Hurwitz action in Coxeter groups and extended Weyl groups with application in representation theory of finite dimensional algebras

Dissertation

zur Erlangung des Grades
Doktor der Mathematik (Dr. math.)

vorgelegt von
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August 2020

Annahme der Promotion durch die Prüfungskommission bestehend aus:

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Datum des Kolloquiums: 16. November 2020

Gedruckt auf altersbeständigem Papier gemäß ISO 9706.

Acknowledgments

First and foremost I thank my supervisor, Barbara Baumeister. I thank her for her ongoing support and for plenty of hours of stimulating discussions. Secondly, I want to thank Henning Krause for his interest in my results and his willingness in answering my basic questions in representation theory.

In addition, I want to thank Dieter Vossiek and Yiyu Cheng and in particular, Patrick Wegener for discussions and fruitful advices.

I would also like to thank Jon McCammond for welcoming me at the UC Santa Barbara and Vic Reiner for welcoming me at the UM Minneapolis. I am very grateful for the opportunity to discuss with them.

I would like to thank Anna Gusakova, Arthur Sinulis and Jonas Jalowy for plenty of funny moments in our shared office that made the time more enjoyable.

Of course I am extremely grateful to my family and friends for supporting me morally and without which I probably would never have finished this thesis. Finally, I want to thank Farina, a very special person to me. I dedicate this work to you.

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Chapter 1

Introduction

The presented thesis examines topics of group theory and representation theory of finite dimensional algebras. The first part deals with reflection groups, more precisely, Coxeter and extended Weyl groups. While Coxeter groups are already intensively studied the extended Weyl groups enjoyed less attention. They can be seen as extensions of Coxeter groups with specific Coxeter diagrams. Both classes of reflection groups appear in various branches of mathematics. For example, finite Coxeter groups are precisely the finite Euclidean reflection groups (see [58]). In particular, the family of symmetry groups of regular polyhedra can be realized by finite Coxeter groups (see [82]). Moreover, they appear in singularity theory (see [41]) and in various classification results of algebraic structures as finite dimensional complex semisimple Lie algebra (see [98]) and representations of quivers of finite type (see [43]). They can also be found in algebraic combinatorics with connection to topology. For example, the poset of non-crossing partition can be defined and described in terms of distinguished elements of finite Coxeter groups (see [4]). The latter has also connections to certain subcategories of module categories (see [56], [60] and [68]).

The extended Weyl groups play a similar role. They are part of singularity theory, where they appear as monodromy groups of simple elliptic and unimodular hyperbolic singularities (see [41]). In the theory of extended affine Lie algebras they are the reflection groups attached to root systems of certain infinite dimensional Lie algebras (see [2]).

In the first part of the thesis we are mainly interested in the so-called Hurwitz action in Coxeter and extended Weyl groups. It is defined as follows. Given any group G and integer $n \geq 2$, the braid group on n strands in its standard presentation

$$\mathcal{B}_n = \langle \sigma_i,\ 1 \leq i \leq n-1 \mid \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| \geq 2;\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ 1 \leq i \leq n-2 \rangle$$

acts on G^n as

$$\sigma_i(g_1, \ldots, g_n) := (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_n)$$
$$\sigma_i^{-1}(g_1, \ldots, g_n) := (g_1, \ldots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \ldots, g_n).$$

This action was first studied by Hurwitz in 1891 in the case of the symmetric group $G = S_n$

(see [59]). In general, the question whether two elements of G^n lie in the same orbit of the Hurwitz action is undecidable. The latter was shown by Liberman and Teicher (see [75]).

From now on let G be a Coxeter or extended Weyl group and T the set of reflections of G that contains a distinguished generating set of G, called simple system. The set T is closed under conjugacy and hence the Hurwitz action can be restricted to T^n. This induces an action on reflection factorizations of fixed length of elements in G. Given an element $w \in G$, we are interested in the set of reduced reflection factorization that is denoted by $\mathrm{Red}_T(w) = \{(t_1, \ldots, t_m) \in T^m \mid w = t_1 \cdots t_m\}$, where m is the minimal number of reflections that are needed to factorize w. Concerning two reflection factorizations $f_1, f_2 \in \mathrm{Red}_T(w)$ one can ask the following important question. Do f_1 and f_2 lie in the same orbit under the Hurwitz action? If the latter is affirmed it is sometimes said that the dual Matsumoto property is satisfied. The term is originated from the so-called classical Matsumoto property that allows to pass from any reduced expression (in generators of the simple system) of an element to any other by successive application of the so-called braid relations, that appear in the definition of Coxeter groups.

For parabolic Coxeter elements in Coxeter groups of finite rank the dual Matsumoto property holds. In other words, the Hurwitz action is transitive on the set of reduced reflection factorizations of parabolic Coxeter elements. This is crucial in the theory of dual braid monoids (see [11]) and in the understanding of thick subcategories of module categories over finite dimensional hereditary algebras over an algebraically closed field (see [56]). For finite Coxeter groups, the Hurwitz action was first shown to act transitively on $\mathrm{Red}_T(c)$ for a Coxeter element c in a letter by Deligne to Looijenga (see [30]). The first published proof is due to Bessis (see [11]). Igusa and Schiffler generalized this result to arbitrary Coxeter groups of finite rank (see [60]). Recently, Baumeister, Dyer, Stump and Wegener gave a simple proof of the same statement (see [8]).

Another important class of elements in Coxeter groups are the so-called quasi-Coxeter elements. These are elements that admit a reduced reflection factorization whose factors generate the whole group. Hence this class of elements contains the Coxeter elements. For finite Coxeter groups Baumeister, Gobet, Roberts and Wegener prove that the Hurwitz action is transitive on the set of reduced reflection factorization of an element if and only if this element is a quasi-Coxeter element in a parabolic subgroup (see [9]). Recently, Wegener shows that for affine Coxeter groups quasi-Coxeter elements admit a transitive Hurwitz action on the set of reduced reflection factorizations (see [108]). In this thesis we direct our interest towards the Hurwitz action on non-reduced reflection factorizations of Coxeter elements in Coxeter groups and state a necessary and sufficient condition under which two reflection factorizations of a Coxeter element lie in the same Hurwitz orbit.

Next we consider an extended Weyl group W, its set of reflections T and distinguished elements that are called Coxeter transformations. Similar to Coxeter groups we investigate the Hurwitz action on reduced reflection factorizations of Coxeter transformations, where the proof of our main theorem is based on results of the Hurwitz action of non-reduced reflection factorization of Coxeter elements, that was investigated previously. These results lead to the second part of the thesis.

The second part deals with examples of hereditary abelian categories with tilting object, namely the module category of finitely generated modules over a finite dimensional hereditary k-algebra for an algebraically closed field k and the category of coherent sheaves over a weighted projective line in the sense of [44]. In fact, by a theorem of Happel there exists up to derived equivalence only the latter two cases if we assume in addition that the category is connected, ext-finite and k-linear (see [51]). One approach to the understanding of these categories is the investigation of certain subcategories. Of course one has to restrict himself to certain subcategories as the so-called thick subcategories, that are also sometimes called wide. These are abelian subcategories that are closed under extensions. The set of these subcategories is naturally equipped with a poset structure and our goal is to understand this structure. Since this is at the moment in general too ambitious we restrict ourselves to those thick subcategories that are generated by a so-called exceptional sequence. These sequences consists of indecomposable objects without self-extension and satisfy among themselves further properties. It turns out that we can attach a reflection group to these categories that is in fact a Coxeter or an extended Weyl group. Then the thick subcategories that are generated by an exceptional sequence can be identified with prefixes of reduced reflection factorizations of Coxeter elements resp. transformations. This approach was already used for the category of finitely generated modules over a finite dimensional hereditary k-algebra (see [56], [60] and [68]). Motivated by the latter references we extend the results to the category of coherent sheaves over a weighted projective line.

1.1 The main results

The first half of this work is dedicated to the study of the Hurwitz action on reflection factorizations in Coxeter groups and extended Weyl groups. For that we consider an arbitrary Coxeter group W of finite rank with set of reflections T and Coxeter element c. We answer a question of Lewis and Reiner [74, Question 6.2] that generalized their main result of [74].

Theorem 3.3.6. *Let (W, S) be a Coxeter system of finite rank. Two reflection factorizations of a Coxeter element lie in the same Hurwitz orbit if and only if they share the same multiset of conjugacy classes.*

Besides new results concerning quasi-Coxeter elements in Coxeter groups of finite rank we reprove uniformly the key ingredient of the main result of [74].

Proposition 4.1.4. [74, Corollary 1.4] *Let (W, S) be a finite Coxeter system and $w = t_1 \cdots t_{m+2k} \in W$ a reflection factorization with $\ell_T(w) = m$ and $k \in \mathbb{Z}_{\geq 0}$. Then there exists* *a braid* $\sigma \in B_{m+2k}$ *such that*

$$\sigma(t_1, \ldots, t_{m+2k}) = (r_1, \ldots, r_m, r_{i_1}, r_{i_1}, \ldots, r_{i_k}, r_{i_k}).$$

Now let W be an extended Weyl group. There is a trichotomy of types of W, namely the domestic, tubular and wild type, that only depend on the signature of the corresponding W-invariant symmetric bilinear form. If W is of domestic type, it is in fact a simply-laced affine Coxeter group, i.e. the type is $\widetilde{A}$, $\widetilde{D}$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. If W is of tubular type, i.e. of type $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ or $E_8^{(1,1)}$, these groups are already partially investigated by Kluitmann (see [65]) and Wegener (see [107]). The results of the infinite family of extended Weyl groups of wild type, that are presented in this thesis are new. In analogy to Coxeter groups our main theorem states that the Hurwitz action is transitive on the set of reduced generating reflection factorization of Coxeter transformations, i.e. of reduced reflection factorizations whose factors generate the whole group.

Theorem 5.4.1 and 5.5.1. *Let W be an extended Weyl group, T its set of reflections and c a Coxeter transformation. The Hurwitz action is transitive on the set of generating factorizations of c. If W is of wild or domestic type any reduced reflection factorization is generating, thus the Hurwitz action is transitive on the set of reduced reflection factorizations of c.*

The previous result is already proven by Kluitmann and by Wegener for the tubular cases. Concretely, Kluitmann considered the types $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ and Wegener the type $D_4^{(1,1)}$. We give a direct proof of these four cases using new methods and avoid therefore computer-based calculations. Since extended Weyl groups of domestic types are affine Coxeter groups and Coxeter transformations admit a reduced reflection factorization that generates the group they belong to the class of the quasi-Coxeter elements. In this particular setting the theorem follows also from the work of Wegener (see [108]). The statement for the infinite family of extended Weyl groups of wild type is a new result.

In the second part we face the problem of understanding the poset of thick subcategories generated by an exceptional sequence of certain abelian categories. For that we associate an extended Weyl group W and a Coxeter transformation c to the category of coherent sheaves over a weighted projective line. This leads due to the theorem of Happel to the following result, where $[1, c]$ is the poset consisting of elements that admit a factorization that is a prefix of reduced reflection factorizations of c.

Theorem 7.2.5. *Let $\mathcal{A}$ be a hereditary connected ext-finite abelian k-category with tilting object that is not derived equivalent to the category of coherent sheaves over a weighted projective line of tubular type, and let Φ be the associated root system, W its reflection group and $c \in W$ a Coxeter transformation. Then there exists an order preserving bijection between*

- *the poset of thick subcategories of $\mathcal{A}$ that are generated by an exceptional sequence, and*

- *the poset* $[1, c]$.

1.1.1 Outline

The thesis is organized as follows. After a short introduction of real reflection groups in Chapter 2 we present the necessary background on Coxeter groups and well-known results concerning reflection subgroups of Coxeter groups in Chapter 3. Based on the latter we study the Hurwitz action on non-reduced reflection factorizations of Coxeter elements which leads to the main Theorem 3.3.6. Chapter 4 is dedicated to the study of the Hurwitz action in finite Coxeter groups. We reprove uniformly well-known and new results with help of the investigations of the previous sections. In Chapter 5 we introduce the extended Weyl groups and Coxeter transformations and prove some of their major properties. Using previous results we first prove in Theorem 5.4.1 the Hurwitz transitivity on the set of reduced reflection factorizations of Coxeter transformations in groups of domestic and wild type. After that we prove the analogous statement for the tubular types in Theorem 5.5.1. Chapter 6 is devoted to some parts of the theory of hereditary abelian and triangulated categories. Two examples are discussed in detail. The discussion of the module category over certain algebras can be found in Section 6.3 and the discussion of the category of coherent sheaves over a weighted projective line in Section 6.4. Finally, in Chapter 7 we state and prove a poset isomorphism between the set of thick subcategories generated by an exceptional sequence and the set of prefixes of generating factorizations of a Coxeter transformation. This result is explicitly formulated in Theorem 7.2.5.

1.1.2 Published parts of this work

This thesis essentially consists of the material of two articles. The first one carries the title *A note on non-reduced reflection factorizations of Coxeter elements* and is published in the *Journal of algebraic combinatorics*. Its major results concerning arbitrary reflection factorizations of Coxeter elements are explained in Section 3.3. The results of Chapter 4 have not been submitted yet. The second article is *Extended and Elliptic Weyl groups, Hurwitz transitivity and a correspondence for the category of coherent sheaves over a weighted projective line* and is available on the arXiv. Its results are included in Section 5.5 and Chapter 7. There we prove the Hurwitz transitivity on the set of generating reflection factorization of Coxeter transformations for tubular cases and explain consequences of the transitivity of the Hurwitz action for the lattice of thick subcategories of certain hereditary abelian categories. The results of Section 5.4, namely the Hurwitz transitivity on the set of reduced reflection factorizations of Coxeter transformations for the domestic and wild cases, have not been submitted yet.

Chapter 2

Reflection groups and Hurwitz action

The theory of real reflection groups plays an important role in several independent areas of mathematics, for instant in group theory, geometry, Lie theory and representation theory of algebras. In this section we introduce the notion of a real reflection group and a root system. Some of their major properties are stated, and in particular, a braid group action on reflection factorizations of elements of reflection groups, the so-called Hurwitz action, is defined. The latter is crucial for this thesis.

Throughout this section let V be a finite dimensional real vector space.

2.1 Reflection groups

Many concepts presented in the following can be generalized to the setting of a finite dimensional complex vector space. We start with the definition of a reflection.

Definition 2.1.1. *A* reflection *s of V is a non-trivial linear isomorphism $s : V \to V$ of finite order such that it fixes a hyperplane pointwise. The fixed hyperplane is called* reflection hyperplane. *A* reflection group *W is a subgroup of $GL(V)$ which is generated by reflections.*

Since the ground field is of characteristic zero and reflections are by definition of finite order Maschke's theorem implies the following properties.

Proposition 2.1.2. [63, Section 14] *Every reflection s is diagonalizable and of order two. Moreover, if H is the reflection hyperplane of s, then there exists $\alpha \in V \setminus \{0\}$ such that $V = H \oplus \mathrm{span}_{\mathbb{R}}(\alpha)$ and $s(v) = v + \Delta(x)\alpha$, for all $v \in V$, where Δ is a linear map $\Delta : V \to \mathbb{R}$ sending α to -2 and which satisfies* $\mathrm{Ker}(s) = H$.

For a finite reflection group W there exists a positive definite W-invariant symmetric bilinear form. Thus each reflection of W can be written in the following form.

Example 2.1.3. *Let $(-,-)$ be a symmetric bilinear form attached to V. Then the assignement $s_\alpha : V \to V;\ v \mapsto v - \frac{2(\alpha,v)}{(\alpha,\alpha)}\alpha$ with α anisotropic is a reflection with reflecting hyperplane $\alpha^\perp = \{v \in V \mid (\alpha, v) = 0\}$.*

Lemma 2.1.4. *Let* $(-,-)$ *be a symmetric bilinear form attached to* V *and* $\alpha \in V$ *anisotropic. Then the following holds:*

(a) The reflection s_α *is uniquely determined by* $Mov(s_\alpha) := (s_\alpha - 1)(V)$*, and it holds* $Mov(s_\alpha) = \text{span}_\mathbb{R}(\alpha)$*.*
(b) Let $g \in O(V,(-,-)) = \{g \in \text{GL}(V) \mid (gu, gv) = (u, v) \text{ for all } u, v \in V\}$*. Then* $gs_\alpha g^{-1} = s_{g(\alpha)}$*.*

Proof. Denote by R the radical of $(-,-)$. As $\alpha \notin R$ and by Proposition 2.1.2 assertion (a) follows directly from the definition of s_α. To prove (b) let $v \in V$ and $u = g^{-1}(v)$, and calculate

$$gs_\alpha g^{-1}(v) - v = g(s_\alpha(u) - u) \in g(\text{Mov}(\alpha)) = \text{span}_\mathbb{R}(g(\alpha)),$$

which implies the assertion. □

Next we state the definition of a root system. They are an important combinatorial tool that helps understanding reflection groups.

Definition 2.1.5. *Let* $(-,-)$ *be a symmetric bilinear form attached to* V *and* $\Phi \subseteq V$ *a non-empty set consisting of anisotropic vectors. The set* Φ *is called* root system *in* V *if the following properties are satisfied*

(a) $\text{span}_\mathbb{R}(\Phi) = V$ *and*
(b) $s_\alpha(\Phi) \subseteq \Phi$ *for all* $\alpha \in \Phi$*.*

The root system is called crystallographic *if in addition holds*

(c) $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ *for all* $\alpha, \beta \in \Phi$*.*

The elements in a root system are called roots*. A root system* Φ *is called* simply-laced *if all roots are of the same length, it is called* reduced *if* $r\alpha \in \Phi$ $(r \in \mathbb{R})$ *implies* $r = \pm 1$ *for all* $\alpha \in \Phi$ *and it is called* irreducible *if there does not exist root systems* Φ_1, Φ_2 *such that* $\Phi = \Phi_1 \sqcup \Phi_2$ *as well as* $\Phi_1 \perp \Phi_2$*, i.e* $(\alpha, \beta) = 0$ *for all* $\alpha \in \Phi_1$ *and* $\beta \in \Phi_2$*. A subset* Φ' *of a root system* Φ *is called* root subsystem *if it is a root system in* $\text{span}_\mathbb{R}(\Phi')$*. Two root systems* Φ_1 *and* Φ_2 *are called isomorphic if there exists an invertible linear transformation* $F : \text{span}_\mathbb{R}(\Phi_1) \longrightarrow \text{span}_\mathbb{R}(\Phi_2)$ *with* $F(\Phi_1) = \Phi_2$ *and that preserves the bilinear form.*

For a root system Φ denote by $W(\Phi)$ the group generated by the reflection s_α for $\alpha \in \Phi$. The set $T_\Phi := \{s_\alpha \mid \alpha \in \Phi\}$ is called set of reflections.

Example 2.1.6. *Let* $n \in \mathbb{N}$*,* $V = \mathbb{R}^{n+1}$ *and* $(-,-)$ *the standard euclidean inner product. The set* $A_n := \{e_i - e_j \mid 1 \leq i \neq j \leq n+1\}$ *is a finite, irreducible, simply-laced and crystallographic root system, where* e_i *is the i-th canonical vector for* $1 \leq i \leq n+1$*. The group* $W := W(A_n) = \langle s_\alpha \mid \alpha \in A_n \rangle$ *permutes the basis* $(e_i)_{1 \leq i \leq n+1}$ *by the usual action on* V*. Thus it is easy to see that* W *is isomorphic to the symmetric group* $\mathcal{S}_{n+1}$ *and that under this identification the set of reflections is the set of transpositions.*

We are mainly interested in Coxeter groups and extended Weyl group and in their corresponding root systems. These are two families of reflection groups that are of general interest.

2.2 Hurwitz action

This section is devoted to the definition and first easy properties of the so-called Hurwitz action, that is crucial for this thesis.

Let Φ be a root system, $W := W(\Phi)$ the corresponding reflection group, i.e. $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$. Since $T := T_\Phi$ is a generating set of W the following definitions make sense.

Definition 2.2.1. *The* reflection length function ℓ_T *is the map*

$$\ell_T : W \longrightarrow \mathbb{N}_0;\ w \longmapsto \min\{m \in \mathbb{N}_0 \mid t_1 \cdots t_m = w,\ t_1, \ldots, t_m \in T\}$$

and the reflection length *of an element* $w \in W$ *is the non-negative integer* $\ell_T(w)$. *A* reflection factorization *of an element* $w \in W$ *is a tuple* $(t_1, \ldots, t_m) \in T^m$ *such that* $t_1 \cdots t_m = w$ *and* $m \in \mathbb{N}$. *The set* $Fac_{T,m}(w)$ *consists of all reflection factorizations with* $m \in \mathbb{N}$ *factors and* $\mathrm{Red}_T(w)$ *is the* set of all reduced reflection factorization, *i.e. whose number of factors is* $\ell_T(w)$.

Definition 2.2.2. *Denote by* $\mathcal{B}_r$ *be the* braid group *on* r *strands, that is the group with (standard) generators* $\sigma_1, \ldots, \sigma_{r-1}$ *and subject to the relations* $\sigma_i \sigma_j = \sigma_j \sigma_i$ *for* $|i-j| \geq 2$ *and* $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ *for* $i = 1, \ldots, r-2$.

Next we define the so-called Hurwitz action on reflection factorizations. It is the braid group action on reflection factorizations defined as follows. Let $w \in W$ and $m \in \mathbb{N}$ such that the set of factorizatiosn $\mathrm{Fac}_{T,m}(w) = \{(t_1, \ldots, t_m) \in T^m \mid t_1 \cdots t_m = w\}$ is non-empty, then it is easy to see that following assignements yield a group action of $\mathcal{B}_m$ on $\mathrm{Fac}_{T,m}(w)$. For $(t_1, \ldots, t_m) \in \mathrm{Fac}_{T,m}(w)$ set

$$\sigma_i(t_1, \ldots, t_m) := (t_1, \ldots, t_{i-1}, t_i t_{i+1} t_i, t_i, t_{i+2}, \ldots, t_m)$$
$$\sigma_i^{-1}(t_1, \ldots, t_m) := (t_1, \ldots, t_{i-1}, t_{i+1}, t_{i+1} t_i t_{i+1}, t_{i+2}, \ldots, t_m).$$

Sometimes we use the symbol $\sim$ if two factorizations lie in the same orbit under the braid group action. The orbits are called Hurwitz orbits.

Example 2.2.3. *Consider the reflection group of Example 2.1.6 identified with the symmetric group* $\mathcal{S}_{n+1}$. *Let* $n = 2$ *and* $c = (1\ 2)(2\ 3)$, *then the Hurwitz action is transitive on* $\mathrm{Fac}_{T,2} = \{(t_1, t_2) \in T^2 \mid t_1 t_2 = c\}$. *For* $s_1 = (1\ 2)$ *and* $s_2 = (2\ 3)$ *the orbit of* (s_1, s_2) *is*

$$\{(s_1, s_2), (s_2, s_2 s_1 s_2), (s_2 s_1 s_2, s_1)\}.$$

There exists a closed formula that counts the number of different (not necessarily reduced) reflection factorization of so-called Coxeter elements in irreducible finite well-generated complex reflection groups (see for example [24]*). This class of reflection groups contains the finite real reflection groups and in particular the symmetric group* S_{n+1}*.*

In Section 3.3 we state a general criterion that describes whether two reflection factorizations of Coxeter elements lie in the same orbit or not.

We close this section with the investigation of the Hurwitz action for arbitrary dihedral groups, i.e. groups generated by two different involutions. It turns out that this case is important in understanding the Hurwitz action in arbitrary Coxeter groups, since the general case can be reduced to it (see the proof of Proposition 3.2.8).

Lemma 2.2.4. [107, Example 3.1.1] *Let* W *be a group,* $t_1, t_2 \in W$ *two different involutions and let* T *the set of alternating products of* t_1, t_2 *with odd factors. Then the pair* $(r, s) \in T^2$ *lies in the Hurwitz orbit* $\mathcal{B}_2(t_1, t_2) \subseteq T^2$ *if and only if* $t_1t_2 = rs$*.*

Proof. A direct computation shows for $m \in \mathbb{Z}_{\geq 0}$ that

$$\begin{aligned} \sigma_1^m(t_1, t_2) &= \left((t_1t_2)^m t_1,\ t_1(t_2t_1)^{m-1}\right) \text{ and} \\ (\sigma_1^{-1})^m(t_1, t_2) &= \left(t_2(t_1t_2)^{m-1},\ (t_2t_1)^m t_2\right). \end{aligned}$$

Since $r \in T$ there exists $m \in \mathbb{Z}_{\geq 0}$ with $r \in \{(t_1t_2)^m t_1,\ (t_2t_1)^m t_2\}$. The latter implies that $(r, s) \sim (t_1, t_2)$ if and only if $rs = t_1t_2$. □

Chapter 3

Hurwitz action in Coxeter groups

The class of Coxeter groups are a widely studied object, since they appear in various branches of mathematics. In this thesis we are mainly interested in reflection factorizations of certain elements of Coxeter groups and especially in extensions of Coxeter groups. The latter leads to the Hurwitz action that has application in for example singularity theory, representation theory and Lie theory.

This chapter is devoted to the study of the Hurwitz action in Coxeter groups. First we introduce Coxeter systems and state their basic properties. Then we explain that reflection subgroups of Coxeter groups are themselves Coxeter groups and prove some consequences for the Bruhat graph of Coxeter groups of finite rank. The latter is based on the work of Matthew Dyer [36]. At the end of the chapter we are in the position to prove one of our main results. It gives a necessary and sufficient condition whether two reflection factorizations of a Coxeter element are in the same Hurwitz orbit.

3.1 Definitions and basic properties

We start by defining Coxeter systems, introducing the Tits representation and stating basic properties of Coxeter groups. Our main sources for the presented facts are [18], [58] and [63].

Definition 3.1.1. *A* Coxeter system *is a pair (W, S) consisting of a group W and a subset $S \subseteq W$ such that W admits the following presentation*

$$\langle S \mid (st)^{m_{st}} = 1,\ s, t \in S\rangle,$$

where $m_{s,t} \in \mathbb{Z}$ with $m_{ss} = 1$ and $m_{st} = m_{ts} \geq 2$ for $s \neq t \in S$. In case no relation occurs for $s, t \in S$ we set $m_{st} = \infty$. The group W is called Coxeter group*, the set S is called* simple system*, the elements of S are called* simple generators *and $|S|$ the* rank *the rank of (W, S). The relations $sts\ldots = tst\ldots$ for $1 < m_{st} < \infty$ are called* braid relations*, where on both sides of the equation are m_{st} factors.*

To a Coxeter system (W, S) one can attach a diagram $\Gamma(W, S)$, called Coxeter diagram. The vertices of $\Gamma(W, S)$ correspond to the simple generators S and two vertices are joined by an edge if for the corresponding generators $s, t \in S$ holds $m_{st} \geq 3$ and for $m_{st} > 3$ it is labelled by m_{st}. Sometimes for $m_{st} = 4$ a double edge is drawn instead of its labelling. The Coxeter system (W, S) is called irreducible if $\Gamma(W, S)$ is connected. If $\Gamma(W, S)$ has connected components $\Gamma_1, \dots, \Gamma_k$ with corresponding subsets $S_1, \dots, S_k$ of S, then by [58, Proposition 6.1] $(\langle S_i \rangle, S_i)$ are themselves Coxeter systems and we have $W \cong \langle S_1 \rangle \times \ldots \times \langle S_k \rangle$. The set $T = \bigcup_{w \in W} wSw^{-1}$ is called the set of reflections for (W, S) and if $\Gamma(W, S)$ has connected components $\Gamma_1, \dots, \Gamma_k$ with corresponding sets of simple generators $S_1, \dots, S_k$ we get $T = \bigsqcup_{i=1}^k T_i$, where T_i is the set of reflections for $(\langle S_i \rangle, S_i)$ (see [46, Lemma 2.8]). Thus in many investigations we can restrict ourselves to irreducible Coxeter systems. A Coxeter system (W, S) is called finite if the group W is finite.

The example $I_2(6) \cong A_1 \times I_2(3)$ shows that, in general, the set of reflections is not determined by the abstract group W alone, but does depend on the simple system S (see for example [42]).

Example 3.1.2. *The reflection group defined in Example 2.1.6 is a Coxeter group with simple generators $S = \{(i \;\; i+1) \mid 1 \leq i \leq n\}$, where $(i \;\; i+1)$ corresponds to the transposition of the vectors e_i and e_{i+1}. The Coxeter system (W, S) is of rank n and its Coxeter diagram with n vertices and $n-1$ edges is*

In fact, all finite Coxeter groups can be classified by their Coxeter diagrams.

Theorem 3.1.3. [58, Theorem 6.4] *The irreducible finite Coxeter systems are those corresponding to the Coxeter diagrams of Figure 3.1*

Definition 3.1.4. *Let Λ be a connected undirected graph. A* spanning tree *Γ is a connected subgraph of Λ that is a tree and contains the set of vertices of Λ.*

The following lemma is standard.

Lemma 3.1.5. *Let (W, S) be a Coxeter system of finite rank such that there exists an odd-labelled spanning tree for $\Gamma(W, S)$, then all reflections are conjugated.*

Proof. Let Γ' be an odd-labelled spanning tree for $\Gamma(W, S)$. Take $s, t \in S$ which correspond to two vertices that are connected in Γ' and let m_{st} the odd integer from the definition of the Coxeter system (W, S). It holds $t = (st)^{m_{st}-1}s$ and thus s and t are conjugated. By the definition of Γ' all the simple generators are conjugated and therefore also all reflections. $\square$

Since S and T are generating sets of the Coxeter group W we can attach to them in an obvious way the length functions ℓ_S and ℓ_T.

Definition 3.1.6. *The function*

$$\ell_T : W \longrightarrow \mathbb{N}_0, \;\; w \longmapsto \min\{m \in \mathbb{N}_0 \mid t_1 \cdots t_m = w, \;\; t_i \in T\}$$

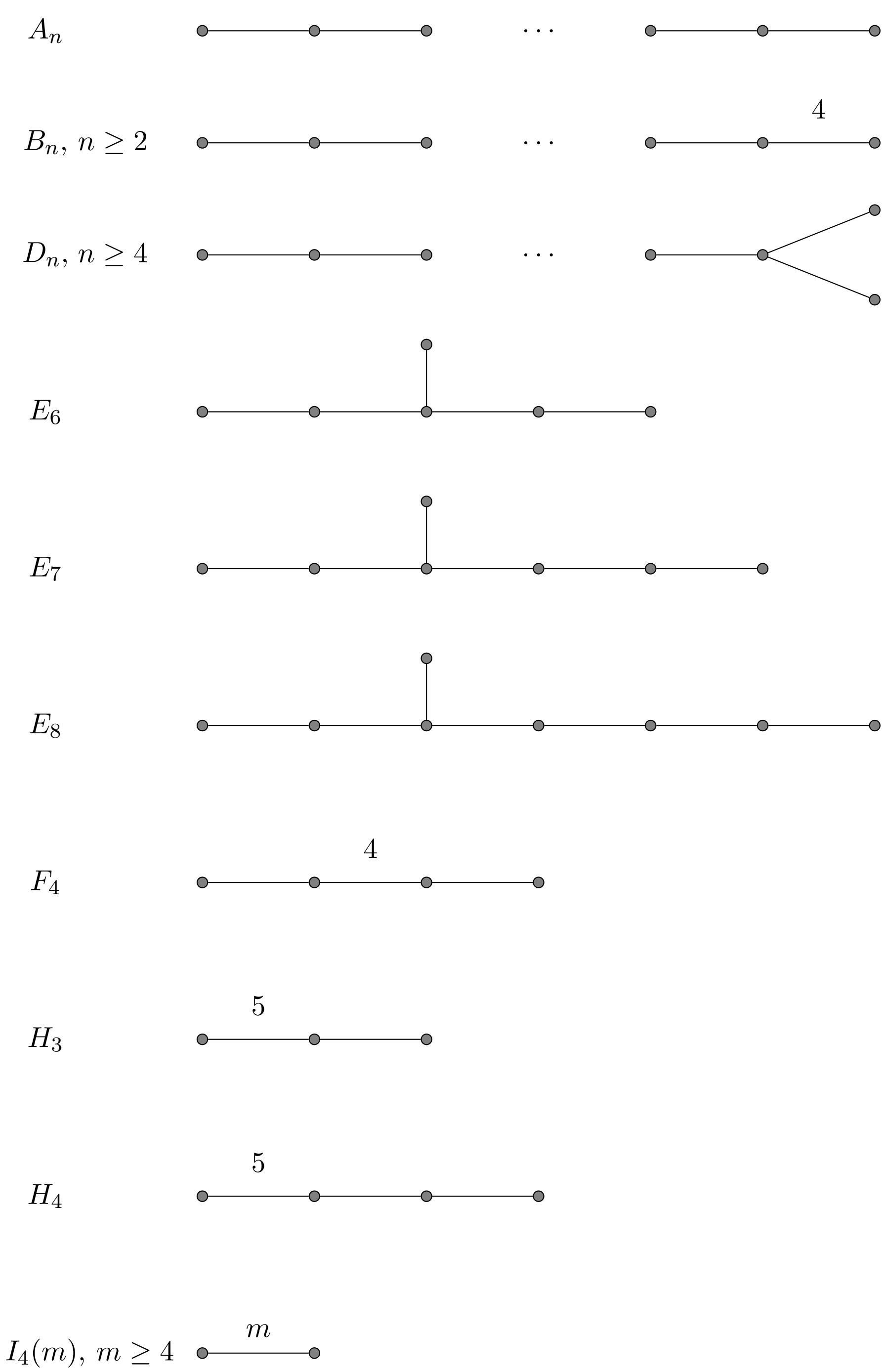


Figure 3.1: Coxeter diagrams of finite Coxeter systems

is called reflection length function *and* $\ell_T(w)$ *is called* reflection length *of* w. *Analogously, let*

$$\ell_S : W \longrightarrow \mathbb{N}_0,\ w \longmapsto \min\{m \in \mathbb{N}_0 \mid t_1 \cdots t_m = w,\ t_i \in S\}.$$

The function ℓ_S *is called* length function *and* $\ell_S(w)$ *is the* length *of* w. *By convention* $\ell_S(e) = 0 = \ell_T(e)$, *where* e *is the identity of* W. *Denote by* $\leq_T$ *the binary relation on* W *defined as follows. For* $v, w \in W$

$$v \leq_T w \Longleftrightarrow \ell_T(w) = \ell_T(v) + \ell_T(v^{-1}w).$$

In other words, we have $v \leq_T w$ *if and only if* v *is a prefix of a reduced reflection factorization of* w.

It will turn out that Coxeter groups are reflection groups and therefore the definition of the function ℓ_T from above coincides with Definition 2.2.1.

The next proposition defines the so-called sign representation and states direct consequences of its existence. The same results can be deduced from the standard representation, that is defined later, with use of Proposition 2.1.2 and the determinant map.

Proposition 3.1.7. [58, Proposition 5.1] *Let* (W, S) *be a Coxeter system. There is a unique epimorphism* $W \longrightarrow \{\pm 1\}$ *sending each generator* $s \in S$ *to* -1. *In particular, every reflection factorization of an element has the same parity.*

Proof. The assignment $S \longrightarrow \{\pm 1\},\ s \longmapsto -1$ can be continued linearly to W since it satisfies the defining relations stated in Definition 3.1.1. Let $w \in W$ be an arbitrary element. The parities of any factorization in simple generators of w is the same and since arbitrary reflections have factorizations in simple generators with an odd number of factors also the parities of arbitrary reflection factorizations of w coincides. □

The next lemma is standard and follows directly from the definitions of the length functions.

Lemma 3.1.8. *For any* $w \in W$ *holds*

(a) $\ell_S(ws) = \ell_S(w) \pm 1$ *and* $\ell_S(sw) = \ell_S(w) \pm 1$ *for all* $s \in S$,
(b) $\ell_T(wt) = \ell_T(w) \pm 1$ *and* $\ell_T(tw) = \ell_T(w) \pm 1$ *for all* $t \in T$ *and*
(c) ℓ_T *is invariant under conjugacy with elements of* W.

Coxeter groups are characterized by the following three equivalent well-known properties, see [1, Theorem 2.49].

Theorem (Exchange condition). *Let* W *be a group with generating set* S *consisting of involutions. Then* (W, S) *is a Coxeter system if and only if the following condition for all* $w \in W$ *holds. If* $w = s_1 \cdots s_m$ $(s_i \in S)$ *with* $\ell_S(ws) < \ell_S(w)$ *for some* $s \in S$, *then there exists* $1 \leq i \leq m$ *such that* $ws = s_1 \cdots \widehat{s_i} \cdots s_m$, *i.e. the i-th entry is omitted.*

Theorem (Strong exchange condition). *Let W be a group with generating set S consisting of involutions and $T = \bigcup_{w\in W} wSw^{-1}$. Then (W,S) is a Coxeter system if and only if the following condition holds for all $w \in W$. If $w = s_1 \cdots s_m$ ($s_i \in S$) with $\ell_S(wt) < \ell_S(w)$ for some $t \in T$, then there exists $1 \leq i \leq m$ such that $wt = s_1 \cdots \widehat{s_i} \cdots s_m$.*

Theorem (Deletion condition). *Let W be a group with generating set S consisting of involutions. Then (W,S) is a Coxeter system if and only if the following condition for all $w \in W$ holds. If $w = s_1 \dots s_m$ ($s_i \in S$) with $\ell_S(w) < m$, then there exists $1 \leq i < j \leq m$ such that $w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_m$..*

A further useful combinatorial property of Coxeter groups is the following result due to Matsumoto, called Matsumoto property (see [78]) and can also be found in [104]. It gives rise to a solution of the word problem in Coxeter groups.

Theorem 3.1.9. [104] *Let (W,S) be a Coxeter system.*

(a) A word in letters of S is reduced if and only if it can not be shortened by a finite sequence of braid-relations and deletion of subwords of the form (s,s).
(b) Two reduced words in letters of S represent the same element in W if and only if they can be transformed in each other by using braid relations.

Tits introduced for any Coxeter system (W,S) a faithful linear representation for W. We call it standard representation and recall its construction. For a detailed discussion see for example [58, Section 5.3].

Let (W,S) be a Coxeter system of finite rank and V be the real vector space generated by the basis $\{e_s \mid s \in S\}$. Next we impose a bilinear on V that is compatible with m_{st} for $s,t \in S$. For $m_{st} < \infty$ set

$$(e_s, e_t) = -\cos\left(\frac{\pi}{m_{st}}\right)$$

and for $m_{st} = \infty$ we interpret (e_s, e_t) to be -1. For each anisotropic vector $v \in V$ we define as in Example 2.1.3 the reflection s_v. By Proposition 2.1.2 its reflection hyperplane is $v^{\perp}$ and s_v is of order 2.

Theorem 3.1.10. [58, Corollary 5.4] *Let (W,S) be a Coxeter system of finite rank and V the corresponding vector space with basis $\{e_s \mid s \in S\}$. Then the assignement*

$$S \longrightarrow GL(V), \ s \longmapsto s_{e_s}$$

can be linearly extended to W and yields a faithful representation of W.

A direct consequence of the previous theorem is that any Coxeter group can be identified with a real reflection group in sense of Definition 2.1.1. By [58, Theorem 6.4] a Coxeter system is finite if and only if the corresponding bilinear form $(-,-)$ is positive definite. If it is positive semidefinite but not positive definite the group W is called affine Coxeter group.

The next lemma attaches to a Coxeter system (W, S) a root system Φ, whose elements encodes useful information on the reflections of the Coxeter system. Concretely, for each reflection there exists exactly two roots α, α' such that $t = s_\alpha = s_{\alpha'}$. For this we identify W with the reflection group described in Theorem 3.1.10, i.e. with its image under the standard representation.

Lemma 3.1.11. *Let (W, S) be a Coxeter system and $\Phi := W(\{e_s \mid s \in S\})$, then Φ is a reduced and simply-laced root system.*

It immediately follows that $W = W(\Phi)$ and $T = \bigcup_{w \in W} wSw^{-1} = \{s_\alpha \mid \alpha \in \Phi\}$. By [1, Proposition 2.69] a Coxeter group is finite if and only if the corresponding root system has finite cardinality, which is the same as saying that the set of reflections T is finite. To give a useful characterization of ℓ_S we need to introduce the concept of positive and negative roots.

Definition 3.1.12. *Let (W, S) be a Coxeter system and Φ its attached root system. The set $\Phi^+ := span_{\mathbb{R}_{\geq 0}}(e_s \mid s \in S) \cap \Phi$ is the set of* positive roots *and $\Phi^- := -\Phi^+$ the set of* negative roots.

Lemma 3.1.13. [58, Chapter 5.4] *Let (W, S) be a Coxeter system and Φ its attached root system, we have $\Phi = \Phi^+ \sqcup \Phi^-$.*

Theorem 3.1.14. [58, Theorem 5.4, Proposition 5.6] *Let (W, S) be a Coxeter system, Φ its attached root system and $w \in W$.*

(a) Let $s \in S$. It holds $\ell_S(ws) < \ell_S(w)$ if and only if $w(e_s) \in \Phi^-$.

(b) The integer $\ell_S(w)$ is the number of positive roots sent by w to negative roots. Explicitly, if $w = s_1 \cdots s_m$ with $\ell_S(w) = m$ these roots are $\{\alpha_m, s_m(\alpha_{m-1}), \ldots, s_m \cdots s_2(\alpha_1)\}$, where $s_{\alpha_i} = s_i$ and $\alpha_i \in \Phi^+$. Moreover, we have $N(w) := \{t \in T \mid \ell_S(wt) < \ell_S(w)\} = \{s_m, s_m s_{m-1} s_m, \ldots, s_m \cdots s_2 s_1 s_2 \cdots s_m\}$.

Important subgroups of Coxeter groups are the so-called parabolic subgroups.

Definition 3.1.15. *Let (W, S) be a Coxeter system and $I \subseteq S$. The group $W_I = \langle I \rangle$ is called* standard parabolic subgroup *and conjugates of $\langle I \rangle$ are called* parabolic subgroup.

The next proposition summarize some of the important facts of parabolic subgroups.

Proposition 3.1.16. *(*[58, Theorem 5.5], [63, Section 5-2], [91] *and* [105]*) Let (W, S) be a Coxeter system and $I \subseteq S$.*

(a) The pair $\left(w\langle I \rangle w^{-1}, wIw^{-1}\right)$ is a Coxeter system for all $w \in W$.

(b) The length functions ℓ_S and ℓ_I coincide on W_I. Moreover, every S-reduced expression of an element of W_I consists only of generators of I.

(c) If W is finite with root system Φ and $V := \operatorname{span}_{\mathbb{R}}(\Phi)$, the parabolic subgroups are

exactly the elementwise stabilizer

$$C_W(E) = \{w \in W \mid w(v) = v \text{ for all } v \in E\},$$

where $E \subseteq V$. *Moreover, the rank of the Coxeter group* $C_W(E)$ *is the codimension of* $\mathrm{span}_{\mathbb{R}}(E)$ *in* V.

(d) *The intersection of parabolic subgroups is itself a parabolic subgroup, thus the notion of the* parabolic closure *is well-defined.*

3.2 The canonical simple system of a reflection subgroup

In this section we summarize well-known facts of the so-called canonical system of reflection subgroups of Coxeter groups of finite rank and its connection to the Bruhat graph and Hurwitz action. It bases mainly on the thesis [36] of Matthew Dyer and can partially also be found in the work of Paul Moszkowski [86] and Vinay Deodhar [31]. The general results dealing with the canonical system of a reflection subgroup can also be found in [37] and the statements concerning the Bruhat graph are published in [38]. Throughout this section let (W, S) be a Coxeter system of finite rank with set of reflections T and for any $w \in W$ let $N(w) = \{t \in T \mid \ell_S(wt) < \ell(w)\}$.

