Capacities, Removable Sets and L^p-Uniqueness on Wiener Spaces

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Abstract

We prove the equivalence of two different types of capacities in abstract Wiener spaces. This yields a criterion for the L^p-uniqueness of the Ornstein-Uhlenbeck operator and its integer powers defined on suitable algebras of functions vanishing in a neighborhood of a given closed set Σ of zero Gaussian measure. To prove the equivalence we show the $W^{r,p}(B, \mu)$-boundedness of certain smooth nonlinear truncation operators acting on potentials of nonnegative functions. We discuss connections to Gaussian Hausdorff measures. Roughly speaking, if L^p-uniqueness holds then the 'removed' set Σ must have sufficiently large codimension, in the case of the Ornstein-Uhlenbeck operator for instance at least $2p$. For $p = 2$ we obtain parallel results on truncations, capacities and essential self-adjointness for Ornstein-Uhlenbeck operators with linear drift. These results apply to the time zero Gaussian free field as a prototype example.

1 Introduction

The present article deals with capacities associated with Ornstein-Uhlenbeck operators on abstract Wiener spaces (B, μ, H), [8, 11, 24, 32, 35–37, 53, 58], and applications to L^p-uniqueness problems for Ornstein-Uhlenbeck operators and their integer powers, endowed with algebras of functions vanishing in a neighborhood of a small closed set.

Our original motivation comes from L^p-uniqueness problems for operators L endowed with a suitable algebra $\mathcal{A}$ of functions, the special case $p = 2$ is the problem of essential

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self-adjointness. For the 'globally defined' operator L on the entire space L^p-uniqueness is well understood, see for instance [18] and the references cited there. If the globally defined operator is L^p-unique one can ask whether the removal of a small set (or, in other words, the introduction of a small boundary) destroys this uniqueness or not. A loss of uniqueness means that extensions to generators of C_0-semigroups, [45], with different boundary conditions exist. The answer to this question depends on the size of the removed set. The most classical example may be the essential self-adjointness problem for the Laplacian Δ on $\mathbb{R}^n$, endowed with the algebra $C_c^\infty(\mathbb{R}^n \setminus \{0\})$ of smooth compactly supported functions on $\mathbb{R}^n$ with the origin $\{0\}$ removed. It is well known that this operator is essential self-adjoint in $L^2(\mathbb{R}^n)$ if and only if $n \geq 4$, [59, p.114] and [47, Theorem X.11, p.161]. Generalizations of this example to manifolds have been provided in [12] and [38], more general examples on Euclidean spaces can be found in [5] and [27], further generalizations to manifolds and metric measure spaces will be discussed in [28]. For the Laplacian on $\mathbb{R}^n$ one main observation is that, if a compact set Σ of zero measure is removed from $\mathbb{R}^n$, the essential self-adjointness of $(\Delta, C_c^\infty(\mathbb{R}^n \setminus \Sigma))$ in $L^2(\mathbb{R}^n)$ implies that $\dim_H \Sigma \leq n - 4$, where $\dim_H$ denotes the Hausdorff dimension. See [5, Theorems 10.3 and 10.5] or [27, Theorem 2]. This necessary 'codimension four' condition can be rephrased by saying that we must have $\mathcal{H}^{n-d}(\Sigma) = 0$ for all $d < 4$, where $\mathcal{H}^{n-d}$ denotes the Hausdorff measure of dimension $n - d$.

Having in mind coefficient regularity or boundary value problems for operators in infinite dimensional spaces, see e.g. [10, 13, 14, 25, 26], one may wonder whether a similar 'codimension four' condition can be observed in infinite dimensional situations. For the case of Ornstein-Uhlenbeck operators on abstract Wiener spaces an affirmative answer to this question follows from the present results in the special case $p = 2$.

The basic tools to describe the critical size of a removed set $\Sigma \subset B$ are capacities associated with the Sobolev spaces $W^{r,p}(B, \mu)$ for the H-derivative respectively the Ornstein-Uhlenbeck semigroup, [8, 11, 24, 32, 35–37, 53, 58]. Such capacities can be introduced following usual concepts of potential theory, [11, 20, 37, 52, 53, 55, 56, 58], see Definition 3.1 below, and they are known to be connected to Gaussian Hausdorff measures, [21]. Uniqueness problems connect easier to another, slightly different definition of capacities, where the functions taken into account in the definition are recruited from the initial algebra $\mathcal{A}$ and, roughly speaking, are required to be equal to one on the set in question, see Definitions 3.2 and 3.3. This type of definition connects them to an algebraic ideal property which is helpful to investigate extensions of operators initially defined on ideals of $\mathcal{A}$. For Euclidean Sobolev spaces these two types of capacities are known to be equivalent, see for instance [2, Section 2.7]. The proofs of these equivalences go back to Maz'ya, Khavin, Adams, Hedberg, Polking and others, [1–3, 41–44], and rely on bounds in Sobolev norms for certain nonlinear composition operators acting on the cone of nonnegative Sobolev functions, see e.g. [1, Theorem 3], or the cone of potentials of nonnegative functions, see e.g. [1, Theorem 2] or [2, Theorem 3.3.3]. Apart from the first order case $r = 1$ this is nontrivial, because in finite dimensions Sobolev spaces are not stable under such compositions, see for instance [2, Theorem 3.3.2]. Apart from the case $p = 2$, where one can also use an integration by parts argument, [1, Theorem 3], the desired bounds are shown using suitable Gagliardo-Nirenberg inequalities, [3, 42], or suitable multiplicative estimates of Riesz or Bessel potential operators involving Hardy-Littelwood maximal functions and the L^p-boundedness of the latter, [2, Theorem 1.1.1, Proposition 3.1.8]. The constants in these estimates are dimension dependent.

Sobolev spaces $W^{r,p}(B, \mu)$ over abstract Wiener spaces (B, μ, H) are stable under compositions with bounded smooth functions, [8, Remark 5.2.1 (i)], but one still needs to

establish quantitative bounds. We establish Sobolev norm bounds for nonlinear composition operators acting on potentials of nonnegative functions, Theorem 3.6. To obtain it, we use the L^p-boundedness of the maximal function in the sense of Rota and Stein for the Ornstein-Uhlenbeck semigroup, [53, Theorem 3.3], this provides a similar multiplicative estimate as in the finite dimensional case, see Lemma 4.2. From the Sobolev norm estimate for compositions we can then deduce the desired equivalence of capacities, Theorem 3.5, where $\mathcal{A}$ is chosen to be the set of smooth cylindrical functions or the space of Watanabe test functions. Applications of this equivalence provide L^p-uniqueness results for the Ornstein-Uhlenbeck operator and, under a sufficient condition that ensures they generate C_0-semigroups, also for its integer powers, see Theorem 5.2. In particular, if $\Sigma \subset B$ is a given closed set of zero Gaussian measure, then the Ornstein-Uhlenbeck operator, endowed with the algebra of cylindrical functions vanishing in a neighborhood of Σ (or the algebra of Watanabe test functions vanishing q.s. on a neighborhood of Σ) is L^p-unique if and only if the $(2, p)$-capacity of Σ is zero, see Theorem 5.2. Combined with results from [21] on Gaussian Hausdorff measures, we then observe that the L^p-uniqueness of this Ornstein-Uhlenbeck operator 'after the removal of Σ' implies that the Gaussian Hausdorff measure $\varrho_d(\Sigma)$ of codimension d of Σ must be zero for all $d < 2p$, see Corollary 6.2. In particular, if the operator is essentially self-adjoint on $L^2(B, \mu)$, then $\varrho_d(\Sigma)$ must be zero for all $d < 4$, what is an analog of the necessary 'codimension four' condition known from the Euclidean case. In Sections 8 and 9 we partial rework the arguments to obtain results on essential self-adjointness of Ornstein-Uhlenbeck operators with linear drift as studied for instance in [4, 7, 51, 52], the prominent example being the Hamiltonian of the time zero Gaussian free field in Euclidean quantum field theory, [46, 48, 49, 54], Example 9.3. Again the characterization of essential self-adjointness, Theorem 9.1, is obtained from a capacitary equivalence, Theorem 8.7, based on a truncation result for potentials, Theorem 8.9.

In the next section we recall standard items from the analysis on abstract Wiener spaces. In Section 3 we define Sobolev capacities and prove their equivalence, based on the norm bound on nonlinear compositions, which is proved in Section 4. Section 5 contains the mentioned L^p-uniqueness results. The connection to Gaussian Hausdorff measures is briefly discussed in Section 6, followed by some remarks on related Kakutani theorems for multiparameter processes in Section 7. Capacities, truncations and essential-selfadjointness of operators with linear drift are discussed in Sections 8 and 9.

2 Preliminaries

Following the presentation in [53], we provide some basic definitions and facts. Let (B, μ, H) be an *abstract Wiener space*. That is, B is a real separable Banach space, H is a real separable Hilbert space which is embedded densely and continuously on B, and μ is a Gaussian measure on B with

$$\int_B \exp\{\sqrt{-1}\langle \varphi, y \rangle\}\mu(dy) = \exp\left\{-\frac{1}{2}|\varphi|^2_{H^*}\right\}, \quad \varphi \in B^*,$$

see for instance [53, Definition 1.2]. Here we identify H^* with H as usual, so that $B^* \subset H \subset B$. Since every $\varphi \in B^*$ is $N(0, |\varphi|^2_H)$-distributed, it is an element of $L^2(B, \mu)$ and the map $\varphi \mapsto \langle \varphi, \cdot \rangle$ is an isometry from B^*, equipped with the scalar product $\langle \cdot, \cdot \rangle_H$, into $L^2(B, \mu)$. It extends uniquely to an isometry

$$h \mapsto \hat{h} \tag{1}$$

from H into $L^2(B, \mu)$. A function $f : B \to \mathbb{R}$ is said to be H-differentiable at $x \in B$ if there exists some $h^* \in H^*$ such that

$$\frac{d}{dt} f(x + th)|_{t=0} = \langle h, h^* \rangle$$

for all $h \in H$. If f is H-differentiable at x then h^* is uniquely determined, denoted by $Df(x)$ and refereed to as the H-derivative of f at x. See [53, Definition 2.6]. For a function f that is H-differentiable at $x \in B$ and an element h of H we can define the *directional derivative* $\partial_h f(x)$ of f at x by

$$\partial_h f(x) := \langle Df(x), h \rangle_H .$$

A function $f : B \to \mathbb{R}$ is said to be k-times H-differentiable at $x \in B$ if there exists a continuous k-linear mapping $\Phi_x : H^k \to \mathbb{R}$ such that

$$\frac{\partial^k}{\partial t_1 \cdots \partial t_k} f(x + t_1 h_1 + \cdots + t_k h_k)|_{t_1 = \cdots = t_k = 0} = \Phi_x(h_1, \ldots h_k)$$

for all $h_1, \ldots, h_k \in H$. If so, Φ_x is unique and denoted by $D^k f(x)$. A function $f : B \to \mathbb{R}$ is called a *(smooth) cylindrical function* if there exist an integer $n \geq 1$, linear functionals $l_1, ..., l_n \in B^*$ and a function $F \in C_b^\infty(\mathbb{R}^n)$ such that

$$f = F(l_1, ..., l_n). \tag{2}$$

The space of all such cylindrical functions on B we denote by $\mathcal{F}C_b^\infty$. Clearly $\mathcal{F}C_b^\infty$ is an algebra under pointwise multiplication and stable under the composition with functions $T \in C_b^\infty(\mathbb{R})$.

A cylindrical function $f \in \mathcal{F}C_b^\infty$ as in Eq. (2) is infinitely many times H-differentiable at any $x \in B$, and for any $k \geq 1$ we have

$$D^k f(x) = \sum_{j_1, \ldots, j_k = 1}^{\infty} \partial_{j_1} \cdots \partial_{j_k} F(\langle x, l_1 \rangle, ..., \langle x, l_n \rangle) \, l_{j_1} \otimes \cdots \otimes l_{j_k}, \tag{3}$$

where ∂_j denotes the j-th partial differentiation in the Euclidean sense. The space $\mathcal{F}C_b^\infty$ is dense in $L^p(B, \mu)$ for any $1 \leq p < +\infty$, see e.g. [7, Lemma 2.1].

We write $\mathcal{H}_0 := \mathbb{R}$, $\mathcal{H}_1 := H$ and generalizing this, denote by $\mathcal{H}_k$ the space of k-linear maps $a : H^k \to \mathbb{R}$ such that

$$|a|_{\mathcal{H}_k}^2 := \sum_{j_1, \ldots, j_k = 1}^{\infty} (a(e_{j_1}, \ldots, e_{j_k}))^2 < +\infty, \tag{4}$$

where $(e_i)_{i=1}^\infty$ is an orthonormal basis in H. The value of this norm does not depend on the choice of this basis. See [9, p.3]. Clearly every such k-linear map a can also be seen as a linear map $a : H^{\otimes k} \to \mathbb{R}$, where $H^{\otimes k}$ denotes the k-fold tensor product of H, with this interpretation we have $a(e_{j_1} \otimes ... \otimes e_{j_k}) = a(e_{j_1}, \ldots, e_{j_k})$ and by Eq. (4) the operator a is a Hilbert-Schmidt operator. For later use we record the following fact.

