}\partial$ and $\mathcal{F}/\sim$ are isometric as Hilbert spaces, and similarly for $\operatorname{Im}\partial^{(m)}$ and $\mathcal{F}^{(m)}/\sim$. Recall also that for each m the pointwise restriction $u \mapsto u|_{X^{(m)}}$ is an isometry from $H_m(X)/\sim$ onto $H_m(X^{(m)})/\sim$. Therefore (10.30) and (10.31) give rise to a well defined restriction of gradients of n-harmonic functions: Given $f \in H_n(X)$ and $m \geq n$ we can define the restriction of ∂f to $X^{(m)}$ by

$$(\partial f)|_{X^{(m)}} := \partial^{(m)}(f|_{X^{(m)}}), \tag{10.42}$$

and this operation is an isometry from $\partial(H_m(X))$ onto $\partial^{(m)}(H_m(X^{(m)}))$, see also Section 11.3. In general it is not straightforward to provide a correct definition for the restriction of a non-gradient field b on $\mathcal{H}$ to $X^{(m)}$.

10.3.4 Accumulation points

Let $a \in C(X)$ satisfy (6.20) with $0 < \lambda < \Lambda$, suppose $M > 0$ is large enough so that $\lambda_0 := \lambda/2 - 1/M > 0$ and that $b^{(m)}, \hat{b}^{(m)} \in \mathcal{H}^{(m)}$ satisfy

$$\sup_m \left\| b^{(m)} \right\|^2_{\mathcal{H}^{(m)}} < +\infty \quad \text{and} \quad \sup_m ||\hat{b}^{(m)}||^2_{\mathcal{H}^{(m)}} < +\infty. \tag{10.43}$$

Let γ_M be as in (10.29) and $\hat{\gamma}_M$ similarly but with the $\hat{b}^{(m)}$ and suppose $c \in C(X)$ satisfies (10.7). Then for each m the form $(\mathcal{Q}^{(m)}, \mathcal{F}^{(m)})$ as in (10.24) is a coercive closed form on $L^2(X^{(m)}, \mu^{(m)})$, and (6.6) holds with $\delta(b^{(m)}) = \delta(\hat{b}^{(m)}) = 1/M$ and with γ_M, $\hat{\gamma}_M$ in place of $\gamma(b)$, $\gamma(\hat{b})$ in (6.7). There is a constant $K > 0$ such that for each m the generator $(\mathcal{L}^{\mathcal{Q}^{(m)}}, \mathcal{D}(\mathcal{L}^{\mathcal{Q}^{(m)}}))$ of $(\mathcal{Q}^{(m)}, \mathcal{F}^{(m)})$ obeys the sector conditions (6.11) with sector constant K. As a consequence, we can observe the following uniform energy bounds on solutions to elliptic and parabolic equations similar to Proposition 10.1.

Proposition 10.2. *Let a, $b^{(m)}$, $\hat{b}^{(m)}$ and c be as above such that (10.43) and (10.7) hold.*

(i) If $f \in L^2(X, \mu)$, and u_m is the unique weak solution to (6.12) with $\mathcal{L}^{\mathcal{Q}^{(m)}}$ in place of $\mathcal{L}$ and $f_m = \Phi_m(f)$ in place of f then we have $\sup_m \mathcal{Q}_1^{(m)}(u_m) < +\infty$.

(ii) If $\mathring{u} \in L^2(X, \mu)$, and u_m is the unique solution to (6.17) in $L^2(X^{(m)}, \mu^{(m)})$ with $\mathcal{L}^{\mathcal{Q}^{(m)}}$ in place of $\mathcal{L}$ and with initial condition $\mathring{u}^{(m)} = \Phi_m(\mathring{u})$ then for any $t > 0$ we have $\sup_m \mathcal{Q}_1^{(m)}(u_m(t)) < +\infty$.

Proof. Since (10.7) and (6.11) hold with the same constants $\underline{c}$ and K for all m, Corollaries 6.2 and 6.3 together with (10.19) yield that

$$\sup_m \mathcal{Q}_1^{(m)}(u_m) \leq \left(\frac{2}{\underline{c}} + \frac{4}{\underline{c}^2} \right) \|f\|_{L^2(X,\mu)}$$

and

$$\sup_m \mathcal{Q}_1^{(m)}(u_m(t)) \leq \left(\frac{C_K}{t} + 1 \right) \|\mathring{u}\|^2_{L^2(X,\mu)}$$

and the results follow. □

By the compactness of X we can find accumulation points in $C(X)$ for extensions to X of linearizations of solutions. The next corollary may be seen as an analog of Corollary 10.3. Recall the definitions of the projections H_m^m and the extension operators E_m.

Corollary 10.8. *Let a, $b^{(m)}$, $\hat{b}^{(m)}$ and c be as above such that (10.43) and (10.7) hold.*

(i) If $f \in L^2(X,\mu)$, and u_m is the unique weak solution to (6.12) with $\mathcal{L}^{\mathcal{Q}^{(m)}}$ in place of $\mathcal{L}^{\mathcal{Q}}$ and $f_m = \Phi_m(f)$ in place of f then each subsequence $(u_{m_k})_k$ of $(u_m)_m$ has a further subsequence $(u_{m_{k_j}})_j$ such that $(E_{m_{k_j}} H_{m_{k_j}}^{m_{k_j}} u_{m_{k_j}})_j$ converges to a limit $\widetilde{u} \in C(X)$ uniformly on X.

(ii) If $\mathring{u} \in L^2(X,\mu)$, and u_m is the unique solution to (6.17) in $L^2(X^{(m)},\mu^{(m)})$ with $\mathcal{L}^{\mathcal{Q}^{(m)}}$ in place of $\mathcal{L}^{\mathcal{Q}}$ and with initial condition $\mathring{u}^{(m)} = \Phi_m(\mathring{u})$ then for any $t > 0$ each subsequence $(u_{m_k}(t))_k$ of $(u_m(t))_m$ has a further subsequence $(u_{m_{k_j}}(t))_j$ such that $(E_{m_{k_j}} H_{m_{k_j}}^{m_{k_j}} u_{m_{k_j}}(t))_j$ converges to a limit $\widetilde{u}_t \in C(X)$ uniformly on X.

10.3.5 Spectral convergence

In the next theorem, we present a uniform approximation result for equations on the target space X with specific coefficients. Recall that the operator $H_{m_k}^{m_k}$ projects a function to m_k-harmonic function on the approximating space $X^{(m_k)}$, the operator E_{m_k} extends a function on $X^{(m_k)}$ to a m_k-harmonic function on X.

Theorem 10.3. *Suppose that*

$$b = \sum_i g_i \cdot \partial f_i \quad \text{and} \quad \hat{b} = \sum_i \hat{g}_i \cdot \partial \hat{f}_i \tag{10.44}$$

are finite linear combinations with $f_i, \hat{f}_i, g_i, \hat{g}_i \in H_n(X)$ as in (10.30) and let

$$b^{(m)} := \sum_i g_i|_{X^{(m)}} \cdot \partial^{(m)}(f_i|_{X^{(m)}}) \quad \text{and} \quad \hat{b}^{(m)} := \sum_i \hat{g}_i|_{X^{(m)}} \cdot \partial^{(m)}(\hat{f}_i|_{X^{(m)}}) \tag{10.45}$$

as in (10.31). Let $a \in H_n(X)$ be such that (6.1) holds and let $c \in C(X)$. Then

$$\lim_{m\to\infty} \mathcal{L}^{\mathcal{Q}^{(m)}} = \mathcal{L}^{\mathcal{Q}}$$

in the KS-generalized resolvent sense, and the following hold.

(i) Let $f \in L^2(X,\mu)$, u be the unique weak solution to (6.12) on X and u_m be the unique weak solution to (6.12) on $X^{(m)}$ with $\mathcal{L}^{\mathcal{Q}^{(m)}}$ and $\Phi_m(f)$ in place of $\mathcal{L}^{\mathcal{Q}}$ and f, respectively. Then we have

$$\lim_{m\to\infty} u_m = u \qquad \text{KS-strongly w.r.t. (10.26).}$$

Moreover, there is a sequence $(m_k)_k$ with $m_k \uparrow +\infty$ such that

$$\lim_{k\to\infty} E_{m_k} \circ H_{m_k}^{m_k}(u_{m_k}) = u \qquad \text{uniformly on } X.$$

(ii) Let $\mathring{u} \in L^2(X,\mu)$, u be the unique solution to (6.17) on X and u_m be the unique weak solution to (6.17) on $X^{(m)}$ with $\mathcal{L}^{\mathcal{Q}^{(m)}}$ and $\Phi_m(\mathring{u})$ in place of $\mathcal{L}^{\mathcal{Q}}$ and $\mathring{u}$, respectively. Then for any $t > 0$ we have

$$\lim_{m\to\infty} u_m = u \qquad \text{KS-strongly w.r.t. (10.26).}$$

Moreover, there is a sequence $(m_k)_k$ with $m_k \uparrow +\infty$ such that for any $t > 0$

$$\lim_{k\to\infty} E_{m_k} \circ H_{m_k}^{m_k}(u_{m_k}(t)) = u(t) \qquad \text{uniformly on } X.$$

The proof of Theorem 10.3 makes use of the following key fact.

Lemma 10.3. *Suppose $(n_k)_k$ is an increasing sequence with $\lim_{k\to\infty} n_k = +\infty$ and $(u_k)_k$ is a sequence with $u_k \in L^2(X^{(n_k)}, \mu^{(n_k)})$ converging to $u \in L^2(X,\mu)$ KS-weakly and satisfying $\sup_k \mathcal{E}_1^{(n_k)}(u_k) < \infty$. Then we have $u \in \mathcal{F}$, and there is a sequence $(k_j)_j$ with $\lim_{j\to\infty} k_j = +\infty$ such that*

(i) $\lim_{j\to\infty} u_{n_{k_j}} = u$ KS-weakly w.r.t. (10.27), and moreover, for $f \in \mathcal{F}$ and $(f_j)_j$ such that $f_j \in \mathcal{F}^{n_{k_j}}$ and $\lim_{j\to\infty} f_j = f$ KS-strongly w.r.t. (10.27) along $(n_{k_j})_j$ we have

$$\lim_{j\to\infty} \mathcal{E}^{n_{k_j}}(f_j, u_{n_{k_j}}) = \mathcal{E}(f,u). \tag{10.46}$$

(ii) $\lim_{j\to\infty} E_{n_{k_j}} H_{n_{k_j}}^{n_{k_j}} u_{n_{k_j}} = u$ uniformly on X.

Proof. Let $v_k := E_{n_k} H_{n_k}^{(n_k)} u_k$. By hypothesis and (10.12) we have

$$\sup_k \mathcal{E}(v_k) = \sup_k \mathcal{E}^{(n_k)}(H_{n_k}^{(n_k)}(u_k)) \leq \sup_k \mathcal{E}^{(n_k)}(u_{n_k}) < +\infty. \tag{10.47}$$

Since $v_k|_{X^{n_k}} = H_{n_k}^{(n_k)} u_k$, (10.47), (10.28) and (10.17) allow to conclude that

$$\lim_{k\to\infty} \left\| v_k|_{X^{(n_k)}} - u_k \right\|_{L^2(X^{(n_k)},\mu^{(n_k)})} = 0, \tag{10.48}$$

what implies that $\lim_{k\to\infty} v_k|_{X^{(n_k)}} = u$ KS-weakly w.r.t. (10.26).

We now claim that for any n and any $w \in H_n(X)$ we have

$$\lim_{k\to\infty} \langle w, v_k \rangle_{L^2(X,\mu)} = \langle w, u \rangle_{L^2(X,\mu)}. \tag{10.49}$$

We clearly have $\lim_{k\to\infty} \Phi_{n_k}(w) = w$ KS-strongly. Therefore

$$\langle w, u \rangle_{L^2(X,\mu)} = \lim_{k\to\infty} \left\langle \Phi_{n_k}(w), v_k|_{X^{(n_k)}} \right\rangle_{L^2(X^{n_k},\mu^{(n_k)})},$$

and using (10.22) and (10.47) this limit is seen to equal

$$\lim_{k\to\infty} \langle \Phi_{n_k}(w), \Phi_{n_k}(v_k) \rangle_{L^2(X^{n_k},\mu^{(n_k)})} = \lim_{k\to\infty} \left\langle \Phi_{n_k}^* \Phi_{n_k}(w), v_k \right\rangle_{L^2(X,\mu)}.$$

Applying (10.21) we arrive at (10.49). By (10.47) and since (10.23) implies

$$\sup_k \|v_k\|_{L^2(X,\mu)} < +\infty$$

we can find a sequence $(k_j)_j$ with $\lim_{j\to\infty} k_j = +\infty$ such that

- $(u_{k_j})_j$ converges KS-weakly w.r.t. (10.27) to a limit $u_\mathcal{E} \in \mathcal{F}$,
- $(v_{k_j})_j$ converges weakly in $L^2(X,\mu)$ to a limit $\overline{u}_\mathcal{E} \in \mathcal{F}$.

Since $\bigcup_{n\geq 0} H_n(X)$ is dense in $L^2(X,\mu)$ we have $\overline{u}_\mathcal{E} = u$ by (10.49), what shows that $u \in \mathcal{F}$. We now verify that

$$\overline{u}_\mathcal{E} = u_\mathcal{E}. \tag{10.50}$$

For any $w \in H_n(X)$ the equalities

$$\begin{aligned}
\mathcal{E}_1(w, \overline{u}_\mathcal{E}) &= \lim_{j\to\infty} \left\{ \mathcal{E}(w, v_{k_j}) + \left\langle w, v_{k_j} \right\rangle_{L^2(X,\mu)} \right\} \\
&= \lim_{j\to\infty} \left\{ \mathcal{E}(w, v_{k_j}) - \left\langle \Phi_{n_{k_j}}^* \Phi_{n_{k_j}} w, v_{k_j} \right\rangle_{L^2(X,\mu)} \right\} \\
&= \lim_{j\to\infty} \left\{ \mathcal{E}^{n_{k_j}}(w|_{X^{(n_{k_j})}}, v_{k_j}|_{X^{(n_{k_j})}}) - \left\langle \Phi_{n_{k_j}} w, \Phi_{n_{k_j}} v_{k_j} \right\rangle_{L^2(X^{(n_{k_j})},\mu^{(n_{k_j})})} \right\}
\end{aligned}$$

hold, the second and third equality due to (10.21) and (10.12). Using (10.22) twice on the second summand, the above limit is seen to equal

$$\lim_{j\to\infty}\left\{\mathcal{E}^{n_{k_j}}(w|_{X^{(n_{k_j})}}, v_{k_j}|_{X^{(n_{k_j})}}) - \left\langle w|_{X^{(n_{k_j})}}, v_{k_j}|_{X^{(n_{k_j})}}\right\rangle_{L^2(X^{(n_{k_j})},\mu^{(n_{k_j})})}\right\}.$$

For j so large that $n_{k_j}\geq n$ the function $w|_{X^{(n_{k_j})}}$ is an element of $H_{n_{k_j}}(X^{n_{k_j}})$, so that by orthogonality in $\mathcal{F}^{(n_{k_j})}$ we can replace $v_{k_j}|_{X^{(n_{k_j})}} = H^{(n_{k_j})}_{n_{k_j}} u_{k_j}$ in the first summand by u_{k_j}. In the second term we can make the same replacement by (10.17) and (10.28), so that the above can be rewritten

$$\begin{aligned}\lim_{j\to\infty}\left\{\mathcal{E}^{n_{k_j}}(w|_{X^{(n_{k_j})}}, u_{k_j}) - \left\langle w|_{X^{(n_{k_j})}}, u_{k_j}\right\rangle_{L^2(X^{(n_{k_j})},\mu^{(n_{k_j})})}\right\} &= \lim_{j\to\infty}\mathcal{E}_1^{(n_{k_j})}(w|_{X^{(n_{k_j})}}, u_{k_j})\\ &= \mathcal{E}_1(w, u_{\mathcal{E}}),\end{aligned}$$

because $\lim_{j\to\infty} w|_{X^{(n_{k_j})}} = w$ KS-strongly w.r.t. (10.27). Since $\bigcup_{n\geq 0} H_n(X)$ is dense in $\mathcal{F}$, this implies (10.50) and therefore the first statement of (i), so far for the sequence $(u_{k_j})_j$. The statement on the limit (10.46) in (i) follows by Corollary 10.5.

To save notation in the proof of (ii) we now write $(u_k)_k$ for the sequence $(u_{k_j})_j$ extracted in (i). Let $x_0\in V_0$. Then (3.1) implies that $(v_k - v_k(x_0))_k$ is an equicontinuous and equibounded sequence of functions on X, so that by Arzelà-Ascoli (see Appendix A) we can find a subsequence $(v_{k_j} - v_{k_j}(x_0))_j$ which converges uniformly on X to a function $w_{x_0}\in C(X)$. Since μ is finite, this implies $\lim_{j\to\infty} v_{k_j} - v_{k_j}(x_0) = w_{x_0}$ in $L^2(X,\mu)$. By (10.22) and (10.47) we also have

$$\lim_{j\to\infty}\left\| v_{k_j}|_{X^{(n_{k_j})}} - v_{k_j}(x_0) - \Phi_{n_{k_j}}(v_{k_j} - v_{k_j}(x_0))\right\|_{L^2(X^{(n_{k_j})},\mu^{(n_{k_j})})} = 0,$$

so that combining, we see that $\lim_{j\to\infty}(v_{k_j}|_{X^{(n_{k_j})}} - u_{k_j}|_{X^{(n_{k_j})}}(x_0)) = w_{x_0}$ KS-strongly w.r.t. (10.26) and therefore also KS-weakly. Since $\lim_{k\to\infty} v_k|_{X^{(n_k)}} = u$ KS-weakly w.r.t. (10.26) by (10.48), we may conclude that $\lim_{k\to\infty} v_k|_{X^{(n_k)}}(x_0) = u - w_{x_0}$ KS-weakly w.r.t. (10.26). In particular, by [KS03, Lemma 2.3],

$$\sup_j |v_{k_j}|_{X^{n_{k_j}}}(x_0)|\mu(X^{n_{k_j}})^{1/2} = \sup_j \left\| v_{k_j}|_{X^{n_{k_j}}}(x_0)\right\|_{L^2(X^{n_{k_j}},\mu^{n_{k_j}})} < +\infty.$$

Since $\lim_{m\to\infty}\mu^{(m)}(X^{(m)}) = \mu(X) > 0$ it follows that $v_{k_j}|_{X^{n_{k_j}}}(x_0)$ is a bounded sequence of real numbers and therefore has a subsequence converging to some limit $z\in\mathbb{R}$. Keeping the same notation for this subsequence, we can use (10.22) and (10.28) to conclude that $\lim_{j\to\infty}\left\| v_{k_j}|_{X^{n_{k_j}}}(x_0) - \Phi_{n_{k_j}} z\right\|_{L^2(X^{n_{k_j}},\mu^{n_{k_j}})} = 0$, hence $\lim_{j\to\infty} v_{k_j}|_{X^{n_{k_j}}}(x_0) = z$ KS-weakly w.r.t. (10.26) and therefore necessarily $z = u - w_{x_0}$. This implies that

$$\lim_{j\to\infty} v_{k_j} = \lim_{j\to\infty}(v_{k_j} - v_{k_j}(x_0)) + \lim_{j\to\infty} v_{k_j}(x_0) = u$$

uniformly on X as stated in (ii). Clearly the statements in (i) remain true for this subsequence. □

We prove Theorem 10.3.

Proof. If $c\in C(X)$ satisfies (10.7), the operators $\mathcal{L}^{\mathcal{Q}^{(m)}}$ obey the sector condition (6.11) with the same sector constant, Theorem 10.1 will imply the desired convergence, provided that the forms $\mathcal{Q}^{(m)}$ and $\mathcal{Q}$ satisfy the conditions in Definition 10.2. Corollary 10.8 then takes care of the claimed uniform convergences.

In the case that c does *not* satisfy (10.7), we use again a shift argument. We take an arbitrary $\breve{c} \in \mathbb{R}$ such that the inequality

$$\operatorname*{ess\,inf}_{x\in X}(-c(x)) + \breve{c} - \frac{\gamma_M + \hat{\gamma}_M}{\lambda - 2/M} > 0$$

holds and, instead of $(\mathcal{Q}^{(m)}, \mathcal{F}^{(m)})$ with associated generator $(\mathcal{L}^{\mathcal{Q}^{(m)}}, \mathcal{D}(\mathcal{L}^{\mathcal{Q}^{(m)}}))$ we consider new bilinear forms $(\breve{\mathcal{Q}}^{(m)}, \mathcal{F}^{(m)})$ with associated generators $(\mathcal{L}^{\breve{\mathcal{Q}}^{(m)}}, \mathcal{D}(\mathcal{L}^{\mathcal{Q}^{(m)}}))$,

$$\breve{\mathcal{Q}}^{(m)}(f,g) := \mathcal{Q}^{(m)}(f,g) + \breve{c}\langle f, g\rangle_{L^2(X,\mu)},\ \ f,g \in \mathcal{F}^{(m)} \qquad \text{and} \qquad \mathcal{L}^{\breve{\mathcal{Q}}^{(m)}} := \mathcal{L}^{\mathcal{Q}^{(m)}} - \breve{c}.$$

Similar as in the proof of Theorem 10.2, we obtain the desired convergence and the statements on uniform convergence follow from Corollary 10.8, since all $\breve{\mathcal{Q}}_{(m)}$ satisfy by Remark 6.3 the inequalities in (6.6) with modified constants $c_0 \mapsto c_0 + \breve{c}$ and $c_\infty \mapsto c_\infty + \breve{c}$ and for all m, for all $u \in \mathcal{D}(\mathcal{Q})$ we have $\mathcal{Q}^{(m)}(u) \le \breve{\mathcal{Q}}^{(m)}(u)$.

