Dissertation

On the long-time behavior of the three-dimensional Dirac–Maxwell equation with zero magnetic field

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Cai Constantin Cloos

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Die Annahme der Dissertation wurde empfohlen von

Prof. Dr. Sebastian Herr (Universität Bielefeld),
Prof. Dr. Mihaela Ifrim (University of Wisconsin–Madison).

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1 Introduction

The aim of this work is to advance the understanding for the long-time behavior of solutions to the Dirac–Maxwell system in three space dimensions. In quantum electrodynamics (QED), the Dirac–Maxwell system (cf. e.g. [11, 12, 86]) arises by coupling Dirac's equation with Maxwell's equations and it models the interaction between an electromagnetic field and a charged fermionic field (e.g. an electron).

The Dirac equation was first stated in Dirac's seminal paper [32] in the 1920s for modeling four-component spinor fields ψ and describing the states of relativistic spin-1/2-particles (e.g. electrons), see also Thaller [95] (and references therein) for more details on the Dirac equation. Maxwell's equations go back to the work of Maxwell [75] in the 1860s as a model to describe the electromagnetic field determined by the electromagnetic potential (A_0, A).

In mathematical research of nonlinear dispersive and wave equations, there is the following hierarchy of general questions for a given Cauchy problem:

a) Is the Cauchy problem locally well-posed for a certain class of initial data?

b) Is the Cauchy problem globally well-posed?

c) What does the long-time behavior of global solutions look like?

In this thesis, we consider a simplified version of Dirac–Maxwell which arises from dropping the magnetic potential A or the magnetic field $\nabla \times A$ (cf. Section 1.2 for details). In some Cauchy problems, global nonlinear solutions behave for large times like linear solutions. This phenomenon is called scattering.

The Dirac–Maxwell equation with zero magnetic field is scattering critical in the sense that the spatial L^2-norm of the nonlinearity decays like t^{-1} where t is the time variable, cf. estimate (72) on page 31. We establish a modified scattering result by showing that solutions decay pointwise like linear solutions but behave asymptotically only like linear solutions multiplied with a correction factor that oscillates logarithmically in time. We refer the reader to Theorem 1.1 on page 12 for a rigorous formulation of our main result.

The oscillating correction factor arises from an asymptotic differential equation for the solution. This equation can be derived by spatial methods (see e.g. Lindblad–Soffer [69] for the one-dimensional cubic Schrödinger equation) or by Fourier methods (see e.g. Hayashi–Naumkin [51] and Kato–Pusateri [59]). We use Ifrim and Tataru's method of testing solutions by wave packets (see e.g. [57]) which interpolates between these two approaches by localizing solutions in space and frequency at a time-dependent scale related to the uncertainty principle.

We also address the asymptotic completeness problem: Provided a sufficiently regular asymptotic state, we construct a solution to the Dirac–Maxwell equation with zero magnetic field which converges to the asymptotic state in a suitable norm as time goes to infinity, see Theorem 4.1 on page 94 for a precise formulation of the asymptotic completeness result.

The question of generalizing our modified scattering result to Dirac–Maxwell with nonzero magnetic field remains open. However, we present some attempts on how our method might be applied to nonzero A_j. In fact, the A_j exhibit a hidden null-structure so that there is some hope that they do not affect the modified scattering behavior and our result would then carry over to Dirac–Maxwell with nonzero magnetic field. We refer to Section 5 for details.

1.1 Notation and preliminaries

This section is devoted to fixing notation and collecting some central definitions and frequently used propositions. General references for this subsection are [45, 46, 4].

$A \lesssim B$ means $A \leq CB$ for some universal constant C. We write $A \lesssim_p B$ to highlight the dependence of C on some parameter p. $A \approx B$ means $A \lesssim B$ and $B \lesssim A$. For integers k, l, we write $k \ll l$ if $k \leq l - 4$, $k \sim l$ if $k - 4 \leq l \leq k + 4$ and $k \prec l$ if $k \leq l + 4$.

For $x \in \mathbb{R}$, we denote by

$$x^+ = \max\{x, 0\}$$

the positive part of x and $x+$, $x-$ stand for arbitrary real numbers $y > x$, $y < x$, respectively.

We use the Kronecker delta notation

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where i, j are usually integers.

Throughout this thesis, $|\cdot|$ is the Euclidean norm on $\mathbb{C}^d$, $d \in \mathbb{N}$, and $B_r(x)$ always denotes the Euclidean ball in $\mathbb{R}^3$ with center $x \in \mathbb{R}^3$ and radius $r > 0$, i.e.

$$B_r(x) = \{y \in \mathbb{R}^3 \colon |y - x| < r\}.$$

L^p spaces

For a Lebesgue-measurable set $\Omega \subseteq \mathbb{R}^n$ and $p \in [1, \infty]$, we consider the function space

$$L^p(\Omega, \mathbb{C}^d) = \{f \colon \Omega \to \mathbb{C}^d \mid f \text{ measurable and } \|f\|_{L^p(\Omega,\mathbb{C}^d)} < \infty\},$$

where

$$\|f\|^p_{L^p(\Omega,\mathbb{C}^d)} = \int_\Omega |f(x)|^p \,\mathrm{d}x, \quad p \in [1, \infty),$$
$$\|f\|_{L^\infty(\Omega,\mathbb{C}^d)} = \operatorname*{ess\,sup}_{\omega \in \Omega} |f(\omega)| = \inf\big\{B > 0 \colon \lambda(x \in \Omega \colon |f(x)| > B) = 0\big\}$$

and λ is the Lebesgue measure on Ω. For any $p, q, r \in [1, \infty]$ and any measurable $f, g \colon \Omega \to \mathbb{C}^d$, $\varphi \colon \Omega \to \mathbb{C}$, we have

$$\|\varphi f\|_{L^p(\Omega,\mathbb{C}^d)} \leq \|\varphi\|_{L^q(\Omega,\mathbb{C})} \|f\|_{L^r(\Omega,\mathbb{C}^d)}, \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r}, \qquad \text{(Hölder's inequality)}$$
$$\|f + g\|_{L^p(\Omega,\mathbb{C}^d)} \leq \|f\|_{L^p(\Omega,\mathbb{C}^d)} + \|g\|_{L^p(\Omega,\mathbb{C}^d)}, \qquad \text{(Minkowski's inequality)}$$
$$\|f\|_{L^p(\Omega,\mathbb{C}^d)} = \sup_{g \in L^{p'}(\Omega,\mathbb{C}^d) \colon \|g\|_{L^{p'}(\Omega,\mathbb{C}^d)} \leq 1} \left| \int_\Omega \langle f(x), g(x) \rangle_{\mathbb{C}^d} \,\mathrm{d}x \right|, \qquad \text{(Duality)}$$

where p' denotes the *dual Hölder exponent* of p, i.e.

$$1 = \frac{1}{p} + \frac{1}{p'}.$$

After identifying functions which only differ on a null set, $L^p(\Omega, \mathbb{C}^d)$ becomes a Banach space and a Hilbert space in the case $p = 2$.

We also work with mixed-norm spaces $L_t^p L_x^q(\Omega_1 \times \Omega_2, \mathbb{C}^d)$ defined via

$$\|f\|_{L_t^p L_x^q(\Omega_1\times\Omega_2,\mathbb{C}^d)} = \Big(\int_{\Omega_1} \Big(\int_{\Omega_2} |f(t,x)|^q \,\mathrm{d}x\Big)^{p/q} \,\mathrm{d}t\Big)^{1/p},$$

where $\Omega_1 \subseteq \mathbb{R}^{n_1}$ and $\Omega_2 \subseteq \mathbb{R}^{n_2}$ are Lebesgue-measurable and $p, q \in [1,\infty]$ with the obvious modifications when $p = \infty$ or $q = \infty$. In the same way, we also define e.g. $L_t^p H_x^s(\Omega_1 \times \mathbb{R}^n, \mathbb{C}^d)$, where H^s is the Bessel-potential space of order $s \in \mathbb{R}$, see below for more details. We write $f(t)$ for the mapping $x \mapsto f(t,x)$, i.e.

$$f(t)(x) := f(t,x).$$

We have

$$\|f\|_{L_t^p L_x^q} \leq \|f\|_{L_x^q L_t^p}, \quad 1 \leq q \leq p \leq \infty. \qquad \text{(Generalized Minkowski's inequality)}$$

The *convolution* of $\varphi \in L^p(\mathbb{R}^n, \mathbb{C})$ and $f \in L^{p'}(\mathbb{R}^n, \mathbb{C}^d)$ is

$$\varphi * f(x) = \int_{\mathbb{R}^n} \varphi(y) f(x-y)\,\mathrm{d}y, \quad x \in \Omega.$$

We have

$$\|\varphi * f\|_{L^p(\mathbb{R}^n,\mathbb{C}^d)} \leq \|\varphi\|_{L^q(\mathbb{R}^n,\mathbb{C})} \|f\|_{L^r(\mathbb{R}^n,\mathbb{C}^d)}, \quad 1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}. \qquad \text{(Young's inequality)}$$

For $1 \leq p < \infty$ and $1 \leq q \leq \infty$, we define the *Lorentz space* $L^{p,q}(\mathbb{R}^n, \mathbb{C}^d)$ by

$$L^{p,q}(\Omega, \mathbb{C}^d) = \{f \colon \Omega \to \mathbb{C}^d \mid f \text{ measurable and } \|f\|_{L^{p,q}(\Omega,\mathbb{C}^d)} < \infty\},$$

where

$$\|f\|^q_{L^{p,q}(\Omega,\mathbb{C}^d)} = \int_0^\infty \Big(t^{1/p} \inf\big\{s > 0 \colon \lambda\big(x \in \Omega \colon |f(x)| > s\big) \leq t\big\}\Big)^q t^{-1}\,\mathrm{d}t, \quad q \in [1,\infty),$$
$$\|f\|_{L^{p,\infty}(\Omega,\mathbb{C}^d)} = \sup_{t>0} t^{1/p} \inf\big\{s > 0 \colon \lambda\big(x \in \Omega \colon |f(x)| > s\big) \leq t\big\}.$$

We have

$$\|\cdot\|_{L^{p,p}(\Omega,\mathbb{C}^d)} = \|\cdot\|_{L^p(\Omega,\mathbb{C}^d)}.$$

As shown in [82, Thm. 3.4], Hölder's inequality can be generalized to

$$\|\varphi f\|_{L^{p,q}(\Omega,\mathbb{C}^d)} \lesssim \|\varphi\|_{L^{p_1,q_1}(\Omega,\mathbb{C})} \|f\|_{L^{p_2,q_2}(\Omega,\mathbb{C}^d)}, \quad p, p_1, p_2 \in (1,\infty), \quad q, q_1, q_2 \in [1,\infty],$$
$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2}, \qquad \text{(Lorentz-type Hölder)}$$

The interpolation theory for Lorentz spaces (cf. Bergh–Löfström [10] for details) yields

$$\|f\|_{L^{p,q}(\Omega,\mathbb{C}^d)} \lesssim \|f\|^{\vartheta}_{L^{p_1}(\Omega,\mathbb{C}^d)} \|f\|^{1-\vartheta}_{L^{p_2}(\Omega,\mathbb{C}^d)}, \quad \vartheta \in (0,1), \quad \frac{1}{p} = \frac{\vartheta}{p_1} + \frac{1-\vartheta}{p_2}. \qquad \text{(Interpolation)}$$

For the following generalizations of Young's inequality, we refer to Grafakos [45, Thm. 1.4.25] and O'Neil [82, Thm. 2.6, Thm. 3.6]:

$$\|\varphi * f\|_{L^p(\mathbb{R}^n,\mathbb{C}^d)} \lesssim \|\varphi\|_{L^{p_1,\infty}(\mathbb{R}^n,\mathbb{C})}\|f\|_{L^{p_2}(\mathbb{R}^n,\mathbb{C}^d)},\ p,p_1,p_2 \in (1,\infty),\quad 1+\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}, \qquad \text{(Weak-type Young)}$$

$$\|\varphi * f\|_{L^{p,q}(\mathbb{R}^n,\mathbb{C}^d)} \lesssim \|\varphi\|_{L^{p_1,q_1}(\mathbb{R}^n,\mathbb{C})}\|f\|_{L^{p_2,q_2}(\mathbb{R}^n,\mathbb{C}^d)},\ p,p_1,p_2 \in (1,\infty),\ q,q_1,q_2\in[1,\infty],\quad 1+\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2},\quad \frac{1}{q}\le\frac{1}{q_1}+\frac{1}{q_2}, \qquad \text{(Lorentz-type Young I)}$$

$$\|\varphi * f\|_{L^{\infty}(\mathbb{R}^n,\mathbb{C}^d)} \lesssim \|\varphi\|_{L^{p_1,q_1}(\mathbb{R}^n,\mathbb{C})}\|f\|_{L^{p_2,q_2}(\mathbb{R}^n,\mathbb{C}^d)},\ p,p_1,p_2 \in (1,\infty),\ q,q_1,q_2\in[1,\infty],\quad 1=\frac{1}{p_1}+\frac{1}{p_2},\quad 1\le\frac{1}{q_1}+\frac{1}{q_2}. \qquad \text{(Lorentz-type Young II)}$$

Schwartz space and Fourier transform

The *Schwartz space* $\mathcal{S}(\mathbb{R}^n,\mathbb{C}^d)$ is defined by

$$\mathcal{S}(\mathbb{R}^n,\mathbb{C}^d)=\big\{f\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{C}^d)\colon \sup_{x\in\mathbb{R}^n}|x^\alpha\partial^\beta f(x)|<\infty\ \forall\,\alpha,\beta\in\mathbb{N}_0^n\big\},$$

where

$$x^\alpha=\prod_{j=1}^n x_j^{\alpha_j},\quad \partial^\beta=\partial_1^{\beta_1}\cdots\partial_n^{\beta_n}.$$

$\mathcal{S}(\mathbb{R}^n,\mathbb{C}^d)$ is dense in $L^p(\mathbb{R}^n,\mathbb{C}^d)$ for any $p\in[1,\infty)$.

The *Fourier transform* of $f\in L^1(\mathbb{R}^n,\mathbb{C}^d)$ is

$$\widehat{f}(\xi)=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}f(x)\,\mathrm{d}x,\quad \xi\in\mathbb{R}^n, \tag{1}$$

where

$$x\cdot\xi=\sum_{j=1}^n x_j\xi_j$$

is the dot product on $\mathbb{R}^n$. The Fourier transform is a homeomorphism on $\mathcal{S}(\mathbb{R}^n,\mathbb{C}^d)$ with inverse

$$\check{f}(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}f(\xi)\,\mathrm{d}\xi.$$

For any $f,g\in\mathcal{S}(\mathbb{R}^n,\mathbb{C}^d)$ and $\varphi\in\mathcal{S}(\mathbb{R}^n,\mathbb{C})$, we have

$$\widehat{\varphi * f}(\xi)=\widehat{\varphi}(\xi)\widehat{f}(\xi),\quad \widehat{\varphi f}(\xi)=(2\pi)^{-n}\big(\widehat{\varphi}*\widehat{f}\big)(\xi),$$

$$\int_{\mathbb{R}^n}\big\langle f(x),g(x)\big\rangle_{\mathbb{C}^d}\,\mathrm{d}x=(2\pi)^{-n}\int_{\mathbb{R}^n}\big\langle\widehat{f}(\xi),\widehat{g}(\xi)\big\rangle_{\mathbb{C}^d}\,\mathrm{d}\xi, \qquad \text{(Parseval)}$$

$$\|f\|_{L^2(\mathbb{R}^n,\mathbb{C}^d)}=(2\pi)^{-n/2}\|\widehat{f}\|_{L^2(\mathbb{R}^n,\mathbb{C}^d)}, \qquad \text{(Plancherel)}$$

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^n,\mathbb{C}^d)}\le(2\pi)^{n/p'}\|f\|_{L^p(\mathbb{R}^n,\mathbb{C}^d)},\quad p\in[1,2],\ \frac{1}{p}+\frac{1}{p'}=1. \qquad \text{(Hausdorff–Young)}$$

Let $\mathcal{S}'(\mathbb{R}^n, \mathbb{C}^d)$ be the *space of tempered distributions*, i.e.

$$\mathcal{S}'(\mathbb{R}^n, \mathbb{C}^d) = \{u\colon \mathcal{S}(\mathbb{R}^n, \mathbb{C}^d) \to \mathbb{C} \mid u \text{ linear and continuous}\}.$$

For defining the continuity of $u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^d)$ on $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^d)$, we equip $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^d)$ with the topology induced by the seminorms

$$\varrho_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)|, \quad f \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^d), \quad \alpha, \beta \in \mathbb{N}_0^n.$$

We denote

$$\langle u, f \rangle = u(f), \quad u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^d),\ f \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^d).$$

We can identify any $g \in L^p(\mathbb{R}^n, \mathbb{C}^d)$, $p \in [1, \infty]$, by $L_g \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^d)$ via

$$\langle L_g, f \rangle = \int_{\mathbb{R}^n} g(x) f(x) \,\mathrm{d}x.$$

The Fourier transform for $u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^d)$ is defined by

$$\langle \widehat{u}, f \rangle = \langle u, \widehat{f} \rangle, \quad f \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^d).$$

In particular, we can identify the *Coulomb potential*

$$V\colon \mathbb{R}^3 \to \mathbb{C}, \quad V(x) = |x|^{-1}$$

as a tempered distribution with Fourier transform

$$\widehat{V}(\xi) = 4\pi |\xi|^{-2}.$$

For $u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^d)$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > -n$, we denote

$$[(-\Delta)^{z/2} u]\widehat{\ } = |\xi|^z \widehat{u}.$$

In particular, for $n = 3$, we have

$$\widehat{V}\widehat{u} = 4\pi[(-\Delta^{-1})u]\widehat{\ }$$

and

$$V * u = 4\pi(-\Delta)^{-1} u.$$

Remark. Standard textbooks like Grafakos [45, 46] usually focus on the case $d = 1$. The d-dimensional case is relevant for this thesis because of the $\mathbb{C}^4$-valued Dirac spinors and the $\mathbb{R}^4$-valued electromagnetic potential. The statements mentioned so far are immediate consequences of the scalar case. For instance, the Fourier transform of $f = (f_1, \dots f_d) \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^d)$ is

$$\widehat{f} = \big(\widehat{f}_1, \dots \widehat{f}_d\big)$$

and we can apply the scalar arguments for each component of f. In fact, the results even remain true after replacing $\mathbb{C}^d$ with a Hilbert space and many of them even for Banach spaces, see e.g. Amann [2, 3], Zimmermann [102] and Schmeißer [87] for more details. For the subsequent statements, we will give references for the vector-valued version as long as it is not an immediate consequence of the scalar case.

Fourier multipliers and Sobolev spaces

Let $m\colon \mathbb{R}^n \to \mathbb{C}^{d\times d}$ be measurable. By $|\cdot|$, we also denote the Euclidean norm on $\mathbb{C}^{d\times d} \cong \mathbb{C}^{d^2}$. We define the *Fourier multiplier* $m(D)$ by

$$[m(D)f]\widehat{\ } = m\widehat{f}, \quad f \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^d).$$

For $m \in \mathcal{C}^k(\mathbb{R}^n \setminus \{0\}, \mathbb{C}^d)$, $k = \max\{\kappa \in \mathbb{N}\colon \kappa \le n/2\} + 1$, $|\partial^\alpha m(\xi)| \lesssim_\alpha |\xi|^{-|\alpha|}$, $\xi \ne 0$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le k$, we have

$$\|m(D)f\|_{L^p(\mathbb{R}^n,\mathbb{C}^d)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n,\mathbb{C}^d)}, \quad p \in (1,\infty), \qquad \text{(Hörmander–Mikhlin)}$$

cf. e.g. Grafakos [45, Thm. 6.2.7] for $d = 1$ and Zimmermann [102, Prop. 3] for a more general UMD space-valued version.

Let

$$\langle \xi \rangle = \left(1 + |\xi|^2\right)^{1/2}, \quad \xi \in \mathbb{R}^n.$$

We define the *Bessel potential space* $H^{s,p}(\mathbb{R}^n, \mathbb{C}^d)$, $s \in \mathbb{R}$, $p \in (1,\infty)$, as the set of all $u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^d)$ such that $\langle D \rangle^s u$ can be identified as an element of $L^p(\mathbb{R}^n, \mathbb{C}^d)$ with norm

$$\|u\|_{H^{s,p}(\mathbb{R}^n,\mathbb{C}^d)} = \|\langle D \rangle^s u\|_{L^p(\mathbb{R}^n,\mathbb{C}^d)}.$$

For $p = 2$, we denote

$$H^s(\mathbb{R}^n, \mathbb{C}^d) = H^{s,2}(\mathbb{R}^n, \mathbb{C}^d).$$

$\mathcal{S}(\mathbb{R}^n, \mathbb{C}^d)$ is dense in $H^{s,p}(\mathbb{R}^n, \mathbb{C}^d)$ and $H^{s,p}(\mathbb{R}^n, \mathbb{C}^d)$ is the completion of $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^d)$ with respect to $\|\cdot\|_{H^{s,p}(\mathbb{R}^n,\mathbb{C}^d)}$. For $s \in \mathbb{N}$, the Bessel potential space $H^{s,p}(\mathbb{R}^n, \mathbb{C}^d)$ coincides with the *Sobolev space* $W^{s,p}(\mathbb{R}^n, \mathbb{C}^d)$ given by

$$W^{s,p}(\mathbb{R}^n, \mathbb{C}^d) = \{f \in L^p(\mathbb{R}^n, \mathbb{C}^d)\colon \partial^\alpha f \in L^p(\mathbb{R}^n, \mathbb{C}^d)\ \forall\, \alpha \in \mathbb{N}_0^n,\ |\alpha| \le s\}$$

and

$$\|f\|^p_{W^{s,p}(\mathbb{R}^n,\mathbb{C}^d)} = \sum_{|\alpha| \le s} \|\partial^\alpha f\|^p_{L^p(\mathbb{R}^n,\mathbb{C}^d)}.$$

More precisely, the norms $\|\cdot\|_{H^{s,p}(\mathbb{R}^n,\mathbb{C}^d)}$ and $\|\cdot\|_{W^{s,p}(\mathbb{R}^n,\mathbb{C}^d)}$ are equivalent. Analogously, we introduce the *homogeneous Sobolev space* $\dot{H}^{s,p}(\mathbb{R}^n, \mathbb{C}^d)$ by replacing $\langle D \rangle^s$ with $|D|^s$, i.e.

$$\|u\|_{\dot{H}^{s,p}(\mathbb{R}^n,\mathbb{C}^d)} = \||D|^s u\|_{L^p(\mathbb{R}^n,\mathbb{C}^d)}.$$

For $p = \infty$, we also consider $W^{s,\infty}(\mathbb{R}^n, \mathbb{C}^d)$ with

$$\|f\|_{W^{s,\infty}(\mathbb{R}^n,\mathbb{C}^d)} = \max_{|\alpha| \le s} \|\partial^\alpha f\|_{L^\infty(\mathbb{R}^n,\mathbb{C}^d)}.$$

For $p \in (1,\infty)$, $0 < s < \frac{n}{p}$ and $\frac{1}{p} - \frac{1}{q} = \frac{s}{n}$, we have

$$H^{s,p}(\mathbb{R}^n, \mathbb{C}^d) \hookrightarrow L^q(\mathbb{R}^n, \mathbb{C}^d), \qquad \text{(Sobolev's embedding theorem I)}$$

i.e.

$$H^{s,p}(\mathbb{R}^n, \mathbb{C}^d) \subseteq L^q(\mathbb{R}^n, \mathbb{C}^d), \quad \|\cdot\|_{L^q(\mathbb{R}^n,\mathbb{C}^d)} \lesssim \|\cdot\|_{H^{s,p}(\mathbb{R}^n,\mathbb{C}^d)}.$$

If $s = \frac{n}{p}$ and $\frac{n}{s} < q < \infty$, then

$$H^{s,p}(\mathbb{R}^n, \mathbb{C}^d) \hookrightarrow L^q(\mathbb{R}^n, \mathbb{C}^d). \qquad \text{(Sobolev's embedding theorem II)}$$

If $s > \frac{n}{p}$, then any $f \in H^{s,p}(\mathbb{R}^n, \mathbb{C}^d)$ coincides almost everywhere with a bounded and continuous function and

$$H^{s,p}(\mathbb{R}^n, \mathbb{C}^d) \hookrightarrow L^\infty(\mathbb{R}^n, \mathbb{C}^d), \qquad \text{(Sobolev's embedding theorem III)}$$

cf. e.g. Grafakos [46, Thm. 1.3.5] for $d = 1$ and Arendt [5, Thm. 5.1] for a Banach space-valued version.

Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \{0\}, [0, \infty))$ be a radial function such that for

$$\chi_j(\xi) = \chi(2^{-j}\xi), \quad j \in \mathbb{Z},$$

we have

$$\operatorname{supp} \chi_j \subseteq \{\xi \in \mathbb{R}^n \colon 2^{j-1} \le |\xi| \le 2^{j+1}\}, \quad \sum_{j \in \mathbb{Z}} \chi_j(\xi) = 1 \ \forall \xi \neq 0.$$

The *Littlewood-Paley-Projector* P_j is defined by

$$\widehat{P_j u} = \chi_j u, \quad u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^d).$$

Note that

$$\begin{aligned}
&\sum_{j \in \mathbb{Z}} P_j u = u, \\
&\check{\chi} = 2^{jn} \check{\chi}(2^j \cdot), \\
&P_j = \tilde{P}_j P_j = P_j \tilde{P}_j, \quad \tilde{P}_j = P_{j-1} + P_j + P_{j+1}.
\end{aligned}$$

We also write

$$\tilde{\chi}(\xi) = \chi(2^{-1}\xi) + \chi(\xi) + \chi(2\xi), \quad \tilde{\chi}_j(\xi) = \tilde{\chi}(2^{-j}\xi)$$

which implies

$$\tilde{P}_j u = \check{\tilde{\chi}}_j * u.$$

For any $f \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^d)$, we have

$$\|P_j f\|_{L^q(\mathbb{R}^n, \mathbb{C}^d)} \lesssim_{p,q} 2^{j(\frac{n}{p} - \frac{n}{q})} \|P_j f\|_{L^p(\mathbb{R}^n, \mathbb{C}^d)}, \ 1 \le p \le q \le \infty, \quad \text{(Bernstein's inequality)}$$

cf. e.g. Lemarié-Rieusset [66, Prop. 3.2].

1.2 The Dirac–Maxwell system and the scalar toy model

In this section, we mainly follow D'Ancona–Foschi–Selberg [25] and Bournaveas [13]. The three-dimensional Dirac–Maxwell system is

$$\gamma^\mu D_\mu \psi + \psi = 0, \tag{2}$$

$$\nabla \cdot E = \rho, \quad \nabla \cdot B = 0, \quad \nabla \times E + \partial_t B = 0, \quad \nabla \times B - \partial_t E = J. \tag{3}$$

Dirac's equation (2) describes the Dirac spinor

$$\psi \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$$

and

$$J^\mu = \langle \gamma^0 \gamma^\mu \psi, \psi \rangle_{\mathbb{C}^4} = \overline{\psi} \gamma^\mu \psi, \quad \mu = 0, 1, 2, 3,$$

is the Dirac four-current density, where

$$\overline{\psi} = \psi^\dagger \gamma^0$$

is the Dirac adjoint and $\psi^\dagger$ the Hermitian adjoint of ψ. In particular,

$$\rho := J^0 = |\psi|^2$$

is the charge density and

$$J := (J^1, J^2, J^3)$$

is the three-current density. By

$$E = (E^1, E^2, E^3) \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3,$$

we denote the electric field and

$$B = (B_1, B_2, B_3) \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$$

is the magnetic field. We use the Einstein summation convention that repeated Greek indices are summed over $\mu = 0, 1, 2, 3$ and repeated Roman indices are summed over $j = 1, 2, 3$. The 4×4-Dirac matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ are given by

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \quad j = 1, 2, 3$$

with Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy the anti-commutation relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} I_4$$

with Minkowski metric

$$(g^{\mu\nu}) = \operatorname{diag}(1, -1, -1, -1).$$

The second and third equation of (3) are equivalent to the existence of a four-potential

$$(A_0, A_1, A_2, A_3) \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^4, \quad A := (A_1, A_2, A_3),$$

such that

$$B = \nabla \times A, \quad E = \nabla A_0 - \partial_t A. \tag{4}$$

Dirac's equation (2) and Maxwell's equations (3) are coupled via

$$D_\mu = D_\mu^{(A_0, A)} = -i\partial_\mu - A_\mu. \tag{5}$$

In the sequel, indices are raised and lowered with respect to the Minkowski metric, i.e.

$$\gamma_\mu = g^{\mu\nu} \gamma^\nu, \quad \partial^\mu = g^{\mu\nu} \partial_\nu \quad \text{etc.},$$

where we set

$$\partial_0 = \partial_t, \quad \partial_j = \partial_{x_j}.$$

After plugging in (4) in (3), we obtain

$$J^\mu = \partial^\mu \partial^\nu A_\nu - \partial_\nu \partial^\nu A^\mu.$$

Plugging in (5) in (2) leads to

$$-i\gamma^\mu \partial_\mu \psi + \psi = \gamma^\mu A_\mu \psi.$$

Therefore, the Dirac–Maxwell system (2)–(3) is equivalent to

$$-i\gamma^\mu \partial_\mu \psi + \psi = \gamma^\mu A_\mu \psi, \tag{6}$$
$$\partial^\mu \partial^\nu A_\nu - \partial_\nu \partial^\nu A^\mu = J^\mu. \tag{7}$$

The initial data are usually denoted by

$$\psi^\pm(0) = \psi_0^\pm, \quad A(0) = a, \quad \partial_t A(0) = \dot{a}.$$

Conservation of charge and gauge invariance

Note that (7) implies $\partial_\mu J^\mu = 0$ and

$$\partial_t \int_{\mathbb{R}^3} J^0(t)\,\mathrm{d}x = -\int_{\mathbb{R}^3} \partial_j J^j(t)\,\mathrm{d}x = 0 \tag{8}$$

which means conservation of charge since

$$\|\psi(t)\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} J^0(t)\,\mathrm{d}x.$$

The Dirac–Maxwell system is invariant under gauge transformations

$$\psi \mapsto e^{i\varphi}\psi, \quad A_\mu \mapsto A_\mu + \partial_\mu \varphi$$

for any $\varphi\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$. This means, if (ψ, A_μ) solve (2)–(3) (or equivalently (6)–(7)), then also $\big(e^{i\varphi}\psi, A_\mu + \partial_\mu\varphi\big)$, where

$$D_\mu^{(A_\nu + \partial_\nu \varphi)}\big(e^{i\varphi}\psi\big) = e^{i\varphi} D_\mu^{(A_0, A)}\psi.$$

Since the observables ρ, J, E, B are invariant under those gauge transformations, solutions (ψ, A_μ) and $\big(e^{i\varphi}\psi, A_\mu + \partial_\mu\varphi\big)$ are physically equivalent. Therefore, we get an equivalence class of solutions and we can choose a representative by adding a feasible gauge condition. One possible choice is

$$\partial^j A_j = 0. \tag{Coulomb gauge}$$

Then, equation (7) can be transformed to

$$J^0 = \Delta A_0, \quad J^j = \Box A_j - \partial_j \partial_t A_0.$$

Therefore, the Dirac–Maxwell system under the Coulomb gauge condition reads

$$
\begin{aligned}
& -i\gamma^\mu\partial_\mu\psi+\psi=\gamma^\mu A_\mu\psi, && (9)\\
& \Delta A_0=|\psi|^2, && (10)\\
& \Box A_j=\overline{\psi}\gamma^j\psi+\partial_j\partial_t A_0, && (11)\\
& \partial^j A_j=0. && (12)
\end{aligned}
$$

Regarding Bournaveas' approach in [13], we can simplify (11) as follows: Let

$$\mathrm{curl}=\nabla\times .$$

By decomposing A into its divergence-free part $-\mathrm{curl}\,\Delta^{-1}\,\mathrm{curl}\,A$ and its rotation-free part $\nabla\Delta^{-1}\partial_j A_j$, equation (11) and the Coulomb gauge condition (12) yield

$$
\begin{aligned}
\Box A &= -\mathrm{curl}\,\Delta^{-1}\,\mathrm{curl}\,(J+\nabla\partial_t A_0)\\
&= -\mathrm{curl}\,\Delta^{-1}\,\mathrm{curl}\,J. \qquad (13)
\end{aligned}
$$

We can diagonalize (9) by decomposing ψ into

$$\psi=\psi^+ + \psi^-,\quad \psi^\pm=\Pi^\pm(D)\psi,$$

with projections $\Pi^\pm$ defined by

$$\Pi^\pm(D)=\frac{1}{2}\big(I_4\pm\langle D\rangle^{-1}\gamma^0(-i\gamma^j\partial_j+I_4)\big). \qquad (14)$$

The properties of the Dirac matrices γ^μ lead to

$$\Pi^\pm(D)(-i\gamma^0\gamma^j\partial_j+\gamma^0)=\pm\langle D\rangle\Pi^\pm(D) \qquad (15)$$

and left-multiplying (9) with γ^0 gives

$$\Pi^\pm(D)(A_0\psi+\gamma^0\gamma^j A_j\psi)=(-i\partial_t\pm\langle D\rangle)\psi^\pm. \qquad (16)$$

By (13) and (16), we conclude that the Dirac–Maxwell system (9)–(12) is equivalent to

$$
\begin{aligned}
& (-i\partial_t\pm\langle D\rangle)\psi^\pm=\Pi^\pm(D)(A_0\psi+\gamma^0\gamma^j A_j\psi), && (17)\\
& \Delta A_0=|\psi|^2, && (18)\\
& \Box A=-\mathrm{curl}\,\Delta^{-1}\,\mathrm{curl}\,\big((\overline{\psi}\gamma^j\psi)_{j=1,2,3}\big), && (19)\\
& \partial^j A_j=0. && (20)
\end{aligned}
$$

Another frequently used gauge is

$$\partial^\mu A_\mu=0. \qquad \text{(Lorenz gauge)}$$

Dirac–Maxwell under the Lorenz gauge condition becomes

$$
\begin{aligned}
& -i\gamma^\mu\partial_\mu\psi+\psi=\gamma^\mu A_\mu\psi, && (21)\\
& \Box A^\mu=-\overline{\psi}\gamma^\mu\psi, && (22)\\
& \partial^\mu A_\mu=0. && (23)
\end{aligned}
$$

Scalar toy model

The systems (17)–(20) and (21)–(23) are closely related to the scalar semi-relativistic Hartree-type equation

$$-i\partial_t u + \langle D\rangle u = c\big(V * |u|^2\big)u, \tag{24}$$

where $c \in \mathbb{R}$, $u\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$, $V(x) = |x|^{-1}$ is the Coulomb potential in $\mathbb{R}^3$ and $*$ denotes convolution with respect to the $\mathbb{R}^3$-variable. In astrophysics (Chandrasekhar theory), equation (24) serves as a model to describe the dynamics and gravitational collapse of boson stars. Therefore, it is often referred to as the boson star equation, see e.g. Lieb–Yau [68], Elgart–Schlein [34], Fröhlich–Jonsson–Lenzmann [42] and Michelangeli–Schlein [76] for details. As already detected by Chadam and Glassey [18], the semi-relativistic equation (24) serves as a toy model for the Dirac–Maxwell system:

For the Coulomb gauge version (17)–(20), note that

$$A_0 = \Delta^{-1}|\psi|^2 = -(4\pi)^{-1}(V * |\psi|^2).$$

Hence, after dropping the A_j, i.e.

$$A = 0,$$

the system (17)–(20) becomes

$$\big(-i\partial_t \pm \langle D\rangle\big)\psi^{\pm}(t) = -(4\pi)^{-1}\Pi^{\pm}(D)\big[\big(V * |\psi(t)|^2\big)\psi(t)\big] \tag{25}$$

which is a vector-valued version of (24).

For the Lorenz gauge version (21)–(23), assume that the magnetic field B vanishes, i.e.

$$\nabla \times A = 0$$

which implies

$$A = \nabla\phi \tag{26}$$

for some $\phi\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$. Let

$$\Psi(t,x) = e^{-i\phi(t,x)}\psi(t,x).$$

Then,

$$\begin{aligned}\partial_t\Psi &= -i\partial_t\phi\Psi + e^{-i\phi}\partial_t\psi \\ &= -i\partial_t\phi\Psi + e^{-i\phi}\big(-\gamma^0\gamma^j\partial_j\psi - i\gamma^0\psi + i\gamma^0\gamma^\mu A_\mu\psi\big) \\ &= -\gamma^0\gamma^j\partial_j\Psi - i\gamma^0\Psi + i(A_0 - \partial_t\phi)\Psi + i\gamma^0\gamma^j(A_j - \partial_j\phi)\Psi,\end{aligned} \tag{27}$$

where we used (21) and $(\gamma^0)^2 = I_4$. From (26) and (23), we obtain

$$\begin{aligned}\Delta\phi &= -\partial^j A_j \\ &= \partial_t A_0\end{aligned}$$

and (22) implies

$$\begin{aligned}\partial_t\phi &= \Delta^{-1}\big(\Delta A_0 - |\psi|^2\big) \\ &= A_0 + (4\pi)^{-1}\big(V * |\Psi|^2\big).\end{aligned} \tag{28}$$

Plugging in (28) and (26) in (27) gives

$$\partial_t \Psi = -\gamma^0\gamma^j\partial_j\Psi - i\gamma^0\Psi - i(4\pi)^{-1}\big(V * |\Psi|^2\big)\Psi$$

or equivalently

$$-i\gamma^0\gamma^\mu\partial_\mu\Psi + \gamma^0\Psi = -(4\pi)^{-1}\big(V * |\Psi|^2\big)\Psi.$$

The diagonalization property (15) leads to

$$(-i\partial_t \pm \langle D\rangle)\Psi^\pm = -(4\pi)^{-1}\Pi^\pm(D)\big[\big(V * |\Psi|^2\big)\Psi\big], \tag{29}$$

where

$$\Psi^\pm = \Pi^\pm(D)\Psi.$$

1.3 Results

We consider the Cauchy problem

$$\begin{cases} \big(-i\partial_t \pm \langle D\rangle\big)\psi^\pm(t) = c\Pi^\pm(D)\big[\big(V * |\psi(t)|^2\big)\psi(t)\big], \\ \psi^\pm(0) = \psi_0^\pm, \end{cases} \tag{30}$$

where $c \in \mathbb{R}$. For $c = -(4\pi)^{-1}$, we obtain problem (25) for Dirac–Maxwell in the Coulomb gauge after dropping the A_j as well as problem (29) for the Dirac spinor of Dirac–Maxwell in the Lorenz gauge provided a zero magnetic field $B = \nabla \times A$.

