Models of degenerate random conductances with stable-like jumps

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Contents

1. Introduction

1.1. Convergence results

Materials appearing to be homogeneous to the naked eye are in fact quite heterogeneous on the microscopic level with deformations that tend to lack describable pattern. In the absence of a better approach their microscopic structure is often modeled as random. From the mathematical point of view, an interesting endeavor is to identify macroscopic properties of a random media which depend only on the statistical property of its random structure. Many classical books of homogenization theory deal with such questions, see [JKO94] for instance.

Take the following problem as an example. Suppose that a symmetric, positive definite $n \times n$ matrix field $a_\omega(x)$ on $\mathbb{R}^n$ is chosen at random (in whatever way). It is known that under the ellipticity assumption $C^{-1}\|v\|^2 \leq (a_\omega(x)v, v) \leq C\|v\|^2$ the parabolic problem

$$\partial_t u_\omega - \operatorname{div}(a_\omega \nabla u_\omega) = 0. \tag{1.1}$$

has a unique solution u_ω. One then wonders if $\lim_{m\to\infty} u_\omega(m^2 t, mx)$ exists almost surely, for which kind of random a_ω and how to identify it. The transformation $(t,x) \to (m^2 t, mx)$ is the so-called parabolic scaling and preserves solutions of Eq. (1.1) in the simplest case when a_ω is just a constant matrix.

Similar questions, posed on the discrete lattice $\mathbb{Z}^n$ and for jump type operators, will occupy our attention throughout this thesis. In order to be more concrete, let us introduce a simplified model of random media known as the random conductance model (see [Bis11] or [Kum18] for an introduction). One takes the lattice $\mathbb{Z}^n$ for any $n \geq 1$ and considers on it a family of non-negative random variables $k(x,y) \geq 0$ indexed by pairs of lattice points, $x, y \in \mathbb{Z}^n$, which are symmetric in the sense that $k(x,y) = k(y,x)$. For every realization of k it is possible to construct a variable speed random walk X_t, starting from 0, corresponding to the generator

$$\mathcal{L}f(x) = \sum_{y\in\mathbb{Z}^n} (f(y) - f(x))k(x,y).$$

(see Section 9.3 for details). The distribution of such a Markov chain satisfies a discrete analogue of Eq. (1.1), that is

$$\partial_t u - \mathcal{L}u = 0, \tag{1.2}$$

and it is natural that questions concerning Eq. (1.2) can be rephrased in terms of X_t. We would like to clarify a possible source of confusion here. For every realization k^ω of random kernel k on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we construct a new Markov chain X_t^ω in order to study Eq. (1.2). This introduces an artificial probability measure $\mathbf{P}_0^\omega$ describing the trajectories of X_t^ω, where subscript 0 indicates the starting point of X_t^ω. The "total" probability, also called the annealed probability (denoted here by P), of an event in the RCM is then computed by

$$P(d\tau) = \int_\Omega \mathbf{P}_0^\omega(d\tau)\mathbb{P}(d\omega).$$

From now on, we will omit the superscript ω in the notation X_t^ω and $\mathbf{P}_0^\omega$ but it is important to keep in mind that these objects will always depend on the realization of k.

If $k(x,y)$ is required to be 0 whenever $|x-y| \neq 1$, we talk about *the nearest neighbor case*, which has been studied extensively. For example, [ABDH13] proves that if $k(x,y)$ are i.i.d. random variable such that $\mathbb{P}(k(x,y)>0)>p_c$ (p_c denotes the critical percolation probability), then, for almost every realization of k, X_{m^2t}/m converges, as m goes to ∞, to a diffusion process weakly on the Skorokhod space $\mathcal{D}([0,T],\mathbb{R}^n)$ for every $T>0$ (see [Bil99], Chapter 3 for details on the Skorokhod space). This convergence statement is known as the quenched invariance principle (QIP) or quenched functional central limit theorem. On the other hand, if $k(x,y)$ are only stationary and ergodic, [ADS15] proves that the QIP holds whenever $\mathbb{E}[k(x,y)^{-q}]+\mathbb{E}[k(x,y)^p]<\infty$ for all $x,y\in\mathbb{Z}^n$, $|x-y|=1$ and some $p,q\in[1,\infty]$ satisfying $\frac{1}{q}+\frac{1}{p}<\frac{2}{n}$. Notice that this implicitly requires $\mathbb{P}(k(x,y)>0)=1$ for all $x,y\in\mathbb{Z}^n$, $|x-y|=1$. It is believed that similar results should hold if the nearest neighbor condition is relaxed to $k(x,y)=0$ whenever $|x-y|>R$ for any finite R or even to $\sum_{y\in\mathbb{Z}^n}k(x,y)|y-x|^2<\infty$ a.s. for every x. Some indications of this can be found in [PZ17, PZ20, FHS19].

The case when, for almost every realization of k, $\sum_{y\in\mathbb{Z}^n}k(x,y)|y-x|^2=+\infty$ is known as *the long range case* and has been less studied. Our focus will be on a particular subclass where k has the special form

$$k(x,y)=\frac{c(x,y)}{|x-y|^{n+2s}} \tag{1.3}$$

for some fixed number $s\in(0,1)$ and a family of random variables $c(x,y)$, $x,y\in\mathbb{Z}^n$. There are several arguments for choosing the weight $|x-y|^{-(n+2s)}$. It is the simplest weight with infinite spatial variance, it is a jump rate of a rotationally symmetric stable process on $\mathbb{R}^n$, for which a central limit theorem type result is known, and the indications are that when $s\to 1$ the transition into finite spatial variance case should occur. A family of random variables $c(x,y)$ is what we would like to call *the conductance* throughout this thesis. The quantity k can then be seen as a weighted conductance but we prefer to call it *the kernel* due to its resemblance to the jumping kernels of rotationally symmetric stable processes. Notice that in the nearest neighbor case $k(x,y)=c(x,y)$ anyway and the distinction between c and k is irrelevant. As far as we know, this particular setup has only been studied in [CKW18b, CKW18a] and [FH20]. A closely related model of long-range percolation was studied in [CS12] and [CS13].

The following two results are the main contributions of this thesis to the long-range random conductance model in dimension greater or equal to two. When random variables $c(x,y)$ have the same distribution (which is the case in all theorems in this introduction) then quantity $\mathbb{E}[c(x,y)^p]$ (for whatever $p\in\mathbb{R}$) does not depend on x or y and we will simply denote it by $\mathbb{E}[c^p]$. The limiting process mentioned in following theorems is a Lévy process of pure jump type given through its Lévy measure, see Chapter 2 of [Sat99]. For the purpose of the next two theorems, let $X_t^{(\infty)}$ be the rotationally symmetric stable process on $\mathbb{R}^n$ with Lévy measure

$$\mu(dy)=\frac{\mathbb{E}[c]}{|y|^{n+2s}}dy.$$

Theorem 1.1.1 (see 12.4.2)**.** *Let c be a symmetrized twofold ergodic conductance on $\mathbb{Z}^n$ ($n\geq 2$) such that $\mathbb{E}[c^{-q}]+\mathbb{E}[c^p]<\infty$ for some $p,q\in[1,\infty]$ satisfying*

$$\frac{1}{p}+\frac{1}{q}<\frac{2s}{n}. \tag{1.4}$$

Then, for a.e. realization of the conductance c, $X_{m^{2s}t}/m\xrightarrow{m\to\infty}X_t^{(\infty)}$ in the sense of finite-dimensional distributions.

Theorem 1.1.2 (see 12.5.3)**.** *Let c be an i.i.d. conductance on $\mathbb{Z}^n$ ($n\geq 2$) that is not identically zero and such that $\mathbb{E}[c^p]<\infty$ for some $p>\frac{n+1}{s}$. Then, for $\mathbb{P}$-a.e. realization of the conductance c, $X_{m^{2s}t}/m\xrightarrow{m\to\infty}X_t^{(\infty)}$ weakly on Skorokhod space $\mathcal{D}([0,T],\mathbb{R}^n)$, for every $T>0$.*

By *twofold ergodic* in Theorem 1.1.1 we mean to say that $c(x,y)$ is stationary and ergodic with respect to independent shifts in x and y variables or, to be precise, with respect to shifts $(x,y)\to(x+z,y)$ and $(x,y)\to(x,y+z)$, for all $x,y,z\in\mathbb{Z}^n$. Such version of ergodicity is used in [FH20] but it is a rather strong assumption. One hopes for an analogous result to be true if stationarity is only assumed under shifts $(x,y) \to (x+z, y+z)$, for all $x,y,z \in \mathbb{Z}^n$. There seems to be no results obtained in this case so far. Note also that in this case the limiting process no longer needs to be rotationally symmetric which makes identifying it much more difficult. Compared to the results known for the nearest neighbor case, the convergence in finite dimension distributions proved in Theorem 1.1.1 is weaker then the weak convergence on the Skorokhod space. However, we believe Theorem 1.1.1 to be among the first results on the long-range ergodic random conductance model.

Theorem 1.1.2 can be seen as an improvement to Theorem 1.1 of [CKW18b]. The result in [CKW18b] also holds when $c(x,y)$ are only independent random variables on more general graphs than $\mathbb{Z}^n$ but, restricted to the i.i.d. conductance used in Theorem 1.1.2 it requires the following assumption in order to obtain the same result. One needs to have $n \geq 4(1-s)$, $\mathbb{P}(c(x,y)=0) < 2^{-4}$ and $\mathbb{E}[c^{-2q}]+\mathbb{E}[c^{2p}] < \infty$ for some $p > \max\{(n+2)/n, (n+1)/(4(1-s))\}$ and $q > (n+2)/n$. In Theorem 1.1.2 we require that $n \geq 2$ and $\mathbb{P}(c(x,y)=0) < 1$. The negative moment condition $\mathbb{E}[c^{-q}]$ is not needed and the upper moment condition is changed to $\mathbb{E}[c^p] < \infty$ for some $p \geq \frac{n+1}{s}$ which is an improvement when $s > 2/3$.

1.2. Regularity results

In order to prove the convergence statements on $X_{m^{2s}t}/m$ we first establish results concerning process X_t that are of independent interest such as weak elliptic and parabolic Harnack inequalities, Hölder regularity (see Theorem 10.4.1, Theorem 11.7.1) and an estimate on the expected exit time (see Theorem 10.5.3 and Theorem 11.8.1). The reader should note that under the uniform pointwise bound

$$A^{-1} \leq c(x,y) \leq A, \tag{1.5}$$

for $A \geq 1$, far better results are already known on general metric measure spaces. The upper and lower pointwise heat kernel estimates have been established in [CK03] on d-sets and later on volume regular metric measure spaces in [GHH17], [GHH18], using mostly analytic methods, and in [CKW19], [CKW16a], [CKW16b] with probabilitic methods. These works are quite extensive and cover relations between heat kernel estimates, Harnack inequalities, Poincaré inequalities, and other conditions in great details. However, the case of kernels not satisfying Ineq. (1.5) remains largely unexplored. This is comparable with the developments in the nearest neighbor case where anomalous behavior of the heat kernel has been discovered for some conductances (for *constant* speed random walk X_t). The defect is caused by emergence of so called "traps" that slow down the propagation of X_t. In the ergodic environment, [ADS16] shows that this can happen when $\frac{1}{p}+\frac{1}{q} < \frac{2}{n}$ is violated and [BBT16] gives an example of an i.i.d. conductance satisfying $\mathbb{E}[c^p]+\mathbb{E}[c^{-p}] < \infty$, $p < 1$ for which even the QIP fails. Furthermore, [BČ11a] (corrected in [BČ11b]) presents an example of i.i.d. conductance where the limiting process is a fractional kinetic process instead of a diffusion.

Here is the statement of the weak elliptic Harnack inequality for the symmetrized twofold ergodic conductance on $\mathbb{Z}^n$ which we prove in Theorem 10.4.1. As mentioned before, the interesting case is when either q or p is different from $+\infty$. Notice also that assumptions on c are the same as in Theorem 1.1.1. The nearest neighbor version of the result can be found in [ADS16].

Theorem 1.2.1 (Weak elliptic Harnack inequality)**.** *Let a symmetrized twofold ergodic conductance c on $\mathbb{Z}^n$ ($n \geq 2$) be such that $\mathbb{E}[c^{-q}]+\mathbb{E}[c^p] < \infty$ for some $p,q \in [1,\infty]$ satisfying*

$$\frac{1}{p}+\frac{1}{q} < \frac{2s}{n}.$$

Then for every $x_0 \in \mathbb{Z}^n$ there exist a $\mathbb{P}$-a.s. finite random variable $R_0(x_0)$ and non random $C_{EH} < \infty$ such that $\mathbb{P}$-a.s. every time-independent solution u of Eq. (1.2) in $2B := B(x_0, 2R)$, for every $R \geq R_0$, *satisfies*

$$\fint_{\frac{1}{2}B} u \le C_{EH} \inf_{\frac{1}{2}B} u.$$

The lower bound $R \ge R_0(x_0)$ is not necessary if Ineq. (1.5) is assumed. It prescribes the minimal radius of the ball on which the weak elliptic Harnack inequality can be expected to hold and its dependence on the realization of c significantly weakens the results. However, such effects are unavoidable and similar radius bounds appear in [Bar04, ADS16] for instance. To understand this, consider, say, an i.i.d. conductance $c \le 1$ which is not uniformly bounded from below, pick a box $Q \subset \mathbb{Z}^n \times \mathbb{Z}^n$ and an arbitrary "desired" configuration of conductance in Q. Then there is a positive probability that c is smaller than the desired configuration in Q which consequently implies that this happens with probability 1 in *some* translate $z + Q$, $z \in \mathbb{Z}^n$. As we are free to choose the desired configuration inconceivably bad (basically meaning that c is very close to 0 everywhere in Q) there is no hope in proving the theorem for balls contained in Q. The role of R_0 in such a situation is to adjust to local peculiarities of c and exclude balls which are too small.

We also obtain an improved version of Theorem 1.2.1 in the case of an i.i.d. conductance. The assumptions on c match those of Theorem 1.1.2.

Theorem 1.2.2 (see Theorem 11.7.1)**.** *Let c be an i.i.d. conductance on $\mathbb{Z}^n$ ($n \ge 2$) such that $\mathbb{E}[c^p] < \infty$ for some $p > \frac{n+1}{s}$. Then there exist non random $\theta \in (0,1)$, $C_{EH} < \infty$ and, for every $x_\star \in \mathbb{Z}^n$, a random variable $R_\star(x_\star)$ such that $\mathbb{P}$-a.s., for all $R_0 \ge R_\star$, $x_0 \in B(x_\star, R_0)$, $R \ge R_0^\theta$, every time-independent solution u of Eq.* (1.2) *in $2B := B(x_0, 2R)$ satisfies*

$$\fint_{\frac{1}{2}B} u \le C_{EH} \inf_{\frac{1}{2}B} u.$$

Clearly, we removed $\mathbb{E}[c^{-q}] < \infty$ condition from Theorem 1.2.1 in Theorem 1.2.2. The other improvement lies hidden behind the claim $\theta < 1$ and "quantifier structure" $\forall x_0 \in B(x_\star, R_0), \forall R \ge R_0^\theta$. Effectively, this means that the random radius that controls the local behavior of the conductance from Theorem 1.2.1 can be chosen locally uniformly. To be precise, the control radius R_0^θ will work for all points x_0 with $|x_\star - x_0| < R_0$ provided that R_0 is large enough. The point is that R_0^θ is on the lower order scale compared to R_0 which, after rescaling, essentially gives a statement for balls of all sizes. To restate it yet again, this mean that the control radius $R_0(x)$ behaves asymptotically like $R_0(x_0) \asymp |x_0 - x_\star|^\theta$. In contrast to this, it is not hard to see that in the ergodic environment one can have $R_0(x) \asymp |x - x_0|$ which is not good enough. The choice of the "quantifier structure" is motivated by [Bar04] where a very similar construction is used for the definition of a "very good ball".

1.3. Techniques

In order to obtain convergence in finite-dimensional distributions in both Theorem 1.1.1 and Theorem 1.1.2 we make use of Mosco convergence results developed in [FH20] and [CKK13] respectively. These results alone imply the convergence in finite-dimensional distributions if the limiting process is started from an absolutely continuous (with respect the Lebesgue measure) initial distribution and the approximating processes from discrete approximations of said initial distribution. See [CKK13] for details. What we contribute here is the regularity result (for both ergodic and i.i.d. conductance) which improves the previous statement by allowing the processes to start from any given point.

The regularity result mentioned above is obtained with the help of De Giorgi, Nash, Moser techniques. To be precise, we modify the version of the nonlocal Moser iteration from [FK13] and adopt it to the setting of general, volume regular, metric measure space. These modifications are in the spirit of [ADS16] and [ACDS18] but further modifications are needed due to the existence of long-range jumps, especially concerning the quantity $\nu(x)$ introduced there. In order to perform the iteration one needs to have Sobolev and Poincaré inequalities as well estimates on the energy density of cutoff functions readily available. For the sake of exposition, let us only consider the Sobolev inequality which, in our opinion, is the most interesting one:

$$\|f^2\|_{L^\rho} \le C \sum_{x,y\in\mathbb{Z}^n} \frac{(f(y)-f(x))^2}{|x-y|^{n+2s}} c(x,y) \tag{1.6}$$

for some $\rho > 1$, $C \ge 0$ and all functions f on $\mathbb{Z}^n$. The main difficulty is that such an estimate can not hold with constants uniform throughout the space without the uniform lower estimate on the conductance. We circumvent this difficulty by considering localized versions of the aforementioned inequality and devote significant effort to proving them for both ergodic and i.i.d. conductance. It turns out that the negative moment condition $\mathbb{E}[c^{-q}]$, for $q > n/(2s)$, is sufficient to inherit the Sobolev inequality from $\mathbb{Z}^n$ through an application of Hölder inequality to the right hand side of Ineq. (1.6). On the other hand, such a method does not work if c is allowed to be zero with positive probability like it is in Theorem 1.1.2. In that case we prove the following version of the Sobolev inequality by adopting the techniques from [DNPV12]:

$$\|f^2\|_{L^\rho} \le C \left\|\lambda^{-1} 1_{\operatorname{supp} f}\right\|_{L^r} \sum_{x,y\in\mathbb{Z}^n} \frac{(f(y)-f(x))^2}{|x-y|^{n+2s}} c(x,y) \tag{1.7}$$

where $\rho > 1$, $C, r < \infty$ and $\lambda : \mathbb{Z}^n \to (0,\infty)$ is a function that depends on the realization of the conductance c. The advantage compared to Ineq. (1.6) is that local random deformations can now be incorporated into the function λ which is then averaged out allowing for the law of large numbers to kick in. Using such localized versions of Sobolev and Poincaré inequalities it is possible to run the iteration machinery and obtain weak elliptic/parabolic Harnack inequalities as well as Hölder regularity type estimates.

We improve on these results using maximum principle techniques from [GHH17] and [GHH18] which allow us to also obtain the expected exit time and survival estimates for process X_t. This is as far as we can get in the ergodic case. On the other hand, if the conductance is i.i.d., we make use of "very good" counterparts of previous estimates in combination with Markov property of X_t and tightness criteria from [Ald78] to prove weak convergence on the Skorokhod space just like in [CKW18b]. Note, however, that we are able to obtain these "very good" estimates only under somewhat stronger conditions $\mathbb{E}[c^p] < \infty$ for some $p > (n+1)/s$.

1.4. Limitations of the method

The reason why the weak convergence on the Skorokhod space works in the nearest neighbor case but not in Theorem 1.1.1 is twofold. Proofs of the quenched invariace principle in nearest neighbor case in [ADS16, MP07], [ABDH13] employ the so called *corrector method* which decomposes

$$X_t = M_t + \varphi(X_t)$$

where M_t is a martingale and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is called *the corrector*. The proof then proceeds on by proving that M_{m^2t}/m coverges to a diffusion and that $\varphi(X_{m^2t})/m$ tends to zero for a.e. realization of c. The main issue is that the corrector is usually constructed using the assumption $\sum_{y\in\mathbb{Z}^n} k(x,y)|y|^2 < \infty$, for all $x \in \mathbb{Z}^n$ which is not available in the long-range case. Even worse, when $s < 1$ in Eq. (1.3) the expectation $\mathbb{E}[X_t]$ does not exist so the martingale part M_t no longer makes sense. Furthermore, supposing that $s > 1$ and assuming one could come up with a way of defining the corrector φ, the usual way of proving $\varphi(X_t)/m$ is through subadditivity estimate

$$\sup_{x\in B(0,R)} \frac{|\varphi(x)|}{R} \xrightarrow{R\to\infty} 0.$$

This estimate is in turn obtained as a consequence of maximal inequality $\sup_B |\varphi| \leq C\|\varphi\|_{L^2(B)}$ which holds because φ solves certain elliptic PDE. The problem is that proving maximal inequalities for irregular non-local kernels (even if they are translation invariant on $\mathbb{R}^n$) is not straightforward as so called *tail terms* pollute the calculation. If the aforementioned maximal inequality is true, then essentially the Harnack inequality has to hold true as well. This is the case for some kernels, as can be seen in [DCKP14] but fails in general as shown in [BS07] where an additional "relative Kato condition" is identified as being decisive. Since random kernels are fundamentally not translation invariant and since we can not see any reasonable way of imposing a kind of relative Kato condition on them, this line of attack looks fairly bleak.

On the other hand, in the i.i.d. case we rely on estimates of the "very good" kind and restarting of the process in the vicinity of the original starting point instead of the corrector method. This however comes at the cost of moment condition because proving estimates of the "very good" kind requires the upper moments of c larger than $\frac{n+1}{s}$ as opposed to $\frac{n}{2s}$ which might be expected from Ineq. (1.4). Furthermore, it is plausible that condition $\mathbb{E}[c^p] < \infty$ in Theorem 1.1.2 is superflous since no such condition is needed for the nearest neighbor case in [ABDH13]. However, the method of dealing with large conductances in [ABDH13] does not seem to be applicable in the long-range case.

1.5. Outline

This thesis consists of two preliminary chapters followed by two main parts. In Chapter 2 we give a summary of basic concepts that will be important throughout the thesis. Chapter 3 contains a list of properties that will be used extensively in Parts I and II.

Part I studies general deterministic jump type symmetric bilinear forms on a metric measure space. It avoids assuming pointwise estimates on the kernel and instead makes use of the assumptions presented in Chapter 3. Part I consists of five chapters and a short summary is given in Chapter 4. In Chapter 5 we introduce the concept of weak solution for nonlocal equations. The Moser iteration in the subsequent chapter will work for such solutions. The iteration itself is performed in Chapter 6 and results in a weak parabolic/elliptic Harnack inequality and a large scale Hölder regularity estimate. We extend on this in Chapter 7, where an expected exit time estimate, a survival estimate and related results are proved. Lastly, Chapter 8 gives a sufficient condition for validity of the Poincaré-Sobolev inequality.

Part II studies the long-range conductance model with stable like jumps on $\mathbb{Z}^n$, for $n \geq 2$. It consists of four chapters. The long-range i.i.d. and symmetrized ergodic conductance are introduced in Chapter 9. Chapters 10 and 11 apply the results of Part I to the symmetrized ergodic and i.i.d. conductance respectively. Finally, the convergence results, which are the main results of this thesis, are obtained in Chapter 12.

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2. Preliminaries

2.1. Notation

$B(x_0, R)$ denotes the open ball of radius R around x_0
$aB(x_0, R) := B(x_0, aR)$ assuming center x_0 and radius R are unique and $a > 0$
$a \wedge b$ and $a \vee b$ denote the minimum and maximum of set $\{a, b\}$ respectively
$\log a$ denotes the natural logarithm of a, i.e. the logarithm to the base e
$f_+ = f \vee 0$, $f_- = -(f \wedge 0)$ denote the positive and negative part of the function f
$\operatorname{ess\,osc}_S f = \operatorname{ess\,sup}_S f - \operatorname{ess\,inf}_S f$ denotes the essential oscillation of function f on set S.
$\operatorname{diam} M$ denotes the diameter of set M
A^c denotes the complement of set A
$|A|$ denotes the measure of set A (the choice of the measure depends on the context)
$\#A$ denotes the number of elements in the set A
$\operatorname{supp} f$ denotes the support of function f
$\mathcal{E}(f)$ is a shorthand notation for $\mathcal{E}(f, f)$ when $\mathcal{E}$ is a bilinear form
$L^p(M, \mu)$ stands for the L^p space on M with respect to measure μ
$L^p(M)$ shortens $L^p(M, \mu)$ if μ is clear from the context
$L^p_+(M)$ contains only $f \in L^p(M)$ such that $f \geq 0$ a.s.
$L^p_{loc}(M)$ denotes the space of functions f such that $f \in L^p(K)$ for every compact $K \subset M$
$L^p(I; L^q(M))$ denotes the space of function $f : I \to L^q(M)$ such that $\int_I \|f\|^p_{L^q(M)} < \infty$
p^* denotes the conjugate exponent of $p \in [1, \infty]$, $\frac{1}{p} + \frac{1}{p^*} = 1$
$⨍_M f(x)\mu(dx) = \mu(M)^{-1} \int_M f(x)\mu(dx)$ denotes the average of f (convention: $1/\infty = 0$)
$\mathcal{B}(M)$ will denote the Borel σ-algebra of topological space M
$C(M)$ denotes the space of continuous functions on M
$C_c(M)$ denotes the space of continuous functions with compact support
$C_0(M)$ denotes the space of continuous functions that tend to 0 at infinity
$C_b(M)$ denotes the space of bounded continuous functions
$(\cdot, \cdot)_H$ or $(\cdot, \cdot)$ denote the scalar product on a Hilbert space H
$\operatorname{Lip} f$ denotes the Lipshitz constant of function f
$C \equiv C(a, b)$ denotes that variable C depends only on a and b. If later we decide to specify $a = \alpha$ and $b = \beta$, we will write $C(\alpha, \beta)$ or $C(a = \alpha, b = \beta)$ or $C(b = \beta, a = \alpha)$.

2.2. Inequalities

Theorem 2.2.1 (Hölder's inequality, Corollary 2.5 [AF03])**.** *Let $(M, \mathcal{M}, m)$ be a measure space and let $f, g : M \to [0, \infty]$ be $\mathcal{M}$-measurable extended functions. Then, for all $p, q, r \in (0, \infty]$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, inequality*

$$\left(\int_M |f(x)g(x)|^r m(dx) \right)^{\frac{1}{r}} \leq \left(\int_M |f(x)|^p m(dx) \right)^{\frac{1}{p}} \left(\int_M |g(x)|^q m(dx) \right)^{\frac{1}{q}}$$

is true. Furthermore, if $f \in L^p(M)$ and $g \in L^q(M)$, this implies that

$$\|fg\|_{L^r(M)} \leq \|f\|_{L^p(M)} \|g\|_{L^q(M)} .$$

Proof. If $f \notin L^p(M)$ or $g \notin L^q(M)$ the inequality trivially holds. If not, then the usual Hölder inequality from Proposition 3.3.2 of [Coh13], with $r/p + r/q = 1$, implies

$$\int_M |f(x)g(x)|^r m(dx) \leq \left(\int_M |f(x)|^p m(dx) \right)^{\frac{r}{p}} \left(\int_M |g(x)|^q m(dx) \right)^{\frac{r}{q}}.$$

□

Theorem 2.2.2 (Jensen's inequality)**.** *Let $(M, \mathcal{M}, m)$ be a measure space with $m(M) = 1$ and $\varphi : \mathbb{R} \to \mathbb{R}$ a convex function. Then, for every integrable function $f : M \to \mathbb{R}$,*

$$\varphi \left(\int_M f(x)m(dx) \right) \leq \int_M \varphi(f(x))m(dx)$$

where the part of the claim is that the integral on the right exists in $(-\infty, \infty]$.

Proof. φ is lower semi-continuous and thus Lebesgue measurable which show that $\varphi \circ f$ is measurable. It is not hard to prove that there exist $a, b > 0$ such that $\varphi(x) \geq -a|x| - b$. Thus, $\int_M (\varphi \circ f)_-(x)m(dx) \leq a \int_M f_-(x)m(dx) + b < \infty$ so the integral of $\varphi(f)$ exists in $(-\infty, \infty]$. The inequality then follows from Theorem 1.5.1 of [Dur10]. □

Corollary 2.2.3. *Let $(M, \mathcal{M}, m)$ be a measure space, $f : M \to [0, \infty)$ be an integrable function and $p \in (-\infty, \infty)$. If $p \in (-\infty, 0) \cup [1, \infty)$, then*

$$\left(\int_M f(x)m(dx) \right)^p \leq \int_M f(x)^p m(dx)$$

and, if $p \in (0, 1]$, then

$$\left(\int_M f(x)m(dx) \right)^p \geq \int_M f(x)^p m(dx).$$

(We use the conventions $0^{-a} = \infty$, $0^a = 0$ for all $a > 0$.)

Theorem 2.2.4 (Minkowski's inequality, Proposition 3.3.3 of [Coh13])**.** *Let (M, m) be a measure space and $p \in [1, \infty]$. Then for all measurable functions $f, g : M \to \mathbb{R}$*

$$\|f + g\|_{L^p(M)} \leq \|f\|_{L^p(M)} + \|g\|_{L^p(M)}.$$

2.3. Bochner integral

We present some selected results concerning Bochner integral. Most of the results are stated without the proof and taken from [Coh13], Appendix E. Additional references can be found in [Fel13], chapter I.1.

Definition 2.3.1. *Let $(X, \mathcal{X}, \mu)$ be a measure space, $(E, |\cdot|)$ a Banach space and denote the Borel sigma algebra on E by $\mathcal{B}$. A function $f : X \to E$ is said to be* Borel measurable *if it is measurable from $\mathcal{X}$ to $\mathcal{B}$ and* strongly measurable *if it is Borel measurable and has seaparate range.*

Theorem 2.3.2. *Function $f : X \to E$ is strongly measurable if and only if it is a pointwise limit of simple Borel measurable functions.*

Corollary 2.3.3. *Let I be a measurable subset of $\mathbb{R}$ and $(E, \|\cdot\|)$ a Banach space. Then every continuous function $f : I \to E$ is strongly measurable.*

Definition 2.3.4. *A function $f : X \to E$ is said to be Bochner integrable if it is measurable and $\int_X \|f(x)\|\mu(dx) < \infty$. In that case the Bochner integral of f is defined to be*

$$\int_X f(x)\mu(dx) := \lim_{k\to\infty} \int_X f_k(x)\mu(dx)$$

where $f_k : X \to E$ is any sequence of simple, Borel measurable functions such that $f_k(x) \to f(x)$ and $\|f_k(x)\| \le \|f(x)\|$ for every $x \in X$.

The following two statements will be implicitly used in Part I. See also [Fel13] Proposition 1.12. for more details and pointers to the literature.

Proposition 2.3.5. *Let $(M, \mathcal{M}, m)$ be a σ-finite measure space, $I \subset \mathbb{R}$ an measurable set and $p \in [1, \infty)$. Denote by λ the Lebesgue measure and by $\mathcal{L}$ the σ-algebra of Lebesgue measurable sets. For every Bochner integrable $f : I \to L^p(M)$ there exist $\mathcal{L} \times \mathcal{M}$ measurable function $g : I \times M \to \mathbb{R}$ such that $g(t, \cdot) = f(t)$ for a.e. $t \in I$.*

Proof. Let us first suppose that M and I have finite measure. Since f is Bochner integrable, there is sequence of simple functions $f_k : I \to L^p(M)$ such that $\int_I \|f(t) - f_k(t)\|_{L^p(M)}\lambda(dt)$ converges to zero. By Hölder inequality this implies that $\int_I \|f(t) - f_k(t)\|_{L^1(M)}\lambda(dt)$ also converges to zero. Functions f_k are simple and can easily be extended to $\mathcal{L}\times\mathcal{M}$ measurable functions from $I\times M$ to $\mathbb{R}$. Thus, Fubini's theorem implies that f_k is a Cauchy sequence in $L^1(I\times M)$. Denote by g its limit, which exist because $L^1(I\times M)$ is complete. Another application of Fubini's theorem shows that $\int_I \|f_k(t)-g(t,\cdot)\|_{L^1(M)}\lambda(dt)$ tends to zero, which implies that $f(t) = g(t, \cdot)$, as elements of $L^1(M)$ or $L^p(M)$, for a.e. $t \in I$.

If I and M are only σ-finite, we can find increasing sequences of measurable sets I_i and M_i which cover I and M respectively and have finite measure. Then $f(t) \in L^p(M_i)$ for every $t \in I_i$ so by first part we know that there exists $\mathcal{L}\times\mathcal{M}$ measurable function $g^{(i)} : I_i \times M_i \to \mathbb{R}$ such that $g^{(i)}(t, \cdot) = f(t)$ for a.e. $t \in I_i$. If we now define $g(t, x) = \lim_{i\to\infty} g^{(i)}(t, x)1_{I_i\times M_i}(t, x)$, then g is $\mathcal{L} \times \mathcal{M}$ measurable as a function $I \times M \to \mathbb{R}$ and $g(t, \cdot) = f(t)$ for a.e. $t \in I$, which is what we wanted to show. □

Proposition 2.3.6. *Let $(M, \mathcal{M}, m)$ be a σ-finite measure space, $I \subset \mathbb{R}$ an measurable set, $p \in [1, \infty)$ and $v : I \to L^p(\mathbb{R})$ a Bochner integrable function such that $v(t) \ge 0$ for all $t \in I$. Then there exists a version of v such that the pointwise integral $\int_I v(t, x)dt$ is in $L^p(M)$ and*

$$\left[\int_I v(t)dt\right](x) = \int_I v(t, x)dt \qquad \text{for a.e. } x \in M.$$

Proof. We consider the version of v which is equal to the positive part of function g from Proposition 2.3.5. Then v is measurable as a function from $I \times M$ to $\mathbb{R}$ and $v \ge 0$. Denote by $V = \int_I v(t)dt \in L^p(M)$ the Bochner integral on the left side of the equality. By [Coh13] Proposition E.11 for every $\varphi \in L^{p^*}(M)$ we know that

$$\|V\|_{L^p(M)}\|\varphi\|_{L^{p^*}(M)} \ge \left|\int_M V(x)\varphi(x)m(dx)\right| = \left|\int_I \left(\int_M v(t, x)\varphi(x)m(dx)\right) dt\right|.$$

If $\varphi \ge 0$, then by Fubini's theorem and measurability of v

$$\|V\|_{L^p(M)}\|\varphi\|_{L^{p^*}(M)} \ge \int_I \left(\int_M v(t, x)\varphi(x)m(dx)\right) dt = \int_M \left(\int_I v(t, x)dt\right) \varphi(x)m(dx).$$

General φ can be split into the positive and negative part $\varphi = \varphi^+ - \varphi^-$ which results in

$$\int_M \left(\int_I v(t, x)dt\right) |\varphi(x)|\, m(dx) \le \|V\|_{L^p} \left(\|\varphi^-\|_{L^{p^*}} + \|\varphi^+\|_{L^{p^*}}\right) = \|V\|_{L^p}\|\varphi\|_{L^{p^*}}$$

and shows that the pointwise integral $\int_I v(t,x)dt$ is an $L^p(M)$ function. By applying Fubini's theorem one more time (this time for signed integrand) we obtain

$$\int_M V(x)\varphi(x)m(dx) = \int_I \left(\int_M v(t,x)\varphi(x)dx\right) dt = \int_M \left(\int_I v(t,x)dt\right) \varphi(x)m(dx)$$

for every $\varphi \in L^{p^*}(M)$. But this implies that $V = \int_I v(t,x)dt$ as elements of $L^p(M)$ and therefore also pointwise a.e. which is what we wanted to show. □

2.4. Dirichlet forms

This section presents a few concepts from the general theory of regular Dirichlet forms on locally compact separable metric measure spaces. The first couple of definitions are taken from [FOT11], Chapter 1.

Definition 2.4.1 (Closed symmetric form)**.** *Let H be a Hilbert space and let $\mathcal{E}$ be a bilinear form on H with domain $\mathcal{D}[\mathcal{E}] \subset H$ which is dense in H. If $\mathcal{E}(u,v) = \mathcal{E}(v,u)$ for all $u, v \in \mathcal{D}[\mathcal{E}]$, the form $(\mathcal{D}[\mathcal{E}], \mathcal{E})$ is said to be* symmetric *and if $\mathcal{D}[\mathcal{E}]$ is complete in the metric $\mathcal{E}_1(\cdot) := \mathcal{E}(\cdot) + \|\cdot\|_H^2$, the form $(\mathcal{D}[\mathcal{E}], \mathcal{E})$ is said to be* closed.

Remark 2.4.2. *We will shorten $\mathcal{E}(f) := \mathcal{E}(f,f)$ for bilinear form $\mathcal{E}$.*

Definition 2.4.3 (Markovian form)**.** *Let (M,m) be a measure space. A closed form $(\mathcal{D}[\mathcal{E}], \mathcal{E})$ on $L^2(M)$ is said to be* Markovian *if, for every $u \in \mathcal{D}[\mathcal{E}]$, $(u \vee 0) \wedge 1 \in \mathcal{D}[\mathcal{E}]$ and*

$$\mathcal{E}((u \vee 0) \wedge 1) \leq \mathcal{E}(u).$$

Definition 2.4.4 (Normal contraction)**.** *Function $v \in L^2(M)$ is called a* normal contraction *of function $u \in L^2(M)$ if, for all $x, y \in M$,*

(i) $|v(x)| \leq |u(x)|$ *and*

(ii) $|v(y) - v(x)| \leq |u(y) - u(x)|$.

Definition 2.4.5 (Dirichlet form)**.** *A bilinear form is said to be a* Dirichlet form *if it is symmetric, closed and Markovian.*

Proposition 2.4.6. *If $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is a Dirichlet form and v is a normal contraction of $u \in \mathcal{D}[\mathcal{E}]$, then $v \in \mathcal{D}[\mathcal{E}]$ and $\mathcal{E}(v) \leq \mathcal{E}(u)$.*

Proof. See Theorem 1.4.1 from [FOT11]. □

Definition 2.4.7. *A metric measure space is a triple (M,d,m), where M is the state space, d is the distance on M and m is the measure on M.*

Definition 2.4.8 (Regular Dirichlet form)**.** *Let (M,d,m) be a metric measure space. A Dirichlet form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ on $L^2(M)$ is said to be regular if $C_c(M) \cap \mathcal{D}[\mathcal{E}]$ is dense in $\mathcal{D}[\mathcal{E}]$ with respect to $\mathcal{E}_1$ norm and at the same time dense in $C_c(M)$ with respect to the uniform norm.*

Definition 2.4.9. *Let (M,d,m) be a metric measure space. For a given Dirichlet form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$, corresponding contractive resolvent, strongly continuous contracive semigroup and generator in $L^2(M,m)$ are denoted by $\{G_\beta : \beta > 0\}$, $\{P_t : t \geq 0\}$ and $\mathcal{L}$ respectively.*

Remark 2.4.10. *Operators P_t and G_β extend to $L^p(M)$ for all $p \in [1, \infty]$.*

Definition 2.4.11. *The semigroup P_t is said to be conservative if $P_t 1 = 1$ m-a.e. for any/every $t > 0$.*

Definition 2.4.12 (Restricted Dirichlet form, see [GHL10]). *Let (M, m, d) be a metric measure space, U an open subspace of M and $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ a Dirichlet form on $L^2(M)$. The space $L^2(U)$ is embedded into $L^2(M)$ by extending functions from $L^2(U)$ by zero outside of U. Denote by $\mathcal{D}_U[\mathcal{E}]$ the closure of $\mathcal{D}[\mathcal{E}] \cap C_c(U)$ in $\mathcal{E}_1$. It is known that $(\mathcal{E}, \mathcal{D}_U[\mathcal{E}])$ is a regular Dirichlet form on $L^2(U)$ and we call it the* restricted form. *The corresponding restricted semigroup, resolvent and generator in the space $L^2(U)$ are denoted by P_t^U, G_β^U and $\mathcal{L}^U$.*

Definition 2.4.13 (see 1.5 of [FOT11]). *Let U be an open subset of M and $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ a Dirichlet form on M. For $f \in L^2(U)$ we define*

$$S_t^U f = \int_0^t P_s^U f ds$$

where the integration is preformed in the Bochner sense in $L^2(U)$. It is known that S_t extends to $L^1(U)$ and we define $G^U f$ to be the pointwise limit $G^U f(x) = \lim_{N\to\infty} S_N^U f(x) \in [0, \infty]$. This makes sense because, for $t_1 < t_2$, $S_{t_1}^U f(x) \leq S_{t_2}^U f(x)$ m-a.e. as a consequence of the Markov property. If

$$G^U f(x) < \infty \quad m\text{-a.e.} \quad \forall f \in L_+^1(U)$$

the semigroup P_t^U is said to be transient, otherwise it is said to be recurrent.

Definition 2.4.14. *For $f \in L^2(M)$ we adopt the convention*

$$P_t^U f = P_t^U (f\big|_U),$$

where $(f\big|_U)$ is the restriction of f to U. The same convention holds for for G^U.

Proposition 2.4.15. *Let $\{U_k\}_{k\in\mathbb{N}} \subset M$ be a sequence of open sets of finite measure, define $U = \bigcup_{k\in\mathbb{N}} U_k$ and let $f \in L_+^\infty(M)$. Then for every $t \in [0, \infty)$ and m-a.e. $x \in M$*

$$P_t^{U_k} f(x) \xrightarrow{k\to\infty} P_t^U f(x).$$

Proof. Lemma 4.17 from [GH08] proves the claim if $f \in L_+^2(U)$. Here we give the agument for extending it to $f \in L^\infty(U)$. Take any increasing sequence $\{f_l\} \subset L_+^2(U)$ converging to f pointwise ($f 1_{U_l}$ is one such sequence). Then m-a.e. $P_t^U f_l \xrightarrow{l\to\infty} P_t^U f$ by definition of extension of P_t^U on $L^\infty(M)$. As f_l is in $L^2(U)$ for every $l \in \mathbb{N}$, Lemma 4.17 of [GH08] proves that $P_t^{U_k} f_l \xrightarrow{k\to\infty} P_t^U f_l$ m-a.e. for every $k \in \mathbb{N}$. Passing to the limit $l \to \infty$ we get

$$\lim_{l\to\infty} \lim_{k\to\infty} P_t^{U_k} f_l = P_t^U f \quad m\text{-a.e.}$$

On the other hand, for every $k, l \in \mathbb{N}$, $P_t^{U_k} f_l \leq P_t^{U_k} f \leq P_t^U f$ m-a.e. which proves that m-a.e. $P_t^{U_k} f \xrightarrow{k\to\infty} P_t^U f$. □

2.5. Markov processes

Definition 2.5.1 (Skorokhod space, see [Bil99], Chapter 3, Section 12). *Let (E, d_E) be a metric space. For $T \in (0, \infty)$ the* Skorokhod space *$\mathcal{D}([0, T]; E)$ is the space of all functions $x : [0, T] \to E$ which are right continuous and have left limits (also called cadlag functions). The topology on $\mathcal{D}([0, T]; E)$ is induced by the metric*

$$d(x, y) = \inf_{\lambda\in\Lambda} \sup_{t\in[0,T]} \{|\lambda(t) - t| \vee d_E(x(t), y(\lambda(t)))\}$$

where Λ is the set of all strictly increasing, continuous bijections from $[0, T]$ onto $[0, T]$.

Remark 2.5.2. *The space $\mathcal{D}([0,T]:\mathbb{R}^n)$ is not complete with respect to d but it is possible to find a metric (which generates the same topology) with respect to which $\mathcal{D}([0,T];\mathbb{R}^n)$ is complete. This and more on the Skorokhod space can be found in Chapter 12 of [Bil99].*

Next, we give the definitions of a Markov process, strong Markov and Hunt process on a state space E and a couple of properties that will be needed later. The definitions that follow are taken from [CF12] Appendix A. For detailed treatment of the subject the reader should consult [CF12] Appendix A, [FOT11] Section A.2 or [CW05] Chapters 1 to 3.

We will be working under the following assumption.

Assumption 2.5.3 (see [FOT11] (1.1.7))**.** *We assume that (M,d,m) is a locally compact and separable metric measure space and that m is a Radon measure of full support (note that this implies that (M,m) is σ-finite).*

Definition 2.5.4. *Let $(\Omega,\mathcal{M},\mathbb{P})$ be a probability space. Any increasing family $\{\mathcal{M}_t; t\in[0,\infty]\}$ of σ-algebras such that $\mathcal{M}_t\subset\mathcal{M}$ for all $t\in[0,\infty]$ is called a* filtration.

Definition 2.5.5. *Let E be a metric space, $E_\partial = E\cup\{\partial\}$ its one point compactification with a* cemetery point *∂ and $\mathcal{B}(E_\partial)$ the Borel σ-algebra on E_∂. A normal Markov process X_t with time parameter $[0,\infty]$ on the state space E is a quadruple*

$$\left(\Omega,\mathcal{M},\{X_t\}_{t\in[0,\infty]},\{\mathbf{P}_x\}_{x\in E_\partial}\right)$$

satisfying:

1. *For every $x\in E_\partial$, $(\Omega,\mathcal{M},\{X_t\}_{t\in[0,\infty]},\mathbf{P}_x)$ is a stochastic process with state space $(E_\partial,\mathcal{B}(E_\partial))$ and time parameter set $[0,\infty]$ such that $X_\infty(\omega)=\partial$ for every $\omega\in\Omega$.*
2. *For every $t\geq 0$ and $B\in\mathcal{B}(E_\partial)$, $\mathbf{P}_x(X_t\in B)$ is $\mathcal{B}(E_\partial)$-measurable as a function of $x\in E_\partial$.*
3. *There exists an admissible filtration $\{\mathcal{M}_t\}_{t\in[0,\infty]}$ such that, for all $x\in E_\partial$, $t_0,t\geq 0$, $A\in\mathcal{B}(E_\partial)$,*
$$\mathbf{P}_x(X_{t_0+t}\in A|\mathcal{M}_{t_0}) = \mathbf{P}_{X_{t_0}}(X_t\in A) \quad \mathbf{P}_x\text{-a.s.} \tag{2.1}$$
4. *$\mathbf{P}_\partial(X_t=\partial)=1$ for every $t\geq 0$.*
5. *$\mathbf{P}_x(X_0=x)=1$ for every $x\in E_\partial$.*

The expectation with respect to the measure $\mathbf{P}_x$ is denoted by $\mathbf{E}_x$.

Definition 2.5.6. *Let $\{\mathcal{M}_t\}_{t\in[0,\infty]}$ be a filtration on measure space $(\Omega,\mathcal{M})$. A function $\tau:\Omega\to[0,\infty]$ is called an $\mathcal{M}_t$-stopping time if $\{\tau\leq t\}\in\mathcal{M}_t$ for every $t\geq 0$. In that case we define the sigma algebra $\mathcal{M}_\tau$ by*

$$\mathcal{M}_\tau = \{\Lambda\in\mathcal{M}_\infty : \Lambda\cap\{\tau\leq t\}\in\mathcal{M}_t \text{ for every } t\geq 0\}.$$

Definition 2.5.7. *Let X_t be a Markov process and $\{\mathcal{M}_t\}$ an admissible filtration. The* first exit time *of X_t from a subset A of the state space, denoted by $\tau_A:\Omega\to[0,\infty]$, is defined by*

$$\tau_A(\omega) := \inf\{t\geq 0 : X_t(\omega)\notin A\}$$

with convention $\inf\emptyset=\infty$.

Definition 2.5.8. *A Markov process X_t is said to be a strong Markov process if there exists a right continuous admissible filtration $\{\mathcal{M}_t\}$ for which the following stronger version of Eq. (2.1) holds. For every $\{\mathcal{M}_t\}$-stopping time τ, every $t\geq 0$, every $A\in\mathcal{B}(E_\partial)$ and every probability measure μ on $(E_\partial,\mathcal{B}(E_\partial))$,*

$$\mathbf{P}_\mu(X_{\tau+t}\in A|\mathcal{M}_\tau) = \mathbf{P}_{X_\tau}(X_t\in A) \quad \mathbf{P}_\mu\text{-a.s.}$$

where $\mathbf{P}_\mu(B) := \int_{x\in E_\partial}\mathbf{P}_x(B)\mu(dx)$ for $B\in\mathcal{B}(E_\partial)$.

Definition 2.5.9. *A* Hunt process X_t *is a normal strong Markov process on a Lusin space* E *that also satisfies:*

6. $X_t(\omega) = \partial$ *for every* $t \geq \zeta(\omega)$ *where* $\zeta(\omega) = \inf\{t \geq 0 : X_T(\omega) = \partial\}$ *(with convention* $\inf \emptyset = \infty$*) is called the* lifetime *of the sample path of* ω.

7. *For every* $t \geq 0$ *there exists a map* $\theta_t : \Omega \to \Omega$ *such that* $X_t \circ \theta_s = X_{t+s}$ *for every* $s \geq 0$. *Moreover* $\theta_0(\omega) = \omega$ *and* $\theta_\infty(\omega) = [\partial]$ *for every* $\omega \in \Omega$*, where* $[\partial]$ *denotes the special element of* Ω *such that* $X_t([\partial]) = \partial$ *for every* $t \geq 0$.

8. *For every* $\omega \in \Omega$ *the sample path* $t \to X_t(\omega) \in E_\partial$ *is right continuous on* $[0, \infty)$ *and has left limits on* $(0, \infty)$ *in* E_∂.

9. *There exists a right continuous admissible filtration* $\{\mathcal{M}_t\}$ *of* X *such that for every sequence of increasing* $\{\mathcal{M}_t\}$*-stopping times* $\{\tau_k\}$ *with* $\tau = \lim_{k\to\infty} \tau_k$ *and every probability measure* μ *on* $(E_\partial, \mathcal{B}(E_\partial))$,
$$\mathbf{P}_\mu\left(\lim_{k\to\infty} X_{\tau_k} = X_\tau, \tau < \infty\right) = \mathbf{P}_\mu(\tau < \infty).$$

Definition 2.5.10. *Let* (M, d, m) *be a metric measure space. A Hunt process* X_t *on* M *is said to be* m*-symmetric if, for all non-negative* $\mathcal{B}(M)$*-measurable functions* $u, v : M \to \mathbb{R}$,
$$\int_M u(x)\mathbf{E}_x[v(X_t)]m(dx) = \int_M \mathbf{E}_x[u(X_t)]v(x)m(dx).$$

Definition 2.5.11. *Let* (M, d, m) *be a metric measure space satisfying Assumption 2.5.3 and* X_t *an* m*-symmetric Hunt process on* (M, d, m). *Then* P_t *defined for* $f \in C_b(M)$ *by*
$$P_t f(x) := \mathbf{E}_x[f(X_t)] \tag{2.2}$$
extends to a strongly continuous symmetric semigroup on $L^2(M, m)$. *The corresponding regular Dirichlet* $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ *form is said to be* the Dirichlet form of X_t. *Furthermore, we say that* X_t *is* properly associated *to* $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$.

Corollary 2.5.12. *Let* X_t *be a* m*-symmetric Hunt process on* (M, d, m). *If* P_t *is conservative then the lifetime* ζ *of* X_t *is* $\mathbf{P}_x$*-a.s. infinite for* m*-a.e.* $x \in M$.

Proof. For $t \in (0, \infty)$ and $x \in M$, $\mathbf{P}_x(\zeta < t) = 1 - \mathbf{E}_x[1_M(X_t)] = 1 - P_t 1_M(x)$. If P_t is conservative, then $P_t 1_M = 1_M$ so, for m-a.e. $x \in M$,
$$\mathbf{P}_x(\zeta < \infty) \leq \sum_{t\in\mathbb{N}} \mathbf{P}_x(\zeta < t) \leq \sum_{t\in\mathbb{N}} (1(x) - P_t 1_M(x)) = 0.$$
Other way around, if $\mathbf{P}_x(\zeta < \infty) = 0$ for m-a.e. x, then $1_M(x) - P_t 1_M(x) \leq \mathbf{P}_x(\zeta < \infty) = 0$, which implies $P_t 1_M = 1_M$ so P_t is conservative. □

Theorem 2.5.13 (Theorem 7.2.1 for [FOT11])**.** *Let* (M, d, m) *be a metric measure space satisfying Assumption 2.5.3 and* $\mathcal{E}$ *a regular Dirichlet form on* $L^2(M, m)$. *Then there exists an* m*-symmetric Hunt process* M *on* $(M, \mathcal{B}(M))$ *whose Dirichlet form is* $\mathcal{E}$.

Theorem 2.5.14. *If there are two Hunt processes associated to* $\mathcal{E}$*, then they transition functions coincide outside of a properly exceptional set.*

Theorem 2.5.15. *Let* (M, d, m) *be a metric measure space satisfying Assumption 2.5.3,* X_t *an* m*-symmetric Hunt process and* P_t *its* $L^2(M, m)$*-semigroup which for all* $t > 0$ *satisfies*
$$P_t\left(L^2(M) \cap C_0(M)\right) \subset C_0(M).$$

Then for all $k \in \mathbb{N}$, $t_1, t_2, \ldots t_k \in (0, \infty)$, $f_1, f_2, \ldots f_k \in C_0(M) \cap L^2(M)$ and m-a.e. $x \in E$,

$$\begin{aligned}\mathbf{E}_x\left[f_1(X_{t_1})f_2(X_{t_2})\ldots f_k(X_{t_k})\right] \\ = P_{t_1}(f_1 P_{t_2-t_1}(f_2 P_{t_3-t_2}(\ldots f_{k-1}P_{t_k-t_{k-1}}(f_k)\ldots)))(x).\end{aligned} \tag{2.3}$$

Proof. By Markov property (Eq. (2.1)) of X_t (used in the third line) and definition of semigroup P_t from Eq. (2.2)

$$\begin{aligned}&\mathbf{E}_x\left[f_1(X_{t_1})f_2(X_{t_2})\ldots f_k(X_{t_k})\right] \\ &= \mathbf{E}_x\left[f_1(X_{t_1})\ldots f_{k-1}(X_{t_{k-1}})\mathbf{E}_x\left[f_k(X_{t_k})|X_1,\ldots,X_{t_{k-1}}\right]\right] \\ &= \mathbf{E}_x\left[f_1(X_{t_1})\ldots f_{k-1}(X_{t_{k-1}})\mathbf{E}_{X_{t_{k-1}}}\left[f_k(X_{t_k-t_{k-1}})\right]\right] \\ &= \mathbf{E}_x\left[f_1(X_{t_1})\ldots f_{k-1}(X_{t_{k-1}})P_{t_k-t_{k-1}}f_k(X_{t_{k-1}})\right].\end{aligned}$$

Since $\widetilde{f}_{k-1} := P_{t_k-t_{k-1}}f_k \in C_0(M) \cap L^2(M)$ by assumption the statement is proved by an induction in k. □

Theorem 2.5.16. *Let X_t be a Hunt process on $\mathbb{R}^n_\partial$ such that its lifetime ζ is $\mathbf{P}_x$-a.s. strictly greater than T for all $x \in \mathbb{R}^n$ and $T \in (0, \infty)$. Then measures $\mathbf{P}_x$, for $x \in \mathbb{R}^n$, can be considered as probability measures on $\mathcal{D}([0,T];\mathbb{R}^n)$.*

Proof. Fix an $x \in \mathbb{R}^n$ and let $\widetilde{\Omega}$ be the set of $\mathbf{P}_x$ probability 1 such that $\zeta(\omega) > T$ for every $\omega \in \widetilde{\Omega}$. Then $X_t(\omega) \in \mathbb{R}^n$ for all $\omega \in \widetilde{\Omega}$ and $t \geq 0$ so the mapping $\mathscr{M} : \widetilde{\Omega} \to D([0,T];\mathbb{R}^n)$, $\mathscr{M}(\omega) = (X_t(\omega))_{t\in[0,T]}$ is well defined. Notice that paths $t \to X_t(\omega)$ are right continuous and have left limits because of Item 8 of Definition 2.5.9. By [Bil99] 12.5 (ii) $\mathscr{M}$ is measurable if and only if $\mathscr{M}_t : \widetilde{\Omega} \to \mathbb{R}^n$ defined by $\mathscr{M}_t(\omega) = X_t(\omega)$ is measurable for every $t \in [0,T]$. Since this is true for every stochastic process we can consider $\mathscr{M}$-pushforward of $\mathbf{P}_x$ on $\mathcal{D}([0,T];\mathbb{R}^n)$ instead of $\mathbf{P}_x$. This is independent of the choice of $\widetilde{\Omega}$ because $\mathbf{P}_x(\Omega \setminus \widetilde{\Omega}) = 0$. □

2.6. Volume regularity of $\mathbb{Z}^n$

In this section we will consider the metric measure space $(\mathbb{Z}^n, d, \#)$, where # denotes the counting measure on $\mathbb{Z}^n$ and d denotes the Euclidean distance on $\mathbb{Z}^n$, for arbitrary $n \in \mathbb{N}$. .

Lemma 2.6.1. *For every ball $B \subset \mathbb{Z}^n$ with radius $R \geq 0$*

$$\#B \geq (2\sqrt{n})^{-n}R^n$$

and hence $(\mathbb{Z}^n, d, \#)$ satisfies $\boldsymbol{V}_{\geq}[\mathbb{Z}^n, [0,\infty); n, (2\sqrt{n})^{-n}]$. On the other hand, for every ball $B \subset \mathbb{Z}^n$ with radius $R \geq 1$

$$\#B \leq 3^n R^n$$

which shows that $(\mathbb{Z}^n, d, \#)$ satisfies $\boldsymbol{V}[\mathbb{Z}^n, [1,\infty); n, (2\sqrt{n})^{-n}, 3^n]$ holds.

Remark 2.6.2. *Notice that the second statement cannot hold for every $R > 0$ because any nonempty ball in $\mathbb{Z}^n$ has the size at least 1.*

Proof. Let $x_0 \in \mathbb{Z}^n$, $R \geq 0$ be arbitrary and set $B = B(x,R)$. We can approximate B with cubes from inside and outside in the following way

$$\left(x - \frac{R}{\sqrt{n}}, x + \frac{R}{\sqrt{n}}\right)^n \cap \mathbb{Z}^n \subset B \subset [x - R, x + R]^n \cap \mathbb{Z}^n.$$

The size of these cubes is not hard to estimate. For the smaller cube take any $a > 1$ and notice that the interval $(x_i + a, x_i + a) \cap \mathbb{Z}$ (where x_i is any coordinate of x) contains at least $1 + 2\lfloor a - 1\rfloor \vee 0$

points (one is always in the center). If $a < 2$, then $1 + 2\lfloor a-1 \rfloor \vee 0 \geq 1 \geq a/2$ and otherwise if $a \geq 2$, then $1 + 2\lfloor a-1 \rfloor \geq 1 + 2\lfloor a/2 \rfloor \geq \lfloor a/2 + 1 \rfloor \geq a/2$ so we get the estimate $1 + 2\lfloor a-1 \rfloor \vee 0 \geq a/2$ in any case. Therefore, using the product structure of cubes in $\mathbb{Z}^n$, for any $R > 0$

$$\#B \geq \left(\#\left(x - \frac{R}{\sqrt{n}}, x + \frac{R}{\sqrt{n}}\right) \cap \mathbb{Z}\right)^n \geq \frac{R^n}{(2\sqrt{n})^n}.$$

On the other hand, if $a \geq 1$, then segment $[x-a, x+a] \cap \mathbb{Z}$ contains at most $2\lfloor a \rfloor + 1 \leq 2a + 1 \leq 3a$ elements and therefore

$$\#B \leq (\#[x-R, x+R] \cap \mathbb{Z})^n \leq 3^n R^n.$$

□

Definition 2.6.3. *We give special names to constants from the previous theorem. Define $C_{VL}(\mathbb{Z}^n) := (2\sqrt{n})^{-n}$ and $C_{VU}(\mathbb{Z}^n) := 3^n$. The $(\mathbb{Z}^n)$ part is sometimes omitted for brevity if it is clear from the context.*

Lemma 2.6.4. *Let $n \in \mathbb{N}$ be arbitrary. For every $\varepsilon > 0$ there exists an $R_V \equiv R_V(n, \varepsilon)$ such that for every ball $B \subset \mathbb{Z}^n$ with radius $R \geq R_V$*

$$(V_n - \varepsilon)R^n \leq |B| \leq (V_n + \varepsilon)R^n$$

where $V_n = \pi^{\frac{n}{2}}\Gamma(n/2+1)^{-1}$ is the volume of the unit ball in $\mathbb{R}^n$. In other words, $(\mathbb{Z}^n, d, \#)$ satisfies $\boldsymbol{V}[\mathbb{Z}^n, [R_V, \infty); n, V_n - \varepsilon, V_n + \varepsilon]$.

Proof. Let $x_0 \in \mathbb{Z}^n$, $R \geq 0$ be arbitrary and set $B = B(x, R)$. We use subscript $\mathbb{R}$ to distinguish $\mathbb{R}^n$ balls from $\mathbb{Z}^n$ balls, $Q(x, a) = [x-a, x+a]^n \subset \mathbb{R}^n$ to denote the cube around x_0 of side length $2a$ in $\mathbb{R}^n$, for $x \in \mathbb{R}^n$, $a > 0$, and λ to denote the Lebesgue measure on $\mathbb{R}^n$. Then

$$B_{\mathbb{R}}(x_0, R - \sqrt{n}/2) \subset \bigcup_{x \in B} Q(x, 1/2) \subset \overline{B_{\mathbb{R}}}(x_0, R + \sqrt{n}/2).$$

Since

$$\#B = \#\bigcup_{x \in B} Q(x, 1/2) = \lambda\left(\bigcup_{x \in B} Q(x, 1/2)\right)$$

using $\lambda(B(x, a)) = V_n a^n$ we end up with

$$V_n\left(R - \sqrt{n}/2\right)^n \leq \#B \leq V_n\left(R + \sqrt{n}/2\right)^n. \tag{2.4}$$

Rewriting Ineq. (2.4) we also see that

$$V_n\left(\frac{R - \sqrt{n}/2}{R}\right)^n R^n \leq \#B \leq V_n\left(\frac{R + \sqrt{n}/2}{R}\right)^n R^n.$$

Since $\frac{R-\sqrt{n}/2}{R}, \frac{R+\sqrt{n}/2}{R} \to 1$ as $R \to \infty$ this shows that for every $\varepsilon > 0$ it is possible to find $R_V > 0$ such that $(\mathbb{Z}^n, d, \#)$ satisfies $\mathbf{V}[\mathbb{Z}^n, [R_V, \infty); n, V_n - \varepsilon, V_n + \varepsilon]$. □

Remark 2.6.5. *The previous lemma gives the simplest estimate concerning the famous Gauß circle problem. Much sharper estimates are know and a good starting place to look them up might be [IKKN06].*

2.7. Symmetric jump type forms

Let (M, d, m) be a metric measure space for the rest of this section.

Definition 2.7.1. *A function $k : M \times M \to [0, \infty]$ is said to be a (jump) kernel on M if it is Borel measurable on $M \times M$. A kernel is said to be symmetric if $k(x, y) = k(y, x)$ $m \times m$-a.e.*

Definition 2.7.2. *A form $\mathcal{E}$ acting on Borel measurable function on M is said to be of* jump type *if for every Borel measurable function f on M*

$$\mathcal{E}(f) = \int_M \int_M (f(x) - f(y))^2 k(x, y) dy dx \tag{2.5}$$

for some jump kernel $k : M \times M \to [0, \infty]$.

Definition 2.7.3 (Energy density)**.** *For symmetric kernel k on M we define* the carré du champ operator Γ *acting on a Borel measurable function f on M by*

$$\Gamma f(x) = \int_M (f(x) - f(y))^2 k(x, y) dy \in [0, \infty]$$

for $x \in M$. $\Gamma f(x)dx$ is then called the energy measure *of f.*

Proposition 2.7.4. *Let $\mathcal{E}$ be a symmetric jump type form on $L^2(M)$ defined by Eq.* (2.5) *for some symmetric kernel k. If we take $\mathcal{D}[\mathcal{E}] = \{f \in L^2(M) : \mathcal{E}(f) < \infty\}$ then the form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is closed.*

Proof. We follow Example 1.2.4 of [FOT11]. Let $u \in L^2(M)$ be arbitrary and $\{u_l\} \subset L^2(M)$ an $\mathcal{E}_1$-Cauchy sequence such that $u_l \to u$ in $L^2(M)$. Because u_l converges to u in $L^2(M)$, we can find a subsequence which u_{l_i} which converges to u a.e. By Fatou's lemma, for every $m \geq 0$,

$$\begin{aligned}\mathcal{E}(u - u_m) &= \int_{M \times M} \lim_{l_i \to \infty} \left[u_{l_i}(x) - u_m(x) - u_{l_i}(y) + u_m(y)\right]^2 k(x, y) dy dx \\ &\leq \liminf_{i \to \infty} \mathcal{E}(u_{l_i} - u_m).\end{aligned}$$

The last term can be made arbitrarily small for m large enough. Thus $u \in \mathcal{D}[\mathcal{E}]$ and $\mathcal{E}_1(u - u_m) \to 0$. □

The next lemma gives an estimate on the pointwise energy density of cutoff functions for kernel $k = d(x, y)^{-(n+2s)}$.

Lemma 2.7.5. *Let (M, d, m) be a metric measure space, $x \in M$ arbitrary and suppose that there is a $C_{VU} > 0$ such that*

$$\forall R > 0 \qquad |B(x, R) \setminus \{x\}| \leq C_{VU} R^n$$

for some $n \geq 2$. Then there exists a $C_{(2.7.5)} \equiv C_{(2.7.5)}(s, n, C_{VU})$ such that, for all $s \in (0, 1)$ and bounded Lipschitz function $\psi : M \to \mathbb{R}$,

$$\int_M \frac{(\psi(x) - \psi(y))^2}{d(x, y)^{n+2s}} dy \leq C_{(2.7.5)} \|\psi\|_{L^\infty(M)}^{2-2s} (\operatorname{Lip} \psi)^{2s}.$$

Relating this to ***CE****, the result of the theorem is equivalent to saying that form $\mathcal{E}$,corresponding to kernel $k(x, y) = d(x, y)^{-(n+2s)}$, satisfies* ***CE***$[M, (0, \infty); s, Q = \infty, \gamma = 0, C_C = C_{(2.7.5)}]$.

Proof. Let us denote $L := \operatorname{Lip} \psi$ and $\|\psi\|_\infty := \|\psi\|_{L^\infty(M)}$. For arbitrary $a > 0$

$$\int_M \frac{(\psi(x) - \psi(y))^2}{d(x, y)^{n+2s}} dy \leq \int_{B(x,a) \setminus \{x_0\}} \frac{L^2 d(x, y)^2}{d(x, y)^{n+2s}} dy + \int_{M \setminus B(x,a)} \frac{4\|\psi\|_\infty^2}{d(x, y)^{n+2s}} dy =: I_1 + I_2.$$

The integrals I_1 and I_2 will be computed separately. For I_1 we use a dyadic decomposition to compute

$$\begin{aligned} I_1 &= L^2 \sum_{k=0}^{\infty} \int_{B(x_0,2^{-k}a)\setminus B(x_0,2^{-(k+1)}a)} d(x,y)^{2-n-2s} dy \\ &\leq L^2 \sum_{k=0}^{\infty} (2^{-(k+1)}a)^{2-n-2s} |B(x_0, 2^{-k}a) \setminus \{x_0\}|. \end{aligned}$$

The volume in the last expression can be estimate using the assumption which gives

$$I_1 \leq 2^{n+2s-2} L^2 a^{2-2s} \sum_{k=0}^{\infty} C_{VU} (2^{2-2s})^{-k} \leq \frac{2^n}{2^{2-2s}-1} L^2 C_{VU} a^{2-2s}$$

where the sum converges because $2^{2s-2} < 1$. The estimate of the second integral can be obtained in a similar way,

$$\begin{aligned} I_2 &= 4\|\psi\|_\infty^2 \sum_{k=0}^{\infty} \int_{B(x_0,2^{k+1}a)\setminus B(x_0,2^{k}a)} d(x,y)^{-n-2s} dy \\ &\leq 4\|\psi\|_\infty^2 \sum_{k=0}^{\infty} (2^k a)^{-n-2s} |B(x_0, 2^{k+1}a) \setminus \{x_0\}|. \end{aligned}$$

The volume on the right can again be estimated using the assumption which leads to

$$I_2 \leq 4\|\psi\|_\infty^2 \sum_{k=0}^{\infty} 2^n a^{-2s} C_{VU} 2^{-2sk} \leq \frac{2^{n+2}}{1-2^{-2s}} \|\psi\|_\infty^2 C_{VU} a^{-2s}.$$

Collecting the estimates for I_1 and I_2 we see that for every $a > 0$

$$\int_{M\setminus\{x_0\}} \frac{(\psi(x)-\psi(y))^2}{d(x,y)^{n+2s}} dy \leq \frac{2^n}{2^{2-2s}-1} L^2 C_{VU} a^{2-2s} + \frac{2^{n+2}}{1-2^{-2s}} \|\psi\|_\infty^2 C_{VU} a^{-2s}.$$

As the first term is increasing in a and explodes for $a \to \infty$ and the second one is decreasing in a and explodes for $a \to 0$ the minimum is obtained when

$$\frac{d}{da}\left(\frac{2^n}{2^{2-2s}-1} L^2 C_{VU} a^{2-2s}\right) + \frac{d}{da}\left(\frac{2^{n+2}}{1-2^{-2s}} \|\psi\|_\infty^2 C_{VU} a^{-2s}\right) = 0.$$

It is not hard to check that this happens for

$$a^2 = \frac{4s(2^{2-2s}-1)}{(1-s)(1-2^{-2s})} \|\psi\|_\infty^2 L^{-2}$$

which gives the bound

$$\begin{aligned} \int_{M\setminus\{x_0\}} \frac{(\psi(x)-\psi(y))^2}{d(x,y)^{n+2s}} dy &\leq 2^n C_{VU} \left[\left(\frac{s}{1-s}\right)^{1-s} + \left(\frac{s}{1-s}\right)^{-s}\right] \left(\frac{4}{1-2^{-2s}}\right)^{1-s} \\ &\quad \times \left(2^{2-2s}-1\right)^{-s} \|\psi\|_\infty^{2-2s} L^{2s} \\ &\leq 2^n C_{VU} \left(\frac{4}{(1-s)(1-2^{-2s})}\right)^{1-s} \left(\frac{1}{s(2^{2-2s}-1)}\right)^{s} \|\psi\|_\infty^{2-2s} L^{2s}. \end{aligned}$$

The claim follows by defining $C_{(2.7.5)} \equiv C_{(2.7.5)}(s, n, C_{VU})$ to be the factor in front of $\|\psi\|_\infty^{2-2s} L^{2s}$. □

Corollary 2.7.6. *Let $n \geq 2$ and let $\psi : \mathbb{Z}^n \to [0,1]$ be a Lipschitz function. Then there exists a $C_{(2.7.6)} \equiv C_{(2.7.6)}(n,s)$ such that for every $s \in (0,1)$*

$$\left\| \sum_{y \in \mathbb{Z}^n} \frac{(\psi(y) - \psi(x))^2}{d(x,y)^{n+2s}} \right\|_{L^\infty(\mathbb{Z}^n)} \leq C_{(2.7.6)} (\operatorname{Lip} \psi)^{2s}.$$

Proof. It is sufficient to notice that $M = \mathbb{Z}^n$ with counting measure $m = \#$ satisfies the estimate

$$|B(x,R) \setminus \{x\}| \leq C_{VU}(\mathbb{Z}^n) R^n$$

for all $x \in \mathbb{Z}^n$, $R > 0$. Indeed, if $R < 1$, then the set on the left is empty so the statement holds and if $R \geq 1$, then this is implied by Lemma 2.6.1. Applying Lemma 2.7.5 leads to the inequality claimed if one takes into account $\|\psi\|_\infty \leq 1$, renames $C_{(2.7.5)}$ to $C_{(2.7.6)} \equiv C_{(2.7.6)}(s, n, C_{VU}(\mathbb{Z}^n))$ and notices that $C_{VU}(\mathbb{Z}^n)$ depends only on n. □

2.8. Generalized Mosco convergence

The concept of Mosco convergence of a sequence of bilinear forms was introduced in [Mos94]. We present here a generalization of this concept, called "generalized Mosco convergence", introduced in [KS03] and also in [Kim06]. The results which follow can be found in [CKK13], appendix 8.

Remark 2.8.1. *It is sometimes convenient to consider symmetric bilinear forms $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ on Hilbert space H to be defined on whole H but take value ∞ outside of $\mathcal{D}[\mathcal{E}]$. In that sense the domain $\mathcal{D}[\mathcal{E}]$ of the form is sometimes not explicitly stated.*

Definition 2.8.2. *Suppose that we are given a sequence of Hilbert spaces $(H_k, \langle \cdot, \cdot \rangle_k)$, for $k \in \mathbb{N}$, and a "limiting" Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Sequences $E_k : H_k \to H$ and $\pi_k : H \to H_k$ of bounded operators such that*

(i) *π_k is the adjoint of E_k, that is $\langle \pi_k f, f_k \rangle_k = \langle f, E_k f_k \rangle$ for all $f \in H$, $f_k \in H_k$,*

(ii) *E_k is the right inverse of π_k, $\pi_k E_k f_k = f_k$ for every $f_k \in H_k$,*

(iii) $\sup_{k \in \mathbb{N}} \|\pi_k\|_{H \to H_k} < \infty$,

(iv) $\lim_{k \to \infty} \|\pi_k f\|_{H_k} = \|f\|_H$ *for every $f \in H$,*

(v) $\langle E_k f, E_k g \rangle = \langle f, g \rangle_k$ *for every $f, g \in H_k$*

are called extension *and* projection *operators respectively.*

Setting 2.8.3. *Suppose that we are given a sequence of Hilbert spaces $(H_k, \langle \cdot, \cdot \rangle_k)$ with a corresponding sequence of densely defined symmetric closed bilinear forms $(\mathcal{E}_k, \mathcal{D}(\mathcal{E}_k))$ and a "limiting" Hilbert space $(H, \langle \cdot, \cdot \rangle)$ together with a closed symmetric bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Suppose also that sequences of extension and projection operator E_k and π_k are given.*

Definition 2.8.4 (Generalized Mosco convergence)**.** *Under Setting 2.8.3 we say that the sequence of forms $\mathcal{E}_k$ converges to $\mathcal{E}$ in the generalized Mosco sense if the following two statements are satisfied:*

(i) *If $u_k \in H_k$ and $u \in H$ are such that $E_k u_k \rightharpoonup u$ weakly in H, then*

$$\liminf_{k \to \infty} \mathcal{E}_k(u_k) \geq \mathcal{E}(u).$$

(ii) *For every $u \in H$ there is a sequence $u_k \in H_k$ such that $E_k u_k \to u$ strongly in H and*

$$\limsup_{k \to \infty} \mathcal{E}_k(u_k) \leq \mathcal{E}(u).$$

The following lemma is a paraphrase of [CKK13] Lemma 8.2 (see also [Kol06] Lemma 2.8).

Theorem 2.8.5. *Under Setting 2.8.3, $\mathcal{E}_k$ converges to $\mathcal{E}$ in the generalized Mosco sense if, in Definition 2.8.4, Item (i) holds and instead of Item (ii) the following three statements are satisfied:*

(i') there exists a set $\mathcal{D} \subset H$ which is $\mathcal{E}_1$-dense in $\mathcal{D}[\mathcal{E}]$,

(ii') $\pi_k(\varphi) \in \mathcal{D}[\mathcal{E}_k]$ for every $\varphi \in \mathcal{D}$,

(iii') for every $\varphi \in \mathcal{D}$,

$$\limsup_{k\to\infty} \mathcal{E}_k(\pi_k\varphi) = \mathcal{E}(\varphi).$$

Proof. The statement is the same as in Lemma 8.2 of [CKK13] up to notational changes. □

Generalized Mosco convergence of densely defined symmetric closed bilinear forms is equivalent to the strong convergence of the corresponding semigroups. The precise statement is given in the following theorem which is taken from [CKK13].

Theorem 2.8.6. *Under Setting 2.8.3 the following are equivalent:*

(i) $\mathcal{E}_k \xrightarrow{k\to\infty} \mathcal{E}$ in the generalized Mosco sense.

(ii) $E_k P_t^{(k)} \pi_k \xrightarrow{k\to\infty} P_t$ strongly on H and the convergence is uniform on any finite time interval of $t \geq 0$.

Proof. See [CKK13], Theorem 8.3. □

3. List of properties

We list here the assumption and properties that will be used throughout the thesis. For the purpose of this chapter let (M, d, m) be metric measure space, $\mathcal{E} : L^2(M) \to [0, \infty]$ a bilinear form on $L^2(M)$, $k : M \times M \to [0, \infty)$ a Borel measurable kernel on $M \times M$ and $c : M \times M \to [0, \infty)$ a Borel measurable "conductance" on $M \times M$.

Whenever a kernel k is mentioned in properties below, it is implicitly assumed that $\mathcal{E}$ is the corresponding energy form on $L^2(M)$ (allowed to take value $+\infty$ for some functions) defined by

$$\mathcal{E}(f) = \int_M \int_M (f(x) - f(y))^2 k(x, y) dy dx \quad \forall f \in L^2(M).$$

In that case we define, for $U \subset M$,

$$\mathcal{E}_U(f) = \int_U \int_U (f(x) - f(y))^2 k(x, y) dy dx \quad \forall f \in L^2(U).$$

Similarly, whenever a conductance c is mentioned in properties below, it is implicit assumed that k is the kernel defined by $k(x, y) = c(x, y) d(x, y)^{-(n+2s)}$.

3.1. Local properties describing jump kernels

The following properties are used to describe a metric measure space (M, d, m), bilinear form $\mathcal{E}$, kernel k or conductance c. As a rule, these properties will apply in certain ball B of M so they will contain parameters $x_0 \in M$ and a radius $R > 0$ indicating the center and the radius of B.

Property 3.1.1 (V). *We say that the measure space (M, d, m) satisfies the volume regularity property on a ball $B := B(x_0, R) \subset M$ with constants $n, C_{VL}, C_{VU} \in (0, \infty)$ if*

$$C_{VL} R^n \leq m\,(B(x_0, R)) \leq C_{VU} R^n.$$

In short we simply say that $\boldsymbol{V}[x_0, R; n, C_{VL}, C_{VU}]$ is satisfied. If only the lower bound holds, we say that (M, d, m) satisfies the lower volume regularity, $\boldsymbol{V}{\geq}[x_0, R; n, C_{VL}]$, and if only the upper bound holds we say that (M, d, m) satisfies upper volume regularity, $\boldsymbol{V}{\leq}[x_0, R; n, C_{VU}]$.

Property 3.1.2 (PSI). *We say that the functional $\mathcal{Q} : L^1(M) \to \mathbb{R}$ satisfies Poincaré-Sobolev inequality with constants $p \in [1, \frac{n}{s})$, $q \in (\frac{n}{sp}, \infty]$, $C_{PS} < \infty$ on a ball $B := B(x_0, R)$ if, for $\rho > 1$ solving $\frac{1}{\rho} = 1 - \frac{sp}{n} + \frac{1}{q}$ and every $f \in L^1(M)$ supported in B, the inequality*

$$\|(f - f_M)^p\|_{L^\rho(M)} \leq C_{PS} R^{\frac{n}{q}} \mathcal{Q}(f)$$

holds, where $f_M = \fint_M f$ (by definition $f_M = 0$ if $|M| = \infty$). In short, we say that $\boldsymbol{PSI}[x_0, R; s, p, q, C_{PS}]$ is satisfied.

Property 3.1.3 (SI). *We say that the kernel k satisfies a version of Sobolev inequality with constants $\rho \in (1, \infty)$, $\zeta \in [1, \infty)$, $s \in (0, 1)$, $\gamma \in [0, 2s)$, $C_{S1}, C_{S2} < \infty$ on a ball $B \equiv B(x_0, R) \subset M$ if, for every $f \in L^1(B)$ and every $\sigma \in (0, 1)$,*

$$\|f^2\|_{L^\rho(\sigma B)} \leq C_{S1} |B|^{\frac{1}{\rho} - 1} R^{2s} \mathcal{E}_B(f) + C_{S2}(1 - \sigma)^{-2s-\gamma} |B|^{\frac{1}{\rho} - \frac{1}{\zeta}} \|f^2\|_{L^\zeta(B)}. \tag{3.1}$$

In short, we say that $\boldsymbol{SI}[x_0, R; s, \rho, \zeta, C_{S1}, C_{S2}, \gamma]$ is satisfied.

Property 3.1.4 (PI). *We say that the kernel k satisfies the L^2-Poincaré inequality with constants $s \in (0,1)$, $C_P < \infty$ on a ball $B := B(x_0, R) \subset M$ if, for every $f \in L^1(B)$,*

$$\|f - f_B\|_2^2 \leq C_P R^{2s} \mathcal{E}_B(f),$$

where $f_B = \fint_B f$. In short, we say that $\boldsymbol{PI}[x_0, R; s, C_P]$ is satisfied.

Property 3.1.5 (CE). *We say that the kernel k satisfies cutoff energy density estimate on a ball $B := B(x_0, R) \subset M$ with constants $Q \in [1, \infty]$, $s \in (0,1)$, $C_C < \infty$, $\gamma \in [0, 2s)$ if for every Lipschitz function $\varphi : M \to [0,1]$*

$$\left(\fint_B \Gamma\varphi(x)^Q dx \right)^{\frac{1}{Q}} \leq C_C R^{-2s} (\xi^{2s-\gamma} \vee \xi^{2s+\gamma})$$

where $\xi = R \operatorname{Lip} \varphi$. In short, we say that $\boldsymbol{CE}[x_0, R; s, Q, \gamma, C_C]$ is satisfied.

Property 3.1.6 (AKB$\geq$). *We say that the kernel $k : M \times M \to [0, \infty)$ is in average bounded from below on the ball $B(x_0, R) \subset M$ with constant $C_K > 0$ if there exists a $y_0 \in M \setminus B(x_0, 6R)$ such that*

$$\fint_{B(x_0,R)} \int_{B(y_0,R)} k(x,y) dy dx \geq C_K R^{-2s}.$$

In short, we say that $\boldsymbol{AKB}{\geq}[x_0, R; s, C_K]$ is satisfied.

Property 3.1.7 (TB). *We say that the kernel k satisfies truncation bound on a ball $B := B(x, R) \subset M$ with constant $s \in (0,1)$, $C_T < \infty$ if*

$$\int_{M \setminus B(x,R)} k(x,y) dy \leq C_T R^{-2s}.$$

In short, we say that $\boldsymbol{TB}[x_0, R; s, C_T]$ is satisfied.

For the following three properties the supersolutions of equations $\mathcal{L}u = 0$ and $\partial_t u - \mathcal{L}u = f$ are defined as in Definition 5.1.5 and Definition 5.1.4.