By (6.4), (6.5), (6.6) and (6.7) together with Proposition 6.2 and Corollaries 10.6 and 10.7 we can find a constant $C > 0$ such that for any sufficiently large m we have

$$C\,\mathcal{E}_1^{(m)}(f) \le \mathcal{Q}_1^{(m)}(f) \le C^{-1}\,\mathcal{E}_1^{(m)}(f), \quad f \in \mathcal{F}^{(m)}. \tag{10.51}$$

To check the condition (1) in Definition 10.2, suppose that $(u_m)_m$ is a sequence with $u_m \in L^2(X^{(m)}, \mu^{(m)})$ converging KS-weakly w.r.t. (10.20) to a function $u \in L^2(X,\mu)$ and such that $\liminf_m \mathcal{Q}_1^{(m)}(u_m) < +\infty$. It has a subsequence $(u_{m_k})_k$ which by (10.51) satisfies $\sup_k \mathcal{E}^{(m_k)}(u_{m_k}) < +\infty$, and by Lemma 10.3 we then know that $u \in \mathcal{F}$, what implies the condition.

To verify condition (2), suppose $u \in \mathcal{F}$, $(m_k)_k$ is a sequence with $\lim_{k\to\infty} m_k = +\infty$ and $u_k \in L^2(X^{(m_k)}, \mu^{(m_k)})$ are such that $\lim_{k\to\infty} u_k = u$ KS-weakly and $\sup_k \mathcal{Q}_1^{(m_k)}(u_k) < +\infty$. By (10.51) we have $\sup_k \mathcal{E}_1^{(m_k)}(u_k) < +\infty$. Now let $w \in H_n(X)$. Clearly $\lim_{m\to\infty} w|_{X^{(m)}} = w$ KS-strongly w.r.t. (10.26). By Lemma 10.1 we may assume that along $(m_k)_k$ we also have

$$\lim_{k\to\infty} a|_{X^{m_k}} u_k = au \qquad \text{and} \qquad \lim_{k\to\infty} (w\hat{g}_i)|_{X^{m_k}} u_k = w\hat{g}_i u \quad \text{KS-weakly w.r.t. (10.26) for all } i,$$

otherwise we pass to a suitable subsequence. By (2.2) also $\sup_k \mathcal{E}_1^{(m_k)}(a|_{X^{m_k}} u_k) < +\infty$ and $\sup_k \mathcal{E}_1^{(m_k)}((w\hat{g}_i)|_{X^{m_k}} u_k) < +\infty$. By Lemma 10.3 we can therefore find a sequence $(k_j)_j$ as stated so that (i) and (ii) in Lemma 10.3 hold simultaneously for the sequences $(u_{k_j})_j$, $(a|_{X^{m_{k_j}}} u_{k_j})_j$ and $((w\hat{g}_i)|_{X^{m_{k_j}}} u_{k_j})_j$ with limits u, au and $w\hat{g}_i u$. Our first claim is that

$$\lim_{j\to\infty} \left\langle \partial^{(m_{k_j})}(w|_{X^{m_{k_j}}}), a|_{X^{m_{k_j}}} \cdot \partial^{(m_{k_j})} u_{k_j} \right\rangle_{\mathcal{H}^{m_{k_j}}} = \langle \partial w, a \cdot \partial u\rangle_{\mathcal{H}}\,. \tag{10.52}$$

To see this note first that by the Leibniz rule for $\partial^{m_{k_j}}$ each element of the sequence on the left hand side equals

$$\left\langle \partial^{(m_{k_j})}(w|_{X^{m_{k_j}}}), \partial^{(m_{k_j})}(a|_{X^{m_{k_j}}} u_{k_j}) \right\rangle_{\mathcal{H}^{m_{k_j}}} - \left\langle \partial^{(m_{k_j})}(w|_{X^{m_{k_j}}}), u_{k_j} \cdot \partial^{(m_{k_j})}(a|_{X^{m_{k_j}}}) \right\rangle_{\mathcal{H}^{m_{k_j}}}.$$

The first term converges to $\langle \partial w, \partial(au)\rangle_{\mathcal{H}}$ by (10.46). In the second summand we can replace u_{k_j} by $H_{m_{k_j}}^{(m_{k_j})} u_{k_j}$, note that by (4.14) and (10.17) we have

$$\lim_{j\to\infty} \left\| (u_{m_{k_j}} - H_{m_{k_j}}^{(m_{k_j})} u_{k_j}) \cdot \partial^{(m_{k_j})}(a|_{X^{m_{k_j}}}) \right\|_{\mathcal{H}^{m_{k_j}}} = 0.$$

By Lemma 10.3 (ii) we also have

$$\lim_{j\to\infty} \left\| (H_{m_{k_j}}^{(m_{k_j})} u_{k_j} - u|_{X^{(m_{k_j})}}) \cdot \partial^{(m_{k_j})}(a|_{X^{m_{k_j}}}) \right\|_{\mathcal{H}^{m_{k_j}}} = 0,$$

so that

$$\begin{aligned}\lim_{j\to\infty} & \left\langle \partial^{(m_{k_j})}(w|_{X^{m_{k_j}}}), u_{k_j} \cdot \partial^{(m_{k_j})}(a|_{X^{m_{k_j}}}) \right\rangle_{\mathcal{H}^{m_{k_j}}} \\ &= \lim_{j\to\infty} \left\langle \partial^{(m_{k_j})}(w|_{X^{m_{k_j}}}), u \cdot \partial^{(m_{k_j})}(a|_{X^{m_{k_j}}}) \right\rangle_{\mathcal{H}^{m_{k_j}}} \\ &= \langle \partial w, u \cdot \partial a \rangle_{\mathcal{H}}\end{aligned}$$

by Corollary 10.7 and polarization. Applying the Leibniz rule for ∂ we arrive at (10.52). We next claim that

$$\lim_{j\to\infty} \left\langle w|_{X^{m_{k_j}}} \cdot \hat{b}^{(m_{k_j})}, \partial^{(m_{k_j})} u_{m_{k_j}} \right\rangle_{\mathcal{H}^{(m_{k_j})}} = \left\langle w \cdot \hat{b}, \partial u \right\rangle_{\mathcal{H}}. \tag{10.53}$$

Each element of the sequence on the left hand side is a finite linear combination with summands

$$\begin{aligned}&\left\langle \partial^{(m_{k_j})}(\hat{f}_i|_{X^{m_{k_j}}}), \partial^{(m_{k_j})}((w\hat{g}_i)|_{X^{m_{k_j}}} u_{k_j}) \right\rangle_{\mathcal{H}^{(m_{k_j})}} \\ &\qquad - \left\langle \partial^{(m_{k_j})}(\hat{f}_i|_{X^{m_{k_j}}}), u_{k_j} \cdot \partial^{(m_{k_j})}((w\hat{g}_i)|_{X^{m_{k_j}}}) \right\rangle_{\mathcal{H}^{(m_{k_j})}}.\end{aligned}$$

The first term converges to $\left\langle \partial \hat{f}_i, \partial(w\hat{g}_i u) \right\rangle_{\mathcal{H}}$ by (10.46). To see that

$$\lim_{j\to\infty} \left\langle \partial^{(m_{k_j})}(\hat{f}_i|_{X^{m_{k_j}}}), u_{k_j} \cdot \partial^{(m_{k_j})}((w\hat{g}_i)|_{X^{m_{k_j}}}) \right\rangle_{\mathcal{H}^{(m_{k_j})}} = \left\langle \partial \hat{f}_i, u \cdot \partial(w\hat{g}_i) \right\rangle_{\mathcal{H}} \tag{10.54}$$

let $\varepsilon > 0$. Choose n' so that by (10.15) we have

$$\mathcal{E}(H_{n'}(w\hat{g}_i) - w\hat{g}_i)^{1/2} < \varepsilon \left\| u \right\|_{\sup}^{-1} \mathcal{E}(\hat{f}_i)^{-1/2}. \tag{10.55}$$

For any j so that $m_{k_j} \geq n'$ we have

$$H_{m_{k_j}}^{(m_{k_j})}((w\hat{g}_i)|_{X^{(m_{k_j})}}) = H_{m_{k_j}}(w\hat{g}_i)|_{X^{(m_{k_j})}} = H_{n'}(w\hat{g}_i)|_{X^{(m_{k_j})}}$$

and by (10.18) therefore

$$\mathcal{E}^{(m_{k_j})}(H_{n'}(w\hat{g}_i)|_{X^{(m_{k_j})}} - (w\hat{g}_i)|_{X^{(m_{k_j})}})^{1/2} < \varepsilon \mathcal{E}(f_i)^{-1/2} \mathcal{E}(u)^{-1/2} \tag{10.56}$$

for large enough j. Since as before we can replace u_{k_j} by $u|_{X^{m_{k_j}}}$, (10.56) shows that

$$\begin{aligned}\lim_{j\to\infty} & \left| \left\langle \partial^{(m_{k_j})}(\hat{f}_i|_{X^{m_{k_j}}}), u_{k_j} \cdot \partial^{(m_{k_j})}((w\hat{g}_i)|_{X^{m_{k_j}}}) \right\rangle_{\mathcal{H}^{(m_{k_j})}} \right. \\ &\qquad \left. - \left\langle \partial^{(m_{k_j})}(\hat{f}_i|_{X^{m_{k_j}}}), u \cdot \partial^{(m_{k_j})}(H_{n'}(w\hat{g}_i)|_{X^{m_{k_j}}}) \right\rangle_{\mathcal{H}^{(m_{k_j})}} \right| < \frac{\varepsilon}{2}.\end{aligned}$$

By Corollary 10.7 and (10.55) we have

$$\lim_{j\to\infty} \left| \left\langle \partial^{(m_{k_j})}(\hat{f}_i|_{X^{m_{k_j}}}), u \cdot \partial^{(m_{k_j})}(H_{n'}(w\hat{g}_i)|_{X^{m_{k_j}}}) \right\rangle_{\mathcal{H}^{(m_{k_j})}} - \left\langle \partial \hat{f}_i, u \cdot \partial(w\hat{g}_i) \right\rangle_{\mathcal{H}} \right| < \frac{\varepsilon}{2}.$$

Since ε was arbitrary, we can combine these two estimates to conclude (10.54) and therefore (10.53). The identity

$$\lim_{j\to\infty} \left\langle u_{k_j} \cdot b^{(m_{k_j})}, \partial^{(m_{k_j})}(w|_{X^{(m_{k_j})}}) \right\rangle_{\mathcal{H}^{(m_{k_j})}} = \langle u \cdot b, \partial w \rangle_{\mathcal{H}} \tag{10.57}$$

follows by linearity from the fact that by Lemma 10.3 (ii) and Corollary 10.7 we have

$$\begin{aligned}&\lim_{j\to\infty}\left\langle (u_{k_j} g_i|_{X^{(m_{k_j})}})\cdot\partial^{(m_{k_j})} f_i, \partial^{(m_{k_j})}(w|_{X^{(m_{k_j})}})\right\rangle_{\mathcal{H}^{(m_{k_j})}}\\&\quad=\lim_{j\to\infty}\left\langle (u g_i|_{X^{(m_{k_j})}})\cdot\partial^{(m_{k_j})} f_i, \partial^{(m_{k_j})}(w|_{X^{(m_{k_j})}})\right\rangle_{\mathcal{H}^{(m_{k_j})}}\\&\quad=\langle (ug_i)\cdot\partial f_i,\partial w\rangle_{\mathcal{H}}.\end{aligned}$$

Together with the obvious identity

$$\lim_{j\to\infty}\left\langle (cw)|_{X^{(m_{k_j})}}, u_{k_j}\right\rangle_{L^2(X^{(m_{k_j})},\mu^{(m_{k_j})})}=\langle cw,u\rangle_{L^2(X,\mu)},$$

formulas (10.52), (10.53) and (10.57) imply

$$\lim_{j\to\infty}\mathcal{Q}^{(m_{k_j})}(w|_{X^{(m_{k_j})}},u_{k_j})=\mathcal{Q}(w,u),$$

what shows condition (2) in Definition 10.2. □

The combination of Theorems 10.2 and 10.3 allows a uniform approximation result for equations on the target space X with more general coefficients.

Theorem 10.4. *Let $a\in\mathcal{F}$ be such that (6.1) holds with $0<\lambda<\Lambda$. Let $b,\hat b\in\mathcal{H}$ and let $c\in C(X)$. Then we can find $a_n^{(m)}\in\mathcal{F}^{(m)}$ and $b_n^{(m)},\hat b_n^{(m)}\in\mathcal{H}^{(m)}$ such that for any n and m the forms*

$$\begin{aligned}\mathcal{Q}^{(n,m)}(f,g):=&\langle a_n|_{X^{(m)}}\cdot\partial f,\partial g\rangle_{\mathcal{H}^{(m)}}-\left\langle g\cdot b_n^{(m)},\partial f\right\rangle_{\mathcal{H}^{(m)}}\\&-\left\langle f\cdot\hat b_n^{(m)},\partial g\right\rangle_{\mathcal{H}^{(m)}}-\langle c|_{X^{(m)}}f,g\rangle_{L^2(X^{(m)},\mu^{(m)})},\quad f,g\in\mathcal{F}^{(m)}\end{aligned}\tag{10.58}$$

are closed in $L^2(X^{(m)},\mu^{(m)})$, respectively. Writing $(\mathcal{L}^{\mathcal{Q}^{(n,m)}},\mathcal{D}(\mathcal{L}^{\mathcal{Q}^{(n,m)}}))$ for the generator of $(\mathcal{Q}^{(n,m)},\mathcal{D}(\mathcal{Q}^{(n,m)}))$, one can observe the following.

(i) Let $f\in L^2(X,\mu)$, u be the unique weak solution to (6.12) on X and $u_n^{(m)}$ be the unique weak solution to (6.12) on $X^{(m)}$ with $\mathcal{L}^{\mathcal{Q}^{(n,m)}}$ and $\Phi_m(f)$ in place of $\mathcal{L}^{\mathcal{Q}}$ and f, respectively. Then there are sequences $(m_k)_k$ and $(n_l)_l$ with $m_k\uparrow+\infty$ and $n_l\uparrow+\infty$ so that

$$\lim_{l\to\infty}\overline{\lim_{k\to\infty}}\left\|E_{m_k}H_{m_k}^{(m_k)}u_{n_l}^{(m_k)}-u\right\|_{\sup}=0.$$

(ii) Let $\mathring{u}\in L^2(X,\mu)$, u be the unique solution to (6.17) on X and $u_n^{(m)}$ be the unique weak solution to (6.17) on $X^{(m)}$ with $\mathcal{L}^{\mathcal{Q}^{(n,m)}}$ and $\Phi_m(\mathring{u})$ in place of $\mathcal{L}^{\mathcal{Q}}$ and $\mathring{u}$, respectively. Then there are sequences $(m_k)_k$ and $(n_l)_l$ with $m_k\uparrow+\infty$ and $n_l\uparrow+\infty$ so that for any $t>0$

$$\lim_{l\to\infty}\overline{\lim_{k\to\infty}}\left\|E_{m_k}H_{m_k}^{(m_k)}u_{n_l}^{(m_k)}(t)-u(t)\right\|_{\sup}=0.$$

In order to handle double indexed sequences we need the following Lemma. It can be seen as an elegant way to apply standard diagonal arguments.

Lemma 10.4. *[Att84, Corollary 1.18] Let (X,d) be a metrizable space and $(x_{\nu,\mu})_{\nu\in\mathbb{N},\mu\in\mathbb{N}}\subset X$ a double indexed sequence in X, $x\in X$, such that*

$$x_{\nu,\mu}\xrightarrow[\nu\to\infty]{d}x_\mu\qquad\text{and}\qquad x_\mu\xrightarrow[\mu\to\infty]{d}x.$$

Then, there exists a mapping $\nu\mapsto\nu(\mu)$ increasing to $+\infty$ such that $x_{\nu,\nu(\mu)}\xrightarrow[\nu\to\infty]{d}x$.

As a consequence of Lemma 10.4 we can find a sequence $(l_k)_k$ with $l_k \uparrow +\infty$ such that

$$\overline{\lim_k} \left\| E_{m_k} H_{m_k}^{(m_k)} u_{n_{l_k}}^{(m_k)} - u \right\|_{\sup} = 0$$

in the situation of Theorem 10.4 (i) and similarly for (ii).

The following is a well known straightforward consequence of the density of $\bigcup_{m\geq 0} H_m(X)$ in $\mathcal{F}$, we omit its short proof.

Lemma 10.5. *The space of finite linear combinations $\sum_i g_i \partial f_i$ with $g_i, f_i \in \bigcup_{m\geq 0} H_m(X)$ is dense in $\mathcal{H}$.*

We prove Theorem 10.4.

Proof. We start with the construction of the closed forms $(\mathcal{Q}^{(n,m)}, \mathcal{D}(\mathcal{Q}^{(n,m)}))$. Let $a \in \mathcal{F}$ such that (6.1) holds with constants $0 < \lambda < \Lambda$ and let $(a_n)_n \subset \bigcup_{\tilde{m}\geq 0} H_{\tilde{m}}(X)$ be a sequence so that $\lim_{m\to\infty} \|a_n - a\|_{\sup} = 0$ is satisfied. Without loss of generality we assume that functions a_n satisfy (6.1) with the same constants λ, Λ. Suppose that $M > 0$ is large enough such that $\lambda_0 := \lambda - \frac{2}{M} > 0$.

Further, let b, $\hat{b} \in \mathcal{H}$ and let $(b_n)_n$, $(\hat{b}_n)_n \subset \mathcal{H}$ be sequences such that

$$\lim_{n\to\infty} \|b_n - b\|_{\mathcal{H}} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|\hat{b}_n - \hat{b}\|_{\mathcal{H}} = 0.$$

In addition, we assume that for each n, b_n and $\hat{b}_n$ are finite linear combinations of the form

$$b_n := \sum_i g_{n_i} \partial f_{n_i} \qquad \text{and} \qquad \hat{b}_n := \sum_i \hat{g}_{n_i} \partial \hat{f}_{n_i},$$

where $f_{n_i}, \hat{f}_{n_i}, g_{n_i}, \hat{g}_{n_i} \in \bigcup_{\tilde{m}\geq 0} H_{\tilde{m}}(X)$. This is possible due Lemma 10.5. Now define for each n the pointwise restriction of b_n and $\hat{b}_n$ to $X^{(m)}$ as in (10.31),

$$b_n|_{X^{(m)}} := \sum_i g_{n_i}|_{X^{(m)}} \cdot \partial^{(m)}(f_{n_i}|_{X^{(m)}}) \quad \text{and} \quad \hat{b}_n|_{X^{(m)}} := \sum_i \hat{g}_{n_i}|_{X^{(m)}} \cdot \partial^{(m)}(\hat{f}_{n_i}|_{X^{(m)}}).$$

From now on we consider the sequences $\left(b_n^{(m)}\right)_{(n,m)}$, $\left(\hat{b}_n^{(m)}\right)_{(n,m)}$ with $b_n^{(m)} \in \mathcal{H}^{(m)}$, $b_n^{(m)} := b_n|_{X^{(m)}}$ and $\hat{b}_n^{(m)} \in \mathcal{H}^{(m)}$, $\hat{b}_n^{(m)} := \hat{b}_n|_{X^{(m)}}$, respectively.

Let γ_M be as in (10.29) with $b_n^{(m)}$ and $\hat{\gamma}_M$ similarly but with the $\hat{b}_n^{(m)}$. Without loss of generality we assume that $c \in C(X)$ satisfies (10.7). Then for each n, m the forms $(\mathcal{Q}^{(n,m)}, \mathcal{D}(\mathcal{Q}^{(n,m)}))$ as in (10.58) with $\mathcal{D}(\mathcal{Q}^{(n,m)}) = \mathcal{F}^{(m)}$ are closed forms in $L^2(X^{(m)}, \mu^{(m)})$.