Our main result is:

Theorem 1.1. *Let $N \geq 103$, $\varepsilon \in (0, 1/16)$. There is $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ and $\psi_0^\pm \in H^N(\mathbb{R}^3, \mathbb{C}^4)$ with*

$$\|\psi_0^\pm\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} + \|\langle\cdot\rangle^2\psi_0^\pm\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} + \|\langle\xi\rangle^9\widehat{\psi_0^\pm}(\xi)\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)} \leq \delta,$$

the Cauchy problem (30) is globally well-posed and its solution $\psi = \psi^+ + \psi^-$ satisfies

$$\begin{aligned} \big|e^{\pm it(1-|x/t|^2)^{1/2}}\psi^\pm(t,x) - t^{-3/2}e^{ic\log(t)(V*|\langle\xi_\bullet\rangle^{5/2}\tilde{W}|^2)(x/t)}\langle\xi_{x/t}\rangle^{5/2}\tilde{W}^\pm(x/t)\big| \\ \lesssim \delta t^{2\varepsilon-7/4}\langle\xi_{x/t}\rangle^{3/2}, \end{aligned} \tag{31}$$

$$\big\|\langle\xi\rangle\big(e^{\pm it\langle\xi\rangle}\widehat{\psi^\pm(t)}(\xi) - (2\pi)^{3/2}e^{ic\log(t)\mathcal{B}^\pm(\xi)}W^\pm(\xi)\big)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \delta\langle t\rangle^{2\varepsilon-1/4}, \tag{32}$$

$$\big|\langle\xi\rangle^8\big(e^{\pm it\langle\xi\rangle}\widehat{\psi^\pm(t)}(\xi) - (2\pi)^{3/2}e^{ic\log(t)\mathcal{B}^\pm(\xi)}W^\pm(\xi)\big)\big| \lesssim \delta\langle t\rangle^{2\varepsilon-1/8} \text{ for } \langle\xi\rangle \leq \langle t\rangle^{1/56}, \tag{33}$$

$$\big|\langle\xi\rangle^8 e^{\pm it\langle\xi\rangle}\widehat{\psi^\pm(t)}(\xi)\big| \lesssim \delta\langle t\rangle^{2\varepsilon-1/8} \text{ for } \langle\xi\rangle \geq \langle t\rangle^{1/56}, \tag{34}$$

where

$$\begin{aligned} \langle\xi_v\rangle &= (1-|v|^2)^{-1/2}, \quad v \in B_1(0), \\ \mathcal{B}^\pm(\xi) &= \int_{\mathbb{R}^3}\Big|\frac{\xi}{\langle\xi\rangle} \mp \frac{\sigma}{\langle\sigma\rangle}\Big|^{-1}\Big(\big|W^+(\sigma)\big|^2 + \big|W^-(-\sigma)\big|^2\Big)\,\mathrm{d}\sigma \end{aligned} \tag{35}$$

for some asymptotic state

$$\tilde{W} = \begin{pmatrix}\tilde{W}^+ \\ \tilde{W}^-\end{pmatrix} : \mathbb{R}^3 \to \mathbb{C}^4 \times \mathbb{C}^4$$

with $\operatorname{supp} \tilde{W} \subseteq B_1(0)$ *and*

$$W = \begin{pmatrix} W^+ \\ W^- \end{pmatrix} : \mathbb{R}^3 \to \mathbb{C}^4 \times \mathbb{C}^4, \quad W(\xi) = \tilde{W}(\pm \langle \xi \rangle^{-1} \xi).$$

We have $W \in \langle \cdot \rangle^{-8} L^\infty(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$ *and*

$$\big\| \langle \xi \rangle^8 W^\pm \big\|_{L^\infty(\mathbb{R}^3, \mathbb{C}^4)} \leq \delta, \tag{36}$$

$$\big\| W^\pm \big\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)} = \big\| \psi_0^\pm \big\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)}. \tag{37}$$

Estimate (31) characterizes the modified scattering property of ψ on the spatial side while estimates (32)–(34) describe the asymptotic behavior on the Fourier side. The factor $e^{\pm it\langle \xi \rangle}$ in front of $[\psi^\pm(t)]^\frown(\xi)$ is the symbol of the inverse linear propagator for (30). In the case of scattering, $e^{\pm it\langle \xi \rangle}[\psi^\pm(t)]^\frown(\xi)$ would behave like W. Our modified scattering behavior in frequency is characterized by the oscillating correction factor $e^{ic\log(t)\mathcal{B}^\pm(\xi)}$ where $\mathcal{B}$ arises from $A_0 = -(4\pi)^{-1}\big(V * |\psi|^2\big)$. Concerning the spatial asymptotics, the factors $e^{\pm it(1-|x/t|^2)^{1/2}}$ and $e^{ic\log(t)(V*|\langle \xi_\bullet \rangle^{5/2}\tilde{W}|^2)(x/t)}$ correspond to $e^{\pm it\langle D \rangle}$ and $e^{ic\log(t)\mathcal{B}^\pm(D)}$, respectively, after localizing on rays $x = tv$.

Since

$$\psi = \psi^+ + \psi^-,$$

we get in particular

$$\begin{aligned} \big[e^{it\langle \xi \rangle}\Pi^+(\xi) + e^{-it\langle \xi \rangle}\Pi^-(\xi)\big]\widehat{\psi(t)}(\xi) &= e^{ic\log(t)\mathcal{B}^+(\xi)}W^+(\xi) + e^{ic\log(t)\mathcal{B}^-(\xi)}W^-(\xi) \\ &\quad + \mathcal{O}_{\langle \cdot \rangle^{-1}L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\big(\delta \langle t \rangle^{2\varepsilon - 1/8}\big) \end{aligned}$$

and analogously for (32) and (33).

By time reversal, one can derive an analogous result for $t \to -\infty$. Throughout this thesis, we will focus on $t \geq 0$ for simplicity.

The choice $N = 103$ comes from the proof of Lemma 3.13 and it is the lowest integer such that our method ensures the $\langle t \rangle^{2\varepsilon - 1/8}$-decay in (33) and (34). One might show a similar result for $N \geq 52$ with a $\langle t \rangle^{-\varepsilon}$-decay for $\varepsilon \in (0, 1/261]$, see also the remarks after Lemma 3.13 for a deeper discussion.

In Section 2.1, we also establish a local well-posedness result (see Theorem 2.2) in a function space being suitable for the proof of Theorem 1.1. Furthermore, we derive an asymptotic completeness result in Section 4, cf. Theorem 4.1.

Previous work

In the 1960s, Gross [48] proved local well-posedness for the three-dimensional Dirac–Maxwell system in a suitable function space. Bournaveas [13] obtained a local result for initial data $(\psi_0, a, \dot{a}) \in H^{s+1/2}(\mathbb{R}^3, \mathbb{C}^4) \times H^{s+1}(\mathbb{R}^3, \mathbb{R}^3) \times H^s(\mathbb{R}^3, \mathbb{R}^3)$, $s > 0$, using null-form estimates established by Klainerman and Machedon [61], see also [62, 63, 7] for applications to Klein–Gordon–Maxwell, Yang–Mills and Dirac–Klein–Gordon. Masmoudi and Nakanishi [73] improved Bournaveas' result to $s = 0$. Null-structure arguments have also been used by D'Ancona, Foschi and Selberg to show first almost optimal local well-posedness for the Dirac–Klein–Gordon system [24] and then local well-posedness for Dirac–Maxwell [25] with almost optimal (due to scaling) initial data $\psi_0 \in H^s(\mathbb{R}^3, \mathbb{C}^4)$, $a \in |D|^{-1}H^{s-1/2}(\mathbb{R}^3, \mathbb{R}^3)$ and $\dot{a} \in H^{s-1/2}(\mathbb{R}^3, \mathbb{R}^3)$, $s > 0$.

Recently, Selberg and Tesfahun [89] also derived an ill-posedness result for initial data $\psi \in H^s(\mathbb{R}^3, \mathbb{C}^4)$, $s < 0$, and lower dimensions.

In one space dimension, Chadam [16] derived global well-posedness in a suitable function space which was improved by Delgado [27]. Chadam and Glassey [17] showed that one-dimensional global solutions do not scatter to linear solutions. We refer the reader to Bachelot [6], Huh [55], Okamoto [80], You–Zhang [99] for recent results in one space dimension. The case of zero magnetic field was first considered by Chadam and Glassey [18] proving global well-posedness in two space dimensions for smooth data with compact support. D'Ancona and Selberg [26] established a global result for the two-dimensional Dirac–Maxwell system provided $\psi_0 \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ and $(a, \dot{a})$ in a suitable Sobolev space by extending Grünrock and Pecher's corresponding result [47] on the two-dimensional Dirac–Klein–Gordon system. In higher dimensions $d \geq 4$, Gavrus and Oh [43] proved almost optimal global well-posedness and scattering for initial data $\psi_0 \in H^s(\mathbb{R}^3, \mathbb{C}^4)$, $a \in H^{s+1/2}(\mathbb{R}^3, \mathbb{R}^3)$ and $\dot{a} \in H^{s-1/2}(\mathbb{R}^3, \mathbb{R}^3)$, $s > 0$, inspired by the work of Krieger–Sterbenz–Tataru [65] and Oh–Tataru [79, 77, 78] for Klein–Gordon–Maxwell in four space dimensions. Independently, Krieger and Lührmann [64] established a similar result for Klein–Gordon–Maxwell.

Global results for the Dirac–Maxwell system in three space dimensions are only available under very strong decay and regularity assumptions. After early results of Chadam [15] for cut-off versions on a fixed bounded space-time region and of Choquet-Bruhat [22] for a zero-charge spinor field, Georgiev [44] showed global well-posedness under the assumption that 28 derivatives of the initial data are small in a weighted $L^\infty(\mathbb{R}^3)$-norm. Flato, Simon and Taflin [39, 40, 41] established global well-posedness and modified scattering for small $\mathcal{C}^\infty(\mathbb{R}^3)$ initial data being images of asymptotic data through a modified wave operator. Psarelli [83] showed global well-posedness and sharper decay estimates for the electromagnetic field A in the case of compactly supported initial data requiring four derivatives for the spinor ψ and three derivatives for the electromagnetic field in the $L^2(\mathbb{R}^3)$-norm.

There are several other directions which are of (current) research interest for the Dirac–Maxwell system and related problems. Bechouche–Mauser–Selberg [8] and Masmoudi–Nakanishi [73] obtained independently results for the nonrelativistic limit of the Dirac–Maxwell system and the latter two authors additionally for Klein–Gordon–Maxwell and Poisson–Schrödinger. Masmoudi and Nakanishi [74] also showed an unconditional uniqueness result for Dirac–Maxwell and Klein–Gordon–Maxwell. Furthermore, Lisi [70] and Esteban–Georgiev–Séré [36, 35] derived stationary soliton-like solutions for Dirac–Maxwell. See also [1, 85, 19, 33, 101, 100, 14, 23] for further developments and [91, 92, 54, 28, 31, 30, 29] for results on semi-classical ground states.

Global well-posedness for the scalar toy model (24) was shown by Lenzmann [67] for small initial data $u_0 \in H^s(\mathbb{R}^3)$, $s \geq 1/2$. The local result was improved by Herr and Lenzmann [52] to the almost optimal case $s > 1/4$ or $s > 0$ for radial data. Cho and Ozawa [21] derived nonexistence of scattering and Pusateri [84, Thm. 1.1] obtained the following modified scattering result by performing asymptotic analysis in the Fourier space:

$$\left\| \langle \xi \rangle^{10} \Big(e^{it\langle \xi \rangle} \widehat{u(t)}(\xi) - (2\pi)^{3/2} e^{-ic\mathcal{B}(t,\xi)} W(\xi) \Big) \right\|_{L^\infty_\xi(\mathbb{R}^3)} \lesssim \langle t \rangle^{-\varepsilon}$$

for $0 < \varepsilon < 1/1000$, initial data

$$\|u_0\|_{H^{1000}(\mathbb{R}^3)} + \|\langle \cdot \rangle^2 u_0\|_{H^2(\mathbb{R}^3)} + \|\langle \cdot \rangle^{10} \widehat{u_0}\|_{L^\infty(\mathbb{R}^3)} \leq \delta,$$

$\delta > 0$ sufficiently small and some asymptotic state W, where

$$\mathcal{B}(t,\xi) = \int_0^t \int_{\mathbb{R}^3} \Big| \frac{\xi}{\langle \xi \rangle} - \frac{\sigma}{\langle \sigma \rangle} \Big|^{-1} \big| \widehat{u(t')}(\sigma) \big|^2 \,\mathrm{d}\sigma\, \chi\big(\xi (t')^{-1/300}\big)(t'+1)^{-1}\,\mathrm{d}t'$$

is the oscillating phase correction term for some smooth cut-off χ on $\mathbb{R}^3$.[1] Furthermore, Cho, Hwang and Yang [20] used Pusateri's method to prove modified scattering for three-dimensional fractional Schrödinger equations with Hartree-type nonlinearity, i.e.

$$i\partial_t u + |D|^\alpha = \big(V * |u|^2\big)u, \quad \alpha \in (1.7, 2),\ V(x) = |x|^{-1}.$$

In this thesis, we apply the method of testing solutions by wave packets to obtain a similar modified scattering result for the Cauchy problem (30) of Dirac–Maxwell with zero magnetic field. This method was first used by Ifrim and Tataru [58] for two-dimensional water wave problems (as a continuation of their joint work with Hunter [56]), see also [57] for a demonstration of that method on the one-dimensional cubic nonlinear Schrödinger equation and [49, 50, 81, 71] for further applications. The method of testing solutions by wave packets is based on localizing in both space and frequency at a time dependent scale related to the uncertainty principle. As a by-product, we can derive a refined version of Pusateri's result for the scalar toy model in the sense that $\mathcal{B}(t,\xi)$ is replaced with $\log(t)\mathcal{B}(\xi)$, where

$$\mathcal{B}(\xi) = \int_{\mathbb{R}^3} \Big| \frac{\xi}{\langle \xi \rangle} - \frac{\sigma}{\langle \sigma \rangle} \Big|^{-1} \big| W(\sigma) \big|^2 \,\mathrm{d}\sigma$$

does no longer depend on t or u.

A closely related problem is

$$-i\gamma^\mu \partial_\mu \psi + \psi = \big(V * \langle \psi, \gamma^0 \psi \rangle_{\mathbb{C}^4}\big)\psi \tag{38}$$

which arises from (9)–(12) after dropping the A_j and replacing the A_0-contribution $V * |\psi|^2$ with $V * \langle \psi, \gamma^0\psi\rangle_{\mathbb{C}^4}$. In contrast to $|\psi|^2$, the expression $\langle \psi, \gamma^0\psi\rangle_{\mathbb{C}^4}$ exhibits a hidden null-structure. In the case of the Yukawa potential

$$V(x) = |x|^{-1} e^{-\mu x}, \quad \mu > 0,$$

Yang [97] and Tesfahun [94] proved independently global well-posedness and scattering for small initial data in $H^s(\mathbb{R}^3)$, $s > 0$, which is almost optimal. There are some further related scattering results, e.g. by Machihara and Tsutaya [72] for the potential

$$V(x) = |x|^{-\gamma}, \quad s > \gamma/6 + 1/2,\ 2 < \gamma < 3$$

and by Yang [98] for smooth potentials V on $\mathbb{R}^3 \setminus \{0\}$ such that

$$\big|\partial_\xi^\alpha \widehat{V}(\xi)\big| \lesssim \begin{cases} |\xi|^{-\gamma_1 - |\alpha|}, & |\xi| \le 1, \\ |\xi|^{-\gamma_2 - |\alpha|}, & |\xi| > 1, \end{cases}$$
$$\gamma_1 \in [0,1), \quad \gamma_2 \in (3/2, 3).$$

Note that the Yukawa potential fulfills these conditions with $\gamma_1 = 0$ and $\gamma_2 = 2$ while the Coulomb potential requires $\gamma_1 = \gamma_2 = 2$.

[1] In his paper [84], Pusateri works with the Fourier transform $\widehat{f}(\xi) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x$, $f \in L^1(\mathbb{R}^3, \mathbb{C})$. For the sake of consistency, we have adjusted his result to our definition (1) of the Fourier transform.

Main ideas

The aim of this work is two-fold. As already mentioned, we refine Pusateri's result for the scalar toy model [84, Thm. 1.1] and carry it over to the vector-valued version (30). We expect that our result in Theorem 1.1 can be transferred to the Dirac–Maxwell system (17)–(20). This would give a more explicit result compared to the work of Flato, Simon and Taflin [41] who consider the Dirac–Maxwell system in the Lorenz gauge, see (21)–(23). They obtained an implicitly defined modified scattering term depending on all A_μ while we expect again an oscillating correction term of the form $\log(t)\mathcal{B}(\xi)$ with $\mathcal{B}$ from (35) depending only on A_0. We refer the reader to Section 5 for some ideas in that direction.

We work with the norm

$$\begin{aligned}\big\|\psi^{\pm}\big\|_{X_T} := \sup_{t\in[0,T]} \Big[\langle t\rangle^{-\varepsilon}\big\|\psi^{\pm}(t)\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} &+ \langle t\rangle^{-\varepsilon}\big\|x\mathrm{w}^{\pm}(t,x)\big\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)} \\ &+ \langle t\rangle^{-2\varepsilon}\big\|\langle x\rangle^2\mathrm{w}^{\pm}(t,x)\big\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)} \\ &+ \big\|\langle\xi\rangle^8\widehat{\psi^{\pm}(t)}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\Big]\end{aligned}$$

for $\varepsilon \in (0, 1/16)$. By the standard contraction mapping principle, we first construct a local solution. We get a global solution bounded in X_T for any $T > 0$ by the following bootstrap argument: Let $0 < \delta < \tilde{\delta}$ be sufficiently small and

$$\big\|\psi_0^{\pm}\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\langle x\rangle^2\psi_0^{\pm}(x)\big\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\langle\xi\rangle^9\widehat{\psi_0^{\pm}}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)} \le \delta.$$

Under the a priori assumption

$$\big\|\psi^{\pm}\big\|_{X_T} \le \tilde{\delta},$$

we will conclude

$$\big\|\psi^{\pm}\big\|_{X_T} \lesssim \delta + \tilde{\delta}^3.$$

The weighted norm $\big\|\langle x\rangle^2\mathrm{w}^{\pm}(t,x)\big\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)}$ plays a crucial role in our whole construction since it enables us to control the remainders in the asymptotic expansions (cf. Section 3 for more details). On the Fourier side, $\langle x\rangle^2$ produces the operator $1-\Delta_\xi$ which hits

$$e^{\pm it\langle\xi\rangle}\widehat{\psi^{\pm}(t)}(\xi) = \widehat{\psi_0^{\pm}}(\xi) + i\int_0^t I(t',\xi)\,\mathrm{d}t',$$

where

$$\begin{aligned}I(t',\xi) &= ce^{\pm_0 it'\langle\xi\rangle}\Pi^{\pm_0}(\xi)\big[\big(V*|\psi(t')|^2\big)\psi(t')\big]\hat{}(\xi) \\ &= 4\pi c\sum_{\pm_1,\pm_2,\pm_3\in\{+,-\}}\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it'(\pm_0\langle\xi\rangle\mp_1\langle\sigma\rangle\pm_2\langle\sigma-\eta\rangle\mp_3\langle\xi-\eta\rangle)} \\ &\qquad\cdot|\eta|^{-2}\big\langle\widehat{\mathrm{w}^{\pm_1}(t')}(\sigma),\widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\big\rangle_{\mathbb{C}^4}\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta.\end{aligned}$$

The most difficult cases arise when derivatives ∂_{ξ_k} hit $e^{it'(\pm_0\langle\xi\rangle\mp_1\langle\sigma\rangle\pm_2\langle\sigma-\eta\rangle\mp_3\langle\xi-\eta\rangle)}$: We have

$$\frac{\mathrm{d}}{\mathrm{d}\xi_k}e^{it'(\pm_0\langle\xi\rangle\mp_1\langle\sigma\rangle\pm_2\langle\sigma-\eta\rangle\mp_3\langle\xi-\eta\rangle)} = it'\Big(\pm_0\frac{\xi_k}{\langle\xi\rangle}\mp_3\frac{\xi_k-\eta_k}{\langle\xi-\eta\rangle}\Big)$$

and we need to recover the loss of t'. For the case $\pm_0 = \pm_3$, as already mentioned in Pusateri's paper [84], the essential idea is to use that

$$\Big|\frac{\xi_k}{\langle\xi\rangle} - \frac{\xi_k - \eta_k}{\langle\xi-\eta\rangle}\Big| \lesssim |\eta|$$

which cancels part of the singularity coming from the Fourier transform $|\eta|^{-2}$ of the Coulomb potential. For the case $\pm_0 = \mp_3$, we can argue similarly by using

$$\big|\Pi^+(\xi)\Pi^-(\xi-\eta)\big| \lesssim |\eta|.$$

In one case, two derivatives ∂_{ξ_k} hit $e^{\pm_0 it'\langle\xi\rangle}$. Here, we also integrate by parts in time to gain a second factor t'.

Testing solutions by wave packets means that we first localize $\psi^\pm(t,x)$ along rays $x = tv$, $v \in B_1(0)$, at a $t^{-1/2}$-scale related to the uncertainty principle. Then, a stationary phase argument suggests that the spatial Fourier transform $\widehat{\psi(t)}(\xi)$ is localized around $\pm\xi_v = \pm(1-|v|^2)^{-1/2}v$. Let

$$\Upsilon^\pm(t,v) := \mathbb{1}_{B_1(0)}(v)\int_{\mathbb{R}^3} \overline{\Omega_v^\pm}(t,x)\,\psi^\pm(t,x)\,\mathrm{d}x,$$

where the wave packet $\Omega_v^\pm(t,x)$ localizes x around tv at a $t^{1/2}$-scale (we refer the reader to Section 3 for a rigorous construction of the wave packets). By Parseval's identity, we obtain

$$\Upsilon^\pm\big(t,\pm\langle\eta\rangle^{-1}\eta\big) = (2\pi)^{-3}\int_{\mathbb{R}^3} \overline{\big[\Omega^\pm_{\pm\langle\eta\rangle^{-1}\eta}(t)\big]^\wedge}(\xi)\widehat{\psi^\pm(t)}(\xi)\,\mathrm{d}\xi,$$

where $\big[\Omega^\pm_{\pm\langle\eta\rangle^{-1}\eta}(t)\big]^\wedge(\xi)$ localizes ξ around $\pm\xi_{\pm\langle\eta\rangle^{-1}\eta} = \eta$. We will show that $\Upsilon^\pm(t,v)$ and $\Upsilon^\pm(t,\pm\langle\eta\rangle^{-1}\eta)$ behave like

$$t^{3/2}\langle\xi_v\rangle^{-5/2}e^{\mp 3\pi i/4 \mp it(1-|v|^2)^{1/2}}\psi^\pm(t,tv)$$

and

$$e^{\pm it\langle\eta\rangle}\widehat{\psi(t)}(\eta),$$

respectively, up to error terms which can be controlled by the weighted energy estimates. In the next step, we derive the approximate ODE

$$\partial_t\Upsilon^\pm(t,v) = ict^{-1}\big(V * |\langle\xi_\bullet\rangle^{5/2}\Upsilon(t)|^2\big)(v)\Upsilon^\pm(t,v) + \mathrm{error}(t,v).$$

By using the weighted energy estimates, we can show that the error term decays faster than t^{-1} and solving the ODE for Υ will lead to the modified scattering result presented in Theorem 1.1.

The remainder of this thesis is organized as follows: In the next section, we first prove the existence of a local solution ψ in a normed space adapted to X_T. Then, we work out the bootstrap argument for $\|\psi(t)\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}$ and the weighted energy terms $\|x\mathrm{w}^\pm(t)\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)}$, $\big\|\langle x\rangle^2\mathrm{w}^\pm(t)\big\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)}$. In Section 3, we construct the wave packets $\Omega_v^\pm$ and calculate their spatial Fourier transforms. After that, we finish the bootstrap argument and prove the modified scattering result of Theorem 1.1 by using the method of testing solutions by wave packets. As a by-product, we get a generalization of Pusateri's result [84, Thm. 1.1] for the scalar toy model. Section 4 treats the asymptotic completeness problem. For suitable asymptotic states, we construct solutions which behave like the asymptotic states as time goes to infinity. In the last section, we sketch how our modified scattering result might be carried over to the general system (17)–(20). In fact, we expect that the A_j are of lower order because of a hidden null-structure so that only A_0 effects the modified scattering factor.

2 Bootstrap argument and weighted estimates

Vector field and profile

Weighted norms play a fundamental role in the derivation of the modified scattering result. Since x-weights do not commute with the linear propagator $e^{\mp it\langle D\rangle}$, we introduce the vector field $L^\pm = (L_1^\pm, L_2^\pm, L_3^\pm)$ defined by

$$\begin{aligned} L_j^\pm &= e^{\mp it\langle D\rangle} x_j e^{\pm it\langle D\rangle} \\ &= x_j \pm it\langle D\rangle^{-1}\partial_j. \end{aligned} \tag{39}$$

We have

$$x e^{\pm it\langle D\rangle} = e^{\pm it\langle D\rangle} L^\pm$$

and

$$\big(-i\partial_t \pm \langle D\rangle\big) L^\pm = L^\pm \big(-i\partial_t \pm \langle D\rangle\big).$$

We define the profile of $\psi^\pm$ by

$$\mathrm{w}^\pm(t) = e^{\pm it\langle D\rangle}\psi^\pm(t). \tag{40}$$

Since $e^{\pm it\langle D\rangle}$ is the inverse linear propagator, the profile would converge to an asymptotic state W in the case of scattering. For our modified scattering result, this will only be true up to a correction factor that oscillates logarithmically in time. Note that

$$i\partial_{\xi_j}\widehat{\mathrm{w}^\pm(t)}(\xi) = e^{\pm it\langle \xi\rangle}\big[L_j^\pm \psi^\pm(t)\big]\widehat{\ }(\xi). \tag{41}$$

Bootstrap argument

Given a local solution on $[0,T]$ (as mentioned before, we focus on $t \geq 0$ for simplicity), we follow the strategy of Pusateri [84] and define

$$\begin{aligned} \big\|\psi^\pm\big\|_{X_T} := \sup_{t\in[0,T]} \Big[&\langle t\rangle^{-\varepsilon}\big\|\psi^\pm(t)\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} + \langle t\rangle^{-\varepsilon}\big\|x\mathrm{w}^\pm(t,x)\big\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)} \\ &+ \langle t\rangle^{-2\varepsilon}\big\|\langle x\rangle^2\mathrm{w}^\pm(t,x)\big\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)} \\ &+ \big\|\langle\xi\rangle^8\widehat{\psi^\pm(t)}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\Big] \end{aligned} \tag{42}$$

for $\varepsilon \in (0, 1/16)$. We will obtain a global solution bounded in X_T for any $T > 0$ by the following bootstrap argument: Let $0 < \delta < \tilde{\delta}$ be sufficiently small and

$$\big\|\psi_0^\pm\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\langle x\rangle^2\psi_0^\pm(x)\big\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\langle\xi\rangle^9\widehat{\psi_0^\pm}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)} \leq \delta. \tag{43}$$

Under the a priori assumption

$$\|\psi^{\pm}\|_{X_T} \le \tilde{\delta}, \tag{44}$$

we will conclude that

$$\|\psi^{\pm}\|_{X_T} \lesssim \delta + \tilde{\delta}^3. \tag{45}$$

Lemma 2.1 (Estimates for $\Pi^{\pm}$)**.** *Let $\xi, \eta \in \mathbb{R}^3$ and $\alpha \in \mathbb{N}_0^3$. Then,*

$$\left|\partial_\xi^\alpha \Pi^{\pm}(\xi)\right| \lesssim \langle\xi\rangle^{-|\alpha|}, \tag{46}$$

$$\left|\Pi^{\pm}(\xi)\Pi^{\mp}(\xi-\eta)\right| \lesssim \langle\xi-\eta\rangle^{-1}|\eta|. \tag{47}$$

Proof. By definition (14) of $\Pi^{\pm}(D)$, we have

$$\Pi^{\pm}(\xi) = \frac{1}{2}\big(I_4 \pm \langle\xi\rangle^{-1}\gamma^0(\gamma^j\xi_j + I_4)\big)$$

which implies (46) since

$$|\partial_\xi^\alpha \langle\xi\rangle^{-1}| \lesssim \langle\xi\rangle^{-1-|\alpha|}, \quad |\partial_\xi^\alpha[\langle\xi\rangle^{-1}\xi]| \lesssim \langle\xi\rangle^{-|\alpha|}.$$

A direct computation shows that

$$\begin{aligned} 4\Pi^{\pm}(\xi)\Pi^{\mp}(\xi-\eta) &= \big(1-\langle\xi-\eta\rangle^{-1}\langle\xi\rangle\big)I_4 \\ &\quad \pm \big(\langle\xi\rangle^{-1} - \langle\xi-\eta\rangle^{-1}\big)\gamma^0(\gamma^j\xi_j + I_4) \\ &\quad + \langle\xi-\eta\rangle^{-1}\big(\pm I_4 + \langle\xi\rangle^{-1}\gamma^0(\gamma^j\xi_j + I_4)\big)\gamma^0\gamma^j\eta_j. \end{aligned} \tag{48}$$

For the first term on the right hand side of (48), note that

$$\left|1-\langle\xi-\eta\rangle^{-1}\langle\xi\rangle\right| = \langle\xi-\eta\rangle^{-1}|\langle\xi-\eta\rangle - \langle\xi\rangle|$$

and

$$\begin{aligned} |\langle\xi-\eta\rangle - \langle\xi\rangle| &= \frac{\big||\xi-\eta|^2 - |\xi|^2\big|}{\langle\xi-\eta\rangle + \langle\xi\rangle} \\ &\le \big||\xi-\eta| - |\xi|\big| \\ &\le |\eta| \end{aligned}$$

which implies

$$\left|1-\langle\xi-\eta\rangle^{-1}\langle\xi\rangle\right| \le \langle\xi-\eta\rangle^{-1}|\eta|.$$

For the second and third term, we observe that

$$\begin{aligned} \left|\langle\xi\rangle^{-1} - \langle\xi-\eta\rangle^{-1}\right| &= \frac{\left|\langle\xi-\eta\rangle - \langle\xi\rangle\right|}{\langle\xi\rangle\langle\xi-\eta\rangle} \\ &\le \frac{|\eta|}{\langle\xi\rangle\langle\xi-\eta\rangle}. \end{aligned} \tag{49}$$

Since

$$\left|\gamma^0\big(\gamma^j\xi_j + I_4\big)\right| \lesssim \langle\xi\rangle, \quad \left|\gamma^0\gamma^j\eta_j\right| \lesssim |\eta|,$$

we obtain (47). □

Remark. Estimate (47) is related to the null-structure of $\Pi^{\pm}(D)$. In the literature, one typically finds the following version of (47):

$$\left|\Pi^{\pm_0}(\xi)\Pi^{\mp_0}(\sigma)\right| \lesssim |\angle(\pm_1\xi,\sigma)| + \langle\xi\rangle^{-1} + \langle\sigma\rangle^{-1}, \quad \xi,\sigma \in \mathbb{R}^3, \quad \pm_0, \pm_1 \in \{+,-\},$$

where $\angle(\pm_1\xi,\sigma)$ denotes the angle between $\pm_1\xi$ and σ, see e.g. Bejenaru–Herr [9, Lemma 2.1].

2.1 Local well-posedness

To initiate our bootstrap argument, we show existence and uniqueness of local solutions $\psi \in \mathcal{C}\big([0,T'], H^N(\mathbb{R}^3,\mathbb{C}^4)\big)$ with

$$\langle\cdot\rangle^2 \mathrm{w} \in \mathcal{C}\big([0,T'], H^2(\mathbb{R}^3,\mathbb{C}^4)\big), \tag{50}$$

$$\langle\cdot\rangle^8 \widehat{\psi} \in \mathcal{C}\big([0,T'], L^\infty(\mathbb{R}^3,\mathbb{C}^4)\big), \tag{51}$$

where $T' > 0$ will be chosen sufficiently small and $\mathrm{w}^\pm(t)$ – as always – denotes the profile $e^{\pm it\langle D\rangle}\psi^\pm(t)$.

Theorem 2.2 (Local well-posedness)**.** *Let $N \geq 103$ and $\varepsilon \in (0,1/16)$. There is $\delta_0 > 0$ such that for any $\delta \in (0,\delta_0]$ and $\psi_0^\pm \in H^N(\mathbb{R}^3,\mathbb{C}^4)$ with*

$$\big\|\psi_0^\pm\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\langle\cdot\rangle^2\psi_0^\pm\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\langle\xi\rangle^9\widehat{\psi_0^\pm}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)} \leq \delta,$$

the Cauchy problem (30) *has a unique local solution $\psi \in \mathcal{C}\big([0,T'], H^N(\mathbb{R}^3,\mathbb{C}^4)\big)$ for some $T' > 0$ which satisfies* (50) *and* (51)*.*

We prove Theorem 2.2 by the standard contraction mapping principle. Let

$$Y_{T',R} = \big\{\psi \in L^\infty_t H^N_x\big([0,T']\times\mathbb{R}^3,\mathbb{C}^4\big) \colon \|\psi\|_{Y_{T'}} \leq R\big\},$$

where

$$\|\psi\|_{Y_{T'}} = \|\psi\|_{L^\infty_t H^N_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} + \|\langle\cdot\rangle^8\widehat{\psi}\|_{L^\infty_t L^\infty_\xi([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}$$

and $R > 0$ will be chosen later. The Duhamel equation for $\psi = \psi^+ + \psi^-$ is

$$\psi^\pm(t) = e^{\mp it\langle D\rangle}\psi_0^\pm + ic\int_0^t e^{\mp i(t-t')\langle D\rangle}\Pi^\pm(D)\big[\big(V * |\psi|^2\big)\psi\big](t')\,\mathrm{d}t'. \tag{52}$$

We first show that

$$\begin{aligned} G(\psi)(t) &= \sum_{\pm_0\in\{+,-\}}\Big(e^{\mp_0 it\langle D\rangle}\psi_0^{\pm_0} + ic\int_0^t e^{\mp_0 i(t-t')\langle D\rangle}\Pi^{\pm_0}(D)\big[\big(V*|\psi(t')|^2\big)\psi(t')\big]\,\mathrm{d}t'\Big) \\ &=: G^+(\psi)(t) + G^-(\psi)(t), \quad \psi \in Y_{T',R} \end{aligned}$$

is a contraction mapping on $Y_{T',R}$ so that there exists a unique solution $\psi \in Y_{T',R}$. Secondly, for that solution ψ, we will conclude that

$$\sum_{\pm_0\in\{+,-\}} \|\langle\cdot\rangle^2 e^{\pm_0 it\langle D\rangle}\psi^{\pm_0}\|_{L^\infty_t H^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \lesssim \delta.$$

The latter norm is not included in $\|\cdot\|_{Y_{T'}}$ for technical reasons (cf. proof of estimate (62)).

The proof of Theorem 2.2 will be a consequence of the subsequent Lemmas 2.3–2.5:

Lemma 2.3. *Assume that $s > 3/2$ and let $\psi, \tilde{\psi} \in Y_{T',R}$. Then,*

$$\big\|G(\psi)\big\|_{L^\infty_t H^s_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \lesssim \delta + T'R^3, \tag{53}$$

$$\big\|G(\psi) - G(\tilde{\psi})\big\|_{L^\infty_t H^s_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \lesssim T'R^2\|\psi - \tilde{\psi}\|_{Y_{T'}}. \tag{54}$$

Proof. Because of $\|G^{\pm}(\psi)(t)\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)} = (2\pi)^{-3/2}\|\langle\xi\rangle^s e^{\pm it\langle\xi\rangle}[G^{\pm}(\psi)(t)]^{\frown}(\xi)\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}$, we have

$$
\begin{aligned}
&\|G^{\pm}(\psi)(t)\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \|\psi_0{}^{\pm}\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)} + \int_0^t \Big\|\langle\xi\rangle^s\int_{\mathbb{R}^3} e^{\pm it'\langle\xi\rangle}\Pi^{\pm}(\xi)|\eta|^{-2}\widehat{|\psi(t')|^2}(\eta)\widehat{\psi(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\
&\lesssim \|\psi_0{}^{\pm}\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)} + \int_0^t \Big\|\int_{\mathbb{R}^3}\langle\eta\rangle^{s-2}\widehat{|\psi(t')|^2}(\eta)\widehat{\psi(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\
&\quad + \int_0^t \Big\|\int_{\mathbb{R}^3}|\eta|^{-2}\widehat{|\psi(t')|^2}(\eta)\langle\xi-\eta\rangle^s\widehat{\psi(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\
&\lesssim \|\psi_0{}^{\pm}\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)} + \int_0^t \big\|\langle\xi\rangle^{s-2}\widehat{|\psi(t')|^2}(\xi)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\|\psi(t')\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\
&\quad + \int_0^t \big\||\xi|^{-2}\widehat{|\psi(t')|^2}(\xi)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\big\|\langle\xi\rangle^s\widehat{\psi(t')}(\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'. \qquad (55)
\end{aligned}
$$

Since

$$
\begin{aligned}
\big\|\langle\xi\rangle^{s-2}\widehat{|\psi(t')|^2}(\xi)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})} &\lesssim \big\|\langle\xi\rangle^{s}\widehat{|\psi(t')|^2}(\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C})}\\
&\lesssim \|\psi(t')\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)}\|\psi(t')\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \|\psi(t')\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)}\|\psi(t')\|_{H^{3/2+}(\mathbb{R}^3,\mathbb{C}^4)} \qquad (56)
\end{aligned}
$$

and

$$
\begin{aligned}
\big\||\xi|^{-2}\widehat{|\psi(t')|^2}(\xi)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})} &\le \Big(\int_{B_1(0)}|\xi|^{-2}\,\mathrm{d}\xi\Big)\big\|\widehat{|\psi(t')|^2}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C})}\\
&\quad + \Big(\int_{B_1(0)^c}|\xi|^{-4}\,\mathrm{d}\xi\Big)^{1/2}\big\|\widehat{|\psi(t')|^2}(\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C})}\\
&\lesssim \|\psi(t')\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \|\psi(t')\|^2_{L^4(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \|\psi(t')\|^2_{H^{3/4}(\mathbb{R}^3,\mathbb{C}^4)}, \qquad (57)
\end{aligned}
$$

we get (53). For deriving estimate (54), we replace $G^{\pm}(\psi)$ in (55), (56) and (57) with $G^{\pm}(\psi) - G^{\pm}(\tilde\psi)$ to obtain

$$
\begin{aligned}
&\widehat{|\psi(t')|^2}(\eta)\widehat{\psi(t')}(\xi-\eta) - \widehat{|\tilde\psi(t')|^2}(\eta)\widehat{\tilde\psi(t')}(\xi-\eta)\\
&\quad = \big(\widehat{|\psi(t')|^2}(\eta) - \widehat{|\tilde\psi(t')|^2}(\eta)\big)\widehat{\psi(t')}(\xi-\eta)\big) + \widehat{|\tilde\psi(t')|^2}(\eta)\big(\widehat{\psi(t')}(\xi-\eta) - \widehat{\tilde\psi(t')}(\xi-\eta)\big).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\|G^{\pm}(\psi)(t) - G^{\pm}(\tilde\psi)\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \int_0^t \|\psi(t') - \tilde\psi(t')\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)}\Big(\|\psi(t')\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)} + \|\tilde\psi(t')\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)}\Big)\|\psi(t')\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\
&\qquad + \int_0^t \|\tilde\psi(t')\|^2_{H^s(\mathbb{R}^3,\mathbb{C}^4)}\|\psi(t') - \tilde\psi(t')\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\
&\qquad + \int_0^t \big\||\xi|^{-2}\big(\widehat{|\psi(t')|^2}(\xi) - \widehat{|\tilde\psi(t')|^2}(\xi)\big)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\|\psi(t')\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\
&\qquad + \int_0^t \big\||\xi|^{-2}\widehat{|\tilde\psi(t')|^2}(\xi)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\|\psi(t') - \tilde\psi(t')\|_{H^s_\xi(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'
\end{aligned}
$$

and

$$\begin{aligned}&\big\||\xi|^{-2}\big(\widehat{|\psi(t')|^2}(\xi)-\widehat{|\tilde\psi(t')|^2}(\xi)\big)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\\&\qquad\lesssim\big\|\psi(t')-\tilde\psi(t')\big\|_{H^{3/4}(\mathbb{R}^3,\mathbb{C}^4)}\Big(\big\|\psi(t')\big\|_{H^{3/4}(\mathbb{R}^3,\mathbb{C}^4)}+\big\|\tilde\psi(t')\big\|_{H^{3/4}(\mathbb{R}^3,\mathbb{C}^4)}\Big)\end{aligned}$$

which leads to estimate (54). □

Remark. As a consequence of that proof, we can conclude local well-posedness in $H^s(\mathbb{R}^3,\mathbb{C}^4)$ for $s>3/2$. We expect that with some extra work, this might be improved to $s>1/4$ and $s>0$ for radial data by following the work of Herr and Lenzmann [52] with some slight modifications.

For the proofs of the next two lemmas, we follow the strategy of Cho, Hwang and Yang [20, Thm. 5.2.1 and Lemma 5.2.2] with some modifications.