Property 3.1.8 (WEHI). *We say that $\mathcal{E}$ satisfies the weak elliptic Harnack inequality on the ball $B := B(x_0, R) \subset M$ with constant $C_{EH} < \infty$ if, for every supersolution u of $\mathcal{L}u = 0$ in $2B$ with $u \geq 0$ on M, the inequality*

$$\fint_{\frac{1}{2}B} u(x) dx \leq C_{EH} \operatorname*{ess\,inf}_{\frac{1}{2}B} u$$

holds. In short, we say that $\boldsymbol{WEHI}[x_0, R; C_{EH}]$ holds.

Property 3.1.9 (WPHI). *We say that $\mathcal{E}$ satisfies the weak parabolic Harnack inequality on a ball $B := B(x_0, R) \subset M$ with constants $s \in (0,1)$, $C_{PH} < \infty$ if the following statement holds. For all $t_0 \in \mathbb{R}$, $f \in L^\infty(I(R); L^Q(2B))$ and for every supersolution u of $\partial_t u - \mathcal{L}u = f$ in $I(R) \times 2B(x_0, R)$ with $u \geq 0$ on M the inequality*

$$\fint_{U_\ominus} u \leq C_{PH} \left(\operatorname*{ess\,inf}_{U_\oplus} u + (2R)^{2s} \sup_{t \in I(R)} \left(\fint_{2B} |f(t)|^Q \right)^{\frac{1}{Q}} \right)$$

is true. Here

$$\begin{aligned} I(R) &= (t_0 - R^{2s}, t_0 + R^{2s}) \\ U_\oplus &:= (t_0 + R^{2s} - (R/2)^{2s}, t_0 + R^{2s}) \times \frac{1}{2}B \\ U_\ominus &:= (t_0 - R^{2s}, t_0 - R^{2s} + (R/2)^{2s}) \times \frac{1}{2}B. \end{aligned}$$

In short, we say that $\boldsymbol{WPHI}[x_0, R; s, C_{PH}, Q]$ is true.

Property 3.1.10 (HR). *We say that $\mathcal{E}$ satisfies Hölder regularity in a ball $B := B(x_0, R) \subset M$ with constants $\eta > 0$ and $C_H < \infty$ if the following statement holds. For all $\mathcal{R} \geq R$, $t_0 \in \mathbb{R}$ and every supersolution u of $\partial_t u - \mathcal{L}u = 0$ in $(t_0 - 2\mathcal{R}^{2s}, t_0) \times B(x_0, 2\mathcal{R})$ with $u \geq 0$ on M,*

$$\operatorname*{ess\,osc}_{[t_0-R^{2s},t_0]\times B(x_0,R)} u \leq C_H \|u\|_{L^\infty((t_0-2\mathcal{R}^{2s},t_0)\times M)} \left(\frac{R}{\mathcal{R}}\right)^\eta.$$

*In short, we say that **HR**$[x_0, R; \eta, C_H]$ is satisfied.*

Property 3.1.11 (ETE). *We say that $\mathcal{E}$ satisfies two-sided expected exit time estimates on $B := B(x_0, R) \subset M$ with constants $s \in (0,1)$, $C_{(E\leq)} < \infty$ and $C_{(E\geq)} > 0$ if*

$$C_{(E\geq)} R^{2s} \leq \operatorname*{ess\,inf}_{x\in\frac{1}{4}B} G^B 1 \leq \operatorname*{ess\,sup}_{x\in B} G^B 1 \leq C_{(E\leq)} R^{2s},$$

*where G^B is the potential operator from Definition 2.4.13. In short, we say that **ETE**$[x_0, R; s, C_{(E\leq)}, C_{(E\geq)}]$ is satisfied.*

Property 3.1.12 (SE). *We say that the semigroup P_t corresponding to the form $\mathcal{E}$ satisfies survival estimate with parameters $s \in (0,1)$, $\varepsilon, \delta > 0$ on a ball $B := B(x_0, R) \subset M$ if, for all $t \in [0, (\delta R)^{2s}]$,*

$$\operatorname*{ess\,inf}_{x\in\frac{1}{4}B} P_t^B 1_B(x) \geq \varepsilon.$$

*In short we say that **SE**$[x_0, R; s, \varepsilon, \delta]$ is satisfied.*

Property 3.1.13 (BA). *For $p, C_M \in \mathbb{R}$ we say that the conductance c on $\mathbb{Z}^n$ has p-average bounded by C_M around $x_0 \in \mathbb{Z}^n$ if*

$$\limsup_{\substack{k,l\to\infty\\ k,l\in\mathbb{N}}} \frac{1}{\#B(x_0,k)} \sum_{x\in B(x_0,k)} \frac{1}{\#B(x,l)} \sum_{y\in B(x_0,l)} c(x,y)^p \leq C_M,$$

*where $\limsup_{k,l\to\infty} f(k,l) = \sup\{\limsup_{i\to\infty} f(k_i, l_i) : \text{ for any } k_i \to \infty, l_i \to \infty\}$. In short, we say that **BA**$[x_0; p, C_M, n]$ is satisfied*

3.2. Conventions

Most of the time form $\mathcal{E}$, kernel k and conductance c will be clear from the context and in that case we will simply say that certain **Property** holds instead of saying that it is satisfied for $\mathcal{E}$, k or c.
Statements of form "**Property**$[A, \ldots; B, \ldots]$ holds", where A and B are sets given in place of a concrete parameters, are understood in the sense that **Property**$[a, \ldots; b, \ldots]$ holds for all choices of parameters in $a \in A$ and $b \in B$.
We will try to suggest the matching of parameters through their notation and not through the order alone. That is, reader will not find statements of the form "**WPHI**$[x_0, R; 1, 2, 3]$ holds" but rather of the form "**WPHI**$[x_0, R; s_1, C_{PH}^{(2)}, Q_3]$ holds", where names s_1, $C_{PH}^{(3)}$ and Q_2 indicate the corresponding parameters in the definition of **WPHI**. If the first situation can not be avoided, we will instead write "**WPHI**$[x_0, R; s = 1, C_{PH} = 2, Q = 3]$ holds".

The construction in the next definition is an imitation of the "very good ball" from [Bar04]. Similar constructions are also used in [CKW18b] and [CKW18a] although they are not stated explicitly.

Definition 3.2.1. *We say that some property holds on scales larger that $\theta \in (0,1)$ in a ball $B(x_\star, R_\star)$, and write $\star$**Property**$[x_\star, R_\star, \theta; \ldots]$ if the following statement is satisfied. For all $R_0 \in \mathbb{N} \cap [R_\star, \infty)$, $x_0 \in B(x_\star, R_0)$, $R \geq R_0^\theta$ **Property**$[x_0, R; \ldots]$ holds.*

Part I.

Deterministic degenerate energy forms of jump type

4. Assumptions and main ideas

Let (M, d, m) be a metric measure space and assume, for the sake of this introduction, that the volume of a ball $B \subset M$ with radius R is comparable to R^n for some $n > 0$. Later on, we will need such comparability of volume only for certain balls.

Remark 4.0.1. *Integration over measure m is denoted simply by dx instead of $m(dx)$ and the m-measure of the set is denote by $|\cdot|$. Since m is the only measure used in this part such conventions should not cause confusion.*

Throughout this part we will be studying the closed symmetric bilinear form $\mathcal{E}$ satisfying the following assumption:

Assumption 4.0.2. *The closed symmetric bilinear form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ on $L^2(M)$ is defined through its action on $f \in L^2(M)$ by*

$$\mathcal{E}(f) = \int_M \int_M (f(x) - f(y))^2 k(x, y) dy dx, \tag{4.1}$$

for some symmetric Borel measurable kernel $k : M \times M \to [0, \infty)$, and $\mathcal{D}[\mathcal{E}]$ is its maximal domain, i.e.

$$\mathcal{D}[\mathcal{E}] = \left\{ f \in L^2(M); \mathcal{E}(f) < \infty \right\}.$$

Furthermore, we assume that $\mathcal{D}[\mathcal{E}]$ contains all Lipschitz functions supported on balls and that all balls are precompact.

Form $\mathcal{E}$ defined in the above way is always closed, see Proposition 2.7.4, and we denote its generator by $(\mathcal{L}, \mathcal{D}[\mathcal{L}])$.

We would like to develop results that will apply to kernels of the form $k(x, y) = c(x, y)d(x, y)^{-(n+2s)}$ for some generic realization of random variables $c(x, y)$, $x, y \in \mathbb{Z}^n$. Such kernels will be studied in Part II. The theory developed in [GHH18, GHH17, GHL14] or [CK03, CKW16b, CKW16a] for kernels satisfying pointwise bounds

$$A^{-1} d(x, y)^{-(n+2s)} \leq k(x, y) \leq A d(x, y)^{-(n+2s)}, \tag{4.2}$$

for some $A \geq 1$, is not directly applicable because we want to allow random variables $c(x, y)$ to be unbounded or to take value zero for some pairs of x and y. Therefore, we need a different way of comparing our kernel to the rotationally stable kernel $d(x, y)^{-(n+2s)}$. One approach, developed in [FK13] and [DK15] for $M = \mathbb{R}^n$, is to assume the comparability of energy forms corresponding to kernels k and $d(x, y)^{-(n+2s)}$. That is, to assume that for some $A \geq 1$, every ball $B \subset \mathbb{R}^n$ and every function $f \in L^2(B)$

$$A^{-1} \int_B \int_B \frac{(f(x) - f(y))^2}{d(x, y)^{n+2s}} dy dx \leq \int_B \int_B (f(x) - f(y))^2 k(x, y) dy dx \leq A \int_B \int_B \frac{(f(x) - f(y))^2}{d(x, y)^{n+2s}} dy dx. \tag{4.3}$$

See also [CS19] for recent developments. We will take a slightly different route here and postulate that the kernel k satisfies certain functional inequalities which are satisfied for the kernel $d(x, y)^{-(n+2s)}$, possibly with different constants. To be more precise, we will assume that energy form $\mathcal{E}$ satisfies Poincaré inequality, Sobolev inequality and estimate of the energy density of cutoff functions given in **PI**, **SI** and **CE** respectively (see Chapter 3). All of these inequalities follow immediately if one assumes Ineq. (4.2) or Ineq. (4.3).

The main results of this part are Theorems 6.5.1, 6.6.3, 7.2.2, 7.3.2 and 8.1.4 and they will be applied in Part II. Theorem 6.5.1 proves the weak parabolic Harnack inequality (**WPHI**) and is used to obtain large scale Hölder regularity (**HR**) in Theorem 6.6.3. The latter result will be used when applying Theorem 12.4.1 to symmetrized twofold ergodic and i.i.d. conductance respectively. Theorem 7.2.2 proves that form $\mathcal{E}$ is conservative which we will use to embed the paths of the corresponding process into Sorokhod space $\mathcal{D}([0,T],\mathbb{R}^n)$. Theorem 7.3.2 gives the lower bound of the decay of restricted semigroup P_t^B and is the crucial ingredient in the proof of tightness for the i.i.d. conductance in Theorem 12.5.2. Finally, Theorem 8.1.4 will be used to obtain **PI** and **SI** for both ergodic and i.i.d. conductance in Theorems 10.2.5 and 11.7.1 respectively.

We believe that Theorems 6.5.2, 7.1.5 and 7.2.1 are interesting by themselves but are not necessary for Part II and will only be briefly mentioned there.

5. Weak solution and testing lemma

The Moser iteration, presented in the next chapter, relies on energy estimates obtained by testing the supersolutions u of the parabolic equation

$$\partial_t u - \mathcal{L}u = f,$$

where $\mathcal{L}$ is the generator of symmetric bilinear form $\mathcal{E}$ from Eq. (4.1), with test functions of the form $\varphi u^{-\beta}$ for some $\beta \in (0, \infty)$ and compactly supported Lipschitz φ. Once such estimates have been obtained, the iteration does not require any additional input from the equation. The goal of this chapter is to establish a definition of weak supersolution (Definition 5.1.4) for the above equation that allows for the derivation of aforementioned energy estimates. Ideally, such definition would require as little as possible a priori regularity of the solution and provide energy estimates at the same time. However, reducing a priori regularity increases technical difficulties and for this reason we will settle on the definition of weak supersolutions similar to the one in [GHL09] which require the a priori existence of weak time derivative $\partial_t u$ (see Definition 5.1.4). This will not give us problems in Part II because we will only be interested in semigroup solutions, which are known to be sufficiently regular. A discussion on the possible solution concepts can be found in [Fel13]. In particular, the a priori existance of $\partial_t u$ can be avoided using Steklov averages techniques.

The main result of this chapter are the weak solution concept in Definition 5.1.4 and the energy estimate in Lemma 5.2.1. Moreover, Lemma 5.1.9 proves that semigroup solutions are weak solutions in the sense of Definition 5.1.4.

5.1. Concept of weak solutions

Definition 5.1.1 (Weak differentiation)**.** *Let H be a Hilbert space and I an interval in $\mathbb{R}$. We say that the function $u : I \to H$ is weakly differentiable at $t \in I$ if there exists a $v \in H$ such that*

$$\frac{u(t+\varepsilon) - u(t)}{\varepsilon} \xrightarrow{\varepsilon \to 0} v$$

where $\rightharpoonup$ stands for the weak convergence in H. Then v is denoted by $u'(t)$ or $\partial_t u(t)$ and called the weak derivative *of u at t.*

Definition 5.1.2 (Solution space)**.** *Let us define, loosely following [FKV15] Definition 2.1, for any ball $B \subset M$ the space*

$$V_B = \{v : M \to \mathbb{R}; v \text{ Borel mesurable and } \mathcal{V}_B(v) < \infty\}$$

where $\mathcal{V}_B$ is the seminorm

$$\mathcal{V}_B(v) = \int_{(B^c \times B^c)^c} (v(x) - v(y))^2 k(x, y) dy dx$$

and $B^c = M \setminus B$ so that $(B^c \times B^c)^c = B \times B \cup B \times (M \setminus B) \cup (M \setminus B) \times B$.

Remark 5.1.3. *For $f \in \mathcal{D}_B[\mathcal{E}]$, $g \in \mathcal{D}[\mathcal{E}]$*

$$\mathcal{E}(f, g) = \mathcal{V}_B(f, g) \leq \mathcal{E}(f)^{1/2} \mathcal{V}_B(g)^{1/2}$$

and V_B can be considered as a subspace of $\mathcal{E}$-dual of $\mathcal{D}_B[\mathcal{E}]$.

Following [GHL09] we now define weak supersolutions (subsolution/solution) of equation $\partial_t u - \mathcal{L}u = f$.

Definition 5.1.4 (Weak parabolic supersolution)**.** *Let $Q \in [1, \infty]$, finite open interval $I \subset \mathbb{R}$, a ball $B \subset M$ and $f \in L^\infty(I; L^Q(M))$ be arbitrary. Denote by $\mathcal{L}$ the L^2-generator corresponding to the closed form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$.*

A function $u : I \to V_B \cap L^2(B)$ is said to be a weak supersolution of the equation

$$\partial_t u - \mathcal{L}u = f \text{ in } I \times B$$

if the weak $L^2(B)$-derivative $\partial_t u$ exists, $\partial_t u \in L^1_{loc}(I; L^2(B))$, $\mathcal{V}_B(u) \in L^1_{loc}(I)$ and for every non-negative $\varphi \in \mathcal{D}_B[\mathcal{E}] \cap L^{Q^*}(B)$,

$$\int_B \partial_t u(t,x)\varphi(x)dx + \mathcal{E}\left(u(t,\cdot), \varphi(\cdot)\right) \geq \int_B f(t,x)\varphi(x)dx \quad \text{for every } t \in I. \tag{5.1}$$

u is said to be a weak subsolution if the same thing holds for non-positive φ and it is called a weak solution if it holds for all φ.

Definition 5.1.5 (Weak elliptic supersolution)**.** *Let $Q \in [1, \infty]$, ball $B \subset M$ and $f \in L^Q(B)$ be arbitrary. Function $u \in V_B \cap L^2(B)$ is said to be a weak supersolution of the equation $\mathcal{L}u = f$ in B if for every $\varphi \in \mathcal{D}_B[\mathcal{E}] \cap L^{Q^*}(B)$, $\varphi \geq 0$,*

$$\mathcal{E}\left(u, \varphi\right) \geq \int_B f(x)\varphi(x)dx. \tag{5.2}$$

u is said to be a weak subsolution if the same thing holds for non-positive φ and it is called a weak solution if it holds for all φ.

Proposition 5.1.6. *Let $Q \in [1, \infty]$, ball B in M and $f \in L^Q(B)$ be arbitrary. Function $u \in L^2(B) \cap V_B$ is a weak super/sub/solution of equation $\mathcal{L}u = f$ in B if and only if the function $v : \mathbb{R} \to L^2(B)$, defined by $v(t) = u$, is a weak super/sub/solution of $\partial_t v - \mathcal{L}v = f$ in $I \times B$ for every open interval $I \subset \mathbb{R}$, where $f(t,x) = f(x)$ for all $t \in \mathbb{R}$.*

Proof. From Definition 5.1.1 it follows immediately that v is weakly differentiable on every open interval I and $\partial_t v(t) = 0$ for all $t \in I$. Due to definitions of $f(t,\cdot)$, $v(t)$ and $\partial_t v = 0$, Ineq. (5.1) collapses to Ineq. (5.2) for every $\varphi \in \mathcal{D}_B[\mathcal{E}] \cap L^{Q^*}(B)$. Suppose u is a weak super/sub/solution of $\mathcal{L}u = f$ and take any open interval $I \subset \mathbb{R}$. Clearly $\int_I \mathcal{V}_B(v(t))dt = \mathcal{V}_B(u)|I| < \infty$ and therefore v is a super/sub/solution of $\partial_t v - \mathcal{L}v = f$ on $I \times B$. Other way around, if v is a weak super/sub/solution of $\partial_t u - \mathcal{L}u = f$ on every finite open interval I, then $\mathcal{V}_B(u) = ⨍_I \mathcal{V}_B(v(t))dt < \infty$ implying $u \in V_B$ which means that u is a super/sub/solution of $\mathcal{L}u = f$. □

The following lemma concerning properties of weak differentiation is borrowed from [GHL09], Lemma 5.1.

Lemma 5.1.7 (Weak differentiation)**.** *Let H be a Hilbert space with inner product $(\cdot,\cdot)$, I and open subset of $\mathbb{R}$ and $u : I \to H$.*

(i) If u is weakly differentiable at $t \in I$, then u is strongly continuous at t.

(ii) (The product rule) If functions $u : I \to H$ and $v : I \to H$ are weakly differentiable at t, then the inner product (u, v) is also weakly differentiable at t and

$$(u,v)' = (u',v) + (u,v').$$

(iii) (The chain rule) Let (X, μ) be a measure space and set $H = L^2(X, \mu)$. Let $u : I \to L^2(X, \mu)$ be weakly differentiable at t and let $\Phi : \mathbb{R} \to \mathbb{R}$ be a smooth real valued function such that

$$\Phi(0) = 0, \quad \sup_{\mathbb{R}} |\Phi'| < \infty \text{ and } \sup_{\mathbb{R}} |\Phi''| < \infty.$$

Then the function $\Phi(u)$ maps I to $L^2(X, \mu)$ and it is also weakly differentiable at t with

$$\Phi(u)' = \Phi'(u)u'.$$

(iv) The conclusion of Item (iii) remain valid if function Φ is defined and smooth only on some open $J \subset \mathbb{R}$ (possibly unbounded), with $\lim_{x\to 0} \Phi(x) = 0$ in case $0 \in \overline{J}$,

$$\sup_J |\Phi'| < \infty, \sup_J |\Phi''| < \infty$$

and $u(t, x) \in J$ for a.e. $t \in I$ and μ-a.e. $x \in X$.

Proof. Just like in [GHL09] Lemma 5.1 except for Item (iv) which is not stated there. The proof of Item (iii) in [GHL09] was corrected by the authors and is available on the webpage of Professor Grigor'yan. The correction is as follows: It is claimed in [GHL09] that the first term on the right of (5.3) converges to zero strongly in L^2. This is not true because the use of Hölder inequality proves that it only converges to zero strongly in L^1. However, the same term is bounded in L^2 by $2 \sup |\Phi'|$ which makes it weakly compact. But then the weak L^2-limit must coincide with the strong L^1-limit and the term converges to zero weakly in L^2 which is sufficient to prove the statement following the remaining arguments from [GHL09].

For Item (iv) we consider first the case $0 \in \overline{J}$. Then one only needs to extend Φ by a smooth function Ψ onto the whole $\mathbb{R}$ with bounded first and second derivatives and apply Item (iii). If $0 \notin \overline{J}$, then one has to, in addition, choose Ψ such that $\Psi(0) = 0$ which is possible because 0 is separated from $\overline{J}$. □

Remark 5.1.8. *Operators of type $L^p(X, \mu) \ni u(x) \to f(x, u(x)) \in L^p(X, \mu)$ are called superposition operators and one can look them up in [AZ90].*

Lemma 5.1.9. *Let $g \in L^2(M)$ be arbitrary and let $U \subset M$ be open. Let P_t^U be a strongly continuous $L^2(U)$ semigroup corresponding to the closed symmetric bilinear form $\mathcal{E}$ on $L^2(U)$, see Definition 2.4.12. Then $P_t^U g$ is a solution of $\partial_t u - \mathcal{L}u = 0$ on $(0, \infty) \times B$, for every ball $B \subset U$, in the sense of Definition 5.1.4. Furthermore,*

$$\lim_{t\to 0} \|P_t^U g - g\|_{L^2(U)} = 0.$$

Proof. Let $\mathcal{L}^U$ denote the generator of P_t^U on $L^2(U)$. Because P_t^U is symmetric and contractive, $P_t^U(L^2(U)) \subset \mathcal{D}[\mathcal{L}^U]$ so the strong $L^2(U)$ Fréchet derivative exists and is equal to $\partial_t P_t^U g = \mathcal{L}^U P_t^U g$. All of this follows from the spectral theorem. Let $\{E_\lambda, \lambda \in (-\infty, \infty)\}$ be a spectral family of $-\mathcal{L}^U$ (which is known to be non-negative and self-adjoint by [FOT11] Lemma 1.3.1) such that

$$\mathcal{L}^U = \int_0^\infty (-\lambda) dE_\lambda \quad \text{and} \quad \mathcal{D}[\mathcal{L}^U] = \left\{ v \in L^2(U) : \int_0^\infty \lambda^2 d(E_\lambda v, v) < \infty \right\}$$

just like in [FOT11] Chapter 1.3, (around Formula (1.3.4)), see also [Kat95] Chapter six, Section 5. Then also $P_t^U = \int_0^\infty e^{-t\lambda} dE_\lambda$ and

$$\int_0^\infty \lambda^2 d(E_\lambda P_t^U g, P_t^U g) = \int_0^\infty \lambda^2 e^{-2\lambda} d(E_\lambda g, g) < \infty$$

implying $P_t^U g \in \mathcal{D}[\mathcal{L}^U]$. This in particular implies that $L^2(B)$-Fréchet deriviative $\partial_t P_t^U g$ exists and

$$\partial_t P_t^U g = \mathcal{L}^U P_t^U f \qquad \text{in } L^2(B)$$

so $P_t^U u$ is also weakly differentiable on $(0,\infty)$ in the sense of Definition 5.1.1. Note that $\mathcal{L}^U$ commutes with P_t^U and that the function $P_t^U \mathcal{L}^U f : (0,\infty) \to L^2(B)$ is continuous on $(0,\infty)$ which implies that $\partial_t P_t^U f \in L^1_{loc}((0,\infty); L^2(B))$. Testing the previous equation with $\varphi \in \mathcal{D}_B[\mathcal{E}]$ we obtain

$$\int_B \partial_t P_t^U g(x)\varphi(x)dx + \mathcal{E}(P_t^U g, \varphi) = 0$$

because $\int_B (-\mathcal{L}^U) P_t^U g(x)\varphi(x)dx = \mathcal{E}(P_t^U g, \varphi)$ (see Corollary 1.3.1 in [FOT11]). This shows that P_t^U satisfies Ineq. (5.1) for all $t \in (0,\infty)$ and $\varphi \in \mathcal{D}_B[\mathcal{E}]$. Furthermore, from

$$\partial_t \mathcal{E}(P_t^U g) = \partial_t(-\mathcal{L}^U P_t g, P_t^U g) = -2(\mathcal{L}^U P_t^U g, \mathcal{L}^U P_t^U g) \leq 0$$

it follow that the function $t \to \mathcal{E}(P_t^U g)$ is decreasing so for any compact $[T_1, T_2] \subset (0,\infty)$,

$$\int_{T_1}^{T_2} \mathcal{V}_B(P_t^U g)dt \leq \int_{T_1}^{T_2} \mathcal{E}(P_t^U g)dt \leq \int_{T_1}^{T_2} \mathcal{E}(P_{T_1}^U g)dt < \infty.$$

Thus $P_t^U g$ is indeed a solution in the sense of Definition 5.1.1 on $(0,\infty) \times B$. The convergence claim is simply a restatement of strong continuity of semigroup P_t^U in $L^2(U)$. □

5.2. Testing lemma

Lemma 5.2.1 (Testing lemma)**.** *Under Assumption 4.0.2 let $I \subset \mathbb{R}$ be a finite interval, B a ball of radius $R > 0$, $Q \geq 1$, and $f \in L^\infty(I; L^Q(B))$. Suppose $\varepsilon > 0$ and a weak supersolution u of $\partial_t u - \mathcal{L}u = f$ in $I \times B$ are such that*

$$u \geq \varepsilon + R^{2s} \operatorname*{ess\,sup}_{t \in I} \left(\fint_B |f(t,x)|^Q dx \right)^{\frac{1}{Q}} \qquad \text{on } I \times B.$$

Define

$$w(t,x) = \begin{cases} \frac{1}{1-\beta} u^{1-\beta}(t,x) & \text{if } \beta \neq 1 \\ \log u(t,x) & \text{if } \beta = 1. \end{cases}$$

Then w is continuous in t and for every $\beta > 0$, every non-negative, bounded and absolutely continuous $\chi : I \to \mathbb{R}$, every non-negative, Lipschitz $\psi : M \to \mathbb{R}$ supported inside of B and every segment $[T_1, T_2] \subset I$

$$\left[\chi(t) \int_B \psi(x) w(t,x) dx\right]_{T_1}^{T_2} + \int_{T_1}^{T_2} \chi(t)\mathcal{E}(u(t), \psi u^{-\beta}(t))dt$$
$$\geq -\|\psi\|_{L^\infty(B)} \int_{T_1}^{T_2} \left[|\chi'(t)| \int_B |w(t,x)|dx + |\chi(t)| R^{-2s} |B|^{\frac{1}{Q}} \left(\int_B u^{(1-\beta)Q^*}(t,x)dx \right)^{\frac{1}{Q^*}} \right] dt,$$

where $\chi'(t)$ is the Radon-Nikodym derivative of χ and $[a(t)]_{T_1}^{T_2}$ is the shorthand notation for $a(T_2) - a(T_1)$.

Furthermore, if $\beta \neq 1$, the last two terms can be combined to give

$$\left[\frac{\chi(t)}{1-\beta} \int_B \psi(x) u^{1-\beta}(t,x) dx\right]_{T_1}^{T_2} + \int_{T_1}^{T_2} \chi(t)\mathcal{E}(u(t), \psi u^{-\beta}(t))dt$$
$$\geq -\|\psi\|_{L^\infty(B)} |B|^{\frac{1}{Q}} \int_{T_1}^{T_2} \left(\frac{|\chi'(t)|}{|1-\beta|} + |\chi(t)| R^{-2s} \right) \left(\int_B u^{(1-\beta)Q^*}(t,x)dx \right)^{\frac{1}{Q^*}} dt.$$

Proof. First of all, notice that $\psi \in \mathcal{D}[\mathcal{E}]$ due to Assumption 4.0.2. We claim that more is true and that in fact $\psi \in \mathcal{D}_B[\mathcal{E}]$. To see this, we approximate $\psi_k := \psi - (\psi \vee (-1/k)) \wedge 1/k \xrightarrow{\mathcal{E}_1, k\to\infty} \psi$ by [FOT11], Lemma 1.4.2 (iv). Every ψ_k is compactly supported inside $L^2(B)$ because ψ is Lipschitz and zero on B^c so ψ_k is in $\mathcal{D}_B[\mathcal{E}]$ by Definition 2.4.12. Since $\mathcal{D}_B[\mathcal{E}]$ is $\mathcal{E}_1$-closed, $\varphi \in \mathcal{D}_B[\mathcal{E}]$ as well. For fixed $t \in I$ we intend to test supersolution u with $\varphi_t = \psi u^{-\beta}(t,\cdot)$ in Ineq. (5.1). For this we have to check that φ_t is a valid test function. Clearly φ_t is a non-negative element of $L^\infty(B) \subset L^{Q^*}(B)$ by assumptions $u \geq \varepsilon$, so we only need to verify that $\varphi_t \in \mathcal{D}_B[\mathcal{E}]$. To do so, notice that $u \geq \varepsilon$ and that for every $a, b \in [\varepsilon, \infty)$

$$|a^{-\beta} - b^{-\beta}| \leq \beta \varepsilon^{-(1+\beta)}|a-b| \leq C(\varepsilon,\beta)|a-b|.$$

Now $\mathcal{E}(\varphi_t) = \mathcal{V}_B(\varphi_t)$ because $\varphi \in D_B(\mathcal{E})$ and thus (with elementary estimate $(ab-cd)^2 \leq 2a^2(b-d)^2 + 2(a-c)^2d^2$ in mind for the first inequality)

$$\begin{aligned}
\mathcal{E}(\varphi_t) = \mathcal{V}_B(\varphi_t) &= \int_{(B^c\times B^c)^c} \Big(\psi(x)u^{-\beta}(t,x) - \psi(y)u^{-\beta}(t,y)\Big)^2 k(x,y)dydx \\
&\leq 2\|\psi\|^2_{L^\infty(B)} \int_{(B^c\times B^c)^c} \Big(u^{-\beta}(t,x) - u^{-\beta}(t,y)\Big)^2 k(x,y)dydx \\
&\qquad + 2\|u^{-\beta}\|^2_{L^\infty(B)} \int_{(B^c\times B^c)^c} (\psi(x)-\psi(y))^2 k(x,y)dydx \\
&\leq C(\varepsilon,\beta)^2\|\psi\|^2_{L^\infty(B)} \int_{(B^c\times B^c)^c} (u(t,x)-u(t,y))^2 k(x,y)dydx + \varepsilon^{-2\beta}\mathcal{E}(\psi) \\
&\leq C(\varepsilon,\beta)^2\|\psi\|^2_{L^\infty(B)}\mathcal{V}_B(u(t)) + \varepsilon^{-2\beta}\mathcal{E}(\psi).
\end{aligned}$$

This shows that $\varphi_t \in \mathcal{D}[\mathcal{E}]$ and eventually that $\varphi_t \in \mathcal{D}_B[\mathcal{E}]$ by application of Lemma 4.4 (ii) of [GT12] because $\varphi_t \leq \varepsilon^{-\beta}\psi(x) \in \mathcal{D}_B[\mathcal{E}]$ together with $\varphi_t \geq 0$. It is therefore justified to test with φ_t which gives, for every $t \in [T_1, T_2]$,

$$\int_B \partial_t u(t,x)\psi(x)u^{-\beta}(t,x)dx + \mathcal{E}\left(u(t), \psi u^{-\beta}(t)\right) \geq \int_B f(t,x)\psi(x)u^{-\beta}(t,x)dx.$$

We deal with the f term using $u \geq R^{2s}\operatorname{ess\,sup}_{t\in I}\left(\fint_B |f(t,x)|^Q dx\right)^{\frac{1}{Q}}$ to estimate

$$\begin{aligned}
&\int_B f(t,x)\psi(x)u^{-\beta}(t,x)dx \\
&\qquad \geq -\left(\int_B \frac{|f(t,x)|^Q}{u^Q(t,x)}\psi^Q(x)dx\right)^{\frac{1}{Q}}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}} \\
&\qquad \geq -\|\psi\|_{L^\infty(B)}R^{-2s}|B|^{\frac{1}{Q}}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}
\end{aligned}$$

and arrive at

$$\begin{aligned}
&\int_B \partial_t u(t,x)\psi(x)u^{-\beta}(t,x)dx + \mathcal{E}\left(u(t,\cdot), \psi u^{-\beta}(t,\cdot)\right) \\
&\qquad \geq -\|\psi\|_{L^\infty(B)}R^{-2s}|B|^{\frac{1}{Q}}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}.
\end{aligned}$$

Multiplying both sides with $\chi(t)$ and integrating in t from T_1 to T_2 results in

$$\begin{aligned}
&\int_{T_1}^{T_2}\chi(t)\int_B \partial_t u(t,x)\psi(x)u^{-\beta}(t,x)dxdt + \int_{T_1}^{T_2}\chi(t)\mathcal{E}\left(u(t,\cdot), \psi u^{-\beta}(t,\cdot)\right)dt \\
&\qquad \geq -\|\psi\|_{L^\infty(B)}\int_{T_1}^{T_2}\chi(t)R^{-2s}|B|^{\frac{1}{Q}}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}.
\end{aligned} \tag{5.3}$$

Notice that two integrals on left exist because a supersolution by definition needs to satisfy $\partial_t u(t) \in L^1_{loc}(I; L^2(B))$ and $\mathcal{V}(u(t)) \in L^1_{loc}(I)$ respectively and the one on the right exists (could be infinite) because its integrand is of constant sign. Let us now attend to the first integral in Ineq. (5.3) in more details. By Item (iv) of Lemma 5.1.7 with $X = B$, $\Phi : (\varepsilon/2, \infty) \to \mathbb{R}$, $\Phi(a) = \frac{a^{1-\beta}}{1-\beta}$ if $\beta \neq 1$ or $\Phi(a) = \log(a)$ if $\beta = 1$ (all derivatives of Φ are bounded on $(\varepsilon/2, \infty)$ in any case) we know that w is $L^2(B)$-weakly differentiable and $\partial_t w = u^{-\beta}\partial_t u$. Together with Item (i) of Lemma 5.1.7 this implies that w is $L^2(B)$-strongly continuous like stated. In addition, calling $(\cdot,\cdot)$ the scalar product on $L^2(U)$, Item (ii) shows that $\partial_t(\psi, w) = (\partial_t\psi, w) + (\psi, \partial_t w) = (\psi, \partial_t w)$ so function $t \to (\psi, w(t))$ is differentiable. It is then also absolutely continuous, just like χ is by assumption, so an application of integration by parts (see [Coh13] Corollary 6.3.9) allows us to rewrite the first integral as

$$\int_{T_1}^{T_2} \chi(t) \int_B \psi(x)\partial_t w(t,x)dxdt = \chi(T_2)\int_B \psi(x)w(T_2,x)dx - \chi(T_1)\int_B \psi(x)w(T_1,x)dx$$
$$- \int_{T_1}^{T_2} \chi'(s)\int_B \psi(x)w(s,x)dxds.$$

Combining this with Ineq. (5.3) we end up at

$$\left[\chi(t)\int_B \psi(x)w(t,x)dx\right]_{T_1}^{T_2} + \int_{T_1}^{T_2} \chi(t)\mathcal{E}\left(u(t,\cdot), \psi(\cdot)u^{-\beta}(t,\cdot)\right)dt$$
$$\geq -\|\psi\|_{L^\infty(B)} \int_{T_1}^{T_2} \left[|\chi'(t)|\int_B |w(t,x)|dx + |\chi(t)|R^{-2s}|B|^{\frac{1}{Q}}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}\right]dt.$$

This proves the first statement. In case $\beta \neq 1$ we know that $u^{(1-\beta)} = (1-\beta)w$ and we can express the last inequality purely in terms of u. Furthermore, by Hölder's inequality

$$\int_B |w(t,x)|dx \leq |B|^{\frac{1}{Q}}\left(\int_B |w(t,x)|^{Q^*}\right)^{\frac{1}{Q^*}}$$

which allows us to combine the last two terms of our main inequality and get

$$\left[\frac{\chi(t)}{1-\beta}\int_B \psi(x)u^{1-\beta}(t,x)dx\right]_{T_1}^{T_2} + \int_{T_1}^{T_2} \chi(t)\mathcal{E}\left(u(t,\cdot), \psi(\cdot)u^{-\beta}(t,\cdot)\right)dt$$
$$\geq -\|\psi\|_{L^\infty(B)}|B|^{\frac{1}{Q}} \int_{T_1}^{T_2} \left(\frac{|\chi'(t)|}{|1-\beta|} + |\chi(t)|R^{-2s}\right)\left(\int_B u(t,x)^{(1-\beta)Q^*}dx\right)^{\frac{1}{Q^*}}dt$$

which proves the second statement. □

5.3. Maximum principle

Here we present two maximum principles from [GHL09] and [GHH17] which we will use in Chapter 7.

Theorem 5.3.1 (Parabolic maximum principle - Proposition 5.2 in [GHL09])**.** *Assume that metric measure space (M, d, m) satisfies Assumption 2.5.3 and $\mathcal{E}$ is a regular Dirichlet form on $L^2(M)$. Let $u \in \mathcal{D}[\mathcal{E}]$ be a weak subsolution of $\partial_t u - \mathcal{L}u = 0$ in $(0,T) \times U$ where $T \in (0,\infty]$ and U is an open subset of M. Assume in addition that u satisfies the following boundary and initial conditions:*

(i) $u_+(t,\cdot) \in D_U[\mathcal{E}]$ for every $t \in (0,T)$;

(ii) $u_+(t,\cdot) \xrightarrow{L^2(U)} 0$ as $t \to 0$.

Then $u(t,x) \leq 0$ for every $t \in (0,T)$ and m-a.e. $x \in U$.

Proof. Notice that Definition 5.1.1 requires some additional integrability on top of solution concept in [GHL09], Chapter 5.2. □

Theorem 5.3.2 (Parabolic maximum principle II - Proposition 6.1 in [GHH17])**.** *Assume (M, d, m) satisfies Assumption 2.5.3 and that $\mathcal{E}$ is a regular Dirichlet form on $L^2(M)$. Suppose we are given $T \in (0, \infty]$, an open set $U \subset M$ and a function $f \in \mathcal{D}[\mathcal{E}] \cap L^\infty(M)$ such that $f_U \equiv \|f\|_{L^\infty(M)}$. If*

(i) $u : (0,T) \to \mathcal{D}[\mathcal{E}]$ is a subsolution of the equation $\partial_t u - \mathcal{L}u = f$ such that $\partial_t u$ from Definition 5.1.1 exists in the strong $L^2(U)$-sense,

(ii) $u_+(t, \cdot) \in \mathcal{D}_U[\mathcal{E}]$ for every $t \in (0,T)$ and

(iii) $u_+(t, \cdot) \to 0$ strongly in $L^2(M)$ as $t \to 0$

then, for every $t \in (0,T)$,

$$\|u(t, \cdot)\|_{L^2(U)} \leq 2 \int_0^t \mathcal{E}(f, u_+(s, \cdot))ds.$$

Proof. It is not difficult to verify that conditions of Lemma 6.1 of [GHH17] hold when u is the subsolution in the sense of Definition 5.1.4. □

6. Parabolic Moser iteration

In this chapter we are going to develop the local version of the parabolic Moser iteration scheme. In Part II this will allow us to obtain regularity-type bounds on the semigroups of long-range random walks in random environments. The local parabolic version of the iteration appeared in [Mos64] (see also [Mos67] and [Mos71]) and generalizes the elliptic version developed a few years earlier in [Mos61]. It was later adjusted to various different setting and in particular extended to discrete spaces in [Del99] and later still to random discrete setting in [ADS16]. The nonlocal elliptic case was recently studied in [DK15] and [CK20] while parabolic case is developed in [FK13]. Our method here is a combination of [ADS16], which provides ideas relating to randomness of environment, and of [FK13], which contains the calculations needed in the nonlocal case. There are four key points the reader should take note of. Firstly, we will be working with a nonlocal energy form which prevents us from obtaining the full Harnack inequality ($\sup \le \inf$) directly from the iteration. In fact, it is not clear if this could be true for our choice of kernels k, see [BS07] for a detailed discussion. Secondly, random kernels that we would like to handle in Part II are fundamentally space dependent (in particular, all space related scalings are destroyed) which forces us to work locally and manifests in constants being space dependent. Thirdly, the lack of a pointwise bound on the kernel causes the energy density Γf of a function f to behave wildly in the pointwise sense and some averaging procedure is needed. In particular, this means that energy measure of a Lipschitz function cannot be bounded pointwise. Instead, we bound it locally in L^Q space for appropriate $Q \in [1, \infty)$. Lastly, the procedure only works if the kernel k is not too degenerate. We can give a sufficient condition for this in terms of parameters in assumed functional inequalities **SI** and **CE**, see Ineq. (6.7). Similar condition appear in [ADS16, FH20].

For the rest of the chapter we will consider weak supersolutions of equation

$$\partial_t u - \mathcal{L} u = f \tag{6.1}$$

in $I \times 2B$ where B is a ball in M of radius R and I is the interval $(t_0 - R^{2s}, t_0 + R^{2s})$ for some $t_0 \in \mathbb{R}$ (recall Definition 5.1.4). To make the main ideas clearer, let us assume that $f = 0$ through this introduction. The definition of sets $Z, Z_\ominus, Z_\oplus$ used below can be found in Definition 6.1.3.

The Moser iteration procedure roughly consist of three steps. In the first step, contained in Section 6.2, one uses an iterative argument to prove, for all $\sigma \in (1/2, 1)$ and $p \in [-1, 0)$, there is a constant C_1 such that, for every supersolution u

$$\operatorname*{ess\,inf}_{Z_\ominus(\sigma R)} u \ge C_1 \left(\fint_{Z_\ominus(R)} u^p \right)^{\frac{1}{p}}.$$

The second step, in Section 6.3, is quite similar and proves that, for all $\sigma \in (1/2, 1)$ and $p \in (0, 1)$, there is a constant C_2 such that, for all supersolutions u,

$$\fint_{Z_\oplus(\sigma R)} u \le C_2 \left(\fint_{Z_\oplus(R)} u^p \right)^{\frac{1}{p}}.$$

If we restrict our attention to solutions instead of supersolutions of $\partial_t u - \mathcal{L} u = 0$, then it is sometimes possible to improve the estimate by replacing the L^1 term on the left hand side with $\operatorname{ess\,sup}_{Z_\oplus(\sigma R)} u$. We will not obtain such a result in this thesis. The third step combines the previous two estimates and proves the weak parabolic Harnack inequality. To be explicit, it proves that there is a constant $C_{PH} < \infty$ such that for every supersolution u

$$\fint_{U_\ominus} u \le C_{PH} \operatorname*{ess\,inf}_{U_\oplus} u.$$

This is done in Sections 6.4 and 6.5. For definition of sets $U_\ominus$ and $U_\oplus$ see theorem Theorem 6.5.1 and Fig. 6.5.1. Once the weak parabolic Harnack inequality is available, a standard argument, see [FK13] for example, proves Hölder regularity estimate of u. We present this in Section 6.6.

6.1. Iteration preparations

We need some preliminary results before we can start with the iteration.

Definition 6.1.1. *Let $R > 0$ and $x_0 \in M$ be arbitrary and set $B := B(x_0, R)$. A* cutoff function *between balls $\sigma B \subset B$, for $\sigma \in (0,1)$, is any function φ that satisfies $\varphi = 1$ on σB, $\varphi = 0$ on B^c and*

$$|\varphi(x) - \varphi(y)| \le \frac{d(x,y)}{(1-\sigma)R}.$$

An example of such function is given by

$$\psi(x) = \left(\frac{R - d(x_0, x)}{(1-\sigma)R} \wedge 1 \right) \vee 0.$$

Assumption 4.0.2 guarantees that all the cutoffs from the above definition are in the energy space of $\mathcal{E}$ and **CE** provides an estimate of its energy which we prove in the following proposition.

Proposition 6.1.2. *Suppose $s \in (0,1)$, $\gamma \in [0, 2s)$, $C_C < \infty$, $Q \ge 1$, $R > 0$ and $x_0 \in M$ are such that $\boldsymbol{CE}[x_0, R; s, Q, \gamma, C_C]$ holds. Let φ be a Lipschitz function supported in $B := B(x_0, R)$ and $\xi = R \operatorname{Lip} \varphi$. Then*

$$\mathcal{E}(\varphi) \le 2C_C(\xi^{2s+\gamma} \vee \xi^{2s-\gamma})|B|R^{-2s}.$$

In particular, if $\boldsymbol{CE}[x_0, [R_0, \infty); s, Q, \gamma, C_C]$ holds for some $R_0 > 0$, then $\mathcal{D}(\mathcal{E})$ (the maximal domain of $\mathcal{E}$) contains all Lipschitz functions with bounded support.

Proof. Start by computing

$$\begin{aligned}
\mathcal{E}(\varphi) &= \int_M \int_M (\varphi(x) - \varphi(y))^2 k(x,y) dy dx \\
&= \left(\int_B \int_B + 2 \int_B \int_{B^c} \right) (\varphi(x) - \varphi(y))^2 k(x,y) dy dx \\
&\le 2 \int_B \int_M (\varphi(x) - \varphi(y))^2 k(x,y) dy dx \le 2 \int_B \Gamma\varphi(x) dx.
\end{aligned}$$

Hölder inequality and $\mathbf{CE}[x_0, R; s, Q, \gamma, C_C]$ imply

$$\mathcal{E}(\varphi) \le 2 \int_B \Gamma\varphi(x) dx \le 2 \left(\int_B 1 dx \right)^{\frac{1}{Q^*}} \left(\int_B \Gamma\varphi(x) dx \right)^{\frac{1}{Q}} \le 2C_C(\xi^{2s-\gamma} \vee \xi^{2s+\gamma})|B|R^{-2s}.$$

If, for some $R_0 > 0$, $\mathbf{CE}[x_0, (R_0, \infty); s, Q, \gamma, C_C]$ holds, then for every Lipschitz function φ' with bounded support it is possible to find $R \ge R_0$ such that $\operatorname{supp} \varphi' \subset B(x_0, R)$. The previous calculation now shows $\mathcal{E}(\varphi') < \infty$ and implies that φ' is in $\mathcal{D}[\mathcal{E}]$. $\square$

The geometry is going to play an important role in what follows which is why we introduce the following definitions, borrowed from [FK13].[1]

Definition 6.1.3. *Let $x_0 \in M$, $t_0 \in \mathbb{R}$ and $R > 0$ be arbitrary. We introduce the following intervals in $\mathbb{R}$ and cylinders in $\mathbb{R} \times M$, see Fig. 6.1.1,*

$$I(t_0, R) = (t_0 - R^{2s}, t_0 + R^{2s}), \qquad Z(t_0, x_0, R) = I(t_0, R) \times B(x_0, R),$$
$$I_\ominus(t_0, R) = (t_0 - R^{2s}, t_0), \qquad Z_\ominus(t_0, x_0, R) = I_\ominus(t_0, R) \times B(x_0, R),$$
$$I_\oplus(t_0, R) = (t_0, t_0 + R^{2s}), \qquad Z_\oplus(t_0, x_0, R) = I_\oplus(t_0, R) \times B(x_0, R).$$

When t_0 and x_0 are clear from context we will omit them in the notation and write just $Z(R)$ instead of $Z(t_0, x_0, R)$.

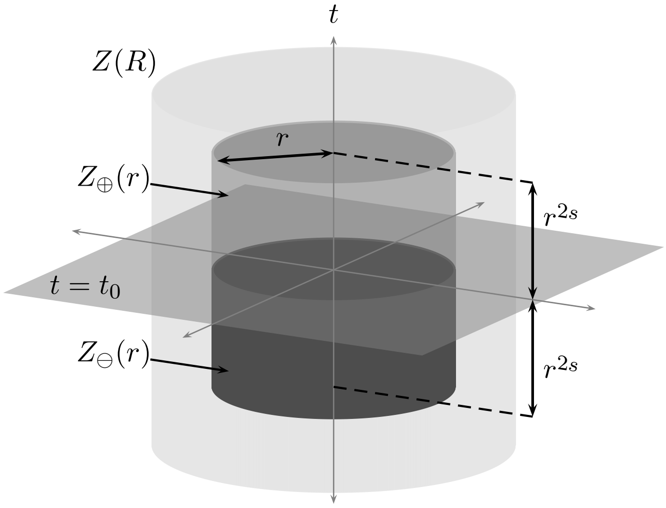

Figure 6.1.1.: “Zylinders”

We also shorten the notation by defining:

Definition 6.1.4. *For $\sigma \in (0, 1)$, $s \in (0, 1)$ and $\gamma \in [0, 2s)$,*

$$K(\sigma) = (1 - \sigma)^{-2s-\gamma} + (1 - \sigma^{2s})^{-1}. \tag{6.2}$$

Lemma 6.1.5. *Function $K : (0, 1) \to (0, \infty)$ is increasing and*

$$K(\sigma) \le 2s^{-1}(1 - \sigma)^{-((2s+\gamma)\vee 1)}.$$

Proof. Both terms on the right of Eq. (6.2) are increasing in σ so K is as well. If $s \ge \frac{1}{2}$, then $1 - \sigma^{2s} \ge 1 - \sigma$ and we can estimate

$$(1 - \sigma)^{-2s-\gamma} + (1 - \sigma^{2s})^{-1} \le 2(1 - \sigma)^{-((2s+\gamma)\vee 1)}.$$

If, on the other hand, $s < \frac{1}{2}$, we resort to convexity of function $1 - \sigma^{2s}$ at $\sigma = 1$ to estimate $1 - \sigma^{2s} \ge 2s(1 - \sigma)$ and eventually get

$$(1 - \sigma)^{-2s-\gamma} + (1 - \sigma^{2s})^{-1} \le s^{-1}(1 - \sigma)^{-((2s+\gamma)\vee 1)}.$$

In either case,

$$K(\sigma) \le (1 - \sigma)^{-2s-\gamma} + (1 - \sigma^{2s})^{-1} \le 2s^{-1}(1 - \sigma)^{-((2s+\gamma)\vee 1)}.$$

□

[1] The author would like to thank Prof. Dr. Moritz Kaßmann for letting him reuse source codes of images from [FK13] in this chapter

Remark 6.1.6. *Due to the γ correction, which is needed in Chapter 10, we are not able to recover the sharp dependence of constants on s. This in particular means it is not possible to use results of this chapter to pass to the limit $s \to 1$. See [FK13] for more details.*

6.1.1. Iteration Norms

In order to get the optimal moment condition we iterate in the following averaged space-time norms.

Definition 6.1.7. *Let $a, b \in \mathbb{R} \setminus \{0\}$, $B \subset \mathbb{R}$ and $A \subset M$ measurable. Define, for measurable $u : B \times A \to \mathbb{R}$, the seminorm*

$$\langle u \rangle_{a,b,B\times A} := \left(⨍_B \left(⨍_A |u(t,x)|^a dx \right)^{\frac{b}{a}} dt \right)^{\frac{1}{b}} \in [0, \infty].$$

Proposition 6.1.8. *Let $a_1, a_2, b_1, b_2 \in \mathbb{R} \setminus \{0\}$, $B \subset \mathbb{R}$ and $A \subset M$. If $a_1 \leq a_2$ and $b_1 \leq b_2$, then*

$$\langle u \rangle_{a_1,b_1,B\times A} \leq \langle u \rangle_{a_2,b_2,B\times A}.$$

Proof. Let us first prove that, for all $c_1, c_2 \neq 0$, $c_1 \leq c_2$,

$$\left(⨍_A |u(t,x)|^{c_1} dx \right)^{\frac{1}{c_1}} \leq \left(⨍_A |u(t,x)|^{c_2} dx \right)^{\frac{1}{c_2}}. \tag{6.3}$$

If $c_1 > 0$, then c_1 and c_2 are both positive and Jensen's inequality with convex function $x \to x^{\frac{c_2}{c_1}}$ proves the claim. If $c_2 < 0$, then both c_1 and c_2 are negative so $x \to x^{c_2/c_1}$ is concave which due to Jensen's inequality gives

$$\left(⨍_A |u(t,x)|^{c_1} dx \right) \geq \left(⨍_A |u(t,x)|^{c_2} dx \right)^{\frac{c_1}{c_2}}$$

but then rising both sides to power $1/c_1 < 0$ reverses the inequality and proves the original claim. In the last case when $c_1 < 0 < c_2$ it is sufficient to prove the claim for $c_1 = -c_2$. For instance, if $c_1 < -c_2$ we can use previously proved case to replace c_1 with $-c_2$. Then Jensen's inequality with convex function $x \to x^{-1}$ proves

$$\left(⨍_A |u(t,x)|^{-c_2} dx \right)^{\frac{1}{-c_2}} \leq \left(⨍_A |u(t,x)|^{c_2} dx \right)^{\frac{1}{c_2}}.$$

Altogether, this proves Ineq. (6.3).

Coming back to proving our main claim we use Ineq. (6.3) to see that

$$\left(⨍_A |u(t,x)|^{a_1} dx \right)^{\frac{1}{a_1}} \leq \left(⨍_A |u(t,x)|^{a_2} dx \right)^{\frac{1}{a_2}}$$

an then, independent of the sign of b_1, we have

$$\left(⨍_B \left(⨍_A |u(t,x)|^{a_1} dx \right)^{\frac{b_1}{a_1}} dt \right)^{\frac{1}{b_1}} \leq \left(⨍_B \left(⨍_A |u(t,x)|^{a_2} dx \right)^{\frac{b_1}{a_2}} dt \right)^{\frac{1}{b_1}}.$$

An application of inequality analogue to Ineq. (6.3) on $\mathbb{R}$ instead of M gives

$$\begin{aligned} \langle u \rangle_{a_1,b_1,B\times A} &\leq \left(⨍_B \left(⨍_A |u(t,x)|^{a_2} dx \right)^{\frac{b_1}{a_2}} dt \right)^{\frac{1}{b_1}} \\ &\leq \left(⨍_B \left(⨍_A |u(t,x)|^{a_2} dx \right)^{\frac{b_2}{a_2}} dt \right)^{\frac{1}{b_2}} = \langle u \rangle_{a_2,b_2,B\times A} \end{aligned}$$

and proves the claim. □

6.1.2. On the choice of Sobolev inequality

The exact form of the inequality given in **SI** does not influence the iteration procedure too much. There are several other alternatives and in the next proposition we prove that one particularly simple form of the inequality implies **SI** under the assumption that **CE** is satisfied.

Lemma 6.1.9. *Suppose $s \in (0,1)$, $\gamma \in [0,2s)$, $C_C < \infty$, $Q \geq 1$, $R > 0$ and $x_0 \in M$ are such that $\boldsymbol{CE}[x_0, R; s, Q, \gamma, C_C]$ holds. For every Lipschitz function $\eta : M \to [0,1]$ supported in $B := B(x_0, R)$ and $\xi := R \operatorname{Lip} \eta$*

$$\mathcal{E}_B(\eta u) \leq \mathcal{E}(\eta u) \leq 2\mathcal{E}_B(u) + 2C_C(\xi^{2s+\gamma} \vee \xi^{2s-\gamma})|B|^{\frac{1}{Q}} R^{-2s} \left(\int_B u^{2Q^*} \right)^{\frac{1}{Q^*}}.$$

Proof. Clearly $\mathcal{E}_B \leq \mathcal{E}$ simply due to the size of the area of integration in definition of forms. For the other inequality we estimate

$$\begin{aligned}
\mathcal{E}(\eta u) &= \int_M \int_M [u(x)\eta(x) - u(y)\eta(y)]^2 \, k(x,y) dy dx \\
&\leq \int_B \int_B [(u(x) - u(y)) \, \eta(x) + u(y) \, (\eta(x) - \eta(y))]^2 \, k(x,y) dy dx \\
&\quad + 2 \int_B \int_{B^c} [u(x)\eta(x)]^2 k(x,y) dy dx \\
&\leq \int_B \int_B 2 \, [u(x) - u(y)]^2 \, k(x,y) dy dx + \int_B \int_B 2u(y)^2 \, [\eta(x) - \eta(y)]^2 \, k(x,y) dy dx \\
&\quad + 2 \int_B \int_{B^c} u(x)^2 [\eta(x) - \eta(y)]^2 k(x,y) dy dx \\
&\leq 2\mathcal{E}_B(u) + 2 \int_B u(x)^2 \Gamma\eta(x) dx \\
&\leq 2\mathcal{E}_B(u) + 2C_C(\xi^{2s+\gamma} \vee \xi^{2s-\gamma})|B|^{\frac{1}{Q}} R^{-2s} \left(\int_B u^{2Q^*} \right)^{\frac{1}{Q^*}}
\end{aligned}$$

where Hölder inequality and $\mathbf{CE}[x_0, R; s, Q, \gamma, C_C]$ were used at the same time in the last inequality. □

Proposition 6.1.10. *Suppose $s \in (0,1)$, $\gamma \in [0,2s)$, $C_C < \infty$, $Q \geq 1$, $R > 0$ and $x_0 \in M$ are such that $\boldsymbol{CE}[x_0, R; s, Q, \gamma, C_C]$ holds. Let $q \in (\frac{n}{2s}, \infty]$ and $C_S < \infty$ be arbitrary and define ρ to be the solution of $\frac{1}{\rho} = 1 - \frac{2s}{n} + \frac{1}{q}$. If, for every $u \in L^1(M)$ with support in $B := B(x_0, R)$,*

$$\|u^2\|_{L^\rho(B)} \leq C_S |B|^{\frac{1}{\rho}-1} R^{2s} \mathcal{E}(u), \tag{6.4}$$

then $\boldsymbol{SI}[x_0, R; s, \rho, \zeta := Q^*, C_{S1} := 2C_S, C_{S2} := 2C_S C_C, \gamma]$ holds.

Proof. Fix $u \in L^1(B)$ and $\sigma \in (0,1)$. Define $\eta : B \to [0,1]$ by

$$\eta(x) = \left(\frac{R - d(x_0, x)}{R(1-\sigma)} \wedge 1 \right) \vee 0.$$

so that $\operatorname{Lip} \eta = \frac{1}{(1-\sigma)R}$. Applying Lemma 6.1.9 (note that $\xi = (1-\sigma)^{-1}$) proves that

$$\mathcal{E}(\eta u) \leq 2\mathcal{E}_B(u) + 2C_C \, (1-\sigma)^{-2s-\gamma} \, |B|^{\frac{1}{Q}} R^{-2s} \left(\int_B u^{2Q^*} \right)^{\frac{1}{Q^*}}.$$

Combining this with Ineq. (6.4) leads to

$$\begin{aligned}\|u^2\|_{L^\rho(\sigma B)} &= \|(\eta u)^2\|_{L^\rho(B)} \le C_S|B|^{\frac{1}{\rho}-1}R^{2s}\mathcal{E}(\eta u)\\ &\le 2C_S|B|^{\frac{1}{\rho}-1}R^{2s}\mathcal{E}_B(u) + 2C_SC_C(1-\sigma)^{-2s-\gamma}|B|^{\frac{1}{\rho}-\frac{1}{Q^*}}\left(\int_B u^{2Q^*}\right)^{\frac{1}{Q^*}}.\end{aligned}$$

The previous inequality holds for all $u \in L^1(M)$, $\sigma \in (0,1)$ proving that $\mathbf{SI}[x_0, R; s, \rho, \zeta, C_{S1}, C_{S_2}, \gamma]$ holds if we take $\zeta := Q^*$, $C_{S1} := 2C_S$ and $C_{S2} := 2C_SC_C$. □

6.2. Iteration for negative exponents

6.2.1. Energy estimate

We follow the approach from [Kas09] and [FK13].

Theorem 6.2.1. *Suppose $s \in (0,1)$, $\gamma \in [0,2s)$, $C_C < \infty$, $Q \ge 1$, $R > 0$ and $x_0 \in M$ are such that $\boldsymbol{CE}[x_0, R; s, Q, \gamma, C_C]$ holds and denote $B := B(x_0, R)$. Let $\varepsilon > 0, t_0 \in \mathbb{R}$, $\sigma \in (1/2, 1)$ and $f \in L^\infty(I_\ominus(R); L^Q(B))$ be arbitrary. Then, for every supersolution u of $\partial_t u - \mathcal{L}u = f$ in $Z_\ominus(R) \equiv Z_\ominus(t_0, x_0, R)$ such that*

$$u \ge \varepsilon + R^{2s} \operatorname*{ess\,sup}_{t\in I_\ominus(R)} \left(\fint_B |f(t,x)|^Q dx\right)^{\frac{1}{Q}} \qquad \text{on } I_\ominus \times M$$

and every $\beta > 1$, both

$$\operatorname*{ess\,sup}_{t\in I_\ominus(\sigma R)} \int_{\sigma B} u^{1-\beta}(t,x)dx \le 8\beta^2(C_C+1)K(\sigma)|B|^{\frac{1}{Q}}R^{-2s}\int_{I_\ominus(R)}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}dt$$

and

$$\int_{I_\ominus(\sigma R)} \mathcal{E}_{\sigma B}(u^{\frac{1-\beta}{2}}(t,\cdot))dt \le 8\beta^2(C_C+1)K(\sigma)|B|^{\frac{1}{Q}}R^{-2s}\int_{I_\ominus(R)}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}dt$$

hold. Here $K(\cdot)$ is is the function from Definition 6.1.4.

Proof. Let us start by introducing cutoffs $\psi : M \to [0,1]$,

$$\psi(x) = \left(\frac{R - d(x_0,x)}{(1-\sigma)R} \wedge 1\right) \vee 0$$

and $\chi : \mathbb{R} \to [0,1]$,

$$\chi(t) = \left(\frac{t + R^{2s}}{R^{2s}(1-\sigma^{2s})} \wedge 1\right) \vee 0.$$

We apply Lemma 5.2.1 with β, χ^2 and $\psi^{\beta+1}$. By the choice of ψ and χ, $|\psi^{\beta+1}| \le 1$, $|\chi^2| \le 1$ and $|(\chi^2)'| \le 2|\chi||\chi'| \le 2R^{-2s}(1-\sigma^{2s})^{-1}$. The second statement of Lemma 5.2.1, justified by $\beta > 1$, implies that for any $T_1, T_2 \in I_\ominus(R)$, $T_1 < T_2$,

$$\begin{aligned}&\left[\frac{\chi^2(t)}{1-\beta}\int_B \psi^{\beta+1}(x)u^{1-\beta}(t,x)dx\right]_{T_1}^{T_2} + \int_{T_1}^{T_2}\chi^2(t)\mathcal{E}(u(t), \psi^{\beta+1}u^{-\beta}(t))dt\\ &\qquad \ge -|B|^{\frac{1}{Q}}R^{-2s}\int_{T_1}^{T_2}\left(\frac{2(1-\sigma^{2s})^{-1}}{\beta-1}+1\right)\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}dt.\end{aligned} \tag{6.5}$$

Let us focus on the second term on the left for a moment. Lemma 2.5 from [Kas09] states that for every $a_1, a_2 > 0$, $b > 1$ and $\tau_1, \tau_2 \geq 0$, defining $\vartheta(b) = \max\left\{4, \frac{6b-5}{2}\right\}$.

$$\begin{aligned}(a_1 - a_2)\left(\tau_1^{b+1}a_1^{-b} - \tau_2^{b+1}a_2^{-b}\right) \leq -\frac{1}{b-1}\tau_1\tau_2 &\left(\left(\frac{a_1}{\tau_1}\right)^{\frac{1-b}{2}} - \left(\frac{a_2}{\tau_2}\right)^{\frac{1-b}{2}}\right)^2 \\ &+ \vartheta(b)(\tau_1 - \tau_2)^2\left(\left(\frac{a_1}{\tau_1}\right)^{1-b} + \left(\frac{a_2}{\tau_2}\right)^{1-b}\right).\end{aligned} \tag{6.6}$$

Using this with $a_1 = u(t,x), a_2 = u(t,y), \tau_1 = \psi(x), \tau_2 = \psi(y)$ and $b = \beta$ we can estimate

$$\begin{aligned}\mathcal{E}(u, \psi^{\beta+1}u^{-\beta}) &= \int_M\int_M (u(t,x) - u(t,y))(\psi^{\beta+1}(x)u^{-\beta}(t,x) - \psi^{\beta+1}(y)u^{-\beta}(t,y))k(x,y)dxdy \\ &\leq -\frac{1}{\beta-1}\int_M\int_M \psi(x)\psi(y)\left[\left(\frac{u(t,x)}{\psi(x)}\right)^{\frac{1-\beta}{2}} - \left(\frac{u(t,y)}{\psi(y)}\right)^{\frac{1-\beta}{2}}\right]^2 k(x,y)dxdy \\ &+ \vartheta(\beta)\int_M\int_M (\psi(x) - \psi(y))^2\left[\left(\frac{u(t,x)}{\psi(x)}\right)^{1-\beta} + \left(\frac{u(t,y)}{\psi(y)}\right)^{1-\beta}\right] k(x,y)dxdy \\ &= I_1 + I_2.\end{aligned}$$

To estimate I_1, notice that the function under the integral is non-negative so reducing the area σB (where $\psi = 1$) we get

$$\begin{aligned}I_1 &= -\frac{1}{\beta-1}\int_M\int_M \psi(x)\psi(y)\left[\left(\frac{u(t,y)}{\psi(y)}\right)^{\frac{1-\beta}{2}} - \left(\frac{u(t,x)}{\psi(x)}\right)^{\frac{1-\beta}{2}}\right]^2 k(x,y)dxdy \\ &\leq -\frac{1}{\beta-1}\int_{\sigma B}\int_{\sigma B}\left[(u(t,y))^{\frac{1-\beta}{2}} - (u(t,x))^{\frac{1-\beta}{2}}\right]^2 k(x,y)dxdy \\ &\leq -\frac{1}{\beta-1}\mathcal{E}_{\sigma B}\left(u^{-\frac{1-\beta}{2}}(t)\right).\end{aligned}$$

For I_2 we use the symmetry of $k(x,y)$ and the fact that $\psi^{\frac{\beta-1}{2}}(x) \leq 1_B(x)$ (because $\beta - 1 \geq 0$) to estimate

$$\begin{aligned}I_2 &= \vartheta(\beta)\int_M\int_M (\psi(x) - \psi(y))^2\left[\left(\frac{u(t,y)}{\psi(y)}\right)^{1-\beta} + \left(\frac{u(t,x)}{\psi(x)}\right)^{1-\beta}\right] k(x,y)dxdy \\ &\leq 2\vartheta(\beta)\int_B u^{1-\beta}(t,x)\int_M (\psi(x) - \psi(y))^2 k(x,y)dydx \\ &\leq 2\vartheta(\beta)\int_B u^{1-\beta}(t,x)\Gamma\psi(x)dx.\end{aligned}$$

Applying Hölder inequality and $\mathbf{CE}[x_0, R; s, Q, \gamma, C_C]$ (with $\xi = (1-\sigma)^{-1}$) to the last expression gives

$$I_2 \leq 2\vartheta(\beta)C_C(1-\sigma)^{-2s-\gamma}|B|^{\frac{1}{Q}}R^{-2s}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}.$$

Collecting the estimates for I_1 and I_2 we come back to

$$\mathcal{E}(u, \psi^{\beta+1}u^{-\beta}) \leq -\frac{1}{\beta-1}\mathcal{E}_{\sigma B}(u^{-\frac{1-\beta}{2}}) + 2\vartheta(\beta)C_C(1-\sigma)^{-2s-\gamma}|B|^{\frac{1}{Q}}R^{-2s}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}.$$

Going even further back to Ineq. (6.5), we end up with

$$
\begin{aligned}
&-\left[\frac{\chi^2(t)}{\beta-1}\int_B \psi^{\beta+1}(x)u^{1-\beta}(t,x)(t,x)dx\right]_{T_1}^{T_2} - \frac{1}{\beta-1}\int_{T_1}^{T_2}\chi^2(t)\mathcal{E}_{\sigma B}(u^{-\frac{1-\beta}{2}})dt \\
&\quad \geq -|B|^{\frac{1}{Q}}R^{-2s}\left(\frac{2(1-\sigma^{2s})^{-1}}{\beta-1}+1+2\vartheta(\beta)C_C(1-\sigma)^{-2s-\gamma}\right)\int_{T_1}^{T_2}\|u^{(1-\beta)}\|_{L^{Q^*}(B)}dt.
\end{aligned}
$$

To keep the size of this expression manageable, let us recall the definition of $K(\sigma)$ from Definition 6.1.4 and slightly overestimate the constant on the right. Since $\vartheta(\beta)=\max\left\{4,\frac{6\beta-5}{2}\right\}\leq 4\beta$ (because $\beta>1$) we can bound

$$
\frac{2(1-\sigma^{2s})^{-1}}{\beta-1}+1+2\vartheta(\beta)C_C(1-\sigma)^{-2s-\gamma}\leq\frac{8\beta^2(C_C+1)K(\sigma)}{\beta-1}.
$$

Rearranging our main inequality now shows that

$$
\begin{aligned}
&\left[\chi^2(t)\int_B \psi^{\beta+1}(x)u^{1-\beta}(t,x)dx\right]_{T_1}^{T_2}+\int_{T_1}^{T_2}\chi^2(t)\mathcal{E}_{\sigma B}(u^{-\frac{1-\beta}{2}})dt \\
&\quad \leq 8\beta^2(C_C+1)K(\sigma)|B|^{\frac{1}{Q}}R^{-2s}\int_{T_1}^{T_2}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}dt.
\end{aligned}
$$

Let us now reduce the integration area of integrals on left to $Z_\ominus(\sigma R)$, where $\chi=\psi=1$, and send $T_1\to -R^{2s}$, to obtain, for $T_2\in(-(\sigma R)^{2s},0]$,

$$
\begin{aligned}
&\int_{\sigma B}u^{1-\beta}(T_2,x)dx+\int_{-(\sigma R)^{2s}}^{T_2}\mathcal{E}_{\sigma B}(u^{-\frac{1-\beta}{2}})dt \\
&\quad \leq 8\beta^2(C_C+1)K(\sigma)|B|^{\frac{1}{Q}}R^{-2s}\int_{-R^{2s}}^{T_2}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}dt
\end{aligned}
$$

The first claim of the theorem now follows by ignoring the second term and taking the sup over all $T_2\in I_\ominus(\sigma R)$, which gives

$$
\operatorname*{ess\,sup}_{t\in I_\ominus(\sigma R)}\int_{\sigma B}u^{1-\beta}(t,x)dx\leq 8\beta^2(C_C+1)K(\sigma)|B|^{\frac{1}{Q}}R^{-2s}\int_{-R^{2s}}^{0}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}dt.
$$

Similarly, the second claim follows by ignoring the first term and letting $T_2\to 0$, which gives

$$
\int_{I_\ominus(\sigma R)}\mathcal{E}_{\sigma B}(u^{\frac{1-\beta}{2}}(t,\cdot))dt\leq 8\beta^2(C_C+1)K(\sigma)|B|^{\frac{1}{Q}}R^{-2s}\int_{-R^{2s}}^{0}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}dt.
$$

□

6.2.2. Elementary step

Theorem 6.2.2 (Elementary step). *Suppose $s\in(0,1)$, $\gamma\in[0,2s)$, $q,Q\in[1,\infty]$, C_C,C_{S1},C_{S2},C_{VL}, $C_{VU},n\in(0,\infty)$, $R>0$ and $x_0\in M$ are such that, with $\rho:=\left(1-\frac{2s}{n}+\frac{1}{q}\right)^{-1}$,*

(i) $\boldsymbol{CE}[x_0,(R/2,R];s,Q,\gamma,C_C]$ holds,

(ii) $\boldsymbol{SI}[x_0,(R/2,R];s,\rho,Q^*,C_{S1},C_{S_2},\gamma]$ holds,

(iii) $\boldsymbol{V}[x_0,(R/2,R];n,C_{VL},C_{VU}]$ holds and

$$
\frac{1}{q}+\frac{1}{Q}<\frac{2s}{n}. \tag{6.7}
$$

Then there exists a

$$C_{(6.2.2)} := C_{(6.2.2)}(s, n, q, Q, C_C, C_{S1}, C_{S2}, \gamma, C_{VL}, C_{VU})$$

fulfilling the following statement.

Set $\kappa = 1+\frac{2s}{n}-\frac{1}{q}-\frac{1}{Q} > 1$, $B := B(x_0, R)$ *and let* $\varepsilon > 0, t_0 \in \mathbb{R}$, $\sigma \in (1/2, 1)$, $f \in L^\infty(I_\ominus(R); L^Q(B))$ *be arbitrary. For every* $b < 0$ *and every supersolution* u *of* $\partial_t u - \mathcal{L}u = f$ *in* $Z_\ominus(R) \equiv Z_\ominus(t_0, x_0, R)$ *such that*

$$u \geq \varepsilon + R^{2s} \operatorname*{ess\,sup}_{t\in I_\ominus(R)} \left(\fint_B |f(t,x)|^Q dx \right)^{\frac{1}{Q}} \quad \text{on } Z_\ominus(R),$$

the inequality

$$\langle u\rangle_{\kappa bQ^*, \kappa b, Z_\ominus(\sigma R)} \geq \left[C_{(6.2.2)}(1-b)^2(1-\sigma)^{-((2s+\gamma)\vee 1)}\right]^{\frac{1}{b}} \langle u\rangle_{bQ^*, b, Z_\ominus(R)} \tag{6.8}$$

is satisfied.

Proof. For $\beta > 1$ we apply Hölder inequality, with exponents $\frac{1}{\kappa-1}$ and ρ for which satisfy

$$\kappa - 1 + \frac{1}{\rho} = 1 + \frac{2s}{n} - \frac{1}{q} - \frac{1}{Q} - 1 + 1 - \frac{2s}{n} + \frac{1}{q} = 1 - \frac{1}{Q} = \frac{1}{Q^*},$$

to get

$$\begin{aligned} A &:= \int_{I_\ominus(\sigma R)} \|u^{(1-\beta)\kappa}(t)\|_{L^{Q^*}(\sigma B)} dt \leq \int_{I_\ominus(\sigma R)} \left(\int_{\sigma B} u^{(1-\beta)}(t,x) dx \right)^{\kappa-1} \left(\int_{\sigma B} u^{(1-\beta)\rho}(t,x) dx \right)^{\frac{1}{\rho}} dt \\ &\leq \left(\operatorname*{ess\,sup}_{t\in I_\ominus(\sigma R)} \int_{\sigma B} u^{(1-\beta)}(t,x)dx \right)^{\kappa-1} \left(\int_{I_\ominus(\sigma R)} \|u^{(1-\beta)}(t)\|_{L^\rho(\sigma B)} dt \right). \end{aligned}$$

Let $\tilde\sigma = \frac{1+\sigma}{2} \in (0,1)$ and let us apply $\mathbf{SI}[x_0, \tilde\sigma R; s, \rho, Q^*, C_{S1}, C_{S_2}, \gamma]$ with larger ball equal to $\tilde\sigma B$ and the smaller equal to σB. Some care is needed because property **SI** also uses a variable σ which we here rename to σ_S. Then $\sigma_S = \sigma/\tilde\sigma \in (0,1)$ and

$$1 - \sigma_S = 1 - \frac{\sigma}{\tilde\sigma} = \frac{\frac{1+\sigma}{2} - \sigma}{\tilde\sigma} \geq \frac{1-\sigma}{2} = 1 - \tilde\sigma.$$

This together with increasing the integration domain $I_\ominus(\sigma R)$ to $I_\ominus(\tilde\sigma R)$ at various points leads us to

$$\begin{aligned} A \leq \left(\operatorname*{ess\,sup}_{t\in I_\ominus(\sigma R)} \int_{\sigma B} u^{(1-\beta)}(t,x)dx \right)^{\kappa-1} &\left[C_{S1} |\tilde\sigma B|^{\frac{1}{\rho}-1} (\tilde\sigma R)^{2s} \int_{I_\ominus(\tilde\sigma R)} \mathcal{E}_{\tilde\sigma B}\left(u^{\frac{1-\beta}{2}}(t)\right) dt \right. \\ &\left. + C_{S2}\,(1-\tilde\sigma)^{-2s-\gamma}\, |\tilde\sigma B|^{\frac{1}{\rho}-\frac{1}{Q^*}} \int_{I_\ominus(\tilde\sigma R)} \|u^{1-\beta}(t)\|_{L^{Q^*}(\tilde\sigma B)} dt \right]. \end{aligned}$$

Resorting to estimates of Theorem 6.2.1 we proceed by bounding

$$\begin{aligned} A \leq &\left(8\beta^2(C_C+1)K(\sigma)|B|^{\frac{1}{Q}} R^{-2s} \int_{I_\ominus(R)} \|u^{1-\beta}(t)\|_{L^{Q^*}(B)} dt \right)^{\kappa-1} \\ &\times \left[8C_{S1}\beta^2(C_C+1)K(\tilde\sigma)|\tilde\sigma B|^{\frac{1}{q}-\frac{2s}{n}} |B|^{\frac{1}{Q}} \int_{I_\ominus(R)} \|u^{1-\beta}(t)\|_{L^{Q^*}(B)} dt \right. \\ &\left. + C_{S2}\,(1-\tilde\sigma)^{-2s-\gamma}\, |\tilde\sigma B|^{\frac{1}{q}+\frac{1}{Q}-\frac{2s}{n}} \int_{I_\ominus(R)} \|u^{1-\beta}(t)\|_{L^{Q^*}(B)} dt \right]. \end{aligned}$$

Now overestimating $(1-\tilde{\sigma})^{2s-\gamma} \le K(\tilde{\sigma})$, $K(\sigma) \le K(\tilde{\sigma})$ (see Definition 6.1.4 and Lemma 6.1.5) and using $\mathbf{V}[x_0,(R/2,R];n,C_{VL},C_{VU}]$ (to get $|\tilde{\sigma}B|^{-\frac{2s}{n}} \le \tilde{\sigma}^{-2s} C_{VL}^{-\frac{2s}{n}} R^{-2s} \le 4C_{VL}^{-\frac{2s}{n}} R^{-2s}$) the expression on the right reduces to

$$A \le \left(32C_{VL}^{-\frac{2s}{n}}\beta^2(C_C+1)(C_{S1}+C_{S2})K(\tilde{\sigma})\right)^{\kappa} |B|^{\frac{1}{q}+\frac{\kappa}{Q}} R^{-2s\kappa} \left(\int_{I_\ominus(R)} \|u^{1-\beta}(t)\|_{L^{Q^*}(B)} dt\right)^{\kappa}.$$

To prevent the constant from eating up the rest of the paper we will estimate it with interest only in its behavior with respect to β and σ. With Lemma 6.1.5 we find

$$\begin{aligned}
&\left(32C_{VL}^{-\frac{2s}{n}}\beta^2(C_C+1)(C_{S1}+C_{S2})K(\tilde{\sigma})\right)^{\kappa}\\
&\quad\le \left(32C_{VL}^{-\frac{2s}{n}}\beta^2(C_C+1)(C_{S1}+C_{S2})s^{-1}(1-\tilde{\sigma})^{-((2s+\gamma)\vee 1)}\right)^{\kappa}\\
&\quad\le \left(32C_{VL}^{-\frac{2s}{n}}\beta^2(C_C+1)(C_{S1}+C_{S2})s^{-1}2^{(2s+\gamma)\vee 1}(1-\sigma)^{-((2s+\gamma)\vee 1)}\right)^{\kappa}.
\end{aligned}$$

Setting $C_1 := C_1(s,C_C,C_{S1},C_{S2},\gamma,n,C_{VL}) = 32C_{VL}^{-\frac{2s}{n}}(C_C+1)(C_{S1}+C_{S2})s^{-1}2^{(2s+\gamma)\vee 1}$ relaxes our notation to

$$\begin{aligned}
\int_{I_\ominus(\sigma R)} \|u^{(1-\beta)\kappa}\|_{L^{Q^*}(\sigma B)} dt &\le \left[C_1\beta^2(1-\sigma)^{-((2s+\gamma)\vee 1)}\right]^{\kappa}\\
&\quad\times |B|^{\frac{1}{q}+\frac{\kappa}{Q}} R^{-2s\kappa} \left(\int_{I_\ominus(R)} \|u^{1-\beta}\|_{L^{Q^*}(B)} dt\right)^{\kappa},
\end{aligned}$$

which, after averaging out all space and time integrals, transforms into

$$\begin{aligned}
\langle u^{(1-\beta)\kappa}\rangle_{Q^*,1,Z_\ominus(\sigma R)} &\le \left[C_1\beta^2(1-\sigma)^{-(2s+\gamma)\vee 1}\right]^{\kappa} \langle u^{1-\beta}\rangle^{\kappa}_{Q^*,1,Z_\ominus(R)}\\
&\quad\times |\sigma B|^{-\frac{1}{Q^*}}(\sigma R)^{-2s}|B|^{\frac{\kappa}{Q^*}} R^{2s\kappa}|B|^{\frac{1}{q}+\frac{\kappa}{Q}} R^{-2s\kappa}.
\end{aligned}$$

Using $\mathbf{V}[x_0,(R/2,R];n,C_{VL},C_{VU}]$ (notice that $\sigma R, R \in (R/2,R]$) the product in the second row can be bounded as (with some comments after the calculation)

$$\begin{aligned}
|\sigma B|^{-\frac{1}{Q^*}}(\sigma R)^{-2s}|B|^{\frac{\kappa}{Q^*}} R^{2s\kappa}|B|^{\frac{1}{q}+\frac{\kappa}{Q}} R^{-2s\kappa} &\le C_{VL}^{-\frac{1}{Q^*}} \sigma^{-(\frac{n}{Q^*}+2s)} R^{-\left(\frac{n}{Q^*}+2s\right)} C_{VU}^{\kappa+\frac{1}{q}} R^{\frac{n}{q}+n\kappa}\\
&\le C_{VL}^{-\frac{1}{Q^*}} 2^{n+2s} C_{VU}^{\kappa+\frac{1}{q}} R^{n(-1+\frac{1}{Q}-2s+\frac{1}{q}+\kappa)} \le 2^{n+2s} C_{VL}^{-\frac{1}{Q^*}} C_{VU}^{\kappa+\frac{1}{q}}.
\end{aligned}$$

where for the last line we need to recall that $\sigma \ge 1/2$, $Q^* \ge 1$ and $\kappa = 1 + \frac{2s}{n} - \frac{1}{q} - \frac{1}{Q}$ by definition. Collecting everything, smuggling $2^{n+2s} C_{VU}^{\kappa+\frac{1}{q}} C_{VL}^{-1}$ into C_1 we find a constant $C_{(6.2.2)} := C_{(6.2.2)}(s,n,q,Q,C_C,C_{S1},C_{S2},\gamma,C_{VL},C_{VU})$ such that

$$\langle u^{(1-\beta)\kappa}\rangle_{Q^*,1,Z_\ominus(\sigma R)} \le \left[C_{(6.2.2)}\beta^2(1-\sigma)^{-(2s+\gamma)\vee 1}\right]^{\kappa} \langle u^{1-\beta}\rangle^{\kappa}_{Q^*,1,Z_\ominus(R)}.$$

Substituting $b = (1-\beta) < 0$, rising everything to power $\frac{1}{\kappa b} < 0$ (which changes the inequality sign) gives

$$\langle u\rangle_{\kappa b Q^*,\kappa b,Z_\ominus(\sigma R)} \ge \left[C_{(6.2.2)}(1-b)^2(1-\sigma)^{-(2s+\gamma)\vee 1}\right]^{\frac{1}{b}} \langle u\rangle_{bQ^*,b,Z_\ominus(R)},$$

which is exactly the statement from the theorem. □

6.2.3. Iteration

Theorem 6.2.3. *Suppose $s \in (0,1)$, $\gamma \in [0,2s)$, $q, Q \in [1,\infty]$, $C_C, C_{S1}, C_{S2}, n, C_{VL}, C_{VU} \in (0,\infty)$, $R > 0$ and $x_0 \in M$ are such that, for $\rho := \left(1 - \frac{2s}{n} + \frac{1}{q}\right)^{-1}$,*

(i) $\boldsymbol{CE}[x_0, (R/2, R]; s, Q, \gamma, C_C]$ holds,

(ii) $\boldsymbol{SI}[x_0, (R/2, R]; s, \rho, Q^*, C_{S1}, C_{S_2}, \gamma]$ holds,

(iii) $\boldsymbol{V}[x_0, (R/2, R]; n, C_{VL}, C_{VU}]$ holds and

$$\frac{1}{q} + \frac{1}{Q} < \frac{2s}{n}. \tag{6.9}$$

Then there is a

$$D_{(6.2.3)} := D_{(6.2.3)}(s, n, q, Q, C_C, C_{S1}, C_{S2}, \gamma, C_{VL}, C_{VU})$$

possessing the following property.