Proposition 2.1 *For any $a \in \mathcal{H}_k$ we have*

$$|a|_{\mathcal{H}_k} \leq 2k^k \sup \{|a(h_1, ..., h_k)| : h_1, \cdots, h_k \text{ are members}$$
$$\text{of an orthonormal system in } H, \text{ not necessarily distinct}\} .$$

Proof By Parseval's identity and Cauchy-Schwarz in $H^{\otimes k}$ we have

$$|a|_{\mathcal{H}_k} = \sup\left\{|ay| : y \in H^{\otimes k} \text{ and } |y|_{H^{\otimes k}} = 1\right\}.$$

Choose an element $y = y_1 \otimes ... \otimes y_k \in H^{\otimes k}$ such that $|y|_{H^{\otimes k}} = 1$ and $|a|_{\mathcal{H}_k} \leq 2|ay|$. Without loss of generality we may assume that $|y_j|_H = 1$, $1 \leq j \leq k$. Choosing an orthonormal basis $(b_i)_{i=1}^n$ in the subspace $\text{span}\{y_1, ..., y_k\}$ of H we observe $n \leq k$ and $y_j = \sum_{i=1}^n b_i \lambda_{ij}$ with some $|\lambda_{ij}| \leq 1$. Since this implies

$$|ay| \leq \sum_{i_1, \cdots, i_k \in \{1, \cdots, n\}} |a(b_{i_1}, \cdots, b_{i_k})|,$$

we obtain the desired result. $\qquad\square$

We recall the definition of Sobolev spaces on B. For any $1 \leq p < +\infty$ and $k \geq 0$ let $L^p(B, \mu, \mathcal{H}_k)$ denote the L^p-space of functions from B into $\mathcal{H}_k$. For any $1 \leq p < +\infty$ and integer $r \geq 0$ set

$$\|f\|_{W^{r,p}(B,\mu)} := \sum_{k=0}^r \left\|D^k f\right\|_{L^p(B,\mu,\mathcal{H}_k)}, \tag{5}$$

$f \in \mathcal{F}C_b^\infty$. The *Sobolev class* $W^{r,p}(B, \mu)$ is defined as the completion of $\mathcal{F}C_b^\infty$ in this norm, see [8, Section 5.2] or [9, Section 8.1]. In particular, $W^{0,p}(B, \mu) = L^p(B, \mu)$. For $f \in W^{r,p}(B, \mu)$ the derivatives $D^k f$, $k \leq r$, are well defined as elements of $L^p(B, \mu, \mathcal{H}_k)$, see [8, Section 5.2]. By definition the spaces $W^{r,p}(B, \mu)$ are Banach spaces. The space W^∞ of *Watanabe test functions* is defined as

$$W^\infty := \bigcap_{r \geq 1, \, 1 \leq p < +\infty} W^{r,p}(B, \mu).$$

We have $\mathcal{F}C_b^\infty \subset W^\infty$, in particular, W^∞ is a dense subset of every $L^p(B, \mu)$ and $W^{r,p}(B, \mu)$.

In contrast to Sobolev spaces over finite dimensional spaces, [2, Theorem 3.3.2], also the Sobolev classes $W^{r,p}(B, \mu)$, $r \geq 2$, are known to be stable under compositions $u \mapsto T(u) = T \circ u$ with functions $T \in C_b^\infty(\mathbb{R})$, as follows from the evaluation of an integration by parts identity together with the chain rule, applied to cylindrical functions. See [8, Remark 5.2.1 (i)] or [9, Proposition 8.7.5]. In particular, the space W^∞ is stable under compositions with functions from $C_b^\infty(\mathbb{R})$. Also, it is an algebra with respect to pointwise multiplication, [37, Corollary 5.8].

Given a bounded (or nonnegative) Borel function $f : B \to \mathbb{R}$ and $t > 0$ set

$$P_t f(x) := \int_B f(e^{-t}x + \sqrt{1 - e^{-2t}}\, y)\mu(dy), \quad x \in B. \tag{6}$$

The function $P_t f$ is again bounded (resp. nonnegative) Borel on B and the operators P_t form a semigroup, i.e. that for any $s, t > 0$ we have $P_{t+s} = P_t P_s$. The semigroup $(P_t)_{t>0}$ is called the *Ornstein-Uhlenbeck semigroup on B*. For any $1 \leq p \leq +\infty$ it extends to a contraction semigroup $(P_t^{(p)})_{t>0}$ on $L^p(B, \mu)$, [53, Proposition 2.4], strongly continuous for $1 \leq p < +\infty$. The semigroup $(P_t^{(2)})_{t>0}$ is a sub-Markovian symmetric semigroup on $L^2(B, \mu)$ in the sense of [11, Definition I.2.4.1]. The infinitesimal generators $(\mathcal{L}^{(p)}, \mathcal{D}(\mathcal{L}^{(p)}))$ of $(P_t^{(p)})_{t>0}$ is called the *Ornstein-Uhlenbeck operator on $L^p(B, \mu)$*, [53, Section 2.1.4]. We will always

write P_t and $\mathcal{L}$ instead of $P_t^{(p)}$ and $\mathcal{L}^{(p)}$, the meaning will be clear from the context. Given $r > 0$ and a bounded (or nonnegative) Borel function $f : B \to \mathbb{R}$, set

$$V_r f := \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} P_t f \, dt, \tag{7}$$

where Γ denotes the Euler Gamma function. The function $V_r f$ is again bounded (resp. nonnegative) Borel, and for any $1 \leq p < \infty$ the operators V_r form a strongly continuous contraction semigroup $(V_r)_{r>0}$ on $L^p(B, \mu)$, see [8, Corollary 5.3.3] or [53, Proposition 4.7], symmetric for $p = 2$. In any of these spaces the operators V_r are the powers $(I - \mathcal{L})^{-r/2}$ of order $r/2$ of the respective 1-resolvent operators $(I - \mathcal{L})^{-1}$. Meyer's equivalence, [9, Theorem 8.5.2], [53, Theorem 4.4], states that for any integer $r \geq 1$ and any $1 < p < +\infty$ and any $u \in W^{r,p}(B, \mu)$ we have

$$c_1 \|u\|_{W^{r,p}(B,\mu)} \leq \left\| (I - \mathcal{L})^{r/2} u \right\|_{L^p(B,\mu)} \leq c_2 \|u\|_{W^{r,p}(B,\mu)} \tag{8}$$

with constants $c_1 > 0$ and $c_2 > 0$ depending only on r and p. By the continuity of the V_r and the density of cylindrical functions we observe $W^{r,p}(B, \mu) = V_r(L^p(B, \mu))$ in the sense of equivalently normed spaces. Note that for $p = 2$ the middle terms in Eq. (8) provide equivalent norms that make the space Hilbert. The operator V_r acts as an isometry from $W^{s,p}(B, \mu)$ onto $W^{s+r,p}(B, \mu)$, [11, Chapter II, Theorem 7.3.1]. For later use we record the following well known fact.

Proposition 2.2 *For any $r > 0$ we have $V_r(\mathcal{F}C_b^\infty) \subset \mathcal{F}C_b^\infty$ and $V_r(W^\infty) \subset W^\infty$.*

Proof From the preceding lines it is immediate that $V_r(W^\infty) \subset W^\infty$. To see the remaining statement suppose $f \in \mathcal{F}C_b^\infty$ with $f = F(l_1, ..., l_n)$, $l_i \in B^*$, $F \in C_b^\infty(\mathbb{R}^n)$, and by applying Gram-Schmidt we may assume $\{l_1, ..., l_n\}$ is an orthonormal system in H. The Ornstein-Uhlenbeck semigroup $(T_t^{(n)})_{t>0}$ on $L^2(\mathbb{R}^n)$, defined by

$$T_t^{(n)} F(\xi) = \int_{\mathbb{R}^n} F(e^{-t}\xi + \sqrt{1-e^{-2t}}\eta)(2\pi)^{-n/2} e^{-|\eta|^2/2} d\eta,$$

preserves smoothness, i.e. $T_t^{(n)} F \in C_b^\infty(\mathbb{R}^n)$ for any $F \in C_b^\infty(\mathbb{R}^n)$. Given $x \in B$ and writing $\xi = (\langle x, l_1 \rangle, ..., \langle x, l_n \rangle)$, we have

$$\begin{aligned}
P_t f(x) &= \int_B F(\langle e^{-t}x + \sqrt{1-e^{-2t}}y, l_1 \rangle, ..., \langle e^{-t}x + \sqrt{1-e^{-2t}}y, l_n \rangle) \mu(dy) \\
&= \int_B F(e^{-t}\xi + \sqrt{1-e^{-2t}}(\langle y, l_1 \rangle, ..., \langle y, l_n \rangle)) \mu(dy) \\
&= \int_{\mathbb{R}^n} F(e^{-t}\xi + \sqrt{1-e^{-2t}}\eta)(2\pi)^{-n/2} e^{-|\eta|^2/2} d\eta \\
&= F_t^{(n)}(\langle x, l_1 \rangle, ..., \langle x, l_n \rangle),
\end{aligned}$$

where $F_t^{(n)} = T_t^{(n)} F$. Consequently $P_t f \in \mathcal{F}C_b^\infty$, and using Eq. (7) and dominated convergence it follows that $V_r f \in \mathcal{F}C_b^\infty$. $\qquad\square$

Although different in nature both $\mathcal{F}C_b^\infty$ and W^∞ can serve as natural replacements in infinite dimensions for algebras of smooth differentiable functions in Euclidean spaces or on manifolds.

3 Capacities and their Equivalence

We define two types of capacities related to $W^{r,p}(B,\mu)$-spaces and verify their equivalence. The following definition is standard, see for instance [20, 23, 33, 52].

Definition 3.1 Let $1 \leq p < +\infty$ and let $r > 0$ be an integer. For open $U \subset B$, let

$$\mathrm{Cap}_{r,p}(U) := \inf\{\|f\|_{L^p}^p \mid f \in L^p(B,\mu),\ V_r f \geq 1\ \mu\text{-a.e. on } U\}$$

and for arbitrary $A \subset B$,

$$\mathrm{Cap}_{r,p}(A) := \inf\{\mathrm{Cap}_{r,p}(U)\mid A \subset U,\ U \text{ open}\}.$$

We give two further definitions of (r,p)-capacities in which we insist on a strict equality on the set to be tested. In finite dimensional spaces such capacities were introduced in [39], see also [2, Definition 2.7.1], [40] and [43, Chapter 13]. The first definition we give is based on cylindrical functions.

Definition 3.2 Let $1 \leq p < +\infty$ and let $r > 0$ be an integer. For an open set $U \subset B$ define

$$\mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(U) := \inf\left\{\|u\|_{W^{r,p}(B,\mu)}^p \mid u \in \mathcal{F}C_b^\infty,\ u = 1 \text{ on } U\right\},$$

and for an arbitrary set $A \subset B$,

$$\mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(A) := \inf\{\mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(U)\mid A \subset U,\ U \text{ open}\}.$$

The capacities $\mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}$ have a more 'algebraic' flavor and are well suited to study operator extensions, see Section 5.

One can give a similar definition based on the space W^∞. To do so, we recall some potential theoretic notions. If a property holds outside a set $E \subset B$ with $\mathrm{Cap}_{r,p}(E) = 0$ then we say it holds (r,p)-*quasi everywhere (q.e.)*. We follow [37, Chapter IV, Section 1.2] and call a set $E \subset B$ *slim* if $\mathrm{Cap}_{r,p}(E) = 0$ for all $1 < p < +\infty$ and all integer $r > 0$, and if a property holds outside a slim set, we say it holds *quasi surely (q.s.)*. A function $u : B \to \mathbb{R}$ is said to be (r,p)-*quasi continuous* if for any $\varepsilon > 0$ we can find an open set $U_\varepsilon \subset B$ such that $\mathrm{Cap}_{r,p}(U) < \varepsilon$ and the restriction $u|_{U_\varepsilon^c}$ of u to U_ε^c is continuous. Every function $u \in W^{r,p}(B,\mu)$ admits a (r,p)-quasi-continuous version $\widetilde{u}$, unique in the sense that two different quasi continuous versions can differ only on a set of zero (r,p)-capacity. Since continuous functions are dense in $W^{r,p}(B,\mu)$ this follows by standard arguments, see for instance [11, Chapter I, Section 8.2]. Now one can follow [37, Chapter IV, Section 2.4] to see that for any $u \in W^\infty$ there exists a function $\widetilde{u} : B \to \mathbb{R}$ such that $u = \widetilde{u}$ μ-a.e. and for all r and p the function $\widetilde{u}$ is (r,p)-quasi continuous. It is referred to as the *quasi-sure redefinition* of u and it is unique in the sense that the difference of two quasi-sure redefinitions of u is zero (r,p)-quasi everywhere for all r and p, [37].

Definition 3.3 Let $1 \leq p < +\infty$ and let $r > 0$ be and integer. For an open set $U \subset B$ define

$$\mathrm{cap}_{r,p}^{(W^\infty)}(U) := \inf\left\{\|u\|_{W^{r,p}(B,\mu)}^p \mid u \in W^\infty,\ \widetilde{u} = 1 \text{ on } U \text{ q.s.}\right\},$$

where $\widetilde{u}$ denotes the quasi-sure redefinition of u with respect to the capacities from Definition 3.1, and for an arbitrary set $A \subset B$,

$$\mathrm{cap}_{r,p}^{(W^\infty)}(A) := \inf\{\mathrm{cap}_{r,p}^{(W^\infty)}(U)\mid A \subset U,\ U \text{ open}\}.$$

For some applications capacities based on the algebra W^∞ may be more suitable that those based on cylindrical functions.

Remark 3.4 In [34, Example 3.13] Kusuoka introduced capacities based on functions $u \in W^\infty$, but requiring that $u \geq 1$ μ-a.e. on U (similarly as in Definition 3.1 above) in place of the condition that $\widetilde{u} = 1$ on U q.s.

The following equivalence can be observed.

Theorem 3.5 *Let* $1 < p < +\infty$ *and let* $r > 0$ *be an integer. Then there are positive constants* c_3 *and* c_4 *depending only on* p *and* r *such that for any set* $A \subset B$ *we have*

$$c_3 \; \mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(A) \leq \mathrm{Cap}_{r,p}(A) \leq c_4 \; \mathrm{cap}_{r,p}^{(\mathcal{F}C_b^\infty)}(A) \tag{9}$$

and

$$c_3 \; \mathrm{cap}_{r,p}^{(W^\infty)}(A) \leq \mathrm{Cap}_{r,p}(A) \leq c_4 \; \mathrm{cap}_{r,p}^{(W^\infty)}(A). \tag{10}$$

Theorem 3.5 is an analog of corresponding results in finite dimensions, [3, Theorem A], [42, Theorem 3.3], see also [2, Section 2.7 and Corollary 3.3.4] or [43, Sections 13.3 and 13.4].