Lemma 2.4. *Let $\psi,\tilde\psi\in Y_{T',R}$. Then,*

$$\big\|\langle\cdot\rangle^8\widehat{G(\psi)}\big\|_{L^\infty_tL^\infty_\xi([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}\lesssim\delta+T'R^3,\tag{58}$$

$$\big\|\langle\cdot\rangle^8\big(\widehat{G(\psi)}-\widehat{G(\tilde\psi)}\big)\big\|_{L^\infty_tL^\infty_\xi([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}\lesssim T'R^2\|\psi-\tilde\psi\|_{Y_{T'}}.\tag{59}$$

Proof. Since $\big\|\langle\xi\rangle^8[G^\pm(\psi)(t)]\widehat{\ }(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}=\big\|\langle\xi\rangle^8e^{\pm it\langle\xi\rangle}[G^\pm(\psi)(t)]\widehat{\ }(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}$, we have

$$\begin{aligned}&\big\|\langle\xi\rangle^8[G^\pm(\psi)(t)]\widehat{\ }(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\&\quad\lesssim\big\|\langle\xi\rangle^8\widehat{\psi_0}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\&\qquad+\int_0^t\Big\|\langle\xi\rangle^8\int_{\mathbb{R}^3}e^{\pm it'\langle\xi\rangle}\Pi^\pm(\xi)|\eta|^{-2}\widehat{|\psi(t')|^2}(\eta)\widehat{\psi(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\&\quad\lesssim\big\|\langle\xi\rangle^8\widehat{\psi_0}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}+\int_0^t\Big\|\int_{\mathbb{R}^3}|\eta|^6\widehat{|\psi(t')|^2}(\eta)\widehat{\psi(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\&\qquad+\int_0^t\Big\|\int_{\mathbb{R}^3}|\eta|^{-2}\widehat{|\psi(t')|^2}(\eta)\langle\xi-\eta\rangle^8\widehat{\psi(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\&\quad\lesssim\big\|\langle\xi\rangle^8\widehat{\psi_0}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}+\int_0^t\big\|\psi(t')\big\|^2_{H^6(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\&\qquad+\int_0^t\big\||\xi|^{-2}\widehat{|\psi(t')|^2}(\xi)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\big\|\langle\xi\rangle^8\widehat{\psi(t')}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'.\end{aligned}\tag{60}$$

Because of

$$\begin{aligned}\big\||\xi|^{-2}\widehat{|\psi(t')|^2}(\xi)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}&\le\Big(\int_{B_1(0)}|\xi|^{-2}\,\mathrm{d}\xi\Big)\big\|\widehat{|\psi(t')|^2}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C})}\\&\quad+\Big(\int_{B_1(0)^c}|\xi|^{-4}\,\mathrm{d}\xi\Big)^{1/2}\big\|\widehat{|\psi(t')|^2}(\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C})}\\&\lesssim\big\|\psi(t')\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|\psi(t')\big\|^2_{L^4(\mathbb{R}^3,\mathbb{C}^4)}\\&\lesssim\big\|\psi(t')\big\|^2_{H^{3/4}(\mathbb{R}^3,\mathbb{C}^4)},\end{aligned}\tag{61}$$

we get (58). For estimate (59), replace $G^\pm(\psi)$ in (60) and (61) with $G^\pm(\psi) - G^\pm(\tilde{\psi})$ and note that

$$\begin{aligned}&\widehat{|\psi(t')|^2}(\eta)\widehat{\psi(t')}(\xi-\eta) - \widehat{|\tilde{\psi}(t')|^2}(\eta)\widehat{\tilde{\psi}(t')}(\xi-\eta)\\&\quad = \big(\widehat{|\psi(t')|^2}(\eta) - \widehat{|\tilde{\psi}(t')|^2}(\eta)\big)\widehat{\psi(t')}(\xi-\eta)\big) + \widehat{|\tilde{\psi}(t')|^2}(\eta)\big(\widehat{\psi(t')}(\xi-\eta) - \widehat{\tilde{\psi}(t')}(\xi-\eta)\big).\end{aligned}$$

Then,

$$\begin{aligned}&\big\|\langle\xi\rangle^8\big([G^\pm(\psi)(t)]\widehat{\ }(\xi) - [G^\pm(\tilde{\psi})(t)]\widehat{\ }(\xi)\big)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\&\lesssim \int_0^t \big\|\psi(t') - \tilde{\psi}(t')\big\|_{H^6(\mathbb{R}^3,\mathbb{C}^4)}\Big(\big\|\psi(t')\big\|_{H^6(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\tilde{\psi}(t')\big\|_{H^6(\mathbb{R}^3,\mathbb{C}^4)}\Big)\big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\&\quad + \int_0^t \big\|\tilde{\psi}(t')\big\|^2_{H^6(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t') - \tilde{\psi}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\&\quad + \int_0^t \big\||\xi|^{-2}\big(\widehat{|\psi(t')|^2}(\xi) - \widehat{|\tilde{\psi}(t')|^2}(\xi)\big)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\big\|\langle\xi\rangle^8\widehat{\psi(t')}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\\&\quad + \int_0^t \big\||\xi|^{-2}\widehat{|\tilde{\psi}(t')|^2}(\xi)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\big\|\langle\xi\rangle^8\big(\widehat{\psi(t')}(\xi) - \widehat{\tilde{\psi}(t')}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\mathrm{d}t'\end{aligned}$$

and

$$\begin{aligned}&\big\||\xi|^{-2}\big(\widehat{|\psi(t')|^2}(\xi) - \widehat{|\tilde{\psi}(t')|^2}(\xi)\big)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\\&\qquad \lesssim \big\|\psi(t') - \tilde{\psi}(t')\big\|_{H^{3/4}(\mathbb{R}^3,\mathbb{C}^4)}\Big(\big\|\psi(t')\big\|_{H^{3/4}(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\tilde{\psi}(t')\big\|_{H^{3/4}(\mathbb{R}^3,\mathbb{C}^4)}\Big)\end{aligned}$$

which leads to estimate (59). □

Lemma 2.5. *The solution $\psi = \psi^+ + \psi^- \in Y_{T',R}$ also satisfies*

$$\|\langle\cdot\rangle^2 \mathrm{w}^\pm\|_{L^\infty_t H^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \lesssim \delta. \tag{62}$$

Proof. Since

$$\langle x\rangle^2\mathrm{w}^\pm = \mathrm{w}^\pm + \sum_{j=1}^3 x_j^2\mathrm{w}^\pm$$

and

$$\|\mathrm{w}^\pm\|_{H^2} = \|\psi^\pm\|_{H^2},$$

it suffices to consider $x_j^2\mathrm{w}^\pm$. We first treat $x_j\mathrm{w}^\pm$. Note that

$$x_j\mathrm{w}^\pm(t) = e^{\pm it\langle D\rangle}\big(\pm it\langle D\rangle^{-1}\partial_j + x_j\big)\psi^\pm(t)$$

and

$$\begin{aligned}\big\|e^{\pm it\langle D\rangle}\big(\pm it\langle D\rangle^{-1}\partial_j\big)\psi^\pm(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} &\le t\big\|\psi(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\\&\le T'R.\end{aligned}$$

For $e^{\pm it\langle D\rangle}x_j\psi^\pm(t)$, we introduce

$$k_n(x) = e^{-|x|^2/n}$$

and consider

$$\big\|k_n(x)x_j\psi^\pm(t,x)\big\|^2_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)} = \int_{\mathbb{R}^3} \big\langle x_j^2 k_n(x)^2\psi^\pm(t,x), \psi^\pm(t,x)\big\rangle_{\mathbb{C}^4}\,\mathrm{d}x.$$

Since

$$|k_n(x)x_j| \lesssim n^{1/2},$$

we have

$$\big\|k_n(x)x_j\psi^\pm(t,x)\big\|_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)} \lesssim n^{1/2}R. \tag{63}$$

Note that

$$\begin{aligned}
&\partial_t\big\|k_n(x)x_j\psi^\pm(t,x)\big\|^2_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)}\\
&= 2\mathrm{Re}\int_{\mathbb{R}^3} \big\langle x_j^2 k_n(x)^2\psi^\pm(t,x), \partial_t\psi^\pm(t,x)\big\rangle_{\mathbb{C}^4}\,\mathrm{d}x\\
&= 2\mathrm{Im}\int_{\mathbb{R}^3} \big\langle x_j^2 k_n(x)^2\psi^\pm(t,x), \mp\langle D\rangle\psi^\pm(t,x)\big\rangle_{\mathbb{C}^4}\\
&\qquad\qquad + \big\langle x_j^2 k_n(x)^2\psi^\pm(t,x), \Pi^\pm(D)\big[\big(V*|\psi(t)|^2\big)\psi(t)\big](x)\big\rangle_{\mathbb{C}^4}\,\mathrm{d}x\\
&=: I_n^\pm(t) + II_n^\pm(t).
\end{aligned}$$

From Parseval's identity, we get

$$\begin{aligned}
I_n^\pm(t) &= \mp\mathrm{Im}\int_{\mathbb{R}^3} \big\langle x_j^2 k_n(x)^2\psi^\pm(t,x), \langle D\rangle\psi^\pm(t,x)\big\rangle_{\mathbb{C}^4}\,\mathrm{d}x\\
&\qquad \mp \mathrm{Im}\int_{\mathbb{R}^3} \big\langle \langle D\rangle\big(x_j^2 k_n^2\psi^\pm(t)\big)(x), \psi^\pm(t,x)\big\rangle_{\mathbb{C}^4}\,\mathrm{d}x\\
&= \mathrm{Im}\int_{\mathbb{R}^3} \big\langle \psi^\pm(t,x), \big[x_j^2 k_n(x)^2, \mp\langle D\rangle\big]\psi^\pm(t,x)\big\rangle_{\mathbb{C}^4}\,\mathrm{d}x,
\end{aligned}$$

where

$$[X,Y] = XY - YX$$

denotes the commutator of operators X and Y. Similarly,

$$\begin{aligned}
I_n^\pm(t) &= 2\mathrm{Im}\int_{\mathbb{R}^3} \big\langle x_j k_n(x)\psi^\pm(t,x), \big[x_j k_n(x), \mp\langle D\rangle^{-1}\big]\langle D\rangle^2\psi^\pm(t,x)\big\rangle_{\mathbb{C}^4}\,\mathrm{d}x\\
&\qquad \mp 2\mathrm{Im}\int_{\mathbb{R}^3} \big\langle \langle D\rangle^{-1}\big(x_j k_n\psi^\pm(t)\big)(x), \big[x_j k_n(x), \langle D\rangle^2\big]\psi^\pm(t,x)\big\rangle_{\mathbb{C}^4}\,\mathrm{d}x\\
&=: I_n^{\pm,1}(t) + I_n^{\pm,2}(t).
\end{aligned}$$

Let K denote the kernel of $\langle D\rangle^{-1}$. Then,

$$\begin{aligned}
\big[x_j k_n(x), \mp\langle D\rangle^{-1}\big]\langle D\rangle^2\psi^\pm(t,x) &= \mp\int_{\mathbb{R}^3} x_j k_n(x)K(x-y)\langle D\rangle^2\psi(t,y)\,\mathrm{d}y\\
&\qquad \pm\int_{\mathbb{R}^3} K(x-y)y_j k_n(y)\langle D\rangle^2\psi^\pm(t,y)\,\mathrm{d}y\\
&= \mp\int_{\mathbb{R}^3} K(x-y)\big(x_j k_n(x) - y_j k_n(y)\big)\langle D\rangle^2\psi^\pm(t,y)\,\mathrm{d}y.
\end{aligned}$$

Hence,

$$\Big|\big[|x|k_n(x), \mp\langle D\rangle^{-1}\big]\langle D\rangle^2\psi^\pm(t,x)\Big| \lesssim \int_{\mathbb{R}^3} |K(x-y)|\,|x-y|\,|\langle D\rangle^2\psi^\pm(t,y)|\,\mathrm{d}y.$$

Since

$$|K(z)| \lesssim |z|^{-2}\mathbb{1}_{B_2(0)}(z) + e^{-|z|/2}\mathbb{1}_{B_2(0)^c}(z),$$

cf. e.g. Grafakos [46, Prop. 1.2.5], we obtain

$$\big|I_n^{\pm,1}(t)\big| \lesssim \big\|k_n x_j \psi^\pm(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi^\pm(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}.$$

For $I_n^{\pm,2}$, note that

$$\big[x_j k_n(x), \langle D\rangle^2\big]\psi^\pm(t,x) = \Delta\big(x_j k_n(x)\big)\,\psi^\pm(t,x) + 2\nabla\big(x_j k_n(x)\big)\cdot\nabla\psi^\pm(t,x)$$

and therefore,

$$\big|I_n^{\pm,2}(t)\big| \lesssim \big\|k_n x_j \psi^\pm(t)\big\|_{H^{-1}(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi^\pm(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}.$$

For $II_n^\pm(t)$, we observe that

$$\begin{aligned}\big|II_n^\pm(t)\big| &\le \big\|k_n x_j \psi^\pm(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|k_n x_j \Pi^\pm(D)\big[\big(V * |\psi(t)|^2\big)\psi(t)\big]\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \big\|k_n x_j \psi^\pm(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|x_j\big(V * |\psi(t)|^2\big)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}.\end{aligned}$$

For $|x| \ge 2|y|$ or $2|x| \le |y|$, we have

$$|x_j|\,|x-y|^{-1} \lesssim 1.$$

For $|y|/2 \le |x| \le 2|y|$, note that

$$|x_j|k_n(x) \le 2|y|k_{4n}(y).$$

Therefore,

$$\begin{aligned}&\big\|x_j\big(V * |\psi(t)|^2\big)(x)\big\|_{L^\infty_x(\mathbb{R}^3,\mathbb{C})}\\ &\quad\lesssim \big\|\psi(t)\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \Big\|\int_{\mathbb{R}^3} k_{4n}(y)|x-y|^{-1}|y|\,|\psi(t,y)|^2\,\mathrm{d}y\Big\|_{L^\infty_x(\mathbb{R}^3,\mathbb{C})}\\ &\quad\lesssim \big\|\psi(t)\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\|k_{4n}x|\psi(t)|^2\big\|_{L^{3/2,1}(\mathbb{R}^3,\mathbb{C})}\\ &\quad\lesssim \big\|\psi(t)\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\|k_{4n}x\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t)\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad\lesssim \big\|\psi(t)\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\|x\psi(t,x)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\end{aligned}$$

and we obtain

$$\begin{aligned}\big|II_n^\pm(t)\big| &\lesssim \big\|k_n x\psi^\pm(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\qquad\cdot\Big(\big\|k_{4n}x\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\psi(t)\big\|^3_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\Big).\end{aligned}$$

We conclude that

$$\begin{aligned}\partial_t\big\|k_n x\psi^\pm(t)\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \big\|k_n x\psi^\pm(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\qquad\cdot\Big(1 + \big\|k_{4n}x\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\psi(t)\big\|^2_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)}\Big)\end{aligned}$$

and therefore,

$$\begin{aligned}
&\big\|k_n x\psi^{\pm}(t)\big\|^2_{L^2_x(\mathbb{R}^3,\mathbb{C}^4))}\\
&\lesssim \big\|x\psi_0^{\pm}\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \int_0^t \big\|k_n x\psi^{\pm}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad\qquad \cdot\Big(1+\big\|k_{4n}x\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|\psi(t')\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\Big)\,\mathrm{d}t'\\
&\lesssim \delta^2 + T'R\big\|k_n x\psi^{\pm}\big\|_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}\Big(1+\big\|k_{4n}x\psi^{\pm}\big\|_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}R+R^2\Big).
\end{aligned}\tag{64}$$

In the case

$$R\big\|k_{4n}x\psi^{\pm}\big\|_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \le 1+R^2,$$

we get

$$\big\|k_n x\psi^{\pm}(t)\big\|^2_{L^2_x(\mathbb{R}^3,\mathbb{C}^4))} \lesssim \delta^2 + T'R(1+R^2)\big\|k_n x\psi^{\pm}\big\|_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}$$

from (64). Without loss of generality, we may assume

$$\big\|k_n x\psi^{\pm}\big\|_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \gtrsim \delta.$$

Then,

$$\big\|k_n x\psi^{\pm}\big\|_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \lesssim \delta + T'R(1+R^2)$$

and choosing $T'>0$ such that

$$T'R(1+R^2)\le\delta$$

leads to

$$\big\|k_n x\psi^{\pm}\big\|_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \lesssim \delta.$$

Now, suppose that

$$R\big\|k_{4n}x\psi^{\pm}\big\|_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \ge 1+R^2.$$

Here, we obtain

$$\big\|k_n x\psi^{\pm}(t)\big\|^2_{L^2_x(\mathbb{R}^3,\mathbb{C}^4))} \lesssim \delta^2 + T'R^2\big\|k_{4n}x\psi^{\pm}\big\|^2_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \tag{65}$$

from (64), where we used the monotonicity of (k_n). Hence,

$$\big\|k_n x\psi^{\pm}\big\|_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \le C\delta + C(T')^{1/2}R\big\|k_{4n}x\psi\big\|_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}.$$

Choose $T'>0$ such that

$$C(T')^{1/2}R\le\frac{1}{4}.$$

Then, iterating (65) yields

$$\begin{aligned}
\big\|k_n x\psi^{\pm}\big\|_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} &\le C\delta\sum_{k=0}^n 4^{-k} + 4^{-n}\big\|k_{4^n n}x\psi\big\|_{L^\infty_t L^2_x([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \delta + n^{1/2}4^{-n/2}\\
&\lesssim \delta
\end{aligned}$$

for sufficiently large $n \in \mathbb{N}$, where we used (63) in the second step. Fatou's lemma implies

$$\big\|x\psi^{\pm}\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \lesssim \delta.$$

The same calculations with $\Delta\psi^{\pm}(t)$ instead of $\psi^{\pm}(t)$ (and applying Leibniz' rule for $II_n^{\pm}$) lead to

$$\begin{aligned}
&\big\|k_n x\Delta\psi^{\pm}(t)\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \big\|x\Delta\psi_0^{\pm}\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad+\int_0^t \big\|k_n x\Delta\psi^{\pm}(t')\big\|_{L_x^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{H^4(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad\qquad\cdot\Big(1+\big\|k_{4n}x\Delta\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|\psi(t')\big\|^2_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\Big)\,\mathrm{d}t'\\
&\lesssim \big\|x\psi_0^{\pm}\big\|^2_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad+\int_0^t \big\|k_n x\Delta\psi^{\pm}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{H^4(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad\qquad\cdot\Big(1+\big\|k_{4n}x\Delta\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|\psi(t')\big\|^2_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\Big)\,\mathrm{d}t'\\
&\lesssim \delta^2+T'R\big\|k_n x\Delta\psi^{\pm}\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}\Big(1+\big\|k_{4n}x\Delta\psi\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}R+R^2\Big)
\end{aligned}$$

and

$$\big\|x\Delta\psi^{\pm}\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \lesssim \delta.$$

Because of

$$\begin{aligned}
\big\|x\psi(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} &\le \big\|x\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|x\Delta\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad+\big\|\Delta(x)\,\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|\nabla(x)\cdot\nabla\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \big\|x\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|x\Delta\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad+\Big(\big\|\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|\psi(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\Big)+\big\|\psi(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \big\|x\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|x\Delta\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|\psi(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)},
\end{aligned}$$

we can conclude that

$$\big\|x\psi(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \delta.$$

Now, consider $x_j^2\mathrm{w}^{\pm}$. We have

$$\begin{aligned}
x_j^2\mathrm{w}^{\pm}(t) &= e^{\pm it\langle D\rangle}\big(\pm it\langle D\rangle^{-1}\partial_j+x_j\big)^2\psi^{\pm}(t)\\
&= e^{\pm it\langle D\rangle}\big(-t^2\langle D\rangle^{-2}\partial_j^2 \pm it\langle D\rangle^{-1}\partial_j x_j \pm it x_j\langle D\rangle^{-1}\partial_j + x_j^2\big)\psi^{\pm}(t).
\end{aligned}$$

Since

$$\big\|t^2\langle D\rangle^{-2}\partial_j^2\psi^{\pm}(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \le (T')^2\big\|\psi^{\pm}(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}$$

and

$$\big\|it\langle D\rangle^{-1}\partial_j x_j\psi^{\pm}(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \le T'\big\|x\psi^{\pm}(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)},$$

it remains to control $(\pm itx_j\langle D\rangle^{-1}\partial_j + x_j^2)\psi^\pm(t)$. For the first part, note that

$$x_j\big(\langle D\rangle^{-1}\partial_j\psi(t)\big) = \big(\langle D\rangle^{-3}\partial_j^2 + \langle D\rangle^{-1} + \langle D\rangle^{-1}\partial_j x_j\big)\psi^\pm(t),$$

whence

$$\big\|itx_j\langle D\rangle^{-1}\partial_j\psi^\pm(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \lesssim T'\Big(\big\|\psi^\pm(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} + \big\|x_j\psi^\pm(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\Big).$$

Finally, consider $x_j^2\psi^\pm(t)$. We only need to replace x with $|x|^2$ in the calculations for $x\psi(t)$ to obtain

$$\begin{aligned}
&\big\|k_n|\cdot|^2\psi^\pm(t)\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \big\||\cdot|^2\psi_0^\pm\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad + \int_0^t \big\|k_n|\cdot|^2\psi^\pm(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad\qquad\cdot\Big(1+\big\|k_{4n}|\cdot|^2\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\psi(t')\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\Big)\,\mathrm{d}t'\\
&\quad\lesssim \delta^2 + T'R\big\|k_n|\cdot|^2\psi\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}\Big(1+\big\|k_{4n}|\cdot|^2\psi\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}R + R^2\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\big\|k_n|\cdot|^2\Delta\psi^\pm(t)\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \big\||\cdot|^2\Delta\psi_0^\pm\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad + \int_0^t \big\|k_n|\cdot|^2\Delta\psi^\pm(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{H^4(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad\qquad\cdot\Big(1+\big\|k_{4n}|\cdot|^2\Delta\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\psi(t')\big\|^2_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\Big)\,\mathrm{d}t'\\
&\quad\lesssim \big\||\cdot|^2\psi_0^\pm\big\|^2_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad + \int_0^t \big\|k_n|\cdot|^2\Delta\psi^\pm(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{H^4(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad\qquad\cdot\Big(1+\big\|k_{4n}|\cdot|^2\Delta\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\psi(t')\big\|^2_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\Big)\,\mathrm{d}t'\\
&\quad\lesssim \delta^2 + T'R\big\|k_n|\cdot|^2\Delta\psi\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad\quad\cdot\Big(1+\big\|k_{4n}|\cdot|^2\Delta\psi\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}R + R^2\Big)
\end{aligned}$$

which leads to

$$\big\|k_n|\cdot|^2\psi^\pm\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \le C\delta + CT'R^2\big\|k_{4n}|\cdot|^2\psi\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}$$

and

$$\big\|k_n|\cdot|^2\Delta\psi^\pm\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} \le C\delta + CT'R^2\big\|k_{4n}|\cdot|^2\Delta\psi\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}. \tag{66}$$

Choose $T'>0$ such that

$$CT'R^2 \le \frac{1}{8}.$$

Then, iterating (66) yields

$$\begin{aligned}\big\|k_n|\cdot|^2\psi^\pm\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} &\le C\delta\sum_{k=0}^n 8^{-k}+8^{-n}\big\|k_{4^n n}|\cdot|^2\psi\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \delta+n2^{-n}\\ &\lesssim\delta\end{aligned}$$

and

$$\begin{aligned}\big\|k_n|\cdot|^2\Delta\psi^\pm\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)} &\le C\delta\sum_{k=0}^n 8^{-k}+8^{-n}\big\|k_{4^n n}|\cdot|^2\Delta\psi\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \delta+n2^{-n}\\ &\lesssim\delta\end{aligned}$$

for sufficiently large n , where we used

$$\big\|k_n\,|\cdot|^2\psi^\pm(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\lesssim n\big\|\psi^\pm(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$$

and

$$\big\|k_n\,|\cdot|^2\Delta\psi^\pm(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\lesssim n\big\|\psi^\pm(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}$$

in the second steps. Fatou's lemma implies

$$\big\||\cdot|^2\psi\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}\lesssim\delta$$

and

$$\big\||\cdot|^2\Delta\psi\big\|_{L_t^\infty L_x^2([0,T']\times\mathbb{R}^3,\mathbb{C}^4)}\lesssim\delta.$$

Since

$$\begin{aligned}\big\||\cdot|^2\psi(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} &\le \big\||\cdot|^2\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\||\cdot|^2\Delta\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad+\big\|\Delta|\cdot|^2\,\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|\nabla|\cdot|^2\cdot\nabla\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \big\||\cdot|^2\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\||\cdot|^2\Delta\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad+\big\|\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|x\psi(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)},\end{aligned}$$

we conclude (62). □

2.2 Weighted energy estimates

From the local theory (cf. estimates (53), (58) and (62)), it follows that $\|\psi\|_{X_{T'}}\le\tilde\delta$ implies

$$\|\psi\|_{X_{T'}}\lesssim\delta+\tilde\delta^3$$

and therefore,

$$\begin{aligned}&\|\psi(t)\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}+\|x\mathrm{w}^\pm(t)\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}+\|\langle x\rangle^2\psi(t)\|_{H_x^2(\mathbb{R}^3,\mathbb{C}^4)}+\|\langle\xi\rangle^8\widehat{\psi^\pm(t)}(\xi)\|_{L_\xi^\infty(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad\lesssim\delta+\tilde\delta^3\end{aligned}\tag{67}$$

whenever $t \le T'$. For (45), we need to show that provided a priori assumption (44),

$$\begin{aligned} \|\psi(t)\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim (\delta+\tilde{\delta}^3)\langle t\rangle^{\varepsilon}, \\ \|x\mathrm{w}^{\pm}(t)\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim (\delta+\tilde{\delta}^3)\langle t\rangle^{\varepsilon}, \\ \|\langle x\rangle^2\psi(t)\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim (\delta+\tilde{\delta}^3)\langle t\rangle^{2\varepsilon}, \\ \|\langle\xi\rangle^8\widehat{\psi^{\pm}(t)}(\xi)\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim (\delta+\tilde{\delta}^3) \end{aligned}$$

for any $t \in [0,T]$. Because of (67), it remains to prove that

$$\begin{aligned} \|\psi(t)\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim (\delta+\tilde{\delta}^3)t^{\varepsilon}, \\ \|x\mathrm{w}^{\pm}(t)\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim (\delta+\tilde{\delta}^3)t^{\varepsilon}, \\ \|\langle x\rangle^2\psi(t)\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim (\delta+\tilde{\delta}^3)t^{2\varepsilon}, \\ \|\langle\xi\rangle^8\widehat{\psi^{\pm}(t)}(\xi)\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim (\delta+\tilde{\delta}^3) \end{aligned}$$

hold true for

$$t \gtrsim 1. \tag{68}$$

Lemma 2.6 (Refined linear decay)**.** *Let $f \in H^N(\mathbb{R}^3,\mathbb{C}^4)$ and $t \in \mathbb{R}$. Then,*

$$\begin{aligned} \big\|e^{\mp it\langle D\rangle}f\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \langle t\rangle^{-3/2}\big\|\langle\xi\rangle^6\widehat{f}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad + \langle t\rangle^{-31/20}\Big[\big\|\langle x\rangle^2 f(x)\big\|_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)} + \big\|f\big\|_{H^{50}(\mathbb{R}^3,\mathbb{C}^4)}\Big]. \end{aligned} \tag{69}$$

In particular, for $\psi^{\pm}$ satisfying a priori assumption (44)*, we have*

$$\big\|\psi^{\pm}(t)\big\|_{W^{2,p}(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}\langle t\rangle^{-3/2+3/p} \tag{70}$$

for any $p \in [2,\infty]$.

Proof. For (69), we can carry over Pusateri's proof in [84, Section B.1]. Statement (70), which is also mentioned in Pusateri's paper without proof, can be seen as follows: By definition (40) of the profile $\mathrm{w}^{\pm}$ and estimate (69), we obtain

$$\begin{aligned} \big\|\psi^{\pm}(t)\big\|_{W^{2,\infty}(\mathbb{R}^3,\mathbb{C}^4)} &= \sum_{\alpha\in\mathbb{N}_0^3:\ |\alpha|\le 2} \big\|e^{\mp it\langle D\rangle}\partial^{\alpha}\mathrm{w}^{\pm}(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \sum_{\alpha\in\mathbb{N}_0^3:\ |\alpha|\le 2}\Bigg(\langle t\rangle^{-3/2}\big\|\langle\xi\rangle^6[\partial_x^{\alpha}\mathrm{w}^{\pm}(t)]\widehat{\ }(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + \langle t\rangle^{-31/20}\Big[\big\|\langle x\rangle^2\partial_x^{\alpha}\mathrm{w}^{\pm}(t,x)\big\|_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad\qquad + \big\|\partial_x^{\alpha}\mathrm{w}^{\pm}(t)\big\|_{H^{50}(\mathbb{R}^3,\mathbb{C}^4)}\Big]\Bigg) \\ &\lesssim \langle t\rangle^{-3/2}\big\|\langle\xi\rangle^8\widehat{\mathrm{w}^{\pm}(t)}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad + \langle t\rangle^{-31/20}\Big[\big\|\langle x\rangle^2\mathrm{w}^{\pm}(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} + \|\mathrm{w}^{\pm}(t)\|_{H^{52}(\mathbb{R}^3,\mathbb{C}^4)}\Big] \end{aligned} \tag{71}$$

and estimate (70) follows for $p = \infty$ by a priori assumption (44). For the case $p = 2$, note that $\|\langle\xi\rangle^{-2}\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C})} \lesssim 1$ and therefore,

$$\begin{aligned}\big\|\psi^\pm(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \big\|\langle\xi\rangle^4\widehat{\psi^\pm(t)}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta},\end{aligned}$$

where we used a priori assumption (44) in the second step. By interpolation, we get (70) for any $p \in [2,\infty]$. □

As a consequence, the nonlinearity of (30) is scattering critical in the sense that

$$\begin{aligned}\big\|\big(V * |\psi(t)|^2\big)\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &\le \big\|V * |\psi(t)|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \big\|V\big\|_{L^{3,\infty}(\mathbb{R}^3,\mathbb{C})}\big\||\psi(t)|^2\big\|_{L^{3/2,1}(\mathbb{R}^3,\mathbb{C})}\delta\\ &\lesssim \big\|\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t)\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\delta\\ &\lesssim \delta^2\tilde{\delta}\langle t\rangle^{-1}, \qquad (72)\end{aligned}$$

where we used Lorentz-type Young's inequality in the second step, interpolation in the third step and estimate (70) in the last step.

Lemma 2.7 (Estimates for $P_l\psi$)**.** *Let $\psi^\pm \in \mathcal{C}\big([0,T], H^N(\mathbb{R}^3,\mathbb{C}^4)\big)$ satisfy a priori assumption* (44)*. Then, for any $l \in \mathbb{Z}$ and $t \in [0,T]$, we have*

$$\big\|P_l|\psi(t)|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \lesssim 2^{3l}\tilde{\delta}^2, \qquad (73)$$

$$\big\|P_l|\psi(t)|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \lesssim 2^{-2l^+}\tilde{\delta}^2\langle t\rangle^{-3}, \qquad (74)$$

$$\big\|P_l|\psi(t)|^2\big\|_{L^2(\mathbb{R}^3,\mathbb{C})} \lesssim 2^{3l/2}2^{-9l^+/2}\tilde{\delta}^2, \qquad (75)$$

$$\big\|P_l\mathrm{w}^\pm(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} = \big\|P_l\psi^\pm(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \lesssim 2^{3l/2}2^{-8l^+}\tilde{\delta}. \qquad (76)$$

Proof. By Bernstein's and Hölder's inequality,

$$\big\|P_l|\psi(t)|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \lesssim 2^{3l}\|\psi(t)\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$$

and estimate (70) implies (73). On the other hand, Bernstein's inequality and decay estimate (70) also give

$$\begin{aligned}\big\|P_l|\psi(t)|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} &\lesssim 2^{-2l^+}\big\||\psi(t)|^2\big\|_{W^{2,\infty}(\mathbb{R}^3,\mathbb{C})}\\ &\lesssim 2^{-2l^+}\tilde{\delta}^2\langle t\rangle^{-3}.\end{aligned}$$

By Plancherel's identity,

$$\begin{aligned}\big\|P_l|\psi(t)|^2\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C})} &= (2\pi)^{-3}\int_{\mathbb{R}^3}\chi_l(\xi)^2\big|\widehat{\psi(t)} * \widehat{\overline{\psi}(t)}(\xi)\big|^2\,\mathrm{d}\xi\\ &\lesssim \int_{\mathbb{R}^3}\chi_l(\xi)^2\|\langle\cdot\rangle^8\widehat{\psi(t)}\|^4_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\Big(\int_{\mathbb{R}^3}\langle\eta\rangle^{-8}\langle\xi-\eta\rangle^{-8}\,\mathrm{d}\eta\Big)^2\,\mathrm{d}\xi\\ &\lesssim 2^{-9l^+}\tilde{\delta}^4\int_{\mathbb{R}^3}\chi_l(\xi)^2\Big(\int_{\mathbb{R}^3}\langle\xi\rangle^{9/2}\langle\eta\rangle^{-8}\langle\xi-\eta\rangle^{-8}\,\mathrm{d}\eta\Big)^2\,\mathrm{d}\xi,\end{aligned}$$

where we used a priori assumption (44) in the last step. For $|\xi| \geq 2|\eta|$, note that

$$\langle\xi\rangle^{9/2}\langle\eta\rangle^{-8}\langle\xi-\eta\rangle^{-8} \lesssim \langle\eta\rangle^{-8}.$$

Now, let $|\xi| \leq 2|\eta|$. Then,

$$\langle\xi\rangle^{9/2}\langle\eta\rangle^{-8}\langle\xi-\eta\rangle^{-8} \lesssim \langle\eta\rangle^{-7/2}.$$

Since $\int_{\mathbb{R}^3}\langle\eta\rangle^{-7/2}\,\mathrm{d}\eta \lesssim 1$, we conclude that

$$\begin{aligned}\big\|P_l|\psi(t)|^2\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C})} &\lesssim 2^{-9l^+}\tilde{\delta}^4\int_{\mathbb{R}^3}\chi_l(\xi)^2\,\mathrm{d}\xi\\ &\lesssim 2^{-9l^+}2^{-3l}\tilde{\delta}^4\end{aligned}$$

which implies (75). Finally, Plancherel's identity leads to

$$\begin{aligned}\big\|P_l\mathrm{w}^{\pm}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &= (2\pi)^{-3/2}\big\|\chi_l(\xi)\widehat{\mathrm{w}^{\pm}(t)}(\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim 2^{3l/2}2^{-8l^+}\big\|\langle\xi\rangle^8\widehat{\mathrm{w}^{\pm}(t)}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}.\end{aligned}$$

Since $\big|\widehat{\mathrm{w}^{\pm}(t)}(\xi)\big| = \big|\widehat{\psi^{\pm}(t)}(\xi)\big|$, we obtain (76) from a priori assumption (44). □

In the Fourier space, the Duhamel formula for the profile $\mathrm{w}^{\pm}$ reads

$$\begin{aligned}\widehat{\mathrm{w}^{\pm_0}(t)}(\xi) &= \widehat{\psi_0^{\pm_0}}(\xi) + ic\int_0^t e^{\pm_0 it'\langle\xi\rangle}\Pi^{\pm_0}(\xi)\big[\big(V*|\psi(t')|^2\big)\psi(t')\big]\widehat{}(\xi)\,\mathrm{d}t'\\ &= \widehat{\psi_0^{\pm_0}}(\xi) + ic\int_0^t I^{\pm_0}(t',\xi)\,\mathrm{d}\xi,\end{aligned}$$

where

$$\begin{aligned}I^{\pm_0}(t',\xi) &= e^{\pm_0 it'\langle\xi\rangle}\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\widehat{V}(\eta)\big[|\psi(t')|^2\big]\widehat{}(\eta)\widehat{\psi(t')}(\xi-\eta)\,\mathrm{d}\eta\\ &= 4\pi e^{\pm_0 it'\langle\xi\rangle}\Pi^{\pm_0}(\xi)\sum_{\pm_1,\pm_2,\pm_3\in\{+,-\}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\eta|^{-2}\Big\langle\widehat{\psi^{\pm_1}(t')}(\sigma),\widehat{\psi^{\pm_2}(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4}\\ &\qquad\cdot\widehat{\psi^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta.\end{aligned}$$

Let

$$\begin{aligned}I^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi) = \Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} & e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\pm_3}(\xi,\eta,\sigma)}|\eta|^{-2}\\ &\cdot\big\langle\widehat{\mathrm{w}^{\pm_1}(t')}(\sigma),\widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\big\rangle_{\mathbb{C}^4}\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta,\end{aligned}\tag{77}$$

where

$$\varphi^{\pm_0,\pm_1,\pm_2,\pm_3}(\xi,\eta,\sigma) = \pm_0\langle\xi\rangle \mp_1 \langle\sigma\rangle \pm_2 \langle\sigma-\eta\rangle \mp_3 \langle\xi-\eta\rangle.\tag{78}$$

Then,

$$I^{\pm_0}(t',\xi) = 4\pi\sum_{\pm_1,\pm_2,\pm_3\in\{+,-\}} I^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)$$

and

$$\widehat{\mathrm{w}^{\pm_0}(t)}(\xi) = \widehat{\psi_0^{\pm_0}}(\xi) + i4\pi c\sum_{\pm_1,\pm_2,\pm_3\in\{+,-\}}\int_0^t I^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t'.\tag{79}$$

2.2.1 Control of $\langle D\rangle^N\psi^\pm$

Lemma 2.8. *Let $\psi^\pm \in \mathcal{C}\big([0,T], H^N(\mathbb{R}^3,\mathbb{C}^4)\big)$ satisfy a priori assumption (44). Then,*

$$\big\|\psi^\pm(t)\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \delta + \langle t\rangle^\varepsilon \tilde{\delta}^3 \tag{80}$$

for any $t \in [0,T]$.

Proof. By Duhamel equation (52) and estimate (46), we have

$$\big\|\psi^\pm(t)\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \big\|\psi_0^\pm\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} + \int_0^t \big\|\big(V * |\psi(t')|^2\big)\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t'. \tag{81}$$

The initial data assumption (43) implies

$$\big\|\psi_0^\pm\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} \le \delta.$$

It remains to control the second term on the right hand side of (81). We have

$$\begin{aligned}\big\|\big(V * |\psi(t')|^2\big)\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \big\|V * |\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad + \big\|V * |\psi(t')|^2\big\|_{H^{N,6}(\mathbb{R}^3,\mathbb{C})}\big\|\psi(t')\big\|_{L^3(\mathbb{R}^3,\mathbb{C}^4)}.\end{aligned} \tag{82}$$

For the first term of (82), Lorentz-type Young's inequality and interpolation give

$$\begin{aligned}\big\|V * |\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} &\lesssim \big\|V\big\|_{L^{3,\infty}(\mathbb{R}^3,\mathbb{C})}\big\||\psi(t')|^2\big\|_{L^{3/2,1}(\mathbb{R}^3,\mathbb{C})}\\ &\lesssim \big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}^2\langle t'\rangle^{-1},\end{aligned}$$

where we used estimate (70) in the last step. For the second term in (82), weak Young's inequality leads to

$$\begin{aligned}\big\|V * |\psi(t')|^2\big\|_{H^{N,6}(\mathbb{R}^3,\mathbb{C})} &\lesssim \big\||\psi(t')|^2\big\|_{H^{N,6/5}(\mathbb{R}^3,\mathbb{C})}\\ &\lesssim \big\|\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^3(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}^2\langle t'\rangle^{\varepsilon-1/2},\end{aligned}$$

where we used a priori assumption (44) and estimate (70) in the last step. Together with (82) and estimates (44), (70), we obtain

$$\big\|\big(V * |\psi(t')|^2\big)\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}^3\langle t'\rangle^{\varepsilon-1}$$

so that the second term on the right hand side of (81) is bounded by $\tilde{\delta}^3\langle t\rangle^\varepsilon$. □

2.2.2 Preliminary estimates for controlling $x\mathrm{w}^\pm$ and $\langle x\rangle^2\mathrm{w}^\pm$

For estimating the weighted terms $\|x\mathrm{w}^\pm(t,x)\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)}$ and $\|\langle x\rangle^2\mathrm{w}^\pm(t,x)\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)}$, we need the following multiplier estimate which is proved is Pusateri's paper [84, Lemma B.1] for the $\mathbb{C}$-valued case. For the sake of completeness, we show the corresponding $\mathbb{C}^4$-valued version:

Lemma 2.9. *Let $m \in L^1(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{C}^{4\times 4})$ and $C > 0$ such that*

$$\Big\| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{iy\cdot\eta} m(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \Big\|_{L^1_{(x,y)}(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{C}^{4\times 4})} \le C \tag{83}$$

or let $m \in L^1(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{C})$ and $C > 0$ such that

$$\Big\| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{iy\cdot\eta} m(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \Big\|_{L^1_{(x,y)}(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{C})} \le C. \tag{84}$$

Then,

$$\begin{aligned} \Big| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Big\langle \widehat{g}(\eta) m(\xi,\eta) \widehat{f}(\xi), \widehat{h}(\xi+\eta) \Big\rangle_{\mathbb{C}^4} \mathrm{d}\xi\, \mathrm{d}\eta \Big| \\ \le C \|f\|_{L^p(\mathbb{R}^3,\mathbb{C}^4)} \|g\|_{L^q(\mathbb{R}^3,\mathbb{C})} \|h\|_{L^r(\mathbb{R}^3,\mathbb{C}^4)} \end{aligned} \tag{85}$$

for any $f \in L^p(\mathbb{R}^3, \mathbb{C}^4)$, $g \in L^q(\mathbb{R}^3, \mathbb{C})$, $h \in L^r(\mathbb{R}^3, \mathbb{C}^4)$ and $p, q, r \in [1, \infty]$ provided $1/p + 1/q + 1/r = 1$. Moreover,

$$\Big\| \int_{\mathbb{R}^3} \widehat{g}(\eta) m(\xi,\eta) \widehat{f}(\xi-\eta)\, \mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \le C \|f\|_{L^p(\mathbb{R}^3,\mathbb{C}^4)} \|g\|_{L^q(\mathbb{R}^3,\mathbb{C})} \tag{86}$$

for any $p, q \in [1, \infty]$ provided $1/p + 1/q = 1/2$.