Set $\kappa = 1 + \frac{2s}{n} - \frac{1}{q} - \frac{1}{Q} > 1$, $B := B(x_0, R)$ and choose any $\varepsilon > 0$, $t_0 \in \mathbb{R}$, $\sigma \in (1/2, 1)$, $-1 \le p_0 < 0$, $f \in L^\infty(I_\ominus(R); L^Q(B))$. Then every supersolution u of $\partial_t u - \mathcal{L}u = f$ in $Z_\ominus(R) \equiv Z_\ominus(t_0, x_0, R)$, such that

$$u \ge \varepsilon + R^{2s} \operatorname*{ess\,sup}_{t \in I_\ominus(R)} \left(\fint_B |f(t,x)|^Q dx \right)^{\frac{1}{Q}} \qquad \text{on } Z_\ominus(R),$$

satisfies

$$\operatorname*{ess\,inf}_{Z_\ominus(\sigma R)} u \ge \left[D_{(6.2.3)} (1-\sigma)^{-\frac{((2s+\gamma)\vee 1)Q^*}{1-\kappa^{-1}}} \right]^{\frac{1}{p_0}} \left(\fint_{Z_\ominus(R)} u^{p_0} \right)^{1/p_0}.$$

Proof. Set $R_k = (\sigma + 2^{-k}(1-\sigma))R$ and $\sigma_k = R_{k+1}/R_k$. We are going to iterate Theorem 6.2.2 over a sequence $b_k = \kappa^k p_0 / Q^*$, R_k and σ_k for k from 0 to $N-1$ for arbitrary $N \in \mathbb{N}$. Notice that this is possible because $R_k \in (R/2, R]$ for every $k \in \mathbb{N}_0$. Taking

$$(1 - \sigma_k) = \frac{R_k - R_{k+1}}{R_k} = 2^{-(k+1)}(1-\sigma)\frac{R}{R_k} \ge 2^{-(k+1)}(1-\sigma)$$

into account the iteration gives

$$\langle u \rangle_{Q^* b_N, b_N, Z_\ominus(R_N)} \ge \prod_{k=0}^{N-1} \left(C_{(6.2.2)} 2^{((2s+\gamma)\vee 1)(k+1)} (1-b_k)^2 (1-\sigma)^{-((2s+\gamma)\vee 1)} \right)^{\frac{1}{b_k}} \langle u \rangle_{Q^* b_0, b_0, Z_\ominus(R)}.$$

The product on the right side can be expressed in term of an exponential as

$$\begin{aligned} &\prod_{k=0}^{N-1} \left(C_{(6.2.2)} 2^{((2s+\gamma)\vee 1)(k+1)} (1-b_k)^2 (1-\sigma)^{-((2s+\gamma)\vee 1)} \right)^{\frac{Q^*}{\kappa^k p_0}} \\ &\quad = \left(C_{(6.2.2)} (1-\sigma)^{-((2s+\gamma)\vee 1)} \right)^{\frac{Q^*}{p_0} \sum_{k=0}^{N-1} \kappa^{-k}} \\ &\qquad \times \exp\left(\frac{Q^*}{p_0} \sum_{k=0}^{N-1} \kappa^{-k} \left(\log 2((2s+\gamma)\vee 1)(k+1) + \log(1-b_k)^2 \right) \right). \end{aligned}$$

Now, since $p_0^2 \le 1$ and $(1-b_k)^2 \le 2 + 2b_k^2 = 2 + 2\kappa^{2k}(p_0/Q^*)^2 \le 2\kappa^{2k}(1+p_0^2) \le 4\kappa^{2k}$, the sum inside the exponential can be bounded from above by

$$\sum_{k=0}^{N-1} \kappa^{-k} \left(\log(2)((2s+\gamma)\vee 1)(k+1) + 2k \log(4\kappa) \right)$$

which converges because $\sum_{k=0}^{\infty} \kappa^{-k} k < +\infty$. This means that independent of N we can find a $D_1 := D_1(q, Q, s, \gamma, n)$ such that the exponential factor is bounded from below by D_1^{1/p_0} (keep in mind that $p_0 < 0$). Estimating $\sum_{k=0}^{N-1} \kappa^{-k} \le \sum_{k=0}^{\infty} \kappa^{-k} \le (1-\kappa^{-1})^{-1}$ and taking into account that p_0 is negative, we obtain

$$\langle u \rangle_{\kappa^{N+1} p_0, \frac{\kappa^{N+1} p_0}{Q^*}, Z_\ominus(R_N)} \ge \left(C_{(6.2.2)} (1-\sigma)^{-((2s+\gamma)\vee 1)} \right)^{\frac{Q^*}{p_0(1-\kappa^{-1})}} D_1^{\frac{1}{p_0}} \langle u \rangle_{p_0, \frac{p_0}{Q^*}, Z_\ominus(R)}. \tag{6.10}$$

On one hand, by Jensen's inequality and Fubini's theorem, the norm on the right can be estimated by (negative p_0 reverses Jensen's inequality)

$$\langle u \rangle_{p_0, \frac{p_0}{Q^*}, Z_\ominus(R)} = \left(\fint_{I_\ominus(\sigma R)} \left(\fint_{\sigma B} u^{p_0} \right)^{\frac{1}{Q^*}} \right)^{\frac{Q^*}{p_0}} \ge \left(\fint_{I_\ominus(\sigma R)} \fint_{\sigma B} u^{p_0} \right)^{\frac{1}{p_0}} = \left(\fint_{Z_\ominus(R)} u^{p_0} \right)^{1/p_0}.$$

On the other hand,

$$\langle u \rangle_{\kappa^N p_0, \frac{\kappa^N p_0}{Q^*}, Z_\ominus(R_N)} \le \left(\frac{(\sigma R)^{2s} |B(x_0, \sigma R)|^{\frac{1}{Q^*}}}{R^{2s} |B(x_0, R_N)|^{\frac{1}{Q^*}}} \right)^{\frac{Q^*}{\kappa^N p_0}} \langle u \rangle_{\kappa^N p_0, \frac{\kappa^N p_0}{Q^*}, Z_\ominus(\sigma R)} \xrightarrow{N\to\infty} \operatorname*{ess\,inf}_{Z_\ominus(\sigma R)} u$$

because, as we will show below,

$$\langle u \rangle_{\kappa^N p_0, \frac{\kappa^N p_0}{Q^*}, Z_\ominus(\sigma R)} = \left(\fint_{I_\ominus(\sigma R)} \left(\fint_{\sigma B} u^{\kappa^N p_0} \right)^{\frac{1}{Q^*}} \right)^{\frac{Q^*}{\kappa^N p_0}} \xrightarrow{N\to\infty} \operatorname*{ess\,inf}_{Z_\ominus(\sigma R)} u. \tag{6.11}$$

and

$$\lim_{N\to\infty} \left(\frac{(\sigma R)^{2s} |B(x_0, \sigma R)|^{\frac{1}{Q^*}}}{R^{2s} |B(x_0, R_N)|^{\frac{1}{Q^*}}} \right)^{\frac{Q^*}{\kappa^N p_0}} = 1. \tag{6.12}$$

Equation (6.12) is due to $\mathbf{V}[x_0, (R/2, R]; n, C_{VL}, C_{VU}]$ together with $R_N \in (R/2, R]$, $\sigma \in (1/2, 1)$ and $Q^* \kappa^{-N} p_0^{-1} \xrightarrow{N\to\infty} 0$. We now turn to proving Eq. (6.11). Plugging $v := \operatorname{ess\,inf}_{Z_\ominus(\sigma R)} u$ instead of u in the middle integral gives the natural inequality (keep in mind $p_0 < 0$)

$$\langle u \rangle_{\kappa^N p_0, \frac{\kappa^N p_0}{Q^*}, Z_\ominus(\sigma R)} \ge v.$$

Observe again that Jensen's inequality applied to the inner integral and Fubini's theorem imply

$$\langle u \rangle_{\kappa^N p_0, \frac{\kappa^N p_0}{Q^*}, Z_\ominus(\sigma R)} = \left(\fint_{I_\ominus(\sigma R)} \left(\fint_{\sigma B} u^{\kappa^N p_0} \right)^{\frac{1}{Q^*}} \right)^{\frac{Q^*}{\kappa^N p_0}} \le \left(\fint_{Z_\ominus(R)} u^{\frac{\kappa^N p_0}{Q^*}} \right)^{\frac{Q^*}{\kappa^N p_0}}.$$

For arbitrary $\varepsilon > 0$ we now find $\delta > 0$ such that $|\{u \le (1+\varepsilon)v\}| \ge \delta |Z_\ominus(\sigma R)|$ and compute

$$\begin{aligned} \left(\fint_{Z_\ominus(\sigma R)} u^{\frac{\kappa^N p_0}{Q^*}} \right)^{\frac{Q^*}{\kappa^N p_0}} &\le \left(|Z_\ominus(\sigma R)|^{-1} \int_{\{u \le (1+\varepsilon)v\}} ((1+\varepsilon)v)^{\frac{\kappa^N p_0}{Q^*}} \right)^{\frac{Q^*}{\kappa^N p_0}} \\ &\le (1+\varepsilon) v \delta^{\frac{Q^*}{\kappa^N p_0}} \xrightarrow{N\to\infty} (1+\varepsilon) v. \end{aligned}$$

Recalling that $\varepsilon > 0$ was arbitrary, $\langle u \rangle_{\kappa^N p_0, \frac{\kappa^N p_0}{Q^*}, Z_\ominus(\sigma R)} \xrightarrow{N\to\infty} v$ follows.

Returning to our main proof, these observations allow us to pass to the limit $N \to \infty$ in Ineq. (6.10) and obtain

$$\operatorname*{ess\,inf}_{Z_\ominus(\sigma R)} u \geq \left[D_1 \left(C_{(6.2.2)} (1-\sigma)^{-((2s+\gamma)\vee 1)} \right)^{\frac{Q^*}{1-\kappa^{-1}}} \right]^{\frac{1}{p_0}} \left(\fint_{Z_\ominus(R)} u^{p_0} \right)^{1/p_0}.$$

The claim of the theorem now follows by taking

$$D_{(6.2.3)} \equiv D_{(6.2.3)}(s,n,q,Q,C_C,C_{S1},C_{S2},\gamma,C_{VL},C_{VU}) = D_1 C_{(6.2.2)}^{\frac{Q^*}{1-\kappa^{-1}}}.$$

□

6.3. Iteration for small positive exponents

6.3.1. Energy estimate

Theorem 6.3.1. *Suppose $s \in (0,1)$, $\gamma \in [0,2s)$, $C_C \in (0,\infty)$, $Q \in [1,\infty]$, $R > 0$ and $x_0 \in M$ are such that $\boldsymbol{CE}[x_0,R;s,Q,\gamma,C_C]$ holds and denote $B := B(x_0,R)$. Let $\varepsilon > 0, t_0 \in \mathbb{R}$, $\sigma \in (1/2,1)$ and $f \in L^\infty(I_\oplus(R);L^Q(B))$ be arbitrary. Then, for every supersolution u of $\partial_t u - \mathcal{L}u = f$ in $Z_\oplus(R) \equiv Z_\oplus(t_0,x_0,R)$ such that*

$$u \geq \varepsilon + R^{2s} \operatorname*{ess\,sup}_{t\in I_\oplus(R)} \left(\fint_B |f(t,x)|^Q dx \right)^{\frac{1}{Q}} \qquad \text{on } Z_\oplus(R)$$

and every $\beta \in (0,1)$, both

$$\operatorname*{ess\,sup}_{t\in I_\oplus(\sigma R)} \int_{\sigma B} u^{1-\beta}(t,x)dx \leq 28(C_C+1)\beta^{-1}K(\sigma)|B|^{\frac{1}{Q}} R^{-2s} \int_{I_\oplus(R)} \left(\int_B u^{(1-\beta)Q^*}(t,x)dx \right)^{\frac{1}{Q^*}} dt$$

and

$$\int_{I_\oplus(\sigma R)} \mathcal{E}_{\sigma B}(u^{\frac{1-\beta}{2}})dt \leq 42(C_C+1)\beta^{-2}K(\sigma)|B|^{\frac{1}{Q}} R^{-2s} \int_{I_\oplus(R)} \left(\int_B u^{(1-\beta)Q^*}(t,x)dx \right)^{\frac{1}{Q^*}} dt$$

hold.

Proof. Let us start by introducing cutoffs

$$\psi(x) = \left(\frac{R - d(x_0,x)}{(1-\sigma)R} \wedge 1 \right) \vee 0$$

and

$$\chi(t) = \left(\frac{R^{2s} - t}{R^{2s}(1-\sigma^{2s})} \wedge 1 \right) \vee 0.$$

We proceed similar to the proof of Theorem 6.2.1 and use Lemma 5.2.1 with β, χ^2 and ψ^2. Notice that by the choice of ψ and χ, $\|\psi^2\|_{L^\infty} < 1$, $|\chi^2| < 1$ and $|(\chi^2)'(t)| \leq 2|\chi(t)||\chi'(t)| \leq 2R^{-2s}(1-\sigma^{2s})^{-1}$. The second statement of Lemma 5.2.1, justified by $\beta \in (0,1)$, implies that for all $T_1, T_2 \in I_\oplus(R)$, $T_1 < T_2$,

$$\begin{aligned} &\left[\frac{\chi^2(t)}{1-\beta} \int_B \psi^2(x) w(t,x)dx \right]_{T_1}^{T_2} + \int_{T_1}^{T_2} \chi^2(t)\mathcal{E}(u(t),\psi^2 u^{-\beta}(t))dt \\ &\geq -|B|^{\frac{1}{Q}} R^{-2s} \int_{T_1}^{T_2} \left(\frac{2(1-\sigma^{2s})^{-1}}{1-\beta} + 1 \right) \left(\int_B u^{(1-\beta)Q^*}(t,x)dx \right)^{\frac{1}{Q^*}} dt. \end{aligned} \tag{6.13}$$

We focus on the second term on the left for a moment. Let us start by using the symmetry of k and $\psi = 0$ on B^c to estimate

$$\begin{aligned}\mathcal{E}(u, \psi^2 u^{-\beta}) &= \int_M \int_M [u(x) - u(y)][\psi^2(x)u^{-\beta}(x) - \psi^2(y)u^{-\beta}(y)]k(x,y)dydx \\ &\leq \int_B \int_B [u(x) - u(y)][\psi^2(x)u^{-\beta}(x) - \psi^2(y)u^{-\beta}(y)]k(x,y)dydx \\ &+ 2\int_B \int_{B^c} [u(x) - u(y)][\psi^2(x)u^{-\beta}(x)]k(x,y)dydx =: I_1 + I_2.\end{aligned}$$

By the positivity of u on M, or more precisely $u(x) - u(y) \leq u(x)$ for all $x, y \in M$, I_2 is bounded by

$$I_2 \leq 2\int_B \int_{B^c} u^{1-\beta}(x)(\psi(x) - \psi(y))^2 k(x,y)dydx \leq 2\int_B u^{1-\beta}(x)\Gamma\psi(x)dx$$

and then $\mathbf{CE}[x_0, R; s, Q, \gamma, C_C]$ (with $\xi = (1-\sigma)^{-1}$) implies that

$$I_2 \leq 2\int_B u^{1-\beta}(x)\Gamma\psi(x)dx \leq 2C_C(1-\sigma)^{-2s-\gamma}|B|^{\frac{1}{Q}}R^{-2s}\left(\int_B u^{(1-\beta)Q^*}(x)dx\right)^{1/Q^*}.$$

Recall now Lemma 3.3 (ii) from [FK13], which states that for every $a_1, a_2 > 0, b \in (0,1)$ and $\tau_1, \tau_2 \geq 0$, setting $\zeta(b) = \frac{4b}{1-b}, \zeta_1(b) = \frac{1}{6}\zeta(b)$ and $\zeta_2(b) = \zeta(b) + \frac{9}{b}$,

$$(a_1 - a_2)(\tau_1^2 a_1^{-b} - \tau_2^2 a_2^{-b}) \leq -\zeta_1(b)\left(\tau_1 a_1^{\frac{1-b}{2}} - \tau_2 a_2^{\frac{1-b}{2}}\right)^2 + \zeta_2(b)(\tau_1 - \tau_2)^2(a_1^{1-b} + a_2^{1-b}). \tag{6.14}$$

Choosing $b = \beta$, $a_1 = u(t,x)$, $a_2 = u(t,y)$, $\tau_1 = \psi(x)$, $\tau_2 = \psi(y)$, we can use this Ineq. (6.14) to estimate I_1 which results in

$$\begin{aligned}I_1 \leq &-\zeta_1(\beta)\int_B \int_B \left[\psi(x)u^{\frac{1-\beta}{2}}(x) - \psi(y)u^{\frac{1-\beta}{2}}(y)\right]^2 k(x,y)dxdy \\ &+ \zeta_2(\beta)\int_B \int_B [\psi(x) - \psi(y)]^2[u^{1-\beta}(x) + u^{1-\beta}(y)]k(x,y)dxdy.\end{aligned}$$

Furthermore, since ψ is identically 1 on σB

$$\begin{aligned}&-\zeta_1(\beta)\int_B \int_B \left[\psi(x)u^{\frac{1-\beta}{2}(x)} - \psi(y)u^{\frac{1-\beta}{2}(y)}\right]^2 k(x,y)dxdy \\ &\leq -\zeta_1(\beta)\int_{\sigma B}\int_{\sigma B}\left[u^{\frac{1-\beta}{2}(x)} - u^{\frac{1-\beta}{2}(y)}\right]^2 k(x,y)dxdy \leq -\zeta_1(\beta)\mathcal{E}_{\sigma B}\left(u^{\frac{1-\beta}{2}}\right).\end{aligned}$$

We can use the symmetry of $k(x,y)$ to get

$$\begin{aligned}\zeta_2(\beta)&\int_B \int_B [\psi(x) - \psi(y)]^2[u^{1-\beta}(x) + u^{1-\beta}(y)]k(x,y)dx \\ &\leq 2\zeta_2(\beta)\int_B u^{1-\beta}(x)\int_M [\psi(x) - \psi(y)]^2 k(x,y)dydx \\ &\leq 2\zeta_2(\beta)\int_B u(x)^{1-\beta}\Gamma\psi(x)dx.\end{aligned}$$

Hölder inequality together with $\mathbf{CE}[x_0, R; s, Q, \gamma, C_C]$ estimate the expression in the last line by

$$2\zeta_2(\beta)\int_B u(x)^{1-\beta}\Gamma\psi(x)dx \leq 2\zeta_2(\beta)C_C(1-\sigma)^{-2s-\gamma}|B|^{\frac{1}{Q}}R^{-2s}\left(\int_B u^{(1-\beta)Q^*}(x)dx\right)^{1/Q^*}$$

which gives

$$I_1 \leq -\zeta_1(\beta)\mathcal{E}_{\sigma B}\left(u^{\frac{1-\beta}{2}}\right) + 2\zeta_2(\beta)C_C(1-\sigma)^{-2s-\gamma}|B|^{\frac{1}{Q}}R^{-2s}\left(\int_B u^{(1-\beta)Q^*}(x)dx\right)^{1/Q^*}.$$

Collecting the estimates for I_1 and I_2 we return to

$$\mathcal{E}(u,\psi^2u^{-\beta}) \leq -\zeta_1(\beta)\mathcal{E}_{\sigma B}\left(u^{\frac{1-\beta}{2}}\right) + 2(\zeta_2(\beta)+1)C_C(1-\sigma)^{-2s-\gamma}|B|^{\frac{1}{Q}}R^{-2s}\left(\int_B u^{(1-\beta)Q^*}(x)dx\right)^{1/Q^*}$$

and then going all the way back to Ineq. (6.13) we end up with

$$\begin{aligned}\left[\frac{\chi^2(t)}{1-\beta}\int_B \psi^2(x)u^{1-\beta}(t,x)dx\right]_{T_1}^{T_2} - \zeta_1(\beta)\int_{T_1}^{T_2}\chi^2(t)\mathcal{E}_{\sigma B}\left(u^{\frac{1-\beta}{2}}\right)dt \\ \geq -|B|^{\frac{1}{Q}}R^{-2s}\left(\frac{2(1-\sigma^{2s})^{-1}}{1-\beta} + 1 + 2(\zeta_2(\beta)+1)C_C(1-\sigma)^{-2s-\gamma}\right)\int_{T_1}^{T_2}\|u^{1-\beta}\|_{L^{Q^*}(B)}dt.\end{aligned}$$

In order to keep the size of expressions manageable let us recall the definition of $K(\sigma)$ from Definition 6.1.4 and overestimate the constant in front of the integral on the right. For $\beta, (1-\beta) \leq 1$, by definition of ζ_2 we have

$$\zeta_2(\beta) + 1 = \frac{4\beta}{1-\beta} + \frac{9}{\beta} + 1 = \frac{4\beta^2 + 9(1-\beta) + \beta(1-\beta)}{\beta(1-\beta)} \leq \frac{14}{\beta(1-\beta)}$$

which we use to estimate

$$\begin{aligned}&\left(\frac{2(1-\sigma^{2s})^{-1}}{1-\beta} + 1 + 2(\zeta_2(\beta)+1)C_C(1-\sigma)^{-2s-\gamma}\right) \\ &\leq \left(\frac{2\beta(1-\sigma^{2s})^{-1} + \beta(1-\beta) + 28C_C(1-\sigma)^{-2s-\gamma}}{\beta(1-\beta)}\right) \\ &\leq \frac{28(C_C+1)K(\sigma)}{\beta(1-\beta)}.\end{aligned}$$

With this in hand we multiply the equation with $-(1-\beta)$ (which changes the sign because $\beta < 1$) to get

$$\begin{aligned}\left[\chi^2(t)\int_B \psi^2(x)u^{1-\beta}(t,x)dx\right]_{T_1}^{T_2} + (1-\beta)\zeta_1(\beta)\int_{T_1}^{T_2}\chi^2(t)\mathcal{E}_{\sigma B}\left(u^{\frac{1-\beta}{2}}\right)dt \\ \leq 28(C_C+1)\beta^{-1}K(\sigma)|B|^{\frac{1}{Q}}R^{-2s}\int_{T_1}^{T_2}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}dt.\end{aligned}$$

If we restrict the integration are of the integral on the left hand side to $Z_\oplus(\sigma R)$ where $\chi = \psi = 1$ and send $T_2 \to R^{2s}$, we obtain, for every $T_1 \in [0, (\sigma R)^{2s}]$,

$$\begin{aligned}\int_{\sigma B} u^{1-\beta}(T_1,x)dx + (1-\beta)\zeta_1(\beta)\int_{T_1}^{(\sigma R)^{2s}}\mathcal{E}_{\sigma B}\left(u^{\frac{1-\beta}{2}}\right)dt \\ \leq 28(C_C+1)\beta^{-1}K(\sigma)|B|^{\frac{1}{Q}}R^{-2s}\int_{T_1}^{R^{2s}}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}dt.\end{aligned}$$

The first claim of Theorem 6.3.1 now follows by ignoring the second term on the left and taking the supremum over $T_1 \in I_\oplus(\sigma R)$ which results in

$$\operatorname*{ess\,sup}_{t\in I_\oplus(\sigma R)}\int_{\sigma B} u^{1-\beta}(t,x)dx \leq 28(C_C+1)\beta^{-1}K(\sigma)|B|^{\frac{1}{Q}}R^{-2s}\int_{I_\oplus(R)}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}dt.$$

To get the second claim, we ignore the first term on the left and let $T_1 \to 0$ to get

$$\int_{I_\oplus(\sigma R)} \mathcal{E}_{\sigma B}(u^{\frac{1-\beta}{2}})dt \leq \frac{28(C_C+1)K(\sigma)}{\beta(1-\beta)\zeta_1(\beta)}|B|^{\frac{1}{Q}}R^{-2s}\int_{I_\oplus(R)}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}} dt.$$

Recalling $\zeta_1(\beta) = \frac{4\beta}{6(1-\beta)}$ it follows that

$$\int_{I_\oplus(\sigma R)} \mathcal{E}_{\sigma B}(u^{\frac{1-\beta}{2}})dt \leq 42(C_C+1)\beta^{-2}K(\sigma)|B|^{\frac{1}{Q}}R^{-2s}\int_{I_\oplus(R)}\left(\int_B u^{(1-\beta)Q^*}(t,x)dx\right)^{\frac{1}{Q^*}} dt.$$

□

6.3.2. Elementary step

Theorem 6.3.2. *Suppose $s \in (0,1)$, $\gamma \in [0,2s)$, $q,Q \in [1,\infty]$, $C_C, C_{S1}, C_{S2}, n, C_{VL}, C_{VU} \in (0,\infty)$, $R > 0$ and $x_0 \in M$ are such that, for $\rho := \left(1 - \frac{2s}{n} + \frac{1}{q}\right)^{-1}$,*

*(i) **CE**$[x_0,(R/2,R];s,Q,\gamma,C_C]$ holds,*

*(ii) **SI**$[x_0,(R/2,R];s,\rho,Q^*,C_{S1},C_{S_2},\gamma]$ holds,*

*(iii) **V**$[x_0,(R/2,R];n,C_{VL},C_{VU}]$ holds and*

$$\frac{1}{q}+\frac{1}{Q}<\frac{2s}{n}$$

are satisfied. Then there exists a constant

$$C_{(6.3.2)} := C_{(6.3.2)}(s,n,q,Q,\gamma,C_C,C_{S1},C_{S2},C_{VL},C_{VU})$$

fulfilling the following statement.

Set $\kappa = 1+\frac{2s}{n}-\frac{1}{q}-\frac{1}{Q} > 1$, $B := B(x_0,R)$ and let $\varepsilon > 0, t_0 \in \mathbb{R}$, $\sigma \in (1/2,1)$, $f \in L^\infty(I_\oplus(R);L^Q(B))$ be arbitrary. For every $b \in (0,1)$ and every supersolution u of $\partial_t u - \mathcal{L}u = f$ in $Z_\oplus(R) \equiv Z_\oplus(t_0,x_0,R)$ such that

$$u \geq \varepsilon + R^{2s}\operatorname*{ess\,sup}_{t\in I_\oplus(R)}\left(\fint_B |f(t,x)|^Q dx\right)^{\frac{1}{Q}} \quad \text{on } Z_\oplus(R),$$

the inequality

$$\langle u\rangle_{\kappa bQ^*,\kappa b,Z_\oplus(\sigma R)} \leq \left[C_{(6.3.2)}(1-b)^{-2}(1-\sigma)^{-(2s+\gamma)\vee 1}\right]^{\frac{1}{b}}\langle u\rangle_{bQ^*,b,Z_\oplus(R)} \tag{6.15}$$

is true.

Proof. For $\beta \in (0,1)$, $\tilde\sigma = \frac{1+\sigma}{2}$ we apply Hölder's inequality with exponents $\frac{1}{\kappa-1}$ and ρ, which satisfy

$$\kappa - 1 + \frac{1}{\rho} = 1 + \frac{2s}{n} - \frac{1}{q} - \frac{1}{Q} - 1 + 1 - \frac{2s}{n} + \frac{1}{q} = 1 - \frac{1}{Q} = \frac{1}{Q^*},$$

followed by $\mathbf{SI}[x_0,\tilde\sigma R;s,\rho,Q^*,C_{S1},C_{S_2},\gamma]$, (see proof of Theorem 6.2.2 for details) to get

$$\begin{aligned} A &:= \int_{I_\ominus(\sigma R)} \|u^{(1-\beta)\kappa}(t)\|_{L^{Q^*}(\sigma B)}dt \\ &\leq \left(\sup_{t\in I_\oplus(\sigma R)}\int_{\sigma B} u^{(1-\beta)}(t,x)dx\right)^{\kappa-1}\left[C_{S1}|\tilde\sigma B|^{\frac{1}{\rho}-1}(\tilde\sigma R)^{2s}\int_{I_\oplus(\tilde\sigma R)}\mathcal{E}_{\tilde\sigma B}\left(u^{\frac{1-\beta}{2}}(t)\right)dt\right. \\ &\qquad \left. + C_{S2}\,(1-\tilde\sigma)^{-2s-\gamma}\,|\tilde\sigma B|^{\frac{1}{\rho}-\frac{1}{Q^*}}\int_{I_\oplus(\tilde\sigma R)}\|u^{1-\beta}(t)\|_{L^{Q^*}(\tilde\sigma B)}dt\right]. \end{aligned}$$

Resorting to the estimates of Theorem 6.3.1, using

$$(1-\tilde{\sigma})^{-2s-\gamma} \leq K(\tilde{\sigma}) \leq 2s^{-1}2^{(2s+\gamma)\vee 1}(1-\sigma)^{-((2s+\gamma)\vee 1)}$$

(see Lemma 6.1.5) and denoting

$$C_1 \equiv C_1(C_C, s, \gamma) = s^{-1}2^{(2s+\gamma)\vee 1}(C_C+1) \geq 1$$

we proceed by bounding

$$\begin{aligned} A \leq & \left(28C_1(1-\sigma)^{-((2s+\gamma)\vee 1)}\beta^{-1}|B|^{\frac{1}{Q}}R^{-2s}\int_{I_\oplus(R)}\|u^{1-\beta}(t)\|_{L^{Q^*}(B)}dt\right)^{\kappa-1} \\ & \times\left[42C_{S1}C_1(1-\sigma)^{-((2s+\gamma)\vee 1)}\beta^{-2}|\tilde{\sigma}B|^{\frac{1}{q}-\frac{2s}{n}}|B|^{\frac{1}{Q}}\int_{I_\oplus(R)}\|u^{1-\beta}(t)\|_{L^{Q^*}(B)}dt\right. \\ & \left.+C_{S2}(1-\sigma)^{-((2s+\gamma)\vee 1)}|\tilde{\sigma}B|^{\frac{1}{q}+\frac{1}{Q}-\frac{2s}{n}}\int_{I_\oplus(R)}\|u^{1-\beta}(t)\|_{L^{Q^*}(B)}dt\right]. \end{aligned}$$

Using $\mathbf{V}[x_0,(R/2,R];n,C_{VL},C_{VU}]$ to bound $|\tilde{\sigma}B|^{-\frac{2s}{n}} \leq 4C_{VL}^{-\frac{2s}{n}}R^{-2s}$, overestimating $1 \leq \beta^{-1} \leq \beta^{-2}$ and defining $C_2 \equiv C_2(C_C,s,C_{S1},C_{S2},\gamma,n,C_{VL}) := 4C_{VL}^{-\frac{2s}{n}}C_1\,(C_{S1}+C_{S2})$ in order to track only the behavior with respect to β and σ we end up with

$$A \leq \left[C_2\beta^{-2}(1-\sigma)^{-((2s+\gamma)\vee 1)}\right]^{\kappa}|B|^{\frac{n}{q}+\frac{\kappa}{Q}}R^{-2s\kappa}\left(\int_{I_\oplus(R)}\|u^{1-\beta}\|_{L^{Q^*}(B)}(t)dt\right)^{\kappa}.$$

After averaging out the integrals, using $\mathbf{V}[x_0,(R/2,R];n,C_{VL},C_{VU}]$ just like in the proof of Theorem 6.2.2, and smuggling volume regularity pollution terms into constant C_2 we end up with

$$\langle u^{(1-\beta)\kappa}\rangle_{Q^*,1,Z_\oplus(\sigma R)} \leq \left[C_{(6.3.2)}\beta^{-2}(1-\sigma)^{-((2s+\gamma)\vee 1)}\right]^{\kappa}\langle u^{1-\beta}\rangle^{\kappa}_{Q^*,1,Z_\oplus(R)}$$

where $C_{(6.3.2)} \equiv C_{(6.3.2)}(s,n,q,Q,\gamma,C_C,C_{S1},C_{S2},C_{VL},C_{VU})$. Substituting $b=(1-\beta)<0$, rising everything to power $\frac{1}{\kappa b}>0$ leads to

$$\langle u\rangle_{\kappa bQ^*,\kappa b,Z_\oplus(\sigma R)} \leq \left[C_{(6.3.2)}(1-b)^{-2}(1-\sigma)^{-((2s+\gamma)\vee 1)}\right]^{\frac{1}{b}}\langle u\rangle_{bQ^*,b,Z_\oplus(R)}$$

which is exactly the statement from the theorem. □

Remark 6.3.3. *Using this theorem one can estimate the* $\langle\cdot\rangle_{Q^*p,p}$ *norm (which is dominating averaged* L^p *norm) of the solution from above by norm of lower power for* $p\in(0,\kappa)$*. Going with* p *above* κ *requires different energy estimates which can be obtained for subsolutions of equation* $\partial_t u-\mathcal{L}u=f$ *as opposed to supersolutions we have been working with. But the iteration is more complicated in that case as certain tail terms have to be included. It is sometimes, see [DCKP14], but not always, see [BS07], possible to get rid of these tail terms and obtain the full Harnack inequality.*

6.3.3. Iteration

Theorem 6.3.4. *Suppose* $s\in(0,1)$, $\gamma\in[0,2s)$, $q,Q\in[1,\infty]$, $C_C,C_{S1},C_{S2},n,C_{VL},C_{VU}\in(0,\infty)$, $R>0$ *and* $x_0\in M$ *are such that for* $\rho := \left(1-\frac{2s}{n}+\frac{1}{q}\right)^{-1}$

(i) $\boldsymbol{CE}[x_0,(R/2,R];s,Q,\gamma,C_C]$ *holds,*

(ii) $\boldsymbol{SI}[x_0,(R/2,R];s,\rho,Q^*,C_{S1},C_{S_2},\gamma]$ *holds,*

(iii) $\boldsymbol{V}[x_0,(R/2,R];n,C_{VL},C_{VU}]$ *holds and*

$$\frac{1}{q}+\frac{1}{Q}<\frac{2s}{n}.$$

Then there is a constant

$$D_{(6.3.4)}\equiv D_{(6.3.4)}(s,n,q,Q,\gamma,C_C,C_{S1},C_{S2},C_{VL},C_{VU})$$

possessing the following property.

Set $\kappa=1+\frac{2s}{n}-\frac{1}{q}-\frac{1}{Q}>1$, $B:=B(x_0,R)$ *and choose any* $t_0\in\mathbb{R}$, $1/2<\sigma<1$, $p_0\in(0,1/2]$, $f\in L^\infty(I_\oplus(R);L^Q(B))$. *Then, for every supersolution* u *of* $\partial_t u-\mathcal{L}u=f$ *in* $Z_\oplus(R)\equiv Z_\oplus(t_0,x_0,R)$ *such that*

$$u\geq\varepsilon+R^{2s}\operatorname*{ess\,sup}_{t\in I_\oplus(R)}\left(\fint_B|f(t,x)|^Q dx\right)^{\frac{1}{Q}}\quad\text{on }Z_\oplus(R),$$

the inequality

$$\fint_{Z_\oplus(\sigma R)}u\leq\left[D_{(6.3.4)}\,(1-\sigma)^{\frac{-2((2s+\gamma)\vee 1)\kappa Q^*}{1-\kappa^{-1}}}\right]^{\left(\frac{1}{p_0}-1\right)}\left(\fint_{Z\oplus(R)}u^{p_0}\right)^{\frac{1}{p_0}}$$

holds.

Proof. Find the smallest integer N such that $\kappa^{-N}\leq p_0/Q^*$, i.e. $N=\lceil-\log_\kappa(p_0/Q^*)\rceil$, and define $p_0'=\kappa^{-N}$. Define also $R_k=\left(\sigma+2^{-k}(1-\sigma)\right)R$ and $\sigma_k=R_{k+1}/R_k$. This time we will iterate Ineq. (6.15) over the sequence $b_k=\kappa^k p_0'$, R_k and σ_k for k from 0 to $N-1$. Noticing that

$$(1-\sigma_k)=\frac{R_k-R_{k+1}}{R_k}=2^{-(k+1)}(1-\sigma)\frac{R}{R_k}\geq 2^{-(k+1)}(1-\sigma)$$

the iteration results in

$$\langle u\rangle_{Q^*b_N,b_N,Z_\oplus(\sigma R)}\leq\prod_{k=0}^{N-1}\left(C_{(6.3.2)}2^{((2s+\gamma)\vee 1)(k+1)}\,(1-b_k)^{-2}\,(1-\sigma)^{-((2s+\gamma)\vee 1)}\right)^{b_k}\langle u\rangle_{Q^*b_0,b_0,Z_\oplus(R)}.$$

As $b_N p_0'=1$ on the left hand side, by Jensen's inequality and Fubini's theorem we have

$$\langle u\rangle_{Q^*,1,Z_\oplus(\sigma R)}=\fint_{I_\oplus(\sigma R)}\left(\fint_{\sigma B}u^{Q^*}(t,x)dx\right)^{\frac{1}{Q^*}}dt\geq\fint_{Z_\oplus(\sigma R)}u.$$

On the right hand side $Q^*b_0=Q^*p_0'\leq p_0$, so Jensen's inequality and Fubini's theorem show that

$$\langle u\rangle_{Q^*p_0',p_0',Z_\oplus(R)}\leq\left(\fint_{I_\ominus(R)}\left(\fint_B u^{Q^*p_0'}(t,x)dx\right)^{\frac{1}{Q^*}}dt\right)^{\frac{1}{p_0'}}\leq\left(\fint_{Z_\oplus(R)}u^{p_0}\right)^{\frac{1}{p_0}}.$$

Combining these two observations with $b_k=\kappa^k p_0'$ we can estimate

$$\fint_{Z_\oplus(\sigma R)}u\leq\prod_{k=0}^{N-1}\left(C_{(6.3.2)}2^{((2s+\gamma)\vee 1)(k+1)}\left(1-\kappa^k p_0'\right)^{-2}(1-\sigma)^{-((2s+\gamma)\vee 1)}\right)^{\frac{1}{\kappa^k p_0'}}\left(\fint_{Z_\oplus(R)}u^{p_0}\right)^{\frac{1}{p_0}}.$$

Implementing the estimates $1-\kappa^k p_0'\geq 1-\kappa^{-1}$ for all $k\leq N-1$ and smuggling the resulting constant $(1-\kappa^{-1})^{-2}$ into $C_1\equiv C_1(s,n,q,Q,\gamma,C_C,C_{S1},C_{S2},C_{VL},C_{VU}):=C_{(6.3.2)}(1-\kappa^{-1})^{-2}$ produces

$$\fint_{Z_\oplus(\sigma R)}u\leq\prod_{k=0}^{N-1}\left(C_1 2^{((2s+\gamma)\vee 1)(k+1)}(1-\sigma)^{-((2s+\gamma)\vee 1)}\right)^{\frac{1}{\kappa^k p_0'}}\left(\fint_{Z_\oplus(R)}u^{p_0}\right)^{\frac{1}{p_0}}.$$

Writing the product in terms of the exponential gives us

$$\fint_{Z_\oplus(\sigma R)} u \le \left(C_1(1-\sigma)^{-((2s+\gamma)\vee 1)}\right)^{\frac{1}{p_0'}\sum_{k=0}^{N-1}\kappa^{-k}}$$
$$\times \exp\left(\frac{1}{p_0'}\sum_{k=0}^{N-1}\kappa^{-k}(\log 2)((2s+\gamma)\vee 1)(k+1)\right)\left(\fint_{Z_\oplus(R)} u^{p_0}\right)^{\frac{1}{p_0}}.$$

We now estimate all sums by utilizing that for every $\delta \in (0,1)$

$$\sum_{k=0}^{N-1}\delta^k = \frac{1-\delta^N}{1-\delta}$$

and

$$\sum_{k=0}^{N-1}(k+1)\delta^k = \frac{d}{dx}\left(\frac{1-(\delta x)^N}{1-\delta x}\right)\Big|_{x=1} = \frac{-N\delta^N(1-\delta)+(1-\delta^N)\delta}{(1-\delta)^2} \le \frac{\delta(1-\delta^N)}{(1-\delta)^2}.$$

Recalling that $p_0'\kappa^N = 1$ we find

$$\frac{1}{p_0'}\sum_{k=0}^{N-1}\kappa^{-k} = \frac{1}{p_0'}\left(\frac{1-\kappa^{-N}}{1-\kappa^{-1}}\right) = \frac{1}{(1-\kappa^{-1})}\left(\frac{1}{p_0'}-\frac{\kappa^{-N}}{p_0'}\right) = \frac{1}{(1-\kappa^{-1})}\left(\frac{1}{p_0'}-1\right)$$

and

$$\frac{1}{p_0'}\sum_{k=0}^{N-1}\kappa^{-k}(\log 2)((2s+\gamma)\vee 1)(k+1) \le \frac{(\log 2)((2s+\gamma)\vee 1)}{p_0'}\left(\sum_{k=0}^{N-1}(k+1)\kappa^{-k}\right)$$
$$\le \frac{(\log 2)((2s+\gamma)\vee 1)}{p_0'}\left(\frac{\kappa^{-1}(1-\kappa^{-N})}{(1-\kappa^{-1})^2}\right) \le \frac{(\log 2)((2s+\gamma)\vee 1)}{\kappa(1-\kappa^{-1})^2}\left(\frac{1}{p_0'}-1\right).$$

Taking into account that $p_0 \le 1/2$ and $Q^*/p_0 \le 1/p_0' \le \kappa Q^*/p_0$, we have $1/(2p_0) \le 1/p_0 - 1$ so we can bound

$$\frac{1}{p_0'} - 1 \le \frac{2\kappa Q^*}{2p_0} \le 2\kappa Q^*\left(\frac{1}{p_0}-1\right).$$

This allow us to rewrite the previous estimates in terms of p_0 instead of p_0', i.e.

$$\frac{1}{p_0'}\sum_{k=0}^{N-1}\kappa^{-k} \le \frac{2\kappa Q^*}{1-\kappa^{-1}}\left(\frac{1}{p_0}-1\right)$$

and

$$\frac{1}{p_0'}\sum_{k=0}^{N-1}\kappa^{-k}(\log 2)((2s+\gamma)\vee 1)(k+1) \le \frac{2(\log 2)((2s+\gamma)\vee 1)Q^*}{(1-\kappa^{-1})^2}\left(\frac{1}{p_0}-1\right).$$

Therefore, we can take

$$D_{(6.3.4)} \equiv D_{(6.3.4)}(s,n,q,Q,\gamma,C_C,C_{S1},C_{S2},C_{VL},C_{VU}) = C_1^{2\kappa Q^*/(1-\kappa^{-1})}2^{2((2s+\gamma)\vee 1)Q^*/(1-\kappa^{-1})^2}$$

to obtain

$$\fint_{Z_\oplus(\sigma R)} u \le \left[D_{(6.3.4)}\,(1-\sigma)^{\frac{-2((2s+\gamma)\vee 1)\kappa Q^*}{1-\kappa^{-1}}}\right]^{\left(\frac{1}{p_0}-1\right)}\left(\fint_{Z_\oplus(R)} u^{p_0}\right)^{1/p_0}$$

which proves the statement of the theorem. □

6.4. Connecting positive and negative exponents

6.4.1. Weighted Poincaré inequality

We now obtain the weighted Poincaré inequality using the result of [DK13].

Theorem 6.4.1 (Weighted Poincaré inequality)**.** *Suppose $x_0 \in M$, $R > 0$, $s \in (0,1)$, C_P, n, C_{VL}, $C_{VU} \in (0,\infty)$ are such that* ***PI***$[x_0,(R,2R];s,C_P]$ *and* ***V***$[x_0,(R,2R];n,C_{VL},C_{VU}]$ *holds, set $B := B(x_0,R)$ and define $\psi : M \to [0,1]$ by*

$$\psi(x) := \left(\frac{3R - 2d(x_0,x)}{R} \wedge 1\right) \vee 0.$$

Then there is a positive constant $C_{\psi P} := C_{\psi P}(C_P, s, n, C_{VL}, C_{VU})$ such that for every $v \in L^2(2B)$

$$\int_{\frac{3}{2}B} [v(x) - v_{\psi,2B}(x)]^2 \psi(x) dx \le C_{\psi P} R^{2s} \int_{\frac{3}{2}B} \int_{\frac{3}{2}B} [v(x) - v(y)]^2 (\psi(x) \wedge \psi(y)) k(x,y) dy dx$$

where

$$v_{\psi,2B} := \frac{\int_{2B} v(x)\psi(x)dx}{\int_{2B} \psi(x)dx}.$$

Proof. The result is a special case of Proposition 4 from [DK13]. In the notation from that paper we take $X = M$ for the space, $\rho(x,y) = d(x,y)/(2R)$ for the metric and $dx = m(dx)$ for the measure. Concerning proposition specific notation we take $p = 2$, $\phi = \Phi(\rho(x_0,\cdot))$ with

$$\Phi(x) = [(3 - 4x) \wedge 1] \vee 0 \qquad \text{for } x \ge 0.$$

By $\mathbf{PI}[x_0,(R,2R];s,C_P]$ we know that for all $r \in (R,2R]$ and all $v \in L^1(2B)$ (using $r^{2s} \le (2R)^{2s}$)

$$\int_{B_r} \left[v(x) - \fint_{B_r} v(y)dy\right]^2 dx \le C_P (2R)^{2s} \int_{B_r} \int_{B_r} [v(x) - v(y)]^2 k(x,y) dy dx.$$

One only needs to translate this to ρ metric to see that it is equivalent to the main assumption of the proposition from [DK13]. Since we verified all the assumptions, the proposition now guaratees that for every $v \in L^2(2B)$ (we immediately translate it to d-metric form)

$$\int_{2B} [v(x) - v_{\psi,2B}(x)]^2 \psi(x) dx \le C_M C_P (2R)^{2s} \int_{2B} \int_{2B} [v(x) - v(y)]^2 (\psi(x) \wedge \psi(y)) k(x,y) dy dx.$$

where

$$C_M = \frac{8^2 |B_{2R}|}{|B_R|} \frac{\Phi(0)}{\Phi(1/2)} = 2^{6+n} C_{VU} C_{VL}^{-1}.$$

Taking $C_{\psi P} := 2^{2s} C_M C_P$ and noticing that $\psi = 0$ outside of $\frac{3}{2}B$ proves the theorem. □

6.4.2. Energy estimate for $\log u$

Our aim is to apply the lemma of Bombieri and Giusti to get the weak Harnack inequality. We start by proving another energy estimate.

Theorem 6.4.2. *Let $x_0 \in M$, $R > 0$ be arbitrary and set $B := B(x_0,R)$. For every strictly positive function $w > 0$ on M, every Lipschitz function $\psi : M \to [0,\infty)$ such that $\psi > 0$ in B, $\psi = 0$ outside of B and $\mathcal{E}(\psi) < \infty$*

$$\mathcal{E}(w, -\psi^2 w^{-1}) \ge \int_B \int_B \psi(x)\psi(y) \left(\log \frac{w(y)}{\psi(y)} - \log \frac{w(x)}{\psi(x)}\right)^2 k(x,y) dx dy - 2\mathcal{E}(\psi). \tag{6.16}$$

If in addition for some $s \in (0,1)$, $\gamma \in [0,2s)$, $Q \in [1,\infty]$, $C_C > 0$ $\boldsymbol{CE}[x_0, R; s, Q, \gamma, C_C]$ is satisfied and $\xi = R \operatorname{Lip} \psi$, then we can additionally estimate

$$\mathcal{E}(\psi) \leq 2C_C(\xi^{2s-\gamma} \vee \xi^{2s+\gamma})|B|R^{-2s}$$

and $\mathcal{E}(\psi) < \infty$ is automatically satisfied.

Proof. To get this, one computes as in [FK13] Section 4,

$$\begin{aligned}
\mathcal{E}(w, -\psi^2 w^{-1}) &= \int_M \int_M [w(y) - w(x)][\psi^2(x)w^{-1}(x) - \psi^2(y)w^{-1}(y)]k(x,y)dydx \\
&\geq \int_B \int_B \psi(x)\psi(y) \left[\frac{\psi(x)w(y)}{\psi(y)w(x)} + \frac{\psi(y)w(x)}{\psi(x)w(y)} - \frac{\psi(y)}{\psi(x)} - \frac{\psi(x)}{\psi(y)}\right] k(x,y)dydx \\
&+ 2\int_B \int_{B^c} [w(y) - w(x)][\psi^2(x)w^{-1}(x) - \psi^2(y)w^{-1}(y)]k(x,y)dydx \\
&=: I_1 + I_2
\end{aligned}$$

where $B^c \times B^c$ term vanished because ψ is supported inside of B. Applying identity

$$\frac{a}{b} + \frac{b}{a} - 2 = (a-b)(b^{-1} - a^{-1}) \geq (\log a - \log b)^2 \qquad \forall a, b \geq 0,$$

which can be found in [DK15] Equation (4.7) for instance, with $a = \frac{w(y)}{\psi(y)}$ and $b = \frac{w(x)}{\psi(x)}$ we can estimate the first integral I_1 on the right by

$$\begin{aligned}
I_1 &\geq \int_B \int_B \psi(x)\psi(y) \left[\log \frac{w(y)}{\psi(y)} - \log \frac{w(x)}{\psi(x)}\right]^2 k(x,y)dydx \\
&\quad - \int_B \int_B \psi(x)\psi(y) \left[\frac{\psi(y)}{\psi(x)} + \frac{\psi(x)}{\psi(y)} - 2\right] k(x,y)dydx \\
&\geq \int_B \int_B \psi(x)\psi(y) \left[\log \frac{w(y)}{\psi(y)} - \log \frac{w(x)}{\psi(x)}\right]^2 k(x,y)dydx \\
&\quad - \int_B \int_B [\psi(x) - \psi(y)]^2 k(x,y)dydx \\
&\geq \int_B \int_B \psi(x)\psi(y) \left[\log \frac{w(y)}{\psi(y)} - \log \frac{w(x)}{\psi(x)}\right]^2 k(x,y)dxdy - \mathcal{E}(\psi).
\end{aligned}$$

Notice that assumption $\mathcal{E}(\psi) < \infty$ guarantees that expression $\infty - \infty$ did not appear in the above computation. For I_2 we use $\psi(y) = 0$ for $y \in B^c$ and the energy density estimate to compute

$$\begin{aligned}
I_2 &= 2\int_B \int_{B^c} [w(y) - w(x)]\psi^2(x)w^{-1}(x)k(x,y)dydx \\
&= 2\int_B \int_{B^c} \frac{\psi^2(x)w(y)}{w(x)} k(x,y)dydx - 2\int_B \int_{B^c} \psi^2(x)k(x,y)dydx \\
&\geq -2\int_B \int_{B^c} (\psi(x) - \psi(y))^2 k(x,y)dydx \geq -\mathcal{E}(\psi).
\end{aligned}$$

Combining the estimates of I_1 and I_2 proves

$$\mathcal{E}(w, -\psi^2 w^{-1}) \geq \int_B \int_B \psi(x)\psi(y) \left[\log \frac{w(y)}{\psi(y)} - \log \frac{w(x)}{\psi(x)}\right]^2 k(x,y)dxdy - 2\mathcal{E}(\psi)$$

which is the first statement of the theorem. The second statement follows from Proposition 6.1.2. $\square$

6.4.3. Weak L^1 estimates on $\log u$

The next step is to prove two weak L^1 estimates of the logarithm of the solution.

Theorem 6.4.3. *Suppose* $x_0 \in M$, $R > 0$, $s \in (0,1)$, $\gamma \in [0, 2s)$, $Q \in [1, \infty]$, $C_P, C_C, n, C_{VL}, C_{VU} \in (0, \infty)$ *are such that*

(i) $\boldsymbol{CE}[x_0, 2R; s, Q, \gamma, C_C]$,

(ii) $\boldsymbol{PI}[x_0, (R, 2R]; s, C_P]$,

(iii) $\boldsymbol{V}[x_0, [R, 2R]; n, C_{VL}, C_{VU}]$

are satisfied and set $B := B(x_0, R)$. *Then there is a constant*

$$D_{(6.4.3)} := D_{(6.4.3)}(C_P, C_C, \gamma, s, Q, n, C_{VL}, C_{VU})$$

fulfilling the following statement.

Let $\varepsilon > 0, t_0 \in \mathbb{R}$, $f \in L^\infty(I(R); L^Q(2B))$ *be arbitrary and set* $I(R) := I(t_0, R)$. *For every supersolution* u *of* $\partial_t u - \mathcal{L}u = f$ *in* $I(R) \times 2B$ *such that*

$$u \geq \varepsilon + (2R)^{2s} \operatorname*{ess\,sup}_{t \in I(R)} \left(\fint_{2B} |f(t,x)|^Q dx \right)^{\frac{1}{Q}} \quad \text{in } I(R) \times 2B,$$

inequalities

$$\forall \xi > 0 \qquad |Z_\oplus(R) \cap \{\log(u^{-1} e^{-a}) \geq \xi\}| \leq \frac{D_{(6.4.3)} R^{n+2s}}{\xi}$$

and

$$\forall \xi > 0 \qquad |Z_\ominus(R) \cap \{\log(u e^{a}) \geq \xi\}| \leq \frac{D_{(6.4.3)} R^{n+2s}}{\xi}$$

are satisfied, where

$$a := \frac{\int_{2B} -\log\left(\frac{u(t_0,x)}{\psi(x)}\right) \psi^2(x) dx}{\int_{2B} \psi^2(x) dx}.$$

Proof. Take the cutoff function $\psi : M \to [0, \infty)$ such that

$$\psi^2(x) = \left(\frac{3R - 2d(x_0, x)}{R} \wedge 1 \right) \vee 0$$

and apply Lemma 5.2.1 on the ball $2B$ with $\beta = 1$ and $\chi \equiv 1$ and ψ^2 for $[t_1, t_2] \subset I(R)$. Implementing the particularities of the current choice of ψ, χ and $\beta = 1$ together with $\mathbf{V}[x_0, 2R; n, C_{VL}, C_{VU}]$ gives

$$\left[\int_{2B} \psi^2(x) \log u(t,x) dx \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \mathcal{E}\left(u(t), \psi^2 u^{-1}(t)\right) dt \geq -C_{VU}(t_2 - t_1)(2R)^{n-2s}.$$

Defining, like in [FK13],

$$v(t,x) := -\log\left(\frac{u(t,x)}{\psi(x)}\right) = \log \psi(x) - \log u(t,x)$$

and implementing the estimate on $\mathcal{E}(\psi)$ from Theorem 6.4.2 (notice that $\mathbf{CE}[x_0, R; s, Q, \gamma, C_C]$ is assumed and $\operatorname{Lip}\psi \geq (R/2)^{-1}$, i.e. $\xi \geq 2$) together with $\mathbf{V}[x_0, 2R; n, C_{VL}, C_{VU}]$ provides us with

$$\begin{aligned}
&\left[\int_{\frac{3}{2}B} \psi^2(x) v(t,x) dx \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{2B} \int_{2B} \psi(x)\psi(y) \left(v(x) - v(y)\right)^2 k(x,y) dx dy dt \\
&\qquad \leq C_{VU} 2^{n-2s}(t_2 - t_1) R^{n-2s} + \int_{t_1}^{t_2} 2 C_C C_{VU} 2^{2s+\gamma} (2R)^{n-2s} dt \\
&\qquad \leq 2^{n-2s} C_{VU} \left[1 + C_C 2^{2s+\gamma+1}\right] (t_2 - t_1) R^{n-2s} =: C_1 (t_2 - t_1) R^{n-2s}.
\end{aligned}$$

At this point we use the weighted Poincaré inequality from Theorem 6.4.1, which requires $\mathbf{PI}[x_0, (R, 2R]; s, C_P]$ and $\mathbf{V}[x_0, (R, 2R]; n, C_{VL}, C_{VU}]$, in order to decrease the second term on the left hand side (note that $\psi(x)^2 \wedge \psi(y)^2 \leq \psi(x)\psi(y)$ for all $x, y \in M$) and get

$$\left[\int_{\frac{3}{2}B} \psi^2(x) v(t,x) dx\right]_{t_1}^{t_2} + C_{\psi P}^{-1} R^{-2s} \int_{t_1}^{t_2} \int_{2B} \left[v(t,x) - V(t)\right]^2 \psi^2(x) dx dt \leq C_1 (t_2 - t_1) R^{n-2s}.$$

Here $V(t)$ is defined to be

$$V(t) := v_{\psi^2, 2B}(t) = \frac{\int_{2B} v(t,x)\psi^2(x) dx}{\int_{2B} \psi^2(x) dx}.$$

A division by $\int_{\frac{3}{2}B} \psi^2(x) dx$, for which we know by $\mathbf{V}[x_0, [R, 2R]; n, C_{VL}, C_{VU}]$ that

$$C_{VL} R^n \leq |B| \leq \int_{\frac{3}{2}B} \psi^2(x) dx \leq \left|\frac{3}{2}B\right| \leq \left(\frac{3}{2}\right)^n C_{VU} R^n,$$

and a restriction of the domain to B, where $\psi = 1$, in the second integral on the left give

$$V(t_2) - V(t_1) + C_2 R^{-n-2s} \int_{[t_1, t_2]} \int_B \left[v(t,x) - V(t)\right]^2 dx dt \leq C_3 R^{-2s} (t_2 - t_1).$$

Here $C_2 = \left(\frac{3}{2}\right)^{-n} C_{VU}^{-1} C_{\psi P}^{-1} > 0$ and $C_3 = C_1 C_{VL}^{-1} = 2^{n-2s} C_{VL}^{-1} \left[C_{VU}^{\frac{1}{Q^*}} + C_C C_{VU} 2^{2s+\gamma+1}\right] > 0$. In the rest of the proof we will mostly deal with the function $V(t)$. It is possible to prove that, when u is $L^2(B)$-weakly differentiable, $V(t)$ is differentiable which would simplify parts of what follows (see [FK13]). Let us however give the more complicated version using only continuity of $V(t)$ in order to show that differentiability of $V(t)$ and consequently $L^2(B)$-weak differentiabitly of $w(t)$ is not crucial and that a priori on the solution could be relaxed.

Recall that Lemma 5.2.1 also states that $\log u(t)$ is $L^2(2B)$-strongly continuous in time which implies that $v(t) = \log \psi - \log u(t)$ is as well and that $V : I \to \mathbb{R}$ is continuous. Thus V is uniformly continuous on $\overline{I_\oplus(r)} = [t_0, t_0 + r^{2s}]$ for any $r < R$ which allows us to find, for an arbitrary $\varepsilon_1 > 0$, an $\delta \equiv \delta(\varepsilon_1, r, u) > 0$ such that $|V(t_1) - V(t_2)| \leq \varepsilon_1$ whenever $t_1, t_2 \in \overline{I_\oplus(r)}$ and $|t_1 - t_2| \leq \delta$. We intend to use this to approximate $V(t)$ by step functions while preserving the inequality. To be more precise, for all $t \in [t_2 - \delta, t_2 + \delta] \cap \overline{I_\oplus(r)}$ and $x \in B$

$$|v(t,x) - V(t_2)|^2 \leq 2|v(t,x) - V(t)|^2 + 2\varepsilon_1^2,$$

which implies that for all $t_1 \in [t_2 - \delta, t_2] \cap \overline{I_\oplus(r)}$

$$V(t_2) - V(t_1) + \frac{C_2}{2} R^{-n-2s} \int_{[t_1, t_2]} \int_B \left(\left[v(t,x) - V(t_2)\right]^2 - 2\varepsilon_1^2\right) dx dt \leq C_3 R^{-2s} (t_2 - t_1).$$

By $\mathbf{V}[x_0, (R, 2R]; n, C_{VL}, C_{VU}]$ the estimate $R^{-n-2s} \int_{[t_1, t_2]} \int_B dx \leq C_{VU} R^{-2s} (t_2 - t_1)$ holds and we use it to move ε_1 to the right side and get

$$V(t_2) - V(t_1) + \frac{C_2}{2} R^{-n-2s} \int_{[t_1, t_2]} \int_B \left[v(t,x) - V(t_2)\right]^2 dx dt \leq \left(C_3 + C_{VU} \varepsilon_1^2\right) R^{-2s} (t_2 - t_1). \quad (6.17)$$

Setting $C_{\varepsilon_1} = C_3 + C_{VU} \varepsilon_1^2 > 0$ and defining

$$w_{\varepsilon_1}(t,x) = v(t,x) - C_{\varepsilon_1} R^{-2s} t, \qquad W_{\varepsilon_1}(t) = V(t) - C_{\varepsilon_1} R^{-2s} t$$

we end up with

$$W_{\varepsilon_1}(t_2) - W_{\varepsilon_1}(t_1) + \frac{C_2}{2} R^{-n-2s} \int_{[t_1, t_2]} \int_B \left[w_{\varepsilon_1}(t,x) - W_{\varepsilon_1}(t_2) + C_{\varepsilon_1}(t_2 - t)\right]^2 dx dt \leq 0. \quad (6.18)$$

Since the last inequality holds for all t_1 and t_2 close enough and the third term is non-negative, this in particular shows that W_{ε_1} is nonincreasing in t.

Let us now define, like in the statement of the theorem,

$$a := W_{\varepsilon_1}(t_0) = V(t_0) = \frac{\int_{2B} -\log\left(\frac{u(t_0,x)}{\psi(x)}\right)\psi^2(x)dx}{\int_{2B}\psi^2(x)dx}.$$

For $\xi > 0$, on the set $\{(t,x) \in [t_1,t_2] \times B : w_{\varepsilon_1}(t,x) \geq a+\xi\}$, we have that $w_{\varepsilon_1}(t,x) - W_{\varepsilon_1}(t_2) \geq \xi + W_{\varepsilon_1}(t_0) - W_{\varepsilon_1}(t_2) \geq \xi > 0$ (because W_{ε_1} is non increasing) which allows us to ignore the $C_{\varepsilon_1}(t_2-t_1)$ part in Ineq. (6.18) and estimate

$$W_{\varepsilon_1}(t_2) - W_{\varepsilon_1}(t_1) + \frac{C_2}{2R^{n+2s}}|\{(t,x) \in [t_1,t_2] \times B : w_{\varepsilon_1}(t,x) \geq \xi + a\}|(\xi + a - W_{\varepsilon_1}(t_2))^2 \leq 0.$$

Since $C_2 > 0$, this implies that

$$\begin{aligned}
|\{(t,x) \in [t_1,t_2] \times B : w_{\varepsilon_1}(t,x) \geq \xi + a\}| &\leq 2C_2^{-1}R^{n+2s}\frac{W_{\varepsilon_1}(t_1) - W_{\varepsilon_1}(t_2)}{(\xi + a - W_{\varepsilon_1}(t_2))^2} \\
&\leq 2C_2^{-1}R^{n+2s}\frac{W_{\varepsilon_1}(t_1) - W_{\varepsilon_1}(t_2)}{(\xi + a - W_{\varepsilon_1}(t_1))(\xi + a - W_{\varepsilon_1}(t_2))} \\
&= 2C_2^{-1}R^{n+2s}\left(\frac{1}{\xi + a - W_{\varepsilon_1}(t_1)} - \frac{1}{\xi + a - W_{\varepsilon_1}(t_2)}\right).
\end{aligned}$$

Here we again used that W_{ε_1} is nonincreasing in the second inequality.

Taking $N \in \mathbb{N}$ large enough such that $R^{2s}/N \leq \delta$ (note that this means that $N \equiv N(R, r, \varepsilon_1, u)$), we can sum up the small intervals and obtain

$$\begin{aligned}
&|\overline{I_\oplus(r)} \times B \cap \{w_{\varepsilon_1} \geq \xi + a\}| \\
&\quad= \sum_{k=0}^{N-1}\left|\left\{(t,x) \in \left[t_0 + \frac{r^{2s}k}{N}, t_0 + \frac{r^{2s}(k+1)}{N}\right] \times B : w_{\varepsilon_1}(t,x) \geq \xi + a\right\}\right| \\
&\quad\leq 2C_2^{-1}R^{n+2s}\sum_{k=0}^{N-1}\left(\frac{1}{\xi + a - W_{\varepsilon_1}(t_0 + r^{2s}k/N)} - \frac{1}{\xi + a - W_{\varepsilon_1}(t_0 + r^{2s}(k+1)/N)}\right) \\
&\quad= 2C_2^{-1}R^{n+2s}\left(\frac{1}{\xi + a - W_{\varepsilon_1}(t_0)} - \frac{1}{\xi + a - W_{t_0+\varepsilon_1}(t_0 + r^{2s})}\right) \\
&\quad\leq \frac{2C_2^{-1}R^{n+2s}}{\xi},
\end{aligned}$$

where in order to get the last line one has to recall that $a = W_{\varepsilon_1}(t_0) = W_{\varepsilon_1}(t_0)$ by definition and ignore the second term $(\xi + a - W_{\varepsilon_1}(t_0 + r^{2s}))^{-1} \geq 0$ in the second to last line. Returning from $w_{\varepsilon_1}(t)$ to $-\log u(t) = w_{\varepsilon_1}(t) + C_{\varepsilon_1}R^{-2s}t$ (keep in mind that $\psi = 1$ on B) we have

$$\begin{aligned}
&|\overline{I_\oplus(r)} \times B \cap \{-\log u \geq \xi + a\}| \\
&\qquad\leq |\overline{I_\oplus(r)} \times B \cap \{w_{\varepsilon_1} \geq \xi/2 + a\}| + |\overline{I_\oplus(r)} \times B \cap \{C_{\varepsilon_1}t \geq r^{2s}\xi/2\}| \\
&\qquad\leq \frac{4C_2^{-1}R^{n+2s}}{\xi} + \left[\left(1 - \frac{\xi}{2C_{\varepsilon_1}}\right) \vee 0\right] r^{2s}|B|.
\end{aligned}$$

Recalling that constants $C_2 \equiv C_2(C_P, n, C_{VL}, C_{VU}) = (3/2)^{-n}C_{VU}^{-1}C_{\psi P}^{-1}$ and $C_{\varepsilon_1} \equiv C_{\varepsilon_1}(\varepsilon_1, Q, \gamma, C_C, s, n, C_{VL}, C_{VU}) = 2^{n-2s}C_{VL}^{-1}[C_{VU}^{1/Q^*} + C_C 2^{2s+\gamma+1}] + C_{VU}\varepsilon_1^2$ do not depend on r and then passing to the limit $r \to R$ results in

$$\begin{aligned}|Z_{\oplus}(R) \cap \{-\log u \geq \xi + a\}| &\leq \left(\frac{4C_2^{-1}}{\xi} + C_{VU}\left[\left(1 - \frac{\xi}{2C_{\varepsilon_1}}\right) \vee 0\right]\right) R^{n+2s} \\ &\leq \frac{\left(4C_2^{-1} + C_{VU}C_{\varepsilon_1}/2\right) R^{n+2s}}{\xi}.\end{aligned}$$

The last line uses an elementary inequality

$$a - b \leq \frac{a^2}{4b}$$

for any $a, b > 0$ (with $a = 1, b = \xi/(2C_{\varepsilon_1})$). Since $C_{\varepsilon_1} \xrightarrow{\varepsilon_1 \to 0} C_3$, passing to the limit $\varepsilon_1 \to 0$ we get the first inequality of the theorem

$$|Z_{\oplus}(R) \cap \{\log\left(u^{-1}e^{-a}\right) \geq \xi\}| \leq \frac{D_{(6.4.3)}R^{n+2s}}{\xi}$$

if we take

$$D_{(6.4.3)} \equiv D_{(6.4.3)}(C_P, Q, \gamma, C_C, s, n, C_{VL}, C_{VU}) = 4C_2^{-1} + C_{VU}C_3/2.$$

The other statement is obtained in a similar way by introducing variable $r < R$, working with $t_1, t_2 \in \overline{I_{\ominus}(r)}$ and analyzing the set $|\{(t,x) \in [t_1,t_2] \times B : w_{\varepsilon_1} \leq -\xi + a\}|$ instead of the set $|\{(t,x) \in [t_1,t_2] \times B : w_{\varepsilon_1} \geq \xi + a\}|$. On has to use the left endpoint of the integral when approximating $V(t)$ on $[t_1,t_2]$ which results in an estimate similar to Ineq. (6.17) but with $V(t_1)$ instead of $V(t_2)$ under the integral. Replacing V with W_{ε_1} like before leads to

$$W_{\varepsilon_1}(t_2) - W_{\varepsilon_1}(t_1) + \frac{C_2}{2}R^{-n-2s}\int_{[t_1,t_2]}\int_B \left[w_{\varepsilon_1}(t,x) - W_{\varepsilon_1}(t_1) - C_{\varepsilon_1}(t-t_1)\right]^2 dxdt \leq 0.$$

This time one ignores the term $-C_{\varepsilon_1}(t - t_1)$ on set $|\{(t,x) \in [t_1,t_2] \times B : w_{\varepsilon_1} \leq -\xi + a\}|$ because $w_{\varepsilon_1}(t,x) - W_{\varepsilon_1}(t_1) \leq -\xi + W_{\varepsilon_1}(t_0) - W_{\varepsilon_1}(t_1) \leq 0$, due to $W_{\varepsilon_1}(t_0) - W_{\varepsilon_1}(t_1) \leq 0$, in order to estimate

$$W_{\varepsilon_1}(t_2) - W_{\varepsilon_1}(t_1) + \frac{C_2}{2R^{n+2s}}|\{(t,x) \in [t_1,t_2] \times B : w_{\varepsilon_1}(t,x) \leq -\xi + a\}|(-\xi + a - W_{\varepsilon_1}(t_1))^2 \leq 0.$$

Note that the inequality

$$\frac{W_{\varepsilon_1}(t_1) - W_{\varepsilon_1}(t_2)}{(-\xi + a - W_{\varepsilon_1}(t_1))^2} \leq \frac{W_{\varepsilon_1}(t_1) - W_{\varepsilon_1}(t_2)}{(-\xi + a - W_{\varepsilon_1}(t_1))(-\xi + a - W_{\varepsilon_1}(t_2))}$$

is still true but for slightly different reasons. The factors in the denominator are now negative and decreasing t_1 increases $-\xi + a - W_{\varepsilon_1}(t_1)$ but decreases its absolute value. The rest of the estimates remain exactly the same and a similar summation procedure produces

$$\begin{aligned}|\overline{I_{\ominus}(R)} \times B \cap \{w_{\varepsilon_1} \leq -\xi + a\}| &\leq 2C_2^{-1}R^{n+2s}\left(\frac{1}{-\xi + a - W_{\varepsilon_1}(t_0 - r^{2s})} - \frac{1}{-\xi + a - W_{\varepsilon_1}(t_0)}\right) \\ &\leq -\frac{2C_2^{-1}}{-\xi}R^{n+2s} \leq \frac{2C_2^{-1}R^{n+2s}}{\xi}.\end{aligned}$$

Translating this into a statement on u and passing to the limits $r \to R$ and $\varepsilon_1 \to 0$ exactly like before we end up with

$$|Z_{\ominus}(R) \cap \{\log\left(ue^a\right) \geq \xi\}| \leq \frac{D_{(6.4.3)}R^{n+2s}}{\xi}.$$

which is the second statement of the theorem. □

6.4.4. Lemma of Bombieri and Giusti

To connect positive and negative exponents we are going to use the idea of Bombieri and Giusti [BG72] in modified version from [SC02] Lemma 2.2.6. We provide the statement of the lemma with reduced and adapted notation for convenience.

Lemma 6.4.4 (Bombieri and Giusti ([SC02], Lemma 2.2.6)). *Fix a collection of measurable subset* $\{U_\sigma \subset M; \sigma \in (0,1]\}$ *such that* $U_{\sigma'} \subset U_\sigma$ *if* $\sigma' \leq \sigma$ *and denote* $U = U_1$. *Let* δ, C *be positive constants and* $0 < \alpha_0 \leq \infty$. *Let also* v *be a positive measurable function on* U *which satisfies*

$$\|v\|_{L^{\alpha_0}(U_{\sigma'})} \leq \left[C(\sigma - \sigma')^{-\delta}|U|^{-1}\right]^{1/\alpha - 1/\alpha_0} \|v\|_{L^\alpha(U_\sigma)} < \infty,$$

for all σ, σ', α *such that* $1/2 \leq \sigma' < \sigma \leq 1$ *and* $0 < \alpha \leq \min\{1, \alpha_0/2\}$. *Assume further that* v *satisfies*

$$|\{\log v > \xi\}| \leq \frac{C|U|}{\xi}$$

for all $\xi > 0$. *Then*

$$\|v\|_{L^{\alpha_0}(U_{1/2})} \leq A|U|^{1/\alpha_0}$$

where A *depends only on* δ, C *and a lower bound on* α_0.

6.5. Weak Harnack inequality

Theorem 6.5.1 (Weak parabolic Harnack inequality). *Suppose* $s \in (0,1)$, $\gamma \in [0,2s)$, $q, Q \in [1,\infty]$, $C_C, C_{S1}, C_{S2}, n, C_{VL}, C_{VU} \in (0,\infty)$, $R > 0$ *and* $x_0 \in M$ *are such that, for* $\rho := \left(1 - \frac{2s}{n} + \frac{1}{q}\right)^{-1}$,

(i) $\boldsymbol{CE}[x_0, (R/2, R] \cup \{2R\}; s, Q, \gamma, C_C]$,

(ii) $\boldsymbol{SI}[x_0, (R/2, R]; s, \rho, Q^*, C_{S1}, C_{S_2}, \gamma]$,

(iii) $\boldsymbol{V}[x_0, (R/2, 2R]; n, C_{VL}, C_{VU}]$,

(iv) $q^{-1} + Q^{-1} < 2s/n$,

(v) $\boldsymbol{PI}[x_0, (R, 2R]; s, C_P]$,

are satisfied and set $B := B(x_0, R)$. *Then there exists a constant*

$$C_{PH} \equiv C_{PH}\left(s, n, q, Q, C_C, C_{S1}, C_{S2}, C_P, \gamma, C_{VL}, C_{VU}\right)$$

such that $\boldsymbol{WPHI}[t_0, x_0, R; s, C_{PH}, Q]$ *is satisfied. To be explicit,* C_{PH} *is such that the following statement holds.*

Let $f \in L^\infty(I(R); L^Q(2B))$ *be arbitrary. For every supersolution* u *of* $\partial_t u - \mathcal{L}u = f$ *in* $I(R) \times B(x_0, 2R)$ *such that* $u \geq 0$ *on* M, *the inequality*

$$⨍_{U_\ominus} u(x)dx \leq C_{PH}\left(\operatorname*{ess\,inf}_{U_\oplus} u + (2R)^{2s} \operatorname*{ess\,sup}_{t \in I(R)} \left(⨍_{2B} |f(t,x)|^Q dx\right)^{\frac{1}{Q}}\right)$$

is satisfied, where $U_\ominus = Z_\oplus(t_0 - R^{2s}, x_0, R/2)$ *and* $U_\oplus = Z_\ominus(t_0 + R^{2s}, x_0, R/2)$ *(see Fig. 6.5.1).*

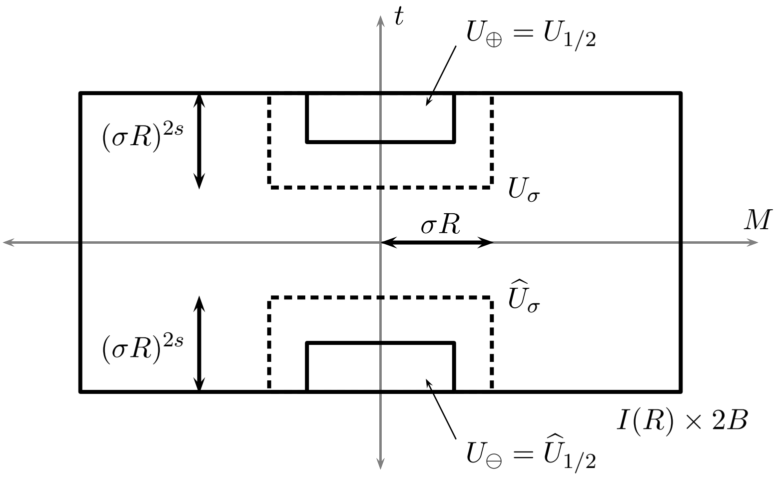

Figure 6.5.1.: Sequences U_σ and $\widehat{U}_\sigma$

Proof. Let us define

$$\tilde{u} = u + \varepsilon + (2R)^{2s} \operatorname*{ess\,sup}_{t \in I(R)} \left(\fint_{2B} |f(t,x)|^Q dx \right)^{\frac{1}{Q}}. \tag{6.19}$$

Notice that we can assume that the $\operatorname{ess\,inf}_{U_\oplus} u$ is finite otherwise the theorem trivially holds. The plan is to apply Lemma 6.4.4 two times, for sequences $\{U_\sigma\}_{\sigma \in [\frac{1}{2},1]}$ and $\{\widehat{U}_\sigma\}_{\sigma \in [\frac{1}{2},1]}$ like in Fig. 6.5.1. Let a be like in Theorem 6.4.3 for function $\tilde{u}$. For the first application take $U_\sigma = Z_\ominus(t_0 + R^{2s}, x_0, \sigma R)$, $\alpha_0 = +\infty$, $v = \tilde{u}^{-1} e^{-a}$,

$$\delta = \frac{((2s+\gamma) \vee 1) Q^*}{(1 - \kappa^{-1})}$$

and $C = 2^{n+2s+\delta} C_{VL}^{-1} C_{VU}^{1} D_{(6.2.3)} \vee C_{VL}^{-1} D_{(6.4.3)}$. For all $\alpha \leq \alpha_0$, $1/2 \leq \sigma' \leq \sigma \leq 1$ we use the iteration for negative exponents in Theorem 6.2.3, applied to $\tilde{u}$ on $U_\sigma = Z_\ominus(t_0 + R^{2s}, x_0, \sigma R)$ with $\sigma_{6.2.3} = \sigma'/\sigma \in (1/2, 1)$ and $p_0 = -\alpha$. Assumptions in Items (i) to (iv) and the definition of $\tilde{u}$ justify the application, which gives

$$\infty > \operatorname*{ess\,inf}_{Z_\ominus(t_0 + R^{2s}, x_0, \sigma' R)} \tilde{u} \geq \left[D_{(6.2.3)} \left(1 - \frac{\sigma'}{\sigma} \right)^{-\frac{((2s+\gamma)\vee 1) Q^*}{1-\kappa^{-1}}} \right]^{-\frac{1}{\alpha}} \left(\fint_{Z_\ominus(t_0 + R^{2s}, x_0, \sigma R)} \tilde{u}^{-\alpha} \right)^{-\frac{1}{\alpha}}.$$

Translating this into statement involving $v = \tilde{u}^{-1} e^{-a}$, U_σ, δ, α and α^0 switches the inequality and shows

$$\|v\|_{L^{\alpha_0}(U_{\sigma'})} \leq \left[D_{(6.2.3)} \left(\frac{(\sigma - \sigma')}{\sigma} \right)^{-\delta} \right]^{\frac{1}{\alpha}} \frac{1}{|U_\sigma|^{\frac{1}{\alpha}}} \|v\|_{L^\alpha(U_\sigma)}.$$

Due to $\sigma^{-\delta} \leq 2^\delta$ and (somewhat deceivingly $\frac{1}{\alpha_0} = 0$)

$$\frac{1}{|U_\sigma|^{\frac{1}{\alpha}}} = \frac{1}{|Z_\ominus(t_0 + R^{2s}, \sigma R)|^{\frac{1}{\alpha}}} \leq \frac{1}{\left(C_{VL} C_{VU}^{-1} \sigma^{n+2s} |U_1| \right)^{\frac{1}{\alpha}}} \leq \left(2^{n+2s} C_{VL}^{-1} C_{VU} |U_1|^{-1} \right)^{\frac{1}{\alpha} - \frac{1}{\alpha_0}},$$

which follows from $\mathbf{V}[x_0, (R/2, R]; n, C_{VL}, C_{VU}]$, we can furthermore estimate

$$\|v\|_{L^{\alpha_0}(U_{\sigma'})} \leq \left[C (\sigma - \sigma')^{-\delta} |U_1|^{-1} \right]^{1/\alpha - 1/\alpha_0} \left(\int_{U_\sigma} v^\alpha \right)^{1/\alpha}.$$

Weirdly enough, U_1 is exactly $Z_\oplus(t_0,x_0,R)$ so weak $L^1(Z_\oplus(R))$ estimate on $\log(\tilde{u}^{-1}e^{-a})$ in Theorem 6.4.3, justified by Items (i), (iii) and (v), proves

$$|U_1 \cap \{\log v > \xi\}| \le \frac{D_{(6.4.3)}R^{n+2s}}{\xi} \le \frac{C_{VL}^{-1}D_{(6.4.3)}|U_1|}{\xi} \le \frac{C|U_1|}{\xi}$$

for all $\xi > 0$, where the second inequality requires $\mathbf{V}[x_0,(R/2,R];n,C_{VL},C_{VU}]$. Lemma 6.4.4 now implies that there is a constant

$$A := A(C,\delta) = A(s,n,C_{VL},C_{VU},q,Q,C_{S1},C_{S2},C_P,C_C,\gamma) > 0$$

such that

$$\operatorname*{ess\,sup}_{U_\oplus} \tilde{u}^{-1}e^{-a} = \operatorname*{ess\,sup}_{U_{\frac{1}{2}}} v \le A$$

or written differently

$$\operatorname*{ess\,inf}_{U_\oplus} \tilde{u} \ge A^{-1}e^{-a}. \tag{6.20}$$

For the second application, let us first suppose that u is in $L^1(I(R),L^2(2B))$ and not only in $L^1_{loc}(I(R),L^2(2B))$ which would follow from Item (i) of Lemma 5.1.7. This implies that $⨍_{I(R)\times 2B} u < \infty$. Take $\widehat{U}_\sigma = Z_\oplus(t_0 - R^{2s}, x_0, \sigma R)$, $\widehat{\alpha}_0 = 1$, $\widehat{v} = \tilde{u}e^a$,

$$\widehat{\delta} = \frac{((2s+\gamma)\vee 1)(1+\kappa)Q^*}{(1-\kappa^{-1})}$$

and $\widehat{C} = 2^{2n+4s+\widehat{\delta}}C_{VL}^{-2}C_{VU}^2 D_{(6.3.4)} \vee C_{VL}^{-1}D_{(6.4.3)}$. The iteration for positive exponents in Theorem 6.3.4 is applicable due to Items (i) to (iv) and shows, for all $\alpha \le \widehat{\alpha}_0/2$, $1/2 \le \sigma' \le \sigma \le 1$,

$$\left(⨍_{\widehat{U}_{\sigma'}} \widehat{v}^{\widehat{\alpha}_0}\right)^{1/\widehat{\alpha}_0} \le \left[\widehat{C}(\sigma-\sigma')^{-\widehat{\delta}}\left|\widehat{U}_1\right|^{-1}\right]^{1/\alpha - 1/\widehat{\alpha}_0}\left(⨍_{\widehat{U}_\sigma} \widehat{v}^{\alpha}\right)^{1/\alpha} < \infty,$$

where $<\infty$ bound comes from assumption $⨍_{I(R)\times 2B} u < \infty$. Due to $|U_{\sigma'}|^{\frac{1}{\widehat{\alpha}_0}} \le |U_1|^{\frac{1}{\widehat{\alpha}_0}}$ and (using $\frac{1}{2\alpha} \le \frac{1}{\alpha} - 1 = \frac{1}{\alpha} - \frac{1}{\widehat{\alpha}_0}$ for $\alpha \in (0,1/2)$)

$$\begin{aligned}\frac{1}{|U_\sigma|^{\frac{1}{\alpha}}} &= \frac{1}{|Z_\ominus(t_0+R^{2s},\sigma R)|^{\frac{1}{\alpha}}} \le \frac{1}{\left(C_{VL}C_{VU}^{-1}\sigma^{n+2s}|U_1|\right)^{\frac{1}{\alpha}}} \\ &\le \left(2^{2n+4s}C_{VL}^{-2}C_{VU}^2\right)^{\frac{1}{2\alpha}}|U_1|^{-\frac{1}{\alpha}} \le \left(2^{2n+4s}C_{VL}^{-2}C_{VU}^2\right)^{\frac{1}{\alpha}-\frac{1}{\widehat{\alpha}_0}}|U_1|^{-\frac{1}{\alpha}},\end{aligned}$$

it follows that

$$\left(\int_{\widehat{U}_{\sigma'}} \widehat{v}^{\widehat{\alpha}_0}\right)^{1/\widehat{\alpha}_0} \le \left[\widehat{C}(\sigma-\sigma')^{-\widehat{\delta}}\left|\widehat{U}_1\right|^{-1}\right]^{1/\alpha-1/\widehat{\alpha}_0}\left(\int_{\widehat{U}_\sigma} f^\alpha\right)^{1/\alpha}.$$

Slightly confusingly, $\widehat{U}_1 = Z_\ominus(t_0,x_0,R)$ and Theorem 6.4.3 shows that for all $\xi > 0$

$$\left|\widehat{U}_1 \cap \{\log(\tilde{u}e^a) > \xi\}\right| \le \frac{D_{(6.4.3)}R^{n+2s}}{\xi} \le \frac{C_{VL}^{-1}D_{(6.4.3)}|\widehat{U}_1|}{\xi} \le \frac{\widehat{C}|\widehat{U}|}{\xi}.$$

Applying Lemma 6.4.4 once more gives us a constant

$$\widehat{A} := \widehat{A}(s,n,C_{VL},C_{VU},q,Q,C_{S1},C_{S2},C_P,C_C,\gamma)$$

such that

$$\int_{U_\ominus} \tilde{u}e^a = \int_{\widehat{U}_{1/2}} v \le \widehat{A}|\widehat{U}_1|.$$

This is now easily transformed into

$$\fint_{U_\ominus} \tilde{u} \leq 2^{n+2s} C_{VU} C_{VL}^{-1} \widehat{A} e^{-a} \tag{6.21}$$

using $\mathbf{V}[x_0, R; n, C_{VL}, C_{VU}]$. On the other hand if u is only in $L^1_{loc}(I(R), L^2(2B))$, then for every $\beta \in (0, R^{2s})$ Ineq. (6.21) can be proved with $\{(t+\beta, x) : (t,x) \in U_\ominus\}$ instead of $U_\ominus$ on the left simply by shifting all sets in time. But the right hand side remains independent of β so sending $\beta \to \infty$ proves Ineq. (6.21) in the original form by monotone convergence theorem.