One ingredient of our proof of Theorem 3.5 is a bound in $W^{r,p}(B, \mu)$-norm for compositions with suitable smooth truncation functions. For the spaces $W^{1,p}(B, \mu)$ such a bound is clear from the chain rule for D respectively from general Dirichlet form theory, see [11]. Norm estimates in $W^{r,p}(B, \mu)$ for compositions $T \circ u$ of elements $u \in W^{r,p}(B, \mu)$ with suitable smooth functions $T : \mathbb{R} \to \mathbb{R}$ can be obtained via the chain rule. For instance, in the special case $r = 2$ the chain and product rules and the definition of the generator $\mathcal{L}$ imply

$$\mathcal{L}T(u) = T'(u)\mathcal{L}u + T''(u) \langle Du, Du \rangle_H \tag{11}$$

for any $u \in W^{2,p}(B, \mu)$. By Eq. (8) it would now suffice to show a suitable bound for $\mathcal{L}T(u)$ in L^p, and the summand more difficult to handle is the one involving the first derivatives Du. In the finite dimensional Euclidean case an L^p-estimate for it follows immediately from a simple integration by parts argument, [1, Theorem 3], or by a use of a suitable Gagliardo-Nirenberg inequality, [3, 42]. Integration by parts for Gaussian measures comes with an additional 'boundary' term involving the direction $h \in H$ of differentiation that spoils the original trick, and the classical proof of the Gagliardo-Nirenberg inequality involves dimension dependent constants. A simple alternative approach, suitable for any integer $r > 0$, is to prove truncation results for potentials in a similar way as in [2, Theorem 3.3.3], so that a quick evaluation of the first order term above follows from estimates in terms of the maximal function, [2, Proposition 3.1.8]. This method can be made dimension independent if the Hardy-Littlewood maximal function is replaced by the maximal function in terms of the semigroup operators Eq. (6) in the sense of Rota and Stein, [53, Theorem 3.3], [57, Chapter III, Section 3], see Lemma 4.2 below. We obtain the following variant of a Theorem due to Maz'ya and Adams, [1, Theorems 2 and 3], [2, Theorem 3.3.3], now for Sobolev spaces $W^{r,p}(B, \mu)$ over abstract Wiener spaces. A proof will be given in Section 4 below.

Theorem 3.6 *Assume* $1 < p < +\infty$ *and let* $r > 0$ *be an integer. Let* $T \in C^\infty(\mathbb{R}^+)$ *and suppose that* T *satisfies*

$$\sup_{t>0} |t^{i-1}T^{(i)}(t)| \leq L < \infty, \quad i = 0, 1, 2, \ldots \tag{12}$$

Then for every nonnegative $f \in L^p(B, \mu)$ the function $T \circ V_r f$ is an element of $W^{r,p}(B, \mu)$, and there is a constant $c_T > 0$ depending only on p, r and L such that for every nonnegative $f \in L^p(B, \mu)$ we have

$$\|T \circ V_r f\|_{W^{r,p}(B,\mu)} \leq c_T \|f\|_{L^p(B,\mu)}. \tag{13}$$

Remark 3.7 To prove Theorem 3.5 the function T can be chosen much more specifically than in Theorem 3.6. However, Eq. (12) is the classical hypothesis introduced by Maz'ya, [40] (see also [2, Theorem 3.3.3]), and since the statement may be of independent interest, we prove Theorem 3.6 in this general form.

Another useful tool in our proof of Theorem 3.5 is the following 'intermediate' description of $\mathrm{Cap}_{r,p}$. By $\mathcal{F}C_{b,+}^\infty$ we denote the cone of nonnegative elements of $\mathcal{F}C_b^\infty$.

Lemma 3.8 *Let* $1 \leq p < +\infty$ *and let* $r > 0$ *be an integer. For any open set* $U \subset B$ *we have*

$$\mathrm{Cap}_{r,p}(U) = \inf \left\{ \|f\|_{L^p(B,\mu)}^p \mid f \in \mathcal{F}C_{b,+}^\infty, \ V_r f \geq 1 \ \text{on} \ U \right\}. \tag{14}$$

Due to Proposition 2.2 the right hand side in Eq. (14) makes sense. The lemma can be proved using standard techniques, we partially follow [36, III. Proposition 3.5].

Proof For $U \subset B$ open let the right hand side of Eq. (14) be denoted by $\mathrm{Cap}_{r,p}'(U)$. Then clearly

$$\mathrm{Cap}_{r,p}'(\{|V_r f| > R\}) \leq R^{-p} \|f\|_{L^p(B,\mu)}^p \tag{15}$$

for all $f \in \mathcal{F}C_b^\infty$ and $R > 0$.

Now let $U \subset B$ open be fixed. The value of $\mathrm{Cap}_{r,p}(U)$ does not change if in its definition we require that $V_r f \geq 1 + \delta$ μ-a.e. on U with an arbitrarily small number $\delta > 0$. It does also not change if in addition we consider only nonnegative $f \in L^p(B, \mu)$ in the definition: For any $f \in L^p(B, \mu)$ the positivity and linearity of V_r imply that $(V_r f)^+ \leq V_r(f^+)$. Consequently, if $f \in L^p(B, \mu)$ is such that $V_r f \geq 1 + \delta$ μ-a.e. on U, then also $V_r(f^+) \geq 1 + \delta$ μ-a.e. on U, and clearly $\|f^+\|_{L^p(B,\mu)} \leq \|f\|_{L^p(B,\mu)}$.

Given $\varepsilon > 0$ choose a nonnegative function $f \in L^p(B, \mu)$ such that $u := V_r f \geq 1 + \delta$ μ-a.e. on U with some $\delta > 0$ and

$$\|f\|_{L^p(B,\mu)}^p \leq \mathrm{Cap}_{r,p}(U) + \frac{\varepsilon}{3}.$$

Approximating f by bounded nonnegative functions in $L^p(B, \mu)$, taking their cylindrical approximations, which are nonnegative as well, and smoothing by convolution in finite dimensional spaces, we can approximate f in $L^p(B, \mu)$ by a sequence of nonnegative functions $(f_n)_{n=1}^\infty \subset \mathcal{F}C_{b,+}^\infty$, see for instance [37, Chapter II, Theorem 5.1] or [35, Theorem 7.4.5]. Clearly the functions $u_n := V_r f_n$ satisfy $\lim_n u_n = u$ in $W^{r,p}(B, \mu)$.

By Eq. (15) and the convergence in $W^{r,p}(B, \mu)$ we can now choose a subsequence $(u_{n_i})_{i=1}^\infty$ such that

$$\mathrm{Cap}_{r,p}'(\{|u_{n_{i+1}} - u_{n_i}| > 2^{-i}\}) \leq 2^{-i} \quad \text{and} \quad \|u_{n_{i+1}} - u_{n_i}\|_{L^p(B,\mu)} \leq 2^{-2i}$$

for all $i = 1, 2, \dots$ For any $k = 1, 2, \dots$ let now

$$A_k := \bigcup_{i \geq k} \{|u_{n_{i+1}} - u_{n_i}| > 2^{-i}\}, \quad k = 1, 2, \dots$$

Then for each k the sequence $(u_{n_i})_{i=1}^{\infty}$ is Cauchy in supremum norm on A_k^c. On the other hand,

$$\mu(\{|u_{n_{i+1}} - u_{n_i}| > 2^{-i}\}) \leq 2^{-ip},$$

so that $\mu(A_k) \leq \sum_{i=k}^{\infty} 2^{-ip}$, what implies

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} \mu(A_k) = 0.$$

Consequently, setting $\overline{u}(x) := \lim_{n \to \infty} u_n(x)$ for all $x \in \bigcup_{k=1}^{\infty} A_k^c$ and $\overline{u}(x) = 0$ for all other x, we obtain a μ-version $\overline{u}$ of u.

Now choose l such that $\mathrm{Cap}_{r,p}'(A_l) < \frac{\varepsilon}{3}$ and then j large enough so that

$$\|f_{n_j} - f\|_{L^p(B,\mu)}^p < \frac{\varepsilon}{3} \quad \text{and} \quad \sup_{x \in A_l^c} |u_{n_j}(x) - \overline{u}(x)| < \delta/2.$$

Then $u_{n_j} \geq 1$ μ-a.e. on some neighborhood V of $U \cap A_l^c$. The topological support of μ is B, see for instance [8, Theorem 3.6.1, Definition 3.6.2 and the remark following it]. Since u_{n_j} is continuous by Proposition 2.2 we therefore have $u_{n_j} \geq 1$ everywhere on V. Now, since $\mathrm{Cap}_{r,p}'$ is clearly subadditive and monotone,

$$\mathrm{Cap}_{r,p}(U) \leq \mathrm{Cap}_{r,p}'(U) \leq \mathrm{Cap}_{r,p}' \ (V) \ + \mathrm{Cap}_{r,p}'(A_l)$$

$$\leq \ \|f_{n_j}\|_{L^p(B,\mu)}^p + \frac{\varepsilon}{3} \leq \|f\|_{L^p(B,\mu)}^p + \frac{2\varepsilon}{3} \leq \mathrm{Cap}_{r,p}(U) + \varepsilon. \qquad \square$$

Using Theorem 3.6 and Lemma 3.8 we can now verify Theorem 3.5.

Proof We show Eq. (9). It suffices to consider open sets U. Since $\mathcal{F}C_b^{\infty} \subset W^{r,p}(B,\mu)$, we have

$$\mathrm{Cap}_{r,p}(U) \leq c_2^p \, \mathrm{cap}_{r,p}^{(\mathcal{F}C_b^{\infty})}(U)$$

with c_2 as in Eq. (8), so that it suffices to show

$$\mathrm{cap}_{r,p}^{(\mathcal{F}C_b^{\infty})}(U) \leq c \, \mathrm{Cap}_{r,p}(U)$$

with a suitable constant $c > 0$ depending only on r and p.

Let $T \in C^{\infty}(\mathbb{R})$ be a function such that $0 \leq T \leq 1$, $T(t) = 0$ for $0 \leq t \leq 1/2$ and $T(t) = 1$ for $t \geq 1$, and let c_T be as in Theorem 3.6. Given $\epsilon > 0$, let $f \in \mathcal{F}C_{b,+}^{\infty}$ be such that $u := V_r f \geq 1$ on U and

$$\|f\|_{L^p(B,\mu)}^p \leq \mathrm{Cap}_{r,p}(U) + \frac{\varepsilon}{c_T^p},$$

due to Lemma 3.8 such f can be found. Clearly $T \circ u \in \mathcal{F}C_b^{\infty}$ and $T \circ u = 1$ on U. Therefore, using Theorem 3.6, we have

$$\mathrm{cap}_{r,p}^{(\mathcal{F}C_b^{\infty})}(U) \leq \|T \circ u\|_{W^{r,p}(B,\mu)}^p \leq c_T^p \|f\|_{L^p(B,\mu)}^p \leq c_T^p \, \mathrm{Cap}_{r,p}(U) + \varepsilon,$$

and we arrive at Eq. (9) with $c_3 := 1/c_T^p$ and $c_4 := c_2^p$. Since $\mathcal{F}C_b^{\infty} \subset W^{\infty} \subset W^{r,p}(B,\mu)$, Eq. (10) is an easy consequence. $\qquad \square$

4 Smooth Truncations

To verify Theorem 3.6 we begin with the following generalization of [8, formula (5.4.4) in Proposition 5.4.8].

Proposition 4.1 *Assume $p > 1$ and $f \in L^p(B, \mu)$. Then for any $t > 0$ and μ-a.e. $x \in B$ the mapping $h \mapsto P_t f(x + h)$ from H to B is infinitely Fréchet differentiable, and given $h_1, ..., h_k \in H$ we have*

$$\partial_{h_1} \cdots \partial_{h_k} P_t f(x) = \left(\frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \right)^k \int_B f(e^{-t} x + \sqrt{1 - e^{-2t}}\, y) Q(\hat{h}_1(y), ..., \hat{h}_k(y)) \mu(dy),$$

where the functions $\hat{h}_i$ are as in Eq. (1) *and $Q : \mathbb{R}^n \to \mathbb{R}$, $n \leq k$, is a polynomial of degree k whose coefficients are constants or products of scalar products $\langle h_i, h_j \rangle_H$. If the $h_1, ..., h_k$ are elements of an orthonormal system $(g_i)_{i=1}^k$ in H, not necessarily distinct, then each coefficient of Q depends only on the multiplicity according to which the respective element of $(g_i)_{i=1}^k$ occurs in $\{h_1, ..., h_k\}$.*

Proof The infinite differentiability was shown in [8, Proposition 5.4.8] as a consequence of the Cameron-Martin formula. By the same arguments we can see that

$$\partial_{h_1} \cdots \partial_{h_k} P_t f(x) = \int_B f(e^{-t} x + \sqrt{1 - e^{-2t}}\, y) \times$$

$$\times \frac{\partial^k}{\partial \lambda_1 \cdots \partial \lambda_k} \varrho(t, \lambda_1 h_1 + ... + \lambda_k h_k, y)|_{\lambda_1 = ... = \lambda_k = 0}\, \mu(dy),$$

where

$$\varrho(t, h, y) = \exp \left\{ \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \hat{h}(y) - \frac{e^{-2t}}{2(1 - e^{-2t})} |h|_H^2 \right\}.$$

A straightforward calculation shows that

$$\frac{\partial^k}{\partial \lambda_1 \cdots \partial \lambda_k} \varrho(t, \lambda_1 h_1 + ... + \lambda_k h_k, y)|_{\lambda_1 = ... = \lambda_k = 0} = \left(\frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \right)^k Q(\hat{h}_1(y), ..., \hat{h}_k(y))$$

with a polynomial Q as stated. $\square$

The next inequality is a counterpart to [2, Proposition 3.1.8]. It provides a pointwise multiplicative estimate for derivatives of potentials in terms of powers of the potential and a suitable maximal function.