The next Lemma is crucial in understanding reflection subgroups, i.e. subgroups of Coxeter groups that are generated by reflections. Because of its plainness the proof is taken from [86], where we corrected several typos to make it understandable.

Theorem 3.2.1. *(*[86, Proposition 6]*) Let* $T' \subseteq T$ *be a set of reflections, then the group* $\langle T' \rangle$ *is a Coxeter group with simple system*

$$S' := \chi(\langle T' \rangle) = \{t \in \langle T' \rangle \cap T \mid N(t) \cap \langle T' \rangle = \{t\}\}.$$

Moreover, if $\ell_{S'}$ *denotes the length function with respect to* S', *for all* $t \in \langle T' \rangle \cap T$, *for all* $w \in \langle T' \rangle$ *we have* $\ell_S(wt) < \ell_S(w)$ *if and only if* $\ell_{S'}(wt) < \ell_{S'}(w)$. *The set of reflections of* $\langle S' \rangle$ *with respect to* S' *is* $\langle S' \rangle \cap T = \langle T' \rangle \cap T$.

Proof. Let t be a reflection in $\langle T' \rangle$ which is not in S' and let $t = s_1 \cdots s_p \cdots s_1$ be an expression of t in simple generators of S. Since $t \notin S'$ there exists a reflection $t' \in \langle S' \rangle \cap T$ such that $\ell_S(tt') < \ell_S(t)$ and therefore by the strong exchange condition there exists $1 \leq i \leq p$ such that $tt' = s_1 \cdots \widehat{s_i} \cdots s_p \cdots s_1$ or $tt' = s_1 \cdots s_p \cdots \widehat{s_i} \cdots s_1$. In both cases, t is generated by $s_1 \cdots s_i \cdots s_1$ and $s_1 \cdots \widehat{s_i} \cdots s_p \cdots \widehat{s_i} \cdots s_1$, which are both in $\langle T' \rangle \cap T$. As these expressions are shorter (although not necessarily reduced), by iterating it follows that $\langle S' \rangle$ contains $\langle T' \rangle \cap T$, i.e. we have $\langle S' \rangle = \langle T' \rangle$.

Let $w \in \langle S' \rangle$ and $t \in \langle T' \rangle \cap T$ such that $\ell_S(wt) < \ell_S(w)$ and let $w = r_1 \dots r_n$ be a reduced expression with respect to S'. For $1 \leq i \leq n$ we put $r_i = s_{i_1} \cdots s_{i_{p_i}}$ with $s_{i_k} \in S$ and

$1 \leq k \leq p_i$ and $\ell_S(r_i) = p_i$. According to the strong exchange condition, we have for some $1 \leq i \leq n$ and some $1 \leq k \leq p_i$ $wt = r_1 \cdots r_{i-1} s_{i_1} \cdots \widehat{s_{i_k}} \cdots s_{i_{p_i}} r_{i+1} \cdots r_n$. We have $r = s_{i_{p_i}} \cdots s_{i_k} \cdots s_{i_{p_i}} \in \langle S' \rangle$ as $wt = r_1 \cdots r_{i-1} r_i r r_{i+1} \cdots r_n \in \langle S' \rangle$. If $r \neq r_i$, then $r_i \notin S'$. It follows $wt = r_1 \cdots \widehat{r_i} \cdots r_n$, and so $\ell_{S'}(wt) \leq \ell_{S'}(w)$. If $\ell_S(wt) \geq \ell_S(w)$, then by Lemma 3.1.7 $\ell_S(wt) > \ell_S(w)$ and thus $\ell_S(wtt) < \ell_S(wt)$. So we have $\ell_S(wt) < \ell_S(w)$ if and only if $\ell_{S'}(wt) < \ell_{S'}(w)$ for all $w \in \langle S' \rangle$ and all $t \in \langle S' \rangle \cap T$. Now it is easy to see that the group $\langle S' \rangle$ with generating set S' satisfies the strong exchange condition and hence the pair $(\langle S' \rangle, S')$ is a Coxeter system.

Let $t \in \langle S' \rangle \cap T$ with $t = r_1 \ldots r_n$ and $r_i \in S'$ for all $1 \leq i \leq n$. Since we have $\ell_{S'}(tt) < \ell_{S'}(t)$ the strong exchange condition implies $t = r_n \cdots r_i \cdots r_n$ for some $1 \leq i \leq n$, and so the set of reflections with respect to S' is $\langle S' \rangle \cap T = \langle T' \rangle \cap T$. □

We call the generating set $\chi(\langle T' \rangle)$ attached to the reflection subgroup $\langle T' \rangle$ for $T' \subseteq T$ and given simple system S the canonical generating system and its elements canonical generators. It is easy to see that for any $I \subseteq S$ we have $\chi(\langle I \rangle) = I$ and for any $T' \subseteq T$ with $s \in S \cap \langle T' \rangle$ we get $s \in \chi(\langle T' \rangle)$.

Next we prove important properties of the canonical generating system of a reflection subgroup. For that we need the following two well-known properties of Coxeter groups.

Lemma 3.2.2. [36, 1.3 Lemma] *Let (W, S) be a Coxeter system and $v, w \in W$, then $N(vw) = w^{-1}N(v)w \;\Delta\; N(w)$, where Δ is the symmetric difference of sets.*

Proposition 3.2.3. [36, Lemma 1.4] *Let (W, S) be a Coxeter system and $t = s_1 \ldots s_{2p+1}$ a reduced factorization of a reflection in simple generators, then $t = s_{2p+1} \cdots s_{p+1} \cdots s_{2p+1}$.*

Proof. Since $\ell_S(tt) < \ell_S(t)$, the strong exchange condition implies that $t = s_{2p+1} \cdots s_j \cdots s_{2p+1}$ for some $1 \leq j \leq 2p+1$. From $e = s_1 \cdots \widehat{s_j} \cdots s_{2p+1}$ follows $s_1 \cdots s_{j-1} = s_{2p+1} \cdots s_{j+1}$. Since both sides of the latter equation are reduced in terms of S we get $j = p + 1$. □

The proof of the first part of the following proposition is taken from [37]. It can also be found with an alternative proof based on root systems of Coxeter groups in [36].

Proposition 3.2.4. *(*[36, Remark 3.14, Corollary 3.16], [37, Proposition 3.5]*)*

(a) Let $T' \subseteq T$, then $N(t) \cap \langle T' \rangle = \{t\}$ for all $t \in T'$ if and only $N(t) \cap \langle t, t' \rangle = \{t\}$ for all $t, t' \in T'$.

(b) Let $T' \subseteq T$ and $s \in S$, then $\chi(s\langle T' \rangle s) = \begin{cases} s\chi(\langle T' \rangle)s & , s \notin \chi(\langle T' \rangle), \\ \chi(\langle T' \rangle) & , s \in \chi(\langle T' \rangle). \end{cases}$

(c) Let $T' \subseteq T$, then $|\chi(\langle T' \rangle)| \leq |T'|$.

Proof. (a) From $N(t) \cap \langle T' \rangle = \{t\}$ for all $t \in T'$ it immediately follows $N(t) \cap \langle t, t' \rangle = \{t\}$ for all $t, t' \in T'$. The remaining implication holds obviously if we have shown $\chi(\langle T' \rangle) \subseteq T'$.

Let $W' = \langle T' \rangle$ with canonical simple system $S' = \chi(W')$ and denote by $\ell_{T'}$ (resp. $\ell_{S'}$) the length function corresponding to the generating set T' (resp. S'). First we will show by induction on $\ell_{T'}(w)$ that for $w \in W'$ and $t \in T'$ we have that $\ell_{T'}(wt) < \ell_{T'}(w)$ (resp. $\ell_{T'}(wt) > \ell_{T'}(w)$) implies $\ell_{S'}(wt) < \ell_{S'}(w)$ (resp. $\ell_{S'}(wt) > \ell_{S'}(w)$). If $\ell_{T'}(w) = 0$ then the implications clearly hold. Let $\ell_{T'}(w) > 0$ and assume first that $\ell_{T'}(wt) < \ell_{T'}(w)$. Then $\ell_{T'}(wt) < \ell_{T'}((wt)t)$ and the induction hypothesis yields the inequality $\ell_{S'}(wt) < \ell_{S'}(w)$. Consider now the case $\ell_{T'}(wt) > \ell_{T'}(w)$ and let $r \in T'$ such that $\ell_{T'}(wr) < \ell_{T'}(w)$. Choose $x \in \langle r, t \rangle$ maximal in terms of $\ell_{T'}$ such that $w = w'x$ with $\ell_{T'}(w) = \ell_{T'}(w') + \ell_{T'}(x)$ and $\ell_{T'}(xr) < \ell_{T'}(x)$. In particular, $r \neq t$. By the maximality of x it follows that $\ell_{T'}(w'r) > \ell_{T'}(w')$ and $\ell_{T'}(w't) > \ell_{T'}(w')$. By induction hypothesis we get $\ell_{S'}(w'r) > \ell_{S'}(w')$ and $\ell_{S'}(w't) > \ell_{S'}(w')$. Thus w' is minimal in the coset $w\langle r, t \rangle = w'\langle r, t \rangle$ with respect to $\ell_{S''}$, where by prerequisite $S'' = \chi(\langle r, t \rangle) = \{r, t\}$ is the canonical generating set of $\langle r, t \rangle$. Thus we get $N(w') \cap \langle r, t \rangle = \emptyset$, and in particular, $xtx^{-1} \notin N(w')$.

Since we have $\ell_{T'}(wt) > \ell_{T'}(w)$ the factorization $w = w'x$ implies $\ell_{S''}(xt) > \ell_{S''}(x)$ and therefore $t \in N(xt)$. By Lemma 3.2.2 we get

$$N(wt) = N(w'xt) = tx^{-1}N(w')xt \;\Delta\; N(xt).$$

Previous calculations show that $t \notin tx^{-1}N(w')xt$, but $t \in N(xt)$. Hence, all together, $t \in N(wt) \cap \langle T' \rangle$ and therefore $\ell_{S'}(wt) > \ell_{S'}(w)$.

Now let $r \in S'$ and $t \in T'$ with $\ell_{T'}(rt) < \ell_{T'}(r)$. Then we get $\ell_{S'}(rt) < \ell_{S'}(r) = 1$ and thus $r = t \in T'$.

(b) If $s \in S$, then $s\langle T' \rangle s = \langle T' \rangle$ and therefore $\chi(\langle T' \rangle) = \chi(s\langle T' \rangle s)$. Assume that $s \notin \chi(\langle T' \rangle)$ and $\chi(\langle T' \rangle) = \{r_1, \ldots, r_m\}$, then we need to show that $\chi(s\langle T' \rangle s) = s\{r_1, \ldots, r_m\}s$. By part (a) of this proposition it is sufficient to show that $N(sr_is) \cap s\langle r_i, r_j \rangle s = \{sr_is\}$ for all $1 \leq i, j \leq m$. By Lemma 3.2.2 we get with $N(s) = \{s\}$

$$N(sr_is) = sr_i\{s\}r_is \;\Delta\; sN(r_i)s \;\Delta\; \{s\}.$$

Since $s \notin \chi(\langle T' \rangle)$ we have $s \notin \langle T' \rangle$ and thus $sr_isr_is \notin s\langle T' \rangle s$. Hence

$$\begin{aligned} N(sr_is) \cap \langle sr_is, sr_js \rangle &\subseteq \big(sr_i\{s\}r_is \cap s\langle T' \rangle s\big) \;\Delta\; \big(sN(r_i)s \cap s\langle T' \rangle s\big) \;\Delta\; \big(\{s\} \cap s\langle T' \rangle s\big) \\ &= sN(r_i)s \cap s\langle T' \rangle s \\ &= \{sr_is\}. \end{aligned}$$

Therefore we have $N(sr_is) \cap \langle sr_is, sr_js \rangle = \{sr_is\}$.

(c) If $|T'|$ is infinite, the statement follows by a standard cardinality argument. For $|T'|$ finite the assertion follows from the algorithm we will describe next. It is based on (a) and the easy fact that the number of canonical generators of a reflection subgroup generated by two different reflections is two (see [36, 1.15 Lemma]). The algorithm determines a canonical simple system of the reflection subgroup $\langle T' \rangle$. Let $T_0 = T'$ and define T_i for $i > 0$ as follows. If $\chi(\langle t, t' \rangle) = \{t, t'\}$ for all $t \neq t' \in T_i$, then define $T_i = T_{i+1}$. Otherwise choose $t, t' \in T_i$ with $t \neq t'$ and $\chi(\langle t, t' \rangle) \neq \{t, t'\}$, and define $T_{i+1} = (T_i \setminus \{t, t'\}) \cup \chi(\langle t, t' \rangle)$. The sequence $(T_i)_{i \in \mathbb{N}_0}$ has to stabilze, since for $T_i \neq T_{i+1}$ it holds $\sum_{t \in T_{i+1}} \ell_S(t) < \sum_{t \in T_i} \ell_S(t)$. Then (a) yields that $\chi(\langle T' \rangle) = T_i$ for $i \gg 0$. Since $|T_0| \geq |T_1| \geq \ldots \geq |T_i| = |\chi(\langle T' \rangle)|$ for $i \gg 0$ it follows $|\chi(\langle T' \rangle)| \leq |T'|$.

□

A generalization of Proposition 3.2.4 (b) can be found in [40, Lemma 1].

An important tool in understanding the Hurwitz action is the so-called Bruhat graph. At the end of this section we will explain the connection to the Hurwitz action. It is investigated by Barbara Baumeister, Matthew Dyer, Christian Stump and Patrick Wegener in [8], and their results are based on [36].

Definition 3.2.5. *The* Bruhat graph *of* (W, S) *is the directed graph* $\Omega_{(W,S)}$ *with vertex set* W *and there is a directed edge from* x *to* y *if there exists* $t \in T$ *such that* $y = xt$ *and* $\ell_S(x) < \ell_S(y)$. *For* $W' \subseteq W$ *denote by* $\Omega_{(W,S)}(W')$ *the full subgraph of* $\Omega_{(W,S)}$ *with vertex set* W'.

As a direct consequence of the properties of the length attached to the canonical generating set described in Theorem 3.2.1 we get the following result.

Proposition 3.2.6. [36, Proposition 1.13 (i)] *Let* W' *be a reflection subgroup of* W, *then* $\Omega_{(W',\chi(W'))} = \Omega_{(W,S)}(W')$.

Proof. Let $T' = T \cap W'$ be the set of reflections attached to (W', S'), where we set $S' = \chi(W')$. Since the vertices of $\Omega_{(W',S')}$ and $\Omega_{(W,S)}(W')$ coincide we need to prove that the graphs have the same directed edges. The set of edges of $\Omega_{(W,S)}(W')$ is

$$
\begin{aligned}
&\{(x, y) \in W' \mid xy^{-1} \in T \text{ with } \ell_S(x) < \ell_S(y)\} \\
=&\{(x, y) \in W' \mid xy^{-1} \in T' \text{ with } \ell_S(x) < \ell_S(y)\} \\
=&\{(x, y) \in W' \mid xy^{-1} \in T' \text{ with } \ell_{S'}(x) < \ell_{S'}(y)\}.
\end{aligned}
$$

But the latter set is also the set of edges of the graph $\Omega_{(W',S')}$. □

Proposition 3.2.7. [36, Proposition 1.13 (ii)] *Let* W' *be the reflection subgroup of* W *generated by* $T' \subseteq T$, $S' = \chi(W')$ *its canonical set of generators and* $x \in W$, *then the Bruhat graph* $\Omega_{(W',S')}$ *and the graph* $\Omega_{(W,S)}(xW')$ *are isomorphic.*

Proof. By Theorem 3.2.1 we have for all $w \in W'$ and $t \in T \cap W'$ $\ell_S(wt) < \ell_S(w)$ if and only if $\ell_{S'}(wt) < \ell_{S'}(w)$. Therefore we get for any $w \in W'$

$$\{t \in T' \mid \ell_{S'}(wt) < \ell_{S'}(w)\} = \{t \in T \mid \ell_S(wt) < \ell_S(w)\} \cap W' = N(w) \cap W'.$$

Let $x_0 \in xW'$ with $\ell_S(x_0)$ minimal. Because of the minimality of $\ell_S(x_0)$ it holds $\ell_S(x_0t) > \ell_S(x_0)$ for all $t \in T'$ and hence $N(x_0) \cap W' = \{t \in T' \mid \ell_{S'}(x_0t) < \ell_{S'}(x_0)\} = \emptyset$. The latter yields with Lemma 3.2.2 for any $w \in W'$

$$\begin{aligned} N(x_0w) \cap W' &= \left(wN(x_0)w^{-1} \cap W'\right) \ \Delta \ \left(N(w) \cap W'\right) \\ &= N(w) \cap W' \\ &= \{t \in T' \mid \ell_{S'}(wt) < \ell_{S'}(w)\}. \end{aligned}$$

The bijection $W' \longrightarrow xW'$; $w \longmapsto x_0w$ induces a bijection between the vertices of $\Omega_{(W',S')}$ and those of $\Omega_{(W,S)}(xW')$. Thus we just need to check that their edges coincide. Let $y, z \in W'$ with $y^{-1}z \in T'$. The pair (y, z) is an edge in $\Omega_{(W',S')}$ if and only if $\ell_{S'}(y) < \ell_{S'}(z)$. The latter is equivalent to $(x_0y)^{-1}(x_0z) = y^{-1}z \notin N(y) \cap W' = N(x_0y) \cap W'$ and this is the same as $\ell_S(x_0y) < \ell_S(x_0z)$, i.e. (y, z) is an edge in $\Omega_{(W',S')}$ if and only if (x_0y, x_0z) is an edge in $\Omega_{(W,S)}(xW')$. □

The following statement is the key result of [8] and yields the promised connection to the Hurwitz action.

Proposition 3.2.8. *([8], [107, Proposition 2.3.6]) Let (W, S) be a Coxeter system of finite rank, $w \in W$ and $t_1, t_2 \in T$ with $t_1 \neq t_2$ such that*

$$w \longrightarrow wt_1 \longleftarrow wt_1t_2$$

in $\Omega_{(W,S)}$. Then there exist $(t_1', t_2') \in B_2(t_1, t_2)$ such that one of the following cases hold:

(a) $w \longrightarrow wt_1' \longrightarrow wt_1't_2' = wt_1t_2$

(b) $w \longleftarrow wt_1' \longleftarrow wt_1't_2' = wt_1t_2$

(c) $w \longleftarrow wt_1' \longrightarrow wt_1't_2' = wt_1t_2$

In particular, in all three cases we have $\ell_S(wt_1') < \ell_S(wt_1)$.

Proof. Let $W' = \langle t_1, t_2 \rangle$ and $S' = \chi(W')$. Since $t_1 \neq t_2$ Proposition 3.2.4 (c) yields that (W', S') is a Coxeter system of rank two. The isomorphisms of the Propositions 3.2.6 and 3.2.7

$$\Omega_{(W,S)}(wW') \cong \Omega_{(W,S)}(W') \cong \Omega_{(W',\chi(W'))} \tag{3.1}$$

map $w \longrightarrow wt_1 \longleftarrow wt_1t_2$ in $\Omega_{(W,S)}$ to $x \longrightarrow xt_1 \longleftarrow xt_1t_2$ in $\Omega_{(W',S')}$ for some $x \in W'$. If

$x = e$, we choose an arbitrary $t'_1 \in S'$. Since $t_1 \neq t_2$ we get with $t'_2 = t'_1 t_1 t_2 \in T \cap \langle t_1, t_2 \rangle$

$$x \longrightarrow xt'_1 \longrightarrow xt'_1 t'_2.$$

If $x \neq e$, there exists $t'_1 \in S'$ such that we have either

$$x \longleftarrow xt'_1 \longleftarrow xt'_1 t'_2 \quad \text{or} \quad x \longleftarrow xt'_1 \longrightarrow xt'_1 t'_2$$

for $t'_2 = t'_1 t_1 t_2 \in T \cap \langle t_1, t_2 \rangle$. Thus the isomorphisms of (3.1) yield one of the paths of $\Omega_{(W,S)}(wW')$ described in (a), (b) and (c). Moreover, Lemma 2.2.4 implies that (t_1, t_2) and (t'_1, t'_2) lie in the same Hurwitz orbit.

Next we compare the length of wt'_1 with the length of wt_1. In the first case (a) we have

$$\ell_S(wt'_1) < \ell_S(wt'_1 t'_2) = \ell_S(wt_1 t_2) < \ell_S(wt_1)$$

and in the other two cases (b) and (c) hold

$$\ell_S(wt'_1) < \ell_S(w) < \ell_S(wt_1).$$

□

Next we prove a statement that extends the connection between the Hurwitz action and the Bruhat graph of the previous proposition. We call it the ordering procedure and it is essentially [8, Proposition 2.2], where we relax its prerequisites.

Proposition 3.2.9. *[Ordering procedure] Let (W, S) be a Coxeter system of finite rank, $x, w \in W$ and $w = t_1 \cdots t_m$ a reflection factorization such that each factorization of $B_m(t_1, \ldots, t_m)$ consists of pairwise different factors. Then there exists a factorization $(t'_1, \ldots, t'_m) \in B_m(t_1, \ldots, t_m)$ such that the corresponding path in the Bruhat graph starting in x and ending in xw is first decreasing and then increasing. More precisely, we have*

$$x \longleftarrow xt'_1 \longleftarrow \ldots \longleftarrow xt'_1 \cdots t'_i \longrightarrow \ldots \longrightarrow xt'_1 \cdots t'_m = xw$$

for a unique $0 \leq i \leq m$.

Proof. Consider the undirected path in $\Omega_{(W,S)}$ corresponding to the reflection factorization $(t_1, \ldots, t_m)$ of $w \in W$

$$x - xt_1 - xt_1 t_2 - \ldots - xt_1 \ldots t_n = xw.$$

Since every factorization of $B_n(t_1, \ldots, t_n)$ contains pairwise different reflections, the Proposition 3.2.8 allows us to change parts of the associated directed path of shape $\star \longrightarrow \star \longleftarrow \star$ to

$$\star \longrightarrow \star \longrightarrow \star, \quad \star \longleftarrow \star \longleftarrow \star \quad \text{or} \quad \star \longleftarrow \star \longrightarrow \star$$

only using the Hurwitz action. Also by Proposition 3.2.8 each replacement reduces the sum of the length of the vertices. Hence we arrive after finitely many steps to a path that is first decreasing and then increasing. □

The next lemma is crucial in the proof of the main theorem of the next section. It is a consequence of the previous result and allows us to control the last factors of a reflection factorization of sufficient length.

Lemma 3.2.10. *Let $w \in W$ with $\ell_S(w) = m$ and $w = t_1 \cdots t_{m+2k}$ with $t_i \in T$ for $1 \leq i \leq m+2k$ and some $k \in \mathbb{Z}_{\geq 0}$. Then there exists a braid $\sigma \in \mathcal{B}_{m+2k}$ and reflections $r_1, \ldots, r_m, r_{i_1}, \ldots, r_{i_k}$ such that*

$$\sigma(t_1, \ldots, t_{m+2k}) = (r_1, \ldots, r_m, r_{i_1}, r_{i_1}, \ldots, r_{i_k}, r_{i_k}).$$

Proof. We proceed by induction on k. The case $k = 0$ is trivially satisfied. Therefore let $k \geq 1$ and assume that all factorizations in $\mathcal{B}_{m+2k}(t_1, \ldots, t_{m+2k})$ consist of pairwise different factors. Consider the path of $\Omega_{(W,S)}$ starting in e and ending in w induced by $(t_1, \ldots, t_{m+2k})$. Then the ordering procedure of Proposition 3.2.9 yields a factorization $(t'_1, \ldots, t'_{m+2k}) \in B_{m+2k}(t_1, \ldots, t_{m+2k})$ such that the corresponding path in the Bruhat graph is

$$e \longleftarrow t'_1 \longleftarrow t'_1 t'_2 \longleftarrow \ldots \longleftarrow t'_1 t'_2 \cdots t'_p \longrightarrow t'_1 t'_2 \cdots t'_p t'_{p+1} \longrightarrow \ldots \longrightarrow t'_1 \cdots t'_{m+2k} = w,$$

that is, the path is first decreasing, then increasing. Since the path starts with e, it holds $p = 0$ and therefore it has no decreasing part. Altogether, we have that

$$e \longrightarrow t'_1 \longrightarrow t'_1 t'_2 \longrightarrow \ldots \longrightarrow t'_1 \cdots t'_{m+2k} = w.$$

Since the length of w is m and $k \geq 1$, there cannot be an increasing path of length $m + 2k$, so we arrive at a contradiction. Thus there exists a factorization $(t'_1, \ldots, t'_{m+2(k-1)}, r_{i_k}, r_{i_k})$ in $\mathcal{B}_{m+2k}(t_1, \ldots, t_{m+2k})$. From the induction hypotheses follows

$$(t'_1, \ldots, t'_{m+2(k-1)}) \sim (r_1, \ldots, r_m, r_{i_1}, r_{i_1}, \ldots, r_{i_{k-1}}, r_{i_{k-1}})$$

and the latter yields the assumption. □

For finite Coxeter groups Lewis and Reiner prove in [74, Corollary 1.4] an even stronger statement of the previous lemma based on the classification of finite Coxeter groups that we state in Theorem 3.1.3. We state a uniform proof of their result in Proposition 4.1.4.

Another useful result is the following, which states that the reflections of a reduced factorization of an element are always contained in the parabolic closure of this element. The proof is a direct consequence of the ordering procedure of Proposition 3.2.9.

Theorem 3.2.11. [8, Theorem 1.4] *Let W' be a parabolic subgroup of W. Then $Red_T(w) = Red_{T'}(w)$ for any $w \in W'$, where $T' = T \cap W'$ is the set of reflections in W'.*

Proof. Without loss of generality we assume that W' is a standard parabolic subgroup of W and let $(t_1, \dots, t_m) \in \mathrm{Red}_T(w)$. By the ordering procedure there exists a braid $\tau \in \mathcal{B}_m$ such that $(t'_1, \dots, t'_m) = \tau(t_1, \dots, t_m)$ corresponds to a directed path in $\Omega_{(W,S)}$. Since W' is a standard parabolic subgroup the strong exchange condition yields that $t'_i \in W' \cap T$ for all $1 \le i \le m$ and thus also $(t_1, \dots, t_m) \in \mathrm{Red}_{T'}(w)$. Altogether, we have $\mathrm{Red}_T(w) \subseteq \mathrm{Red}_{T'}(w)$ and since $T' \subseteq T$ it also holds $\mathrm{Red}_T(w) = \mathrm{Red}_{T'}(w)$. □

Another consequence of the ordering procedure is the following result. As before let $N(w) = \{t \in T \mid \ell_S(wt) < \ell_S(w)\}$ for $w \in W$.

Lemma 3.2.12. *Let $w \in W$ and $(t_1, \dots, t_n) \in Red_T(w)$, then there exists a braid $\tau \in B_n$ such that $\{t'_1, \dots, t'_n\} \subseteq N(w)$, where $(t'_1, \dots, t'_n) := \tau(t_1, \dots, t_n)$. Moreover, if $w = xy$ with $\ell_S(w) = \ell_S(x) + \ell_S(y)$, then there exists $0 \le r \le n$ such that $t'_{r+1}, \dots, t'_n \in y^{-1}N(x)y$ and $t'_1, \dots, t'_r \in N(y)$.*

Proof. Let $w = s_1 \cdots s_m$ be a $S-$reduced factorization. Hence we have by Theorem 3.1.14 (b)

$$N(w) = \{t \in T \mid \ell_S(wt) < \ell_S(w)\} = \{s_m, s_m s_{m-1} s_m, \dots, s_m \cdots s_1 \cdots s_m\},$$

which is independent of the initial reduced factorization. The ordering procedure implies the existence of a factorization $(t'_1, \dots, t'_n) \sim (t_1, \dots, t_n)$ such that $\ell_S(wt'_n \cdots t'_i) < \ell_S(wt'_n \cdots t'_{i+1})$ for $1 \le i \le n$. Thus the strong exchange condition yields $1 \le i_{n-1} \ne i_n \le m$ such that $wt'_n = s_1 \cdots \widehat{s_{i_n}} \cdots s_m$ and $wt'_n t'_{n-1} = s_1 \cdots \widehat{s_{i_n}} \cdots \widehat{s_{i_{n-1}}} \cdots s_m$ or $wt'_n t'_{n-1} = s_1 \cdots \widehat{s_{i_{n-1}}} \cdots \widehat{s_{i_n}} \cdots s_m$. If $i_n < i_{n-1}$, i.e. $wt'_n t'_{n-1} = s_1 \cdots \widehat{s_{i_n}} \cdots \widehat{s_{i_{n-1}}} \cdots s_m$, then $t'_{n-1}, t'_n \in N(w)$. Otherwise apply $\sigma_{n-1}^{-1} \in B_n$ to $(t'_1, \dots, t'_n)$ and obtain

$$\sigma_{n-1}^{-1}(t'_1, \dots, t'_{n-1}) = (t'_1, \dots, t'_{n-2}, t'_n, t'_n t'_{n-1} t'_n).$$

Since $wt'_n = s_1 \cdots \widehat{s_{i_n}} \cdots s_m$, $wt'_n t'_{n-1} t'_n = s_1 \cdots \widehat{s_{i_{n-1}}} \cdots s_m$ and $i_{n-1} < i_n$ we get $t'_n, t'_n t'_{n-1} t'_n \in N(w)$.

Now we proceed in this way for all the neighbors of the resulting factorization until we obtain (after finitely many steps) a factorization whose reflections are in $N(w)$.

With Lemma 3.2.2 the equality $\ell_S(w) = \ell_S(x) + \ell_S(y)$ implies

$$N(w) = N(xy) = y^{-1}N(x)y \,\Delta\, N(y) = y^{-1}N(x)y \,\sqcup\, N(y).$$

The latter proves the second part of the lemma. □

From the proof of the previous Lemma one can easily deduce the following fact. Let $w \in W$, $w = s_1 \cdots s_n$ be a reduced expression in simple factors and $(t_1, \dots, t_m) \in \mathrm{Red}_T(w)$. Then there exists a factorization $(t'_1, \dots, t'_m) \in \mathrm{Red}_T(w)$ and $1 \leq i_m < i_{m-1} < \ldots < i_1 \leq m$ such that $(t_1, \dots, t_m) \sim (t'_1, \dots, t'_m)$ and $wt'_m \cdots t'_k = s_1 \cdots \widehat{s_{i_m}} \cdots \widehat{s_{i_k}} \cdots s_m$ for all $1 \leq k \leq m$. The latter implies that $t'_k = s_m \cdots s_{i_k - 1} s_{i_k} s_{i_k - 1} \cdots s_m$ for all $1 \leq k \leq m$.

3.3 Hurwitz action on arbitrary reflection factorizations of Coxeter elements

In this section we will investigate the Hurwitz action on reflection factorization of Coxeter elements of arbitrary length. We will generalize a result of Joel Lewis and Vic Reiner [74] from finite to arbitrary Coxeter groups of finite rank. We start with the definition of a Coxeter element.

Definition 3.3.1. *Let (W, S) be a Coxeter system of rank n. The elements of the form $s_1 \cdots s_m$ with pairwise different $s_i \in S$ are called* Coxeter elements *for $m = n$ and for $0 \leq m \leq n$ they are called* parabolic Coxeter elements.

Observe that there is an alternative description of parabolic Coxeter elements which is given by the following lemma and its proof is based on an investigation of the Bruhat path corresponding to a reduced reflection factorization of a parabolic Coxeter element.

Lemma 3.3.2. [39, Theorem 1.1] *Let (W, S) be a Coxeter system of finite rank. An element $w \in W$ is a parabolic Coxeter element if and only if $\ell_S(w) = \ell_T(w)$.*

Proof. Let $w = s_1 \cdots s_m$ be a reduced factorization in simple generators. If $\ell_T(w) = m$ we get immediantely $s_i \neq s_j$ for all $1 \leq i \neq j \leq m$ and thus w is a parabolic Coxeter element. Let w be a parabolic Coxeter element. Thus by Theorem 3.1.9 $s_i \neq s_j$ for all $1 \leq i \neq j \leq m$ implies $\ell_S(w) = m$. Let $w = t_1 \cdots t_k$ be a reduced reflection factorization with $\ell_T(w) = k \leq m = \ell_S(w)$. By Lemma 3.2.12 we can assume that $t_i \in N(w)$ for all $1 \leq i \leq k$. Thus there exist $1 \leq i_1 < \ldots < i_k \leq m$ such that $e = wt_k \cdots t_1 = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_k}} \cdots s_m$. Since the latter expression contains pairwise different simple factors, Theorem 3.1.9 implies as above $\ell_T(w) = k = m = \ell_S(w)$. □

The Hurwitz action on the set of reduced reflection factorizations of Coxeter elements is already well-investigated by many authors. The first occurrence is in a letter by Pierre Deligne to Eduard Looijenga in 1974 (see [30]), where a proof of Jacques Tits is stated, that shows that for any finite Coxeter group the Hurwitz action is transitive on the set of reduced reflection factorizations of Coxeter elements. In 2003 David Bessis published another proof of the same statement in [11] and Kiyoshi Igusa and Ralf Schiffler extended it in [60] to arbitrary Coxeter groups of finite rank in 2010. Barbara Baumeister, Matthew

Dyer, Christian Stump and Patrick Wegener give another proof in [8], which also hold for parabolic Coxeter elements.

Theorem 3.3.3. [8, Theorem 1.3] *Let (W, S) be a Coxeter system of finite rank, T its set of reflections and c a parabolic Coxeter element. The Hurwitz action is transitive on the set of reduced reflection factorizations of c.*

Proof. Let $c = s_1 \cdots s_m$ be a reduced factorization in simple reflections and by Lemma 3.3.2 it is a reduced reflection factorization of c. Let $(t_1, \ldots, t_m) \in \mathrm{Red}_T(c)$ be an arbitrary reduced reflection factorization of c. Thus by Lemma 3.2.12 and

$$N(c) = \{s_m, s_m s_{m-1} s_m, \ldots, s_m \cdots s_1 \cdots s_m\}$$

we get that

$$(t_1, \ldots, t_m) \sim (s_m, s_m s_{m-1} s_m, \ldots, s_m s_{m-1} \cdots s_1 \cdots s_{m-1} s_m).$$

Let $\tau_k = \sigma_k \cdots \sigma_2 \sigma_1$ for $1 \leq k \leq m-1$. Now it is easy to see that

$$\tau_1 \cdots \tau_{m-2} \tau_{m-1} (s_m, s_m s_{m-1} s_m, \ldots, s_m s_{m-1} \cdots s_1 \cdots s_{m-1} s_m) = (s_1, \ldots, s_m).$$

□

Remark 3.3.4. *The previous result also holds for arbitrary conjugates of parabolic Coxeter elements.*

Definition 3.3.5. *Let (W, S) be a Coxeter system of finite rank, T its set of reflections and $(t_1, \ldots, t_n)$, $(t_1', \ldots, t_n') \in T^n$ two reflection factorizations of the same element of W. We say that $(t_1, \ldots, t_n)$ and $(t_1', \ldots, t_n')$* share the same multiset of conjugacy classes *if there exists a permutation $\tau \in S_n$ such that $[t_{\tau(i)}] = [t_i']$ for all $1 \leq i \leq n$, where $[t]$ is the conjugacy class of $t \in T$ in W.*

The main theorem of this section give a criteria under which two arbitrary reflection factorizations lie in the same Hurwitz orbit.

Theorem 3.3.6. *Let (W, S) be a Coxeter system of finite rank. Two arbitrary reflection factorizations of a Coxeter element lie in the same Hurwitz orbit if and only if they share the same multiset of conjugacy classes.*

If W is finite this theorem is already proven by Joel Lewis and Vic Reiner in [74], but relies on a case-based calculation. Here we will provide a case-free proof for all Coxeter groups of finite rank. Thus we answer the question [74, Question 6.2] in the affirmative.

The last ingredient of the proof of Theorem 3.3.6 is the following easy result.

Lemma 3.3.7. [74] *Let* $t_1, \dots, t_n, t \in T$. *Then* $(t_1, \dots, t_n, t, t) \sim (t_1, \dots, t_n, t^w, t^w)$ *for all* $w \in \langle t_1, \dots, t_n \rangle$.

Proof. Let $i \in \{1, \dots, n\}$ be arbitrary and assume $w = t_i$ (the general assertion follows by induction). Denoting an omitted entry by $\widehat{t_i}$, we obtain

$$\begin{aligned}(t_1, \dots, t_n, t, t) &\sim (t_1, \dots, t_{i-1}, \widehat{t_i}, t_{i+1}^{t_i} \dots, t_n^{t_i}, t^{t_i}, t^{t_i}, t_i)\\ &\sim (t_1, \dots, t_{i-1}, \widehat{t_i}, t_{i+1}^{t_i} \dots, t_n^{t_i}, t_i, t^{t_i}, t^{t_i})\\ &\sim (t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n, t^{t_i}, t^{t_i}).\end{aligned}$$
□

Proof of Theorem 3.3.6. Let $n = |S|$ and $c \in W$ be a Coxeter element with

$$c = t'_1 \cdots t'_{n+2k} = r'_1 \cdots r'_{n+2k}$$

two reflection factorizations of c for some $k \in \mathbb{Z}_{\geq 0}$. It is obviously true that two reflection factorization of the same Hurwitz orbit share the same multiset of conjugacy classes. Thus assume that the initial factorizations share the same multiset of conjugacy classes. By Lemma 3.2.10 we have

$$\begin{aligned}(t'_1, \dots, t'_{n+2k}) &\sim (t_1, \dots, t_n, t_{i_1}, t_{i_1}, \dots, t_{i_k}, t_{i_k})\\ \text{and } (r'_1, \dots, r'_{n+2k}) &\sim (r_1, \dots, r_n, r_{i_1}, r_{i_1}, \dots, r_{i_k}, r_{i_k}).\end{aligned}$$

Since $c = t_1 \cdots t_n = r_1 \cdots r_n$ and $\ell_S(c) = \ell_T(c) = n$ by Lemma 3.3.2, $(t_1, \dots, t_n)$ and $(r_1, \dots, r_n)$ are reduced reflection factorizations of c. Hence we have $(t_1, \dots, t_n) \sim (r_1, \dots, r_n)$ by Theorem 3.3.3. In particular $(t_1, \dots, t_n)$ and $(r_1, \dots, r_n)$ share the same multiset of conjugacy classes. Hence $t_{i_1}, \dots, t_{i_k}$ and $r_{i_1}, \dots, r_{i_k}$ have to share the same multiset of conjugacy classes. Since $(t, t, r, r) \sim (r, r, t, t)$ for all $r, t \in T$, we can assume after a possible renumbering that there exists $w_j \in W$ such that $t_{i_j}^{w_j} = r_{i_j}$ for all $j \in \{1, \dots, k\}$. We proceed by induction on k. As we have seen above, the case $k = 0$ is precisely Theorem 3.3.3. Therefore let $k > 0$. By induction we have

$$(t_1, \dots, t_n, t_{i_1}, t_{i_1}, \dots, t_{i_{k-1}}, t_{i_{k-1}}, t_{i_k}, t_{i_k}) \sim (r_1, \dots, r_n, r_{i_1}, r_{i_1}, \dots, r_{i_{k-1}}, r_{i_{k-1}}, t_{i_k}, t_{i_k}).$$

As a consequence of Theorem 3.3.3, we have $W = \langle r_1, \dots, r_n \rangle$. By what we have pointed out before, there exists $w_k \in \langle r_1, \ldots, r_n \rangle$ such that $t_{i_k}^{w_k} = r_{i_k}$. We conclude

$$\begin{aligned}
(t'_1, \ldots, t'_{n+2k}) &\sim (r_1, \ldots, r_n, r_{i_1}, r_{i_1}, \ldots, r_{i_{k-1}}, r_{i_{k-1}}, t_{i_k}, t_{i_k}) \\
&\sim (r_1, \ldots, r_n, t_{i_k}, t_{i_k}, r_{i_1}, r_{i_1}, \ldots, r_{i_{k-1}}, r_{i_{k-1}}) \\
&\overset{3.3.7}{\sim} (r_1, \ldots, r_n, t_{i_k}^{w_k}, t_{i_k}^{w_k}, r_{i_1}, r_{i_1}, \ldots, r_{i_{k-1}}, r_{i_{k-1}}) \\
&= (r_1, \ldots, r_n, r_{i_k}, r_{i_k}, r_{i_1}, r_{i_1}, \ldots, r_{i_{k-1}}, r_{i_{k-1}}) \\
&\sim (r_1, \ldots, r_n, r_{i_1}, r_{i_1}, \ldots, r_{i_{k-1}}, r_{i_{k-1}}, r_{i_k}, r_{i_k}) \\
&\sim (r'_1, \ldots, r'_{n+2k}).
\end{aligned}$$

□

An easy consequence of the main theorem is the following result.

Corollary 3.3.8. *If the Coxeter graph of (W, S) is connected and contains an odd-labelled spanning tree, then two reflection factorizations of the same length of a Coxeter element in W lie in the same Hurwitz orbit.*

Proof. Since the Coxeter graph of (W, S) contains an odd-labelled spanning tree all reflections are conjugate by Lemma 3.1.5. Thus the assertion follows by Theorem 3.3.6. □

It is easy to see that Theorem 3.3.6 and Corollary 3.3.8 also hold for any conjugate of a Coxeter element.

Chapter 4

Hurwitz action in finite Coxeter groups

This chapter is dedicated to the study of the Hurwitz action in finite Coxeter groups. The first part deals with non-reduced reflection factorizations of arbitrary elements. If restricted to finite Coxeter groups its main result generalizes Lemma 3.2.10 . We reprove uniformly the key ingredient of the main result of [74].

The second part investigates the quasi-Coxeter elements and in particular, reproves and generalizes uniformly some of the results of [9]. The required tools of this chapter are developed in Section 3.2.

4.1 Hurwitz action on the set of non-reduced reflection factorizations

The goal of this section is to prove [74, Corollary 1.4] that is stated in Proposition 4.1.4. We start with the following useful result that characterizes reduced reflection factorizations in finite Coxeter groups.

Lemma 4.1.1 (Carter's Lemma)**.** [23, Lemma 3] *Let (W, S) be a finite Coxeter system and $s_{\beta_1}, \dots, s_{\beta_m} \in W$ reflections attached to the roots $\beta_1, \dots, \beta_m \in \Phi$, where Φ is the finite root system that is induced by the standard representation. Then $s_{\beta_1} \cdots s_{\beta_m}$ is a reduced reflection factorization if and only if $\{\beta_1, \dots, \beta_m\}$ is linearly independent.*

For later use we need the following two statements about the parabolic closures of elements.

Lemma 4.1.2. *Let (W, S) be a finite Coxeter system and $w = t_1 \cdots t_m \in W$ a reduced reflection factorization of w. Then the group $W' = \langle t_1, \dots, t_m \rangle$ is a Coxeter group of rank m and its rank coincides with the rank of the parabolic closure $P(\{t_1, \dots, t_m\})$ of $t_1, \dots, t_m$.*

Proof. By Proposition 3.2.4 (c) the group W' is a Coxeter group of rank at most m. Since the factorization $t_1 \cdots t_m$ is reduced Carter's lemma implies that the rank is at least m. By [35, Lemma 2.1] the rank of the parabolic closure of $t_1, \dots, t_m$ coincides with the rank of W'. □

Proposition 4.1.3. [55, Proposition 2.5] *Let (W, S) be a Coxeter system of finite rank, $w \in W$ and $(t_1, \dots, t_m) \in \mathrm{Red}_T(w)$. Then $P(w) = P(\{t_1, \dots, t_m\})$.*

Proof. Since $w = t_1 \cdots t_m$ we have that $P(w) \subseteq P(\{t_1, \dots, t_m\})$. It remains to show that $P(\{t_1, \dots, t_m\}) \subseteq P(w)$. By Theorem 3.2.11 we have that $\mathrm{Red}_T(w) = \mathrm{Red}_{T \cap P(w)}(w)$ and hence $t_1, \dots, t_m \in P(w)$. In particular, it holds $P(\{t_1, \dots, t_m\}) \subseteq P(w)$. □

The next result is the key ingredient of the main theorem of [74] and is a generalization of Lemma 3.2.10 if the corresponding group is a finite Coxeter group. Here we state a uniform proof.