Combining Ineq. (6.20) and Ineq. (6.21) gives the parabolic weak Harnack inequality for $\tilde{u}$,

$$\fint_{U_\ominus} \tilde{u} \leq C_{PH} \operatorname*{ess\,inf}_{U_\oplus} \tilde{u}$$

where we take

$$C_{PH} := C_{PH}(s, n, C_{VL}, C_{VU}, q, Q, C_{S1}, C_{S2}, C_P, C_C, \gamma) = 2^{n+2s} C_{VU} C_{VL}^{-1} A \widehat{A}.$$

Substituting $\tilde{u}$ with u through Eq. (6.19) this is equivalent to

$$\begin{aligned}\fint_{U_\ominus} u + \varepsilon + (2R)^{2s} \operatorname*{ess\,sup}_{t \in I(R)} \left(\fint_{2B} |f(t,x)|^Q dx \right)^{\frac{1}{Q}} \\ \leq C_{PH} \left(\operatorname*{ess\,inf}_{U_\oplus} u + \varepsilon + (2R)^{2s} \operatorname*{ess\,sup}_{t \in I(R)} \left(\fint_{2B} |f(t,x)|^Q dx \right)^{\frac{1}{Q}} \right).\end{aligned}$$

If we ignore term $\operatorname{ess\,sup}_{t \in I(R)} \left(\fint_{2B} |f(t,x)|^Q dx \right)^{\frac{1}{Q}} > 0$ on the left hand side and send $\varepsilon \to 0$ (note that C_{PH} does not depend on ε), we obtain exactly the statement of the theorem, i.e.

$$\fint_{U_\ominus} u \leq C_{PH} \left(\operatorname*{ess\,inf}_{U_\oplus} u + (2R)^{2s} \operatorname*{ess\,sup}_{t \in I(R)} \left(\fint_{2B} |f(t,x)|^Q dx \right)^{\frac{1}{Q}} \right).$$

□

Theorem 6.5.2. *Let $x_0 \in M$, $R > 0$, $Q \in [1, \infty]$, $C_{PH} \in (0, \infty)$ be such that $\boldsymbol{WPHI}[x_0, R; s, C_{PH}, Q]$ is satisfied and set $B := B(x_0, R)$. Then for every $f \in L^Q(2B)$, every supersolution u of $\mathcal{L}u = f$ in $2B$ with $u \geq 0$ on M*

$$\fint_{\frac{1}{2}B} u \leq C_{EH} \left(\operatorname*{ess\,inf}_{\frac{1}{2}B} u + (2R)^{2s} \left(\fint_{2B} |f|^Q \right)^{\frac{1}{Q}} \right).$$

In particular, $\boldsymbol{WEHI}[x_0, R; C_{PH}]$ also hold.

Proof. By Proposition 5.1.6 we know that every supersolution u of $\mathcal{L}u = f$ in $2B$ solves $\partial_t u - \mathcal{L}u = f$ in $I(t_0, R) \times 2B$ for all $t_0 \in \mathbb{R}$. Therefore $\mathbf{WPHI}[x_0, R; s, C_{PH}, Q]$ gives

$$\fint_{U_\ominus} u \leq C_{PH} \left(\operatorname*{ess\,inf}_{U_\oplus} u + (2R)^{2s} \operatorname*{ess\,sup}_{t \in I(R)} \left(\fint_{2B} |f(t,x)|^Q dx \right)^{\frac{1}{Q}} \right)$$

which is equivalent to the statement of the theorem because u and f are time independent and $U_\oplus$, $U_\ominus$ are cylinders over $B/2$. □

6.6. Hölder regularity estimate

Definition 6.6.1. *For $x_0 \in M, t_0 \in \mathbb{R}$ and $R >$ define*

$$D(t_0, x_0, R) = (t_0 - 2R^{2s}, t_0) \times B(x_0, 2R),$$
$$\hat{D}(t_0, x_0, R) = (t_0 - 2R^{2s}, t_0) \times B(x_0, 3R),$$
$$D_{\ominus}(t_0, x_0, R) = \left(t_0 - 2R^{2s}, t_0 - 2R^{2s} + (R/2)^{2s}\right) \times B(x_0, R/2),$$
$$D_{\oplus}(t_0, x_0, R) = \left(t_0 - (R/2)^{2s}, t_0\right) \times B(x_0, R/2)$$

like in Fig. 6.6.1. We will leave t_0 and x_0 implicit whenever possible.

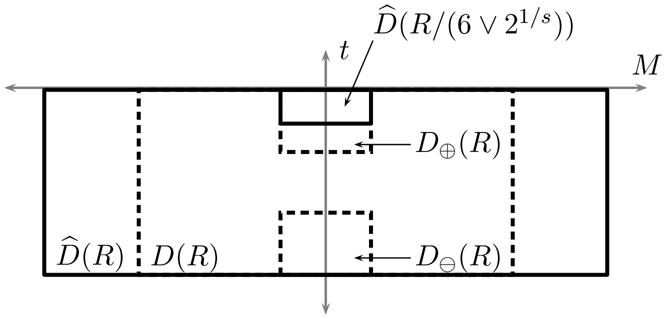

Figure 6.6.1.: Sets $D(R), \hat{D}(R), D_{\ominus}$ and $D_{\oplus}$

Lemma 6.6.2 (Increase of inf)**.** *Let $x_0 \in M$, $t_0 \in \mathbb{R}$, $R > 0$, $s \in (0,1)$, $\gamma \in [0, 2s)$, $C_{PH}, C_C \in (0, \infty)$, $Q \in [1, \infty]$ be such that $\mathcal{E}$ satisfies* ***WPHI***$[t_0, x_0, R; s, C_{PH}, Q]$, ***CE***$[x_0, R; s, Q, \gamma, C_C]$ *and set $\hat{B} = B(x_0, 3R)$, $\delta = \frac{1}{8C_{PH}} > 0$. Then there exists a constant $\beta := \beta(C_{PH}, s, \gamma, C_C) > 0$ with the following property. If $w : M \to \mathbb{R}$ is a supersolution of $\partial_t u - \mathcal{L}u = 0$ in $D(R)$ such that*

$$w \geq 1 - 3^{j\beta} \qquad \text{in } (t_0 - (2R)^{2s}, t_0) \times 3^j \hat{B} \qquad \forall j \in \mathbb{N}_0, \tag{6.22}$$

and

$$\left| D_{\ominus}(R) \cap \left\{ w \geq \frac{1}{2} \right\} \right| \geq \frac{|D_{\ominus}(R)|}{2}, \tag{6.23}$$

then w is separated from 0 in $D_{\oplus}(R)$ i.e.

$$w \geq \delta \qquad \text{in } D_{\oplus}(R).$$

Proof. Define $B_2 := B(x_0, 2R)$ so that $D(R) = (t_0 - 2R^{2s}, t_0) \times B_2$. We only need to find an appropriate $\beta > 0$ as small as necessary. Let us write $w = w^+ - w^-$ and observe that $w(t) \in V_{B_2}$ implies $w^+(t), w^-(t) \in V_{B_2}$ (recall Definition 5.1.2). This allows us to consider w^+ as a supersolution of $\partial_t u - \mathcal{L}u = \mathcal{E}(w^-, \cdot)$ in $D(R)$. Functional $\mathcal{E}(w^-(t), \cdot)$ on $\mathcal{D}_{B_2}(\mathcal{E})$ appears because $\mathcal{E}$ is nonlocal and $w^- = 0$ on $\hat{D}(R) \supset D(R)$ is not enough to make it trivial. Let us denote this functional by $F_t : \mathcal{D}_{B_2}[\mathcal{E}] \to \mathbb{R}$, $F_t(\varphi) := \mathcal{E}(w^-(t), \varphi)$. For $t \in (t_0 - 2R^{2s}, t_0)$ we find the estimate for $\|F_t\|_{L^{Q^*}(B_2) \to \mathbb{R}}$ by calculating, for arbitrary $\varphi \in C_c(B_2)$,

$$\begin{aligned}
|F_t(\varphi)| = |\mathcal{E}(w^-(t), \varphi)| &\leq \left| \int_M \int_M \frac{(w^-(t,x) - w^-(t,y))(\varphi(x) - \varphi(y))}{d(x,y)^{n+2s}} c(x,y) dy dx \right| \\
&= 2 \int_{B_2} \varphi(x) \int_{\hat{B}^c} \frac{w^-(t,y)}{d(x,y)^{n+2s}} c(x,y) dy dx \\
&\leq 2 \left(\int_{B_2} \left(\int_{\hat{B}^c} \frac{w^-(t,y)}{d(x,y)^{n+2s}} c(x,y) dy \right)^Q dx \right)^{\frac{1}{Q}} \left(\int_{B_2} \varphi^{Q^*}(x) dx \right)^{\frac{1}{Q^*}}.
\end{aligned}$$

Let us, for $j \in \mathbb{N}$, introduce cutoffs

$$\psi_j(x) = \left(\frac{3^j R - d(x_0, x)}{(3^j - 2)R} \wedge 1 \right) \vee 0$$

which have the property that $\psi_j = 1$ on B_2 and $\psi_j = 0$ outside of $3^{j-1}\hat{B}$. These function can be used together with $\mathbf{CE}[x_0, R; s, Q, \gamma, C_C]$ and Ineq. (6.22) to estimate

$$\begin{aligned}
\frac{\|F_t\|_{L^{Q^*}(B_2)\to\mathbb{R}}}{|B_2|^{\frac{1}{Q}}} &\leq 2|B_2|^{-\frac{1}{Q}} \left\| \sum_{j=1}^{\infty} \int_{3^j\hat{B}\setminus 3^{j-1}\hat{B}} \frac{(3^{j\beta}-1)(\psi_j(x)-\psi_j(y))^2}{d(x,y)^{n+2s}} \right\|_{L^Q(B_2)} \\
&\leq 2\sum_{j=1}^{\infty}(3^{j\beta}-1)|B_2|^{-\frac{1}{Q}} \|\Gamma\psi_j\|_{L^Q(B_2)} \leq C_C(2R)^{-2s}\sum_{j=1}^{\infty}\frac{3^{j\beta}-1}{(3^j-2)^{2s-\gamma}} \\
&\leq 2C_C(2R)^{-2s}\sum_{j=1}^{\infty}\frac{3^{j\beta}-1}{3^{(j-1)(2s-\gamma)}}.
\end{aligned}$$

For $\beta < (2s-\gamma)/2$, the series is dominated by

$$3^{-(2s-\gamma)}\sum_{j=1}^{\infty}\frac{3^{j\beta}-1}{3^{(2s-\gamma)j}} \leq 3^{-(2s-\gamma)}\sum_{j=1}^{\infty}3^{j(\beta-2s+\gamma)} \leq 3^{-(2s-\gamma)}\sum_{j=1}^{\infty}3^{-\frac{(2s-\gamma)j}{2}} < \infty$$

so dominated converegence theorem implies

$$(2R)^{2s}|B_2|^{\frac{1}{Q}}\|F_t\|_{L^{Q^*}(B_2)\to\mathbb{R}} \leq 2C_C\sum_{j=1}^{\infty}\frac{3^{j\beta}-1}{3^{(j-1)(2s-\gamma)}} =: \zeta(\beta) \xrightarrow{\beta\to 0} 0$$

because $3^{j\beta} - 1 \xrightarrow{\beta\to 0} 0$ pointwise. In particular, $\|F_t\|_{L^{Q^*}(B_2)\to\mathbb{R}}$ is bounded so, for every $t \in (t_0 - 2R^{2s}, t_0)$ there exist an $f(t) \in L^{Q^*}(B_2)$ such that $\mathcal{E}(w^-(t), \varphi) = (f(t), \varphi)$ and

$$(2R)^{2s}\left(\fint_{B_2}|f(t,x)|^Q dx\right)^{\frac{1}{Q}} = \|F_t\|_{L^{Q^*}(B_2)\to\mathbb{R}} \leq \zeta(\beta) \xrightarrow{\beta\to 0} 0.$$

If we take $\beta < s$ small enough such that $\zeta(\beta) < C_{PH}^{-1}/8$, then $\mathbf{WPHI}[t_0 - R^{2s}, x_0, R; s, C_{PH}, Q]$ applied to w^+ in $D(R)$ together with Ineq. (6.23) gives

$$\begin{aligned}
\operatorname*{ess\,inf}_{D_\oplus} w = \operatorname*{ess\,inf}_{D_\oplus} w^+ &\geq C_{PH}^{-1}\left(\fint_{D_\ominus} w^+(t,x)dxdt\right) - (2R)^{2s}\operatorname*{ess\,sup}_{t\in(t_0-(2R)^{2s},t_0)}\left(\fint_{B_2}|f(t,x)|^Q dx\right)^{\frac{1}{Q}} \\
&\geq \frac{1}{4C_{PH}} - \zeta(\beta) \geq \frac{1}{8C_{PH}} \geq \delta.
\end{aligned}$$

At the end, observe that ζ depended only on β, s, γ, C_C and since β was chosen such that $\zeta(\beta) \leq C_{PH}^{-1}/8$ we have $\beta \equiv \beta(C_{PH}, s, \gamma, C_C)$. □

Theorem 6.6.3 (Hölder regularity estimate)**.** *Suppose that $x_0 \in M$, $t_0 \in \mathbb{R}$, R_0, $s \in (0,1)$, $\gamma \in [0, 2s)$, $C_{PH} < \infty$, $Q \in [1, \infty]$ are such that $\mathcal{E}$ satisfies **WPHI**$[x_0, t_0, [R_0, \infty); s, C_{PH}, Q]$ and **CE**$[x_0, [R_0, \infty); s, Q, \gamma, C_C]$. Then there exists*

$$\eta \equiv \eta(C_{PH}, C_C, s, \gamma) > 0$$

such that the following is true. For every $\mathcal{R}, R > 0$ such that $\mathcal{R} \geq R \geq R_0$, every supersolution $u : (t_0 - 2\mathcal{R}^{2s}, t_0) \times M \to (0, \infty)$ of $\partial_t u - \mathcal{L}u = 0$ in $D(\mathcal{R})$,

$$\operatorname*{ess\,osc}_{(t_0-R^{2s},t_0)\times B(x_0,R)} u \leq \left(12 \vee 2^{1+\frac{1}{s}}\right)\|u\|_{L^\infty((t_0-2R^{2s},t_0)\times M)}\left(\frac{R}{\mathcal{R}}\right)^\eta.$$

*In other words, **HR**$[x_0, [R_0, \infty); \eta, C_H = 12 \vee 2^{1+1/s}]$ holds.*

Proof. We will use sets $D, \hat{D}, D_\ominus$ and $D_\oplus$ from Definition 6.6.1. If $u = 0$, the statement is trivially true. If not, dividing both sides with $2\|u\|_{L^\infty(I\times M)}$ it is sufficient to consider only u such that $-1/2 \leq u \leq 1/2$ (the statement is trivial for $u = 0$). Let δ' and β be constants from Lemma 6.6.2. Take $\mathfrak{c} := 6 \vee 2^{\frac{1}{s}}$ and notice that this is exactly the condition needed to have $\hat{D}(\mathfrak{c}^{-1}\mathcal{R}) \subset D_\oplus(\mathcal{R})$. Set $\delta = (1 - \mathfrak{c}^{-(\beta\wedge 1)}) \wedge \delta'$ and define

$$\eta \equiv \eta(C_{PH}, C_C, s, \gamma) := -\log_{\mathfrak{c}}(1-\delta) > 0$$

which in particular implies that $(1-\delta) = \mathfrak{c}^{-\eta}$ and $\eta \leq \beta \wedge 1$.

We will construct a increasing sequence $\{m_k\}$ and a decreasing sequence $\{M_k\}$ in $[-1/2, 1/2]$, for $k \in \mathbb{N}_0$, such that

$$m_k \leq u \leq M_k \qquad \text{in } \hat{D}(\mathfrak{c}^{-k}\mathcal{R}),$$

$$M_k - m_k = \mathfrak{c}^{-k\eta}.$$

We set $m_0 = -1/2$ and $M_0 = 1/2$ and proceed by induction supposing $k \geq 1$ and m_{k-1}, M_{k-1} are already defined. There are two possibilities, either

$$\left| D_\ominus(\mathfrak{c}^{-(k-1)}\mathcal{R}) \cap \left\{ u \geq \frac{M_{k-1} + m_{k-1}}{2} \right\} \right| \geq \frac{|D_\ominus(\mathfrak{c}^{-(k-1)}\mathcal{R})|}{2} \qquad \text{or} \tag{6.24}$$

$$\left| D_\ominus(\mathfrak{c}^{-(k-1)}\mathcal{R}) \cap \left\{ u \leq \frac{M_{k-1} + m_{k-1}}{2} \right\} \right| \geq \frac{|D_\ominus(\mathfrak{c}^{-(k-1)}\mathcal{R})|}{2}. \tag{6.25}$$

In the first case we take $v = \frac{u - m_{k-1}}{M_{k-1} - m_{k-1}}$ and verify that it satisfies the assumptions of Lemma 6.6.2. Firstly, since u is a supersolution of $\partial_t u - \mathcal{L}u = f$ on $D(\mathcal{R})$, clearly v is a supersolution of the same equation on $D(\mathfrak{c}^{-k}\mathcal{R}) \subset D(\mathcal{R})$. Secondly, for every $j \in \mathbb{N}_0, j \leq k-1$ and every

$$(t, x) \in \left[t_0 - (\mathfrak{c}^{-(k-1)}\mathcal{R})^{2s}, t_0 \right] \times \hat{B}(x_0, \mathfrak{c}^{-(k-1)+j}\mathcal{R}) \subset \hat{D}(\mathfrak{c}^{-(k-1-j)}\mathcal{R})$$

we have, by induction hypothesis

$$\begin{aligned} v(t,x) &\geq \frac{u - m_{k-1}}{M_{k-1} - m_{k-1}} \geq \frac{m_{k-j-1} - m_{k-1}}{M_{k-1} - m_{k-1}} \\ &\geq \frac{M_{k-1} - M_{k-1-j} - m_{k-1} + m_{k-1-j}}{M_{k-1} - m_{k-1}} = \frac{\mathfrak{c}^{-(k-1)\eta} - \mathfrak{c}^{-(k-1-j)\eta}}{\mathfrak{c}^{-(k-1)\eta}} \\ &= 1 - \mathfrak{c}^{j\eta} \geq 1 - \mathfrak{c}^{j\beta}. \end{aligned}$$

If $j \geq k$, the same computation as for $j = k-1$ applies because $m_0 \leq u \leq M_0$ on the whole space M and one only needs to estimate $1 - 3^{(k-1)\beta} \geq 1 - 3^{j\beta}$ in the very end. The final condition in Lemma 6.6.2 is fulfilled by Eq. (6.24). Provided that $\mathbf{WPHI}[x_0, t_0, \mathfrak{c}^{-k}\mathcal{R}; s, C_{PH}, Q]$ and $\mathbf{CE}[x_0, \mathfrak{c}^{-k}\mathcal{R}; s, Q, \gamma, C_C]$ are satisfied, Lemma 6.6.2 implies that $v \geq \delta$ in $D_\oplus(\mathfrak{c}^{-(k-1)}\mathcal{R})$ or equivalently

$$u \geq m_{k-1} + \delta(M_{k-1} - m_{k-1}) = m_{k-1} + \delta\mathfrak{c}^{(k-1)\eta} \qquad \text{in } D_\oplus(\mathfrak{c}^{-k}\mathcal{R}).$$

The same then also holds in $\hat{D}(\mathfrak{c}^{-k}\mathcal{R}) \subset D_\oplus(\mathfrak{c}^{-(k-1)}\mathcal{R})$. Setting $m_k = m_{k-1} + \delta\mathfrak{c}^{-(k-1)\eta}$ and $M_k = M_{k-1}$ we end up with

$$M_k - m_k = M_{k-1} - (m_{k-1} + \delta\mathfrak{c}^{-(k-1)\eta}) = \mathfrak{c}^{-(k-1)\eta}(1-\delta) = \mathfrak{c}^{-k\eta}$$

because $1 - \delta = \mathfrak{c}^{-\eta}$ by our choice of η.

In case Eq. (6.25) is true we instead take $v = \frac{M_{k-1} - u}{M_{k-1} - m_{k-1}}$ and prove in the analogue way that

$$u \leq M_{k-1} - \delta(M_{k-1} - m_{k-1}) = M_{k-1} - \delta\mathfrak{c}^{-(k-1)\eta} \qquad \text{in } \hat{D}(\mathfrak{c}^{-k}\mathcal{R}).$$

Contrary to the previous case this time we set $m_k = m_{k-1}$ and $M_k = M_{k-1} - \delta\mathfrak{c}^{-(k-1)\eta}$ but this again results in

$$M_k - m_k = M_{k-1} - m_{k-1} - \delta\mathfrak{c}^{-(k-1)\eta} = \mathfrak{c}^{(k-1)\eta}(1-\delta) = \mathfrak{c}^{-k\eta}.$$

Supposing that we are able to repeat this procedure all the way down to $\hat{D}(\mathfrak{c}^{-(N+1)}\mathcal{R})$ (for some large $N \in \mathbb{N}$) we would get

$$\operatorname*{ess\,osc}_{\hat{D}(\mathfrak{c}^{-(N+1)}\mathcal{R})} \leq \mathfrak{c}^{-(N+1)\eta}.$$

In order to repeat it, however, lemma Lemma 6.6.2 needs $\mathbf{WPHI}[x_0, t_0, \mathfrak{c}^{-k}\mathcal{R}; s, C_{PH}, Q]$ and $\mathbf{CE}[x_0, \mathfrak{c}^{-k}R; s, Q, \gamma, C_C]$ to be satisfied for all $k \leq N$ which, under assumptions of the theorem, is equivalent to $\mathfrak{c}^{-N}\mathcal{R} \geq R_0$. Additionally, in order to estimate the oscilation of u in $(t - R^{2s}, t_0) \times B(x_0, R)$ it is necessary that $(t_0 - R^{2s}, t_0) \times B(x_0, R) \subset \hat{D}(\mathfrak{c}^{-(N+1)}\mathcal{R})$ that is, $3\mathfrak{c}^{-(N+1)}\mathcal{R} \geq R$ and $2\mathfrak{c}^{-2s(N+1)}\mathcal{R}^{2s} \geq R^{2s}$. Since $R \geq R_0$ and $\mathfrak{c} \geq 6 \vee 2^{\frac{1}{s}}$, it only makes sense to apply the procedure

$$N + 1 = \left\lfloor \log_{\mathfrak{c}} \left(\frac{(3 \wedge 2^{\frac{1}{2s}})\mathcal{R}}{R} \right) \right\rfloor$$

times which together with $\eta \leq 1$, elementary bound $\mathfrak{c}^{-\lfloor y \rfloor \eta} \leq \mathfrak{c}^{-(y-1)\eta} \leq \mathfrak{c}\mathfrak{c}^{-y\eta}$ ($y \in \mathbb{R}$) and a few trivial estimates results in

$$\operatorname*{ess\,osc}_{[t,t_0]\times B(x_0,r)} u \leq \mathfrak{c} \left(\frac{R}{\mathcal{R}} \right)^{\eta}.$$

Recalling the definition of $\mathfrak{c}$ and assumption $\|u\|_{L^\infty(I\times M)} = 1/2$ from the beginning this proves the theorem. □

7. Exit time estimates and conservativeness

In this chapter we obtain several estimates concerning the semigroup P_t, restricted semigroup P_t^B and operator G^B corresponding to the bilinear form $\mathcal{E}$, defined by Eq. (4.1), restricted to $L^2(B)$ for some ball $B \subset M$ (see Section 2.4). All of these are obtained under assumption that $\mathcal{E}$ is Dirichlet and under conditions on the kernel that we can prove for certain random conductance models in Part II. As a consequence, we also obtain that the $\mathcal{E}$ is conservative under the same conditions. If the kernel k is pointwise comparable to the kernel $d(x,y)^{-n-2s}$, like in Ineq. (4.2), then the results presented here have already been established in the context of metric measure spaces. See for instance [GHH17, GHH18] and [CK03, CKW16a] and references therein. Our arguments are not very different and most of the time we simply localize the results from [GHH18, GHH17], but this is not always possible.

The main results in this chapter are Theorem 7.1.5, which proves the expected exit time estimates (**ETE**) of exiting a given ball, Theorem 7.2.1, which proves the survival estimate (**SE**), Theorem 7.2.2, which proves that semigroup P_t is conservative and Theorem 7.3.2, which proves the short time estimates on the restricted semigrop P_t^B. The last theorem requires kernel k to satisfies and additional truncation condition (**TB**) that we can verify for an i.i.d. conductance but not for a symmetrized ergodic conductance in Part II. Estimates on the expected exit time are obtained from weak elliptic Harnack inequality (**WEHI**) and condition **AKB**$\geq$. The latter replaces the lower bound $k(x,y) \geq A^{-1}d(x,y)^{-(n+2s)}$, for all $x, y \in M$, $A \geq 1$, used in [GHH18]. Proofs of the survival estimate and conservativeness of $\mathcal{E}$ follow the arguments of [GHH18] and [GHH17] and rely on maximum principles from Section 5.3. The estimate on P_t^B for small times is proved by iterating the survival estimate (**SE**) through space with the help of truncation bound (**TB**).

Assumption 2.5.3 and Assumption 4.0.2 are assumed to hold for the rest of chapter.

7.1. Estimate of the expected exit time

Let us start by recalling Lemma 4.1 from [GHH17].

Lemma 7.1.1 ([GHH17], Lemma 4.1). *Assume that (M,d,m) is a metric measure space satisfying Assumption 2.5.3 and $(\mathcal{E},\mathcal{D}[\mathcal{E}])$ is a regular Dirichlet form on $L^2(M)$. Let also V be an open subset of M, h a non-negative function in $L^1(M)\cap L^2(M)$ and $\phi \in \mathcal{D}[\mathcal{E}]$ such that $0 \leq \phi \leq 1$ and $\phi = 0$ on V. Then for every $T > 0$*

$$(1 - P_T^V 1, h) \geq \int_0^T -\mathcal{E}(\phi, P_t^V h)dt.$$

The next lemma reformulates the previous statement in terms of G^V.

Lemma 7.1.2. *Assume that metric measure space (M,d,m) satisfies Assumption 2.5.3 and $(\mathcal{E},\mathcal{D}[\mathcal{E}])$ is a regular Dirichlet form satisfying Assumption 4.0.2. Let V be open subset of M and $B(y_0,R)$ a ball in M such that $B(y_0,2R) \subset M \setminus V$. Then for every non-negative $h \in L^1(M)\cap L^2(M)$*

$$\|h\|_{L^1(M)} \geq 2\int_V \int_{B(y_0,R)} G^V h(x)k(x,y)dydx.$$

Proof. Define a cutoff function

$$\phi(y) = \frac{2 - d(y_0,y)}{R} \wedge 1 \vee 0$$

between balls $B(y_0, R)$ and $B(y_0, 2R)$. Then ϕ in $\mathcal{D}[\mathcal{E}]$ by Assumption 4.0.2. Taking this ϕ is in Lemma 7.1.1 we know that for every non-negative $h \in L^1 \cap L^2(M)$ and every $T > 0$

$$\|h\|_{L^1(M)} \geq (1 - P_T^V 1, h) \geq \int_0^T -\mathcal{E}(\phi, P_t^V h)dt. \tag{7.1}$$

Since, for every $t > 0$, $P_t^V h$ and ϕ are supported inside disjoint sets $\overline{V}$ and $B(y_0, 2R)$ respectively and $\phi = 1$ on $B(y_0, R)$, the energy on the right can be estimated by

$$\begin{aligned} -\mathcal{E}(\phi, P_t^V h) &= -\int_M \int_M (\phi(x) - \phi(y))(P_t^V h(x) - P_t^V h(y))k(x, y)dydx \\ &\geq 2\int_V \int_{B(y_0,R)} P_t^V h(x)k(x, y)dydx \geq 0. \end{aligned}$$

This makes the integral on the right side of Ineq. (7.1) strictly increasing in T. Thus sending $T \to \infty$ and using Fubini-Tonelli's theorem ($P_t(x)$ has a version which is measurable as a function from $I \times M$ to $\mathbb{R}$ by Corollary 2.3.3 and Proposition 2.3.5) results in

$$\begin{aligned} \|h\|_{L^1(M)} &\geq \int_0^\infty -\mathcal{E}(\phi, P_t^V h)dt \geq 2\int_0^\infty \int_B \int_{B(y_0,R)} P_t^V h(x)k(x, y)dydxdt \\ &\geq 2\int_B \int_{B(y_0,R)} \lim_{T\to\infty} \int_0^T P_t^V h(x)dtk(x, y)dydx. \end{aligned}$$

Definition of G^V in Definition 2.4.13 and Proposition 2.3.6 assure that the limit of Bochner integrals corresponds to the limit of pointwise integrals so for a.e. $x \in V$

$$\lim_{T\to\infty} \int_0^T P_t^V h(x)dt = G^V h(x)$$

which proves

$$\|h\|_{L^1(M)} \geq 2\int_B \int_{B(y_0,R)} G^V h(x)k(x, y)dydx.$$

□

Theorem 7.1.3. *Assume that (M, d, m) satisfies Assumption 2.5.3 and that $\mathcal{E}$ is a regular Dirichlet form on $L^2(M)$ satisfying Assumption 4.0.2. Let $B \subset M$ be an arbitrary ball and suppose that*

$$\int_{M\setminus\overline{B}} k(x, y)dy > 0 \quad \text{for } m\text{-a.e. } x \in B. \tag{7.2}$$

Then $G^B 1 < \infty$ m-a.e.

Proof. For $y_0 \in M$, $R, \varepsilon > 0$ consider the sets

$$W(y_0, R, \varepsilon) = \left\{ x \in B : \int_{B(y_0,R)} k(x, y)dy \geq \varepsilon \right\}.$$

We know that M is separable by Assumption 2.5.3, so let D be a dense set in $M \setminus \overline{B}$. We claim that there is a set N of measure 0 such that

$$\bigcup_{y\in D} \bigcup_{\substack{R\in\mathbb{Q}_+ \\ B(y_0,2R)\subset M\setminus\overline{B}}} \bigcup_{\varepsilon\in\mathbb{Q}_+} W(y_0, R, \varepsilon) = B \setminus N. \tag{7.3}$$

This is true because we can take N to be the exceptional set of Ineq. (7.2) and then for every $x \in B \setminus N$ it holds that $\int_{M\setminus\overline{B}} k(x,y)dy > 0$. The countable family of balls

$$\{B(y_0,R); y_0 \in D, R \in \mathbb{Q}_+, B(y_0,2R) \subset M \setminus \overline{B}\}$$

covers $M \setminus \overline{B}$. Thus, $\int_{B(y_0,R)} k(x,y) = 0$ for all such balls implies $\int_{M\setminus\overline{B}} k(x,y)dy = 0$ and therefore $x \in N$ which proves Eq. (7.3). On the other hand, applying Lemma 7.1.2 with $V = B$ and $h = 1_B$ and arbitrary ball $B(y_0,R) \subset M$ such that $B(y_0,2R) \subset M \setminus \overline{B}$ results in

$$\|1\|_{L^1(B)} \geq \int_B \int_{B(y_0,R)} G^B 1(x) k(x,y) dy dx.$$

Now, for fixed y_0, R, ε and for $x \in W(y_0,R,\varepsilon)$,

$$\|1\|_{L^1(B)} \geq \varepsilon \int_{W(y_0,R,\varepsilon)} G^B 1(x) dx$$

which implies that $G^B 1 < \infty$ m-a.s. in $W(y_0,R,\varepsilon)$. But since B is as contained in the countable union of $W(y_0,R,\varepsilon)$ up to a null set N we can conclude that $G^B < \infty$ m-a.s. on B. □

Remark 7.1.4. *Notice that on a discrete space, in particular on $\mathbb{Z}^n$, $\|G^B 1\|_{L^\infty} < \infty$ is equivalent to $G^B 1 < \infty$ although the first statement is stronger in general. As we only intend to apply the results in the case of $\mathbb{Z}^n$ this deficiency in the current chapter will not be a problem. However, it would be desirable to have a condition on the kernel which would guarantee $\|G^B 1\|_{L^\infty} < \infty$.*

Theorem 7.1.5 (Expected exit time)**.** *Assume that metric measure space (M,d,m) and regular Dirichlet form $\mathcal{E}$ on $L^2(M)$ satisfy Assumption 2.5.3 and Assumption 4.0.2. Let $x_0 \in M$, $R > 0$ and denote $B := B(x_0,R)$. Suppose that $\|G^{4B} 1\|_{L^\infty(4B)} < \infty$ and that there exist $s \in (0,1)$, $\gamma \in [0,2s)$, $n, C_{VL}, C_{VU}, C_{EH}, C_K \in (0,\infty)$ and $Q \in [1,\infty]$ such that*

(i) ***WEHI***$[x_0, \{R/2, 2R\}; C_{EH}]$,

(ii) ***AKB***$\geq[x_0, R; s, C_K]$,

(iii) ***V***$[x_0, R/4; n, C_{VL}, C_{VU}]$,

(iv) ***CE***$[x_0, R; s, Q, \gamma, C_C]$

are satisfied. Then there exist $C_{(E\geq)} \equiv C_{(E\geq)}(C_{EH}, C_C, \gamma, s, n, C_{VL}, C_{VU}) \in (0,\infty)$ and $C_{(E\leq)} \equiv C_{(E\leq)}(C_{EH}, C_K) \in (0,\infty)$ such that ***ETE***$[x_0, R; s, C_{(E\leq)}, C_{(E\leq)}]$ *holds. To be explicit,*

$$C_{(E\geq)} R^{2s} \leq \operatorname*{ess\,inf}_{x\in\frac{1}{4}B} G^B 1 \leq \operatorname*{ess\,sup}_{x\in B} G^B 1 \leq C_{(E\leq)} R^{2s}.$$

The constants can be taken to be $C_{(E\geq)} := 2^{-3-2s-\gamma-2n} C_{EH}^{-1} C_{VL} C_{VU}^{-1} C_C^{-1}$ and $C_{(E\leq)} := \frac{C_{EH}}{2C_K}$.

The proof uses the ideas from Lemmas 5.4 and 5.5 of [GHH18] but replaces the pointwise bound on the jumping kernel $A^{-1} d(x,y)^{-(n+2s)} \leq k(x,y) \leq A d(x,y)^{-(n+2s)}$ with **AKB**$\geq[x_0, R; s, C_K]$, **CE**$[x_0, R; s, Q, \gamma, C_C]$ and $\|G^B 1\|_{L^\infty(M)} < \infty$. The hardest part consists of obtaining weak elliptic Harnack inequality without the pointwise bound of the kernel which is what Chapter 6 was dedicated to.

Proof. Applying Lemma 7.1.2 with $V = 4B$, $y_0 \in M \setminus B(x_0, 6R)$ that satisfies **AKB**$\geq[x_0, R; s, C_K]$ and an arbitrary $h \geq 0$ in $L^1(B) \cap L^2(B)$ gives us

$$\|h\|_{L^1(B)} \geq 2 \int_{4B} \int_{B(y_0,R)} G^{4B} h(x) k(x,y) dy dx.$$

We have assumed that $\|G^{4B}1\|_{L^\infty(B)} < \infty$ so Lemma 3.2 of [GT12] implies that G^{4B} is a bounded operator on $L^2(4B)$ and $G^{4B} = (-\mathcal{L}^{4B})^{-1}$. This implies that the function $G^{4B}h$ is a supersolution of $\mathcal{L}u = 0$ in $4B$ because

$$\mathcal{E}(G^{4B}h, \phi) = (-\mathcal{L}^{4B}G^{4B}h, \phi)_{L^2(M)} = (h, \phi)_{L^2(M)} \geq 0$$

for every $\phi \in \mathcal{D}_{4B}(E)$, $\phi \geq 0$. Therefore, the application of **WEHI**$[x_0, 2R; C_{EH}]$ with $G^{4B}h$ is justified and gives

$$\operatorname*{ess\,inf}_{x\in B} G^{4B}h(x) \geq C_{EH}^{-1} \fint_B G^{4B}h(x)dx$$

which leads to

$$\begin{aligned}\|h\|_{L^1(B)} &\geq 2\int_B \int_{B(y_0,R)} G^{4B}h(x)k(x,y)dydx \\ &\geq 2C_{EH}^{-1}\frac{\|G^{4B}h\|_{L^1(B)}}{|B|}\int_B\int_{B(y_0,R)} k(x,y)dydx.\end{aligned}$$

Combining this with **AKB**$\geq[x_0, R; s, C_K]$ we end up with

$$\|h\|_{L^1(B)} \geq 2C_{EH}^{-1}C_K\frac{\|G^{4B}h\|_{L^1(B)}}{R^{2s}}.$$

Now using $G^B 1_B \leq G^{4B}1_B$, $h \geq 0$ and symmetry of G^{4B} we conclude that

$$(h, G^B 1_B) \leq (h, G^{4B}1_B) = (G^{4B}h, 1_B) = \|G^{4B}h\|_{L^1(B)} \leq \frac{C_{EH}}{2C_K}R^{2s}\|h\|_{L^1(B)}.$$

Recalling that h was an arbitrary non-negative function from $L^1(B) \cap L^2(B)$ this provides us with sufficiently many test function to conclude

$$\operatorname*{ess\,sup}_B G^B 1_B \leq \frac{C_{EH}}{2C_K}R^{2s},$$

and proves claimed upper bound with $C_{(E\leq)} = \frac{C_{EH}}{2C_K}$.

For the lower bound let us call $u = G^B 1$. Then, in the same way as for $G^{4B}1$, $\|G^B 1\|_{L^\infty(B)} < \infty$ and Lemma 3.2 from [GT12] imply that $u = G^B 1$ is positive and superharmonic in B. Therefore **WEHI**$[x_0, R/2; C_{EH}]$ applies and gives (the second inequality being the consequence of Jensen's inequality)

$$\operatorname*{ess\,inf}_{\frac{1}{4}B} u \geq C_{EH}^{-1}\fint_{\frac{1}{4}B} u \geq C_{EH}^{-1}\left(\fint_{\frac{1}{4}B}\frac{1}{u}\right)^{-1}. \tag{7.4}$$

Due to **CE**$[x_0, R; s, Q, \gamma, C_C]$ and Proposition 6.1.2, any Lipschitz cutoff function φ between $\frac{1}{4}B$ and $\frac{3}{4}B$ has energy bounded by

$$\mathcal{E}(\varphi) \leq C(\varphi)|B|R^{2s}$$

with $C(\varphi)$ depending on its Lipschitz constant. In order to be unambiguous, let us take

$$\varphi(y) = \left(\frac{3R - 4d(x_0, y)}{2R} \wedge 1\right) \vee 0$$

which gives $C(\varphi) = 2^{2s+\gamma+1}C_C|B|R^{-2s}$ because $\mathrm{Lip}(\varphi) = 2R^{-1}$. It is now possible to use Proposition A.2 (iii) from [GHH18] to see that for every $\varepsilon > 0$

$$\frac{\varphi^2}{u+\varepsilon} \in \mathcal{D}_B[\mathcal{E}] \tag{7.5}$$

but let us give some more details here. Function $a \to a^{-1}$ is Lipschitz on $[\varepsilon, \infty)$ with Lipschitz constant ε^{-2}. This implies that $\varepsilon^2(u+\varepsilon)^{-1}-\varepsilon$ is a normal contraction of u and therefore $\varepsilon^2(u+\varepsilon)^{-1}-\varepsilon \in \mathcal{D}_B(\mathcal{E})$ due to Proposition 2.4.6. This function is bounded, as is φ^2, so Theorem 1.4.2 (ii) from [FOT11] implies that $(\varepsilon^2(u+\varepsilon)^{-1}-\varepsilon)\varphi^2 \in \mathcal{D}_B(\mathcal{E})$ and, in particular, $\varphi^2(u+\varepsilon)^{-1} \in \mathcal{D}_B(\mathcal{E})$ (because $\varepsilon\varphi^2 \in \mathcal{D}_B[\mathcal{E}]$ by Assumption 4.0.2) which proves Eq. (7.5).

Encouraged by this fact write

$$\int_{\frac{1}{4}B} \frac{1}{u+\varepsilon} \leq \left(1_B, \frac{\varphi^2}{u+\varepsilon}\right) \leq \mathcal{E}\left(u, \frac{\varphi^2}{u+\varepsilon}\right)$$

and estimate the energy term on the right using Theorem 6.4.2 (alternatively, one can look into Lemma 3.7 in [GHH18]). In Ineq. (6.16) we take $\psi = \varphi$ and ignore the first term to get

$$\int_{\frac{1}{4}B} \frac{1}{u+\varepsilon} \leq \mathcal{E}\left(u, \frac{\varphi^2}{u+\varepsilon}\right) \leq 3\mathcal{E}(\varphi) \leq 3 \cdot 2^{1+2s+\gamma} C_C |B| R^{-2s}.$$

We now divide both sides with $|B/4|$, use volume regularity $\mathbf{V}[x_0, R/4; n, C_{VL}, C_{VU}]$ and pass to the limit $\varepsilon \to 0$, using monotone convergence theorem, to obtain

$$⨍_{\frac{1}{4}B} \frac{1}{u} \leq \frac{2^{3+2s+\gamma} C_C |B|}{|B/4|} R^{-2s} \leq 2^{3+2s+\gamma+2n} C_{VL}^{-1} C_{VU} C_C R^{-2s}.$$

Inverting the inequality and combining it with Ineq. (7.4) leads to

$$\operatorname*{ess\,inf}_{\frac{1}{4}B} u \geq C_{EH}^{-1} \left(⨍_{\frac{1}{4}B} \frac{1}{u} \right)^{-1} \geq 2^{-3-2s-\gamma-2n} C_{EH}^{-1} C_{VL} C_{VU}^{-1} C_C^{-1} R^{2s}$$

and proves the lower bound with $C_{(E\geq)} = 2^{-3-2s-\gamma-2n} C_{EH}^{-1} C_{VL} C_{VU}^{-1} C_C^{-1}$. □

7.2. Survival estimate and conservativeness

Theorem 7.2.1. *Assume that metric measure space (M, d, m) and regular Dirichlet form $\mathcal{E}$ on $L^2(M)$ satisfy Assumption 2.5.3 and Assumption 4.0.2. Let $x_0 \in M$ and $R > 0$ be arbitrary, denote $B := B(x_0, R)$ and suppose that, for some $C_{(E\geq)}, C_{(E\leq)} \in (0, \infty)$, $\mathcal{E}$ satisfies*

$$\boldsymbol{ETE}[x_0, R; s, C_{(E\geq)}, C_{(E\leq)}].$$

Then, taking $\varepsilon \equiv \varepsilon(C_{(E\geq)}, C_{(E\leq)}) := C_{(E\leq)}^{-1} C_{(E\geq)}/2 > 0$ and $\delta \equiv \delta(C_{(E\geq)}, s) := (C_{(E\geq)}/2)^{\frac{1}{2s}} > 0$, $\mathcal{E}$ also satisfies

$$\boldsymbol{SE}[x_0, R; s, \varepsilon, \delta].$$

To be explicit, for all $t \leq (\delta R)^{2s}$,

$$\operatorname*{ess\,inf}_{x \in \frac{1}{4}B} P_t^B 1_B(x) \geq \varepsilon.$$

The proof is again very close to Lemma 5.6 of [GHH18], however, in [GHH18] the expected exit time condition (E) is uniform throughout the space while we are working with a local condition **ETE** on some ball B. Also, Lemma 5.6 implicitly uses arguments of the proof of Theorem 6.13 from [GHL14] which has assumption slightly incompatible with our setting. Let us, for these reasons, present the full proof here.

Proof. We will prove the inequality

$$P_t^B 1_B(x) \geq \frac{G^B 1(x) - t}{\|G^B 1\|_{L^\infty(B)}} \quad \text{for } m\text{-a.e. } x \in B \tag{7.6}$$

for every $t \geq 0$ using parabolic maximum principle from Theorem 5.3.1. Denote $u = G^B 1$, take $\varphi \geq 0$ to be any cutoff function between B and $2B$ and define

$$w(t) = u - \varphi t - \|u\|_{L^\infty(B)} P_t^B 1_B.$$

Note that by $\mathbf{ETE}[x_0, R; s, C_{(E\geq)}, C_{(E\leq)}]$ we know that $\|G^B 1\|_{L^\infty(B)} < \infty$. Lemma 3.2 of [GT12] thus implies that $G^B = (-\mathcal{L}^B)^{-1}$ so in particular $G^B 1 \in \mathcal{D}[\mathcal{E}]$ and $\mathcal{E}(G^B 1, \psi) = (1_B, \psi)$ for every $\psi \in \mathcal{D}[\mathcal{E}]$. By Lemma 5.1.9, we know that $P_t^B 1$ is a weak solution of $\partial_t u - \mathcal{L}u = 0$ in $[0, \infty) \times B$ with values in $\mathcal{D}_B[\mathcal{E}] \subset L^2(B)$. Therefore, w takes values in $\mathcal{D}_B[\mathcal{E}]$ and for every non-negative $\psi \in C_c(B)$ we have

$$\begin{aligned}
(\partial_t w(t), \psi) + \mathcal{E}(w, \psi) &= -\left(\varphi + \|u\|_{L^\infty(B)} \partial_t P_t^B 1_B, \psi\right) + \mathcal{E}\left(u - \varphi t - \|u\|_{L^\infty(B)} P_t^B 1_B, \psi\right) \\
&= -\left(\varphi, \psi\right) + \mathcal{E}\left(u - \varphi t, \psi\right) - \|u\|_{L^\infty(B)} \left[(\partial_t P_t^B 1_B, \psi) + \mathcal{E}\left(P_t^B 1_B, \psi\right)\right] \\
&\leq -(\varphi, \psi) + \mathcal{E}(u, \psi) - t\mathcal{E}(\varphi, \psi).
\end{aligned}$$

Because $\varphi \equiv 1$ on B and $\operatorname{supp} \psi \subset B$ we can compute

$$\mathcal{E}(\varphi, \psi) = 2 \int_B \int_{M \setminus B} \psi(x) \left(1 - \varphi(y)\right) k(x, y) dy dx \geq 0,$$

which leads to

$$(\partial_t w, \psi) + \mathcal{E}(w, \psi) \leq -(\varphi, \psi) + \mathcal{E}(G^B 1, \psi) \leq (1_B - \varphi, \psi) \leq 0.$$

Thus, w is a weak subsolution of equation $\partial_t u - \mathcal{L}u = 0$ in $[0, \infty) \times B$. Furthermore, $w_+(t, \cdot) \in \mathcal{D}_B[\mathcal{E}]$ since $\mathcal{E}$ is Markovian so

$$\lim_{L^2, t \to 0} \|u\|_{L^\infty(B)} P_t^B 1_B = \|u\|_{L^\infty(B)} 1_B \geq u$$

shows that

$$\lim_{t \to 0} \|w_+(t, \cdot)\|_{L^2(B)} = 0.$$

This justifies the application of the parabolic maximum principle from Theorem 5.3.1 and proves that $w \leq 0$ on $[0, \infty) \times B$ which proves Ineq. (7.6) by definition of w if we take the account that $\varphi \equiv 1$ on B. Recall that $\mathbf{ETE}[x_0, R; s, C_{(E\geq)}, C_{(E\leq)}]$ states that

$$C_{(E\geq)} R^{2s} \leq \operatorname*{ess\,inf}_{\frac{1}{4}B} G^B 1(x) \leq \|G^B 1\|_{L^\infty(B)} \leq C_{(E\leq)} R^{2s}.$$

Combined with Ineq. (7.6), this implies that for all $t \leq C_{(E\geq)} R^{2s}/2$ and m-a.e. $x \in \frac{1}{4}B$

$$P_t^B 1(x) \geq \frac{C_{(E\geq)} R^{2s} - \frac{1}{2} C_{(E\geq)} R^{2s}}{C_{(E\leq)} R^{2s}} \geq \frac{C_{(E\geq)}}{2 C_{(E\leq)}}.$$

Taking $\varepsilon = C_{(E\leq)}^{-1} C_{(E\geq)}/2$ and $\delta = (C_{(E\geq)}/2)^{\frac{1}{2s}}$ gives the statement of the theorem. □

Theorem 7.2.2. *Assume that metric measure space (M, d, m) and regular Dirichlet form $\mathcal{E}$ on $L^2(M)$ satisfy Assumption 2.5.3 and Assumption 4.0.2. Let $x_0 \in M$, $R_0 > 0$, $\varepsilon, \delta \in (0, \infty)$ be such that $\boldsymbol{SE}[x_0, [R_0, \infty); s, \varepsilon, \delta]$ holds. Then the semigroup P_t corresponding to Dirichlet form $\mathcal{E}$ is conservative, i.e. $P_t 1 = 1$ for every $t > 0$.*

This time we follow the proof of [GHH17], Lemma 4.6. Unfortunately, the original statement is not directly applicable in the current setting because condition (S) is assumed to be uniform through the whole space. However, going through the proof it is quite clear that a localized version of this condition works just as well and the changes are mostly cosmetic.

Proof. Let $r, R \geq R_0$, $r < R$ be variables which we will later use to optimize the estimates. Define $U = B(x_0, r)$, $\Omega = B(x_0, 8R)$ and let ψ be a Lipschitz cutoff between $B(x_0, R)$ and $B(x_0, 2R)$ such that $\psi = 1$ on $B(x_0, R)$ and $\psi = 0$ on $B(x_0, 2R)$. Then ψ is in $\mathcal{D}[\mathcal{E}]$ by Assumption 4.0.2. For fixed $r \leq R$ we consider the function

$$u(t, x) \equiv u(t, x, r, R) = P_t^U 1(x) - \frac{P_t^\Omega 1(x) - \varepsilon\psi(x)}{1 - \varepsilon}$$

and apply on it the parabolic maximum principle from Theorem 5.3.2. To verify that this application is valid, notice that $u(t, \cdot) \in \mathcal{D}(\mathcal{E})$ by definition (see Lemma 5.1.9) and $u(t, \cdot) \leq P_t^U 1 \in \mathcal{D}_U(\mathcal{E})$ for all $t \leq (8\delta R)^{2s}$ due to

$$\frac{P_t^\Omega 1 - \varepsilon\psi}{1 - \varepsilon} \geq 0$$

which follows from $\mathbf{SE}[x_0, 8R; s, \varepsilon, \delta]$. By Lemma 4.4 of [GH08], $u(t, \cdot) \leq P_t^U 1 \in \mathcal{D}_B[\mathcal{E}]$ implies that $u^+(t, \cdot) \in \mathcal{D}_U(\mathcal{E})$ for $t \leq (8\delta R)^{2s}$. The strong continuity of P_t^U ensures that when $t \to 0$, $P_t^U 1 \xrightarrow{L^2(U)} 1$ and which together with $\psi \equiv 1$ on $U \subset B(x_0, R)$ shows that $u^+(t, \cdot) \xrightarrow{L^2(U)} 0$. On the other hand, knowing that both P_t^U and P_t^Ω solve the equation $\partial_t u - \mathcal{L}u = 0$ in $[0, \infty) \times U$ we can compute, for non-negative $\varphi \in \mathcal{D}_U(\mathcal{E})$,

$$(\partial_t u, \varphi) + \mathcal{E}(u, \varphi) = \frac{\varepsilon}{1 - \varepsilon}\mathcal{E}(\psi, \varphi).$$

Notice also that $f := \frac{\varepsilon\psi}{1-\varepsilon} \in L^\infty(M)$ and $f|_U = \|f\|_{L^\infty(M)}$ by construction of ψ. This verifies the assumptions of Theorem 5.3.2 and its application yields, for $t \leq (8\delta R)^{2s}$,

$$\|u_+(t, \cdot)\|_{L^2(U)} \leq \frac{2\varepsilon}{1 - \varepsilon} \int_0^t \mathcal{E}(\psi, u^+(\tau, \cdot))d\tau. \tag{7.7}$$

We proceed by computing

$$\begin{aligned} \mathcal{E}(\psi, u_+(\tau, \cdot)) &= \int_M \int_M (\psi(x) - \psi(y))(u_+(\tau, x) - u_+(\tau, y)k(x, y)dydx \\ &= 2\int_U \int_{B(x_0,R)^c} u_+(\tau, x)(1 - \psi(y))k(x, y)dydx. \end{aligned}$$

As already know from the beginning that $u \leq P_t^U 1 \leq 1$, thinking of k as measure on $M \times M$, $k(A, B) = \int_A \int_B k(x, y)dydx$, we estimate

$$\mathcal{E}(\psi, u_+(\tau, \cdot)) \leq 2k(U, B(x_0, R)^c).$$

Returning to Ineq. (7.7) we find

$$\|u_+(t, \cdot)\|_{L^2(U)} \leq \frac{4\varepsilon t}{1 - \varepsilon}k(U, B(x_0, R)^c).$$

Assumption 4.0.2 ensures that Lipschitz cutoff η between $U = B(x_0, r)$ and $B(x_0, R)$ has finite energy meaning that

$$k(U, B(x_0, R)^c) \leq \int_U \int_{B(x_0,R)} (\eta(x) - \eta(y))^2 k(x, y)dydx \leq \mathcal{E}(\eta) < \infty.$$

Therefore, by continuity from above of measure k on $M \times M$,

$$k(U, B(x_0, R)^c) \to 0 \quad \text{as } R \to 0$$

which in combination with Fatou's lemma shows that

$$\| \liminf_{R\to\infty} u_+(t,\cdot)\|_{L^2(U)} \leq \frac{4\varepsilon t}{1-\varepsilon} \liminf_{R\to\infty} k(U, B(x_0,R)^c) \leq 0 \tag{7.8}$$

for all $t > 0$ because condition $t \leq (8\delta R)^{2s}$ is always satisfied for R large enough. Let now R_i be an increasing sequence tending to ∞, $R_i \to \infty$. Because $\bigcup_i \Omega_i = M$, Proposition 2.4.15 proves that $P_t^{\Omega_i} \to P_t$ pointwise m-a.e. This together with $\psi \equiv \psi(R) = 1$ on U for every $R \geq r$ and the definition of u gives

$$\liminf_{R\to\infty} u(t,x) = P_t^U 1(x) - \frac{P_t 1(x) - \varepsilon}{1-\varepsilon} \quad \text{for } m\text{-a.e. } x \in U.$$

On the other hand, Ineq. (7.8) proved that $\liminf_{R\to\infty} u^+(t,x) = 0$ for m-a.e. x and every $t > 0$ so

$$P_t^U 1 \leq \frac{P_t 1 - \varepsilon}{1-\varepsilon} \quad \text{on } U$$

also for every $t > 0$. Furthermore, taking any incresing sequence r_j tending to ∞, $r_j \to \infty$, Proposition 2.4.15 shows that $P_t^{U_j} 1 \to P_t 1$ pointwise m-a.e. implying that for every $t > 0$ and m-a.e. every $x \in M$

$$P_t 1(x) \leq \frac{P_t 1(x) - \varepsilon}{1-\varepsilon}.$$

But this is only possible if $P_t 1 = 1$, for $t > 0$ which proves that the semigroup P_t is conservative. □

7.3. Truncation and survival probabilities

Here we present a way of self improving the survival condition

$$\operatorname*{ess\,inf}_{\frac{1}{4}B} P_t^B 1 \geq \varepsilon \qquad \text{for } t \leq (\delta r)^{2s} \text{ with } \delta, \varepsilon > 0 \text{ fixed}$$

using ideas similar to [GHH17] Lemma 4.6 together with L^∞ bound on the tail. The result is not new, but we feel that the proof is interesting because it gives a way of obtaining the estimate without explicitly truncating the kernel.

Lemma 7.3.1 (Iteration lemma)**.** *Assume that metric measure space (M, d, m) and regular Dirichlet form $\mathcal{E}$ on $L^2(M)$ satisfy Assumption 2.5.3 and Assumption 4.0.2. Let ball $B_\star \subset M$ be given and suppose that there are constants $C_T \equiv C_T(B_\star) > 0$, $\varepsilon \equiv \varepsilon(B_\star) \in (0,1)$, $\delta = \delta(B_\star) > 0$ and $\mathcal{R}_0 \equiv \mathcal{R}_0(B_\star)$ such that, for all $x \in B_\star$ and $\mathcal{R} \geq \mathcal{R}_0$ satisfying $B(x,\mathcal{R}) \subset B_\star$,*

(i) $\boldsymbol{SE}[x, \mathcal{R}; s, \varepsilon, \delta]$ *and*

(ii) $\boldsymbol{TB}[x, \mathcal{R}; C_T]$

hold. Then the following statement is true. Choose any $x_0 \in B_\star$, $r, t, H > 0$, such that $B_r := B(x_0, r) \subset B_\star$ and take

$$R \geq r + 3\left(\mathcal{R}_0 \vee \delta^{-1} t^{\frac{1}{2s}} \vee t^{\frac{1}{2s}} H^{-\frac{1}{2s}} (4C_T)^{\frac{1}{2s}}\right). \tag{7.9}$$

If $B_R := B(x_0, R) \subset B_\star$, then for every $U \subset B_r$ the implication

$$1 - P_t^{B_r} 1 \leq H \text{ } m\text{-a.e. on } U \quad \Longrightarrow \quad 1 - P_t^{B_R} 1 \leq \left(1 - \frac{\varepsilon}{2}\right) H \text{ } m\text{-a.e. on } U \tag{7.10}$$

holds.

Proof. Let us suppose that for some $x_0 \in M, r > 0$ such that $B_r := B(x_0, r) \subset B_\star$ and some $t > 0$, $U \subset B_r$ we have

$$1 - P_t^{B_r} 1 \leq H \quad m\text{-a.e. on } U.$$

Take variable $R \geq r$, $\beta > 0$ and $\psi \in \mathcal{D}(\mathcal{E})$ such that $B_R := B(x_0, R) \subset B_\star$, $\psi \leq 1$ and consider the function

$$u(\tau, x) = P_\tau^{B_r} 1(x) - \frac{P_\tau^{B(x_0,R)} 1(x) - \varepsilon\psi(x)}{1-\varepsilon} - \beta\tau\psi(x)$$

for $\tau \in [0, \infty)$ and $x \in M$. We intend to adjust variables R, β and ψ as the proof progresses. For start, let us choose R and β so that the parabolic maximum principle from Theorem 5.3.1 applies to u in B_r. First of all, notice that $u \in \mathcal{D}(\mathcal{E})$ as it is a linear combination of functions from $\mathcal{D}(\mathcal{E})$. If $\frac{R-r}{3} > \mathcal{R}_0 \vee \delta^{-1} t^{\frac{1}{2s}}$, then $B(y, (R-r)/3) \subset B_R \subset B_\star$ for every $y \in B(x_0, r + 2(R-r)/3)$ so $\mathbf{SE}[y, (R-r)/3; s, \varepsilon, \delta]$ holds by assumption implying that, for all $\tau \in [0, \left(\frac{\delta(R-r)}{3}\right)^{2s}]$,

$$P_\tau^{B\left(y, \frac{R-r}{3}\right)} 1 \geq \varepsilon \quad m\text{-a.e. on } B\left(y, \frac{R-r}{12}\right) \text{ for every } y \in B\left(x_0, r + \frac{2(R-r)}{3}\right).$$

By Assumption 2.5.3, M is separable so taking countably many y such that $B(y, (R-r)/3)$ cover $B(x_0, r + 2(R-r)/3)$ we obtain

$$P_\tau^{B_R} 1 \geq \varepsilon \quad m\text{-a.e. on } B\left(x_0, r + \frac{2(R-r)}{3}\right).$$

Choosing ψ to be a cutoff between $B(x, r + \frac{R-r}{3})$ and $B(x_0, r + \frac{2(R-r)}{3})$ results in $P_\tau^{B_R} 1 - \varepsilon\psi \geq 0$ m-a.e. on B_r for τ as before. Hence $u \leq P_\tau^{B_r} 1 \in \mathcal{D}_{B_r}[\mathcal{E}]$ and Lemma 4.4 of [GH08] leads to $u^+(\tau, \cdot) \in \mathcal{D}_{B_r}[\mathcal{E}]$. Furthermore, semigroups $P_\tau^{B_r}$ and $P_\tau^{B_R}$ are strongly continuous so, letting $\tau \to 0$, $P_\tau^{B_r} 1 \xrightarrow{L^2(B_r)} 1$ and $\frac{P_\tau^{B_R} 1 - \varepsilon\psi}{1-\varepsilon} \xrightarrow{L^2(B_r)} 1$ (recall that $\psi = 1$ on B) which implies $u^+(\tau, \cdot) \xrightarrow{L^2(U)} 0$. Finally, let us take $\varphi \in \mathcal{D}_{B_r}(\mathcal{E})$, $\varphi \geq 0$, and compute, keeping in mind that $P_\tau^{B_r} 1$ and $P_\tau^{B_R} 1$ solve the equation $\partial_t u - \mathcal{L}u = 0$ in $[0, \infty) \times B_r$,

$$(\partial_t u, \varphi) + \mathcal{E}(u, \varphi) = \frac{\varepsilon}{1-\varepsilon} \mathcal{E}(\psi, \varphi) - \beta(\psi, \varphi) - \tau\beta\mathcal{E}(\psi, \varphi).$$

Recall now that ψ is identically 1 on $B(x_0, r + \frac{R-r}{3})$ and φ is non-negative and supported in B_r, which means that $(\psi, \varphi) = (1, \varphi)$ and

$$\begin{aligned} \mathcal{E}(\psi, \varphi) &= \int_M \int_M (\psi(y) - \psi(x))(\varphi(y) - \varphi(x)) k(x, y) dx dy \\ &= 2 \int_{B_r} \int_{M \setminus B\left(x_0, r + \frac{R-r}{3}\right)} \varphi(x)(1 - \psi(y)) k(x, y) dy dx \geq 0. \end{aligned}$$

Plugging these two observations in allows us to estimate

$$(\partial_t u, \varphi) + \mathcal{E}(u, \varphi) \leq \frac{\varepsilon}{1-\varepsilon} \mathcal{E}(\psi, \varphi) - \beta(1, \varphi).$$

Moving on, we use $\mathbf{TB}[x, (R-r)/3; C_T]$ for $x \in B_r$, justified by $\frac{R-r}{3} \geq \mathcal{R}_0$ and $B(x, (R-r)/3) \subset B_R \subset B_\star$, to estimate

$$\begin{aligned} \mathcal{E}(\psi, \varphi) &= 2 \int_{B_r} \int_{M \setminus B\left(x_0, r + \frac{R-r}{3}\right)} \varphi(x)(1 - \psi(y)) k(x, y) dy dx \\ &\leq 2 \int_{B_r} \varphi(x) \int_{M \setminus B\left(x_0, r + \frac{R-r}{3}\right)} k(x, y) dx dy \\ &\leq 2 \int_{B_r} \varphi(x) \int_{M \setminus B\left(x, \frac{R-r}{3}\right)} k(x, y) dx dy \leq 2C_T \left(\frac{R-r}{3}\right)^{-2s} (1, \varphi). \end{aligned}$$

Thus, for $\tau \in [0, (\delta(R-r)/3)^{2s}]$, choosing

$$\beta = \frac{2\varepsilon C_T}{1-\varepsilon}\left(\frac{R-r}{3}\right)^{-2s}$$

results in

$$(\partial_t u(\tau,\cdot),\varphi) + \mathcal{E}(u,\varphi) \le \frac{2C_T\varepsilon}{1-\varepsilon}\left(\frac{R-r}{3}\right)^{-2s}(1,\varphi) - \frac{2C_T\varepsilon}{1-\varepsilon}\left(\frac{R-r}{3}\right)^{-2s}(1,\varphi) \le 0.$$

The arguments presented so far verified the assumptions of parabolic maximum principle in Theorem 5.3.1. Its application proves that, for $\tau \in [0, (\delta(R-r)/3)^{2s}]$ and m-a.e. $x \in B_r$, $u(\tau, x) \le 0$. The choice of R assures that $t \le (\delta(R-r)/3)^{2s}$ so, taking $\tau = t$ in particular, we get $u(t,x) \le 0$, which translates into

$$P_t^{B_r}1 - \frac{2C_T\varepsilon t}{1-\varepsilon}\left(\frac{R-r}{3}\right)^{-2s} \le \frac{P_t^{B_R}1-\varepsilon}{1-\varepsilon} \qquad m\text{-a.e. in } B_r.$$

Previous inequality holds also m-a.e. on $U \subset B_r$ and, since we assumed $P_t^U 1 \ge (1-H)$ m-a.e. on U, this leads to (recall $P_t^U 1 \le P_t^{B_r}1$ because $U \subset B_r$)

$$P_t^{B_R}1 \ge \varepsilon + (1-\varepsilon)\left((1-H) - \frac{2\cdot 3^{2s}C_T\varepsilon}{1-\varepsilon}(R-r)^{-2s}t\right) \qquad m\text{-a.e. on } U$$

where the only assumption on R so far was that $R - r > 3(\mathcal{R}_0 \vee \frac{t^{1/2s}}{\delta})$. The previous line is equivalent to

$$1 - P_t^{B_R}1 \le (1-\varepsilon)\left(H + \frac{2\cdot 3^{2s}C_T\varepsilon}{1-\varepsilon}(R-r)^{-2s}t\right) \qquad m\text{-a.e. on } U.$$

Let us take R large enough such that also

$$(R-r)^{-2s}t \le \frac{H}{4\cdot 3^{2s}C_T}.$$

In this case, m-a.e. on U,

$$\begin{aligned} 1 - P_t^{B_R}1 &\le (1-\varepsilon)\left(H - \frac{\varepsilon}{2(1-\varepsilon)}H\right) \\ &\le (1-\varepsilon)\left(\frac{1-\frac{\varepsilon}{2}}{1-\varepsilon}\right)H \le \left(1-\frac{\varepsilon}{2}\right)H, \end{aligned}$$

which is what we have promised to obtain. In the end, let us collect the assumptions on the range of R. Firstly, to be able to use survival estimate and truncation inequalities we needed to assume $R - r > 3(\mathcal{R}_0 \vee \frac{t^{1/2s}}{\delta})$ and secondly, to get the multiplicative decrease in the end, we needed to assume that $R - r \ge 3t^{\frac{1}{2s}}H^{-\frac{1}{2s}}(4C_T)^{\frac{1}{2s}}$. Collectively, it suffices to assume

$$R \ge r + 3\left(\mathcal{R}_0 \vee \delta^{-1}t^{\frac{1}{2s}} \vee t^{\frac{1}{2s}}H^{-\frac{1}{2s}}(4C_T)^{\frac{1}{2s}}\right)$$

just like we assumed in the theorem. □

Theorem 7.3.2 (Iteration procedure). *Assume that metric measure space (M,d,m) and regular Dirichlet form $\mathcal{E}$ on $L^2(M)$ satisfy Assumption 2.5.3 and Assumption 4.0.2. Let ball $B_\star := B(x_\star, R_\star)$ be given and suppose that there are constants $C_T \equiv C_T(B_\star) > 0$, $\varepsilon \equiv \varepsilon(B_\star) \in (0,1)$, $\delta = \delta(B_\star) > 0$ and $\mathcal{R}_0 \equiv \mathcal{R}_0(B_\star)$ such that for all $x_0 \in B_\star$, $\mathcal{R} \ge \mathcal{R}_0$ satisfying $B(x_0, \mathcal{R}) \subset B_\star$,*

(i) $\boldsymbol{SE}[x_0, \mathcal{R}; s, \varepsilon, \delta]$ *and*

(ii) $\boldsymbol{TB}[x_0, \mathcal{R}; C_T]$

hold. Then there is a constant $C_{(7.3.2)} \equiv C_{(7.3.2)}(\varepsilon, s)$ *such that for all* $x_0 \in B(x_\star, R_\star/2)$, $t > 0$ *and* $\mathcal{R} \in (0, R_\star/2]$

$$1 - P_t^{B(x_0,\mathcal{R})} 1 \leq C_{(7.3.2)} \frac{t}{\mathcal{R}^{2s}} \cdot \left(\frac{\mathcal{R}_0^{2s}}{t} \vee \delta^{-2s} \vee 4C_T \right) \quad \text{in } B\left(x_0, \mathcal{R}_0 \vee \delta^{-1} t^{\frac{1}{2s}}\right).$$

Proof. The assumption in the theorem are exactly the same as in Lemma 7.3.1 allowing us to use Implication (7.10) which we plan to iterate in this proof. Fix an arbitrary $x_0 \in B(x_\star, R_\star/2)$. We start with $R^{(0)} := \mathcal{R}_0 \vee \delta^{-1} t^{\frac{1}{2s}}$ and trivial estimate

$$1 - P_t^{B(x_0, R^{(0)})} 1 \leq 1 \quad m \text{ a.e. on } B(x_0, R^{(0)}).$$

Define $\kappa \equiv \kappa(\varepsilon) := \left(1 - \frac{\varepsilon}{2}\right) < 1$ for readability. Iterating Implication (7.10) over $k \in \mathbb{N}$ with $U = B(x_0, R^{(0)})$, $r = R^{(k-1)}$, $R = R^{(k)}$ we obtain

$$1 - P_t^{B(x_0, R^{(k)})} 1 \leq \kappa^k \quad m\text{-a.e. on } B(x_0, R^{(0)})$$

where

$$R^{(k)} = R^{(k-1)} + 3 \left(\frac{\mathcal{R}_0}{t^{\frac{1}{2s}}} \vee \delta^{-1} \vee (4C_T)^{\frac{1}{2s}} \right) t^{\frac{1}{2s}} \kappa^{-\frac{k}{2s}}$$

satisfies Ineq. (7.9) from Lemma 7.3.1 for every $k \in \mathbb{N}$. Telescoping this rule together with

$$R^{(0)} = \mathcal{R}_0 \vee \delta^{-1} t^{\frac{1}{2s}} \leq 3 \left(\frac{\mathcal{R}_0}{t^{\frac{1}{2s}}} \vee \delta^{-1} \vee (4C_T)^{\frac{1}{2s}} \right) t^{\frac{1}{2s}} \kappa^0$$

and substitution $C_1 \equiv C_1(t, s, \mathcal{R}_0, \delta, C_T) := 3 \left(\frac{\mathcal{R}_0}{t^{\frac{1}{2s}}} \vee \delta^{-1} \vee (4C_T)^{\frac{1}{2s}} \right) t^{\frac{1}{2s}}$ gives

$$R^{(k)} \leq C_1 \sum_{l=0}^{k} \kappa^{-\frac{l}{2s}} \leq C_1 \frac{\kappa^{-\frac{k+1}{2s}} - 1}{\kappa^{-\frac{1}{2s}} - 1} \leq C_2 \kappa^{-\frac{k}{2s}}$$

with $C_2 \equiv C_2(\kappa, t, s, \mathcal{R}_0, \delta, C_T) = C_1 \kappa^{-\frac{1}{2s}} / (\kappa^{-\frac{1}{2s}} - 1)$. This is valid as long as $B(x_0, R^{(k)}) \subset B(x_\star, R_\star)$ which is satisfied if $R^{(k)} \leq R_\star/2$ due to $x_0 \in B(x_\star, R_\star/2)$. Let us thus stop the iteration at the smallest number $k \in \mathbb{N}$ such that $R^{(k+1)} > \mathcal{R}$ (recall that $\mathcal{R} \leq R_\star/2$ by assumption). As a consequence $R^{(k)} \leq \mathcal{R}$ and $k \geq -2s \log_\kappa \left(\frac{\mathcal{R}}{C_2}\right) - 1$ implying

$$1 - P_t^{B(x_0,\mathcal{R})} 1 \leq 1 - P_t^{B(x_0, R^{(k)})} 1 \leq \kappa^k \leq \kappa^{-1} \left(\frac{\mathcal{R}}{C_2} \right)^{-2s} \quad m\text{-a.e. on } B(x_0, R^{(0)}).$$

Recalling what C_2 was we end up, m-a.e. on $B(x_0, \mathcal{R}_0 \vee \delta^{-1} t^{\frac{1}{2s}})$, with

$$1 - P^{B(x_0,\mathcal{R})} 1 \leq \kappa^{-1} \frac{t}{\mathcal{R}^{2s}} \cdot \left(3^{2s} \kappa^{-1} (\kappa^{-\frac{1}{2s}} - 1)^{-2s} \left(\frac{\mathcal{R}_0^{2s}}{t} \vee \delta^{-2s} \vee 4C_T \right) \right)$$

which after taking (recall $\kappa = \left(1 - \frac{\varepsilon}{2}\right)$)

$$C_{(7.3.2)} \equiv C_{(7.3.2)}(\varepsilon, s) = 3^{2s} \kappa^{-2} (\kappa^{-\frac{1}{2s}} - 1)^{-2s}$$

gives exactly the claim from the theorem. □

Remark 7.3.3. *We intend to use the previous result for random conductance model on* $\mathbb{Z}^n$ *with independent conductances. The same result was obtain in [CKW18b] using truncation techniques.*

8. Local Poincaré-Sobolev inequality

In this chapter we employ the method from [DNPV12], section 6, to obtain the weighted version of Sobolev-Poincaré inequality in Theorem 8.1.4 under relatively weak assumption on the kernel given in Assumption 8.1.1. This will be used to prove Sobolev and Poincaré inequalities in Part II. At the end of the chapter, in theorem Theorem 8.2.4, we also present a consequence of Theorem 8.1.4 in $\mathbb{Z}^n$.

For the rest of the chapter we will be working on an arbitrary measure space (M, m) (we will not use the distance d anywhere) together with an arbitrary $p \in [1, \infty)$ and a symmetric form $\mathcal{Q}_p : L^1(M) \to [0, \infty]$ (allowed to be $+\infty$ for some functions) of jump-type:

$$\mathcal{Q}_p(f) = \int_M \int_M |f(x) - f(y)|^p k_p(x, y) dx dy$$

for some symmetric $k \equiv k_p : M \times M \to [0, \infty)$ which is Borel measurable on $M \times M$. All these notation will be fixed for the rest of the chapter.

Note that form $\mathcal{E}$ defined by Eq. (4.1) fits into this setting with $p = 2$.

8.1. Abstract inequality

The following assumption suffices to obtain the Poincaré-Sobolev inequality for a jump-type form.

Assumption 8.1.1. *Assume that one can find a function $\lambda : M \to (0, \infty)$ and a $\nu > 0$ such that for every $E \subset M$, which satisfies $|E| < \infty$ and $|E| \leq |M \setminus E|$, and m-a.e. $x \in E$*

$$\int_{M \setminus E} k_p(x, y) dy \geq \lambda(x) |E|^{-\nu p}. \tag{8.1}$$

Definition 8.1.2. *For $f \in L^1(M)$ let us define*

$$f_M = \fint_M f(x) dx := \begin{cases} 0 \text{ if } |M| = \infty \\ \frac{1}{|M|} \int_M f(x) dx \text{ otherwise.} \end{cases}$$

Lemma 8.1.3. *Let $f \in L^1(M)$ and $r \in [1, \infty)$ be arbitrary. Then*

$$\left(\int_M |f(x) - f_M|^r \, dx \right)^{\frac{1}{r}} \leq 2 \inf_{a \in \mathbb{R}} \left(\int_M |f(x) - a|^r \, dx \right)^{\frac{1}{r}}.$$

Proof. If $|M| = \infty$, then $f - a$ is in $L^1(M)$ only when $a = 0$ so the infimum is obtained for $a = 0$. On the other hand $f_M = 0$, and the inequality is trivially true. In case $|M| < \infty$, for every $a \in \mathbb{R}$, by Minkowski inequality for $L^r(M)$ norm and Jensen's inequality

$$\begin{aligned} \left(\int_M |f(x) - f_M|^r \, dx \right)^{\frac{1}{r}} &\leq \left(\int_M \left| \fint_M f(y) - a dy \right|^r dx \right)^{\frac{1}{r}} + \left(\int_M |f(x) - a|^r \, dx \right)^{\frac{1}{r}} \\ &\leq \left(\int_M \fint_M |a - f(y)|^r \, dy dx \right)^{\frac{1}{r}} + \left(\int_M |f(x) - a|^r \, dx \right)^{\frac{1}{r}} \\ &\leq 2 \left(\int_M |f(x) - a|^r \, dx \right)^{\frac{1}{r}}. \end{aligned} \tag{8.2}$$

□

Theorem 8.1.4 (Weighted Poincaré-Sobolev inequality). *Suppose Assumption 8.1.1 is satisfied and set* $I = [(\nu p)^{-1}, \infty]$ *if* $\nu p < 1$, $I = [1, \infty)$ *if* $\nu p = 1$ *and* $I = [1, (\nu p - 1)^{-1})$ *if* $\nu p > 1$. *Choose any* $q \in I$ *and find* $r \in [1/p, \infty]$ *such that* $\frac{p}{r} = 1 - \nu p + \frac{1}{q}$. *Then there exists a constant* $C_{(8.1.4)} \equiv C_{(8.1.4)}(\nu, p, q)$ *(but independent of* M*) such that for every* $f \in L^1(M)$ *supported on the set of finite measure, setting* $A = M$ *if* $|M| < \infty$ *and* $A = \operatorname{supp} f$ *otherwise,*

$$\|f - f_M\|^p_{L^r(M)} \le C_{(8.1.4)} \|\lambda^{-1}\|_{L^q(\mathrm{A})} \mathcal{Q}_p(f). \tag{8.3}$$

Proof. The proof is given after preliminary Lemmas 8.1.8 to 8.1.10. □

An immediate consequence of the previous theorem is

Theorem 8.1.5 (Weighted Sobolev inequality). *Suppose Assumption 8.1.1 is satisfied and in addition* $|M| = \infty$. *Choose any* $q \in I$ *(where* I *is the interval depending on* ν, p *from Theorem 8.1.4) and find* $r \in [p, \infty]$ *such that* $\frac{p}{r} = 1 - \nu p + \frac{1}{q}$. *Then*

$$\|f\|^p_r \le C_{(8.1.4)} \|\lambda^{-1}\|_{L^q(F)} \mathcal{Q}_p(f) \tag{8.4}$$

for every bounded measurable function f *with* $F := \operatorname{supp} f$ *such that* $|F| < \infty$.

Proof. $|M| = \infty$ makes $f_M = 0$ and Theorem 8.1.4 then automatically gives the claim. □

Remark 8.1.6. *Notice that the previous theorem generalizes the classical Sobolev inequality for the fractional Laplacian* Δ^{2s} *(it can be found in Theorem 6.5, Chapter 6 of [DNPV12] for instance) if one chooses* $M = \mathbb{R}^n$, $sp > n$, $k_p(x, y) = C_s |x - y|^{-(n+2s)}$, $\nu = s/n$, $\lambda(x) = 1$ *and* $q = \infty$ *where* C_s *is an explicit constant depending on* s.