Proof. We essentially follow the arguments in Pusateri's paper [84, Lemma B.1] for the scalar case. For $m \in L^1(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{C})$, we can define

$$\mathbb{R}^3 \times \mathbb{R}^3 \ni (\xi,\eta) \mapsto m(\xi,\eta) I_4 \in \mathbb{C}^{4\times 4}$$

and (84) immediately implies (83). So, it suffices to consider $m \in L^1(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{C}^{4\times 4})$ satisfying (83). We have

$$\begin{aligned} &\Big| \int_{\mathbb{R}^3\times\mathbb{R}^3} \Big\langle \widehat{g}(\eta) m(\xi,\eta) \widehat{f}(\xi), \widehat{h}(\xi+\eta) \Big\rangle_{\mathbb{C}^4} \mathrm{d}(\xi,\eta) \Big| \\ &= \Big| \int_{\mathbb{R}^3\times\mathbb{R}^3} \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \Big\langle e^{-ix\cdot\xi} e^{-iy\cdot\eta} e^{iz\cdot(\xi+\eta)} g(y) m(\xi,\eta) f(x), h(z) \Big\rangle_{\mathbb{C}^4} \mathrm{d}(x,y,z)\, \mathrm{d}(\xi,\eta) \Big| \\ &= \Big| \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \Big\langle \int_{\mathbb{R}^3\times\mathbb{R}^3} e^{i(z-x)\cdot\xi} e^{i(z-y)\cdot\eta} m(\xi,\eta)\, \mathrm{d}(\xi,\eta)\ g(y) f(x), h(z) \Big\rangle_{\mathbb{C}^4} \mathrm{d}(x,y,z) \Big| \end{aligned}$$

and assumption (83) leads to

$$\begin{aligned} &\Big| \int_{\mathbb{R}^3\times\mathbb{R}^3} \Big\langle \widehat{g}(\eta) m(\xi,\eta) \widehat{f}(\xi), \widehat{h}(\xi+\eta) \Big\rangle_{\mathbb{C}^4} \mathrm{d}(\xi,\eta) \Big| \\ &\quad \le C \operatorname*{ess\,sup}_{x,y\in\mathbb{R}^3} \int_{\mathbb{R}^3} \big| g(z-y) f(z-x) \big| \big| h(z) \big|\, \mathrm{d}z \\ &\quad \le C \|f\|_{L^p(\mathbb{R}^3,\mathbb{C}^4)} \|g\|_{L^q(\mathbb{R}^3,\mathbb{C})} \|h\|_{L^r(\mathbb{R}^3,\mathbb{C}^4)}. \end{aligned}$$

By duality and Plancherel's identity,

$$\begin{aligned} &\Big\| \int_{\mathbb{R}^3} \widehat{g}(\eta) m(\xi,\eta) \widehat{f}(\xi-\eta)\, \mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\ &= \sup_{\|\widehat{h}\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \le 1} \Big| \int_{\mathbb{R}^3} \Big\langle \int_{\mathbb{R}^3} \widehat{g}(\eta) m(\xi,\eta) \widehat{f}(\xi-\eta)\, \mathrm{d}\eta, \widehat{h}(\xi) \Big\rangle_{\mathbb{C}^4} \mathrm{d}\xi \Big| \\ &= \sup_{\|h\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \le (2\pi)^{-3/2}} \Big| \int_{\mathbb{R}^3\times\mathbb{R}^3} \Big\langle \widehat{g}(\eta) \tilde{m}(\xi,\eta) \widehat{f}(\xi), \widehat{h}(\xi+\eta) \Big\rangle_{\mathbb{C}^4} \mathrm{d}(\xi,\eta) \Big|, \end{aligned} \tag{87}$$

where

$$\tilde{m}(\xi,\eta) = m(\xi+\eta,\eta).$$

Since

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}^3\times\mathbb{R}^3} e^{ix\cdot\xi} e^{iy\cdot\eta} \tilde{m}(\xi,\eta)\,\mathrm{d}(\xi,\eta)\Big\|_{L^1_{(x,y)}(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{C}^{4\times4})} \\
&\quad = \Big\| \int_{\mathbb{R}^3\times\mathbb{R}^3} e^{ix\cdot(\xi-\eta)} e^{iy\cdot\eta} m(\xi,\eta)\,\mathrm{d}(\xi,\eta)\Big\|_{L^1_{(x,y)}(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{C}^{4\times4})} \\
&\quad = \Big\| \int_{\mathbb{R}^3\times\mathbb{R}^3} e^{ix\cdot\xi} e^{i(y-x)\cdot\eta} m(\xi,\eta)\,\mathrm{d}(\xi,\eta)\Big\|_{L^1_{(x,y)}(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{C}^{4\times4})} \\
&\quad = \Big\| \int_{\mathbb{R}^3\times\mathbb{R}^3} e^{ix\cdot\xi} e^{iy\cdot\eta} m(\xi,\eta)\,\mathrm{d}(\xi,\eta)\Big\|_{L^1_{(x,y)}(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{C}^{4\times4})} \\
&\quad \le C,
\end{aligned}$$

estimate (86) follows from (87) and (85). □

Lemma 2.10. *The assumption* (83) *of Lemma 2.9 is satisfied with* $C \lesssim \tilde{C}$ *whenever* $m \in \mathcal{C}^4\big((\mathbb{R}^3\setminus\{0\})\times(\mathbb{R}^3\setminus\{0\}),\mathbb{C}^{4\times4}\big)$ *fulfills*

$$\begin{aligned}
&\big|\partial_\xi^\alpha\partial_\eta^\beta m(\xi,\eta)\big| \\
&\qquad \le \tilde{C}2^{-k|\alpha|}2^{-l|\beta|}\tilde{\chi}_k(\xi)\tilde{\chi}_l(\eta) \quad \forall\, \xi,\eta\in\mathbb{R}^3\setminus\{0\},\ \alpha,\beta\in\mathbb{N}_0^3,\ |\alpha|,|\beta|\le 4
\end{aligned} \tag{88}$$

for some $k,l\in\mathbb{Z}$. *Analogously for the assumption* (84) *of Lemma 2.9.*

Proof. Let $n_1,n_2\in\{2,4\}$. Integration by parts leads to

$$\Big|\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{ix\cdot\xi}e^{iy\cdot\eta}m(\xi,\eta)\,\mathrm{d}(\xi,\eta)\Big| \lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3}|x|^{-n_1}|y|^{-n_2}\max_{|\alpha|=n_1,|\beta|=n_2}\big|\partial_\xi^\alpha\partial_\eta^\beta m(\xi,\eta)\big|\,\mathrm{d}(\xi,\eta)$$

and from (88), we obtain

$$\begin{aligned}
\Big|\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{ix\cdot\xi}e^{iy\cdot\eta}m(\xi,\eta)\,\mathrm{d}(\xi,\eta)\Big| &\lesssim \tilde{C}|x|^{-n_1}|y|^{-n_2}2^{-n_1k}2^{-n_2l}\int_{\mathbb{R}^3\times\mathbb{R}^3}\tilde{\chi}_k(\xi)\tilde{\chi}_l(\eta)\,\mathrm{d}(\xi,\eta) \\
&\lesssim \tilde{C}2^{(3-n_1)k}2^{(3-n_2)l}|x|^{-n_1}|y|^{-n_2}.
\end{aligned} \tag{89}$$

We first consider the $L^1_{(x,y)}$-norm on $B_{2^{-k}}(0)\times B_{2^{-l}}(0)$. Estimate (89) with $n_1=n_2=2$ gives

$$\begin{aligned}
&\int_{B_{2^{-k}}(0)\times B_{2^{-l}}(0)}\Big|\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{ix\cdot\xi}e^{iy\cdot\eta}m(\xi,\eta)\,\mathrm{d}(\xi,\eta)\Big|\,\mathrm{d}(x,y) \\
&\qquad \lesssim \tilde{C}2^k2^l\int_{B_{2^{-k}}(0)\times B_{2^{-l}}(0)}|x|^{-2}|y|^{-2}\,\mathrm{d}(x,y) \\
&\qquad \lesssim \tilde{C}.
\end{aligned} \tag{90}$$

For the $L^1_{(x,y)}$-norm on $B_{2^{-k}}(0)\times B_{2^{-l}}(0)^c$, we use estimate (89) with $n_1=2$ and $n_2=4$ to get

$$\begin{aligned}
&\int_{B_{2^{-k}}(0)\times B_{2^{-l}}(0)^c}\Big|\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{ix\cdot\xi}e^{iy\cdot\eta}m(\xi,\eta)\,\mathrm{d}(\xi,\eta)\Big|\,\mathrm{d}(x,y) \\
&\qquad \lesssim \tilde{C}2^k2^{-l}\int_{B_{2^{-k}}(0)\times B_{2^{-l}}(0)^c}|x|^{-2}|y|^{-4}\,\mathrm{d}(x,y) \\
&\qquad \lesssim \tilde{C}.
\end{aligned} \tag{91}$$

Analogously, estimate (89) with $n_1 = 4$ and $n_2 = 2$ leads to

$$\int_{B_{2^{-k}}(0)^c \times B_{2^{-l}}(0)} \Big| \int_{\mathbb{R}^3\times\mathbb{R}^3} e^{ix\cdot\xi} e^{iy\cdot\eta} m(\xi,\eta)\,\mathrm{d}(\xi,\eta) \Big| \chi_{k'}(x)\chi_{l'}(y)\,\mathrm{d}(x,y) \lesssim \tilde{C} \tag{92}$$

and estimate (89) with $n_1 = n_2 = 4$ yields

$$\int_{B_{2^{-k}}(0)^c \times B_{2^{-l}}(0)^c} \Big| \int_{\mathbb{R}^3\times\mathbb{R}^3} e^{ix\cdot\xi} e^{iy\cdot\eta} m(\xi,\eta)\,\mathrm{d}(\xi,\eta) \Big| \chi_{k'}(x)\chi_{l'}(y)\,\mathrm{d}(x,y) \lesssim \tilde{C}. \tag{93}$$

By (90)–(93), we can conclude estimate (88). □

Lemma 2.11. *Assume that* $\psi^\pm \in \mathcal{C}\big([0,T], H^N(\mathbb{R}^3,\mathbb{C}^4)\big)$ *satisfies a priori assumption* (44) *and* $m_{l,l_{12}} \in L^1(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{C}^{4\times4})$, $m_{l,l_{12}} \in L^1(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{C})$ *fulfill* (83), (84), *respectively, with* $C \lesssim 2^{-nl_{12}}2^{-l^+}$, $l, l_{12} \in \mathbb{Z}$, $n \in \{0,1\}$ *and*

$$\big|m_{l,l_{12}}(\xi,\eta)\big| \lesssim 2^{-nl_{12}}2^{-l^+}\chi_l(\xi)\chi_{l_{12}}(\eta). \tag{94}$$

Let

$$I(t,\xi) = e^{i\phi(t,\xi)} \sum_{l,l_{12},l_3\in\mathbb{Z}} \int_{\mathbb{R}^3} \chi_l(\xi)\big[P_{l_{12}}|\psi(t)|^2\big]^\wedge(\eta) m_{l,l_{12}}(\xi,\eta)\big[P_{l_3}\psi^\pm(t)\big]^\wedge(\xi-\eta)\,\mathrm{d}\eta$$

for some real-valued phase function ϕ. *Then,*

$$\big\|\langle\xi\rangle^2 I(t,\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}^3\langle t\rangle^{\varepsilon+n-3}.$$

Proof. Denote

$$I_{l,l_{12},l_3} = e^{i\phi(t,\xi)} \int_{\mathbb{R}^3} \chi_l(\xi)\big[P_{l_{12}}|\psi(t)|^2\big]^\wedge(\eta) m_{l,l_{12}}(\xi,\eta)\big[P_{l_3}\psi^\pm(t)\big]^\wedge(\xi-\eta)\,\mathrm{d}\eta.$$

From (86), we obtain

$$\begin{aligned}\big\|\langle\xi\rangle^2 I_{l,l_{12},l_3}(t,\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim 2^{l^+}2^{-nl_{12}}\big\|P_{l_{12}}|\psi(t)|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|P_{l_3}\psi^\pm(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \tilde{\delta}^3 2^{l^+}2^{-nl_{12}}\min\big\{2^{3l_{12}}, 2^{-2l_{12}^+}\langle t\rangle^{-3}\big\}2^{3l_3/2}2^{-8l_3^+},\end{aligned} \tag{95}$$

where we used estimates (73), (74) and (76) in the second step. On the other hand, estimate (94), Hölder's inequality and Hausdorff–Young's inequality yield

$$\begin{aligned}&\big\|\langle\xi\rangle^2 I_{l,l_{12},l_3}(t,\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad\lesssim 2^{l^+}2^{-nl_{12}}\big\|\chi_l\big\|_{L^2(\mathbb{R}^3,\mathbb{C})}\big\|P_{l_{12}}|\psi(t)|^2\big\|_{L^2(\mathbb{R}^3,\mathbb{C})}\big\|P_{l_3}\psi^\pm(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad\lesssim \tilde{\delta}^3 2^{l^+}2^{3l/2}2^{(3/2-n)l_{12}}2^{-9l_{12}^+/2}2^{3l_3/2}2^{-8l_3^+},\end{aligned} \tag{96}$$

where we used estimates (75) and (76) in the second step.

In order to get nonzero contributions, at least one of the following four cases must hold:

a) $l \sim l_3 \succ l_{12}$,
b) $l \ll l_3 \sim l_{12},\ 2^l \le \langle t\rangle^{-2}$,
c) $l \ll l_3 \sim l_{12},\ 2^l \ge \langle t\rangle^{-2}$,
d) $l_{12} \sim l \gg l_3$.

a) Estimate (95) gives

$$\begin{aligned}\sum_{l\sim l_3\succ l_{12}}\big\|\langle\xi\rangle^2 I_{l,l_{12},l_3}(t,\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \tilde\delta^3\sum_{l\sim l_3\succ l_{12}} 2^{-nl_{12}}\min\big\{2^{3l_{12}},2^{-2l_{12}^+}\langle t\rangle^{-3}\big\}2^{3l/2}2^{-7l^+}\\ &\lesssim \tilde\delta^3\Big(\sum_{2^{l_{12}}\le\langle t\rangle^{-1}}2^{(3-n)l_{12}}+\sum_{2^{l_{12}}\ge\langle t\rangle^{-1}}2^{-nl_{12}}2^{-2l_{12}^+}\langle t\rangle^{-3}\Big)\\ &\lesssim \tilde\delta^3\langle t\rangle^{\varepsilon+n-3}.\end{aligned}$$

b) By estimate (96), we have

$$\begin{aligned}&\sum_{l\ll l_3\sim l_{12},\,2^l\le\langle t\rangle^{-2}}\big\|\langle\xi\rangle^2 I_{l,l_{12},l_3}(t,\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\qquad\lesssim \tilde\delta^3\sum_{l\ll l_3\sim l_{12},\,2^l\le\langle t\rangle^{-2}}2^{3l/2}2^{(3/2-n)l_{12}}2^{-9l_{12}^+/2}2^{3l_3/2}2^{-8l_3^+}\\ &\qquad\lesssim \tilde\delta^3\sum_{2^l\le\langle t\rangle^{-2}}2^{3l/2}\\ &\qquad\lesssim \tilde\delta^3\langle t\rangle^{-3}.\end{aligned}$$

c) Estimate (95) implies

$$\begin{aligned}&\sum_{l\ll l_3\sim l_{12},\,2^l\ge\langle t\rangle^{-2}}\big\|\langle\xi\rangle^2 I_{l,l_{12},l_3}(t,\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde\delta^3\sum_{l\ll l_3\sim l_{12},\,2^l\ge\langle t\rangle^{-2}}2^{l^+}2^{-nl_{12}}\min\big\{2^{3l_{12}},2^{-2l_{12}^+}\langle t\rangle^{-3}\big\}2^{3l_3/2}2^{-8l_3^+}\\ &\lesssim \tilde\delta^3\sum_{2^{l_{12}}\le\langle t\rangle^{-1}}2^{(3-n)l_{12}}\sum_{\langle t\rangle^{-2}\le 2^l\le\langle t\rangle^{-1}}1+\tilde\delta^3\sum_{2^{l_{12}}\ge\langle t\rangle^{-1}}2^{-nl_{12}}2^{-l_{12}^+}\langle t\rangle^{-3}\sum_{2^l\ge\langle t\rangle^{-2}}2^{-l^+}\\ &\lesssim \tilde\delta^3\langle t\rangle^{\varepsilon+n-3}.\end{aligned}$$

d) By estimate (95), we have

$$\begin{aligned}&\sum_{l_{12}\sim l\gg l_3}\big\|\langle\xi\rangle^2 I_{l,l_{12},l_3}(t,\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\qquad\lesssim \tilde\delta^3\sum_{l_{12}\sim l\gg l_3}2^{l^+}2^{-nl}\min\big\{2^{3l},2^{-2l^+}\langle t\rangle^{-3}\big\}2^{3l_3/2}2^{-8l_3^+}\\ &\qquad\lesssim \tilde\delta^3\Big(\sum_{2^l\le\langle t\rangle^{-1}}2^{(3-n)l}+\sum_{2^l\ge\langle t\rangle^{-1}}2^{-nl}2^{-l^+}\langle t\rangle^{-3}\Big)\sum_{l_3}2^{3l_3/2}2^{-8l_3^+}\\ &\qquad\lesssim \tilde\delta^3\langle t\rangle^{\varepsilon+n-3}.\end{aligned}$$

□

Lemma 2.12 (Estimate for $\pm_0=\mp_3$). *Let $\psi^\pm\in\mathcal{C}\big([0,T],H^N(\mathbb{R}^3,\mathbb{C}^4)\big)$ satisfy a priori assumption* (44) *and $m_{l,l_{12}}\in L^1(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{C}^{4\times4})$, $m_{l,l_{12}}\in L^1(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{C})$ fulfill* (83), (84), *respectively, with $C\lesssim 2^{-n_0 l_{12}}$, $l,l_{12}\in\mathbb{Z}$, $n_0\in\{1,2\}$, and*

$$\big|m_{l,l_{12}}(\xi,\eta)\big|\lesssim 2^{-n_0 l_{12}}\tilde\chi_l(\xi)\tilde\chi_{l_{12}}(\eta). \tag{97}$$

Assume that

$$\begin{aligned} &I^{\pm_0,\pm_1,\pm_2}(t',\xi) \\ &\quad = \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2}} \sum_{l_{12}\in-\mathbb{N}} \sum_{l_3\in\mathbb{Z}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (t')^n e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)} \chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_3}(\xi-\eta)\chi_{l_1}(\sigma) \\ &\qquad \cdot \left\langle \widehat{\mathrm{w}^{\pm_1}(t')}(\sigma), \widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\right\rangle_{\mathbb{C}^4} m_{l,l_{12}}(\xi,\eta)\widehat{\mathrm{w}^{\mp_0}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta \end{aligned}$$

for some $n\in\mathbb{N}$, $\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}$ *from* (78) *and*

$$\mathbb{I}^{\pm_1,\pm_2} = \begin{cases} \mathbb{Z}^2 & \text{if} \quad \pm_1=\pm_2, \\ \{(l,l_1)\in\mathbb{Z}^2:\ l\geq 4,\ l_1\not\sim l\} & \text{if} \quad \pm_1=\mp_2. \end{cases}$$

Then,

$$\left\|\langle\xi\rangle^2\int_0^t I^{\pm_0,\pm_1,\pm_2}(t',\xi)\,\mathrm{d}t'\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim n\tilde{\delta}^3\langle t\rangle^{\varepsilon+n-3+n_0}.$$

Proof. Since

$$e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)} = \partial_{t'}\frac{e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)}}{i\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)},$$

integration by parts in t' leads to

$$\int_0^t I(t',\xi)\,\mathrm{d}t' = \sum_{j=1}^5 \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2}} \sum_{l_{12}\in-\mathbb{N}} \sum_{l_3\in\mathbb{Z}} I_{j;l,l_1,l_{12},l_3}(t,\xi),$$

where

$$\begin{aligned} I_{1;l,l_1,l_{12},l_3}(t,\xi) &= t^n \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{it\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)}}{i\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)} \chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_1}(\sigma)\chi_{l_3}(\xi-\eta) \\ &\quad \cdot \left\langle \widehat{\mathrm{w}^{\pm_1}(t)}(\sigma), \widehat{\mathrm{w}^{\pm_2}(t)}(\sigma-\eta)\right\rangle_{\mathbb{C}^4} m_{l,l_{12}}(\xi,\eta)\widehat{\mathrm{w}^{\mp_0}(t)}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta, \\ I_{2;l,l_1,l_{12},l_3}(t,\xi) &= \int_0^t n(t')^{n-1} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)}}{i\varphi^{\pm_0,\pm_1,\pm_1,\mp_0}(\xi,\eta,\sigma)} \chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_1}(\sigma)\chi_{l_3}(\xi-\eta) \\ &\quad \cdot \left\langle \widehat{\mathrm{w}^{\pm_1}(t')}(\sigma), \widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\right\rangle_{\mathbb{C}^4} m_{l,l_{12}}(\xi,\eta)\widehat{\mathrm{w}^{\mp_0}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}t', \\ I_{3;l,l_1,l_{12},l_3}(t,\xi) &= \int_0^t (t')^n \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)}}{i\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)} \chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_1}(\sigma)\chi_{l_3}(\xi-\eta) \\ &\quad \cdot \left\langle \partial_{t'}\widehat{\mathrm{w}^{\pm_1}(t')}(\sigma), \widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\right\rangle_{\mathbb{C}^4} m_{l,l_{12}}(\xi,\eta)\widehat{\mathrm{w}^{\mp_0}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}t', \\ I_{4;l,l_1,l_{12},l_3}(t,\xi) &= \int_0^t (t')^n \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)}}{i\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)} \chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_1}(\sigma)\chi_{l_3}(\xi-\eta) \\ &\quad \cdot \left\langle \widehat{\mathrm{w}^{\pm_1}(t')}(\sigma), \partial_{t'}\widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\right\rangle_{\mathbb{C}^4} m_{l,l_{12}}(\xi,\eta)\widehat{\mathrm{w}^{\mp_0}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}t', \\ I_{5;l,l_1,l_{12},l_3}(t,\xi) &= \int_0^t (t')^n \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)}}{i\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)} \chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_1}(\sigma)\chi_{l_3}(\xi-\eta) \\ &\quad \cdot \left\langle \widehat{\mathrm{w}^{\pm_1}(t')}(\sigma), \widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\right\rangle_{\mathbb{C}^4} m_{l,l_{12}}(\xi,\eta)\partial_{t'}\widehat{\mathrm{w}^{\mp_0}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}t'. \end{aligned}$$

Consider $\xi, \eta, \sigma \in \mathbb{R}^3$ with $\chi_l(\xi), \chi_{l_{12}}(\eta), \chi_{l_1}(\sigma) \neq 0$ for $l_{12} \in -\mathbb{N}$ and $(l, l_1) \in \mathbb{I}^{\pm_1,\pm_2}$, i.e. in particular

$$|\eta| \leq 2^{l_{12}+1} \leq 1 \tag{98}$$

and additionally for the case $\pm_1 = \mp_2$, we also have

$$|\xi| \geq 2^{l-1} \geq 8, \tag{99}$$

$$|\sigma| \leq 2^{l_1+1} \leq 2^{l-3} \leq \frac{|\xi|}{2} \quad \text{if} \quad l_1 \ll l, \tag{100}$$

$$|\xi| \leq \frac{\sigma}{2} \quad \text{if} \quad l_1 \gg l. \tag{101}$$

For $\pm_1 = \pm_2$, we observe that

$$\begin{aligned} \left|\varphi^{\pm_0,\pm_1,\pm_1,\mp_0}(\xi,\eta,\sigma)\right| &\geq \langle\xi\rangle + \langle\xi-\eta\rangle - \left|\langle\sigma\rangle - \langle\sigma-\eta\rangle\right| \\ &\geq \langle\xi\rangle + \langle\xi-\eta\rangle - |\eta| \\ &\geq \langle\xi\rangle, \end{aligned}$$

where we used estimate (98) in the last step. For $\pm_1 = \mp_2$ and $l_1 \ll l$, note that

$$\begin{aligned} \left|\varphi^{\pm_0,\pm_1,\mp_1,\mp_0}(\xi,\eta,\sigma)\right| &\geq \langle\xi\rangle + \langle\xi-\eta\rangle - \langle\sigma\rangle - \langle\sigma-\eta\rangle \\ &\geq 2|\xi| - 2|\eta| - 2|\sigma| - 2 \\ &\geq |\xi| - 4 \\ &\geq \frac{|\xi|}{2} \\ &\gtrsim \langle\xi\rangle, \end{aligned}$$

where we used (98), (100) in the third step and (99) in the last two steps. In the $l_1 \gg l$-case, we get analogously

$$\begin{aligned} \left|\varphi^{\pm_0,\pm_1,\mp_1,\mp_0}(\xi,\eta,\sigma)\right| &\geq \langle\sigma\rangle + \langle\sigma-\eta\rangle - \langle\xi\rangle - \langle\xi-\eta\rangle \\ &\geq |\sigma| - 4 \\ &\geq \frac{31|\sigma|}{32} \\ &\gtrsim \langle\sigma\rangle \\ &\gtrsim \langle\xi\rangle, \end{aligned}$$

where we used (98), (101) in the second step and (99), (101) in the last three steps. Hence, for any of the cases mentioned above, we have

$$\left|\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)\right| \gtrsim \langle\xi\rangle. \tag{102}$$

From Bernstein's inequality, Hölder's inequality, a priori assumption (44) and estimate (70), we obtain

$$\begin{aligned} \left\|P_{l_{12}}\left\langle P_{l_1}\psi^{\pm_1}(t), \psi^{\pm_2}(t)\right\rangle_{\mathbb{C}^4}\right\|_{L^2(\mathbb{R}^3,\mathbb{C})} &\lesssim 2^{(3/p-3/2)l_{12}}\left\|\left\langle P_{l_1}\psi^{\pm_1}(t), \psi^{\pm_2}(t)\right\rangle_{\mathbb{C}^4}\right\|_{L^p(\mathbb{R}^3,\mathbb{C})} \\ &\lesssim 2^{(3/p-3/2)l_{12}}\left\|P_{l_1}\psi^{\pm_1}(t)\right\|_{L^q(\mathbb{R}^3,\mathbb{C}^4)}\left\|\psi^{\pm_2}(t)\right\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim 2^{(3/p-3/2)l_{12}}2^{(3/2-3/q)l_1}2^{-l_1^+}\left\|P_{l_1}\psi^{\pm_1}(t)\right\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad \cdot \left\|\psi^{\pm_2}(t)\right\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim 2^{(3/2-\varepsilon/4)l_{12}}2^{\varepsilon l_1/4}2^{-l_1^+}\tilde{\delta}^2, \end{aligned} \tag{103}$$

where we set $p = 3/(3-\varepsilon/4)$ and $1/q = 1/p - 1/2$ in the last step. Similarly,

$$\begin{aligned}\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t), \psi^{\pm_2}(t)\big\rangle_{\mathbb{C}^4}\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} &\lesssim 2^{3l_{12}/p}\big\|\big\langle P_{l_1}\psi^{\pm_1}(t), \psi^{\pm_2}(t)\big\rangle_{\mathbb{C}^4}\big\|_{L^p(\mathbb{R}^3,\mathbb{C})}\\ &\lesssim 2^{3l_{12}/p}\big\|P_{l_1}\psi^{\pm_1}(t)\big\|_{L^q(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi^{\pm_2}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim 2^{3l_{12}/p}2^{(3/2-3/q)l_1}2^{-l_1^+}\big\|P_{l_1}\psi^{\pm_1}(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\ &\qquad\cdot\big\|\psi^{\pm_2}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim 2^{(3-\varepsilon/4)l_{12}}2^{\varepsilon l_1/4}2^{-l_1^+}\tilde{\delta}^2 \end{aligned} \tag{104}$$

by choosing again $p = 3/(3-\varepsilon/4)$ and $1/q = 1/p - 1/2$. Furthermore,

$$\begin{aligned}&\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t), \psi^{\pm_2}(t)\big\rangle_{\mathbb{C}^4}\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\\ &\quad\lesssim 2^{(\varepsilon/2-2)l_{12}^+}\big\|\langle D\rangle^{2-\varepsilon/2}\big\langle P_{l_1}\psi^{\pm_1}(t), \psi^{\pm_2}(t)\big\rangle_{\mathbb{C}^4}\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\\ &\quad\lesssim 2^{(\varepsilon/2-2)l_{12}^+}2^{-\varepsilon l_1^+/2}\big\|\langle D\rangle^2P_{l_1}\psi^{\pm_1}(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\big\|\langle D\rangle^{2-\varepsilon/2}\psi^{\pm_2}(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad\lesssim 2^{(\varepsilon/2-2)l_{12}^+}2^{-\varepsilon l_1^+/2}2^{3l_1/p}\big\|\langle D\rangle^2P_{l_1}\psi^{\pm_1}(t)\big\|_{L^p(\mathbb{R}^3,\mathbb{C}^4)}\big\|\langle D\rangle^{2-\varepsilon/2}\psi^{\pm_2}(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad\lesssim 2^{(\varepsilon/2-2)l_{12}^+}2^{-\varepsilon l_1^+/2}2^{\varepsilon l_1/4}\langle t\rangle^{\varepsilon/4-3}\tilde{\delta}^2, \end{aligned} \tag{105}$$

where we choose $p = 12/\varepsilon$ in the last step.

In order to get nonzero contributions, at least one of the following four cases must be fulfilled:

a) $l \sim l_3 \succ l_{12}$,
b) $l \ll l_3 \sim l_{12},\ 2^l \leq \langle t\rangle^{-2}$,
c) $l \ll l_3 \sim l_{12},\ 2^l \geq \langle t\rangle^{-2}$,
d) $l_{12} \sim l \gg l_3$.

Estimate for $I_{1;l,l_1,l_{12},l_3}$: Denote

$$\begin{aligned}\langle\xi\rangle^2 I_{1;l,l_1,l_{12},l_3}(t,\xi) = t^n\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}&\frac{\langle\xi\rangle^2 e^{\pm_0 it\langle\xi\rangle}}{i\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)}\chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_1}(\sigma)\chi_{l_3}(\xi-\eta)\\ &\cdot\Big\langle\widehat{\psi^{\pm_1}(t)}(\sigma), \widehat{\psi^{\pm_2}(t)}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} m_{l,l_{12}}(\xi,\eta)\widehat{\psi^{\mp_0}(t)}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta.\end{aligned}$$

From estimates (97), (102), Hölder's inequality and Hausdorff–Young's inequality,

$$\begin{aligned}\big\|\langle\xi\rangle^2 I_{1;l,l_1,l_{12},l_3}(t,\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim t^n 2^{l^+}2^{-n_0 l_{12}}\big\|\chi_l\big\|_{L^2(\mathbb{R}^3,\mathbb{C})}\\ &\quad\cdot\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t), \psi^{\pm_2}(t)\big\rangle_{\mathbb{C}^4}\big\|_{L^2(\mathbb{R}^3,\mathbb{C})}\big\|P_{l_3}\psi^{\mp_0}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim t^n 2^{l^+}2^{-n_0 l_{12}}2^{3l/2}\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t), \psi^{\pm_2}(t)\big\rangle_{\mathbb{C}^4}\big\|_{L^2(\mathbb{R}^3,\mathbb{C})}\\ &\quad\cdot\big\|P_{l_3}\psi^{\mp_0}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}. \end{aligned} \tag{106}$$

Furthermore, estimates (97), (102) and Lemma 2.9 give

$$\begin{aligned}\big\|\langle\xi\rangle^2 I_{1;l,l_1,l_{12},l_3}(t,\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim t^n 2^{l^+}2^{-n_0 l_{12}}\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t), \psi^{\pm_2}(t)\big\rangle_{\mathbb{C}^4}\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\\ &\quad\cdot\big\|P_{l_3}\psi^{\mp_0}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}. \end{aligned} \tag{107}$$

a) From (107), (104), (105) and (76), we conclude

$$
\begin{aligned}
&\sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\sim l_3\succ l_{12}} \left\| \langle\xi\rangle^2 I_{1,l,l_1,l_{12},l_3}(t,\xi)\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \tilde\delta^3 t^n \sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\sim l_3\succ l_{12}} 2^{l^+}2^{-n_0 l_{12}}2^{\varepsilon l_1/4}2^{-\varepsilon l_1^+/2} \\
&\qquad\qquad \cdot \min\left\{2^{(3-\varepsilon/4)l_{12}}, \langle t\rangle^{\varepsilon/4-3}\right\}2^{3l_3/2}2^{-8l_3^+} \\
&\quad\lesssim \tilde\delta^3 t^n \sum_{l_1\in\mathbb{Z}} 2^{\varepsilon l_1/4}2^{-\varepsilon l_1^+/2}\Big(\sum_{2^{l_{12}}\le\langle t\rangle^{-1}} 2^{(3-n_0-\varepsilon/4)l_{12}} + \sum_{2^{l_{12}}\ge\langle t\rangle^{-1}} 2^{-n_0 l_{12}}\langle t\rangle^{\varepsilon/4-3}\Big) \\
&\qquad\qquad \cdot \sum_{l\in\mathbb{Z}} 2^{3l/2}2^{-7l^+} \\
&\quad\lesssim \tilde\delta^3 t^{n-3+n_0-1+\varepsilon/4}.
\end{aligned}
$$

b) Estimates (106), (103) and (76) lead to

$$
\begin{aligned}
&\sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\ll l_3\sim l_{12},\, 2^l\le\langle t\rangle^{-2}} \left\| \langle\xi\rangle^2 I_{1,l,l_1,l_{12},l_3}(t,\xi)\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \tilde\delta^3 t^n \sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\ll l_3\sim l_{12},\, 2^l\le\langle t\rangle^{-2}} 2^{l^+}2^{(3/2-n_0-\varepsilon/4)l_{12}}2^{3l/2}2^{-l_1^+}2^{\varepsilon l_1/4}2^{3l_3/2}2^{-8l_3^+} \\
&\quad\lesssim \tilde\delta^3 t^n \sum_{l_1\in\mathbb{Z}} 2^{-l_1^+}2^{\varepsilon l_1/4} \sum_{l_{12}\in-\mathbb{N}} 2^{(3-n_0-\varepsilon/4)l_{12}} \sum_{2^l\le\langle t\rangle^{-2}} 2^{3l/2} \\
&\quad\lesssim \tilde\delta^3 t^{n-3}.
\end{aligned}
$$

c) By estimates (107), (104), (105) and (76),

$$
\begin{aligned}
&\sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\ll l_3\sim l_{12},\, 2^l\ge\langle t\rangle^{-2}} \left\| \langle\xi\rangle^2 I_{1,l,l_1,l_{12},l_3}(t,\xi)\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \tilde\delta^3 t^n \sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\ll l_3\sim l_{12},\, 2^l\ge\langle t\rangle^{-2}} 2^{l^+}2^{-n_0 l_{12}}2^{\varepsilon l_1/4}2^{-\varepsilon l_1^+/2} \\
&\qquad\qquad \cdot \min\left\{2^{(3-\varepsilon/4)l_{12}}, \langle t\rangle^{\varepsilon/4-3}\right\}2^{3l_3/2}2^{-8l_3^+} \\
&\quad\lesssim \tilde\delta^3 t^n \sum_{2^{l_{12}}\le\langle t\rangle^{-1}} 2^{(3-n_0-\varepsilon/4)l_{12}} \sum_{\langle t\rangle^{-2}\le 2^l\le\langle t\rangle^{-1}} 1 + \tilde\delta^3 t^n \sum_{2^{l_{12}}\ge\langle t\rangle^{-1}} 2^{-n_0 l_{12}}\langle t\rangle^{\varepsilon/4-3} \sum_{2^l\ge\langle t\rangle^{-2}} 2^{-7l^+} \\
&\quad\lesssim \tilde\delta^3 t^{n-3+n_0+\varepsilon}.
\end{aligned}
$$

d) We use estimates (107), (104), (105) and (76) to obtain

$$
\begin{aligned}
&\sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l_{12}\sim l\gg l_3} \left\| \langle\xi\rangle^2 I_{1,l,l_1,l_{12},l_3}(t,\xi)\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \tilde\delta^3 t^n \sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l_{12}\sim l\gg l_3} 2^{-n_0 l_{12}}2^{\varepsilon l_1/4}2^{-\varepsilon l_1^+/2}\min\left\{2^{(3-\varepsilon/4)l_{12}}, \langle t\rangle^{\varepsilon/4-3}\right\}2^{3l_3/2} \\
&\quad\lesssim \tilde\delta^3 t^n \sum_{l_1\in\mathbb{Z}} 2^{\varepsilon l_1/4}2^{-\varepsilon l_1^+/2} \sum_{l_3\in-\mathbb{N}} 2^{3l_3/2}\Big(\sum_{2^l\le\langle t\rangle^{-1}} 2^{(3-n_0-\varepsilon/4)l} + \sum_{2^l\ge\langle t\rangle^{-1}} 2^{-n_0 l}\langle t\rangle^{\varepsilon/4-3}\Big) \\
&\quad\lesssim \tilde\delta^3 t^{n-3+n_0+\varepsilon/4}.
\end{aligned}
$$

Estimate for $I_{2;l,l_1,l_{12},l_3}$: Since

$$\begin{aligned}
&\langle\xi\rangle^2 I_{2;l,l_1,l_{12},l_3}(t,\xi)\\
&\quad= \int_0^t n(t')^{n-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\langle\xi\rangle^2 e^{\pm_0 it'\langle\xi\rangle}}{i\varphi^{\pm_0,\pm_1,\pm_1,\mp_0}(\xi,\eta,\sigma)}\chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_1}(\sigma)\chi_{l_3}(\xi-\eta)\\
&\qquad\qquad \cdot\Big\langle \widehat{\psi^{\pm_1}(t')}(\sigma),\widehat{\psi^{\pm_2}(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} m_{l,l_{12}}(\xi,\eta)\widehat{\psi^{\mp_0}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}t'\\
&\quad= \int_0^t n(t')^{-1} I_{1;l,l_1,l_{12},l_3}(t',\xi)\,\mathrm{d}t',
\end{aligned}$$

the estimates for $I_{1;l,l_1,l_{12},l_3}$ imply

$$\begin{aligned}
\big\|\langle\xi\rangle^2 I_{2;l,l_1,l_{12},l_3}(t,\xi)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \int_0^t n(t')^{-1}\tilde{\delta}^3(t')^{n-1+\varepsilon}\,\mathrm{d}t'\\
&\lesssim n\tilde{\delta}^3 t^{n-3+n_0+\varepsilon}.
\end{aligned}$$