To prove (i), suppose that $f \in L^2(X, \mu)$ and u is the unique weak solution to (6.12) on X. For each n, m, let $u_n^{(m)}$ be the unique weak solution to (6.12) on $X^{(m)}$ with $\mathcal{L}^{\mathcal{Q}^{(n,m)}}$ and $\Phi_m(f)$ in place of $\mathcal{L}^{\mathcal{Q}}$ and f. Then Theorem 10.3 yields that there exists a sequence $(m_k)_k$ with $m_k \uparrow +\infty$ so that $\lim_{k\to\infty} E_{m_k} H_{m_k}^{(m_k)} u_1^{(m_k)} = u_1$ uniformly on X. Repeated applications of Theorem 10.3 allow to thin out $(m_k)_k$ further so that for any n we have

$$\left\| E_{m_k} H_{m_k}^{(m_k)} u_j^{(m_k)} - u_j \right\|_{\sup} < 2^{-n}, \quad j \leq n,$$

provided that k is greater than some integer k_n depending on n. On the other hand Theorem 10.2 allows to find a sequence $(n_l)_l$ with $n_l \uparrow +\infty$ such that $\lim_{l\to\infty} u_{n_l} = u$ uniformly on X. Combining these facts with Lemma 10.4, we obtain (i). Statement (ii) is proved in the same manner. □

Remark 10.7. Regarding the convergence scheme we use one might ask whether there is an easier way to obtain convergence results on varying spaces for solutions to partial differential equations of, for example, type

$$\operatorname{div}(a\nabla u) + b\cdot\nabla u - \operatorname{div}(ub) + cu = f. \tag{10.59}$$

In [GRS01], the authors approximate solutions to equations of type

$$\Delta u + qu = f$$

on a post-critically finite fractal K. The functions q and f are given continuous functions on K. Their approximation scheme uses the finite element method based on piecewise harmonic or piecewise biharmonic splines. This work relies on [SU00] where the general theory of piecewise multiharmonic splines for post-critically finite fractals was constructed. However, we cannot use the method of spline approximation on varying spaces for equations involving dynamics like (10.59).

10.4 Approximations

In this section we verify that graph approximations for finitely ramified spaces and metric graph approximations for p.c.f. self-similar spaces satisfy the assumptions stated in the previous section.

10.4.1 Discrete graph approximations for finitely ramified spaces

We specify to the case where the approximating spaces $X^{(m)}$ are finite point sets.

Let $(\mathcal{E},\mathcal{F})$ be a resistance form on a nonempty set X so that (X,R) is compact. For simplicity we restrict to compact resistance spaces that are in addition metrically doubling with doubling constant $K_R > 1$. We also assume that $(\mathcal{E},\mathcal{F})$ is local. We consider the spaces $X^{(m)} := V_m$ and the resistance forms $\mathcal{E}^{(m)} := \mathcal{E}_{V_m}$ with domains $\mathcal{F}^{(m)} = \ell(V_m)$, respectively. Therefore each $\mathcal{E}^{(m)}$ is of the form (3.4). It is well known that $R^{(m)}(p,q) = R(p,q)$ for $p,q\in V_m$, [Kig01, Definition 2.2.1 and Theorem 2.3.7]. This implies that a ball in $(V_m, R^{(m)})$ of radius $r > 0$ centered at $p\in V_m$ coincides with $B(p,r)\cap V_m$, hence also the spaces $(V_m, R^{(m)})$ are metrically doubling with doubling constant K_R.

Clearly Assumption 5.1 is satisfied, note that for every $u\in H_m(X)$ we have

$$\mathcal{E}_{V_m}(u|_{V_m}) = \mathcal{E}(u)$$

and (10.13) is immediate from (4.5). Since every element of $\ell(V_m)$ is the pointwise restriction of a function in $H_m(X)$, the operator $H_m^{(m)}$ is the identity operator $\mathrm{id}_{\mathcal{F}^{(m)}}$, so that Assumption 10.2 is trivial, as pointed out in Remark 10.5.

In what follows let μ be a finite Borel measure on (X,R) which admits a uniform lower bound V. Similar as in [PS18a] we define, for each m, a measure $\mu^{(m)}$ on V_m by

$$\mu^{(m)}(\{p\}) := \int_X \psi_{p,m}(x)d\mu(x), \quad p\in V_m, \tag{10.60}$$

where $\psi_{p,m}$ is the (unique) harmonic extension to X of the function $\mathbf{1}_{\{p\}}$ on V_m. For fixed m and any $\rho > 0$ we have

$$\inf_{p\in V_m} \mu^{(m)}(B^{(m)}(p,\rho)) = \inf_{p\in V_m} \mu^{(m)}(\{p\}) > 0,$$

so the first part in Assumption 10.3 is satisfied.

For each m let Φ_m be a linear operator $\Phi_m : L^2(X,\mu) \to \ell^2(V_m, \mu^{(m)})$ defined by

$$\Phi_m f(p) := \frac{1}{\mu^{(m)}(p)} \langle f, \psi_{p,m} \rangle_{L^2(X,\mu)}, \quad p \in V_m, \quad f \in L^2(X,\mu). \tag{10.61}$$

In [PS18a, proof of Theorem 1.1] it was proved that for each m its adjoint operator Φ_m^* satisfies $\|\Phi_m^* f\|_{L^2(X,\mu)} \le \|f\|_{\ell^2(V_m,\mu^{(m)})}$ for all $f \in \ell^2(V_m, \mu^{(m)})$. Thus it follows that

$$\sup_m \|\Phi_m\|_{L^2(X,\mu)\to\ell^2(V_m,\mu^{(m)})} < \infty \tag{10.62}$$

and (10.19) is fulfilled.

For each m the operator $E_m : \ell^2(V_m, \mu^{(m)}) \to H_m(X)$ is the harmonic extension operator

$$E_m v(x) = \sum_{p\in V_m} v(p)\psi_{p,m}(x), \quad v \in \ell^2(V_m, \mu^{(m)}).$$

In [PS18a, proof of Theorem 1.1] it was also shown that

$$\|E_m v\|^2_{L^2(X,\mu)} \le \|v\|^2_{\ell^2(V_m,\mu^{(m)})}, \quad v \in \ell^2(V_m, \mu^{(m)}),$$

what proves (10.23).

To verify (10.20), (10.22) and (10.21) in Assumption 10.3 we need an additional assumption on the decay of the support of $\psi_{p,m}$ as m goes to infinity.

Assumption 10.4. In what follows we assume that

$$\lim_{m\to\infty} \sup_{p\in V_m} \operatorname{diam}_R (\operatorname{supp} \psi_{p,m}) = 0.$$

Remark 10.8. If we consider discrete graph approximations for a finitely ramified space X, the validity of Assumption 10.4 follows from Definition 5.1.

As a particular consequence of Assumption 10.4 we obtain that

$$\lim_{m\to\infty} \sum_{p\in V_m} \int_X |u(p) - u(x)|\psi_{p,m}(x) d\mu(x) = 0. \tag{10.63}$$

Now let $(u_m)_m \subset \ell(V_m)$ be an arbitrary sequence with $\sup_m \mathcal{E}(u_m) < \infty$. Then

$$\begin{aligned}\|\Phi_m u_m - u_{m|V_m}\|^2_{\ell^2(V_m,\mu^{(m)})} &\le \sum_{p\in V_m} \frac{1}{\mu^{(m)}(p)} \left(\int_X |u_m(p) - u_m(x)|\psi_{p,m}(x) d\mu(x) \right)^2 \\ &\le 2 \sup_m \|u_m\|_{\sup} \sum_{p\in V_m} \int_X |u(p) - u(x)|\psi_{p,m}(x) d\mu(x)\end{aligned}$$

and therefore (10.63) yields that

$$\lim_{m\to\infty} \|\Phi_m u_m - u_{m|V_m}\|_{\ell^2(V_m,\mu^{(m)})} = 0, \tag{10.64}$$

what shows (10.22). For every $u \in \mathcal{F}$ it follows that

$$\begin{aligned}\lim_{m\to\infty} \|u_{|V_m}\|^2_{\ell^2(V_m,\mu^{(m)})} &= \lim_{m\to\infty} \left(\sum_{p\in V_m} \int_X \left[(u(p) - u(x))(u(p) + u(x)) + u^2(x) \right] \psi_{p,m}(x) d\mu(x) \right) \\ &= \|u\|^2_{L^2(X,\mu)},\end{aligned}$$

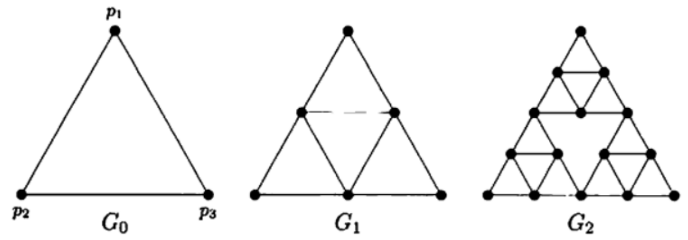

Figure 10.2: Approximation of the Sierpiński gasket by graphs G_m,[Kig01, Fig. 0.3 on page 3]

since $\sum_{p\in V_m} \psi_{p,m} \equiv 1$ for all m and by (10.63),

$$\sum_{p\in V_m} \int_X (u(p)-u(x))(u(p)+u(x))\psi_{p,m}(x)d\mu(x) \leq 2\|u\|_{\sup} \sum_{p\in V_m} \int_X (u(p)-u(x))\psi_{p,m}(x)d\mu(x) \xrightarrow{m\to\infty} 0,$$

so (10.20) is fulfilled. Moreover, if $u \in H_n(X)$ we obtain

$$\begin{aligned}\|\Phi_m^*\Phi_m u - u\|_{L^2(X,\mu)} \leq \|\Phi_m^*\|_{\ell^2(V_m,\mu^{(m)})\to L^2(X,\mu)} \|\Phi_m u - u_{|V_m}\|_{\ell^2(V_m,\mu^{(m)})} \\ + \|\Phi_m^*(u_{|V_m}) - u\|_{L^2(X,\mu)}.\end{aligned}$$

Note that $\Phi_m^*(u_{|V_m}) = H_m u$, hence the last term is bounded from above by $\mathcal{E}(H_m u - u)\mu(X)$. Due to Assumption 5.1, (10.22) and (10.19) the right hand side vanishes as m goes to infinity and (10.21) is also satisfied.

10.4.2 Metric graph approximations for p.c.f. self-similar spaces

We specify to the case where X is a post-critically finite self-similar set and the approximating spaces $X^{(m)}$ are metric graphs as introduced in Section 5.1.

Throughout this section let $(X, S, \{F_j\}_{j\in S})$ be a self-similar post critically finite structure, see also Definitions 5.2 and 5.3. Recall that $S = \{1, \dots, N\}$. We also assume that X is connected. Let $(\mathcal{E}, \mathcal{F})$ be constructed as in (5.16) and (5.17). For simplicity we assume that all energy renormalisation parameters r_j of the energy forms are the same, i.e., there exists $r \in (0,1)$ such that $r = r_j$ for all $j = 1, \dots, N$. Further, let $(\mathcal{E}, \mathcal{F})$ be a local regular resistance form on X so that (X, R) is compact and metrically doubling with doubling constant $K_R > 1$.

We construct the approximating spaces $X^{(m)}$. For each $m \geq 0$ we consider V_m as the vertex set of a (discrete) graph $G_m = (V_m, E_m)$ with vertices $p, q \in V_m$ being the endpoints of an edge $e \in E_m$ connecting them if there is a word w of length $|w| = m$ such that $F_w^{-1}p, F_w^{-1}q \in V_0$ so that $c(0; p, q) > 0$. In this case we write $p \sim_m q$.

Now let $(\Gamma_m)_{m\geq 0}$ be the sequence of metric graphs $\Gamma_m = (E_m, V_m, i_m, j_m)$ naturally defined by the graphs G_m, endowed with an orientation, arbitrary but fixed. For simplicity we write $i = i_m$ and $j = j_m$ when m is fixed. As an example, one may consider the graph approximation for the Sierpiński gasket as shown in Figure 10.2. For simplicity we restrict our attention to post-critically finite self-similar sets that can be approximated by metric graphs Γ_m such that all edges $e \in E_m$ have the same length l_m.

On the space $X^{(m)} := X_{\Gamma_m}$, defined as in (5.1), we consider the local resistance form $(\mathcal{E}^{(m)}, \mathcal{F}^{(m)}) := (\mathcal{E}_{\Gamma_m}, \dot{W}^{1,2}(X_{\Gamma_m}))$, where

$$\mathcal{E}_{\Gamma_m}(f) := l_m r^{-m} \sum_{e\in E_m} \mathcal{E}_e(f_e) \quad \text{and} \quad \mathcal{E}_e(f_e) = \int_0^{l_m} (f_e'(t))^2 dt.$$

Recall from Section 5.1 that

$\dot{W}^{1,2}(X_{\Gamma_m}) := \{f \in C(X_{\Gamma_m}) : \text{for any } e \in E_m \text{ we have } f_e \in \dot{W}^{1,2}(e), \text{ and } \mathcal{E}_{\Gamma_m}(f) < +\infty\}.$

Similar as in [Str06, Sec. 1.6], we observe that it is extremely difficult to compute $R_{\Gamma_m}(x, y)$, but it is rather easy to obtain approximate values. From now on, let $\deg_m(p)$ denote the degree of the vertex $p \in V_m$ in the graph G_m. Let $x, y \in V_m$ be connected by an edge $e \in E_m$. Then the m-piecewise harmonic function $\psi_{x,m}$ with $\psi_{x,m}(x) = 1$ and $\psi_{x,m}(y) = 0$ is linear on all line segments and satisfies $\mathcal{E}_{\Gamma_m}(\psi_{x,m}) = \deg_m(x) \cdot r^{-m}$. We obtain the lower bound

$$R_{\Gamma_m}(x, y) \geq (\mathcal{E}_{\Gamma_m}(\psi_{x,m}))^{-1} \geq \frac{1}{\deg_m(x)} r^m. \tag{10.65}$$

On the other hand, every energy finite function u with $u(x) = 1$ and $u(y) = 0$ satisfies $\mathcal{E}_{\Gamma_m}(u) \geq r^{-m}$ and this yields the upper bound

$$R_{\Gamma_m}(x, y) \leq r^m. \tag{10.66}$$

Using these bounds for the resistance metric we show that there is a finite constant $K_\rho > 0$ such that any ball $B(x, 2\rho)$ can be covered by K_ρ balls of radius ρ. This will imply that the spaces (X, R) and $(X_{\Gamma_m}, R_{\Gamma_m})$, $m \in \mathbb{N}$, are metrically doubling with doubling constant $\max(K_R, K_\rho)$. To see this, we consider four cases for a ball $B^{(m)}(x, \rho)$ in $(X_{\Gamma_m}, R^{(m)})$ centered at x with radius ρ, i.e. $B^{(m)}(x, \rho) = \{y \in X_{\Gamma_m} : R_{\Gamma_m}(x, y) < \rho\}$.

(i) Macroscopic picture, case that $x \in V_m$.
We consider balls centered in a junction point $x \in V_m$ with radius ρ such that

$$2^k r^m < \rho \leq 2^{k+1} r^m, \qquad 2 \leq k \leq m-2,\, k \in \mathbb{N} \tag{10.67}$$

holds. Clearly, we have $2\rho \leq 2^{k+2} r^m$. If $y \in X_{\Gamma_m}$ satisfies

$$R_{\Gamma_m}(x, y) \leq 2\rho < 2^{k+2} r^m,$$

then by (10.66) the point y has to be either in one of the cells containing x or in one of the adjacent cells having edges with length $2^{k+2} \cdot l_m$. Now we use a very coarse estimate for the number of cells with edges of length l_m needed to cover union of all such big cells:
The number of such big cells that are adjacent to the big cell containing x is bounded by N and each of this big cells is covered by N^4 smaller cells with edges of length $2^{k-2} \cdot l_m$. Note that each small cell is covered by a resistance ball of radius $2^k r^m \leq \rho$. Increasing the factor to $N^4 \cdot N^2 = N^6$ one can also allow $k = 0, 1$ in (10.67). In this case, it would be sufficient to choose $K_\rho = (N+1) \cdot N^6$.

(ii) Macroscopic picture, case that $x \notin V_m$.
We consider balls with radius ρ such that

$$2^k r^m < \rho \leq 2^{k+1} r^m, \qquad 0 \leq k \leq m-2,\, k \in \mathbb{N}$$

holds, but now not centered in a junction point. Then the center x is located on an edge $e \in E_m$ with endpoints $i(e)$ and $j(e)$. Take $\tilde{\rho} = 2\rho$. We consider the new ball $B(i(e), \tilde{\rho})$ and proceed as in the previous case. Here, it would be sufficient to increase the constant K_ρ to $(N+1)^2 \cdot N^{12}$.

(iii) Microscopic picture, case that $x \in V_m$.
We consider the ball that is centered in a junction point $x \in V_m$ and such that the radius ρ satisfies

$$0 < \rho \leq r^m.$$

– If it also holds that $2\rho \leq r^m$, then every point y in the ball is located on an edge adjacent to the center x. We can cover the ball by itself and $\deg_m(x)$ balls of the same size located on the adjacent edges.

– Now we suppose that the following holds,

$$\frac{r^m}{2} < \rho \leq r^m < 2\rho \leq 2r^m.$$

Then the resistance ball $B(x, 2\rho)$ is covered by the resistance ball $B(x, 2r^m)$ which contains N^2 cells with edges of length l_m. Since the number of vertices and the number of edges in one cell at level m is bounded by N, we can cover the ball $B(x, 2r^m)$ by $2 \cdot N^3$ balls of radius $\frac{r^m}{2} < \rho$.

(iv) Microscopic picture, case that $x \notin V_m$.
We consider balls with radius ρ such that

$$0 < \rho \leq r^m.$$

holds, but now not centered in a junction point.

– If it holds that $2\rho \leq r^m$, two cases can appear: Either the whole ball $B(x, 2\rho)$ is located on one single edge or the center x is very close to a junction point and the ball $B(x, 2\rho)$ is included in the union of edges adjacent to x. In the first case, 3 balls of radius ρ would be sufficient to cover $B(x, 2\rho)$. In the second case, we can cover this ball by $\deg_m(x) + 1$ balls of radius ρ.

– If it holds that $\frac{r^m}{2} < \rho \leq r^m$ we simply use the same method as in the second part of case (iii) to cover the ball $B(x, 2\rho)$.

All in all, it is sufficient to choose $K_\rho = (N + 1)^2 N^{12}$.

We check whether the conditions in Assumptions 10.1, 10.2 and 10.3 are fulfilled. To a function $f \in \dot{W}^{1,2}(X_{\Gamma_m})$ which is linear on each edge $e \in E_m$ we refer as *edge-wise linear* function, and we denote the subspace of $\dot{W}^{1,2}(X_{\Gamma_m})$ of such functions by EL_m. If $f \in EL_m$, then its derivative on e is the constant function $f_e' = l_m^{-1}(f(j(e)) - f(i(e)))$, so that

$$\mathcal{E}_e(f_e) = \int_0^{l_m} (f_e'(t))^2 dt = \frac{1}{l_m}(f(j(e)) - f(i(e)))^2 \tag{10.68}$$

on each $e \in E_m$ and consequently $\mathcal{E}_{\Gamma_m}(f) = \mathcal{E}_m(f|_{V_m})$. For a general function $f \in \dot{W}^{1,2}(X_{\Gamma_m})$ formula (10.68) becomes an inequality in which the left hand side dominates the right hand side. This implies

$$\mathcal{E}_m(f|_{V_m}) \leq \mathcal{E}_{\Gamma_m}(f), \quad f \in \dot{W}^{1,2}(X_{\Gamma_m}). \tag{10.69}$$

By H_{Γ_m} we denote the linear operator $H_{\Gamma_m} : \dot{W}^{1,2}(X_{\Gamma_m}) \to EL_m$ that assigns to a function $f \in \dot{W}^{1,2}(X_{\Gamma_m})$ the unique edge-wise linear function on X_{Γ_m} that interpolates $f|_{V_m}$. For metric graph approximations we have $H_m^{(m)} f = H_{\Gamma_m} f$, $f \in \dot{W}^{1,2}(X_{\Gamma_m})$.

Given a function $f \in \mathcal{F}$ on X, we can interpret its pointwise restriction to the line segment connecting two neighbor points $p \sim_m q$ from V_m as a continuous function f_e on the edge $e \in E_m$ of Γ_m with $i(e) = p$ and $j(e) = q$. This defines a continuous function on X_{Γ_m}, which we denote by $f|_{X_{\Gamma_m}}$. Since a function $f \in H_m(X)$ is linear on all line segments connecting two neighbor points $p \sim_m q$, the above interpretation $f|_{X_{\Gamma_m}}$ of f is a function in EL_m which satisfies (10.68) on each edge and $\mathcal{E}_{\Gamma_m}(f|_{X_{\Gamma_m}}) = \mathcal{E}_m(f|_{V_m}) = \mathcal{E}(f)$, so (10.12) in Assumption 10.1 is satisfied. Moreover, we have $H_{\Gamma_m}(f|_{X_{\Gamma_m}}) = H_m(f)|_{X_{\Gamma_m}}$ for any $f \in \mathcal{F}$. Since

$$\lim_{m\to\infty} \mathcal{E}(H_m(f) - f) = 0 \tag{10.70}$$

for any $f \in \mathcal{F}$, see for instance [Str06, Theorem 1.4.4], we observe that

$$\mathcal{E}(f) = \sup_m \mathcal{E}_{\Gamma_m}(H_m(f)|_{X_{\Gamma_m}}), \quad f \in \mathcal{F}.$$

We verify (10.13) in the next lemma.