Theorem 8.1.7. *Suppose* $|M| < \infty$, $1 \le p < r < \infty$ *and that for some* $H > 0$ *and* $f \in L^1(M)$ *the inequality*

$$\|f - f_M\|^p_{L^r(M)} \le H \mathcal{Q}_p(f) \tag{8.5}$$

is satisfied on M. *Then for every* $\zeta \in [1, r]$

$$\|f\|^p_{L^r(M)} \le 2^p H \mathcal{Q}_p(f) + 2^p |M|^{p\left(\frac{1}{r} - \frac{1}{\zeta}\right)} \|f\|^p_{L^\zeta(M)} \tag{8.6}$$

and

$$\|f - f_M\|^p_{L^p(M)} \le H |M|^{1 - \frac{p}{r}} \mathcal{Q}_p(f). \tag{8.7}$$

Proof. To prove Ineq. (8.6) we compute

$$\begin{aligned}
\|f\|^p_{L^r(M)} &\le 2^p \|f - f_M\|^p_{L^r(M)} + 2^p \|f_M\|^p_{L^r(M)} \\
&\le 2^p H \mathcal{Q}_p(f) + 2^p \left(|M| \left| \fint_M f(x) dx \right|^r \right)^{\frac{p}{r}} \\
&\le 2^p H \mathcal{Q}_p(f) + 2^p |M|^{\frac{p}{r}} \left(\left(\fint_M |f(x)|^\zeta dx \right)^{\frac{r}{\zeta}} \right)^{\frac{p}{r}} \\
&\le 2^p H \mathcal{Q}_p(f) + 2^p |M|^{p\left(\frac{1}{r} - \frac{1}{\zeta}\right)} \|f\|^p_{L^\zeta(M)}
\end{aligned}$$

where we used Jensen's inequality in the second to last line. For Ineq. (8.7), Hölder inequality followed by Ineq. (8.5) produces

$$\|f - f_M\|^p_{L^p(M)} \le \|f - f_M\|^p_{L^r(M)} \|1\|^p_{L^{\left(\frac{1}{p} - \frac{1}{r}\right)^{-1}}(M)} \le H |M|^{1 - \frac{p}{r}} \mathcal{Q}_p(f).$$

□

The rest of the section contains the proof of Theorem 8.1.4 arranged in a sequence of lemmas.

Lemma 8.1.8. *Suppose Assumption 8.1.1 is satisfied and $q > 0$. Let f be a bounded measurable function with support $F := \operatorname{supp} f$ such that $|F| \le |M|/2$ and $|F| < \infty$ and define, for $i \in \mathbb{Z}$,*

$$a_i := |\left\{|f| \ge 2^i\right\}|.$$

Then

$$\|\lambda^{-1}\|_{L^q(F)}\mathcal{Q}_p(f) \ge \frac{2^p-1}{2^{p-1}} \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_{i+1}^{1+1/q} a_i^{-\nu p}.$$

Proof. It is not hard, using the formula for $\mathcal{Q}_p$, to verify $\mathcal{Q}_p(|f|) \le \mathcal{Q}_p(f)$ so the worst case for the proof of the estimate arises when f is non-negative and we can therefore assume that this is the case.

Let us, for $i \in \mathbb{Z}$, define:

$$A_i := \left\{f \ge 2^i\right\}, \qquad a_i := |\left\{|f| \ge 2^i\right\}|, \qquad a_i^{(\lambda)} := \int_{A_i} \lambda(x)dx,$$

$$D_i := A_i \setminus A_{i+1}, \qquad d_i := |D_i|, \qquad d_i^{(\lambda)} = \int_{D_i} \lambda(x)dx$$

and

$$S^{(\lambda)} = \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_i^{-\nu p} d_{i+1}^{(\lambda)}.$$

We now repeat the calculation of line (6.14) from [DNPV12] in two steps (our $S^{(\lambda)}$ is marginally different then their S but conceptually the same). First compute

$$\sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} \sum_{\substack{l\in\mathbb{Z}\\ l\ge i+1}} 2^{pi} a_i^{-\nu p} d_{l+1}^{(\lambda)} = \sum_{\substack{l\in\mathbb{Z}\\ a_l\neq 0}} \sum_{\substack{i\in\mathbb{Z}\\ i\le l-1}} 2^{pi} a_i^{-\nu p} d_{l+1}^{(\lambda)} \tag{8.8}$$

where we changed the order of summation and reformulated the set of indices. For the change in the indices, notice that it is sufficient to consider only $i+1 \le l$ such that $a_l \neq 0$, because then $a_i \ge a_l > 0$ and the summands with $a_l = 0$ do not contribute due to $0 \le d_{l+1}^{(\lambda)} \le a_l = 0$. In the second step we estimate $a_i \ge a_l$ and get rid of the additional summation to get

$$\begin{aligned}\sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} \sum_{\substack{l\in\mathbb{Z}\\ l\ge i+1}} 2^{pi} a_i^{-\nu p} d_{l+1}^{(\lambda)} &\le \sum_{\substack{l\in\mathbb{Z}\\ a_l\neq 0}} \sum_{\substack{i\in\mathbb{Z}\\ i\le l-1}} 2^{pi} a_l^{-\nu p} d_{l+1}^{(\lambda)} \le \sum_{\substack{l\in\mathbb{Z}\\ a_l\neq 0}} 2^{p(l-1)} a_l^{-\nu p} d_{l+1}^{(\lambda)} \sum_{k=0}^{\infty} 2^{-pk}\\ &\le \frac{2^p}{2^p-1} \sum_{\substack{l\in\mathbb{Z}\\ a_l\neq 0}} 2^{p(l-1)} a_l^{-\nu p} d_{l+1}^{(\lambda)} = \frac{1}{2^p-1} S^{(\lambda)}.\end{aligned} \tag{8.9}$$

Observe that sets D_i, $i \in \mathbb{Z}$ are disjoint and $\bigcup_{k\ge i} D_k = A_i$ which implies that

$$d_i^{(\lambda)} = a_i^{(\lambda)} - \sum_{k\ge i}^{\infty} d_{k+1}^{(\lambda)}.$$

This, together with Ineq. (8.9) (used in the last line), allows us to estimate

$$\begin{aligned}S^{(\lambda)} = \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_i^{-\nu p} d_{i+1}^{(\lambda)} &\ge \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} \left[2^{pi} a_i^{-\nu p} a_{i+1}^{(\lambda)} - \sum_{\substack{l\in\mathbb{Z}\\ l\ge i+1}} 2^{pi} a_i^{-\nu p} d_{l+1}^{(\lambda)}\right]\\ &\ge \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_i^{-\nu p} a_{i+1}^{(\lambda)} - \frac{1}{2^p-1} S^{(\lambda)}\end{aligned}$$

which implies that

$$S^{(\lambda)} \geq \left(1 + \frac{1}{2^p - 1}\right)^{-1} \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_i^{-\nu p} a_{i+1}^{(\lambda)}.$$

Let us call the constant in front of the sum on the right $c_1/2$.

On the other hand, sets A_i are defined so that for all $i \in \mathbb{Z}$, $x \in A_{i+1}$ and $y \in M \setminus A_i$

$$|f(x) - f(y)| \geq 2^i.$$

If i is such that $a_i \neq 0$, then we also know that $|A_i| \leq |\operatorname{supp} f| < \infty$ and $|A_i| \leq |\operatorname{supp} f| \leq |M|/2$. Because of Assumption 8.1.1 this makes Ineq. (8.1) applicable with $E = A_i$ resulting in

$$\int_{M\setminus A_i} |f(x) - f(y)|^p k_p(x, y) dy \geq 2^{pi} \int_{M\setminus A_i} k_p(x, y) dy \geq \lambda(x) 2^{pi} a_i^{-\nu p}$$

for m-a.e. $x \in A_{i+1}$. Integrating in $x \in D_{i+1} \subset A_{i+1}$ and summing over all i with $a_i \neq 0$ gets us to

$$\begin{aligned}\sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} \int_{D_{i+1}\times(M\setminus A_i)} |f(x) - f(y)|^p k_p(x, y) dx dy &\geq \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_i^{-\nu p} d_{i+1}^{(\lambda)} = S^{(\lambda)} \\ &\geq \frac{c_1}{2} \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_i^{-\nu p} a_{i+1}^{(\lambda)}.\end{aligned}$$

The last estimate leads to the lower estimate of $\mathcal{Q}_p(f)$ because

$$\begin{aligned}\mathcal{Q}_p(f) &= \int_M \int_M |f(x) - f(y)|^p k_p(x, y) dx dy \\ &\geq 2 \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} \int_{D_{i+1}\times(M\setminus A_i)} |f(x) - f(y)|^p k_p(x, y) dx dy \\ &\geq c_1 \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_i^{-\nu p} a_{i+1}^{(\lambda)}.\end{aligned}$$

Furthermore, for every $i \in \mathbb{Z}$ (recall $A_i \subset F := \operatorname{supp} f$) we can use Hölder's inequality to get

$$\|\lambda^{-1}\|_{L^q(F)} a_i^{(\lambda)} \geq \|\lambda^{-1}\|_{L^q(A_i)} \int_{A_i} \lambda(x) dx \geq \left(\int_{A_i} (\lambda\lambda^{-1})^{\frac{1}{1+1/q}} dx\right)^{1+1/q} = a_i^{1+1/q}.$$

The version of the Hölder inequality applied in the previous line is perhaps not so standard but can be found in Corollary 2.5 of [AF03] (see Theorem 2.2.1 for the statement). This results in the estimate

$$\begin{aligned}\|\lambda^{-1}\|_{L^q(F)} \mathcal{Q}_p(f) &\geq c_1 \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_i^{-\nu p} \|\lambda^{-1}\|_{L^q(F)} a_{i+1}^{(\lambda)} \\ &\geq c_1 \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_{i+1}^{1+1/q} a_i^{-\nu p}\end{aligned}$$

and proves the result since

$$c_1 = 2\left(1 + \frac{1}{2^p - 1}\right)^{-1} = \frac{2^p - 1}{2^{p-1}}.$$

□

The following lemma is inspired by Lemma 6.2 from [DNPV12] Chapter 5.

Lemma 8.1.9. *Suppose Assumption* 8.1.1 *is satisfied and choose any* $q > 0$ *such that* $1 - \nu p + \frac{1}{q} > 0$. *Fix* $T > 1$ *and let* $(a_i)_{i\in\mathbb{Z}}$ *be a bounded, non-negative and decreasing two-way sequence which is identically* 0 *for all* i *large enough. Then*

$$\sum_{i\in\mathbb{Z}} a_i^{1-\nu p+\frac{1}{q}} T^i \leq T^{\left(1+\frac{1}{q}\right)\left(1-\nu p+\frac{1}{q}\right)^{-1}} \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} a_{i+1}^{1+\frac{1}{q}} a_i^{-\nu p} T^i.$$

Proof. First of all, the conditions on q and the sequence a_i guarantee that both series are finite. Tails for i going to $-\infty$ converge because T^i is decreasing exponentially and a_i is bounded while tails for i going to ∞ do not exist because $a_i = 0$ for large enough i. For $\gamma, \delta > 0$ (to be chosen later) define $\gamma^* := \frac{\gamma}{\gamma-1}$ (i.e. $\frac{1}{\gamma} + \frac{1}{\gamma^*} = 1$) and use Hölder's inequality to get

$$\begin{aligned}
\frac{1}{T}\sum_{i\in\mathbb{Z}} a_i^{1-\nu p+\frac{1}{q}} T^i &= \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} a_{i+1}^{1-\nu p+\frac{1}{q}} T^i = \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} \left(a_i^{\delta} T^{i\left(1-\frac{1}{\gamma}\right)}\right)\left(a_{i+1}^{1-\nu p+\frac{1}{q}} a_i^{-\delta} T^{\frac{i}{\gamma}}\right) \\
&\leq \left(\sum_{i\in\mathbb{Z}} a_i^{\delta\gamma^*} T^i\right)^{\frac{1}{\gamma^*}} \left(\sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} a_{i+1}^{\gamma\left(1-\nu p+\frac{1}{q}\right)} a_i^{-\delta\gamma} T^i\right)^{\frac{1}{\gamma}}.
\end{aligned}$$

It is now possible to choose γ and δ such that $\gamma\left(1 - \nu p + \frac{1}{q}\right) = 1 + \frac{1}{q}$, $\gamma^*\delta = 1 - \nu p + \frac{1}{q}$ and $\delta\gamma = \nu p$. The correct choice turns out to be

$$\gamma = \left(1 + \frac{1}{q}\right)\left(1 - \nu p + \frac{1}{q}\right)^{-1} \geq 1, \qquad \gamma^* = \left(1 + \frac{1}{q}\right)(\nu p)^{-1} \geq 1,$$

$$\delta = \nu p\left(1 - \nu p + \frac{1}{q}\right)\left(1 + \frac{1}{q}\right)^{-1} > 0,$$

which is not hard to verify. Notice that $\gamma \geq 1$ and $\delta > 0$ crucially depend on $(1 - \nu p + 1/q) > 0$. Our choice was made so that

$$\frac{1}{T}\sum_{i\in\mathbb{Z}} a_i^{1-\nu p+\frac{1}{q}} T^i \leq \left(\sum_{i\in\mathbb{Z}} a_i^{1-\nu p+\frac{1}{q}} T^i\right)^{\frac{1}{\gamma^*}} \left(\sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} a_{i+1}^{1+\frac{1}{q}} a_i^{-\nu p} T^i\right)^{\frac{1}{\gamma}},$$

with the intention of making the first factor on the right a multiple of the term on the left. For this reason

$$\left(\sum_{i\in\mathbb{Z}} a_i^{1-\nu p+\frac{1}{q}} T^i\right)^{\left(1-\frac{1}{\gamma^*}\right)\gamma} \leq T^{\gamma} \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} a_{i+1}^{1+\frac{1}{q}} a_i^{-\nu p} T^i,$$

which proves the lemma because $(1 - 1/\gamma^*)\,\gamma = 1$. □

Lemma 8.1.10. *Suppose Assumption 8.1.1 is satisfied, choose any* $q \in [(\nu p)^{-1}, \infty]$ *such that* $\frac{1}{q} > \nu p - 1$ *and set let* r *be the unique number solving* $\frac{p}{r} = 1 - \nu p + \frac{1}{q}$. *Then there is a constant* $C_{(8.1.10)} \equiv C_{(8.1.10)}(\nu, p, q)$ *with the following property. For every bounded, measurable* $f : M \to \mathbb{R}$ *such that, setting* $F_+ := \operatorname{supp}(f_+)$, *(i)* $\displaystyle\int_M f_+(y)^r dy \ge \frac{1}{2}\int_M |f(y)|^r dy,$

(ii) $|F_+| \le \dfrac{|M|}{2}$ and $|F_+| < \infty$

it holds that

$$\|f\|^p_{L^r(M)} \le C_{(8.1.10)}\|\lambda^{-1}\|_{L^q(F_+)}\mathcal{Q}_p(f).$$

Proof. The first assumption on f allows us to estimate

$$\|f\|^p_{L^r(M)} \le 2^{\frac{p}{r}}\|f_+\|^p_{L^r(M)}.$$

Furthermore, $\mathcal{Q}_p(f_+) \le \mathcal{Q}_p(f)$ follows directly from formula for $\mathcal{Q}_p$. It is therefore sufficient (up to changing the constant by a factor of $2^{p/r}$) to prove the statement for f_+ instead of f. To do so, define $a_i = |\{f_+ \ge 2^i\}|$ and estimates the left hand side by

$$\|f_+\|^p_{L^r(M)} = \left(\sum_{i\in\mathbb{Z}}\int_{A_i\setminus A_{i+1}}|f_+(x)|^r dx\right)^{\frac{p}{r}} \le \left(\sum_{i\in\mathbb{Z}}2^{(i+1)r}a_i\right)^{\frac{p}{r}} \le \sum_{i\in\mathbb{Z}}2^{(i+1)p}a_i^{p/r}.$$

The last inequality follows by reading elementary inequality

$$\forall\beta\ge 1, \forall\alpha_i>0, \quad \left(\sum_{i\in\mathbb{Z}}\alpha_i\right)^{\beta} = \sum_{i\in\mathbb{Z}}\left[\alpha_i\left(\sum_{i\in\mathbb{Z}}\alpha_i\right)^{\beta-1}\right] \ge \sum_{i\in\mathbb{Z}}\alpha_i\cdot\alpha_i^{\beta-1} \ge \sum_{i\in\mathbb{Z}}\alpha_i^{\beta},$$

with $\alpha_i = 2^{(i+1)p}a_i^{p/r}$, $\beta = \frac{r}{p} \ge 1$ (which is equivalent to $q \ge (\nu p)^{-1}$), backwards. From Lemma 8.1.8 we already know that

$$\|\lambda^{-1}\|_{L^q(F_+)}\mathcal{Q}_p(f_+) \ge \frac{2^p-1}{2^{p-1}}\sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}}2^{pi}a_{i+1}^{1+1/q}a_i^{-\nu p}$$

and we just need to compare $\sum_{i\in\mathbb{Z}}2^{(i+1)p}a_i^{p/r}$ with $\sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}}2^{pi}a_{i+1}^{1+1/q}a_i^{-\nu p}$ up to a multiplicative constant. We will use Lemma 8.1.9 with $T = 2^p$, and the sequence $a_i = |\{f_+ \ge 2^i\}|$ as defined before to do so. Notice that $\frac{p}{r} = 1 - \nu p + \frac{1}{q} > 0$ because of the assumption $\frac{1}{q} \ge \nu p - 1$. Let us verify that a_i indeed satisfies all the assumptions of the lemma. It is bounded because $a_i \le |F_+| < \infty$, decreasing by definition and identically zero for i large enough because f was assumed to be bounded. Lemma 8.1.9 thus implies

$$\sum_{i\in\mathbb{Z}}2^{ip}a_i^{p/r} \le 2^{p\left(1+\frac{1}{q}\right)\left(1-\nu p+\frac{1}{q}\right)^{-1}}\sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}}2^{pi}a_{i+1}^{1+1/q}a_i^{-\nu p}.$$

Collecting everything we get that

$$\begin{aligned}\|\lambda^{-1}\|_{L^q(F_+)}\mathcal{Q}_p(f) &\ge \frac{2^p-1}{2^{p-1}}\sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}}2^{pi}a_{i+1}^{1+1/q}a_i^{-\nu p}\\ &\ge \frac{2^p-1}{2^{p-1}}2^{-p\left(1+\frac{1}{q}\right)\left(1-\frac{sp}{n}+\frac{1}{q}\right)^{-1}}\sum_{i\in\mathbb{Z}}2^{ip}a_i^{p/r}\\ &\ge \frac{2^p-1}{2^{p-1}}2^{-r\left(1+\frac{1}{q}\right)}2^{-p}2^{-\frac{p}{r}}\|f\|^p_{L^r(M)}\end{aligned}$$

which proves the inequality taking

$$C_{(8.1.10)} := \left(\frac{2^p-1}{2^{p-1}}2^{-r\left(1+\frac{1}{q}\right)}2^{-p}2^{-\frac{p}{r}}\right)^{-1} = \frac{2^{p-1+p+\frac{p}{r}+r\left(1+\frac{1}{q}\right)}}{2^p-1}.$$

At the end let us remark that this constant clearly depends only on ν, p and q (because r was defined depending on ν, p and q) and is finite. □

All the preparations are now complete and we are ready to prove Theorem 8.1.4.

Proof of Theorem 8.1.4. Let us first show that it is sufficient to prove the claim for bounded functions f. Suppose that the *Ineq.* (8.3) holds for bounded f. Define $f^{(N)} = (f \vee -N) \wedge N$ and notices that $f^{(N)}_M \to f_M$ due to the dominated convergence theorem and $f \in L^1(M)$. Following this up with Fatou's lemma shows that

$$\|f - f_M\|^p_{L^r(M)} \le \varliminf_{N\to\infty} \|f^{(N)} - f^{(N)}_M\|^p_{L^r(M)} \le \varliminf_{N\to\infty} C_{(8.1.4)} \|\lambda^{-1}\|_{L^q\left(A^{(N)}\right)} \mathcal{Q}_p(f^{(N)})$$

where we denoted by $A^{(N)}$ the set from Theorem 8.1.4 corresponding to function $f^{(N)}$. Now using $\mathcal{Q}_p(f^{(N)}) \le \mathcal{Q}_p(f)$, owing to the shape of $\mathcal{Q}_p$, and $A^{(N)} \subset A$, because the support of $f^{(N)}$ is contained in the support of f, it follows that

$$\|f - f_M\|^p_{L^r(M)} \le C_{(8.1.4)} \|\lambda^{-1}\|_{L^q(A)} \mathcal{Q}_p(f)$$

so Ineq. (8.3) holds for f as well. Hence, without the loss of generality, we will from now on assume that f is bounded.

If f is constant on M, then there is nothing to prove because it is equal to its average and the inequality trivially holds. The proofs of $|M| < \infty$ and $|M| = \infty$ diverge slightly at this point. Let us first deal with the case $|M| < \infty$. We start by finding a number $\xi \equiv \xi(f) \in \mathbb{R}$ such that

$$\int_M (f(y) - \xi)^r_+ dy = \int_M (f(y) - \xi)^r_- dy = \frac{1}{2} \int_M (f(y) - \xi)^r dy.$$

This is possible because function

$$\xi \to \int_M (f(y) - \xi)^r_+ dy$$

is continuous via dominated convergence theorem since M has finite measure. This function clearly tends to ∞ when ξ goes to $-\infty$ and to 0 when ξ goes to ∞. Having chosen ξ, $|\operatorname{supp}(f - \xi)_+|$ or $|\operatorname{supp}(f - \xi)_-|$ (or both) is smaller or equal than $|M|/2$ which means that at least one of function $f - \xi$ and $-(f - \xi)$ satisfies assumptions of Lemma 8.1.10. Applying Lemma 8.1.10 in either case gives us the estimate

$$\|f - \xi\|^p_{L^r(M)} \le C_{(8.1.10)} \|\lambda^{-1}\|_{L^q(M)} \mathcal{Q}_p(f)$$

which, due to Lemma 8.1.3, implies that

$$\|f - f_M\|^p_{L^r(M)} \le 2^p \|f - \xi\|^p_{L^r(M)} \le 2^p C_{(8.1.10)} \|\lambda^{-1}\|_{L^q(M)} \mathcal{Q}_p(f).$$

Recalling that we defined $A := M$ in case $|M| < \infty$ produces the statement of the theorem in case $|M| < \infty$ with constant $C_{(8.1.4)} := 2^p C_{(8.1.10)}$ which depends only on s, p, q and n just like $C_{(8.1.10)}$.

Suppose now that $|M| = \infty$. Then

$$|\operatorname{supp} f_+| \vee |\operatorname{supp} f_-| \le |\operatorname{supp} f| < |M|/2 = \infty$$

since f is assumed to be supported on a set of finite measure. Furthermore,

$$\int_M f_+(y)^r dy \ge \frac{1}{2} \int_M f(y)^r dy \quad \text{or} \quad \int_M f_-(y)^r dy \ge \frac{1}{2} \int_M f(y)^r dy$$

which means that at least one of f_+, f_- qualifies for the application of Lemma 8.1.10. Applying Lemma 8.1.10 in either case results in

$$\|f\|^p_{L^r(M)} \le C_{(8.1.10)} \|\lambda^{-1}\|_{L^q(\operatorname{supp} f)} \mathcal{Q}_p(f).$$

Since in this case A was defined to be $\operatorname{supp} f$, taking $C_{(8.1.4)} := C_{(8.1.10)}$ gives exactly the statement from the theorem. Notice that $C_{(8.1.4)}$ depends only on ν, p and q as claimed. □

8.2. Examples of the inequality

Volume regularity from below is a sufficient condition for a metric measure space (M, d, m) to satisfy *Assumption* 8.1.1 with kernel

$$k_p(x, y) = d(x, y)^{-(n+sp)}.$$

A bit more is shown in the following lemma.

Lemma 8.2.1. *Let (M, d, m) be a metric measure space, $p \in [1, \infty)$ and $s > 0$. Suppose that $\eta : M \to [0, \infty)$, $n, C_{VL}, D_1 > 0$ are such that $k_p(x, y) \geq \eta(x) d(x, y)^{-(n+sp)}$ and $\mathbf{V}\geq[M, (0, D_1 \operatorname{diam}(M)]; n, C_{VL}]$ hold. Then there is a constant $C_{(8.2.1)} \equiv C_{(8.2.1)}(n, s, p, D_1, C_{VL})$ such that for every $E \subset M$ satisfying $|E| < \infty$, $|E| \leq |M \setminus E|$, and every point $x \in E$*

$$\int_{M\setminus E} k_p(x, y) dy \geq C_{(8.2.1)} \eta(x) |E|^{-sp/n}. \tag{8.10}$$

In other words, Assumption 8.1.1 holds for $\nu = s/n$ and $\lambda(x) = C_{(8.2.1)}\eta(x)$.

Proof. The plan is to treat "large" and "small" sets separately. If $E \subset M$ is small, that is, if

$$|E| \leq D_1^n C_{VL} 4^{-n} (\operatorname{diam} M)^n,$$

there exists $\mathcal{R} > 0$ such that $|E| = C_{VL}\mathcal{R}^n$. The bound on $|E|$ implies that $4\mathcal{R} \leq D_1 \operatorname{diam} M$ so, by assumption, $\mathbf{V}\geq[M, (0, 4\mathcal{R}]; n, C_{VL}]$ holds true. Thus for every $r \in [2\mathcal{R}, 4\mathcal{R}]$:

$$|B(x, r) \setminus E| \geq |B(x, r)| - |E| \geq C_{VL} r^n - C_{VL}\mathcal{R}^n \geq C_{VL} \left(r^n - 2^{-n} r^n\right) \geq (1 - 2^{-n}) r^n.$$

Keeping this in mind, let us use Fubini's theorem, or more precisely Cavalieri's principle, to compute

$$\begin{aligned}
\int_{M\setminus E} k_p(x, y) dy &\geq \eta(x) \int_{M\setminus E} \frac{dy}{d(x, y)^{n+sp}} \\
&= \eta(x)(n + sp) \int_{M\setminus E} \int_{[d(x,y),\infty)} r^{-n-sp-1} dr dy \\
&= \eta(x)(n + sp) \int_{[0,\infty)} r^{-n-sp-1} |(B(x, r) \setminus E)| dr.
\end{aligned}$$

Reducing the area of integration on the right side to $[2\mathcal{R}, 4\mathcal{R}]$ it follows that

$$\begin{aligned}
\int_{M\setminus E} k_p(x, y) dy &\geq \eta(x)(n + sp) \int_{2\mathcal{R}}^{4\mathcal{R}} r^{-n-sp-1} |B(x, r) \setminus E| dr \\
&\geq \eta(x)(n + sp) \int_{2\mathcal{R}}^{4\mathcal{R}} C_{VL}(1 - 2^{-n}) r^{-sp-1} dr \\
&\geq C_{VL}(1 - 2^{-n})\eta(x) \left(\frac{n + sp}{sp}\right) \left((2\mathcal{R})^{-sp} - (4\mathcal{R})^{-sp}\right) \\
&\geq C_{VL}^{1+\frac{sp}{n}} (1 - 2^{-n}) 2^{-sp} (1 - 2^{-sp}) \left(\frac{n + sp}{sp}\right) \eta(x) C_{VL}^{-\frac{sp}{n}} \mathcal{R}^{-sp} \\
&\geq C_{VL}^{1+\frac{sp}{n}} (1 - 2^{-n}) 2^{-sp} (1 - 2^{-sp}) \left(\frac{n + sp}{sp}\right) \eta(x) |E|^{-\frac{sp}{n}}.
\end{aligned}$$

On the other hand, if the set E is large, that is, if $|E| > D_1^n C_{VL} 4^{-n} (\operatorname{diam} M)^n$ (notice that this only happens if $\operatorname{diam} M < \infty$), then in particular we have

$$(\operatorname{diam} M)^{-n-sp} \geq D_1^{n+sp} 4^{-(n+sp)} C_{VL}^{1+\frac{sp}{n}} |E|^{-1-\frac{sp}{n}}.$$

Recalling that $|E| \leq |M \setminus E|$ by assumption, we proceed to calculate

$$\begin{aligned}\int_{M\setminus E} k_p(x,y)dy &\geq \eta(x)\int_{M\setminus E}\frac{dy}{d(x,y)^{n+sp}}\\ &\geq \eta(x)(\operatorname{diam} M)^{-n-sp}|M\setminus E|\\ &\geq D_1^{n+sp}4^{-(n+sp)}C_{VL}^{1+\frac{sp}{n}}\eta(x)|E|^{-1-\frac{sp}{n}}|E|\\ &\geq D_1^{n+sp}4^{-(n+sp)}C_{VL}^{1+\frac{sp}{n}}\eta(x)|E|^{-\frac{sp}{n}}.\end{aligned}$$

Combining the estimate for small and large sets we see that Assumption 8.1.1 holds with $\nu = \frac{s}{n}$ and $\lambda(x) = C_{(8.2.1)}\eta(x)$ where

$$C_{(8.2.1)} = C_{VL}^{1+\frac{sp}{n}}(1-2^{-n})2^{-sp}(1-2^{-sp})\left(\frac{n+sp}{sp}\right) \vee D_1^{n+sp}4^{-(n+sp)}C_{VL}^{1+\frac{sp}{n}}$$

which in particular depends only on s, p, n, D_1 and C_{VL} and not on the space M in any other way. □

Remark 8.2.2. *The fact that the lemma is stated in the setting of metric measure spaces makes it quite useful because subsets of metric measure space are metric measure spaces themselves. Thus, this automatically implies that all compact and smooth domains of $\mathbb{R}^n$ as well as all uniform d sets satisfy Poincaré-Sobolev inequality. The only truly restrictive condition is the uniform lower volume regularity of the space.*

With the help of Lemma 8.2.1, the abstract Theorem 8.1.4 proves the Poincaré-Sobolev inequality on any ball in $\mathbb{Z}^n$ with constants independent of the ball. This is quite convenient since it circumvents the need for proving uniform extensions estimates. We start with a lemma.

Lemma 8.2.3. *Let $R_0 > 0$ and $x_0 \in \mathbb{Z}^n$ be fixed. Consider $M = B(x_0, R_0) \cap \mathbb{Z}^n$, d to be the Euclidean distance on $\mathbb{Z}^n$ and m to be the counting measure. Then M satisfies $\boldsymbol{V}\geq[M,(0,R_0];n,C_{VL}(M)]$ with*

$$C_{VL}(M) \equiv C_{VL}(B(x_0,R_0)\cap\mathbb{Z}^n) = \frac{C_{VL}(\mathbb{Z}^n)}{4^n} \wedge (2\sqrt{n})^{-n}$$

which depends only on n.

Proof. Let $x \in B(x_0, R_0)$, $r \in (0, R_0)$ be arbitrary and embed $\mathbb{Z}^n$ into $\mathbb{R}^n$. In $\mathbb{R}^n$ it is true that

$$B\left(x - \frac{r(x-x_0)}{2R_0}, \frac{r}{2}\right) \subset B(x_0,R_0)\cap B(x,r)$$

but $x - \frac{r(x-x_0)}{2R_0}$ does not have to be in $\mathbb{Z}^n$. Nevertheless, we can find the point $y \in \mathbb{Z}^n$ which is closest to it. Then $d(x - \frac{r(x-x_0)}{2R_0}, y)$ is at most $\sqrt{n}/2$ so

$$B\left(y, \frac{r}{2} - \frac{\sqrt{n}}{2}\right) \subset B\left(x - \frac{r(x-x_0)}{2R_0}, \frac{r}{2}\right) \subset B(x_0,R_0)\cap B(x,r).$$

Therefore, if $r \geq 2\sqrt{n}$,

$$|B(x_0,R_0)\cap B(x,r)| \geq \left|B\left(y,\frac{r}{4}\right)\right| \geq \frac{C_{VL}(\mathbb{Z}^n)}{4^n}r^n.$$

When r is small (comparable to the discrete structure), that is $r < 2\sqrt{n}$, we instead use the property of the counting measure, $|\{z\}| = 1$ for any $z \in \mathbb{Z}^n$, to obtain

$$|B(x_0,R_0)\cap B(x,r)| \geq |\{x\}| \geq (2\sqrt{n})^{-n}r^n.$$

In both cases it suffices to take

$$C_{VL}(B(x_0,R_0)\cap\mathbb{Z}^n) = \frac{C_{VL}(\mathbb{Z}^n)}{4^n} \wedge (2\sqrt{n})^{-n}.$$

□

Theorem 8.2.4 (Uniform Poincaré-Sobolev inequality on balls in $\mathbb{Z}^n$)**.** *For fixed $n \in \mathbb{N}$, $s > 0$ and $p \geq 1$ such that $n > sp$ there is a constant $C_{(8.2.4)} \equiv C_{(8.2.4)}(n, s, p)$ such that, fixing r to be the solution of $\frac{p}{r} = 1 - \frac{sp}{n}$, for every ball B in $\mathbb{Z}^n$ and every $f : B \to \mathbb{R}$,*

$$\|f - f_B\|_{L^r(B)} \leq C_{(8.2.4)} \sum_{x,y \in B} \frac{|f(x) - f(y)|^p}{d(x,y)^{n+2s}}.$$

In combination with Theorem 8.1.7 this also provides the estimate

$$\|f\|^p_{L^r(B)} \leq 2^p C_{(8.2.4)} \sum_{x,y \in B} \frac{|f(x) - f(y)|^p}{d(x,y)^{n+sp}} + 2^p C_{VL}(\mathbb{Z}^n) R^{-sp} \|f\|^p_{L^p(B)}.$$

Proof. Set $M = B$, $k_p(x,y) = d(x,y)^{-(n+sp)}$ and take m to be the counting measure. Lemma 8.2.3 showed that M satisfies $\mathbf{V}{\geq}[M, (0, \operatorname{diam} M/2); n, C_{VL}]$ with

$$C_{VL} = \frac{C_{VL}(\mathbb{Z}^n)}{4^n} \wedge (2\sqrt{n})^{-n}$$

independent of the ball B. Plugging this into Lemma 8.2.1 with $\eta(x) = 1$ and $D_1 = 1/2$ proves that Assumption 8.1.1 is valid on M with $\lambda \equiv \lambda(n, s, p) := C_{(8.2.1)}$ and allows us to apply Theorem 8.1.4 with $\nu = s/n$. Note that due to assumption $n > sp$ we are allowed to choose any $q \in (\frac{n}{sp}, \infty]$. Taking $q = \infty$ results in $\frac{p}{r} = 1 - \frac{sp}{n}$ and for $f \in L^1(B)$ we obtain

$$\|f - f_B\|_{L^r(B)} \leq C_{(8.1.4)} C^{-1}_{(8.2.1)} \mathcal{Q}_p(f).$$

Defining $C_{(8.2.4)} \equiv C_{(8.2.4)}(n, s, p) := C_{(8.1.4)} C^{-1}_{(8.2.1)}$ and writing out the definition of $\mathcal{Q}_p$ proves the first inequality. The second is the consequence of Theorem 8.1.7 which implies that

$$\|f\|^p_{L^r(B)} \leq 2^p C_{(8.2.4)} \sum_{x,y \in B} \frac{|f(x) - f(y)|^p}{d(x,y)^{n+2s}} + 2^p |B|^{-\frac{sp}{n}} \|f\|^p_{L^p(B)},$$

and combining this with $\mathbf{V}{\geq}[\mathbb{Z}^n, [0, \infty); n, C_{VL}(\mathbb{Z}^n)]$ property of $\mathbb{Z}^n$ gives

$$\|f\|^p_{L^r(B)} \leq 2^p C_{(8.2.4)} \sum_{x,y \in B} \frac{|f(x) - f(y)|^p}{d(x,y)^{n+2s}} + \frac{2^p}{C_{VL}(\mathbb{Z}^n)} R^{-sp} \|f\|^p_{L^p(B)},$$

which gives the required statement. □

Remark 8.2.5. *Alternative way of approaching the proof of the previous theorem would be to bound the extension operator from $W^{p,s}(B)$ to $W^{p,s}(\mathbb{Z}^n)$ and use the Sobolev inequality for the full space $\mathbb{Z}^n$. This is a classical approach in $\mathbb{R}^n$ and a similar approach on $\mathbb{Z}^n$ is taken in [FH20]. However, while this works nicely for a fixed ball or domain, the lack of scaling makes it difficult to preserve the same constant under dilations. Additional work is required as one can see in Definition 16 and what follows in [FH20]. Notice that we obtained constants which are independent of the ball in the previous theorem.*

Part II.

Long-range random conductance model

9. Motivation and definitions

For the rest of the thesis we will only be working on a metric measure space $(\mathbb{Z}^n, d, \#)$, for $n \geq 2$, but we will allow the conductance c to be random. Here d is the Euclidean distance on $\mathbb{R}^n$ and $\#$ is the counting measure. We also fix an arbitrary number $s \in (0,1)$. We will define what we mean by terms symmetrized twofold ergodic conductance (Definition 9.1.6) and i.i.d. conductance (Definition 9.1.4). Furthermore, in Section 9.3 we will explain how to construct the variable speed random walk X_t for almost every realization of conductance that is either symmetrized ergodic or i.i.d.

Let, from now on, c be either an i.i.d. or symmetrized ergodic conductance. Our plan is to verify that assumptions from Part I are satisfied for almost every realization of conductance c, under certain conditions on its distribution, and then apply deterministic results obtained in Part I for every such realization. In case of the symmetrized ergodic conductance, these assumptions are verified in Sections 10.2 and 10.3 for an arbitrary but fixed point $x_0 \in \mathbb{Z}^n$ and all large enough radii. We wish to point out that the exact meaning of phrase "large enough" in the previous sentence depend on the concrete realization of conductance c. The consequences of applying Part I are stated in Theorems 10.4.1 and 10.5.3. In case of the i.i.d. conductance, we can verify that the assumptions of Part I hold, with uniform parameters for almost every realization of conductance c and "large enough" radii, not only for arbitrary fixed point $x_\star \in \mathbb{Z}^n$ but also for points in the vicinity of $x_\star$. What we mean by vicinity is explained in Definition 3.2.1 and is inspired by the very good ball definition from [Bar04]. This uniformity of parameters is a big advantage compared to the symmetrized ergodic case because it means that results of Part I apply also for balls in the vicinity of $x_\star$ and not only for balls with center at $x_\star$. This technical detail will play the crucial role in the proof of tightness on the Skorokhod space in Theorem 12.5.2, where we make use the Markov property to restart the process $X_{m^{2s}t}/m$ in the vicinity of point $x_\star = 0$. The consequences of applying Part I are collected in Theorems 11.7.1 and 11.8.1.

In particular, we obtain the large scale Hölder regularity estimate (**HR**) around point $0 \in \mathbb{Z}^n$ for both choices of conductance c. Under rescaling $X_t \to X_{m^{2s}t}/m$, which is discussed in Section 12.1, this estimate together with Mosco convergence results from [FH20] (ergodic) and [CKK13] (i.i.d.) can be used to prove that the process $X_{m^{2s}t}/m$, starting from $X_0 = 0$, converges to a rotationally symmetric alpha stable process in finite-dimensional distributions. We present the proof in Theorem 12.4.1. In case of the i.i.d. conductance, we can in addition verify that the truncation estimate (**TB**), needed for short time estimates of P_t^B in Theorem 7.3.2, holds. This is the second crucial ingredient that allows us to prove the tightness in Theorem 12.5.2.

The main results of this part, and the thesis in whole, are Corollary 12.4.2, which proves the convergence of $X_{m^{2s}t}/m$ in finite-dimensional distributions for almost every realization of symmetrized ergodic conductance, and Theorem 12.5.2, which proves the weak convergence of $X_{m^{2s}t}/m$ on the Skorokhod space $\mathcal{D}([0,T], \mathbb{R}^n)$ for every $T > 0$ and almost every realization of i.i.d. conductance. The argument used in the proof of Theorem 12.5.2 is due to [CKW18b], Theorem 4.5 which proves the same statement but under different moment assumptions.

Remark 9.0.1. *Throughout this part $C_{VL} \equiv C_{VL}(n)$ and $C_{VU} \equiv C_{VU}(n)$ will denote the lower and upper volume regularity constants of $\mathbb{Z}^n$ (see Lemma 2.6.1). To be precise*

$$C_{VL}R^n \leq \#B \leq C_{VU}R^n$$

for all balls B with radius $R \geq 1$. Here $\#$ denotes the counting measure. In case we need to use these constants for $\mathbb{Z}^{n_1}$ and $\mathbb{Z}^{n_2}$ at the same time, we will write $C_{VL}(\mathbb{Z}^{n_1})$ and $C_{VL}(\mathbb{Z}^{n_2})$ respectively.

The integration in $\mathbb{Z}^n$ will always be performed with respect to the counting measure $\#$. That is, for $f \in L^1(\mathbb{Z}^n, \#)$, we define

$$\int_{\mathbb{Z}^n} f(x)dx := \int_{\mathbb{Z}^n} f(x)\#(dx) = \sum_{x\in\mathbb{Z}^n} f(x).$$

Furthermore, for a set $A \subset \mathbb{Z}^n$ we will denote $|A| := \#A$.

9.1. Definition of random conductance

Definition 9.1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.* Random conductance *c on $\mathbb{Z}^n$ is a function*

$$c : \Omega \times \mathbb{Z}^n \times \mathbb{Z}^n \to [0, \infty)$$

which is $\mathcal{F} \times \mathcal{B}(\mathbb{Z}^n) \times \mathcal{B}(\mathbb{Z}^n)$ to $\mathcal{B}(\mathbb{R})$ measurable (with $\mathcal{B}$ denoting Borel sigma algebra). Random conductance c is said to be symmetric if

$$\forall\omega \in \Omega, \forall x, y \in \mathbb{Z}^n \qquad c(\omega, x, y) = c(\omega, y, x).$$

Remark 9.1.2. *As is common when working with random variables, we will suppress ω dependence in the notation of c.*

To a conductance c we associate the kernel k and the energy form $\mathcal{E}$ in the following way:

Definition 9.1.3. *For a random conductance c on $\mathbb{Z}^n$ and $s \in (0, 1)$ we define a random kernel*

$$k(x, y) = \frac{c(x, y)}{d(x, y)^{n+2s}}$$

and a random form $\mathcal{E}$, trough its action on a measurable $f : \mathbb{Z}^n \to \mathbb{R}$, by

$$\mathcal{E}(f) = \sum_{x\in\mathbb{Z}^n} \sum_{y\in\mathbb{Z}^n} (f(x) - f(y))k(x, y) \tag{9.1}$$

Unless explicitly stated otherwise, we will always consider on its maximal domain $\mathcal{D}[\mathcal{E}] = \{f \in L^2(\mathbb{Z}^n), \mathcal{E}(f) < \infty\}$. (Both c, s are suppressed in the notation but should be clear from the context.)

Definition 9.1.4. *Symmetric random conductances c is said to be* independent and identically distributed (i.i.d.) *if the family of random variables*

$$\{c(x, y) : x, y \in \mathbb{Z}^n, x \prec y\}$$

is independent and identically distributed. Here $\prec$ is any total ordering on $\mathbb{Z}^n$. (The ordering is needed to avoid the conflict with the symmetry of c.)

Definition 9.1.5. *Random conductance c is said to be* twofold *ergodic if there exists a family of invertible, commuting and measurable mappings $\{\tau_i^{(j)} : \Omega \to \Omega : i \in \{1, 2, \dots, n\}, j \in \{1, 2\}\}$ (also called shifts) such that dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\hat{\tau}_i^{(j)}\})$ is ergodic and for every $i \in \{1, 2, \dots n\}$,*

$$c(\tau_i^{(1)}(\omega), x, y) = c(\omega, x - e_i, y) \quad \& \quad c(\tau_i^{(2)}(\omega), x, y) = c(\omega, x, y - e_i)$$

where $e_i \in \mathbb{Z}^n$ is the unit vector having 1 only at i-th coordinate.

Recall that the dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, T)$ (here T denotes a family of shifts) is said to be ergodic if all shifts are measure preserving, that is

$$\forall\tau \in T, \forall A \in \mathcal{F} \qquad \mathbb{P}\left(\tau^{-1}(A)\right) = \mathbb{P}(A),$$

and for every $A \in \mathcal{F}$

$$[\tau(A) = A \quad \forall\tau \in T] \quad \Longrightarrow \quad \mathbb{P}(A) = 0 \text{ or } 1.$$

Definition 9.1.6. *Random conductance c on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be* symmetrized (twofold) ergodic *if there exists an ergodic conductance c' such that*

$$\forall x, y \in \mathbb{Z}^n \qquad c(x,y) = \frac{c'(x,y) + c'(y,x)}{2}.$$

Remark 9.1.7. *Let c be a symmetrized ergodic conductance and c' an ergodic conductance from the previous definition. Denote by $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ and $(\mathcal{E}', \mathcal{D}[\mathcal{E}'])$ corresponding bilinear forms with maximal domains on $L^2(\mathbb{Z}^n)$. It is not hard to see that $\mathcal{D}[\mathcal{E}] = \mathcal{D}[\mathcal{E}']$ and that $\mathcal{E}$ is equal to the symmetric part of $\mathcal{E}'$, that is, for all $f, g \in \mathcal{D}[\mathcal{E}]$*

$$\mathcal{E}(f,g) = \frac{1}{2}\left(\mathcal{E}'(f,g) + \mathcal{E}'(g,f)\right).$$

In particular, forms $\mathcal{E}$ and $\mathcal{E}'$ coincide on the diagonal and $\mathcal{E}(f) = \mathcal{E}'(f)$ for all $f \in \mathcal{D}[\mathcal{E}]$. Therefore, any statement on c' that contains only symmetric part of $\mathcal{E}'$ immediately translates to the statement on c. For instance, Poincaré inequality, Sobolev inequality or Mosco convergece (assuming we are given sequences $\{c_m\}$, $\{c'_m\}$) are valid for c if and only if they are valid for c'.

Definition 9.1.8. *Let c be an i.i.d. or ergodic or symmetrized ergodic conductance. In any of these cases the distribution of variables $c(x,y)$, for $x, y \in \mathbb{Z}^n$, is the same so the value $\mathbb{E}[f(c(x,y))]$, for whatever $f : \mathbb{R} \to \mathbb{R}$, does not depend on x or y and we will simply denote it by $\mathbb{E}[f(c)]$.*

9.2. Dirichlet form property

We prove that, if conductance c is symmetrized ergodic or i.i.d., then the random bilinear form introduced in Definition 9.1.3 with its maximal domain is $\mathbb{P}$-a.s. a Dirichlet form.

Theorem 9.2.1 (cf. [CKK13], Theorem 3.2)**.** *Let $A > 0$ and a symmetric conductance c on $\mathbb{Z}^n$ be such that $\mathbb{E}[c(x,y)] \leq A$ for all $x, y \in \mathbb{Z}^n$. Then, $\mathbb{P}$-almost surely, $\mathcal{E}$ with its maximal domain*

$$\mathcal{D}[\mathcal{E}] = \left\{v \in L^2(\mathbb{Z}^n) : \mathcal{E}(v) < \infty\right\}$$

is a regular Dirichlet form on $L^2(\mathbb{Z}^n)$ containing $C_c(\mathbb{Z}^n)$ in its domain. In particular, $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ $\mathbb{P}$-a.s. satisfies Assumption 4.0.2.

Proof. We essentially repeat the proof of Theorem 3.2 in [CKK13]. By monotone convergence theorem, for every $x \in \mathbb{Z}^n$,

$$\mathbb{E}\left[\sum_{y \in \mathbb{Z}^n \setminus \{x\}} k(x,y)\right] = \mathbb{E}\left[\sum_{y \in \mathbb{Z}^n \setminus \{x\}} \frac{c(x,y)}{d(x,y)^{n+2s}}\right] \leq \sum_{y \in \mathbb{Z}^n \setminus \{x\}} \frac{A}{d(x,y)^{n+2s}} < \infty.$$

Therefore there exists a a $\mathbb{P}$-null set N_x outside of which $\sum_{y \in \mathbb{Z}^n \setminus \{x\}} k(x,y) < \infty$. Then $N := \bigcup_{x \in \mathbb{Z}^n} N_x$ is also a $\mathbb{P}$-null set outside of which

$$\forall x \in \mathbb{Z}^n \qquad \sum_{y \in \mathbb{Z}^n \setminus \{x\}} k(x,y) < \infty.$$

Let us now fix an $\omega \notin N$ and work only on realization $c(\omega)$. That $\mathcal{E}$ is a positive definite bilinear form follows from formula for $\mathcal{E}$ in Eq. (9.1) and symmetry is inherited from the symmetry of c. To prove it is closed, take an arbitrary $\mathcal{E}_1$-Cauchy sequence f_m. Then $f_m \xrightarrow{L^2(\mathbb{Z}^n)} f$ for some $f \in L^2(\mathbb{Z}^n)$ and we can find a subsequence f'_m (either because $\mathbb{Z}^n$ is σ-finite or because counting measure is atomized) which converges to f pointwise. Fatou's lemma together with the Cauchy property of sequence f_m now implies

$$\begin{aligned}\mathcal{E}(f-f_m) &= \sum_{x\in\mathbb{Z}^n}\sum_{y\in\mathbb{Z}^n}\lim_{l\to\infty}(f_l'(x)-f_m(y))^2k(x,y)\\ &\le \lim_{l\to\infty}\sum_{x\in\mathbb{Z}^n}\sum_{y\in\mathbb{Z}^n}(f_l'(x)-f_m(y))^2k(x,y) = \lim_{l\to\infty}\mathcal{E}(f_l'-f_m)\xrightarrow{m\to\infty}0,\end{aligned}$$

which proves that the form is closed. The form is also Markov because $\mathcal{E}((f\vee 1)\wedge 0)\le\mathcal{E}(f)$ follows straight from formula for $\mathcal{E}$ in Eq. (9.1) and therefore a Dirichlet form. Furthermore, by symmetry of k, for a function g compactly supported in some set $K\subset\mathbb{Z}^n$ we have

$$\begin{aligned}\mathcal{E}(g) &= \sum_{x\in\mathbb{Z}^n}\sum_{y\in\mathbb{Z}^n}(g(x)-g(y))^2k(x,y)\le 2\sum_{x\in\mathbb{Z}^n}g(x)^2\sum_{y\in\mathbb{Z}^n\setminus\{x\}}k(x,y)\\ &= 2\sum_{x\in K}g(x)^2\sum_{y\in\mathbb{Z}^n\setminus\{x\}}k(x,y)<\infty,\end{aligned}$$

where the last inequality works only because compact set $K\subset\mathbb{Z}^n$ has finitely many elements. Hence $C_c(\mathbb{Z}^n)\subset\mathcal{D}[\mathcal{E}]$. By [FOT11] Theorem 1.4.2 (iv) any function $f\in\mathcal{D}[\mathcal{E}]$ is approximated in $\mathcal{E}_1$ norm by functions $f_m = f-(f\vee -1/m)\wedge 1/m$. But sets $\{|f|>1/m\}$, on which f_m are supported, are finite because $f\in L^2(M)$ (keep in mind that we are working with counting measure). Hence f_m are compactly supported and since any function on $\mathbb{Z}^n$ (with discrete topology) is continuous, $f\in C_c(\mathbb{Z}^n)$ which proves the regularity of $\mathcal{E}$. Since the set of Lipschitz functions supported in balls is contained in $C_c(\mathbb{Z}^n)$ the Assumption 4.0.2 is satisfied. All of this holds outside of $\mathbb{P}$-null set N which completes the proof. □

Corollary 9.2.2. *Let c be an i.i.d. or a symmetrized ergodic conductance on $\mathbb{Z}^n$ such that $\mathbb{E}[c]<\infty$. Then $\mathbb{P}$-almost surely $\mathcal{E}$ with its maximal domain is a regular Dirichlet form on $L^2(\mathbb{Z}^n)$ satisfying Assumption 4.0.2.*

Proof. As mentioned when defining $\mathbb{E}[c]$, in both cases the distribution of $c(x,y)$ does not depend on $x,y\in\mathbb{Z}^n$. Thus $\mathbb{E}[c(x,y)]\le\mathbb{E}[c]$ for all $x,y\in\infty$ so it suffices to apply Theorem 9.2.1 with $A=\mathbb{E}[c]<\infty$. □

Lemma 9.2.3. *Let c be a symmetric random conductance such that $\mathbb{P}$.a.s. $(\mathcal{E},\mathcal{D}[\mathcal{E}])$ is a regular Dirichlet form and for every ball $B\subset\mathbb{Z}^n$*

$$\sum_{y\in\mathbb{Z}^n\setminus B}\frac{c(x,y)}{d(x,y)^{n+2s}}>0. \tag{9.2}$$

Then $\mathbb{P}$-a.s., for every ball $B\subset\mathbb{Z}^n$, $\|G^B1\|_{L^\infty(\mathbb{Z}^n)}<\infty$.

Proof. Fix some ball $B\subset\mathbb{Z}^n$. Let $N(B)\in\mathcal{F}$, $\mathbb{P}(N(B))=0$, be a set such that Ineq. (9.2) is true and $(\mathcal{E},\mathcal{D}[\mathcal{E}])$ is a regular Dirichlet form on $\Omega\setminus N(B)$. This verifies Assumption 4.0.2. Assumption 2.5.3 is satisfied because $(\mathbb{Z}^n,d,\#)$ is a σ-finite, locally compact, separable measure space and counting measure $\#$ has full support. By Theorem 7.1.3 this implies that $G^B1<\infty$ $\#$-a.e. in B so in fact everywhere on B because $\emptyset$ is the only $\#$-null set. We also know that $G^B1=0$ on $\mathbb{Z}^n\setminus B$ and as the ball B has only finitely many elements we must have $\|G_B1\|_{L^\infty(\mathbb{Z}^n)}<\infty$ for every $\omega\in\Omega\setminus N(B)$. Since $\mathbb{Z}^n$ is separable and discrete it contains only finitely many different balls which allows us to define $\mathbb{P}$-null set $N=\bigcup_{x\in\mathbb{Z}^n}\bigcup_{R\in\mathbb{Q}}N(B(x,R))$ outside of which $\|G^B1\|_{L^\infty(\mathbb{Z}^n)}<\infty$ for every ball $B\subset\mathbb{Z}^n$. □

9.3. Random walk

Let μ be a Radon measure of full support on $\mathbb{Z}^n$. If the conductance c is such that $\mathbb{P}$-a.s. $\mathcal{E}$ is a regular Dirichlet form on $L^2(\mathbb{Z}^n, \mu)$, the existence of the Hunt process X_t for every such realization of c follows from Theorem 2.5.13. Note that we leave the dependence of X_t on the realization of c out of the notation, as is common in the literature. The discreteness of state space $\mathbb{Z}^n$ simplifies the general results relating Dirichlet forms to their symmetric Hunt process considerably. Firstly, there is no need to care about μ-a.e. or q.e. notions because $\emptyset$ is the only μ-null subset of $\mathbb{Z}^n$. Thus the transition functions of any two Hunt process coincide everywhere instead of q.e. like claimed in Theorem 2.5.14. Secondly, the Hunt process X_t is the continuous time Markov chain with the generator

$$\mathcal{L}f(x) = \frac{1}{\mu(x)} \sum_{y \in \mathbb{Z}^n} (f(y) - f(x))k(x, y)$$

and can be constructed directly, just like in [Nor98] Chapter 2. We summarize this in the following theorem.

Theorem 9.3.1. *Let c be a symmetric random conductance on $\mathbb{Z}^n$ such that $\mathbb{P}$-a.e. $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is a regular Dirichlet form on $L^2(\mathbb{Z}^n, \mu)$. Then for $\mathbb{P}$-a.e. realization of c there exist a Hunt process $(\Omega, \mathcal{F}, X_t, \{\mathbf{P}_x\}_{x \in \mathbb{Z}^n_\partial})$ on $\mathbb{Z}^n_\partial$ (where $\mathbb{Z}^n_\partial = \mathbb{Z}^n \cup \{\partial\}$ and ∂ is the* cemetery point*) such that*

$$P_t f(x) = \mathbf{E}_x[f(X_t)] \quad \forall x \in \mathbb{Z}^n. \tag{9.3}$$

Here P_t is the strongly continuous contractive semigroup corresponding to $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ on $L^2(\mathbb{Z}^n, \mu)$.

Proof. Directly from Theorem 2.5.13 applied to every realization of c for which $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is a regular Dirichlet form. This proves that Eq. (9.3) holds for q.e. x but, because the space $\mathbb{Z}^n$ is discrete and μ has full support, it also holds for every $x \in \mathbb{Z}^n$. □

In particular, the previous theorem combined with Corollary 9.2.2 proves that the continuous time Markov chain exists when μ is the counting measure on $\mathbb{Z}^n$ and conductance c is either i.i.d. or symmetrized ergodic. In general, measure μ governs the jumping rates of X_t at different points in space and there are two standard choices in the literature. If μ is the counting measure on $\mathbb{Z}^n$, the process X_t is called the *variable speed random walk* and if μ is given by

$$\mu(A) = \sum_{x \in A} \sum_{y \in \mathbb{Z}^n} \frac{c(x, y)}{d(x, y)^{n+2s}} \quad \forall A \subset \mathbb{Z}^n,$$

it is called the *constant speed random walk*. Other choices are possible but they are not as common.

For the rest of Part II we will only consider the variable speed random walk, that is, we will only analyze the form $\mathcal{E}$ on $L^2(\mathbb{Z}^n, \#)$ where $\#$ is the counting measure. Thus we have the following corollary.

Theorem 9.3.2. *Let c be an i.i.d. or symmetrized ergodic conductance on $\mathbb{Z}^n$. Then for $\mathbb{P}$-a.e. realization of c there exist a Hunt process $(\Omega, \mathcal{F}, X_t, \{\mathbf{P}_x\}_{x \in \mathbb{Z}^n_\partial})$ on $\mathbb{Z}^n_\partial$ such that*

$$P_t f(x) = \mathbf{E}_x[f(X_t)] \quad \forall x \in \mathbb{Z}^n$$

where P_t is the strongly continuous contractive semigroup corresponding to $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ on $L^2(\mathbb{Z}^n, \#)$ and $\#$ denotes the counting measure.

10. Symmetrized ergodic conductance

In this chapter we will study the regular Dirichlet form $\mathcal{E}$, with its maximal domain, corresponding to symmetrized ergodic conductance introduced in Definition 9.1.6 on $L^2(\mathbb{Z}^n, \#)$. The main results are Theorem 10.4.1, which gives the weak parabolic Harnack inequality (**WPHI**) and large scale Hölder regularity (**HR**), and Theorem 10.5.3, which gives the expected exit time estimate (**ETE**), survival estimate (**SE**) and conservativeness of $\mathcal{E}$. Both of these theorems are valid at every point $x_0 \in \mathbb{Z}^n$ in space but only on the scale large enough depending on x_0 and the realization of c.

Let $p > 0$ be arbitrary. In Section 10.1 we prove, by requiring that $\mathbb{E}[c^{p'}] < \infty$ for whatever $p' > p$ and using the ergodic theorem of Zygmund and Fava, that the space averages of c^p are bounded in the limit. To be more precise, there is a $C_M < \infty$ such that $\mathbb{P}$-a.s.

$$\limsup_{a,b\to\infty} \frac{1}{\#B(0,a)} \sum_{x\in B(0,a)} \frac{1}{\#B(x,b)} \sum_{y\in B(x,b)} c^p(x,y) \le C_M. \tag{10.1}$$

where a and b are allowed to tend to infinity independently (see **BA**). This is slightly stronger than the ergodic theorem of Tempel'man (see [Kre85] Theorem 2.8) because the set of $\mathbb{P}$-full measure does not depend of the choice of sequences $\{a_k\}$ and $\{b_k\}$ tending to infinity. However, a stronger assumption $\mathbb{E}[c^p(\log^+ c)^{2n-1}] < \infty$ is needed instead of $\mathbb{E}[c^p] < \infty$ required in the theorem of Tempel'man. Sobolev and Poincaré inequalities are proved in Section 10.2 with the help of Ineq. (10.1) applied to c^{-1} instead of c and $q > \frac{n}{2s}$ instead of p. In Section 10.3, Ineq. (10.1), with $Q > 0$ in place of p, is used to prove the estimate on the energy density of cutoff functions. Finally, the application of method from Part I demands the moment condition

$$\frac{1}{q} + \frac{1}{Q} < \frac{2s}{n}$$

and results in Theorems 10.4.1 and 10.5.3.

10.1. Estimates on spatial averages

The main and only tool from ergodic theory that we are going to use is the theorem of Zygmund and Fava which we paraphrase from [Fav72], Corollary after Theorem 3. It can also be found in Theorem 1.1, Section 6.1 in [Kre85]. It improves on the theorem 10 of [DS56] that is, in fact, already sufficient for the arguments in this chapter.

Theorem 10.1.1 (Zygmund-Fava)**.** *Let (Ω, μ) be a measure space of finite total measure and $k \in \mathbb{N}$. If each of linear operators T_i $(i = 1, 2, \dots k)$ is at the same time a contraction of $L^1(\Omega)$ and of $L^\infty(\Omega)$, then for every $f \in L(\log^+ L)^{k-1}(\Omega)$ the multiple averages*

$$\frac{1}{(m_1+1)\dots(m_k+1)} \sum_{i_1=0}^{m_1} \dots \sum_{i_k=0}^{m_k} T_1^{i_1} \dots T_k^{i_k} f(\omega)$$

converge when $m_1, \dots m_k \to \infty$ independently (meaning that they converge for every sequence of k-tuples that converge to $+\infty$ componentwise) for μ-a.e. $\omega \in \Omega$.

In particular, if $p \in (1, \infty)$, the same conclusion holds for $f \in L^p(\Omega) \subset L(\log^+ L)^{k-1}(\Omega)$.

Proposition 10.1.2. *Let c be an ergodic conductance and $\tilde{c}$ its symmetrized version. Let $p \in \mathbb{R}$ be such that $\mathbb{E}[c^{p'}] < \infty$ for whatever $p' \in \mathbb{R}$ with $\frac{p'}{p} > 1$. Then, $\mathbb{P}$-a.s., for every $x_0 \in \mathbb{Z}^n$ $c(\omega)$ satisfies $\boldsymbol{BA}[x_0; p, \mathbb{E}[c^p], n]$ and $\tilde{c}$ satisfies $\boldsymbol{BA}[x_0; p, 2\mathbb{E}[c^p], n]$.*

Proof. Fix an $x_0 \in \mathbb{Z}^n$. Let us use shifts $\tau_i^{(1)}, \tau_i^{(2)}$, $(i = 1, 2, \dots n)$ from Definition 9.1.5 to define $T_1, T_2, \dots T_{2n}$, through its action on a random variable f, by $T_i f(\omega) = f(\tau_i^{(1)} \tau_i^{(2)} \omega)$ and $T_{n+i} f = f(\tau_i^{(2)} \omega)$ for $i = 1, 2, \dots n$. Then each of the operators $T_1, T_2, \dots T_{2n}$ is a contraction in both $L^1(\Omega)$ and $L^\infty(\Omega)$ so Theorem 10.1.1, applied to $c^p(\omega, x_0, x_0) \in L^{\frac{p'}{p}}(\Omega)$, proves that

$$\frac{1}{(m_1 + 1) \dots (m_{2n} + 1)} \sum_{i_1=0}^{m_1} \dots \sum_{i_{2n}=0}^{m_{2n}} T_1^{i_1} \dots T_{2n}^{i_{2n}} c^p(\omega, x_0, x_0)$$

converges independently for $\mathbb{P}$-a.e. $\omega \in \Omega$. Changing the notation from sums to integrals with respect to the counting measure, this implies that

$$\frac{1}{\#[0, m_1] \dots \#[0, m_{2n}]} \int_{[0,m_1] \times \dots \times [0,m_n]} \int_{[0,m_{n+1}] \times \dots \times [0,m_{2n}]} c^p(\omega, x_0 + x, x_0 + x + y) dy dx.$$

also converges when $m_1, \dots m_n \to \infty$ independently. Because shifts $\tau_i^{(1)}, \tau_i^{(2)}$ are invertible by definition, this remains true if one or more segments of the form $[0, m_i]$ are replaced by segments $[-m_i, 0]$. From this it is not hard to see that there is a $\mathbb{P}$-null set $N(x_0) \subset \Omega$ such that for every $\omega \in \Omega \setminus N(x_0)$

$$\fint_{[-m_1,m_1] \times \dots \times [-m_n,m_n]} \fint_{[-m_{n+1},m_{n+1}] \times \dots \times [-m_{2n},m_{2n}]} c^p(\omega, x_0 + x, x_0 + x + y) dy dx$$

converges when $m_1, \dots, m_{2n} \to \infty$ independently. Specifying $m_1 = m_2 = \dots = m_n := l$ and $m_{n+1} = \dots = m_{2n} := k$ we find that for $\omega \in \Omega \setminus N(x_0)$

$$A(\omega) := \lim_{\substack{k,l \to \infty \\ \text{indepenently}}} \fint_{x_0 + [-l,l]^n} \fint_{x + [-k,k]^n} c^p(\omega, x, y) dy dx < \infty.$$

But then

$$\limsup_{k,l \to \infty} \fint_{B(x_0,k)} \fint_{B(x,l)} c^p(\omega, x, y) dy dx \le 2^{2n} C_{VL}^{-2} A(\omega) < \infty.$$

Clearly lim sup is invariant under shifts $\tau_i^{(1)}$, $\tau_i^{(2)}$ for $i = 1, \dots n$ so it has to be equal to a constant, call it $E < \infty$, by ergodicity of $(\Omega, \mathcal{F}, \mathbb{P})$. On the other hand, we can find a subsequence $k_i \to \infty, l_i \to \infty$ such that

$$E = \lim_{i \to \infty} \fint_{B(x_0,k_i)} \fint_{B(x,l_i)} c^p(\omega, x, y) dy dx$$

so Fatou's lemma proves that

$$E = \mathbb{E}\left[\liminf_{i \to \infty} \fint_{B(x_0,k_i)} \fint_{B(x,l_i)} c^p(x, y)\right] \le \liminf_{i \to \infty} \mathbb{E}\left[\fint_{B(x_0,k_i)} \fint_{B(x,l_i)} c^p(x, y)\right] = \mathbb{E}[c^p].$$

Therefore $c(\omega)$ satisfies $\mathbf{BA}[x_0; p, C_M = \mathbb{E}[c^p], n]$ for $\omega \in \Omega \setminus N(x_0)$. Flipping operators $T_1, \dots, T_n$ and $T_{n+1}, \dots, T_{2n}$, exactly the same proof (with variables x and y flipped) gives a $\mathbb{P}$-null set $N'(x_0)$ such that for $\omega \in \Omega \setminus N'(x_0)$

$$\limsup_{k,l \to \infty} \fint_{B(x_0,k)} \fint_{B(x,l)} c^p(\omega, y, x) dy dx \le \mathbb{E}[c^p].$$

Now, for the symmetrized ergodic conductance $\tilde{c}$, we can estimate

$$\tilde{c}(x, y)^p \le \left(\frac{c(x, y) + c(y, x)}{2}\right)^p \le c(x, y)^p \vee c(y, x)^p \le c(x, y)^p + c(y, x)^p.$$

Thus for every $l, k \in \mathbb{N}$ and every $\omega \in \Omega \setminus (N(x_0) \cup N'(x_0))$

$$\fint_{B(x_0,k)} \fint_{B(x,l)} \tilde{c}(x, y)^p dy dx \le \fint_{B(x_0,k)} \fint_{B(x,l)} c(x, y)^p + c(y, x)^p dy dx \le 2\mathbb{E}[c^p]$$

proving that $\tilde{c}$ satisfies $\mathbf{BA}[x_0; p, C_M = 2\mathbb{E}[c^p], n]$. Since $\mathbb{Z}^n$ is countable, we can easily find a null set N which is good for all $x_0 \in \mathbb{Z}^n$. □

We now introduce the family of sets on which the averages of conductance will later be required to converge.

Definition 10.1.3. *Fix an $x_0 \in \mathbb{Z}^n$ and define, for $a, b, R > 0$,*

$$A_{x_0}(R,a,b) = \{x, y \in \mathbb{Z}^n : x \in B(x_0, 2^{a-1}bR), d(x,y) < 2^{b-1}aR\}.$$

Proposition 10.1.4. *Let x_0 be fixed and take arbitrary $R, a, b > 0$. If $R', a', b' > 0$ are such that $a \le a'$, $b \le b'$ and $R \le R'$, then*

$$A_{x_0}(R,a,b) \subset A_{x_0}(R',a,b) \cap A_{x_0}(R,a',b) \cap A_{x_0}(R,a,b')$$

and

$$\bigcup_{R\in\mathbb{N}} A_{x_0}(R,a,b) \cap \bigcup_{a\in\mathbb{N}} A_{x_0}(R,a,b) \cap \bigcup_{b\in\mathbb{N}} A_{x_0}(R,a,b) \supset \mathbb{Z}^n \times \mathbb{Z}^n.$$

Furthermore,

$$C_{VL}^2 (R^2ab)^n 2^{(a+b-2)n} \le \# A_{x_0}(R,a,b) \le C_{VU}^2 (R^2ab)^n 2^{(a+b-2)n}.$$

Proof. Everything but the volume estimate follows directly from the definition because

$$\bigcup_{\zeta\in\mathbb{N}} \{x, y \in \mathbb{Z}^n : x \in B(x_0,\zeta), d(x,y) \le \zeta\} = \mathbb{Z}^n \times \mathbb{Z}^n.$$

For volume estimate we use $\mathbf{V}(\mathbb{Z}^n, [1,\infty]; n, C_{VL}, C_{VU})$ of $\mathbb{Z}^n$ with counting measure $\#$ to estimate, for every $x \in \mathbb{Z}^n$,

$$C_{VL} R^n 2^{n(b-1)} a^n \le \#\{y \in B(x, 2^{b-1}aR)\} \le C_{VU} R^n 2^{n(b-1)} a^n.$$

Summing this inequality in $x \in B(x_0, R2^{a-1}b)$ and using $\mathbf{V}$ once again results in

$$C_{VL}^2 R^{2n} 2^{(a+b-2)n} (ab)^n \le \sum_{x\in B(x_0, R2^{a-1}b)} \#\{y \in B(x, 2^{b-1}aR)\} \le C_{VU}^2 R^{2n} 2^{(a+b-2)n} (ab)^n.$$

Since the middle term is equal to $\# A_{x_0}(R,a,b)$, the claim follows. □

Lemma 10.1.5. *Suppose $x_0 \in \mathbb{Z}^n$, $p, C_M \in \mathbb{R}$ and conductance c are such that c satisfies $\boldsymbol{BA}[x_0; p, C_M, n]$ $\mathbb{P}$-a.s. Then for every $\delta > 0$ there exist $\mathbb{P}$-a.s. finite random variables $R_{(10.1.5)} \equiv R_{(10.1.5)}(x_0, \omega, c, \delta, p, C_M)$, $a_{(10.1.5)} \equiv a_{(10.1.5)}(x_0, \omega, c, \delta, p, C_M)$ and $b_{(10.1.5)} \equiv b_{(10.1.5)}(x_0, \omega, c, \delta, p, C_M)$ such that*

$$⨍_{A_{x_0}(R,a,b)} c(\omega, x, y)^p dxdy \le C_M + \delta \tag{10.2}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $R, a, b \in \mathbb{N}$ such that $R \ge R_{(10.1.5)}(\omega)$ <u>or</u> $a \ge a_{(10.1.5)}(\omega)$ <u>or</u> $b \ge b_{(10.1.5)}(\omega)$, where the integration is with respect to the counting measure.

Proof. Choose arbitrary x_0, p, δ and fix them for the rest of the proof. Let us denote by $\chi(S)$ the indicator function of a set $S \subset \Omega$ and define

$$R_{(10.1.5)}(x_0, c, \delta, p, C_M) := \sup_{R\in\mathbb{N}} \left[R \cdot \chi\left(\bigcup_{a\in\mathbb{N}, b\in\mathbb{N}} \left\{ ⨍_{A_{x_0}(R,a,b)} c^p(x,y) dxdy > C_M + \delta \right\} \right) \right] + 1.$$

Since sets $\{⨍_{A_{x_0}(R,a,b)} c(x,y)^p dydx > C_M + \delta\}$ are measurable for every $R, a, b \in \mathbb{N}$, $R_{(10.1.5)}$ is also measurable as an extended real valued function, possibly taking value $+\infty$ on a measurable set. $a_{(10.1.5)}$ and $b_{(10.1.5)}$ are defined in an analogue way and they are also measurable for the same reason.

Let $N \subset \Omega$ be a $\mathbb{P}$-null set such that for every $\omega \in \Omega \setminus N$ conductance $c(\omega)$ satisfies $\mathbf{BA}[x_0; p, C_M, n]$. We will prove that $R_{(10.1.5)}, a_{(10.1.5)}, b_{(10.1.5)}$ are finite on $\Omega \setminus N$ by fixing $\omega \in \Omega \setminus N$ and proving that there are only finitely many triples (R, a, b) not satisfying Ineq. (10.2). So, let us fix an $\omega \in \Omega \setminus N$ and suppose the opposite, that the there is infinitely many triples $(R, a, b) \in \mathbb{N}^3$ not satisfying Ineq. (10.2). We can then find a infinite sequence $(R_i, a_i, b_i) \in \mathbb{N}^3$ such that

$$\fint_{A_{x_0}(R_i,a_i,b_i)} c(\omega, x, y)^p dxdy > C_M + \delta. \tag{10.3}$$

First we prove that one can extract a strictly increasing subsequence of (R_i, a_i, b_i) (using ordering $(R_1, a_1, b_1) \leq (R_2, a_2, b_2)$ if and only if $R_1 \leq R_2$, $a_1 \leq a_2$ and $b_1 \leq b_2$). $(R_i)_{i\in\mathbb{N}}$, $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ cannot all be bounded because the sequence is infinite. Without the loss of generality suppose $(R_i)_{i\in\mathbb{N}}$ is unbounded and (by passing to the subsequence if necessary) that R_i are strictly increasing. If now the sequences a_i and b_i are bounded, then at least one of the values, say (a^*, b^*), has to appear infinitely many times in $(a_i, b_i)_{i\in\mathbb{N}}$ allowing us to extract a strictly increasing subsequence (R_i, a^*, b^*) (the subsequence notation is omitted) which satisfies Ineq. (10.3). Otherwise, one of the sequences a_i and b_i has to be unbounded, so let us assume (without the loss of generality) that it is a_i, and (up to passing to subsequence again) that a_i is strictly increasing. The same reasoning applies again. If b_i is bounded, then at least one of the values, say b^* has to appear infinitely many times in $(b_i)_{i\in\mathbb{N}}$. In this case we can again extract an increasing sequence (R_i, a_i, b^*) which satisfies Ineq. (10.3). Otherwise, if b_i is not bounded, we can take it to be strictly increasing (by passing to the subsequence one more time) so that (R_i, a_i, b_i) is again strictly increasing and satisfying Ineq. (10.3). Therefore, if *Ineq.* (10.2) were wrong, we would be able to find a strictly increasing infinite sequence (R_i, a_i, b_i) satisfying Ineq. (10.3). Now set $k_i := 2^{a_i-1} b_i R_i, l_i := 2^{b_i-1} a_i R_i$ and notice that $l_i, k_i \to \infty$ because at least one of the sequences R_i, a_i or b_i is strictly increasing. We would then have

$$\fint_{B(x_0,k_i)} \fint_{B(x,l_i)} c(\omega, x, y)^p dydx = \fint_{A_{x_0}(R_i,a_i,b_i)} c(\omega, x, y)^p dydx > C_M + \delta$$

which contradicts $\mathbf{BA}[x_0; p, C_M, n]$ of $c(\omega)$ if we pass to the limit $k_i, l_i \to \infty$. □

We now define the minimal radius that will guarantee that the averages of c^p are finite on sets $A_{x_0}(2^{-a}R, a, b)$ for all $R \geq R_{(10.1.6)}$, $a, b \in \mathbb{N}$, which is of technical importance for the rest of the chapter.

Definition 10.1.6. *Suppose $x_0 \in \mathbb{Z}^n$, $p, C_M \in \mathbb{R}$ and conductance c are such that c satisfies $\boldsymbol{BA}[x_0; p, C_M, n]$ $\mathbb{P}$-a.s. Define*

$$R_{(10.1.6)}(x_0, \omega, c, p, C_M) := R_{(10.1.5)}(x_0, \omega, c, \delta = 1, p, C_M) 2^{a_{(10.1.5)}(x_0,\omega,c,\delta=1,p,C_M)}.$$

Lemma 10.1.5 implies that $R_{(10.1.6)}$ is measurable and $\mathbb{P}$-a.s. finite.

10.2. Functional inequalities in ergodic environment

The next theorem, written in a slightly different way, appears in [FH20], see Lemma 28.

Lemma 10.2.1. *Let $x_0 \in \mathbb{Z}^n$ be arbitrary and suppose that conductance c on $\mathbb{Z}^n$ $\mathbb{P}$-a.s. satisfies $\boldsymbol{BA}[x_0; -q, C_M, n]$ for some $q, C_M \in (0, \infty)$. Then $\mathbb{P}$-a.s., for all $R \geq R_{(10.1.6)}(x_0, \omega, c, p = -q, C_M)$ and $\beta > 0$,*

$$\sum_{\substack{B(x_0,R)^2 \\ x\neq y}} \frac{c(x,y)^{-q}}{d(x,y)^{n-\beta}} \leq \left[\mathcal{B}_{(10.2.1)}(\beta, n)\right] (C_M + 1) R^{n+\beta}.$$

Function $\mathcal{B}_{(10.2.1)}$ *can be taken to be*

$$\mathcal{B}_{(10.2.1)}(\beta,n)=2^{3n}C_{VU}^2\sum_{l=0}^{\infty}(l+1)^n2^{-(l+1)\beta},$$

which explodes when β *goes to* 0.

Proof. Let us rewrite the sums using integration with respect to counting measure, $\int f(x)dx := \int f(x)\#(dx)=\sum_x f(x)$. For $R\geq R_{(10.1.6)}(x_0,\omega,c,p=-q,C_M)$ we compute

$$\begin{aligned}&\int_{\substack{x,y\in B(x_0,R)\\x\neq y}}\frac{c(x,y)^{-q}}{d(x,y)^{n-\beta}}dxdy\\&\quad\leq\sum_{l=0}^{\lfloor\log_2 R\rfloor}\int_{\substack{x,y\in B(x_0,R)\\2^{-(l+1)}R\leq d(x,y)<2^{-l}R}}c(x,y)^{-q}2^{(l+1)(n-\beta)}R^{n-\beta}dxdy\\&\quad\leq\sum_{l=0}^{\lfloor\log_2 R\rfloor}R^{\beta-n}2^{(l+1)(n-\beta)}\int_{A_{x_0}(\lceil 2^{-l}R\rceil,l+1,1)}c(x,y)^{-q}dxdy.\end{aligned}$$

The definition of $R_{(10.1.6)}$ guarantees that for every $l\geq 0$ either $2^{-l}R\geq R_{(10.1.5)}$ or $l\geq a_{(10.1.5)}$. It is therefore legitimate to use the estimate of Lemma 10.1.5. Moreover, when $l\leq\lfloor\log_2 R\rfloor$ we have $2^{-l}R\geq 1$ and we can estimate $\lceil 2^{-l}R\rceil\leq 2^{1-l}R$, which, together with the upper estimate of volume of set A_{x_0} from Proposition 10.1.4 in the second inequality, allows us to proceed with

$$\begin{aligned}\int_{\substack{x,y\in B(x_0,R)\\x\neq y}}\frac{c(x,y)^{-q}}{d(x,y)^{n-\beta}}dxdy&\leq(C_M+1)\sum_{l=0}^{\lfloor\log_2 R\rfloor}R^{\beta-n}2^{(l+1)(n-\beta)}\#\left[A_{x_0}(\lceil 2^{-l}R\rceil,l+1,1)\right]\\&\leq C_{VU}^2(C_M+1)\sum_{l=0}^{\lfloor\log_2 R\rfloor}R^{\beta-n}2^{(l+1)(n-\beta)}\lceil 2^{-l}R\rceil^{2n}2^{nl}(l+1)^n\\&\leq 2^{2n}C_{VU}^2(C_M+1)\sum_{l=0}^{\lfloor\log_2 R\rfloor}R^{\beta-n}2^{(l+1)(n-\beta)}R^{2n}2^{-ln}(l+1)^n\\&\leq 2^{3n}C_{VU}^2(C_M+1)R^{n+\beta}\sum_{l=0}^{\infty}(l+1)^n2^{-(l+1)\beta}.\end{aligned}$$

This proves the statement because the series in the last line is converging and we simply have to define $\mathcal{B}_{(10.2.1)}(\beta,n)$ like stated in the theorem. □

The following result is the same as one obtained in [FH20], where the variable r corresponds to variable p' used used below.