Estimates for $I_{3;l,l_1,l_{12},l_3}$ and $I_{4;l,l_1,l_{12},l_3}$: By symmetry, it suffices to consider $I_{3;l,l_1,l_{12},l_3}$. Note that

$$\begin{aligned}
&\langle\xi\rangle^2 I_{3;l,l_1,l_{12},l_3}(t,\xi)\\
&\quad= \int_0^t (t')^n\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\langle\xi\rangle^2 e^{it'(\pm_0\langle\xi\rangle\mp_1\langle\sigma\rangle)}}{i\varphi^{\pm_0,\pm_1,\pm_1,\mp_0}(\xi,\eta,\sigma)}\chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_1}(\sigma)\chi_{l_3}(\xi-\eta)\\
&\qquad\qquad \cdot\Big\langle \partial_{t'}\widehat{\mathrm{w}^{\pm_1}(t')}(\sigma),\widehat{\psi^{\pm_2}(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} m_{l,l_{12}}(\xi,\eta)\widehat{\psi^{\mp_0}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}t'\\
&\quad=: \int_0^t \tilde{I}_{3;l,l_1,l_{12},l_3}(t',\xi)\,\mathrm{d}t'
\end{aligned}$$

and

$$\begin{aligned}
\partial_t \mathrm{w}^\pm &= -i\partial_t\psi^\pm + \langle D\rangle\psi^\pm\\
&= \Pi^\pm(D)\big[(V*|\psi|^2)\psi\big]. \qquad (108)
\end{aligned}$$

With the same arguments as in (106) and (107), we get

$$\begin{aligned}
&\big\|\tilde{I}_{3;l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim (t')^n 2^{l^+}2^{-n_0 l_{12}}2^{3l/2}\Big\|P_{l_{12}}\big\langle P_{l_1}\Pi^{\pm_1}(D)\big[(V*|\psi(t')|^2)\psi(t')\big],\psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\Big\|_{L^2(\mathbb{R}^3,\mathbb{C})}\\
&\qquad\qquad \cdot\big\|P_{l_3}\psi^{\mp_0}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \qquad (109)
\end{aligned}$$

and

$$\begin{aligned}
&\big\|\tilde{I}_{3;l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim (t')^n 2^{l^+}2^{-n_0 l_{12}}\Big\|P_{l_{12}}\big\langle P_{l_1}\Pi^{\pm_1}(D)\big[(V*|\psi(t')|^2)\psi(t')\big],\psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\Big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\\
&\qquad\qquad \cdot\big\|P_{l_3}\psi^{\mp_0}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}. \qquad (110)
\end{aligned}$$

Similarly to estimates (103)–(105), we observe that

$$\begin{aligned}
&\Big\| P_{l_{12}} \big\langle P_{l_1}\Pi^{\pm_1}(D)\big[(V*|\psi(t')|^2)\psi(t')\big], \psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\Big\|_{L^2(\mathbb{R}^3,\mathbb{C})} \\
&\quad \lesssim 2^{(3/p-3/2)l_{12}} \Big\| P_{l_1}\Pi^{\pm_1}(D)\big[(V*|\psi(t')|^2)\psi(t')\big]\Big\|_{L^q(\mathbb{R}^3,\mathbb{C}^4)} \big\|\psi^{\pm_2}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad \lesssim 2^{(3/p-3/2)l_{12}} 2^{(3/2-3/q)l_1} 2^{-l_1^+} \big\|\psi(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)} \\
&\qquad\qquad \cdot \big\|\psi^{\pm_2}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad \lesssim 2^{(3/2-\varepsilon/4)l_{12}} 2^{\varepsilon l_1/4} 2^{-l_1^+} \langle t'\rangle^{-1}\tilde{\delta}^4
\end{aligned} \tag{111}$$

and

$$\begin{aligned}
&\Big\| P_{l_{12}} \big\langle P_{l_1}\Pi^{\pm_1}(D)\big[(V*|\psi(t')|^2)\psi(t')\big], \psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\Big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \\
&\quad \lesssim 2^{3l_{12}/p} \Big\| P_{l_1}\Pi^{\pm_1}(D)\big[(V*|\psi(t')|^2)\psi(t')\big]\Big\|_{L^q(\mathbb{R}^3,\mathbb{C}^4)} \big\|\psi^{\pm_2}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad \lesssim 2^{(3-\varepsilon/4)l_{12}} 2^{\varepsilon l_1/4} 2^{-l_1^+} \langle t'\rangle^{-1}\tilde{\delta}^4
\end{aligned} \tag{112}$$

by choosing $p = 3/(3-\varepsilon/4)$ and $1/q = 1/p - 1/2$. Furthermore,

$$\begin{aligned}
&\Big\| P_{l_{12}} \big\langle P_{l_1}\Pi^{\pm_1}(D)\big[(V*|\psi(t')|^2)\psi(t')\big], \psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\Big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \\
&\quad \lesssim 2^{(\varepsilon/2-2)l_{12}^+} 2^{-\varepsilon l_1^+/2} \Big\| P_{l_1}\Pi^{\pm_1}(D)\big[(V*|\psi(t')|^2)\psi(t')\big]\Big\|_{W^{2,\infty}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\qquad\qquad \cdot \big\|\psi^{\pm_2}(t')\big\|_{W^{2,\infty}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad \lesssim 2^{(\varepsilon/2-2)l_{12}^+} 2^{-\varepsilon l_1^+/2} 2^{3l_1/p} \big\|\psi(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \big\|\psi(t')\big\|_{W^{2,6}(\mathbb{R}^3,\mathbb{C}^4)} \big\|\psi(t')\big\|_{W^{2,p}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\qquad\qquad \cdot \big\|\psi^{\pm_2}(t')\big\|_{W^{2,\infty}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad \lesssim 2^{(\varepsilon/2-2)l_{12}^+} 2^{-\varepsilon l_1^+/2} 2^{\varepsilon l_1/4} \langle t'\rangle^{\varepsilon/4-4}\tilde{\delta}^4,
\end{aligned} \tag{113}$$

where $p = 12/\varepsilon$.

a) By estimates (110), (112), (113) and (76), we have

$$\begin{aligned}
&\sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\sim l_3\succ l_{12}} \big\|\tilde{I}_{3,l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim \tilde{\delta}^5 (t')^n \sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\sim l_3\succ l_{12}} 2^{l^+} 2^{-n_0 l_{12}} 2^{\varepsilon l_1/4} 2^{-\varepsilon l_1^+/2} \\
&\qquad\qquad \cdot \min\big\{2^{(3-\varepsilon/4)l_{12}}\langle t'\rangle^{-1}, \langle t'\rangle^{\varepsilon/4-4}\big\} 2^{3l_3/2} 2^{-8l_3^+} \\
&\lesssim \tilde{\delta}^5 (t')^{n-1} \sum_{l_1\in\mathbb{Z}} 2^{\varepsilon l_1/4} 2^{-\varepsilon l_1^+/2} \Big(\sum_{2^{l_{12}}\le\langle t'\rangle^{-1}} 2^{(3-n_0-\varepsilon/4)l_{12}} + \sum_{2^{l_{12}}\ge\langle t'\rangle^{-1}} 2^{-n_0 l_{12}} \langle t'\rangle^{\varepsilon/4-3}\Big) \\
&\qquad\qquad \cdot \sum_{l\in\mathbb{Z}} 2^{3l/2} 2^{-7l^+} \\
&\lesssim \tilde{\delta}^5 (t')^{n-4+n_0+\varepsilon/4}.
\end{aligned}$$

b) We use estimates (109), (111) and (76) to obtain

$$\begin{aligned}
&\sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\ll l_3\sim l_{12},\, 2^l\le\langle t'\rangle^{-2}} \big\|\tilde{I}_{3,l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \tilde{\delta}^5 (t')^n \sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\ll l_3\sim l_{12},\, 2^l\le\langle t'\rangle^{-2}} 2^{l^+} 2^{(3/2-n_0-\varepsilon/4)l_{12}} 2^{3l/2} 2^{-l_1^+} 2^{\varepsilon l_1/4} \langle t'\rangle^{-1}\\
&\qquad\qquad \cdot 2^{3l_3/2} 2^{-8l_3^+}\\
&\lesssim \tilde{\delta}^5 (t')^{n-1} \sum_{l_1\in\mathbb{Z}} 2^{-l_1^+} 2^{\varepsilon l_1/4} \sum_{l_{12}\in-\mathbb{N}} 2^{(3-n_0-\varepsilon/4)l_{12}} \sum_{2^l\le\langle t'\rangle^{-2}} 2^{3l/2}\\
&\lesssim \tilde{\delta}^5 (t')^{n-4}.
\end{aligned}$$

c) Estimates (110), (112), (113) and (76) give

$$\begin{aligned}
&\sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\ll l_3\sim l_{12},\, 2^l\ge\langle t'\rangle^{-2}} \big\|\tilde{I}_{3,l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \tilde{\delta}^5 (t')^n \sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\ll l_3\sim l_{12},\, 2^l\ge\langle t'\rangle^{-2}} 2^{l^+} 2^{-n_0 l_{12}} 2^{\varepsilon l_1/4} 2^{-\varepsilon l_1^+/2}\\
&\qquad\qquad \cdot \min\big\{2^{(3-\varepsilon/4)l_{12}}\langle t'\rangle^{-1}, \langle t'\rangle^{\varepsilon/4-4}\big\} 2^{3l_3/2} 2^{-8l_3^+}\\
&\quad\lesssim \tilde{\delta}^5 (t')^{n-1} \sum_{2^{l_{12}}\le\langle t'\rangle^{-1}} 2^{(3-n_0-\varepsilon/4)l_{12}} \sum_{\langle t'\rangle^{-2}\le 2^l\le\langle t'\rangle^{-1}} 1\\
&\qquad + \tilde{\delta}^5 (t')^{n-1} \sum_{2^{l_{12}}\ge\langle t'\rangle^{-1}} 2^{-n_0 l_{12}} \langle t'\rangle^{\varepsilon/4-3} \sum_{2^l\ge\langle t'\rangle^{-2}} 2^{-7l^+}\\
&\quad\lesssim \tilde{\delta}^5 (t')^{n-4+n_0+\varepsilon}.
\end{aligned}$$

d) We use estimates (110), (112), (113) and (76) to get

$$\begin{aligned}
&\sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l_{12}\sim l\gg l_3} \big\|\tilde{I}_{3,l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \tilde{\delta}^5 (t')^n \sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l_{12}\sim l\gg l_3} 2^{-n_0 l_{12}} 2^{\varepsilon l_1/4} 2^{-\varepsilon l_1^+/2}\\
&\qquad\qquad \cdot \min\big\{2^{(3-\varepsilon/4)l_{12}}\langle t'\rangle^{-1}, \langle t'\rangle^{\varepsilon/4-4}\big\} 2^{3l_3/2}\\
&\lesssim \tilde{\delta}^5 (t')^{n-1} \sum_{l_1\in\mathbb{Z}} 2^{\varepsilon l_1/4} 2^{-\varepsilon l_1^+/2} \sum_{l_3\in-\mathbb{N}} 2^{3l_3/2} \Big(\sum_{2^l\le\langle t\rangle^{-1}} 2^{(3-n_0-\varepsilon/4)l} + \sum_{2^l\ge\langle t\rangle^{-1}} 2^{-n_0 l} \langle t'\rangle^{\varepsilon/4-3}\Big)\\
&\lesssim \tilde{\delta}^5 (t')^{n-4+n_0+\varepsilon/4}.
\end{aligned}$$

Estimate for $I_{5;l,l_1,l_{12},l_3}$: By (108),

$$\begin{aligned}
I_{5;l,l_1,l_{12},l_3}(t,\xi) = \int_0^t (t')^n \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} &\frac{e^{\pm_0 it'(\langle\xi\rangle+\langle\xi-\eta\rangle)}}{i\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)} \chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_1}(\sigma)\chi_{l_3}(\xi-\eta)\\
&\cdot \Big\langle \widehat{\psi^{\pm_1}(t')}(\sigma), \widehat{\psi^{\pm_2}(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} m_{l,l_{12}}(\xi,\eta)\\
&\cdot \Pi^{\mp_0}(\xi-\eta)\big[\big(V*|\psi(t')|^2\big)\psi(t')\big]\widehat{\ }(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}t'\\
=: \tilde{I}_{5;l,l_1,l_{12},l_3}(t',\xi)\,\mathrm{d}t'.&
\end{aligned}$$

With the same arguments as in (106) and (107),

$$\begin{aligned}\big\|\tilde{I}_{5;l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim (t')^n 2^{l^+}2^{-n_0 l_{12}}2^{3l/2}\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t'),\psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\big\|_{L^2(\mathbb{R}^3,\mathbb{C})}\\ &\quad\cdot\Big\|P_{l_3}\Pi^{\mp_0}(D)\big[(V*|\psi(t')|^2)\psi(t')\big]\Big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \qquad (114)\end{aligned}$$

and

$$\begin{aligned}\big\|\tilde{I}_{5;l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim (t')^n 2^{l^+}2^{-n_0 l_{12}}\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t'),\psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\\ &\quad\cdot\Big\|P_{l_3}\Pi^{\mp_0}(D)\big[(V*|\psi(t')|^2)\psi(t')\big]\Big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}. \qquad (115)\end{aligned}$$

While estimates (103)–(105) are still applicable, we replace (76) with

$$\begin{aligned}&\Big\|P_{l_3}\Pi^{\mp_0}(D)\big[(V*|\psi(t')|^2)\psi(t')\big)\Big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim 2^{-2l_3^+}\big\|(V*|\psi(t')|^2)\psi(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim 2^{-2l_3^+}\big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}^3\langle t'\rangle^{-1}2^{-2l_3^+}. \qquad (116)\end{aligned}$$

a) By estimates (115), (104), (105) and (116),

$$\begin{aligned}&\sum_{l_{12}\in-\mathbb{N}}\ \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\,l_3\in\mathbb{Z},\,l\sim l_3\succ l_{12}}\big\|\tilde{I}_{5,l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}^5(t')^n\sum_{l_{12}\in-\mathbb{N}}\ \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\,l_3\in\mathbb{Z},\,l\sim l_3\succ l_{12}} 2^{l^+}2^{-n_0 l_{12}}2^{\varepsilon l_1/4}2^{-\varepsilon l_1^+/2}\min\big\{2^{(3-\varepsilon/4)l_{12}},\langle t'\rangle^{\varepsilon/4-3}\big\}\\ &\qquad\cdot 2^{-2l_3^+}\langle t'\rangle^{-1}\\ &\lesssim \tilde{\delta}^5(t')^{n-1}\sum_{l_1\in\mathbb{Z}}2^{\varepsilon l_1/4}2^{-\varepsilon l_1^+/2}\Big(\sum_{2^{l_{12}}\le\langle t'\rangle^{-1}}2^{(3-n_0-\varepsilon/4)l_{12}}+\sum_{2^{l_{12}}\ge\langle t'\rangle^{-1}}2^{-n_0 l_{12}}\langle t'\rangle^{\varepsilon/4-3}\Big)\\ &\qquad\cdot\sum_{l\in\mathbb{Z}}2^{\varepsilon l/4}2^{-l^+}\\ &\lesssim \tilde{\delta}^5(t')^{n-4+n_0+\varepsilon/4}.\end{aligned}$$

b) We use estimates (114), (103) and (116) to obtain

$$\begin{aligned}&\sum_{l_{12}\in-\mathbb{N}}\ \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\,l_3\in\mathbb{Z},\,l\ll l_3\sim l_{12},\,2^l\le\langle t'\rangle^{-2}}\big\|\tilde{I}_{5,l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}^5(t')^n\sum_{l_{12}\in-\mathbb{N}}\ \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\,l_3\in\mathbb{Z},\,l\ll l_3\sim l_{12},\,2^l\le\langle t'\rangle^{-2}}2^{l^+}2^{(3/2-n_0-\varepsilon/4)l_{12}}2^{3l/2}2^{-l_1^+}2^{\varepsilon l_1/4}\\ &\qquad\cdot 2^{-2l_3^+}\langle t'\rangle^{-1}\\ &\lesssim \tilde{\delta}^5(t')^{n-1}\sum_{l_1\in\mathbb{Z}}2^{-l_1^+}2^{\varepsilon l_1/4}\sum_{l_{12}\in-\mathbb{N}}2^{(3/2-n_0-\varepsilon/4)l_{12}}\sum_{2^l\le\langle t'\rangle^{-2}}2^{3l/2}\\ &\lesssim \tilde{\delta}^5(t')^{n-2}\\ &\lesssim\begin{cases}\tilde{\delta}^5(t')^{n-1}\sum_{l_{12}\in-\mathbb{N}}2^{(3/2-n_0-\varepsilon/4)l_{12}}\sum_{2^l\le\langle t'\rangle^{-2}}2^{3l/2}\lesssim\tilde{\delta}^5(t')^{n-4}, & n_0=1,\\ \tilde{\delta}^5(t')^{n-1}\sum_{l_{12}\in-\mathbb{N}}2^{(5/2-n_0-\varepsilon/4)l_{12}}\sum_{2^l\le\langle t'\rangle^{-2}}2^{l/2}\lesssim\tilde{\delta}^5(t')^{n-2}, & n_0=2.\end{cases}\end{aligned}$$

c) Estimates (115), (104), (105) and (116) lead to

$$\begin{aligned}
&\sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\ll l_3\sim l_{12},\, 2^l\geq\langle t'\rangle^{-2}} \big\|\tilde{I}_{5,l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \tilde{\delta}^5(t')^n \sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l\ll l_3\sim l_{12},\, 2^l\geq\langle t'\rangle^{-2}} 2^{l^+}2^{-n_0 l_{12}}2^{\varepsilon l_1/4}2^{-\varepsilon l_1^+/2} \\
&\qquad\qquad \cdot \min\big\{2^{(3-\varepsilon/4)l_{12}}, \langle t'\rangle^{\varepsilon/4-3}\big\}2^{-2l_3^+}\langle t'\rangle^{-1} \\
&\quad\lesssim \tilde{\delta}^5(t')^{n-1}\sum_{2^{l_{12}}\leq\langle t'\rangle^{-1}} 2^{(3-n_0-\varepsilon/4)l_{12}} \sum_{\langle t'\rangle^{-2}\leq 2^l\leq\langle t'\rangle^{-1}} 1 \\
&\qquad + \tilde{\delta}^5(t')^{n-1}\sum_{2^{l_{12}}\geq\langle t'\rangle^{-1}} 2^{-n_0 l_{12}}\langle t'\rangle^{\varepsilon/4-3}\sum_{2^l\geq\langle t'\rangle^{-2}} 2^{-l^+} \\
&\quad\lesssim \tilde{\delta}^5(t')^{n-4+n_0+\varepsilon}.
\end{aligned}$$

d) In order to obtain a factor $2^{3\varepsilon l_3/4}$, we refine estimate (115) as follows: From Lemma 2.9, we have

$$\begin{aligned}
\big\|\tilde{I}_{5,l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim (t')^n 2^{l^+}2^{-n_0 l_{12}}\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t'),\psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\big\|_{L^{4/\varepsilon}(\mathbb{R}^3,\mathbb{C})} \\
&\qquad\cdot\Big\|P_{l_3}\Pi^{\mp_0}(D)\big((V*|\psi(t')|^2)\psi(t')\big)\Big\|_{L^{4/(2-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim (t')^n 2^{l^+}2^{-n_0 l_{12}}\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t'),\psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\big\|_{L^{4/\varepsilon}(\mathbb{R}^3,\mathbb{C})} \\
&\qquad\cdot 2^{3\varepsilon l_3/4}\big\|(V*|\psi(t')|^2)\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim (t')^n 2^{l^+}2^{-n_0 l_{12}}2^{3\varepsilon l_3/4}\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t'),\psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\big\|_{L^{4/\varepsilon}(\mathbb{R}^3,\mathbb{C})} \\
&\qquad\cdot\big\|\psi(t')\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim \tilde{\delta}^3(t')^{n-1}2^{l^+}2^{-n_0 l_{12}}2^{3\varepsilon l_3/4} \\
&\qquad\cdot\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t'),\psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\big\|_{L^{4/\varepsilon}(\mathbb{R}^3,\mathbb{C})},
\end{aligned}\tag{117}$$

where we used Bernstein's inequality in the second step. Similarly to (104) and (105), we get

$$\begin{aligned}
&\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t'),\psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\big\|_{L^{4/\varepsilon}(\mathbb{R}^3,\mathbb{C})} \\
&\quad\lesssim 2^{(3-\varepsilon)l_{12}}\big\|\big\langle P_{l_1}\psi^{\pm_1}(t'),\psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\big\|_{L^{3/(3-\varepsilon/4)}(\mathbb{R}^3,\mathbb{C})} \\
&\quad\lesssim 2^{(3-\varepsilon)l_{12}}\big\|P_{l_1}\psi^{\pm_1}(t')\big\|_{L^{12/(6-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi^{\pm_2}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim 2^{(3-\varepsilon)l_{12}}2^{\varepsilon l_1/4}2^{-2l_1^+}\big\|\psi^{\pm_1}(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi^{\pm_2}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \tilde{\delta}^2 2^{(3-\varepsilon)l_{12}}2^{\varepsilon l_1/4}2^{-2l_1^+}
\end{aligned}\tag{118}$$

and

$$\begin{aligned}
\big\|P_{l_{12}}\big\langle P_{l_1}\psi^{\pm_1}(t'),\psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\big\|_{L^{4/\varepsilon}(\mathbb{R}^3,\mathbb{C})} &\lesssim \big\|P_{l_1}\psi^{\pm_1}(t')\big\|_{L^{4/\varepsilon}(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi^{\pm_2}(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim 2^{-2l_1^+}2^{\varepsilon l_1/4}\big\|\psi^{\pm_1}(t')\big\|_{L^{3/\varepsilon}(\mathbb{R}^3,\mathbb{C}^4)}\tilde{\delta}\langle t'\rangle^{-3/2} \\
&\lesssim \tilde{\delta}^2\langle t'\rangle^{-3+\varepsilon}2^{-2l_1^+}2^{\varepsilon l_1/4}.
\end{aligned}\tag{119}$$

Finally, by estimates (117), (118) and (119),

$$
\begin{aligned}
&\sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l_{12}\sim l\gg l_3} \big\|\tilde{I}_{5,l,l_1,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim \tilde{\delta}^5 \langle t'\rangle^{n-1} \sum_{l_{12}\in-\mathbb{N}} \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2},\, l_3\in\mathbb{Z},\, l_{12}\sim l\gg l_3} 2^{-n_0 l_{12}} 2^{3\varepsilon l_3/4} 2^{\varepsilon l_1/4} 2^{-2l_1^+} \min\big\{2^{(3-\varepsilon/4)l_{12}}, \langle t'\rangle^{\varepsilon-3}\big\} \\
&\lesssim \tilde{\delta}^5 \langle t'\rangle^{n-1} \sum_{l_1\in\mathbb{Z}} 2^{\varepsilon l_1/4} 2^{-2l_1^+} \sum_{l_3\in-\mathbb{N}} 2^{3\varepsilon l_3/4} \Big(\sum_{2^l\leq\langle t\rangle^{-1}} 2^{(3-n_0-\varepsilon/4)l} + \sum_{2^l\geq\langle t\rangle^{-1}} 2^{-n_0 l}\langle t'\rangle^{\varepsilon-3} \Big) \\
&\lesssim \tilde{\delta}^5 \langle t'\rangle^{n-4+n_0+\varepsilon}.
\end{aligned}
$$

□

2.2.3 Control of $x\mathrm{w}^\pm$

Lemma 2.13. *Let $\psi^\pm \in \mathcal{C}\big([0,T], H^N(\mathbb{R}^3,\mathbb{C}^4)\big)$ satisfy a priori assumption (44). For any $t\in[0,T]$, we have*

$$\big\|x\mathrm{w}^\pm(t,x)\big\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \delta + \tilde{\delta}^3\langle t\rangle^\varepsilon. \tag{120}$$

Proof. Because of Plancherel's identity, we need to control $\big\|\langle\xi\rangle^2\nabla_\xi\widehat{\mathrm{w}^\pm(t)}(\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}$. Duhamel equation (79) for $\mathrm{w}^\pm$ gives

$$\partial_{\xi_k}\widehat{\mathrm{w}^{\pm_0}(t)}(\xi) = \partial_{\xi_k}\widehat{\psi_0^{\pm_0}}(\xi) + i4\pi c \sum_{\pm_1,\pm_2,\pm_3\in\{+,-\}} \int_0^t \partial_{\xi_k} I^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t' \tag{121}$$

with $I^{\pm_0,\pm_1,\pm_2,\pm_3}$ coming from (77). For the first term on the right hand side of (121), note that

$$
\begin{aligned}
\big\|\langle\xi\rangle^2\partial_{\xi_k}\widehat{\psi_0^{\pm_0}}(\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &= (2\pi)^{3/2}\big\|x_k\psi_0^{\pm_0}(x)\big\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)} \\
&\leq (2\pi)^{3/2}\delta,
\end{aligned}
$$

where we used the initial data assumption (43) in the last step. For the second term, we have

$$
\begin{aligned}
&\partial_{\xi_k} I^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi) \\
&\quad= \partial_{\xi_k}\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\pm_3}(\xi,\eta,\sigma)}|\eta|^{-2} \\
&\qquad\qquad \cdot\Big\langle \widehat{\mathrm{w}^{\pm_1}(t')}(\sigma), \widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} \widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta \\
&\qquad + \Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\pm_3}(\xi,\eta,\sigma)}|\eta|^{-2} \\
&\qquad\qquad \cdot\Big\langle \widehat{\mathrm{w}^{\pm_1}(t')}(\sigma), \widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} \partial_{\xi_k}\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta \\
&\qquad + \Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_{\xi_k}\Big[e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\pm_3}(\xi,\eta,\sigma)}\Big]|\eta|^{-2} \\
&\qquad\qquad \cdot\Big\langle \widehat{\mathrm{w}^{\pm_1}(t')}(\sigma), \widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} \widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta \\
&\quad=: \sum_{j=1}^3 I_{k,j}^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi).
\end{aligned} \tag{122}
$$

Let

$$I_{k,j}^{\pm_0,\pm_3}(t,\xi) := \sum_{\pm_1,\pm_2\in\{+,-\}} I_{k,j}^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi). \tag{123}$$

Then,

$$\sum_{\pm_1,\pm_2,\pm_3\in\{+,-\}} \partial_{\xi_k} I^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi) = \sum_{\pm_3\in\{+,-\}} I_{k,j}^{\pm_0,\pm_3}(t',\xi).$$

Estimate for $I_{k,1}^{\pm_0,\pm_3}$: By plugging in the definitions (40) of w and (78) of φ, we observe that

$$\begin{aligned} I_{k,1}^{\pm_0,\pm_3}(t',\xi) &= \partial_{\xi_k}\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{\pm_0 it'\langle\xi\rangle}|\eta|^{-2}\big\langle\widehat{\psi(t')}(\sigma),\widehat{\psi(t')}(\sigma-\eta)\big\rangle_{\mathbb{C}^4} \\ &\qquad\cdot\widehat{\psi^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta \\ &= (4\pi)^{-1}\partial_{\xi_k}\Pi^{\pm_0}(\xi)e^{\pm_0 it'\langle\xi\rangle}\big[\big(V*|\psi(t')|^2\big)\psi^{\pm_3}(t')\big]^\frown(\xi). \end{aligned}$$

By using estimate (46), we conclude that

$$\begin{aligned} \big\|\langle\xi\rangle^2 I_{k,1}^{\pm_0,\pm_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \big\|\big(V*|\psi(t')|^2\big)\psi^{\pm_3}(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \big\|\big(\langle D\rangle|D|^{-2}|\psi(t')|^2\big)\psi^{\pm_3}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + \big\|V*|\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|\psi^{\pm_3}(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \big\||D|^{-1}|\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|\psi^{\pm_3}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + \big\|V*|\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|\psi^{\pm_3}(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}. \end{aligned} \tag{124}$$

By Lorentz-type Young's inequality, interpolation and

$$|D|^{-1}|\psi(t')|^2 = (2\pi^2)^{-1}\big(|\cdot|^{-2}*|\psi(t')|^2\big),$$

we obtain

$$\begin{aligned} \big\|\langle\xi\rangle^2 I_{k,1}^{\pm_0,\pm_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \big\||\psi(t')|^2\big\|_{L^{3,1}(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi^{\pm_3}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad + \big\||\psi(t')|^2\big\|_{L^{3/2,1}(\mathbb{R}^3,\mathbb{C})}\big\|\psi^{\pm_3}(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \big\|\psi(t')\big\|_{L^4(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^{12}(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi^{\pm_3}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad + \big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi^{\pm_3}(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \tilde{\delta}^3\langle t'\rangle^{-1}, \end{aligned} \tag{125}$$

where the last step follows from a priori assumption (44), decay estimate (70) and

$$\big\|\psi^{\pm_3}(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \big\|\langle\xi\rangle^3\widehat{\psi^{\pm_3}(t')}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}.$$

Estimate for $I_{k,2}^{\pm_0,\pm_3}$: Plugging in the definitions (40) of w and (78) of φ yields

$$\begin{aligned} I_{k,2}^{\pm_0,\pm_3}(t',\xi) &= \Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it'(\pm_0\langle\xi\rangle\mp_3\langle\xi-\eta\rangle)}|\eta|^{-2}\Big\langle\widehat{\psi(t')}(\sigma),\widehat{\psi(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} \\ &\qquad\cdot\partial_{\xi_k}\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta \\ &= (4\pi)^{-1}\Pi^{\pm_0}(\xi)e^{\pm_0 it'\langle\xi\rangle}\big[\big(V*|\psi(t')|^2\big)e^{\mp_3 it'\langle D\rangle}x_k\mathrm{w}^{\pm_3}(t')\big]^\frown(\xi). \end{aligned} \tag{126}$$

By estimate (46), we have

$$\big\|\langle\xi\rangle^2 I_{k,2}^{\pm_0,\pm_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \big\|\big(V*|\psi(t')|^2\big)e^{\mp_3 it'\langle D\rangle}x_k \mathrm{w}^{\pm_3}(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}.$$

Analogously to estimate (124) and (125), we get

$$\begin{aligned}
\big\|\langle\xi\rangle^2 I_{k,2}^{\pm_0,\pm_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \big\|\big(V*|\psi(t')|^2\big)x_k \mathrm{w}^{\pm_3}(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \big\|\big(\langle D\rangle^2|D|^{-2}|\psi(t')|^2\big)e^{\mp_3 it'\langle D\rangle}x_k \mathrm{w}^{\pm_3}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad + \big\|V*|\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|x_k \mathrm{w}^{\pm_3}(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \big\||\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|x_k \mathrm{w}^{\pm_3}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad + \big\|V*|\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|x_k \mathrm{w}^{\pm_3}(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \big\|\psi(t')\big\|^2_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\big\|x_k \mathrm{w}^{\pm_3}(t',x)\big\|_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad + \big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\big\|x_k \mathrm{w}^{\pm_3}(t',x)\big\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \tilde\delta^3\langle t'\rangle^{\varepsilon-1},
\end{aligned}$$

where we used Lorentz-type Young's inequality and interpolation in the fourth step and a priori assumption (44) and estimate (70) in the last step.

Estimate for $I_{k,3}^{\pm_0,\pm_3}$: By definitions (40) of w and (78) of φ, we have

$$\begin{aligned}
I_{k,3}^{\pm_0,\pm_3}(t',\xi) &= \Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} it'\bigg(\pm_0\frac{\xi_k}{\langle\xi\rangle}\pm_3\frac{\eta_k-\xi_k}{\langle\eta-\xi\rangle}\bigg)e^{\pm_0 it'\langle\xi\rangle}|\eta|^{-2}\\
&\qquad\cdot\Big\langle\widehat{\psi(t')}(\sigma),\widehat{\psi(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4}\widehat{\psi^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\\
&= \Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3} it'|\eta|^{-2}\bigg(\pm_0\frac{\xi_k}{\langle\xi\rangle}\pm_3\frac{\eta_k-\xi_k}{\langle\eta-\xi\rangle}\bigg)e^{\pm_0 it'\langle\xi\rangle}\big[|\psi(t')|^2\big]\widehat{\ }(\eta)\\
&\qquad\cdot\widehat{\psi^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\eta.
\end{aligned}\tag{127}$$

Dyadic decomposition in ξ, η and $\xi-\eta$ leads to

$$\begin{aligned}
I_{k,3}^{\pm_0,\pm_3}(t',\xi) &= \sum_{l,l_{12},l_3\in\mathbb{Z}}\Pi^{\pm_0}(\xi)it'\int_{\mathbb{R}^3} e^{\pm_0 it'\langle\xi\rangle}\big[P_{l_{12}}|\psi(t')|^2\big]\widehat{\ }(\eta)\\
&\qquad\cdot m_{k,l,l_{12}}^{\pm_0,\pm_3}(\xi,\eta)\big[P_{l_3}\psi^{\pm_3}(t')\big]\widehat{\ }(\xi-\eta)\,\mathrm{d}\eta\\
&=: \sum_{l,l_{12},l_3\in\mathbb{Z}} I_{k,3;l,l_{12},l_3}^{\pm_0,\pm_3}(t',\xi),
\end{aligned}$$

where

$$m_{k,l,l_{12}}^{\pm_0,\pm_3}(\xi,\eta) = |\eta|^{-2}\bigg(\pm_0\frac{\xi_k}{\langle\xi\rangle}\pm_3\frac{\eta_k-\xi_k}{\langle\eta-\xi\rangle}\bigg)\chi_l(\xi)\tilde\chi_{l_{12}}(\eta).\tag{128}$$

For the **case $\pm_0=\pm_3$**, we are in the same situation as in Pusateri's paper [84] for the scalar toy model so that we essentially adopt his argument. For any $\alpha,\beta\in\mathbb{N}_0^3$, we have

$$\begin{aligned}
\big|\partial_\xi^\alpha\partial_\eta^\beta m_{k,l,l_{12}}^{\pm_0,\pm_0}(\xi,\eta)\big| &\lesssim_{\alpha,\beta}\min\{|\eta|^{-1}\langle\xi-\eta\rangle^{-1},|\eta|^{-2}\}\chi_l(\xi)\tilde\chi_l(\eta)2^{-l|\alpha|}2^{-l_{12}|\beta|}\\
&\lesssim_{\alpha,\beta} 2^{-l_{12}}2^{-l^+}\chi_l(\xi)\tilde\chi_{l_{12}}(\eta)2^{-l|\alpha|}2^{-l_{12}|\beta|}.
\end{aligned}$$

Therefore, Lemma 2.10 is applicable with $C \lesssim 2^{-l_{12}}2^{-l^+}$ and from Lemma 2.11, we obtain

$$\left\|\langle\xi\rangle^2 I^{\pm_0,\pm_0}_{k,3}(t',\xi)\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}^3\langle t'\rangle^{\varepsilon-1}.$$

Finally, we consider the **case** $\boldsymbol{\pm_0 = \mp_3}$. Since

$$\begin{aligned}\mathrm{w}^\pm(t') &= \Pi^\pm(D)e^{\pm it'\langle D\rangle}\psi(t'),\\ (\Pi^\pm)^2 &= \Pi^\pm,\end{aligned}$$

we have

$$\begin{aligned}I^{\pm_0,\mp_0}_{k,3;l,l_{12},l_3}(t',\xi) = it'\int_{\mathbb{R}^3}\Pi^{\pm_0}(\xi)\Pi^{\mp_0}(\xi-\eta)m^{\pm_0,\mp_0}_{k,l,l_{12}}(\xi,\eta)e^{\pm_0 it'\langle\xi\rangle}\big[P_{l_{12}}|\psi(t')|^2\big]\widehat{\ }(\eta)\\ \cdot\big[P_{l_3}\psi^{\mp_0}(t')\big]\widehat{\ }(\xi-\eta)\,\mathrm{d}\eta.\end{aligned}$$

For the multiplier

$$M^{\pm_0}_{k,l,l_{12}}(\xi,\eta) := \Pi^{\pm_0}(\xi)\Pi^{\mp_0}(\xi-\eta)m^{\pm_0,\mp_0}_{k,l,l_{12}}(\xi,\eta),$$

we obtain

$$\begin{aligned}\left|\partial^\alpha_\xi\partial^\beta_\eta M^{\pm_0}_{k,l,l_{12}}(\xi,\eta)\right| &\lesssim_\alpha \min\{|\eta|^{-1}\langle\xi-\eta\rangle^{-1},|\eta|^{-2}\}\chi_l(\xi)\tilde{\chi}_{l_{12}}(\eta)2^{-l|\alpha|}2^{-l_{12}|\beta|}\\ &\lesssim_\alpha 2^{-l_{12}}2^{-l^+}\chi_l(\xi)\tilde{\chi}_{l_{12}}(\eta)2^{-l|\alpha|}2^{-l_{12}|\beta|},\end{aligned}$$

see also (46) and (47). Hence, Lemma 2.10 remains applicable with $C \lesssim 2^{-l_{12}}2^{l^+}$ and Lemma 2.11 gives

$$\left\|\langle\xi\rangle^2 I^{\pm_0,\mp_0}_{k,3}(t',\xi)\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}^3\langle t'\rangle^{\varepsilon-1}.$$

□

Remark. In the case $\pm_0 = \mp_3$, we could also use Lemma 2.12 for $(l,l_1) \in \mathbb{I}^{\pm_1,\pm_2}$, $l_{12} \in \mathbb{N}$, $l_3 \in \mathbb{Z}$. We will pursue this idea for the estimates of $\langle x\rangle^2\mathrm{w}^\pm(t,x)$, where we need to gain a factor $(t')^2$.

2.2.4 Control of $\langle x\rangle^2\mathrm{w}^\pm$

Lemma 2.14. *Let $\psi^\pm \in \mathcal{C}\big([0,T],H^N(\mathbb{R}^3,\mathbb{C}^4)\big)$ satisfy a priori assumption (44). Then,*

$$\left\|\langle x\rangle^2\mathrm{w}^\pm(t,x)\right\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \delta + \tilde{\delta}^3\langle t\rangle^{2\varepsilon} \tag{129}$$

for any $t \in [0,T]$.