Proposition 4.1.4. [74, Corollary 1.4] *Let (W, S) be a finite Coxeter system and $w = t_1 \cdots t_{m+2k} \in W$ a reflection factorization with $\ell_T(w) = m$ and $k \in \mathbb{Z}_{\geq 0}$. Then there exists a braid $\sigma \in B_{m+2k}$ such that*

$$\sigma(t_1, \dots, t_{m+2k}) = (r_1, \dots, r_m, r_{i_1}, r_{i_1}, \dots, r_{i_k}, r_{i_k}).$$

Proof. We proceed by induction on k. If $k = 0$ there is nothing to prove. Assume that $k > 0$ and let $1 \leq l \leq m + 2k - 1$ be maximal such that $w' := t_1 \cdots t_l$ is a reduced reflection factorization. By Lemma 3.1.8 we have $\ell_T(w' t_{l+1}) = l - 1$ and hence there exists a factorization $w' = x^{-1} t_{l+1}$ with $x^{-1} \in W$ and $\ell_T(w') = \ell_T(x^{-1}) + 1$. By Lemma 4.1.2 and Proposition 4.1.3 the rank of $P(x^{-1}) = P(x)$ is $l - 1$. Without loss of generality, we can assume that $P(x)$ is a standard parabolic subgroup of W.

Consider the (non-directed) path in the Bruhat graph starting in x and ending in e that is induced by the non-reduced factorization $(t_1, \dots, t_{l+1})$

$$x - xt_1 - xt_1t_2 - \ldots - xt_1t_2 \cdots t_l = xw' = t_{l+1} - e.$$

If there exists a factorization in $B_{l+1}(t_1, \dots, t_{l+1})$ with two identical factors, then we can shift them to the end of the factorization by using the Hurwitz action and apply the induction hypothesis. Hence let us assume to the contrary that each factorization in $B_{l+1}(t_1, \dots, t_{l+1})$ consists of pairwise different factors. The ordering procedure of Proposition 3.2.9 yields the existence of a braid $\sigma \in B_{l+1}$ such that the factorization $\sigma(t_1, \dots, t_{l+1}) = (\bar{t}_1, \dots, \bar{t}_{l+1})$ induces the following directed path in the Bruhat graph

$$x \longleftarrow x\bar{t}_1 \longleftarrow x\bar{t}_1\bar{t}_2 \longleftarrow \ldots \longleftarrow x\bar{t}_1 \cdots \bar{t}_l = \bar{t}_{l+1} \longleftarrow e.$$

Since $P(x)$ is a standard parabolic subgroup, the strong exchange condition yields that $\bar{t}_1, \dots, \bar{t}_{l+1} \in P(x)$ and therefore also $t_1, \dots, t_{l+1} \in P(x)$. In particular, $w' \in P(x)$ and by Theorem 3.2.11 we have that

$$l = \ell_T(w') = \ell_{T \cap P(x)}(w'),$$

where $T \cap P(x)$ is the set of reflections of the standard parabolic subgroup $P(x)$. But by Carter's lemma the length $\ell_{T\cap P(x)}$ is bounded by the rank of $P(x)$, that is $l-1$. Hence we arrive to a contradiction. Thus there exists a braid $\sigma \in B_{m+2k}$ such that

$$\sigma(t_1, \dots, t_{m+2k}) = (t'_1, \dots, t'_{m+2(k-1)}, r_{i_k}, r_{i_k}).$$

The induction hypothesis yields the assertion. □

The following calculation shows that the previous result does not hold for arbitrary Coxeter groups. It is proposed by Patrick Wegener.

Remark 4.1.5. *We use the notation of* [58, Chapter 4]. *Consider the affine Coxeter group of type* $\widetilde{B}_2$. *Let* $\alpha_1 = e_1 - e_2, \widetilde{\alpha} = e_1 + e_2, \alpha_2 = e_1$ *be roots of the finite root system* $B_2 \subseteq \mathbb{R}^2$, *where* e_1, e_2 *are the canonical unit vectors. It holds*

$$s_{\alpha_1,1}s_{\alpha_1}s_{\widetilde{\alpha},1}s_{\widetilde{\alpha}} = s_{\alpha_2,1}s_{\alpha_2}.$$

Since $\alpha_1 \in \widetilde{\alpha}^\perp$ *every factorization of the Hurwitz orbit* $B_4(s_{\alpha_1,1}, s_{\alpha_1}, s_{\widetilde{\alpha},1}, s_{\widetilde{\alpha}})$ *consists of pairwise different factors.*

4.2 Investigation of quasi-Coxeter elements in finite Coxeter groups

In this section we direct our attention to (parabolic) quasi-Coxeter elements in finite Coxeter groups. We reprove uniformly some of the properties that are stated in [9] and give new insights to this class of elements. We start with the definition of quasi-Coxeter elements.

Definition 4.2.1. *Let* (W, S) *be a Coxeter system of rank* n. *An element* $w \in W$ *is called* quasi-Coxeter element *if it admits a reduced reflection factorization with* n *factors that generate* W *and it is called* parabolic quasi-Coxeter element *if it admits a reduced reflection factorization with* $k \leq n$ *factors that generate a parabolic subgroup of rank* k.

Example 4.2.2. *Every conjugate of a (parabolic) quasi-Coxeter element is a (parabolic) quasi-Coxeter element. Parabolic Coxeter elements are by definition parabolic quasi-Coxeter elements, but there exist quasi-Coxeter elements that are not conjugated to Coxeter elements. Let* (W, S) *be a finite Coxeter system of type* D_4 *and* $S = \{s_1, s_2, s_3, s_4\}$. *Assume that* s_2 *is the simple reflection that commutes with non of the other simple reflections, then the element* $w = s_1(s_3s_2s_3)s_4s_2$ *is a quasi-Coxeter element, that is not conjugated to a Coxeter element.*

Remark 4.2.3. *By Proposition 4.1.3 a parabolic quasi-Coxeter element is a quasi-Coxeter element in its corresponding parabolic closure.*

Quasi-Coxeter elements are already investigated in [9] and their study is justified by the following result.

Theorem 4.2.4. [9, Theorem 1.1] *Let (W, S) be a finite Coxeter system and $w \in W$. The Hurwitz action is transitive on the set of reduced reflection factorization of w if and only if w is a parabolic quasi-Coxeter element.*

Currently we are working on a uniform proof of the previous theorem by reproducing the results of [9]. The rest of this section describes the current state of our investigations.

We start with a useful statement that helps understanding the canonical simple system of parabolic subgroups.

Lemma 4.2.5. *Let (W, S) be a Coxeter system of finite rank, P a standard parabolic subgroup of W such that there exists $t \in P \cap T$ and $t' \in T \setminus P$, then $t \in \chi(< t, t' >)$.*

Proof. Let $w \in P$ such that $s' := wtw^{-1} \in S$ and such that $\ell_S(w)$ is minimal among all w. Therefore we get

$$s' \in \chi(< s', wt'w^{-1} >) = \chi(< wtw^{-1}, wt'w^{-1} >) = \chi(w < t, t' > w^{-1}).$$

If $w = e$ the previous equation yields the assertion. Thus assume that $\ell_S(w) \geq 1$ and set $t'' := wt'w^{-1}$. Let $w = s_1 \cdots s_n$ be a reduced factorization in simple reflections. Because of the minimality of $\ell_S(w)$ we have $s_i s_{i-1} \cdots s_1 s' s_1 \cdots s_{i-1} s_i \notin S$ for all $1 \leq i \leq n$. The latter can be verified as follows. Assume that $s_i s_{i-1} \cdots s_1 s' s_1 \cdots s_{i-1} s_i =: s'' \in S$ for some $1 \leq i \leq n$. Then

$$s_n \cdots s_{i+1} s'' s_{i+1} \cdots s_n = s_n \cdots s_{i+1} s_i \cdots s_1 s' s_1 \cdots s_i s_{i+1} \cdots s_n = w^{-1} s' w = t$$

and thus

$$s_{i+1} \cdots s_n t s_n \cdots s_{i+1} = s'' \in S.$$

The latter contradicts the minimality of $\ell_S(w)$.

In the following we will show by induction that

$$s_i \cdots s_1 \chi(\langle s', t'' \rangle) s_1 \cdots s_i = \chi(s_i \cdots s_1 \langle s', t'' \rangle s_1 \cdots s_i)$$

for all $1 \leq i \leq n$. Consider the situation for $i = 1$. Since $s' \in S \cap P$ and $t'' \notin P$ we have $\chi(\langle s', t'' \rangle) = \{s', \bar{t}\}$ for some $\bar{t} \notin P$. Since $s_1 s' s_1 \notin S$ and $\bar{t} \notin P$ we have $s_1 \notin \{s', \bar{t}\} = \chi(\langle s', t'' \rangle)$. Hence by Proposition 3.2.4 (b) we get $s_1 \chi(\langle s', t'' \rangle) s_1 = \chi(s_1 \langle s', t'' \rangle s_1)$. Assume that $i \geq 2$. By induction hypothesis it holds

$$\begin{aligned}\chi(s_{i-1} \cdots s_1 < s', t'' > s_1 \cdots s_{i-1}) &= s_{i-1} \cdots s_1 \chi(< s', t'' >) s_1 \cdots s_{i-1} \\ &= s_{i-1} \cdots s_1 \{s', \bar{t}\} s_1 \cdots s_{i-1}.\end{aligned}$$

As before we have that $s_i \notin s_{i-1} \cdots s_1 \{s', \bar{t}\} s_1 \cdots s_{i-1} = \chi(s_{i-1} \cdots s_1 \langle s', t'' \rangle s_1 \cdots s_{i-1})$ and thus Proposition 3.2.4 (b) implies

$$s_i \chi(s_{i-1} \cdots s_1 \langle s', t'' \rangle s_1 \cdots s_{i-1}) s_i = \chi(s_i \cdots s_1 \langle s', t'' \rangle s_1 \cdots s_i).$$

The latter yields by the induction hypothesis

$$\chi(s_i \cdots s_1 \langle s', t'' \rangle s_1 \cdots s_i) = s_i \cdots s_1 \chi(\langle s', t'' \rangle) s_1 \cdots s_i.$$

Altogether, we have with $s' \in \chi(\langle s', t'' \rangle)$

$$s_i \cdots s_1 s' s_1 \cdots s_i \in \chi(s_i \cdots s_1 < s', t'' > s_1 \cdots s_i)$$

for all $1 \leq i \leq n$. In particular, it holds for $i = n$

$$t = w^{-1} s' w \in \chi(w^{-1} < s', t'' > w) = \chi(< t, t' >).$$

□

The following lemma will not be used in this work. It describes the canonical simple system of a (non-trivial) intersection of two parabolic subgroups and could be interesting in future investigations.

Lemma 4.2.6. *Let (W, S) be a Coxeter system of finite rank, P a standard parabolic subgroup and P' a parabolic subgroup. If $P \cap P' \neq \{e\}$, then $\chi(P') \cap P = \chi(P' \cap P)$. In particular, $P \cap P' \neq \{e\}$ if and only if $\chi(P') \cap P \neq \emptyset$.*

Proof. By assumption $P \cap P' \neq \{e\}$ and thus by Proposition 3.1.16 (d) the group $P \cap P'$ is a reflection subgroup of W. Let $t \in \chi(P') \cap P$ and $t' \in (P' \cap P) \cap T$ such that $\ell_S(tt') < \ell_S(t)$. The fact

$$t' \in \left\{ \bar{t} \in (P' \cap P) \cap T \mid \ell_S(t\bar{t}) < \ell_S(t) \right\} \subseteq \left\{ \bar{t} \in P' \cap T \mid \ell_S(t\bar{t}) < \ell_S(t) \right\} = \{t\}$$

implies $t = t'$ and therefore we have by definition $\chi(P') \cap P \subseteq \chi(P \cap P')$.

Let $t \in \chi(P \cap P')$ and $t' \in P' \cap T$ such that $\ell_S(tt') < \ell_S(t)$. If $t' \in P$ we get

$$t' \in \{ \bar{t} \in (P' \cap P) \cap T \mid \ell_S(t\bar{t}) < \ell_S(t) \} = \{t\}$$

and therefore $t = t'$. If $t' \notin P$, Lemma 4.2.5 implies $t \in \chi(\langle t, t' \rangle)$ and since $t \neq t'$ we get that $\ell_{\chi(\langle t, t' \rangle)}(tt') > \ell_{\chi(\langle t, t' \rangle)}(t)$. The latter is by Theorem 3.2.1 equivalent to $\ell_S(tt') > \ell_S(t)$, a contradiction. □

Another consequence of Lemma 4.2.5 is the following useful result about prefixes of quasi-Coxeter elements.

Proposition 4.2.7. *Let (W, S) be a Coxeter system of of rank n, $w \in W$ a quasi-Coxeter element and $w = tx$ such that $\ell_T(w) = \ell_T(x) + 1$ and $P(x) \neq W$. Then the element x is a parabolic quasi-Coxeter element and $P(x)$ is parabolic subgroup of rank $n - 1$.*

Proof. Let n be the rank of W, $(t_1, \ldots, t_n)$ be a reduced factorization of w such that $\langle t_1, \ldots, t_n \rangle = W$, $w = tx$ with $\ell_T(w) = \ell_T(x) + 1$. Without loss of generality, we can assume that $P(x)$ is a standard parabolic subgroup of W.

Assume that each factorization of $B_{n+1}(t, t_1, \ldots, t_n) \subseteq \mathrm{Fac}_{T,n+1}(x)$ contains pairwise different factors. By Proposition 3.2.9 there exists a factorization $(t', t'_1, \ldots, t'_n) \in B_{n+1}(t, t_1, \ldots, t_n)$ that corresponds to the following directed path in the Bruhat graph

$$x \longleftarrow xt'_n \longleftarrow xt'_n t'_{n-1} \longleftarrow \ldots \longleftarrow xt'_n t'_{n-1} \cdots t'_1 = t' \longleftarrow e.$$

Since $P(x)$ is a standard parabolic subgroup, the strong exchange condition yields

$$W = \langle t, t_1, \ldots, t_n \rangle = \langle t', t'_1, \ldots, t'_n \rangle \subseteq P(x),$$

a contradiction. Thus there exists a factorization $(t'_1, \ldots, t'_{n-1}, t', t') \in B_{n+1}(t, t_1, \ldots, t_n)$, where $(t'_1, \ldots, t'_{n-1}) \in \mathrm{Red}_T(x)$. By Theorem 3.2.11 we have that $t'_1, \ldots, t'_{n-1} \in P(x)$ and $\langle t'_1, \ldots, t'_{n-1}, t' \rangle = W$. Set $T_0 = \{t'_1, \ldots, t'_{n-1}, t'\}$. We use the algorithm that is described in the proof of Proposition 3.2.4 (c) (or see [37, Proposition 3.7]) with starting set T_0. In each step $i \geq 0$ of the algorithm, Lemma 4.2.5 yields that $|T_i \cap P(x)| = n - 1$. In particular, we get for $i \gg 0$ (large enough)

$$\chi(\langle t'_1, \ldots, t'_{n-1} \rangle) \subseteq T_i = S.$$

Furthermore, we have that $\langle \chi(\langle t'_1, \ldots, t'_{n-1} \rangle) \rangle = P(x)$. □

The following result states that prefixes of quasi-Coxeter elements in finite Coxeter groups are parabolic quasi-Coxeter elements. For an analogous statement about Coxeter elements we refer to [11, Lemma 1.4.3].

Lemma 4.2.8. [9, Corollary 6.11] *Let (W, S) be a finite Coxeter system and $v \in W$. The element v is a parabolic quasi-Coxeter element if and only if there exists a quasi-Coxeter element w such that $v \leq_T w$.*

Proof. If v is a parabolic quasi-Coxeter element, then [9, Proposition 6.2] yields uniformly that $v \leq_T w$ for a quasi-Coxeter element $w \in W$. Let n be the rank of (W, S) and $v \leq_T w$ for a quasi-Coxeter element $w \in W$. Then there exist reflections $t_{k+1}, \ldots, t_n \in T$ with $k := \ell_T(v) \leq n$ and $vt_{k+1} \cdots t_n = w$. If $k = n$ we have that $v = w$ is a parabolic quasi-Coxeter element. Thus assume that $k < n$. By applying Proposition 4.2.7 and Lemma 4.1.2 the element $vt_{k+1} \cdots t_{n-1}$ is a parabolic quasi-Coxeter element. Moreover, by Proposition

4.1.3 it is a quasi-Coxeter element of the parabolic subgroup $P(vt_{k+1}\cdots t_{n-1})$ that has rank $n-1$. Inductively we get that v is a parabolic quasi-Coxeter element. □

The previous result is not true for arbitrary Coxeter groups. For a suitable example we refer to [56, Example 5.7].

We immediately get the following.

Corollary 4.2.9. [9, Theorem 1.5] *Let (W,S) be a finite Coxeter system of rank n and $t_1,\dots,t_n \in T$ with $< t_1,\dots,t_n >= W$, then $< t_1,\dots,t_{n-1} >$ is a parabolic subgroup.*

Since the Coxeter diagram of a finite Coxeter group is bipartite (see Figure 3.1) we immediately get that Coxeter elements are a product of two parabolic Coxeter elements (see [11, Section 1.2]). The next proposition generalizes the latter fact to quasi-Coxeter elements, but before we state two auxiliary results.

Lemma 4.2.10. *Let (W,S) a Coxeter system of finite rank with root system Φ induced by the standard representation, linear independent roots $\beta_1,\dots,\beta_m \in \Phi$ and $v \in \mathrm{span}_{\mathbb{R}}(\Phi)$. Then $s_{\beta_1}\cdots s_{\beta_m}(v) = v$ if and only if $s_{\beta_i}(v) = v$ for all $1 \leq i \leq m$.*

Proof. A direct calculation yields

$$-(\beta_1, v)\beta_1 = s_{\beta_2}\cdots s_{\beta_m}(v) - v \in \mathrm{span}_{\mathbb{R}}(\beta_2,\dots,\beta_m)$$

and the linear independence of $\{\beta_1,\dots,\beta_m\}$ implies $(\beta_1, v) = 0$. The latter is equivalent to $s_{\beta_1}(v) = v$. In the same manner we get inductively that $s_{\beta_i}(v) = v$ for all $2 \leq i \leq m$. □

Next we prove that any reduced reflection factorization of an involution of a finite Coxeter group consists of pairwise commuting reflections. The proof uses [93, Theorem A] and is essentially [23, Lemma 4].

Lemma 4.2.11. *Let (W,S) be a finite Coxeter system and $w \in W$ an involution. Then any reduced reflection factorization of w consists of pairwise commuting factors.*

Proof. We identify W with its image under the standard presentation. Let Φ be the corresponding root system and $(-,-)$ the corresponding positive definite symmetric bilinear form. By [93, Theorem A] there exist $u \in W$ and $J \subseteq S$ such that $uwu^{-1} = -1$ is the longest element in the standard parabolic subgroup W_J. Let $(t_1,\dots,t_k) \in \mathrm{Red}_T(w)$ be a reduced reflection factorization of w. Obviously, we have that the factors t_i and t_j are commuting for all $1 \leq i,j \leq k$ if and only if ut_iu^{-1} and ut_ju^{-1} are commuting for all $1 \leq i,j \leq k$. Thus by Theorem 3.2.11 we can assume without loss of generality that $W = W_J$, $S = J$ and $w = -1$. By [23, Lemma 2] we get that $k = \ell_T(w) = |S| = \dim_{\mathbb{R}}(V)$, where $V = \mathrm{span}_{\mathbb{R}}(\Phi)$.

Let $s_{\beta_i} = t_i$ for all $1 \leq i \leq k$ and by Carter's Lemma $\{\beta_1, \dots, \beta_k\}$ is linearly independent. Let $v \in \beta_1^{\perp}$, then

$$s_{\beta_1} \cdots s_{\beta_k}(v) = -v.$$

Thus we have

$$s_{\beta_2} \cdots s_{\beta_k}(v) = -v$$

and therefore

$$2v = v - s_{\beta_2} \cdots s_{\beta_k}(v) \in \operatorname{span}_{\mathbb{R}}(\beta_2, \dots, \beta_k).$$

The latter implies that $\beta_1^{\perp} \subseteq \operatorname{span}_{\mathbb{R}}(\beta_2, \dots, \beta_k)$ and since the dimensions of both spaces are $k-1$ they have to coincide. In particular, we have that $\beta_2, \dots, \beta_k \in \beta_1^{\perp}$. A repetition of the same argument shows that $(\beta_i, \beta_j) = 0$ for all $1 \leq i \neq j \leq k$ and thus the corresponding reflections are mutually commuting. □

The next result gives a new description of quasi-Coxeter elements in finite Coxeter groups.

Proposition 4.2.12. *Let (W, S) be a finite Coxeter system and $w \in W$ a quasi-Coxeter element. There exist two parabolic Coxeter elements $x, y \in W$ such that $w = xy$ and $\ell_T(w) = \ell_T(x) + \ell_T(y)$. Moreover, the element x (resp. y) is a Coxeter element of a parabolic subgroup of type $A_1^{\ell_T(w)}$ (resp. $A_1^{\ell_T(y)}$), i.e. the corresponding Coxeter diagrams contains no edges.*

Proof. We identify W with its image under the standard presentation. Let Φ be its corresponding finite root system and $w = s_{\beta_1} \cdots s_{\beta_1}$ a reduced reflection factorization of w with $\beta_i \in \Phi$ for $1 \leq i \leq n$. By Carter's Lemma the roots $\beta_1, \dots, \beta_n$ are linearly independent, thus Lemma 4.2.10 shows for $v \in V$ that $w(v) = v$ if and only if $s_{\beta_i}(v) = v$ for all $1 \leq i \leq n$.

By [23, Theorem C] (for Weyl groups) and [53, Lemma 2.4] every element of a finite Coxeter group is an involution or is a product of two involutions. Thus $w = xy$ with involution x and either y is also an involution or $y = e$. If y is an involution, consider an arbitrary vector v in the intersection of the eigenspaces to the eigenvalue -1, i.e. $v \in \operatorname{Eig}(x, -1) \cap \operatorname{Eig}(y, -1)$. Hence we have $w(v) = xy(v) = v$ and therefore by the first part of the proof $(\beta_i, v) = 0$ for all $1 \leq i \leq n$. Since $v \in \operatorname{span}_{\mathbb{R}}(\beta_1, \dots, \beta_n)$ and by the positive definiteness of the bilinear form attached to W we get $v = 0$. Then [23, Lemma 6] implies $\ell_T(w) = \ell_T(x) + \ell_T(y)$.

By Lemma 4.2.8 the element x is a parabolic quasi-Coxeter element, and by Lemma 4.2.11 every factorization of $\operatorname{Red}_T(x)$ consists of pairwise commuting reflections. In particular, there exists a reduced reflection factorization of x with pairwise commuting reflections whose factors generate the parabolic subgroup $P(x)$. The same is true for y, hence the assertion follows.

If $y = e$ we have that $w = x$ is an involution. Again by Lemma 4.2.11 there exists a reduced reflection factorization of w whose factors generate the group W and that are pairwise commuting. Thus W is itself of type $A_1^{\ell_T(w)}$ and hence the assertion follows. □

Remark 4.2.13. *The proofs of the results* [23, Theorem C] *and* [53, Lemma 2.4] *that are used in the previous proof are case-based. It is desirable to have a uniform proof of the fact that each element of a finite Coxeter group is an involution or a product of two involutions.*

Chapter 5

Hurwitz action in extended Weyl groups

In this chapter we introduce the extended Weyl groups and investigate their structure. They are reflection groups of infinite order which are defined by so-called extended Coxeter diagrams and can be viewed as extensions of Coxeter groups with certain Coxeter diagrams. They find application in the theory of unimodal singularities, where they appear as monodromy groups of simple elliptic and hyperbolic singularities. In this thesis we are mainly interested in their appearance in the theory of finite dimensional algebras. There they are realized as groups of isometries of the Grothendieck group of certain hereditary abelian category. As for Coxeter groups we fix distinguished elements, the so-called Coxeter transformations. This class of elements are induced by an auto equivalence of the categories, the so-called Serre functor. The main theorem of this chapter is a statement about the Hurwitz action on the set of reduced reflection factorizations of Coxeter transformations.

5.1 Definitions and basic properties

The goal of this section is to define the extended Weyl groups and Coxeter transformations. We start by introducing a diagram, the so-called extended Coxeter diagram, that depends on the non-negative integers $r \in \mathbb{Z}_{\geq 0}$ and $p_i \in \mathbb{N}$ for $1 \leq i \leq r$ (see Figure 5.1).

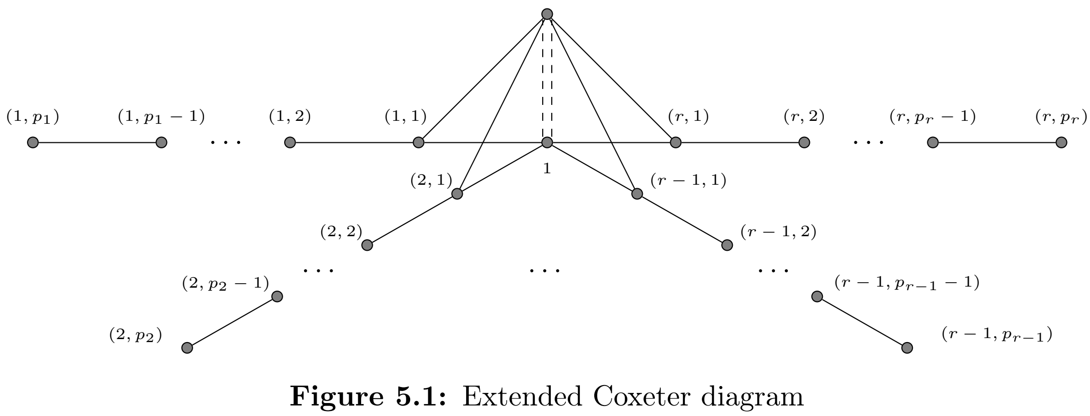


Figure 5.1: Extended Coxeter diagram

We attach analogously to the standard representation for Coxeter groups (see Theorem 3.1.10) a geometric datum to the extended Coxeter diagram. This will lead to the definition of the extended Weyl group and a Coxeter transformation.

Let Q be the vertex set of the diagram in Figure 5.1. Define V to be the real vector space with basis $B := \{\alpha_\nu \mid \nu \in Q\}$. We further define a symmetric bilinear form $(-,-)$ attached to V induced by

$$(\alpha_\nu, \alpha_\omega) = \begin{cases} 2 & \nu, \omega \in Q \text{ are connected by a dotted double bound or } \nu = \omega, \\ 0 & \nu, \omega \in Q \text{ are disconnected,} \\ -1 & \nu, \omega \in Q \text{ are connected by a single edge.} \end{cases}$$

A direct calculation yields the signature of the symmetric bilinear form $(-,-)$.

Lemma 5.1.1. *Let V be the vector space induced by an extended Coxeter diagram and $(-,-)$ its attached symmetric bilinear form. The only possible signatures of $(-,-)$ are $(|B|-2,1,1)$, $(|B|-2,0,2)$ or $(|B|-1,0,1)$ where the first (resp. second, resp. third) entry is the geometric dimension of the positive (resp. negative, resp. zero) eigenvalues.*

Proof. Let V_+ (resp. V_-) be the maximal subspace of V such that the corresponding restriction of $(-,-)$ is positive definite (resp. negative definite). Sylvester's law of inertia yields that $V = V_+ \oplus V_- \oplus R$, where R is the radical of $(-,-)$. Thus with $n_+ := \dim_{\mathbb{R}}(V_+)$, $n_- := \dim_{\mathbb{R}}(V_-)$ and $n_0 := \dim_{\mathbb{R}}(R)$ we get $|B| = n_+ + n_- + n_0$. By the classification of positive definite graphs in [58, Chapter 2.4] we get $\text{span}_{\mathbb{R}}(B \setminus \{\alpha_1, \alpha_{1^*}\}) \subseteq V_+$ and thus $n_+ \geq |B| - 2$. Since $\alpha_{1^*} - \alpha_1 \in R$ we have $n_0 \geq 1$. Altogether, the only possible signatures are $(|B|-2,1,1)$, $(|B|-2,0,2)$ or $(|B|-1,0,1)$. □

Remark 5.1.2. *(a) Notice that not all possible triples actually appear as signatures of $(-,-)$, see Proposition 5.2.1 (a).*

(b) Lemma 5.1.1 can also be deduced for $r \geq 3$ in terms of the tilting theory. In fact, it is a combination of [3, Section 10] *and* [73, Proposition 18.8]. *For the so-called $T_{p,q,r}$ cases it is also proven in* [41, Chapter 5.11].

5.2 The extended Weyl group

Proposition and Definition 5.2.1. *Let V and $(-,-)$ be as in the beginning of the previous section.*

(a) The group $W := \langle s_\alpha \mid \alpha \in B\rangle$ is called extended Weyl group, *where s_α is defined as in Example 2.1.3. It is called to be* wild *(resp.* tubular*, resp.* domestic*) type if the signature of $(-,-)$ is $(|B|-2,1,1)$ (resp. $(|B|-2,0,2)$, resp. $(|B|-1,0,1)$). We denote by R the radical of the form $(-,-)$. By help of the classification of positive semidefinite graphs in* [58, Chapter 2.5] *it is easy to see that the signatures that occur*

are exactly the following.

$$\begin{aligned} \textit{tubular: } & (m,0,2) \textit{ for } m = 4,6,7,8 \\ \textit{domestic: } & (m,0,1) \textit{ for } m \in \mathbb{N} \\ \textit{wild: } & (m,1,1) \textit{ for } m \geq 6 \end{aligned}$$

(b) *If W is of domestic or wild type, then dim $R = 1$ and $R = \operatorname{span}_{\mathbb{R}}(a)$ where $a := \alpha_{1^*} - \alpha_1$. If W is of tubular type, then $\operatorname{span}_{\mathbb{R}}(a) \subset R$.*

(c) *The set $S := \{s_\alpha \mid \alpha \in B\}$ is called* simple system *and its elements* simple reflections*. We call (W, S) an* extended Weyl system*. The set $T := \bigcup_{w \in W} wSw^{-1}$ is the* set of reflections *for (W, S).*

(d) *Let $\Phi \subseteq V$ be the minimal set that contains B and is closed under the action of W, i.e. $w(\beta) \in \Phi$ for all $w \in W$ and $\beta \in \Phi$. Then Φ is a reduced, irreducible and crystallographic root system in the sense of Definition 2.1.5 and $\Phi = W(B)$. The elements of B are called* simple roots*. The root system Φ is called to be of* wild *(resp.* domestic*, resp.* tubular*) type if the corresponding extended Weyl group is of wild (resp. domestic, resp. tubular) type.*

(e) *Let $\overline{\Phi} \subseteq V$ be the minimal set that contains $\overline{B} := B \backslash \{\alpha_{1^*}\}$ and is closed under the action of $\overline{W} = \langle s_\alpha \mid \alpha \in \overline{B} \rangle$. Then $\overline{\Phi}$ is a root system which is irreducible, crystallographic and reduced, and it holds $\overline{\Phi} = \overline{W}(\overline{B})$. We call the set $\overline{\Phi}$ the* projected root system*. By construction, $(\overline{W}, \overline{S})$ is a Coxeter system of finite rank, where $\overline{S} = S \setminus \{s_{1^*}\}$. We call it the* projected Coxeter system*.*

(f) *As $(\overline{W}, \overline{S})$ is a Coxeter system whose Coxeter diagram is a tree and whose simple roots are all of the same length, the action of $\overline{W} \subseteq W$ is transitive on $\overline{\Phi}$ (see Lemma 3.1.5). This also implies that all the roots in $\overline{\Phi}$, and therefore also in Φ, are of the same length.*

(g) *The elements $c := \left(\prod_{\alpha \in B \backslash \{\alpha_1, \alpha_{1^*}\}} s_\alpha\right) \cdot s_{\alpha_1} s_{\alpha_{1^*}}$ are called* Coxeter transformations*, where we take the first $|B| - 2$ factors in arbitrary order.*

Remark 5.2.2. (a) *The set $\overline{\Phi}$ is a root subsystem of Φ, and $\overline{W}$ is a subgroup of W. Let p be the natural projection of V onto $V/\operatorname{span}_{\mathbb{R}}(a)$. Then $p(\Phi)$ and $\overline{\Phi}$ are isomorphic root systems. If it is convenient we will abbreviate $p(v)$ by $\overline{v}$ for $v \in V$. In particular, in our setting $\alpha = \overline{\alpha}$ for $\alpha \in \overline{\Phi}$.*

(b) *Notice that the extended Weyl groups W of domestic type are precisely the affine simply-laced irreducible Coxeter groups. We claim: if W is an extended Weyl group of domestic type then there is $S' \subseteq T$ such that (W, S') is an affine Coxeter system of type $\widetilde{X}$ where X is the type of the Coxeter system $(\overline{W}, \overline{S})$. This can be seen as follows. If W is domestic then $(-,-)$ restricted to $\operatorname{span}_{\mathbb{R}}(\overline{B})$ is positive definite, let us say $(\overline{W}, \overline{S})$ is of type X. Let $\widetilde{\alpha}$ be the highest root in the positive subsystem of $\overline{\Phi}$ containing $\overline{B}$.*

By 5.2.1 (f) there exists $w \in \overline{W}$ *that maps* α_1 *onto* $-\widetilde{\alpha}$. *This shows that* $s_{-\widetilde{\alpha}+a} \in W$. *Then* W *is generated by* $S' := \overline{S} \cup \{s_{-\widetilde{\alpha}+a}\} \subseteq T$. *As* S' *is a simple system for* W, *see* [58, Theorem 4.6], *it follows that* (W, S') *is a Coxeter system of affine type.*

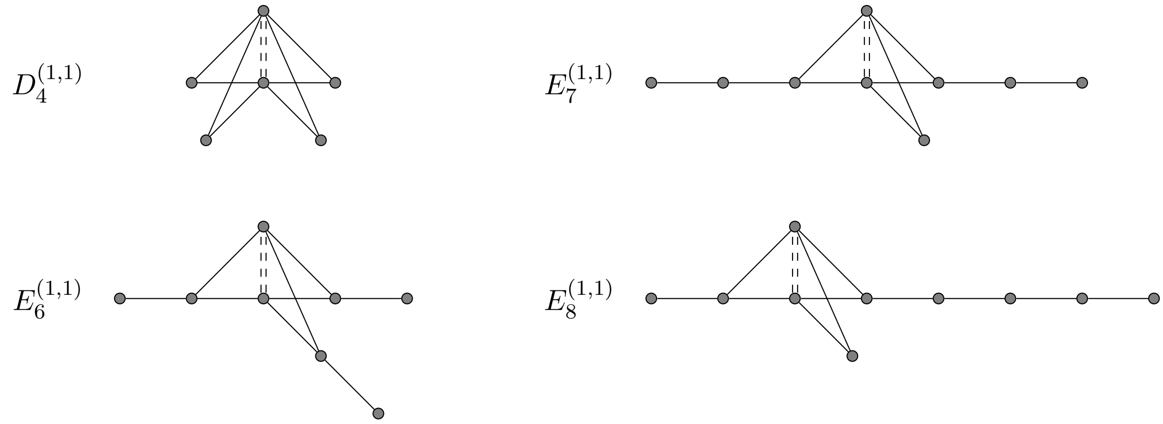


Figure 5.2: Extended Coxeter diagram of tubulare type

The next lemma describes the root system with help of the underlying projected root system.

Lemma 5.2.3. *Let* Φ *be the root system attached to* W *and* R *the radical of the corresponding bilinear form.*

(a) We have $\Phi = \{\overline{\alpha} + ka \mid \overline{\alpha} \in \overline{\Phi},\ k \in \mathbb{Z}\}$.

(b) If Φ *is of tubular type, then there exists* $b \in \operatorname{span}_{\mathbb{Z}}(\Phi) \cap R$ *such that* $\Phi = \{\alpha' + ka + lb \mid \alpha' \in \Phi_{\text{fin}},\ k, l \in \mathbb{Z}\}$, *where* Φ_{fin} *is a fixed finite root subsystem of* Φ. *We call* Φ_{fin} *the* canonical finite root subsystem.

(c) The set $R \cap \operatorname{span}_{\mathbb{Z}}(\Phi)$ *is a full lattice in* R. *More concretely, if* Φ *is of wild or domestic type we have* $R \cap \operatorname{span}_{\mathbb{Z}}(\Phi) = \operatorname{span}_{\mathbb{Z}}(a)$ *and else* $R \cap \operatorname{span}_{\mathbb{Z}}(\Phi) = \operatorname{span}_{\mathbb{Z}}(a, b)$, *where* b *is defined as in (b).*

Proof. (a) The group $\langle s_{\alpha_1}, s_{\alpha_{1^*}} \rangle$ is the dihedral group of type $\widetilde{A_1}$, thus the corresponding root system is $\Phi' = \{\pm\alpha_1 + ka \mid k \in \mathbb{Z}\}$ by [62, Proposition 6.3]. Set $\Lambda = \{\overline{\alpha} + ka \mid \overline{\alpha} \in \overline{\Phi},\ k \in \mathbb{Z}\}$. Next we will prove that $\Phi = \Lambda$. Therefore observe that $B \subseteq \Lambda$. Further it is easy to check that $W(\Lambda) \subseteq \Lambda$. Therefore 5.2.1 (d) implies that $\Phi = W(B) \subseteq W(\Lambda) \subseteq \Lambda$. Since $\{\pm\alpha_1 + ka \mid k \in \mathbb{Z}\} \subseteq \Phi$ and since the transitivity of that $\overline{W} \subseteq W$ on $\overline{\Phi}$ (see 5.2.1 (f)) yields $\Lambda \subseteq \Phi$, we get the assertion.

(b) Let Φ be of type $X^{(1,1)} \in \left\{D_4^{(1,1)}, E_n^{(1,1)} \mid n = 6, 7, 8\right\}$, i.e. Φ is tubular (see Figure 5.2). We fix subdiagrams of type X of the extended Coxeter diagram which are shown in Figure 5.3, and let Φ_{fin} be the finite root subsystem of Φ that is generated by a suitable subset $B_{fin} \subseteq \overline{B}$ and corresponds to the fixed subdiagram. Observe that B_{fin} has not to be unique, e.g. for $D_4^{(1,1)}$ there are exactly four different choices.
Let $b = \alpha_0 + \sum_{\nu \in Q} a_\nu \alpha_\nu \in R$, where $\alpha_0 \in \overline{B} \setminus B_{fin}$ and Q is the set of vertices of the Dynkin diagram attached to the fixed subdiagram of type X and a_ν the labelling of

the vertices (see Figure 5.3). Then [62, Proposition 6.3] yields $\overline{\Phi} = \{\alpha' + lb \mid \alpha' \in \Phi_{\text{fin}},\ l \in \mathbb{Z}\}$ and hence (a) implies $\Phi = \{\alpha' + ka + lb \mid \alpha' \in \Phi_{\text{fin}},\ k, l \in \mathbb{Z}\}$.

(c) Let Φ be of tubular type. Since $\Phi = \{\alpha' + ka + lb \mid \alpha' \in \Phi_{\text{fin}},\ k, l \in \mathbb{Z}\}$, $R = \text{span}_{\mathbb{R}}(a, b)$ and $(-,-)$ restricted to $\text{span}_{\mathbb{R}}(\Phi_{\text{fin}})$ is positive definite, we get $R \cap \text{span}_{\mathbb{Z}}(\Phi) = \text{span}_{\mathbb{Z}}(a, b)$. If Φ is of wild or domestic type we have $\Phi = \{\overline{\alpha} + ka \mid \overline{\alpha} \in \overline{\Phi},\ k \in \mathbb{Z}\}$ and $(-,-)$ restricted to $\text{span}_{\mathbb{R}}(\overline{\Phi})$ has trivial radical, we get $R \cap \text{span}_{\mathbb{Z}}(\Phi) = \text{span}_{\mathbb{Z}}(a)$.

□

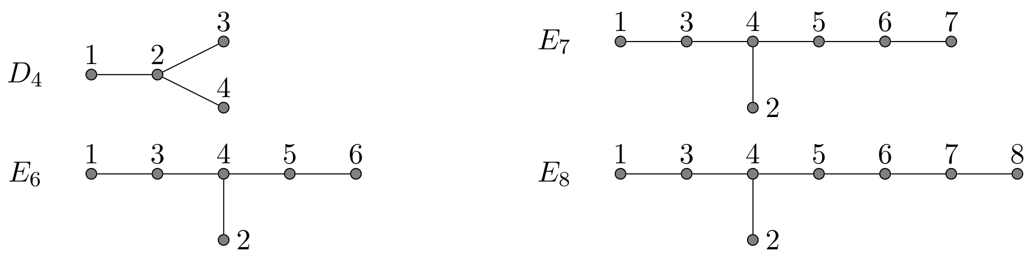


Figure 5.3: Dynkin diagrams with the coefficient of the highest root

The following definition due to Saito is helpful for the understanding of W.

Definition 5.2.4 ([97, (1.14) Definition 1])**.** *Define the following map:*

$$E : V \otimes_{\mathbb{R}} (V/R) \longrightarrow \text{End}_{\mathbb{R}}(V),\ \sum_i u_i \otimes \overline{v}_i \longmapsto \left[x \mapsto x - \sum_i (v_i, x) u_i \right],$$

the Eichler-Siegel map.

We define a binary operation $\circ$ on $V \otimes_{\mathbb{R}} (V/R)$ by setting for $x_1, x_2 \in V \otimes_{\mathbb{R}} (V/R)$:

$$x_1 \circ x_2 = x_1 + x_2 - I(x_1, x_2)$$

where

$$I : (V \otimes_{\mathbb{R}} (V/R)) \times (V \otimes_{\mathbb{R}} (V/R)) \longrightarrow V \otimes_{\mathbb{R}} (V/R),\ (x_1, x_2) \longmapsto I(x_1, x_2)$$

with

$$I(x_1, x_2) := \sum_{i_1, i_2} u_{1i_1} \otimes (v_{1i_1}, u_{2i_2}) \overline{v}_{2i_2}$$

for

$$x_j = \sum_{i_j} u_{ji_j} \otimes \overline{v}_{ji_j} \in V \otimes_{\mathbb{R}} (V/R),\ (j = 1, 2).$$

The operation $\circ$ yields a semi-group structure on $V \otimes_{\mathbb{R}} (V/R)$. We obtain directly:

Proposition 5.2.5. [97, 1.14]

(a) The map E is injective. It is bijective if and only if $R = 0$.

(b) The map E is a homomorphism of semi-groups, i.e. $E(x_1 \circ x_2) = E(x_1)E(x_2)$.

(c) For an anisotropic $v \in V$, the reflection s_v is given by $s_v = E(v \otimes \overline{v})$.

(d) The inverse of the Eichler-Siegel map on W is well defined

$$E^{-1} : W \longrightarrow V \otimes_{\mathbb{R}} (V/R).$$

(e) The subspace $R \otimes_{\mathbb{R}} (V/R)$ is closed under $\circ$, and $\circ$ coincides on $(R \otimes_{\mathbb{R}} (V/R)) \times (R \otimes_{\mathbb{R}} (V/R)$ with the additive structure of $V \otimes_{\mathbb{R}} (V/R)$.

(f) Let $r \in R$ and $v \in V$ anisotropic. Then

$$E((v + r) \otimes \overline{v}) = E(v \otimes \overline{v})E(r \otimes \overline{v}) = s_v E(r \otimes \overline{v}),$$

where $\overline{v}$ is the element $v + R \in V/R$.