Lemma 4.2 *Let $1 < q < +\infty$, let $r \geq 2$ be an integer and let $k < r$. Then for any nonnegative Borel function f on B and μ-a.e. $x \in B$ we have*

$$\left| D^k V_r f(x) \right|_{\mathcal{H}_k} \leq c(k, q, r)\, (V_r f(x))^{1 - \frac{k}{r}} \left(\sup_{t > 0} P_t(f^q)(x) \right)^{\frac{k}{rq}}. \tag{16}$$

Proof Suppose $h_1, ..., h_k \in H$ are members of an orthonormal system in H, not necessarily distinct. Then for any $\delta > 0$ we have, by dominated convergence,

$$
\begin{aligned}
D^k V_r f(x)(h_1, ..., h_k) &= \partial_{h_1} \cdots \partial_{h_k} V_r f(x) \\
&= \int_0^\delta \int_B e^{-t} t^{r/2-1} \left(\frac{e^{-t}}{\sqrt{1-e^{-2t}}} \right)^k f(e^{-t}x + \sqrt{1-e^{-2t}}y) \times \\
&\quad \times Q(\hat{h}_1(y), ..., \hat{h}_k(y)) \, \mu(dy) dt \\
&\quad + \int_\delta^\infty \int_B e^{-t} t^{r/2-1} \left(\frac{e^{-t}}{\sqrt{1-e^{-2t}}} \right)^k f(e^{-t}x + \sqrt{1-e^{-2t}}y) \times \\
&\quad \times Q(\hat{h}_1(y), ..., \hat{h}_k(y)) \, \mu(dy) dt \\
&=: I_1(\delta) + I_2(\delta)
\end{aligned}
$$

with a polynomial Q of degree k as in Proposition 4.1. Now let $\beta > 1$ be a real number such that

$$
\frac{r}{2k} < \beta < \frac{r}{k}. \tag{17}
$$

Hölder's inequality yields

$$
\begin{aligned}
|I_1(\delta)| &\leq \left(\int_0^\delta \int_B e^{-t} t^{r/2-1} \left(\frac{e^{-t}}{\sqrt{1-e^{-2t}}} \right)^{\beta k} f(e^{-t}x + \sqrt{1-e^{-2t}}y) \right. \\
&\quad \left. \times |Q(\hat{h}_1(y), ..., \hat{h}_k(y))|^\beta \mu(dy) dt \right)^{1/\beta} \\
&\quad \times \left(\int_0^\delta \int_B e^{-t} t^{r/2-1} f(e^{-t}x + \sqrt{1-e^{-2t}}y) \, \mu(dy) dt \right)^{1/\beta'}. \tag{18}
\end{aligned}
$$

Using the fact that $r \geq 2$, the elementary inequality $e^{-t}t \leq 1 - e^{-2t}$, $t \geq 0$, and the left inequality in Eq. (17),

$$
e^{-t} t^{r/2-1} \left(\frac{e^{-t}}{\sqrt{1-e^{-2t}}} \right)^{\beta k} \leq (1-e^{-2t})^{r/2-k\beta/2-1} e^{-2t},
$$

so that another application of Hölder's inequality, now with q, shows that the first factor on the right hand side of Eq. (18) is bounded by

$$
\begin{aligned}
&\left(\int_0^\delta \int_B f(e^{-t}x + \sqrt{1-e^{-2t}}y)^q \mu(dy)(1-e^{-2t})^{r/2-k\beta/2-1} e^{-2t} dt \right)^{1/(\beta q)} \times \\
&\quad \times \left(\int_B |Q(\hat{h}_1(y), ..., \hat{h}_k(y))|^{\beta q'} \mu(dy) \int_0^\delta (1-e^{-2t})^{r/2-k\beta/2-1} e^{-2t} dt \right)^{1/(\beta q')}.
\end{aligned}
$$

According to Proposition 4.1 the coefficients of the polynomial Q are bounded for fixed k, and since its degree does not exceed k, it involves only finitely many distinct products of powers of the functions $\hat{h}_i$. Together with the fact that each $\hat{h}_i$ is $N(0,1)$-distributed, this implies that there is a constant $c_1(k, q, \beta) > 0$, depending on k but not on the particular choice of the elements $h_1, ..., h_k$, such that

$$
\left(\int_B |Q(\hat{h}_1(y), ..., \hat{h}_k(y))|^{\beta q'} \mu(dy) \right)^{1/(\beta q')} < c_1(k, q, \beta).
$$

Taking into account Eq. ([17](#)), we therefore obtain

$$|I_1(\delta)| \leq c_1(k, q, \beta) \left(\frac{r}{2} - \frac{\beta k}{2}\right)^{-1/\beta} (1 - e^{-2\delta})^{r/(2\beta)-k/2} (V_r f(x))^{1/\beta'} \left(\sup_{t>0} P_t(f^q)(x)\right)^{\frac{1}{\beta q}}. \quad (19)$$

To estimate $I_2(\delta)$ let

$$\frac{r}{k} < \gamma. \quad (20)$$

In a similar fashion we can then obtain the estimate

$$|I_2(\delta)| \leq c_2(k, q, \gamma) \left(\frac{r}{2} - \frac{\gamma k}{2}\right)^{-1/\gamma} (1 - e^{-2\delta})^{r/(2\gamma)-k/2} (V_r f(x))^{1/\gamma'} \left(\sup_{t>0} P_t(f^q)(x)\right)^{\frac{1}{\gamma q}}, \quad (21)$$

where $c_2(k, q, \gamma) > 0$ is a constant depending on n but not on the particular choice of $h_1, ..., h_k$.

We finally choose suitable $\delta > 0$. The function

$$\delta \mapsto (1 - e^{-2\delta}), \quad \delta > 0,$$

can attain any value in $(0, 1)$. Since Jensen's inequality implies

$$(V_r f(x))^q \leq \sup_{t>0}(P_t(f)(x))^q \leq \sup_{t>0} P_t(f^q)(x), \quad (22)$$

we have $\sup_{t>0}(P_t(f^q)(x))^{1/q} \geq V_r f(x)$ and can choose $\delta > 0$ such that

$$(1 - e^{-2\delta}) = \frac{V_r f(x)^{2/r}}{2\sup_{t>0}(P_t(f^q)(x))^{2/(qr)}}, \quad (23)$$

note that the denominator cannot be zero unless f is zero μ-a.e. Combining with Eqs. ([19](#)) and ([21](#)) we obtain

$$|D^k V_r f(x) \quad (h_1, ..., h_k)|$$
$$\leq \left\{ c_1'(k, q, \beta) \left(\frac{r}{2} - \frac{\beta k}{2}\right)^{-1/\beta} + c_2'(k, q, \gamma) \left(\frac{r}{2} - \frac{\gamma k}{2}\right)^{-1/\gamma} \right\} \times$$
$$\times (V_r f(x))^{1-k/r} \left(\sup_{t>0} P_t(f^q)(x)\right)^{k/(qr)}$$

for some constants $c_1'(k, q, \beta)$, $c_2'(k, q, \gamma)$. For any given r there exist only finitely many numbers $k < r$ and for any such k numbers β and γ as in Eqs. ([17](#)) and ([20](#)) can be fixed. Using Proposition 2.1 we can therefore find a constant $c(k, q, r)$ depending only on k, q and r such that Eqs. ([16](#)) holds. □

We prove Theorem 3.6, basically following the method of proof used for [2, Theorem 3.3.3].

Proof If $r = 1$ then T has a bounded first derivative, and the desired bound is immediate from the definition of the norm $\|\cdot\|_{W^{1,p}(B,\mu)}$, the chain rule for the gradient D and Meyer's equivalence, [53, Theorem 4.4]. In the following we therefore assume $r \geq 2$.

We verify that for any $k \leq r$ the inequality

$$\left\| D^k(T \circ V_r f) \right\|_{L^p(B,\mu,\mathcal{H}^k)} \leq c(k, L, p, r) \|f\|_{L^p(B,\mu)} \quad (24)$$

holds with a constant $c(k, L, p, r) > 0$ depending only on k, L, p and r. If so, then summing up yields

$$\|T \circ V_r f\|_{W^{r,p}(B,\mu)} = \sum_{k=0}^{r} \left\| D^k (T \circ V_r f) \right\|_{L^p(B,\mu,\mathcal{H}^r)} \leq c_T \ \|f\|_{L^p(B,\mu)}$$

with a constant $c_T > 0$ depending on L, p and r, as desired.

To see Eq. (24) suppose $k \leq r$ and that $h_1, ..., h_k$ are members of an orthonormal system $(g_i)_{i=1}^k$, not necessarily distinct. To simplify notation, we use multiindices with respect to this orthonormal system: Given a multiindex $\alpha = (\alpha_1, ..., \alpha_k)$ we write $D^\alpha := \partial_{g_1}^{\alpha_1} \cdots \partial_{g_k}^{\alpha_k}$, where for $\beta = 0, 1, 2, ...$, a function $u : B \to \mathbb{R}$ and an element $g \in H$ we define $\partial_g^\beta u$ as the image of u under the application of β differentiations in direction g,

$$\partial_g^\beta u(x) := \partial_g \cdots \partial_g u(x) = D^\beta u(x)(g, ..., g).$$

Now let α be a multiindex such that $D^\alpha = \partial_{h_1} \cdots \partial_{h_k}$. Then clearly $|\alpha| = k$. Moreover, we have

$$D^\alpha (T \circ V_r f)(x) = \sum_{j=1}^{k} T^{(j)} \circ V_r f(x) \sum C_{\alpha^1, ..., \alpha^j} D^{\alpha^1} V_r f(x) \cdots D^{\alpha^j} V_r f(x)$$

by the chain rule, where the interior sum is over all j-tuples $(\alpha^1, ..., \alpha^j)$ of multiindices α^i such that $|\alpha^i| \geq 1$ for all i and $\alpha^1 + \alpha^2 + ... + \alpha^j = \alpha$. The interior sum has $\binom{k-1}{j-1}$ summands. The $C_{\alpha^1, ..., \alpha^j}$ are real valued coefficients, and since there are only finitely many different $C_{\alpha^1, ..., \alpha^j}$, there exists a constant $C(k) > 0$ which for all multiindices α with $|\alpha| = k$ dominates these constants, $C_{\alpha^1, ..., \alpha^j} \leq C(k)$. In particular, $C(k)$ does not depend on the particular choice of the elements $h_1, ..., h_k$. More explicit computations can for instance be obtained using [19].

The hypothesis Eq. (12) on T implies

$$|D^\alpha (T \circ V_r f)(x)| \leq c(k) L \sum_{j=1}^{k} (V_r f(x))^{1-j} \sum |D^{\alpha^1} V_r f(x) \cdots D^{\alpha^j} V_r f(x)|$$

with a constant $c(k) > 0$ depending only on k and with L being as in Eq. (12). Since $\sum_{i=1}^{j}(1 - |\alpha^i|/k) = j - |\alpha|/k = j - 1$ and

$$|D^{\alpha^i} V_r f(x)| \leq \left| D^{|\alpha^i|} V_r f(x) \right|_{\mathcal{H}_{|\alpha^i|}},$$

Lemma 4.2 implies that

$$\sum_{j=2}^{k} (V_r f(x))^{1-j} \sum |D^{\alpha^1} V_r f(x) \cdots D^{\alpha^j} V_r f(x)|$$

$$\leq \sum_{j=2}^{k} (V_r f(x))^{1-j} \sum \left| D^{|\alpha^1|} V_r f(x) \right|_{\mathcal{H}_{|\alpha^1|}} \cdots \left| D^{|\alpha^j|} V_r f(x) \right|_{\mathcal{H}_{|\alpha^j|}}$$

$$\leq c(k, q, r) \sum_{j=2}^{k} \binom{k-1}{j-1} (\sup_{t>0} P_t(f^q)(x))^{1/q},$$

where $1 < q < +\infty$ is arbitrary and $c(k, q, r) > 0$ is a constant depending only on k, q and r. For the case $j = 1$ we have

$$|D^\alpha V_r f(x)| \leq \left| D^k V_r f(x) \right|_{\mathcal{H}_k}.$$

Taking the supremum over all $h_1, ..., h_k \in H$ as above we obtain

$$\left| D^k T \circ V_r f(x) \right|_{\mathcal{H}^k} \leq c(k, L, q, r) \left[(\sup_{t>0} P_t(f^q)(x))^{1/q} + \left| D^k V_r f(x) \right|_{\mathcal{H}_k} \right]$$

with a constant $c(k, L, q, r) > 0$ by Proposition 2.1.

Fixing $1 < q < p$ and using the boundedness of the semigroup maximal function, [53, Theorem 3.3], we see that there is a constant $c(p, q) > 0$ depending only on p and q such that

$$\left\| (\sup_{t>0} P_t(f^q))^{1/q} \right\|_{L^p(B,\mu)} \leq c(p, q) \, \|f\|_{L^p(B,\mu)}.$$

On the other hand, by Eq. (8), we have

$$\left\| D^k V_r f \right\|_{L^p(B,\mu,\mathcal{H}_k)} \leq \frac{1}{c_1} \, \|f\|_{L^p(B,\mu)}.$$

Combining, we arrive at Eq. (24). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

5 L^p-uniqueness of Powers of the Ornstein-Uhlenbeck Operator

We discuss related uniqueness problems for the Ornstein Uhlenbeck operator $\mathcal{L}$ and its integer powers.

Recall first that a densely defined operator $(L, \mathcal{A})$ on $L^p(B, \mu)$, $1 \leq p < +\infty$ is said to be L^p-*unique* if there is only one C_0-semigroup on $L^p(B, \mu)$ whose generator extends $(L, \mathcal{A})$, see e.g. [18, Chapter I b), Definition 1.3]. If $(L, \mathcal{A})$ has an extension generating a C_0-semigroup on $L^p(B, \mu)$ then $(L, \mathcal{A})$ is L^p-unique if and only if the closure of $(L, \mathcal{A})$ generates a C_0-semigroup on $L^p(B, \mu)$, see [18, Chapter I, Theorem 1.2 of Appendix A].

From Eq. (8) it follows that for any $m = 1, 2, ...$ and $1 < p < +\infty$ we have $\mathcal{D}((-\mathcal{L})^m) = W^{2m,p}(B, \mu)$. The density of $\mathcal{F}C_b^\infty$ and W^∞ in the spaces $W^{2m,p}(B, \mu)$ and the completeness of the latter imply that $((-\mathcal{L})^m, W^{2m,p}(B, \mu))$ is the closure in $L^p(B, \mu)$ of $((-\mathcal{L})^m, \mathcal{F}C_b^\infty)$ and also of $((-\mathcal{L})^m, W^\infty)$.

Since obviously $(P_t)_{t>0}$ is a C_0-semigroup, $(\mathcal{L}, \mathcal{F}C_b^\infty)$ and $(\mathcal{L}, W^\infty)$ are L^p-unique in all $L^p(B, \mu)$, $1 \leq p < +\infty$. To discuss the its powers $-(-\mathcal{L})^m$ for $m \geq 2$ we quote well known facts to provide a sufficient condition for them to generate C_0-semigroups. Since $(P_t)_{t>0}$ is a symmetric Markov semigroup on $L^2(B, \mu)$, for any $1 < p < +\infty$ the operator $\mathcal{L} = \mathcal{L}^{(p)}$ generates a bounded holomorphic semigroup on $L^p(B, \mu)$ with angle θ satisfying $\frac{\pi}{2} - \theta \leq \frac{\pi}{2}|\frac{2}{p} - 1|$, see for instance [15, Theorem 1.4.2]. On the other hand [16, Theorem 4] tells that if L is the generator of a bounded holomorphic semigroup with angle θ satisfying $\frac{\pi}{2} - \theta < \frac{\pi}{2m}$, then also $-(-L)^m$ generates a bounded holomorphic semigroup. Combining, we can conclude that $-(-\mathcal{L})^m$ generates a bounded holomorphic semigroup on $L^p(B, \mu)$ and therefore in particular a (bounded) C_0-semigroup if

$$|\frac{2}{p} - 1| < \frac{1}{m}. \tag{25}$$

[17, Theorem 8] shows that (up to a discussion of limit cases) this is a sharp condition for $-(-\mathcal{L})^m$ to generate a bounded C_0-semigroup. For $1 < p < +\infty$ this also recovers the L^p-uniqueness in the case $m = 1$. For $p = 2$ condition Eq. (25) is always satisfied.