Theorem 10.2.2 (Embedding of random Sobolev-Slobodeckij spaces cf. [FH20] Section 3.4)**.** *Suppose* $x_0\in\mathbb{Z}^n$*,* $q,C_M>0$ *and conductance* c *on* $\mathbb{Z}^n$ *are such that* $\mathbb{P}$*-a.s.* c *satisfies* $\boldsymbol{BA}[x_0;-q,C_M,n]$*. Let* $1\leq p\leq\infty$ *and* $0<s'<s<1$ *be arbitrary and set* $p'=pq/(q+1)$*. Then for* $\mathbb{P}$*-a.e.* ω*, every* $R\geq R_{(10.1.6)}(x_0,\omega,c,p=-q,C_M)$ *and every measurable* $f:B(x_0,R)\to\mathbb{R}$*, writing for short* $B:=B(x_0,R)$*,*

$$\left(\sum_{x,y\in B}\frac{|f(x)-f(y)|^{p'}}{d(x,y)^{n+s'p'}}\right)^{\frac{1}{p'}}\leq(C_M+1)^{\frac{1}{qp}}\mathcal{B}_{10.2.1}((s-s')pq,n)^{\frac{1}{qp}}$$

$$\times R^{\frac{n}{qp}+(s-s')}\left(\sum_{x,y\in B}\frac{|f(x)-f(y)|^p}{d(x,y)^{n+sp}}c(x,y)\right)^{\frac{1}{p}}.$$

Proof. Fix $\omega \in \Omega$ such that $c(\omega)$ satisfies $\mathbf{BA}[x_0; -q, C_M, n]$ and $R \geq R_{(10.1.6)}(x_0, \omega, c, p = -q, C_M)$. Starting from the quantity on the left and applying Hölder inequality with exponents $(q+1)/q$ and $(q+1)$ gives

$$\sum_{x,y\in B} \frac{|f(x)-f(y)|^{p'}}{d(x,y)^{n+s'p'}} = \sum_{x,y\in B} \frac{|f(x)-f(y)|^{p'} c(x,y)^{\frac{q}{q+1}}}{d(x,y)^{n+s'p'}} c(x,y)^{-\frac{q}{q+1}}$$
$$\leq \left(\sum_{x,y\in B} \frac{|f(x)-f(y)|^{\frac{p'(q+1)}{q}} c(x,y)}{d(x,y)^{n+\frac{sp'(q+1)}{q}}} \right)^{\frac{q}{q+1}} \left(\sum_{x,y\in B} \frac{c(x,y)^{-q}}{d(x,y)^{n-(s-s')p'(q+1)}} \right)^{\frac{1}{q+1}}.$$

The second factor in the last expression can be estimated using Lemma 10.2.1, which require $\mathbf{BA}[x_0; -q, C_M, n]$ assumption, with $\beta = (s-s')p'(q+1) > 0$. Rising at the same time both sides to power $1/p'$ and recalling that $p'(q+1)/q = p$, we proceed with

$$\left(\sum_{x,y\in B} \frac{|f(x)-f(y)|^{p'}}{d(x,y)^{n+s'p'}} \right)^{\frac{1}{p'}} \leq \Big(\left[\mathcal{B}_{10.2.1}((s-s')pq, n)\right] (C_M+1) R^{n+(s-s')pq} \Big)^{\frac{1}{qp}}$$
$$\times \left(\sum_{x,y\in B} \frac{|f(x)-f(y)|^{p} c(x,y)}{d(x,y)^{n+sp}} \right)^{\frac{1}{p}}$$

whenever $R \geq R_{(10.1.6)}(x_0, \omega, c, p = -q, C_M)$. □

Remark 10.2.3. *From now on we will state our results in term of ergodic conductance for concreteness. However, they could be reformulated for conductance c satisfying appropriate **BA** conditions $\mathbb{P}$-a.s. without much problem.*

Theorem 10.2.4 (Ergodic Poincaré-Sobolev inequality)**.** *Suppose c is an ergodic or symmetrized ergodic conductance on $\mathbb{Z}^n$ such that $\mathbb{E}[c^{-q}] < \infty$ for some $q > 1$. Let $p \geq 1$, $s \in (0,1)$ and $\varepsilon > 0$ be arbitrary but such that $n > sp$, $q > \frac{n}{sp}$, $pq/(q+1) > 1$ and $\varepsilon < (\frac{sp}{n} - \frac{1}{q})$. Define $r \equiv r(\varepsilon, p, q, s, n)$ implicitly by $\frac{p}{r} := 1 - \frac{sp}{n} + \frac{1}{q} + \varepsilon$. Then, for every $x_0 \in \mathbb{Z}^n$, there is a non random $C_{PS}^{erg} \equiv C_{PS}^{erg}(n, s, p, q, \mathbb{E}[c^{-q}], \varepsilon)$ and a random $R_{(10.2.4)} \equiv R_{10.2.4}(x_0, \omega, c, n, s, p, q, \varepsilon)$ such that the following Poincaré-Sobolev inequality holds $\mathbb{P}$-a.s.: For all $R \geq R_{(10.2.4)}$, $f \in L^1(B)$, writing $B = B(x_0, R)$ for short,*

$$\|f - f_B\|_{L^r(B)}^p \leq C_{PS}^{erg} R^{n\left(\frac{1}{q}+\varepsilon\right)} \sum_{x,y\in B} \frac{|f(x)-f(y)|^p}{d(x,y)^{n+sp}} c(x,y) \tag{10.4}$$

where $f_B = (\#B)^{-1} \sum_{x\in B} f(x)$. Constant C_{PS}^{erg} can be taken to be

$$C_{PS}^{erg}(n, s, p, q, \mathbb{E}[c^{-q}], \varepsilon) := \left(1 + C_{(8.2.4)}\right)^{\frac{(1+n/sp)}{n/s}} \left(1 + \mathcal{B}_{(10.2.1)}(\varepsilon sp/2, n)\right)^{\frac{n}{sp^2}} \left(\mathbb{E}[c^{-q}] + 2\right)^{\frac{n}{sp}}.$$

Proof. The right hand side in of Ineq. (10.4) is symmetric in x, y and it doesn't change if the conductance c is symmetrized. Thus it is enough to prove the statement for ergodic c, which we will now do. By Proposition 10.1.2 we know that for every $q' \in (0, q)$ there is a $\mathbb{P}$-null set $N_{q'} \subset \Omega$ such that for all $\omega \in \Omega \setminus N_{q'}$ conductance $c(w)$ satisfies $\mathbf{BA}[x_0; -q', 1 + \mathbb{E}[c^{-q}], n]$ (note the estimate $\mathbb{E}[c^{-q'}] \leq 1 + \mathbb{E}[c^{-q}]$). Fix a $q' \in (0, q)$ such that $q' > \frac{n}{sp}$, $pq'/(q'+1) > 1$ and $\varepsilon + 1/q - 1/q' > \varepsilon/2$. This is possible because all inequalities are strict and assumed to hold for $q' = q$, leading to the following dependency $q' \equiv q'(n, s, p, q, \varepsilon)$.

Let us now fix $\omega \in \Omega \setminus N_{q'}$ for the rest of the proof. We define $p' := pq'/(q'+1)$, $\varepsilon' := \varepsilon + 1/q - 1/q'$, $s' := s - \varepsilon' n/p$ and notice that the choice of q', q and ε guarantees $p' > 1$, $\varepsilon' \in (\frac{\varepsilon}{2}, \frac{sp}{n} - \frac{1}{q'})$ and

$s' \in (0,s)$. We start with Sobolev-Poincaré inequality provided in Theorem 8.2.4 applied with n, p', s' and $r := \frac{np'}{n-s'p'}$ i.e. $\frac{p'}{r} = 1 - \frac{s'p'}{n}$:

$$\|f - f_B\|^{p'}_{L^r(B)} \le C_{(8.2.4)} \sum_{x,y\in B} \frac{|f(x)-f(y)|^{p'}}{d(x,y)^{n+s'p'}}$$

where $C_{(8.2.4)} \equiv C_{(8.2.4)}(n,p',s') \equiv C_{(8.2.4)}(n,s,p,q,\varepsilon)$. Taking the p'-th root of both sides, using Theorem 10.2.2 (with $q_{(10.2.2)} = q'$, $C_M = 1 + \mathbb{E}[c^{-q}]$ and all other variables matched by name) and renaming the constant popping out to C_1, we find that for every $R \ge R_{(10.1.6)}(x_0,\omega,c,p=-q',C_M = 1+\mathbb{E}[c^{-q}])$ one has

$$\|f - f_B\|_{L^r(B)} \le C_1(\mathbb{E}[c^{-q}]+2)^{\frac{1}{pq'}} R^{\frac{n}{q'p}+(s-s')} \left(\sum_{x,y\in B} \frac{|f(x)-f(y)|^p}{d(x,y)^{n+sp}} c(x,y)\right)^{\frac{1}{p}}.$$

Using $s - s' = \frac{\varepsilon' n}{p} > \frac{\varepsilon n}{2p}$, $p' = \frac{pq'}{q'+1}$, $q' > \frac{sp}{n}$ and the fact that function $\mathcal{B}_{(10.2.1)}(\cdot,\cdot)$ is decreasing in the first variable, we can estimate

$$C_1 = C^{\frac{1}{p'}}_{(8.2.4)} \left(\mathcal{B}_{(10.2.1)}((s-s')pq',n)\right)^{\frac{1}{pq'}} \le \left(1 + C_{(8.2.4)}\right)^{\frac{(1+n/sp)}{n/s}} \left(1 + \mathcal{B}_{(10.2.1)}(\varepsilon sp/2,n)\right)^{\frac{n}{sp^2}}$$

which which leads to $C_1 \equiv C_1(n,s,p,q,\varepsilon)$. Now rising both sides to power p results in

$$\|f - f_B\|^p_{L^r(B)} \le C_1^p(\mathbb{E}[c^{-q}]+2)^{\frac{n}{sp}} R^{\frac{n}{q'}+\varepsilon' n} \sum_{x,y\in B} \frac{|f(x)-f(y)|^p}{d(x,y)^{n+sp}} c(x,y).$$

Finally, we can define $C^{erg}_{PS} \equiv C^{erg}_{PS}(n,s,p,q,\mathbb{E}[c^{-q}],\varepsilon) := C_1^p(\mathbb{E}[c^{-q}]+2)^{\frac{n}{sp}}$, $R_{10.2.4} \equiv R_{10.2.4}(x_0,\omega,c,n,$ $s,p,q,\varepsilon) := R_{(10.1.6)}(x_0,\omega,c,p=-q',C_M = 1+\mathbb{E}[c^{-q}])$ and use the definition of ε' to get

$$\|f - f_B\|^p_{L^r(B)} \le C^{erg}_{PS} R^{n\left(\frac{1}{q}+\varepsilon\right)} \sum_{x,y\in B} \frac{|f(x)-f(y)|^p}{d(x,y)^{n+sp}} c(x,y)$$

which proves Ineq. (10.4). Here

$$\frac{1}{r} = \frac{1}{p'} - \frac{s'}{n} = \left(\frac{q'+1}{q'}\right)\frac{1}{p} - \frac{s}{n} + \frac{s-s'}{n} = \frac{1}{p} + \frac{1}{qp} - \frac{s}{n} + \frac{\varepsilon}{p}$$

or

$$\frac{p}{r} = 1 - \frac{sp}{n} + \frac{1}{q} + \varepsilon.$$

In order to get the increase of regularity we need that $\frac{p}{r} < 1$ which is equivalent to $\frac{1}{q} + \varepsilon < \frac{sp}{n}$. i.e. $q > \frac{n}{sp}$ and $\varepsilon < \frac{sp}{n} - \frac{1}{q}$ and this is exactly what we assume in the theorem. □

Theorem 10.2.5 (Functional inequalities). *Let $x_0 \in \mathbb{Z}^n$, $n \ge 2$, $s \in (0,1)$, $\varepsilon \in (0, (\frac{2s}{n} - \frac{1}{q}))$, $q > \frac{n}{2s}$ be arbitrary and let c be an ergodic or symmetrized ergodic conductance on $\mathbb{Z}^n$ such that $\mathbb{E}[c^{-q}] < \infty$. Let us also shorten $R_{(10.2.5)} \equiv R_{(10.2.5)}(x_0,\omega,c,n,s,q,\varepsilon) := R_{(10.2.4)}(x_0,\omega,c,n,s,p=2,q,\varepsilon)$ and define ρ^{erg} by $\frac{1}{\rho^{erg}} = 1 - \frac{2s}{n} + \frac{1}{q} + \varepsilon$. Then it is possible to find non random $C^{erg}_P \equiv C^{erg}_P(n,s,q,\mathbb{E}[c^{-q}],\varepsilon)$, $C^{erg}_{S1} \equiv C^{erg}_{S1}(n,s,q,\mathbb{E}[c^{-q}],\varepsilon)$ and $C^{erg}_{S2} \equiv C^{erg}_{S2}(\emptyset)$ such that $\mathbb{P}$-a.s. $\mathcal{E}$ satisfies*

$$\mathbf{PI}\left[x_0, [R_{(10.2.5)},\infty); s, C^{erg}_P\right] \tag{10.5}$$

and, for every $\zeta \in [2,\rho^{erg}]$, also

$$\mathbf{SI}\left[x_0, [R_{(10.2.5)},\infty); s, \rho^{erg}, \zeta, C^{erg}_{S1}, C^{erg}_{S2}, \gamma = 0\right]. \tag{10.6}$$

Proof. Poincaré and Sobolev inequalities follow by combining Poincaré-Sobolev inequality, for $p = 2$, in Ineq. (10.4) with Theorem 8.1.7. Choosing $p = 2$ in Theorem 10.2.4, Ineq. (10.4) holds $\mathbb{P}$-a.s. for $C_{PS}^{erg} \equiv C_{PS}^{erg}(n, s, p = 2, q, \mathbb{E}[c^{-q}], \varepsilon)$, r defined by $\frac{2}{r} := 1 - \frac{2s}{n} + \frac{1}{q} + \varepsilon < 1$ and every $R \geq R_{(10.2.4)}$. Theorem 8.1.7 (with $p = 2$) now gives

$$\|f - f_B\|_{L^2(B)}^2 \leq C_{VU}^{\frac{2s}{n} - \frac{1}{q} - \varepsilon} C_{PS}^{erg} R^{2s} \mathcal{E}_B(f) \tag{10.7}$$

and for every $\zeta \in [1, r/2]$

$$\|f\|_{L^r(B)}^2 \leq 4 C_{PS}^{erg} R^{n\left(\frac{1}{q} + \varepsilon\right)} \mathcal{E}_B(f) + 4|B|^{1 - \frac{2s}{n} + \frac{1}{q} + \varepsilon - \frac{2}{2\zeta}} \|f\|_{L^{2\zeta}(B)}^2$$

where $f \in L^1(B)$ is arbitrary. Using $\|f\|_{L^{2\beta}}^2 = \|f^2\|_{L^\beta(B)}$ (for $\beta > 0$) leads us to

$$\|f^2\|_{L^{\frac{r}{2}}(B)} \leq 4 C_{PS}^{erg} R^{n\left(\frac{1}{q} + \varepsilon\right)} \mathcal{E}_B(f) + 4|B|^{\frac{1}{\rho^{erg}} - \frac{1}{\zeta}} \|f^2\|_{L^\zeta(B)}. \tag{10.8}$$

Ineq. (10.7) shows that $\mathbf{PI}\big[x_0, [R_{(10.2.5)}, \infty); s, C_P^{erg}\big]$ is satisfied $\mathbb{P}$-a.s. with $C_P^{erg} \equiv C_P^{erg}(n, s, q, \mathbb{E}[c^{-q}], \varepsilon) := C_{VU}^{\frac{2s}{n} - \frac{1}{q} - \varepsilon} C_{PS}^{erg}$. On the other hand, Ineq. (10.8) remains true if the domain on the left hand side is reduced to σB for any $\sigma \in (0, 1)$ so $\mathbf{SI}\big[x_0, [R_{(10.2.5)}, \infty); s, \rho^{erg}, \zeta, C_{S1}^{erg}, C_{S2}^{erg}(\zeta), \gamma = 0\big]$ holds $\mathbb{P}$-a.s. with constants $\rho^{erg} \equiv \rho^{erg}(n, s, q, \varepsilon) := r/2$, $C_{S1}^{erg} \equiv C_{S1}^{erg}(n, s, q, \mathbb{E}[c^{-q}], \varepsilon) := 4 C_{PS}^{erg} C_{VL}^{\frac{2}{r} - 1}$, $C_{S2}^{erg} \equiv C_{S2}^{erg} := 4$. This completes the proof. □

10.3. Energy density of cutoff functions

Theorem 10.3.1. *Suppose $x_0 \in \mathbb{Z}^n$, $Q \geq 1$, $C_M < \infty$ and a random conductance c on $\mathbb{Z}^n$ are such that $\mathbb{P}$-a.s. c satisfies $\boldsymbol{BA}[x_0; Q, C_M, n]$. Then $\mathbb{P}$-a.s. for all $R \geq R_{(10.1.6)}(x_0, \omega, c, p = Q, C_M)$ and Lipschitz function $\varphi : \mathbb{Z}^n \to [0, 1]$, with $\xi := R \operatorname{Lip} \varphi$ and $B := B(x_0, R)$,*

$$\sum_{x \in B} \sum_{y \in \mathbb{Z}^n} \frac{(\varphi(x) - \varphi(y))^2 c(x, y)^Q}{d(x, y)^{d + 2s}} \leq C_{(10.3.1)} (C_M + 1) \xi^{2s} (\log_2(\xi^{-1}) \vee \log_2 \xi + 1)^n R^{n - 2s}$$

where $C_{(10.3.1)} \equiv C_{(10.3.1)}(n, s)$ can be taken to be

$$C_{(10.3.1)} := 2 \cdot 2^{5n} C_{VU}^2 \left(\sum_{k=0}^{\infty} 2^{-2(k+1)(1-s)} (k+1)^n + \sum_{k=1}^{\infty} 2^{-2(k-1)s} (k+1)^n \right).$$

Proof. We will prove the theorem pointwise in ω and all statements should be understood in the $\mathbb{P}$-a.e. sense. $\varphi = 0$ would make the right hand side of the inequality zero but this is alright because the left side is also zero in that case. Let us assume that $\varphi \neq 0$ so $\xi = R \operatorname{Lip} \varphi > 0$. The argument below will require introduction of an additional summation, which is why we prefer to write the original summations as integrals over counting measure $\#$, i.e. denote $\int_x f(x) dx := \int_x f(x) \#(dx) = \sum_x f(x)$. The proof consist of estimating the sum/integral on the left side of stated inequality,

$$I := \int_B \int_{\mathbb{Z}^n} \frac{(\varphi(x) - \varphi(y))^2 c(x, y)^Q}{d(x, y)^{d + 2s}} dy dx,$$

but the computation is slightly different for $\xi \geq 1$ and $\xi < 1$. However, we will squash these two cases together by adding integral terms which might have trivial integration areas in one of the cases. We split I into four integrals which are to be estimated separately. Let us denote the integrand in I by $\mathscr{A}$ for brevity, only for the next line.

$$I = \int_B \int_{d(x,y) < \frac{R}{\xi} \wedge R} \mathscr{A} + \int_B \int_{\frac{R}{\xi} \leq d(x,y) < R} \mathscr{A} + \int_B \int_{R \leq d(x,y) < \frac{R}{\xi}} \mathscr{A} + \int_B \int_{\frac{R}{\xi} \vee R \leq d(x,y)} \mathscr{A} =:$$
$$=: I_1 + I_2 + I_3 + I_4.$$

The computations for I_1, I_2, I_3, I_4 that follow are very similar except in the choice of sets $A_{x_0}(\cdot,\cdot,\cdot)$ (For sets $A(R,a,b)$ see Definition 10.1.3) and the choice of the bound for $(\varphi(x)-\varphi(y))^2$ term. The first step is always to decompose the integral in dyadic way and then use sets $A_{x_0}(R,a,b)$ in combination with Lemma 10.1.5 to estimate every summand in the decomposition. One has to keep in mind that this is only possible when at lest one of the variables R, a, b is larger than its lower bound from Lemma 10.1.5.

Making an example of I_1, we can, without the loss of generality (by replacing ξ with 1), assume that $\xi \geq 1$ so that $\frac{R}{\xi} \wedge R = \frac{R}{\xi}$. Using bound $|\varphi(x)-\varphi(y)| \leq d(x,y)\xi/R$ we decompose

$$\begin{aligned} I_1 &\leq \sum_{k=0}^{\lfloor \log_2(R/\xi)\rfloor} \int_B \int_{2^{-k-1}R/\xi \leq d(x,y) < 2^{-k}R/\xi} \frac{(\xi d(x,y)/R)^2\, c(x,y)^Q}{d(x,y)^{n+2s}} dydx \\ &= \left(\frac{R}{\xi}\right)^{-(n+2s)} \sum_{k=0}^{\lfloor \log_2(R/\xi)\rfloor} 2^{(k+1)(n-2(1-s))} \int_{A(\lceil 2^{-k}R/\xi\rceil, k+\lceil \log_2 \xi\rceil+1,1)} c(x,y)^Q dxdy. \end{aligned}$$

where the summation range is limited to $k \leq \lfloor \log_2(R/\xi)\rfloor$ because $B(x_0,1)\setminus\{x_0\} = \emptyset$. Due to assumption $R \geq R_{(10.1.6)}$ either $2^{-k}R/\xi \geq R_{(10.1.5)}$ or $k+\log_2\xi+1 \geq a_{(10.1.5)}$. Therefore Lemma 10.1.5 applies and proves

$$(\forall k \in \mathbb{N}_0) \qquad \fint_{A(\lceil 2^{-k}R/\xi\rceil, k+\lceil \log_2 \xi\rceil+1,1)} c(x,y)^Q dxdy \leq C_M + 1. \tag{10.9}$$

Additionally, the upper bound on k implies that $2^{-k}R/\xi \geq 1$, which allows us to estimate $\lceil 2^{-k}R/\xi\rceil \leq 2^{-k}R/\xi + 1 \leq 2^{1-k}R/\xi$, and $\log_2\xi \geq 0$ implies that

$$k + \lceil \log_2 \xi\rceil + 1 \leq 2(k+1)(\log_2\xi + 1) \qquad (\text{using } a+b \leq 2ab \quad \forall a,b \geq 1).$$

Keeping this in mind, Proposition 10.1.4 allows us to bound

$$\begin{aligned} \#A(\lceil 2^{-k}R/\xi\rceil, k+\lceil \log_2\xi\rceil+1,1) &\leq C_{VU}^2 \left(\lceil 2^{-k}R/\xi\rceil^2 (k+\lceil \log_2\xi\rceil+1)\right)^n 2^{(k+\lceil \log_2\xi\rceil)n} \\ &\leq 2^{4n} C_{VU}^2 R^{2n}\xi^{-n}(k+1)^n(\log_2\xi+1)^n 2^{-nk}. \end{aligned}$$

Thus we can now estimate

$$\begin{aligned} I_1 &\leq \left(\frac{R}{\xi}\right)^{-(n+2s)} \sum_{k=0}^{\lfloor \log_2(R/\xi)\rfloor} 2^{(k+1)(n-2(1-s))}(C_M+1)\#A_{x_0}\left(\lceil 2^{-k}R/\xi\rceil, k+\lceil \log_2\xi\rceil+1,1\right) \\ &\leq 2^{5n}C_{VU}^2(C_M+1)\xi^{2s}(\log_2\xi+1)^n R^{n-2s}\sum_{k=0}^{\infty} 2^{-2(k+1)(1-s)}(k+1)^n \\ &\leq C_1(C_M+1)\xi^{2s}(\log_2(\xi^{-1}) \vee \log_2\xi + 1)^n R^{n-2s}. \end{aligned}$$

In the last line we used that the sum converges to define

$$C_1 \equiv C_1(n,s) := 2^{5n}C_{VU}^2 \sum_{k=0}^{\infty} 2^{-2(k+1)(1-s)}(k+1)^n < \infty.$$

The term $\log_2(\xi^{-1}) \vee \log_2\xi$ is added to account for the case when $\xi < 1$. Then the whole computation goes through by replacing ξ with 1 but the final estimate is also valid for the original ξ.

I_2 is bounded in a similar way, but using $|\varphi(x)-\varphi(y)| \leq 1$ instead of $|\varphi(x)-\varphi(y)| \leq \frac{\xi}{R}d(x,y)$. Notice also that we might again assume that $\xi \geq 1$ because otherwise $I_2 = 0$. Hence

$$\begin{aligned} I_2 &\leq \sum_{k=1}^{\infty} \int_B \int_{2^{k-1}R/\xi \leq d(x,y) < 2^k R/\xi} \frac{c(x,y)^Q}{d(x,y)^{n+2s}} dxdy \\ &\leq (R/\xi)^{-(n+2s)} \sum_{k=1}^{\infty} 2^{-(k-1)(n+2s)} \int_{A(\lceil R/\xi\rceil, \lceil \log_2\xi\rceil+1,k+1)} c(x,y)^Q dxdy. \end{aligned}$$

The volume of the set A can again be estimated using Proposition 10.1.4 together with $\lceil \log_2 \xi \rceil + 1 \leq 2(\log_2 \xi + 1)$, which results in

$$\begin{aligned}\#A(\lceil R/\xi \rceil^2, \lceil \log_2 \xi \rceil + 1, k+1) &\leq C_{VU}^2 \left(\lceil R/\xi \rceil^2 \left(\lceil \log_2 \xi \rceil + 1 \right) (k+1) \right)^n 2^{n \lceil \log_2 \xi \rceil + nk} \\ &\leq 2^{4n} C_{VU}^2 \xi^{-n} (\log_2 \xi + 1)^n R^{2n} (k+1)^n 2^{nk}.\end{aligned}$$

As either $R/\xi \geq R_{(10.1.5)}$ or $\log_2 \xi + 1 \geq a_{(10.1.5)}$, by Lemma 10.1.5 we have

$$\begin{aligned}I_2 &\leq 2^{4n} C_{VU}^2 (C_M + 1) \left(\frac{R}{\xi} \right)^{-(n+2s)} \sum_{k=1}^{\infty} 2^{-(k-1)(n+2s)} \xi^{-n} (\log_2 \xi + 1)^n R^{2n} (k+1)^n 2^{nk} \\ &\leq 2^{5n} C_{VU}^2 (C_M + 1) \xi^{2s} (\log_2 \xi + 1)^n R^{n-2s} \sum_{k=1}^{\infty} 2^{-2(k-1)s} (k+1)^n \\ &\leq C_2 (C_M + 1) \xi^{2s} (\log_2(\xi^{-1}) \vee \log_2 \xi + 1)^n R^{n-2s},\end{aligned}$$

where the sum in the second to last line is again converges and

$$C_2 \equiv C_2(n,s) := 2^{5n} C_{VU}^2 \sum_{k=1}^{\infty} 2^{-2(k-1)s} (k+1)^n < \infty.$$

For I_3 we may assume $\xi < 1$ because otherwise $I_3 = 0$ and then the same procedure gives

$$\begin{aligned}I_3 &\leq \sum_{k=0}^{\lfloor \log_2(\xi^{-1}) \rfloor} \int_B \int_{2^{-k-1} R/\xi \leq d(x,y) < 2^{-k} R/\xi} \frac{(\xi d(x,y)/R)^2 \, c(x,y)^Q}{d(x,y)^{n+2s}} dy dx \\ &= \left(\frac{R}{\xi} \right)^{-(n+2s)} \sum_{k=0}^{\lfloor \log_2(\xi^{-1}) \rfloor} 2^{(k+1)(n-2(1-s))} \int_{A(\lceil R \rceil, 1, \lceil \log_2(\xi^{-1}) \rceil - k + 1)} c(x,y)^Q dx dy.\end{aligned}$$

This time $\#A(\lceil R \rceil, 1, \lceil \log_2(\xi^{-1}) \rceil - k + 1) \leq 2^{4n} C_{VU}^2 R^{2n} \xi^{-n} (\log_2(\xi^{-1}) + 1)^n 2^{-nk} (k+1)^n$ which means that Lemma 10.1.5 implies

$$\begin{aligned}I_3 &\leq 2^{5n} C_{VU}^2 (C_M + 1) \xi^{2s} (\log_2(\xi^{-1}) + 1)^n R^{n-2s} \sum_{k=0}^{\infty} 2^{-2(k+1)(1-s)} (k+1)^n \\ &\leq C_3 (C_M + 1) \xi^{2s} (\log_2(\xi^{-1}) \vee \log_2 \xi + 1)^n R^{n-2s}\end{aligned}$$

where $C_3 \equiv C_3(n,s) := 2^{5n} C_{VU}^2 \sum_{k=0}^{\infty} 2^{-2(k+1)(1-s)} (k+1)^n$.

Finally we arrive at I_4. Here we can, without the loss of generality, assume $\xi < 1$ (otherwise we replace ξ with 1 in what follows). In the same way we estimate

$$\begin{aligned}I_4 &\leq \sum_{k=1}^{\infty} \int_B \int_{2^{k-1} R/\xi \leq d(x,y) < 2^k R/\xi} \frac{c(x,y)^Q}{d(x,y)^{n+2s}} dx dy \\ &\leq \left(\frac{R}{\xi} \right)^{-(n+2s)} \sum_{k=1}^{\infty} 2^{-(k-1)(n+2s)} \int_{A(\lceil R \rceil, 1, \lceil \log_2(\xi^{-1}) \rceil + k + 1)} c(x,y)^Q dx dy.\end{aligned}$$

The volume of the set A is bounded by

$$\#A(\lceil R \rceil, 1, \lceil \log_2(\xi^{-1}) \rceil + k + 1) \leq 2^{4n} C_{VU}^2 R^{2n} \xi^{-n} (\log_2(\xi^{-1}) + 1)^n (k+1)^n 2^{nk}$$

so

$$\begin{aligned}I_4 &\leq 2^{5n} C_{VU}^2 (C_M + 1) \xi^{2s} (\log_2(\xi^{-1}) + 1)^n R^{n-2s} \sum_{k=1}^{\infty} 2^{-2(k-1)s} (k+1)^n \\ &\leq C_4 (C_M + 1) \xi^{2s} (\log_2(\xi^{-1}) \vee \log_2 \xi + 1)^n R^{n-2s},\end{aligned}$$

where $C_4 \equiv C_4(n,s) := 2^{5n} C_{VU}^2 \sum_{k=1}^{\infty} 2^{-2(k-1)s}(k+1)^n$.

Collecting the computations for I_1, I_2, I_3 and I_4 we finally get

$$I_1 \le (C_1 + C_2 + C_3 + C_4)(C_M + 1)\xi^{2s}(\log_2(\xi^{-1}) \vee \log_2 \xi + 1)^n R^{n-2s}$$

which proves the claim with

$$\begin{aligned} C_{(10.3.1)} \equiv C_{(10.3.1)}(n,s) &:= C_1 + C_2 + C_3 + C_4 \\ &= 2 \cdot 2^{5n} C_{VU}^2 \left(\sum_{k=0}^{\infty} 2^{-2(k+1)(1-s)}(k+1)^n + \sum_{k=1}^{\infty} 2^{-2(k-1)s}(k+1)^n \right) \end{aligned}$$

because, as it turns out, $C_1 = C_3$ and $C_2 = C_4$. □

Lemma 10.3.2. *Suppose $x \in \mathbb{Z}^n$, $R > 0$, $s \in (0,1)$, $Q \ge 1$, $\mathcal{H} < \infty$, conductance c on $\mathbb{Z}^n$ and a Lipschitz function $\varphi : \mathbb{Z}^n \to [0,1]$, with $\xi = R \operatorname{Lip} \varphi$, are such that*

$$\sum_{x \in B} \sum_{y \in \mathbb{Z}^n} \frac{(\varphi(x) - \varphi(y))^2}{d(x,y)^{n+2s}} c^Q(x,y) \le \mathcal{H} \xi^{2s} R^{n-2s},$$

where we shortened $B := B(x_0, R)$. Then, recalling $C_{(2.7.6)} \equiv C_{(2.7.6)}(s,n)$,

$$\left(\sum_{x \in B} (\Gamma\varphi(x))^Q dx \right)^{\frac{1}{Q}} \le C_{(2.7.6)}^{\frac{1}{Q^*}} \mathcal{H}^{\frac{1}{Q}} \xi^{2s} R^{\frac{n}{Q} - 2s}.$$

Proof. Let us again replace the summation in the statement with integrals over the counting measure #, $\int_x f(x)dx := \int_x f(x)\#(dx) = \sum_x f(x)$. Take $u \in L^{Q^*}(B)$, where Q^* is the conjugated exponent of Q i.e. $\frac{1}{Q} + \frac{1}{Q^*} = 1$, and compute using Hölder inequality

$$\begin{aligned} \int_B u(x)\Gamma\varphi(x)dx &= \int_B \int_{\mathbb{Z}^n} u(x) \frac{(\varphi(x) - \varphi(y))^2}{d(x,y)^{n+2s}} c(x,y) dy dx \\ &\le \left(\int_B u^{Q^*}(x) \int_{\mathbb{Z}^n} \frac{(\varphi(x) - \varphi(y))^2}{d(x,y)^{n+2s}} dy dx \right)^{1/Q^*} \\ &\quad \times \left(\int_B \int_{\mathbb{Z}^n} \frac{(\varphi(x) - \varphi(y))^2}{d(x,y)^{n+2s}} c(x,y)^Q dy dx \right)^{1/Q}. \end{aligned}$$

According to Corollary 2.7.6, the first factor can be estimated by

$$\left(\int_B u^{Q^*}(x) \int_{\mathbb{Z}^n} \frac{(\varphi(x) - \varphi(y))^2}{d(x,y)^{n+2s}} dy dx \right)^{1/Q^*} \le \left(C_{2.7.6} \xi^{2s} R^{-2s} \int_B u^{Q^*}(x) dx \right)^{1/Q^*}$$

and the second factor is immediately estimated by the assumption of this lemma. Combining these two gives

$$\begin{aligned} \int_B u(x)\Gamma\varphi(x)dx &\le C_{(2.7.6)}^{\frac{1}{Q^*}} \xi^{\frac{2s}{Q^*}} R^{\frac{-2s}{Q^*}} \left(\int_B u^{Q^*}(x)dx \right)^{1/Q^*} \mathcal{H}^{\frac{1}{Q}} \xi^{\frac{2s}{Q}} R^{\frac{n-2s}{Q}} \\ &\le C_{(2.7.6)}^{\frac{1}{Q^*}} \mathcal{H}^{\frac{1}{Q}} \xi^{2s} R^{\frac{n}{Q} - 2s} \|u\|_{L^{Q^*}(B)}. \end{aligned}$$

Since $u \in L^{Q^*}(B)$ was arbitrary, the duality between L^Q and L^{Q^*} proves the theorem. □

Theorem 10.3.3. *Suppose $x_0 \in \mathbb{Z}^n$, $Q \geq 1$ and a symmetrized ergodic conductance c on $\mathbb{Z}^n$ are such that $\mathbb{E}[c^{Q'}] < \infty$ for whatever $Q' > Q$. Let us shorten $R_{(10.3.3)} \equiv R_{(10.3.3)}(x_0, \omega, c, Q) := R_{(10.1.6)}(x_0, \omega, c, p = Q, C_M = \mathbb{E}[c^Q])$. Then it is possible to find a non random $C_C^{erg}(\gamma) \equiv C_C^{erg}(\gamma, n, s, Q, \mathbb{E}[c^Q])$, for every $\gamma \in (0, 2s)$, such that symmetric form $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies*

$$\mathbf{CE}\left[x_0, [R_{(10.3.3)}, \infty); s, Q, \gamma, C_C^{erg}(\gamma)\right].$$

A little bit stronger statement is true: for all $R \geq R_{(10.3.3)}$ and Lipschitz function $\varphi : \mathbb{Z}^n \to [0,1]$ with $\xi := R \operatorname{Lip} \varphi$, shortening $B := B(x_0, R)$,

$$\left(\int_B (\Gamma\varphi(x))^Q dx\right)^{\frac{1}{Q}} \leq C_{(2.7.6)}^{\frac{1}{Q^*}} C_{(10.3.1)}^{\frac{1}{Q}} (\mathbb{E}[2c^Q + 1])^{\frac{1}{Q}} \xi^{2s} R^{\frac{n}{Q} - 2s} \left(\log_2(\xi^{-1}) \vee \log_2 \xi + 1\right)^{\frac{n}{Q}}.$$

Proof. By Proposition 10.1.2 we know that $\mathbb{P}$-a.s. c satisfies $\mathbf{BA}[x_0; Q, 2\mathbb{E}[c^Q], n]$ so Theorem 10.3.1 produces a constant $C_{(10.3.1)} \equiv C_{(10.3.1)}(n, s)$ such that, for every $R \geq R_{(10.3.3)}$,

$$\sum_{x \in B} \sum_{y \in \mathbb{Z}^n} \frac{(\varphi(x) - \varphi(y))^2}{d(x,y)^{d+2s}} c(x,y)^Q \leq C_{(10.3.1)} \mathbb{E}[2c^Q + 1] \xi^{2s} (\log_2(\xi^{-1}) \vee \log_2 \xi + 1)^n R^{n-2s}.$$

Following this up with an application of Lemma 10.3.2 results in

$$\left(\int_B (\Gamma\varphi(x))^Q dx\right)^{\frac{1}{Q}} \leq C_{(2.7.6)}^{\frac{1}{Q^*}} \left[C_{(10.3.1)} \mathbb{E}[2c^Q + 1] (\log_2(\xi^{-1}) \vee \log_2 \xi + 1)^n\right]^{\frac{1}{Q}} \xi^{2s} R^{\frac{n}{Q} - 2s}$$

and proves the inequality. Furthermore, for every $\gamma \in (0, 2s)$, it is possible to find a $C_\gamma \equiv C_\gamma(\gamma, n)$ such that

$$(\log_2 a + 1)^n \leq C_\gamma a^\gamma \quad \forall a \in [1, \infty]$$

so we can estimate

$$\left(\fint_B (\Gamma\varphi(x))^Q dx\right)^{\frac{1}{Q}} \leq C_{VL}^{-\frac{1}{Q}} C_{(2.7.6)}^{\frac{1}{Q^*}} \left(C_{(10.3.1)} C_\gamma \mathbb{E}[2c^Q + 1]\right)^{\frac{1}{Q}} \left(\xi^{2s-\gamma} \vee \xi^{2s+\gamma}\right) R^{-2s}$$

This is now equivalent, by definition, to the statement in the theorem if we take $C_C^{erg}(\gamma) \equiv C_C^{erg}(\gamma, n, s, Q, \mathbb{E}[c^Q]) := C_{VL}^{-\frac{1}{Q}} C_{(2.7.6)}^{\frac{1}{Q^*}} \left(C_{(10.3.1)} C_\gamma \mathbb{E}[2c^Q + 1]\right)^{\frac{1}{Q}}$. □

10.4. Weak Harnack inequality and Hölder regularity

In this section we present the weak parabolic Harnack inequality and Hölder regularity obtained by applying the results of Part I for almost every realization of symmetrized ergodic conductance.

Theorem 10.4.1. *Suppose $q, Q \geq 1$ and a symmetrized ergodic conductance c on $\mathbb{Z}^n$ ($n \geq 2$) are such that*

$$\frac{1}{q} + \frac{1}{Q} < \frac{2s}{n} \tag{10.10}$$

and $\mathbb{E}[c^{-q}] + \mathbb{E}[c^Q] < \infty$. Then for every $x_0 \in \mathbb{Z}^n$ there exist a $\mathbb{P}$-a.s. finite random variable $R_{(10.4.1)} \equiv R_{(10.4.1)}(x_0, \omega, c, n, s, q, Q)$ and non random $C_{PH}^{erg} \equiv C_{PH}^{erg}(s, n, q, Q, \mathbb{E}[c^{-q}], \mathbb{E}[c^Q]) < \infty$, $\eta^{erg} \equiv \eta^{erg}(s, n, q, Q, \mathbb{E}[c^{-q}], \mathbb{E}[c^Q]) \in (0,1)$, $C_H^{erg} \equiv C_H^{erg}(s)$ such that form $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies

$$\boldsymbol{WPHI}[x_0, [R_{(10.4.1)}, \infty); s, C_{PH}^{erg}, Q],$$

$$\boldsymbol{WEHI}[x_0, [R_{(10.4.1)}, \infty); C_{EH} = C_{PH}^{erg}]$$

and

$$\boldsymbol{HR}[x_0, [R_{(10.4.1)}, \infty); \eta^{erg}, C_H^{erg}].$$

Proof. We fix an arbitrary $x_0 \in \mathbb{Z}^n$ and work pointwise for $\mathbb{P}$-a.e. $\omega \in \Omega$. We will use results from Chapter 6 on (M, d, m) with $M = \mathbb{Z}^n$, d the Euclidean distance and $m = \#$ the counting measure. First of all, notice that Lemma 2.6.1 implies that $(\mathbb{Z}^n, d, \#)$ satisfies

$$\mathbf{V}[x_0, [1, \infty), n, C_{VL}, C_{VU}].$$

Because of the strict inequality in Ineq. (10.10), we can find $\varepsilon \in (0, (\frac{2s}{n} - \frac{1}{q}))$ small enough such that for $q' := (1/q + \varepsilon)^{-1}$, $Q' := (1/Q + \varepsilon)^{-1}$ we also have

$$\frac{1}{q'} + \frac{1}{Q'} = \frac{1}{q} + \frac{1}{Q} + 2\varepsilon < \frac{2s}{n}.$$

To be explicit, we take $\varepsilon = (\frac{2s}{n} - \frac{1}{q} - \frac{1}{Q})/4$ which results in $\varepsilon \equiv \varepsilon(n, s, q, Q)$, $q' \equiv q'(n, s, q, Q)$ and $Q' \equiv Q'(n, s, q, Q)$. Because of $q > \frac{n}{2s}$ and $\varepsilon \in (0, \frac{2s}{n} - \frac{1}{q})$, we can use Theorem 10.2.5 to find random variable $R_{(10.2.5)} \equiv R_{(10.2.5)}(x_0, \omega, c, n, s, q, Q)$ and non random $C_P^{erg} \equiv C_P^{erg}(n, s, q', \mathbb{E}[c^{-q}], \varepsilon)$, $C_{S1}^{erg} \equiv C_{S1}^{erg}(n, s, q', \mathbb{E}[c^{-q}], \varepsilon)$, $C_{S2}^{erg} = 4$, $\rho^{erg} = \left(1 - \frac{2s}{n} + \frac{1}{q'}\right)^{-1}$ such that, for every $\zeta \in [2, \rho^{erg}]$, form $\mathcal{E}$ satisfies

$$\mathbf{PI}\left[x_0, [R_{(10.2.5)}, \infty); s, C_P^{erg}\right],$$

$$\mathbf{SI}\left[x_0, [R_{(10.2.5)}, \infty); s, \rho^{erg}, \zeta, C_{S1}^{erg}, C_{S2}^{erg}, \gamma = 0\right].$$

Note that the last line then also trivially holds for every $\gamma \in (0, 2s)$. By Theorem 10.3.3 (with roles of Q and Q' reversed), due to $Q' < Q$ and $\mathbb{E}[c^Q] < \infty$, it is possible to find a random variable $R_{(10.3.3)} \equiv R_{(10.3.3)}(x_0, \omega, c, Q')$ and, for every $\gamma \in (0, 2s)$, a non random $C_C^{erg}(\gamma) \equiv C_C^{erg}(\gamma, n, s, Q', \mathbb{E}[1 + c^Q])$ such that $\mathcal{E}$ satisfies

$$\mathbf{CE}\left[x_0, [R_{(10.3.3)}, \infty); Q', \gamma, C_C^{erg}(\gamma)\right].$$

Let us now specify $\zeta = (Q')^*$, $\gamma = s$, $R_{(10.4.1)} \equiv R_{(10.4.1)}(x_0, \omega, c, n, s, q, Q) := 1 \vee R_{(10.2.5)} \vee R_{(10.3.3)}$ and notice that Chapter 6, with exponents q' and Q' in place of q and Q, applies to $\mathcal{E}$ in $B(x_0, R)$ for all $R \geq R_{(10.4.1)}$. Theorem 6.5.1 then proves that there is a constant C_{PH}^{erg} such that $\mathcal{E}$ satisfies

$$\mathbf{WPHI}[x_0, [R_{(10.4.1)}, \infty); s, C_{PH}^{erg}, Q],$$

with the following dependence of the parameter

$$C_{PH}^{erg} \equiv C_{PH}\left(s, n, q', Q', C_C^{erg}, C_{S1}^{erg}, C_{S2}^{erg}, C_P^{erg}, \gamma, C_{VL}^{erg}, C_{VU}^{erg}\right),$$

which reduces to

$$C_{PH}^{erg} \equiv C_{PH}\left(s, n, q, Q, \mathbb{E}[c^{-q}], \mathbb{E}[c^Q]\right).$$

By Theorem 6.5.2 this also proves $\mathbf{WEHI}[x_0, [R_{(10.4.1)}, \infty); C_{EH} = C_{PH}^{erg}]$ for $\mathcal{E}$. Furthermore, Theorem 6.6.3 implies that there exist a non random $\eta^{erg} \equiv \eta^{erg}$ and $C_H^{erg} = 12 \vee 2^{1+1/s}$ such that $\mathcal{E}$ satisfies $\mathbf{HR}[x_0, [R_{(10.4.1)}, \infty); \eta^{erg}, C_H^{erg}]$, with dependence

$$\eta^{erg} \equiv \eta^{erg}(C_{PH}, C_C^{erg}, s, n, C_{VL}, C_{VU}) > 0,$$

which reduces to

$$\eta^{erg} \equiv \eta^{erg}(s, n, q, Q, \mathbb{E}[c^Q], \mathbb{E}[c^{-q}]).$$

□

10.5. Exit time estimates

We present the expected exit time estimate, survival estimate and conservativeness of $\mathcal{E}$ obtained by applying the results of Part I for almost every realization of symmetrized ergodic conductance.

Lemma 10.5.1. *Let c be a symmetrized ergodic conductance on $\mathbb{Z}^n$ such that $\mathbb{E}[c^{-q}] < \infty$ and $\mathbb{E}[c] < \infty$ for some $q > 0$. Then $\mathbb{P}$-a.s., for every ball $B \subset \mathbb{Z}^n$, $\|G^B 1\|_{L^\infty(\mathbb{Z}^n)} < \infty$.*

Proof. Let us fix some ball $B \subset \mathbb{Z}^n$. $\mathbb{E}[c^{-q}]$ implies that $c(x, y) > 0$ $\mathbb{P}$-a.s. for every pair $x, y \in \mathbb{Z}^n$. There are only countably many such pairs so it is possible to find a $\mathbb{P}$-null set $N \in \mathcal{F}$, such that $c(\omega, x, y) > 0$ for all $\omega \in \Omega \setminus N$ and all $x, y \in \mathbb{Z}^n$. But this implies that for all $\omega \in \Omega \setminus N$ and $x \in B$

$$\sum_{y \in \mathbb{Z}^n \setminus B} \frac{c(\omega, x, y)}{d(x, y)^{n+2s}} > 0.$$

Since the form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is $\mathbb{P}$-a.s. is regular and Dirichlet by Corollary 9.2.2, the statement now follows from Lemma 9.2.3. □

Theorem 10.5.2. *Suppose c is a symmetrized ergodic conductance on $\mathbb{Z}^n$ such that $\mathbb{E}[c^{-q}] < \infty$ for whatever $q > 1$. Then, for every $x_0 \in \mathbb{Z}^n$, there exist a $\mathbb{P}$-a.s. finite random variable $R_{(10.5.2)} \equiv R_{(10.5.2)}(x_0, \omega, c, \mathbb{E}[c^{-1}])$ and a non random $C_K^{erg} \equiv C_K^{erg}(n, s, \mathbb{E}[c^{-1}]) > 0$ such that $\mathbb{P}$-a.s. for every $R \geq R_0$ there exist $y_0 \equiv y_0(R, x_0, \omega, c) \in M \setminus B(x_0, 6R)$ with the property*

$$\fint_{B(x_0,R)} \int_{B(y_0,R)} k(x, y) dy dx \geq C_K^{erg} R^{-2s},$$

where the integration is with respect to the counting measure. In other words, form $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies

$$\mathbf{AKB}{\geq}[x_0, [R_{(10.5.2)}, \infty); s, C_K^{erg}].$$

Proof. Fix an arbitrary $x_0 \in \mathbb{Z}^n$. By Proposition 10.1.2, c $\mathbb{P}$-a.s. satisfies $\mathbf{BA}[x_0; -1, 2\mathbb{E}[c^{-1}], n]$ so in particular

$$\begin{aligned}
&\limsup_{k\to\infty} \fint_{B(x_0,k)} \fint_{B(x_0,k)} c(x, y)^{-1} dy dx \\
&\quad \leq \limsup_{k\to\infty} \frac{B(x_0, 2k)}{B(x_0, k)} \fint_{B(x_0,2k)} \frac{B(x, 2k)}{B(x_0, k)} \fint_{B(x,2k)} c(x, y)^{-1} dy dx \leq 2^{2n+1} C_{VL}^{-2} C_{VU}^2 \mathbb{E}[2c^{-1}].
\end{aligned}$$

Hence, using $\chi(A)$ to denote a characteristic function of the set $A \subset \Omega$, the random variable

$$\begin{aligned}
&R_{(10.5.2)}(x_0, \omega, c, \mathbb{E}[c^{-1}]) \\
&\quad := \sup_{R\in\mathbb{N}} \left[R \cdot \chi\left(\left\{ \fint_{B(x_0,R)} \fint_{B(x_0,R)} c(x, y)^{-1} dy dx > 2^{2n+1} C_{VL}^{-2} C_{VU}^2 \mathbb{E}[2c^{-1} + 1] \right\} \right) \right] + 2
\end{aligned}$$

is $\mathbb{P}$-a.s. finite. Therefore $\mathbb{P}$-a.s. for every $R \geq R_{(10.5.2)}$, denoting $B := B(x_0, R)$,

$$\fint_{8B} \fint_{8B} c(x, y)^{-1} dy dx \leq 2^{2n+1} C_{VL}^{-2} C_{VU}^2 \mathbb{E}\left[2c^{-1} + 1\right].$$

Taking any $y \in B(x_0, 7R) \setminus B(x_0, 6R)$ (this is not an empty subset of $\mathbb{Z}^n$ because $R > 1$) we can increase the area of integration to estimate

$$\begin{aligned}
\fint_{B} \fint_{B(y_0,R)} c(x, y)^{-1} dy dx &\leq 2^{2n+1} C_{VL}^{-2} C_{VU}^2 \frac{|8B|^2}{|B||B(y_0, R)|} \fint_{8B} \fint_{8B} c(x, y)^{-1} dy dx \\
&\leq 2^{2n+1} C_{VL}^{-2} C_{VU}^2 \frac{|8B|^2}{|B||B(y_0, R)|} \mathbb{E}\left[2c^{-1} + 1\right] \leq 2^{8n+1} C_{VU}^4 C_{VL}^{-4} \mathbb{E}\left[2c^{-1} + 1\right].
\end{aligned}$$

On the other hand, Jensen's inequality implies that

$$\left(\fint_{B\times B(y_0,R)} c(x,y)dxdy\right)^{-1} \leq \fint_{B\times B(y_0,R)} c(x,y)^{-1}dxdy$$

so the last two inequalities combined prove

$$\fint_{B\times B(y_0,R)} c(x,y)dxdy \geq \left(\fint_{B\times B(y_0,r)} c(x,y)^{-1}\right)^{-1} \geq 2^{-8n-1}C_{VL}^4 C_{VU}^{-4}\mathbb{E}\left[2c^{-1}+1\right]^{-1}.$$

Since $d(x,y)\leq 9R$ for all $x\in B, y\in B(y_0,R)$, we can use

$$k(x,y)=\frac{c(x,y)}{d(x,y)^{n+2s}} \geq 9^{-(n+2s)}R^{-(n+2s)}c(x,y)$$

to estimate

$$\begin{aligned}\int_{B\times B(y_0,R)} k(x,y)dxdy &\geq 9^{-(n+2s)}R^{-(n+2s)}\int_{B\times B(y_0,R)} c(x,y)dxdy\\ &\geq 9^{-(n+2s)}R^{-(n+2s)}2^{-8n-1}C_{VL}^4 C_{VU}^{-4}\mathbb{E}\left[2c^{-1}+1\right]^{-1}|B||B(y_0,R)|\\ &\geq 9^{-(n+2s)}2^{-8n-1}C_{VL}^5 C_{VU}^{-4}\mathbb{E}\left[2c^{-1}+1\right]^{-1}|B|R^{-2s}\end{aligned}$$

Dividing both sides with $|B|$ and defining

$$C_K^{erg}\equiv C_K^{erg}(n,s,\mathbb{E}[c^{-1}]) := 9^{-(n+2s)}2^{-8n-1}C_{VL}^5 C_{VU}^{-4}\mathbb{E}\left[2c^{-1}+1\right]^{-1}$$

leads to the claimed $\mathbb{P}$-a.s. statement and finishes the proof. □

Theorem 10.5.3. *Suppose symmetrized ergodic conductance c on $\mathbb{Z}^n$ ($n\geq 2$) is such that $\mathbb{E}[c^{-q}]+\mathbb{E}[c^Q]<\infty$, for some $q,Q\geq 1$ satisfying $\frac{1}{q}+\frac{1}{Q}<\frac{2s}{n}$. Then, for every $x_0\in\mathbb{Z}^n$, there exist a $\mathbb{P}$-a.s. finite random variable $R_{(10.5.3)}\equiv R_{(10.5.3)}(x_0,\omega,c,q,Q)$ and non random $C_{(E\leq)}^{erg}\equiv C_{(E\leq)}^{erg}(s,n,q,Q,\mathbb{E}[c^Q],\mathbb{E}[c^{-q}])<\infty$, $C_{(E\geq)}^{erg}\equiv C_{(E\geq)}^{erg}(s,n,q,Q,\mathbb{E}[c^Q],\mathbb{E}[c^{-q}])>0$, $\varepsilon^{erg}\equiv\varepsilon^{erg}(s,n,q,Q,\mathbb{E}[c^{-q}],\mathbb{E}[c^Q])$ and $\delta^{erg}\equiv\delta^{erg}(s,n,q,Q,\mathbb{E}[c^{-q}],\mathbb{E}[c^Q])$ such that symmetric form $\mathcal{E}$ on $(\mathbb{Z}^n,d,\mu)$ satisfies*

$$\mathbf{ETE}\left[x_0,[R_{(10.5.3)},\infty);s,C_{(E\leq)}^{erg},C_{(E\geq)}^{erg}\right]$$

and

$$\boldsymbol{SE}\left[x_0,[R_{(10.5.3)},\infty);s,\delta^{erg},\varepsilon^{erg}\right].$$

In particular, the semigroup corresponding to $\mathcal{E}$ is $\mathbb{P}$-a.s. conservative.

Proof. Observe that $\mathbb{Z}^n$ with the Euclidean distance and counting measure # is a locally compact, separable metric space and # has full support which verifies Assumption 2.5.3. Furthermore, $(\mathcal{E},\mathcal{D}[\mathcal{E}])$ is a $\mathbb{P}$-a.s. a regular Dirichlet form satisfying Assumption 4.0.2 by Corollary 9.2.2. Let us fix an $x_0\in\mathbb{Z}^n$. We will prove the theorem pointwise for $\mathbb{P}$-a.s. ω. By Theorem 10.4.1 we know that there exist a random variable $R_{(10.4.1)}\equiv R_{(10.4.1)}(x_0,\omega,c,q,Q)$ and a non random $C_{PH}^{erg}\equiv C_{PH}^{erg}(s,n,q,Q,\mathbb{E}[c^{-q}],\mathbb{E}[c^Q])$ such that $\mathcal{E}$ satisfies

$$\mathbf{WEHI}\left[x_0,[R_{(10.4.1)},\infty);C_{PH}^{erg}\right].$$

Furthermore, Lemma 10.5.1 shows that $\|G^B 1\|_{L^\infty(B)}<\infty$ $\mathbb{P}$-a.s. Because of $q>1$ and $\mathbb{E}[c^{-q}]<\infty$, Theorem 10.5.2 shows that there is a random variable $R_{(10.5.2)}\equiv R_{(10.5.2)}(x_0,\omega,c,\mathbb{E}[c^{-1}])$ and a non random $C_K^{erg}\equiv C_K^{erg}(n,s,\mathbb{E}[c^{-q}])>0$ (having a look in the proof it is not hard to see that $\mathbb{E}[c^{-1}]$ dependency can be replaced with $\mathbb{E}[c^{-q}]$) such that symmetric kernel k satisfies

$$\mathbf{AKB}\geq\left[x_0,[R_{(10.5.2)},\infty);s,C_K^{erg}\right].$$

By Theorem 10.3.3 we also know that there exists a random variable $R_{(10.3.3)} \equiv R_{(10.3.3)}(x_0, \omega, c, Q, \mathbb{E}[c^Q])$ and a non random $C_C^{erg} \equiv C_C^{erg}(\gamma, n, s, Q, \mathbb{E}[c^Q])$ such that for every $\gamma \in (0, 2s)$ $\mathcal{E}$ satisfies

$$\mathbf{CE}\left[x_0, [R_{(10.3.3)}, \infty); Q, \gamma = s, C_C^{erg}(\gamma)\right].$$

Finally, we also know that counting measure # on $\mathbb{Z}^n$ satisfies

$$\mathbf{V}\left[x_0, [1, \infty), n, C_{VL}, C_{VU}\right]$$

by Lemma 2.6.1. These conditions (along with Assumption 2.5.3 and Assumption 4.0.2 verified in the beginning) are sufficient for Theorem 7.1.5 to apply and conclude that $\mathcal{E}$ satisfies

$$\mathbf{ETE}\left[x_0, [2R_{(10.4.1)} \vee R_{(10.5.2)} \vee R_{(10.3.3)}, \infty); s, C_{(E\leq)}^{erg}, C_{(E\geq)}^{erg}\right]$$

with $C_{(E\geq)}^{erg} \equiv C_{(E\geq)}^{erg}(C_{EH}, C_C^{erg}, \gamma, s, n, C_{VL}, C_{VU})$ and $C_{(E\leq)}^{erg} \equiv C_{(E\leq)}^{erg}(C_{EH}, C_K) > 0$. If we choose $\gamma = s$, dependencies change to $C_{(E\geq)}^{erg} \equiv C_{(E\geq)}^{erg}(s, n, q, Q, \mathbb{E}[c^Q], \mathbb{E}[c^{-q}])$ and $C_{(E\leq)}^{erg} \equiv C_{(E\leq)}^{erg}(s, n, q, Q, \mathbb{E}[c^Q], \mathbb{E}[c^{-q}]) > 0$. The first statement now follows by defining $R_{(10.5.3)} \equiv R_{(10.5.3)}(x_0, \omega, c, q, Q) := 2R_{(10.4.1)} \vee R_{(10.5.2)} \vee R_{(10.3.3)}$. Moreover, using Theorem 7.2.1 we can find constants $\varepsilon^{erg} \equiv \varepsilon^{erg}(C_{(E\geq)}^{erg}, C_{(E\leq)}^{erg})$ and $\delta^{erg} \equiv \delta^{erg}(C_{(E\geq)}^{erg}, s)$ such that $\mathcal{E}$ satisfies

$$\mathbf{SE}\left[x_0, [R_{(10.5.3)}, \infty); s, \delta^{erg}, \varepsilon^{erg}\right].$$

In the current case dependencies transform into $\varepsilon^{erg} \equiv \varepsilon^{erg}(s, n, q, Q, \mathbb{E}[c^{-q}], \mathbb{E}[c^Q])$ and $\delta^{erg} \equiv \delta^{erg}(s, n, q, Q, \mathbb{E}[c^{-q}], \mathbb{E}[c^Q])$ which proves the second statement. Finally, the conservativeness follows from the last property through Theorem 7.2.2. □

11. i.i.d. conductance

In this chapter we will study the regular Dirichlet form $\mathcal{E}$, with its maximal domain, corresponding to the i.i.d. conductance introduced in Definition 9.1.4 on $L^2(\mathbb{Z}^n, \#)$. Similar to Chapter 10, the main results of this chapter are Theorem 11.7.1, which gives the weak parabolic Harnack inequality (**WPHI**) and large scale Hölder regularity (**HR**), and Theorem 11.8.1, which gives the expected exit time estimate (**ETE**), survival estimate (**SE**), conservativeness of $\mathcal{E}$ and also the short time estimate on the restricted semigroup P_t^B. In contrast to Chapter 10, the results of Theorem 11.7.1 and Theorem 11.8.1 hold with uniform constants in the vicinity of the point $x_\star$. To be more precise, there is a $\theta \in (0, 1)$ such that, for every fixed $x_\star \in \mathbb{Z}^n$, all the results hold in balls $B(x_0, R)$ whenever $R \geq (|x_0 - x_\star| \vee R_\star)^\theta$, where $R_\star$ is a minimal radius depending on the realization of c (see Definition 3.2.1 for more details). For the sake of this exposition, let us say that a statement *fails in* $\star(\theta, R_\star)$*-way* if it fails to hold in at least one of the balls described in the previous sentence. The improvement we mentioned is possible because in the i.i.d. case we are able to estimate the probability that **SI**, **PI**, **CE** or **TB** fails in $\star(\theta, R_\star)$-way. If these probabilites are summable in $R_\star$ for some choice of θ, then Borel-Cantelli lemma can be used to prove the existence of the minimal radius $R_\star < \infty$.

In Sections 11.2 and 11.3, respectively, we estimate the probability that Sobolev or Poincaré inequality fails in $\star(\theta, R_\star)$-way. To do so, we couple the conductance c with a family $\{\xi_{xy} : x, y \in \mathbb{Z}^n\}$ of symmetric i.i.d. Bernoulli random variables such that $c(x, y) \geq \nu \xi_{xy}$, for some $\nu > 0$ and all $x, y \in \mathbb{Z}^n$. Then the bilinear form corresponding to the conductance $\nu\xi$, call it $\widehat{\mathcal{E}}$, is dominated by $\mathcal{E}$, that is, for every $f \in L^2(\mathbb{Z}^n)$

$$\widehat{\mathcal{E}}(f) \leq \mathcal{E}(f).$$

Because of this, it is enough to estimate the probability that Sobolev or Poincaré inequality with $\mathcal{E}$ replaced by $\widehat{\mathcal{E}}$ fails in $\star(\theta, R_\star)$-way. The fact that ξ_{xy} are Bernoulli random variable allows us to use Chernoff's bound (see Theorem 11.1.2) and prove that these probabilities decay exponentially in $R_\star$. The result then follows from Borel-Cantelli lemma and does not require an assumption on the moments of c^{-1}. On the other hand, in Sections 11.5 and 11.6, we make use of Rosenthal's inequality (see Theorem 11.1.3) to estimate the probability that **CE** or **TB** fail in $\star(\theta, R_\star)$-way under the assumption that certain positive moment of c is finite. For appropriate choice of $\theta \in (0, 1)$, these probabilities turn out to be summable in $R_\star$ and **CE** and **TB** are then proved by another application of Borel-Cantelli lemma. Finally, an application of method from Part I results in Theorems 11.7.1 and 11.8.1 and the moment condition boils down to

$$p > \frac{n+1}{s}.$$

Assumption 11.0.1. *In this chapter we will consider an i.i.d. conductance c which is allowed to zero but not $\mathbb{P}$-a.s. (which is the trivial case)*

Lemma 11.0.2. *Since all $c(x, y) \neq 0$ have the same distribution, we can find numbers $\mathfrak{p} > 0$ and $\nu > 0$ such that $\mathbb{P}(c(x, y) \geq \nu) \geq \mathfrak{p}$.*

Proof. By assumption $\mathbb{P}(c(x, y) \neq 0) > 0$, so we can take ν to be the median of $c(x, y)$ on the set $\{c(x, y) > 0\}$. Defining $\mathfrak{p} := \mathbb{P}(c(x, y) \geq \nu) > 0$ proves the claim. □

Definition 11.0.3. *For the rest of this chapter, $\mathfrak{p}$ and ν will denote any choice of numbers which existence is claimed in previous lemma. We use ν to define a family of i.i.d. random variables*

$$\xi_{xy} := 1_{\{c(x,y) \geq \nu\}}, \quad x, y \in \mathbb{Z}^n.$$

They are all Bernoulli distributed with parameter $\mathfrak{p}$.

11.1. Basic estimates

Most of the proofs in this chapter rely heavily on the first Borel-Cantelli lemma (see [Dur10], Chapter 2.3).

Theorem 11.1.1 (Borel-Cantelli lemma). *Let $E_1, E_2 \dots$ be a sequence of events in $(\Omega, \mathcal{F}, \mathbb{P})$. Then*

$$\sum_{i=1}^{\infty} \mathbb{P}(E_i) < \infty \implies \mathbb{P}(\limsup_{i\to\infty} E_i) = 0$$

where $\limsup_{i\to\infty} E_i = \bigcap_{i\geq 1}\bigcup_{j\geq i} E_i$ is the subset of Ω where infinitely many events E_i happen.

To estimate the tails of different sums of ξ_{xy}, we will use Chernoff bound.

Theorem 11.1.2 (Chernoff's bound). *Let X be a binomial random variable. Then for every $\delta \in (0,1)$*

$$\mathbb{P}\left(X \leq (1-\delta)\mathbb{E}[X]\right) \leq e^{-\frac{\delta^2 \mathbb{E}[X]}{2}}.$$

Proof. Suppose X is binomial with parameters $m \in \mathbb{N}, p \in [0,1]$ and set $\mu = \mathbb{E}[X] = mp$. Taking any $t < 0$, by Markov inequality for e^{tX}, we find

$$\mathbb{P}\left(X \leq (1-\delta)\mu\right) \leq \frac{\mathbb{E}[e^{tX}]}{e^{t(1-\delta)\mu}} = \frac{\left(pe^t + (1-p)\right)^m}{e^{t(1-\delta)mp}}.$$

Estimating the numerator using $p(e^t - 1) + 1 \leq \exp(p(e^t-1))$ and specifying $t = \log(1-\delta) < 0$ gives

$$\mathbb{P}\left(X \leq (1-\delta)\mu\right) \leq \frac{\exp(-\delta\mu)}{(1-\delta)^{(1-\delta)\mu}} = \exp(-\mu(\delta + (1-\delta)\log(1-\delta))).$$

To complete the proof it suffices to check that $\delta + (1-\delta)\log(1-\delta) \geq \delta^2/2$ for $\delta \in (0,1)$. Setting $x = -\log(1-\delta) > 0$ this is equivalent to $1 - e^{-x} - xe^{-x} - (1-e^{-x})^2/2 \geq 0$ and reduces to

$$\frac{e^{2x}}{2} - xe^x - \frac{1}{2} = \sum_{i=2}^{\infty} \frac{(2^{i-1} - i)x^i}{i!} > 0.$$

□

When the summands are not Bernoulli distributed we rely on Rosenthal's inequality to obtain the deviation estimate. Here is the statement of the Rosenthal inequality paraphrased from from Theorem 3 of [Ros70]:

Theorem 11.1.3 (Rosenthal's inequality). *Let $w \geq 2$. There exists a constant $C_R \equiv C_R(w)$ such that for every $N \in \mathbb{N}$ and every sequence X_i, $i = 1, \dots, N$ of independent random variables with finite absolute w moments and $\mathbb{E}[X_i] = 0$*

$$\mathbb{E}[|S_N|^w] \leq C_R(w) \max\left(\left(\mathbb{E}\left[S_N^2\right]\right)^{\frac{w}{2}}, \sum_{i=1}^{N} \mathbb{E}[|X_i|^w]\right), \tag{11.1}$$

where $S_N = \sum_{i=1}^N X_i$.

We will use this inequality to calculate the probability that S_N deviates from its mean. The idea is similar to the one used in the proof of the weak law of large numbers except that it uses moments higher than the second one.

Lemma 11.1.4 (Deviation estimate)**.** *Let $w \geq 2$, $N \in \mathbb{N}$ be arbitrary and let X_i, $i = 1, \dots, N$, be a sequence of i.i.d. random variables with finite absolute w moment. Then for every $\delta > 0$*

$$\mathbb{P}\left(\sum_{i=1}^{N} \frac{X_i - \mathbb{E}[X_i]}{N} > \delta\right) \leq 2^w C_R(w) \delta^{-w} N^{-\frac{w}{2}} \mathbb{E}[|X_1|^w]. \tag{11.2}$$

Proof. We start by estimating the probability using Markov inequality and then applying Rosenthal's inequality (Theorem 11.1.3) to a centered sequence $\frac{X_i - \mathbb{E}[X_i]}{N}$ of independent random variables. This results in

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{N} \frac{X_i - \mathbb{E}[X_i]}{N} > \delta\right) &\leq \delta^{-w} \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N} X_i - \mathbb{E}[X_i]\right|^w\right] \\ &\leq C_R(w)\delta^{-w} \max\left(\left(\sum_{i=1}^{N} \frac{\operatorname{Var}(X_i)}{N^2}\right)^{\frac{w}{2}}, \sum_{i=1}^{N} \mathbb{E}\left[\left|\frac{X_i - \mathbb{E}[X_i]}{N}\right|^w\right]\right) \\ &\leq C_R(w)\delta^{-w} \max\left(\frac{\operatorname{Var}(X_1)^{\frac{w}{2}}}{N^{\frac{w}{2}}}, \frac{\mathbb{E}\left[|X_1 - \mathbb{E}[X_1]|^w\right]}{N^{(w-1)}}\right).\end{aligned}$$

The centered moment can be estimated by the normal one with the same trick used in Ineq. (8.2) as

$$\mathbb{E}[|X_1 - \mathbb{E}[X_1]|^w] \leq 2^w \mathbb{E}[|X_1|^w].$$

This, in turn, gives

$$\mathbb{P}\left(\sum_{i=1}^{N} \frac{X_i - \mathbb{E}[X_i]}{N} > \delta\right) \leq C_R(w)\delta^{-w} \max\left(\frac{\mathbb{E}[X_1^2]^{\frac{w}{2}}}{N^{\frac{w}{2}}}, 2^w \frac{\mathbb{E}\left[|X_1|^w\right]}{N^{(w-1)}}\right).$$

Taking into account that $w > 2$, on one hand $N^{-(w-1)} \leq N^{-\frac{w}{2}}$ and on the other Jensen's inequality allows for the estimate

$$\mathbb{E}[X_1^2]^{\frac{w}{2}} \leq \mathbb{E}[|X_1|^w]$$

because $w/2 > 1$ makes the power convex. The last two observations lead to

$$\mathbb{P}\left(\sum_{i=1}^{N} \frac{X_i - \mathbb{E}[X_i]}{N} > \delta\right) \leq 2^w C_R(w)\delta^{-w} N^{-\frac{w}{2}} \mathbb{E}[|X_1|^w]$$

and complete the proof. □

Lemma 11.1.5. *Let c be a non-zero i.i.d. conductance on $\mathbb{Z}^n$. Then $\mathbb{P}$-a.s. $\|G^B 1\|_{L^\infty(\mathbb{Z}^n)} < \infty$ for every ball $B \subset \mathbb{Z}^n$.*

Proof. Let a ball $B \subset \mathbb{Z}^n$ be arbitrary. For fixed $x \in B$ we have

$$\mathbb{P}\left(\sum_{y \in \mathbb{Z}^n \setminus B} \frac{c(x,y)}{d(x,y)^{n+2s}} = 0\right) \leq \lim_{L \to \infty} \prod_{y \in B(x,L) \setminus B} \mathbb{P}(\xi_{xy} = 0) \leq \lim_{L \to \infty} (1 - \mathfrak{p})^{|B(x,L) \setminus B|} = 0.$$

As there are only finitely many $x \in B$, this implies that $\mathbb{P}$-a.s. for all $x \in B$

$$\sum_{y \in \mathbb{Z}^n} \frac{c(x,y)}{d(x,y)^{n+2s}} > 0$$

and the claim follows from Corollary 9.2.2 and Lemma 9.2.3. □

11.2. Sobolev inequality

We will prove Assumption 8.1.1 using the independence of $\{\xi_{xy}\}_{y\in\mathbb{Z}^n}$ for fixed $x \in \mathbb{Z}^n$.

Theorem 11.2.1. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$, $\mathfrak{p}, \nu, \xi_{xy}$ like in Definition 11.0.3 and $p \geq 1$ arbitrary. There is a family $\{\lambda(x) \equiv \lambda(x, \omega, c, n, s, p, \mathfrak{p}, \nu)\}_{x\in\mathbb{Z}^n}$ of i.i.d. random variables on Ω such that, for every $E \subset \mathbb{Z}^n$, $|E| < \infty$ and every $x \in E$, $\mathbb{P}$-a.s.*

$$\int_{\mathbb{Z}^n\setminus E} k(x,y)dy \geq \lambda(x)|E|^{-sp/n}. \tag{11.3}$$

In addition, $\lambda(x) > 0$ $\mathbb{P}$-a.s., there exist $\zeta_{(11.2.1)} \equiv \zeta_{(11.2.1)}(n, s, p) > 0$ and $C_{(11.2.1)} \equiv C_{(11.2.1)}(\mathfrak{p}, n, s, q, p) > 0$ such that

$$\forall \zeta \in (-\infty, \zeta_{(11.2.1)}) \qquad \mathbb{E}\left[\exp\left(\zeta\lambda(x)^{-\frac{n}{n+sp}}\right)\right] < \infty$$

and for all $q \in \mathbb{R}$

$$\mathbb{E}[\lambda(x)^{-q}] < C_{(11.2.1)}\nu^{-q}.$$

Proof. Let us fix an arbitrary unit vector e_i in $\mathbb{Z}^n$ and define, for an $x \in \mathbb{Z}^n$,

$$H^+_{e_i}(x) = \{y \in \mathbb{Z}^n : y \cdot e_i > x \cdot e_i\}.$$

We will construct $\lambda(x)$ from random variables $\{\xi_{xy}\}_{y\in H^+_{e_i}(x)}$ which will automatically make $\{\lambda(x)\}_{x\in\mathbb{Z}^n}$ independent since sets $\{\xi_{xy} : y \in H^+_{e_i}(x)\}_{x\in\mathbb{Z}^n}$ are disjoint. The symmetry $\xi_{xy} = \xi_{yx}$ prevents us from using $\mathbb{Z}^n$ instead of $H^+_{e_i}(x)$ because ξ_{xy} would be involved in constructions of both $\lambda(x)$ and $\lambda(y)$ and these variables would not be independent.

Moving on to the construction, we fix x in $\mathbb{Z}^n$ and $\omega \in \Omega$ and find the largest number $\mathfrak{c} \equiv \mathfrak{c}(n) > 1$ such that

$$C_{VL}(\mathbb{Z}^n)\mathfrak{c}^n - C_{VU}(\mathbb{Z}^n) - C_{VU}(\mathbb{Z}^{n-1})\mathfrak{c}^{n-1} = 2. \tag{11.4}$$

This is possible because the above expression is continuous in $\mathfrak{c}$, $C_{VL} \leq C_{VU}$ implies that it is non-positive for $\mathfrak{c} = 1$ and it tends to $+\infty$ when $\mathfrak{c}$ goes to $+\infty$. Define now a sequence of annuli

$$A_l = B(x, \mathfrak{c}^{l+1}) \setminus B(x, \mathfrak{c}^l)$$

and two sequences of half annuli

$$A^+_l = A_l \cap H^+_{e_i}(x), \qquad A^-_l := A_l \cap H^+_{-e_i}(x).$$

We claim that $|A^+_l| := \#A^+_l \geq \mathfrak{c}^{nl}$ for every $l \in \mathbb{N}$. Indeed,

$$A_l = A^+_l + A^-_l + A_l \cap \{y : y_i \cdot e_i = x \cdot e_i\}$$

so using volume regularity of both $\mathbb{Z}^n$ and $\mathbb{Z}^{n-1}$ (note that $\mathfrak{c} \geq 1$) we know that

$$|A_l| \geq |B(x, \mathfrak{c}^{l+1})| - |B(x, \mathfrak{c}^l)| \geq C_{VL}(\mathbb{Z}^n)\mathfrak{c}^{(l+1)n} - C_{VU}(\mathbb{Z}^n)\mathfrak{c}^{ln}$$

and

$$|A_l \cap \{y : y_i \cdot e_i = x \cdot e_i\}| \leq |B(x, \mathfrak{c}^{l+1}) \cap \{y : y_i \cdot e_i = x \cdot e_i\}| \leq C_{VU}(\mathbb{Z}^{n-1})\mathfrak{c}^{(l+1)(n-1)}.$$

By the symmetry of $\mathbb{Z}^n$ under reflection over plane $\{y, y \cdot e_i = x \cdot e_i\}$, we deduce that $|A^+_l| = |A^-_l|$. This together with the previous estimate implies that

$$\begin{aligned}
|A^+_l| &= \frac{|A_l| - |A_l \cap \{y : y \cdot e_i = x \cdot e_i\}|}{2} \\
&\geq \frac{C_{VL}(\mathbb{Z}^n)\mathfrak{c}^{(l+1)n} - C_{VU}(\mathbb{Z}^n)\mathfrak{c}^{ln} - C_{VU}(\mathbb{Z}^{n-1})\mathfrak{c}^{(l+1)(n-1)}}{2} \\
&\geq \frac{C_{VL}(\mathbb{Z}^n)\mathfrak{c}^n - C_{VU}(\mathbb{Z}^n) - C_{VU}(\mathbb{Z}^{n-1})\mathfrak{c}^{n-1}}{2}\mathfrak{c}^{ln} = \mathfrak{c}^{ln}
\end{aligned}$$

where we have intentionally estimated $\mathfrak{c}^{(l+1)(n-1)} \leq \mathfrak{c}^{(l+1)n-1}$ and used Eq. (11.4) in the last line.

Let us now find the smallest $l_0 \equiv l_0(x, \omega, c, \mathfrak{p}, \nu) \in \mathbb{N} \cup \{\infty\}$ such that

$$|\{y \in A_l^+ : \xi_{xy} = 1\}| \geq \frac{\mathfrak{p}}{2}|A_l^+| \qquad \forall l \geq l_0.$$

We will show, using Borel-Cantelli lemma, that such l_0 is finite $\mathbb{P}$-almost surely. To do so, we estimate the tail of binomial random variable $X := \sum_{y \in A_l^+} \xi_{xy}$ using the Chernoff bound from Theorem 11.1.2,

$$\mathbb{P}(X \leq (1-\delta)\mathbb{E}X) \leq e^{-\frac{\delta^2 \mathbb{E}X}{2}},$$

to get (note that $\mathbb{E}X = \mathfrak{p}|A_l^+|$)

$$\begin{aligned}\mathbb{P}\left(\sum_{y \in A_l^+} \xi_{xy} \leq \frac{\mathfrak{p}}{2}|A_l^+|\right) &= \mathbb{P}\left(\sum_{y \in A_l^+} \xi_{xy} \leq \left(1 - \frac{1}{2}\right)\mathfrak{p}|A_l^+|\right) \\ &\leq e^{-\frac{(1/2)^2|A_l^+|\mathfrak{p}}{2}} \leq e^{-\frac{\mathfrak{p}}{8}\mathfrak{c}^{nl}}.\end{aligned}$$

This is summable in l so Borel-Cantelli lemma implies that

$$\mathbb{P}\left(\limsup_{l \to \infty}\left\{\sum_{y \in A_l^+} \xi_{xy} \leq \frac{\mathfrak{p}}{2}|A_l^+|\right\}\right) = 0$$

which means that l_0 is finite $\mathbb{P}$-a.s. It is measurable because, for every $l \in \mathbb{N}_0$,

$$\{l_0 > l\} = \bigcup_{i \geq l}\left\{\sum_{y \in A_i^+} \xi_{xy} < \frac{\mathfrak{p}}{2}|A_i^+|\right\} \in \mathcal{F}$$

and its distribution is independent of x since the family $\{\xi_{xy}\}_{y \in H_{e_i}^+(x)}$ is i.i.d. Bernoulli with parameter independent of x. Using this l_0 we now define random variables

$$\lambda(x) \equiv \lambda(x, \omega, c, s, n, \mathfrak{p}, \nu) = \nu\left(\frac{\mathfrak{p}}{4}\right)^{(n+sp)/n} \mathfrak{c}^{-(l_0+2)(n+sp)}$$

and observe that $\lambda(x) > 0$ $\mathbb{P}$-a.s. because l_0 is $\mathbb{P}$-a.s. finite.

To verify Ineq. (11.3), we fix a set E and find the smallest $l_1 \in \mathbb{N}_0$ such that $|E| \leq \frac{\mathfrak{p}}{4}\mathfrak{c}^{l_1 n}$, i.e. $l_1 = \lceil \frac{1}{n}\log_{\mathfrak{c}}\left(\frac{4|E|}{\mathfrak{p}}\right)\rceil$. Notice that $l_1 > 0$ because (by special property of the counting measure) $E \neq \emptyset$ implies $|E| \geq 1$. With l_1 defined in this way, $|E| \leq \frac{\mathfrak{p}}{4}\mathfrak{c}^{(l_0+l_1)n} \leq \frac{\mathfrak{p}}{4}|A_{l_0+l_1}^+|$ and we can calculate

$$\begin{aligned}\int_{\mathbb{Z}^n \setminus E} k(x,y)dy \geq \int_{A_{l_0+l_1}^+ \setminus E} k(x,y)dy &\geq \int_{\{y \in A_{l_0+l_1}^+ : \xi_{xy}=1\} \setminus E} \frac{c(x,y)}{(\mathfrak{c}^{l_0+l_1+1})^{n+sp}}dy \\ &\geq \nu\mathfrak{c}^{-(l_0+1)(n+sp)}\mathfrak{c}^{-l_1(n+sp)}|\{y \in A_{l_0+l_1}^+ : \xi_{xy} = 1\} \setminus E| \\ &\geq \nu\mathfrak{c}^{-(l_0+1)(n+sp)}\mathfrak{c}^{-l_1(n+sp)}|E|,\end{aligned}$$

where for the last line one needs to take into account that, by definition of l_0,

$$|\{y \in A_{l_0+l_1}^+ : \xi_{xy} = 1\} \setminus E| \geq \sum_{y \in A_{l_0+l_1}^+} \xi_{xy} - |E| \geq \left(\frac{1}{2} - \frac{1}{4}\right)\mathfrak{p}|A_{l_0+l_1}^+| \geq |E|.$$

Furthermore, clearly

$$l_1 \le \frac{1}{n}\log_{\mathfrak{c}}\left(\frac{4|E|}{\mathfrak{p}}\right)+1$$

and therefore

$$\begin{aligned}\int_{\mathbb{Z}^n\setminus E} k(x,y)dy &\ge \nu\mathfrak{c}^{-(l_0+2)(n+sp)}\left(\frac{4|E|}{\mathfrak{p}}\right)^{-(n+sp)/n}|E| \\ &= \nu\left(\frac{\mathfrak{p}}{4}\right)^{(n+sp)/n}\mathfrak{c}^{-(l_0+2)(n+sp)}|E|^{-sp/n}.\end{aligned}$$

Now plugging in the definition of $\lambda(x)$ we get exactly Ineq. (11.3).