Denote by $\overline{L}$ the lattice $\overline{L} = \text{span}_{\mathbb{Z}}(\overline{B})$, $L_a = \text{span}_{\mathbb{Z}}(\{a \otimes \overline{\alpha} \mid \overline{\alpha} \in \overline{\Phi}\})$ and $\mathrm{L}_{\text{fin}} = \text{span}_{\mathbb{Z}}(\Phi_{\text{fin}})$.

Lemma 5.2.6. *Let $\alpha \in \Phi$ and $r \in R$. Then the following holds.*

(a) $s_\alpha s_{\alpha+r} = E(r \otimes \overline{\alpha})$

(b) $wE(r \otimes \overline{\alpha})w^{-1} = E(r \otimes \overline{w(\alpha)})$ for all $w \in W$.

(c) $E(L_a)$ is an abelian group that is normalized by W.

Proof. Assertion (a) follows from Proposition 5.2.5 (c) and (f), and (b) is a consequence of Lemma 2.1.4 (b) and of (a).

By Proposition 5.2.5 (b) and (e) $E(L_a)$ is an abelian group, and it is normalized by W by (b). □

Next we make a first statement on the structure of W.

Lemma 5.2.7. *Let G be a linear subspace of R such that $G \cap \text{span}_{\mathbb{R}}(\Phi)$ is a full lattice in G. Then the natural projection $p : V \to V/G$ induces an epimorphism $\rho : W \to \overline{W}_G$, where $\overline{W}_G = \langle s_{p(\alpha)} \mid \alpha \in \Phi\rangle$ is the induced reflection group acting on V/G. For $G = \text{span}_{\mathbb{R}}(a)$ we get $\overline{W}_G \cong \overline{W}$, and if $G = R$ and W is of tubular type we have $\overline{W}_G \cong W(\Phi_{\text{fin}})$, where $W(\Phi_{\text{fin}})$ is the finite Coxeter group attached to the canonical finite root subsystem Φ_{fin}.*

Proof. As R is the radical of the bilinear form $(-,-)$ attached to V, the form induces a symmetric bilinear form on V/G. Denote by $\overline{\Phi}_G$ the image of Φ in V/G. Since $G \cap \text{span}_{\mathbb{Z}}(\Phi)$ is a full lattice the set $\overline{\Phi}_G$ is a root system in V/G (see [97, (1.8)]). The corresponding reflection group is $\overline{W}_G = \langle s_{\alpha_G} \mid \alpha_G \in \overline{\Phi}_G\rangle$, where s_{α_G} is the unique reflection with $\text{Mov}(s_{\alpha_G}) = \text{span}_{\mathbb{R}}(\alpha_G) \subseteq V/G$ (see Lemma 2.1.4). Now the homomorphism ρ is the continuation of the assignment $s_\alpha \mapsto s_{p(\alpha)}$ for $\alpha \in \Phi$ on W, where $G \subseteq R$ implies that this map is well-defined.

Let $G = \mathrm{span}_{\mathbb{R}}(a)$. We have $V = \mathrm{span}_{\mathbb{R}}(\overline{\Phi}) \oplus G$. As $\overline{W}$ acts faithfully on $\mathrm{span}_{\mathbb{R}}(\overline{\Phi})$, it also acts faithfully on V/G and therefore $\mathrm{Ker}(\rho) \cap \overline{W} = \{1\}$. Since $W = \langle s_\alpha \mid \alpha \in B\rangle$ and since $E(a \otimes \overline{\alpha}_1) = s_{\alpha_1} s_{\alpha_{1*}} \in \ker(\rho)$, it follows by the Dedekind identity that $\ker(\rho)$ equals $E(a \otimes \overline{\alpha}_1)^W$, the normal closure M of $E(a \otimes \overline{\alpha}_1)$ in W, and that $W = \overline{W} \ltimes M$. This yields an isomorphism between $\rho(W)$ and $\overline{W}$.

If Φ is of tubular type we have $V = \mathrm{span}_{\mathbb{R}}(\Phi_{\mathrm{fin}}) \oplus R$, where $R = \mathrm{span}_{\mathbb{R}}(a, b)$ (see Lemma 5.2.3). Analogously, the subgroup $W(\Phi_{\mathrm{fin}})$ of W acts faithfully on V/R and therefore $\mathrm{Ker}(\rho) \cap W(\Phi_{\mathrm{fin}}) = \{1\}$. Again by the Dedekind identity $\mathrm{Ker}(\rho)$ equals the normal subgroup $M = \langle E(a \otimes \alpha_1), E(b \otimes -\widetilde{\alpha})\rangle^W$ of W, and that $W = W(\Phi_{\mathrm{fin}}) \ltimes M$. The latter yields an isomorphism between $\rho(W)$ and $W(\Phi_{\mathrm{fin}})$. □

Proposition 5.2.8 ([97, 1.15], [99, Theorem 3.5]). *Let G be a linear subspace of R such that $G \cap \mathrm{span}_{\mathbb{Z}}(\Phi)$ is a full lattice in G. The sequence*

$$0 \longrightarrow E^{-1}(W) \cap (G \otimes_{\mathbb{R}} (V/R)) \xrightarrow{E} W \xrightarrow{\rho} \overline{W}_G \longrightarrow 1$$

is exact. Further, if $G = \mathrm{span}_{\mathbb{R}}(a)$ it splits and we have $E^{-1}(W) \cap (G \otimes_{\mathbb{R}} (V/R)) = L_a \cong \overline{L}$ and $W = \overline{W} \ltimes E(L_a)$. If $G = R$ and Φ of tubular type we get

$$E^{-1}(W) \cap (R \otimes_{\mathbb{R}} (V/R)) = \{(ka + lb) \otimes \alpha' \mid \alpha' \in \Phi_{\mathrm{fin}},\ k, l \in \mathbb{Z}\} \cong \mathrm{L_{fin}}^2$$

and $W \cong W(\Phi_{\mathrm{fin}}) \ltimes \mathrm{L_{fin}}^2$.

Proof. Let $N = E^{-1}(W) \cap (G \otimes_{\mathbb{R}} (V/R))$. The following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{E} & W & \xrightarrow{\rho} & \overline{W}_G & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle E^{-1}} & & \big\downarrow{\scriptstyle E^{-1}} & & \\
0 & \longrightarrow & G \otimes_{\mathbb{R}} (V/R) & \longrightarrow & V \otimes_{\mathbb{R}} (V/R) & \longrightarrow & (V/G) \otimes_{\mathbb{R}} (V/R) & \longrightarrow & 0,
\end{array}$$

and the second row is exact, which implies the exactness of the first row. In particular, this also implies that $E(N) = \ker(\rho)$. Since the root system is $\Phi = \{\overline{\alpha} + ka \mid \overline{\alpha} \in \overline{\Phi},\ k \in \mathbb{Z}\}$ the continuation of the assignment

$$\overline{W} = \overline{W}_G \longrightarrow W,\ s_{\overline{\alpha}} \longmapsto s_{\overline{\alpha}} \text{ for } \overline{\alpha} \in \overline{\Phi}$$

yields for $G = \mathrm{span}_{\mathbb{R}}(a)$ that ρ is a retraction. Thus the sequence splits. Analogously for $G = R$ and Φ of tubular type it holds with $\Phi = \{\alpha' + ka + lb \mid \alpha' \in \Phi_{\mathrm{fin}},\ k, l \in \mathbb{Z}\}$ that the continuation of the assignment

$$W(\Phi_{\mathrm{fin}}) = \overline{W}_G \longrightarrow W,\ s_{\alpha'} \longmapsto s_{\alpha'} \text{ for } \alpha' \in \Phi_{\mathrm{fin}}$$

yields that ρ is a retraction. Thus the sequence splits.

If $G = \mathrm{span}_{\mathbb{R}}(a)$ the proof of Lemma 5.2.7 shows that $\ker(\rho)$ is $\langle E(a \otimes \overline{\alpha}_1)\rangle^W$, the normal closure of $E(a\otimes\overline{\alpha}_1)$ in W. Since $\overline{W}$ acts transitively on $\overline{\Phi}$ we get that $E(a\otimes\overline{\alpha}) \in \langle E(a\otimes\overline{\alpha}_1)\rangle^W$ for every $\alpha \in \overline{B}$. Therefore $E(L_a) \subseteq \ker(\rho)$, and $\ker(\rho) = E(L_a)$ follows with Lemma 5.2.6 (c).

Let Φ to be of tubular type and $R = \mathrm{span}_{\mathbb{R}}(a, b)$, where b is chosen as in Lemma 5.2.3. As in the proof of Lemma 5.2.7 we get

$$\begin{aligned} W \cap E(R \otimes_{\mathbb{R}} V/R) = E(N) &= \mathrm{Ker}(\rho) \\ &= \langle E(a \otimes \alpha_1), E(b \otimes -\widetilde{\alpha})\rangle^W \\ &= \langle E(r \otimes \alpha') \mid \alpha' \in \Phi_{\mathrm{fin}},\ r \in \mathrm{span}_{\mathbb{Z}}(a, b)\rangle. \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{L_{fin}}^2 &\cong \{(ka + lb) \otimes \alpha' \mid \alpha' \in \Phi_{\mathrm{fin}},\ k, l \in \mathbb{Z}\} \\ &= \mathrm{span}_{\mathbb{Z}}(r \otimes \alpha' \mid \alpha' \in \Phi_{\mathrm{fin}},\ r \in \mathrm{span}_{\mathbb{Z}}(a, b)) \\ &= E^{-1}(W) \cap (R \otimes_{\mathbb{R}} V/R). \end{aligned}$$

□

The following result provides information about the Hurwitz orbits of the element $s_1 s_{1^*}$ and is needed later.

Lemma 5.2.9. *Let $s_{\beta_1} s_{\beta_2} = s_1 s_{1^*}$ with $\beta_1 = \overline{\beta_1} + r_1$ and $\beta_2 = \overline{\beta_2} + r_2$ with $\overline{\beta_1}, \overline{\beta_2} \in \overline{\Phi}$ and $r_1, r_2 \in \mathrm{span}_{\mathbb{Z}}(a)$, then $(s_{\beta_1}, s_{\beta_2})$ and (s_1, s_{1^*}) lie in the same Hurwitz orbit.*

Proof. By Lemma 2.2.4 it suffices to show that s_{β_1}, s_{β_2} are reflections of the Coxeter system $(\langle s_1, s_{1^*}\rangle, \{s_1, s_{1^*}\})$ of type $\widetilde{A_1}$. We can assume that $\overline{\beta_1}, \overline{\beta_2} \in \overline{\Phi}^+ = \overline{\Phi} \cap \mathrm{span}_{\mathbb{Z}_{\geq 0}}(\overline{B})$. It holds by Lemma 5.2.6 and Proposition 5.2.5 (e)

$$E(a \otimes \alpha_1) = s_1 s_{1^*} = s_{\beta_1} s_{\beta_2} = s_{\overline{\beta_1}} s_{\overline{\beta_2}} E\left(r_1 \otimes s_{\overline{\beta_2}}(\overline{\beta_1}) + r_2 \otimes \overline{\beta_2}\right)$$

and hence $s_{\overline{\beta_1}} = s_{\overline{\beta_2}}$. Since $\overline{\Phi}$ is reduced we have $\overline{\beta} := \overline{\beta_1} = \overline{\beta_2}$. The latter implies that

$$a \otimes \alpha_1 = r_1 \otimes s_{\overline{\beta_2}}(\overline{\beta_1}) + r_2 \otimes \overline{\beta_2} = (r_2 - r_1) \otimes \overline{\beta}.$$

Again by the reducibility of $\overline{\Phi}$ we get $\overline{\beta} = \alpha_1$ and $a = r_2 - r_1$. Thus there exists $k \in \mathbb{Z}$ such that $\beta_1 = \alpha_1 + (k-1)a$ and $\beta_2 = \alpha_1 + ka$. Hence by [62, Proposition 6.3] s_{β_1} and s_{β_2} are reflections of $(\langle s_1, s_{1^*}\rangle, \{s_1, s_{1^*}\})$. □

5.2.1 A remark on Coxeter groups whose diagram is a star

In this subsection we present some information on the root system $\overline{\Phi}$ for a Coxeter system $(\overline{W}, \overline{S})$ whose diagram is a simply-laced star. We use the notation as introduced in Definition 5.2.1. In particular, $\overline{B}$ is a simple system for $\overline{\Phi}$, and we use the numbering of the roots in $\overline{B}$ as given in Figure 5.1. The reflection with respect to the root $\alpha_{(i,j)} \in \overline{B}$ will again be abbreviated by s_{ij}. The results obtained in this section will be used in the proof of Theorem 5.4.1.

Lemma 5.2.10. *Let $\alpha \in \overline{\Phi}$ such that*

$$\alpha = \alpha_1 + \sum_{i=1}^{r}\sum_{j=1}^{p_i} \lambda_{ij}\alpha_{(i,j)}$$

where $\lambda_{ij} \in \mathbb{N}_0$. Then for each $i \in \{1, \ldots r\}$ either

(a) $\lambda_{ij} = 0$ for $1 \leq j \leq p_i$, or

(b) there is $m_i \in \{1, \ldots, p_i\}$ such that $\lambda_{ij} = 1$ for $1 \leq j \leq m_i$ and $\lambda_{ij} = 0$ for $m_i < j \leq p_i$.

Proof. First suppose that $\lambda_{\ell 1} = 0$ for some $\ell \in \{1, \ldots, r\}$. Furthermore, assume that $\lambda_{\ell m} \neq 0$ for some $m \in \{2, \ldots, p_\ell\}$. Without loss of generality we may assume that m is maximal with this property, that is, $\lambda_{\ell j} = 0$ for all $j > m$. We calculate

$$s_{\ell 2} \ldots s_{\ell m}(\alpha) = \alpha_1 - \lambda_{\ell m}\alpha_{(\ell,1)} + \sum_{j=2}^{m}(\lambda_{\ell,j-1} - \lambda_{\ell m})\alpha_{(\ell,j)} + \sum_{i\neq\ell}^{r}\sum_{j=1}^{p_i} \lambda_{ij}\alpha_{(i,j)}.$$

Since $s_{\ell 2} \ldots s_{\ell m}(\alpha)$ is a positive root it follows $\lambda_{\ell m} = 0$, contrary to our assumption. This shows (a).

Now assume $\lambda_{\ell 1} \neq 0$. Further let $m \in \{2, \ldots, p_\ell\}$ such that $\lambda_{\ell m} \neq 0$, but $\lambda_{\ell j} = 0$ for all $j > m$. Then

$$s_{\ell 1} \ldots s_{\ell m}(\alpha) = \alpha_1 + (1 - \lambda_{\ell m})\alpha_{(\ell,1)} + \sum_{j=2}^{m}(\lambda_{\ell,j-1} - \lambda_{\ell m})\alpha_{(\ell,j)} + \sum_{i\neq\ell}^{r}\sum_{j=1}^{p_i} \lambda_{ij}\alpha_{(i,j)}.$$

Since $s_{\ell 2} \ldots s_{\ell m}(\alpha)$ is a positive root and $\lambda_{\ell m} \geq 1$ we obtain $\lambda_{\ell m} = 1$, hence $(1 - \lambda_{\ell m}) = 0$. As in the first part of this proof, this yields $\lambda_{\ell,j-1} - \lambda_{\ell m} = 0$ for $j \in \{2, \ldots, m\}$, which shows (b). ☐

Corollary 5.2.11. *The parabolic subgroup $P := \langle s_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq p_i \rangle$ of $\overline{W}$ acts transitively on the set of roots $(\alpha_1 + \Lambda) \cap \overline{\Phi}$, where $\Lambda := \operatorname{span}_{\mathbb{Z}}(\alpha_{(1,1)}, \ldots, \alpha_{(r,p_r)})$.*

Proof. Let $\beta = \sum_{i=1}^{r}\sum_{j=1}^{p_i} \lambda_{ij}\alpha_{(i,j)} \in \Lambda$ and $\alpha = \alpha_1 + \beta$. If $\lambda_{i1} \neq 0$ for some $1 \leq i \leq r$, then there is $m_i \in \{1, \ldots, p_i\}$ such that $\lambda_{ij} = 1$ for $1 \leq j \leq m_i$ and $\lambda_{ij} = 0$ for $m_i < j \leq p_i$

by Lemma 5.2.10. If we set $w := s_{\ell 1} \dots s_{\ell m}$ then we get, as in the proof of Lemma 5.2.10, that $w(\alpha) = \alpha_1 + \sum_{i=1,\ i\neq l}^{r} \sum_{j=1}^{p_i} \lambda_{ij}\alpha_{(i,j)}$ is a linear combination of simple roots that do not contain a root $\alpha_{(l,j)}$ for some $1 \leq j \leq p_l$. Now the assertion follows by induction on r. □

5.3 The conjugacy class and reflection length of Coxeter transformations

5.3.1 The conjugacy class of Coxeter transformations

We use the notation as introduced in Figure 1 and abbreviate the simple reflection with respect to the root $\alpha_{(i,j)}$ by s_{ij}. As defined in 5.2.1 (g), the element

$$c = (\prod_{i=1}^{r} \prod_{j=1}^{p_i} s_{ij}) s_1 s_1^* \in W$$

is a Coxeter transformation of (W, S). We show that the ordering of the reflections s_{ij} where $1 \leq i \leq r$ and $1 \leq j \leq p_i$ does not matter up to conjugacy.

Lemma 5.3.1. *Let (W, S) be an extended Weyl system. Then in W all the Coxeter transformations are conjugated.*

Proof. Let d be another Coxeter transformation of (W, S). Because of the shape of the diagram and the definition of a Coxter transformation, we have

$$d = (\prod_{i=1}^{r} \prod_{j=1}^{p_i} s_{i\pi_i(j)}) s_1 s_1^*,$$

where π_i is in $\mathrm{Sym}(p_i)$ for $1 \leq i \leq r$.

Thus $c_i := \prod_{j=1}^{p_i} s_{ij}$ and $d_i := \prod_{j=1}^{p_i} s_{i\pi_i(j)}$ are Coxeter elements in a Coxeter system Λ_i of type A_{p_i} with set of simple reflections $S_i := \{s_{ij} \mid 1 \leq j \leq p_i\}$ for every $1 \leq i \leq r$.

We claim that $P_i := \langle s_{i2}, \dots, s_{ip_i} \rangle$ acts transitively on the set of Coxeter elements in $W_i := \langle S_i \rangle$, more precisely that there is $x_i \in P_i$ such that $c_i^{x_i} = d_i$. As (W_i, S_i) is a Coxeter system of type A_{p_i}, the Coxeter elements c_i and d_i are (p_i+1)-cycles in $W_i \cong \mathrm{Sym}(p_i + 1)$ (see [4]). Without loss of generality we can assume that $c_i = (1\ 2\ \dots\ p_i+1)$ and $d_i = (1\ k_2\ \cdots\ k_{p_i+1})$. Then $x_i \in P_i$ defined by $x_i(j) := k_j$ for $2 \leq j \leq p_i + 1$ maps c_i onto d_i. Now it follows, as every element of P_i commutes with s_1, s_1^* as well as with every element in S_k for $k \neq i$, that $x := x_1 \cdots x_r$ conjugates c onto d. □

Remark 5.3.2. *Let (W, S) be an extended Weyl system of domestic type. Direct calculations show that the Coxeter transformation c defined above is conjugate to a Coxeter element in the affine Coxeter system (W, S'), see Remark 5.2.2. This yields that every Coxeter* *transformation of the extended Weyl system (W, S) is also a Coxeter element of the affine Coxeter system (W, S'), but not vice versa.*

5.3.2 The reflection length of Coxeter transformations

In this section we prove that the factorization of the Coxeter transformation c of the extended Weyl Group W in pairwise different simple reflections is $T-$reduced, i.e. the reflection length is given by

$$\ell_T(c) = \min\{n \in \mathbb{N}_0 \mid t_1 \cdots t_n = c,\ t_i \in T \text{ for } 1 \leq i \leq n\} = |S|.$$

A proof for the tubular cases can be found in [107, Proposition 7.4.7]. The proof stated in this section is a consequence of Scherk's Theorem. In order to formulate Scherk's theorem we like to recall some notation. Let k be a field of characterstic different from 2, and U a finite dimensional k-vector space that is equipped with a symmetric bilinear form $(-,-)$. A null-space (totally isotropic subspace) of $(U, (-,-))$ is a subspace that consists only of isotropic vectors, that is, as $\mathrm{char}(k) \neq 2$, a subspace where $(-,-)$ vanishes.

Theorem 5.3.3 (Scherk, [100, Theorem 260.1])**.** *Let k be a field of characterstic different from 2, U a finite dimensional $k-$vector space with symmetric bilinear form $(-,-)$. Further let $\sigma \neq 1_U$ be an isometry of U. Let $F = Fix(\sigma)$ be the fixed space of σ in U, and n the dimension of $F^\perp$, the orthogonal complement of F in U. If $F^\perp$ is not a null space, then σ is the product of n and not less than n reflections. If $F^\perp$ is a null space, then σ is the product of $n + 2$ and not less than $n + 2$ reflections.*

Let n be the rank of W, i.e. $|S| = n$ and recall that we denote the radical of the bilinear form $(-,-)$ attached to W by R and let Φ the root system attached to W. The following lemma is a generalization of Lemma 4.2.10. It can be proven in the same way.

Lemma 5.3.4. *Let $\beta_1, \ldots, \beta_m \in \Phi$ be linear independent and $v \in V$, then $s_{\beta_1} \cdots s_{\beta_m}(v) = v$ if and only if $s_{\beta_i}(v) = v$ for all $1 \leq i \leq m$.*

Lemma 5.3.5. *Let c be a Coxeter transformation of an extended Weyl group W. The fixed space of c coincides with R.*

Proof. As usual, denote by Φ the corresponding root system and $V = \mathrm{span}_{\mathbb{R}}(\Phi)$. The proof is based on Lemma 5.3.4 and the following fact. Let $\alpha \in \Phi$, then

$$\mathrm{Fix}(s_\alpha) := \{v \in V \mid s_\alpha(v) = v\} = \alpha^\perp.$$

The latter implies $\mathrm{Fix}(c) = \bigcap_{\alpha \in B} \mathrm{Fix}(s_\alpha) = \bigcap_{\alpha \in B} \alpha^\perp$. Since B is a basis of V the assertion $\mathrm{Fix}(c) = R$ follows. □

Now we obtain as a corollary the main result of this section.

Corollary 5.3.6. *Let W be an extended Weyl group of arbitrary type of rank n. The Coxeter transformations are the product of at least n reflections. In particular, $\ell_T(c) = n$ for each Coxeter transformation c.*

Proof. Let c be a Coxeter transformation and $\mathrm{Fix}(c)$ its fixed space. By Lemma 5.3.5 it holds $\dim_{\mathbb{R}}(\mathrm{Fix}(c)^{\perp}) = n$ and since $\mathrm{Fix}(c)^{\perp}$ is not a null space, as for instance $\alpha_1 \in \mathrm{Fix}(c)^{\perp}$, Scherk's Theorem 5.3.3 yields the assertion. □

5.4 Hurwitz action for the wild and domestic cases

Let (W, S) be an extended Weyl system of rank m and of wild or domestic type with simple system S, set of reflections T and $c \in W$ a Coxeter transformation, that is c admits a factorization into pairwise different simple reflections such that $s_1 s_{1^*}$ is a suffix of this factorization. The goal of this section is to prove the Hurwitz transitivity on the set of reduced reflection factorizations of c. As usual we denote the set of reduced reflection factorizations of c by $\mathrm{Red}_T(c) = \{(t_1, \dots, t_m) \in T^m \mid c = t_1 \cdots t_m\}$.

Theorem 5.4.1. *Let (W, S) be an extended Weyl system of rank m of wild or domestic type with set of reflections T and Coxeter transformation $c \in W$. The Hurwitz action is transitive on the set of reduced reflection factorization of c, namely on the set $Red_T(c)$.*

We first start with the definition of a normal form in W.

5.4.1 The normal form in W

Since the $R = \mathrm{span}_{\mathbb{R}}(a)$ with $a = \alpha_{1^*} - \alpha_1$ Proposition 5.2.8 yields a normal form for the elements in W. Observe if $w \in W$, then $w = \overline{w}P(w)$ where $\overline{w} = \rho(w) \in \overline{W}$ and P is the projection of W onto $\ker(\rho)$. It is $P(w) = E(a \otimes \overline{\beta})$ for some $\beta \in \overline{L}$ by Proposition 5.2.8.

Definition 5.4.2. *Define a map* $\mathrm{tr} : W \longrightarrow \overline{L}$ *by setting for* $w \in W$

$$\mathrm{tr}(w) = \beta \text{ if } P(w) = E(a \otimes \overline{\beta}) \text{ where } \beta \in \overline{L}.$$

We call the pair $(\overline{w}, \mathrm{tr}(w))$ *the* normal form *of* w *and* $\mathrm{tr}(w)$ *the* translation vector *of* w.

Lemma 5.4.3. *The following holds.*

(a) Let $w \in W$ such that $\mathrm{tr}(w) = \sum_{\beta \in \overline{B}} m_\beta \cdot \beta$ where $m_\beta \in \mathbb{Z}$. Then

$$w = \overline{w} \prod_{\beta \in \overline{B}} (s_\beta s_{\beta + a})^{m_\beta}.$$

(b) Let $\alpha = \overline{\alpha} + ka \in \Phi$ where $\overline{\alpha} \in \overline{\Phi}$ and $k \in \mathbb{Z}$. Then $\mathrm{tr}(s_\alpha) = k\overline{\alpha}$.

(c) For $\gamma \in \overline{B}$, $m_\gamma \in \mathbb{Z}$ and $y \in \overline{W}$ we have

$$\operatorname{tr}(y^{-1} \prod_{\gamma \in \overline{B}} (s_\gamma s_{\gamma+a})^{m_\gamma} y) = \sum_{\gamma \in \overline{B}} m_\gamma \cdot y^{-1}(\gamma) = y^{-1}(\operatorname{tr}\left(\prod_{\gamma \in \overline{B}} (s_\gamma s_{\gamma+a})^{m_\gamma}\right)).$$

Proof. Lemma 5.2.6 (a) and Proposition 5.2.5 (e) yield (a), and Proposition 5.2.5 (f) assertion (b). The third assertion is a direct consequence of Lemma 5.2.6. □

Lemma 5.4.4. *The translation vector satisfies the following properties.*

(a) $\operatorname{tr}(s_{1^*}) = \alpha_1$,
(b) $\operatorname{tr}(s) = 0$ *for all* $s \in S \setminus \{s_{1^*}\}$ *and*
(c) $\operatorname{tr}(xy) = y^{-1}\operatorname{tr}(x) + \operatorname{tr}(y)$ *for all* $x, y \in W$.

The translation vector of an element of W is uniquely determined by the properties (a) – (c).

Proof. Lemma 5.4.3 (b) yields $\operatorname{tr}(s_{\alpha_{1^*}}) = \operatorname{tr}(s_{\alpha_1+a}) = \alpha_1$, which is (a). Assertion (b) follows from Proposition 5.2.5 (c) and (f). Next we prove (c). Let $x, y \in W$ and $\beta_x = \operatorname{tr}(x)$, $\beta_y = \operatorname{tr}(y)$. Then by Lemma 5.2.6 (b)

$$xy = \overline{x}E(a \otimes \beta_x)\overline{y}E(a \otimes \beta_y) = \overline{x}\overline{y}\overline{y}^{-1}E(a \otimes \beta_x)\overline{y}E(a \otimes \beta_y) = \overline{x}\overline{y}E(a \otimes (\overline{y}^{-1}(\beta_x)) + \beta_y),$$

which yields (c). □

Corollary 5.4.5. *Let $t \in W$. Then $t \in T$ if and only if t has normal form $(s_\alpha, k\alpha)$ for some $k \in \mathbb{Z}$ and $\alpha \in \overline{\Phi}$.*

Proof. If $t \in T$, then $t = ws_\beta w^{-1} = s_{w(\beta)}$ for some $\beta \in B$ and $w \in W$ by the definition of T and by Lemma 2.1.4 (b). Then $\alpha := w(\beta) \in \Phi$ and $\alpha = \overline{\alpha} + ka$ where $\overline{\alpha} \in \overline{\Phi}$ by Lemma 5.2.3 (a). Thus $t = s_\alpha = s_{\overline{\alpha}}E(ka \otimes \overline{\alpha})$ by 5.2.5 (f), and the normal form of t is $(s_{\overline{\alpha}}, k\overline{\alpha})$.

If on the other hand $t \in W$ has normal form $(s_\alpha, k\alpha)$ for some $\alpha \in \overline{\Phi}$ and $k \in \mathbb{Z}$, then

$$t = s_\alpha E(ka \otimes \overline{\alpha}) = E((\alpha + ka) \otimes \overline{\alpha}) = E((\alpha + ka) \otimes (\overline{\alpha + ka})) = s_{\alpha+ka}$$

by Proposition 5.2.5 (c) and (f). □

The proof of the following statement uses a generalization of a result from affine simply-laced Coxeter groups (see [108, Lemma 2.11]).

Lemma 5.4.6. *Let $\beta_1, \dots, \beta_n \in \Phi$ such that $\overline{\beta_1}, \dots, \overline{\beta_n} \in \overline{\Phi}$ are linear independent. Then* $\operatorname{tr}(s_{\beta_1} \cdots s_{\beta_n}) = 0$ *if and only if* $\operatorname{tr}(s_{\beta_1}) = \ldots = \operatorname{tr}(s_{\beta_n}) = 0$.

Proof. It is easy to prove that (see [108, Lemma 2.11])

$$\mathrm{tr}(s_{\beta_1}\cdots s_{\beta_n}) = \sum_{i=0}^{n-2} s_{\overline{\beta_n}}\cdots s_{\overline{\beta_{n-i}}}\left(\mathrm{tr}\left(s_{\beta_{n-i-1}}\right)\right) + \mathrm{tr}\left(s_{\beta_n}\right).$$

Using the previous formula in an easy induction on the number of factors, the assertion follows. □

5.4.2 The proof of the Hurwitz transitivity: Theorem 5.4.1

Notice that by Corollary 5.3.6 the elements of $\mathrm{Red}_T(c) = \{(t_1,\ldots,t_m) \in T^m \mid t_1\cdots t_m = c\}$ are the shortest possible reflection factorizations of c.

Proof of Theorem 5.4.1 Let $c = s_1'\cdots s_{m-2}'s_1s_{1^*}$ be a factorization of a Coxeter transformation c in (W,S) in the pairwise different simple reflections of the extended Weyl group (W,S), that is $S = \{s_1',\ldots,s_{m-2}',s_1,s_{1^*}\}$. By Corollary 5.3.6 this factorization is $T-$reduced. Let $\alpha_i' \in \overline{\Phi}$ be the corresponding simple roots such that $s_{\alpha_i'} = s_i'$ for $1 \le i \le m-2$ and $s_{\alpha_1} = s_1,\ s_{\alpha_{1^*}} = s_{1^*}$. Further fix a reduced factorization $(t_1,\ldots,t_m) \in \mathrm{Red}_T(c)$. We prove the theorem by showing that there exists a braid $\tau \in \mathcal{B}_m$ such that

$$\tau(t_1,\ldots,t_m) = (s_1',\ldots,s_{m-2}',s_1,s_{1^*}).$$

Consider the reflection factorization $(\overline{t_1},\ldots,\overline{t_m}) \in \overline{T}^m$ induced by $(t_1,\ldots,t_m) \in W^m$ (see Corollary 5.4.5). In $\overline{W}$ we calculate

$$\overline{t_1}\cdots\overline{t_m} = \overline{c} = s_1'\cdots s_{m-2}'s_1\overline{s_{1^*}} = s_1'\cdots s_{m-2}'$$

where $\overline{s_i'} = s_i'$ for $1 \le i \le m$ and $\overline{s_{1^*}} = s_1$. Since $\ell_{\overline{S}}(\overline{c}) = m-2$, Lemma 3.2.10 yields the existence of a braid $\tau_1 \in \mathcal{B}_{m+2}$ and a reflection $t \in \overline{T}$ such that

$$\tau_1(\overline{t_1},\ldots,\overline{t_m}) = (\overline{t_1'},\ldots,\overline{t_{m-2}'},t,t).$$

Thus we have

$$\overline{t_1'}\cdots\overline{t_{m-2}'} = \overline{t_1'}\cdots\overline{t_{m-2}'}tt = \overline{c} = s_1'\cdots s_{m-2}'.$$

Therefore $\overline{t_1'}\cdots\overline{t_{m-2}'}$ is a reduced reflection factorization of the parabolic Coxeter element $\overline{c}$ of the Coxeter system $(\overline{W},\overline{S})$. By Theorem 3.3.3 there exists a braid $\tau_2 \in \mathcal{B}_m$ such that

$$\tau_2(\overline{t_1'},\ldots,\overline{t_{m-2}'},t,t) = (s_1',\ldots,s_{m-2}',t,t).$$

Applying the braid $\tau_2\tau_1$ to the initial factorization and using Lemma 5.2.7, we obtain

$$\tau_2\tau_1(t_1,\ldots,t_m) = (t_1'',\ldots,t_{m-2}'',t_a,t_b)$$

where $t''_1, \dots, t''_{m-2}, t_a, t_b \in T$ such that $\overline{t''_i} = s'_i$ for $1 \leq i \leq m-2$ and $\overline{t_a} = \overline{t_b} = t$. Let $\alpha \in \overline{\Phi}^+$ such that $t = s_\alpha$. Then $t_a t_b$ has normal form $(1, \lambda\alpha)$ for some $\lambda \in \mathbb{Z}$ by Lemma 5.2.6 (a) and Corollary 5.4.5. Therefore we obtain using Lemma 5.4.4 (c), and setting $\beta := \text{tr}(t''_1 \cdots t''_{m-2})$:

$$\alpha_1 = \text{tr}(c) = \text{tr}(t''_1 \cdots t''_{m-2} t_a t_b) = \beta + \text{tr}(t_a t_b) = \beta + \lambda\alpha.$$

The fact that $\overline{t''_i} = s'_i$ for $1 \leq i \leq m-2$ yields $\overline{t''_1 \cdots t''_{m-2}} \in P := \langle s'_1, \dots, s'_{m-2} \rangle$ and $\beta \in \text{span}_{\mathbb{Z}}(\alpha'_1, \dots, \alpha'_{m-2})$. As $\{\alpha_1, \alpha'_1, \dots, \alpha'_{m-2}\}$ is linearly independent, it follows $\lambda \neq 0$ and therefore

$$\alpha = \frac{1}{\lambda}\alpha_1 - \frac{1}{\lambda}\beta \in \overline{\Phi}^+.$$

The latter implies, as $\alpha \in \overline{\Phi}^+$, that $\lambda = 1$. Hence $\alpha = \alpha_1 - \beta$.

By Corollary 5.2.11 there exists $x \in P$ such that $t^x = s_\alpha^x = s_1$ and therefore we get $\overline{t_a^x} = \overline{t_a}^x = s_1$. Since $\overline{t''_i} = s'_i$ for $1 \leq i \leq m-2$, Lemma 3.3.7 yields the existence of a braid $\tau_3 \in \mathcal{B}_m$ such that

$$\tau_3(t''_1, \dots, t''_{m-2}, t_a, t_b) = (t''_1, \dots, t''_{m-2}, t_a^x, t_b^x).$$

Set $t'_a = t_a^x$ and $t'_b = t_b^x$, and observe that $\text{tr}(t'_a t'_b) = \lambda' \alpha_1$ for some $\lambda' \in \mathbb{Z}$.

Similar as above we obtain by Lemma 5.4.4(c)

$$\alpha_1 = \text{tr}(c) = \text{tr}(t''_1 \cdots t''_{m-2} t'_a t'_b) = \beta + \text{tr}(t'_a t'_b) = \beta + \lambda'\alpha_1,$$

which yields, as α_1 and β are linearly independent, that $\beta = 0$ and $\lambda' = 1$.

Since the roots related to the reflections $\overline{t''_i}$ for $1 \leq i \leq m-2$ are linearly independent, Lemma 5.4.6 yields that

$$\text{tr}(t''_1) = \ldots = \text{tr}(t''_{m-2}) = 0,$$

and therefore $t''_i = s'_i$ for $1 \leq i \leq m-2$. From $\overline{t'_a} = \overline{t'_a} = s_1$ and $\lambda' = 1$ follows $t'_a t'_b = s_1 s_{1^*}$. Therefore by Lemma 5.2.9 there exists a braid $\tau_4 \in \mathcal{B}_m$ such that

$$\tau_4(s'_1, \dots, s'_{m-2}, t'_a, t'_b) = (s'_1, \dots, s'_{m-2}, s_1, s_{1^*}).$$

Altogether, setting $\tau := \tau_4 \tau_3 \tau_2 \tau_1 \in \mathcal{B}_m$, we obtain

$$\tau(t_1, \dots, t_m) = (s'_1, \dots, s'_{m-2}, s_1, s_{1^*}),$$

the assertion. □

Corollary 5.4.7. *Let (W, S) an extended Weyl system of rank m of wild or domestic type. For any Coxeter transformation c and any reduced reflection factorization $(t_1, \dots, t_m) \in Red_T(c)$ we have $W = \langle t_1, \dots, t_m \rangle$.*

Proof. Two reflection factorizations in the same Hurwitz orbit generate the same group and since there exists exactly one Hurwitz orbit any reduced reflection factorization generates W. □

5.5 Hurwitz action for the tubular cases

Let (W, S) be an extended Weyl system of tubular type of rank m and Φ its root system. In this section we prove the following theorem about the Hurwitz action on the set of reduced reflection factorizations that generate the group W.

Theorem 5.5.1. *Let (W, S) be an extended Weyl system of rank m of tubular type with simple system S, set of reflections T and Coxeter transformation $c \in W$. The Hurwitz action is transitive on the set of reduced reflection factorization of c that generates the group W, namely on the set $\underline{\mathrm{Red}}_T(c) := \{(t_1, \dots, t_m) \in T^m \mid c = t_1 \cdots t_m,\ W = \langle t_1, \dots, t_m \rangle\}$.*

In the following we use the notation of [18]. Since Φ is of type $X^{(1,1)} \in \{D_4^{(1,1)}, E_n^{(1,1)} \mid n = 6, 7, 8\}$ the canonical finite root subsystem Φ_{fin} of $\Phi = \{\alpha' + ka + lb \mid \alpha' \in \Phi_{\text{fin}},\ k, l \in \mathbb{Z}\}$ (see Lemma 5.2.3 (b)) is of type $X \in \{D_4, E_n \mid n = 6, 7, 8\}$. Let $\Phi_{\text{af}} := \{\alpha' + lb \mid \alpha' \in \Phi_{\text{fin}},\ l \in \mathbb{Z}\}$, then Φ_{af} is by [62, Proposition 6.3] an affine root system of type $\widetilde{X}$. Put $\alpha_0 = -\widetilde{\alpha} + b$, where $\widetilde{\alpha}$ is the highest root of the finite root system Φ_{fin}. Let $s_i = s_{\alpha_i}$ and $s_{t^*} = s_{\alpha_{t^*}}$ the simple reflections of (W, S) for $t = 2$ in the case $X = D_4$ and $t = 4$ for $X \in \{E_n \mid n = 6, 7, 8\}$ and $i \geq 0$, where we use the numbering described in [18].

The next example shows that the generating property of the reduced reflection factorizations in Theorem 5.5.1 is necessary and therefore can not be omitted. With other words, if W is tubular we have $\mathrm{Red}_T(c) \neq \underline{\mathrm{Red}}_T(c)$ for every Coxeter transformation c.

Example 5.5.2. *Consider the root system of type $E_6^{(1,1)}$ and let $\{\alpha_1, \dots, \alpha_6\}$ be a simple system for Φ_{fin} of type E_6. Since by Lemma 5.3.1 all Coxeter transformations are conjugated it is sufficient to investigate $c = s_1 s_2 s_3 s_5 s_6 s_0 s_4 s_{4^*}$. Consider the roots*

$$\beta_1 = \alpha_1 - a,\ \beta_2 = \alpha_2 + 2a,\ \beta_3 = \alpha_3 + 2a,\ \beta_4 = \alpha_4 - 3a,$$
$$\beta_5 = \alpha_5 + 2a,\ \beta_6 = \alpha_6 - a,\ \beta_7 = \widetilde{\alpha} - a - b,\ \beta_8 = \alpha_4 + a,$$

then a direct calculation yields $c = s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\beta_5} s_{\beta_6} s_{\beta_7} s_{\beta_4} s_{\beta_8}$, but this is not a generating factorization. In particular, $(s_{\beta_1}, s_{\beta_2}, s_{\beta_3}, s_{\beta_5}, s_{\beta_6}, s_{\beta_7}, s_{\beta_4}, s_{\beta_8})$ and $(s_1, s_2, s_3, s_5, s_6, s_0, s_4, s_{4^*})$ do not lie in the same Hurwitz orbit. The calculations can be found in Calculation A5.5.2.

Non-generating reduced factorizations of Coxeter transformations in the remaining root systems of tubular type can also be found in Example A5.5.2.

For the rest of this section we fix the Coxeter transformation

$$c = s_1 s_3 s_4 s_0 s_2 s_{2^*}$$

in the case $\Phi = D_4^{(1,1)}$ and

$$c = s_1 \cdots \widehat{s_4} \cdots s_n s_0 s_4 s_{4^*}$$

in the cases $\Phi = E_n^{(1,1)}$ $(n = 6, 7, 8)$. Let $n := m - 2$ and $(t_1, \dots, t_{n+2}) \in \underline{\mathrm{Red}}_T(c)$ be a factorization of c such that $W = \langle t_1, \dots, t_{n+2} \rangle$, and for $1 \leq i \leq n+2$ let

$$k_i, \ell_i \in \mathbb{Z} \text{ and } \beta_i \in \Phi_{\text{fin}}^+ := \mathrm{span}_{\mathbb{Z}_{\geq 0}}(\alpha_1, \dots, \alpha_n) \cap \Phi_{\text{fin}}$$

such that $t_i = s_{\beta_i + k_i a + \ell_i b}$.

First we study $\rho(c)$ in the projection $\overline{W}_G$, where we identified $W_a := \overline{W}$ with $\overline{W}_G$ for $G = \mathrm{span}_{\mathbb{R}}(a)$ (see Lemma 5.2.7). Since W_a is the group corresponding to the root system $\Phi_{\text{af}} = \{\alpha' + lb \mid \alpha' \in \Phi_{\text{fin}},\ l \in \mathbb{Z}\}$ it is an affine Coxeter group with simple system $S_a = \{s_0, s_1, \dots, s_n\}$, where $s_0 = s_{-\widetilde{\alpha}+b}$ and $\widetilde{\alpha}$ is the highest root in Φ_{fin} in terms of the simple system B_{fin} that is defined in Lemma 5.2.3 (b). In the following we denote by $\overline{w}$ the image of w in W_a, i.e. for $s_{\beta_i + k_i a + l_i b}$ with $\beta_i \in \Phi_{\text{fin}}$, $k_i, l_i \in \mathbb{Z}$ we have $\overline{t}_i = s_{\beta_i + l_i b}$ for $1 \leq i \leq n+2$.