Lemma 10.6. *Let $f \in \dot{W}^{1,2}(X_{\Gamma_m})$. Then $\nu_f = \lim_{m\to\infty} \nu^{(m)}_{H_m(f)|_{X_{\Gamma_m}}}$ in the sense of weak convergence of measures on X.*

Proof. Let $g \in C(X)$. First, we show that

$$\int_X g d\nu_f = \int_{X_{\Gamma_m}} g d\nu_f^{(m)}, \qquad f \in H_n(X). \tag{10.71}$$

In the metric graph case, we can rewrite (4.5) to

$$\int_X g\, d\nu_f = \frac{1}{2} \lim_{m\to\infty} \sum_{e\in E_m, e\sim p, e\sim q} c(m,p,q) g_e(p)(f_e(p) - f_e(q))^2.$$

Since $f \in H_n(X)$ is in particular edge-wise linear, we have that

$$f_e(p) - f_e(q) = f_e'(p-q) = \int_q^p f_e' dt,$$

where $f_e' \in \mathbb{R}$. Moreover, for each m it holds that

$$\int_{X_{\Gamma_m}} g\, d\nu_f^{(m)} = r^{-m} \sum_{e\in E_m} l_m \int_0^{l_m} g(t) \left(f(t)'\right)^2 dt.$$

Thus, for $f \in H_n(X)$, $m \geq n$, it suffices to compare

$$|l_m g_e(p) \int_0^{l_m} f_e' dt - l_m \int_0^{l_m} g(t) \left(f(t)'\right)^2 dt| \leq l_m (f_e')^2 \int_0^{l_m} |g_e(p) - g_e(t)| dt. \tag{10.72}$$

Recall that the space X is assumed to be compact, therefore for arbitrary $\varepsilon > 0$ we can find $\tilde{m}$ large enough such that

$$\sup_{e\in E_{\tilde{m}}} \sup_{t,s\in e} |g(s) - g(t)| < \varepsilon,$$

so (10.72) is bounded from above by $\varepsilon l_m \mathcal{E}_e(f_e)$, what implies (10.71).

Now it suffices to show that for any $f \in \mathcal{F}$ and for any $g \in C(X)$ it holds that

$$\int_X g d\nu_f = \lim_m \int_X g d\nu_{H_m(f)}.$$

Without loss of generality we assume that $g \geq 0$, otherwise we just consider g^+, g^-, $g = g^+ - g^-$ separately. Since $\nu_{f,H_m(f)}$ is bilinear in f, $H_m(f)$ and ν_f is a positive measure, we have that

$$\left| \left(\int_X g d\nu_f\right)^{\frac{1}{2}} - \left(\int_X g d\nu_{H_m(f)}\right)^{\frac{1}{2}} \right| \leq \int_X g d\nu_{f-H_m(f)} \leq \|g\|_{\sup} \mathcal{E}(f - H_m(f)),$$

see also [FOT94, Chapter 3.2]. Now the desired statement follows from (10.70), see also [Str06, Thm. 1.4.4]. ☐

To verify (10.17) in Assumption 10.2 we need the following lemma.

Lemma 10.7. *For each m suppose that $f_m \in \dot{W}^{1,2}(X_{\Gamma_m})$ are such that the following holds,*

$$\sup_m \mathcal{E}_{\Gamma_m}(f_m) < +\infty \qquad \textit{and} \qquad f_{m|V_m} = 0.$$

Then it holds that

$$\lim_{m\to\infty} \|f_m\|_{\sup,X_{\Gamma_m}} = 0.$$

Proof. By resistance estimate (5.2) on each edge $e = [0, l_m]$ in X_{Γ_m} we have

$$\sup_{t\in[0,l_m]} |\left(f_m\right)_e(t)| \leq l_m^{\frac{1}{2}} \left(\int_0^{l_m} \left(\left(f_m'\right)_e(t)\right)^2 dt\right)^{\frac{1}{2}}$$

and consequently

$$\|f_m\|^2_{sup,X_{\Gamma_m}} \leq \sum_{e\in E_m} \sup_{t\in[0,l_m]} |\left(f_m\right)_e(t)|^2 \leq r^m \sup_n \mathcal{E}_{\Gamma_n}(f_n) < \infty.$$

□

Note that for each m, $f_m - H_m^{(m)} f_m \in \dot{W}^{1,2}(X_{\Gamma_m})$ we have $\left(f_m - H_m^{(m)} f_m\right)|_{V_m} = 0$ and $\mathcal{E}_{\Gamma_m}(f_m - H_m^{(m)} f_m) \leq 2\,\mathcal{E}_{\Gamma_m}(f_m)$. The next lemma yields a proof of (10.18) in Assumption 10.2.

Lemma 10.8. *Given $f, g \in H_n(X)$, we have*

$$\lim_{m\to\infty} \mathcal{E}_{\Gamma_m}\left(f|_{X_{\Gamma_m}} g|_{X_{\Gamma_m}} - H_m^{(m)}\left(f|_{X_{\Gamma_m}} g|_{X_{\Gamma_m}}\right)\right) = 0.$$

Proof. We first note that for any $m \geq n$ the functions f_e and g_e are linear on any fixed $e \in E_m$, in particular they are for all $t \in [0, l_m]$ of the form

$$f_e(t) = f_e(0) + f_e' \cdot t \qquad \text{and} \qquad g_e(t) = g_e(0) + g_e' \cdot t$$

with slopes $f_e' \in \mathbb{R}$ and $g_e' \in \mathbb{R}$, respectively. Therefore $\mathcal{E}_e(f_e) = l_m \left(f_e'\right)^2$ for each such e and

$$l_m^2 \left(f_e'\right)^2 \leq \sum_{|w|=m} \sum_{e\in E_m, e\in K_w} l_m^2 \left(f_e'\right)^2 \leq r^m \sup_{m\geq n} \mathcal{E}_{\Gamma_m}\left(f|_{X_{\Gamma_m}}\right) = r^m\, \mathcal{E}(f), \tag{10.73}$$

similarly for the function g. Since

$$\left(fg\right)_e(t) = f_e(t)g_e(t) = f_e(0)g_e(0) + g_e(0)f_e' \cdot t + f_e(0)g_e' \cdot t + f_e'g_e' \cdot t^2$$

and

$$\begin{aligned} H_m^{(m)}\left((fg)|_e\right)(t) &= f_e(0)g_e(0) + \frac{t}{l_m}\left(f_e(l_m)g_e(l_m) - f_e(0)g_e(0)\right) \\ &= f_e(0)g_e(0) + \frac{t}{l_m}\left(f_e'g_e'l_m^2 + \left(f_e(0)g_e' + g_e(0)f_e'\right)l_m\right) \end{aligned}$$

we obtain for $t \in [0, l_m]$

$$\left(\left(fg\right)_e - H_m^{(m)}\left((fg)|_e\right)\right)(t) = f_e'g_e't^2 - f_e'g_e'l_mt.$$

This implies that for any edge $e \in E_m$ we have

$$\mathcal{E}_e\left(\left((fg)_e - H_m^{(m)}\left((fg)|_e\right)\right)(t)\right) = \left(f'_e g'_e\right)^2 \int_0^{l_m} (2t - l_m)^2\, dt = \frac{1}{3}\left(f'_e g'_e\right)^2 l_m^3.$$

Summing up over $e \in E_m$ and using (10.73), we see that

$$\begin{aligned}
&\mathcal{E}_{\Gamma_m}\left(f|_{X_{\Gamma_m}} g|_{X_{\Gamma_m}} - H_m^{(m)}\left(f|_{X_{\Gamma_m}} g|_{X_{\Gamma_m}}\right)\right) \\
&= r^{-m} l_m \sum_{e \in E_m} \mathcal{E}_e\left(\left((fg)_e - H_m^{(m)}\left((fg)|_e\right)\right)(t)\right) \\
&\leq \frac{1}{3} r^{-m} l_m^4 \sum_{e \in E_m} \left(f'_e g'_e\right)^2 \\
&\leq \frac{1}{3}\mathcal{E}(f) \sum_{e \in E_m} g_e'^2 l_m^2 \leq \frac{1}{3} r^m\, \mathcal{E}(f)\, \mathcal{E}(g).
\end{aligned}$$

□

In what follows let μ be a finite Borel measure on (X, R) so that $\mu(B(x,r)) > 0$ for any $x \in X$ and $r > 0$. Given an edge $e \in E_m$ we set

$$\psi_{e,m}(x) := \frac{1}{\deg_m(i(e))}\psi_{i(e),m}(x) + \frac{1}{\deg_m(j(e))}\psi_{j(e),m}(x), \quad x \in X, \tag{10.74}$$

to obtain a function $\psi_{e,m}$ which satisfies

$$\sum_{e \in E_m} \langle \psi_{e,m}, \mathbf{1}\rangle_{L^2(X,\mu)} = \sum_{p \in V_m} \psi_{p,m}(x) = 1, \quad x \in X. \tag{10.75}$$

We endow the space X_{Γ_m} with the measure $\mu^{(m)} := \mu_{\Gamma_m}$ defined as in (5.5) with constants

$$c_e := \frac{1}{l_m}\left(\int_X \psi_{e,m}(x)\mu(dx)\right), \quad e \in E_m,$$

so that $\mu_{\Gamma_m}(dt) := \sum_{e \in E_m} c_e \lambda_e(dt)$.

For fixed m and any $0 < \rho < r^m$ we have by (5.5)

$$\inf_{x \in X_{\Gamma_m}} \mu_{\Gamma_m}(B^{(m)}(x,\rho)) = \inf_{x \in X_{\Gamma_m}} \sum_{e \in E_m} c_e \lambda^1|_e\left(B^{(m)}(x,\rho)\right).$$

If the center x of the ball is a vertex point, i.e. $x \in V_m$, then all points in the ball $B(x,\rho)$ are included in the euclidean ball $B(x, \rho_{euklid.})$ with $\rho_{euklid.} = r^{-m}\rho l_m \deg_m(x)$. On edges $e \in E_m$ such that $e \cap B^{(m)}(x,\rho) \neq \emptyset$ we obtain the lower bound

$$\begin{aligned}
c_e \lambda^1|_e\left(B^{(m)}(x,\rho)\right) &= c_e \rho_{euklid.} \\
&= r^{-m}\rho \deg_m(x)\left(\int_X \psi_{e,m}(y)\mu(dy)\right) \\
&\geq r^{-m}\rho \frac{2}{\max_{p \in V_m}\deg_m(p)} \inf_{p \in V_m}\left(\int_X \psi_{p,m}(y)\mu(dy)\right) > 0.
\end{aligned}$$

If the center x of the ball is not a vertex point, i.e. $x \notin V_m$, and ρ is sufficiently small, all points in the ball $B(x,\rho)$ are included in the euclidean ball $B(x, \rho_{euklid.})$. Here, the euclidean ball is located on a single edge $e \in E_m$ and has radius $\rho_{euklid.} = r^{-m}\rho l_m$. As before, we obtain

$$\begin{aligned}\mu_{\Gamma_m}(B^{(m)}(x,\rho)) = \mu_{\Gamma_m}|_e(B^{(m)}(x,\rho)) &= 2c_e\rho_{euklid.}\\ &= 2r^{-m}\rho\left(\int_X \psi_{e,m}(y)\mu(dy)\right)\\ &\geq r^{-m}\rho\frac{2}{\max_{p\in V_m}\deg_m(p)}\inf_{p\in V_m}\left(\int_X \psi_{p,m}(y)\mu(dy)\right)\\ &> 0,\end{aligned}$$

so the first part in Assumption 10.3 is satisfied.

For each m let Φ_m be a linear operator $\Phi_m : L^2(X,\mu) \to L^2(X_{\Gamma_m}, \mu_{\Gamma_m})$ defined by

$$\Phi_m u(t) = \sum_{e\in E_m} \mathbf{1}_e(t)\,\frac{\langle u, \psi_{e,m}\rangle_{L^2(X,\mu)}}{\left(\int_X \psi_{e,m}d\mu\right)}, \quad u \in L^2(X,\mu).$$

Later in Section 11.2, we introduce Φ_m as $J^*_{0,m}$. The reason is that we follow [Pos12; PS18a] there and we also use their notation. The rest of Assumption 10.3 is satisfied, this will follow from several results presented in Chapter 11 and in the appendix of this part. More precisely, the conditions (10.19) and (10.20) are satisfied by Proposition 11.1. A proof of condition (10.21) is provided in Lemma A3.3. Condition (10.22) follows from Lemma A3.2 (ii). In this lemma, the pointwise restriction of m-harmonic functions to $X^{(m)}$ is denoted by $\widetilde{J}_{1,m}$. For each m the composition of the operators E_m and H_{Γ_m}, $E_m \circ H_{\Gamma_m} : W^{1,2}(X_{\Gamma_m}, \mu_{\Gamma_m}) \to H_m(X)$ is the harmonic extension of an edge-wise linear function and has the following form

$$E_m \circ H_{\Gamma_m} f(x) = \sum_{e\in E_m} f_e\,\psi_{e,m}(x), \quad f \in W^{1,2}(X_{\Gamma_m}, \mu_{\Gamma_m}).$$

In Lemma A3.2 we denote this extension by $J_{1,m}$. Let $(f_m)_m$, $f_m \in W^{1,2}(X_{\Gamma_m}, \mu_{\Gamma_m})$ be an arbitrary sequence with $\sup_m \mathcal{E}_{\Gamma_m}(f_m) < \infty$. Using Lemma A3.2 and Proposition 11.1 we see that

$$\|E_m \circ H_{\Gamma_m} f_m\|_{L^2(K,\mu)} \leq \|f_m\|_{L^2(X_{\Gamma_m},\mu_{\Gamma_m})} + Cr^{m/2}\sup_m \mathcal{E}_{\Gamma_m}(f_m)^{1/2}$$

with a positive constant C depending only on N. Consequently also (10.23) is satisfied.

Chapter 11

Generalized norm resolvent convergence and metric graph approximation for Cole-Hopf solutions to the Burgers equation

In this chapter we consider metric graph approximations to a connected p.c.f. self-similar structure in the sense of Kigami, [Kig93a; Kig01]. We provide metric graph approximation results for solutions of the heat equation (7.4). Moreover, we show in Theorem 11.2 that Cole-Hopf solutions to Burgers equations (7.8) on metric graphs converge in a appropriate weak sense to a Cole-Hopf solution to the Burgers equation (7.12) on a connected p.c.f. self-similar structure.

Although it would be sufficient to verify convergence of the semigroup operators in the strong sense, we do not verify generalized Mosco convergence, [KS03, Section 2.5], which would be equivalent to convergence of operators in a suitable strong sense, [KS03, Theorem 2.4]. Our approximation scheme basically follows the methods in [PS18a] and [PS18b]. The reason is that in practice it seems easier to verify generalized norm resolvent convergence in the sense of [Pos12; PS18a; PS18b] than to verify generalized Mosco convergence.

The results of this chapter are based on the joint work [HM20b].

11.1 Generalized norm resolvent convergence

We review the notion of generalized norm resolvent convergence of self-adjoint operators on varying Hilbert spaces as developed by Olaf Post in [Pos12, Section 4.2] and recall some key definitions needed in our later results and proofs. In this section we omit proofs and give detailed references to the literature.

Let $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}_2$, ... be Hilbert spaces which are dense in H, H_1, H_2, ..., respectively, and such that with a universal constant $c > 0$ we have

$$\|u\|_H \leq c\,\|u\|_{\mathcal{D}}\,, \quad u \in \mathcal{D}, \qquad \text{and} \qquad \|u\|_{H_m} \leq c\,\|u\|_{\mathcal{D}_m}\,, \quad u \in \mathcal{D}_m.$$

Crucial to Post's concept is the existence of bounded identification operators J_m and J_m' from all of H_m into H and vice versa. We recall key notions from [Pos12, Definitions 4.1.1, 4.2.1, 4.2.3 and 4.2.6] to define the notion of convergence of nonnegative operators acting in different Hilbert spaces.

Definition 11.1.

(i) *Let m be fixed. Suppose that we are given a bounded linear operator*

$$J_{0,m} : H_m \to H,$$

*nonnegative definite self-adjoint operators $A_m : H_m \to H_m$ and $A : H \to H$ and $\delta_m \geq 0$. The operators A_m and A are said to be δ_m-*close with identification operator *$J_{0,m}$ if their 1-resolvent operators $G_{m,1}$ and G_1 satisfy*

$$\|G_1 J_{0,m} - J_{0,m} G_{m,1}\| \leq \delta_m.$$

(ii) *Let m be fixed. Suppose that we are given bounded linear operators*

$$J_{0,m} : H_m \to H \quad \text{and} \quad \widetilde{J}_{0,m} : H \to H_m \tag{11.1}$$

*and $\delta_m \geq 0$. The operators $J_{0,m}$ and $\widetilde{J}_{0,m}$ are said to be δ_m-*quasi-unitary *with respect to $\mathcal{D}_m$ and $\mathcal{D}$ if*

$$\|J_{0,m}\| \leq 2, \qquad \left\| J_{0,m} - \widetilde{J}^*_{0,m} \right\| \leq \delta_m$$

and

$$\left\| \mathrm{id}_{H_m} - \widetilde{J}_{0,m} J_{0,m} \right\|_{\mathcal{D}_m \to H_m} \leq \delta_m \quad \text{as well as} \quad \left\| \mathrm{id}_H - J_{0,m} \widetilde{J}_{0,m} \right\|_{\mathcal{D} \to H} \leq \delta_m.$$

(iii) *Let m be fixed. Two nonnegative definite self-adjoint operators $A_m : H_m \to H_m$ and $A : H \to H$ are said to be δ_m-*quasi-unitarily equivalent *if there exist operators $J_{0,m}$ and $\widetilde{J}_{0,m}$ as in (11.1) and $\delta_m \geq 0$ so that $J_{0,m}$ and $\widetilde{J}_{0,m}$ are δ_m-quasi-unitary and A_m and A are δ_m-close with identification operator $J_{0,m}$.*

(iv) *Suppose that $(A_m)_m$ is a sequence of nonnegative definite self-adjoint operators $A_m : H_m \to H_m$ and that $A : H \to H$ is a nonnegative definite self-adjoint operator. We say that the sequence $(A_m)_m$* converges *to A* in the P-generalized norm resolvent sense *if there is a sequence of non-negative numbers $(\delta_m)_m$ converging to zero as $m \to \infty$ and for each m the operators A_m and A are δ_m-quasi-unitarily equivalent.*

Remark 11.1. In the classical case when $H_m \equiv H$ and $J_m = J'_m = \mathrm{id}_H$ the convergence of the sequence of operators $(A_m)_m$ to A as in (iv) in the P-generalized norm resolvent sense recovers exactly the convergence of $(A_m)_m$ to A in the usual norm resolvent sense, [Pos12, Lemma 4.2.7].

We extend the notion of convergence of non-negative operators to their associated quadratic forms and quote from [Pos12, Def. 4.4.11, Prop.4.4.12], see also [PS18a, Def. 2.1]. To do so, we need also identification operators 'of order 1', respecting the quadratic form domains $\mathcal{D}$ and $\mathcal{D}_m$. Although the assumptions are stronger than the assumptions for δ_m-quasi-unitary equivalence of operators, it is easier for us to deal with the first order domains.

Definition 11.2.

(i) *Let m be fixed. Suppose that we are given bounded linear operators*

$$J_{0,m} : H_m \to H \quad \text{and} \quad \widetilde{J}_{0,m} : H \to H_m \tag{11.2}$$

on the Hilbert spaces H_m, H that are δ_m-quasi-unitary and suppose that we are given bounded linear operators

$$J_{1,m} : \mathcal{D}_m \to \mathcal{D} \quad \text{and} \quad \widetilde{J}_{1,m} : \mathcal{D} \to \mathcal{D}_m \tag{11.3}$$

*on energy form domains $\mathcal{D}_m$, $\mathcal{D}$ and $\delta_m \geq 0$. The operators $J_{1,m}$ and $\widetilde{J}_{1,m}$ are said to be δ_m-*compatible *with the identification operators $J_{0,m}$ and $\widetilde{J}_{0,m}$ if*

$$\|J_{1,m}f - J_{0,m}f\| \leq \delta_m \|f\|_{\mathcal{D}_m} \quad \text{and} \quad \|\widetilde{J}_{1,m}u - \widetilde{J}_{0,m}u\| \leq \|u\|_{\mathcal{D}} \tag{11.4}$$

for all $f \in \mathcal{D}_m$ and for all $u \in \mathcal{D}$.

(ii) Let m be fixed. Suppose that we are given bounded linear operators

$$J_{1,m} : \mathcal{D}_m \to \mathcal{D} \quad \text{and} \quad \widetilde{J}_{1,m} : \mathcal{D} \to \mathcal{D}_m,$$

energy forms $\mathcal{E}_m : \mathcal{D}_m \to \mathcal{D}_m$, $\mathcal{E} : \mathcal{D} \to \mathcal{D}$ and $\delta_m \geq 0$. The energy forms $\mathcal{E}_m$ and $\mathcal{E}$ are said to be δ-close if

$$|\,\mathcal{E}(J_{1,m}f, u) - \mathcal{E}_m(f, \widetilde{J}_{1,m}u)| \leq \|u\|_{\mathcal{D}} \|f\|_{\mathcal{D}_m} \tag{11.5}$$

for all $f \in \mathcal{D}_m$ and for all $u \in \mathcal{D}$.