We still have to find $\zeta_{(11.2.1)}$ such that the moment generating function of $\lambda(x)^{-\frac{n}{n+sp}}$ is finite on $(-\infty,\zeta_{(11.2.1)})$ and that random variable $\lambda(x)$ has all moments. Notice first that random variable l_0 has a doubly exponentially decaying tail at infinity which can be seen from

$$\begin{aligned}\mathbb{P}(l_0>l) &\le \sum_{i=l}^{\infty}\mathbb{P}\left(\sum_{y\in A_i^+}\xi_{xy}\le\frac{\mathfrak{p}}{2}|A_i^+|\right)\le\sum_{i=l}^{\infty}e^{-\frac{\mathfrak{p}}{8}\mathfrak{c}^{ni}}\le e^{-\frac{\mathfrak{p}}{8}\mathfrak{c}^{nl}}\sum_{i=0}^{\infty}e^{-\frac{\mathfrak{p}}{8}\mathfrak{c}^{ni}} \\ &\le C_1(\mathfrak{p},\mathfrak{c})e^{-\frac{\mathfrak{p}}{8}c^{nl}},\end{aligned}$$

where $C_1\equiv C_1(\mathfrak{p},n)$ is defined by

$$C_1 := \sum_{i=0}^{\infty}e^{-\frac{\mathfrak{p}}{8}\mathfrak{c}^{ni}},$$

which is summable since it has a sub-exponential tail. It is elementary to see that, for a.s. finite random variable $X\in\mathbb{N}$ and increasing non-negative measurable function $g:\mathbb{R}\to\mathbb{R}$,

$$\mathbb{E}g(X)=\sum_{i=0}^{\infty}g(i+1)\mathbb{P}(X=i+1)\le\sum_{i=0}^{\infty}g(i+1)\mathbb{P}(X>i). \tag{11.5}$$

For $\zeta>0$, Ineq. (11.5) implies

$$\begin{aligned}\mathbb{E}\left[\exp\left(\zeta\nu^{\frac{n}{n+sp}}\lambda(x)^{-\frac{n}{n+sp}}\right)\right] &\le \sum_{l=0}^{\infty}\exp\left(\zeta\left(\frac{\mathfrak{p}}{4}\right)\mathfrak{c}^{(l+3)n}\right)\mathbb{P}(l_0>l) \\ &\le \sum_{l=0}^{\infty}\exp\left(\zeta\left(\frac{\mathfrak{p}}{4}\right)\mathfrak{c}^{(l+3)n}\right)C_1e^{-\frac{\mathfrak{p}}{8}\mathfrak{c}^{nl}} \\ &\le C_1\sum_{l=0}^{\infty}\exp\left[\frac{\mathfrak{p}}{4}\left(\zeta\mathfrak{c}^{3n}-\frac{1}{2}\right)\mathfrak{c}^{nl}\right].\end{aligned}$$

The sum clearly converges if the expression inside of the exponential is negative which happens for $\zeta<2^{-1}\mathfrak{c}^{-3n}$. For $\zeta\le 0$ the same claim follows because $\nu,\lambda(x)$ are non-negative and

$$\exp\left(\zeta\nu^{\frac{n}{n+sp}}\lambda(x)^{-\frac{n}{n+sp}}\right)\le 1.$$

Taking $\zeta_{(11.2.1)}\equiv\zeta_{(11.2.1)}(n):=2^{-1}\mathfrak{c}^{-3n}$ proves the estimate of the moment-generating function.

We can also use Ineq. (11.5) to compute the moments of $\lambda^{-1}(x)$. For $q>0$ we get

$$\begin{aligned}\mathbb{E}[\lambda(x)^{-q}] &\le \nu^{-q}\left(\frac{\mathfrak{p}}{4}\right)^{-q(n+sp)/n}\sum_{l=0}^{\infty}\mathfrak{c}^{q(l+3)(n+sp)}\mathbb{P}(l_0>l) \\ &\le \nu^{-q}\left(\frac{\mathfrak{p}}{4}\right)^{-q(n+sp)/n}\sum_{l=0}^{\infty}\mathfrak{c}^{q(l+3)(n+sp)}C_1e^{-\frac{\mathfrak{p}}{8}\mathfrak{c}^{nl}} \\ &\le C_1\nu^{-q}\left(\frac{\mathfrak{p}}{4}\right)^{-q(n+sp)/n}\sum_{l=0}^{\infty}e^{\log(\mathfrak{c})q(l+3)(n+sp)-\frac{\mathfrak{p}}{8}\mathfrak{c}^{nl}}.\end{aligned}$$

The last series is finite because

$$\log(\mathfrak{c})q(l+3)(n+sp) - \frac{\mathfrak{p}}{8}\mathfrak{c}^{nl} \leq -\frac{\mathfrak{p}}{16}\mathfrak{c}^{nl}$$

for sufficiently large l. Defining $C_2 \equiv C_2(\mathfrak{p}, n, q)$ to be

$$C_2 := \sum_{l=0}^{\infty} e^{\log(\mathfrak{c})q(l+3)(n+sp) - \frac{\mathfrak{p}}{8}\mathfrak{c}^{nl}} < \infty$$

we get

$$\mathbb{E}[\lambda(x)^{-q}] \leq C_1 C_2 \nu^{-q} \left(\frac{\mathfrak{p}}{4}\right)^{-q(n+sp)/n}.$$

Taking $C_{(11.2.1)} \equiv C_{(11.2.1)}(\mathfrak{p}, n, s, q, p) := C_1 C_2 (\mathfrak{p}/4)^{-q(n+sp)/n}$ we get the last claim of the theorem. □

Theorem 11.2.2. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$, $\mathfrak{p}, \nu$ as defined in Definition 11.0.3. For all $p \in (1, n/s)$, $\theta \in (0,1)$ and $q > \frac{n}{sp}$ there exists a family of random variables $\{R_\star(x_\star) \equiv R_\star(x_\star, \omega, c, \theta, n, s, q, p, \mathfrak{p}, \nu)\}_{x_\star \in \mathbb{Z}^n}$ such that, for every $x_\star \in \mathbb{Z}^n$, the following two claims hold.*

(i) There exists a non random $C_{PS}^{iid} \equiv C_{PS}^{iid}(n, s, q, p, \mathfrak{p}, \nu)$ such that $\mathbb{P}$-a.s. c satisfies

$$\star\boldsymbol{PSI}[x_\star, R_\star(x_\star), \theta; s, p, q, C_{PS}^{iid}].$$

Explicitly, for all $R_0 \geq R_\star$, $x_0 \in B(x_\star, R_0)$, $R \geq R_0^\theta$ and $f \in L^1(B(x_0, R))$, with ρ being the unique solution of $\frac{1}{\rho} = 1 - \frac{sp}{n} + \frac{1}{q}$,

$$\|f^p\|_{L^\rho(B(x_0,R))} \leq C_{PS}^{iid} R^{n/q} \sum_{x,y \in \mathbb{Z}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} c(x,y). \tag{11.6}$$

(ii) Let $p = 2$ and define ρ as in Item (i). If for some $\zeta \in [1, \rho]$, $\gamma \in [0, 2s]$, $C_C < \infty$ and random variable $R_\star^{(E)}$ form $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies $\star\boldsymbol{CE}[x_\star, R_\star^{(E)}(x_\star), \theta; s, \zeta, \gamma, C_C]$, then $\mathbb{P}$-a.s. $\mathcal{E}$ satisfies

$$\star\boldsymbol{SI}[x_\star, R_\star(x_\star) \vee R_\star^{(E)}(x_\star), \theta; s, \rho, \zeta, C_{S1}, C_{S2}, \gamma]$$

with non random $C_{S1} \equiv C(n, s, q, \mathfrak{p}, \nu)$ and $C_{S2} \equiv C_{S2}(n, s, q, \mathfrak{p}, \nu, C_C)$.

Proof. By Theorem 11.2.1 we can find a family $\{\lambda(x) \equiv \lambda(x, \omega, c, p, n, s, \mathfrak{p}, \nu)\}_{x \in \mathbb{Z}^n}$ such that Assumption 8.1.1 from Chapter 8 is satisfied $\mathbb{P}$-a.s. with space $M = \mathbb{Z}^n$, counting measure # and $\nu_{(8.1.1)} = s/n$. Keeping the assumption $sp/n < 1$ in mind, Theorem 8.1.4 proves that, for all $q > \frac{n}{sp}$, $x_0 \in \mathbb{Z}^n$, $R > 0$ and $f : L^1(\mathbb{Z}^n)$ supported in $B(x_0, R)$, $\mathbb{P}$-a.s.

$$\|f\|^p_{L^r(B(x_0,R))} \leq C_{(8.1.4)} \left(\int_{B(x_0,R)} \lambda(x)^{-q} dx \right)^{\frac{1}{q}} \mathcal{E}_p(f)$$

where $C_{(8.1.4)} \equiv C_{(8.1.4)}(s, n, p, q)$ is the constant from Theorem 8.1.4 and $f_{\mathbb{Z}^n} = 0$ by convention since $|\mathbb{Z}^n| = \infty$. We intend to estimate the integral on the right side uniformly for x_0 close to $x_\star$ and large R. What exactly is meant by "close" and "large" should become clear as the proof goes on.

Let $w \geq 2$ be arbitrary. Recall that Theorem 11.2.1 proved that all negative moments of λ are finite, which allows us to use deviation estimate from Lemma 11.1.4 (with $\delta = 1$) together with $\mathbb{E}[\lambda^{-wq}] < \infty$ as follows. For arbitrary $x_0 \in \mathbb{Z}^n$, $R \in \mathbb{N}_0$ set $N = |B(x_0, R)|$ and estimate

$$\begin{aligned} \mathbb{P}\left(\fint_{B(x_0,R)} \lambda(x)^{-q} dx > \mathbb{E}\left[\lambda^{-q}\right] + 1 \right) &= \mathbb{P}\left(\int_{B(x_0,R)} \frac{\lambda(x)^{-q} - \mathbb{E}\left[\lambda^{-q}\right]}{N} dx > 1 \right) \\ &\leq 2^w C_R(w) N^{-\frac{w}{2}} \mathbb{E}[\lambda^{-wq}]. \end{aligned} \tag{11.7}$$

Now fix arbitrary $x_\star \in \mathbb{Z}^n$, $R_0 > 0$ and shorten $B_\star = B(x_\star, R_0)$. We can use the previous estimate to calculate the probability that there exists an $R > R_0^\theta$ such that $\mathbf{PSI}[x \in B(x_\star, R_0), [R_0^\theta, \infty); s, p, q, A]$ fails to hold for some $A > 0$ chosen appropriately. To be precise, let us define

$$P(R_0, A) := \mathbb{P}\left(\exists x_0 \in B_\star, \exists R \geq R_0^\theta : \fint_{B(x_0,R)} \lambda(x)^{-q} dx > A\right).$$

We will now show that $P(R_0, A)$ can be estimated by only considering R of the form $R = 2^l$ for $l \in \mathbb{N}_0$, that is,

$$P(R_0, A) \leq \mathbb{P}\left(\exists x_0 \in B_\star, \exists l \in \mathbb{N}_0, 2^l \geq R_0^\theta : \fint_{B(x_0,2^l)} \lambda(x)^{-q} dx > C_{VD}^{-1} A\right) \tag{11.8}$$

where $C_{VD} \equiv C_{VD}(n)$ is the volume doubling constant of $\mathbb{Z}^n$ (one can take $C_{VD} := 2^n C_{VL}^{-1} C_{VU}$). If $R \leq 1$, then $B(x_0, R) = B(x_0, 1)$ and examining $R = 1$ instead will suffice. If $R \in (2^l, 2^{(l+1)})$ for some $l \in \mathbb{N}_0$, by volume regularity of $\mathbb{Z}^n$ it follows that

$$\fint_{B(x_0,R)} \lambda^{-q}(x) dx \leq \frac{|B(x_0, 2^{(l+1)})|}{|B(x_0, 2^l)|} \fint_{B(x_0,2^{(l+1)})} \lambda(x)^{-q} dx \leq C_{VD} \fint_{B(x_0,2^{(l+1)})} \lambda^{-q}(x) dx.$$

This shows that, if $\fint_{B(x_0,R)} \lambda^{-q} > A$ is true for some $R \geq R_0^\theta$, then also $\fint_{B(x_0,2^l)} \lambda^{-q} > C_{VD}^{-1} A$ is true for some $l \in \mathbb{N}_0$, $2^l \geq R_0^\theta$, which proves Ineq. (11.8).

Defining $A = C_{VD} \mathbb{E}[\lambda^{-q} + 1]$ and using Ineq. (11.7) we obtain, for $R_0 \geq 1$,

$$\begin{aligned}
P(R_0, A) &\leq \sum_{x_0 \in B_\star} \sum_{l=\lceil \log_2 R_0^\theta \rceil}^{\infty} \mathbb{P}\left(\fint_{B(x_0,2^l)} \lambda(x)^{-q}(x) dx > \mathbb{E}[\lambda^{-q} + 1]\right) \\
&\leq 2^w C_R(w) \mathbb{E}[\lambda^{-qw}] \sum_{x_0 \in B_\star} \sum_{l=\lceil \log_2 R_0^\theta \rceil}^{\infty} C_{VL}^{-\frac{w}{2}} 2^{-\frac{lnw}{2}} \\
&\leq C_{VL}^{-\frac{w}{2}} C_{VU} 2^w C_R(w) \mathbb{E}[\lambda^{-qw}] R_0^n \left(1 - 2^{-\frac{nw}{2}}\right)^{-1} 2^{-\frac{\lceil \log_2 R_0^\theta \rceil nw}{2}} \\
&\leq \frac{C_{VL}^{-\frac{w}{2}} C_{VU} 2^w C_R(w)}{1 - 2^{-\frac{nw}{2}}} \mathbb{E}[\lambda^{-qw}] R_0^{n(1-\frac{\theta w}{2})}.
\end{aligned}$$

If $n(1 - \frac{\theta w}{2}) < -1$, then probabilities $P(R_0, A)$ are summable in $R_0 \in \mathbb{N}$. Since Theorem 11.2.1 proved that all negative moments of λ are finite, we can take w large enough, say $w = 2(1 + 1/n)/\theta + 1$, to assure that this happens. Then Borel-Cantelli lemma implies that random variable $R_\star(x_\star) \equiv R_\star(x_\star, \omega, c, \theta, n, s, q, \mathfrak{p}, \nu)$ defined by

$$\begin{aligned}
R_\star(x_\star) = \sup\Big\{ R_* > 0 : \exists R_0 \geq R_*^\theta, \exists x_0 \in B(x_\star, R_0), \exists R \geq R_0^\theta \text{ s.t.} \\
\fint_{B(x_0,R)} \lambda(x)^{-q} dx > C_{VD} \mathbb{E}[\lambda^{-q} + \delta] \Big\} + 1
\end{aligned}$$

is $\mathbb{P}$-a.s. finite. This proves that $\mathbb{P}$-a.s., for all $R_0 > R_\star(x_\star)$, $x_0 \in B(x_\star, R_0)$, $R > R_0^\theta$ and $f \in L^1(B(x_0, R))$,

$$\|f^p\|_{L^\rho(B(x_0,R))} \leq C_{(8.1.4)} C_{VD} \left(\mathbb{E}\left[\lambda^{-q} + 1\right]\right)^{\frac{1}{q}} R^{\frac{n}{q}} \mathcal{E}_p(f).$$

Defining $C_{PS}^{iid} \equiv C_{PS}^{iid}(n, s, p, q, \mathfrak{p}, \nu) := C_{(8.1.4)} C_{VD} (\mathbb{E}[\lambda^{-q} + 1])^{1/q}$ (see Theorem 11.2.1 for dependence) this is equivalent to saying that $\mathbb{P}$-a.s. $\mathcal{E}_p$ satisfies $\star\mathbf{PSI}[x_\star, R_\star, \theta; s, p, q, C_{PS}^{iid}]$ which proves Item (i).

For Item (ii) we restrict to $p = 2$. By assumption we know that there exists a random variable $R_\star^{(E)}$ such that $\mathbf{CE}[x_0, R; \zeta, \gamma, C_C]$ holds $\mathbb{P}$-a.s. for every $R_0 \in \mathbb{N}$, $R_0 \geq R_\star^{(C)}(x_\star)$, $x_0 \in B(x_\star, R_0), R \geq$

R_0^θ. This, combined with Ineq. (11.6) and Proposition 6.1.10, implies that, for all $R_0 \in \mathbb{N}$, $R_0 \geq R_\star(x_\star) \vee R_\star^{(C)}(x_\star)$, $x_0 \in B(x_\star, R_0)$ $R > R_0^\theta$ $\mathbf{SI}[x_0, R; s, \rho, \zeta, C_{S1}, C_{S2}, \gamma]$ holds $\mathbb{P}$-a.s. with non random $C_{S1} \equiv C_{S1}(n, s, q, \mathfrak{p}, \nu) = 2C_{PS}^{iid}$ and $C_{S2} \equiv C_{S2}(n, s, q, \mathfrak{p}, \nu, C_C) = 2C_{PS}^{iid} C_C$. But quantifiers on R_0, x_0 and R are exactly the ones needed to prove $\star\mathbf{SI}[x_\star, R_\star(x_\star) \vee R_\star^{(C)}(x_\star), \theta; s, \rho, \zeta, C_{S1}, C_{S2}, \gamma]$. □

11.3. Poincare inequality

Theorem 11.3.1. *Let c be an i.i.d. conductance and $\mathfrak{p}, \nu$ numbers from Definition 11.0.3. Then there is a non random $C_P^{iid} \equiv C_P^{iid}(n, \mathfrak{p}, \nu)$ such that for all $\theta \in (0, 1)$, $x_\star \in \mathbb{Z}^n$ there exists a random variable $R_\star(x_\star) \equiv R_\star(x_\star, \omega, c, \theta, \mathfrak{p}, \nu, n)$ such that $\mathbb{P}$-a.s. form $\mathcal{E}$ satisfies $\star\boldsymbol{PI}[x_\star, R_\star, \theta; s, C_P^{iid}]$. To be explicit, for all $R_0 \geq R_\star$, $x_0 \in B(x_\star, R_0)$, $R \geq R_0^\theta$ and every $f \in L^1(B(x, R))$, with $f_B := \fint_{B(x_0,R)} f$,*

$$\int_{B(x_0,R)} (f(x) - f_B(x))^2 dx \leq C_P^{iid} R^{2s} \mathcal{E}_B(f).$$

The proof is postponed for the end of the chapter because we need two preliminary results in Theorem 11.3.2 and Lemma 11.3.3. Theorem 11.2.1 proves that all L^p-Poincaré inequalities for $p \in (1, \infty)$ follow from a sort of fractional isoperimetric inequality given in Ineq. (11.9). The computations we use are not very different from the ones in [Kum18] Chapter 3.3, which deals mostly with cases $p = 1, 2$.

Theorem 11.3.2. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$, $\mathfrak{p}, \nu > 0$ and a family of Bernoulli random variables $\{\xi_{xy}\}_{x,y \in \mathbb{Z}^n}$ like in Definition 11.0.3. Let $B \subset \mathbb{Z}^n$ be a ball of radius $R \geq 1$ and suppose, similar to isoperimetric inequality, that there exists a parameter $\beta \in (0, 1)$ such that $\mathbb{P}$-a.s. for every set $A \subset B$*

$$|\{\xi_{xy} = 1; x \in A, y \in A^c\}| \geq \beta |A||A^c| \tag{11.9}$$

where A^c stands for $B \setminus A$.

Then, $\mathbb{P}$-a.s., for every $p \in [1, \infty)$ the L^p-Poincaré inequality holds on B. That is, there is a function

$$\mathcal{A}(p) \equiv \mathcal{A}(p, \beta, \nu, C_{VL}, C_{VU}) = 8^p C_{VU}^{p-1} C_{VL}^{-p} p^p \beta^{-p} \nu^{-1}$$

such that for every $f \in L^1(B)$, setting $f_B = \fint_B f$,

$$\int_B |f(x) - f_B|^p dx \leq \mathcal{A}(p) R^{sp} \int_B \int_B \frac{|f(x) - f(y)|^p}{d(x, y)^{n+sp}} c(x, y) dx dy.. \tag{11.10}$$

Proof. We work pointwise only for $\omega \in \Omega$ for which Ineq. (11.9) holds. Suppose $h : B \to [0, \infty)$ is a function with $|\operatorname{supp}(h)| \leq |B|/2$ and define $H_t := \{h \geq t\}$. Our isoperimetric assumption implies that for every $t > 0$

$$\beta |H_t||H_t^c| \leq |\{\xi_{xy} = 1; x \in H_t, y \in H_t^c\}|$$

and allows for the following computation (the exchange of integrals is justified by Fubini's theorem since the integrand is non-negative)

$$\begin{aligned} \int_B h(x) dx = \int_B \int_0^\infty 1_{\{t \leq h(x)\}}(t, x) dt dx &= \int_0^\infty |H_t| dt \\ &\leq \int_0^\infty \beta^{-1} \frac{|\{\xi_{xy} = 1 : x \in H_t, y \in H_t^c\}|}{|H_t^c|} dt. \end{aligned}$$

Since, by volume regularity of $\mathbb{Z}^n$, $|H_t^c| \geq |B|/2 \geq \frac{C_{VL}}{2} R^n$ we can proceed with (again switching the integrals)

$$\begin{aligned}\int_B h(x)dx &\leq 2C_{VL}^{-1}\beta^{-1}R^{-n} \int_0^\infty \int_B \int_B \xi_{xy} 1_{\{h(y)<t\leq h(x)\}} dxdydt \\ &= 2C_{VL}^{-1}\beta^{-1}R^{-n} \int_{h(x)>h(y)} \xi_{xy}(h(x)-h(y))dxdy \\ &= C_{VL}^{-1}\beta^{-1}R^{s} \int_B \int_B \xi_{xy} \frac{|h(x)-h(y)|}{R^{n+s}} dxdy.\end{aligned}$$

The last equality is true because the integrand is zero when $h(x) = h(y)$ and symmetric in x and y.

Now take an arbitrary $f \in L^1(B)$ and find an $a \equiv a(f) \in \mathbb{R}$ such that

$$\int_B (f-a)_+^p = \int_B (f-a)_-^p = \frac{1}{2}\int_B |f-a|^p.$$

That such a exists follows from the dominated convergence theorem which shows that $\int_B (f-a)_+^p$ is continuous in a and tends to 0 or ∞ when a goes to ∞ or $-\infty$ respectively. Choose g to be either $(f-a)_+, (f-a)_-$, depending on which of them has the smallest support, and apply the calculation from before with $h = g^p$, together with the elementary inequality

$$|g(x)^p - g(y)^p| \leq p(g^{p-1}(x) + g^{p-1}(y))|g(x)-g(y)|,$$

to get

$$\begin{aligned}\int_B g(x)^p dx &\leq C_{VL}^{-1}\beta^{-1}R^s \int_B\int_B \xi_{xy} \frac{|g(x)^p - g(y)^p|}{R^{n+s}} dxdy \\ &\leq C_{VL}^{-1}\beta^{-1}pR^s \int_B\int_B \xi_{xy} \frac{(g^{p-1}(x)+g^{p-1}(y))|g(x)-g(y)|}{R^{n+s}} dxdy.\end{aligned}$$

Applying Hölder's inequality with exponents p and $\frac{p}{p-1}$ ($\frac{p}{p-1} = \infty$ by definition if $p = 1$) we obtain

$$\begin{aligned}\int_B g(x)^p dx \leq \frac{p}{C_{VL}\beta} R^s &\left(\int_B\int_B \frac{(g(x)^{p-1}+g(y)^{p-1})^{\frac{p}{p-1}}}{R^n} dxdy \right)^{\frac{p-1}{p}} \\ &\times \left(\int_B\int_B \xi_{xy}^p \frac{|g(x)-g(y)|^p}{R^{n+sp}} dxdy \right)^{\frac{1}{p}}.\end{aligned}$$

The first factor can be estimated using the inequality

$$\left(g(x)^{p-1} + g(y)^{p-1}\right)^{\frac{p}{p-1}} \leq 2^{\frac{p}{p-1}} \left(g(x)^p + g(y)^p\right),$$

symmetry and volume regularity of $\mathbb{Z}^n$ (note that $R \geq 1$ was assumed) in the following way:

$$\int_B\int_B \frac{(g(x)+g(y))^p}{R^n} dxdy \leq 2^{\frac{p}{p-1}} R^{-n} \cdot 2 \int_B\int_B g(x)^p dxdy \leq 2^{\frac{2p-1}{p-1}} C_{VU} \int_B g(x)^p dx.$$

Inserting this into our main inequality (notice that $\xi_{xy}^p = \xi_{xy}$) gives

$$\int_B g(x)^p dx \leq \frac{p 2^{\frac{2p-1}{p}} C_{VU}^{\frac{p-1}{p}}}{C_{VL}\beta} R^s \left(\int_B g(x)^p dx\right)^{\frac{p-1}{p}} \left(\int_B\int_B \xi_{xy} \frac{|g(x)-g(y)|^p}{R^{n+sp}} dxdy\right)^{\frac{1}{p}}.$$

Dividing by $\left(\int_B g(x)^p dx\right)^{\frac{p-1}{p}}$, rising everything to power p and taking into account $\xi_{xy} \leq \nu^{-1} c(x,y)$ finally results in

$$\int_B g(x)^p dx \leq \frac{\mathcal{A}(p)}{2^{p+1}} R^{sp} \int_B \int_B \frac{|g(x)-g(y)|^2}{R^{n+sp}} c(x,y) dx dy$$

with $\mathcal{A}(p) \equiv \mathcal{A}(p,\beta,\nu,C_{VL},C_{VU}) = 2^{p+1} C_{VU}^{p-1} C_{VL}^{-p} p^p 2^{2p-1} \beta^{-p} \nu^{-1}$. Recall that constant a and function g were chosen in such a way so that

$$2\int_B g(x)^p dx = \int_B |f(x)-a|^p dx,$$

which allows for the following line of estimates

$$\begin{aligned}\int_B |f(x)-a|^p dx &= 2\int_B g(x)^p dx \\ &\leq \frac{\mathcal{A}(p)}{2^p} R^{sp} \int_B \int_B \frac{|g(x)-g(y)|^p}{R^{n+2s}} c(x,y) dx dy \\ &\leq \frac{\mathcal{A}(p)}{2^p} R^{sp} \int_B \int_B \frac{|f(x)-f(y)|^p}{d(x,y)^{n+sp}} c(x,y) dx dy.\end{aligned}$$

Therefore, like in Ineq. (8.2),

$$\int_B |f - f_B|^p dx \leq 2^p \int_B (f-a)^p dx \leq \mathcal{A}(p) R^{sp} \int_B \int_B \frac{|f(x)-f(y)|^p}{d(x,y)^{n+sp}} c(x,y) dx dy.$$

which proves the theorem. □

Let us now show that Ineq. (11.9) is satisfied if R is large enough (depending on ω).

Lemma 11.3.3. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$, $\mathfrak{p}, \nu > 0$ and $\{\xi_{xy}\}_{x,y\in\mathbb{Z}^n}$ Bernoulli random variables from Definition 11.0.3. For every $\delta \in (0,1)$ there exist non random $R_* \equiv R_*(\delta,\mathfrak{p},n)$, $C_{(11.3.3)} \equiv C_{(11.3.3)}(\delta,\mathfrak{p},n)$ and $D_{(11.3.3)} \equiv D_{(11.3.3)}(\delta,\mathfrak{p},n)$ such that for every ball B of radius $R \geq R_*$*

$$\mathbb{P}\left(\exists A \subset B : |\{\xi_{xy} = 1; x \in A, y \in A^c\}| < (1-\delta)\mathfrak{p}|A||A^c|\right) \leq C_{(11.3.3)} e^{-D_{(11.3.3)} R^n}.$$

Proof. Let us set $N := |B|$, fix a $\delta \in [0,1)$ and denote

$$P := \mathbb{P}\left(\exists A \subset B : |\{\xi_{xy} = 1; x \in A, y \in A^c\}| < (1-\delta)\mathfrak{p}|A||A^c|\right).$$

It suffices to prove the statement of the lemma for $|A| \leq |B|/2$ due to symmetry of A and A^c. This implies that $|A^c| \geq |B|/2$ so using Chernoff's bound (Theorem 11.1.2) to estimate the tail of binomial variable $\sum \xi_{xy}$ we can bound

$$\mathbb{P}\left(\sum_{x\in A, y\in A^c} \xi_{xy} < (1-\delta)\mathfrak{p}|A||A^c|\right) \leq e^{-\frac{\delta^2 \mathfrak{p}|A||A^c|}{2}} \leq e^{-\frac{\delta^2 \mathfrak{p}|B|}{4}|A|}.$$

Summing these probabilities over all $A \subset B$ gives the estimate

$$P \leq \sum_{A\subset B} \mathbb{P}\left(\sum_{x\in A, y\in A^c} \xi_{xy} < (1-\delta)\mathfrak{p}|A||A^c|\right) \leq \sum_{i=1}^{\lfloor N/2 \rfloor} \sum_{\substack{A\subset B\\ |A|=i}} e^{-\frac{\delta^2 \mathfrak{p}}{4} N i}.$$

If we overestimate the number of subsets of B with exactly i elements by N^i and recall that for $R \geq 1$, $C_{VL}R^n \leq N \leq C_{VU}R^n$, we end up with

$$P \leq \sum_{i=1}^{\lfloor N/2 \rfloor} N^i e^{-\frac{\delta^2 \mathfrak{p}}{4} Ni} \leq \sum_{i=1}^{\infty} e^{-\left(\frac{\delta^2 \mathfrak{p} C_{VL}}{4} R^n - \log(C_{VU}R^n)\right) i}.$$

Choosing $R_* \equiv R_*(\delta, \mathfrak{p}, n)$ large enough such that $R_* \geq 1$ and

$$\frac{\delta^2 \mathfrak{p} C_{VL}}{4} R_*^n > 2\log(C_{VU}R_*^n),$$

for $R \geq R_*$ we can further estimate

$$P \leq \sum_{i=1}^{\infty} e^{-\frac{1}{8}\delta^2 \mathfrak{p} C_{VL} R^n i} \leq e^{-\frac{1}{16}\delta^2 \mathfrak{p} C_{VL} R^n} \sum_{i=1}^{\infty} e^{-\frac{1}{16}\delta^2 \mathfrak{p} C_{VL} i} \leq C_{(11.3.3)} e^{-D_{(11.3.3)} R^n}.$$

which proves the theorem with constants $C_{(11.3.3)}$ and $D_{(11.3.3)}$ taken as $C_{(11.3.3)} = \sum_{i=1}^{\infty} e^{-\frac{1}{16}\delta^2 \mathfrak{p} C_{VL} i}$ and $D_{(11.3.3)} = -\frac{1}{16}\delta^2 \mathfrak{p} C_{VL}$. □

The previous lemma gives the exponentially decreasing bound on the probability that fractional isoperimetric inequality does not hold in an arbitrary ball B with large enough radius. By Theorem 11.3.2 this probability is greater than the probability that L^2-Poincaré inequality fails in B. We will use this exponential decay together with Borel-Cantelli lemma in order to find $R_\star(x_\star)$ in which $\star$**PI** will hold.

Proof of Theorem 11.3.1. For variable $\theta, \delta \in (0,1)$, $R_0 \geq 1$ we interested in the probability that there exist some $x_0 \in B(x_\star, R_0)$, $R \geq R_0$ such that $\mathbf{PI}[x_0, R; s, C_P^{iid}]$ fails where $C_P^{iid} \equiv C_P^{iid}(n, \delta, \mathfrak{p}, \nu) := \mathcal{A}(p = 2, \beta = 2^{-1}C_{VD}^{-1}(1-\delta)\mathfrak{p}, \nu, C_{VL}, C_{VU})$ ($\mathcal{A}$ is the function from Theorem 11.3.2). The choice of $\beta \equiv \beta(\delta, \mathfrak{p}, n) := 2^{-1}C_{VD}^{-1}(1-\delta)\mathfrak{p}$ is a technicality and will make sense shortly.

Due to Theorem 11.3.2, this probability is bounded by probability, call it $P(R_0)$, that isoperimetric inequality Ineq. (11.9) fails in $B(x_0, R)$ for some x and R as above, i.e. it is bounded by

$$\begin{aligned} P(R_0) := \mathbb{P}\Big(&\exists x_0 \in B(x_\star, R_0), \exists R \geq R_0^\theta, \exists A \subset B(x_0, R) : \\ &: |\{\xi_{xy} = 1; x \in A, y \in A^c\}| < \beta |A||A^c|\Big). \end{aligned}$$

We now show that it is possible to have a similar bound in terms of integer radii $R \in \mathbb{N} \cap [R_0^\theta, \infty)$ only, at the expense of constant β. Suppose $2^l \leq R \leq 2^{l+1}$ for some $l \in \mathbb{N}$. For $A \subset B(x_0, R)$ such that $|B(x_0,R) \setminus A| \geq |B(x_0,R)|/2$ we can estimate

$$\begin{aligned} &\frac{|\{\xi_{xy} = 1; x \in A, y \in B(x_0,R) \setminus A\}|}{|A||B(x_0,R) \setminus A|} \\ &\leq \frac{|B(x_0, 2^{l+1})|}{|B(x_0, 2^{l+1}) \setminus A|} \cdot \frac{|\{\xi_{xy} = 1; x \in A, y \in B(x_0, 2^{l+1}) \setminus A\}|}{|A||B(x_0,R)|/2} \\ &\leq 2C_{VD} \frac{|\{\xi_{xy} = 1; x \in A, y \in B(x_0, 2^{l+1}) \setminus A\}|}{|A||B(x_0, 2^{l+1}) \setminus A|}. \end{aligned}$$

Therefore, if isoperimetric inequality fails in $B(x_0, R)$ with constant $\beta = 2^{-1}C_{VD}^{-1}(1-\delta)\mathfrak{p}$, then isoperimetric inequality fails in $B(x_0, 2^{l+1})$ with constant $(1-\delta)\mathfrak{p}$ and we get the estimate

$$P(R_0) \leq \sum_{x_0 \in B(x_\star, R_0)} \sum_{R = \lceil \log_2 R_0^\theta \rceil}^{\infty} \mathbb{P}\left(\exists A \subset B(x_0, R) : |\{\xi_{xy} = 1; x \in A, y \in A^c\}| < (1-\delta)\mathfrak{p}|A||A^c|\right).$$

By Lemma 11.3.3 with any $\delta \in (0,1)$, say $\delta = 1/2$ for the sake of concreteness, previous expression for $R \geq R_*$ can be estimated further by

$$P(R_0) \leq C_{VU} R_0^n \sum_{l=\lceil \log_2 R_0^\theta \rceil}^{\infty} C_{(11.3.3)} e^{-D_{(11.3.3)} 2^{ln}}$$
$$\leq C_{VU} C_{(11.3.3)} R_0^n e^{-D_{(11.3.3)} R_0^{n\theta}} \sum_{l=0}^{\infty} e^{-D_{(11.3.3)} 2^{ln}}.$$

The last series converges because it has a double exponential tail so we can find constants $C_1 \equiv C_1(n, \mathfrak{p})$ and $C_2 \equiv C_2(n, \mathfrak{p})$ such that, for $R_0 \geq R_*$,

$$P(R_0) \leq C_1 R_0^n e^{-C_2 R_0^\theta}.$$

$P(R_0)$ it therefore is summable in $R_0 \in \mathbb{N}$ so, by Borel-Cantelli lemma, random variable $R_\star(x_\star) \equiv R_\star(x_\star, \omega, c, \theta, \mathfrak{p}, \nu, n)$ (recall that $\beta \equiv \beta(n, \delta, \mathfrak{p})$ and that we fixed $\delta = 1/2$ for dependence), defined by

$$R_\star(x_\star) := \sup \big\{ R_0 \in \mathbb{N} : \exists x_0 \in B(x_\star, R_0), \exists R \geq R_0^\theta, \exists A \subset B(x_0, R) :$$
$$: |\{\xi_{xy} = 1; x \in A, y \in A^c\}| < \beta |A||A^c| \big\} + 1$$

is $\mathbb{P}$-a.s. finite. Thus, due to Theorem 11.3.2, there is a non random $C_P^{iid} \equiv C_P^{iid}(n, \mathfrak{p}, \nu)$ such that $\star\mathbf{PI}[x_\star, R_\star, \theta; s, C_P^{iid}]$ also holds. □

11.4. Lower estimates on the kernel

Lemma 11.4.1. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$ and let $\mathfrak{p}, \nu, \{\xi_{xy}\}_{x,y \in \mathbb{Z}^n}$ be like in Definition 11.0.3. There exist $C_K \equiv C_K(\nu, \mathfrak{p}, n, s) > 0$ such that for all $\theta \in (0,1)$, $x_\star \in \mathbb{Z}^n$ there exists an $\mathbb{P}$-a.s. finite random variable $R_\star(x_\star) \equiv R_\star(x_\star, \omega, c, \theta, n, \mathfrak{p})$ such that $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies*

$$\star\boldsymbol{AKB}{\geq}[x_\star, R_\star(x_\star), \theta; s, C_K].$$

To be explicit, for all $R_0 \geq R_\star(x_\star)$, $x_0 \in B(x_\star, R_0)$ and $R \geq R_0^\theta$ there exists $y_0 \in M \setminus B(x_0, 6R)$ such that

$$⨍_{B(x_0,R)} \int_{B(y_0,R)} k(x,y) dy dx \geq C_K R^{-2s}. \tag{11.11}$$

In fact, a stronger statement with $⨍_{B(x_0,R)}$ replaced by $\sup_{B(x_0,R)}$ is also true.

Proof. We relay on Chernoff's bound from Theorem 11.1.2 and Borel-Cantelli lemma from Theorem 11.1.1 once again. Fix arbitrary $\theta, \delta \in (0,1)$. For $R \geq 1$, $x \in \mathbb{Z}^n$ and $y_0 \in B(x, 8R)$ we consider the probability

$$P_1(x, y_0, R) := \mathbb{P}\left(\int_{B(y_0,R)} k(x,y) dy \leq (1-\delta) 9^{-(n+2s)} C_{VU}^{-1} \nu \mathfrak{p} R^{-2s} \right).$$

By assumption on y_0, $d(x,y) \leq 9R$ for all $y \in B(y_0, R)$, which gives

$$k(x,y) = c(x,y) d(x,y)^{-n-2s} \geq (9R)^{-n-2s} c(x,y) \geq \frac{\nu \xi_{xy} R^{-2s}}{9^{n+2s} C_{VU} |B(y_0,R)|}.$$

Hence

$$P_1(x, y_0, R) \leq \mathbb{P}\left(\sum_{y \in B(y_0,R)} \xi_{xy} \leq (1-\delta) \mathfrak{p} |B(y_0,R)| \right)$$

so Chernoff's bound from Theorem 11.1.2 applies, with $X = \sum_{y \in B(y_0,R)} \xi_{xy}$, $\mathbb{E}[X] = |B(y_0,R)|\mathfrak{p}$, and proves

$$P_1(x, y_0, R) \le e^{-\frac{\delta^2 \mathfrak{p} |B(y_0,R)|}{2}} \le e^{-\frac{C_{VL}\delta^2 \mathfrak{p} R^n}{2}}.$$

Let now $v \in \mathbb{Z}^n$ be an arbitrary unit vector and notice that, if for some $x_0 \in M$ and $R > 0$, $\mathbf{AKB}{\ge}[x_0, R; s, 9^{-(n+2s)}(1-\delta)C_{VU}^{-1}\nu\mathfrak{p}R^{-2s}]$ fails, then for every $y \in M \setminus B(x_0, 6R)$

$$\fint_{B(x_0,R)} \int_{B(y_0,R)} k(x,y)dy \le 9^{-(n+2s)}(1-\delta)C_{VL}^{-1}\nu\mathfrak{p}R^{-2s}.$$

Therefore, taking $y = x_0 + 6Rv$, there must exists an $x \in B(x_0, R)$ such that $\int_{B(x_0+6Rv,R)} k(x,y)dy \le 9^{-(n+2s)}(1-\delta)C_{VL}\nu\mathfrak{p}R^{-2s}$.

Defining $C_K \equiv C_K(\delta, \nu, \mathfrak{p}, n, s) = 9^{-(n+2s)}(1-\delta)C_{VL}^{-1}\nu\mathfrak{p}$ and taking the previous argument one step further, for arbitrary $R_* \ge 1$, $x_\star \in \mathbb{Z}^n$, we can estimate

$$\begin{aligned} P_2 &:= \mathbb{P}\left(\star\mathbf{AKB}{\ge}[x_\star, R_*, \theta; s, C_K] \text{ fails}\right) \\ &\le \mathbb{P}\Big(\exists R_0 \in \mathbb{N}, R_0 \ge R_*, \exists x_0 \in B(x_\star, R_0), \exists R \ge R_0^\theta, \exists x \in B(x_0, R) : \\ &\qquad : \int_{y \in B(x_0+6Rv,R)} k(x,y)dy \le C_K R^{-2s}\Big). \end{aligned}$$

A similar technique like the one in Theorem 11.3.1 can be used to show that it is enough to consider only $R \in \mathbb{N}$ (or R of the form 2^l for some $l \in \mathbb{N}$). Hence

$$\begin{aligned} P_2 &\le \sum_{R_0=R_*}^{\infty} \sum_{x_0 \in B(x_\star,R_0)} \sum_{R=R_0^\theta}^{\infty} \sum_{x \in B(x_0,R)} P_1(x, x_0+6Rv, R) \\ &\le \sum_{R_0=R_*}^{\infty} C_{VU}R_0^n \sum_{R=R_0^\theta}^{\infty} C_{VU}R^n e^{-\frac{C_{VL}\delta^2\mathfrak{p}}{2}R^n} \\ &\le C_{VU}^2 \sum_{R_0=R_*}^{\infty} R_0^n e^{-\frac{C_{VL}\delta^2\mathfrak{p}}{4}R_0^{\theta n}} \sum_{R=R_0}^{\infty} R^n e^{-\frac{C_{VL}\delta^2\mathfrak{p}}{4}R^n}. \end{aligned}$$

Introducing an auxiliary constant $C_2 \equiv C_2(\delta, \mathfrak{p}, n) = C_{VL}\delta^2\mathfrak{p}/8 > 0$, we can proceed with

$$P_2 \le C_{VU}^2 e^{-C_2 R_*^{\theta n}} \sum_{R_0=1}^{\infty} R_0^n e^{-C_2 R_0^{\theta n}} \sum_{R=1}^{\infty} R^n e^{-2C_2 R^n} \le C_3 e^{-C_2 R_*^{\theta n}}.$$

where

$$C_3 \equiv C_3(\delta, \mathfrak{p}, n, \theta) = C_{VU}^2 \sum_{R_0=1}^{\infty} R_0^n e^{-C_2 R_0^{\theta n}} \sum_{R=1}^{\infty} R^n e^{-2C_2 R^n} < \infty$$

is finite because both series converge. Since $C_2 > 0$,

$$\sum_{R_*=1}^{\infty} \mathbb{P}\left(\star\mathbf{AKB}{\ge}[x_\star, R_*, \theta; s, C_K] \text{ fails}\right) \le \sum_{R_*=1}^{\infty} C_3 e^{-C_2 R_*^{\theta n}} < \infty$$

so Borel-Cantelli lemma from Theorem 11.1.1 proves that $R_\star(x_\star) \equiv R_\star(x_\star, \omega, c, \theta, \delta, n, \mathfrak{p})$ (it does not depend on ν in fact) defined by

$$R_\star(x_\star) := \sup\{R_* \ge 1 : \mathbf{AKB}{\ge}[x_\star, R_*, \theta; s, (1-\delta)C_K] \text{ fails}\} + 1$$

is $\mathbb{P}$-a.s. finite. Fixing $\delta = 1/2$, for instance, proves the lemma. □

11.5. Energy density of cutoff functions

In this section we prove, $\mathbb{P}$-a.s., an estimate on $L^Q(B)$ norm of $\Gamma(\varphi)$ depending on the radius of the ball B and Lipschitz constant of φ for large enough balls around a point $x_\star \in \mathbb{Z}^n$. The minimal size of the ball depends on the realization of c close to $x_\star$. The computation is very similar to the one performed in Section 10.3 but in case of i.i.d. conductance it can be improved to work for $\gamma = 0$ and not only for $\gamma > 0$.

Definition 11.5.1. *Set* $\mathfrak{f} \equiv \mathfrak{f}(C_{VL}, C_{VU}) = \left(\frac{C_{VU}}{C_{VL}} + 1\right)^{1/n}$. *Let* $B \subset \mathbb{Z}^n$ *be an arbitrary ball with radius* $R \geq 1$. *For* $i \in \mathbb{Z}$ *define*

$$F_i(B) = \left\{x, y \in \mathbb{Z}^n : x \in B \quad \& \quad \mathfrak{f}^{i-1}R < d(x,y) \leq \mathfrak{f}^i R\right\}.$$

$F_i(B)$ is denoted just by F_i if the ball B is clear from the context. Note that $F_i = \emptyset$ when $i < -\log_{\mathfrak{f}} R$.

Lemma 11.5.2. *Let $B \subset \mathbb{Z}^n$ be an arbitrary ball with radius $R \geq 1$. Then for every $i \in \mathbb{Z}, i \geq -\log_{\mathfrak{f}} R$*

$$C_{VL}\mathfrak{f}^{(i-1)n}R^{2n} \leq |F_i| \leq C_{VU}^2 \mathfrak{f}^{in} R^{2n}.$$

Proof. Notice that for $i \geq -\log_{\mathfrak{f}} R + 1$, $\mathfrak{f}^{i-1}R \geq 1$ and we can use volume regularity of $\mathbb{Z}^n$ to get

$$\begin{aligned} |F_i| &\geq \int_B |B(x, \mathfrak{f}^i R)| - |B(x, \mathfrak{f}^{i-1}R)| dx \geq \int_B \left(C_{VL}\mathfrak{f}^{in}R^n - C_{VU}\mathfrak{f}^{(i-1)n}R^n\right) dx \\ &\geq C_{VL}\mathfrak{f}^{(i-1)n}R^{2n}\left(\mathfrak{f}^n - C_{VU}C_{VL}^{-1}\right) \geq C_{VL}\mathfrak{f}^{(i-1)n}R^{2n} \end{aligned}$$

where the choice $\mathfrak{f} = \left(\frac{C_{VU}}{C_{VL}} + 1\right)^{1/n}$ is crucial for the last inequality. If $-\log_{\mathfrak{f}} R \leq i < -\log_{\mathfrak{f}} R + 1$, then the upper volume regularity might not be available for the ball $B(x, \mathfrak{f}^{i-1}R)$ but the set $B(x, \mathfrak{f}^i R) \setminus B(x, \mathfrak{f}^{i-1}R)$ contains at least one element because $1 \in \left(\mathfrak{f}^{i-1}R, \mathfrak{f}^i R\right]$. Thus, taking into account that in this case $\mathfrak{f}^{i-1}R < 1$,

$$|F_i| \geq \int_B 1 dx \geq |B| \geq C_{VL}R^n \geq C_{VL}\mathfrak{f}^{(i-1)n}R^{2n}$$

so the same statement is true.

The other estimate is easier as one has $\mathfrak{f}^i R \geq 1$ so volume regularity of $\mathbb{Z}^n$ implies

$$|F_i| \leq \int_B |B(x, \mathfrak{f}^i R)| dx \leq C_{VU}\mathfrak{f}^{in}R^n \int_B 1 dx \leq C_{VU}^2 \mathfrak{f}^{in} R^{2n}.$$

□

Lemma 11.5.3. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$, $B \subset \mathbb{Z}^n$ an arbitrary ball with radius $R \geq 1$ and shorten $F_i \equiv F_i(B)$. Suppose that there exist $Q \geq 1$ and $\mathcal{G} < \infty$ such that $\mathbb{P}$-a.s., for all $i \in \mathbb{Z}$,*

$$\sum_{(x,y) \in F_i} c(x,y)^Q \leq \mathcal{G}|F_i|. \tag{11.12}$$

Then $\mathbb{P}$-a.s. for every Lipschitz $\varphi : \mathbb{Z}^n \to [0,1]$, with $\xi := R \operatorname{Lip} \varphi$,

$$\sum_{x \in B} \sum_{y \in \mathbb{Z}^n} \frac{(\varphi(x) - \varphi(y))^2}{d(x,y)^{n+2s}} c^Q(x,y) \leq C_{(11.5.3)} \mathcal{G} \xi^{2s} R^{n-2s},$$

where

$$C_{(11.5.3)} \equiv C_{(11.5.3)}(n) := \left(\frac{\mathfrak{f}^n}{1 - \mathfrak{f}^{-2(1-s)}} + \frac{\mathfrak{f}^n}{1 - \mathfrak{f}^{-2s}}\right) C_{VU}^2.$$

Proof. Again, we work pointwise for $\mathbb{P}$-a.e. $\omega \in \Omega$ where the assumption holds. Let us also write the summation as the integration with respect to the counting measure $\#$, i.e. $\int f(x)dx := \int f(x)\#(dx) = \sum_x f(x)$. With this change of notation in mind, we compute

$$\begin{aligned}\int_B \int_{\mathbb{Z}^n} & \frac{(\varphi(x)-\varphi(y))^2}{d(x,y)^{d+2s}} c(x,y)^Q dydx \\ &= \int_B \int_{d(x,y)\le \mathfrak{f}^{\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil} R} \frac{(\varphi(x)-\varphi(y))^2}{d(x,y)^{d+2s}} c(x,y)^Q dydx \\ &\quad + \int_B \int_{d(x,y)> \mathfrak{f}^{\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil} R} \frac{(\varphi(x)-\varphi(y))^2}{d(x,y)^{d+2s}} c(x,y)^Q dydx =: I_1 + I_2.\end{aligned}$$

I_1 is estimated using the Lipschitz constant of φ ($|\varphi(x)-\varphi(y)| \le \xi R d(x,y)$),

$$\begin{aligned} I_1 &= \int_B \int_{d(x,y)\le \mathfrak{f}^{\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil} R} \frac{(\varphi(x)-\varphi(y))^2}{d(x,y)^{d+2s}} c(x,y)^Q dydx \\ &\le \sum_{i=-\lceil \log_{\mathfrak{f}} R\rceil}^{\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil} \int_B \int_{\mathfrak{f}^{i-1}R < d(x,y) \le \mathfrak{f}^i R} \frac{(\xi d(x,y)/R)^2}{d(x,y)^{n+2s}} c(x,y)^Q dydx \\ &\le \left(\frac{R}{\xi}\right)^{-2} \sum_{i=-\lceil \log_{\mathfrak{f}} R\rceil}^{\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil} \mathfrak{f}^{-(i-1)(n-2(1-s))} R^{-(n-2(1-s))} \int_{F_i} c(x,y)^Q dxdy \\ &\le \mathcal{G} \left(\frac{R}{\xi}\right)^{-2} \sum_{i=-\lceil \log_{\mathfrak{f}} R\rceil}^{\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil} \mathfrak{f}^{-(i-1)(n-2(1-s))} R^{-(n-2(1-s))} |F_i|, \end{aligned}$$

where Ineq. (11.12) is used in the last line. By Lemma 11.5.2, we know that $|F_i| \le C_{VU}^2 \mathfrak{f}^{ni} R^{2n}$ which gives

$$I_1 \le \mathfrak{f}^n C_{VU}^2 \mathcal{G} \xi^2 R^{n-2s} \sum_{i=-\lceil \log_{\mathfrak{f}} R\rceil}^{\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil} \mathfrak{f}^{2(i-1)(1-s)}.$$

Using $\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil - 1 \le \log_{\mathfrak{f}}(\xi^{-1})$, the sum can be estimated by

$$\begin{aligned}\sum_{i=-\lceil \log_{\mathfrak{f}} R\rceil}^{\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil} \mathfrak{f}^{2(i-1)(1-s)} &= \sum_{i=-\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil}^{\lceil \log_{\mathfrak{f}} R\rceil} \mathfrak{f}^{-2(i+1)(1-s)} \le \sum_{j=0}^{\infty} \mathfrak{f}^{-2(j-\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil+1)(1-s)} \\ &\le \xi^{2(s-1)} \sum_{j=0}^{\infty} \mathfrak{f}^{-2(1-s)j} \le \frac{\xi^{2(s-1)}}{1-\mathfrak{f}^{-2(1-s)}},\end{aligned}$$

which leads to

$$I_1 \le \frac{\mathfrak{f}^n C_{VU}^2}{1-\mathfrak{f}^{-2(1-s)}} \mathcal{G} \xi^{2s} R^{n-2s}.$$

The computation for I_2 is similar but uses $|\varphi(x)-\varphi(y)| \le 1$ instead of the Lipschitz estimate:

$$\begin{aligned} I_2 &= \int_B \int_{d(x,y)> \mathfrak{f}^{\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil} R} \frac{(\varphi(x)-\varphi(y))^2}{d(x,y)^{n+2s}} c(x,y)^Q dydx \\ &\le \sum_{i=\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil+1}^{\infty} \int_B \int_{\mathfrak{f}^{i-1}R < d(x,y) \le \mathfrak{f}^i R} \frac{c(x,y)^Q}{d(x,y)^{n+2s}} dydx \\ &\le \sum_{i=\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil+1}^{\infty} \mathfrak{f}^{-(i-1)(n+2s)} R^{-(n+2s)} \int_{F_i} c(x,y)^Q dydx. \end{aligned}$$

Due to Ineq. (11.12) and the bound $|F_i| \le C_{VU}^2 \mathfrak{f}^{ni} R^{2n}$ from Lemma 11.5.2, we arrive at

$$I_2 \le \mathfrak{f}^n C_{VU}^2 \mathcal{G} R^{n-2s} \sum_{i=\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil+1}^{\infty} \mathfrak{f}^{-2(i-1)s}.$$

Estimating the sum using $-\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil \le -\log_{\mathfrak{f}}(\xi^{-1})$ provides us with

$$\sum_{i=\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil+1}^{\infty} \mathfrak{f}^{-2(i-1)s} \le \sum_{j=0}^{\infty} \mathfrak{f}^{-2s(j+\lceil \log_{\mathfrak{f}}(\xi^{-1})\rceil)} \le \xi^{2s} \sum_{j=0}^{\infty} \mathfrak{f}^{-2sj} \le \frac{\xi^{2s}}{1-\mathfrak{f}^{-2s}},$$

which we insert into the previous computation to get

$$I_2 \le \frac{\mathfrak{f}^n C_{VU}^2}{1-\mathfrak{f}^{-2s}} \mathcal{G} \xi^{2s} R^{n-2s}.$$

Combining estimates for I_1 and I_2 leads to

$$\begin{aligned} \int_B \int_{\mathbb{Z}^n} \frac{(\varphi(x)-\varphi(y))^2}{d(x,y)^{d+2s}} c(x,y)^Q dy dx &\le I_1 + I_2 \\ &\le \left(\frac{\mathfrak{f}^n}{1-\mathfrak{f}^{-2(1-s)}} + \frac{\mathfrak{f}^n}{1-\mathfrak{f}^{-2s}} \right) C_{VU}^2 \mathcal{G} \xi^{2s} R^{n-2s} \end{aligned}$$

and proves the lemma. □

Lemma 11.5.4. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$ such that $\mathbb{E}[c^{wQ}]$ for some $w \ge 2, Q > 1$. Let $B \subset \mathbb{Z}^n$ be a ball with radius $R \ge 1$. Then for every $\delta > 0$*

$$\mathbb{P}\left(\exists i \in \mathbb{Z}, \sum_{(x,y)\in F_i} c(x,y)^Q > \mathbb{E}[c^Q + \delta]|F_i| \right) \le C_{(11.5.4)} \mathbb{E}[c^{wQ}] R^{-\frac{nw}{2}},$$

with $C_{(11.5.4)} \equiv C_{(11.5.4)}(w,\delta,n)$ and $F_i \equiv F_i(B)$.

Proof. We will us use deviation estimate from Lemma 11.1.4 on the i.i.d. variables $\{c(x,y)^Q\}_{(x,y)\in F_i}$. There is exactly $|F_i| := \#F_i$ of them and this number is greater or equal to $C_{VL}\mathfrak{f}^{(i-1)n}R^{2n}$ by Lemma 11.5.2. Therefore

$$\begin{aligned} \mathbb{P}\left(\sum_{(x,y)\in F_i} c(x,y)^Q > \mathbb{E}[c^Q + \delta]|F_i| \right) &\le 2^w C_R(w) \delta^{-w} \mathbb{E}[c^{wQ}] |F_i|^{-\frac{w}{2}} \\ &\le 2^w C_R(w) \delta^{-w} \mathbb{E}[c^{wQ}] C_{VL}^{-\frac{w}{2}} R^{-nw} \mathfrak{f}^{-\frac{(i-1)nw}{2}}. \end{aligned}$$

Summing this over $i \in \mathbb{Z}^n$ implies (recall $F_i = \emptyset$ for $i < -\log_{\mathfrak{f}} R$)

$$\begin{aligned} &\mathbb{P}\left(\exists i \in \mathbb{Z}, \sum_{(x,y)\in F_i} c(x,y)^Q > (\mathbb{E}[c^Q] + \delta)|F_i| \right) \\ &\quad \le \sum_{i=-\lceil \log_{\mathfrak{f}} R\rceil}^{\infty} \mathbb{P}\left(\sum_{y\in F_i} c(x,y)^Q > (\mathbb{E}[c^Q] + \delta)|F_i| \right) \\ &\quad \le 2^w C_R(w) C_{VL}^{-\frac{w}{2}} \delta^{-w} \mathbb{E}[c^{wQ}] R^{-nw} \sum_{i=-\lceil \log_{\mathfrak{f}} R\rceil}^{\infty} \mathfrak{f}^{-\frac{(i-1)nw}{2}}. \end{aligned}$$

The only thing left to do is to estimate the sum by

$$\sum_{i=-\lceil \log_{\mathfrak{f}} R\rceil}^{\infty} \mathfrak{f}^{-\frac{(i-1)nw}{2}} \leq \mathfrak{f}^{\frac{(\lceil \log_{\mathfrak{f}} R\rceil+1)nw}{2}} \sum_{j=0}^{\infty} \mathfrak{f}^{-\frac{jnw}{2}} \leq \frac{\mathfrak{f}^{nw}}{1-\mathfrak{f}^{-\frac{nw}{2}}} R^{\frac{nw}{2}}$$

and obtain

$$\mathbb{P}\left(\exists i \in \mathbb{Z}, \sum_{(x,y)\in F_i} c(x,y)^Q > \mathbb{E}[c^Q+\delta]|F_i|\right) \leq \frac{\mathfrak{f}^{nw}}{1-\mathfrak{f}^{-\frac{nw}{2}}} C_R(w) C_{VL}^{-\frac{w}{2}} \delta^{-w} \mathbb{E}[c^{wQ}] R^{-\frac{nw}{2}}.$$

Taking

$$C_{(11.5.4)} \equiv C_{(11.5.4)}(w,\delta,n) := \frac{\mathfrak{f}^{nw}}{1-\mathfrak{f}^{-\frac{nw}{2}}} C_R(w) C_{VL}^{-\frac{w}{2}} \delta^{-w}$$

proves the claim. □

Theorem 11.5.5. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$ such that $\mathbb{E}[c^{wQ}] < \infty$ for some $Q \geq 1$ and $w > 2 + \frac{2}{n}$. Then there exist non random $\theta \equiv \theta(w,n) \in (0,1)$, $C_C^{iid} \equiv C_C^{iid}(\mathbb{E}[c^{wQ}], Q, s, n)$ and, for every $x_\star \in \mathbb{Z}^n$, a random variable $R_\star \equiv R_\star(x_\star, c, Q, s, n, \omega)$ such that $\mathcal{E}$ $\mathbb{P}$-a.e. satisfies $\star\boldsymbol{CE}[x_\star, R_\star, \theta; s, Q, \gamma = 0, C_C^{iid}]$.*

Written out explicitly, for all $R_0 > R_\star, x_0 \in B(x_\star, R_0)$, $R > R_0^\theta$ and Lipschitz $\varphi : \mathbb{Z}^n \to [0,1]$, with $\xi := R \operatorname{Lip}\varphi$,

$$\left(\int_{B(x_0,R)} \Gamma\varphi(x)^Q dx\right)^{\frac{1}{Q}} \leq C_C^{iid} \xi^{2s} R^{\frac{n}{Q}-2s}.$$

Proof. For $\theta \in (0,1)$ and $R_0 \geq 1$ we are interested in estimating the probability that

$$\left(\int_{B(x_0,R)} \Gamma\varphi(x)^Q dx\right)^{\frac{1}{Q}} \leq A \xi^{2s} R^{\frac{n}{Q}-2s} \tag{11.13}$$

fails for certain $A > 0$ and some admissible choice of x_0, R_0, R and φ. Using Lemma 11.5.4 we intend to guarantee that this probability is small for the correct choice of A, that is, we are looking to bound

$$P(\theta, R_0, A) := \mathbb{P}\left(\exists x_0 \in B(x_\star, R_0), \exists R \geq R_0^\theta, \exists \varphi : \text{ Ineq. (11.13) fails on } B(x,R)\right).$$

First of all, we claim that this probability can be estimated considering only R of the form 2^l for $l \in \mathbb{N}$. To see this, take some $x_0 \in B(x_\star, R_0)$ and suppose that Ineq. (11.13) holds for all $R = 2^l$, $l \in \mathbb{N}$ and all φ. For arbitrary $R \geq R_0^\theta$, Lipschitz φ, set $\xi := R \operatorname{Lip}\varphi$ and find $l \in \mathbb{N}$ such that $2^l \leq R \leq 2^{l+1}$. Then Ineq. (11.13) applied with $R = 2^{l+1}$ and $\xi = 2^{l+1} \operatorname{Lip}\varphi$ implies that

$$\begin{aligned}\left(\int_{B(x_0,R)} (\Gamma\varphi)^Q(y)dy\right)^{\frac{1}{Q}} &\leq \left(\int_{B(x_0,2^{l+1})} (\Gamma\varphi)^Q(y)dy\right)^{\frac{1}{Q}} \leq A(\operatorname{Lip}\varphi)^{2s} 2^{\frac{(l+1)n}{Q}} \\ &\leq 2^{\frac{n}{Q}} A (\operatorname{Lip}\varphi)^{2s} 2^{\frac{ln}{Q}} \leq 2^{\frac{n}{Q}} A \xi^{2s} R^{\frac{n}{Q}-2s},\end{aligned}$$

which shows that Ineq. (11.13) is also satisfied in $B(x_0, R)$ but with constant $2^{\frac{n}{Q}} A$ instead of A on the right. Hence, $P(\theta, R_0, 2^{\frac{n}{Q}} A)$ can be estimated by looking only at $R \geq R_0$ of the form 2^l for $l \in \mathbb{N}$. For $x_0 \in M, R = 2^l$, $l \in \mathbb{N}$, Lemma 11.5.3 and Lemma 10.3.2 combined show that the following implication holds (for $F_i := F_i(B(x_0,R))$):

$$(\forall i \in \mathbb{Z}) \quad \sum_{y \in F_i} c(x,y)^Q \leq \mathbb{E}[c^Q+1]|F_i| \implies$$

$$\implies \forall\varphi \left(\int_{B(x_0,R)} (\Gamma\varphi)^Q(y)dy\right)^{\frac{1}{Q}} \leq C_{(2.7.5)}^{\frac{1}{Q^*}} C_{(11.5.3)}^{\frac{1}{Q}} \mathbb{E}[c^Q+1]^{\frac{1}{Q}} \xi^{2s} R^{\frac{n}{Q}-2s}$$

where, in the second line, $\xi = R\,\mathrm{Lip}\,\varphi$. Setting $A = 2^{\frac{n}{Q}} C_{(2.7.5)}^{\frac{1}{Q^*}} C_{(11.5.3)}^{\frac{1}{Q}} \mathbb{E}[c^Q+1]^{\frac{1}{Q}}$ this proves that

$$P(\theta, R_0, A) \leq \sum_{x \in B(x_\star, R_0)} \sum_{l=\lceil \log_2 R_0^\theta \rceil}^{\infty} \mathbb{P}\left(\exists i \in \mathbb{Z} : \sum_{y \in F_i} c(x,y)^Q > \mathbb{E}[c^Q+1]|F_i| \right).$$

But $P(\theta, R_0, A)$ is decreasing in A so defining

$$C_C^{iid} \equiv C_C^{iid}(\mathbb{E}[c^{wQ}], Q, n, s) = 2^{\frac{n}{Q}} C_{(2.7.5)}^{\frac{1}{Q^*}} C_{(11.5.3)}^{\frac{1}{Q}} \mathbb{E}[c^{wQ}+2]^{\frac{1}{Q}} > A$$

the previous estimate also holds with C_C^{iid} instead of A. Using Lemma 11.5.4 with $\delta = 1$ gives

$$\begin{aligned} P(\theta, R_0, C_C^{iid}) &\leq C_{(11.5.4)} \mathbb{E}[c^{wQ}] \sum_{x \in B(x_\star, R_0)} \sum_{l=\lceil \log_2 R_0^\theta \rceil}^{\infty} 2^{-\frac{lnw}{2}} \\ &\leq C_{(11.5.4)} \mathbb{E}[c^{wQ}] R_0^{n - \frac{\theta n w}{2}} \sum_{l=0}^{\infty} 2^{-\frac{lnw}{2}}. \end{aligned}$$

The last expression is summable in R_0 whenever $n - \frac{\theta n w}{2} < -1$ which is equivalent to $\frac{\theta n w}{2} > n + 1$. If $w > 2 + \frac{2}{n}$, then $\frac{n+1}{nw/2-1} < 1$ and it is possible to choose any $\theta \in \left(\frac{2(1+1/n)}{w}, 1\right)$, say $\theta \equiv \theta(w,n) := \frac{1}{2} + \frac{2(1+1/n)}{w}$ to be specific. Borel-Cantelli lemma now proves that random variable $R_\star(x_\star) \equiv R_\star(x_\star, c, Q, C_C^{iid}, \theta, s, n, \omega)$ defined by

$$R_\star = \sup\{R_* : \star\mathbf{CE}[x_\star, R_*, \theta; s, Q, \gamma = 0, C_C^{iid}] \text{ fails}\} + 1$$

is almost surely finite. Dependencies reduce to $R_\star(x_\star) \equiv R_\star(x_\star, c, p, s, n, \omega)$ and $C_C^{iid} \equiv C_C^{iid}(\mathbb{E}[c^{wQ}], Q, n, s)$, which proves the theorem. □

11.6. Tail estimates

Lemma 11.6.1. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$ such that $\mathbb{E}[c^w] < \infty$ for some $w \geq 2$. Let $R > 0$, $x \in \mathbb{Z}^n$ be arbitrary, set $B = B(x,R)$ and let $\mathfrak{c} > 0$ be such that $\mathfrak{c} C_{VL} \geq C_{VU} + 1$. Then for every $\delta > 0$*

$$\mathbb{P}\left(\sum_{y \in \mathbb{Z}^n \setminus B} \frac{c(x,y)}{d(x,y)^{n+2s}} > \frac{C_{VU} \mathfrak{c}^n \mathbb{E}[c+\delta]}{1 - \mathfrak{c}^{-2s}} R^{-2s} \right) \leq \frac{2^w \delta^{-w} C_R(w)}{1 - \mathfrak{c}^{-\frac{nw}{2}}} R^{-\frac{nw}{2}} \mathbb{E}[c^w].$$

Proof. The plan is again to make use of Lemma 11.1.4 which is based on Rosenthal's inequality. We again write the sums in $\mathbb{Z}^n$ as integrals over counting measure $\#$, $\int_x f(x)dx := \int_x f(x)\#(dx) = \sum_x f(x)$ for the rest of the proof. Define annuli $A_i := \mathfrak{c}^{i+1} B \setminus \mathfrak{c}^i B$ which, owing to the assumption on $\mathfrak{c}$ and volume regularity of $\mathbb{Z}^n$, have the property that

$$R^n \mathfrak{c}^{ni} \leq |c^{i+1} B| - |c^i B| \leq |A_i| \leq |c^{i+1} B| \leq C_{VU} R^n \mathfrak{c}^{n(i+1)}.$$

If for every $i \geq 0$

$$⨍_{A_i} c(x,y)dy \leq \mathbb{E}[c+\delta],$$

then

$$\int_{B^c} \frac{c(x,y)}{d(x,y)^{n+2s}} dy \le \sum_{i=0}^{\infty} \int_{A_i} \frac{c(x,y)}{d(x,y)^{n+2s}} dy \le \sum_{i=0}^{\infty} (R\mathfrak{c}^i)^{-n-2s} \int_{A_i} c(x,y)dy$$
$$\le C_{VU}\mathfrak{c}^n R^{-2s} \sum_{i=0}^{\infty} \mathfrak{c}^{-2is} \fint_{A_i} c(x,y)dy \le C_{VU}\mathfrak{c}^n \mathbb{E}[c+\delta] R^{-2s} \sum_{i=0}^{\infty} \mathfrak{c}^{-2is}$$
$$\le \frac{C_{VU}\mathfrak{c}^n}{1-\mathfrak{c}^{-2s}} \mathbb{E}[c+\delta] R^{-2s}.$$

Therefore

$$\mathbb{P}\left(\int_{B^c} \frac{c(x,y)}{d(x,y)^{n+2s}} dy > \frac{C_{VU}\mathfrak{c}^n \mathbb{E}[c+\delta]}{1-\mathfrak{c}^{-2s}} R^{-2s}\right) \le \sum_{i=0}^{\infty} \mathbb{P}\left(\fint_{A_i} c(x,y)dy > \mathbb{E}[c+\delta]\right).$$

Deviation estimate from Lemma 11.1.4 applies easily to probabilities on the right and gives

$$\mathbb{P}\left(\fint_{A_i} c(x,y)dy > \mathbb{E}[c+\delta]\right) \le 2^w C_R(w)\delta^{-w}|A_i|^{-\frac{w}{2}}\mathbb{E}[c^w]$$
$$\le 2^w C_R(w)\delta^{-w} R^{-\frac{nw}{2}} \mathfrak{c}^{-\frac{niw}{2}} \mathbb{E}[c^w].$$

Summing up in i we come back to

$$\mathbb{P}\left(\int_{B^c} \frac{c(x,y)}{d(x,y)^{n+2s}} dy > \frac{C_{VU}\mathfrak{c}^n \mathbb{E}[c+\delta]}{1-\mathfrak{c}^{-2s}} R^{-2s}\right) \le \frac{2^w\delta^{-w}C_R(w)}{1-\mathfrak{c}^{-\frac{nw}{2}}} R^{-\frac{nw}{2}}\mathbb{E}[c^w],$$

which proves the statement. □

Theorem 11.6.2. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$ such that $\mathbb{E}[c^w] < \infty$ for some $w > 2 + \frac{2}{n}$. Then there exist non random $\theta \equiv \theta(n,w) \in (0,1)$, $C_T \equiv C_T(\mathbb{E}[c], n, s)$ such that, for every $x_\star \in \mathbb{Z}^n$, we can find a random variable $R_\star \equiv R_\star(x_\star, \omega, \theta, c, n, s)$ such that $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies $\star\boldsymbol{TB}[x_\star, R_\star(x_\star), \theta; C_T]$. Explicitly,*

$$\forall R_0 \ge R_\star, \forall x \in B(x_\star, R_0), \forall R \ge R_0^\theta, \quad \int_{B(x,R)^c} \frac{c(x,y)}{d(x,y)^{n+2s}} dy \le C_T R^{-2s}.$$

Proof. Consider, for $R_0 \ge 1$ and $A > 0$, the probability

$$P(R_0,\theta,A) := \mathbb{P}\left(\exists R \ge R_0^\theta, \exists x \in B(x_\star, R_0) : \int_{B(x,R)^c} k(x,y)dy > AR^{-2s}\right)$$

and notice that it can be bounded in terms of $R \ge R_0^\theta$ of the form $R = 2^l$ for $l \in \mathbb{N}_0$. Let us explain how this works. If $R < 1$, then $B(x,R)^c = B(x,1)^c$ and we can just use $R = 1$. On the other hand, if $R \ge 1$, let us find $l \in \mathbb{N}_0$ such that $2^l \le R < 2^{l+1}$ and observe that, if $\int_{B(x,R)^c} k(x,y)dy > 4AR^{-2s}$ for some $A > 0$, then

$$\int_{B(x,2^l)^c} k(x,y)dy \ge \int_{B(x,R)^c} k(x,y)dy \ge 4AR^{-2s} > 4A\left(\frac{R}{2^l}\right)^{-2s} 2^{-2ls} \ge A2^{-2ls}.$$

Because of this we can now estimate

$$P(R_0,\theta,4A) \le \sum_{x\in B(x_\star,R_0)} \sum_{l=\lceil \log_2 R_0^\theta \rceil}^{\infty} \mathbb{P}\left(\int_{B(x,2^l)^c} k(x,y)dy > A2^{-2ls}\right).$$

Taking $\mathfrak{c} := \frac{C_{VU}+1}{C_{VL}}$, $\delta = 1$, $A = \frac{C_{VU}\mathfrak{c}^n\mathbb{E}[c+1]}{1-\mathfrak{c}^{-2s}}$ and using Lemma 11.6.1 results in

$$\begin{aligned} P\left(R_0, \theta, \frac{4C_{VU}\mathfrak{c}^n\mathbb{E}[c+1]}{1-\mathfrak{c}^{-2s}}\right) &\leq \sum_{x\in B(x_\star,R_0)} \sum_{l=\lceil \log_2 R_0^\theta \rceil}^{\infty} \frac{2^w C_R(w)}{1-\mathfrak{c}^{\frac{nw}{2}}}\mathbb{E}[c^w]2^{-\frac{lnw}{2}} \\ &\leq \frac{2^w C_R(w)}{1-\mathfrak{c}^{\frac{nw}{2}}}\mathbb{E}[c^w]R_0^{n-\frac{\theta nw}{2}} \sum_{l=0}^{\infty} 2^{\frac{lnw}{2}} \\ &\leq \frac{2^w C_R(w)\mathbb{E}[c^w]}{(1-\mathfrak{c}^{\frac{nw}{2}})(1-2^{\frac{nw}{2}})} R_0^{n-\frac{\theta nw}{2}}. \end{aligned}$$

The right side is summable in R_0 if we can find θ such that the exponent is smaller than -1. This is possible if and only if $w > 2 + \frac{2}{n}$ just like in the proof of Theorem 11.5.5. Thus, we can define $C_T \equiv C_T(n, s, \mathbb{E}[c]) := 4A$ and make use of Borel-Cantelli lemma (Theorem 11.1.1) to conclude that random variable $R_\star(x_\star) \equiv R_\star(x_\star, \omega, c, \theta, n, s) > 0$, defined by

$$R_\star(x_\star) := \sup\{R_* \in \mathbb{N} : \star\mathbf{TB}[x_\star, R_*, \theta; C_T] \text{ fails}\} + 1,$$

is $\mathbb{P}$-a.s. finite, which proves the theorem. □

11.7. Weak Harnack inequality and Hölder regularity

Theorem 11.7.1. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$ ($n \geq 2$) such that $\mathbb{E}[c^p] < \infty$ for some $p > \frac{n+1}{s}$. Then there exist non random $\theta \equiv \theta(p, n, s) \in (0, 1)$, $Q \equiv Q(p, s, n) \in [1, \infty)$, $C_C^{iid} \equiv C_C^{iid}(p, \mathbb{E}[c^p], s, n)$, $C_{S1}^{iid} \equiv C_{S1}^{iid}(p, \mathfrak{p}, \nu, s, n)$, $C_{S2}^{iid} \equiv C_{S2}^{iid}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n)$, $C_P^{iid} \equiv C_P^{iid}(n, \mathfrak{p}, \nu)$, $C_{PH}^{iid} \equiv C_{PH}^{iid}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n)$, $C_H^{iid} \equiv C_H^{iid}(s)$, $\eta^{iid} \equiv \eta^{iid}(s, n, p, \mathbb{E}[c^p], \mathfrak{p}, \nu)$ and, for every $x_\star \in \mathbb{Z}^n$, a random variable $R_\star(x_\star) \equiv R_\star(x_\star, \omega, c, p, \mathfrak{p}, \nu, n, s)$ such that $\mathbb{P}$-a.s. $\mathcal{E}$ satisfies*

(i) $\star\boldsymbol{CE}[x_\star, R_\star(x_\star), \theta; s, Q, \gamma = 0, C_C^{iid}]$,

(ii) $\star\boldsymbol{SI}[x_\star, R_\star(x_\star), \theta; s, \rho = (1 - 2s/n + 1/q)^{-1}, \zeta = Q, C_{S1}^{iid}, C_{S2}^{iid}, \gamma = 0]$,

(iii) $\star\boldsymbol{PI}[x_\star, R_\star(x_\star), \theta; s, C_P^{iid}]$,

(iv) $\star\boldsymbol{WPHI}[x_\star, R_\star(x_\star), \theta; s, C_{PH}^{iid}, Q]$,

(v) $\star\boldsymbol{HR}[x_\star, R_\star(x_\star), \theta; \eta^{iid}, C_H^{iid}]$.

Proof. Assumption on p allows us to find $Q, w \geq 1$ such that $Q > \frac{n}{2s}$, $w > 2 + \frac{2}{n}$ and $Qw = p$. For such $Q \equiv Q(p, s, n)$ and $w \equiv w(p, s, n)$ let us choose any $q > \left(\frac{2s}{n} - \frac{1}{Q}\right)^{-1}$ which assures that

$$\frac{1}{q} + \frac{1}{Q} < \frac{2s}{n}. \tag{11.14}$$

For concreteness let us fix $q \equiv q(p, s, n) := 2\left(\frac{2s}{n} - \frac{1}{Q}\right)^{-1}$. Due to Theorem 11.5.5 there exist $\theta^{(E)} \equiv \theta^{(E)}(n, w) \equiv \theta^{(E)}(n, s, p) \in (0, 1)$, $C_C^{iid} \equiv C_C^{iid}(\mathbb{E}[c^{wQ}], Q, s, n) \equiv C_C^{iid}(\mathbb{E}[c^p], p, s, n)$ and a $\mathbb{P}$-a.s. finite random variables $R_\star^{(E)}(x_\star) \equiv R_\star^{(E)}(x_\star, \omega, c, Q, s, n) \equiv R_\star^{(E)}(x_\star, \omega, c, p, s, n)$ such that $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies

$$\star\mathbf{CE}[x_\star, R_\star^{(E)}(x_\star), \theta^{(E)}; s, Q, \gamma = 0, C_C^{iid}].$$

By Item (ii) of Theorem 11.2.2 there exist $C_{S1}^{iid} \equiv C_{S1}^{iid}(n, s, q, \mathfrak{p}, \nu) \equiv C_{S1}^{iid}(n, s, p, \mathfrak{p}, \nu)$, $C_{S2}^{iid} \equiv C_{S2}^{iid}(n, s, q, \mathfrak{p}, \nu, C_C^{iid}) \equiv C_{S2}^{iid}(n, s, p, \mathfrak{p}, \nu, \mathbb{E}[c^p])$ and a $\mathbb{P}$-a.s. finite $R_\star^{(S)}(x_\star) \equiv R_\star(x_\star, \omega, c, \theta^{(E)}, n, s, q, \mathfrak{p}, \nu) \equiv R_\star(x_\star, \omega, c, n, s, p, \mathfrak{p}, \nu)$ such that $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies

$$\star\mathbf{SI}[x_\star, R_\star^{(S)}(x_\star) \vee R_\star^{(E)}(x_\star), \theta^{(E)}; s, \rho = (1 - 2s/n + 1/q)^{-1}, \zeta = Q, C_{S1}^{iid}, C_{S2}^{iid}, \gamma = 0].$$

Moving on, Theorem 11.3.1 proved that there exist $C_P^{iid} \equiv C_P^{iid}(n, \mathfrak{p}, \nu)$ and an $\mathbb{P}$-a.s. finite random variables $R_\star^{(P)}(x_\star) \equiv R_\star^{(P)}(x_\star, \omega, c, \theta^{(E)}, n, \mathfrak{p}, \nu) \equiv R_\star^{(P)}(x_\star, \omega, c, p, s, n, \mathfrak{p}, \nu)$ such that $\mathbb{P}$-a.s. $\mathcal{E}$ satisfies

$$\star\mathbf{PI}[x_\star, R_\star^{(P)}(x_\star), \theta^{(E)}; s, C_P^{iid}].$$

This verifies Items (i) to (iii) for all $\theta \geq \theta^{(E)}$ and $R_\star(x_\star) \geq R_\star^{(E)}(x_\star) \vee R_\star^{(S)}(x_\star) \vee R_\star^{(P)}(x_\star)$.

To get $\star$**WPHI** we take $\theta^{(H)} \equiv \theta^{(H)}(n, s, p) = \frac{1+\theta^{(E)}}{2}$ and

$$R_\star^{(H)}(x_\star) \equiv R_\star^{(H)}(x_\star, \omega, c, p, s, n, \mathfrak{p}, \nu) = 4^{(1-\theta^{(E)})^{-1}} \vee R_\star^{(E)}(x_\star) \vee R_\star^{(S)}(x_\star) \vee R_\star^{(P)}(x_\star).$$

Then for all $R_0 \geq R_\star^{(H)}(x_\star)$, $x_0 \in B(x_\star, R_0)$ and $R \geq R_0^{\theta^{(H)}}$ we have

$$R/2 \geq R_0^{\theta^{(H)}}/2 \geq R_0^{\theta^{(E)}} R_\star^{(H)}(x_\star)^{\theta^{(H)}-\theta^{(E)}}/2 \geq R_0^{\theta^{(E)}} 4^{(1-\theta^{(E)})^{-1}(1-\theta^{(E)})/2}/2 \geq R_0^{\theta^{(E)}}$$

which means that $\mathbb{P}$-a.s. $\mathcal{E}$ satisfies

- $\mathbf{CE}[x_0, [R/2, \infty); s, Q, 2s, C_C^{iid}]$,
- $\mathbf{SI}[x_0, [R/2, \infty); s, \rho = (1 - 2s/n + 1/q)^{-1}, \zeta = Q, C_{S1}^{iid}, C_{S2}^{iid}, \gamma = 0]$ and
- $\mathbf{PI}[x_0, [R/2, \infty); s, C_P^{iid}]$.

This verifies assumptions Items (i), (ii) and (v) of Theorem 6.5.1. Item (iii) is verified by Lemma 2.6.1 which confirms that $\mathbb{Z}^n$ with counting measure satisfies $\mathbf{V}[x_0, [R/2, \infty), n, C_{VL}, C_{VU}]$ and Item (iv) is identical to Ineq. (11.14). Therefore Theorem 6.5.1 implies that there exists

$$C_{PH}^{iid} \equiv C_{PH}^{iid}\left(s, n, q, Q, C_C^{iid}, C_{S1}^{iid}, C_{S2}^{iid}, C_P^{iid}, \gamma, C_{VL}, C_{VU}\right) \equiv C_{PH}^{iid}(s, n, p, \mathbb{E}[c^p], \mathfrak{p}, \nu)$$

such that $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies $\mathbf{WPHI}[x_0, [R, \infty); s, C_{PH}^{iid}, Q]$. This is sufficient to conclude that $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies

$$\star\mathbf{WPHI}[x_0, R_\star^{(H)}(x_\star), \theta^{(H)}; s, C_{PH}^{iid}, Q]$$

which proves Item (iv). An immediate consequence, due to Theorem 6.5.2, is that

$$\star\mathbf{WEHI}[x_0, R_\star^{(H)}(x_\star), \theta^{(H)}; C_E = C_{PH}^{iid}]$$

also holds $\mathbb{P}$-a.s. Finally, Theorem 6.6.3, with the help of Items (i) and (iv), implies that there exists an

$$\eta^{iid} \equiv \eta^{iid}(C_{PH}^{iid}, C_C^{iid}, s, n, C_{VL}, C_{VU}) \equiv \eta(s, n, p, \mathbb{E}[c^p], \mathfrak{p}, \nu)$$

and $C_H^{iid} \equiv C_H^{iid}(s)$ such that

$$\star\mathbf{HR}[x_0, R_\star(x_\star), \theta; \eta^{iid}, C_H^{iid}]$$

is satisfied $\mathbb{P}$-a.s., which proves Item (v).