It follows that

$$\overline{c} = \overline{t_1} \cdots \overline{t_{n+2}} = s_1 \cdots \widehat{s_t} \cdots s_n s_0$$

where $t = 2$ for $\Phi = D_4^{(1,1)}$ and $t = 4$ for $\Phi = E_n^{(1,1)}$ $(n = 6, 7, 8)$. Thus $\overline{c}$ is a standard parabolic Coxeter element in (W_a, S_a). Therefore $\ell_{S_a}(\overline{c}) = n = \ell_{T_a}(\overline{c})$, where T_a is the set of reflections in W_a, and ℓ_{S_a} (resp. ℓ_{T_a}) is the canonical length function attached to the generating set S_a (resp. T_a) (see Lemma 3.3.2). By Lemma 3.2.10 there exists a braid $\tau_1 \in \mathcal{B}_{n+2}$ such that

$$\tau_1(\overline{t_1}, \dots, \overline{t_{n+2}}) = (\overline{r_1}, \dots, \overline{r_n}, \overline{r_{n+1}}, \overline{r_{n+1}}),$$

where $\overline{r_i} \in T_a$ for $1 \leq i \leq n+1$. Note that $\overline{c} = \overline{r_1} \cdots \overline{r_n}$ is a reduced reflection factorization of a parabolic Coxeter element in W_a. By Theorem 3.3.3 there exists a braid $\tau_2 \in \mathcal{B}_{n+2}$ such that

$$\tau_2(\overline{r_1}, \dots, \overline{r_n}, \overline{r_{n+1}}, \overline{r_{n+1}}) = (s_1, \dots, \widehat{s_t}, \dots, s_n, s_0, \overline{r_{n+1}}, \overline{r_{n+1}}).$$

Let $\beta \in \Phi_{\text{fin}}^+$ and $k \in \mathbb{Z}$ such that $\overline{r_{n+1}} = s_{\beta + kb}$, and recall that $s_0 = s_{\alpha_0} = s_{-\widetilde{\alpha}+b}$, where $\widetilde{\alpha} = \sum_{i=1}^n m_i \alpha_i$ is the highest root in Φ_{fin}. In particular, there are $k_1, \dots, k_n, \ell, \ell', \overline{\ell} \in \mathbb{Z}$ such that

$$\tau_2\tau_1(t_1, \dots, t_{n+2}) = (s_{\alpha_1 + k_1 a}, \dots, \widehat{s_{\alpha_t + k_t a}}, \dots, s_{\alpha_n + k_n a}, s_{-\widetilde{\alpha} + b + \overline{\ell} a}, s_{\beta + \ell a + kb}, s_{\beta + \ell' a + kb}). \tag{5.1}$$

Before we investigate the root β we need the following useful lemma.

Lemma 5.5.3. *Let (W, S) be an extended Weyl system, c a Coxeter transformation, Φ the corresponding root system and $\langle s_{\beta_1}, \dots, s_{\beta_m} \rangle = W$ with $\beta_i \in \Phi$ for all $1 \leq i \leq m$, then*

$\text{span}_{\mathbb{Z}}(\Phi) = \text{span}_{\mathbb{Z}}(\beta_1, \dots, \beta_m)$.

Proof. Since the extended Coxeter diagram in Figure 5.1 has a spanning tree with simple edges all the simple reflections are conjugated and thus all reflections lie in the same conjugacy class. Let α be a root. Then by the previous fact there exists $w \in W = \langle s_{\beta_1}, \dots, s_{\beta_m} \rangle$ such that $w(\beta_1) = \alpha$. Thus by the definition of the reflections $\alpha = w(\beta_1) \in \text{span}_{\mathbb{Z}}(\beta_1, \dots, \beta_m)$. Hence we get $\text{span}_{\mathbb{Z}}(\beta_1, \dots, \beta_m) = \text{span}_{\mathbb{Z}}(\Phi)$. □

Lemma 5.5.4. *Let $\lambda_j \in \mathbb{Z}_{\geq 0}$ such that $\beta = \sum_{i=1}^{n} \lambda_i \alpha_i \in \Phi^+_{\text{fin}}$. Then*

(a) $(\lambda_t, k) \in \{(1, 0), (m_t - 1, -1)\}$ and

(b) $|\ell - \ell'| = 1$.

Proof. The reflections $t_1, \dots, t_{n+2}$ generate W by assumption and the Hurwitz action preserves this property. In particular, the tuple obtained by applying the braid $\tau_2\tau_1$ to the tuple $(t_1, \dots, t_{n+2})$ generates W. Therefore by Lemma 5.5.3 the equation (5.1) yields another generating set for the lattice $Z := \text{span}_{\mathbb{Z}}(\Phi)$

$$\begin{aligned} Z &= \text{span}_{\mathbb{Z}}(\alpha_1 + k_1 a, \dots, \widehat{\alpha_t + k_t} a, \dots, \alpha_n + k_n a, -\widetilde{\alpha} + b + \overline{\ell} a, \beta + \ell a + kb, \beta + \ell' a + kb) \\ &= \text{span}_{\mathbb{Z}}(\alpha_1 + k_1 a, \dots, \widehat{\alpha_t + k_t} a, \dots, \alpha_n + k_n a, -\widetilde{\alpha} + b + \overline{\ell} a, \beta + \ell a + kb, (\ell - \ell')a). \end{aligned}$$

In particular, the rank of Z is $n+2$ and since $a \in Z \cap \text{span}_{\mathbb{R}}(B)$ we get that $\ell - \ell' \in \mathbb{Z}^* = \{\pm 1\}$, which shows (b). Therefore

$$Z = \text{span}_{\mathbb{Z}}(\alpha_1, \dots, \widehat{\alpha}_t, \dots, \alpha_n, -\widetilde{\alpha} + b, \beta + kb, a).$$

Let us show part (a). First assume that $\lambda_t = 0$. Since $\alpha_t \in Z$ there exist $\mu_1, \dots, \mu_n, \mu, \mu' \in \mathbb{Z}$ with

$$\alpha_t = \sum_{i=1,\ i \neq t}^{n} \mu_i \alpha_i + \mu(-\widetilde{\alpha} + b) + \mu'(\beta + kb).$$

Therefore we get $1 = -m_t \mu + \lambda_t \mu' = -m_t \mu$ and in particular $m_t \in \{\pm 1\}$. By [18] the following holds:

Φ_{fin}	D_4	E_6	E_7	E_8
m_t	2	3	4	6

which yields a contradiction. Thus we can assume that $\lambda_t > 0$. The fact that $\alpha_t \in Z$ implies

$$\alpha_t = \sum_{i=1,\ i \neq t}^{n} \mu_i \alpha_i + \mu(-\widetilde{\alpha} + b) + \mu'(\beta + kb),$$

and therefore

$$0 = \mu + \mu' k \text{ and } 1 = -m_t \mu + \lambda_t \mu'.$$

Thus

$$1 = m_t\mu' k + \lambda_t\mu' = \mu'(m_t k + \lambda_t).$$

Hence $\mu', m_t k + \lambda_t \in \{\pm 1\}$. As $1 \leq \lambda_t \leq m_t$ (see [18]), we get $(k, \lambda_t) = (0, 1)$ if $m_t k + \lambda_t = 1$ and $(k, \lambda_t) = (-1, m_t - 1)$ if $m_t k + \lambda_t = -1$, which is the assertion. □

Lemma 5.5.5. *The reflection s_β is conjugate to s_t under the group $H := \langle s_{\widetilde{\alpha}}, s_1, \ldots, \widehat{s_t}, \ldots, s_n\rangle$.*

Proof. By Lemma 5.5.4 we have $\beta = \sum_{i=1}^{n} \lambda_i\alpha_i$ where $\lambda_t \in \{1, m_t - 1\}$.

Assume $\lambda_t > 1$. Then $\lambda_t = m_t - 1$ and Φ_{fin} is of type E_n for some $n \in \{6, 7, 8\}$. According to [18, Plates V- VI] the highest root $\tilde{\alpha}$ is perpendicular to $\alpha_1, \ldots, \alpha_n$ beside one α_j. Moreover in loc. cit. all the roots β are listed that have the property $\lambda_t = m_t - 1$. It is easily checked that for all these roots $m_j = 1$. Therefore it follows that $s_{\tilde{\alpha}}(\beta) = \sum_{i=1}^{n} \mu_i\alpha_i$ with $\mu_i \in \mathbb{Z}$ and $\mu_t = 1$. Thus we may assume that $\lambda_t = 1$. Then Lemma 5.2.10 yields $\lambda_j \in \{0, 1\}$ for all $1 \leq j \leq n$. In this case it is easy to see that s_β is conjugated to s_t under H. □

By Lemma 5.5.5 and Lemma 3.3.7 we obtain that there exists a braid $\tau_3 \in \mathcal{B}_{n+2}$ such that

$$\tau_3(s_{\alpha_1+k_1a}, \ldots, \widehat{s_{\alpha_t+k_ta}}, \ldots, s_{\alpha_n+k_na}, s_{-\widetilde{\alpha}+b+\bar{\ell}a}, s_{\beta+\ell a+kb}, s_{\beta+\ell' a+kb}) = \\ (s_{\alpha_1+k_1a}, \ldots, \widehat{s_{\alpha_t+k_ta}}, \ldots, s_{\alpha_n+k_na}, s_{-\widetilde{\alpha}+b+\bar{\ell}a}, s_{\alpha_t+\overline{\ell'}a+\overline{k}b}, s_{\alpha_t+\overline{\ell''}a+\overline{k}b}),$$

where $\overline{\ell'}, \overline{\ell''}, \overline{k} \in \mathbb{Z}$. By Lemma 5.5.4 (b) we have $|\overline{\ell'} - \overline{\ell''}| = 1$. By part (a) of the same lemma, we have

1. either $\overline{k} = 0$,
2. or $\overline{k} = -1$ and Φ_{fin} is of type D_4 (as in this case $m_t = 2$).

In the next lemma we state in general the existence of a Hurwitz move such that the last two reflections of the resulting factorization do not depend on b.

Lemma 5.5.6. *There exists a braid $\tau_4 \in \mathcal{B}_{n+2}$ such that*

$$\tau_4(s_{\alpha_1+k_1a}, \ldots, \widehat{s_{\alpha_t+k_ta}}, \ldots, s_{\alpha_n+k_na}, s_{-\widetilde{\alpha}+b+\bar{\ell}a}, s_{\alpha_t+\overline{\ell'}a+\overline{k}b}, s_{\alpha_t+\overline{\ell''}a+\overline{k}b}) = \\ (s_{\alpha_1+k_1a}, \ldots, \widehat{s_{\alpha_t+k_ta}}, \ldots, s_{\alpha_n+k_na}, s_{-\widetilde{\alpha}+b+\bar{\ell}a}, s_{\alpha_t+\widetilde{\ell'}a}, s_{\alpha_t+\widetilde{\ell''}a}),$$

with $\widetilde{\ell'}, \widetilde{\ell''} \in \mathbb{Z}$ and $|\widetilde{\ell'} - \widetilde{\ell''}| = 1$.

Proof. As we have seen right before this lemma, we have $\overline{k} = 0$ or $\overline{k} = -1$ and Φ_{fin} is of type D_4. So assume the latter case. Then

$$(s_{\alpha_1+k_1a}, \ldots, \widehat{s_{\alpha_t+k_ta}}, \ldots, s_{\alpha_n+k_na}, s_{-\widetilde{\alpha}+b+\bar{\ell}a}, s_{\alpha_2+\overline{\ell'}a+\overline{k}b}, s_{\alpha_2+\overline{\ell''}a+\overline{k}b}) = \\ (s_{\alpha_1+k_1a}, s_{\alpha_3+k_3a}, s_{\alpha_4+k_4a}, s_{-\widetilde{\alpha}+b+\bar{\ell}a}, s_{\alpha_2+\overline{\ell'}a-b}, s_{\alpha_2+\overline{\ell''}a-b}).$$

Since $s_4s_3s_1s_{-\widetilde{\alpha}+b}(\alpha_2 - b) = -\alpha_2$, Lemma 3.3.7 yields the existence of a braid $\tau_4 \in \mathcal{B}_6$ such that

$$\begin{aligned}\tau_4(&s_{\alpha_1+k_1a}, s_{\alpha_3+k_3a}, s_{\alpha_4+k_4a}, s_{-\widetilde{\alpha}+b+\bar{\ell}a}, s_{\alpha_2+\overline{\ell'}a-b}, s_{\alpha_2+\overline{\ell''}a-b}) = \\ &(s_{\alpha_1+k_1a}, s_{\alpha_3+k_3a}, s_{\alpha_4+k_4a}, s_{-\widetilde{\alpha}+b+\bar{\ell}a}, s_{\alpha_2+\widetilde{\ell'}a}, s_{\alpha_2+\widetilde{\ell''}a}),\end{aligned}$$

and Lemma 5.5.4 (b) yields that $|\widetilde{\ell'} - \widetilde{\ell''}| = 1$. □

We want to show that after suitable Hurwitz moves we can assume that $k_i = 0 = \bar{\ell}$ for $1 \leq i \leq n, i \neq t$. To achieve this, we will introduce in the following Kluitmanns's notation [65, Section 3.1]. By Proposition 5.2.8 the group W decomposes as $W = W(\Phi_{\text{fin}}) \ltimes E(r \otimes v')$ with $r \in \text{span}_{\mathbb{Z}}(a, b)$ and $v' \in \text{span}_{\mathbb{Z}}(\Phi_{\text{fin}})$. Thus every element w can be uniquely written as $w = w_1E(a \otimes x_1 + b \otimes x_2)$ for $x_1, x_2 \in \text{span}_{\mathbb{Z}}(\Phi_{\text{fin}})$. We will write the element w in vector notation $\begin{bmatrix} w_1 \\ x_1 \\ x_2 \end{bmatrix}$. Define as follows a binary operation of two vectors

$$\begin{bmatrix} w_1 \\ x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} w_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} w_1w_2 \\ w_2^{-1}(x_1) + y_1 \\ w_2^{-1}(x_2) + y_2 \end{bmatrix}.$$

This corresponds to the group operation of W, since by Lemma 5.2.6 we obtain for $w_1, w_2 \in W(\Phi_{\text{fin}})$ and $x_1, x_2, y_1, y_2 \in V/R$

$$\begin{aligned}&w_1E(a \otimes x_1 + b \otimes x_2) \cdot w_2E(a \otimes y_1 + b \otimes y_2) \\ &= (w_1w_2)(w_2^{-1}E(a \otimes x_1 + b \otimes x_2)w_2E(a \otimes y_1 + b \otimes y_2) \\ &= (w_1w_2)E(a \otimes w_2^{-1}(x_1) + b \otimes w_2^{-1}(x_2))E(a \otimes y_1 + b \otimes y_2) \\ &= (w_1w_2)E(a \otimes (w_2^{-1}(x_1) + y_1) + b \otimes (w_2^{-1}(x_2) + y_2)).\end{aligned}$$

Lemma 5.5.7. *Up to Hurwitz action it holds $k_1 = \ldots = k_{t-1} = k_{t+1} = \ldots = k_n = \bar{\ell} = 0$.*

Proof. For $\Phi = D_4^{(1,1)}$ we get

$$c = \begin{bmatrix} s_1s_3s_4s_{-\widetilde{\alpha}} \\ -\alpha_2 \\ \widetilde{\alpha} \end{bmatrix} = \begin{bmatrix} s_1 \\ -k_1\alpha_1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_3 \\ -k_3\alpha_3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_4 \\ -k_4\alpha_4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_{-\widetilde{\alpha}} \\ \overline{\ell}\widetilde{\alpha} \\ \widetilde{\alpha} \end{bmatrix} \cdot \begin{bmatrix} s_2 \\ -\widetilde{\ell'}\alpha_2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_2 \\ -\widetilde{\ell''}\alpha_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} s_1s_3s_4s_{-\widetilde{\alpha}} \\ -k_1\alpha_1 - k_3\alpha_3 - k_4\alpha_4 + \overline{\ell}\widetilde{\alpha} + (\widetilde{\ell'} - \widetilde{\ell''})\alpha_2 \\ \widetilde{\alpha} \end{bmatrix}.$$

Since $\widetilde{\alpha} = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ it holds

$$-1 = 2\overline{\ell} + \widetilde{\ell'} - \widetilde{\ell''} \text{ and } \overline{\ell} = k_1 = k_3 = k_4.$$

As already discussed before, the only possible cases are $\widetilde{\ell'} - \widetilde{\ell''} = \pm 1$.

If $\widetilde{\ell'} - \widetilde{\ell''} = -1$ it follows directly $0 = \overline{\ell} = k_1 = k_3 = k_4$. Therefore let $\widetilde{\ell'} - \widetilde{\ell''} = 1$. We get $-1 = \overline{\ell} = k_1 = k_3 = k_4$ and hence

$$\begin{aligned} &(s_{\alpha_1+k_1a}, s_{\alpha_3+k_3a}, s_{\alpha_4+k_4a}, s_{-\widetilde{\alpha}+b+\overline{\ell}a}, s_{\alpha_2+\widetilde{\ell'}a}, s_{\alpha_2+\widetilde{\ell''}a}) \\ =&(s_{\alpha_1-a}, s_{\alpha_3-a}, s_{\alpha_4-a}, s_{-\widetilde{\alpha}+b-a}, s_{\alpha_2+\widetilde{\ell'}a}, s_{\alpha_2+\widetilde{\ell''}a}). \end{aligned}$$

Since $s_1s_3s_4s_{-\widetilde{\alpha}}$ is in the center of the Weyl group W_{D_4}, the braid

$$\tau := (\sigma_4\sigma_5)(\sigma_3\sigma_4)(\sigma_2\sigma_3)(\sigma_1\sigma_2)(\sigma_2\sigma_3\sigma_4\sigma_5)(\sigma_1\sigma_2\sigma_3\sigma_4)$$

yields with $\widetilde{\ell'} - \widetilde{\ell''} = 1$

$$\begin{aligned} &\tau(s_{\alpha_1-a}, s_{\alpha_3-a}, s_{\alpha_4-a}, s_{-\widetilde{\alpha}+b-a}, s_{\alpha_2+\widetilde{\ell'}a}, s_{\alpha_2+\widetilde{\ell''}a}) \\ =&(\sigma_4\sigma_5)(\sigma_3\sigma_4)(\sigma_2\sigma_3)(\sigma_1\sigma_2)(s_{\alpha_2+\widetilde{\ell'}a}, s_{\alpha_2+\widetilde{\ell''}a}, s_{\alpha_1}, s_{\alpha_3}, s_{\alpha_4}, s_{-\widetilde{\alpha}+b}) \\ =&(s_1, s_3, s_4, s_0, s_{\alpha_2+\widetilde{\ell'}a-b}, s_{\alpha_2+\widetilde{\ell''}a-b}). \end{aligned}$$

In this situation we can apply the braid of Lemma 5.5.6 and get the factorization

$$(s_1, s_3, s_4, s_0, s_{\alpha_2+\widetilde{\ell'''}a}, s_{\alpha_2+\widetilde{\ell''''}a})$$

with $|\widetilde{\ell'''} - \widetilde{\ell''''}| = 1$, that is, up to Hurwitz action it holds

$$k_1 = \ldots = k_{t-1} = k_{t+1} = \ldots = k_n = \overline{\ell} = 0.$$

Let $\Phi = E_6^{(1,1)}$. We get analogously

$$c = \begin{bmatrix} s_1 s_2 s_3 s_5 s_6 s_{-\widetilde{\alpha}} \\ -\alpha_4 \\ \widetilde{\alpha} \end{bmatrix}$$

$$= \begin{bmatrix} s_1 \\ -k_1\alpha_1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_2 \\ -k_2\alpha_2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_3 \\ -k_3\alpha_3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_5 \\ -k_5\alpha_5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_6 \\ -k_6\alpha_6 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_{-\widetilde{\alpha}} \\ \overline{\ell}\widetilde{\alpha} \\ \widetilde{\alpha} \end{bmatrix} \cdot \begin{bmatrix} s_4 \\ -\widetilde{\ell}'\alpha_4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_4 \\ -\widetilde{\ell}''\alpha_4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} s_1 s_2 s_3 s_5 s_6 s_{-\widetilde{\alpha}} \\ -k_1\alpha_1 - k_2\alpha_2 - (k_1+k_3)\alpha_3 - k_5\alpha_5 - (k_5+k_6)\alpha_6 - (k_2 - \overline{\ell})\widetilde{\alpha} + (\widetilde{\ell}' - \widetilde{\ell}'')\alpha_4 \\ \widetilde{\alpha} \end{bmatrix}.$$

Since $\widetilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ it holds

$$k_1 = \frac{1}{3}\overline{\ell},\ k_2 = -\frac{2}{3}\overline{\ell},\ k_3 = -\overline{\ell},\ k_5 = -\frac{2}{3}\overline{\ell},\ k_6 = \frac{1}{3}\overline{\ell},\ \widetilde{\ell}' - \widetilde{\ell}'' = -1 + \overline{\ell}.$$

Since $\widetilde{\ell}' - \widetilde{\ell}'' = \pm 1$ it follows that $\overline{\ell} \in \{0, 2\}$. If $\overline{\ell} = 2$ we get $k_1 = \frac{2}{3} \in \mathbb{Z}$, a contradiction. Thus $\overline{\ell} = 0$ and hence $0 = \overline{\ell} = k_1 = k_2 = k_3 = k_5 = k_6$.

The proofs of the other cases $E_7^{(1,1)}$ and $E_8^{(1,1)}$ are similar and their calculations can be found in Calculation A5.5.7. □

Proof of Theorem 5.5.1. By what we have shown so far in this section, there exists a braid $\sigma \in \mathcal{B}_{n+2}$ such that

$$\sigma(t_1, \ldots, t_{n+2}) = (s_1, \ldots, \widehat{s_t}, \ldots, s_n, s_0, s_{\alpha_t + \widetilde{\ell}' a}, s_{\alpha_t + \widetilde{\ell}'' a}).$$

Therefore $(s_{\alpha_t + \widetilde{\ell}' a}, s_{\alpha_t + \widetilde{\ell}'' a})$ is a reduced factorization of $s_t s_{t^*} = s_{\alpha_t} s_{\alpha_t + a}$. Finally, Lemma 5.2.9 implies that $(t_1, \ldots, t_{n+2})$ and $(s_1, \ldots, \widehat{s_t}, \ldots, s_n, s_0, s_t, s_{t^*})$ lie in the same Hurwitz orbit. By Lemma 5.3.1 all Coxeter transformations are conjugated. Thus the Hurwitz transitivity on the set of reduced factorizations which generate the group is valid for all Coxeter transformations. □

Corollary 5.5.8. *Let (W, S) be an extended Weyl system of rank m, c a Coxeter transformation, Φ the corresponding root system and $(s_{\alpha_1}, \ldots, s_{\alpha_m}) \in \mathrm{Red}_T(c)$ with $\alpha_i \in \Phi$ for all $1 \leq i \leq n$. Then $\langle s_{\alpha_1}, \ldots, s_{\alpha_m} \rangle = W$ if and only if* $\mathrm{span}_{\mathbb{Z}}(\alpha_1, \ldots, \alpha_m) = \mathrm{span}_{\mathbb{Z}}(\Phi)$.

Proof. Assume it holds $\langle s_{\alpha_1}, \dots, s_{\alpha_m} \rangle = W$. Hence Lemma 5.5.3 implies $\text{span}_{\mathbb{Z}}(\alpha_1, \dots, \alpha_m) = \text{span}_{\mathbb{Z}}(\Phi)$. Since the proof of Theorem 5.5.1 only uses the fact that the roots of the initial reduced reflection factorization span the root lattice, we get that $(s_{\alpha_1}, \dots, s_{\alpha_m})$ and $(s_1, \dots, \widehat{s_t}, \dots, s_{m-2}, s_0, s_t, s_{t*})$ lie in the same Hurwitz orbit and therefore we have $\langle s_{\alpha_1}, \dots, s_{\alpha_m} \rangle = W$. □

Chapter 6

Exceptional sequences in abelian categories and thick subcategories

This chapter provides the theoretical requirements for an investigations of the so-called exceptional sequences and thick subcategories of certain abelian categories. First we introduce abelian and then triangulated categories with the important example of the derived category. After that we explain the concept of exceptional sequences and attach to them a reflection group. We discuss two examples in detail. For the category of finitely generated modules over a finite dimensional hereditary algebra it turns out that the associated reflection group is a Coxeter group. The category of coherent sheaves over a weighted projective line, that will also be defined in this section, leads to an extended Weyl group. In fact, by a theorem of Dieter Happel these are (up to derived equivalence) the main examples for connected ext-finite hereditary abelian k-categories with tilting object and algebraically closed field k.

6.1 Abelian categories

This section is devoted to the definition and basic properties of abelian categories. For notions that are not properly defined in this work we refer to [77]. We start with the definition of an additive category.

Definition 6.1.1. *A category $\mathcal{A}$ is called* additive *if the following conditions are satisfied.*

(a) The morphism sets $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ are abelian groups for all $X, Y \in \mathrm{Ob}(\mathcal{A})$ and the composition maps

$$\mathrm{Hom}_{\mathcal{A}}(Y, Z) \times \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

are biadditive for $X, Y, Z \in \mathrm{Ob}(\mathcal{A})$,

(b) it exists a zero object 0, i.e. $\mid \mathrm{Hom}_{\mathcal{A}}(0, X) \mid = 1 = \mid \mathrm{Hom}_{\mathcal{A}}(X, 0) \mid$ for every $X \in \mathrm{Ob}(\mathcal{A})$, and

(c) every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{A})$ admits a product $X \times Y$.

Definition 6.1.2. *An additive category $\mathcal{A}$ is called* abelian *if the following properties are satisfied.*

(a) *Every morphism* $\varphi : X \to Y$ *has a kernel and a cokernel, and*

(b) *the canonical factorization* $\overline{\varphi}$ *of* φ *pictured in the following diagram is an isomorphism.*

$$\begin{array}{ccccccc} \mathrm{Ker}(\varphi) & \xrightarrow{\;i\;} & X & \xrightarrow{\;\varphi\;} & Y & \xrightarrow{\;p\;} & \mathrm{Coker}(\varphi) \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{Coker}(i) & \xrightarrow{\;\overline{\varphi}\;} & \mathrm{Ker}(p) & & \end{array}$$

Example 6.1.3. *The prototype of an abelian category is the category of modules over a ring which is denoted by Mod(A). The latter is justified by Mitchell's embedding theorem* [85, Chapter VI, Theorem 7.2]. *It says that every small abelian category admits a full exact (covariant) embedding into a category of modules over an appropriate ring.*

Definition 6.1.4. *Let* $\mathcal{A}$ *be an abelian category. An object* $X \in \mathrm{Ob}(\mathcal{A})$ *is called* simple *if* $X \neq 0$ *and if* $X' \subseteq X$ *implies* $X' = X$ *or* $X' = 0$. *An object* $X \in \mathrm{Ob}(\mathcal{A})$ *has* finite length *if it has a finite composition series*

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X,$$

that is, each X_i/X_{i-1} *is simple for all* $1 \leq i \leq n$. *The largest index of the objects in a composition series is called* length.

Theorem 6.1.5 (Jordan-Hölder Theorem)**.** [89, Section 8.4 Theorem 1] *Let* $\mathcal{A}$ *be an abelian category and* $X \in \mathrm{Ob}(\mathcal{A})$ *an object of finite length. The composition series of* X *all have the same length and isomorphic factors up to the order.*

Definition 6.1.6. *An additive category* $\mathcal{A}$ *is called* Krull-Schmidt category *if every object decomposes into a finite direct sum of objects whose endomorphism rings are local.*

Definition 6.1.7. *An abelian category* $\mathcal{A}$ *is called* hom-finite *if there exists a field* k *such that all morphism spaces are finite dimensional* k*-vector spaces and the composition maps are* k*-bilinear.*

In the following we state a sufficient condition under which an abelian category is a Krull-Schmidt category due to Michael Atiyah.

Theorem 6.1.8. [5, Section 4 Theorem 1] *Let* $\mathcal{A}$ *be a hom-finite abelian category, then* $\mathcal{A}$ *is a Krull-Schmidt category.*

A characterization of Krull-Schmidt categories in ring theoretic terms is given in [69, Corollary 4.4].

Example 6.1.9. *The category* $\mathrm{mod}(A)$ *of finitely generated modules over a finite dimensional* k*-algebra is a Krull-Schmidt category.*

A class of important groups attached to abelian categories are the so-called extension groups. They provide useful information about exact sequences. This will be made precise in the following. For the definitions and basic properties of extensions we follow the textbooks [76], [85] and [109]. We start with the definition of an exact sequence in abelian categories. For that fix an abelian category $\mathcal{A}$.

Definition 6.1.10. *A Sequence E of the following form*

$$E: \quad 0 \xrightarrow{\varphi_{n+1}} A \xrightarrow{\varphi_n} B_{n-1} \xrightarrow{\varphi_{n-1}} B_{n-2} \xrightarrow{\varphi_{n-2}} \dots \xrightarrow{\varphi_2} B_1 \xrightarrow{\varphi_1} B_0 \xrightarrow{\varphi_0} C \xrightarrow{\varphi_{-1}} 0$$

with $A, B_{n-1}, \dots, B_0, C \in \mathrm{Ob}(\mathcal{A})$ and $n \in \mathbb{N}$ is called exact *if $\mathrm{Ker}(\varphi_i) = \mathrm{Im}(\varphi_{i+1})$ for all $-1 \leq i \leq n$. The integer n is called the* length *of the exact sequence E and if the length is one, the sequence E is called a* short exact sequence. *A* morphism of exact sequences *E, E' of the same length is a family of morphisms $(\alpha_i)_{-1\leq i\leq n}$ such that the following diagram commutes*

$$\begin{array}{lccccccccccccc}
E: & 0 & \longrightarrow & A & \longrightarrow & B_{n-1} & \longrightarrow & \dots & \longrightarrow & B_0 & \longrightarrow & C & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n-1}} & & & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_{-1}} & & \\
E': & 0 & \longrightarrow & A' & \longrightarrow & B'_{n-1} & \longrightarrow & \dots & \longrightarrow & B'_0 & \longrightarrow & C' & \longrightarrow & 0.
\end{array}$$

Next we define the extension classes in the sense of Yoneda, i.e. without assuming the existence of projective or injective objects.

Definition 6.1.11. *Two exact sequences E and E' with same length and same left end A and right end B are called* equivalent *if there exists a finite sequence $E_1, \dots, E_{2k-1}$ $(k \in \mathbb{Z}_{\geq 0})$ of exact sequences of same length and left end A and right end B such that*

$$E \to E_1 \leftarrow E_2 \to E_3 \leftarrow \dots \to E_{2k-2} \leftarrow E_{2k-1} \to E',$$

where the arrows are morphisms of sequences. For two equivalent exact sequences E, E' we write $E \sim E'$.

A direct calculation shows that the relation $\sim$ is an equivalence relation on the class of exact sequences with constant length and constant left and right ends (see [76, Chapter III, Section 5]).

Definition 6.1.12. *Let $\mathrm{Ext}^n_{\mathcal{A}}(C, A)$ be the class of all exact sequences of length $n \in \mathbb{N}$ with left end A and right end C modulo the relation $\sim$. The class $\mathrm{Ext}^n_{\mathcal{A}}(C, A)$ is called* extension. *In addition, one defines $\mathrm{Ext}^0_{\mathcal{A}}(C, A) = \mathrm{Hom}_{\mathcal{A}}(A, C)$ and $\mathrm{Ext}^i_{\mathcal{A}}(C, A) = 0$ for all $i < 0$.*

If the abelian category $\mathcal{A}$ has enough projectives or injectives the extensions can be defined in terms of the derived hom-functor (see [109, Chapter 2]).

We will see that to the extension classes are attached a binary operation which yields an abelian group structure, where we have to relax the definition of a group since $\mathrm{Ext}^n_{\mathcal{A}}$ can be a proper class. Therefore in [85] it is called a big abelian group, but in the following we will omit the adjective big.

Definition 6.1.13. *Let*

$$E : 0 \to A \xrightarrow{\varphi_n} B_{n-1} \xrightarrow{\varphi_{n-1}} B_{n-2} \xrightarrow{\varphi_{n-2}} \dots \xrightarrow{\varphi_2} B_1 \xrightarrow{\varphi_1} B_0 \xrightarrow{\varphi_0} C \to 0 \text{ and}$$
$$E' : 0 \to A' \xrightarrow{\varphi'_n} B'_{n-1} \xrightarrow{\varphi'_{n-1}} B'_{n-2} \xrightarrow{\varphi'_{n-2}} \dots \xrightarrow{\varphi'_2} B'_1 \xrightarrow{\varphi'_1} B'_0 \xrightarrow{\varphi'_0} C' \to 0$$

be two exact sequences of the same length. The direct sum $E \oplus E'$ *of* E *and* E' *is defined as follows*

$$E \oplus E' : 0 \to A \oplus A' \xrightarrow{\varphi_n \oplus \varphi'_n} B_{n-1} \oplus B'_{n-1} \xrightarrow{\varphi_{n-1} \oplus \varphi'_{n-1}} \dots \xrightarrow{\varphi_1 \oplus \varphi'_1} B_0 \oplus B'_0 \xrightarrow{\varphi_0 \oplus \varphi'_0} C \oplus C' \to 0.$$

For an object C of an abelian category we denote by $\Delta : C \longrightarrow C \oplus C$ the diagonal map that is defined by $p \circ \Delta = 1_C$, where $p : C \oplus C \longrightarrow C$ is the natural projection. The codiagonal map is defined dually.

The existence of push-outs and pull-backs in abelian categories yields the following basic result.

Lemma 6.1.14. [85, Chapter VII, Section 3] *Let* E *and* E' *be two exact sequences of length* n *with left end* A *and right end* C*. The pull-back along the diagonal map* $\Delta : C \longrightarrow C \oplus C$ *and the push-out along the codiagonal map* $\nabla : A \oplus A \to A$ *yield the commutative diagram*

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & PO & \longrightarrow & \dots & \longrightarrow & B_1 \oplus B'_1 & \longrightarrow & PB & \longrightarrow & C & \longrightarrow & 0 \\
 & & \nabla \uparrow & & \uparrow & & & & \| & & \uparrow & & \| & & \\
0 & \longrightarrow & A \oplus A & \longrightarrow & B_{n-1} \oplus B'_{n-1} & \longrightarrow & \dots & \longrightarrow & B_1 \oplus B'_1 & \longrightarrow & PB & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \| & & & & \| & & \downarrow & & \downarrow \Delta & & \\
0 & \longrightarrow & A \oplus A & \longrightarrow & B_{n-1} \oplus B'_{n-1} & \longrightarrow & \dots & \longrightarrow & B_1 \oplus B'_1 & \longrightarrow & B_0 \oplus B'_0 & \longrightarrow & C \oplus C & \longrightarrow & 0.
\end{array}$$

It holds $\nabla((E \oplus E')\Delta) = (\nabla(E \oplus E'))\Delta$.

Theorem 6.1.15. [85, Chapter VII Theorem 3.3] *Let* E *and* E' *be two exact sequences of length* n *with left end* A *and right end* C*. The assignment* $E + E' := \nabla(E \oplus E')\Delta$ *makes* $\mathrm{Ext}^n(C, A)$ *to an abelian group. Moreover,* $\mathrm{Ext}^n_{\mathcal{A}}(C, A)$ *is an* $\mathrm{End}_{\mathcal{A}}(A)$*-*$\mathrm{End}_{\mathcal{A}}(C)-$*bimodule for all* $n \in \mathbb{Z}$ *and there exists an* $\mathrm{End}_{\mathcal{A}}(C)$*-*$\mathrm{End}_{\mathcal{A}}(A)$*-bimodule homomorphism*

$$\mathrm{Ext}^n_{\mathcal{A}}(B, C) \otimes_{\mathrm{End}_{\mathcal{A}}(B)} \mathrm{Ext}^m_{\mathcal{A}}(A, B) \longrightarrow \mathrm{Ext}^{n+m}_{\mathcal{A}}(A, C),$$

which is induced by the concatenation of exact sequences for all $m, n \in \mathbb{N}_0$ *and* $L, M, N \in \mathrm{Ob}(\mathcal{A})$.

Remark 6.1.16. *Given an exact sequence*

$$E : 0 \to A \xrightarrow{\varphi_n} B_{n-1} \to B_{n-2} \to \ldots \to B_1 \to B_0 \xrightarrow{\varphi_0} C \to 0$$

in an abelian category $\mathcal{A}$, the complex E is the identity element in $\operatorname{Ext}^n_{\mathcal{A}}(C, A)$ if and only if φ_n is a split mono or φ_0 is a split epi.

Theorem 6.1.17. [85, Chapter VII Theorem 5.1] *Every short exact sequence $E : \ 0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ and every object $X \in \operatorname{Ob}(\mathcal{A})$ induces a (long) exact sequence*

$$\begin{aligned}
0 &\longrightarrow \operatorname{Hom}_{\mathcal{A}}(X, A) \xrightarrow{\alpha^*} \operatorname{Hom}_{\mathcal{A}}(X, B) \xrightarrow{\beta^*} \operatorname{Hom}_{\mathcal{A}}(X, C) \\
&\xrightarrow{\theta_0} \operatorname{Ext}^1_{\mathcal{A}}(X, A) \xrightarrow{\widehat{\alpha}_1} \operatorname{Ext}^1_{\mathcal{A}}(X, B) \xrightarrow{\widehat{\beta}_1} \operatorname{Ext}^1_{\mathcal{A}}(X, C) \\
\ldots &\xrightarrow{\theta_{n-1}} \operatorname{Ext}^n_{\mathcal{A}}(X, A) \xrightarrow{\widehat{\alpha}_n} \operatorname{Ext}^n_{\mathcal{A}}(X, B) \xrightarrow{\widehat{\beta}_n} \operatorname{Ext}^n_{\mathcal{A}}(X, C) \\
&\xrightarrow{\theta_n} \operatorname{Ext}^{n+1}_{\mathcal{A}}(X, A) \longrightarrow \ldots,
\end{aligned}$$

where $\theta_n : \operatorname{Ext}^{n-1}(X, C) \longrightarrow \operatorname{Ext}^n(X, A)$ is induced by the concatenation of exact sequences, θ_0 by the pull-back and $\widehat{\alpha}_n$ (resp. $\widehat{\beta}_n$) by the push-out along α (resp. β) for all $n \in \mathbb{N}$.

Let $X \in \operatorname{Ob}(\mathcal{A})$ be an object. We call $\operatorname{proj.dim}(X) = \inf\{n \geq 0 \mid \operatorname{Ext}^{n+1}_{\mathcal{A}}(X, -) = 0\}$ the projective dimension of X and set $\operatorname{proj.dim}(X) = \infty$ if such $n \in \mathbb{N}$ does not exist. A standard result concerning the projective dimension of an object of an abelian category is the following (see for example [25, Lemma 1.5.1]).

Lemma 6.1.18. *For $X \in \operatorname{Ob}(\mathcal{A})$ and $n \in \mathbb{N}$ the following are equivalent.*

(a) $\operatorname{Ext}^n_{\mathcal{A}}(X, -)$ *is a right exact functor,*
(b) $\operatorname{Ext}^{n+1}_{\mathcal{A}}(X, -) = 0$*,*
(c) $\operatorname{proj.dim}(X) \leq n$ *and*
(d) $\operatorname{Ext}^m_{\mathcal{A}}(X, -) = 0$ *for all $m > n$.*

Definition 6.1.19. *Let k be a field. An abelian category $\mathcal{A}$ is called* k-category *(or k-*abelian*) if all its extension groups are k-vector spaces. It is called* ext-finite *if $\mathcal{A}$ is a k-category and all the spaces $\operatorname{Ext}^n_{\mathcal{A}}(X, Y)$ are finite dimensional for all $n \in \mathbb{Z}_{\geq 0}$ and $X, Y \in \operatorname{Ob}(\mathcal{A})$. An abelian k-category is of* finite type *if the vector space $\bigoplus_{p \in \mathbb{Z}_{\geq 0}} \operatorname{Ext}^p_{\mathcal{A}}(X, Y)$ is finite dimensional for all $X, Y \in \operatorname{Ob}(\mathcal{A})$. An abelian category $\mathcal{A}$ is called* hereditary *if $\operatorname{Ext}^2_{\mathcal{A}}(X, Y) = 0$ for all $X, Y \in \operatorname{Ob}(\mathcal{A})$.*

If $\mathcal{A}$ is hereditary, Lemma 6.1.18 yields that $\operatorname{Ext}^m_{\mathcal{A}}(X, Y) = 0$ for all $X, Y \in \operatorname{Ob}(\mathcal{A})$ and $m \geq 2$.

Example 6.1.20. *If k is a field and A a k-algebra, the category of left A-modules is* *hereditary if and only if A is (left-)hereditary, i.e. every (left-)ideal of A is projective or equivalent every (left-)submodule of a projective (left-)module is projective.*

Important hereditary k-algebras are the quiver algebras. For definitions and basic properties we refer to [6].

We close this section by defining thick subcategories in abelian categories and state an alternative description if the category is hereditary. Throughout this thesis all subcategories are strictly full, i.e. the class of objects is closed under isomorphisms.

Definition 6.1.21. *A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called* thick *if it is abelian and closed under extensions.*

The definition of thick subcategories in hereditary abelian categories can be rephrased.

Lemma 6.1.22 ([32, Theorem 3.3.1] and [61, Proposition 9.1])**.** *Let $\mathcal{A}$ be a hereditary abelian category. A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is thick if and only if it is closed under direct summands and if it fulfils the following property. Given an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ and two of the three objects are in $\mathcal{C}$ then the third also lies in $\mathcal{C}$. The latter property is called 'two out of three' property.*

6.2 Triangulated categories and the derived category of an abelian category

In this section we introduce triangulated categories and as a special case the bounded derived category of an abelian category. The notions of a triangulated category and triangulated subcategory first appear in the Ph.D. thesis of Jean-Louis Verdier [106] and independent of that, in the work of Dieter Puppe [90], where the so-called octahedron axiom is omitted. Verdier was influenced by ideas of Alexander Grothendieck and introduced triangulated categories to find an axiomatic way to describe the notion of the derived category of an abelian category.

We start with the definition of an additive functor between additive categories.

Definition 6.2.1. *A functor $T : \mathcal{C} \longrightarrow \mathcal{D}$ between additive categories is called* additive *if it induces for every $A, B \in \mathrm{Ob}(\mathcal{C})$ a group homorphism*

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(TA, TB).$$

Definition 6.2.2. *A* triangulated category *is an additive category $\mathcal{T}$ with additive automorphism T endowed with a family of triangles, called* distinguished triangles*. A triangle is a sequence of morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX \ .$$

In addition, we require that the class of distinguished triangles satisfy the following axioms.

(TR0) *A triangle isomorphic to a distinguished triangle is a distinguished triangle.*

(TR1) *The triangle* $X \xrightarrow{1} X \longrightarrow 0 \longrightarrow TX$ *is a distinguished triangle.*

(TR2) *For all* $f : X \to Y$, *there exists a distinguished triangle* $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX$.

(TR3) *A triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ *is a distinguished triangle if and only if* $Y \xrightarrow{-g} Z \xrightarrow{-h} TX \xrightarrow{-Tf} TY$ *is a distinguished triangle.*

(TR4) *For any commutative diagram of the form*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX' \end{array}$$

where the rows are distinguished triangles, there is a morphism $h : Z \to Z'$, *not necessarily unique, which makes the diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX' \end{array}$$

commute.

(TR5) *Given three distinguished triangles*

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \longrightarrow TX$$

$$Y \xrightarrow{g} Z \xrightarrow{k} X' \longrightarrow TY$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Y' \longrightarrow TX$$

there exists a distinguished triangle

$$Z' \xrightarrow{u} Y' \xrightarrow{v} X' \longrightarrow TZ'$$

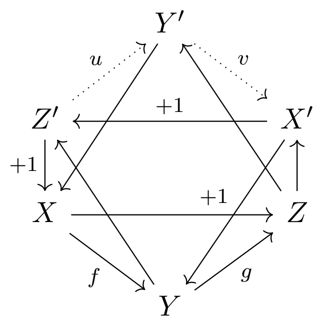
making the following diagram commute

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{h} & Z' & \longrightarrow & TX \\
\big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle g} & & \vdots{\scriptstyle u} & & \big\downarrow{\scriptstyle id} \\
X & \xrightarrow{f\circ g} & Z & \xrightarrow{l} & Y' & \longrightarrow & TX \\
\big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle id} & & \vdots{\scriptstyle v} & & \big\downarrow{\scriptstyle Tf} \\
Y & \xrightarrow{g} & Z & \xrightarrow{k} & X' & \longrightarrow & TY \\
\big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle l} & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle Th} \\
Z' & \overset{u}{\dashrightarrow} & Y' & \overset{v}{\dashrightarrow} & X' & \overset{w}{\dashrightarrow} & TZ',
\end{array}$$

that is called octahedron diagram*.*

The object Z of an distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ *is called* cone *and will be denoted by* $\mathrm{cone}(f)$*. The functor T is called* translation functor*.*

The axiom (TR5) is called octahedron axiom since it can be arranged in a diagram having the shape of an octahedron [64, Diagram 10.1.2].