*(iii) Let m be fixed. Two quadratic forms $\mathcal{E}_m : \mathcal{D}_m \to \mathcal{D}_m$ and $\mathcal{E} : \mathcal{D} \to \mathcal{D}$ are said to be δ_m-*quasi-unitarily equivalent *if there exist operators $J_{0,m}$ and $\widetilde{J}_{0,m}$ as in (11.1), operators $J_{1,m}$ and $\widetilde{J}_{1,m}$ as in* (11.4) *and $\delta_m \geq 0$ so that $J_{0,m}$ and $\widetilde{J}_{0,m}$ are δ_m-quasi-unitary, $J_{1,m}$ and $\widetilde{J}_{1,m}$ are δ_m-compatible and $\mathcal{E}_m$ and $\mathcal{E}$ are as in* (11.5)*.*

(iv) Suppose that $(\mathcal{E}_m)_m$ is a sequence of quadratic forms $\mathcal{E}_m : \mathcal{D}_m \to \mathcal{D}_m$ and that $\mathcal{E} : \mathcal{D} \to \mathcal{D}$ is a quadratic form . We say that the sequence $(\mathcal{E}_m)_m$ converges to $\mathcal{E}$ in the P-generalized norm resolvent sense if there is a sequence of nonnegative numbers $(\delta_m)_m$ converging to zero as $m \to \infty$ and for each m the forms $\mathcal{E}_m$ and $\mathcal{E}$ are δ_m-quasi-unitarily equivalent.

Remark 11.2. Actually, the δ_m-quasi unitarily equivalence of the forms $\mathcal{E}_m$ is equivalent to the δ_m-quasi unitarily equivalence of the associated operators A_m, see [Pos12, Prop. 4.4.15].

11.2 Metric graph approximation of solutions to the heat equation

We specify to quadratic forms on metric graphs approximating a connected p.c.f. self-similar structure in the sense of Kigami, [Kig93a; Kig01].

Throughout this section let $(K, S, \{F_j\}_{j \in S})$ be a post-critically finite, self-similar structure associated with a regular harmonic structure. We will assume that K is connected. Let $(\mathcal{E}, \mathcal{F})$ be a local regular resistance form on K. For the construction of such a resistance form we refer to the p.c.f. part of Section 5.2. In what follows let μ be an atom free Radon measure on K with full support. Then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with $\mathcal{D}(\mathcal{E}) = \mathcal{F}$ is a local regular Dirichlet form on $L^2(K, \mu)$, see [Kig01, Theorem 3.4.6]. We assume μ is such that the associated Markov semigroup $(e^{t\mathcal{L}})_{t>0}$ is conservative.

Let $(\Gamma_m)_{m \geq 0}$ be the sequence of metric graphs $\Gamma_m = (E_m, V_m, i_m, j_m)$ as defined in Subsection 10.4.2. On the space X_{Γ_m}, defined as in (5.1), we consider the bilinear form $(\mathcal{E}_{\Gamma_m}, \dot{W}^{1,2}(X_{\Gamma_m}))$, where

$$\mathcal{E}_{\Gamma_m}(f) := \sum_{w \in W_m} r_w^{-1} \sum_{e \in E_m,\, e \subset K_w} l_e \mathcal{E}_e(f_e) \quad \text{and} \quad \mathcal{E}_e(f_e) = \int_0^{l_e} (f_e'(t))^2 dt.$$

As in Subsection 10.4.2, we denote by $\deg_m(p)$ the degree of the vertex $p \in V_m$ in the graph G_m. Given an edge $e \in E_m$ we set

$$\psi_{e,m}(x) := \frac{1}{\deg_m(i(e))} \psi_{i(e),m}(x) + \frac{1}{\deg_m(j(e))} \psi_{j(e),m}(x), \quad x \in K, \tag{11.6}$$

to obtain a function $\psi_{e,m}$ which satisfies

$$\sum_{e\in E_m} \langle \psi_{e,m}, \mathbf{1}\rangle_{L^2(K,\mu)} = \sum_{p\in V_m} \psi_{p,m}(x) = 1, \quad x \in K. \tag{11.7}$$

We endow the space X_{Γ_m} with the measure μ_{Γ_m} defined as in (5.5) with constants

$$c_e := \frac{1}{l_e}\left(\int_K \psi_{e,m}(x)\mu(dx)\right), \quad e \in E_m,$$

so that $\mu_{\Gamma_m}(dt) := \sum_{e\in E_m} c_e\lambda_e(dt)$. We write $W^{1,2}(X_{\Gamma_m}, \mu_{\Gamma_m})$ for the space $\dot{W}^{1,2}(X_{\Gamma_m}, \mu_{\Gamma_m})$ endowed with the Hilbert norm as in (5.3) and consider the strongly local regular Dirichlet form $(\mathcal{E}_{\Gamma_m}, W^{1,2}(X_{\Gamma_m}, \mu_{\Gamma_m}))$ on $L^2(X_{\Gamma_m}, \mu_{\Gamma_m})$.

Following [Pos12; PS18a], we show that the quadratic forms $\mathcal{E}_{\Gamma_m}$ and $\mathcal{E}$ are δ_m-quasi unitary equivalent on $L^2(X_{\Gamma_m}, \mu_{\Gamma_m})$ and $L^2(K,\mu)$ in the sense of Definition 11.2 for each m. This will imply the P-generalized norm resolvent convergence of the associated operators, see Remark 11.2.

The average of a function $f \in L^2(X_{\Gamma_m}, \mu_{\Gamma_m})$ on an edge $e \in E_m$ we denote by

$$\overline{f_e} := \frac{1}{l_e}\int_0^{l_e} f_e(s)\, ds.$$

We define identification operators $J_{0,m} : L^2(X_{\Gamma_m}, \mu_{\Gamma_m}) \to L^2(K,\mu)$ by

$$J_{0,m}f(x) := \sum_{e\in E_m} \overline{f_e}\, \psi_{e,m}(x), \quad x \in K.$$

Proposition 11.1. *The operators $J_{0,m}$ satisfy $\|J_{0,m}f\|_{L^2(K,\mu)} \le \|f\|_{L^2(X_{\Gamma_m},\mu_{\Gamma_m})}$ for any $f \in L^2(X_{\Gamma_m}, \mu_{\Gamma_m})$. The adjoint $J^*_{0,m} : L^2(K,\mu) \to L^2(X_{\Gamma_m}, \mu_{\Gamma_m})$ of $J_{0,m}$ is given by*

$$J^*_{0,m}u(t) = \sum_{e\in E_m} \mathbf{1}_e(t)\, \frac{\langle u, \psi_{e,m}\rangle_{L^2(K,\mu)}}{\left(\int_K \psi_{e,m}d\mu\right)}, \quad u \in L^2(K,\mu).$$

Proof. By Cauchy-Schwarz and (11.7)

$$\begin{aligned}
\|J_{0,m}f\|^2_{L^2(K,\mu)} &= \int_K \sum_{e,e'\in E_m} \frac{1}{l_e}\frac{1}{l_{e'}} \int_0^{l_e}\int_0^{l_{e'}} f_e(s)f_{e'}(s')\psi_{e,m}(x)\psi_{e',m}(x)dsds'\mu(dx) \\
&\le \frac{1}{2}\sum_{e\in E_m} \frac{1}{l_e}\int_0^{l_e} f_e(s)^2ds\left(\int_K \psi_{e,m}(x)\mu(dx)\right) \\
&\quad + \frac{1}{2}\sum_{e'\in E_m} \frac{1}{l_{e'}}\int_0^{l_{e'}} f_{e'}(s')^2ds'\left(\int_K \psi_{e',m}(x)\mu(dx)\right) \\
&= \|f\|^2_{L^2(X_{\Gamma_m},\mu_{\Gamma_m})}.
\end{aligned}$$

The second statement follows because for any $f \in L^2(X_{\Gamma_m}, \mu_{\Gamma_m})$ and $u \in L^2(K,\mu)$ we have

$$\langle J_{0,m}f, u\rangle_{L^2(K,\mu)} = \sum_{e\in E_m} \frac{1}{l_e}\int_0^{l_e} f_e(s)ds\, \langle \psi_{e,m}, u\rangle_{L^2(K,\mu)}.$$

□

The next spectral convergence statement is a special case of [PS18a, Theorem 1.1]. It can be used to see that in some way the solutions to the heat equations on the approximating spaces X_{Γ_m} converge to the solution to the heat equation on K.

Theorem 11.1. *For any $t > 0$ we have*

$$\lim_{m\to\infty} \left\| e^{t\mathcal{L}} - J_{0,m} e^{t\mathcal{L}_{\Gamma_m}} J_{0,m}^* \right\|_{L^2(K,\mu)\to L^2(K,\mu)} = 0.$$

Theorem 11.1 will follow from the spectral convergence results in [Pos12; PS18a; PS18b]. The validity of the hypotheses of these results are verified in Section A3.2.

We first collect some prerequisites.

Lemma 11.1. *Let $w_0 \in L^2(K,\mu)$. For any $m \geq 1$ let $w_m(t)$ denote the unique solution to (7.4) for $\mathcal{L}_{\Gamma_m}$ in $L^2(X_{\Gamma_m}, \mu_{\Gamma_m})$ with initial condition $J_{0,m}^* w_0$. Then we have $\sup_m \mathcal{E}_{\Gamma_m}(w_m(t)) < +\infty$ for any $t > 0$.*

Proof. There is a constant $c > 0$ independent of m and t such that for any $t > 0$ we have

$$\left\| \sqrt{\mathcal{L}_{\Gamma_m}} e^{t\mathcal{L}_{\Gamma_m}} \right\|_{L^2(X_{\Gamma_m},\mu_{\Gamma_m})\to L^2(X_{\Gamma_m},\mu_{\Gamma_m})} \leq c\, t^{-1/2}, \tag{11.8}$$

as follows from the spectral theorem: Since the metric graphs Γ_m are compact, the operators $\mathcal{L}_{\Gamma_m}$ have pure point spectrum, [Pos12, Proposition 2.2.14]. Consequently the eigenvalues of $-\mathcal{L}_{\Gamma_m}$, ordered with multiplicities taken into account, are $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$ with only accumulation point $+\infty$, and

$$-\mathcal{L}_{\Gamma_m} f = \sum_{k=0}^{\infty} \lambda_k(m) \left\langle \varphi_k(m), f \right\rangle_{L^2(X_{\Gamma_m},\mu_{\Gamma_m})} \varphi_k(m), \quad f \in L^2(X_{\Gamma_m}, \mu_{\Gamma_m}),$$

where $\varphi_k(m)$ are the eigenfunction of $-\mathcal{L}_{\Gamma_m}$ for the eigenvalue $\lambda_k(m)$. This yields

$$\left\| \sqrt{-\mathcal{L}_{\Gamma_m}} e^{t\mathcal{L}_{\Gamma_m}} f \right\|^2_{L^2(X_{\Gamma_m},\mu_{\Gamma_m})} = t^{-1} \sum_{k=0}^{\infty} t\lambda_k(m) e^{-2t\lambda_k(m)} \left| \left\langle \varphi_k(m), f \right\rangle_{L^2(X_{\Gamma_m},\mu_{\Gamma_m})} \right|^2,$$

and since the function $s \mapsto s\, e^{-2s}$ is bounded on $[0, +\infty)$, this implies (11.8). By (11.8),

$$\begin{aligned} \sup_m \mathcal{E}_{\Gamma_m}(w_m(t)) &= \sup_m \left\| \sqrt{-\mathcal{L}_{\Gamma_m}} e^{t\mathcal{L}_{\Gamma_m}} J_{0,m}^* w_0 \right\|^2_{L^2(X_{\Gamma_m},\mu_{\Gamma_m})} \\ &\leq c\, t^{-1/2} \sup_m \left\| J_{0,m}^* w_0 \right\|^2_{L^2(X_{\Gamma_m},\mu_{\Gamma_m})} \\ &\leq c\, t^{-1/2} \left\| w_0 \right\|^2_{L^2(K,\mu)} . \end{aligned}$$

□

Recall that a function f on K is called *m-piecewise harmonic* if it minimizes all energies $\mathcal{E}_n$, $n \geq m+1$, amongst all functions on K which coincide with $f|_{V_m}$ on V_m. If f is m-piecewise harmonic, then it is also n-piecewise harmonic for any $n \geq m$, $f \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(f) = \mathcal{E}_m(f|_{V_m})$. Throughout this chapter we write PH_m for the subspace of all m-piecewise harmonic functions on K. Our notation here differs from the notation used in Chapter 10. There we denote the target space by X instead of K and the space of m-harmonic functions on X by $H_m(X)$ instead of PH_m.

As usual we denote by $\psi_{p,m}$ the function in PH_m satisfying $\psi_{p,m}(q) = \delta_{pq}$, $q \in V_m$. Given a function f on V_m we write $h_m(f) \in \mathcal{D}(\mathcal{E})$ to denote its unique extension to an m-piecewise harmonic function,

$$h_m f(x) := \sum_{p \in V_m} f(p) \psi_{p,m}(x), \quad x \in K. \tag{11.9}$$

We use the symbol H_m to denote the linear operator $H_m : \mathcal{D}(\mathcal{E}) \to PH_m$ defined by $H_m(f) := h_m(f|_{V_m})$, $f \in \mathcal{D}(\mathcal{E})$.

Corollary 11.1. *Let w_0 and $w_m(t)$ be as in Lemma 11.1 and let $w(t)$ be the unique solution to (7.4) for $\mathcal{L}$ in $L^2(K,\mu)$ with initial condition w_0.*

(i) For any $t>0$ we have $\lim_{m\to\infty} H_m(w_m(t)|_{V_m}) = w(t)$ uniformly on K, strongly in $L^2(K,\mu)$ and weakly in $\mathcal{D}(\mathcal{E})$.

(ii) If $w_0 \in \mathcal{D}(\mathcal{E})$ and w_0 is strictly positive on K then for any $t>0$ we also have $\lim_{m\to\infty} H_m(\log w_m(t)|_{V_m}) = \log w(t)$ strongly in $L^2(K,\mu)$ and weakly in $\mathcal{D}(\mathcal{E})$.

Remark 11.3.

(i) Considering $H_m(w_m(t)|_{V_m})$ we implicitely linearize $w_m(t)$ along the edges E_m of Γ_m and compare the resulting function to $w(t)$. Doing so, we discard information, but since we rely on approximation by functions from PH_m anyway, (10.70), it is natural to proceed this way.

(ii) For the special case that μ is the natural self-similar Hausdorff measure on K one can use higher order splines to approximate functions in, roughly speaking, the graph norm of the associated Laplacian, [SU00, Theorem 7.5]. See also [Str00] for related results. It will be a future project to try to combine this with a metric graph approximation scheme to obtain (strong) convergence in $\mathcal{D}(\mathcal{E})$ instead of in $L^2(K,\mu)$.

We prove Corollary 11.1.

Proof. From Lemma A3.2 (i) it follows that

$$\left\|H_m(w_m(t)|_{V_m}) - J_{0,m}w_m(t)\right\|_{L^2(K,\mu)} \le N^6 r_{\max}^m \max_{|w|=m} \mu(K_w)\, \mathcal{E}_{\Gamma_m}(f),$$

and combining with Lemma 11.1 we obtain

$$\lim_{m\to\infty} \left\|H_m(w_m(t)|_{V_m}) - J_{0,m}w_m(t)\right\|_{L^2(K,\mu)} = 0.$$

The $L^2(K,\mu)$ limit relation in Corollary 11.1 (i) now follows from Theorem 11.1. By (10.69) and Lemma 11.1 we also have

$$\sup_m \mathcal{E}(H_m(w_m(t)|_{V_m})) = \sup_m \mathcal{E}_m(w_m(t)) < +\infty.$$

Combining, we obtain

$$\sup_m \|H_m(w_m(t)|_{V_m})\|_{\mathcal{D}(\mathcal{E})} < +\infty. \tag{11.10}$$

Consequently any fixed subsequence of $(H_m(w_m(t)|_{V_m}))_m$ has a further subsequence converging weakly in $\mathcal{D}(\mathcal{E})$, we denote the limit by $\tilde{w} \in \mathcal{D}(\mathcal{E})$. By the Banach-Saks theorem, it has a subsequence whose convex combinations converge strongly to $\tilde{w}$ in $\mathcal{D}(\mathcal{E})$, hence also strongly in $L^2(K,\mu)$, which implies that $\tilde{w}$ must equal $w(t)$. This argument also shows that $(H_m(w_m(t)|_{V_m}))_m$ cannot have any other weak accumulation point than $w(t)$. To see the pointwise convergence note that (3.1) and (11.10) together imply the equicontinuity of $(H_m(w_m(t)|_{V_m}))_m$, and since $\sup_m \|w_m(t)\|_{\sup} \le c \sup_m \|w_m(t)\|_{\mathcal{D}(\mathcal{E})}$ by [Kig01, Lemma 5.2.8], the sequence is also equibounded. Arzela-Ascoli implies that it cannot have a subsequence that does not converge uniformly, and since μ is finite, the only possible limit a subsequence can have is $w(t)$. This shows (i).

To see (ii) suppose that there exists $\gamma>0$ such that $\inf_{x\in K} w_0(x) \ge \gamma$. As $(e^{t\mathcal{L}})_{t>0}$ is conservative and $w(t) \in \mathcal{D}(\mathcal{E})$ continuous we also have $\inf_{x\in K} w(t,x) \ge \gamma$ for any $t \ge 0$. The definition of the operators $J_{0,m}^*$, the conservativity of the semigroups $(e^{t\mathcal{L}_{\Gamma_m}})_{t>0}$ and the continuity of the functions $w_m(t) \in W^{1,2}(X_{\Gamma_m}, \mu_{\Gamma_m})$ imply $\inf_{x\in K} w_m(t,x) \geq \gamma$ for any m and any $t \geq 0$. These lower bounds imply that

$$\mathcal{E}(H_m(\log w_m(t)|_{V_m}) = \mathcal{E}_m(\log w_m(t)) \leq \gamma^{-2}\mathcal{E}_m(w_m(t)|_{V_m}) \leq \sup_m \gamma^{-2}\mathcal{E}_{\Gamma_m}(w_m(t)). \tag{11.11}$$

Now let $\varepsilon > 0$ be arbitrary and m large enough so that

$$\max_{|w|=m} r_w \leq \varepsilon\gamma \left\{ \sup_m \mathcal{E}_{\Gamma_m}(w(t))^{1/2} + \mathcal{E}(w(t))^{1/2} \right\}^{-1}.$$

For any word w with $|w| = m$ and any $x \in K_w$ estimate (5.18) then yields that

$$\begin{aligned}
&|H_m(\log w_m(t)|_{V_m})(x) - \log w(t)(x)| \\
&\qquad \leq |H_m(\log w_m(t)|_{V_m})(x) - H_m(\log w_m(t)|_{V_m})(p)| + |\log w(t)(x) - \log w(t)(p)| \\
&\qquad \leq \operatorname{diam}(K_w) \left\{ \mathcal{E}(H_m(\log w_m(t)|_{V_m}))^{1/2} + \mathcal{E}(w(t))^{1/2} \right\} \\
&\qquad \leq \varepsilon,
\end{aligned}$$

where p is a point from $V_m \cap K_w$. We have used (11.11) and that $H_m(\log w_m(t)|_{V_m})(p) = \log w_m(t)(p)$ for all $p \in V_m$. As a consequence,

$$\|H_m(\log w_m(t)|_{V_m}) - \log w(t)\|^2_{L^2(K,\mu)} = \sum_{|w|=m} \int_{K_w} |H_m(\log w_m(t)|_{V_m}) - \log w(t)|^2 d\mu \leq \varepsilon^2$$

whenever m is sufficiently large. Using (11.11) we can proceed similarly as in (i) to see the weak convergence in $\mathcal{D}(\mathcal{E})$. □

11.3 Metric graph approximation of Cole-Hopf solutions to the Burgers equation

In this section, we present an approximation result for solutions to the Burgers equations, the main result of Chapter 11. We work under the assumptions of Section 11.2.

Recall from Subsection 10.4.2 that we denote by EL_m the subspace of $\dot{W}^{1,2}(X_{\Gamma_m})$ of edge-wise linear functions and by H_{Γ_m} the linear operator $H_{\Gamma_m} : \dot{W}^{1,2}(X_{\Gamma_m}) \to EL_m$ that assigns to a function $f \in \dot{W}^{1,2}(X)$ the unique edge-wise linear function on X_{Γ_m} that interpolates $f|_{V_m}$.