Proof. Obviously,

$$\begin{aligned}\left\|\mathrm{w}^\pm(t)\right\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} &= \left\|\psi^\pm(t)\right\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \delta + \tilde{\delta}^3\langle t\rangle^\varepsilon,\end{aligned}$$

where we used estimate (80) in the last step. It remains to consider the $H_x^2(\mathbb{R}^3, \mathbb{C}^4)$-norm of $|x|^2 \mathrm{w}^{\pm}(t,x)$ which is the inverse Fourier transform of $-\Delta_\xi[\mathrm{w}^{\pm}(t)]\widehat{\ }(\xi)$. By Duhamel equation (79), we have

$$\partial_{\xi_k}^2 \widehat{\mathrm{w}^{\pm_0}(t)}(\xi) = \partial_{\xi_k}^2 \widehat{\psi^{\pm_0}}(\xi) + i4\pi c \sum_{\pm_1,\pm_2,\pm_3 \in \{+,-\}} \int_0^t \partial_{\xi_k}^2 I^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t' \tag{130}$$

with $I^{\pm_0,\pm_1,\pm_2,\pm_3}$ coming from (77). For the first term on the right hand side of (130), note that

$$\begin{aligned}\big\|\langle\xi\rangle^2 \partial_{\xi_k}^2 \widehat{\psi^{\pm_0}}(\xi)\big\|_{L_\xi^2(\mathbb{R}^3,\mathbb{C}^4)} &= (2\pi)^{3/2}\big\||x|^2\psi_0^{\pm_0}(x)\big\|_{H_x^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\le (2\pi)^{3/2}\delta,\end{aligned}$$

where we used the initial data assumption (43) in the last step. For the second term, we write

$$\partial_{\xi_k}^2 I^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi) = \sum_{j=1}^3 \partial_{\xi_k} I_{k,j}^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi),$$

where the $I_{k,j}^{\pm_0,\pm_1,\pm_2,\pm_3}$ are coming from (122). As in (123), we also denote

$$\sum_{\pm_1,\pm_2 \in \{+,-\}} \partial_{\xi_k}^2 I_k^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi) =: \sum_{j=1}^3 \partial_{\xi_k} I_{k,j}^{\pm_0,\pm_3}(t',\xi).$$

Estimate for $\boldsymbol{\partial_{\xi_k} I_{k,1}^{\pm_0,\pm_3}}$: We calculate

$$\begin{aligned}
&\partial_{\xi_k} I_{k,1}^{\pm_0,\pm_3}(t',\xi) \\
&\quad= \sum_{\pm_1,\pm_2 \in \{+,-\}} \partial_{\xi_k}^2 \Pi^{\pm_0}(\xi) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\pm_3}(\xi,\eta,\sigma)} |\eta|^{-2} \\
&\qquad\qquad \cdot \Big\langle \widehat{\mathrm{w}^{\pm_1}(t')}(\sigma), \widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} \widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta \\
&\qquad + \sum_{\pm_1,\pm_2 \in \{+,-\}} \partial_{\xi_k} \Pi^{\pm_0}(\xi) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\pm_3}(\xi,\eta,\sigma)} |\eta|^{-2} \\
&\qquad\qquad \cdot \Big\langle \widehat{\mathrm{w}^{\pm_1}(t')}(\sigma), \widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} \partial_{\xi_k}\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta \\
&\qquad + \sum_{\pm_1,\pm_2 \in \{+,-\}} \partial_{\xi_k} \Pi^{\pm_0}(\xi) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\mathrm{d}}{\mathrm{d}\xi_k}\Big[e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\pm_3}(\xi,\eta,\sigma)}\Big] |\eta|^{-2} \\
&\qquad\qquad \cdot \Big\langle \widehat{\mathrm{w}^{\pm_1}(t')}(\sigma), \widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} \widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta \\
&\quad=: \sum_{j=1}^3 I_{k,1,j}^{\pm_0,\pm_3}(t',\xi).
\end{aligned}$$

Since

$$I_{k,1,1}^{\pm_0,\pm_3}(t',\xi) = (4\pi)^{-1}\partial_{\xi_k}^2 \Pi^{\pm_0}(\xi) e^{\pm_0 it'\langle\xi\rangle}\big[\big(V * |\psi(t')|^2\big)\psi^{\pm_3}(t')\big]\widehat{\ }(\xi)$$

and, by (46),

$$|\partial_{\xi_k}^2 \Pi^{\pm_0}(\xi)| \lesssim \langle\xi\rangle^{-2},$$

Lorentz-type Young's inequality and interpolation lead to

$$\begin{aligned}\big\|\langle\xi\rangle^2 I^{\pm_0,\pm_3}_{k,1,1}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \big\|\big(V*|\psi(t')|^2\big)\psi^{\pm_3}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \|\psi(t')\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi^{\pm_3}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde\delta^3\langle t'\rangle^{-1}.\end{aligned}$$

For $I^{\pm_0,\pm_3}_{k,1,2}$, we have

$$I^{\pm_0,\pm_3}_{k,1,2}(t',\xi) = (4\pi)^{-1}\partial_{\xi_k}\Pi^{\pm_0}(\xi)e^{\pm_0 it'\langle\xi\rangle}\Big[\big(V*|\psi(t')|^2\big)e^{\mp_3 it'\langle D\rangle}x_k \mathrm{w}^{\pm_3}(t')\Big]\widehat{\ }(\xi)$$

and by the same arguments as in (124) and (125), we obtain

$$\begin{aligned}\big\|\langle\xi\rangle^2 I^{\pm_0,\pm_3}_{k,1,2}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \big\|\big(V*|\psi(t')|^2\big)e^{\mp_3 it'\langle D\rangle}x_k \mathrm{w}^{\pm_3}(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \big\|\psi(t')\big\|_{L^4(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^{12}(\mathbb{R}^3,\mathbb{C}^4)}\big\|x_k \mathrm{w}^{\pm_3}(t',x)\big\|_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad + \big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\big\|x_k \mathrm{w}^{\pm_3}(t',x)\big\|_{H^1_x(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde\delta^3\langle t'\rangle^{\varepsilon-1}.\end{aligned}$$

Finally,

$$\begin{aligned}I^{\pm_0,\pm_3}_{k,1,3}(t',\xi) = \partial_{\xi_k}\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3} it'|\eta|^{-2}\bigg(\pm_0\frac{\xi_k}{\langle\xi\rangle}\pm_3\frac{\eta_k-\xi_k}{\langle\eta-\xi\rangle}\bigg)e^{it'(\pm_0\langle\xi\rangle\mp_3\langle\xi-\eta\rangle)}\big[|\psi(t')|^2\big]\widehat{\ }(\eta)\\ \cdot\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\eta.\end{aligned}$$

In the **case $\pm_0 = \pm_3$**, we use the same arguments as for (127) to get

$$\big\|\langle\xi\rangle^2 I^{\pm_0,\pm_0}_{k,1,3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde\delta^3\langle t'\rangle^{\varepsilon-1}.$$

However, in the case **case $\pm_0 = \mp_3$**, we cannot carry over the method for (127) since Lemma 2.9 is no longer applicable for the multiplier

$$(\xi,\eta)\mapsto \partial_{\xi_k}\Pi^{\pm_0}(\xi)\Pi^{\mp_0}(\xi-\eta)m^{\pm_0,\mp_0}_{k,l,l_{12}}(\xi,\eta).$$

Instead, we apply Lemma 2.12. Note that

$$\begin{aligned}I^{\pm_0,\pm_1,\pm_2,\mp_0}_{k,1,3}(t',\xi) &= \sum_{l,l_1,l_{12},l_3\in\mathbb{Z}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} t' e^{it'\varphi^{\pm_0,\pm_1,\pm_2,\mp_0}(\xi,\eta,\sigma)}\chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_3}(\xi-\eta)\chi_{l_1}(\sigma)\\ &\qquad\cdot\Big\langle\widehat{\mathrm{w}^{\pm_1}(t')}(\sigma),\widehat{\mathrm{w}^{\pm_2}(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} m^{\pm_0}_{k,l,l_{12}}(\xi,\eta)\\ &\qquad\cdot\widehat{\mathrm{w}^{\mp_0}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\\ &=: \sum_{l,l_1,l_{12},l_3\in\mathbb{Z}} I^{\pm_0,\pm_1,\pm_2,\mp_0}_{k,1,3;l,l_1,l_{12},l_3}(t',\xi),\end{aligned}$$

where

$$m^{\pm_0}_{k,l,l_{12}}(\xi,\eta) = \pm_0 i|\eta|^{-2}\bigg(\frac{\xi_k}{\langle\xi\rangle}+\frac{\xi_k-\eta_k}{\langle\xi-\eta\rangle}\bigg)\tilde\chi_l(\xi)\tilde\chi_{l_{12}}(\eta)\partial_{\xi_k}\Pi^{\pm_0}(\xi).$$

Since $m^{\pm_0}_{k,l,l_{12}}$ fulfills the conditions of Lemma 2.12 with $n = 1$ and $n_0 = 2$ (cf. also Lemma 2.10), we have

$$\Big\| \langle\xi\rangle^2 \int_0^t \sum_{(l,l_1)\in\mathbb{I}^{\pm_1,\pm_2}} \sum_{l_{12}\in-\mathbb{N}} \sum_{l_3\in\mathbb{Z}} I^{\pm_0,\pm_1,\pm_2,\mp_0}_{k,1,3;l,l_1,l_{12},l_3}(t',\xi)\,\mathrm{d}t' \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}^3 \langle t\rangle^\varepsilon.$$

Hence, it remains to consider

$$\sum_{\pm_1,\pm_2\in\{+,-\}} \Bigg(\sum_{(l,l_1)\in\mathbb{Z}^2} \sum_{l_{12}\in\mathbb{N}_0} \sum_{l_3\in\mathbb{Z}} I^{\pm_0,\pm_1,\pm_2,\mp_0}_{k,1,3;l,l_1,l_{12},l_3}(t',\xi) + \sum_{(l,l_1)\in\mathbb{Z}^2} \sum_{l_{12}\in-\mathbb{N}} \sum_{l_3\in\mathbb{Z}} I^{\pm_0,\pm_1,\mp_1,\mp_0}_{k,1,3;l,l_1,l_{12},l_3}(t',\xi) \Bigg). \tag{131}$$

We start with the first term in (131). Note that

$$\begin{aligned}
\sum_{\pm_1,\pm_2\in\{+,-\}} \sum_{(l,l_1)\in\mathbb{Z}^2} \sum_{l_{12}\in\mathbb{N}_0} \sum_{l_3\in\mathbb{Z}} & I^{\pm_0,\pm_1,\pm_2,\mp_0}_{k,1,3;l,l_1,l_{12},l_3}(t',\xi) \\
&= \sum_{l,l_3\in\mathbb{Z}} \sum_{l_{12}\in\mathbb{N}_0} \int_{\mathbb{R}^3} t' e^{\pm_0 it'\langle\xi\rangle} \chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_3}(\xi-\eta)\big[|\psi(t')|^2\big]^\frown(\eta) \\
&\qquad\qquad \cdot m^{\pm_0}_{k,l,l_{12}}(\xi,\eta)\widehat{\psi^{\mp_0}(t')}(\xi-\eta)\,\mathrm{d}\eta \\
&=: \sum_{l,l_3\in\mathbb{Z}} \sum_{l_{12}\in\mathbb{N}_0} I^{\pm_0}_{k,1,3;l,l_{12},l_3}(t',\xi).
\end{aligned}$$

Since

$$\begin{aligned}
|m^{\pm_0}_{k,l,l_{12}}(\xi,\eta)| &\lesssim \langle\xi\rangle^{-1}|\eta|^{-2}\tilde{\chi}_l(\xi)\tilde{\chi}_{l_{12}}(\eta) \\
&\lesssim 2^{-l^+}2^{-2l_{12}}\tilde{\chi}_l(\xi)\tilde{\chi}_{l_{12}}(\eta)
\end{aligned} \tag{132}$$

and

$$|\partial^\alpha_\xi \partial^\beta_\eta m^{\pm_0}_{k,l,l_{12}}(\xi,\eta)| \lesssim_{\alpha,\beta} 2^{-l^+}2^{-2l_{12}}2^{-l|\alpha|}2^{-l_{12}|\beta|}\tilde{\chi}_l(\xi)\tilde{\chi}_{l_{12}}(\eta), \tag{133}$$

Lemma 2.10 is applicable with $C \lesssim 2^{-l^+}2^{-2l_{12}}$ and we obtain

$$\begin{aligned}
\big\|\langle\xi\rangle^2 I^{\pm_0}_{k,1,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim t'2^{l^+}2^{-2l_{12}}\big\|P_{l_{12}}|\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|P_{l_3}\psi^{\mp_0}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim \tilde{\delta}^3\langle t'\rangle^{-2}2^{-l^+}2^{-2l_{12}}2^{3l_3/2}2^{-8l_3^+},
\end{aligned} \tag{134}$$

where we used estimates (74) and (76) in the second step. On the other hand, estimate (132), Hölder's inequality and Hausdorff–Young's inequality give

$$\begin{aligned}
&\big\|\langle\xi\rangle^2 I^{\pm_0}_{k,1,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim t'2^{l^+}2^{-2l_{12}}\big\|\chi_l\big\|_{L^2(\mathbb{R}^3,\mathbb{C})}\big\|P_{l_{12}}|\psi(t')|^2\big\|_{L^2(\mathbb{R}^3,\mathbb{C})}\big\|P_{l_3}\psi^{\mp_0}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \tilde{\delta}^3 t'2^{l^+}2^{3l/2}2^{-l_{12}/2}2^{-9l_{12}^+/2}2^{3l_3/2}2^{-8l_3^+},
\end{aligned} \tag{135}$$

where we used estimates (75) and (76) in the second step.

a) Estimate (134) implies

$$\begin{aligned}\sum_{l\sim l_3\succ l_{12}\geq 0}\big\|\langle\xi\rangle^2 I^{\pm_0}_{k,1,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \tilde\delta^3\langle t'\rangle^{-2}\sum_{l\sim l_3\succ l_{12}\geq 0}2^{-2l_{12}}2^{3l/2}2^{-9l^+}\\ &\lesssim \tilde\delta^3\langle t'\rangle^{-2}\sum_{l_{12}\in\mathbb{N}_0}2^{-2l_{12}}\sum_{l\in\mathbb{Z}}2^{3l/2}2^{-9l^+}\\ &\lesssim \tilde\delta^3\langle t'\rangle^{-2}.\end{aligned}$$

b) By estimate (135), we have

$$\begin{aligned}&\sum_{l\ll l_3\sim l_{12},\, l_{12}\geq 0,\, 2^l\leq\langle t'\rangle^{-2}}\big\|\langle\xi\rangle^2 I^{\pm_0}_{k,1,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\qquad\lesssim \tilde\delta^3 t'\sum_{l\ll l_3\sim l_{12},\, l_{12}\geq 0,\, 2^l\leq\langle t'\rangle^{-2}}2^{3l/2}2^{l_{12}}2^{-25l_{12}^+/2}\\ &\qquad\lesssim \tilde\delta^3 t'\sum_{2^l\leq\langle t'\rangle^{-2}}2^{3l/2}\\ &\qquad\lesssim \tilde\delta^3\langle t'\rangle^{-2}.\end{aligned}$$

c) Estimate (134) gives

$$\begin{aligned}&\sum_{l\ll l_3\sim l_{12},\, l_{12}\geq 0,\, 2^l\geq\langle t'\rangle^{-2}}\big\|\langle\xi\rangle^2 I^{\pm_0}_{k,1,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\qquad\lesssim \tilde\delta^3\langle t'\rangle^{-2}\sum_{l\ll l_3\sim l_{12},\, l_{12}\geq 0,\, 2^l\geq\langle t'\rangle^{-2}}2^{-l^+}2^{-2l_{12}}2^{3l_3/2}2^{-8l_3^+}\\ &\qquad\lesssim \tilde\delta^3\langle t'\rangle^{-2}\sum_{l_{12}\in\mathbb{N}_0}2^{-2l_{12}}\sum_{2^l\geq\langle t'\rangle^{-2},\, l\leq l_{12}}2^{-l^+}\\ &\qquad\lesssim \tilde\delta^3\langle t'\rangle^{-2+\varepsilon}.\end{aligned}$$

d) By estimate (134), we get

$$\begin{aligned}\sum_{l_{12}\sim l\gg l_3,\, l_{12}\geq 0}\big\|\langle\xi\rangle^2 I^{\pm_0}_{k,1,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \tilde\delta^3\langle t'\rangle^{-2}\sum_{l_{12}\sim l\gg l_3,\, l_{12}\geq 0}2^{-l^+}2^{-2l}2^{3l_3/2}2^{-8l_3^+}\\ &\lesssim \tilde\delta^3\langle t'\rangle^{-2}\sum_{2^{l_{12}}\in\mathbb{N}_0}2^{-2l}\sum_{l_3\in\mathbb{Z}}2^{3l_3/2}2^{-8l_3^+}\\ &\lesssim \tilde\delta^3\langle t'\rangle^{-2}.\end{aligned}$$

We continue with the second term in (131). Let

$$\tilde m^{\pm_0}_{k,l,l_{12}}(\xi,\eta) = |\eta|\, m^{\pm_0}_{k,l,l_{12}}(\xi,\eta).$$

Then,

$$\begin{aligned}I^{\pm_0,\pm_1}_{k,1,3;l,l_{12},l_3}(t',\xi) &= \sum_{l,l_1,l_3\in\mathbb{Z}}\sum_{l_{12}\in-\mathbb{N}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} t' e^{\pm_0 it'\langle\xi\rangle}\chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_3}(\xi-\eta)\tilde m^{\pm_0}_{k,l,l_{12}}(\xi,\eta)\\ &\qquad\cdot|\eta|^{-1}\Pi^{\mp_1}(\sigma-\eta)\Pi^{\pm_1}(\sigma)\Big\langle\widehat{\psi(t')}(\sigma),\widehat{\psi(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4}\\ &\qquad\cdot\widehat{\psi^{\mp_0}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\\ &=: I^{\pm_0,\pm_1}_{k,1,3;l,l_{12},l_3}(t',\xi).\end{aligned}$$

We take $p = 6/\varepsilon$ and $1/q = 1/2 - 1/p$. Since

$$\left|\partial_\eta^\alpha\left[|\eta|^{-1}\Pi^{\pm_1}(\sigma)\Pi^{\mp_1}(\sigma-\eta)\right]\right| \lesssim_\alpha |\eta|^{-|\alpha|},$$

Hörmander–Mikhlin's multiplier theorem gives

$$\left\|P_{l_{12}}|D|^{-1}\left\langle P_{l_1}\psi^{\pm_1}(t'),\psi^{\mp_1}(t')\right\rangle_{\mathbb{C}^4}\right\|_{L^p(\mathbb{R}^3,\mathbb{C})} \lesssim \left\|P_{l_{12}}\left\langle P_{l_1}\psi(t'),\psi(t')\right\rangle_{\mathbb{C}^4}\right\|_{L^p(\mathbb{R}^3,\mathbb{C})}. \tag{136}$$

Because of

$$\left|\partial_\xi^\alpha\partial_\eta^\beta\tilde{m}^{\pm_0}_{k,l,l_{12}}(\xi,\eta)\right| \lesssim_{\alpha,\beta} 2^{l^+}2^{-l_{12}}2^{-l|\alpha|}2^{-l_{12}|\beta|}\tilde{\chi}_l(\xi)\tilde{\chi}_{l_{12}}(\eta), \tag{137}$$

we obtain

$$\begin{aligned}\left\|\langle\xi\rangle^2 I^{\pm_0,\pm_1}_{k,k,3;l,l_{12},l_3}\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim t'2^{l^+}2^{-l_{12}}&\left\|P_{l_{12}}\left\langle P_{l_1}\psi(t'),\psi(t')\right\rangle_{\mathbb{C}^4}\right\|_{L^p(\mathbb{R}^3,\mathbb{C})}\\ &\cdot\left\|P_{l_3}\psi^{\mp_0}(t')\right\|_{L^q(\mathbb{R}^3,\mathbb{C}^4)}\end{aligned} \tag{138}$$

from Lemmata 2.10, 2.9 and estimate (136). Furthermore,

$$\begin{aligned}\left\|\langle\xi\rangle^2 I^{\pm_0,\pm_1}_{k,k,3;l,l_{12},l_3}\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim t'2^{l^+}2^{-l_{12}}2^{3l/2}&\left\|P_{l_{12}}\left\langle P_{l_1}\psi(t'),\psi(t')\right\rangle_{\mathbb{C}^4}\right\|_{L^2(\mathbb{R}^3,\mathbb{C})}\\ &\cdot\left\|P_{l_3}\psi^{\mp_0}(t')\right\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\end{aligned} \tag{139}$$

by using estimate (137), Bernstein's inequality and Hölder's inequality. Similarly,

$$\begin{aligned}\left\|P_{l_{12}}\left\langle P_{l_1}\psi(t'),\psi(t')\right\rangle_{\mathbb{C}^4}\right\|_{L^p(\mathbb{R}^3,\mathbb{C})} &\lesssim 2^{(3/2-3/p)l_{12}}\left\|\psi(t')\right\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\left\|\psi(t')\right\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}^2 2^{(3/2-\varepsilon/2)l_{12}}\langle t'\rangle^{-3/2},\end{aligned} \tag{140}$$

$$\begin{aligned}\left\|P_{l_{12}}\left\langle P_{l_1}\psi(t'),\psi(t')\right\rangle_{\mathbb{C}^4}\right\|_{L^p(\mathbb{R}^3,\mathbb{C})} &\lesssim 2^{-2l_{12}^+}\left\|\psi(t')\right\|^2_{W^{2,p}(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}^2 2^{-2l_{12}^+}\langle t'\rangle^{\varepsilon/2-3}\end{aligned} \tag{141}$$

and

$$\begin{aligned}\left\|P_{l_{12}}\left\langle P_{l_1}\psi(t'),\psi(t')\right\rangle_{\mathbb{C}^4}\right\|_{L^2(\mathbb{R}^3,\mathbb{C})} &\lesssim 2^{3l_{12}/2}\left\|\left\langle P_{l_1}\psi(t'),\psi(t')\right\rangle_{\mathbb{C}^4}\right\|_{L^1(\mathbb{R}^3,\mathbb{C})}\\ &\lesssim \tilde{\delta}^2 2^{3l_{12}/2},\end{aligned} \tag{142}$$

$$\begin{aligned}\left\|P_{l_3}\psi^{\mp_0}(t')\right\|_{L^q(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim 2^{(3/2-3/q)l_3}\left\|P_{l_3}\psi^{\mp_0}(t')\right\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}2^{(3/2+\varepsilon)l_3}2^{-8l_3^+}.\end{aligned} \tag{143}$$

a) Estimates (138), (140), (141) and (143) lead to

$$\begin{aligned}&\sum_{l\sim l_3\succ l_{12},\, l_{12}\le -1}\left\|\langle\xi\rangle^2 I^{\pm_0,\pm_1}_{k,1,3;l,l_{12},l_3}(t',\xi)\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}^3 t'\sum_{l\sim l_3\succ l_{12},\, l_{12}\le -1} 2^{l^+}2^{-l_{12}}\min\{2^{(3/2-\varepsilon/2)l_{12}}\langle t'\rangle^{-3/2}, 2^{-2l_{12}^+}\langle t'\rangle^{\varepsilon/2-3}\}2^{(3/2+\varepsilon)l_3}2^{-8l_3^+}\\ &\lesssim \tilde{\delta}^3 t'\Big(\sum_{2^{l_{12}}\le\langle t'\rangle^{-1}} 2^{(1/2-\varepsilon/2)l_{12}}\langle t'\rangle^{-3/2} + \sum_{1/2\ge 2^{l_{12}}\ge\langle t'\rangle^{-1}} 2^{-l_{12}}\langle t'\rangle^{\varepsilon/2-3}\Big)\sum_{l_3\in\mathbb{Z}} 2^{(3/2+\varepsilon/2)l_3}2^{-7l_3^+}\\ &\lesssim \tilde{\delta}^3\langle t'\rangle^{\varepsilon/2-1}.\end{aligned}$$

b) By estimates (139), (142) and (76), we have

$$\begin{aligned}
&\sum_{l\ll l_3\sim l_{12}\le -1,\, 2^l\le\langle t'\rangle^{-2}} \big\|\langle\xi\rangle^2 I^{\pm_0,\pm_1}_{k,1,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad\lesssim \tilde\delta^3 t' \sum_{l\ll l_3\sim l_{12}\le -1,\, 2^l\le\langle t'\rangle^{-2}} 2^{3l/2}2^{l_{12}/2}2^{3l_3/2}2^{-8l_3^+}\\
&\qquad\lesssim \tilde\delta^3 t' \sum_{2^l\le\langle t'\rangle^{-2}} 2^{3l/2}\\
&\qquad\lesssim \tilde\delta^3 \langle t'\rangle^{-2}.
\end{aligned}$$

c) Estimates (138), (140), (141) and (143) give

$$\begin{aligned}
&\sum_{l\ll l_3\sim l_{12}\le -1,\, 2^l\ge\langle t'\rangle^{-2}} \big\|\langle\xi\rangle^2 I^{\pm_0,\pm_1}_{k,1,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \tilde\delta^3 t' \sum_{l\ll l_3\sim l_{12}\le -1,\, 2^l\ge\langle t'\rangle^{-2}} 2^{-l_{12}}\min\{2^{(3/2-\varepsilon/2)l_{12}}\langle t'\rangle^{-3/2}, \langle t'\rangle^{\varepsilon/2-3}\}2^{(3/2+\varepsilon)l_3}\\
&\quad\lesssim \tilde\delta^3 t' \sum_{2^{l_{12}}\le\langle t'\rangle^{-1}} 2^{(1/2-\varepsilon/2)l_{12}}\langle t'\rangle^{-3/2} \sum_{1/2\ge 2^l\ge\langle t'\rangle^{-2}} 1\\
&\qquad+ \tilde\delta^3 t' \sum_{2^{l_{12}}\ge\langle t'\rangle^{-1}} 2^{-l_{12}}\langle t'\rangle^{\varepsilon/2-3} \sum_{1/2\ge 2^l\ge\langle t'\rangle^{-2}} 1\\
&\quad\lesssim \tilde\delta^3 \langle t'\rangle^{\varepsilon-1}.
\end{aligned}$$

d) By estimates (138), (140), (141) and (143), we have

$$\begin{aligned}
&\sum_{-1\ge l_{12}\sim l\gg l_3} \big\|\langle\xi\rangle^2 I^{\pm_0,\pm_1}_{k,1,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \tilde\delta^3 t' \sum_{-1\ge l_{12}\sim l\gg l_3} 2^{-l}\min\{2^{(3/2-\varepsilon/2)l}\langle t'\rangle^{-3/2}, \langle t'\rangle^{\varepsilon/2-3}\}2^{(3/2+\varepsilon/2)l_3/2}\\
&\quad\lesssim \tilde\delta^3 t' \Big(\sum_{2^l\le\langle t'\rangle^{-1}} 2^{(1/2-\varepsilon/2)l}\langle t'\rangle^{-3/2} + \sum_{1/2\ge 2^l\ge\langle t'\rangle^{-1}} 2^{-l}\langle t'\rangle^{\varepsilon/2-3}\Big)\\
&\quad\lesssim \tilde\delta^3 \langle t'\rangle^{\varepsilon/2-1}.
\end{aligned}$$

Estimate for $\boldsymbol{\partial_{\xi_k} I^{\pm_0,\pm_3}_{k,2}}$: We calculate

$$\begin{aligned}
\partial_{\xi_k} I^{\pm_0,\pm_3}_{k,2}(t',\xi) &= \partial_{\xi_k}\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it'(\pm_0\langle\xi\rangle\mp_3\langle\xi-\eta\rangle)}|\eta|^{-2}\Big\langle\widehat{\psi(t')}(\sigma),\widehat{\psi(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4}\\
&\qquad\cdot\partial_{\xi_k}\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\\
&\quad+\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{it'(\pm_0\langle\xi\rangle\mp_3\langle\xi-\eta\rangle)}|\eta|^{-2}\Big\langle\widehat{\psi(t')}(\sigma),\widehat{\psi(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4}\\
&\qquad\cdot\partial^2_{\xi_k}\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\\
&\quad+\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_{\xi_k}\big[e^{it'(\pm_0\langle\xi\rangle\mp_3\langle\xi-\eta\rangle)}\big]|\eta|^{-2}\Big\langle\widehat{\psi(t')}(\sigma),\widehat{\psi(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4}\\
&\qquad\cdot\partial_{\xi_k}\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\\
&=: \sum_{j=1}^{3} I^{\pm_0,\pm_3}_{k,2,j}(t',\xi).
\end{aligned}$$

Note that $I^{\pm_0,\pm_3}_{k,2,1} = I^{\pm_0,\pm_3}_{k,1,2}$ has already been treated above. For $I^{\pm_0,\pm_3}_{k,2,2}$, we can use the same arguments as for (126) in the previous proof just by replacing $\partial_{\xi_k}\widehat{\mathrm{w}^{\pm_3}(t')}$ with $\partial^2_{\xi_k}\widehat{\mathrm{w}^{\pm_3}(t')}$, i.e.

$$\begin{aligned}\big\|\langle\xi\rangle^2 I^{\pm_0,\pm_3}_{k,2,2}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \big\|\psi(t')\big\|^2_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\big\|x_k^2\mathrm{w}^{\pm_3}(t')\big\|_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad+\big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\big\|x_k^2\mathrm{w}^{\pm_3}(t')\big\|_{H^2_x(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \tilde\delta^3\langle t'\rangle^{2\varepsilon-1}.\end{aligned}$$

For $I^{\pm_0,\pm_3}_{k,2,3}$, we write

$$\begin{aligned}I^{\pm_0,\pm_3}_{k,2,3}(t',\xi) &= it'\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_3}(\xi-\eta)\widehat{|\psi(t')|^2}(\eta) \\ &\qquad\qquad\cdot m^{\pm_0,\mp_3}_{k,l,l_{12}}(\xi,\eta)\partial_{\xi_k}\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\eta \\ &= \sum_{l,l_{12},l_3\in\mathbb{Z}} it'\int_{\mathbb{R}^3}\chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_3}(\xi-\eta)\widehat{|\psi(t')|^2}(\eta) \\ &\qquad\qquad\cdot\Pi^{\pm_0}(\xi)\Pi^{\pm_3}(\xi-\eta)m^{\pm_0,\mp_3}_{k,l,l_{12}}(\xi,\eta)\partial_{\xi_k}\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\eta \\ &=: \sum_{l,l_{12},l_3\in\mathbb{Z}} I^{\pm_0,\pm_3}_{k,2,3;l,l_{12},l_3}(t',\xi),\end{aligned}$$

where

$$m^{\pm_0,\mp_3}_{k,l,l_{12}}(\xi,\eta) = |\eta|^{-2}\left(\pm_0\frac{\xi_k}{\langle\xi\rangle}\mp_3\frac{\xi_k-\eta_k}{\langle\xi-\eta\rangle}\right)\tilde\chi_l(\xi)\tilde\chi_{l_{12}}(\eta).$$

In the **case** $\pm_0 = \pm_3$, we can apply Lemma 2.9 with $C\lesssim 2^{-l^+}2^{-l_{12}}$ since

$$\big|\partial_\xi^\alpha\partial_\eta^\beta m^{\pm_0,\pm_0}_{k,l,l_{12}}(\xi,\eta)\big| \lesssim_{\alpha,\beta} 2^{-l^+}2^{-l_{12}}2^{-l|\alpha|}2^{-l_{12}|\beta|}\tilde\chi_l(\xi)\tilde\chi_{l_{12}}(\eta),$$

see also Lemma 2.10. For the **case** $\pm_0 = \mp_3$, Lemma 2.9 remains applicable with $C\lesssim 2^{-l^+}2^{-l_{12}}$ because of

$$\big|\partial_\xi^\alpha\partial_\eta^\beta \Pi^{\pm_0}(\xi)\Pi^{\mp_0}(\eta)m^{\pm_0,\mp_0}_{k,l,l_{12}}(\xi,\eta)\big| \lesssim_{\alpha,\beta} 2^{-l^+}2^{-l_{12}}2^{-l|\alpha|}2^{-l_{12}|\beta|}\tilde\chi_l(\xi)\tilde\chi_{l_{12}}(\eta)$$

and Lemma 2.10. Therefore,

$$\begin{aligned}\big\|\langle\xi\rangle^2 I^{\pm_0,\pm_3}_{k,2,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim t'2^{l^+}2^{-l_{12}}\big\|P_{l_{12}}|\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \\ &\quad\cdot\big\|P_{l_3}x_k\mathrm{w}^{\pm_3}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\end{aligned}\tag{144}$$

and

$$\begin{aligned}\big\|\langle\xi\rangle^2 I^{\pm_0,\pm_3}_{k,2,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim t'2^{l^+}2^{-l_{12}}2^{3l/2}\big\|P_{l_{12}}|\psi(t')|^2\big\|_{L^2(\mathbb{R}^3,\mathbb{C})} \\ &\quad\cdot\big\|P_{l_3}x_k\mathrm{w}^{\pm_3}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}.\end{aligned}\tag{145}$$

From Bernstein's inequality,

$$\begin{aligned}\big\|P_{l_3}\big(x_k\mathrm{w}^{\pm_3}(t')\big)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim 2^{l_3/2}\big\|x_k\mathrm{w}^{\pm_3}(t')\big\|_{L^{3/2}(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim 2^{l_3/2}\big\||x|\mathrm{w}^{\pm_3}(t')\big\|^{1/2}_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\||x|^2\mathrm{w}^{\pm_3}(t')\big\|^{1/2}_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \tilde\delta 2^{l_3/2}\langle t'\rangle^{3\varepsilon/2},\end{aligned}\tag{146}$$

where the second step comes from the following interpolation argument: Let

$$R = \frac{\||x|^2 \mathrm{w}^{\pm_3}(t')\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}}{\||x|\mathrm{w}^{\pm_3}(t')\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}}.$$

Then,

$$\begin{aligned}
&\big\||x|\mathrm{w}^{\pm_3}(t')\big\|_{L^{3/2}(\mathbb{R}^3,\mathbb{C}^4)} \\
&= \Big(\int_{B_R(0)} |y\mathrm{w}^{\pm_3}(t',y)|^{3/2}\,\mathrm{d}y + \int_{B_R(0)^c} |y|^{-3/2}\big||y|^2\mathrm{w}^{\pm_3}(t',y)\big|^{3/2}\,\mathrm{d}y\Big)^{2/3} \\
&\lesssim \Big(\big\|1\big\|_{L^4(B_R(0),\mathbb{C})}\big\||x|\mathrm{w}^{\pm_3}(t')\big\|^{3/2}_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\||x|^{-3/2}\big\|_{L^4(B_R(0)^c,\mathbb{C})}\big\||x|^2\mathrm{w}^{\pm_3}(t')\big\|^{3/2}_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\Big)^{2/3} \\
&\lesssim \Big(R^{3/4}\big\||x|\mathrm{w}^{\pm_3}(t')\big\|^{3/2}_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + R^{-3/4}\big\||x|^2\mathrm{w}^{\pm_3}(t')\big\|^{3/2}_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\Big)^{2/3} \\
&\lesssim \big\||x|\mathrm{w}^{\pm_3}(t')\big\|^{1/2}_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\||x|^2\mathrm{w}^{\pm_3}(t')\big\|^{1/2}_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\big\|P_{l_3}\big(x_k\mathrm{w}^{\pm_3}(t')\big)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim 2^{-2l_3^+}\big\|x_k\mathrm{w}^{\pm_3}(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim \tilde{\delta}2^{-2l_3^+}\langle t'\rangle^{\varepsilon}.
\end{aligned} \tag{147}$$

a) Estimates (144), (73), (74), (146) and (147) give

$$\begin{aligned}
&\sum_{l\sim l_3\succ l_{12}} \big\|\langle\xi\rangle^2 I^{\pm_0,\pm_3}_{k,2,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim t'\tilde{\delta}^3 \sum_{l\sim l_3\succ l_{12}} 2^{l^+}2^{-l_{12}}\min\big\{2^{3l_{12}},\langle t'\rangle^{-3}\big\}\min\{2^{l_3/2}\langle t'\rangle^{3\varepsilon/2}, 2^{-2l_3^+}\langle t'\rangle^{\varepsilon}\} \\
&\quad\lesssim t'\tilde{\delta}^3\Big(\sum_{2^{l_{12}}\le\langle t'\rangle^{-1}} 2^{2l_{12}} + \sum_{2^{l_{12}}\ge\langle t'\rangle^{-1}} 2^{-l_{12}}\langle t\rangle^{-3}\Big)\sum_{l_3\in\mathbb{Z}}\min\{2^{l_3/2}, 2^{-l_3^+}\}\langle t'\rangle^{3\varepsilon/2} \\
&\quad\lesssim \tilde{\delta}^3\langle t'\rangle^{3\varepsilon/2-1}.
\end{aligned}$$

b) By estimates (145), (75) and (147), we have

$$\begin{aligned}
\sum_{l\ll l_3\sim l_{12},\,2^l\le\langle t'\rangle^{-2}} &\big\|\langle\xi\rangle^2 I^{\pm_0,\pm_3}_{k,2,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim t'\tilde{\delta}^3 \sum_{l\ll l_3\sim l_{12},\,2^l\le\langle t'\rangle^{-2}} 2^{3l/2}2^{l_{12}/2}2^{-9l_{12}^+/2}\langle t'\rangle^{\varepsilon} \\
&\lesssim \tilde{\delta}^3\langle t'\rangle^{1+\varepsilon}\sum_{2^l\le\langle t\rangle^{-2}} 2^{3l/2} \\
&\lesssim \tilde{\delta}^3\langle t'\rangle^{\varepsilon-2}.
\end{aligned}$$

c) Estimates (144), (73), (74) and (147) lead to

$$\begin{aligned}
\sum_{l\ll l_3\sim l_{12},\,2^l\ge\langle t'\rangle^{-2}} &\big\|\langle\xi\rangle^2 I^{\pm_0,\pm_3}_{k,2,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim t'\tilde{\delta}^3 \sum_{l\ll l_3\sim l_{12},\,2^l\ge\langle t'\rangle^{-2}} 2^{l^+}2^{-l_{12}}\min\big\{2^{3l_{12}}, 2^{-2l^+}\langle t'\rangle^{-3}\big\}2^{-2l_3^+}\langle t'\rangle^{\varepsilon}
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\sum_{l\ll l_3\sim l_{12},\, 2^l\geq\langle t'\rangle^{-2}} \big\|\langle\xi\rangle^2 I^{\pm_0,\pm_3}_{k,2,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad\lesssim \tilde\delta^3\langle t'\rangle^{1+\varepsilon}\sum_{2^{l_{12}}\leq\langle t'\rangle^{-1}} 2^{2l_{12}}\sum_{\langle t'\rangle^{-2}\leq 2^l\leq\langle t'\rangle^{-1}} 1\\
&\qquad\quad+\tilde\delta^3\langle t'\rangle^{1+\varepsilon}\sum_{2^{l_{12}}\geq\langle t'\rangle^{-1}} 2^{-l_{12}}\langle t'\rangle^{-3}\sum_{2^l\geq\langle t'\rangle^{-2}} 2^{-3l^+}\\
&\qquad\lesssim \tilde\delta^3\langle t'\rangle^{3\varepsilon/2-1}.
\end{aligned}$$

d) By estimates (144), (73), (74), (146) and (147), we obtain

$$\begin{aligned}
&\sum_{l_{12}\sim l\gg l_3} \big\|\langle\xi\rangle^2 I^{\pm_0,\pm_3}_{k,2,3;l,l_{12},l_3}(t',\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim t'\tilde\delta^3\sum_{l_{12}\sim l\gg l_3} 2^{l^+}2^{-l}\min\big\{2^{3l},2^{-2l^+}\langle t'\rangle^{-3}\big\}\min\{2^{l_3/2},2^{-2l_3}\}\langle t'\rangle^{3\varepsilon/2}\\
&\quad\lesssim \tilde\delta^3\langle t'\rangle^{1+3\varepsilon/2}\Big(\sum_{2^l\leq\langle t'\rangle^{-1}} 2^{2l}+\sum_{2^l\geq\langle t'\rangle^{-1}} 2^{-l}\langle t'\rangle^{-3}\Big)\sum_{l_3\in\mathbb{Z}}\min\{2^{l_3/2},2^{-2l_3^+}\}\\
&\quad\lesssim \tilde\delta^3\langle t'\rangle^{3\varepsilon/2-1}.
\end{aligned}$$

Estimate for $\boldsymbol{\partial_{\xi_k} I^{\pm_0,\pm_3}_{k,3}}$: We have

$$\begin{aligned}
&\partial_{\xi_k} I^{\pm_0,\pm_3}_{k,3}(t',\xi)\\
&\quad=\partial_{\xi_k}\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}}{\mathrm{d}\xi_k}\big[e^{it'(\pm_0\langle\xi\rangle\mp_3\langle\xi-\eta\rangle)}\big]\,|\eta|^{-2}\Big\langle\widehat{\psi(t')}(\sigma),\widehat{\psi(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4}\\
&\qquad\qquad\cdot\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\\
&\qquad+\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}}{\mathrm{d}\xi_k}\big[e^{it'(\pm_0\langle\xi\rangle\mp_3\langle\xi-\eta\rangle)}\big]\,|\eta|^{-2}\Big\langle\widehat{\psi(t')}(\sigma),\widehat{\psi(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4}\\
&\qquad\qquad\cdot\partial_{\xi_k}\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\\
&\qquad+\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big(\frac{\mathrm{d}}{\mathrm{d}\xi_k}\Big)^2\big[e^{it'(\pm_0\langle\xi\rangle\mp_3\langle\xi-\eta\rangle)}\big]\,|\eta|^{-2}\Big\langle\widehat{\psi(t')}(\sigma),\widehat{\psi(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4}\\
&\qquad\qquad\cdot\widehat{\mathrm{w}^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\\
&\quad=:\sum_{j=1}^3 I^{\pm_0,\pm_3}_{k,3,j}(t',\xi).
\end{aligned}$$