Here, for example, $X' \xrightarrow{+1} Y$ means a morphism $X' \longrightarrow TY$.

In the literature, there are proven several equivalent reformulations of the axioms of triangulated categories, see for example [79] or [87]. Nevertheless, it is not known yet whether (TR5) is redundant.

The next example will be the starting point in defining the derived category of an abelian category. We will define the so-called homotopy category with a triangulated structure on it.

Example 6.2.3. [64, Section 11.2] *Let $\mathcal{A}$ be an additive category and $C(\mathcal{A})$ the corresponding category of cochain complexes. The objects of $C(\mathcal{A})$ are cochain complexes and the morphisms are morphisms of complexes. Let $X, Y \in \mathrm{Ob}(C(\mathcal{A}))$ and $\varphi : X \longrightarrow Y$ a morphism, i.e. we* *have the commutative diagram*

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} & \xrightarrow{d_X^{n+1}} & \dots \\
& & \downarrow{\scriptstyle\varphi^{n-1}} & & \downarrow{\scriptstyle\varphi^n} & & \downarrow{\scriptstyle\varphi^{n+1}} & & \\
\dots & \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} & \xrightarrow{d_Y^{n+1}} & \dots
\end{array}$$

The morphism φ is called null-homotopic *if there are morphisms $h^n : X^n \longrightarrow Y^{n-1}$ such that $\varphi^n = d_Y^{n-1}h^n + h^{n+1}d_X^n$ for all $n \in \mathbb{Z}$. The null-homotopic morphisms form an ideal $\mathcal{N}$ in $C(\mathcal{A})$, that is, for each pair $X, Y \in \mathrm{Ob}(C(\mathcal{A}))$ a subgroup $\mathcal{N} \subseteq \mathrm{Hom}_{C(\mathcal{A})}(X,Y)$ such that any composition $\psi\varphi$ of morphisms in $C(\mathcal{A})$ belongs to $\mathcal{N}$ if φ or ψ belongs to $\mathcal{N}$. The* homotopy category *$K(\mathcal{A})$ is the quotient of $C(\mathcal{A})$ with respect to this ideal, i.e. the objects of $K(\mathcal{A})$ are precisely those of $C(\mathcal{A})$ and $\mathrm{Hom}_{K(\mathcal{A})}(X,Y) := \mathrm{Hom}_{C(\mathcal{A})}(X,Y)/\mathcal{N}$ for complexes $X, Y \in \mathrm{Ob}(C(\mathcal{A}))$.*

The functor $[1] : K(A) \longrightarrow K(A),\ X \longmapsto X[1]$ with $(X[1])^n = X^{n+1}$ and $d^n_{X[1]} = -d_X^{n+1}$ is called translation functor*. To get a class of distinguished triangles we need to construct cones for arbitrary morphisms between complexes. Let $X \xrightarrow{f} Y$ be a morphism in $K(\mathcal{A})$. We define the complex C_f by*

$$C_f^n = X^{n+1} \oplus Y^n; \quad d^n_{C_f} = \begin{bmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{bmatrix} : C_f^n \longrightarrow C_f^{n+1}.$$

The object C_f is called cone *of the morphism f. The obvious maps*

$$i_f : Y \longrightarrow C_f \text{ with } i_f^n : Y^n \longrightarrow C_f^n$$

and

$$p_f : C_f \longrightarrow X[1] \text{ with } p_f^n : C_f^n \longrightarrow X^{n+1}$$

are morphisms of complexes and the diagram $X \xrightarrow{f} Y \xrightarrow{i_f} C_f \xrightarrow{p_f} X[1]$ is called standard triangle*. We call diagrams isomorphic to a standard one* distinguished*. Then the additive category $K(\mathcal{A})$ equipped with the translation functor $[1]$ and the class of distinguished triangles is a triangulated category.*

We need the following basic properties of triangulated categories which can be found for example in [10], [48], [64], [88] or [106].

Proposition 6.2.4. *Let $\mathcal{T}$ be a triangulated category with translation functor T. The following properties hold.*

(a) For any $W \in \mathrm{Ob}(\mathcal{T})$ the functor $\mathrm{Hom}_{\mathcal{T}}(W,-)$ is cohomological, that is, for a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ the functor $F = \mathrm{Hom}_{\mathcal{T}}(W,-)$

yields the exact sequences

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z)$$

and axiom (TR3) gives rise to the long exact sequence

$$\ldots \longrightarrow FT^{-1}X \longrightarrow FX \longrightarrow FY \longrightarrow FZ \longrightarrow FTX \longrightarrow \ldots .$$

The dual holds for the functor $\mathrm{Hom}_{\mathcal{T}}(-, W)$.

(b) *Consider the following morphism of distinguished triangles.*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle T\alpha} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX'. \end{array}$$

If the morphisms α *and* β *are isomorphisms, then so is* γ.

(c) *Cones are essentially unique, that is, given a morphism* $f \in \mathrm{Ob}(\mathcal{T})$ *the object* $\mathrm{cone}(f)$ *is unique up to isomorphism.*

(d) *Let*

$$X_1 \longrightarrow Y_1 \longrightarrow Z_1 \longrightarrow TX_1$$

and

$$X_2 \longrightarrow Y_2 \longrightarrow Z_2 \longrightarrow TX_2$$

be distinguished triangles. Then

$$X_1 \oplus X_2 \longrightarrow Y_1 \oplus Y_2 \longrightarrow Z_1 \oplus Z_2 \longrightarrow TX_1 \oplus TX_2$$

is a distinguished triangle.

Next we will introduce triangulated and thick subcategories of triangulated categories.

Definition 6.2.5. *Let* $\mathcal{T}$ *be a triangulated category with translation functor* T. *A non-empty full subcategory* $\mathcal{S}$ *is a* triangulated subcategory *if the following conditions hold.*

(a) $T^n X \in \mathcal{S}$ *for all* $X \in \mathrm{Ob}(\mathcal{T})$ *and* $n \in \mathbb{Z}$.

(b) *Let* $X \longrightarrow Y \longrightarrow Z \longrightarrow TX$ *be a distinguished triangle in* $\mathcal{T}$. *If the objects* X *and* Y *belong to* $\mathcal{S}$, *then also* Z.

A triangulated subcategory $\mathcal{S}$ *is called* thick *if in addition the following condition holds.*

(c) *Every direct summand of an object in* $\mathcal{S}$ *belongs to* $\mathcal{S}$.

An equivalent formulation of thick subcategories is given in [106]. There these subcategories are called *épaisse* and the equivalence of these notions are proven in [94, Proposition 1.3].

By [103, Remark 1.5] the intersection of thick subcategories is itself thick. So it makes sense to define the smallest thick subcategory containing the full subcategory $\mathcal{S}$ of $\mathcal{T}$, that we denote by Thick($\mathcal{S}$). We say the thick subcategory Thick($\mathcal{S}$) is generated by $\mathcal{S}$.

Henning Krause states in [67, Section 3.3] a recursive way to describe the triangulated subcategory generated by a class of objects $\mathcal{S}_0$ of $\mathcal{T}$. Denote by $\mathcal{U} * \mathcal{V}$ for two classes of objects $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{T}$ the class of objects X occuring in a distinguished triangle

$$U \longrightarrow X \longrightarrow V \longrightarrow TU$$

with $U \in \mathrm{Ob}(\mathcal{U})$ and $V \in \mathrm{Ob}(\mathcal{V})$. Now let $\mathcal{S}_1$ be the class of all $T^n X$ with $X \in \mathcal{S}_0$ and $n \in \mathbb{Z}$. For $r > 0$, let $\mathcal{S}_r = \mathcal{S}_1 * \mathcal{S}_1 * \ldots * \mathcal{S}_1$ be the product r times $\mathcal{S}_1$.

Lemma 6.2.6. [67, Section 3.3] *Let $\mathcal{S}_0$ be a class of objects in $\mathcal{T}$.*

(a) The full subcategory of objects in $\mathcal{S} = \bigcup_{r\geq 0} \mathcal{S}_r$ is the smallest full triangulated subcategory of $\mathcal{T}$ containing $\mathcal{S}_0$.

(b) The full subcategory of direct summands of objects of $\mathcal{S}$ is the smallest full thick subcategory of $\mathcal{T}$ containing $\mathcal{S}_0$.

It is no coincidence that there is a notion of thick subcategories in abelian and in triangulated categories. The connection of these two different subcategories is provided by a result of [22] that is stated in Theorem 6.2.26.

From now on let $\mathcal{T}$ be essentially small, i.e. it is equivalent to a small category.

Definition 6.2.7. *Let $\mathcal{T}$ be a triangulated category with translation functor T. The free abelian group $K_0(\mathcal{T})$ with set of generators $\{[X] \mid X \in \mathcal{T}\}$ modulo the subgroup generated by $[X] - [Y] + [Z]$ for all distinguished triangles $X \to Y \to Z \to TX$ is called the* Grothendieck group *of $\mathcal{T}$. The relations $[Y] = [X] + [Z]$ are called* Euler relations.

In many situations the Grothendieck group does not have finite rank. Later we will see that the finiteness of the rank of the Grothendieck group depends on the existence of a so-called generating exceptional sequence.

The following lemma states well-known facts of the Grothendieck group that are taken from [103]. They can easily be deduced from Proposition 6.2.4. Classical sources for the definition and properties of the Grothendieck group are [10, IV, Section 1] and [48, VIII, Section 2].

Lemma 6.2.8. [103, 1.6] *The Grothendieck group satisfies the universal mapping property, i.e. any function from the set of isomorphism classes of objects of $\mathcal{T}$ to an abelian group G such that the Euler relations hold in G factors through a unique homomorphism $K_0(\mathcal{T}) \longrightarrow G$.*

It holds $[A] + [B] = [A \oplus B]$ for all $A, B \in \mathrm{Ob}(\mathcal{T})$, $[0] = 0$ and $[A] + [TA] = [0] = 0$. In addition $A \cong B$ implies $[A] = [B]$ and every element in $K_0(\mathcal{T})$ is of the form $[C]$ for an object $C \in \mathrm{Ob}(\mathcal{T})$. Every exact equivalence between triangulated categories, i.e. an equivalence that maps distinguished triangles to distinguished triangles, induces an isomorphism on the level *of Grothendieck groups.*

In abelian categories there is a similar notion of a Grothendieck group, where instead of distinguished triangles the short exact sequences are considered. It will turn out that the group attached to an abelian category is the Grothendieck group of a suitable triangulated category, the so-called bounded derived category. One advantage to work with Grothendieck groups associated to triangulated categories instead of its analogue in abelian categories is that every element of the Grothendieck group of a triangulated category is the class of an object of the triangulated category. The latter property can fail for abelian categories.

Definition 6.2.9. *A triangulated k-category $\mathcal{T}$ with translation functor T is of* finite type *if $\bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, T^pY)$ is finite dimensional for all $X, Y \in \mathrm{Ob}(\mathcal{T})$.*

The next result states a necessary and sufficient condition of two object representing the same element in the Grothendieck group.

Lemma 6.2.10. [71] *Let $\mathcal{T}$ be a triangulated category with translation functor T and let $X, Y \in \mathrm{Ob}(\mathcal{T})$. Then $[X] = [Y]$ in $K_0(\mathcal{T})$ if and only if there are objects $A, B, C \in \mathrm{Ob}(\mathcal{T})$ and two distinguished triangles $A \xrightarrow{\alpha_U} B \oplus U \xrightarrow{\beta_U} C \longrightarrow TA$ with $U \in \{X, Y\}$.*

Next we define the so-called Euler form that is attached to the Grothendieck group of a triangulated category of finite type.

Definition 6.2.11. *Let $\mathcal{T}$ be a triangulated k-category of finite type with Grothendieck group $K_0(\mathcal{T})$, the assignment*

$$\chi : K_0(\mathcal{T}) \times K_0(\mathcal{T}) \longrightarrow \mathbb{Z}, \;\; ([X],[Y]) \longmapsto \sum_{p\in\mathbb{Z}} (-1)^p \dim_k \mathrm{Hom}_{\mathcal{T}}(X, T^pY)$$

is called Euler form*. The map χ^s defined by $\chi^s([X],[Y]) = \chi([X],[Y]) + \chi([Y],[X])$ for all $X, Y \in \mathrm{Ob}(\mathcal{T})$ is the* symmetrized Euler form*.*

Lemma 6.2.12. *The Euler form is bilinear. In particular, the symmetrized Euler form is a symmetric bilinear form.*

Proof. Since the bilinearity is obviously true we only prove that χ is well-defined. Let $X, Y, Y' \in \mathrm{Ob}(\mathcal{T})$ with $[Y] = [Y'] \in K_0(\mathcal{T})$ and set $F = \mathrm{Hom}(X, -)$. By Lemma 6.2.10 there exists two distinguished triangles of the form

$$A \xrightarrow{\alpha_U} B \oplus U \xrightarrow{\beta_U} C \longrightarrow TA$$

with $U \in \{Y, Y'\}$. Thus Proposition 6.2.4 yields the exact sequence of finite dimensional vector spaces

$$\ldots \to F(T^{-1}C) \to F(A) \to F(B\oplus U) \to F(C) \to F(TA) \to F(T(B\oplus U)) \to F(TC) \to \ldots$$

for $U \in \{Y, Y'\}$. Since $\mathcal{T}$ is of finite type we get

$$0 = \sum_{p \in \mathbb{Z}} (-1)^p \left(\dim_k(F(T^p A)) - \dim_k(F(T^p(B \oplus U))) + \dim_k(F(T^p C))\right),$$

where again $U \in \{Y, Y'\}$. The latter implies $\chi([X],[Y]) = \chi([X],[Y'])$.
Dually, we have $\chi([X],[Y]) = \chi([X'],[Y])$ for $X, X', Y \in \mathrm{Ob}(\mathcal{T})$ with $[X] = [X']$. □

Next we define an important functor on additive categories that has further properties if the underlying category is triangulated.

Definition 6.2.13. *Let $\mathcal{T}$ be an additive hom-finite k-category. A* Serre functor *is an additive covariant equivalence of categories $\mathcal{S} : \mathcal{T} \longrightarrow \mathcal{T}$ with isomorphisms*

$$\eta_{A,B} : \mathrm{Hom}_{\mathcal{T}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(B, \mathcal{S}\, A)^* := \mathrm{Hom}_k(\mathrm{Hom}_{\mathcal{T}}(B, \mathcal{S}\, A), k)$$

for any $A, B \in \mathcal{T}$ which are natural in A and B.

The following are well-known basic properties of Serre functors.

Proposition 6.2.14. [92, Section I.1] *Let $\mathcal{T}$ be an additive hom-finite k-category. If $\mathcal{S}$ and $\mathcal{S}'$ are Serre functors, then $\mathcal{S}$ and $\mathcal{S}'$ are naturally isomorphic.*

If $\mathcal{A}$ has a triangulated structure the Serre functor has the following additional properties.

Proposition 6.2.15. [15, Proposition 1.3, 1.4] *Let $\mathcal{T}$ be a hom-finite triangulated k-category with Serre functor $\mathcal{S}$.*

(a) Any auto-equivalence $U : \mathcal{T} \longrightarrow \mathcal{T}$ commutes with $\mathcal{S}$, i.e. there is an isomorphism of functors $U \circ \mathcal{S} \cong \mathcal{S} \circ U$.
(b) $\mathcal{S}$ is exact, i.e. it maps distinguished triangles to distinguished triangles.

6.2.1 The derived category of an abelian category and tilting theory for abelian categories

The theory of derived categories builds a bridge between abelian and triangulated categories. It is the foundation of the tilting theory, that can be seen as a generalization of Morita theory.

Before defining the derived category of an abelian category, we introduce the notion of the localisation of categories that leads to the Verdier localisation.

Definition 6.2.16. *Let $\mathcal{T}$ be a category and S a class of morphisms in $\mathcal{T}$. The* localisation *of $\mathcal{A}$ with respect to S is a category $\mathcal{T}[S^{-1}]$ together with a functor $Q : \mathcal{T} \to \mathcal{T}[S^{-1}]$ satisfying the following properties.*

(a) For every morphism f of S, the morphism Qf is invertible.

(b) *For every functor* $F : \mathcal{T} \longrightarrow \mathcal{D}$ *such that* Ff *is invertible for all morphisms* f *from* S*, there exists a unique functor* $\overline{F} : \mathcal{T}[S^{-1}] \to \mathcal{D}$ *such that* $F = \overline{F} \circ Q$.

Since the localisation solves a universal property it is unique, up to a unique isomorphism. If, in addition, the chosen class of morphisms satisfies the properties of the so-called calculus of left fraction, the morphisms of $\mathcal{C}[S^{-1}]$ have an explicit description.

Definition 6.2.17. *Let* $\mathcal{T}$ *be a category and* S *a class of morphisms in* $\mathcal{T}$*. The class* S *admits a* calculus of left fractions *if the following conditions are satisfied.*

(a) *The identity morphism for each object is in* S*. The composition of two morphisms of* S *is again in* S.

(b) *Each pair of morphisms* $X' \xleftarrow{f} X \longrightarrow Y$ *with* f *in* S *can be completed to a commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow f & & \big\downarrow g \\ X' & \longrightarrow & Y' \end{array}$$

such that g *is in* S.

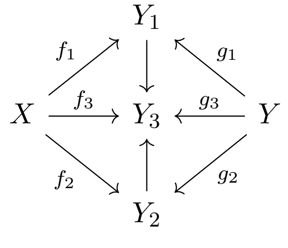
(c) *Let* $f, g : X \longrightarrow Y$ *be morphisms in* $\mathcal{T}$*. If there is* $h : X' \longrightarrow X$ *in* S *such that* $fh = gh$*, then there is a morphism* $j : Y \longrightarrow Y'$ *in* S *such that* $jf = jg$.

Next we describe the category $\mathcal{T}[S^{-1}]$ where S admits a calculus of left fraction.

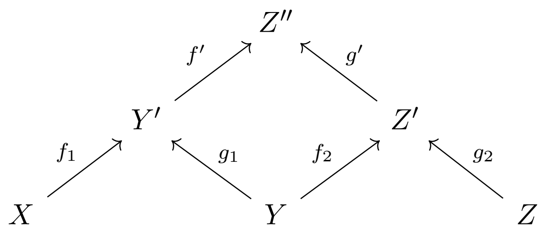
Theorem 6.2.18. *Let* $\mathcal{T}$ *be a category and* S *a class of morphisms that admits a calculus of left fractions. The objects of the localisation* $\mathcal{T}[S^{-1}]$ *are those of* $\mathcal{T}$*, and the morphisms are the classes of pairs* (f, g) *of morphisms in* $\mathcal{T}$ *such that* g *is in* S*,*

$$X \xrightarrow{f} Y' \xleftarrow{g} Y'$$

and modulo the following relation. Two pairs (f_1, g_1)*,* (f_2, g_2) *are equivalent if there exists a commutative diagram*



with f_3 *in* S*. The composition of two classes* $[f_1, g_1]$ *and* $[f_2, g_2]$ *is then given by* $[f'f_1, g'g_2]$*, where* f' *and* g' *are obtained from the following commutative diagram.*



The canonical functor $\mathcal{T} \longrightarrow \mathcal{T}[S^{-1}]$ *is the identity on objects and sends a morphism* $f : X \longrightarrow Y$ *to* $[f, 1_Y]$.

If $\mathcal{T}$ is a triangulated category and S chosen in a suitable way the localisation $\mathcal{T}[S^{-1}]$ inherits a natural triangulated structure.

Proposition 6.2.19. [106] *Let* $\mathcal{T}$ *be a triangulated category and* $\mathcal{S}$ *a triangulated subcategory. Let* $S := S(\mathcal{S})$ *be the class of morphisms that consists of those morphisms* f *that satisfy* $\operatorname{cone}(f) \in \operatorname{Ob}(\mathcal{S})$*. The class* S *admits a calculus of left fraction. Moreover, the localisation* $\mathcal{T}[S^{-1}]$ *carries a unique triangulated structure such that the canonical functor* $Q : \mathcal{T} \longrightarrow \mathcal{T}[S^{-1}]$ *is exact and annihilates* S.

The category $\mathcal{T} / \mathcal{S} := \mathcal{T}[S^{-1}]$ together with the canonical functor $Q : \mathcal{T} \longrightarrow \mathcal{T} / \mathcal{S}$ is called Verdier localisation.

Let $\mathcal{A}$ be an abelian category and $K(\mathcal{A})$ the corresponding homotopy category defined in Example 6.2.3. Let $K^b(\mathcal{A})$ be the full triangulated subcategory of $K(\mathcal{A})$, that consists only of bounded cochain complexes. We write $K^*(\mathcal{A})$ if we do not want to distinguish between $K(\mathcal{A})$ and $K^b(\mathcal{A})$. A morphism $X \xrightarrow{f} Y$ in $K^*(\mathcal{A})$ is called quasi-isomorphism if for a representative of f and therefore for all, holds $H^p f : H^p X \longrightarrow H^p Y$ is an isomorphisms for all $p \in \mathbb{Z}$, where H^p denotes the usual cohomology functor. Let $\mathcal{S}^*$ be the triangulated subcategory consisting of $X \in \operatorname{Ob}(K^*(\mathcal{A}))$ such that $H^p X \cong 0$ for all $p \in \mathbb{Z}$. Then $\operatorname{Qis} := S(\mathcal{S}^*)$ consists of all quasi-isomorphisms of $K^*(\mathcal{A})$.

Definition 6.2.20. *Let* $\mathcal{A}$ *be an abelian category. The category* $\mathcal{D}(\mathcal{A}) = K(\mathcal{A}) / \mathcal{S}$ *is called* derived category *of* $\mathcal{A}$ *and* $\mathcal{D}^b(\mathcal{A}) = K^b(\mathcal{A}) / \mathcal{S}^b$ *is called* bounded derived category *of* $\mathcal{A}$*. The translation functor is denoted in both cases by* [1]*. Two abelian categories are called* derived equivalent *if their corresponding bounded derived categories are triangulated equivalent.*

The natural functors $\mathcal{A} \longrightarrow \mathcal{D}^b(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ are fully faithful (see [64, Proposition 13.1.10, 13.1.12]), hence $\mathcal{D}^b(\mathcal{A})$ can be identified with a full subcategory of $\mathcal{D}(\mathcal{A})$ whose complexes have bounded cohomology and $\mathcal{A}$ can be identified with the complexes that have cohomology centered in degree zero. By Proposition 6.2.19 the categories $\mathcal{D}^*(\mathcal{A})$ are triangulated categories. Moreover, every short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathcal{A}$ induces a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

in $\mathcal{D}^b(\mathcal{A})$, where X, Y, Z are interpreted as cochains with cohomology concentrated in degree zero (see [64, Proposition 13.1.13]).

The next proposition lists useful properties of $\mathcal{D}^*(\mathcal{A})$.

Proposition 6.2.21. *(*[25, Lemma 2.1.1], [64, Corollary 13.1.20], [3, Chapter 6, Section 1]*) Let $\mathcal{A}$ be an abelian category.*

(a) Let $A, B \in \mathrm{Ob}(\mathcal{A})$. Then $\mathrm{Ext}^n_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A, B[n])$ for all $n \in \mathbb{Z}$, where on the right hand side A and B are interpreted as complexes whose cohomology is concentrated in degree zero. In particular, $\mathrm{Ext}^k_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(X, Y[k])$ for all $k \in \mathbb{Z}$ and $X, Y \in \mathrm{Ob}(\mathcal{A})$, where $[k]$ is the k'th times iterated (resp. inverse) translation functor.
(b) If $\mathcal{A}$ is hereditary, then for every object $X \in \mathrm{Ob}(\mathcal{D}^b(\mathcal{A}))$ we have a (non-canonical) isomorphism $X \cong \bigoplus_{n\in\mathbb{Z}} H^n X[-n]$.

From the second part of the previous proposition we get a description of the indecomposable objects of $\mathcal{D}^b(\mathcal{A})$ in terms of the indecomposable objects of $\mathcal{A}$ if $\mathcal{A}$ is hereditary.

Corollary 6.2.22. *Let $\mathcal{A}$ be a hereditary abelian category. The indecomposable objects of $\mathcal{D}^b(\mathcal{A})$ and those of $\mathcal{A}$ coincide up to shifts.*

Definition 6.2.23. *Let $\mathcal{A}$ be an ext-finite abelian k-category. An auto-equivalence $\tau : \mathcal{A} \longrightarrow \mathcal{A}$ is called* Serre duality *if $\mathrm{Ext}^1_{\mathcal{A}}(X, Y)^* \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(Y, \tau X)$ is satisfied.*

A direct calculation based on 6.2.21 (b) shows that a Serre duality induces a Serre functor on the level of the corresponding bounded derived category.

Corollary 6.2.24. *Let $\mathcal{A}$ be an ext-finite hereditary abelian k-category with Serre duality τ. Then $\tau(-)[1]$ is a Serre functor on $\mathcal{D}^b(\mathcal{A})$.*

Example 6.2.25. *(*[49, Section 3.7], [56, Section 6]*) Let k be an algebraically closed field and A a finite dimensional hereditary k-algebra. The Auslander-Reiten translate τ is a Serre duality of* $\mathrm{mod}(A)$ *and induces a Serre functor on $\mathcal{D}^b(\mathrm{mod}(A))$.*

The notion of a thick subcategory is defined in abelian categories and triangulated categories. The connection between these apparently two different concepts is stated in the next result.

Theorem 6.2.26. [22, Theorem 5.1] *Let $\mathcal{A}$ be a hereditary abelian category. The assignments*

$$\mathcal{C} \longmapsto \{H^0(C) \mid C \in \mathcal{C}\} \text{ and } \mathcal{M} \longmapsto \{C \in \mathcal{D}^b(\mathcal{A}) \mid H^n C \in \mathcal{M} \text{ for all } n \in \mathbb{Z}\}$$

induce mutually inverse bijections between

1. *the class of thick subcategories in $\mathcal{D}^b(\mathcal{A})$, and*
2. *the class of thick subcategories in $\mathcal{A}$.*

Thus the study of thick subcategories of abelian categories can be shifted to its bounded derived category.

We close this subsection with the summary of some important aspects of tilting theory. Tilting theory can be seen as generalization of Morita theory. It is the connection between algebraic geometry and representation theory and started in 1978 with [12], where a derived equivalence between the category of coherent sheaves over the projective n-space and the bounded derived category of a module category is established. Further contributions in this area are due to Happel in [50]. He used the concepts of triangulated categories and tilting theory to study the modules over a finite dimensional algebra over a field. Until now it became an active field of research with various contributions by many mathematicians.

We start with the definition of a tilting object in abelian categories.

Definition 6.2.27. *Let $\mathcal{A}$ be an abelian category. An object $T \in \mathrm{Ob}(\mathcal{A})$ is called* tilting object *if*

(a) $\mathrm{proj.\,dim}(T) \leq 1$,
(b) $\mathrm{Ext}^1_{\mathcal{A}}(T, T) = 0$ *and*
(c) $\mathrm{Hom}_{\mathcal{A}}(T, A) = 0 = \mathrm{Ext}^1_{\mathcal{A}}(T, A)$ *implies* $A = 0$ *for all objects* $A \in \mathrm{Ob}(\mathcal{A})$.

The following result will be central in our later investigations.

Theorem 6.2.28. [52, Chapter I, Theorem 4.6] *Let $\mathcal{A}$ be an ext-finite abelian k-category, $T \in \mathrm{Ob}(\mathcal{A})$ a tilting object and $\Lambda = \mathrm{End}_{\mathcal{A}}(T)$. Then the functor*

$$-\otimes^L_\Lambda : \mathcal{D}^b(\mathrm{mod}(\Lambda)) \longrightarrow \mathcal{D}^b(\mathcal{A})$$

is an equivalence of triangulated categories and its right adjoint $R\,\mathrm{Hom}_{\mathcal{A}}(T, -)$ is a quasi-inverse.

6.2.2 Exceptional sequences in abelian and triangulated categories

In this section we define the notion of exceptional objects in triangulated and abelian categories. We state their major properties and point out their connections. We will see that exceptional objects of hereditary abelian categories can be identified up to shifts with those of the corresponding bounded derived category.

All categories considered here are essentially small and k is always an algebraically closed field of arbitrary characteristic.

Definition 6.2.29. *Let $\mathcal{T}$ be a hom-finite triangulated k-category with translation functor T. An object $E \in \mathrm{Ob}(\mathcal{T})$ is called* exceptional *if $\mathrm{End}_{\mathcal{T}}(E) \cong k$ and $\mathrm{Hom}_{\mathcal{T}}(E, T^i E) = 0$ for all $i \neq 0$. For $n \in \mathbb{N}$ a sequence $E = (E_1, \ldots, E_n)$ of exceptional objects is called* exceptional *if $\mathrm{Hom}_{\mathcal{T}}(E_k, T^i E_j) = 0$ for all $1 \leq j < k \leq n$ and all $i \in \mathbb{Z}$. It is called* complete *if $n = \mathrm{rk}(K_0(\mathcal{T}))$ and* generating *if its objects generate $\mathcal{T}$, i.e. the triangulated subcategory stated in Lemma 6.2.6 (a) for $\mathcal{S}_0 = \{E_1, \ldots, E_n\}$ is already $\mathcal{T}$.*

Remark 6.2.30. *Instead of generating sometimes the term* full *is used (see* [99]*).*

For the triangulated categories of our interest, namely the bounded derived category of the category of coherent sheaves over a weighted projective line and the bounded derived category of the module category over a finite dimensional hereditary k-algebra, the notions of complete exceptional sequence and generating exceptional sequence coincide (see the Lemmas 6.3.12 and 6.4.5).

Definition 6.2.31. *Given a triangulated k-category of finite type with a generating exceptional sequence $E = (E_1, \dots, E_n)$ and symmetrized Euler form χ^s, the diagram attached to E defined as follows is called* generalized Coxeter diagram. *The set of vertices is in bijection to the set $\{E_1, \dots, E_n\}$. Let E_i and E_j be two different exceptional objects, that is $i \neq j$. If $\chi^s([E_i], [E_j]) = -l$ with $l \in \mathbb{N}_0$ we draw l edges between the corresponding vertices, and these edges are dotted if $\chi^s([E_i], [E_j]) = l$.*

The existence of exceptional sequences implies many properties for the category, such as the following.

Proposition 6.2.32. [14, Corollary 3.5] *Let $\mathcal{T}$ be a triangulated hom-finite category that contains a generating exceptional sequence. Then $\mathcal{T}$ admits a Serre functor.*

The following connection between the number of objects in an exceptional sequences and the rank of the Grothendieck group is a well-known fact.

Lemma 6.2.33. *Let $\mathcal{T}$ be a triangulated k-category of finite type with translation functor T and Grothendieck group $K_0(\mathcal{T})$. The number of objects of any exceptional sequence is bounded by the rank of $K_0(\mathcal{T})$. The objects of any generating exceptional sequence induce a basis of the Grothendieck group. In particular, every generating exceptional sequence is complete.*

Proof. Let $E = (E_1, \dots, E_n)$ be an exceptional sequence of $\mathcal{T}$. We will show that $\{[E_1], \dots, [E_n]\}$ is linear independent over $\mathbb{Z}$. Thus let $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ such that $0 = \sum_{i=1}^n \lambda_i [E_i]$. By Lemma 6.2.8 we have $0 = \left[\bigoplus_{i=1}^n T^{\varepsilon_i} E_i^{|\lambda_i|}\right]$, where $E_i^{|\lambda_i|}$ is the $|\lambda_i|$'th direct sum of E_i and $\epsilon_i = 1$ if $\lambda_i < 0$ and $\epsilon_i = 0$ else. Set $\overline{E} = \bigoplus_{i=1}^n T^{\varepsilon_i} E_i^{|\lambda_i|}$. By the definition of exceptional sequences we get

$$\begin{aligned} 0 &= \chi(0, [E_1]) \\ &= \chi([\overline{E}], [E_1]) \\ &= \sum_{p \in \mathbb{Z}} (-1)^p \dim_k \left(\bigoplus_{i=1}^n \operatorname{Hom}_{\mathcal{T}}(T^{\varepsilon_i} E_i^{|\lambda_i|}, T^p E_1) \right) \\ &= \begin{cases} \mid \lambda_1 \mid & , \ \varepsilon_1 = 0 \\ - \mid \lambda_1 \mid & , \ \varepsilon_1 = 1. \end{cases} \end{aligned}$$

The latter implies $\lambda_1 = 0$ and inductively we get $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$.

Assume now that E is a generating exceptional sequence. We get with $\mathcal{S}_0 := \{E_1, \ldots, E_n\}$ by Lemma 6.2.6 that $\mathcal{T} = \bigcup_{r \geq 0} \mathcal{S}_r$. Let $x \in K_0(\mathcal{T})$, thus there exist by Lemma 6.2.8 a $r \in \mathbb{N}_0$ and $X \in \mathcal{S}_r$ with $[X] = x$. The goal is to show that $x \in \bigoplus_{i=1}^n \mathbb{Z}[E_i]$. If $r = 0$, then X is an object of the sequence E, and thus $x \in \bigoplus_{i=1}^n \mathbb{Z}[E_i]$. Let $r > 0$, there exists $X' \in \mathcal{S}_1$ and $X'' \in \mathcal{S}_1 * \ldots * \mathcal{S}_1$ ($r-1$ factors) and a distinguished triangle

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow TX' \ .$$

By the induction hypothesis $[X'], [X''] \in \bigoplus_{i=1}^n \mathbb{Z}[E_i]$ and therefore also $x = [X] = [X'] + [X''] \in \bigoplus_{i=1}^n \mathbb{Z}[E_i]$. Altogether, the set $\{[E_1], \ldots, [E_n]\}$ is a basis for the Grothendieck group $K_0(\mathcal{T})$. □

Now we are in the position to define an action of $\mathcal{B}_n \ltimes \mathbb{Z}^n$ on the isomorphism classes of exceptional sequences, where $\mathcal{B}_n$ is the braid group on n strands that is given by the following presentation

$$\mathcal{B}_n = \langle \sigma_i, \ 1 \leq i \leq n-1 \mid \sigma_i \sigma_j = \sigma_j \sigma_i, \ |i-j| \geq 2; \ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ i = 1, \ldots, n-2 \rangle$$

and $\mathcal{B}_n \ltimes \mathbb{Z}^n$ is defined by the natural group homomorphism $\mathcal{B}_n \longrightarrow S_n \longrightarrow \mathrm{Aut}(\mathbb{Z}^n)$, given by $\sigma_i \longmapsto (i, i+1)$ (the transposition in the symmetric group S_n) and the natural action of S_n on $\mathbb{Z}^n$. This group action is called mutation and a detailed treatment can be found in [13, Chapter 2] and [84, Chapter 4.2].

Two exceptional sequences $(E_1, \ldots, E_n)$ and $(E'_1, \ldots, E'_n)$ with the same number of objects are called isomorphic if $E_i \cong E'_i$ for all $1 \leq i \leq n$. In the rest of this thesis we only consider exceptional sequences up to isomorphism, although we do not always mention it.

Recall that $\mathcal{T}$ is a triangulated k-category of finite type with translation functor T. Given an object $E \in \mathrm{Ob}(\mathcal{T})$ and a finite dimensional k-vector space V, we write $V \otimes E$ for $E^{\dim_k(V)}$.

Lemma 6.2.34. [3, Chapter 6, Lemma 4.1] *For each pair of objects $E, X \in \mathrm{Ob}(\mathcal{T})$ there is a canonical morphism $\mathrm{Hom}_{\mathcal{T}}(E, X) \otimes E \longrightarrow X$ such that for any $F \in \mathrm{Ob}(\mathcal{T})$ the application of $\mathrm{Hom}_{\mathcal{T}}(F, -)$ induces the composition map*

$$\mathrm{Hom}_{\mathcal{T}}(E, X) \otimes_k \mathrm{Hom}_{\mathcal{T}}(F, E) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(F, X), \ u \otimes v \longmapsto u \circ v.$$

Proof. Let $u_1, \ldots, u_n$ be a basis of $\mathrm{Hom}_{\mathcal{T}}(E, X)$. Then the universal property of the coproduct induces the canonical morphism

$$(u_1, \ldots, u_n) : \mathrm{Hom}_{\mathcal{T}}(E, X) \otimes E = E^n \longrightarrow X.$$

□

Given the objects $E, F \in \mathcal{T}$, put

$$\mathrm{Hom}^\bullet(E,F)\otimes E = \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(E, T^jF)\otimes T^{-j}E$$

and

$$\mathrm{Hom}^\bullet(E,F)^*\otimes F = \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(E, T^jF)\otimes T^{j}F.$$

Since $\mathcal{T}$ is of finite type the latter sums have only finitely many non-trivial summands. For defining the (left-) mutation of exceptional pairs we need the combination of the canonical morphisms $\mathrm{Hom}_{\mathcal{T}}(E, T^jF)\otimes T^{-j}E = \mathrm{Hom}_{\mathcal{T}}(T^{-j}E, F)\otimes T^{-j}E \longrightarrow F$ for $j \in \mathbb{Z}$, that are described in Lemma 6.2.34. They yield the morphism $\mathrm{can} : \mathrm{Hom}^\bullet(E,F)\otimes E \longrightarrow F$, that is independent of the chosen basis.

The object $\mathcal{L}_E F$ is defined by means of the distinguished triangle

$$T^{-1}\mathcal{L}_E F \longrightarrow \mathrm{Hom}^\bullet(E,F)\otimes E \xrightarrow{\mathrm{can}} F \longrightarrow \mathcal{L}_E F\ .$$

By dualizing Lemma 6.2.34, we get the canonical map $\mathrm{can}^* : E \longrightarrow \mathrm{Hom}^\bullet(E,F)^*\otimes F$ and the right mutation $\mathcal{R}_F E$ is defined in means of the following distinguished triangle

$$\mathcal{R}_F E \longrightarrow E \xrightarrow{\mathrm{can}^*} \mathrm{Hom}^\bullet(E,F)^*\otimes F \longrightarrow T\mathcal{R}_F E\ .$$

With other words, we have $\mathcal{L}_E F \cong \mathrm{cone}(\ \mathrm{Hom}^\bullet(E,F)\otimes E \xrightarrow{-\mathrm{can}} F)$ and $\mathcal{R}_F E \cong T^{-1}\,\mathrm{cone}(\ E \xrightarrow{-\mathrm{can}^*} \mathrm{Hom}^\bullet(E,F)^*\otimes F\ )$. The left (resp. right) mutation of an exceptional pair (E, F) is the pair $(\mathcal{L}_E F, E)$ (resp. $(F, \mathcal{R}_F E)$). A mutation of an exceptional sequence $(E_1, \ldots, E_r)$ is defined as a mutation of a pair of adjacent objects which fixes the remaining objects in the sequence.

Proposition 6.2.35. *(*[13], [47]*) The group $\mathcal{B}_n \ltimes \mathbb{Z}^n$ acts on the isomorphism classes of exceptional sequences by*

$$\begin{aligned}
\sigma_i(E_1,\ldots,E_n) &= (E_1,\ldots,E_{i-1},\mathcal{L}_{E_i}E_{i+1},E_i,E_{i+2}\ldots,E_n)\\
\sigma_i^{-1}(E_1,\ldots,E_n) &= (E_1,\ldots,E_{i-1},E_{i+1},\mathcal{R}_{E_{i+1}}E_i,E_{i+2},\ldots,E_n)\\
e_i(E_1,\ldots,E_n) &= (E_1,\ldots,E_{i-1},TE_i,E_{i+1},\ldots,E_n),
\end{aligned}$$

where $(E_1, \ldots, E_r)$ is an exceptional sequence and $e_i \in \mathbb{Z}$ the i-th standard basis vector.

If E is a generating (resp. complete) exceptional sequence, then any mutated exceptional sequence of E is generating (resp. complete). In particular, $\mathcal{B}_n \ltimes \mathbb{Z}^n$ acts on the set of isomorphism classes of generating (resp. complete) exceptional sequences.

Remark 6.2.36. *The proof of the previous proposition is mainly based on the fact that the functor* $\mathrm{Hom}_{\mathcal{T}}$ *is cohomological. One needs to consider various long exact sequences induced* *by* $\mathrm{Hom}_{\mathcal{T}}$ *applied to the following two distinguished triangles*

$$T^{-1}\mathcal{L}_E F \longrightarrow \mathrm{Hom}^\bullet(E,F)\otimes E \xrightarrow{\text{can}} F \longrightarrow \mathcal{L}_E F$$

and

$$\mathcal{R}_F E \longrightarrow E \xrightarrow{\text{can}^*} \mathrm{Hom}^\bullet(E,F)^*\otimes F \longrightarrow T\mathcal{R}_F E\ .$$

The following is well-known.

Lemma 6.2.37. [99] *Let $\mathcal{T}$ be a triangulated k-category of finite type. For $X \in \mathcal{T}$ the map*

$$s_{[X]} : K_0(\mathcal{T}) \longrightarrow K_0(\mathcal{T}),\ [Y] \longmapsto [Y] - \chi^s([X],[Y])[X]$$

is a linear isometric involution and is called reflection*. For $X, Y \in \mathrm{Ob}(\mathcal{T})$ hold*

$$s_{[X]} \circ s_{[Y]} \circ s_{[X]} = s_{s_{[X]}([Y])}.$$

Proof. For Grothendieck groups of finite rank the statements follow from the basic facts of Section 2.1. The general case can be proven by a direct calculation. □

Lemma 6.2.38. *Let $\mathcal{T}$ be a triangulated k-category of finite type with translation functor T. Let (E,F) be an exceptional pair, then $[\mathcal{L}_E F] = s_{[E]}([F])$ and $[\mathcal{R}_F E] = s_{[F]}([E])$.*

Proof. In the Grothendieck group $K_0(\mathcal{T})$ it holds, as $\mathrm{Hom}_{\mathcal{T}}(F, T^iE) = 0$ for all $i \in \mathbb{Z}$, that

$$[\mathrm{Hom}^\bullet_{\mathcal{T}}(E,F)\otimes E] = \sum_{i\in\mathbb{Z}}(-1)^i \dim_k(\mathrm{Hom}_{\mathcal{T}}(E,T^iF))[E] = \chi^s([E],[F])[E] = [\mathrm{Hom}^\bullet_{\mathcal{T}}(E,F)^*\otimes E].$$

Thus the defining distinguished triangles of the mutation yield

$$[\mathcal{L}_E F] = [F] - \chi^s([E],[F])[E] = s_{[E]}([F])$$

and

$$[\mathcal{R}_F E] = [E] - \chi^s([E],[F])[F] = s_{[F]}([E]).$$

□

Now we define the notion of an exceptional sequence in an abelian category.