To formulate an approximation result for Cole-Hopf solutions to the Burgers equation (7.8) on K by corresponding solutions to Burgers equations (7.12) on the metric graphs Γ_m we define the operators H_{Γ_m} and H_m on gradient fields. Since for any $c \in \mathbb{R}$ we have

$$H_{\Gamma_m}(f + c) = H_{\Gamma_m}(f) + c, \qquad f \in \dot{W}^{1,2}(X_{\Gamma_m}),$$

and

$$H_m(f + c) = H_m(f) + c, \qquad f \in \mathcal{D}(\mathcal{E}),$$

these operators may be interpreted as linear operators on $\dot{W}^{1,2}(X_{\Gamma_m})/\mathbb{R}$ and $\mathcal{D}(\mathcal{E})/\mathbb{R}$, respectively. According to Remarks 5.1 and 4.3 these spaces are isometrically isomorphic to $\operatorname{Im} d$ and $\operatorname{Im} \partial$, respectively, so that we obtain well-defined operators $H_{\Gamma_m} : \operatorname{Im} d \to d(EL_m)$ and $H_m : \operatorname{Im} \partial \to \partial(PH_m)$ by setting

$$H_{\Gamma_m}(df) := dH_{\Gamma_m}(f) \quad \text{ and } \quad H_m(\partial f) := \partial H_m(f).$$

$$\begin{array}{ccc} \mathcal{D}(\mathcal{E})/\mathbb{R} \cong \operatorname{Im}\partial & & \operatorname{Im} d \cong \dot{W}^{1,2}(X_{\Gamma_m})/\mathbb{R} \\ \Big\downarrow{\scriptstyle H_m} & & \Big\downarrow{\scriptstyle H_{\Gamma_m}} \\ \partial(PH_m) & \underset{\mathfrak{E}_m}{\overset{\mathfrak{E}_m^{-1}}{\rightleftarrows}} & d(EL_m) \end{array}$$

Figure 11.1: Vector space isomorphism $\mathfrak{E}_m$ and its inverse

Moreover, for any m we can define a vector space isomorphism $\mathfrak{E}_m : EL_m \to PH_m$ by $\mathfrak{E}_m(f) := H_m(f|_{V_m})$, its inverse $\mathfrak{E}_m^{-1}$ is given by $f|_{X_{\Gamma_m}}$. It satisfies $\mathfrak{E}_m(f+c) = \mathfrak{E}_m(f)+c$, $c \in \mathbb{R}$, and therefore also induces a well defined linear map $\mathfrak{E}_m : d(EL_m) \to \partial(PH_m)$ by

$$\mathfrak{E}_m(df) := \partial\mathfrak{E}_m(f), \quad f \in EL_m.$$

Since

$$\|\partial\mathfrak{E}_m(f)\|_{\mathcal{H}}^2 = \mathcal{E}(H_m(f|_{V_m})) = \mathcal{E}_{\Gamma_m}(f) = \|df\|_{L^2(X_{\Gamma_m},\mu_{\Gamma_m})}^2$$

for any $f \in \dot{W}^{1,2}(X_{\Gamma_m})$, the map $\mathfrak{E}_m$ is seen to be an isometric isomorphism, see also Figure 11.1.

We are now ready to prove one of our main results in this thesis.

Theorem 11.2. *Assume $u_0 = \partial h_0$ with $h_0 \in \mathcal{D}(\mathcal{E})$. Let $u(t)$ denote the unique solution to (7.12) with initial condition u_0 and for any $m \geq 1$ let $u_m(t)$ denote the unique solution to (7.8) with initial condition $-2d\log J_{0,m}^* e^{-h_0/2}$. Then we have*

$$\lim_{m\to\infty} \langle \mathfrak{E}_m \circ H_{\Gamma_m}(u_m(t)) - u(t), v\rangle_{\mathcal{H}} = 0 \tag{11.12}$$

for any $t \geq 0$ and $v \in \mathcal{H}$.

Considering $\mathfrak{E}_m \circ H_{\Gamma_m}(u_m(t))$ we implicitly linearize $u_m(t)$ along the edges E_m of metric graphs Γ_m and compare the resulting function to $u(t)$. One can conclude from Theorem 11.2 that the abstract Burgers equation (7.12) can be seen as a natural limit of more familiar equations (7.8) on metric graphs.

Proof. For any $m \geq 1$ we have

$$\begin{aligned}\mathfrak{E}_m \circ H_{\Gamma_m}(u_m(t)) &= -2\mathfrak{E}_m(d(H_{\Gamma_m}(\log w_m(t))) \\ &= -2\partial\mathfrak{E}_m(H_{\Gamma_m}(\log(w_m(t))) = -2\partial H_m(\log(w_m(t)|_{V_m}),\end{aligned}$$

and according to Corollary 11.1 (ii),

$$\lim_{m\to\infty} \langle \partial H_m(\log w_m(t)|_{V_m}) - \partial \log w(t), \partial\varphi\rangle_{\mathcal{H}} = \lim_{m\to\infty} \mathcal{E}(H_m(\log w_m(t)|_{V_m}) - \log w(t), \varphi) = 0$$

for any $\varphi \in \mathcal{D}(\mathcal{E})$. Since $\mathfrak{E}_m \circ H_{\Gamma_m}(u_m(t))$ and $u(t)$ are elements of $\operatorname{Im}\partial$, it follows from the orthogonal Helmholtz-Hodge type decomposition in (4.17) that we may use general test vector fields $v \in \mathcal{H}$ in place of $\partial\varphi$. □

Remark 11.4. The space $\mathcal{H}$ can be rewritten as the closure of the union of an increasing sequence of finite dimensional subspaces, [IRT12, Definition 5.2, Lemmas 5.3 and 5.5 and Theorem 5.6]. Then (11.12) can also be expressed using these subspaces.

Chapter 12

Discrete graph approximation for continuity equations on finitely ramified spaces

Using the concept of vanishing viscosity, we proved in Chapter 8 that a sequence of solutions $(u_n(t))_n$ to the viscous continuity equations with diffusion parameter σ_n converges weakly to a solution u to the continuity equation on fixed space X.

In this chapter, we combine the vanishing diffusion argument with the concept of KS-generalized Mosco convergence as introduced in Chapter 10. Using in addition a diagonal compactness argument, we show that a solution u to the continuity equation (8.1) on a finitely ramified space X can be approximated in a suitable weak sense by a sequence $\left(u_n^{(m)}\right)_{m,n}$ of solutions to the viscous continuity equation (8.3) with viscosity parameter σ_n on graphs $X^{(m)}$ approximating X.

The results of this chapter are experimental and based on joint work in progress [HMS20].

In what follows let the target space X be a finitely ramified set with finitely ramified cell structure and equipped with a local resistance form $(\mathcal{E}, \mathcal{F})$. Further, let Assumption 5.1 be in force. Let (X, R) be metrically doubling with doubling constant $K_R > 1$ and let μ be a finite Borel measure on (X, R) which admits a uniform lower bound V.

We discuss the case where the approximating spaces $X^{(m)}$ are finite point sets. Let $X^{(m)} := V_m$. We consider the resistance forms $\mathcal{E}^{(m)} := \mathcal{E}_{V_m}$ of the form (3.4) with domains $\mathcal{F}^{(m)} = \ell(V_m)$, respectively. Recall from Section 10.4.1 that also the spaces $(V_m, R^{(m)})$ are metrically doubling with doubling constant K_R. Moreover, let $\mu^{(m)}$ on V_m be defined as in (10.60), i.e.

$$\mu^{(m)}(\{p\}) := \int_X \psi_{p,m}(x) d\mu(x), \quad p \in V_m,$$

where $\psi_{p,m}$ is the (unique) harmonic extension to X of the function $\mathbf{1}_{\{p\}}$ on V_m.

12.1 Convergence in the sense of Kuwae and Shioya

It follows from Subsection 10.4.1 that the assumptions 10.1, 10.2 and 10.3 are satisfied, note that the additional Assumption 10.4 is satisfied by Remark 10.8. Using them, we obtain convergence of Bochner spaces $L^2\left((0,T), \ell^2(V_m, \mu^{(m)})\right)$ and $L^2\left((0,T), \ell(V_m)\right)$ in the sense of Definition 10.1.

Lemma 12.1.

(i) For each $m \geq 1$, let Φ_m be the identification operator $\Phi_m : L^2(X,\mu) \to \ell^2(V_m, \mu^{(m)})$ defined as in (10.61)*. Then we have*

$$\lim_{m\to\infty} L^2\left((0,T), \ell^2(V_m,\mu^{(m)})\right) = L^2\left((0,T), L^2(X,\mu)\right) \tag{12.1}$$

in the KS-sense with identification operators $\varphi(t)\cdot\psi \mapsto \varphi(t)\cdot\Phi_m(\psi)$ mapping from $C([0,T])\otimes L^2(X,\mu)$ into $C([0,T])\otimes \ell^2(V_m,\mu^{(m)})$ respectively.

(ii) We have

$$\lim_{m\to\infty} L^2\left((0,T), \ell(V_m)\right) = L^2\left((0,T), \mathcal{F}\right) \tag{12.2}$$

in the KS-sense with identification operators $\varphi(t)\cdot\psi \mapsto \varphi(t)\cdot(H_m\psi)|_{V_m}$ mapping from $C([0,T])\otimes\mathcal{F}$ into $C([0,T])\otimes\ell(V_m)$ respectively.

(iii) If $f(t) = f_1(t)\cdot f_2 \in L^2\left((0,T),\mathcal{F}\right)$ and $(f_m(t))_m$ is a sequence of functions $f_m(t) = f_1(t)\cdot f_{m,2} \in L^2\left((0,T),\ell(V_m)\right)$ such that $\lim_{m\to\infty} f_m(t) = f(t)$ KS-strongly w.r.t. (12.2) then we also have $\lim_{m\to\infty} f_m(t) = f(t)$ KS-strongly w.r.t. (12.1).

The proof is quite similar to the proof of Corollary 10.5.

Proof. Let $u(t) = u_1(t)\cdot u_2$ be in $C([0,T])\otimes L^2(X,\mu)$. In Subsection 10.4.1, we have shown that the operator Φ_m,

$$\Phi_m f(p) := \frac{1}{\mu^{(m)}(p)} \langle f, \psi_{p,m}\rangle_{L^2(X,\mu)}, \quad p\in V_m, \quad f\in L^2(X,\mu),$$

satisfies all conditions in Assumption 10.3, for proofs we refer to Subsection 10.4.1. Together with the monotone convergence theorem we obtain

$$\begin{aligned}
\lim_{m\to\infty} \|u_1(t)\Phi_m(u_2)\|^2_{L^2\left((0,T),\ell^2(V_m,\mu^{(m)})\right)} &= \lim_{m\to\infty}\int_0^T \|u_1(t)\cdot\Phi_m(u_2)\|^2_{\ell^2(V_m,\mu^{(m)})}dt \\
&= \int_0^T |u_1(t)|^2 \lim_{m\to\infty}\|\Phi_m(u_2)\|^2_{\ell^2(V_m,\mu^{(m)})}dt \\
&= \int_0^T |u_1(t)|^2 \int_X |u_2|^2 d\mu\, dt \\
&= \|u_1(t)\cdot u_2\|^2_{L^2((0,T),L^2(X,\mu))}.
\end{aligned}$$

and statement (i) follows.

To see statement (ii) let $u = u_1(t)\cdot u_2 \in C([0,T])\otimes\mathcal{F}$. If $x_0\in V_0$ is fixed, we have $H_m u_2(x_0) = u_2(x_0)$ for any m and therefore, by (3.1) and (10.15),

$$\begin{aligned}
\lim_{m\to\infty}\|u_2 - H_m u_2\|^2_{L^2(X,\mu)} &\leq \mu(X)\lim_{m\to\infty}\|u_2 - H_m u_2\|^2_{\sup} \\
&\leq \mu(X)\operatorname{diam}(X)\lim_{m\to\infty}\mathcal{E}(u_2 - H_m u_2) = 0.
\end{aligned}$$

Using (10.19), we obtain

$$\lim_{m\to\infty}\|u_1(t)\left(\Phi_m(H_m u_2) - \Phi_m(u_2)\right)\|_{L^2(0,T),L^2(X,\mu))} = 0,$$

and combining with (10.22) and (10.20),

$$\begin{aligned}
&\lim_{m\to\infty}\|u_1(t)\cdot(H_m u_2)|_{V_m}\|_{L^2\left((0,T),\ell^2(V_m,\mu^{(m)})\right)} \\
&= \lim_{m\to\infty}\|u_1(t)\cdot((H_m u_2)|_{V_m} - \Phi_m(H_m u_2))\|_{L^2\left((0,T),\ell^2(V_m,\mu^{(m)})\right)} \\
&+ \lim_{m\to\infty}\|u_1(t)\cdot\Phi_m(H_m u_2)\|_{L^2\left((0,T),\ell^2(V_m,\mu^{(m)})\right)} \\
&= \lim_{m\to\infty}\|u_1(t)\cdot\Phi_m(u_2)\|_{L^2\left((0,T),\ell^2(V_m,\mu^{(m)})\right)} \\
&= \|u_1(t)\cdot u_2\|_{L^2(0,T),L^2(X,\mu))}.
\end{aligned}$$

Together with (10.14) this shows that $\lim_{m\to\infty}\mathcal{E}_{V_m,1}(u_1(t)\cdot(H_m u_2)|_{V_m})=\mathcal{E}_1(u_1(t)\cdot u_2)$ for all $u(t)=u_1(t)\cdot u_2\in\mathcal{F}$.

To see (iii) note that according to the hypothesis, there exists some $\varphi(t)=\varphi_1(t)\cdot\varphi_{n,2}\in C([0,T])\otimes\mathcal{F}$ such that

$$\lim_{n\to\infty}\int_0^T \mathcal{E}_1\left(\varphi_1(t)\cdot\varphi_{n,2}-f_1(t)\cdot f_2\right)dt=0$$

and

$$\lim_{n\to\infty}\varlimsup_{m\to\infty}\int_0^T \mathcal{E}_{V_m,1}\left(\varphi_1(t)\cdot(H_m\varphi_n)|_{V_m}-f_1(t)\cdot f_{m,2}\right)dt=0.$$

This implies

$$\lim_{n\to\infty}\int_0^T\int_X |\varphi_1(t)\cdot\varphi_{n,2}-f_1(t)\cdot f_2|^2\,d\mu dt=0$$

and

$$\lim_{n\to\infty}\varlimsup_{m\to\infty}\int_0^T \|\varphi_1(t)\cdot(H_m\varphi_{n,2})|_{V_m}-f_1(t)\cdot f_{m,2}\|^2_{\ell^2(V_m,\mu^{(m)})}\,dt=0.$$

Conditions (10.20) and (10.22), applied to the constant function $\mathbf{1}$, yield $\lim_{m\to\infty}\mu^{(m)}(V_m)=\mu(X)$, and in particular,

$$\sup_m \mu(V_m)<+\infty.$$

We may therefore use (10.17) to conclude

$$\lim_{m\to\infty}\|\varphi_1(t)\left((H_m\varphi_{n,2})|_{V_m}-\Phi_m(H_m\varphi_{n,2})\right)\|_{L^2\left((0,T),\ell^2(V_m,\mu^{(m)})\right)}=0$$

for any n, so that

$$\lim_{n\to\infty}\varlimsup_{m\to\infty}\|\varphi_1(t)\cdot\Phi_m(H_m\varphi_{n,2})-f_1(t)\cdot f_{m,2}\|_{L^2\left((0,T),\ell^2(V_m,\mu^{(m)})\right)}=0.$$

Let $x_0\in V_0$. Then, since $\varphi_1(t)\cdot H_m\varphi_{n,2}(x_0)=\varphi_1(t)\cdot\varphi(x_0)$ for all m and n, the resistance estimate (3.1) implies $\varphi_1(t)\lim_{m\to\infty}\|H_m\varphi_{n,2}-\varphi_{n,2}\|_{L^2(X,\mu)}=0$ for all n. Together with (10.19) and the monotone convergence theorem it follows that

$$\begin{aligned}\lim_{n\to\infty}\varlimsup_{m\to\infty}&\|\varphi_1(t)\left(\Phi_m(H_m\varphi_{n,2})-\Phi_m(\varphi_{n,2})\right)\|_{L^2\left((0,T),\ell^2(V_m,\mu^{(m)})\right)}\\ &\le\sup_m\|\Phi_m\|_{L^2(X,\mu)\to\ell^2(V_m,\mu^{(m)})}\lim_{n\to\infty}\varlimsup_{m\to\infty}\|\varphi_1(t)\left(H_m\varphi_{n,2}-\varphi_{n,2}\right)\|_{L^2\left((0,T),\ell^2(V_m,\mu^{(m)})\right)}\\ &=0,\end{aligned}$$

what entails $\lim_{n\to\infty}\varlimsup_{m\to\infty}\|\varphi_1(t)\cdot\Phi_m(\varphi_{n,2})-f_1(t)\cdot f_{m,2}\|_{L^2\left((0,T),\ell^2(V_m,\mu^{(m)})\right)}=0.$ □

Recall that, given $f\in H_n(X)$ and $m\ge n$, we can define the restriction of ∂f to V_m by

$$(\partial f)|_{V_m}:=\partial^{(m)}(f|_{V_m}).\tag{12.3}$$

12.2 Choice of vector fields

Since our main interest is a first approximation of continuity equations on a finitely ramified set X, it seems convenient to restrict our attention to vector fields b on X and $b^{(m)}$ on V_m suitable to allow an approximation procedure. For simplicity we consider time-independent vector fields b. To be able to achieve results from Chapter 8 uniformly in m the vector fields $b^{(m)}$ should satisfy the condition

$$\sup_m\|\partial^* b^{(m)}\|_{\ell^\infty(V_m,\mu^{(m)})}<\infty.\tag{12.4}$$

Therefore we only consider vector fields that are m-harmonic 1-forms. With this in hand, generalizations of the application of Lions-Lax-Milgram Lemma (8.1) and of the *a priori* estimates (Theorem 8.2) shown in Chapter 8 can be obtained easily.

Since X is assumed to be a finitely ramified set with finitely ramified cell structure we can also fix a suitable vector field b on X and obtain approximating vector fields $b^{(m)}$ on V_m by a well defined restriction operation. As in Section 10.3, we follow [IRT12] and define subspaces $\mathcal{H}_m$ of $\mathcal{H}$ by

$$\mathcal{H}_m := \left\{ \sum_{\alpha \in \mathcal{A}_m} \mathbf{1}_{X_\alpha} \partial h_\alpha : \quad h_\alpha \in H_m(X) \text{ for all } \alpha \in \mathcal{A}_m \right\}.$$

From Definition 5.1 it follows that $\mathcal{H}_m \subset \mathcal{H}_{m+1}$ for all m, see [IRT12, Lemma 5.3] for a proof. For a particular element $\sum_{\alpha \in \mathcal{A}_m} \mathbf{1}_{X_\alpha} \partial h_\alpha$ of $\mathcal{H}_m$ we have

$$\left\| \sum_{\alpha \in \mathcal{A}_m} \mathbf{1}_{X_\alpha} \partial h_\alpha \right\|_{\mathcal{H}}^2 = \sum_{\alpha \in \mathcal{A}_m} \mathcal{E}_\alpha(h_\alpha, h_\alpha), \tag{12.5}$$

[IRT12, Theorem 5.4]. Moreover, $\bigcup_{m \geq 0} \mathcal{H}_m$ is dense in $\mathcal{H}$.

To generalize this we also defined a pointwise restriction of elements of $\mathcal{H}_m$ to V_m by

$$\left(\sum_{\alpha \in \mathcal{A}_m} \mathbf{1}_{X_\alpha} \partial h_\alpha \right)|_{V_m} := \sum_{\alpha \in \mathcal{A}_m} \mathbf{1}_{X_\alpha \cap V_m} \partial^{(m)}(h_\alpha|_{V_m}), \tag{12.6}$$

and clearly this restriction operation maps $\mathcal{H}_m$ into $\mathcal{H}^{(m)}$.

In this section we need even more structure as introduced in the work [IRT12] by Ionesco, Rogers and Teplyaev. They also introduced the subspace $\mathcal{P}^\perp \mathcal{H}_m \subset \mathcal{H}_m$ as the *space of m-harmonic 1-forms* via

$$\begin{aligned} \mathcal{P}^\perp \mathcal{H}_m = \{ & \sum_{\alpha \in \mathcal{A}_m} \mathbf{1}_{X_\alpha} \partial h_\alpha : \text{the } h_\alpha \text{ are } m\text{-harmonic and the values of the} \\ & \text{normal derivatives } d_n h_\alpha \text{ sum to zero at every vertex in } V_m \}, \end{aligned}$$

where the normal derivatives d_n are defined in [Kig03, Thms. 6.6 and 6.8]. Note that the restriction operation in (12.6) also maps $\mathcal{P}^\perp \mathcal{H}_m$ into a subspace of $\mathcal{H}^{(m)}$.