Note that $I^{\pm_0,\pm_3}_{k,3,1}=I^{\pm_0,\pm_3}_{k,1,3}$ and $I^{\pm_0,\pm_3}_{k,3,2}=I^{\pm_0,\pm_3}_{k,2,3}$ have already been treated above. For $I^{\pm_0,\pm_3}_{k,3,3}$, we define

$$\begin{aligned}
M^{\pm_0,\pm_3}_{l,l_{12}}(\xi,\eta)&=|\eta|^{-2}\partial_{\xi_k}\Big(\pm_0\frac{\xi_k}{\langle\xi\rangle}\mp_3\frac{\xi_k-\eta_k}{\langle\xi-\eta\rangle}\Big)\tilde\chi_l(\xi)\tilde\chi_{l_{12}}(\eta),\\
\tilde M^{\pm_0,\pm_3}_{l,l_{12}}(\xi,\eta)&=|\eta|^{-2}\Big(\pm_0\frac{\xi_k}{\langle\xi\rangle}\mp_3\frac{\xi_k-\eta_k}{\langle\xi-\eta\rangle}\Big)^2\tilde\chi_l(\xi)\tilde\chi_{l_{12}}(\eta).
\end{aligned}$$

Then,

$$
\begin{aligned}
I^{\pm_0,\pm_3}_{k,3,3}(t',\xi) &= \sum_{l,l_{12},l_3\in\mathbb{Z}} it'\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3} e^{\pm_0 it'\langle\xi\rangle}\chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_3}(\xi-\eta) \\
&\qquad\cdot \widehat{|\psi(t')|^2}(\eta) M^{\pm_0,\pm_3}_{l,l_{12}}(\xi,\eta)\widehat{\psi^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\eta \\
&\quad - \sum_{l,l_{12},l_3\in\mathbb{Z}} (t')^2\Pi^{\pm_0}(\xi)\int_{\mathbb{R}^3} e^{\pm_0 it'\langle\xi\rangle}\chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_3}(\xi-\eta) \\
&\qquad\cdot \widehat{|\psi(t')|^2}(\eta)\tilde{M}^{\pm_0,\pm_3}_{l,l_{12}}(\xi,\eta)\widehat{\psi^{\pm_3}(t')}(\xi-\eta)\,\mathrm{d}\eta \\
&=: \sum_{l,l_{12},l_3\in\mathbb{Z}} I^{\pm_0,\pm_3}_{k,3,3,1;l,l_{12},l_3}(t',\xi) - \sum_{l,l_{12},l_3\in\mathbb{Z}} I^{\pm_0,\pm_3}_{k,3,3,2;l,l_{12},l_3}(t',\xi) \\
&=: I^{\pm_0,\pm_3}_{k,3,3,1}(t',\xi) - I^{\pm_0,\pm_3}_{k,3,3,2}(t',\xi).
\end{aligned}
$$

Estimate for $\boldsymbol{I^{\pm_0,\pm_3}_{k,3,3,1}}$: We first consider the **case $\pm_0 = \pm_3$**. Since

$$
\left|\partial_\xi^\alpha\partial_\eta^\beta M^{\pm_0,\pm_0}_{l,l_{12}}(\xi,\eta)\right| \lesssim_{\alpha,\beta} 2^{-l^+}2^{-l_{12}}2^{-l|\alpha|}2^{-l_{12}|\beta|}\tilde{\chi}_l(\xi)\tilde{\chi}_{l_{12}}(\eta),
$$

Lemmata 2.10 and 2.11 give

$$
\left\|\langle\xi\rangle^2 I^{\pm_0,\pm_0}_{k,3,3,1}(t',\xi)\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}^3\langle t'\rangle^{\varepsilon-1}.
$$

For the **case $\pm_0 = \mp_3$**, note that

$$
\begin{aligned}
I^{\pm_0,\mp_0}_{k,3,3,1;l,l_{12},l_3}(t',\xi) = it'\int_{\mathbb{R}^3} &\Pi^{\pm_0}(\xi)\Pi^{\mp_0}(\xi-\eta)e^{\pm_0 it'\langle\xi\rangle}\chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_3}(\xi-\eta) \\
&\cdot \widehat{|\psi(t')|^2}(\eta)M^{\pm_0,\mp_0}_{l,l_{12}}(\xi,\eta)\widehat{\psi^{\mp_0}(t')}(\xi-\eta)\,\mathrm{d}\eta
\end{aligned}
$$

and

$$
\left|\partial_\xi^\alpha\partial_\eta^\beta\left[\Pi^{\pm_0}(\xi)\Pi^{\mp_0}(\xi-\eta)M^{\pm_0,\mp_0}_{l,l_{12}}(\xi,\eta)\right]\right| \lesssim_{\alpha,\beta} 2^{-l^+}2^{-l_{12}}2^{-l|\alpha|}2^{-l_{12}|\beta|}\tilde{\chi}_l(\xi)\tilde{\chi}_{l_{12}}(\eta).
$$

Hence, we can apply Lemmas 2.10 and 2.11 to obtain

$$
\left\|\langle\xi\rangle^2 I^{\pm_0,\mp_0}_{k,3,3,1}(t',\xi)\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}^3\langle t'\rangle^{\varepsilon-1}.
$$

Estimate for $\boldsymbol{I^{\pm_0,\pm_3}_{k,3,3,2}}$: The most challenging term is $I^{\pm_0,\pm_3}_{k,3,3,2}$ where we need to gain two factors of t'. For the **case $\pm_0 = \pm_3$**, note that

$$
\left|\partial_\xi^\alpha\partial_\eta^\beta \tilde{M}^{\pm_0,\pm_0}_{l,l_{12}}(\xi,\eta)\right| \lesssim_{\alpha,\beta} 2^{-2l^+}|\xi|^{-|\alpha|}|\eta|^{-|\beta|}\tilde{\chi}_l(\xi)\tilde{\chi}_{l_{12}}(\eta)
$$

and Lemmas 2.10 and 2.11 lead to

$$
\left\|\langle\xi\rangle^2 I^{\pm_0,\pm_0}_{k,3,3,2}(t',\xi)\right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}^3\langle t'\rangle^{\varepsilon-1}.
$$

In the **case $\pm_0 = \mp_3$**, we have

$$
\begin{aligned}
I^{\pm_0,\mp_0}_{k,3,3,2;l,l_{12},l_3}(t',\xi) = it'\int_{\mathbb{R}^3} &e^{\pm_0 it'(\langle\xi\rangle+\langle\xi-\eta\rangle)}\chi_l(\xi)\chi_{l_{12}}(\eta)\chi_{l_3}(\xi-\eta) \\
&\cdot \widehat{|\psi(t')|^2}(\eta)\Pi^{\pm_0}(\xi)\Pi^{\mp_0}(\xi-\eta)\tilde{M}^{\pm_0,\mp_0}_{l,l_{12}}(\xi,\eta)\widehat{\mathrm{w}^{\mp_0}(t')}(\xi-\eta)\,\mathrm{d}\eta
\end{aligned}
$$

and

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \left[\Pi^{\pm_0}(\xi)\Pi^{\mp_0}(\xi-\eta)\tilde{M}_{l,l_{12}}^{\pm_0,\mp_0}(\xi,\eta)\right]\right| \lesssim_{\alpha,\beta} 2^{-l^+} 2^{-l_{12}} 2^{-l|\alpha|} 2^{-l_{12}|\beta|} \tilde{\chi}_l(\xi)\tilde{\chi}_{l_{12}}(\eta).$$

Hence, we can apply Lemmas 2.10 and 2.12 with $n = 2$ and $n_0 = 1$ to get

$$\left\| \langle \xi \rangle^2 \int_0^t I_{k,3,3,2}^{\pm_0,\mp_0}(t',\xi)\, \mathrm{d}t' \right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}^3 \langle t \rangle^\varepsilon.$$

□

3 Testing solutions by wave packets

The key innovation of the method of testing solutions by wave packets is to localize in both space and frequency at a time-dependent scale. In order to get some intuition for the definition of our wave packets, the following result of Hörmander (cf. [53, Thm. 7.2.5]) might be helpful:

Proposition 3.1. *Let $t > 0$, $\xi \in \mathbb{R}^3$, and*

$$\widehat{u(t)}(\xi) = e^{it\langle \xi \rangle} \widehat{\varphi}(\xi)$$

for some $\varphi \in \mathcal{S}(\mathbb{R}^3)$. Then,

$$u(t,x) = U_0(t,x) + U_1(t,x) e^{i(t^2-|x|^2)^{1/2}}, \quad x \in \mathbb{R}^3,$$

where $U_0 \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^3)$ and U_1 is a polyhomogeneous symbol of order $-3/2$ with support in $\{(t,x) \in \mathbb{R}\times\mathbb{R}^3 \colon |x| < t\}$. More precisely, $U_1(t,x)$ behaves like

$$e^{3\pi i/4} t^{-3/2} \big(1-|x/t|^2\big)^{-3/4} \sum_{j\in\mathbb{N}_0} t^{-j} \big(1-|x/t|^2\big)^{-j/2} p_j(t,x)$$

in the sense that

$$U_1(t,x) - e^{3\pi i/4} t^{-3/2} \big(1-|x/t|^2\big)^{-3/4} \sum_{j=0}^{N} t^{-j} \big(1-|x/t|^2\big)^{-j/2} p_j(t,x) \in S_{1,0}^{-3/2-N-1}$$

for any $N \in \mathbb{N}$, where $S_{1,0}^m$, $m \in \mathbb{R}$, denotes the space of symbols $a \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^3)$ such that

$$|\partial_t^\alpha \partial_\xi^\beta a(t,x)| \lesssim_{\alpha,\beta} \langle x \rangle^{m-|\beta|}, \quad \alpha \in \mathbb{N}_0, \quad \beta \in \mathbb{N}_0^3$$

and p_j are symbols of order $-3/2-j$. The leading term is

$$p_0(t,x) = (2\pi)^{-3/2} \big(1-|x/t|^2\big)^{-1/2} \widehat{\varphi}\big(-(t^2-|x|^2)^{-1/2} x\big) \mathbb{1}_{B_t(0)}(x).$$

The critical frequencies

$$\xi_{\mathrm{cr}}(t,x) = (t^2-|x|^2)^{-1/2} x, \quad |x| < t,$$

are obtained by considering the critical points of

$$\phi(t,x,\xi) = x\cdot\xi + t\langle \xi \rangle$$

since

$$\begin{aligned} u(t,x) &= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\widehat{u(t)}(\xi)\,\mathrm{d}\xi \\ &= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i\phi(t,x,\xi)}\widehat{\varphi}(\xi)\,\mathrm{d}\xi. \end{aligned}$$

Note that there are no critical points for $|x| \geq t$ which implies fast decay outside the light cone.

We define our wave packets $\Omega_v^\pm$ as follows: Let

$$\mathcal{X} \in \mathcal{C}_c^\infty(\mathbb{R}^3, [0,\infty)) \quad \text{radial,} \tag{148}$$

$$\operatorname{supp}\mathcal{X} \subseteq \overline{B_{r^{1/2}/4}(0)}, \tag{149}$$

$$\int_{\mathbb{R}^3} \mathcal{X}(x)\,\mathrm{d}x = 1, \tag{150}$$

where $r > 0$ denotes the implicit constant of (68). Then,

$$\Omega_v^\pm(t,x) := (1-|v|^2)^{-7/4} e^{\pm i\Phi(t,x)} \mathcal{X}\big(t^{-1/2}(1-|v|^2)^{-1}(x-tv)\big), \quad v \in B_1(0),$$

where

$$\Phi(t,x) = -\frac{3\pi}{4} - (t^2-|x|^2)^{1/2}, \quad x \in B_t(0). \tag{151}$$

Note that

$$\mathcal{X}\big(t^{-1/2}(1-|v|^2)^{-1}(x-tv)\big) \neq 0$$

only if

$$t^{1/2}(1-|v|^2)^{-1}|x/t-v| < r^{1/2}/4$$

which implies

$$\begin{aligned} |x/t| &\leq \frac{1-|v|^2}{4t^{1/2}} r^{1/2} + |v| \\ &< 1 \end{aligned}$$

for $t \geq r$ and $v \in B_1(0)$. Hence, $\Phi(t,x)$ is well-defined for

$$x \in \operatorname{supp}\mathcal{X}\big(t^{-1/2}(1-|v|^2)^{-1}(\cdot - tv)\big)$$

and $t \geq r$. On the spatial side, $\Omega_v^\pm$ localizes to rays

$$x = tv.$$

The localization on the Fourier side is around frequencies $\pm\xi_v$, where

$$\begin{aligned} \xi_v &= \xi_{\mathrm{cr}}(t,tv) \\ &= (1-|v|^2)^{-1/2} v. \end{aligned}$$

Note that

$$\langle \xi_v \rangle = (1-|v|^2)^{-1/2}, \tag{152}$$

$$v = \langle \xi_v \rangle^{-1} \xi_v, \tag{153}$$

$$\xi_{\langle \eta \rangle^{-1}\eta} = \eta, \quad \eta \in \mathbb{R}^3. \tag{154}$$

In the sequel, it always suffices to consider $t \geq r$, cf. (68).

Lemma 3.2. *Let $t \geq r$, $v \in B_1(0)$ and $\xi \in \mathbb{R}^3$. We have*

$$\widehat{\Omega_v^{\pm}(t)}(\xi) = (2\pi t)^{3/2} \langle \xi_v \rangle^{-6} e^{\mp it\langle \xi \rangle} \tilde{\mathcal{X}}_{t,v}\big(t^{1/2} \langle \xi_v \rangle^{-2} (\xi \mp \xi_v)\big), \tag{155}$$

where $\tilde{\mathcal{X}}_{t,v} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$ satisfies

$$\sup_{\zeta \in \mathbb{R}^3} \big| \langle \zeta \rangle^k \partial^\alpha \tilde{\mathcal{X}}_{t,v}(\zeta) \big| \lesssim_{k,\alpha} 1 \tag{156}$$

for any $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^3$. Furthermore,

$$\int_{\mathbb{R}^3} \tilde{\mathcal{X}}_{t,v}(\zeta) \, \mathrm{d}\zeta = 1 + \mathcal{O}\big(t^{-1/2} \langle \xi_v \rangle^{-2} |\xi_v|\big). \tag{157}$$

Proof. We perform a quadratic Taylor approximation for $\Phi(t, \cdot)$ in $x = tv$. Note that

$$\begin{aligned}
\partial_j \Phi(t,x) &= (t^2 - |x|^2)^{-1/2} x_j \\
&= \xi_{\mathrm{cr}}(t,x)_j, \\
\partial_j \partial_k \Phi(t,x) &= (t^2 - |x|^2)^{-3/2} x_j x_k + (t^2 - |x|^2)^{-1/2} \delta_{j,k}, \\
\partial_i \partial_j \partial_k \Phi(t,x) &= 3(t^2 - |x|^2)^{-5/2} x_i x_j x_k + (t^2 - |x|^2)^{-3/2} x_i (\delta_{i,j} + \delta_{i,k}) \\
&\quad + (t^2 - |x|^2)^{-3/2} x_j \delta_{j,k}
\end{aligned}$$

for $i, j, k = 1, 2, 3$. Hence,

$$\begin{aligned}
\Phi(t, tv) &= -\frac{3\pi}{4} - t(1 - |v|^2)^{1/2} \\
&= -\frac{3\pi}{4} - t\langle \xi_v \rangle^{-1}, && (158) \\
\partial_j \Phi(t, tv) &= \xi_{v,j}, && (159) \\
\partial_k \partial_k \Phi(t, tv) &= t^{-1}(1 - |v|^2)^{-3/2} v_j v_k + t^{-1}(1 - |v|^2)^{-1/2} \delta_{j,k} \\
&= t^{-1} \langle \xi_v \rangle (\xi_{v,j} \xi_{v,k} + \delta_{j,k})
\end{aligned}$$

for $j, k = 1, 2, 3$. The Hessian of $\Phi(t, \cdot)$ in $x = tv$ is consequently

$$\begin{aligned}
\mathrm{Hess}_x \Phi(t, tv) &= t^{-1} \langle \xi_v \rangle \begin{pmatrix} 1 + \xi_{v,1}^2 & \xi_{v,1}\xi_{v,2} & \xi_{v,1}\xi_{v,3} \\ \xi_{v,1}\xi_{v,2} & 1 + \xi_{v,2}^2 & \xi_{v,2}\xi_{v,3} \\ \xi_{v,1}\xi_{v,3} & \xi_{v,2}\xi_{v,3} & 1 + \xi_{v,3}^2 \end{pmatrix} \\
&= \lambda_{t,v}^2 H_v, && (160)
\end{aligned}$$

where

$$\lambda_{t,v} = t^{-1/2} \langle \xi_v \rangle^2 \tag{161}$$

and

$$H_v = \langle \xi_v \rangle^{-3} \begin{pmatrix} 1 + \xi_{v,1}^2 & \xi_{v,1}\xi_{v,2} & \xi_{v,1}\xi_{v,3} \\ \xi_{v,1}\xi_{v,2} & 1 + \xi_{v,2}^2 & \xi_{v,2}\xi_{v,3} \\ \xi_{v,1}\xi_{v,3} & \xi_{v,2}\xi_{v,3} & 1 + \xi_{v,3}^2 \end{pmatrix}.$$

Note that H_v is invertible with

$$H_v^{-1} = \langle \xi_v \rangle \big((1 + \langle \xi_v \rangle^2) I_3 - \langle \xi_v \rangle^3 H_v \big)$$

and

$$\det H_v = \langle \xi_v \rangle^{-7}. \tag{162}$$

The Fourier transform of $\Omega_v^{\pm}(t)$ is

$$\begin{aligned}\widehat{\Omega_v^{\pm}(t)}(\xi) &= \langle \xi_v \rangle^{7/2} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} e^{\pm i\Phi(t,x)} \mathcal{X}\big(\lambda_{t,v}(x-tv)\big)\,\mathrm{d}x \\ &= \langle \xi_v \rangle^{7/2} \int_{\mathbb{R}^3} e^{\pm i\mathcal{R}(t,x-tv)} e^{-ix\cdot\xi} e^{\pm i\Phi(t,tv)} e^{\pm i\nabla_x\Phi(t,tv)\cdot(x-tv)} \\ &\qquad \cdot e^{\pm\frac{i}{2}(x-tv)^T \mathrm{Hess}_x\Phi(t,tv)(x-tv)} \mathcal{X}\big(\lambda_{t,v}(x-tv)\big)\,\mathrm{d}x,\end{aligned}$$

where

$$\mathcal{R}(t,y) = \sum_{\alpha\in\mathbb{N}_0^3,\,|\alpha|=3} \frac{y^\alpha}{\alpha!}\partial_x^\alpha \Phi(t, tv + \vartheta_{t,y} y)$$

for some $\vartheta_{t,y} \in [0,1]$ with $|tv + \vartheta_{t,y} y| < t$. The identities (158), (159), (160) and the substitution $z = \lambda_{t,v}(x - tv)$ give

$$\begin{aligned}\widehat{\Omega_v^{\pm}(t)}(\xi) &= \langle \xi_v \rangle^{7/2} \lambda_{t,v}^{-3} \int_{\mathbb{R}^3} e^{\pm i\mathcal{R}(t,\lambda_{t,v}^{-1}z)} e^{-i\lambda_{t,v}^{-1}z\cdot\xi} e^{-itv\cdot\xi} e^{\mp 3\pi i/4} e^{\mp it\langle\xi_v\rangle^{-1}} e^{\pm i\lambda_{t,v}^{-1}\xi_v\cdot z} \\ &\qquad \cdot e^{\pm\frac{i}{2}\lambda_{t,v}^{-2} z^T \mathrm{Hess}_x\Phi(t,tv)z} \mathcal{X}(z)\,\mathrm{d}z \\ &= \langle \xi_v \rangle^{7/2} \lambda_{t,v}^{-3} e^{\mp 3\pi i/4} e^{-it(v\cdot\xi \pm \langle\xi_v\rangle^{-1})} \int_{\mathbb{R}^3} e^{\pm i\mathcal{R}(t,\lambda_{t,v}^{-1}z)} e^{-i\lambda_{t,v}^{-1}z\cdot(\xi\mp\xi_v)} \\ &\qquad \cdot e^{\pm\frac{i}{2}z^T H_v z}\mathcal{X}(z)\,\mathrm{d}z \\ &= \langle \xi_v \rangle^{7/2} \lambda_{t,v}^{-3} e^{-it(v\cdot\xi \pm \langle\xi_v\rangle^{-1})} \widehat{\tilde{\mathcal{X}}_{t,v,0}}\big(\lambda_{t,v}^{-1}(\xi \mp \xi_v)\big),\end{aligned} \tag{163}$$

where

$$\tilde{\mathcal{X}}_{t,v,0}(z) = e^{\mp 3\pi i/4} e^{\pm i\mathcal{R}(t,\lambda_{t,v}^{-1}z)} e^{\pm\frac{i}{2}z^T H_v z}\mathcal{X}(z). \tag{164}$$

Let

$$\tilde{\mathcal{X}}_{t,v}(\zeta) = (2\pi)^{-3/2} \langle \xi_v \rangle^{7/2} e^{\pm\frac{i}{2}\zeta^T H_v^{-1}\zeta} \widehat{\tilde{\mathcal{X}}_{t,v,0}}(\zeta). \tag{165}$$

Then, $\tilde{\mathcal{X}}_{t,v} \in \mathcal{S}(\mathbb{R}^3,\mathbb{C})$ and (156) follow from (165), (164), (162) and $\mathcal{X} \in \mathcal{S}(\mathbb{R}^3,\mathbb{C})$. For (157), we calculate

$$\begin{aligned}&\int_{\mathbb{R}^3} \tilde{\mathcal{X}}_{t,v}(\zeta)\,\mathrm{d}\zeta \\ &\quad = (2\pi)^{-3/2}\langle\xi_v\rangle^{7/2} e^{\mp 3\pi i/4} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{\pm\frac{i}{2}\zeta^T H_v^{-1}\zeta} e^{-iz\cdot\zeta} e^{\pm i\mathcal{R}(t,\lambda_{t,v}^{-1}z)} e^{\pm\frac{i}{2}z^T H_v z}\mathcal{X}(z)\,\mathrm{d}z\,\mathrm{d}\zeta \\ &\quad = (\det H_v)^{-1/2} e^{\mp 3\pi i/4} \int_{\mathbb{R}^3} e^{\pm i\mathcal{R}(t,\lambda_{t,v}^{-1}z)} e^{\pm\frac{i}{2}z^T H_v z}\mathcal{X}(z)(2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-iz\cdot\zeta} e^{\pm\frac{i}{2}\zeta^T H_v^{-1}\zeta}\,\mathrm{d}\zeta\,\mathrm{d}z \\ &\quad = (\det H_v)^{-1/2} e^{\mp 3\pi i/4} \int_{\mathbb{R}^3} e^{\pm i\mathcal{R}(t,\lambda_{t,v}^{-1}z)} e^{\pm\frac{i}{2}z^T H_v z}\mathcal{X}(z)\big(\det(\pm iH_v)\big)^{1/2} e^{\mp\frac{i}{2}z^T H_v z}\,\mathrm{d}z \\ &\quad = \int_{\mathbb{R}^3} e^{\pm i\mathcal{R}(t,\lambda_{t,v}^{-1}z)}\mathcal{X}(z)\,\mathrm{d}z.\end{aligned} \tag{166}$$

For the remainder, note that

$$
\begin{aligned}
&\mathcal{R}(t,\lambda_{t,v}^{-1}z)\\
&\quad= \lambda_{t,v}^{-3} \sum_{\alpha\in\mathbb{N}_0^3,\,|\alpha|=3} \frac{z^\alpha}{\alpha!}\partial_x^\alpha \Phi(t, tv + \vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z)\\
&\quad= 3t^{3/2}\langle\xi_v\rangle^{-6} \sum_{\alpha\in\mathbb{N}_0^3,\,|\alpha|=3} \frac{z^\alpha}{\alpha!}\Big(t^2 - \big|tv + \vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z\big|^2\Big)^{-5/2}\big(tv_j + \vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z_j\big)^\alpha\\
&\qquad+ 3t^{3/2}\langle\xi_v\rangle^{-6}\sum_{j=1}^3 z_j^3\Big(t^2 - \big|tv + \vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z\big|^2\Big)^{-3/2}\big(tv_j + \vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z_j\big)\\
&\qquad+ 3t^{3/2}\langle\xi_v\rangle^{-6}\sum_{j\neq k=1}^3 z_j z_k^2\Big(t^2 - \big|tv + \vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z\big|^2\Big)^{-3/2}\big(tv_j + \vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z_j\big)\\
&\quad= 3t^{-1/2}\langle\xi_v\rangle^{-6} \sum_{\alpha\in\mathbb{N}_0^3,\,|\alpha|=3} \frac{z^\alpha}{\alpha!}\Big(1 - \big|v + t^{-1/2}(1-|v|^2)\vartheta_{t,\lambda_{t,v}^{-1}z}z\big|^2\Big)^{-5/2}\\
&\qquad\qquad\qquad\cdot\big(v_j + t^{-1/2}(1-|v|^2)\vartheta_{t,\lambda_{t,v}^{-1}z}z_j\big)^\alpha\\
&\qquad+ 3t^{-1/2}\langle\xi_v\rangle^{-6}\sum_{j=1}^3 z_j^3\Big(1 - \big|v + t^{-1/2}(1-|v|^2)\vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z\big|^2\Big)^{-3/2}\\
&\qquad\qquad\qquad\cdot\big(v_j + t^{-1/2}(1-|v|^2)\vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z_j\big)\\
&\qquad+ 3t^{-1/2}\langle\xi_v\rangle^{-6}\sum_{j\neq k=1}^3 \frac{z_j z_k^2}{2}\Big(1 - \big|v + t^{-1/2}(1-|v|^2)\vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z\big|^2\Big)^{-3/2}\\
&\qquad\qquad\qquad\cdot\big(v_j + t^{-1/2}(1-|v|^2)\vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z_j\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{\alpha\in\mathbb{N}_0^3,\,|\alpha|=3} \frac{z^\alpha}{\alpha!}\Big(1 - \big|v + t^{-1/2}(1-|v|^2)\vartheta_{t,\lambda_{t,v}^{-1}z}z\big|^2\Big)^{-5/2}\big(v_j + t^{-1/2}(1-|v|^2)\vartheta_{t,\lambda_{t,v}^{-1}z}z_j\big)^\alpha\\
&\quad= \mathcal{O}\big(\langle\xi_v\rangle^5|v|^3\big)\\
&\quad= \mathcal{O}\big(\langle\xi_v\rangle^2|\xi_v|^3\big),\\
&z_j z_k^2\Big(1 - \big|v + t^{-1/2}(1-|v|^2)\vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z\big|^2\Big)^{-3/2}\big(v_j + t^{-1/2}(1-|v|^2)\vartheta_{t,\lambda_{t,v}^{-1}z}\lambda_{t,v}^{-1}z_j\big)\\
&\quad= \mathcal{O}\big(\langle\xi_v\rangle^3|v|\big)\\
&\quad= \mathcal{O}\big(\langle\xi_v\rangle^2|\xi_v|\big), \quad j,k = 1,2,3.
\end{aligned}
$$

Hence,

$$
\mathcal{R}(t,\lambda_{t,v}^{-1}z) = \mathcal{O}\big(t^{-1/2}\langle\xi_v\rangle^{-2}|\xi_v|\big)
$$

and (157) follows from (166). It remains to derive (155). Because of (163), (164) and (165), we get

$$
\widehat{\Omega_v^\pm(t)}(\xi) = (2\pi)^{3/2}\lambda_{t,v}^{-3}e^{-it(v\cdot\xi\pm\langle\xi_v\rangle^{-1})}e^{\mp\frac{i}{2}\lambda_{t,v}^{-2}(\xi\mp\xi_v)^T H_v^{-1}(\xi\mp\xi_v)}\tilde{\mathcal{X}}_{t,v}\big(\lambda_{t,v}^{-1}(\xi\mp\xi_v)\big). \tag{167}
$$

From (156), we obtain

$$
\begin{aligned}
|\xi\mp\xi_v|\,\big|\tilde{\mathcal{X}}_{t,v}\big(\lambda_{t,v}^{-1}(\xi\mp\xi_v)\big)\big| &\lesssim \lambda_{t,v}\\
&= t^{-1/2}\langle\xi_v\rangle^2.
\end{aligned} \tag{168}
$$

We have

$$v \cdot \xi \pm \langle \xi_v \rangle^{-1} = \frac{\pm \xi_v \cdot \xi + 1}{\pm \langle \xi_v \rangle}$$

and

$$\begin{aligned}
&\pm \frac{1}{2} \lambda_{t,v}^{-2} \cdot (\xi \mp \xi_v)^T H_v^{-1} (\xi \mp \xi_v) \\
&\qquad = \pm \frac{1}{2} t \langle \xi_v \rangle^{-4} \cdot \langle \xi_v \rangle \big(|\xi|^2 + |\xi_v|^2 + |\xi|^2 |\xi_v|^2 \mp 2\xi \cdot \xi_v - (\xi \cdot \xi_v)^2 \big) \\
&\qquad = \pm \frac{1}{2} t \langle \xi_v \rangle^{-3} \big(|\xi|^2 + |\xi_v|^2 + |\xi|^2 |\xi_v|^2 - (\pm \xi_v \cdot \xi + 1)^2 + 1 \big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&it\big(v \cdot \xi \pm \langle \xi_v \rangle^{-1}\big) \pm \frac{1}{2} \lambda_{t,v}^{-2} (\xi \mp \xi_v)^T H_v^{-1} (\xi \mp \xi_v) \\
&\quad = \pm \frac{1}{2} t \langle \xi_v \rangle^{-3} \big(2\langle \xi_v \rangle^2 (\pm \xi_v \cdot \xi + 1) + |\xi|^2 + |\xi_v|^2 + |\xi|^2 |\xi_v|^2 - (\pm \xi_v \cdot \xi + 1)^2 + 1 \big) \\
&\quad = \pm \frac{1}{2} t \langle \xi_v \rangle^{-3} \big(2\langle \xi_v \rangle^4 \pm 2\langle \xi_v \rangle^2 (\xi_v \cdot (\xi \mp \xi_v)) + |\xi|^2 + |\xi_v|^2 + |\xi|^2 |\xi_v|^2 \\
&\qquad - (\langle \xi_v \rangle^2 \pm \xi_v \cdot (\xi \mp \xi_v))^2 + 1 \big) \\
&\quad = \pm \frac{1}{2} t \langle \xi_v \rangle^{-3} \big(\langle \xi_v \rangle^4 + |\xi|^2 + |\xi_v|^2 + |\xi|^2 |\xi_v|^2 + 1 \big) \\
&\quad = \pm \frac{1}{2} t \langle \xi_v \rangle^{-3} \big(2\langle \xi_v \rangle^4 - |\xi|^2 + |\xi_v|^2 - |\xi|^4 + |\xi|^2 |\xi_v|^2 \big) \\
&\quad = \pm t \langle \xi_v \rangle \pm t \langle \xi_v \rangle^{-3} \langle \xi \rangle^2 (|\xi_v|^2 - |\xi|^2) \\
&\quad = \pm t \langle \xi \rangle \pm t \big(\langle \xi_v \rangle - \langle \xi \rangle \big) \pm t \langle \xi_v \rangle^{-3} \langle \xi \rangle^2 \big(|\xi| + |\xi_v| \big) \big(|\xi_v| - |\xi| \big).
\end{aligned}$$

Since

$$\begin{aligned}
\big| \langle \xi \rangle - \langle \xi_v \rangle \big| &= \frac{\big| |\xi|^2 - |\xi_v|^2 \big|}{\langle \xi \rangle + \langle \xi_v \rangle} \\
&\leq \frac{|\xi| + |\xi_v|}{\langle \xi \rangle + \langle \xi_v \rangle} \big| \xi \mp \xi_v \big|
\end{aligned}$$

and

$$\big| |\xi| - |\xi_v| \big| \leq |\xi \mp \xi_v|,$$

we obtain (155) from (167) and (168). □

3.1 Approximation by wave packets

We test the solution $\psi^\pm$ against the wave packet $\Omega_v^\pm$ as follows: Let

$$\begin{aligned}
\Upsilon^\pm(t, v) :&= \mathbb{1}_{B_1(0)}(v) \int_{\mathbb{R}^3} \overline{\Omega_v^\pm}(t, x)\, \psi^\pm(t, x)\, \mathrm{d}x \\
&= \mathbb{1}_{B_1(0)}(v) \int_{\mathbb{R}^3} \langle \xi_v \rangle^{7/2} \mathcal{X}(t^{-1/2} \langle \xi_v \rangle^2 (x - tv))\, w^\pm(t, x)\, \mathrm{d}x, \qquad (169)
\end{aligned}$$

where

$$w^{\pm}(t,x) = e^{\mp i\Phi(t,x)}\psi^{\pm}(t,x), \quad x \in B_t(0), \tag{170}$$

with Φ from (151). Parseval's identity and (155) give

$$\begin{aligned} &\Upsilon^{\pm}\big(t, \pm\langle\eta\rangle^{-1}\eta\big) \\ &= (2\pi)^{-3/2} \int_{\mathbb{R}^3} \lambda^{-3}_{t,\pm\langle\eta\rangle^{-1}\eta} \tilde{\mathcal{X}}_{t,\pm\langle\eta\rangle^{-1}\eta}\big(\lambda^{-1}_{t,\pm\langle\eta\rangle^{-1}\eta}(\xi-\eta)\big)\, \widehat{\mathrm{w}^{\pm}(t)}(\xi)\,\mathrm{d}\xi, \end{aligned} \tag{171}$$

where the profile $\mathrm{w}^{\pm}$ comes from (40) and the scaling factor $\lambda_{t,v}$ from (161).