Definition 6.2.39. *Let $\mathcal{A}$ be an abelian k-category. An object $E \in \mathrm{Ob}(\mathcal{A})$ is called* exceptional *if $\mathrm{End}_{\mathcal{A}}(E) \cong k$ and $\mathrm{Ext}^i_{\mathcal{A}}(E,E) = 0$ for all $i \geq 1$. A sequence $E = (E_1,\dots,E_n)$ of exceptional objects is called* exceptional *if $\mathrm{Ext}^i_{\mathcal{A}}(E_k,E_j) = 0$ for all $1 \leq j \leq k \leq n$ and all $i \in \mathbb{N}_0$ and it is* complete *(resp.* generating*) if E is complete (resp. generating) in the bounded derived category $\mathcal{D}^b(\mathcal{A})$.*

If $\mathcal{A}$ is a hereditary abelian k-category, Proposition 6.2.21 implies that exceptional sequences of $\mathcal{A}$ coincide with those of $\mathcal{D}^b(\mathcal{A})$ up to shift. Hence for an exceptional pair (E, F) in $\mathcal{A}$ the left mutation, that is denoted by $L_E F$, is uniquely determined by one of the following exact sequences

$$0 \to L_E F \to \mathrm{Hom}_{\mathcal{A}}(E, F) \otimes E \to F \to 0$$
$$0 \to \mathrm{Hom}_{\mathcal{A}}(E, F) \otimes E \to F \to L_E F \to 0$$
$$0 \to F \to L_E F \to \mathrm{Ext}^1_{\mathcal{A}}(E, F) \otimes E \to 0.$$

The right mutation $R_F E$ can be reformulated similarly. In particular, this induces naturally a braid group action on the isomorphism classes of exceptional sequences in $\mathcal{A}$.

We define the Grothendieck group $K_0(\mathcal{A})$ of an abelian category $\mathcal{A}$ as the Grothendieck group $K_0(\mathcal{D}^b(\mathcal{A}))$ attached to the bounded derived category of $\mathcal{A}$. In case of an abelian k-category of finite type the (symmetrized) Euler Form attached to $K_0(\mathcal{A})$ is by definition the (symmetrized) Euler form attached to $K_0(\mathcal{D}^b(\mathcal{A}))$.

Remark 6.2.40. *The notions of a Grothendieck group and (symmetrized) Euler form can be defined directly in hereditary abelian k-categories of finite type without using the corresponding bounded derived category. Since there is an isometric isomorphism between these lattices and most proofs in the derived categories are less technical, we defined these notions in the triangulated setting.*

Now we will state well-known facts about exceptional objects in ext-finite hereditary abelian k-categories.

Proposition 6.2.41. *(*[3, Chapter 6, Proposition 5.3], [25, Proposition 6.4.2]*) Let $\mathcal{A}$ be a hereditary ext-finite abelian k-category.*

(a) An object $X \in \mathrm{Ob}(\mathcal{A})$ is exceptional if and only if X is indecomposable and has no self-extension.
(b) Let $E, F \in \mathrm{Ob}(\mathcal{A})$ be exceptional objects with $[E] = [F] \in K_0(A)$. Then $E \cong F$.

6.2.3 Root data attached to triangulated k-categories of finite type

Let k be an algebraically closed field. In this subsection we attach to essentially small triangulated k-categories of finite type, a root system, a reflection group and a distinguished element, the so-called Coxeter transformation. This will be the starting point of the combinatorial approach to exceptional sequences and to the investigation of the poset of thick subcategories generated by an exceptional sequence.

The following statements and their proofs are essentially [99, Proposition 2.10, Lemma 2.11, 2.13]. Here we give a slightly modified versions such that it can be applied to a larger class of triangulated categories.

Theorem 6.2.42. [99, Proposition 2.10] *Let $\mathcal{T}$ be a triangulated k-category of finite type with translation functor T. Assume that $\mathcal{T}$ satisfies the following conditions.*

(a) There exists a generating exceptional sequence $(E_1, \dots, E_n)$ in $\mathcal{T}$.
(b) The action of $\mathcal{B}_n \ltimes \mathbb{Z}^n$ on the set of isomorphism classes of generating exceptional sequences in $\mathcal{T}$ is transitive.

Let $K_0(\mathcal{T})$ be the Grothendieck group of $\mathcal{T}$, χ^s the symmetrized Euler form, $W := \langle s_{[E_i]} \mid 1 \leq i \leq n\rangle$ the group generated by the reflections corresponding to the generating exceptional sequence and $\Phi = W(\{[E_1], \dots, [E_n]\})$.

Then Φ is a simply-laced crystallographic root system in sense of Definition 2.1.5 with root lattice $K_0(A) = \operatorname{span}_{\mathbb{Z}}(\Phi)$. The root system Φ does not depend on the choice of the generating exceptional sequence.

Proof. We will check that the defining properties of a root system are satisfied for Φ. By Lemma 6.2.33 we have $K_0(\mathcal{T}) = \bigoplus_{i=1}^n \mathbb{Z}[E_i]$ for the generating exceptional sequence $(E_1, \dots, E_n)$. Therefore by the definition of W it holds $\operatorname{span}_{\mathbb{Z}}(\Phi) = K_0(\mathcal{T})$. By the definition of exceptional objects we observe that for every $1 \leq i \leq n$ we have

$$\chi^s([E_i],[E_i]) = \chi([E_i],[E_i]) + \chi([E_i],[E_i]) = 2\sum_{p\in\mathbb{Z}}(-1)^p \dim_k(\operatorname{Hom}_{\mathcal{T}}(E_i, T^p E_i)) = 2.$$

Hence for any $X, Y \in \operatorname{Ob}(\mathcal{T})$ and $1 \leq i \leq n$ by the bilinearity of the symmetrized Euler form we get

$$\chi^s\left(s_{[E_i]}([X]), s_{[E_i]}([Y])\right) = \chi^s([X],[Y]).$$

Therefore the group W preserves the bilinear form χ^s and any $\alpha \in \Phi$ satisfies $\chi^s(\alpha, \alpha) = 2$. Hence

$$\frac{2\chi^s(\alpha,\beta)}{\chi^s(\alpha,\alpha)} = \chi^s(\alpha,\beta) \in \mathbb{Z}.$$

A direct consequence of the definition is $s_\alpha(\Phi) \subseteq \Phi$ for all $\alpha \in \Phi$. Altogether, Φ is a simply-laced crystallographic root system.

Let E' be another generating exceptional sequence, then Lemma 6.2.33 yields that the number of objects of E' is $n = \operatorname{rk}(K_0(\mathcal{T}))$. Thus let $E' = (E_1', \dots, E_n')$ and by property (b) there exists $\tau \in \mathcal{B}_n \ltimes \mathbb{Z}^n$ such that $\tau(E_1', \dots, E_n') = (E_1, \dots, E_n)$. Hence Lemma 6.2.38 yields that the root system induced by E' is a subset of Φ. Dually, using τ^{-1} we get that the root systems induced by $(E_1, \dots, E_n)$ and $(E_1', \dots, E_n')$ coincide. Therefore Φ does not depend on the initial generating exceptional sequence. □

Lemma 6.2.43. [99, Lemma 2.13] *Let $\mathcal{T}$ be a triangulated k-category of finite type satisfying the conditions of Theorem 6.2.42, and assume in addition that for any exceptional object E' in $\mathcal{T}$ there exists a complete exceptional sequence in $\mathcal{T}$ that contains E'. Then $[E']$ is a root in Φ.*

Proof. Let E' be an exceptional object and E a generating exceptional sequence containing E'. Since the root system Φ does not depend on the initial generating exceptional sequence it can be also induced by E. Therefore $[E'] \in \Phi$. □

The elements of

$$\Phi^s := \{[E] \mid E \text{ is an exceptional object in } \mathcal{T}\}$$

are called Schur roots. Under the conditions of Lemma 6.2.43 it always holds $\Phi^s \subseteq \Phi$. This property is investigated for the (bounded derived) category of finitely generated modules over certain algebras in [56, Section 4]. For an investigation of the Schur roots attached to the category of coherent sheaves over a weighted projective line of tubular type we refer to the work [7].

Since $\mathcal{T}$ is of finite type and generated by an exceptional sequence, Proposition 6.2.32 implies the existence of a Serre functor $\mathcal{S}$. The latter induces a distinguished element of W that can be described in terms of a generating exceptional sequence.

Lemma 6.2.44. [99, Lemma 2.11] *Let $\mathcal{T}$ be a triangulated k-category of finite type with translation functor T that satisfies the conditions of Theorem 6.2.42. Then the so-called Coxeter functor $C_{\mathcal{T}} := T^{-1} \circ \mathcal{S}$ induces the automorphism $c = s_{[E_1]} \cdots s_{[E_n]} \in W$.*

Proof. Let $(E_1, \ldots, E_n)$ be a fixed generating exceptional sequence in $\mathcal{T}$ and consider the assignment $\Theta : K_0(\mathcal{T}) \longrightarrow K_0(\mathcal{T}),\ [X] \longmapsto [C_{\mathcal{T}} X]$. It is well-defined since for $A, B \in \mathrm{Ob}(\mathcal{T})$ with $[A] = [B]$ there exist distinguished triangles

$$X \longrightarrow Y \oplus U \longrightarrow Z \longrightarrow TX$$

with $U \in \{A, B\}$ by Lemma 6.2.10. Because of Proposition 6.2.15, i.e. the exactness of $\mathcal{S}$, the following are also distinguished triangles

$$\mathcal{S}\, X \longrightarrow \mathcal{S}(Y \oplus U) \longrightarrow \mathcal{S}\, Z \longrightarrow T\, \mathcal{S}\, X$$

with $U \in \{A, B\}$. Hence we get the distinguished triangles

$$T^{-1}\, \mathcal{S}\, X \longrightarrow T^{-1}\, \mathcal{S}(Y \oplus U) \longrightarrow T^{-1}\, \mathcal{S}\, Z \longrightarrow \mathcal{S}\, X$$

for $U \in \{A, B\}$ and therefore $\Theta([A]) = [T^{-1} \circ \mathcal{S}\, A] = [T^{-1} \circ \mathcal{S}\, B] = \Theta([B])$. Since T^{-1} and $\mathcal{S}$ are additive functors the map Θ is a group homomorphism. Using the so-called helix of period n defined in [16, Section 1, Definition], the identification stated in [16, Page 223] and Lemma 6.2.38 we get with a direct calculation the assertion $c = s_{[E_1]} \cdots s_{[E_n]}$. □

Remark 6.2.45. *For more details of helices we refer to the unfinished preprint of Brav and Ploog* [20].

6.3 The category of finitely generated modules over a finite dimensional k-algebra

In this section we are interested in the category mod(A) of finitely generated modules over a finite dimensional k-algebra A with algebraically closed field k and global dimension at most two.

We start with basic statements about (left-) Artinian algebras and their module categories. Throughout this section let A be a unital associated ring. All the A-modules, that we consider, are left A-modules.

Definition 6.3.1. *The ring A is called (left-)* Artinian *if every descending chain of (left-) ideals stabilises. An algebra A is called* Artinian *if A is Artinian as a ring. It is called* basic *if the regular module ${}_AA$ is isomorphic to a finite sum of pairwise different indecomposable A-modules. We say that an A-module M has* finite length *if M as an object of the abelian category Mod(A) has finite length (see Definition 6.1.4).*

In the following we will omit the adjective left in front of Artinian, ideal, module, etc.

Important examples of Artinian algebras are the bound quiver algebras for finite and acyclic quivers. For basic definitions we refer to [6, Chapter III, Section 1].

Example 6.3.2. *Every finite dimensional k-algebra A is Artinian. In particular, if Q is a finite and acyclic quiver and $I \subseteq kQ$ an admissible ideal, then the algebra kQ/I is Artinian.*

The following theorem shows the importance of the theory of quivers.

Theorem 6.3.3. [6, Chapter III, Corollary 1.10] *Let A be a finite dimensional basic k-algebra with algebraically closed field k. There exists a finite acyclic quiver Q and an admissible ideal $I \subseteq kQ$ such that $A \cong kQ/I$.*

In fact, if we are only interested in the module category of a finite dimensional algebra, it suffices to consider basic algebras. It is well-known that any finite dimensional algebra over an algebraically closed field is Morita equivalent to a bound quiver algebra, i.e. their module categories are equivalent (see [6, Chapter II, Corollary 2.6]).

Let A be a ring. The ideal $J(A) := \{r \in A \mid Sr = 0$ for all simple A-modules $S\}$ is called Jacobson radical.

Example 6.3.4. *If $A = kQ/I$ is a finite dimensional bound quiver algebra with finite and acyclic quiver Q and admissible ideal I, then the Jacobson radical $J(A)$ is generated by the arrows of Q (see* [6, Chapter III, Proposition 1.6]*).*

Proposition 6.3.5. [6, Chapter I, Proposition 1.6, Proposition 3.1] *The algebra A is Artinian if and only if the regular module ${}_AA$ has finite length. In this situation, the Jacobson radical $J(A)$ is a two-sided nilpotent ideal and $A/J(A)$ is a semisimple algebra.*

With other words, Artinian rings belong to the larger class of semiprimary rings. For a better understanding of the simple modules we need the following standard result about projective objects.

Proposition 6.3.6. *Let A be a k-algebra and P an A-module. The following is equivalent.*

(a) P is projective,
(b) every short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow 0$ splits,
(c) the functor $\mathrm{Hom}_A(P,-)$ is exact and
(d) P is a direct summand of a free A-module.

The following collection of facts is central in understanding hereditary Artinian k-algebras.

Theorem 6.3.7. [6, Chapter I, II and III] *Let A be an Artinian k-algebra.*

(a) The regular module ${}_AA$ decomposes in a finite sum of indecomposable projectives ${}_AA = \bigoplus_i P_i$. For each of the indecomposables P_i exists an idempotent $\varepsilon_i \in A$ such that $P_i = A\varepsilon_i$. In this case, the idempotent is called primitive.
(b) Every simple A-module is isomorphic to some $S_i := P_i/JP_i$, where $J = J(A)$ is the Jacobson radical of A.
(c) We have $\mathrm{Ext}^1_A(S_i, S_j) \cong \varepsilon_j \mathrm{Hom}_{A/J}(J/J^2, A/J)\varepsilon_i$, where $\mathrm{Ext}^1_A := \mathrm{Ext}^1_{\mathrm{mod}(A)}$.
(d) If A is hereditary, then $\mathrm{End}_A(P_i) = \mathrm{End}_A(S_i)$ and $A = R \oplus J$, where R is a semisimple subalgebra of A.
(e) If A is hereditary, then A is a triangular algebra, i.e. if $\{P_1, \ldots, P_n\}$ is a set of representatives for the isomorphism classes of indecomposable projective modules, then, up to reordering, we may assume that $\mathrm{Hom}_A(P_j, P_i) = 0$ for $1 \leq i < j \leq n$.

It is well-known that the global dimension of the module category $\mathrm{mod}(A)$ for a finite dimensional k-algebra A can be calculated by taking the supremum of the projective dimensions of the simple A-modules.

Proposition 6.3.8. *Let A be a finite dimensional k-algebra and $\{S_1, \ldots, S_n\}$ its set of representatives for the isomorphism classes of simple A-modules. The global dimension is*

$$\mathrm{glob.\,dim}(A) = \sup_{M \in \mathrm{Ob}(\mathrm{mod}(A))} \mathrm{proj.\,dim}(M) = \sup_{1 \leq i \leq n} \mathrm{proj.\,dim}(S_i).$$

A well-known and crucial result is the following.

Proposition 6.3.9. [56, Lemma 4.1] *Let A be a finite dimensional hereditary k-algebra. There exists a complete exceptional sequence consisting of the complete set of representatives of simple modules.*

The next result is central for this thesis. Later we will see that it is an analogy to the Hurwitz transitivity of reduced reflection factorization of Coxeter elements in Coxeter groups.

Theorem 6.3.10. *(*[26]*,* [56]*,* [96]*) Let A be an arbitrary hereditary Artinian algebra. The braid group action is transitive on the set of complete exceptional sequences in* $\mathrm{mod}(A)$*.*

In 1993 William Crawley-Boevey proved a weaker version of the previous theorem. He assumes A to be quiver algebra over a finite quiver and an algebraically closed field with n simples. Then in 1994 Claus Ringel proved the theorem for arbitrary hereditary Artinian algebras. Recently, Andrew Hubery and Henning Krause gave in [56] an alternative proof of the transitivity by using the Hurwitz transitivity of reduced reflection factorizations in Coxeter groups and a reduction theorem by Schofield in the case of finite dimensional algebras.

Another important fact about exceptional sequences is that they can be enlarged to complete exceptional sequences.

Lemma 6.3.11. [26, Lemma 1] *Let A be a finite dimensional hereditary k-algebra and E an exceptional sequence in* $\mathrm{mod}(A)$*. Then E can be enlarged to a complete exceptional sequence of* $\mathrm{mod}(A)$*.*

The following lemma is also well-known and can for example be deduced from the previous theorem.

Lemma 6.3.12. *Let A be a finite dimensional hereditary k-algebra. Then an exceptional sequence of* $\mathrm{mod}(A)$ *is complete if and only if it is generating.*

The next proposition describes the reflection group attached to the category $\mathrm{mod}(A)$ for a finite dimensional hereditary k-algebra A.

Proposition 6.3.13. [56, Section 3] *Let A be a finite dimensional hereditary k-algebra with algebraically closed field k and $(S_1, \ldots, S_n)$ a complete exceptional sequence consisting of simple A-modules. For the Grothendieck group holds $K_0(A) := K_0(\mathrm{mod}(A)) = \bigoplus_{i=1}^n \mathbb{Z}[S_i]$ and the Euler form χ is non-degenerated. The reflection group $W = \langle s_{[S_i]} \mid 1 \leq i \leq n \rangle$ attached to* $\mathrm{mod}(A)$ *is a Coxeter group with simple system $S = \{s_{[S_i]} \mid 1 \leq i \leq n\}$ and the Coxeter transformation is a Coxeter element.*

Proof. By Proposition 6.3.9 there exists a complete exceptional sequence consisting of simple A-modules. Since exceptional sequences of $\mathrm{mod}(A)$ and $\mathcal{D}^b(\mathrm{mod}(A))$ coincides (up to shifts), Lemma 6.2.33 implies that $K_0(A) = \bigoplus_{i=1}^n \mathbb{Z}[S_i]$. Since A is triangular, the Euler form is non-degenerated and the reflection group W attached to $\mathrm{mod}(A)$ due to Theorem 6.2.42 is by [56, Section 3] a Coxeter group with simple system $\{s_{[S_i]} \mid 1 \leq i \leq n\}$. Therefore the Coxeter transformation $c = s_{[S_1]} \cdots s_{[S_n]}$ is a Coxeter element. □

The Jordan-Hölder Theorem 6.1.5 implies directly the following result.

Corollary 6.3.14. *Let A be a finite dimensional hereditary k-algebra and $X \in \mathrm{Ob}(\mathrm{mod}(A))$. Then $0 = [X] \in K_0(A)$ if and only if $X = 0$.*

If we are in the situation of the previous proposition, i.e. A is a finite dimensional hereditary k-algebra, only Coxeter groups as Weyl groups of a symmetrizable Kac-Moody algebra appear. These groups can be found in [56, Theorem B.2].

Next we explain the idea of the proof of Theorem 6.3.10 that is based on the Hurwitz transitivity on the set of reduced reflection factorizations of Coxeter elements attached to finite dimensional k-algebras.

Remark 6.3.15. [56, Theorem 4.7] *Let A be a finite dimensional hereditary k-algebra. In* [56, Proposition 4.6] *Krause and Hubery establish a bijection between complete exceptional sequences of* $\mathrm{mod}(A)$ *and the set of reduced reflection factorizations of a Coxeter element in the Coxeter group attached to* $\mathrm{mod}(A)$. *Moreover, they show that this map is invariant under the braid group action. Thus the transitivity of the braid group action on complete exceptional sequences follows from the Hurwitz transitivity on the set of reduced reflection factorizations of Coxeter elements.*

For later purpose we investigate an example of a module category whose algebra is not hereditary, i.e. its global dimension is at least two. For that we introduce the so-called one point extension of the star quiver and its induced quiver algebra.

Definition 6.3.16. *Let k be an algebraically closed field, $t \geq 3$ a positive integer, $p = (p_1, \ldots, p_t)$ a t-tuple of positive integers greater than one, $\lambda = ([1:0], [0:1], [1:1], [\lambda_4 : 1], \ldots, [\lambda_t : 1])$ a t-tuple of pairwise distinct elements of $\mathbb{P}(k^2)$. Denote by $\mathbb{T}_p$ the quiver in Figure 6.1, that is called* one point extension of the star quiver. *We denote its set of vertices by Q_0.*

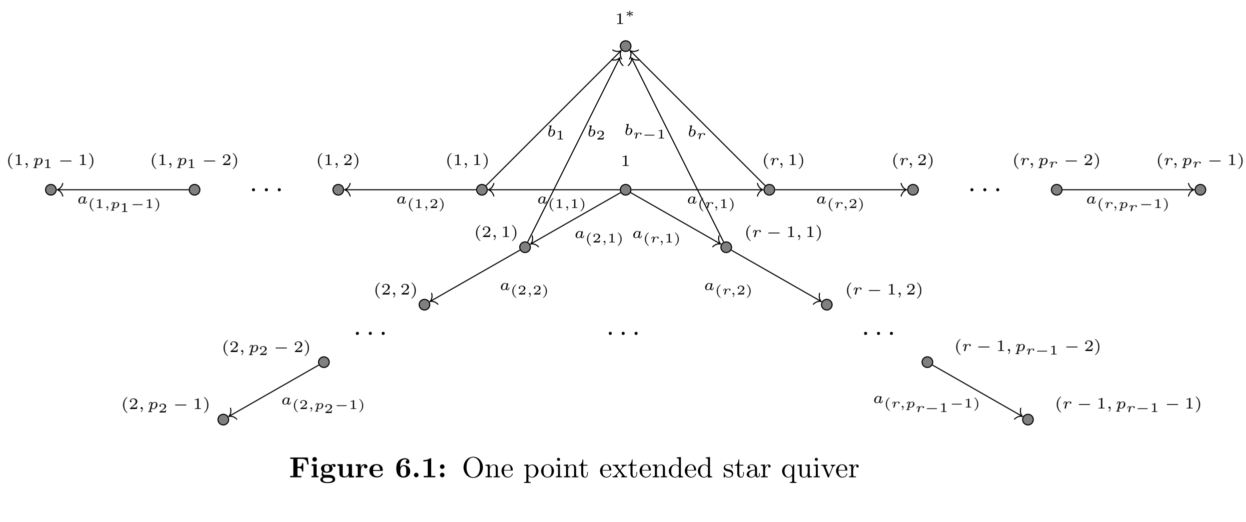


Figure 6.1: One point extended star quiver

Let $n = 2 + \sum_{i=1}^{r} p_i - 1$ the number of vertices of $\mathbb{T}_p$, $k\mathbb{T}_p$ be the corresponding path algebra and

$$I_\lambda = \langle b_1 a_{(1,1)} + b_3 a_{(3,1)} + \sum_{i=4}^{t} \lambda_i b_i a_{(i,1)}, \ \sum_{i=2}^{t} b_i a_{(i,1)} \rangle \subseteq k\mathbb{T}_p$$

an admissible ideal. The bound quiver algebra $\mathfrak{T}_{(p,\lambda)} = k\mathbb{T}_p / I_\lambda$ is called octopus of type (p, λ).

Remark 6.3.17. [99, Remark 2.20] *The octopus of type* (p, λ) *is derived equivalent to the squid of the same type. The latter was first studied by Brenner and Butler in* [21] *(or see* [27]*).*

Lemma 6.3.18. *The algebra* $\mathfrak{T}_{(p,\lambda)}$ *is a unital, associated, basic and finite dimensional* k*-algebra with global dimension* $\operatorname{glob.dim}(\mathfrak{T}_{(p,\lambda)}) = 2$.

Proof. Since $\mathbb{T}_{\mathbb{p}}$ is a finite acyclic quiver the algebra $A := \mathfrak{T}_{(p,\lambda)}$ is unital, associated, basic and finite dimensional over k. Let Q_0 the set of vertices of $\mathbb{T}_{\mathbb{p}}$. Since A is Artinian, the algebra decomposes by Theorem 6.3.7 as

$$_A A = \bigoplus_{v \in Q_0} P_v,$$

where $P_v = A\varepsilon_v$ with a complete set of primitive idempotents ε_v for $v \in Q_0$. Without loss of generality we can assume that the idempotents corresponds to the n trivial paths of $\mathbb{T}_p$. The simples (up to isomorphism) are $S_v = P_v/J(A)P_v = A\varepsilon_v/J(A)\varepsilon_v$ for $v \in Q_0$. By [6, Chapter III, Lemma 1.11] the global dimension of A is at least two. To state the exact global dimension it suffices to know the projective dimensions for all the simples by Proposition 6.3.8. We immediately get $S_{1^*} = P_{1^*}$ and $S_{(i,p_i-1)} = P_{(i,p_i-1)}$ for $1 \leq i \leq r$. A direct calculation yields for $1 \leq i \leq r$ the following projective resolution

$$0 \longrightarrow P_{1^*} \oplus \bigoplus_{j=2}^{p_i-1} P_{(i,j)} \longrightarrow P_{(i,1)} \longrightarrow S_{(i,1)} \longrightarrow 0.$$

A projective resolution for S_1 is given by

$$0 \longrightarrow P_{1^*}^2 \longrightarrow \bigoplus_{i=1}^{r} P_{(i,1)} \longrightarrow P_1 \longrightarrow S_1 \longrightarrow 0$$

and projective resolutions for the remaining simples, i.e. for $1 \leq i \leq r$ and $2 \leq j \leq p_i - 2$, are

$$0 \longrightarrow \bigoplus_{k=j+1}^{p_i-1} P_{(i,k)} \longrightarrow P_{(i,j)} \longrightarrow S_{(i,j)} \longrightarrow 0.$$

The projective resolutions yields that $\operatorname{glob.dim}(A) \leq 2$, which yields the assertion. $\square$

Proposition 6.3.19. *(a) The sequence*

$$\mathcal{E}' = (S_1, S_{(1,1)}, S_{(1,2)}, \ldots, S_{(1,p_1-1)}, \ldots, S_{(r,1)}, S_{(r,2)}, \ldots, S_{(r,p_r-1)}, S_{1^*})$$

consisting of a complete set of simple $\mathfrak{T}_{(p,\lambda)}$*-modules is a complete exceptional sequence.*

(b) The symmetrized Euler form χ^s *on* $K_0(\mathfrak{T}_{(p,\lambda)})$ *satisfies*

$$\chi^s([X],[Y]) = q\big((X_\nu)_{\nu\in Q_0} + (Y_\omega)_{\omega\in Q_0}\big) - q\big((X_\nu)_{\nu\in Q_0}\big) - q\big((Y_\omega)_{\omega\in Q_0}\big)$$

for all $X, Y \in mod(\mathfrak{T}_{(p,\lambda)})$ *with corresponding dimension vectors* $(X_\nu)_{\nu\in Q_0}$ *resp.* $(Y_\nu)_{\nu\in Q_0}$ *where* q *is the Tits form given by*

$$q((Z_\nu)_{\nu\in Q_0}) = \sum_{\nu\in Q_0} Z_\nu^2 - \sum_{\nu\to\omega} Z_\nu Z_\omega + 2Z_{1^*}Z_1$$

for $(Z_\nu)_{\nu\in Q_0} \in \mathbb{N}^{|Q_0|} \setminus \mathbf{0}$.

Proof. Denote by S_ν (up to isomorphism) the simple $\mathfrak{T}_{(p,\lambda)}$-modules corresponding to the vertices of the quiver $\mathbb{T}_p$ in Figure 6.1. Each simple module S_ν is an exceptional object in the category $\mathrm{mod}(\mathfrak{T}_{(p,\lambda)})$. By Theorem 6.3.7 the k-dimension of $\mathrm{Ext}^1_{\mathfrak{T}_{(p,\lambda)}}(S_\nu, S_\omega)$ for vertices ν, ω is equal to the number of arrows from ν to ω and since the global dimension of $\mathrm{mod}(\mathfrak{T}_{(p,\lambda)})$ is two by Lemma 6.3.18, the k-dimension of $\mathrm{Ext}^2_{\mathfrak{T}_{(p,\lambda)}}(S_\nu, S_\omega)$ can easily be calculated by using [17, Proposition 1]. This implies that the sequence

$$\mathcal{E}' = (S_1, S_{(1,1)}, S_{(1,2)}, \dots, S_{(1,p_1-1)}, \dots, S_{(r,1)}, S_{(r,2)}, \dots, S_{(r,p_r-1)}, S_{1^*})$$

is a complete exceptional sequence in $\mathrm{mod}(\mathfrak{T}_{(p,\lambda)})$. A direct calculation yields that the Tits form on $\mathrm{mod}(\mathfrak{T}_{(p,\lambda)})$ is given by

$$q((X_\nu)_{\nu\in Q_0}) = \sum_{\nu\in Q_0} X_\nu^2 - \sum_{\nu\to\omega} X_\nu X_\omega + 2X_{1^*}X_1$$

for $X \in \mathrm{mod}(\mathfrak{T}_{(p,\lambda)})$ with corresponding dimension vectors $(X_\nu)_{\nu\in Q_0}$ (see [17, Definition 2]). By [17, Proposition 2.2] the Euler quadratic form and the Tits form coincide. The latter yields due to polarization the symmetrized Euler form

$$\chi^s([X],[Y]) = q\big((X_\nu)_{\nu\in Q_0} + (Y_\omega)_{\omega\in Q_0}\big) - q\big((X_\nu)_{\nu\in Q_0}\big) - q\big((Y_\omega)_{\omega\in Q_0}\big)$$

for all $X, Y \in \mathrm{mod}(\mathfrak{T}_{(p,\lambda)})$ with corresponding dimension vectors $(X_\nu)_{\nu\in Q_0}$ resp. $(Y_\nu)_{\nu\in Q_0}$. □

Corollary 6.3.20. *The category* $\mathrm{mod}(\mathfrak{T}_{(p,\lambda)})$ *contains a complete exceptional sequence* $\mathcal{E} = (E_2, \dots, E_{n-1}, E_1, E_{1^*})$ *whose generalized Coxeter diagram is an extended Coxeter diagram, that is illustrated in Figure 5.1, where the arms are shortened. In particular, the corresponding reflection group* $W = \langle s_{[E_v]} \mid v \in \{1, \dots, n-1, 1^*\}\rangle$ *is an extended Weyl group with Coxeter transformation* $c = s_{[E_2]} \cdots s_{[E_{n-1}]} s_{[E_1]} s_{[E_{1^*}]}$.

Proof. Let $\mathcal{E}'$ be the sequence defined in the previous result. By applying the braid

$\sigma_{n-2} \dots \sigma_2 \sigma_1$ to $\mathcal{E}'$ we obtain a new complete exceptional sequence $\mathcal{E} = (E_2, \dots, E_{n-1}, E_1, E_{1^*})$. Due to Lemma 6.2.38 the roots corresponding to $\mathcal{E}$ induce a generalized Coxeter diagram that is in fact an extended Coxeter diagram. □

6.4 Category of coherent sheaves over a weighted projective line

The first part of this section is devoted to the definition of the category of coherent sheaves over a weighted projective line and its basic properties. The weighted projective line first appeared in a work of Werner Geigle and Helmut Lenzing in 1987 (see [44]). They provide a connection between the so-called canonical algebras defined by Claus Michael Ringel (see [95]) and a graded sheaf theory. We start with the classical approach described in [44] and summarize later all the necessary properties in Theorem 6.4.3.

The relevance of the category of coherent sheaves over a weighted projective line is also justified by the following important theorem by Happel.

Theorem. *(Happel's Theorem)*[51] *Let $\mathcal{A}$ be a connected hereditary ext-finite k-category with tilting object and k an algebraically closed field. Then $\mathcal{A}$ is derived equivalent to* $\operatorname{mod}(A)$ *for a finite dimensional hereditary k-algebra A or to the category of coherent sheaves over a weighted projective line.*

Let k be an algebraically closed field, $p = (p_1, \dots, p_t)$ be a $t-$tuple of positive integers, called weight sequence. Denote by $L(p)$ the rank one abelian group on generators $\vec{x_1}, \dots, \vec{x_t}$ with relations $\vec{c} := p_1\vec{x_1} = \dots = p_t\vec{x_t}$ and define the partially order that is induced by the relation $\vec{x} \geq 0$ if and only if $\vec{x} \in L(p)_+ := \sum_{i=1}^{t} \mathbb{Z}_{\geq 0}\vec{x_i}$. The sequence $\lambda = (\lambda_1, \dots, \lambda_t)$ with elements of $\mathbb{P}(k^2)$ is called parameter sequence. Since $\mathrm{PGL}(k^2)$ acts 3-transitively on $\mathbb{P}(k^2)$ we can assume with $\lambda_i = [a_i : b_i]$ that $a_1 = 1$, $b_1 = 0$, $a_2 = 0$, $b_2 = 1$, $a_3 = 1$ and $b_3 = 1$. Consider the algebra

$$S(p, \lambda) = \frac{k[X_1, \dots, X_t]}{(X_i^{p_i} - X_2^{p_2} + \lambda_i X_1^{p_1} \mid i = 3, 4, \dots, t)}$$

with $L(p)$-grading that is defined by $\deg(X_i) = \vec{x}_i$, where we use the identification $\lambda_i = \frac{a_i}{b_i}$. A weighted projective line $\mathbb{X}(p, \lambda)$ with weight sequence p and parameter sequence λ is the set of $L(p)$-graded homogeneous prime ideals $\mathfrak{p}$ such that $S(\mathbf{p}, \lambda)_+ := \bigoplus_{\mathbf{0} < \vec{x}} \mathbf{S}(\mathbf{p}, \lambda)_{\vec{x}} \not\subseteq \mathfrak{p}$. The set $\mathbb{X}(p, \lambda)$ is equipped with the Zariski topology, i.e. with the system of closed sets of the form

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \mathbb{X}(p, \lambda) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

for a homogeneous ideal $\mathfrak{a} \subseteq S(p, \lambda)$, and a $L(p)$-graded structure sheaf of rings $\mathcal{O} = \mathcal{O}_{\mathbb{X}}$, that arises as follows. To any homogeneous $f \in S(p, \lambda)$ there is attached the open set

$$D(f) := \mathbb{X}(p, \lambda) \setminus V(\langle f \rangle) = \{\mathfrak{p} \in \mathbb{X}(p, \lambda) \mid f \notin \mathfrak{p}\},$$

that is called standard open set. The sets $D(f)$ form a basis of the Zariski topology. Now set

$$\mathcal{O}(D(f)) := S(p,\lambda)_f := \left\{ \frac{g}{f^l} \mid g \in S(p,\lambda),\ l \in \mathbb{N} \right\}$$

with the natural grading, and analogous to the construction in [54, page 76] we get that $\mathcal{O}$ is a sheaf of $L(p)$-graded rings on the topological space $\mathbb{X}(p,\lambda)$. The category $\operatorname{coh}(\mathbb{X}(p,\lambda))$ of coherent sheaves over a weighted projective line $\mathbb{X}(p,\lambda)$ contains as objects the $L(p)$-graded $\mathcal{O}$-modules that satisfy the following condition. For each point in $\mathbb{X}$ there is a neighbourhood U and an exact sequence

$$\bigoplus_{j\in J} \mathcal{O}(\vec{l_j})_{|U} \longrightarrow \bigoplus_{i\in I} \mathcal{O}(\vec{l_i})_{|U} \longrightarrow M_{|U} \longrightarrow 0$$

with I, J finite, where $\mathcal{O}(\vec{l})$ for $\vec{l} \in L(p)$ is twisted structure sheaf that arises by translating the grading structure on $S(p,\lambda)$ by $\vec{l}$ (see a similar construction in [54, Chapter II, Section 5]).

Example 6.4.1. [3, Chater 6, Example 2.3] *Let $R = k[X,Y,Z]/(h)$ with $h = X^2 + Y^3 + Z^5$. By attaching the degrees 15, 10 and 6 to X, Y and Z the algebra R turns into a positively $\mathbb{Z}$-graded k-algebra. Let $\operatorname{mod}^{\mathbb{Z}}(R)$ be the category of finitely generated $\mathbb{Z}$-graded R-modules and $\operatorname{mod}_0^{\mathbb{Z}}(R)$ be the subcategory consisting of modules of finite length. Let S be the class of morphisms f of $\operatorname{mod}^{\mathbb{Z}}(R)$ such that $\operatorname{Ker}(f), \operatorname{Coker}(f) \in \operatorname{Ob}(\operatorname{mod}_0^{\mathbb{Z}}(R))$. Since $\operatorname{mod}_0^{\mathbb{Z}}(R)$ is a Serre subcategory we can form the quotient that can be defined by $\operatorname{mod}^{\mathbb{Z}}(R)/\operatorname{mod}_0^{\mathbb{Z}}(R) = \operatorname{mod}^{\mathbb{Z}}(R)[S^{-1}]$. The category $\operatorname{mod}^{\mathbb{Z}}(R)/\operatorname{mod}_0^{\mathbb{Z}}(R)$ is equivalent to $\operatorname{coh}(\mathbb{X}(2,3,5))$.*

The representation type of the category is connected to the so-called genus of the line $\mathbb{X}(p,\lambda)$.

Definition 6.4.2. [84, Definition 3.1.10] *The genus $g_{\mathbb{X}}$ of a weighted projective line $\mathbb{X} = \mathbb{X}(p,\lambda)$ is defined by*

$$g_{\mathbb{X}} = 1 + \frac{1}{2}\left((t-2)p - \sum_{i=1}^{t} \frac{p}{p_i}\right),$$

where $p = lcm(p_1, \ldots, p_t)$.

A weighted projective line of genus smaller than one (resp. equal to one, resp. greater than one) is called of domestic (resp. tubular, resp. wild) type.

Up to permutations the domestic weight types are (p) with $p \geq 1$, (p,q) with $p,q \geq 2$, $(2,2,n)$ with $n \geq 2$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$. The tubular weight types are up to permutations $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ and $(2,3,6)$.

In 1997 Lenzing stated in [72] conditions that determine exactly the category of coherent sheaves over a weighted projective line. In the following we mostly suppress the weight sequence p and the parameter family λ and denote the category shortly by $\operatorname{coh}(\mathbb{X})$.

Theorem 6.4.3. *([25, Theorem 6.8.1], [72, Theorem 1]) Let $\mathcal{A}$ be a k-abelian category.*

The categories $\mathcal{A}$ and $\mathrm{coh}(\mathbb{X})$ are equivalent if and only if $\mathcal{A}$ satisfies the following properties.

- *(a) $\mathcal{A}$ is skeletally small, connected, and ext-finite.*
- *(b) $\mathcal{A}$ is noetherian, i.e. each object is noetherian.*
- *(c) $\mathcal{A}$ is hereditary and has no non-zero projective object.*
- *(d) $\mathcal{A}$ has a tilting object.*
- *(e) The Euler form associated to $\mathcal{A}$ is non-degenerated and has discriminant ± 1.*

From the latter we can deduce all the important facts we need for the last section of this thesis. These information are summarized in the following proposition.

Proposition 6.4.4. *(*[25, Proposition 6.2.1], [25, Proposition 6.3.7], [44, Proposition 4.1], [83, Lemma 9.1.3]*)*

- *(a) The category $\mathrm{coh}(\mathbb{X})$ admits a Serre duality that induces by Corollary 6.2.24 a Serre functor. The duality is given by*

$$\mathrm{Ext}^1_{\mathrm{coh}(\mathbb{X})}(X,Y)^* \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{coh}(\mathbb{X})}(Y, X(\vec{w}))$$

  for $X, Y \in \mathrm{Ob}(\mathrm{coh}(\mathbb{X}))$ and $\vec{w} := (n-2)\vec{c} - \sum_i \vec{x}_i \in L(p)$.
- *(b) The Grothendieck group is finitely generated.*
- *(c) Every object in $\mathrm{coh}(\mathbb{X})$ is a direct sum of an object in $\mathrm{coh}(\mathbb{X})_0$ and $\mathrm{coh}(\mathbb{X})_+$, where $\mathrm{coh}(\mathbb{X})_0$ is the subcategory consisting of all finite length objects and*

$$\mathrm{coh}(\mathbb{X})_+ = \{A \in \mathrm{coh}(\mathbb{X}) \mid \mathrm{Hom}_{\mathrm{coh}(\mathbb{X})}(A_0, A) = 0 \text{ for } A_0 \in \mathcal{A}_0\}.$$

  The objects of $\mathrm{coh}(\mathbb{X})_0$ are called torsion *and those of $\mathrm{coh}(\mathbb{X})_+$* vector bundle *or* torsion-free*. The vector bundles are the locally finite sheaves and those of rank one are called* line bundle*. They are exactly of the form $\mathcal{O}(\vec{x})$ for $\vec{x} \in L(p)$.*
- *(d) Each non-zero object $A \in \mathrm{Ob}(\mathrm{coh}(\mathbb{X}))$ satisfies $[A] \neq 0$ in the Grothendieck group $K_0(\mathrm{coh}(\mathbb{X})$.*
- *(e) There is a canonical tilting object $T = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} \mathcal{O}(\vec{x})$, called* tilting bundle*, whose endomorphism algebra is isomorphic to a canonical algebra Λ with the same parameters p and λ in sense of* [95]*. In particular, T yields a triangulated equivalence $\mathcal{D}^b(\mathrm{mod}(\Lambda)) \cong \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$.*
- *(f) The category $\mathrm{coh}(\mathbb{X})$ contains a complete exceptional sequence.*

By Lemma 6.2.33 we know that every generating exceptional sequence is complete. The next result shows that the reverse is also true for the category $\mathrm{coh}(\mathbb{X})$.

Lemma 6.4.5. *(*[84, Lemma 4.1.2]*) An exceptional sequence E in $\mathrm{coh}(\mathbb{X})$ is generating if and only if it is complete.*

The following theorem yields the connection between the category $\mathrm{coh}(\mathbb{X})$ and the extended Weyl groups.

Theorem 6.4.6. [99, Proposition 2.24] *Let $\mathfrak{T}_{p,\lambda}$ be the algebra defined in section 6.3. There exists an equivalence of triangulated categories*

$$\mathcal{D}^b(\operatorname{coh}(\mathbb{X}(p,\lambda))) \cong \mathcal{D}^{\mathbf{b}}(\mathfrak{T}_{\mathbf{p},\lambda}).$$

Sketch of proof. Let $p = (p_1, \ldots, p_t)$ be the weight sequence, $\lambda = (\lambda_1, \ldots, \lambda_t)$ the parameter sequence and denote as usual $\mathbb{X} = \mathbb{X}(p, \lambda)$. Since $\operatorname{coh}(\mathbb{X})_0$ is a hom-finite length k-category that admits a Serre duality τ, it is by [25, Proposition 1.8.2] uniserial, i.e. each indecomposable object has a unique composition series. Thus by [25, Proposition 1.8.2] $\operatorname{coh}(\mathbb{X})_0 = \coprod_{i\in I} \mathcal{A}_i$ decomposes in connected uniserial categories $\mathcal{A}_i$ with index set I consisting of the τ-orbits of the simple objects of $\operatorname{coh}(\mathbb{X})_0$. Each simple object S of $\operatorname{coh}(\mathbb{X})_0$ has a finite chain of monomorphisms

$$S = S_{[1]} \hookrightarrow S_{[2]} \hookrightarrow S_{[3]} \hookrightarrow \ldots,$$

where $S_{[i]} \in \operatorname{Ob}(\operatorname{coh}(\mathbb{X})_0)$ are indecomposable objects of length i with socle $\operatorname{soc}(S_{[i]}) = S$ for all i, where $\operatorname{soc}(S_{[i]})$ is the sum of all maximal subobjects of $S_{[i]}$.