Alternatively we can conclude the generation of C_0-semigroups on $L^2(B, \mu)$ directly from the spectral theorem.

For later use we fix the following fact.

Proposition 5.1 *Let* $1 < p < +\infty$ *and let* $m > 0$ *be an integer satisfying* Eq. (25). *Then the operators* $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty)$ *and* $(-(-\mathcal{L})^m, W^\infty)$ *are* L^p-*unique in* $L^p(B, \mu)$. *In particular, they are essentially self-adjoint in* $L^2(B, \mu)$ *for all* $m > 0$.

The last statement is true because a semi-bounded symmetric operator (L, A) on $L^2(B, \mu)$ is L^2-unique if and only if it is essential self-adjoint, see [18, Chapter I c), Corollary 1.2].

Here we are interested in L^p-uniqueness after the removal of a small closed set $\Sigma \subset B$ of zero measure. This is similar to our discussion in [27] and, in a sense, similar to a removable singularities problem, see for instance [42] or [43] or [2, Section 2.7].

Let $\Sigma \subset B$ be a closed set of zero Gaussian measure and $N := B \setminus \Sigma$. We define

$$\mathcal{F}C_b^\infty(N) := \{f \in \mathcal{F}C_b^\infty \mid f = 0 \text{ on an open neighborhood of } \Sigma\}$$

and

$$W^\infty(N) := \{f \in W^\infty \mid \widetilde{f} = 0 \text{ q.s. on an open neighborhood of } \Sigma\}.$$

The L^p-uniqueness of $-(-\mathcal{L})^m$, restricted to $\mathcal{F}C_b^\infty(N)$ and $W^\infty(N)$, respectively, now depends on the size of the set Σ. If it is small enough not to cause additional boundary effects then from the point of view of operator extensions it is removable.

Theorem 5.2 *Let* $1 < p < +\infty$, *let* $m > 0$ *be an integer and assume that* $\Sigma \subset B$ *is a closed set of zero measure* μ. *Write* $N := B \setminus \Sigma$.

(i) *If* $\mathrm{Cap}_{2m, p}(\Sigma) = 0$ *then the closure of* $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ *in* $L^p(B, \mu)$ *is*

$$(-(-\mathcal{L})^m, W^{2m, p}(B, \mu)).$$

If in addition m *satisfies* Eq. (25) *then* $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ *is* L^p-*unique.*
(ii) *If* $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ *is* L^p-*unique, then* $\mathrm{Cap}_{2m, p}(\Sigma) = 0$.

The same statements are true with $W^\infty(N)$ *in place of* $\mathcal{F}C_b^\infty(N)$.

Proof To see (i) suppose that $\mathrm{Cap}_{2m, p}(\Sigma) = 0$. Let $((-\mathcal{L})^m, \mathcal{D}((-\mathcal{L})^m))$ denote the closure of $((-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ in $L^p(B, \mu)$. Since $\mathcal{F}C_b^\infty(N) \subset \mathcal{F}C_b^\infty$ we trivially have

$$\mathcal{D}((-\mathcal{L})^m) \subset W^{2m, p}(B, \mu),$$

and it remains to show the converse inclusion.

Given $u \in W^{2m, p}(B, \mu)$, let $(u_j)_{j=1}^\infty \subset \mathcal{F}C_b^\infty$ be a sequence approximating u in $W^{2m, p}(B, \mu)$. By Theorem 3.5 there is a sequence $(v_l)_{l=1}^\infty \subset \mathcal{F}C_b^\infty$ such that $\lim_{l\to\infty} v_l = 0$ in $W^{2m, p}(B, \mu)$ and for each l the function v_l equals one on an open neighborhood of Σ. Set $w_{jl} := (1 - v_l)u_j$ to obtain functions $w_{jl} \in \mathcal{F}C_b^\infty(N)$. Now let j be fixed. For any $1 \le k \le 2m$ let $h_1, ..., h_k$ be members of an orthonormal system $(g_i)_{i=1}^k$, not necessarily distinct. As in the proof of Theorem 3.6 we use multiindex notation with respect to this orthonormal system. Let α be such that $D^\alpha = \partial_{h_1} \cdots \partial_{h_k}$. Then, by the general Leibniz rule,

$$D^\alpha(u_j - w_{jl})(x) = D^\alpha(u_j v_l)(x) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u_j(x) D^{\alpha-\beta} v_l(x),$$

where for two multiindices α and β we write $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, ..., k$. For any such β we clearly have

$$|D^\beta u_j(x)| \leq \left|D^{|\beta|}u_j(x)\right|_{\mathcal{H}_{|\beta|}} \text{ and } |D^{\alpha-\beta}v_l(x)| \leq \left|D^{|\alpha-\beta|}v_l(x)\right|_{\mathcal{H}_{|\alpha-\beta|}},$$

and taking the supremum over all $h_1, ..., h_k$ as above,

$$\left|D^k(u_j - w_{jl})(x)\right|_{\mathcal{H}_k} \leq c(k) \max_{n \leq k}\left|D^n u_j(x)\right|_{\mathcal{H}_n} \max_{n \leq k}\left|D^n v_l(x)\right|_{\mathcal{H}_n}$$

with a constant $c(k) > 0$ depending only on k. Taking into account that

$$\sup_{x \in B}\left|D^n u_j(x)\right|_{\mathcal{H}_n} < +\infty$$

for any $n \geq 1$ and summing up, we see that

$$\lim_l \sum_{k=1}^{2m}\left\|D^k(u_j - w_l)\right\|_{L^p(B,\mu,\mathcal{H}_k)} \leq c(m) \max_{n \leq 2m} \sup_{x \in B}\left|D^n u_j(x)\right|_{\mathcal{H}_n} \lim_l \|v_l\|_{W^{2m,p}} = 0,$$

here $c(m) > 0$ is a constant depending on m only. Since u_j is bounded, we also have $\lim_l(u_j - w_{jl}) = \lim_l u_j v_l = 0$ in $L^p(B, \mu)$ so that

$$\lim_l w_{jl} = u_j \quad \text{in } W^{2m,p}(B, \mu),$$

what implies $u \in \mathcal{D}((-\mathcal{L})^m)$ and therefore

$$W^{2m,p}(B, \mu) \subset \mathcal{D}((-\mathcal{L})^m).$$

To see (ii) suppose that $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ is L^p-unique in $L^p(B, \mu)$. Then its unique extension must be $(-(-\mathcal{L})^m, W^{2m,p}(B, \mu))$. Let $u \in \mathcal{F}C_b^\infty$ be a function that equals one on a neighborhood of Σ. Since $\mathcal{F}C_b^\infty \subset W^{2m,p}(B, \mu)$ and by hypothesis $\mathcal{F}C_b^\infty(N)$ is dense in $W^{2m,p}(B, \mu)$, we can find a sequence $(u_l)_l \subset \mathcal{F}C_b^\infty(N)$ approximating u in $W^{2m,p}(B, \mu)$. The functions $e_l := u - u_l$ then are in $\mathcal{F}C_b^\infty$, each equals one on an open neighborhood of Σ, and they converge to zero in $W^{2m,p}(B, \mu)$, so that by Theorem 3.5 we have

$$\text{Cap}_{2m,p}(\Sigma) \leq c_2 \lim_l \|e_l\|_{W^{2m,p}(B,\mu)} = 0.$$

The proof for W^∞ is similar. □

6 Comments on Gaussian Hausdorff Measures

For finite dimensional Euclidean spaces the link between Sobolev type capacities and Hausdorff measures is well known and the critical size of a set Σ in order to have (r, p)-capacity zero or not is, roughly speaking, determined by its Hausdorff codimension, see e.g. [2, Chapter 5]. For Wiener spaces one can at least provide a partial result of this type.

Hausdorff measures on Wiener spaces of integer codimension had been introduced in [21, Section 1]. We briefly sketch their method but allow non-integer codimensions, this is an effortless generalization and immediate from their arguments.

Given an m-dimensional Euclidean space F and a real number $0 \leq d \leq m$ the spherical Hausdorff measure $\mathcal{S}^d$ of dimension d can be defined as follows: For any $\varepsilon > 0$ set

$$\mathcal{S}_\varepsilon^d(A) := \inf \left\{ \sum_{i=1}^\infty r_i^d : \{B_i\}_{i=1}^\infty \text{ is a collection of balls} \right.$$

$$\left. \text{of radius } r_i < \varepsilon/2 \text{ such that } A \subset \bigcup_{i=1}^\infty B_i \right\},$$

and finally, $\mathcal{S}^d(A) := \sup_{\varepsilon > 0} \mathcal{S}_\varepsilon^d(A)$, $A \subset F$. A priori $\mathcal{S}^d$ is an outer measure, but its σ-algebra of measurable sets contains all Borel sets. For any $0 \leq d \leq m$ we define

$$\theta_d^F(A) := (2\pi)^{-m/2} \int_A \exp\left(-\frac{|y|_F^2}{2} \right) \mathcal{S}^{m-d}(dy),$$

for Borel sets $A \subset F$, [21, 1. Definition], by approximation from outside it extends to an outer measure on F, defined in particular for any analytic set. Recall that a set $A \subset F$ is called *analytic* if it is a continuous image of a Polish space.

We return to the abstract Wiener space (B, μ, H). Let $d \geq 0$ be a real number and let F be a subspace of H of finite dimension $m \geq d$. Let p^F denote the orthogonal projection from H onto F, it extends to a linear projection p^F from B onto F which is (r, p)-quasi continuous for all r and p, [20, 11. Théorème]. We write $\widetilde{F}$ for the kernel of p^F. The spaces B and $F \times \widetilde{F}$ are isomorphic under the map $p^F \times (I - p^F)$. If $A \subset B$ is analytical and for any $x \in \widetilde{F}$ the section with respect to the above product is denotes by $A_x \subset F$, then for any $a \in \mathbb{R}$ the set $\{x \in \widetilde{F} : \theta_d^F(A_x) > a\}$ is analytic up to a slim set, as shown in [21, 4. Lemma]. We follow [21, 5. Definition] and set $\mu^F(B) := \mu((I - p^F)^{-1}(B))$ for any analytic subset B of F. Then by [21, 4. Lemma] we can define

$$\varrho_d^F(A) := \int_B \theta_d^F(A_x) \mu(dx)$$

for any analytic subset A of B. As in [21, 8. Definition] we define the *Gaussian Hausdorff measure ϱ_d of codimension* $d \geq 0$ by

$$\varrho_d(A) := \sup \left\{ \varrho_d^F(A) : F \subset H \text{ and } d \leq \dim F < +\infty \right\}$$

for any analytic set $A \subset B$. Restricted to the Borel σ-algebra it is a Borel measure. The next result follows in the same way as [21, 9. Theorem] from [20, 32. Théorème] and [44], see also [2, Theorem 5.1.13].

Theorem 6.1 *If a Borel set $A \subset B$ satisfies $\text{Cap}_{r,p}(A) = 0$, then $\varrho_d(A) = 0$ for all $d < rp$.*

Combined with Theorem 5.2 this yields a necessary codimension condition which is similar as in the case of Laplacians on Euclidean spaces, [5, 27].

Corollary 6.2 *Assume $1 < p < +\infty$. Let $\Sigma \subset B$ be a closed set of zero measure and $N := B \setminus \Sigma$.*
If $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ is L^p-unique, then

$$\varrho_d(\Sigma) = 0 \quad \text{for all } d < 2mp.$$

In particular, if $(\mathcal{L}, \mathcal{F}C_b^\infty(N))$ is essentially self-adjoint, then

$$\varrho_d(\Sigma) = 0 \quad \text{for all } d < 4.$$

The same is true with $W^\infty(N)$ in place of $\mathcal{F}C_b^\infty(N)$.

7 Comments on Stochastic Processes

We finally like to briefly point out connections to known Kakutani type theorems for related multiparameter Ornstein-Uhlenbeck processes. The connection between Gaussian capacities, [20], and the hitting behaviour of multiparameter processes, [29–31], has for instance been investigated in [6, 55, 56]. We briefly sketch the construction and main result of [56], later generalized in [6].

Let $\Theta^{(0)} := B$ and for integer $k \geq 1$, $\Theta^{(k+1)}(B) := C(\mathbb{R}_+, \Theta^{(k)}(B))$. The space $\Theta^k(B)$ can be identified with $C(\mathbb{R}_+^k, B)$. Moreover, set $\mu^{(0)} := \mu$, $T_t^{(0)} := P_t$, $t > 0$, and let $Z^{(1)}$ be the Ornstein-Uhlenbeck process taking values in $\Theta^{(0)}(B) = B$ with semigroup $T_t^{(0)}$ and initial law $\mu^{(0)}$. Let $\mu^{(1)}$ denote the law of the process $Z^{(1)}$, clearly a centered Gaussian measure on $\Theta^{(1)}(B)$. Next, let $(T_t^{(1)})_{t>0}$ be the Ornstein-Uhlenbeck semigroup on $\Theta^{(1)}(B)$ defined by

$$T_t^{(1)} f(x) = \int_{\Theta^{(1)}(B)} f(e^{-t}x + \sqrt{1 - e^{-2t}} y)\mu^{(1)}(dy), \quad x \in \Theta^{(1)}(B),$$

for any bounded Borel function f on $\Theta^{(1)}(B)$, and let $Z^{(2)}$ be the Ornstein-Uhlenbeck process taking values in $\Theta^{(1)}(B)$ with semigroup $(\Theta^{(1)})_{t>0}$ and initial law $\mu^{(1)}$. Iterating this construction yields, for any integer $r \geq 1$, an Ornstein-Uhlenbeck process $Z^{(r)}$ taking values in $\Theta^{(r-1)}(B)$. This process may also be viewed as an r-parameter process $Z^{(r)} = (Z_{\mathbf{t}}^{(r)})_{\mathbf{t}\in\mathbb{R}_+^r}$ taking values in B. Now [56, §6, Théorème 1] tells that a Borel set $A \subset B$ has zero $(r, 2)$-capacity $\mathrm{Cap}_{r,2}(A) = 0$ if and only if the event

$$\left\{ \text{there exists some } \mathbf{t} \in \mathbb{R}_+^r \text{ such that } Z_{\mathbf{t}}^{(r)} \in A \right\}$$

has probability zero. See also [6, 13. Corollary].