Lemma 12.2. *Let $b \in \mathcal{P}^\perp \mathcal{H}_m$, where $\mathcal{P}^\perp \mathcal{H}_m$ denotes the space of m-harmonic 1-forms defined as in [IRT12, Thm.5.6, Cor.5.7]. Then it follows for each $m \geq 1$ that $b^{(m)} \in$ $\ker(\partial^*)^{(m)}$, where $b^{(m)}$ is of the form*

$$b^{(m)} := \sum_{\alpha \in \mathcal{A}_m} \mathbb{1}_{X_\alpha \cap V_m} \partial^{(m)} \left(h_\alpha|_{V_m} \right). \tag{12.7}$$

Proof. Let $b \in \mathcal{P}^\perp \mathcal{H}_m$. Then b is of the form $b = \sum_{\alpha \in \mathcal{A}_m} \mathbb{1}_{X_\alpha} \partial h_\alpha$, where $h_\alpha \in H_m(X)$ and it holds that $\mathcal{P}^\perp \mathcal{H}_m \subset \mathcal{P}^\perp \mathcal{H}$, see [IRT12, Theorem 5.6, Corollary 5.7]. Moreover, for every function $\phi \in H_n(X)$, $m \geq n$, it holds that

$$0 = \langle \partial \phi, b \rangle_{\mathcal{H}}.$$

Using polarization and (12.5) we obtain

$$\langle \partial \phi, b \rangle_{\mathcal{H}} = \langle \partial \phi, \sum_{\alpha \in \mathcal{A}_m} \mathbb{1}_{X_\alpha} \partial h_\alpha \rangle_{\mathcal{H}} = \sum_{\alpha \in \mathcal{A}_m} \mathcal{E}_\alpha \left(\phi, h_\alpha \right)$$

and definition of $\mathcal{E}_\alpha$ as in [IRT12, Sec. 2] yields that

$$\sum_{\alpha\in\mathcal{A}_m}\mathcal{E}_\alpha(\phi,h_\alpha)=\sum_{\alpha\in\mathcal{A}_m}\sum_{x,y\in V_\alpha}c_{xy}\left(\phi(x)-\phi(y)\right)\left(h_\alpha(x)-h_\alpha(y)\right).$$

Note that $V_\alpha = V_m \cap V_\alpha$ because V_m is just $\bigcup_{m\geq 0} V_\alpha$. Therefore the right hand side can be rewritten to

$$\sum_{\alpha\in\mathcal{A}_m}\sum_{x,y\in V_\alpha}c_{xy}\mathbb{1}_{V_m}(x)\mathbb{1}_{V_m}(y)\left(\phi(x)-\phi(y)\right)\left(h_\alpha(x)-h_\alpha(y)\right)=\sum_{\alpha\in\mathcal{A}_m}\mathcal{E}_\alpha\left(\phi_{|V_m},h_{\alpha_{|V_m}}\right)$$

and again by polarization and (12.5) it holds that

$$\begin{aligned}\sum_{\alpha\in\mathcal{A}_m}\mathcal{E}_\alpha\left(\phi_{|V_m},h_{\alpha_{|V_m}}\right)&=\langle\partial^{(m)}\left(\phi_{|V_m}\right),\sum_{\alpha\in\mathcal{A}_m}\mathbb{1}_{X_\alpha\cap V_m}\partial^{(m)}\left(h_{\alpha_{|V_m}}\right)\rangle_{\mathcal{H}^{(m)}}\\&=\langle(\partial\phi)^{(m)},b^{(m)}\rangle_{\mathcal{H}^{(m)}}.\end{aligned}$$

Combining, we obtain

$$\langle(\partial\phi)^{(m)},b^{(m)}\rangle_{\mathcal{H}^{(m)}}=0.$$

□

Remark 12.1.

(i) An even simpler choice for the vector field would be $b=\partial h$ such that the function h satisfies $h\in\mathcal{D}(\mathcal{L})$, where $\mathcal{L}$ is defined via $\mathcal{L}=-\partial^*\partial$.

(ii) If $b\in\mathcal{P}^\perp\mathcal{H}_m$ is absolutely continuous with respect to the volume measure μ, see Definition 8.2, and for each m $b^{(m)}$ is of the form (12.7), then $b^{(m)}$ is absolutely continuous w.r.t. $\mu^{(m)}$.

Lemma 12.3. *Suppose that $b\in\mathcal{P}^\perp\mathcal{H}_n$ and let $b^{(m)}$ be as in (12.7). Further, suppose that $\left(f^{(m)}(t)\right)_m$ is a sequence with $f^{(m)}(t)\in L^2((0,T),\ell^2(V_m,\mu^{(m)}))$ converging to $f(t)\in L^2((0,T),L^2(X,\mu))$ KS-weakly. Then for every $g(t)\in C[0,T]\otimes\bigcup H_n(X)$ and every sequence $\left(g^{(m)}(t)\right)_m$ such that $g(t):=g_1(t)\cdot g_2$, $g_1(t)\in C[0,T]$, $g_2\in\bigcup H_n(X)$, $g^{(m)}(t):=g_1(t)\cdot\left(g_{2|V_m}\right)\in C([0,T])\otimes\ell(V_m)$ we have that $\lim_{m\to\infty}g^{(m)}(t)=g(t)$ KS-strongly w.r.t. to* (12.2) *and*

$$\lim_{m\to\infty}\int_0^T g_1(t)\langle f^{(m)}(t)\cdot b^{(m)},\partial^{(m)}g_2^{(m)}\rangle_{\mathcal{H}^{(m)}}dt=\int_0^T g_1(t)\langle f(t)\cdot b,\partial g_2\rangle_{\mathcal{H}}dt. \tag{12.8}$$

Proof. The desired convergence is a consequence of the construction of $g(t)$ and $g^{(m)}(t)$. The identity (12.8) follows from the fact that $b^{(m)}$ is absolutely continuous w.r.t. $\mu^{(m)}$, monotone convergence theorem and by linearity from the fact that by Lemma 10.3 (i) and Corollary 10.7 we have

$$g_1(t)\lim_{m\to\infty}\left\langle(f^{(m)}(t)\mathbf{1}_{X_\alpha})\cdot\partial^{(m)}\left(h_{\alpha_{|V_m}}\right),\partial^{(m)}g_2^{(m)}\right\rangle_{\mathcal{H}^{(m)}}=g_1(t)\left\langle(f(t)\mathbf{1}_{X_\alpha})\cdot\partial\left(h_\alpha\right),\partial g_2\right\rangle_{\mathcal{H}}.$$

□

12.3 Uniform bounds

The next corollary is a generalization of Theorem 8.1.

Corollary 12.1. *Let $u_0 \in L^2(X,\mu)$ and let $b \in \mathcal{P}^\perp\mathcal{H}_m$. For each $m \geq 1$ let $b^{(m)}$ be the pointwise restriction of b to V_m as in (12.6).*

(i) For every $\sigma_n \in (0, \frac{1}{2}]$, $n \in \mathbb{N}$, and any $m \geq 0$ there exists a weak solution $u_n^{(m)}(t) \in L^2((0,T);\ell(V_m))$ to the Cauchy problem

$$\begin{cases} \partial_t u_n^{(m)}(t) & = -\sigma_n \mathcal{L}^{(m)} u_n^{(m)}(t) + (\partial^*)^{(m)} \left(u_n^{(m)}(t) \cdot b^{(m)} \right), \qquad t \in (0,T), \\ u_n^{(m)}(0) & = u_0^{(m)} \end{cases} \tag{12.9}$$

with initial condition $u_0^{(m)} = \Phi_m u_0 \in \ell^2(V_m, \mu^{(m)})$ and such that

$$\|e^{-\sigma_n t} u_n^{(m)}(t)\|_{L^2((0,T);\ell(V_m))} \leq \frac{1}{\sigma_n} \|u_0^{(m)}\|_{\ell^2(V_m,\mu^{(m)})}.$$

(ii) Moreover, we have the uniform estimate

$$\sup_m \|e^{-\sigma_n t} u_n^{(m)}(t)\|_{L^2((0,T);\ell(V_m))} \leq \frac{1}{\sigma_n} \|u_0\|_{L^2(X,\mu)}. \tag{12.10}$$

Proof. The proof of (i) follows the same arguments as that of Theorem 8.1.

Boundedness of Φ_m leads to the stronger estimate

$$\sup_m \|e^{-\sigma_n t} u_n^{(m)}\|_{L^2(I,\mathcal{D}(\mathcal{E}^{(m)}))} \leq \frac{1}{\sigma_n} \sup_m \|u_0^{(m)}\|_{L^2(X^{(m)},\mu^{(m)})} \leq \frac{1}{\sigma_n} \|u_0\|_{L^2(X,\mu)},$$

what shows (ii). □

As a consequence of the a-priori estimates stated in Theorem 8.2 we obtain the following. Let $I = [0,T]$.

Corollary 12.2. *Let $u_0 \in L^2(X,\mu)$ and let $b \in \mathcal{P}^\perp\mathcal{H}_m$. For each $m \geq 1$ let $b^{(m)}$ be the pointwise restriction of b to V_m as in (12.6).*

(i) For any $m \geq 0$ there exists a weak solution

$$u_n^{(m)}(t) \in L^2((0,T);\ell(V_m) \cap L^\infty \left((0,T); \ell^2(V_m, \mu^{(m)}) \right)$$

to (12.9) with initial condition $u_0^{(m)} = \Phi_m u_0 \in \ell^2(V_m, \mu^{(m)})$ and such that

$$\operatorname*{ess\,sup}_{t \in I} \|u(t)_n^{(m),\pm}(t)\|_{\ell^2(V_m,\mu^{(m)})} \leq \|u_0^{(m),\pm}\|_{\ell^2(V_m,\mu^{(m)})}.$$

(ii) We have the uniform bound

$$\sup_m \operatorname*{ess\,sup}_{t \in I} \|u_n^{(m)}(t)\|_{\ell^2(V_m,\mu^{(m)})} < \infty. \tag{12.11}$$

Proof. The proof of (i) follows the same arguments as that of Theorem 8.2. By Lemma 12.2 it holds that $b^{(m)} \in \ker(\partial^*)^{(m)}$ for each $m \geq 0$ and therefore the calculation in the proof of Lemma 8.2 is even simplified. Using again the boundedness of Φ_m we obtain

$$\begin{aligned} \sup_m \operatorname*{ess\,sup}_{t \in I} \|u_n^{(m),\pm}(t)\|_{L^2(X^{(m)},\mu^{(m)})} &\leq \|u_0^{(m),\pm}\|_{L^2(X^{(m)},\mu^{(m)})} \\ &\leq \sup_m \|\Phi_m\|_{L^2(X,\mu)\to L^2(X^{(m)},\mu^{(m)})} \|u_0^\pm\|_{L^2(X,\mu)} < \infty, \end{aligned}$$

what shows (ii). □

In fact the solutions $u_n^{(m)}(t)$ satisfy

$$\begin{aligned}\sup_n \sup_m \|u_n^{(m)}(t)\|_{L^2((0,T),\ell(V_m))} &\leq \frac{1}{\sigma_n}\|u_0\|_{L^2(X,\mu)},\\ \sup_n \sup_m \|u_n^{(m)}(t)\|_{L^\infty((0,T),\ell^2(V_m,\mu^{(m)}))} &< \infty\end{aligned} \tag{12.12}$$

and for the special case that $n(m) = m$ we obtain

$$\begin{aligned}\sup_m \|\frac{u_m^{(m)}(t)}{m}\|_{L^2((0,T),\ell(V_m))} &\leq \|u_0\|_{L^2(X,\mu)},\\ \sup_m \|u_m^{(m)}(t)\|_{L^\infty((0,T),\ell^2(V_m,\mu^{(m)}))} &< \infty.\end{aligned} \tag{12.13}$$

Finite $T > 0$ and (12.13) yield that

$$\sup_m \|u_m^{(m)}(t)\|_{L^2((0,T),\ell^2(V_m,\mu^{(m)}))} \leq T^{\frac{1}{2}} \sup_m \|u_m^{(m)}(t)\|_{L^\infty((0,T),\ell^2(V_m,\mu^{(m)}))} < \infty. \tag{12.14}$$

12.4 Accumulation point along a subsequence to the solution of the continuity equation

Let $\mathcal{A}$ be the space of test functions defined by

$$\mathcal{A} := \big(C([0,T]) \cap C^1((0,T))\big) \otimes \bigcup_{n\geq 0} H_n(X).$$

The next Theorem 12.1 shows, given the special case that $n = m$, that the solutions $u_m^{(m)}(t)$ to the continuity equations with diffusion on approximating graphs converge along a subsequence to the solution to the continuity equation on a finitely ramified set X. This might be regarded as a first piece of evidence that our proposed formulation of the continuity equation is physically meaningful.

We obtain the following new result.

Theorem 12.1. *Let $u_0 \in L^2(X,\mu)$ and let $b \in \mathcal{P}^\perp \mathcal{H}_m$ be absolutely continuous w.r.t. μ. For each $m \geq 1$ let $b^{(m)}$ be the pointwise restriction of b to V_m as in* (12.6). *Moreover, for any $m \geq 1$ let $u_m^{(m)}(t)$ denote the weak solution to* (12.9) *with $\sigma_m = \frac{1}{m}$ and initial condition $\Phi_m u_0$ and let $u(t)$ be the weak solution to* (8.1) *with initial condition u_0. Then there exists a sequence $(m_k)_k$ with $m_k \uparrow \infty$ such that the subsequence $\left(u_{m_k}^{(m_k)}(t)\right)_k$ converges weakly to $u(t)$.*

Proof. Suppose that $u_m^{(m)}(t)$ is a weak solution to (12.9) with $\sigma = \frac{1}{m}$ for every $m \geq 0$.

By Corollary 12.1 we know that $\frac{u_m^{(m)}(t)}{m}$ is bounded in $L^2((0,T),\ell(V_m))$ for every m. Hence $\left(\frac{u_m^{(m)}(t)}{m}\right)_m$ has a subsequence $\left(\frac{u_{m_k}^{(m_k)}(t)}{m_k}\right)_k$ that converges KS-weakly to some limit $u_\varepsilon(t) \in L^2((0,T),\mathcal{F})$, i.e.

$$\lim_{k\to\infty} \int_0^T \mathcal{E}_{V_{m_k},1}\left(\frac{u_{m_k}^{(m_k)}(t)}{m_k}, w_k(t)\right) dt = \int_0^T \mathcal{E}_1\left(u_\varepsilon(t), w_k(t)\right) dt \tag{12.15}$$

for every sequence $(w_k(t))_k$, $w_k(t) \in L^2((0,t),\ell(V_m))$ KS-strongly convergent to $w(t) \in L^2((0,T),\mathcal{F})$ w.r.t. (12.2). Note that by Lemma 12.1 (iii) these sequences are also KS-strongly convergent w.r.t. (12.1). By Corollary 12.2 and (12.14) we know that $\left(u_m^{(m)}(t)\right)_m$

is bounded in $L^2((0,T),\ell^2(V_m,\mu^{(m)}))$ for every m. Hence we can extract a subsequence $(u_{m_k}^{(m_k)})_{m_k}$, w.l.o.g. the same subsequence as above, converging KS-weakly w.r.t. (12.1) to some limit $u(t)\in L^2((0,T),L^2(X,\mu))$, i.e.

$$\lim_{k\to\infty}\int_0^T \langle u_{m_k}^{(m_k)}(t),v_k(t)\rangle_{\ell^2(V_{m_k},\mu^{(m_k)})}dt=\int_0^T\langle u(t),v(t)\rangle_{L^2(X,\mu)}dt \tag{12.16}$$

for every sequence $(v_k(t))_k$, $v_k(t)\in L^2((0,T),\ell^2(V_{m_k},\mu^{(m_k)}))$ KS-strongly convergent to $v(t)\in L^2((0,T),L^2(X,\mu))$ w.r.t. (12.1).

For any $w\in\mathcal{A}$ such that $w(t):=w_1(t)\cdot w_2$, $w_1(t)\in\big(C([0,T])\cap C^1((0,T))\big)$ with $w_1(T)=0$, $w_2\in\bigcup_{n\geq 0}H_n(X)$ we have $\lim_{k\to\infty}w_1(t)\left(w_{2|V_{m_k}}\right)=w(t)$ KS-strongly. Fix such $w\in\mathcal{A}$. From (12.15) and (12.16) it follows that

$$\int_0^T\langle u_\varepsilon(t),w(t)\rangle_{L^2(X,\mu)}dt=\lim_{k\to\infty}\frac{1}{m_k}\int_0^T\langle u_{m_k}^{(m_k)}(t),w_k(t)\rangle_{\ell^2(V_{m_k},\mu^{(m_k)})}dt=0.$$

Consequently, we obtain $u_\varepsilon(t)=0$ $\mu-$a.e. (choice of $w(t)$ is arbitrary) and in particular

$$\lim_{k\to\infty}\int_0^T\mathcal{E}_{V_{m_k}}\left(\frac{u_{m_k}^{(m_k)}(t)}{m_k},w_k(t)\right)dt=0.$$

We conclude from the formulation of a weak solution to (12.9) and Lemma 12.3 that

$$\begin{aligned}
0&=\lim_{k\to\infty}\int_0^T\mathcal{E}_{V_{m_k}}\left(\frac{u_{m_k}^{(m_k)}(t)}{m_k},w_1(t)w_{2|X^{(m_k)}}\right)dt\\
&=\lim_{k\to\infty}\int_0^T\langle u_{m_k}^{(m_k)}(t),w_1'(t)w_{2|V_{m_k}}\rangle_{\ell^2(V_{m_k},\mu^{(m_k)})}+w_1(t)\langle u_{m_k}^{(m_k)}(t)\cdot b^{(m_k)},\partial^{(m_k)}w_{2|V_{m_k}}\rangle_{\mathcal{H}^{(m_k)}}dt\\
&+\lim_{k\to\infty}\langle u_{m_k}^{(m_k)}(0),w_1(0)w_{2|V_{m_k}}\rangle_{\ell^2(V_{m_k},\mu^{(m_k)})}\\
&=\int_0^T\int_X u(t)w_1'(t)w_2d\mu+w_1(t)\left\langle u(t)\cdot b,\partial w_2\right\rangle_{\mathcal{H}}dt+\int_X u(0)w_1(0)w_2d\mu.
\end{aligned}$$

This yields that $u(t)$ is a weak solution to (8.1). □

Chapter A3

Appendix to Part III

A3.1 KS-generalized strong resolvent convergence and P-generalized norm resolvent convergence

In this section we provide a useful observation regarding the connection between convergence in the KS-generalized strong resolvent sense and convergence in the P-generalized norm resolvent sense. We present this auxiliary result for the interesting reader in Theorem A3.1, it is not used in the main text of the thesis.

In view of Remarks 10.1 and 11.1 together with the fact that on a single Hilbert space norm resolvent convergence implies strong resolvent convergence it seems natural to expect that generalized norm resolvent convergence in the sense of Post implies generalized strong resolvent convergence in the sense of Kuwae and Shioya. Under mild compatibility assumptions this is true.

Theorem A3.1. *Suppose that* $\lim_{m\to\infty} A_m = A$ *in the P-generalized norm resolvent sense with identification operators* $J'_m : H \to H_m$*. Suppose further that there exists a dense subset* $\mathcal{C}$ *of* H *such that*

$$\lim_{m\to\infty} \left\| J'_m w \right\|_{H_m} = \|w\|_H , \quad w \in \mathcal{C}. \tag{A3.1}$$

Then we have $\lim_{m\to\infty} H_m = H$ *in the KS-sense and* $\lim_{m\to\infty} A_m = A$ *in the KS-generalized strong resolvent sense.*

Remark A3.1. Note that if the H_m converge to H in the KS-sense and (10.2) holds for some $\mathcal{C}$ and Φ_m as in (10.1), then the condition

$$\lim_{m\to\infty} \left\| \Phi_m w - J'_m w \right\|_{H_m} = 0, \quad w \in \mathcal{C}, \tag{A3.2}$$

implies (A3.1). We would like to remark that condition (A3.2) might be seen as a simple statement saying that Φ_m and J'_m are 'asymptotically close to each other'.

We quote the following consequence of convergence in the P-generalized norm resolvent sense from Theorem [Pos12, Theorem 4.2.14].

Proposition A3.1. *Suppose that* $(A_m)_m$ *is a sequence of nonnegative definite self-adjoint operators* $A_m : H_m \to H_m$ *which converges to a nonnegative definite self-adjoint operator* $A : H \to H$ *in the P-generalized norm resolvent sense. Then we have*

$$\lim_{m\to\infty} \left\| G_1 J'_m - J'_m G_{m,1} \right\| = 0, \tag{A3.3}$$

where J_m *and* J'_m *are two operators as in Definition 11.1 (iii) and* G_1 *and* $G_{m,1}$ *denote the* 1*-resolvent operators of* A *and* A_m*.*

The short proof of Theorem A3.1 is straightforward from Definition 10.1 and (A3.3).