Note in the end that replacing $R_\star^{(E)}, R_\star^{(S)}, R_\star^{(P)}$ with the larger $R_\star^{(H)}$ and $\theta^{(E)}, \theta^{(S)}, \theta^{(P)}$ with larger $\theta^{(H)}$ preserves corresponding $\star$**Property** which reduces some of the notation from the proof by taking $R_\star := R_\star^{(H)}$ and $\theta := \theta^{(H)}$. □

11.8. Exit time estimates

Theorem 11.8.1. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$ ($n \geq 2$) such that $\mathbb{E}[c^p] < \infty$ for some $p > \frac{n+1}{s}$. Then there exist non random $\theta \equiv \theta(p, s, n) \in (0, 1)$, $C_{(E\geq)}^{iid} = C_{(E\geq)}^{iid}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n)$, $C_{(E\leq)}^{iid} \equiv C_{(E\leq)}^{iid}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n)$, $\varepsilon^{iid} \equiv \varepsilon^{iid}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n)$, $\delta_{iid} \equiv \delta_{iid}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n)$, $C_{EP}^{iid} \equiv C_{EP}^{iid}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n)$, $C_T \equiv C_T(\mathbb{E}[c^p], n, s)$ and, for every fixed $x_\star \in \mathbb{Z}^n$, a random variable $R_\star(x_\star) \equiv R_\star(x_\star, \omega, c, p, \mathfrak{p}, \nu, n, s)$ such that $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies* *(i)* $\star\boldsymbol{ETE}[x_\star, R_\star(x_\star), \theta; s, C^{iid}_{(E\geq)}, C^{iid}_{(E\leq)}]$,

(ii) $\star\boldsymbol{SE}[x_\star, R_\star(x_\star), \theta; s, \varepsilon^{iid}, \delta_{iid}]$ *and*

(iii) for all $x_\star \in \mathbb{Z}^n$, $t > 0$, $R_0 \geq R_\star(x_\star)$, $x_0 \in B(x_\star, R_0/2)$, $R \leq R_0/2$

$$1 - P_t^{B(x_0,R)} 1 \leq C^{iid}_{EP} \frac{t}{R^{2s}} \left(\frac{R_0^{2s\theta}}{t} \vee \delta_{iid}^{-2s} \vee 4C^{iid}_T \right) \text{ in } B\left(x_0, R_0^\theta \vee \delta_{iid}^{-1} t^{\frac{1}{2s}}\right). \tag{11.15}$$

In particular, the semigroup corresponding to $\mathcal{E}$ *is* $\mathbb{P}$*-a.s. conservative.*

Proof. From Theorem 11.7.1 we know that there exist $\theta^{(H)} \equiv \theta^{(H)}(p, n, s) \in (0, 1)$, $Q \equiv Q(p, n, s) \geq 1$, $C^{iid}_{PH} \equiv C^{iid}_{PH}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n), C^{iid}_C \equiv C^{iid}_C(\mathbb{E}[c^p], s, n)$ and $R^{(H)}_\star(x_\star) \equiv R^{(H)}_\star(x_\star, \omega, c, p, \mathfrak{p}, \nu, n, s)$ such that $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies

- $\star\mathbf{CE}[x_\star, R_\star, \theta^{(H)}; s, Q, \gamma = 0, C^{iid}_C]$ and
- $\star\mathbf{WEHI}[x_0, R^{(H)}_\star(x_\star), \theta^{(H)}; C^{iid}_{PH}]$.

On the other hand, due to Lemma 11.4.1, for this $\theta^{(H)}$ there exist $C^{iid}_K \equiv C^{iid}_K(\nu, \mathfrak{p}, n, s) > 0$ and $R^{(K)}_\star(x_\star) \equiv R^{(K)}_\star(x_\star, \omega, c, \theta^{(H)}, n, \mathfrak{p})$ such that $\mathbb{P}$-a.s. $\mathcal{E}$ satisfies

$$\star\mathbf{AKB}{\geq}[x_\star, R^{(K)}_\star(x_\star), \theta^{(H)}; s, C^{iid}_K].$$

Writing $\star$-quantifiers out, for all $R_0 \geq R^{(H)}_\star \vee R^{(K)}_\star$, $x_0 \in B(x_0, R_0)$, we have

- $\mathbf{CE}[x_0, [R_0^{\theta^{(H)}}, \infty); s, Q, \gamma = 0, C^{iid}_C]$,
- $\mathbf{WEHI}[x_0, [R_0^{\theta^{(H)}}, \infty); C^{iid}_{EH}]$,
- $\mathbf{AKB}{\geq}[x_0, [R_0^{\theta^{(H)}}, \infty); s, C^{iid}_K]$,
- $\mathbf{V}[x_0, [1, \infty); n, C_{VL}(\mathbb{Z}^n), C_{VU}(\mathbb{Z}^n)]$.

(where the last statement comes from Lemma 2.6.1). As the reader might suspect, the aim is to use Theorem 7.1.5 but there are three more assumptions that need to be verified. Firstly, $\mathbb{Z}^n$ with counting measure is a separable metric measure space with Radon measure of full support which verifies Assumption 2.5.3. Secondly, that $\mathcal{E}$ is a regular Dirichlet form on $L^2(M)$ $\mathbb{P}$-a.s. satisfying Assumption 4.0.2 we know from Corollary 9.2.2. This verifies Assumption 2.5.3. Finally, $\|G^B 1\|_{L^2(B)} < \infty$ by Lemma 11.1.5 for every ball $B \subset \mathbb{Z}^n$. With the application of Theorem 7.1.5 now justified, we can find

$$C^{iid}_{(E\geq)} \equiv C^{iid}_{(E\geq)}(C^{iid}_{PH}, C^{iid}_C, \gamma, s, n, C_{VL}, C_{VU}) \equiv C_{(E\geq)}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n) > 0$$

and

$$C^{iid}_{(E\leq)} \equiv C^{iid}_{(E\leq)}(C^{iid}_{PH}, C^{iid}_K) \equiv C_{(E\leq)}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n) < \infty$$

such that $\mathbf{ETE}[x_0, [2R_0^{\theta^{(H)}}, \infty); s, C^{iid}_{(E\geq)}, C^{iid}_{(E\leq)}]$ is satisfied. Redefining $\theta^{(S)} \equiv \theta(p, n, s) := \frac{\theta^{(H)}+1}{2}$ and

$$R^{(S)}_\star(x_\star) \equiv R^{(S)}_\star(x_\star, \omega, c, p, \mathfrak{p}, \nu, n, s) := R^{(H)}_\star(x_\star) \vee R^{(K)}_\star(x_\star) \vee 4^{(\theta - \theta^{(H)})^{-1}}$$

implies that for $R_0 \geq R^{(S)}_\star(x_\star)$

$$R_0^{\theta^{(S)}} \geq R_0^{\theta^{(H)}} \left(R^{(S)}_\star(x_\star)\right)^{\theta^{(S)} - \theta^{(H)}} \geq 2R_0^{\theta^{(H)}}$$

which shows that $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies

$$\star\mathbf{ETE}[x_\star, R_\star^{(S)}(x_\star), \theta^{(S)}; s, C^{iid}_{(E\geq)}, C^{iid}_{(E\leq)}].$$

Now Theorem 7.2.1 proves that there exist $\varepsilon^{iid} \equiv \varepsilon^{iid}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n)$, $\delta_{iid} \equiv \delta_{iid}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n)$ such that $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies

$$\star\mathbf{SE}[x_\star, R_\star^{(S)}(x_\star), \theta^{(S)}; s, \varepsilon^{iid}, \delta_{iid}].$$

The conservativeness of $\mathcal{E}$ follows from the last property through Theorem 7.2.2.

On an independent argument line, Theorem 11.6.2 proved that there exists $\theta^{(T)} \equiv \theta^{(T)}(n, p)$, $C_T \equiv C_T(\mathbb{E}[c], n, s)$ and $R_\star^{(T)}(x_\star) \equiv R_\star^{(T)}(x_\star, \omega, \theta^{(T)}, c, n, s)$ such that $\mathbb{P}$-a.s. $\mathcal{E}$ satisfies

$$\star\mathbf{TB}[x_\star, R_\star^{(T)}(x_\star), \theta^{(T)}; C_T].$$

(Checking the proof of Theorem 11.6.2 the previous statement, due to monotonicity in C_T, remains true if $\mathbb{E}[c^p + 2]$ is used instead of $\mathbb{E}[c + 1]$ when defining C_T). This allows us to consider C_T as depending on $\mathbb{E}[c^p]$ and not on $\mathbb{E}[c]$.)

Let us define $\theta \equiv \theta(n, p, s) = \theta^{(S)} \vee \theta^{(T)}$,

$$R_\star(x_\star) \equiv R_\star(x_\star, \omega, c, p, \mathfrak{p}, \nu, n, s) = R_\star^{(S)}(x_\star) \vee R_\star^{(T)}(x_\star)$$

and take arbitrary $R_0 \geq R_\star(x_\star)$, $x_0 \in B(x_\star, R_0/2)$. Denoting $B_\star = B(x_0, R_0)$ and $\mathcal{R}_0 = R_0^\theta$ in Theorem 7.3.2 we find

$$C^{iid}_{EP} \equiv C^{iid}_{EP}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n) := C_{(7.3.2)}(\varepsilon^{iid}, s)$$

such that $\mathbb{P}$-a.s. the following estimate holds. For all $t > 0$, $\mathcal{R} \in (0, R_0/2]$,

$$1 - P_t^{B(x_0, \mathcal{R})} 1 \leq C^{iid}_{EP} \left(\frac{\mathcal{R}}{t^{1/2s}} \right)^{-2s} \left(\frac{R_0^{2s\theta}}{t} \vee \delta^{-2s} \vee 4C_T \right) \quad \text{in } B\left(x_0, R_0^\theta \vee \delta^{-1} t^{\frac{1}{2s}}\right),$$

which proves Ineq. (11.15). □

12. Convergence results

In this chapter we investigate the consequences that results from Chapters 10 and 11 have on the rescaled process $X_{m^{2s}t}/m$. In Section 12.1, we introduce the rescaled versions of $\mathcal{E}$ and P_t which we denote by $\mathcal{E}^{(m)}$ and $P_t^{(m)}$ respectively. Theorem 4 from [FH20], in the ergodic case, and Theorem 8.3 from [CKK13](in the i.i.d. case) prove that $\mathcal{E}^{(m)}$ converges in the generalized Mosco sense to a Dirichlet form of a rotationally symmetric stable process. As a consequence, $P_t^{(m)}f$ converges strongly in L^2-sense, for all $t > 0$ and $f \in L^2$. With the help of large scale Hölder regularity from Theorems 10.4.1 and 11.7.1 this L^2-convergence can be improved to the pointwise convergence of $P_t^{(m)}f(0)$ in case of both ergodic and i.i.d. conductance. Using arguments similar to the ones in Theorem 4.5 of [CKW18b] the pointwise convergence can be used to prove that $X_{m^{2s}t}/m$ converges in the sense of finite-dimensional distributions.

If c is an i.i.d. conductance, we can also prove the convergence of $X_{m^{2s}t}/m$ in Skorokhod space $\mathcal{D}([0,T],\mathbb{R}^n)$ for every $T > 0$. The limit can be identified from the convergence in the finite-dimensional distributions so one only need to prove the tightness of distributions of $X_{m^{2s}t}/m$. This is again done with the help of arguments from Theorem 4.5 of [CKW18b] and relies on the tightness criteria from [Ald78].

12.1. Rescaling

Definition 12.1.1. *For $n, m \in \mathbb{N}$ denote by Z_m the refinement of the lattice $\mathbb{Z}^n$,*

$$Z_m := \left(\frac{\mathbb{Z}}{m}\right)^n = \{(x_1, x_2, \dots x_n) \in \mathbb{R}^n | m x_i \in \mathbb{Z}, \forall i = 1, 2, \dots n\}.$$

Define $\mu_m = m^{-n}\#$ ($\#$ the counting measure on $\mathbb{Z}^n$) to be our default measure on Z_m and take $L^2(Z_m)$ to be the shorthand notation for $L^2(Z_m, \mu_m)$.
Furthermore, define dilation operators $D_m : L^2(\mathbb{Z}^n) \to L^2(Z_m)$ by $D_m f(x) = f(mx)$ for $x \in Z_m$.

Proposition 12.1.2. *For every $m \in \mathbb{N}$, $m^{n/2} D_m$ is a bijective isometry between $L^2(\mathbb{Z}^n)$ and $L^2(Z_m)$.*

Proof. For $g \in L(Z_m)$ the inverse of D_m is given by $D_m^{-1} g(x) = g(x/m)$, for $x \in \mathbb{Z}^n$. Moreover, for every $f \in L^2(\mathbb{Z}^n)$ we have

$$\|m^{n/2} D_m f\|^2_{L^2(Z_m)} = \sum_{x \in Z_m} m^n f(mx)^2 \mu_m = \sum_{x \in \mathbb{Z}^n} f(x)^2 = \|f\|^2_{L^2(\mathbb{Z}^n)}.$$

□

Definition 12.1.3. *Let c be a random conductance on $\mathbb{Z}^n$ such that $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is $\mathbb{P}$-a.s. a conservative regular Dirichlet form on $L^2(\mathbb{Z}^n)$. Let X_t denote the random walk, properly associated to Dirichlet form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$, from Theorem 9.3.1. Because $\mathcal{E}$ is conservative, we can assume that X_t takes values in $\mathbb{Z}^n$ instead of $\mathbb{Z}^n_\partial$ (see Corollary 2.5.12).*

For $m \in \mathbb{N}$, we define $X_t^{(m)}$ by $X_t^{(m)} = X_{m^{2s}t}/m$. Then $X_t^{(m)}$ takes values in Z_m. The distributions of $X_t^{(m)}$ on $\mathcal{D}([0,\infty],\mathbb{R}^n)$ under $X_t^{(m)} = x_0$ are denoted by $\mathbf{P}^{(m)}_{x_0}$ for every $x_0 \in Z_m$. The expectation with respect to $\mathbf{P}^{(m)}_{x_0}$ is denoted by $\mathbf{E}^{(m)}_{x_0}$. We denote by $(\mathcal{E}^{(m)}, \mathcal{D}[\mathcal{E}^{(m)}])$ and $P_t^{(m)}$ the regular Dirichlet form and the symmetric semigroup associated with $X_t^{(m)}$ on $L^2(Z_m)$ in the sense of Definition 2.5.11. For every $U \subset Z_m$, $P_t^{U,(m)}$ stands for the restricted semigroup corresponding to $(\mathcal{E}^{(m)}, \mathcal{D}[\mathcal{E}])$ in the sense of Definition 2.4.12.

Proposition 12.1.4 (Scaling relations)**.** *For $m \in \mathbb{N}$ and $f \in L^2(Z_m)$ we have the following three scaling relations:*

(i) $X_t^{(m)} = \dfrac{X_{m^{2s}t}}{m}$,

(ii) $P_t^{(m)} = D_m P_{m^{2s}t} D_m^{-1}$, *equivalently* $P_t^{(m)} f(x) = P_{m^{2s}t}\left[f\left(\frac{\cdot}{m}\right)\right](mx)$, *and*

(iii) $\mathcal{D}[\mathcal{E}^{(m)}] = D_m \mathcal{D}[\mathcal{E}]$ *and* $\mathcal{E}^{(m)}(f) = m^{-2n} \sum_{x,y \in Z_m} \dfrac{(f(x)-f(y))^2}{d(x,y)^{n+2s}} c(mx,my)$ *for every* $f \in L^2(Z_m)$.

Proof. Item (i) is just the definition of $X_t^{(m)}$. For all $f \in C_c(Z_m)$ and $x \in Z_m$

$$\begin{aligned} P_t^{(m)} f(x) &= \mathbf{E}_x[f(X_t^{(m)})] = \mathbf{E}_{mx}[D_m^{-1} f(X_{m^{2s}t})] = P_{m^{2s}t} D_m^{-1} f(mx) \\ &= D_m P_{m^{2s}t} D_m^{-1} f(x). \end{aligned}$$

Since both semigroups are bounded and $C_c(Z_m)$ is dense in $L^2(Z_m)$, we have $P_t^{(m)} = D_m P_{m^{2s}t} D_m^{-1}$, which proves Item (ii). Furthermore $m^{-n/2} D_m^{-1}$ is an isometry between $L^2(Z_m)$ and $L^2(\mathbb{Z}^n)$ by Proposition 12.1.2 so, for every $t > 0$,

$$\begin{aligned} \frac{1}{t}\left(P_t^{(m)} f - f, f\right)_{L^2(Z_m)} &= \frac{m^{-n}}{t}\left(D_m^{-1} P_t^{(m)} f - D_m^{-1} f, D_m^{-1} f\right)_{L^2(Z_m)} \\ &= \frac{m^{-n+2s}}{m^{2s} t}\left(P_{m^{2s}t} D_m^{-1} f - D_m^{-1} f, D_m^{-1} f\right)_{L^2(\mathbb{Z}^n)}. \end{aligned}$$

Passing to the limit $t \to \infty$ and using Lemma 1.3.4 of [FOT11] we find that $\mathcal{D}[\mathcal{E}] = D_m^{-1}(\mathcal{D}[\mathcal{E}^{(m)}])$ and

$$\begin{aligned} \mathcal{E}^{(m)}(f) = m^{-n+2s} \mathcal{E}(D_m^{-1} f) &= m^{-n+2s} \sum_{x,y \in \mathbb{Z}^n} \frac{(f(x/m) - f(y/m))^2}{d(x,y)^{n+2s}} c(x,y) \\ &= m^{-2n} \sum_{x',y' \in Z_m} \frac{(f(x') - f(y'))^2}{d(x',y')^{n+2s}} c(mx',my') \end{aligned}$$

where we used $d(mx,my) = md(x,y)$ property of the Euclidean distance on $\mathbb{R}^n$ in the last line. This shows Item (iii) and completes the proof. □

Definition 12.1.5. *Let c be a symmetrized ergodic or i.i.d. conductance such that $\mathbb{E}[c] < \infty$. We define $X_t^{(\infty)}$ to be the pure jump Lévy process determined by its Lévy measure (see [Ber96], Chapter I, Theorem 1)*

$$\Pi(dy) = \frac{\mathbb{E}[c]}{|y|^{n+2s}} dy.$$

We denote by $\mathcal{E}^{(\infty)}$ and $P_t^{(\infty)}$ the regular Dirichlet form and semigroup of the process $X_t^{(\infty)}$. It is known that for every $f \in L^2(\mathbb{R}^n)$

$$\mathcal{E}^{(\infty)}(f) = \mathbb{E}[c] \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(f(x) - f(y))^2}{|x-y|^{n+2s}} dxdy.$$

The distributions of $X^{(\infty)}$ on $\mathcal{D}([0,\infty], \mathbb{R}^n)$ under $X_t^{(\infty)} = x_0$ are denoted by $\mathbf{P}_{x_0}^{(\infty)}$ for every $x_0 \in \mathbb{R}^n$. The expectation with respect to $\mathbf{P}_{x_0}^{(\infty)}$ is denoted by $\mathbf{E}_{x_0}^{(\infty)}$.

Proposition 12.1.6. *Function $g(t,x) := P_t^{(\infty)} f(x)$ is continuous in $(0,\infty) \times \mathbb{R}^n$ for every $f \in L^2(\mathbb{R}^n)$.*

Proof. See [FK13] Theorem 1.2 which proves Hölder continuity. □

Proposition 12.1.7. *Semigroup $P_t^{(\infty)}$ has the Feller property. That is, for every $f \in C_0(\mathbb{R}^n)$,*

$$\forall t > 0 \qquad P_t^{(\infty)} f \in C_0(\mathbb{R}^n)$$

and

$$\lim_{t \to 0} \|P_t f - f\|_{L^\infty(\mathbb{R}^n)} = 0.$$

Proof. Semigroups of Lévy processes are always Feller which can be found in [Ber96] Chapter I Proposition 5 for instance. □

As foreshadowed by notation we aim to prove that $\mathcal{E}^{(m)}$ converges to $\mathcal{E}^{(\infty)}$ in Mosco sense. As the classical Mosco convergence requires forms to be defined on the same Hilbert space, which is not the case here, we instead turn to the generalized Mosco convergence introduced in Section 2.8. In Setting 2.8.3 we take $H_m = L^2(Z_m)$, $H = L^2(\mathbb{Z}^n)$ and the projection and extension operators π_m, E_m defined below. These are the natural projection and extension operators when approximating $\mathbb{R}^n$ by a square mesh.

Definition 12.1.8. *Define the* projection operator $\pi_m : L^2(\mathbb{R}^n) \to L^2(Z_m)$ *by*

$$\pi_m f(z) = \fint_{z+[-\frac{1}{2m}, \frac{1}{2m})^n} f(x)dx$$

and the extension operator $E_m : L^2(Z_m) \to L^2(\mathbb{R}^n)$ by

$$E_m f(x) = f([x]_m)$$

where $[x]_m \in Z_m$ is the unique point such that $x \in [x]_m + [-\frac{1}{2m}, \frac{1}{2m})^n$.

Almost the same definition of projection and extension operators is given in the beginning of Section 2 of [FH20]. For more general limit spaces and tiling patterns the definition can be found in [CKK13], see Equations (2.13) and (2.14), or in [CKW18b], Section 4.3.

Proposition 12.1.9. *π_m and E_m are projection and extension operators in the sense of Definition 2.8.2.*

Proof. Items (i), (ii) and (v) of Definition 2.8.2 follow by straightforward computations. For $f \in L^2(\mathbb{Z}^n)$, Jensen's inequality implies

$$\|\pi_m f\|_{L^2(Z_m)} = \sum_{z \in Z_m} m^n \left(\fint_{z+[-\frac{1}{2m}, \frac{1}{2m})^n} f(x)dx \right)^2 \leq \|f\|_{L^2(\mathbb{Z}^n)},$$

which proves remaining Items (iii) and (iv) of Definition 2.8.2. □

Definition 12.1.10. *Define the sequence of semigroups $\tilde{P}^{(m)}$ on $L^2(\mathbb{R}^n)$ by $\tilde{P}^{(m)} = E_m P^{(m)} \pi_m$.*

The following lemma will be needed in Theorem 12.4.1.

Lemma 12.1.11. *Let $t_1, M > 0$, $f \in L^2(\mathbb{R}^n)$ and a sequence $\{f_m\} \subset L^2(\mathbb{R}^n)$ be arbitrary but such that $\sup_m \|f_m\|_{L^\infty} \leq M$. If $\mathbb{P}$-a.s. $\mathcal{E}$ is a Dirichlet form satisfying $\boldsymbol{HR}[0, [R_0, \infty); \eta, C_H]$, for some $R_0 > 0$, $\eta, C_H \in (0, \infty)$, and*

$$\tilde{P}_{t_1}^{(m)} f_m \xrightarrow{m \to \infty} P_{t_1}^{(\infty)} f \qquad \text{in } L^2(\mathbb{R}^n)\text{-sense},$$

then $\mathbb{P}$-a.s. $\tilde{P}_{t_1}^{(m)} f_m(0) \xrightarrow{m \to \infty} P_{t_1}^{(\infty)} f(0)$.

Proof. We will prove the claim pointwise on the set of $\mathbb{P}$ probability 1 where all the assumptions are satisfied. Recall that by definition $\tilde{P}_t^{(m)} = E_m P_t^{(m)} \pi_m = E_m D_m P_{m^{2s}t} D_m^{-1} \pi_m$ where $D_m : L^2(\mathbb{Z}^n) \to L^2(Z_m)$ are dilation operators $D_m g(x) := g(mx)$. We know from Lemma 5.1.9 that, for every $g \in L^2(\mathbb{Z}^n)$, function $P_t g(x)$ solves $\partial_t u - \mathcal{L}u = 0$ on $(0,\infty) \times B$ for every ball $B \subset \mathbb{Z}^n$. Hence $\mathbf{HR}[0,[R_0,\infty);\eta,C_H]$ implies that for all $R_0 \le R \le \mathcal{R} \le (t_0/2)^{\frac{1}{2s}}$, denoting $B_R := B(0,R)$,

$$\operatorname*{ess\,osc}_{(t,x)\in[t_0-R^{2s},t_0]\times B_R\cap\mathbb{Z}^n} P_t D_m^{-1}\pi_m f_m(x) \le C_H \left(\frac{R}{\mathcal{R}}\right)^\eta \|D_m^{-1}\pi_m f_m\|_{L^\infty(\mathbb{Z}^n)}.$$

Let us first show that essential oscillation on the left can be replaced with the classical oscillation. This works on any countable measure spaces where the algebra of null sets is trivial, so in particular it works on $\mathbb{Z}^n$ with counting measure $\#$. To elaborate, denote $g := D_m^{-1}\pi_m f_m$ for a short while and notice that P_t is strongly continuous in $L^2(\mathbb{Z}^n)$. This allows us to estimate, for all $y \in \mathbb{Z}^n$, $\delta > 0$,

$$|P_{t+\delta}g(y) - P_t g(y)| \le \frac{1}{\#(\{y\})}\left(\sum_{z\in\mathbb{Z}^n} |P_{t+\delta}g(z) - P_t g(z)|^2\right)^{1/2} \le \|P_{t+\delta}g - P_t g\|_{L^2(\mathbb{Z}^n)},$$

implying that $P_t g(y)$ is continuous in t for every $y \in \mathbb{Z}^n$. Let us denote $S := [t_0 - R^{2s}, t_0] \times B(0,R) \cap \mathbb{Z}^n$ and suppose that $N \subset S$, $\lambda \times \#(N) = 0$ (λ being the Lebesgue measure on $\mathbb{R}$) is an null set such that

$$\operatorname*{ess\,osc}_{(t,x)\in S} P_t g(x) = \operatorname*{osc}_{(t,x)\in S\setminus N} P_t g(x).$$

Since $\mathbb{Z}^n$ is countable and $\emptyset$ is the only $\#$-null subset of $\mathbb{Z}^n$ we can decompose N into

$$N = \bigcup_{y\in B(0,R)\cap\mathbb{Z}^n} N_y,$$

where $\lambda(N_y) = 0$ for every $y \in B(0,R) \cap \mathbb{Z}^n$. Therefore

$$\begin{aligned}\operatorname*{ess\,osc}_{(t,x)\in S} P_t g(x) &= \operatorname*{osc}_{(t,x)\in S\setminus N} P_t g(x)\\ &= \sup_{x\in B(0,R)\cap\mathbb{Z}^n}\ \sup_{t\in[t_0-R^{2s},t_0]\setminus N_x} P_t g(x) - \inf_{x\in B(0,R)\cap\mathbb{Z}^n}\ \inf_{t\in[t_0-R^{2s},t_0]\setminus N_x} P_t g(x).\end{aligned}$$

Due to the continuity of $P_t g(x)$ for every $x \in B(0,R) \cap \mathbb{Z}^n$, null sets N_x in the last line can be ignored giving

$$\operatorname*{ess\,osc}_{(t,x)\in S} P_t g(x) = \operatorname*{osc}_{(t,x)\in S} P_t g(x)$$

just like we promised. This proves that for all $R_0 \le R \le \mathcal{R} \le (t_0/2)^{\frac{1}{2s}}$

$$\operatorname*{osc}_{(t,x)\in[t_0-R^{2s},t_0]\times B_R\cap\mathbb{Z}^n} P_t D_m^{-1}\pi_m f_m(x) \le C_H \left(\frac{R}{\mathcal{R}}\right)^\eta \|D_m^{-1}\pi_m f_m\|_{L^\infty(\mathbb{Z}^n)}.$$

Since both π_m and D_m^{-1} do not increase the $L^\infty(\mathbb{R}^n)$ norm, the last factor is bounded by $\|f_m\|_{L^\infty(R^n)} \le M$. Rescaling variables $R_0, R, \mathcal{R}$ and t_0 we can translate this into the following statement concerning $P_t^{(m)}\pi_m f_m = D_m P_{m^{2s}t} D_m^{-1}\pi_m f_m$: For $\frac{R_0}{m} \le R \le \mathcal{R} \le (t_0/2)^{\frac{1}{2s}}$,

$$\operatorname*{osc}_{(t,x)\in[t_0-R^{2s},t_0]\times B_R\cap Z_m} P_t^{(m)}\pi_m f_m(x) \le C_H M \left(\frac{R}{\mathcal{R}}\right)^\eta. \tag{12.1}$$

Recall that for every $g \in L^2(Z_m)$ by definition $E_m g(x) := g([x]_m)$ where $d([x]_m, x) \le \frac{\sqrt{n}}{2m}$ and we can estimate, for every $S' \subset \mathbb{R}^n$ and $g \in L^2(Z_m)$,

$$\sup_{S'} E_m g \le \sup_{\{x\in Z_m : d(x,S')\le\frac{\sqrt{n}}{2m}\}} g \quad \text{and} \quad \inf_{S'} E_m g \ge \inf_{\{x\in Z_m : d(x,S')\le\frac{\sqrt{n}}{2m}\}} g.$$

Translating Ineq. (12.1) to a statement on $\tilde{P}_t^{(m)} = E_m P_t^{(m)} \pi_m f_m$ we find that for all $\frac{R_0}{m} \le R \le R + \frac{\sqrt{n}}{2m} \le \mathcal{R} \le (t_0/2)^{\frac{1}{2s}}$

$$\operatorname*{osc}_{(t,x)\in[t_0-R^{2s},t_0]\times B_R} \tilde{P}_t^{(m)} f_m(x) \le C_H M \left(\frac{R + \frac{\sqrt{n}}{2m}}{\mathcal{R}} \right)^\eta .$$

The left hand side is increasing in R and the right hand side is decreasing in $\mathcal{R}$ which makes it possible to allow for small values of R and $\mathcal{R}$ by slightly modifying the inequality. That is, for all $R, \mathcal{R} \in [0, (t_0/2)^{\frac{1}{2s}}]$

$$\operatorname*{osc}_{(t,x)\in[t_0-R^{2s},t_0]\times B_R} \tilde{P}_t^{(m)} f_m(x) \le (C_H \vee 2) M \left(\frac{R + \frac{R_0}{m} + \frac{\sqrt{n}}{2m}}{\mathcal{R}} \right)^\eta .$$

Note that if $R + \frac{\sqrt{n}}{2m} \ge \mathcal{R}$ the above inequality follows from L^∞-contractiveness of $\tilde{P}_t^{(m)}$ explaining the change to $C_H \vee 2$. Let us specify $t_0 := t_1$, $\mathcal{R} = (t_0/2)^{\frac{1}{2s}}$ in what follows. Then in particular

$$\operatorname*{osc}_{x\in B_R} \tilde{P}_{t_1}^{(m)} f_m(x) \le C_1 \left(R + \frac{R_0}{m} + \frac{\sqrt{n}}{2m} \right)^\eta$$

where $C_1 := (C_H \vee 2) M (t_1/2)^{-\frac{\eta}{2s}}$. For arbitrary $\delta \in (0, (t_1/2)^{\frac{1}{2s}})$ this implies that

$$\left| \fint_{B(0,\delta)} \tilde{P}_{t_1}^{(m)} f_m(x) dx - \tilde{P}_{t_1}^{(m)} f_m(0) \right| \le \operatorname*{osc}_{x\in B(0,\delta)} \tilde{P}_{t_1}^{(m)} f_m(x) \le C_1 \left(\delta + \frac{R_0}{m} + \frac{\sqrt{n}}{2m} \right)^\eta .$$

On the other hand, by continuity of $P_{t_1}^{(\infty)} f(x)$ from Proposition 12.1.6, for every $\varepsilon > 0$ there is a $\delta \equiv \delta(\varepsilon) \in (0, \varepsilon)$ such that

$$\left| \fint_{B(0,\delta)} P_{t_1}^{(\infty)} f(x) dx - P_{t_1}^{(\infty)} f(0) \right| < \varepsilon.$$

The last two observations, with $\varepsilon \equiv \varepsilon(t_1)$ small enough, lead to

$$|\tilde{P}_{t_1}^{(m)} f_m(0) - P_{t_1}^{(\infty)} f(0)| \le \fint_{B(0,\delta)} |\tilde{P}_{t_1}^{(m)} f_m(x) - P_{t_1}^{(\infty)} f(x)| dx + C_1 \left(\delta + \frac{R_0}{m} + \frac{\sqrt{n}}{2m} \right)^\eta + \varepsilon.$$

From $L^2(\mathbb{R}^n)$-convergence $\tilde{P}_{t_1}^{(m)} f_m \xrightarrow{m\to\infty} P_{t_1}^{(\infty)} f$ we now conclude that

$$\fint_{B(0,\delta)} |\tilde{P}_{t_1}^{(m)} f_m(x) - P_{t_1}^{(\infty)} f(x)| dx \le |B(0,\delta)|^{-\frac{1}{2}} \| \tilde{P}_{t_1}^{(m)} f_m - P_{t_1}^{(\infty)} f \|_{L^2(M)} \xrightarrow{m\to\infty} 0.$$

Thus

$$\limsup_{m\to\infty} |\tilde{P}_{t_1}^{(m)} f_m(0) - P_{t_1}^{(\infty)} f(0)| \le \varepsilon + C_1 \varepsilon^\eta,$$

which proves the claim because the left hand side does not depend on ε which can be taken arbitrarily small. □

12.2. Mosco convergence for symmetrized ergodic conductance

We will prove that $\mathcal{E}^{(m)}$ converges to $\mathcal{E}^{(\infty)}$ in the generalized Mosco sense when c is symmetrized ergodic conductance. The result which we need is proved in Theorem 4 of [FH20] and only minor modifications are needed to adopt it into our setting. Here is a paraphrase of their result.

Theorem 12.2.1. *Let $s \in (0,1)$ and $n \geq 2$ be arbitrary and let $q > \frac{n}{2s}$ be such that ergodic conductance c has finite expectation and negative q moment, i.e. $\mathbb{E}[c] + \mathbb{E}\left[c^{-q}\right] < \infty$. Then the following two statements hold $\mathbb{P}$-a.s.*

(i) For every sequence $u_m \in L^2(Z_m)$ such that $\sup_m \mathcal{E}^{(m)}(u_m) < \infty$ there exist a $u \in W^{s,2}(\mathbb{R}^n)$ and a subsequence u_m such that $E_m u_m \to u$ a.e. on $L^2(\mathbb{R}^n)$ and

$$\liminf_{m\to\infty} \mathcal{E}^{(m)}(u_m) \geq \mathcal{E}^{(\infty)}(u).$$

(ii) For every $u \in C_c(\mathbb{R}^n)$ consider the sequence $\pi_m u$. Then $\pi_m u \in \mathcal{D}[\mathcal{E}^{(m)}]$, $E_m \pi_m u \to u$ strongly in $L^2(\mathbb{R}^n)$ and

$$\lim_{m\to\infty} \mathcal{E}^{(m)}(\pi_m u) = \mathcal{E}^{(\infty)}(u).$$

Proof of paraphrase. Since the differences that we need to address are mostly cosmetic we advise the reader to have a copy of [FH20] nearby. Note that [FH20] does not state the condition $\mathbb{P}$-a.s. explicitly. We restrict the definition of the form $\mathscr{E}_{p,s,\varepsilon}$ from the introduction of [FH20] by specifying $p = 2$, $G = 0$, $V(\xi) = \xi^2$ and $f_\varepsilon = 0$. Taking $\varepsilon = 1/m$ these forms coincide with $\mathcal{E}^{(m)}$ and the $\mathcal{R}_\varepsilon$, $\mathcal{R}^*_\varepsilon$ coincide with π_m, E_m respectively. The result is a combination of Theorem 5 and Lemma 36 from [FH20]. The second conclusion of Theorem 5 does not quite fit with our definition of generalized Mosco convergence because the approximating sequence is not claimed to be in $L^2(Z_m)$ and the convergence is pointwise. But let us not get ahead of ourselves and first find r such that Assumptions 1 and 3 from [FH20] are satisfied. For Assumption 1, c is ergodic in $\mathbb{Z}^n \times \mathbb{Z}^n$ by assumption, and we need to find $r \in (1,p)$ such that $q \geq \frac{r}{p-r} > \frac{n}{sp}$. We know already that $q > \frac{n}{sp} > 1$, $p = 2$, $\mathbb{E}[c] < \infty$ and $\mathbb{E}[c^{-q}] < \infty$. Plugging $r = 1$ into expression $\frac{r}{p-r}$ gives $\frac{1}{2-1} = 1$ while for $r \to p$ the expression goes to infinity. Since $\frac{r}{p-r}$ is continuous in r, we can find an r such that $\frac{r}{p-r} \in \left(\frac{n}{sp}, q\right]$ so Assumption 1 is satisfied. Assumption 3 is satisfied because $V(\xi) = \xi^2$ is continuous and one can take $\alpha = 1$, $\beta = 1$, $c = 0$ and $p = 2$ to get

$$\alpha|\xi|^p \leq V(\xi) \leq c + \beta|\xi|^p.$$

Furthermore, $|\xi|^{-p}V(\xi) = 1$ is clearly continuous at 0. The second condition 2 in Theorem 5 from [FH20] is satisfied because $G = \alpha|\xi|^r + \tilde{G}$ with $\alpha = 0$, $\tilde{G} = 0$ (which is clearly non-negative and convex) and in addition $f_m = 0$. The first claim now follows from statement 1 of Theorem 5 of [FH20].

To prove the second statement we turn to Lemma 36 of [FH20]. Take an arbitrary $u \in C_c(\mathbb{R}^n)$, find a bounded set $\mathcal{Q} \subset \mathbb{R}^n$ large enough such that $d(\mathrm{supp}(u), \mathcal{Q}^c) > 2\sqrt{n}$ and set $u_m = \pi_m u$. We already checked Assumptions 1 and 3 of [FH20] and a simple computation

$$|u_m(x) - u_m(y)| \leq \fint_{[-\frac{1}{2m},\frac{1}{2m})^n} |u(x+z) - u(y+z)|dz \leq \fint_{[-\frac{1}{2m},\frac{1}{2m})^n} (\mathrm{Lip}\, u)d(x,y)dz \leq (\mathrm{Lip}\, u)d(x,y)$$

shows that u_m are Lipschitz functions on Z_m with the same Lipshitz constant as u. In a similar way, $\|u_m\|_\infty \leq \|u\|_\infty$. Furthermore, the cubes over which π_m averages and E_m extends are contained in a ball of radius $\frac{\sqrt{n}}{m}$ around its center so

$$\mathrm{supp}(\pi_m u) \subset \left\{y \in Z_m : d(y, \mathrm{supp} u) < \sqrt{n}\right\}$$

and

$$\mathrm{supp}(E_m \pi_m u) \subset \left\{y \in \mathbb{R}^n : d(y, \mathrm{supp}(\pi_m u) \leq \sqrt{n}\right\}$$

which means that $\mathrm{supp}(\pi_m u) \subset \mathcal{Q}$ and $\mathrm{supp}(E_m \pi_m u) \subset \mathcal{Q}$ by definition of $\mathcal{Q}$. Continuity of u implies that $E_m u_m \to u$ a.e. and since we know that u and all $E_m u_m$ are bounded and supported on $\mathcal{Q}$ the dominated convergence theorem implies that $E_m u_m$ converges to u strongly in $L^2(\mathcal{Q})$ and hence also in $L^2(\mathbb{R}^n)$. This verifies the assumptions of Lemma 35 from [FH20] and its second to last claim (Equation (43)) gives, in slightly adjusted notation,

$$\lim_{m\to\infty} m^{-2n} \sum_{x,y\in Z_m} \frac{(u_m(x)-u_m(y))^2}{d(x,y)^{n+2s}} c(mx,my) = \mathbb{E}[c] \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{d(x,y)^{n+2s}} dydx.$$

Since $u_m \in C_c(Z_m)$, $u_m \in \mathcal{D}[\mathcal{E}^{(m)}]$. By definition of $\mathcal{E}^{(m)}$ and $\mathcal{E}^{(\infty)}$ this is equivalent to

$$\lim_{m\to\infty} \mathcal{E}^{(m)}(u_m) = \mathcal{E}^{(\infty)}(u)$$

which, together with the strong convergence, proves the claim. □

Theorem 12.2.2 (Mosco convergence)**.** *Let c be a symmetrized ergodic conductance on $\mathbb{Z}^n$ ($n \geq 2$) such that $\mathbb{E}[c] + \mathbb{E}\left[c^{-q}\right] < \infty$ for some $q > \frac{n}{2s}$. Then $\mathbb{P}$-a.s. $\mathcal{E}^{(m)}$ converges to $\mathcal{E}^{(\infty)}$ in generalized Mosco sense.*

Proof. Recall that by Remark 9.1.7 the fact that c is symmetrized plays no role since our statement only depends on the symmetric part of $\mathcal{E}$. Thus we can assume that c is not symmetrized and we only need to check the assumptions of Theorem 2.8.5. Let $u \in L^2(\mathbb{R}^n)$ and let $u_m \in L^2(Z^n)$ be a sequence such that $E_m u_m \rightharpoonup u$ weakly in $L^2(\mathbb{R}^n)$. If $\liminf_{m\to\infty} \mathcal{E}^{(m)}(u_m) = \infty$, then trivially

$$\liminf_{m\to\infty} \mathcal{E}^{(m)}(u_m) \geq \mathcal{E}(u). \tag{12.2}$$

Otherwise Item (i) of Theorem 12.2.1 implies that every subsequence u'_m of u_m has a sub-subsequence u''_m such that $E_m u''_m$ converges a.e. to some v and

$$\liminf_{m\to\infty} \mathcal{E}^{(m)}(u''_m) \geq \mathcal{E}^{(\infty)}(v).$$

In fact $v = u$ λ-a.e. as we will now show (λ denotes the Lebesgue measure). Take any $\varphi \in C_c^\infty(\mathbb{R}^n)$ and a bounded open set $U \subset \mathbb{R}^n$ such that φ is supported inside of U. Then weak convergence $u''_m \rightharpoonup u$ in $L^2(U)$ implies that $\int_{\mathbb{R}^n} E_m u''_m(x)\varphi(x)dx \xrightarrow{m\to\infty} \int_{\mathbb{R}^n} u(x)\varphi(x)dx$. On the other hand $E_m u''_m \xrightarrow{a.e.} v$ implies that $E_m u''_m$ converges to v in measure on $L^2(U)$ and moreover that it converges as distribution on $C_b(U)$ so $\int_U E_m u''_m(x)\varphi(x)dx \xrightarrow{m\to\infty} \int_U v(x)\varphi(x)dx$. Thus $E_m u''_m$ converge as a distribution on $C_c^\infty(\mathbb{R}^n)$ to both u and v which is only possible if $u = v$ λ-a.e. But this means that

$$\liminf_{m\to\infty} \mathcal{E}^{(m)}(u''_m) \geq \mathcal{E}^{(\infty)}(u).$$

Such u''_m exists for every subsequence u'_m of u_m which implies that Eq. (12.2) holds in this case as well. Hence Item (i) of Definition 2.8.4 is satisfied. Let us now take $\mathcal{D} := C_c(\mathbb{R}^n)$ and verify the remaining conditions in Theorem 2.8.5. We know that $C_c(\mathbb{R}^n)$ is dense in $(\mathcal{D}[\mathcal{E}^{(\infty)}], \mathcal{E}_1^{(\infty)})$ because $\mathcal{E}^{(\infty)}$ is a regular Dirichlet form of a Lévy process. Item (ii) of Theorem 12.2.1 immediately verifies remaining conditions and therefore Theorem 2.8.5 implies that $\mathcal{E}^{(m)} \to \mathcal{E}^{(\infty)}$ in Mosco sense. □

12.3. Mosco convergence for i.i.d. conductance

If $c(x,y)$ is allowed to be zero for some $x, y \in \mathbb{Z}^n$ we are only able to deal with the i.i.d. case. The Mosco convergence result which we require was obtained in [CKK13] Proposition 7.1. Here we present a particular realization of that theorem in $\mathbb{Z}^n$ and explain how it is obtained from the result of [CKK13].

Theorem 12.3.1 ([CKK13] Proposition 7.1)**.** *Let $c(x,y)$ be a non-negative i.i.d. conductance on $\mathbb{Z}^n$ ($n \geq 2$) such that* $\mathrm{Var}[c] < \infty$*. Then $\mathbb{P}$-a.s. $\mathcal{E}^{(m)}$ converges to $\mathcal{E}^{(\infty)}$ in generalized Mosco sense.*

Proof. Proposition 7.1 of [CKK13] is stated in a more general way allowing the limiting spaces to differ from $\mathbb{R}^n$, limiting kernel to differ from $d(x,y)^{-(n+2s)}$ and allowing for sequences of graph approximations other than Z_m. We will now indicate, among other things, how these more general notation collapse in our setting. In our setting $c(x,y)$ replaces $\xi_{x,y}$ from [CKK13]. The assumption $\mathbb{E}[\xi_{x,y}] = 1$ is benign since we can redefine $c'(x,y) := \mathbb{E}[c]^{-1}c(x,y)$ and c' remains non-negative, i.i.d. and $\mathrm{Var}[c'] < \infty$. Forms $\mathcal{E}^{(m)}$ and $\mathcal{E}^{(\infty)}$ then have to be multiplied by a factor $\mathbb{E}[c]^{-1}$ which is of little relevance for the generalized Mosco convergence. Let us suppose thus that $\mathbb{E}[c] = 1$. We take the sequence of graphs $V_m := Z_m$ with partitions $U_m(z) = z + [-1/(2m), 1/(2m))^n$. It is not hard to verify that this choice satisfies (AG.1), (AG.2), (AG.3). With such choice of approximating graphs $j^{(m)}(x,y)$ and $k^{(m)}(x,y)$ coincide as well as forms $\mathcal{E}^{(m)}$ (which happen to be denoted in exactly the same way) and $\mathcal{E}^{(\infty)}$ and $\mathcal{E}$. Lastly, [CKK13] only states the consequences of generalized Mosco convergence although Mosco convergence is used in the proof. The conditions (A2), (A3)* and (A4) are proved to be satisfied, which together with Theorem 4.7 of [CKK13] proves that $\mathbb{P}$-a.s. $\mathcal{E}^{(m)} \to \mathcal{E}^{(\infty)}$ in generalized Mosco sense. □

12.4. Convergence in finite-dimensional distributions

Theorem 12.4.1 gives the convergence of the finite-dimensional distributions when $X_t^{(m)}$ and $X_t^{(\infty)}$ are started from 0. The proof relies on the Mosco convergence, conservativeness of the limiting process and large scale Hölder regularity only around point 0. One can also start $X_0^{(\infty)}$ from initial distribution $\varphi\lambda(dx)$ for some $\varphi \in C_c(\mathbb{R}^n)$, $\varphi > 0$, $\int_{\mathbb{R}^n} 1\varphi\lambda(dx)$ (λ is the Lebesgue measure) and $X_t^{(m)}$ from $\varphi\mu_m(dx)$. In that case the proof of the vague convergence (tested with $C_c(\mathbb{R}^n)$) requires no Hölder regularity at all and can be found in [Kol06], Section 7. If the limiting process is conservative, the convergence is also true in the weak sense (tested with $C_b(\mathbb{R}^n)$) and the proof can be found in [CKK13], Theorem 5.1. On the other hand, if large scale Hölder regularity for $X_t^{(m)}$ is available, it is possible to obtain the same results when $X_t^{(m)}$ and $X_t^{(\infty)}$ are started from the origin $0 \in \mathbb{Z}^n$. The vague convergence in this case can be found in [HK07], end of proof of Theorem 5.1 or [CKW18b] Theorem 4.5. Here we slightly relax Hölder regularity assumption by requiring that it only holds at point 0. This is necessary because in the ergodic case we do not have the control of the minimal scale of Hölder regularity in the neighborhood of 0.

Theorem 12.4.1 (Weak convergence of finite-dimensional distributions)**.** *Let c be a symmetric random conductance such that $\mathcal{E}$ is a regular Dirichlet form $\mathbb{P}$-a.s. Suppose also that*

(i) $P_t^{(\infty)}$ is conservative,

(ii) $\mathcal{E}^{(m)} \xrightarrow{m\to\infty} \mathcal{E}^{(\infty)}$ in generalized Mosco sense $\mathbb{P}$-a.s. and

(iii) there exist $\eta > 0$, $C_H < \infty$ and a random variable $R_0 > 0$ such that $\mathcal{E}$ $\mathbb{P}$-a.s. satisfies

$$\boldsymbol{HR}[0, [R_0, \infty); \eta, C_H].$$

Then, $\mathbb{P}$-a.s., finite-dimensional distributions of $X_t^{(m)}$ under $\mathbf{P}_0^{(m)}$ weakly converge to those of $X_t^{(\infty)}$ under $\mathbf{P}_0^{(\infty)}$. Explicitly, we will prove that, $\mathbb{P}$-a.s., for every $k \in \mathbb{N}$, any sequence of times $0 < t_1 < t_2 < \ldots < t_k < \infty$ and any sequence of bounded continuous functions $f_1, f_2, \ldots, f_k \in C_b(\mathbb{R}^n)$

$$\lim_{m\to\infty} \mathbf{E}_0^{(m)} \left[f_1(X_{t_1}^{(m)}) f_2(X_{t_2}^{(m)}) \ldots f_k(X_{t_k}^{(m)}) \right] = \mathbf{E}_0^{(\infty)} \left[f_1(X_{t_1}^{(\infty)}) f_2(X_{t_2}^{(\infty)}) \ldots f_k(X_{t_k}^{(\infty)}) \right].$$

Proof. We will work with semigroups $\tilde{P}_t^{(m)} := E_m P_t^{(m)} \pi_m$ from Definition 12.1.10. First of all, notice that generalized Mosco convergence $\mathcal{E}^{(m)} \to \mathcal{E}^{(\infty)}$ implies, through Theorem 2.8.6, that for all $v \in L^2(\mathbb{R}^n)$ and $t > 0$

$$\tilde{P}_t^{(m)} v \to P_t^{(\infty)} v \tag{12.3}$$

strongly in $L^2(\mathbb{R}^n)$. Fix an arbitrary $k \in \mathbb{N}$, an arbitrary sequence of times $0 < t_1 < t_2 \ldots < t_k < \infty$ and an arbitrary sequence of functions $f_1, f_2, \ldots, f_k \in C_b(\mathbb{R}^n)$. Let us first prove that we can require $f_1, f_2, \ldots f_k$ to have compact support. By conservativeness of $P_t^{(\infty)}$, for every $\varepsilon > 0$ we can find large enough ball B, with radius at least 1, such that for $i = 1, 2, \ldots k$, $P_{t_i}^{(\infty)} 1_B(0) \geq 1 - \varepsilon$. Take any $h \in C_c(\mathbb{R}^n)$ such that $1_B \leq h \leq 1_{2B}$. By Proposition 12.1.7 we know that the semigroup $P_t^{(\infty)}$ is Feller so $P_{t_i}^{(\infty)} h \in C_0(\mathbb{R}^n)$. By Eq. (12.3) we know that $\tilde{P}_{t_i}^{(m)} h \to P_{t_i}^{(\infty)} h$ strongly in $L^2(\mathbb{R}^n)$. In addition, Lemma 12.1.11 combined with $\mathbf{HR}[0, [R_0, \infty); \theta, C_H]$ implies that $\lim_{m\to\infty} \tilde{P}_{t_i}^{(m)} h(0) = P_{t_i}^{(\infty)} h(0) \geq P_{t_i}^{(\infty)} 1_B(0) \geq 1 - \varepsilon$ for every $i = 1, 2, \ldots k$. By increasing the ball B if necessary, we can find $m_0 \equiv m_0(\varepsilon)$ such that $\mathbf{P}_0^{(m)}(X_{t_i}^{(m)} \in B) = P_t^{(m)} 1_B(0) \geq 1 - 2\varepsilon$ for all $m \geq m_0$ and $i = 1, 2, \ldots k$. Let us now take a compactly supported Lipschitz function $\varphi : \mathbb{R}^n \to [0, 1]$ such that $\varphi = 1$ on B. Then for all $m \geq m_0(\varepsilon)$

$$\begin{aligned}
&\mathbf{E}_0^{(m)} \left[f_1(X_{t_1}^{(m)}) f_2(X_{t_2}^{(m)}) \ldots f_k(X_{t_k}^{(m)}) \right] \\
&\quad = \mathbf{E}_0^{(m)} \left[f_1(X_{t_1}^{(m)}) f_2(X_{t_2}^{(m)}) \ldots f_k(X_{t_k}^{(m)}) \Big| \bigcap_{i=1}^{k} \{X_{t_i}^{(m)} \in B\} \right] \mathbf{P}_0^{(m)} \left(\bigcap_{i=1}^{k} \{X_{t_i}^{(m)} \in B\} \right) \\
&\qquad + \mathbf{E}_0^{(m)} \left[f_1(X_{t_1}^{(m)}) f_2(X_{t_2}^{(m)}) \ldots f_k(X_{t_k}^{(m)}) \Big| \bigcup_{i=1}^{k} \{X_{t_i}^{(m)} \notin B\} \right] \mathbf{P}_0^{(m)} \left(\bigcup_{i=1}^{k} \{X_{t_i}^{(m)} \notin B\} \right).
\end{aligned}$$

The size of the second term is comparable to ε when $m \geq m_0(\varepsilon)$ because

$$\begin{aligned}
&\left| \mathbf{E}_0^{X^{(m)}} \left[f_1(X_{t_1}^{(m)}) f_2(X_{t_2}^{(m)}) \ldots f_k(X_{t_k}^{(m)}) \Big| \bigcup_{i=1}^{k} \{X_{t_i}^{(m)} \notin B\} \right] \right| \mathbf{P}_0^{(m)} \left(\bigcup_{i=1}^{k} \{X_{t_i}^{(m)} \notin B\} \right) \\
&\qquad \leq \left(\prod_{i=1}^{k} \|f_i\|_\infty \right) \sum_{i=1}^{k} \mathbf{P}_0^{(m)}(X_{t_i}^{(m)} \notin B) \leq \left(\prod_{i=1}^{k} \|f_i\|_\infty \right) 2k\varepsilon.
\end{aligned}$$

Since the previous calculations work in the same way if f_i are replaced with φf_i, it follows that for $m \geq m(\varepsilon)$

$$\left| \mathbf{E}_0^{(m)} \left[f_1(X_{t_1}^{(m)}) \ldots f_k(X_{t_k}^{(m)}) \right] - \mathbf{E}_0^{(m)} \left[(f_1\varphi)(X_{t_1}^{(m)}) \ldots (f_k\varphi)(X_{t_k}^{(m)}) \right] \right| \leq 2k\varepsilon \prod_{i=1}^{k} \|f_1\|_\infty.$$

An analogue estimate holds for $X_t^{(\infty)}$ where we get

$$\left| \mathbf{E}_0^{(\infty)} \left[f_1(X_{t_1}^{(\infty)}) \ldots f_k(X_{t_k}^{(\infty)}) \right] - \mathbf{E}_0^{(\infty)} \left[(f_1\varphi)(X_{t_1}^{(\infty)}) \ldots (f_k\varphi)(X_{t_k}^{(\infty)}) \right] \right| \leq k\varepsilon \prod_{i=1}^{k} \|f_1\|_\infty.$$

Thus, for every $\varepsilon > 0$, it is possible to find $\varphi \in C_c(\mathbb{R}^n)$ such that

$$\begin{aligned}
&\limsup_{m\to\infty} \left| \mathbf{E}_0^{(m)} \left[f_1(X_{t_1}^{(m)}) \ldots f_k(X_{t_k}^{(m)}) \right] - \mathbf{E}_0^{(\infty)} \left[f_1(X_{t_1}^{(\infty)}) \ldots f_k(X_{t_k}^{(\infty)}) \right] \right| \\
&\quad \leq \limsup_{m\to\infty} \left| \mathbf{E}_0^{(m)} \left[(f_1\varphi)(X_{t_1}^{(m)}) \ldots (f_k\varphi)(X_{t_k}^{(m)}) \right] - \mathbf{E}_0^{(\infty)} \left[(f_1\varphi)(X_{t_1}^{(\infty)}) \ldots (f_k\varphi)(X_{t_k}^{(\infty)}) \right] \right| \\
&\qquad + 3k\varepsilon \prod_{i=1}^{k} \|f_1\|_\infty,
\end{aligned}$$

which show that up to an arbitrary small error in the final result we can replace functions f_i by their compactly supported alternatives φf_i.

Let us therefore assume that $f_1, f_2, \dots f_k$ were compactly supported to begin with. This guarantees that they are all uniformly continuous so we can find a modulus of continuity $\mathcal{U}(\cdot) : \mathbb{R} \to \mathbb{R} \equiv \mathcal{U}[f_1, f_2, \dots f_k](\cdot)$ such that

$$\forall 1 \le i \le k \qquad |f_i(x) - f_i(y)| \le \mathcal{U}(|x-y|) \xrightarrow{|x-y|\to 0} 0.$$

Moving on, notice that $L^2(Z_m) \subset C_0(Z_m)$, for every $m \in \mathbb{N}$, because the measure on Z_m is a multiple of the counting measure (see Definition 12.1.1). This means that $P_t^{(m)}(L^2(Z_m) \cap C_0(Z_m)) \subset C_0(Z_m)$ for every $t \ge 0$. Therefore, by Theorem 2.5.15, we have

$$\begin{aligned}\mathbf{E}_0^{(m)} &\left[f_1(X_{t_1}^{(m)}) f_2(X_{t_2}^{(m)}) \dots f_k(X_{t_k}^{(m)}) \right] \\ &= P_{t_1}^{(m)}(f_1 P_{t_2-t_1}^{(m)}(f_2 P_{t_3-t_2}^{(m)}(\dots f_{k-1} P_{t_k-t_{k-1}}^{(m)}(f_k)\dots)))(0).\end{aligned}$$

We now claim that replacing $P_t^{(m)}$ with $\tilde{P}_t^{(m)} = E_m P_t^{(m)} \pi_m$ will not make any difference in the limit. More precisely, we claim that

$$P_{t_1}^{(m)}(f_1 P_{t_2-t_1}^{(m)}(\dots f_{k-1} P_{t_k-t_{k-1}}^{(m)}(f_k)\dots)))(0) \xrightarrow{m\to\infty} \tilde{P}_{t_1}^{(m)}(f_1 \tilde{P}_{t_2-t_1}^{(m)}(\dots f_{k-1} \tilde{P}_{t_k-t_{k-1}}^{(m)}(f_k)\dots)))(0). \tag{12.4}$$

To prove this, notice first that for every $h, g \in L^2(\mathbb{R}^n)$, by definition of extension and restriction operators in Definition 12.1.8, $E(\pi(h))$ is constant on the averaging domains of π so $\pi(gE(\pi(h))) = \pi(g)\pi(h)$. Secondly, notice that $(Eh)(0) = h(0)$ since $0 \in \mathbb{Z}^n$. The extension operators therefore play no role in our current computation and

$$\begin{aligned}&\tilde{P}_{t_1}^{(m)}(f_1 \tilde{P}_{t_2-t_1}^{(m)}(\dots f_{k-1} \tilde{P}_{t_k-t_{k-1}}^{(m)}(f_k)\dots)))(0) \\ &\quad = P_{t_1}^{(m)}(\pi^{(m)}(f_1) P_{t_2-t_1}^{(m)}(\dots \pi^{(m)}(f_{k-1}) P_{t_k-t_{k-1}}^{(m)}(\pi^{(m)}(f_k))\dots)))(0).\end{aligned}$$

Furthermore, for all $h \in L^\infty(Z_m)$, $i = 1, \dots k$,

$$\left\| P_{t_i-t_{i-1}}^{(m)}(f_i h) - P_{t_i-t_{i-1}}^{(m)}(\pi_m(f_i)h) \right\|_\infty \le \left\| P_{t_i-t_{i-1}}^{(m)}((f_i - \pi_m f_i)h) \right\|_\infty \le \mathcal{U}\left(\frac{\sqrt{n}}{2m}\right) \|h\|_\infty$$

because for all $x \in \mathbb{R}^n$

$$|f_i(x) - \pi_m f_i(x)| \le \fint_{[x]_m + \left[\frac{1}{2m}, \frac{1}{2m}\right)^n} |f(x) - f(y)| dy \le \mathcal{U}\left(\frac{\sqrt{n}}{2m}\right).$$

Let us now shorten

$$\mathcal{P}[a_1, a_2, \dots, a_k] := P_{t_1}^{(m)}(a_1 P_{t_2-t_1}^{(m)}(a_2 P_{t_3-t_2}^{(m)}(\dots a_{k-1} P_{t_k-t_{k-1}}^{(m)}(a_k)\dots)))$$

and notice that $\mathcal{P}$ is multilinear. Defining, for $1 \le i < k$,

$$h_i := P_{t_{i+1}-t_i}^{(m)}(f_{i+1} \dots P_{t_k-t_{k-1}}^{(m)}(\pi_m f_k)\dots)$$

and $h_k = 1$ we know that $\|h_i\|_\infty \le \prod_{j=i+1}^k \|f_j\|_\infty$ (because $P_t^{(m)}$ and π_m are L^∞-contractions) so we can estimate

$$\begin{aligned}&\|\mathcal{P}[f_1, f_2, \dots, f_k] - \mathcal{P}[\pi_m f_1, \pi_m f_2, \dots, \pi_m f_k]\|_\infty \\ &\quad \le \sum_{i=1}^k \|\mathcal{P}[f_1, \dots, f_{i-1}, f_i, \pi_m f_{i+1}, \dots, \pi_m f_k] - \mathcal{P}[f_1, \dots f_{i-1}, \pi_m f_i, \pi_m f_{i+1}, \dots, \pi_m f_k]\|_\infty \\ &\quad \le \sum_{i=1}^k \left\| P_{t_i-t_{i-1}}^{(m)}(f_i h_i) - P_{t_i-t_{i-1}}^{(m)}(\pi_m(f_i)h_i) \right\|_\infty \prod_{j=1}^{i-1} \|f_j\|_\infty \\ &\quad \le \mathcal{U}\left(\frac{\sqrt{n}}{2m}\right) \sum_{i=1}^k \|h_i\|_\infty \prod_{j=1}^{i-1} \|f_j\|_\infty \le \mathcal{U}\left(\frac{\sqrt{n}}{2m}\right) \sum_{i=1}^k \prod_{j=1}^k \|f_j\|_\infty \xrightarrow{m\to\infty} 0.\end{aligned}$$

This proves Eq. (12.4). In a similar way, by applying Eq. (12.3) several times (keeping in mind that $\|f_i\|_\infty < \infty$ and that $\tilde{P}_t^{(m)}, P_t^{(\infty)}$ are L^2-contractions), we find that

$$\begin{aligned}\tilde{P}_{t_1}^{(m)} v_m := \tilde{P}_{t_1}^{(m)}(f_1 \tilde{P}_{t_2-t_1}^{(m)}(\dots f_{k-1}\tilde{P}_{t_k-t_{k-1}}^{(m)}(f_k)\dots))) \xrightarrow{L^2, m\to\infty} \\ \xrightarrow{L^2, m\to\infty} P_{t_1}^{(\infty)}(f_1 P_{t_2-t_1}^{(\infty)}(\dots f_{k-1}P_{t_k-t_{k-1}}^{(\infty)}(f_k)\dots))) =: P_{t_1}^{(\infty)} v\end{aligned}$$

in $L^2(\mathbb{R}^n)$ where v_m and v are implicitly defined. Functions v and $P_{t_1}^{(\infty)} v$ are continuous by Feller property of $P_t^{(\infty)}$. Noticing that $\|v_m\|_{L^\infty} \le \prod_{i=1}^k \|f_i\|_\infty$ (because $\tilde{P}_t^{(m)}$ is a L^∞-contraction) allows us to use Lemma 12.1.11 and conclude that

$$\tilde{P}_{t_1}^{(m)} h_m(0) \xrightarrow{m\to\infty} P_{t_1}^{(\infty)} h(0).$$

With the last statement and Eq. (12.4) in hand it follows that

$$\begin{aligned}\mathbf{E}_0^{(m)}\left[f_1(X_{t_1}^{(m)})\dots f_k(X_{t_k}^{(m)})\right] = P_{t_1}^{(m)}(f_1 P_{t_2-t_1}^{(m)}(\dots f_{k-1}P_{t_k-t_{k-1}}^{(m)}(f_k)\dots)))(0) \\ \xrightarrow{m\to\infty} P_{t_1}^{(\infty)}(f_1 P_{t_2-t_1}^{(\infty)}(\dots f_{k-1}P_{t_k-t_{k-1}}^{(\infty)}(f_k)\dots)))(0) = \mathbf{E}_0^{(\infty)}\left[f_1(X_{t_1}^{(\infty)})\dots f_k(X_{t_k}^{(\infty)})\right].\end{aligned}$$

In the last line we again used Theorem 2.5.15 and the fact that $P_t(C_0(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$ due to $P_t^{(\infty)}$ being Feller, see Proposition 12.1.7. This completes the proof. □

Corollary 12.4.2. *Let a symmetrized twofold ergodic conductance c on $\mathbb{Z}^n$ ($n \ge 2$) be such that $\mathbb{E}[c^{-q}] + \mathbb{E}[c^p] < \infty$ for some $p, q \in [1, \infty]$ satisfying*

$$\frac{1}{p} + \frac{1}{q} < \frac{2s}{n}. \tag{12.5}$$

Then, for $\mathbb{P}$-a.e. realization of conductance, $X_t^{(m)}$, started from $X_0^{(m)} = 0$, converges to X_t^∞, started from $X_0^{(\infty)} = 0$, in the sense of finite-dimensional distributions.

Proof. Theorem 12.2.2 shows that $\mathbb{P}$-a.s. $\mathcal{E}^{(m)} \to \mathcal{E}^{(\infty)}$ in Mosco sense because $q > \frac{n}{2s}$ from Ineq. (12.5). Since $X_t^{(\infty)}$ is a Lévy process, we know that $P_t^{(\infty)}$ is conservative. On the other hand, Theorem 10.4.1, which requires the same moment assumption as in Ineq. (12.5), for $x_0 := 0$ provides us with non random $\theta^{erg} \equiv \theta^{erg}(s, n, q, p, \mathbb{E}[c^{-q}], \mathbb{E}[c^p])$, $C_H^{erg} \equiv C_H^{erg}(s)$ and a random variable $R_0 \equiv R_{(10.4.1)}(x_0 = 0, \omega, c, q, p, \mathbb{E}[c^{-q}], \mathbb{E}[c^p])$ such that $\mathbb{P}$-a.s. $\mathcal{E}$ satisfies $\mathbf{HR}[x_0 = 0, [R_0(\omega), \infty); \theta^{erg}, C_H^{erg}]$. This means that assumptions of Theorem 12.4.1 are $\mathbb{P}$-a.s. satisfied and thus $\mathbb{P}$-a.s. $X_t^{(m)} \xrightarrow{m\to\infty} X_t^{(\infty)}$ in finite-dimensional distributions when all processes are started from 0. □

Corollary 12.4.3. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$ ($n \ge 2$) such that $\mathbb{E}[c^p] < \infty$ for some $p > \frac{n+1}{s}$. Then $\mathbb{P}$-a.s. $X_t^{(m)}$, started from $X_0^{(m)} = 0$, converges to $X_t^{(\infty)}$, started from $X_0^{(\infty)} = 0$, in the sense of finite-dimensional distributions.*

Proof. We check the assumptions of Theorem 12.4.1. Semigroup $P_t^{(\infty)}$ is conservative because $X_t^{(\infty)}$ is a Lévy process. By Theorem 12.3.1, we know that $\mathcal{E}^{(m)}$ converges in generalized Mosco sense to $\mathcal{E}^{(\infty)}$ for $\mathbb{P}$-a.e. realization of c. Finally, by Theorem 11.7.1 there exist $\theta \equiv \theta(p, n, s)$, $C_H \equiv C_H(s)$, $\eta \equiv \eta(s, n, p, \mathbb{E}[c^p], \mathfrak{p}, \nu)$ and a random variable $R_\star \equiv R_\star(x_0 = 0, \omega, c, p, s, n, \mathfrak{p}, \nu)$ such that $\star\mathbf{HR}[x_\star = 0, R_\star(0), \theta; \eta, C_H]$ holds $\mathbb{P}$-a.s. In particular, $\mathcal{E}$ satisfies $\mathbf{HR}[x_0 = 0, [R_\star^\theta, \infty); \eta, C_H]$. This verifies all assumptions of Theorem 12.4.1 which implies that $X_t^{(m)}$ converges to $X_t^{(\infty)}$ in sense of finite-dimensional distributions when all processes are started from 0. □

12.5. Tightness in the i.i.d. case

Having control of exit probabilities for process started in a neighborhood $B(0, R_0)$ of 0 on scales of any order strictly lower than R_0 together with tightness criteria from [Ald78] implies that probability measures $\mathbf{P}^{(m)}_{x_m}$ are tight on Skorokhod space $\mathcal{D}([0,T], \mathbb{R}^n)$ for every $T > 0$ and every sequence $x_m \subset Z_m$ such that $x_m \to 0$.

We start with a paraphrase of Theorem 1 from [Ald78].

Theorem 12.5.1 (Tightness criteria)**.** *Let $T > 0$ be arbitrary and denote by $\pi_t : \mathcal{D}([0,T]; \mathbb{R}^n) \to \mathbb{R}$ projections $x \to x(t)$ (see Definition 2.5.1 for Skorokhod space $\mathcal{D}$). A sequence of probability measures $\{\mathbf{P}_i\}_{i\in\mathbb{N}}$ is tight on $\mathcal{D}([0,T]; \mathbb{R}^n)$ if*

(i) $\mathbf{P}_i(\pi_t^{-1})$ is tight on $\mathbb{R}^n$ for every $t \in [0,T]$ and

(ii) for every sequence of stopping times $\{\tau_i\}$ taking only finitely many values, every sequence of constants $\{\varepsilon_i\} \subset [0,1]$ tending to 0 and every $\delta > 0$

$$\mathbf{P}_i\left(|x(\tau_i + \varepsilon_i) - x(\tau_i)| > \delta\right) \xrightarrow{i\to\infty} 0.$$

Proof. Define a sequence of random elements $\{X_i\}$ of $\mathcal{D}([0,T], \mathbb{R}^n)$ that have distributions $\{P_i\}$ and apply Theorem 1 from [Ald78]. □

Theorem 12.5.2. *Let c an i.i.d. conductance on $\mathbb{Z}^n$ ($n \geq 2$) that is not identically zero and has finite p moment, i.e. $\mathbb{E}[c^p] < \infty$, for some $p > \frac{n+1}{s}$. Let $\{x_m\}_{m\in\mathbb{N}} \subset \mathbb{R}^n$ be an arbitrary sequence such that $x_m \in Z_m$ and $x_m \to 0$. Then $\mathbb{P}$-a.s. the sequence of random measures $\{\mathbf{P}^{(m)}_{x_m}\}_{m\in\mathbb{N}}$ is tight on the Skorokhod space $\mathcal{D}([0,T]; \mathbb{R}^n)$ for every $T > 0$.*

Proof. The following argument is borrowed from the proof of Theorem 4.5 from [CKW18b]. Let us fix an arbitrary $T > 0$. The plan is to verify Items (i) and (ii) from Theorem 12.5.1 for the sequence $\{\mathbf{P}^{(m)}_{x_m}\}$. Since c is a nonzero conductance, we can find $\mathfrak{p} > 0$ and $\nu > 0$ such that $\mathbb{P}(c(x,y) > \nu) \geq \mathfrak{p}$ just like in Definition 11.0.3. Using Theorem 11.8.1 we find $\theta \equiv \theta(p,s,n) \in (0,1)$, $C_{EP} \equiv C_{EP}(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n)$, $\delta \equiv \delta(p, \mathbb{E}[c^p], \mathfrak{p}, \nu, s, n)$, $C_T \equiv C_T(\mathbb{E}[c^p], s, n)$ and a random variable $R_\star := R_\star(0) \equiv R_\star(0, \omega, c, p, \mathfrak{p}, \nu, n)$ such that, for all $t > 0$, $R_0 \geq R_\star$, $x_0 \in B(0, R_0/2)$, $R \leq R_0/2$, Ineq. (11.15) holds, that is,

$$1 - P_t^{B(x_0,R)} 1 \leq C_{EP} \frac{t}{R^{2s}} \left(\frac{R_0^{2s\theta}}{t} \vee \delta^{-2s} \vee 4C_T \right) \qquad \text{in } B\left(x_0, R_0^\theta \vee \delta^{-1} t^{\frac{1}{2s}}\right).$$

On the other hand, for every ball $B \subset \mathbb{Z}^n$ (which is a nearly Borel set) we know by [CF12] Theorem 3.3.8 that process X_t^B, obtained from X_t by killing it upon exiting B, is properly associated with Dirichlet form $(\mathcal{E}, \mathcal{D}_B[\mathcal{E}])$. Thus [FOT11], Theorem 4.2.3 implies that for every function $v : \mathbb{Z}^n \to \mathbb{R}$ (all such functions are automatically universally measurable because the space is discrete) we have

$$\mathbf{E}_x(v(X_t^B)) = P_t^B v(x) \quad \forall x \in \mathbb{Z}^n. \tag{12.6}$$

Notice that because we are dealing with the discrete measure "for every x" in the last statement is equivalent to "for a.e. x" and "for q.e. x". Our estimate therefore translates into

$$\sup_{x \in B\left(x_0, R_0^\theta \vee \delta^{-1} t^{\frac{1}{2s}}\right)} \mathbf{P}_x\left(X_t^{B(x_0,R)} \notin B(x_0,R)\right) \leq C_{EP} \frac{t}{R^{2s}} \left(\frac{R_0^{2s\theta}}{t} \vee \delta^{-2s} \vee 4C_T \right).$$

The probability on the left is equal to the probability that X_t does not exit ball $B(x_0, R)$ at any time $\tau \in [0,t]$ because otherwise $X_t^{B(0,R)}$ is killed upon exit and never comes back to $B(0,R)$. Thus we can also write

$$\sup_{x\in B\left(x_0,R_0^\theta\vee\delta^{-1}t^{\frac{1}{2s}}\right)} \mathbf{P}_x\left(\sup_{\tau\in[0,t]}|X_\tau - x_0|>R\right)\le C_{EP}\frac{t}{R^{2s}}\left(\frac{R_0^{2s\theta}}{t}\vee\delta^{-2s}\vee 4C_T\right) \tag{12.7}$$

for all $t\geq 0$, $R_0\geq R_\star$, $x_0\in B(0,R_0/2)$, $R\leq R_0/2$. The fact that Ineq. (12.7) holds uniformly in $x_0\in B(0,R_0/2)$ will be of crucial importance for the rest of the proof.

Choosing $x_0=0$, $R_0=2R=mr$ in Ineq. (12.7), for arbitrary $r>0$ and $m\in\mathbb{N}$, and recalling the scaling relation $X_t^{(m)}=X_{m^{2s}t}/m$ from Definition 12.1.3 we obtain

$$\begin{aligned}\sup_{x\in B(0,\delta^{-1}t^{1/2s})}\mathbf{P}_x(X_t^{(m)}\notin B(0,r)) &\le \sup_{x\in B(0,m\delta^{-1}t^{1/2s})}\mathbf{P}_x\left(\sup_{\tau\in[0,t]}|X_{m^{2s}\tau}|>mr/2\right)\\ &\le C_{EP}\frac{t}{(r/2)^{2s}}\left(\frac{r^{2s\theta}}{tm^{2s(1-\theta)}}\vee\delta^{-2s}\vee 4C_T\right).\end{aligned}$$

Due to $\theta<1$, when passing to $\limsup_{m\to\infty}$ the expression reduces to

$$\limsup_{m\to\infty}\sup_{x\in B(0,\delta^{-1}t^{1/2s})}\mathbf{P}_x(X_t^{(m)}\notin B(0,r))\le C_{EP}\left(\delta^{-2s}\vee 4C_T\right)\frac{t}{(r/2)^{2s}}$$

and then letting $r\to\infty$ we find that, for every $t>0$,

$$\lim_{r\to\infty}\limsup_{m\to\infty}\sup_{x\in B(0,\delta^{-1}t^{1/2s})}\mathbf{P}_x(X_t^{(m)}\notin B(0,r))=0.$$

Since $x_m\to 0$, $x_m\in B(0,\delta^{-1}t^{\frac{1}{2s}})$ for m large enough so in particular we have

$$\lim_{r\to\infty}\limsup_{m\to\infty}\mathbf{P}_{x_m}(X_t^{(m)}\notin B(0,r))=0$$

which proves that $\mathbf{P}_{x_m}(X_t\in\cdot)$ is tight on $\mathbb{R}^n$ for every $t>0$ and checks Item (i) of Theorem 12.5.1. To prove Item (ii) take an arbitrary $\xi>0$, an arbitrary sequence of stopping time $\tau_m\leq T$, an arbitrary sequence of numbers $\varepsilon_m>0$, $\varepsilon_m\to 0$, and consider the probability

$$\mathbf{P}_0\left(|X^{(m)}_{\tau_m+\varepsilon_m}-X^{(m)}_{\tau_m}|>\xi\right).$$

For $r_0\geq 0$ we can estimate

$$\begin{aligned}\mathbf{P}_0\left(|X^{(m)}_{\tau_m+\varepsilon_m}-X^{(m)}_{\tau_m}|>\xi\right)&\le\mathbf{P}_0\left(\sup_{t\in[0,T]}|X_t^{(m)}|>r_0/2\right)\\ &\quad+\mathbf{P}_0\left(|X^{(m)}_{\tau_m+\varepsilon_m}-X^{(m)}_{\tau_m}|>\xi\ \Big|\ \sup_{t\in[0,T]}|X_t^{(m)}|\le r_0/2\right):=P_1+P_2.\end{aligned}$$

P_1 and P_2 can be expressed in terms of X_t using the scaling relation $X_t^{(m)}=X_{m^{2s}t}/m$ from Definition 12.1.3. Taking $R=R_0/2=mr_0/2$, $x_0=0\in B(0,R_0/2)$ in Ineq. (12.7) and assuming that $r_0\geq 2$ and m is large enough so that $R_0=mr_0>R_\star$ we can estimate

$$\begin{aligned}P_1=\mathbf{P}_0\left(\sup_{t\in[0,m^{2s}T]}|X_t|>mr_0/2\right)&\le C_{EP}\frac{T}{(r_0/2)^{2s}}\left(\frac{r_0^{2s\theta}}{Tm^{2s(1-\theta)}}\vee\delta^{-2s}\vee 4C_T\right)\\ &\le C_{EP}(r_0/2)^{-2s(1-\theta)}\left(1\vee\delta^{-2s}T\vee 4C_TT\right).\end{aligned}$$

On the other hand, Markov property of process $X_t^{(m)}$ under condition $X_t^{(m)} \in B(0, r_0/2)$ implies that

$$\mathbf{P}_0\left(|X^{(m)}_{\tau_m+\varepsilon_m} - X^{(m)}_{\tau_m}| > \xi \quad \Big| \sup_{t\in[0,T]} |X_t^{(m)}| \le r_0/2\right)$$
$$= \mathbf{E}_0\left[\mathbf{P}_{X_{\tau_m}}\left(|X^{(m)}_{\tau_m+\varepsilon_m} - X^{(m)}_{\tau_m}| > \xi\right) \quad \Big| \sup_{t\in[0,T]} |X_t^{(m)}| \le r_0/2\right].$$

This allows us to estimate

$$P_2 \le \sup_{x_0\in B(0,r_0)} \mathbf{P}_{x_0}\left(\sup_{t_1\in[0,\varepsilon_m]} |X^{(m)}_{t_1} - x_0| > \xi\right)$$
$$\le \sup_{x_0\in B(0,mr_0)} \mathbf{P}_{x_0}\left(\sup_{t_1\in[0,m^{2s}\varepsilon_m]} |X_{t_1} - x_0| > m\xi\right).$$

This time we would like to apply Ineq. (12.7) with $R := m\xi$, $R_0 := 2mr_0$, $t = m^{2s}\varepsilon_m$ and $x_0 \in B(0, R_0/2) = B(0, mr_0)$. We can assume $\xi \le 1$ (without loss of generality) and $r_0 \ge 2$ like before to get $R = \xi m \le m \le mr_0 \le R_0/2$. If m is large enough so that $R_0 = 2mr_0 > R_\star$, the estimate is uniform in $x_0 \in B(0, mr_0)$ so

$$P_2 \le C_{EP}\frac{\varepsilon_m}{\xi^{2s}}\left(\frac{(2r_0)^{2s\theta}}{m^{1-\theta}\varepsilon_m} \vee \delta^{-2s} \vee 4C_T\right) \le C_{EP}\left(\frac{(2r_0)^{2s\theta}}{m^{1-\theta}\xi^{2s}} \vee \frac{\varepsilon_m}{\delta^{2s}\xi^{2s}} \vee \frac{4C_T\varepsilon_m}{\xi^{2s}}\right) \xrightarrow{m\to\infty} 0.$$

where we used $\varepsilon_m \to 0$ in the last inequality. The estimate on P_1 does not depend on m, as long as it is large enough ($m > R_\star/2$ to be precise), and hence

$$\mathbf{P}_0\left(|X^{(m)}_{\tau_m+\varepsilon_m} - X^{(m)}_{\tau_m}| > \delta\right) \le P_1 + P_2 \xrightarrow{m\to\infty} C_{EP}(r_0/2)^{-2s(1-\theta)}\left(1 \vee \delta^{-2s}T \vee 4C_T T\right)$$

for every $r_0 \ge 2$. But as we are still free to choose $r_0 \ge 2$ arbitrary large it follows that

$$\lim_{m\to\infty} \mathbf{P}_0\left(|X^{(m)}_{\tau_m+\varepsilon_m} - X^{(m)}_{\tau_m}| > \delta\right) \le \lim_{r_0\to\infty} C_{EP}(r_0/2)^{-2s(1-\theta)}\left(1 \vee \delta^{-2s}T \vee 4C_T T\right) = 0$$

which proves Item (ii) of Theorem 12.5.1. Hence Theorem 12.5.1 implies that the sequence of measures $\mathbf{P}_{x_m}$ is tight on $\mathcal{D}([0,T], \mathbb{R}^n)$ for every $T > 0$. □

Corollary 12.5.3. *Let c be an i.i.d. conductance on $\mathbb{Z}^n$ ($n \ge 2$) such that $\mathbb{E}[c^p] < \infty$ for some $p > \frac{n+1}{s}$. Then, for $\mathbb{P}$-a.e. realization of conductance c, $X_t^{(m)}$ started at $X_0^{(m)} = 0$ converges weakly on Skorohod space $\mathcal{D}([0,T], \mathbb{R}^n)$ to $X_t^{(\infty)}$, started from $X_0^{(\infty)} = 0$, for every $T > 0$.*

Proof. By Theorem 12.5.2 we know that distributions of $X_t^{(m)}$ are tight on $\mathcal{D}([0,T]; \mathbb{R}^n)$ for every $T > 0$. On the other hand, Corollary 12.4.3 proves that $X_t^{(m)}$ converges in finite-dimensional distributions to $X_t^{(\infty)}$ which identifies the limit point of sequence $X_t^{(m)}$ as $X_t^{(\infty)}$. Hence $X_t^{(m)}$ converges to $X_t^{(\infty)}$ weakly on $\mathcal{D}([0,T]; \mathbb{R}^n)$ for every $T > 0$. □

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