Combined with Theorem 5.2 this result gives a preliminary characterization of L^2-uniqueness (that is, essential self-adjointness) in terms of the hitting behaviour of the $2m$-parameter Ornstein-Uhlenbeck process $(Z_{\mathbf{t}}^{(m)})_{\mathbf{t}\in\mathbb{R}_+^{2m}}$.

Corollary 7.1 *Let $m > 0$ be an integer. Let $\Sigma \subset B$ be a closed set of zero measure and $N := B \setminus \Sigma$. The operators $(-(-\mathcal{L})^m, \mathcal{F}C_b^\infty(N))$ and $(-(-\mathcal{L})^m, W^\infty(N))$ are L^2-unique (resp. essentially self-adjoint) if and only if $Z^{(2m)}$ does not hit Σ with positive probability.*

A more causal connection between uniqueness problems for operators and classical probability should involve certain branching diffusions rather than multiparameter processes, but even for finite dimensional Euclidean spaces the problem is not fully settled and remains a future project.

8 Capacities and Truncations for Ornstein-Uhlenbeck Operators with Linear Drift

In this section we investigate Ornstein-Uhlenbeck semigroups with linear drift as considered for instance in [4, 7, 51]. The main example we have in mind is the time zero Gaussian

free field, [46, 48, 49], see Example 9.3. We therefore restrict attention to the special cases $r = 1, 2$, $p = 2$ and $m = 1$ (in the notation of Section 5) and follow [7] and [51].

Let (E, H, μ) be an abstract Wiener space and let $(A, \mathcal{D}(A))$ be a strictly positive self-adjoint operator on H such that the operators e^{-tA}, $t > 0$, extend to a strongly continuous contraction semigroup on E. We assume that $K \subset E^* \cap \mathcal{D}(A)$ is a dense subspace of E^*, dense in $\mathcal{D}(A)$ w.r.t. the graph norm, that $A(K) \subset K$ and that

$$e^{-tA}(K) = K, \quad t > 0. \tag{26}$$

Remark 8.1 If H is a real separable Hilbert space and $(A, \mathcal{D}(A))$ a strictly positive self-adjoint operator on H then one can find an inner product norm q, continuous on H and such that the embedding of H into the completion E of H w.r.t. q is Hilbert-Schmidt and the operators e^{-tA}, $t > 0$, behave as stated. One can also find a space K as above. This is part of the statement of [7, Theorem 3.1]. If in this situation μ^* is a standard Gaussian cylindrical measure on H then it induces a Gaussian measure μ on E, [7, Remark 3.2 (ii)].

For the special cases $r = 1, 2$, $p = 2$ and $m = 1$ under consideration it is convenient to use the corresponding semigroup as the starting point for subsequent developments. Given a bounded (or nonnegative) Borel function $f : E \to \mathbb{R}$ and $t > 0$ set

$$P_{A,t} f(x) := \int_E f(e^{-tA} x + \sqrt{1 - e^{-2tA}} y) \mu(dy), \quad x \in E. \tag{27}$$

Since the operators $\sqrt{1 - e^{-2tA}}$ are bounded on E, [7, Lemma 3.5], this definition makes sense. The family of operators $(P_{A,t})_{t>0}$ is a symmetric sub-Markovian semigroup on $L^2(E, \mu)$, referred to as the *Mehler semigroup corresponding to* (E, H, A, μ). Actually, it is a Feller semigroup on E, as shown in [7, Corollary 3.6] and commented in [51, Section 2, p. 732]. If A is the identity operator on H then we recover Eq. (6) form Eq. (27). Let $(\mathcal{L}_A, \mathcal{D}(\mathcal{L}_A))$ denote the infinitesimal generator in $L^2(E, \mu)$ of $(P_{A,t})_{t>0}$, that is, the unique non-positive definite self-adjoint operator on $L^2(E, \mu)$ such that $P_{A,t} = e^{tA}$, $t > 0$, see for instance [11]. It is referred to as the *Ornstein-Uhlenbeck operator on* $L^2(E, \mu)$ *with linear drift* A. Similarly as in Eq. (7) we define

$$V_{A,r} f := \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2 - 1} e^{-t} P_{A,t} f \, dt$$

for any $r > 0$ and bounded (or nonnegative) Borel function $f : E \to \mathbb{R}$. Being symmetric and Markovian, the semigroup $(P_{A,t})_{t>0}$ also induces (unique) strongly continuous contraction semigroups on the spaces $L^p(E, \mu)$, $1 \le p < +\infty$ (as mentioned in Section 2), and for simplicity we denote them by the same symbol; likewise for their generators and the contractive operators $V_{A,r} = (I - \mathcal{L}_A)^{-r/2}$. For any $1 \le p < +\infty$ and $r > 0$ we define the Sobolev spaces

$$\mathcal{W}_A^{r,p}(E, \mu) := V_{A,r}(L^p(E, \mu)).$$

Endowed with the norms

$$\|u\|_{\mathcal{W}_A^{r,p}(E,\mu)} := \left\| (I - \mathcal{L}_A)^{r/2} u \right\|_{L^p(E,\mu)}, \quad u \in \mathcal{W}_A^{r,p}(E, \mu), \tag{28}$$

these spaces are Banach, and for $p = 2$ Hilbert. We also consider the space

$$\mathcal{W}_A^\infty := \bigcap_{r \ge 1, 1 \le p < +\infty} \mathcal{W}_A^{r,p}(E, \mu).$$

Let $\mathcal{F}C_{b,K}^\infty$ denote the collection of functions on E of form $f = F(l_1, ..., l_n)$, where $n \ge 1$, $F \in C_b^\infty(\mathbb{R}^n)$ and $l_1, ..., l_n \in K$. Clearly this space is an algebra, and it is dense

in $L^p(E, \mu)$, $1 \le p < +\infty$ (as can be seen using arguments as in [7, Lemma 2.1 and Proposition 5.4]).

With D defined on $\mathcal{F}C_{b,K}^\infty$ by formula Eq. (3) with $k = 1$ we now set $D_A := \sqrt{A}D$. Then for $f = F(l_1, ..., l_n) \in \mathcal{F}C_{b,K}^\infty$ with $F \in C_b^\infty(\mathbb{R}^n)$ and $l_i \in K$ and $x \in E$ we obtain

$$D_A f(x) = \sum_{j=1}^\infty \partial_j F(\langle x, l_1 \rangle, ..., \langle x, l_n \rangle) \sqrt{A} \, l_j. \tag{29}$$

The following was proved in [7, Theorem 5.3, Proposition 5.4 and Theorem 5.5].

Proposition 8.2

(i) *The domain $\mathcal{D}(\mathcal{L}_A) \subset L^2(E, \mu)$ of $\mathcal{L}_A$ contains $\mathcal{F}C_{b,K}^\infty$. Moreover, $\mathcal{L}_A|_{\mathcal{F}C_{b,K}^\infty}$ is essentially self-adjoint with unique self-adjoint extension $(\mathcal{L}_A, \mathcal{D}(\mathcal{L}_A))$.*

(ii) *The Dirichlet form $(\mathcal{E}_A, \mathcal{D}(\mathcal{E}_A))$ generated by $(\mathcal{L}_A, \mathcal{D}(\mathcal{L}_A))$ is the closure of*

$$\mathcal{E}_A(u, v) = \int_E \langle D_A u(x), D_A v(x) \rangle_H \, \mu(dx), \quad u, v \in \mathcal{F}C_{b,K}^\infty. \tag{30}$$

For background on Dirichlet form theory see for instance [11] or in the present context, [7]. Here we only point out that $\mathcal{E}_A$ and $\mathcal{L}_A$ are uniquely associated by the identity

$$\mathcal{E}_A(u, v) = -\langle \mathcal{L}_A u, v \rangle_{L^2(E, \mu)}, \quad u \in \mathcal{D}(\mathcal{L}_A), \quad v \in \mathcal{D}(\mathcal{E}_A).$$

In terms of the Sobolev type spaces defined above, we observe that $\mathcal{D}(\mathcal{L}_A) = \mathcal{W}_A^{2,2}(E, \mu)$ and $\mathcal{D}(\mathcal{E}_A) = \mathcal{W}_A^{1,2}(E, \mu)$.

Remark 8.3 For systematical reasons we mention the following results from [51], although we will not use them explicitly. Consider the norms on $\mathcal{F}C_{b,K}^\infty$ defined by

$$\|f\|_{W_A^{1,2}(E,\mu)} := \|f\|_{L^2(E,\mu)} + \|D_A f\|_{L^2(E,\mu,H)}$$

and

$$\|f\|_{W_A^{2,2}(E,\mu)} := \|f\|_{L^2(E,\mu)} + \left\|(1 + A)^{1/2} D_A f\right\|_{L^2(E,\mu,H)} + \left\|D_A^2 f\right\|_{L^2(E,\mu,\mathcal{H}_2)},$$

where $\mathcal{H}_2$ is as in Eq. (4), and let $W_A^{1,2}(E, \mu)$ and $W_A^{2,2}(E, \mu)$ denote the completions of $\mathcal{F}C_{b,K}^\infty$ in these norms, respectively. By the Meyer equivalence proved in [51, Theorems 3.1 and 3.6] the spaces $\mathcal{W}_A^{r,2}(E, \mu)$ and $W_A^{r,2}(E, \mu)$, $r = 1, 2$, coincide in the sense of equivalently normed spaces. We remark that in [51] not the space $\mathcal{F}C_{b,K}^\infty$ was used, but a space of polynomial functions based on $E^* \cap \bigcap_{k=1}^\infty \mathcal{D}(A^k)$. However, for the cases $r = 1, 2$ the necessary modifications in the proof are inessential.

To discuss capacities based on the space $\mathcal{W}_A^\infty$ below, we have to import two implications of the Meyer equivalence in [51]: The first is the fact that the space $\mathcal{W}_A^\infty$ is an algebra (which can be seen as in the proof of [51, Theorem 4.3]) and the second is the fact that $\mathcal{F}C_{b,K}^\infty \subset \mathcal{W}_A^{r,p}(E, \mu)$ for all r and p, so that in particular,

$$\mathcal{F}C_{b,K}^\infty \subset \mathcal{W}_A^\infty. \tag{31}$$

The following is an analog of Proposition 2.2.

Proposition 8.4 *For any $r > 0$ we have $V_{A,r}(\mathcal{F}C_{b,K}^\infty) \subset \mathcal{F}C_{b,K}^\infty$ and $V_{A,r}(\mathcal{W}_A^\infty) \subset \mathcal{W}_A^\infty$.*

Proof Again the statement for $\mathcal{W}_A^\infty$ is immediate. The statement for $\mathcal{F}C_{b,K}^\infty$ can be proved similarly as in Proposition 2.2: If $f = F(l_1, ..., l_n) \in \mathcal{F}C_{b,K}^\infty$ with $F \in C_b^\infty(\mathbb{R}^n)$ and $l_1, ..., l_n \in K$ orthogonal in H, we have, for any $x \in E$,

$$
\begin{aligned}
P_{A,t} f(x) &= \int_E F(\langle e^{-tA}x, l_1\rangle + \langle\sqrt{1 - e^{-2tA}}y, l_1\rangle, ..., \langle e^{-tA}x, l_n\rangle + \langle\sqrt{1 - e^{-2tA}}y, l_n\rangle)\mu(dy) \\
&= \int_E F(\langle x, e^{-tA}l_1\rangle + \langle y, \sqrt{1 - e^{-2tA}}l_1\rangle, ..., \langle x, e^{-tA}l_n\rangle + \langle y, \sqrt{1 - e^{-2tA}}l_n\rangle)\mu(dy) \\
&= \int_{\mathbb{R}^n} F(\xi_t + \eta)\, \mathcal{N}(0, D(t, A; l_1, ..., l_n))(d\eta),
\end{aligned}
$$

where $\xi_t = (\langle x, e^{-tA}l_1\rangle, ..., \langle x, e^{-tA}l_n\rangle)$ and the symbol $\mathcal{N}(0, D(t, A; l_1, ..., l_n))$ denotes the centered normal distribution with diagonal covariance matrix $D(t, A; l_1, ..., l_n)$ having diagonal entries $|\sqrt{1 - e^{-2tA}}l_i|_H^2$. This can be rewritten as $F_t^{(n)}(\langle x, e^{-tA}l_1\rangle, ..., \langle x, e^{-tA}l_n\rangle)$ with the $C_b^\infty(\mathbb{R}^n)$ function

$$
F_t^{(n)}(\xi) = \int_{\mathbb{R}^n} F(\xi + \eta)\, \mathcal{N}(0, D(t, A; l_1, ..., l_n))(d\eta), \quad \xi \in \mathbb{R}^n,
$$

and by Eq. (26) we have $e^{-tA}l_i \in K$ for all i. $\qquad\square$

We turn to consider related capacities. Let

$$
\mathrm{Cap}_{A,r,p}, \quad 1 \le p < +\infty, \quad r > 0,
$$

be the capacities defined in the same way as $\mathrm{Cap}_{r,p}$ in Definition 3.1, but with E and $V_{A,r}$ in place of B and V_r. Likewise, let

$$
\mathrm{cap}_{A,r,2}^{(\mathcal{F}C_{b,K}^\infty)}, \quad r = 1, 2,
$$

be defined as $\mathrm{cap}_{r,2}^{(\mathcal{F}C_b^\infty)}$ in Definition 3.2 but with E, $\mathcal{W}_A^{r,2}(E, \mu)$ and $\mathcal{F}C_{b,K}^\infty$ in place of B, $W^{r,2}(B, \mu)$ and $\mathcal{F}C_b^\infty$. To define capacities based on $\mathcal{W}_A^\infty$ we now make the following assumption.

Assumption 8.5 For any $1 < p < +\infty$ and $r > 0$ the capacities $\mathrm{Cap}_{A,r,p}$ are tight, i.e., there exists an increasing sequence $(F_n)_n$ of compact sets $F_n \subset E$ such that

$$
\lim_n \mathrm{Cap}_{A,r,p}(E \setminus F_n) = 0.
$$

Remark 8.6 In [7, Theorem 6.7] it was shown that one can always find a Banach space E_1 such that E is continuously and densely embedded into E_1, the operators e^{-tA}, $t > 0$, extend to a strongly continuous contraction semigroup on E_1, and when μ is considered as a measure on E_1, the capacities $\mathrm{Cap}_{A,r,p}$, $1 < p < +\infty, r > 0$, associated with the Mehler semigroup corresponding to (E_1, H, A, μ) are tight. The key items in the proof of this fact were the density of K in E^* and Eq. (26). In [4, Corollary 1.5], quoted to prove [7, Theorem 6.7], the space E_1 was constructed as the completion of H w.r.t. the norm $\|x\|_{E_1} := \|e^{-sA}x\|_E$ for fixed $s > 0$. Using Eq. (26) one can then conclude that $\|e^{-sA}l\|_{E_1^*} \le \|l\|_{E^*}$, for all $l \in K$, so that even the initial assumptions involving the space K remain valid.