Proof. Obviously condition (A3.1) implies $\lim_{m\to\infty} H_m = H$ in KS-sense. Suppose that $u_m \to u$ KS-strongly and that $(\widetilde{u}_n)_n \subset \mathcal{C}$ is as in 10.3 with J'_m in place of Φ_m. By the (uniform) boundedness of the resolvent operators we then have

$$\lim_{n\to\infty} \|G_1\widetilde{u}_n - G_1 u\|_H = 0 \tag{A3.4}$$

and

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}} \left\|G_{m,1}J'_m\widetilde{u}_n - G_{m,1}u_m\right\|_{H_m} = 0. \tag{A3.5}$$

Now (A3.3) implies

$$\lim_{m\to\infty} \sup_n \left\|G_{m,1}J'_m\widetilde{u}_n - J'_mG_1\widetilde{u}_n\right\|_{H_m} \leq (\sup_n \|\widetilde{u}_n\|_H) \lim_{m\to\infty} \left\|G_{m,1}J'_m - J'_mG_1\right\| = 0,$$

and together with (A3.5) we obtain

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}} \left\|J'_mG_1\widetilde{u}_n - G_{m,1}u_m\right\|_{H_m} = 0.$$

For each n let now $\widetilde{w}_n \in \mathcal{C}$ be such that $\|\widetilde{w}_n - G_1\widetilde{u}_n\|_H < 2^{-n}$. Then also

$$\sup_m \left\|J'_m\widetilde{w}_n - J'_mG_1\widetilde{u}_n\right\| < 2^{-n+1}(\sup_m \delta_m)$$

because $\|J'_m\| \leq 2 + \delta_m$ for all m by Definition 11.1 (ii). Consequently

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}} \left\|J'_m\widetilde{w}_n - G_{m,1}u_m\right\|_{H_m} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\widetilde{w}_n - G_1 u\|_H = 0,$$

that is, $G_{m,1}u_m \to G_1 u$ KS-strongly. □

A3.2 Proof of Theorem 11.1

To verify the validity of the hypotheses in Theorem 11.1 we need the following statements that are versions of results established earlier in [PS18a; PS18b].

We write $r_{\max} := \max_{i=1,\ldots,N} r_i$. According to our hypotheses that the associated harmonic structure is regular, it holds that $0 < r_{\max} < 1$. Recall that $\deg_m(p)$ denotes the degree of the vertex $p \in V_m$ in the graph G_m. For all m and all $p \in V_m$ the total number of words w such that the corresponding cells K_w meet in vertex p admits the bound

$$|\{w \in W_m \mid p \in K_w\}| \leq N. \tag{A3.6}$$

Since for any m and any $w \in W_m$ also $|V_w| \leq N$, we have

$$\deg_m(p) \leq N(N-1) \tag{A3.7}$$

for all $m \geq 0$.

Lemma A3.1. *For any $f \in W^{1,2}(X_{\Gamma_m}, \mu_{\Gamma_m})$ we have*

$$\left\|f - J^*_{0,m}J_{0,m}f\right\|^2_{L^2(X_{\Gamma_m},\mu_{\Gamma_m})} \leq 150N^5\, r^m_{\max} \max_{|w|=m} \mu(K_w)\, \mathcal{E}_{\Gamma_m}(f).$$

Given two edges $e, e' \in E_m$ we write $e \sim e'$ if $e \neq e'$ and e and e' have a common vertex.

Proof. Since $f(s) = \sum_{e \in E_m} \mathbf{1}_e(s) f_e(s)$ for μ_{Γ_m}-a.e. $s \in X_{\Gamma_m}$ we have

$$f(s) - J_{0,m}^* J_{0,m} f(s) = \sum_{e \in E_m} \mathbf{1}_e(s) \sum_{e' \in E_m} \frac{1}{l_{e'}} \int_0^{l_{e'}} (f_e(s) - f_{e'}(s')) ds' \frac{\langle \psi_{e,m}, \psi_{e',m} \rangle_{L^2(K,\mu)}}{\int_K \psi_{e,m} d\mu}$$

for μ_{Γ_m}-a.e. s and therefore

$$\begin{aligned}
&\|f - J_{0,m}^* J_{0,m} f\|^2_{L^2(X_{\Gamma_m}, \mu_{\Gamma_m})} \\
&= \sum_{e \in E_m} \frac{1}{l_e} \int_0^{l_e} \left(\sum_{e' \in E_m} \frac{1}{l_{e'}} \int_0^{l_{e'}} (f_e(s) - f_{e'}(s')) ds' \frac{\langle \psi_{e,m}, \psi_{e',m} \rangle_{L^2(K,\mu)}}{\int_K \psi_{e,m} d\mu} \right)^2 ds \left(\int_K \psi_{e,m} d\mu \right) \\
&\leq \sum_{e \in E_m} \frac{1}{l_e \int_K \psi_{e,m} d\mu} \int_0^{l_e} \sum_{e' \in U(e)} \frac{1}{l_{e'}} \int_0^{l_{e'}} (f_e(s) - f_{e'}(s'))^2 ds' ds \sum_{\tilde{e} \in E_m} \langle \psi_{\tilde{e},m}, \psi_{e,m} \rangle^2_{L^2(K,\mu)},
\end{aligned}$$

where $U_1(e)$ is the set of all $e' \in E_m$ such that $e' \sim e$, $U_2(e)$ is the set of all $e' \in E_m$ such that $e' \neq e$ and there exists $e'' \in E_m$, such that $e'' \sim e'$ and $e'' \sim e$, and

$$U(e) = \{e\} \cup U_1(e) \cup U_2(e).$$

Note that for $\tilde{e} \in E_m \setminus U(e)$ we have $\langle \psi_{\tilde{e},m}, \psi_{e,m} \rangle_{L^2(K,\mu)} = 0$. In case that $e' = e$ estimate (5.2) yields

$$(f_e(s) - f_{e'}(s'))^2 \leq l_e \mathcal{E}_e(f_e).$$

If $e' \in U_1(e)$ and p is the common vertex of e' and e then, using the elementary inequality $|a+b|^2 \leq 2\left(|a|^2 + |b|^2\right)$ and the triangle inequality,

$$(f_e(s) - f_{e'}(s'))^2 \leq 2 \cdot \left\{ (f_e(s) - f_e(p))^2 + (f_{e'}(p) - f_{e'}(s'))^2 \right\} \leq 2 \cdot \{ l_e \mathcal{E}_e(f_e) + l_{e'} \mathcal{E}_{e'}(f_{e'}) \}.$$

For $e' \in U_2(e)$ with (unique) e'' such that $e'' \sim e'$ and $e'' \sim e'$ we similarly obtain

$$(f_e(s) - f_{e'}(s'))^2 \leq 4 \cdot \{ l_e \mathcal{E}_e(f_e) + l_{e''} \mathcal{E}_{e''}(f_{e''}) \} + 2 \cdot l_{e'} \mathcal{E}_{e'}(f_{e'}).$$

Inserting into the above yields

$$\begin{aligned}
&\|f - J_{0,m}^* J_{0,m} f\|^2_{L^2(X_{\Gamma_m}, \mu_{\Gamma_m})} \\
&\leq \sum_{e \in E_m} \frac{1}{\int_K \psi_{e,m} d\mu} \sum_{\tilde{e} \in E_m} \langle \psi_{\tilde{e},m}, \psi_{e,m} \rangle^2_{L^2(K,\mu)} \Bigg[l_e \mathcal{E}_e(f_e) + 2 \sum_{e' \in U_1(e)} \{ l_e \mathcal{E}_e(f_e) + l_{e'} \mathcal{E}_{e'}(f_{e'}) \} \\
&\quad + \sum_{e' \in U_2(e)} \{ 4 l_e \mathcal{E}_e(f_e) + 4 l_{e''} \mathcal{E}_{e''}(f_{e''}) + 2 l_{e'} \mathcal{E}_{e'}(f_{e'}) \} \Bigg],
\end{aligned}$$

where in the last sum for each fixed e and e' the edge e'' is one possible connecting edge. This is less or equal

$$15 \left(\max_{e \in E_m} \frac{1}{\int_K \psi_{e,m} d\mu} \sum_{\tilde{e} \in E_m} \langle \psi_{\tilde{e},m}, \psi_{e,m} \rangle^2_{L^2(K,\mu)} \right) r_{\max}^m \mathcal{E}_{\Gamma_m}(f).$$

From the definition (11.6) of $\psi_{e,m}$ and the fact that $(\deg_m(p))^{-1}$ is bounded from above by 1 for all m and the bound (A3.7) we obtain

$$\begin{aligned}
\langle \psi_{\tilde{e},m}, \psi_{e,m} \rangle_{L^2(K,\mu)} &\leq \int_K \psi_{e,m} d\mu \\
&= \frac{1}{\deg_m(i(e))} \sum_{\substack{|w|=m \\ i(e) \in K_w}} \int_{K_w} \psi_{i(e),m} d\mu + \frac{1}{\deg_m(j(e))} \sum_{\substack{|w|=m \\ j(e) \in K_w}} \int_{K_w} \psi_{j(e),m} d\mu \\
&\leq 2\, N \max_{|w|=m} \mu(K_w).
\end{aligned}$$

The term in brackets can be estimated as follows

$$\begin{aligned}
&\max_{e\in E_m}\frac{1}{\int_K \psi_{e,m}d\mu}\sum_{\widetilde{e}\in E_m}\left\langle \psi_{\widetilde{e},m},\psi_{e,m}\right\rangle^2_{L^2(K,\mu)}\\
&\le \max_{e\in E_m}\frac{1}{\int_K \psi_{e,m}d\mu}\sum_{\widetilde{e}\in E_m}\left\langle \psi_{\widetilde{e},m},\psi_{e,m}\right\rangle_{L^2(K,\mu)}\int_K \psi_{e,m}d\mu\\
&= \max_{e\in E_m}\sum_{\widetilde{e}\in U(e)}\left\langle \psi_{\widetilde{e},m},\psi_{e,m}\right\rangle_{L^2(K,\mu)}\\
&\le 2N\,\left(1+2N^2+2N^4\right)\max_{|w|=m}\mu\left(K_w\right),
\end{aligned}$$

note that from (A3.7) we easily see the rough bound $|U(e)|\le(1+2N^2+2N^4)$ for the cardinality of $U(e)$ for each $e\in E_m$. □

We define operators $J_{1,m}:W^{1,2}(X_{\Gamma_m},\mu_{\Gamma_m})\to\mathcal{D}(\mathcal{E})$ and $\widetilde{J}_{1,m}:\mathcal{D}(\mathcal{E})\to W^{1,2}(X_{\Gamma_m},\mu_{\Gamma_m})$ by

$$J_{1,m}f:=H_m(f|_{V_m}),\quad f\in W^{1,2}(X_{\Gamma_m},\mu_{\Gamma_m}),\quad\text{and}\quad \widetilde{J}_{1,m}u:=H_m(u)|_{X_{\Gamma_m}},\quad u\in\mathcal{D}(\mathcal{E}).$$

Lemma A3.2.

(i) For any $f\in W^{1,2}(X_{\Gamma_m},\mu_{\Gamma_m})$ we have

$$\|J_{1,m}f-J_{0,m}f\|^2_{L^2(K,\mu)}\le N^6\,r^m_{\max}\max_{|w|=m}\mu(K_w)\,\mathcal{E}_{\Gamma_m}(f).$$

(ii) For any $u\in\mathcal{D}(\mathcal{E})$ we have

$$\|\widetilde{J}_{1,m}u-J^*_{0,m}u\|^2_{L^2(X_{\Gamma_m},\mu_{\Gamma_m})}\le 2\,N^4\,r^m_{\max}\max_{|w|=m}\mu(K_w)\,\mathcal{E}(u).$$

Proof. To see (i) note first that

$$\begin{aligned}
(J_{1,m}f(x)-J_{0,m}f(x))^2&=\left(\sum_{p\in V_m}\frac{1}{\deg_m(p)}\psi_{p,m}(x)\sum_{e\in E_m:\,p\sim e}(f_e(p)-\overline{f_e})\right)^2\\
&\le\left(\sum_{p\in V_m}\frac{1}{\deg_m(p)}\psi_{p,m}(x)\sum_{e\in E_m:\,p\sim e}(l_e\mathcal{E}_e(f_e))^{1/2}\right)^2
\end{aligned}$$

for any $x\in K$ by (5.2). Consequently, writing $e\sim K_w$ if there is some $p\in V_w$ incident to e,

$$\begin{aligned}
&\|J_{1,m}f-J_{0,m}f\|^2_{L^2(K,\mu)}\\
&\quad\le\sum_{|w|=m}\int_{K_w}\left(\sum_{p\in V_m\cap K_w}\frac{1}{\deg_m(p)}\psi_{p,m}(x)\sum_{e\in E_m:\,p\sim e}(l_e\mathcal{E}_e(f_e))^{1/2}\right)^2\mu(dx)\\
&\quad\le\sum_{|w|=m}\int_{K_w}\left(\sum_{e\in E_m,e\sim K_w}(l_e\mathcal{E}_e(f_e))^{1/2}\sum_{p\in V_m\cap K_w}\frac{1}{\deg_m(p)}\psi_{p,m}(x)\right)^2\mu(dx)\\
&\quad\le\sum_{|w|=m}\left(\sum_{e\in E_m,e\sim K_w}(l_e\mathcal{E}_e(f_e))^{1/2}\right)^2\int_{K_w}\left(\sum_{p\in V_m\cap K_w}\frac{1}{\deg_m(p)}\psi_{p,m}(x)\right)^2\mu(dx)\\
&\quad\le(N\cdot N(N-1))^2\sum_{e\in E_m}l_e\mathcal{E}_e(f_e)\,\max_{|w|=m}\mu(K_w)
\end{aligned}$$

by (A3.7), what implies (i). To show (ii) we use that by (11.7) we have

$$\widetilde{J}_{1,m}u(s) - J^*_{0,m}u(s) = \sum_{e\in E_m} \mathbf{1}_e(s) \frac{\langle (H_m(u)_e(s) - u, \psi_{e,m}\rangle_{L^2(K,\mu)}}{\int_K \psi_{e,m} d\mu}$$

and that for fixed $e \in E_m$ and $s \in e$,

$$\begin{aligned}\frac{\langle H_m(u)_e(s) - u, \psi_{e,m}\rangle^2_{L^2(K,\mu)}}{\int_K \psi_{e,m} d\mu} &\leq \int_{\operatorname{supp}\psi_{e,m}} (H_m(u)_e(s) - u(x))^2 \psi_{e,m}(x)\mu(dx) \\ &\leq \sum_{|w|=m, e\cap K_w \neq 0} r_w \mathcal{E}_{K_w}(u) \int_{\operatorname{supp}\psi_{e,m}} \psi_{e,m} d\mu.\end{aligned}$$

Integrating, we obtain

$$\begin{aligned}\left\| \widetilde{J}_{1,m}u - J^*_{0,m}u \right\|^2_{L^2(X_{\Gamma_m}, \mu_{\Gamma_m})} &\leq \sum_{e\in E_m} \frac{1}{l_e} \int_0^{l_e} \langle H_m(u)_e(s) - u, \psi_{e,m}\rangle^2_{L^2(K,\mu)} \, ds \frac{1}{\int_K \psi_{e,m} d\mu} \\ &\leq \sum_{|w|=m} r_w \mathcal{E}_{K_w}(u) \sum_{e\in E_m : e\cap K_w \neq \emptyset} \mu(\operatorname{supp}\psi_{e,m}).\end{aligned}$$

Using again (A3.7) we obtain

$$\left\| \widetilde{J}_{1,m}u - J^*_{0,m}u \right\|^2_{L^2(X_{\Gamma_m}, \mu_{\Gamma_m})} \leq N^3 \cdot 2N\, r^m_{\max} \max_{|w|=m} \mu(K_w)\mathcal{E}(u).$$

□

Lemma A3.3. *For any $u \in \mathcal{D}(\mathcal{E})$ we have*

$$\left\| u - J_{0,m} J^*_{0,m} u \right\|^2_{L^2(K,\mu)} \leq 4\, N^4 r^m_{\max} \max_{|w|=m} \mu(K_w)\mathcal{E}(u).$$

Proof. We follow [PS18a, Lemma 2.3] and prove that for any $u \in \mathcal{D}(\mathcal{E})$ we have

$$\left\| u - J_{0,m} \widetilde{J}_{1,m} u \right\|^2_{L^2(K,\mu)} \leq r^m_{\max} \max_{|w|=m} \mu(K_w)\mathcal{E}(u). \tag{A3.8}$$

Together with the triangle inequality, Proposition 11.1 and Lemma A3.2 (ii) we then obtain the result. To see (A3.8) note that for any $x \in K$ we have

$$u(x) - J_{0,m}\widetilde{J}_{1,m}u(x) = \sum_{e\in E_m} \frac{1}{l_e} \int_0^{l_e} (u(x) - H_m(u)_e(s) ds\, \psi_{e,m}(x).$$

Therefore

$$\begin{aligned}\left\| u - J_{0,m} \widetilde{J}_{1,m} u \right\|^2_{L^2(K,\mu)} &\leq \sum_{|w|=m} \sum_{e\in E_m} \sum_{e'\in E_m} \int_{K_w} \left(\frac{1}{l_e} \int_0^{l_e} (u(x) - H_m(u)_e(s)) ds \right) \times \\ &\quad \times \left(\frac{1}{l_e} \int_0^{l_e} (u(x) - H_m(u)_e(s')) ds' \right) \psi_{e,m}(x) \psi_{e',m}(x) \mu(dx) \\ &\leq \sum_{|w|=m} r_w \mathcal{E}_{K_w}(u) \mu(K_w)\end{aligned}$$

and (A3.8) follows. □

Lemma A3.4. *For any* $f \in W^{1,2}(X_{\Gamma_m}, \mu_{\Gamma_m})$ *and* $u \in \mathcal{D}(\mathcal{E})$ *we have*

$$\mathcal{E}_{\Gamma_m}(f, \widetilde{J}_{1,m}u) - \mathcal{E}(J_{1,m}f, u) = 0.$$

Proof. Using the operators H_{Γ_m} and H_m,

$$\mathcal{E}_{\Gamma_m}(f, \widetilde{J}_{1,m}u) = \mathcal{E}_{\Gamma_m}(H_{\Gamma_m}f, H_{\Gamma_m}(u|_{X_{\Gamma_m}}) = \mathcal{E}(H_m(f|_{V_m}), H_m(u|_{V_m})) = \mathcal{E}(J_{1,m}f, u).$$

□

To see Theorem 11.1 it now suffices to note that by Proposition 11.1 and Lemmas A3.1, A3.2, A3.3 and A3.4 the quadratic forms $\mathcal{E}_{\Gamma_m}$ and $\mathcal{E}$ are δ_m-quasi unitarily equivalent on $L^2(X_{\Gamma_m}, \mu_{\Gamma_m})$ and $L^2(K, \mu)$ in the sense of [PS18a, Definition 2.1] resp. [Pos12, Definition 4.4.11] with $\delta_m = 150N^6\, r_{\max}^m \max_{|w|=m} \mu(K_w)$. Therefore [PS18a, Corollary 1.2] implies that for any $t > 0$ there exists some $C_t > 0$ such that

$$\left\| e^{t\mathcal{L}} - J_{0,m} e^{t\mathcal{L}_m} J_{0,m}^* \right\|_{L^2(K,\mu) \to L^2(K,\mu)} \leq C_t\, \delta_m,$$

see also [Pos12, Theorem 4.2.10 and Proposition 4.4.15].

Appendix A

Auxiliary results from functional analysis

In this chapter we recall some well-known results from the area of functional analysis. One can find detailed accounts and proofs of the mentioned subjects in the book of Werner, [Wer00]. We also refer to classical textbooks, for example [RS80] and [Yos80].

We start with recording a useful compactness criterion for uniform convergence.

Theorem A.1 (Arzelà-Ascoli)**.** *Let X be a compact metric space and let $C(X)$ be the Banach space of real-valued continuous functions $f(x)$ normed by $\|f\| = \sup_{x\in X}|f(x)|$. Then a sequence $(f_n(x))_{n\in\mathbb{N}} \subset C(X)$ is relatively compact in $C(X)$ if the following two conditions are satisfied:*

1. *$f_n(x)$ is* equi-bounded *(in n), i.e.*

$$\sup_{n\in\mathbb{N}}\sup_{x\in X}|f_n(x)| < \infty.$$

2. *$f_n(x)$ is* equi-continuous *(in n), i.e. for each $\varepsilon > 0$, there exists $\delta > 0$, such that $|x-y| < \delta$ implies $|f_n(x) - f_n(y)| < \varepsilon$, for all $x, y \in X$, $n \in \mathbb{N}$.*

For a proof see [Yos80, Section III.3].

The weak-$*$topologies have a very important compactness property to which we now turn our attention.

Theorem A.2 (Banach-Alaoglu)**.** *Let X be a Banach space with norm $\|\,\|$ and X^* its dual (with operator norm). Then the unit ball in X^* is compact in the weak-$*$topology (i.e. the topology generated by the maps $l \mapsto l(x)$, $l \in X^*$, and x running through X).*

The proof of this theorem can be found in [RS80, Theorem IV.21].

Theorem A.3 (Banach-Saks)**.** *Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\,,\,\rangle$ and norm $\|\,\| := \sqrt{\langle\,,\,\rangle}$. Let u, $u_n \in \mathcal{H}$, $n \in \mathbb{N}$, with $u_n \to u$ weakly in $\mathcal{H}$ as $n \to \infty$. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that the* Cesaro mean

$$v_N := \frac{1}{N}\sum_{k=1}^{N} u_{n_k}, \qquad N \in \mathbb{N},$$

converges strongly to u in $\mathcal{H}$.

For the proof see [RS55, Section 38].

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