Let $1 \leq i \leq t$. The so-called exceptional simple sheaves $S_{i,j}$ for $j \in \mathbb{Z}/p_i\mathbb{Z}$ arise as cokernel terms of the exact sequence

$$0 \longrightarrow \mathcal{O}(j\vec{x}_i) \xrightarrow{X_i} \mathcal{O}((j+1)\vec{x}_i) \longrightarrow S_{i,j} \longrightarrow 0$$

(see [44, Equation (2.5.2)]) and by Serre duality we have $S_{i,j-1} = \tau S_{i,j} = S_{i,j}(\vec{w})$ ([44, Equation (2.5.4)]) for all $j \in \mathbb{Z}/p_i\mathbb{Z}$. Thus we get the chain of maps

$$\mathcal{O}(\vec{x_i}) \to (S_{i,1})_{[2]} \hookrightarrow (S_{i,1})_{[3]} \hookrightarrow \ldots \hookrightarrow (S_{i,1})_{[p_i-1]}$$

for all $1 \leq i \leq t$ and therefore the diagram

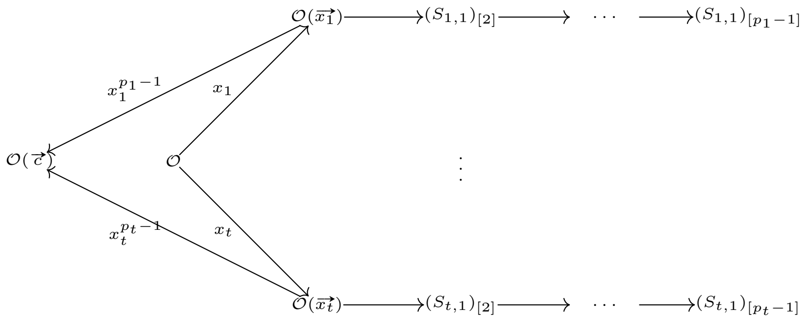


Figure 6.2: Extended star quiver

A direct calculation using the structure of the $S_{i,j}$ yields that the sum of the objects that appears in Figure 6.2 is a tilting object. The defining relations of $\mathfrak{T}_{p,\lambda}$ can be easily deduced from the line bundles in Figure 6.2. □

Remark 6.4.7. *By* [52, Chapter II, Theorem 2.3] *the Theorem 6.4.6 implies directly that* *the category* $\operatorname{mod}(\mathfrak{T}_{p,\lambda})$ *has global dimension at most two.*

Analogously to Theorem 6.3.10 and Lemma 6.3.11 we get the following results.

Theorem 6.4.8. *(*[70, Theorem 1.1], [84, Theorem 4.31]*) Let* $\mathbb{X}$ *be a weighted projective line. Then the braid group acts transitively on the ismorphism classes of complete exceptional sequences in* $\operatorname{coh}(\mathbb{X})$*. In particular, we have a transitive action on the isomorphism classes of complete exceptional sequences in* $\mathcal{D}^b(\operatorname{coh}(\mathbb{X}))$*.*

Lemma 6.4.9. [84, Lemma 4.1.3] *Every exceptional sequence in* $\operatorname{coh}(\mathbb{X})$ *(resp. in* $\mathcal{D}^b(\operatorname{coh}(\mathbb{X}))$*) can be enlarged to a complete exceptional sequence.*

Proposition 6.4.4 (f) and Theorem 6.4.8 are exactly the prerequisites of Theorem 6.2.42. Therefore the complete exceptional sequences of $\operatorname{coh}(\mathbb{X})$ induce a simply-laced, crystallographic and reduced root system. In fact, the corresponding reflection group is an extended Weyl group and the Coxeter transformation is induced by the Serre functor.

Theorem 6.4.10. [99, Section 2] *Let* $\operatorname{coh}(\mathbb{X}(p,\lambda))$ *be the category of coherent sheaves over a weighted projective line* $\mathbb{X}(p,\lambda)$*. There exists a complete exceptional sequence* E *such that the corresponding generalized Coxeter diagram is an extended Coxeter diagram as given in Figure 5.1, that only depends on the weight sequence* p*. The reflection group is the associated extended Weyl group and the element induced by the Serre functor is a Coxeter transformation in the sense of Definition 5.2.1 and Lemma 6.2.44.*

Proof. By Theorem 6.4.6 we have a triangulated equivalence $\mathcal{D}^b(\operatorname{coh}(\mathbb{X}(\mathbf{p},\lambda)) \cong \mathcal{D}^{\mathbf{b}}(\mathfrak{T}_{\mathbf{p},\lambda})$, and since triangulated equivalences preserves exceptional sequences and induce isometric isomorphisms on the level of the Grothendieck groups the assertion follows from Corollary 6.3.20. □

Remark 6.4.11. *If* $\mathbb{X}$ *is of tubular type the reflection group attached to* $\operatorname{coh}(\mathbb{X})$ *is an elliptic Weyl group of type* $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ *or* $E_8^{(1,1)}$*. A uniform investigation of elliptic Weyl groups can be found in* [97].

If $\mathbb{X}$ *is of domestic type the reflection group attached to* $\operatorname{coh}(\mathbb{X})$ *is an affine Coxeter group. We get the following assignments.*

weight sequence	*type of the affine Dynkin diagram*
(p,q), $p,q \geq 1$	$\widetilde{A}_{p+q}$
$(2,2,n)$, $n \geq 2$	$\widetilde{D}_{n+2}$
$(2,3,3)$	$\widetilde{E}_6$
$(2,3,4)$	$\widetilde{E}_7$
$(2,3,5)$	$\widetilde{E}_8$

An important tool in later investigations will be the so-called right perpendicular category defined as follows.

Definition 6.4.12. *Let $\mathcal{A}$ be a hereditary abelian category and C a class of objects. The* right-perpendicular *of C is the following full subcategory of $\mathcal{A}$*

$$C^{\perp} = \{X \in \mathrm{Ob}(\mathcal{A}) \mid \mathrm{Hom}_{\mathcal{A}}(A, X) = 0 = \mathrm{Ext}^1_{\mathcal{A}}(A, X) \text{ for all } A \in C\}.$$

The left-perpendicular *of C is defined dually.*

The following result is well-known.

Lemma 6.4.13. *Let $\mathcal{A}$ be a hereditary abelian category and C a full subcategory. Then $C^{\perp}$ is a thick subcategory. If $\mathcal{A}$ is in addition a k-category of finite type, C a thick subcategory and $E = (E_1, \dots, E_n)$ an exceptional sequence in C with $n \in \mathbb{Z}_{>0}$ objects, then $\tau E \in C$ for all $\tau \in \mathcal{B}_n$, i.e. thick subcategories are closed under mutations of exceptional sequences.*

Proof. Following Lemma 6.1.22 it suffices to show that $C^{\perp}$ is closed under direct summands and fulfils the 'two out of three property'. The category $C^{\perp}$ is obviuosly closed under direct summands and the 'two out of three' property follows from the exact sequence described in Theorem 6.1.17. Concretely, let $X, Y \in \mathrm{Ob}(C^{\perp})$. By definition it holds $0 = \mathrm{Hom}_{\mathcal{A}}(A, X) = \mathrm{Ext}^1_{\mathcal{A}}(A, X) = \mathrm{Hom}_{\mathcal{A}}(A, Y) = \mathrm{Ext}^1_{\mathcal{A}}(A, Y)$ for any $A \in \mathrm{Ob}(C)$. Let $Z \in \mathrm{Ob}(\mathcal{A})$ such that $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is exact. Since $\mathcal{A}$ is hereditary, applying the functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ yields the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, X) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, Z)$$

$$\longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(A, X) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(A, Y) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(A, Z) \longrightarrow 0$$

The latter implies that $\mathrm{Hom}_{\mathcal{A}}(A, Z) = \mathrm{Ext}^1_{\mathcal{A}}(A, Z) = 0$, i.e. $Z \in \mathrm{Ob}(C^{\perp})$. The other two conditions can be proven analogously.

Next we will show that for an exceptional pair (E, F) in C we have that $L_E F$ and $R_F E$ are objects of C. We only consider the short exact sequence

$$0 \longrightarrow L_E F \longrightarrow \mathrm{Hom}_{\mathcal{A}}(E, F) \otimes E \longrightarrow F \longrightarrow 0.$$

The argumentation for the other sequences is analogous. Since thick subcategories are closed under extensions we have $\mathrm{Hom}_{\mathcal{A}}(E, F) \otimes E \in \mathrm{Ob}(C)$ and the 'two out of three' property implies $L_E F \in \mathrm{Ob}(C)$. □

We are mainly interested in $E^{\perp}$ for an exceptional object E in $\mathrm{coh}(\mathbb{X})$. Since exceptional objects are indecomposable they are either torsion-free or vector bundles by Proposition 6.4.4 (c).

Theorem 6.4.14. [102, Lemma 3.16] *Let* $E \in \mathrm{Ob}(\mathrm{coh}(\mathbb{X}(p,\lambda)))$ *be a torsion exceptional object. Then* $E^{\perp}$ *is equivalent to* $\mathrm{coh}(\mathbb{X}') \prod \mathrm{mod}(k\overrightarrow{A}_{l(E)-1})$*, where* $\mathbb{X}$ *is the weighted projective line with weight sequence* $p = (p_1, \ldots, p_i - l(E), \ldots, p_t)$ *for a suitable* $1 \leq i \leq t$ *and* $k\overrightarrow{A}_{l(E)-1}$ *is the path algebra of the equioriented* A_l *quiver, where* $l(E)$ *is the length of* E*.*

Theorem 6.4.15. [57] *Let* $E \in \mathrm{Ob}(\mathrm{coh}(\mathbb{X}(p,\lambda)))$ *be an exceptional vector bundle and* n *the rank of the associated Grothendieck group. Then* $E^{\perp}$ *is equivalent to* $\mathrm{mod}(\Lambda)$ *for a finite-dimensional hereditary algebra* Λ *with* $n-1$ *simple modules.*

For a proof of the previous result we refer to [28, Theorem 4.2]

For finite dimensional hereditary algebras the situation is more straightforward.

Theorem 6.4.16. [45] *Let* A *be a finite-dimensional hereditary* k*-algebra such that* $\mathrm{mod}(A)$ *consists of* n *simples. Let* E *be an exceptional object in* $\mathrm{mod}(A)$*, then the category* $E^{\perp}$ *is equivalent to* $\mathrm{mod}(\Lambda)$ *for a finite dimensional hereditary* k*-algebra* Λ *with* $n-1$ *simples.*

Chapter 7

Thick subcategories generated by an exceptional sequence and the interval poset

This section is devoted to the study of exceptional sequences and thick subcategories of hereditary ext-finite abelian k-categories with tilting object, where k is an algebraically closed field. For connected categories Happel's theorem states that there exists up to derived equivalence only the cases $\text{mod}(A)$ for a finite dimensional hereditary k-algebra A and $\text{coh}(\mathbb{X})$ for a weighted projective line $\mathbb{X}$. The latter leads to the following table, that pictures all the possible cases.

	$\text{coh}(\mathbb{X})$ type: wild	
	$\text{coh}(\mathbb{X})$ type: tubular	
$\text{mod}(A)$ A representation-finite	$\text{mod}(A) \sim_{der} \text{coh}(\mathbb{X})$ A tame hereditary type: domestic	$\text{mod}(A)$ A wild hereditary

Table 7.1: Hereditary categories with tilting object [3, Chapter 6]

Since a derived equivalence preserves exceptional sequences as well as thick subcategories, Happel's theorem allows us to restrict ourselves to these cases.

If the category is derived equivalent to $\text{mod}(A)$ for a finite dimensional k-algebra A, the thick subcategories generated by an exceptional sequence have been combinatorially described by Kiyoshi Igusa, Ralf Schiffler and Hugh Thomas in [60] as well as Krause in [68]. For that they equip the category $\text{mod}(A)$ with a Coxeter group W and establish a bijection between exceptional sequences in $\text{mod}(A)$ and prefixes of reduced reflection factorizations of a suitable Coxeter element in W, that is induced by the Auslander-Reiten translate (for instance see [56]). This leads to the important classification of thick subcategories generated by an exceptional sequence in terms of the generalized non-crossing partitions. The latter approach gives a new perspective on these interesting combinatorial structures that already appeared in other mathematical disciplines. In fact, for algebras A of finite representation type the latter description is a combinatorial classification of the lattice of thick subcategories. For algebras A of tame representation type Köhler gives a complete classification of the thick subcategories that partially depends on the generalized non-crossing partitions (see [66, Theorem 1.3]).

We have already seen that the category of coherent sheaves over a weighted projective line induces an extended Weyl group. As in the module case it turns out that there exists a distinguished element, induced by the Serre functor, whose set of prefixes of generating reflection factorizations is in bijection with the exceptional sequences. This leads to a similar description of the thick subcategories generated by exceptional sequences in terms of prefixes of Coxeter transformations. The latter approach is deeply influenced by [68].

7.1 Exceptional sequences and prefixes of reduced generating reflection factorizations of Coxeter transformations

In this section we link exceptional sequences of hereditary ext-finite abelian k-categories with tilting object to prefixes of reduced reflection factorizations of certain elements of suitable reflection groups. For that we use the results about the Hurwitz transitivity of reduced generating reflection factorizations of Coxeter transformations of Section 5 as well as the transitivity of the mutation of complete exceptional sequences.

We start by associating a reflection group to the categories of our interest. Let $\mathcal{A}$ be a connected hereditary ext-finite abelian k-category with tilting object. By Happel's theorem we can assume that $\mathcal{A}$ is $\mathrm{mod}(A)$ for a hereditary finite dimensional k-algebra or $\mathrm{coh}(\mathbb{X})$, the category of coherent sheaves over a weighted projective line $\mathbb{X}$. By Theorem 6.3.10 and Theorem 6.4.8 the triangulated category $\mathcal{D}^b(\mathcal{A})$ satisfies the prerequisite of Theorem 6.2.42. Thus it yields the existence of a root system Φ attached to $\mathcal{A}$, a reflection group W and a Coxeter transformation c that is induced by the Serre functor. If $\mathcal{A} = \mathrm{mod}(A)$ for a finite dimensional k-algebra A we have seen in Proposition 6.3.13 that W is a Coxeter group and c a Coxeter element. For $\mathcal{A} = \mathrm{coh}(\mathbb{X})$ Theorem 6.4.10 implies that W is an extended Weyl group and c a Coxeter transformation in the sense of Chapter 5.

The next lemma defines a map that attaches an exceptional object uniquely to a reflection.

Lemma 7.1.1. *Let $\mathcal{A}$ be a connected hereditary ext-finite abelian k-category with tilting object and algebraically closed field k. The assignment $E \longmapsto s_{[E]}$ is an injection from the set of (isomorphism classes) of exceptional objects of $\mathcal{A}$ and the reflection group W that is attached to $\mathcal{A}$.*

Proof. By Happel's theorem we can restrict ourselves to the cases $\mathcal{A} \in \{\mathrm{mod}(A), \mathrm{coh}(\mathbb{X})\}$. The case $\mathcal{A} = \mathrm{mod}(A)$ is already discussed in [68, Lemma 6.2]. Let E, F be two exceptional objects with $s_{[E]} = s_{[F]}$. By Lemma 6.2.43 we have $[E], [F] \in \Phi$ and thus $s_{[E]} = s_{[F]} \in W$. Since Φ is reduced we get $[E] = \pm[F]$. Assume that $[E] = -[F]$, then $0 = [E \oplus F]$ and the latter implies due to Corollary 6.3.14 and Proposition 6.4.4 (d) that $E \oplus F = 0$, a contradiction. Hence the classes $[E]$ and $[F]$ coincides and Proposition 6.2.41 (b) yields $E \cong F$. □

As noted in Section 5.5 let

$$\underline{\mathrm{Red}}_T(c) = \{(t_1, \ldots, t_n) \in T^n \mid c = t_1 \cdots t_n,\ W = \langle t_1, \ldots, t_n \rangle\}$$

be the set of reduced reflection factorizations with $\ell_T(c) = n$ and that generate the group W, where we have $n = \mathrm{rk}(K_0(\mathcal{A}))$ by Corollary 5.3.6. The factorizations of $\underline{\mathrm{Red}}_T(c)$ are called generating.

Proposition 7.1.2. *Let $\mathcal{A}$ be a hereditary connected ext-finite abelian k-category with tilting object and algebraically closed field k. Let W be the associated reflection group and $c \in W$ the Coxeter transformation that is induced by the Serre functor and $n = \ell_T(c) = \mathrm{rk}(K_0(\mathcal{A}))$. The map*

$$\mathcal{E} \longrightarrow \{(t_1, \ldots, t_r) \mid t_1, \ldots, t_r \in T,\ 1 \leq r \leq n, \\ \text{there exist } t_{r+1}, \ldots, t_n \in T \text{ such that } (t_1, \ldots, t_n) \in \underline{\mathit{Red}}_T(c)\}$$

defined by

$$(E_1, \ldots, E_r) \longmapsto (s_{[E_1]}, \ldots, s_{[E_r]})$$

is a bijection between the set of isomorphism classes of exceptional sequences in $\mathcal{A}$ and the set of prefixes of reduced generating reflection factorizations of the Coxeter transformation c. Moreover, this map is invariant under the braid group action and exceptional sequences of length k are mapped to prefixes with k factors, where $1 \leq k \leq n$.

Proof. The case $\mathcal{A} = \mathrm{mod}(A)$ is discussed in [56, Proposition 4.6]. Hence let $\mathcal{A} = \mathrm{coh}(\mathbb{X})$ and $\mathcal{E}$ be the set of isomorphism classes of exceptional sequences. Consider an exceptional sequence $E = (E_1, \ldots, E_r) \in \mathcal{E}$ with $1 \leq r \leq n$. We assign to E the tuple $(s_{[E_1]}, \ldots, s_{[E_r]})$. Lemma 6.2.43 implies that $s_{[E_i]} \in T$ for all $1 \leq i \leq r$. By Lemma 6.4.9 we can enlarge E to a complete exceptional sequence $(E_1, \ldots, E_n)$. By Lemma 6.2.44 we have that $(s_{[E_1]}, \ldots, s_{[E_n]}) \in \underline{\mathrm{Red}}_T(c)$ and thus $(s_{[E_1]}, \ldots, s_{[E_r]})$ is an element of

$$\{(t_1, \ldots, t_r) \mid t_1, \ldots, t_r \in T,\ 1 \leq r \leq n, \text{ there exist } t_{r+1}, \ldots, t_n \in T \\ \text{such that } (t_1, \ldots, t_n) \in \underline{\mathrm{Red}}_T(c)\}.$$

Therefore the assignment is well-defined. The injectivity follows from Lemma 7.1.1 and by Lemma 6.2.38 it is compatible with the braid group action. The latter together with Lemma

6.4.9 and Theorems 6.4.8, 5.4.1 and 5.5.1 yield the surjectivity of the map. Obviously, by Corollary 5.3.6 sequences of length k are mapped to prefixes of length k. □

Remark 7.1.3. *Let $\mathcal{A}$, W and c be as in Proposition 7.1.2. If $\mathcal{A}$ is not derived equivalent to the category of coherent sheaves over a weighted projective line of tubular type, then any reflection factorization of the Coxeter transformation is generating, i.e. $Red_T(c) = \underline{Red}_T(c)$. Hence we have a bijection between exceptional sequences and prefixes of reduced reflection factorizations of the Coxeter transformation.*

7.2 Thick subcategories generated by an exceptional sequence

Now we turn to the investigation of thick subcategories generated by an exceptional sequence. We use in the following the same notions as in the previous section. We start with the definition of the interval poset that is already known from the context of Coxeter groups.

Definition 7.2.1. *Denote by T the set of reflections of W.*

(a) Define a partial order on W by

$$x \leq y \text{ if and only if } \ell_T(y) = \ell_T(x) + \ell_T(x^{-1}y)$$

for $x, y \in W$, called absolute order*, where ℓ_T is the* length function *on W with respect to T.*

(b) For $x, w \in W$ the element x is called a prefix *of w if $x \leq w$.*

(c) For $w \in W$ the interval

$$[1, w] = \{x \in W \mid 1 \leq x \leq w\}$$

is called the interval poset *of w with respect to the partial order $\leq$. The subposet $[1, c]^{gen} \subseteq [1, w]$ consists of prefixes $x \leq c$ such that there exists a reduced reflection factorization $(t_1, \ldots, t_n)$ of c with $\langle t_1, \ldots, t_n \rangle = W$ and $x = t_1 \cdots t_k$ for some $0 \leq k \leq n$.*

In the case of the module category the interval poset is the intensively studied set of generalized non-crossing partitions attached to Coxeter elements of Coxeter groups. For an introduction to this topic we refer to [4] and [80].

The next theorems collect results of the poset of thick subcategories in the case of the module category over a hereditary finite dimensional k-algebra.

Theorem 7.2.2. *([66, Theorem 1.3], [68, Remark 6.8]) Let A be a hereditary finite dimensional k-algebra with algebraically closed field k. If A is representation finite, then every thick subcategory of $\mathrm{mod}(A)$ is generated by an exceptional sequence. If A has tame domestic type, the thick subcategories of $\mathrm{mod}(A)$ are either generated by an exceptional sequence or subcategories of the subcategory consisting of regular objects.*

Remark 7.2.3. *Let $\mathbb{X}$ be a weighted projective line of domestic type. In this situation the category of coherent sheaves over $\mathbb{X}$ is derived equivalent to* $\text{mod}(A)$ *for finite dimensional k-algebra of tame representation type. In particular, the work of Köhler yields that there exist thick subcategories of* $\text{coh}(\mathbb{X})$ *that are not generated by an exceptional sequence.*

Theorem 7.2.4. *([56], [60], [68]) Let A be a hereditary finite dimensional k-algebra. The poset of thick subcategories of* $\text{mod}(A)$ *generated by an exceptional sequence is isomorphic (as posets) to the poset of generalized non-crossing partitions $[1, c]$ attached to a Coxeter group associated to A and c a Coxeter element. If A is of finite representation type, then by Theorem 7.2.2 the poset of thick subcategories is a lattice. In particular, the corresponding set of generalized non-crossing partitions is a lattice.*

The generalized non-crossing partitions attached to Coxeter groups have already been investigated by many mathematicians. One motivation are the Artin-Tits groups (of spherical type). They can be understood by Garside theory if the corresponding generalized set of non-crossing partitions is a lattice (see for example [29]). The fact that generalized non-crossing partitions have a link to thick subcategories of certain hereditary categories yields an easy and uniform proof of the lattice properties of generalized non-crossing partitions attached to finite Coxeter groups. The lattice property was first uniformly proven by Thomas Brady and Colum Watt in [19] using topological methods. For affine Coxeter groups the situation was lately complete described by Jon McCammond who proves in [81] that the corresponding generalized non-crossing partitions are no lattices for all types beside $\widetilde{A}_n$, $\widetilde{C}_n$ and $\widetilde{G}_2$, namely the types $\widetilde{B}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ and $\widetilde{F}_4$. The proof of the positivity of the lattice property of the cases $\widetilde{A}_n$, $\widetilde{C}_n$ and $\widetilde{G}_2$ was done by François Digne in [33] and [34] and Craig Squier in [101].

The main result of this section links thick subcategories of the category of coherent sheaves over a weighted projective line of wild and domestic type generated by an exceptional sequence to the interval poset of the Coxeter transformation that is induced by the Serre functor.

Theorem 7.2.5. *Let $\mathcal{A}$ be a hereditary connected ext-finite abelian k-category with a tilting object that is not derived equivalent to the category of coherent sheaves over weighted projective line of tubular type, and let Φ be the associated root system, W its reflection group and $c \in W$ a Coxeter transformation. Then there exists an order preserving bijection between*

- *the poset of thick subcategories of $\mathcal{A}$ that are generated by an exceptional sequence ordered by inclusion, and*
- *the poset $[1, c]$.*

We strongly believe that the previous theorem also holds for the remaining four tubular cases with weight types $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$ if we replace the interval poset $[1, c]$ by the subposet $[1, c]^{\text{gen}}$ (see Conjecture 7.2.7). For the category of coherent sheaves over a weighted projective line of wild or domestic type we have due to the Hurwitz transitivity on the set of reduced reflection factorizations of Coxeter transformations $[1, c]^{\text{gen}} = [1, c]$.

The rest of this section is devoted to the proof of Theorem 7.2.5.

The next result compares thick subcategories of the category $\text{coh}(\mathbb{X})$ for a weighted projective line that are generated by an exceptional sequence.

Lemma 7.2.6. *Let $(E_1, \ldots, E_n)$ and $(F_1, \ldots, F_n)$ be complete exceptional sequences in* $\text{coh}(\mathbb{X})$ *and* $U = \text{Thick}(E_1, \ldots, E_r)$, $V = \text{Thick}(F_1, \ldots, F_s)$ *for some $r, s \leq n$. Then $U = V$ if and only if $r = s$ and $(F_1, \ldots, F_r, E_{r+1}, \ldots, E_n)$ is a complete exceptional sequence.*

Proof. First assume that $(F_1, \ldots, F_s, E_{r+1}, \ldots, E_n)$ is a complete exceptional sequence. Then it holds $r = s$. Consider the right perpendicular category $\mathcal{H} := (E_{r+1}, \ldots, E_n)^\perp$ in $\text{coh}(\mathbb{X})$. By applying successively the Theorems 6.4.14, 6.4.15 and 6.4.16 the category $\mathcal{H}$ is equivalent to a coproduct of categories of the form $\text{coh}(\mathbb{X}')$ for a weighted projective line $\mathbb{X}'$ of reduced weight and $\text{mod}(A)$ for a hereditary finite dimensional k-algebra A. By the same theorems we get that $(E_1, \ldots, E_r)$ and $(F_1, \ldots, F_r)$ are complete exceptional sequences in $\mathcal{H}$. By Theorem 6.4.8 and Theorem 6.3.10 the exceptional sequences $(E_1, \ldots, E_r)$ and $(F_1, \ldots, F_r)$ lie in the same orbit of the mutation of exceptional sequences. By Lemma 6.4.13 thick subcategories are closed under left and right mutation, so $\text{Thick}(E_1, \ldots, E_r) = \text{Thick}(F_1, \ldots, F_r)$ holds.

Assume that $U = V$. Thus by Lemma 6.4.13 we have

$$F_1, \ldots, F_s \in \text{Thick}(F_1, \ldots, F_s) = \text{Thick}(E_1, \ldots, E_r) \subseteq (E_{r+1}, \ldots, E_n)^\perp.$$

Therefore $(F_1, \ldots, F_s, E_{r+1}, \ldots, E_n)$ is an exceptional sequence and hence Lemma 6.2.33 implies $s \leq r$. Analogous, by

$$E_1, \ldots, E_r \in \text{Thick}(E_1, \ldots, E_r) = \text{Thick}(F_1, \ldots, F_s) \subseteq (F_{s+1}, \ldots, F_n)^\perp$$

we have $r \leq s$. Altogether, $(F_1, \ldots, F_r, E_{r+1}, \ldots, E_n)$ is a complete exceptional sequence. □

Proof of Theorem 7.2.5. By Happel's theorem we can restrict ourselves to the cases $\mathcal{A} = \text{mod}(A)$ for a hereditary finite dimensional k-algebra A or $\mathcal{A} = \text{coh}(\mathbb{X})$ for a weighted projective line $\mathbb{X}$ that is by assumption not of tubular type. The case $\mathcal{A} = \text{mod}(A)$ is well-known and is discussed for example in [56]. Thus assume that $\mathcal{A} = \text{coh}(\mathbb{X})$ and let $(E_1, \ldots, E_n)$ be a complete exceptional sequence in $\mathcal{A}$. Let $U = \text{Thick}(E_1, \ldots, E_r)$ for some $r \leq n$ and put $\text{cox}(U) := s_{[E_1]} \cdots s_{[E_r]}$. By Proposition 7.1.2 we have that $\text{cox}(U) \in [1, c]$.

Let us first point out that $\text{cox}(-)$ is well-defined. Therefore choose another complete exceptional sequence $(F_1, \ldots, F_n)$ in $\text{coh}(\mathbb{X})$ such that $U = \text{Thick}(F_1, \ldots, F_s)$ for some $s \leq n$. Then $(F_1, \ldots, F_s, E_{r+1}, \ldots, E_n)$ is a complete exeptional sequence by Lemma 7.2.6 and $r = s$. Since c is independent of the chosen complete exceptional sequence (see Lemma 6.2.44) we get

$$c = s_{[F_1]} \cdots s_{[F_s]} s_{[E_{s+1}]} \cdots s_{[E_n]} = s_{[E_1]} \cdots s_{[E_s]} s_{[E_{s+1}]} \cdots s_{[E_n]}.$$

Thus we obtain $s_{[F_1]} \cdots s_{[F_s]} = s_{[E_1]} \cdots s_{[E_s]}$.

Next we show that the map $\mathrm{cox}(-)$ is injective. Let $(E_1, \ldots, E_n)$ and $(F_1, \ldots, F_n)$ be two complete exceptional sequences such that $U = \mathrm{Thick}(E_1, \ldots, E_r)$ and $V = \mathrm{Thick}(F_1, \ldots, F_s)$ for some $r, s \leq n$ and such that $\mathrm{cox}(U) = \mathrm{cox}(V)$. That is, $s_{[E_1]} \cdots s_{[E_r]} = s_{[F_1]} \cdots s_{[F_s]}$ and since c is of reflection length n (see Corollary 5.3.6) we obtain $r = s$. In particular

$$c = s_{[F_1]} \cdots s_{[F_s]} s_{[E_{s+1}]} \cdots s_{[E_n]}. \tag{7.1}$$

Since every reduced reflection factorization is generating for $\mathbb{X}$ of wild and domestic type, Proposition 7.1.2 yields that $(F_1, \ldots, F_s, E_{s+1}, \ldots, E_n)$ is a complete exceptional sequence. By Lemma 7.2.6 we get $U = V$.

The surjectivity of $\mathrm{cox}(-)$ follows directly from Proposition 7.1.2.

It remains to show that $\mathrm{cox}(-)$ is order preserving. Therefore let $V \subseteq U$ be thick subcategories with $U = \mathrm{Thick}(E_1, \ldots E_r)$ and $V = \mathrm{Thick}(F_1, \ldots, F_s)$. By applying successively Theorems 6.4.14, 6.4.15 and 6.4.16 the category $\mathcal{C} := (E_{r+1}, \ldots, E_n)^{\perp}$ is equivalent to a coproduct of categories of the form $\mathrm{coh}(\mathbb{X}')$ for a weighted projective line $\mathbb{X}'$ of reduced weight and $\mathrm{mod}(A)$ for a hereditary finite dimensional k-algebra A and $(F_1, \ldots, F_s)$ is an exceptional sequence in $\mathcal{C}$. The rank of the Grothendieck group $K_0(\mathcal{C})$ implies $s \leq r$. By Lemma 6.4.9 there exist exceptional objects $F'_{s+1}, \ldots, F'_r$ such that $(F_1, \ldots, F_s, F'_{s+1}, \ldots, F'_r)$ is a complete exceptional sequence in $\mathcal{C}$. Therefore by Theorem 6.4.8 $(E_1, \ldots E_r)$ and $(F_1, \ldots, F_s, F'_{s+1}, \ldots, F'_r)$ lie in the same orbit of the mutation of exceptional sequences and by Lemma 6.2.38 $s_{[F_1]} \ldots s_{[F_s]} s_{[F'_{s+1}]} \ldots s_{[F'_r]} = s_{[E_1]} \ldots s_{[E_r]}$. Since $c = s_{[E_1]} \ldots s_{[E_n]}$ is reduced by Corollary 5.3.6, we have $s_{[F_1]} \ldots s_{[F_s]} \leq s_{[E_1]} \ldots s_{[E_r]}$. □

We close this section with a conjecture that states that Theorem 7.2.5 is also true for the category of coherent sheaves over a weighted projective line of tubular type, i.e. it holds for every hereditary connected ext-finite abelian k-category with a tilting object.

Conjecture 7.2.7. *Let $\mathcal{A}$ be a hereditary connected ext-finite abelian k-category with a tilting object, and let Φ be the associated root system, W its reflection group and $c \in W$ a Coxeter transformation. Then there exists an order preserving bijection between*

- *the poset of thick subcategories of $\mathcal{A}$ that are generated by an exceptional sequence ordered by inclusion, and*
- *the poset $[1, c]^{gen}$.*

In order to prove Conjecture 7.2.7 the only thing one needs to show is that the factorization of the equation (7.1) is generating. For that one can possibly use Corollary 5.5.8 that states that reduced reflection factorizations are generating if and only if the corresponding roots span the root lattice. All the other steps in the proof of Theorem 7.2.5 are valid without any restrictions.

CHAPTER 8

Auxiliary calculations

Here we collect auxiliary calculations.

Calculation A5.5.2. *Here we offer a detailed calculation of Example 5.5.2, where we use the notation introduced in Example 5.5.2. Assume that* $W = \langle s_{\beta_1}, \ldots, s_{\beta_8}\rangle$*. By Lemma 5.5.3 we get* $\mathrm{span}_{\mathbb{Z}}(\Phi) = \langle \beta_1, \ldots, \beta_8\rangle$*. Therefore there exist* $\lambda_i \in \mathbb{Z}$ $(1 \leq i \leq 8)$ *such that* $a = \sum_{i=1}^{8} \lambda_i\beta_i$*. Since b only appears in the expression for* β_7*, we have* $\lambda_7 = 0$ *and the linear independence of* $a, \alpha_1, \ldots, \alpha_6$ *yields* $\lambda_i = 0$ *for* $i = 1, 2, 3, 5, 6$ *and* $\lambda_4 + \lambda_8 = 0$*. The equation* $\lambda_4\beta_4 + \lambda_8\beta_8 = a$ *then leads to the contradiction* $\lambda_4 = -\frac{1}{4}$*.*

Example A5.5.2. *In the following we use the notation of Subsection 5.5.*

Let Φ *of type* $D_4^{(1,1)}$ *and since by Lemma 5.3.1 all Coxeter transformations are conjugated we restrict ourselves to* $c = s_1s_3s_4s_0s_2s_{2^*}$*. Consider the roots*

$$\beta_1 = \alpha_1 + a,\ \beta_2 = \alpha_2 - a,\ \beta_3 = \alpha_3 + a,\ \beta_4 = \alpha_4 + a,$$
$$\beta_5 = -\widetilde{\alpha} + a + b,\ \beta_6 = \alpha_2 + 2a$$

A direct calculation shows that $c = s_{\beta_1}s_{\beta_3}s_{\beta_4}s_{\beta_5}s_{\beta_2}s_{\beta_6}$*. Assume that* $W = \langle s_{\beta_1}, \ldots, s_{\beta_6}\rangle$*. Hence we get* $\mathrm{span}_{\mathbb{Z}}(\Phi) = \mathrm{span}_{\mathbb{Z}}(\beta_1, \ldots, \beta_6)$*. Let* $\lambda_i \in \mathbb{Z}$ $(1 \leq i \leq 6)$ *such that* $a = \sum_{i=1}^{6} \lambda_i\beta_i$*. Since b only appears in the expression of* β_5 *we have* $\lambda_5 = 0$*. Using the linear independence of* $\alpha_1, \ldots, \alpha_4, a$ *we get* $\lambda_i = 0$ *for* $i = 1, 3, 4$ *and* $\lambda_2 + \lambda_6 = 0$*. The equation* $\lambda_2\beta_2 + \lambda_6\beta_6 = a$ *then leads to the contradiction* $\lambda_6 = \frac{1}{3}$*. Thus the factorization* $(s_{\beta_1}, s_{\beta_3}, s_{\beta_4}, s_{\beta_5}, s_{\beta_2}, s_{\beta_6})$ *of c does not generate the group.*

Now let Φ *be of type* $E_7^{(1,1)}$*. Here we also fix* $c = s_1s_2s_3s_5s_6s_7s_0s_4s_{4^*}$*. Consider the roots*

$$\beta_1 = \alpha_1 + 2a,\ \beta_2 = \alpha_2 + 2a,\ \beta_3 = \alpha_3 + a,\ \beta_4 = \alpha_4 - a,\ \beta_5 = \alpha_5 + 3a$$
$$\beta_6 = \alpha_6 - a,\ \beta_7 = \alpha_7 - a,\ \beta_8 = -\widetilde{\alpha} - a + b,\ \alpha_9 = \alpha_4 + 4a.$$

A direct calculation shows that $c = s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5}s_{\beta_6}s_{\beta_7}s_{\beta_8}s_{\beta_4}s_{\beta_9}$ *and assume that* $W = \langle s_{\beta_1}, \ldots, s_{\beta_9}\rangle$*. Hence we get* $\mathrm{span}_{\mathbb{Z}}(\Phi) = \mathrm{span}_{\mathbb{Z}}(\beta_1, \ldots, \beta_9)$*. Let* $\lambda_i \in \mathbb{Z}$ $(1 \leq i \leq 9)$ *such that* $a = \sum_{i=1}^{9} \lambda_i\beta_i$*. Since b only appears in the expression of* β_8 *we have* $\lambda_8 = 0$*. As in the previous case,* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = \lambda_6 = \lambda_7 = 0$ *and* $\lambda_4 + \lambda_9 = 0$*. The* *equation* $\lambda_4\beta_4 + \lambda_9\beta_9 = a$ *then leads to the contradiction* $\lambda_9 = \frac{1}{5}$. *Thus the factorization* $(s_{\beta_1}, s_{\beta_2}, s_{\beta_3}, s_{\beta_5}, s_{\beta_6}, s_{\beta_7}, s_{\beta_8}, s_{\beta_4}, s_{\beta_9})$ *of* c *does not generate the group.*

Let Φ *be of type* $E_8^{(1,1)}$ *and* $c = s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5}\cdots s_{\beta_9}s_{\beta_4}s_{\beta_{10}}$. *Consider the roots*

$$\beta_1 = \alpha_1 + 2a,\ \beta_2 = \alpha_2 + 3a,\ \beta_3 = \alpha_3 + 2a,\ \beta_4 = \alpha_4 - a,\ \beta_5 = \alpha_5 + 5a$$
$$\beta_6 = \alpha_6 - a,\ \beta_7 = \alpha_7 - a,\ \beta_8 = \alpha_8 - a,\ \alpha_9 = -\widetilde{\alpha} - a + b,\ \alpha_{10} = \alpha_4 + 6a.$$

As above the reflection factorization $(s_{\beta_1}, s_{\beta_2}, s_{\beta_3}, s_{\beta_5}, \cdots, s_{\beta_9}, s_{\beta_4}, s_{\beta_{10}})$ *is a reduced factorization of* c *and assume that* $W = \langle s_{\beta_1}, \ldots, s_{\beta_{10}}\rangle$. *Then there exists* $\lambda_i \in \mathbb{Z}$ $(1 \le i \le 10)$ *such that* $a = \sum_{i=1}^{10}\lambda_i\beta_i$. *As above, we get* $\lambda_i = 0$ *for* $i = 1, 2, 3, 5, 6, 7, 8, 9$ *and* $\lambda_4 + \lambda_{10} = 0$. *The equation* $\lambda_4\beta_4 + \lambda_{10}\beta_{10} = a$ *then leads to the contradiction* $\lambda_{10} = \frac{1}{7}$.

Calculation A5.5.7. *In the following the cases for* E_7 *and* E_8 *of Lemma 5.5.7 are calculated.*

Let $\Phi = E_7^{(1,1)}$ *and we get with Kluitmann's notation*

$$c = \begin{bmatrix} s_1s_2s_3s_5s_6s_7s_{-\widetilde{\alpha}} \\ -\alpha_4 \\ \widetilde{\alpha} \end{bmatrix}$$
$$= \begin{bmatrix} s_1 \\ -k_1\alpha_1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_2 \\ -k_2\alpha_2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_3 \\ -k_3\alpha_3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_5 \\ -k_5\alpha_5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_6 \\ -k_6\alpha_6 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_7 \\ -k_7\alpha_7 \\ 0 \end{bmatrix}$$
$$\cdot \begin{bmatrix} s_{-\widetilde{\alpha}} \\ \overline{\ell}\widetilde{\alpha} \\ \widetilde{\alpha} \end{bmatrix} \cdot \begin{bmatrix} s_4 \\ -\widetilde{\ell'}\alpha_4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_4 \\ -\widetilde{\ell''}\alpha_4 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} s_1s_2s_3s_5s_6s_7s_{-\widetilde{\alpha}} \\ -k_1\alpha_1 - k_2\alpha_2 - (k_1+k_3)\alpha_3 - k_5\alpha_5 - (k_5+k_6)\alpha_6 - (k_5+k_6+k_7)\alpha_7 \\ -(k_1 - \overline{\ell})\widetilde{\alpha} + (\widetilde{l'} - \widetilde{l''})\alpha_4 \\ \widetilde{\alpha} \end{bmatrix}.$$

Since $\widetilde{\alpha} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ *it holds*

$$k_1 = \frac{2}{3}\overline{\ell},\ k_2 = -\frac{10}{3}\overline{\ell},\ k_3 = -\frac{17}{3}\overline{\ell},\ k_5 = \overline{\ell},\ k_6 = -\frac{1}{3}\overline{\ell},\ k_7 = \frac{1}{3}\overline{\ell},\ \widetilde{\ell'} - \widetilde{\ell''} = -1 - \frac{4}{3}\overline{\ell}.$$

Since $\widetilde{\ell'} - \widetilde{\ell''} = \pm 1$ *it follows that* $\overline{\ell} \in \{0, -\frac{3}{2}\}$. *Thus* $\overline{\ell} = 0$ *and hence* $0 = \overline{\ell} = k_1 = k_2 = k_3 = k_5 = k_6 = k_7$.

Let $\Phi = E_8^{(1,1)}$ *and we get analogously*

$$c = \begin{bmatrix} s_1 s_2 s_3 s_5 s_6 s_7 s_8 s_{-\widetilde{\alpha}} \\ -\alpha_4 \\ \widetilde{\alpha} \end{bmatrix}$$

$$= \begin{bmatrix} s_1 \\ -k_1\alpha_1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_2 \\ -k_2\alpha_2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_3 \\ -k_3\alpha_3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_5 \\ -k_5\alpha_5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_6 \\ -k_6\alpha_6 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_7 \\ -k_7\alpha_7 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_8 \\ -k_8\alpha_8 \\ 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} s_{-\widetilde{\alpha}} \\ \overline{\ell}\widetilde{\alpha} \\ \widetilde{\alpha} \end{bmatrix} \cdot \begin{bmatrix} s_4 \\ -\widetilde{\ell'}\alpha_4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s_4 \\ -\widetilde{\ell''}\alpha_4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} s_1 s_2 s_3 s_5 s_6 s_7 s_8 s_{-\widetilde{\alpha}} \\ -k_1\alpha_1 - k_2\alpha_2 - (k_1 + k_3)\alpha_3 - k_5\alpha_5 - (k_5 + k_6)\alpha_6 - (k_5 + k_6 + k_7)\alpha_7 \\ -(k_5 + k_6 + k_7 + k_8)\alpha_8 - (k_5 + k_6 + k_7 + k_8 - \overline{\ell})\widetilde{\alpha} + (\widetilde{\ell'} - \widetilde{\ell''})\alpha_4 \\ \widetilde{\alpha} \end{bmatrix}.$$

Since $\widetilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + \alpha_8$ *it holds*

$$k_1 = \frac{2}{3}\overline{\ell},\ k_2 = -\overline{\ell},\ k_3 = \frac{2}{3}\overline{\ell},\ k_5 = \frac{5}{3}\overline{\ell},\ k_6 = -\frac{1}{3}\overline{\ell},\ k_7 = -\frac{1}{3}\overline{\ell},\ k_8 = -\frac{1}{3}\overline{\ell},\ \widetilde{\ell'} - \widetilde{\ell''} = -1 - 2\overline{\ell}.$$

Since $\widetilde{\ell'} - \widetilde{\ell''} = \pm 1$ *it follows that* $\overline{\ell} \in \{0, -1\}$. *If* $\overline{\ell} = -1$ *we get* $k_1 = -\frac{2}{3}$, *a contradiction. Thus* $\overline{\ell} = 0$ *and hence* $0 = \overline{\ell} = k_1 = k_2 = k_3 = k_5 = k_6 = k_7 = k_8$.

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