Under Assumption 8.5 one can now define the notions (r, p)-*quasi everywhere (q.e.)*, *slim*, *quasi surely (q.s.)* and (r, p)-*quasi continuous* in the same manner as in Section 3 (see [22], [37, Chapter IV] and the comments in [4, Section 1]) and also prove that any $u \in \mathcal{W}_A^\infty$ has a quasi-sure redefinition $\widetilde{u}$. Let

$$\mathrm{cap}_{r,2}^{(\mathcal{W}_A^\infty)}, \quad r = 1, 2,$$

be defined as in Definition 3.3 but with E, $\mathcal{W}_A^{r,2}(E, \mu)$ and $\mathcal{W}_A^\infty$ in place of B, $W^{r,2}(B, \mu)$ and W^∞.

We have the following partial analog of Theorem 3.5.

Theorem 8.7 *Let Assumption 8.5 be in force. Then for $r = 1, 2$ there exist positive constants c_3 and c_4 depending only on r, such that for any set $C \subset E$ we have*

$$c_3 \ \mathrm{cap}_{A,r,2}^{(\mathcal{F}C_{b,K}^\infty)}(C) \le \mathrm{Cap}_{A,r,2}(C) \le c_4 \ \mathrm{cap}_{A,r,2}^{(\mathcal{F}C_{b,K}^\infty)}(C)$$

and

$$c_3 \ \mathrm{cap}_{r,2}^{(\mathcal{W}_A^\infty)}(C) \le \mathrm{Cap}_{A,r,2}(C) \le c_4 \ \mathrm{cap}_{r,2}^{(\mathcal{W}_A^\infty)}(C).$$

Remark 8.8 (i) The left inequalities remain valid without Assumption 8.5.

(ii) Using the theory in [51] one can study capacities of type $\mathrm{cap}_{A,r,p}^{(\mathcal{F}C_{b,K}^\infty)}$ and $\mathrm{cap}_{r,p}^{(\mathcal{W}_A^\infty)}$ for general r and p and establish a more general version of Theorem 8.7. However, as this is not needed to discuss our main example and since in the presence of a drift A the corresponding Sobolev spaces of higher order are considerably more complicated to handle, we leave it to the interested reader.

We provide two results from which Theorem 8.7 is obtained by similar arguments as Theorem 3.5. The first is the following Theorem 8.9 which is a partial analog of Theorem 3.6 and of course interesting only for $r = 2$.

Theorem 8.9 *Let $r = 1, 2$ and let $T \in C^2(\mathbb{R})$ be as in Theorem 8.9, so that* Eq. (12) *holds. Then for any nonnegative $f \in L^2(E, \mu)$ the function $T \circ V_{A,r} f$ is in $\mathcal{W}_A^{r,2}(E, \mu)$, and there is a constant $c_T > 0$, depending only on r and on L in* Eq. (12) *such that for any nonnegative $f \in L^2(E, \mu)$ we have*

$$\left\| T \circ V_{A,r} f \right\|_{\mathcal{W}_A^{r,2}(E,\mu)} \le c_T \left\| f \right\|_{L^2(E,\mu)}.$$

The second result employed to prove Theorem 8.7 is the following Lemma 8.10, and since it can be shown in the same manner as Lemma 3.8, we omit its proof. The symbol $\mathcal{F}C_{b,K,+}^\infty$ denotes the cone of nonnegative elements of $\mathcal{F}C_{b,K}^\infty$.

Lemma 8.10 *Let Assumption 8.5 be in force and $r = 1, 2$. For any open set $U \subset E$ we have*

$$\mathrm{Cap}_{A,r,2}(U) = \inf \left\{ \|f\|_{L^p(E,\mu)}^p \mid f \in \mathcal{F}C_{b,K,+}^\infty, \ V_{A,r} f \ge 1 \ \text{on} \ U \right\}.$$

In the sequel we provide a proof for Theorem 8.9. To do so we use a spectral theoretic refinement of the arguments used to show Theorem 3.6. Recall that the self-adjoint operator $(A, \mathcal{D}(A))$ on H is assumed to be strictly positive, hence we can find $\lambda_0 > 0$ so that

$$\inf_{0 \neq k \in \mathcal{D}(A)} \frac{\langle Ak, k \rangle_H}{|k|_H^2} > \lambda_0. \tag{32}$$

Let $(\Pi_\lambda)_{\lambda \geq \lambda_0}$ denote the family of spectral projectors in H uniquely associated with $(A, \mathcal{D}(A))$, so that

$$A = \int_{\lambda_0}^{\infty} \lambda \, d\Pi_\lambda.$$

Note that by the above bound we have $\Pi_{\lambda_0}(H) = \{0\}$. Given $t > 0$ let now $\varphi_t : (0, +\infty) \to (0, +\infty)$ be the function defined by

$$\varphi_t(\lambda) := \frac{e^{-\lambda t}}{\sqrt{1 - e^{-2\lambda t}}}, \quad \lambda > 0,$$

obviously nonnegative, continuous and decreasing with $\lim_{\lambda \to \infty} \varphi_t(\lambda) = 0$. Recall from Section 2 that $h \mapsto \hat{h}$ denotes the isometry from H into $L^2(E, \mu)$. Clearly its range $\hat{H}$ is a closed subspace of $L^2(E, \mu)$, we denote the orthogonal projection in $L^2(E, \mu)$ onto $\hat{H}$ by $\Pi_{\hat{H}}$, and for the inverse of the bijection $\hat{\ } : H \to \hat{H}$ we write $\check{\ } : \hat{H} \to H$. We fix some straightforward consequences of the spectral representation and the isometry.

Lemma 8.11

(i) *For any $t > 0$ the self-adjoint operator $\varphi_t(A) = \int_{\lambda_0}^{\infty} \varphi_t(\lambda) \, d\Pi_\lambda$ on H is bounded, more precisely, $\|\varphi_t(A)\|_{H \to H} \leq \varphi_t(\lambda_0)$.*

(ii) *For any $t > 0$, any $h \in H$ and any $g \in L^2(E, \mu)$ we have*

$$\left\langle (\varphi_t(A)h)^\wedge, g \right\rangle_{L^2(E,\mu)} = \int_{\lambda_0}^{\infty} \varphi_t(\lambda) \, d \left\langle (\Pi_\lambda h)^\wedge, g \right\rangle_{L^2(E,\mu)},$$

where the integral over $(\lambda_0, +\infty)$ on the right hand side is taken w.r.t. the signed measure

$$d \left\langle (\Pi_\lambda h)^\wedge, g \right\rangle_{L^2(E,\mu)} = d \left\langle \Pi_\lambda h, (\Pi_{\hat{H}} g)^\vee \right\rangle_H. \tag{33}$$

(iii) *For any $\beta > 1$, $t > 0$, $h \in H$ and nonnegative $g \in L^2(E, \mu)$, we have*

$$\left| \left\langle (\varphi_t(A)h)^\wedge, g \right\rangle_{L^2(E,\mu)} \right| \leq \varphi_t(\lambda_0) \left(\int_E |\hat{h}|^\beta g \, d\mu \right)^{1/\beta} \left(\int_E g \, d\mu \right)^{1/\beta'},$$

where $\frac{1}{\beta} + \frac{1}{\beta'} = 1$.

Proof Statement (i) is clear from the spectral theorem and since φ_t is decreasing. To see (ii) note that by the isometry and the spectral theorem,

$$\begin{aligned}
\left\langle (\varphi_t(A)h)^\wedge, g \right\rangle_{L^2(E,\mu)} &= \left\langle (\varphi_t(A)h)^\wedge, \Pi_{\hat{H}} g \right\rangle_{L^2(E,\mu)} \\
&= \left\langle \varphi_t(A)h, (\Pi_{\hat{H}} g)^\vee \right\rangle_H \\
&= \int_{\lambda_0}^{\infty} \varphi_t(\lambda) \, d \left\langle \Pi_\lambda h, (\Pi_{\hat{H}} g)^\vee \right\rangle_H \\
&= \int_{\lambda_0}^{\infty} \varphi_t(\lambda) \, d \left\langle (\Pi_\lambda h)^\wedge, \Pi_{\hat{H}} g \right\rangle_{L^2(E,\mu)} \\
&= \int_{\lambda_0}^{\infty} \varphi_t(\lambda) \, d \left\langle (\Pi_\lambda h)^\wedge, g \right\rangle_{L^2(E,\mu)}.
\end{aligned}$$

To see (iii) note that by Hölder's inequality,

$$\left| \int_{\lambda_0}^{\infty} \varphi_t(\lambda) d \left\langle \Pi_\lambda h, (\Pi_{\hat{H}} g)^\vee \right\rangle_H \right| \leq \varphi_t(\lambda_0) \int_{\lambda_0}^{\infty} d | \left\langle \Pi_\lambda h, (\Pi_{\hat{H}} g)^\vee \right\rangle_H |$$

$$= \varphi_t(\lambda_0) | \left\langle \Pi_\lambda h, (\Pi_{\hat{H}} g)^\vee \right\rangle_H |$$

$$= \varphi_t(\lambda_0) \left| \int_E \hat{h} g \, d\mu \right|$$

$$\leq \varphi_t(\lambda_0) \left(\int_E |\hat{h}|^\beta g \, d\mu \right)^{1/\beta} \left(\int_E g \, d\mu \right)^{1/\beta'},$$

where $d | \left\langle \Pi_\lambda h, (\Pi_{\hat{H}} g)^\vee \right\rangle_H |$ denotes the total variation of Eq. (33). $\qquad\square$

The next fact is probably well known, but as we could not locate it in the literature, we sketch it briefly.

Proposition 8.12 *For any $t > 0$ and any $f \in L^2(E, \mu)$ the function $P_{A,t} f$ is H-differentiable at μ-a.e. $x \in E$, and we have*

$$\partial_h P_{A,t} f(x) = \int_E f(e^{-tA} x + \sqrt{1 - e^{-2tA}} y) \, (\varphi_t(A) h)^\wedge (y) \mu(dy)$$

for any $h \in H$ and μ-a.e. $x \in E$.

Proof Similarly as in [8, Proposition 5.4.8] Fubini implies that for μ-a.e. $x \in E$ the function $y \mapsto f(e^{-tA} x + \sqrt{1 - e^{-2tA}} y)$ is in $L^2(E, \mu)$. We have

$$P_{A,t} f(x + \lambda h) = \int_E f \left(e^{-tA} x + \sqrt{1 - e^{-2tA}} \left(\frac{e^{-tA}}{\sqrt{1 - e^{-2tA}}} \lambda h + y \right) \right) \mu(dy)$$

$$= \int_E f(e^{-tA} x + \sqrt{1 - e^{-2tA}} z) \exp \left\{ \lambda \, (\varphi_t(A) h)^\wedge - \frac{\lambda^2}{2} |\varphi_t(A) h|_H^2 \right\} \mu(dz)$$

by the Cameron-Martin theorem, and proceeding as in Proposition 4.1 we obtain the result. $\qquad\square$

A suitable partial analog of the multiplicative estimate in Lemma 4.2 is as follows.

Lemma 8.13 *For any $1 < q < +\infty$, any nonnegative $f \in L^2(E, \mu)$ and μ-a.e. $x \in E$ we have*

$$\left| D_A V_{A,2} f(x) \right|_H \leq c(q) \, (V_{A,2} f(x))^{\frac{1}{2}} \left(\sup_{t>0} P_{A,t}(f^q)(x) \right)^{\frac{1}{2q}}.$$

We use the fact that $\mathcal{D}(\sqrt{A})$, endowed with $\langle \cdot, \cdot \rangle_{\mathcal{D}(\sqrt{A})} := \left\langle \sqrt{A} \cdot, \sqrt{A} \cdot \right\rangle_H$, is Hilbert space and the fact that

$$\left| D_A V_{A,2} f(x) \right|_H = \left\| D V_{A,2} f(x) \right\|_{\mathcal{D}(\sqrt{A})} = \sup_{h \in \mathcal{D}(\sqrt{A}), \, \|h\|_{\mathcal{D}(\sqrt{A})} = 1} |D V_{A,2} f(x)(h)|.$$

Note also that by Eq. (6) we have

$$P_{A,t} \mathbf{1} = \mathbf{1}, \quad t > 0, \quad \text{and} \quad V_{A,2} \mathbf{1} = \mathbf{1}. \tag{34}$$

Proof To shorten notation we will use the abbreviation $g_{t,x}(y) := f(e^{-tA}x + \sqrt{1 - e^{-2tA}}y)$. We first assume that f is bounded. Let $h \in \mathcal{D}(\sqrt{A})$ be such that $\|h\|_{\mathcal{D}(\sqrt{A})} = 1$, and let $\delta > 0$ be arbitrary. By Proposition 8.12 we see that for μ-a.e. $x \in E$ we have

$$
\begin{aligned}
DV_{A,2}f(x)(h) = \partial_h V_{A,2}f(x) &= \int_0^\infty e^{-t} \int_E g_{t,x}(y)(\varphi_t(A)h)^\wedge(y)\, \mu(dy)\, dt \\
&= \int_0^\delta e^{-t} \int_{\lambda_0}^\infty \varphi_t(\lambda)\, d\left\langle (\Pi_\lambda h)^\wedge, g_{t,x} \right\rangle_{L^2(E,\mu)} dt \\
&\quad + \int_\delta^\infty e^{-t} \int_{\lambda_0}^\infty \varphi_t(\lambda)\, d\left\langle (\Pi_\lambda h)^\wedge, g_{t,x} \right\rangle_{L^2(E,\mu)} dt \\
&=: I_1(\delta) + I_2(\delta).
\end{aligned}
$$

Now let

$$
\max\left\{ 1, 2 - \frac{1}{\lambda_0} \right\} < \beta < 2.
$$

Then by Lemma 8.11 and Hölder's inequality,

$$
\begin{aligned}
|I_1(\delta)| &\leq \int_0^\delta e^{-t}\varphi_t(\lambda_0)\left(\int_E |\hat{h}|^\beta g_{t,x}\, d\mu \right)^{1/\beta}\left(\int_E g_{t,x}\, d\mu \right)^{1/\beta'} dt \\
&\leq \left(\int_0^\delta e^{-t}\varphi_t(\lambda_0)^\beta \int_E |\hat{h}|^\beta g_{t,x}\, d\mu\, dt \right)^{1/\beta}\left(\int_0^\delta e^{-t} \int_E g_{t,x}\, d\mu\, dt \right)^{1/\beta'} \quad (35)
\end{aligned}
$$

with $\frac{1}{\beta} + \frac{1}{\beta'} = 1$, and by Hölder, now with $\frac{1}{q} + \frac{1}{q'} = 1$, the first factor in Eq. (35) admits the bound

$$
\begin{aligned}
&\left(\int_0^\delta \int_E f(e^{-tA}x + \sqrt{1 - e^{-2tA}}y)^q\, \mu(dy)\,(1 - e^{-2\lambda_0 t})^{\beta/2} e^{-(1+\beta\lambda_0)t}\, dt \right)^{1/(\beta q)} \\
&\quad \times \left(\int_E |\hat{h}(y)|^{\beta q'}\, \mu(dy) \int_0^\delta (1 - e^{-2\lambda_0 t})^{\beta/2} e^{-(1+\beta\lambda_0)t}\, dt \right)^{1/(\beta q')}.
\end{aligned}
$$

Since our choice of β guarantees $1 + \beta\lambda_0 > 2\lambda_0$, we have

$$
\begin{aligned}
\int_0^\delta (1 - e^{-2\lambda_0 t})^{\beta/2} e^{-(1+\beta\lambda_0)t}\, dt &= \frac{1}{2\lambda_0} \int_{e^{-2\lambda_0\delta}}^1 (1-s)^{-\beta/2} s^{(1+\beta\lambda_0)/(2\lambda_0)-1}\, ds \\
&\leq \frac{(1 - e^{-2\lambda_0\delta})^{1-\beta/2}}{2\lambda_0(1 - \frac{\beta}{2})}.
\end{aligned}
$$

Using the facts that

$$
\int_E |\hat{h}(y)|^{\beta q'}\, \mu(dy) = \int_{\mathbb{R}} |\eta|^{\beta q'} \mathcal{N}(0, |h|_H^2)(d\eta) = |h|_H^{\beta q'} \int_{\mathbb{R}} |\xi|^{\beta q'} \mathcal{N}(0,1)(d\xi),
$$

$$
\left\| A^{-1/2} \right\|_{H \to H} \leq \frac{1}{\sqrt{\lambda_0}}
$$

and

$$
|h|_H \leq \left\| A^{-1/2} \right\|_{H \to H} |\sqrt{A}h|_H,
$$

where $|\sqrt{A}h|_H = \|h\|_{\mathcal{D}(\sqrt{A})} = 1$ by the hypothesis on H, we arrive at the bound

$$
|I_1(\delta)| \leq c_1(\lambda_0, \beta, q)\left(1 - \frac{\beta}{2} \right)^{-1/\beta} (1 - e^{-2\lambda_0\delta})^{1/\beta - 1/2} \left(V_{A,r}f(x) \right)^{1/\beta'} \left(\sup_{t>0} P_{A,t}(f^q)(x) \right)^{1/(\beta q)},
$$

where $c_1(\lambda_0, \beta, q) > 0$ is a constant depending only on λ_0, β and q. In a similar fashion we can obtain a bound

$$|I_2(\delta)| \le c_2(\lambda_0, \gamma, q) \left(1 - \frac{\gamma}{2}\right)^{-1/\gamma} (1 - e^{-2\lambda_0\delta})^{1/\gamma - 1/2} \left(V_{A,r} f(x)\right)^{1/\gamma'} \left(\sup_{t>0} P_{A,t}(f^q)(x)\right)^{1/(\gamma q)}$$

with arbitrary $\gamma > 2$, $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$ and a constant $c_2(\lambda_0, \gamma, q) > 0$ depending only on λ_0, γ and q. Choosing suitable $\delta > 0$ as in the proof of Lemma 4.2, we obtain the statement for bounded f. For general nonnegative $f \in L^2(E, \mu)$ consider $f_N := f \wedge N$, for which we obtain

$$\left|D_A((V_{A,2}f) \wedge N)(x)\right|_H = \left|D_A V_{A,2} f_N(x)\right|_H$$

$$\le c(q) \left(V_{A,2}f(x)\right)^{\frac{1}{2}} \left(\sup_{t>0} P_{A,t}(f^q)(x)\right)^{\frac{1}{2q}}$$

by the positivity of the operators $V_{A,2}$ and P_t and Eq. (34). This allows to conclude the result using standard Dirichlet form theory. $\qquad\square$

We prove Theorem 8.9.

Proof We provide a proof only for the case $r = 2$. Let $u = V_{A,2}f$. Since $T(0) = 0$ and $|T'|$ is bounded by L, we have

$$\|T(u)\|_{L^2(E,\mu)} \le L \|V_{A,2}f\|_{L^2(E,\mu)} \le L \|f\|_{L^2(E,\mu)}.$$

By Eq. (28) and the triangle inequality it therefore suffices to obtain a suitable bound for $\|\mathcal{L}_A T(u)\|_{L^2(E,\mu)}$. For any $\varepsilon > 0$ we also consider $u_\varepsilon := V_{A,2}(f + \varepsilon)$. By Eq. (34) we have $u_\varepsilon \ge \varepsilon$ and, because it implies $\mathbf{1} \in \ker \mathcal{L}_A$, also $\mathcal{L}_A u_\varepsilon = \mathcal{L}_A u$. By Eq. (29) clearly also $D_A u_\varepsilon = D_A u$. The analog of the chain rule Eq. (11) for the case of linear drift, applied to u_ε, yields

$$\|\mathcal{L}_A T(u_\varepsilon)\|_{L^2(E,\mu)} \le L \|\mathcal{L}_A u\|_{L^2(E,\mu)} + \left\| \frac{1}{u_\varepsilon} \langle D_A u_\varepsilon, D_A u_\varepsilon \rangle_H \right\|_{L^2(E,\mu)}.$$

Clearly $\|\mathcal{L}_A u\|_{L^2(E,\mu)} \le \|f\|_{L^2(E,\mu)}$. On the other hand, by Lemma 8.13 with some $1 < q < 2$ and the boundedness of the maximal function in $L^{2/q}(E, \mu)$, [57, Chapter III, Section 3, Maximal Theorem], we can mimic the original idea of Maz'ya, [1, Theorem 3], and estimate

$$\int_E \frac{|D_A u_\varepsilon(x)|_H^4}{u_\varepsilon(x)^2} \mu(dx) \le c(q) \int_E \left(\sup_{t>0} P_{A,t}((f+\varepsilon)^q)(x)\right)^{2/q} \mu(dx)$$

$$\le c \int_E (f + \varepsilon)^2 \, d\mu$$

$$\le c' \left(\int_E f^2 \, d\mu + \varepsilon^2\right).$$

Together with the preceding argument this implies that for any $0 < \varepsilon < \|f\|_{L^2(E,\mu)}$ we have

$$\|\mathcal{L}_A T(u_\varepsilon)\|_{L^2(E,\mu)} \le c \|f\|_{L^2(E,\mu)}.$$

Accordingly we can find a sequence $(\varepsilon_k)_k$ with $\lim_k \varepsilon_k = 0$ and an element $w \in L^2(E, \mu)$ such that $\lim_k \mathcal{L}_A T(u_{\varepsilon_k}) = w$ weakly in $L^2(E, \mu)$ and such that with $w_N :=$

$\sum_{k=1}^{N} T(u_{\varepsilon_k}) \in \mathcal{D}(\mathcal{L}_A)$ also $w = \lim_N \mathcal{L}_A w_N = \lim_N \sum_{k=1}^{N} \mathcal{L}_A T(u_{\varepsilon_k})$ strongly in $L^2(E, \mu)$. Since

$$\lim_\varepsilon \|T(u_\varepsilon) - T(u)\|_{L^2(E,\mu)} \leq L \lim_\varepsilon \|u_\varepsilon - u\|_{L^2(E,\mu)} = 0,$$

we also have $\lim_N w_N = T(u)$ strongly in $L^2(E, \mu)$. Since $(\mathcal{L}_A, \mathcal{D}(\mathcal{L}_A))$ is closed, it follows that $T(u) \in \mathcal{D}(\mathcal{L}_A)$ and $\mathcal{L}_A T(u) = w$. We can therefore conclude that

$$\|\mathcal{L}_A T(u)\|_{L^2(E,\mu)} \leq \liminf_k \|\mathcal{L}_A T(u_{\varepsilon_k})\|_{L^2(E,\mu)} \leq c \|f\|_{L^2(E,\mu)},$$

and combined with the above this yields $\|T \circ u\|_{\mathcal{W}_A^{2,2}(E,\mu)} \leq c_T \|f\|_{L^2(E,\mu)}$, as desired. $\square$

9 Essential Self-Adjointness of Ornstein-Uhlenbeck Operators with Linear Drift

In this section we consider the essential self-adjointness of $\mathcal{L}_A$ in $L^2(E, \mu)$, endowed with subspaces of $\mathcal{F}C_{b,K}^\infty$ or $\mathcal{W}_A^\infty$ after the removal of a small set from E. Let $(A, \mathcal{D}(A))$ and K be as in in the preceding section. In [7, Proposition 5.4] it was shown that $\mathcal{L}_A$, endowed with $\mathcal{F}C_{b,K}^\infty$, is essentially self-adjoint in $L^2(E, \mu)$, and that its unique self-adjoint extension is $(\mathcal{L}_A, \mathcal{D}(\mathcal{L}_A))$ with $\mathcal{D}(\mathcal{L}_A) = \mathcal{W}_A^{2,2}(E, \mu)$ as discussed above. By Eq. (31) then also $\mathcal{L}_A$, endowed with $\mathcal{W}_A^\infty$, is essentially self-adjoint with the same unique self-adjoint extension.

Similarly as before let now $\Sigma \subset E$ be a closed set of zero Gaussian measure and write $N := E \setminus \Sigma$. Let

$$\mathcal{F}C_{b,K}^\infty(N) := \{f \in \mathcal{F}C_{b,K}^\infty \mid f = 0 \text{ on an open neighborhood of } \Sigma\}$$

and

$$\mathcal{W}_A^\infty(N) := \{f \in \mathcal{W}_A^\infty \mid \widetilde{f} = 0 \text{ q.s. on an open neighborhood of } \Sigma\}.$$

Theorem 9.1 *Let Assumption 8.5 be in force. Let $\Sigma \subset E$ be a closed set of zero measure μ and write $N := E \setminus \Sigma$.*

(i) *If $\mathrm{Cap}_{A,2,2}(\Sigma) = 0$ then $(\mathcal{L}_A, \mathcal{F}C_{b,K}^\infty(N))$ is essentially self-adjoint on $L^2(E, \mu)$ with unique self-adjoint extension $(\mathcal{L}_A, \mathcal{W}_A^{2,2}(E, \mu))$.*

(ii) *If $(\mathcal{L}_A, \mathcal{F}C_{b,K}^\infty(N))$ is essentially self-adjoint on $L^2(E, \mu)$ then $\mathrm{Cap}_{A,2,2}(\Sigma) = 0$.*

The same is true with $\mathcal{W}_A^\infty(N)$ in place of $\mathcal{F}C_{b,K}^\infty(N)$.

Theorem 9.1 follows by (a simpler version of) the same arguments as Theorem 5.2.

Remark 9.2 Of course an analogous statement is true for capacities of type $\mathrm{Cap}_{A,1,2}$ and with essential self-adjointness replaced by Markov uniqueness, but this is the special case of a well known standard result in Dirichlet form theory.

Theorem 9.1 can be applied to the time zero Gaussian free field, [54], as discussed in [4, Examples 3.3 (ii)], [7, Examples 5.6. (ii) and 6.6.(i)] and [48, 49]. We follow [50, Examples 3.5], referred to as the 'second approach to the free Dirichlet form' in [7, Examples 5.6.(ii)].

Example 9.3 For any $s \in \mathbb{R}$ let $H^s(\mathbb{R}^d) = \left\{f \in \mathcal{S}'(\mathbb{R}^d) \mid (1 + |\xi|^2)^{s/2} \mathcal{F} f \in L^2(\mathbb{R}^d)\right\}$ as usual, where $\mathcal{S}'(\mathbb{R}^d)$ is the topological dual of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{F}$ denotes

the Fourier transform. Now let $m \in (0, +\infty)$ and consider $A = (m^2 - \Delta)^{1/2}$ as a self-adjoint operator on $H := H^{-1/2}(\mathbb{R}^d)$ with $\mathcal{D}(\sqrt{A}) = L^2(\mathbb{R}^d)$. Obviously A is strictly positive and we may choose any $0 < \lambda_0 < m$ in Eq. (32). Let E and K be as constructed in [7, Theorem 3.1]. Alternatively, let E be the space B_α as described in [50, Examples 3.5] and constructed in [48, 49] and $K = \mathcal{S}(\mathbb{R}^d)$. Let E be extended according to [7, Theorem 6.7] and Remark 8.6 above (we keep the same symbol E). The mean zero Gaussian measure μ on E with covariance

$$\int_E l_1(y)l_2(y)\mu(dy) = \langle l_1, l_2 \rangle_H, \quad l_1, l_2 \in E',$$

makes (E, H, μ) into an abstract Wiener space. It is called the *time zero Gaussian free field with mass m*. This setup satisfies all assumptions made in the beginning of Section 8, and it satisfies Assumption 8.5. The corresponding generator $(\mathcal{L}_A, \mathcal{D}(\mathcal{L}_A))$ and Dirichlet form $(\mathcal{E}_A, \mathcal{D}(\mathcal{E}_A))$ as in Eq. (30) are called the *free Hamiltonian* and the *free-field Dirichlet form*, respectively.

If a closed set $\Sigma \subset E$ of zero Gaussian measure is removed from E and $N := E \setminus \Sigma$, then the free Hamiltonian $\mathcal{L}_A$, endowed with $\mathcal{F}C_{b,K}^\infty(N)$ or $\mathcal{W}_A^\infty(N)$ as defined above, is essentially self-adjoint on $L^2(N, \mu) = L^2(E, \mu)$ with unique self-adjoint extension $(\mathcal{L}_A, \mathcal{D}(\mathcal{L}_A))$ if and only if $\mathrm{Cap}_{A,2,2,}(\Sigma) = 0$. In other words, small 'boundaries' Σ of zero $\mathrm{Cap}_{A,2,2,}$-capacity are not seen when extending the operator, and if a small boundary is not seen, it must have zero $\mathrm{Cap}_{A,2,2,}$-capacity.

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