Lemma 3.3 (Estimate for $\Upsilon^{\pm}$)**.** *Let $t \in [r,T]$ and $\psi^{\pm} \in \mathcal{C}\big([0,T], H^N(\mathbb{R}^3,\mathbb{C}^4)\big)$ satisfy a priori assumption* (44)*. Then,*

$$\big\|\langle\xi_v\rangle^{5/2}\Upsilon^{\pm}(t,v)\big\|_{L^\infty_v(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}, \tag{172}$$

$$\big\|\langle\xi_v\rangle^{3}\Upsilon^{\pm}(t,v)\big\|_{L^\infty_v(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}t^{3/4}, \tag{173}$$

$$\big\|\langle\xi_v\rangle^{5/2}\Upsilon^{\pm}(t,v)\big\|_{L^2_v(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}, \tag{174}$$

$$\big\|v\langle\xi_v\rangle^{5/2}\Upsilon^{\pm}(t,v)\big\|_{L^2_v(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}, \tag{175}$$

Proof. By definition (169) of Υ, we have

$$\begin{aligned} \big|\Upsilon^{\pm}(t,v)\big| &\le \langle\xi_v\rangle^{7/2}\big\|\mathcal{X}(t^{-1/2}\langle\xi_v\rangle^2 y)\big\|_{L^1_y(\mathbb{R}^3,\mathbb{C})}\big\|\psi^{\pm}(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \langle\xi_v\rangle^{7/2}\, t^{3/2}\langle\xi_v\rangle^{-6}\, t^{-3/2}\tilde{\delta} \\ &= \tilde{\delta}\langle\xi_v\rangle^{-5/2}. \end{aligned}$$

From (171), we get

$$\begin{aligned} \big|\Upsilon^{\pm}(t,v)\big| &\lesssim \big\|\lambda^{-3}_{t,v}\mathcal{X}(\lambda^{-1}_{t,v}y)\big\|_{L^2_y(\mathbb{R}^3,\mathbb{C})}\big\|\psi^{\pm}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \lambda^{-3/2}_{t,v}\tilde{\delta} \\ &= \tilde{\delta}t^{3/4}\langle\xi_v\rangle^{-3}. \end{aligned}$$

Since

$$\xi_{\langle\eta\rangle^{-1}\eta} = \eta,$$

the substitution

$$F\colon \mathbb{R}^3 \to B_1(0), \quad \eta \mapsto \langle\eta\rangle^{-1}\eta$$

with

$$\det DF(\eta) = \langle\eta\rangle^{-5}$$

yields

$$\begin{aligned} &\big\|\langle\xi_v\rangle^{5/2}\Upsilon^{\pm}(t,v)\big\|_{L^2_v(\mathbb{R}^3,\mathbb{C}^4)} \\ &= \bigg(\int_{\mathbb{R}^3}\Big|\langle\eta\rangle^{5/2}\Upsilon^{\pm}\big(t,\langle\eta\rangle^{-1}\eta\big)\Big|^2\langle\eta\rangle^{-5}\,\mathrm{d}\eta\bigg)^{1/2} \\ &= \bigg(\int_{\mathbb{R}^3}\Big|\int_{\mathbb{R}^3}\langle\eta\rangle^{7/2}\mathcal{X}\Big(t^{-1/2}\langle\eta\rangle^2\big(x-\langle\eta\rangle^{-1}\eta\big)\Big)w^{\pm}(t,x)\,\mathrm{d}x\Big|^2\,\mathrm{d}\eta\bigg)^{1/2}. \end{aligned}$$

Substituting

$$y = t^{-1/2}\langle\eta\rangle^2\big(x - \langle\eta\rangle^{-1}\eta\big)$$

shows

$$\begin{aligned}
&\big\|\langle\xi_v\rangle^{5/2}\Upsilon^{\pm}(t,v)\big\|_{L^2_v(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad= \bigg(\int_{\mathbb{R}^3}\Big|\int_{\mathbb{R}^3}\mathcal{X}(y)w^{\pm}\big(t, t^{1/2}\langle\eta\rangle^{-2}y + t\langle\eta\rangle^{-1}\eta\big)t^{3/2}\langle\eta\rangle^{-5/2}\,\mathrm{d}y\Big|^2\,\mathrm{d}\eta\bigg)^{1/2} \\
&\quad\le \int_{\mathbb{R}^3}\bigg(\int_{\mathbb{R}^3}\Big|\psi^{\pm}\big(t, t^{1/2}\langle\eta\rangle^{-2}y + t\langle\eta\rangle^{-1}\eta\big)\Big|^2 t^3\langle\eta\rangle^{-5}\,\mathrm{d}\eta\bigg)^{1/2}\mathcal{X}(y)\,\mathrm{d}y,
\end{aligned}$$

where we used Minkowski's inequality in the last step. Another substitution

$$G\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad \eta \mapsto t^{1/2}\langle\eta\rangle^{-2}y + t\langle\eta\rangle^{-1}\eta$$

with

$$\det DG(\eta) = t^3\langle\eta\rangle^{-5}\big(1 - 2t^{-1/2}\langle\eta\rangle^{-1}\eta\cdot y\big)$$

leads to

$$\begin{aligned}
\big\|\langle\xi_v\rangle^{5/2}\Upsilon^{\pm}(t,v)\big\|_{L^2_v(\mathbb{R}^3,\mathbb{C}^4)} &\le 2\int_{\mathbb{R}^3}\bigg(\int_{\mathbb{R}^3}\big|\psi^{\pm}(t,z)\big|^2\,\mathrm{d}z\bigg)^{1/2}\mathcal{X}(y)\,\mathrm{d}y \\
&= 2\big\|\psi^{\pm}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}
\end{aligned}$$

because of

$$\big|2t^{-1/2}\langle\eta\rangle^{-1}\eta\cdot y\big| \le 2r^{-1/2}|y| \le \frac{1}{2}$$

for any $y \in \operatorname{supp}\mathcal{X}$ and $t \ge r$. Now, estimate (174) follows from (70). Estimate (175) is an immediate consequence of (174) since $\Upsilon^{\pm}(t,v) = 0$ for $|v| \ge 1$. □

We want to approximate both, $w^{\pm}$ and $\widehat{\mathrm{w}^{\pm}}$, by $\Upsilon^{\pm}$. First, we need:

Lemma 3.4. *For any $f \in \mathcal{S}(\mathbb{R}^n,\mathbb{C}^d)$ and $s \in \mathbb{R}$ provided $n/2 < s < 1 + n/2$, we have*

$$|f(x) - f(y)| \lesssim_{s,n} |x-y|^{s-n/2}\|f\|_{\dot{H}^s(\mathbb{R}^n,\mathbb{C}^d)}. \tag{176}$$

Proof. This result is a homogeneous version of the well-known embeddings between Hölder and Sobolev spaces (cf. e.g. Evans [37, Thm. 6 in Chapter 5.6.3] for the inhomogeneous case). But for the sake of completeness, we present a proof, here. Since

$$\begin{aligned}
|f(x) - f(y)| &= (2\pi)^{-n}\Big|\int_{\mathbb{R}^n}\frac{e^{ix\cdot\xi} - e^{iy\cdot\xi}}{|\xi|^s}\,|\xi|^s\widehat{f}(\xi)\,\mathrm{d}\xi\Big| \\
&\le (2\pi)^{-n}\Big(\int_{\mathbb{R}^n}\frac{|e^{ix\cdot\xi} - e^{iy\cdot\xi}|^2}{|\xi|^{2s}}\,\mathrm{d}\xi\Big)^{1/2}\Big(\int_{\mathbb{R}^n}|\xi|^{2s}\big|\widehat{f}(\xi)\big|^2\,\mathrm{d}\xi\Big)^{1/2},
\end{aligned}$$

we need to show that

$$\int_{\mathbb{R}^n}\frac{|e^{ix\cdot\xi} - e^{iy\cdot\xi}|^2}{|\xi|^{2s}}\,\mathrm{d}\xi \lesssim_{s,n} |x-y|^{2s-n}.$$

Because of

$$\big|e^{ix\cdot\xi} - e^{iy\cdot\xi}\big| \le |\xi|\,|x-y|,$$

we have

$$\int_{|\xi|\le|x-y|^{-1}} \frac{|e^{ix\cdot\xi}-e^{iy\cdot\xi}|^2}{|\xi|^{2s}}\,\mathrm{d}\xi \le \int_{|\xi|\le|x-y|^{-1}} |\xi|^{2-2s}|x-y|^2\,\mathrm{d}\xi$$
$$= |x-y|^2 \int_0^{|x-y|^{-1}} \int_{\mathbb{S}^{n-1}} |r\omega|^{2-2s}\,\mathrm{d}\sigma(\omega)\,r^{n-1}\,\mathrm{d}r$$
$$\lesssim |x-y|^2 \int_0^{|x-y|^{-1}} r^{1-2s+n}\,\mathrm{d}r$$
$$\lesssim_{s,n} |x-y|^{2s-n},$$

where we used $s<1+n/2$ in the last step. On the other hand, the inequality

$$\left|e^{ix\cdot\xi}-e^{iy\cdot\xi}\right|\le 2$$

leads to

$$\int_{|\xi|>|x-y|^{-1}} \frac{|e^{ix\cdot\xi}-e^{iy\cdot\xi}|^2}{|\xi|^{2s}}\,\mathrm{d}\xi \lesssim \int_{|\xi|\le|x-y|^{-1}} |\xi|^{-2s}\,\mathrm{d}\xi$$
$$= \int_{|x-y|^{-1}}^{\infty} \int_{\mathbb{S}^{n-1}} |r\omega|^{-2s}\,\mathrm{d}\sigma(\omega)\,r^{n-1}\,\mathrm{d}r$$
$$\lesssim \int_{|x-y|^{-1}}^{\infty} r^{-2s+n-1}\,\mathrm{d}r$$
$$\lesssim_{s,n} |x-y|^{2s-n},$$

where we used $s>n/2$ in the last step. □

We are now in position to show:

Lemma 3.5 (Approximation of $w^\pm$ and $\widehat{\mathrm{w}^\pm}$ by $\Upsilon^\pm$)**.** *Assume that $t\in[r,T]$ and $\psi^\pm\in\mathcal{C}\big([0,T],H^N(\mathbb{R}^3,\mathbb{C}^4)\big)$ satisfies a priori assumption (44). For any $\eta\in\mathbb{R}^3$ and $v\in B_1(0)$, we have*

$$\left|\widehat{\mathrm{w}^\pm(t)}(\eta)-(2\pi)^{3/2}\Upsilon^\pm(t,\pm\langle\eta\rangle^{-1}\eta)\right| \lesssim (\delta+\tilde\delta^3)t^{2\varepsilon-1/4}\langle\eta\rangle^{-1}, \tag{177}$$
$$\left|w^\pm(t,tv)-\langle\xi_v\rangle^{5/2}t^{-3/2}\Upsilon^\pm(t,v)\right| \lesssim (\delta+\tilde\delta^3)t^{2\varepsilon-7/4}\langle\xi_v\rangle^{-1}. \tag{178}$$

Proof. Note that

$$\lambda_{t,\pm\langle\eta\rangle^{-1}\eta} = t^{-1/2}\langle\eta\rangle^2 =: \tilde\lambda_{t,\eta}$$

and

$$\int_{\mathbb{R}^3} \tilde\lambda_{t,\eta}^{-3}\mathcal{X}_{t,\pm\langle\eta\rangle^{-1}\eta}\big(\tilde\lambda_{t,\eta}(\xi-\eta)\big)\,\mathrm{d}\xi = 1+\mathcal{O}(t^{-1/2}\langle\xi_v\rangle^{-2}|\xi_v|).$$

We have

$$\begin{aligned}
&\left|\widehat{\mathrm{w}^\pm(t)}(\eta)-\Upsilon^\pm(t,\pm\langle\eta\rangle^{-1}\eta)\right| \\
&\lesssim \int_{\mathbb{R}^3} \tilde\lambda_{t,\eta}^{-3}\left|\tilde{\mathcal{X}}_{t,\pm\langle\eta\rangle^{-1}\eta}\big(\tilde\lambda_{t,\eta}^{-1}(\xi-\eta)\big)\right|\left|\widehat{\mathrm{w}^\pm(t)}(\eta)-\widehat{\mathrm{w}^\pm(t)}(\xi)\right|\mathrm{d}\xi \\
&\le \int_{\mathbb{R}^3} \langle\eta\rangle^{-2}\tilde\lambda_{t,\eta}^{-3}\left|\tilde{\mathcal{X}}_{t,\pm\langle\eta\rangle^{-1}\eta}\big(\tilde\lambda_{t,\eta}^{-1}(\xi-\eta)\big)\right|\left|\langle\eta\rangle^2\widehat{\mathrm{w}^\pm(t)}(\eta)-\langle\xi\rangle^2\widehat{\mathrm{w}^\pm(t)}(\xi)\right|\mathrm{d}\xi \\
&\quad+\int_{\mathbb{R}^3} \langle\eta\rangle^{-2}\tilde\lambda_{t,\eta}^{-3}\left|\tilde{\mathcal{X}}_{t,\pm\langle\eta\rangle^{-1}\eta}\big(\tilde\lambda_{t,\eta}^{-1}(\xi-\eta)\big)\right|\left|\langle\xi\rangle^2-\langle\eta\rangle^2\right|\left|\widehat{\mathrm{w}^\pm(t)}(\xi)\right|\mathrm{d}\xi \\
&=: I_1^\pm(t,\eta)+I_2^\pm(t,\eta).
\end{aligned}$$

Estimates (176) and (129) give

$$\begin{aligned}I_1^{\pm}(t,\eta) &\lesssim \int_{\mathbb{R}^3} \langle\eta\rangle^{-2}|\xi-\eta|^{1/2}\,\tilde{\lambda}_{t,\eta}^{-3}\left|\tilde{\mathcal{X}}_{t,\pm\langle\eta\rangle^{-1}\eta}\big(\tilde{\lambda}_{t,\eta}^{-1}(\xi-\eta)\big)\right|\mathrm{d}\xi\,\big\|\langle\xi\rangle^2\widehat{\mathrm{w}^{\pm}(t)}(\xi)\big\|_{\dot{H}^2_{\xi}(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \langle\eta\rangle^{-2}\tilde{\lambda}_{t,\eta}^{1/2}\,(\delta+\tilde{\delta}^3)t^{2\varepsilon}.\end{aligned}$$

For $I_2^{\pm}(t,\eta)$, we first note that

$$\begin{aligned}\left|\langle\xi\rangle^2-\langle\eta\rangle^2\right| &= \big||\xi|-|\eta|\big|\,\big(|\xi|+|\eta|\big)\\ &\le 2|\xi-\eta|\,|\xi|+|\xi-\eta|^2.\end{aligned} \tag{179}$$

Therefore,

$$\begin{aligned}I_2^{\pm}(t,\eta) &\lesssim \int_{\mathbb{R}^3}\langle\eta\rangle^{-2}\tilde{\lambda}_{t,\eta}^{-3}\left|\tilde{\mathcal{X}}_{t,\pm\langle\eta\rangle^{-1}\eta}\big(\tilde{\lambda}_{t,\eta}^{-1}(\xi-\eta)\big)\right||\xi-\eta|\,\left|\langle\xi\rangle\widehat{\mathrm{w}^{\pm}(t)}(\xi)\right|\mathrm{d}\xi\\ &\quad+\int_{\mathbb{R}^3}\langle\eta\rangle^{-2}\tilde{\lambda}_{t,\eta}^{-3}\left|\tilde{\mathcal{X}}_{t,\pm\langle\eta\rangle^{-1}\eta}\big(\tilde{\lambda}_{t,\eta}^{-1}(\xi-\eta)\big)\right||\xi-\eta|^2\left|\widehat{\mathrm{w}^{\pm}(t)}(\xi)\right|\mathrm{d}\xi\\ &=: I_{21}^{\pm}(t,\eta)+I_{22}^{\pm}(t,\eta).\end{aligned}$$

Let $k=1,2$. For $\langle\eta\rangle\le t^{1/4}$, we have

$$\begin{aligned}I_{2k} &\lesssim \tilde{\lambda}_{t,\eta}^{k}\langle\eta\rangle^{-2}\big\|\langle\xi\rangle\widehat{\psi^{\pm}(t)}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}t^{-k/2}\langle\eta\rangle^{2k-2}\\ &\le \tilde{\delta}t^{-1/4}\langle\eta\rangle^{-1}\\ &\lesssim \tilde{\delta}^3t^{2\varepsilon-1/4}\langle\eta\rangle^{-1}.\end{aligned}$$

For $\langle\eta\rangle\ge t^{1/4}$, note that

$$\begin{aligned}I_{2k} &\lesssim \tilde{\lambda}_{t,\eta}^{k}\langle\eta\rangle^{-2}\tilde{\lambda}_{t,\eta}^{-3}\big\|\langle\xi\rangle\widehat{\psi^{\pm}(t)}(\xi)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\lambda}_{t,\eta}^{k-3}\langle\eta\rangle^{-2}\big\|\langle\xi\rangle^5\widehat{\psi^{\pm}(t)}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}t^{(3-k)/2}\langle\eta\rangle^{2k-8}\\ &\le \tilde{\delta}t^{-1/4}\langle\eta\rangle^{-1}\\ &\lesssim \tilde{\delta}^3t^{2\varepsilon-1/4}\langle\eta\rangle^{-1}.\end{aligned}$$

Because of

$$\langle\xi_v\rangle^{5/2}t^{-3/2}=\langle\xi_v\rangle^{-7/2}\lambda_{t,v}^3$$

and

$$\lambda_{t,v}^3t^3\int_{\mathbb{R}^3}\mathcal{X}\big(\lambda_{t,v}t(y-v)\big)\,\mathrm{d}y=1,$$

estimate (176) gives

$$\begin{aligned}&\left|w^{\pm}(t,tv)-\langle\xi_v\rangle^{5/2}t^{-3/2}\Upsilon^{\pm}(t,v)\right|\\ &\quad=\Big|\int_{\mathbb{R}^3}\lambda_{t,v}^3\mathcal{X}(\lambda_{t,v}(x-tv))\big(w^{\pm}(t,tv)-w^{\pm}(t,x)\big)\,\mathrm{d}x\Big|\\ &\quad=\Big|\int_{\mathbb{R}^3}\lambda_{t,v}^3\mathcal{X}(\lambda_{t,v}(x-tv))\big(w^{\pm}(t,tv)-w^{\pm}(t,x)\big)\,\mathrm{d}x\Big|\\ &\quad\lesssim\int_{\mathbb{R}^3}|x-tv|^{1/2}\lambda_{t,v}^3\mathcal{X}(\lambda_{t,v}(x-tv))\,\mathrm{d}x\,\big\|w^{\pm}(t)\big\|_{\dot{H}^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad\lesssim\lambda_{t,v}^{-1/2}\big\|w^{\pm}(t)\big\|_{\dot{H}^2(\mathbb{R}^3,\mathbb{C}^4)}.\end{aligned}$$

Since

$$\lambda_{t,v}^{-1/2} = t^{1/4}\langle \xi_v\rangle^{-1},$$

it remains to show that

$$\big\|w^{\pm}(t)\big\|_{\dot H^2(\mathbb{R}^3,\mathbb{C}^4)} \lesssim (\delta + \tilde\delta^3)t^{2\varepsilon-2}. \tag{180}$$

To see this, we first note that

$$\begin{aligned}\partial_j w^{\pm}(t,x) &= \partial_j\big(e^{\pm i(t^2-|x|^2)^{1/2}}\psi^{\pm}(t,x)\big)\\ &= e^{\pm i(t^2-|x|^2)^{1/2}}\big(\mp i(t^2-|x|^2)^{-1/2}x_j + \partial_j\big)\psi^{\pm}(t,x)\\ &= e^{\pm i(t^2-|x|^2)^{1/2}}\Big(\mp i\langle\xi_{x/t}\rangle\frac{x_j}{t} + \partial_j\Big)\psi^{\pm}(t,x).\end{aligned}$$

A first order Taylor approximation on the Fourier side in $\xi = \pm\xi_{x/t}$ shows

$$\langle\xi_{x/t}\rangle = \langle\xi\rangle + \nabla\langle\cdot\rangle(\eta)\cdot(\xi \mp \xi_{x/t})$$

for some

$$\eta = \eta(\xi, x/t) \in \{\pm s\xi_{x/t} + (1-s)\xi \colon s\in[0,1]\}.$$

Since

$$L_j^{\pm}\psi^{\pm}(t,x) = \big(x_j \pm it\langle D\rangle^{-1}\partial_j\big)\psi^{\pm}(t,x),$$

we have

$$\begin{aligned}&\partial_j w^{\pm}(t,x)\\ &= \mp i e^{\pm i(t^2-|x|^2)^{1/2}}\Big(t^{-1}\langle D\rangle L_j^{\pm} + \frac{x_j}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\big(\pm\nabla - i\xi_{x/t}\big)\Big)\psi^{\pm}(t,x)\\ &= \mp i e^{\pm i(t^2-|x|^2)^{1/2}}\Big(t^{-1}\langle D\rangle L_j^{\pm} \pm \frac{x_j}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\Big(\nabla \mp i\langle\xi_{x/t}\rangle\frac{x}{t}\Big)\Big)\psi^{\pm}(t,x)\\ &= \mp i t^{-1}e^{\pm i(t^2-|x|^2)^{1/2}}\langle D\rangle L_j^{\pm}\psi^{\pm}(t,x) \pm \frac{x_j}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\nabla w^{\pm}(t,x). \end{aligned}\tag{181}$$

Analogously,

$$\begin{aligned}&\partial_j^2 w^{\pm}(t,x)\\ &\quad= \mp i\partial_j\Big(t^{-1}e^{\pm i(t^2-|x|^2)^{1/2}}\langle D\rangle L_j^{\pm}\psi^{\pm}(t,x) \pm \frac{x_j}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\nabla w^{\pm}(t,x)\Big)\\ &\quad= -e^{\pm i(t^2-|x|^2)^{1/2}}t^{-2}\big(\langle D\rangle L_j^{\pm}\big)^2\psi^{\pm}(t,x)\\ &\qquad - it^{-1}\frac{x_j}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\nabla\Big(e^{\pm i(t^2-|x|^2)^{1/2}}\langle D\rangle L_j^{\pm}\psi^{\pm}(t,x)\Big)\\ &\qquad \pm\partial_j\Big(\frac{x_j}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\nabla w^{\pm}(t,x)\Big)\\ &\quad= -e^{\pm i(t^2-|x|^2)^{1/2}}t^{-2}\langle D\rangle^2(L_j^{\pm})^2\psi^{\pm}(t,x) + e^{\pm i(t^2-|x|^2)^{1/2}}t^{-2}\partial_j L_j^{\pm}\psi^{\pm}(t,x)\\ &\qquad - it^{-1}\frac{x_j}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\nabla\Big(e^{\pm i(t^2-|x|^2)^{1/2}}\langle D\rangle L_j^{\pm}\psi^{\pm}(t,x)\Big)\\ &\qquad \pm\Big(\frac{x_j}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\nabla\partial_j w^{\pm}(t,x)\Big)\\ &\qquad \pm\partial_j\Big(\frac{x_j}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\Big)\cdot\nabla w^{\pm}(t,x),\end{aligned}\tag{182}$$

where we used

$$\begin{aligned} L_j^{\pm}\langle D\rangle &= \big(x_j \pm it\langle D\rangle^{-1}\partial_j\big)\langle D\rangle \\ &= \langle D\rangle^{-1}\partial_j + \langle D\rangle x_j \pm it\partial_j \\ &= \langle D\rangle^{-1}\partial_j + \langle D\rangle L_j^{\pm} \end{aligned} \tag{183}$$

for the first line of the last step. From (181), we obtain

$$\begin{aligned} \big\|\nabla w^{\pm}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &\le t^{-1}\big\|L^{\pm}\psi^{\pm}(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad + \Big\|\frac{x}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\nabla w^{\pm}(t)\Big\|_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)} \\ &\le t^{-1}\big\|L^{\pm}\psi^{\pm}(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} + d\big\|\nabla w^{\pm}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \end{aligned}$$

for some $d < 1$ and therefore,

$$\big\|\nabla w^{\pm}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \lesssim t^{-1}\big\|L^{\pm}\psi^{\pm}(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}. \tag{184}$$

From (182), we conclude that

$$\begin{aligned} &\big\|\Delta w^{\pm}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad\le 2t^{-2}\big\|(L^{\pm})^2\psi^{\pm}(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + t^{-1}\Big\|\frac{x}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\nabla\Big(e^{\pm i(t^2-|x|^2)^{1/2}}\langle D\rangle L^{\pm}\psi^{\pm}(t)\Big)\Big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + \Big\|\frac{x}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\Delta w^{\pm}(t)\Big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + Ct^{-1}\big\|\nabla w^{\pm}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad\le 2t^{-2}\big\|(L^{\pm})^2\psi^{\pm}(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + t^{-1}\Big\|\frac{x}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\nabla\Big(e^{\pm i(t^2-|x|^2)^{1/2}}\langle D\rangle L^{\pm}\psi^{\pm}(t)\Big)\Big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + d\big\|\Delta w^{\pm}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + Ct^{-2}\big\|L^{\pm}\psi^{\pm}(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \end{aligned}$$

for some $C > 0$ and $d < 1$, where we used (184) in the last step. Hence,

$$\begin{aligned} &\big\|\Delta w^{\pm}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad\lesssim t^{-2}\big\|(L^{\pm})^2\psi^{\pm}(t)\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} + t^{-2}\big\|L^{\pm}\psi^{\pm}(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + t^{-1}\Big\|\frac{x}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\nabla\Big(e^{\pm i(t^2-|x|^2)^{1/2}}\langle D\rangle L^{\pm}\psi^{\pm}(t)\Big)\Big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}. \end{aligned}$$

For the second term, a similar argument to (181) leads to

$$\begin{aligned} &\nabla\Big(e^{\pm i(t^2-|x|^2)^{1/2}}\langle D\rangle L^{\pm}\psi^{\pm}(t)\Big) \\ &\qquad = \mp it^{-1}e^{\pm i(t^2-|x|^2)^{1/2}}\big(\langle D\rangle L^{\pm}\big)^2\psi^{\pm}(t,x) \\ &\qquad\quad \pm \frac{x}{t}\langle\eta(D,x/t)\rangle^{-1}\eta(D,x/t)\cdot\nabla\Big(e^{\pm i(t^2-|x|^2)^{1/2}}\langle D\rangle L^{\pm}\psi^{\pm}(t)\Big) \end{aligned}$$

and from (183), we get

$$\begin{aligned}
&\Big\| \frac{x}{t} \langle \eta(D, x/t) \rangle^{-1} \eta(D, x/t) \cdot \nabla \Big(e^{\pm i (t^2 - |x|^2)^{1/2}} \langle D \rangle L^\pm \psi^\pm(t) \Big) \Big\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \\
&\qquad \le t^{-1} \big\| (L^\pm)^2 \psi^\pm(t) \big\|_{H^2(\mathbb{R}^3, \mathbb{C}^4)} + t^{-1} \big\| \nabla L^\pm \psi^\pm(t) \big\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \\
&\qquad\quad + \Big\| \Big| \frac{x}{t} \langle \eta(D, x/t) \rangle^{-1} \eta(D, x/t) \Big|^2 \nabla \Big(e^{\pm i (t^2 - |x|^2)^{1/2}} \langle D \rangle L^\pm \psi^\pm(t) \Big) \Big\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \\
&\qquad \le t^{-1} \big\| (L^\pm)^2 \psi^\pm(t) \big\|_{H^2(\mathbb{R}^3, \mathbb{C}^4)} + t^{-1} \big\| \nabla L^\pm \psi^\pm(t) \big\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \\
&\qquad\quad + d \Big\| \frac{x}{t} \langle \eta(D, x/t) \rangle^{-1} \eta(D, x/t) \cdot \nabla \Big(e^{\pm i (t^2 - |x|^2)^{1/2}} \langle D \rangle L^\pm \psi^\pm(t) \Big) \Big\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)}
\end{aligned}$$

for some $d < 1$. Therefore,

$$\begin{aligned}
&\Big\| \frac{x}{t} \langle \eta(D, x/t) \rangle^{-1} \eta(D, x/t) \cdot \nabla \Big(e^{\pm i (t^2 - |x|^2)^{1/2}} \langle D \rangle L^\pm \psi^\pm(t) \Big) \Big\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \\
&\qquad \lesssim t^{-1} \big\| (L^\pm)^2 \psi^\pm(t) \big\|_{H^2(\mathbb{R}^3, \mathbb{C}^4)} + t^{-1} \big\| \nabla L^\pm \psi^\pm(t) \big\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)}
\end{aligned}$$

and finally

$$\big\| \Delta w^\pm(t) \big\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \lesssim t^{-2} \big\| (L^\pm)^2 \psi^\pm(t) \big\|_{H^2(\mathbb{R}^3, \mathbb{C}^4)} + t^{-2} \big\| L^\pm \psi^\pm(t) \big\|_{H^1(\mathbb{R}^3, \mathbb{C}^4)}.$$

Because of

$$L^\pm \psi^\pm(t) = x \mathrm{w}^\pm(t)$$

and

$$(L^\pm)^2 \psi^\pm(t) = |x|^2 \mathrm{w}^\pm(t),$$

estimate (180) follows from (129). □

3.2 Asymptotic expansion

In the previous section, we have seen that we need to determine the asymptotics of $\Upsilon^\pm(t, \pm\langle\xi\rangle^{-1}\xi)$ in order to describe the asymptotics of $[\mathrm{w}^\pm(t)]^\frown(\xi) = e^{\pm it\langle\xi\rangle}[\psi^\pm(t)]^\frown(\xi)$. We consider the following ODE for $\Upsilon^\pm$:

$$\begin{aligned}
\partial_t \Upsilon^\pm(t, v) &= \int_{\mathbb{R}^3} \overline{\Omega_v^\pm}(t, x)\, \partial_t \psi^\pm(t, x) + \partial_t \overline{\Omega_v^\pm}(t, x)\, \psi^\pm(t, x)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \overline{\Omega_v^\pm}(t, x) \Big(\mp i \langle D \rangle \psi^\pm(t, x) + ic \Pi^\pm(D) \big[\big(V * |\psi(t)|^2 \big) \psi(t) \big](x) \Big) \\
&\qquad\qquad + \partial_t \overline{\Omega_v^\pm}(t, x)\, \psi^\pm(t, x)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} -i\, \overline{(-i\partial_t \pm \langle D \rangle) \Omega_v^\pm}(t, x)\, \psi^\pm(t, x)\, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} ic \overline{\Omega_v^\pm}(t, x)\, \Pi^\pm(D) \big[\big(V * |\psi(t)|^2 \big) \psi(t) \big](x)\, \mathrm{d}x \\
&=: \mathcal{L}^\pm(t, v) + \mathcal{N}^\pm(t, v).
\end{aligned}$$

For the linear part $\mathcal{L}^\pm$, we use that the wave packet $\Omega^\pm$ is an approximate solution to the linear equation. More precisely, we have:

Lemma 3.6 (Estimates for $\mathcal{L}^{\pm}$)**.** *Let $t \geq r$ and $v \in B_1(0)$. Then,*

$$|\mathcal{L}^{\pm}(t,v)| \lesssim (\delta + \tilde{\delta}^3) t^{2\varepsilon - 5/4} \langle \xi_v \rangle^{-1}, \tag{185}$$

$$|\mathcal{L}^{\pm}(t,v)| \lesssim (\delta + \tilde{\delta}^3) t^{2\varepsilon - 1} \langle \xi_v \rangle^{-2}. \tag{186}$$

Proof. Parseval's identity and (155) give

$$\begin{aligned}
&\mathcal{L}^{\pm}(t,v) \\
&= \int_{\mathbb{R}^3} \big(\partial_t \mp i\langle\xi\rangle\big) \overline{\widehat{\Omega_v^{\pm}(t)}}(\xi)\, \widehat{\psi^{\pm}(t)}(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{3/2} \int_{\mathbb{R}^3} \Big(\frac{3}{2} t^{1/2} \langle \xi_v \rangle^{-6} e^{\pm it\langle\xi\rangle}\, \overline{\tilde{\mathcal{X}}_{t,v}}\big(t^{1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\big) \\
&\qquad \pm i t^{3/2} \langle\xi_v\rangle^{-6} \langle\xi\rangle e^{\pm it\langle\xi\rangle}\, \overline{\tilde{\mathcal{X}}_{t,v}}\big(t^{1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\big) \\
&\qquad + t^{3/2}\langle\xi_v\rangle^{-6} e^{\pm it\langle\xi\rangle}\, \nabla \overline{\tilde{\mathcal{X}}_{t,v}}\big(t^{1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\big) \cdot \Big(\frac{1}{2} t^{-1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\Big) \\
&\qquad \mp i\langle\xi\rangle\, t^{3/2}\langle\xi_v\rangle^{-6} e^{\pm it\langle\xi\rangle}\, \overline{\tilde{\mathcal{X}}_{t,v}}\big(t^{1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\big)\Big)\, \widehat{\psi^{\pm}(t)}(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{3/2} \int_{\mathbb{R}^3} \Big(\frac{3}{2} t^{1/2}\langle\xi_v\rangle^{-6}\, \overline{\tilde{\mathcal{X}}_{t,v}}\big(t^{1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\big) \\
&\qquad + \frac{t}{2}\langle\xi_v\rangle^{-8}\, \nabla\overline{\tilde{\mathcal{X}}_{t,v}}\big(t^{1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\big) \cdot (\xi \mp \xi_v)\Big)\, \widehat{\mathrm{w}^{\pm}(t)}(\xi)\,\mathrm{d}\xi \\
&= \frac{(2\pi)^{3/2}}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^3} \Big(t^{1/2}\langle\xi_v\rangle^{-6}\, \overline{\tilde{\mathcal{X}}_{t,v}}\big(t^{1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\big) \\
&\qquad + t\langle\xi_v\rangle^{-8}\, \partial_{\xi_j}\overline{\tilde{\mathcal{X}}_{t,v}}\big(t^{1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\big)\, (\xi_j \mp \xi_{v,j})\Big)\, \widehat{\mathrm{w}^{\pm}(t)}(\xi)\,\mathrm{d}\xi \\
&= \frac{(2\pi)^{3/2}}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^3} \partial_{\xi_j} \Big(t^{1/2}\langle\xi_v\rangle^{-8}\, \overline{\tilde{\mathcal{X}}_{t,v}}\big(t^{1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\big)\, (\xi_j \mp \xi_{v,j})\Big) \langle\xi_v\rangle^2 \widehat{\mathrm{w}^{\pm}(t)}(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Note that

$$\begin{aligned}
\langle\xi_v\rangle^2 &= |\xi_v|^2 - |\xi|^2 + \langle\xi\rangle^2 \\
&= \big(|\xi_v| - |\xi|\big)\big(|\xi_v| + |\xi|\big) + \langle\xi\rangle^2 \\
&\leq 2|\xi_v - \xi|\,|\xi| + |\xi_v - \xi|^2 + \langle\xi\rangle^2.
\end{aligned}$$

Integration by parts leads to

$$\begin{aligned}
&\big|\mathcal{L}^{\pm}(t,v)\big| \\
&\lesssim \sum_{j=1}^{3} \int_{\mathbb{R}^3} \Big| t^{1/2}\langle\xi_v\rangle^{-8}\, \overline{\tilde{\mathcal{X}}_{t,v}}\big(t^{1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\big)\, (\xi_j \mp \xi_{v,j})\, \partial_{\xi_j}\big(\langle\xi\rangle^2 \widehat{\mathrm{w}^{\pm}(t)}(\xi)\big)\Big|\,\mathrm{d}\xi \\
&\quad + \sum_{j=1}^{3} \int_{\mathbb{R}^3} \Big| t^{1/2}\langle\xi_v\rangle^{-8}\, \overline{\tilde{\mathcal{X}}_{t,v}}\big(t^{1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\big)\, |\xi \mp \xi_v| (\xi_j \mp \xi_{v,j})\, \partial_{\xi_j}\big(|\xi| \widehat{\mathrm{w}^{\pm}(t)}(\xi)\big)\Big|\,\mathrm{d}\xi \\
&\quad + \sum_{j=1}^{3} \int_{\mathbb{R}^3} \Big| t^{1/2}\langle\xi_v\rangle^{-8}\, \overline{\tilde{\mathcal{X}}_{t,v}}\big(t^{1/2}\langle\xi_v\rangle^{-2}(\xi \mp \xi_v)\big)\, |\xi \mp \xi_v|^2 (\xi_j \mp \xi_{v,j})\, \partial_{\xi_j} \widehat{\mathrm{w}^{\pm}(t)}(\xi)\Big|\,\mathrm{d}\xi \\
&=: \sum_{k=1}^{3} I_k^{\pm}(t,v).
\end{aligned}$$

Hölder's inequality gives

$$\begin{aligned} I_1^{\pm}(t,v) &\lesssim \sum_{j=1}^{3} \big\| t^{1/2} \langle \xi_v \rangle^{-8} \tilde{\mathcal{X}}_{t,v}(t^{1/2}\langle \xi_v \rangle^{-2}\xi)\xi_j \big\|_{L_\xi^{6/5}(\mathbb{R}^3,\mathbb{C})} \big\| \langle \xi \rangle^2 \partial_{\xi_j} \widehat{\mathrm{w}^{\pm}(t)}(\xi) \big\|_{L_\xi^{6}(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \sum_{j=1}^{3} t^{-5/4} \langle \xi_v \rangle^{-1} \big\| \tilde{\mathcal{X}}_{t,v}(\xi)\xi_j \big\|_{L_\xi^{6/5}(\mathbb{R}^3,\mathbb{C})} \big\| \langle \xi \rangle^2 \partial_{\xi_j} \widehat{\mathrm{w}^{\pm}(t)}(\xi) \big\|_{H_\xi^{1}(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim t^{-5/4} \langle \xi_v \rangle^{-1} \big\| \langle \cdot \rangle^2 \mathrm{w}^{\pm}(t) \big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}, \end{aligned}$$

where we used Sobolev's embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ in the second step. For estimate (186), note that Hölder's inequality also yields

$$\begin{aligned} I_1^{\pm}(t,v) &\lesssim \sum_{j=1}^{3} \big\| t^{1/2} \langle \xi_v \rangle^{-8} \tilde{\mathcal{X}}_{t,v}(t^{1/2}\langle \xi_v \rangle^{-2}\xi)\xi_j \big\|_{L_\xi^{3/2}(\mathbb{R}^3,\mathbb{C})} \big\| \langle \xi \rangle^2 \partial_{\xi_j} \widehat{\mathrm{w}^{\pm}(t)}(\xi) \big\|_{L_\xi^{3}(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \sum_{j=1}^{3} t^{-1} \langle \xi_v \rangle^{-2} \big\| \tilde{\mathcal{X}}_{t,v}(\xi)\xi_j \big\|_{L_\xi^{3/2}(\mathbb{R}^3,\mathbb{C})} \big\| \langle \xi \rangle^2 \partial_{\xi_j} \widehat{\mathrm{w}^{\pm}(t)}(\xi) \big\|_{H_\xi^{1/2}(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim t^{-1} \langle \xi_v \rangle^{-2} \big\| \langle \cdot \rangle^{3/2} \mathrm{w}^{\pm}(t) \big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}, \end{aligned}$$

where we used Sobolev's embedding $H^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ in the second step. Similarly,

$$\begin{aligned} I_2^{\pm}(t,v) &\lesssim \sum_{j=1}^{3} \big\| t^{1/2} \langle \xi_v \rangle^{-8} \tilde{\mathcal{X}}_{t,v}(t^{1/2}\langle \xi_v \rangle^{-2}\xi)|\xi|\xi_j \big\|_{L_\xi^{2}(\mathbb{R}^3,\mathbb{C})} \big\| \partial_{\xi_j} \big(|\xi| \widehat{\mathrm{w}^{\pm}(t)}(\xi) \big) \big\|_{L_\xi^{2}(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \sum_{j=1}^{3} t^{-5/4} \langle \xi_v \rangle^{-1} \big\| \tilde{\mathcal{X}}_{t,v}(\xi)|\xi|\xi_j \big\|_{L_\xi^{2}(\mathbb{R}^3,\mathbb{C})} \big\| \langle x \rangle \mathrm{w}^{\pm}(t) \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim t^{-5/4} \langle \xi_v \rangle^{-1} \big\| \langle \cdot \rangle^2 \mathrm{w}^{\pm}(t) \big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \end{aligned}$$

and

$$\begin{aligned} I_2^{\pm}(t,v) &\lesssim \sum_{j=1}^{3} \big\| t^{1/2} \langle \xi_v \rangle^{-8} \tilde{\mathcal{X}}_{t,v}(t^{1/2}\langle \xi_v \rangle^{-2}\xi)|\xi|\xi_j \big\|_{L_\xi^{3}(\mathbb{R}^3,\mathbb{C})} \big\| \partial_{\xi_j} \big(|\xi| \widehat{\mathrm{w}^{\pm}(t)}(\xi) \big) \big\|_{L_\xi^{3/2}(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \sum_{j=1}^{3} t^{-1} \langle \xi_v \rangle^{-2} \big\| \tilde{\mathcal{X}}_{t,v}(\xi)|\xi|\xi_j \big\|_{L_\xi^{2}(\mathbb{R}^3,\mathbb{C})} \big\| \langle \xi \rangle \partial_{\xi_j} \big(|\xi| \widehat{\mathrm{w}^{\pm}(t)}(\xi) \big) \big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim t^{-1} \langle \xi_v \rangle^{-2} \big\| \langle \cdot \rangle \mathrm{w}^{\pm}(t) \big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}. \end{aligned}$$

For the last term, we have

$$\begin{aligned} I_3^{\pm}(t,v) &\lesssim \sum_{j=1}^{3} \big\| t^{1/2} \langle \xi_v \rangle^{-8} \tilde{\mathcal{X}}_{t,v}(t^{1/2}\langle \xi_v \rangle^{-2}\xi)|\xi|^2\xi_j \big\|_{L_\xi^{6/5}(\mathbb{R}^3,\mathbb{C})} \big\| \partial_{\xi_j} \widehat{\mathrm{w}^{\pm}(t)}(\xi) \big\|_{L_\xi^{6}(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \sum_{j=1}^{3} t^{-9/4} \langle \xi_v \rangle^{3} \big\| \tilde{\mathcal{X}}_{t,v}(\xi)|\xi|^2\xi_j \big\|_{L_\xi^{2}(\mathbb{R}^3,\mathbb{C})} \big\| \partial_{\xi_j} \widehat{\mathrm{w}^{\pm}(t)}(\xi) \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim t^{-9/4} \langle \xi_v \rangle^{3} \big\| \langle \cdot \rangle^2 \mathrm{w}^{\pm}(t) \big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \end{aligned}$$

which is fine for (185) in the case $\langle \xi_v \rangle \le t^{1/4}$. On the other hand,

$$
\begin{aligned}
I_3^{\pm}(t,v) &\lesssim \sum_{j=1}^{3} \big\| t^{1/2} \langle \xi_v \rangle^{-8} \tilde{\mathcal{X}}_{t,v}(t^{1/2}\langle \xi_v \rangle^{-2}\xi) |\xi|^2 \xi_j \big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C})} \big\| \partial_{\xi_j} \widehat{\mathrm{w}^{\pm}(t)}(\xi) \big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim \sum_{j=1}^{3} t^{-1} \langle \xi_v \rangle^{-2} \big\| \tilde{\mathcal{X}}_{t,v}(\xi) |\xi|^2 \xi_j \big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C})} \big\| \langle \xi \rangle^2 \partial_{\xi_j} \widehat{\mathrm{w}^{\pm}(t)}(\xi) \big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim t^{-1} \langle \xi_v \rangle^{-2} \big\| \langle x \rangle \mathrm{w}^{\pm}(t) \big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}
\end{aligned}
$$

which is sufficient for (185) in the case $\langle \xi_v \rangle \ge t^{1/4}$ and for (186). □

For the nonlinear part, denote

$$
\begin{aligned}
\mathcal{N}^{\pm}(t,v) &= ic \int_{\mathbb{R}^3} \overline{\Omega_v^{\pm}}(t,x) \big(V * |\Psi(t)|^2 \big)(x)\, \psi^{\pm}(t,x)\, \mathrm{d}x \\
&\quad + 2ic \int_{\mathbb{R}^3} \overline{\Omega_v^{\pm}}(t,x) \big(V * \mathrm{Re}\langle \psi^+(t), \psi^-(t) \rangle_{\mathbb{C}^4} \big)(x)\, \psi^{\pm}(t,x)\, \mathrm{d}x \\
&\quad + ic \int_{\mathbb{R}^3} \overline{\Omega_v^{\pm}}(t,x) \Big(\Pi^{\pm}(D) \big[\big(V * |\psi(t)|^2 \big) \psi(t) \big] - \big(V * |\psi(t)|^2 \big) \psi^{\pm}(t) \Big)(x)\, \mathrm{d}x \\
&=: c \sum_{j=1}^{3} \mathcal{N}_j^{\pm}(t,v),
\end{aligned}
$$

where

$$
\Psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}
$$

and in particular

$$
|\Psi|^2 = |\psi^+|^2 + |\psi^-|^2.
$$

We further decompose $\mathcal{N}_1^{\pm}$ into

$$
\begin{aligned}
\mathcal{N}_1^{\pm}(t,v) &= i \int_{\mathbb{R}^3} \overline{\Omega_v^{\pm}}(t,x) \big[\big(V * |\Psi(t)|^2 \big)(x) - \big(V * |\Psi(t)|^2 \big)(tv) \big] \psi^{\pm}(t,x)\, \mathrm{d}x \\
&\quad + i \Big(\big(V * |\Psi(t)|^2 \big)(tv) - t^{-1} \big(V * |\langle \xi_\bullet \rangle^{5/2} \Upsilon(t)|^2 \big)(v) \Big) \Upsilon^{\pm}(t,v) \\
&\quad + i t^{-1} \big(V * |\langle \xi_\bullet \rangle^{5/2} \Upsilon(t)|^2 \big)(v)\, \Upsilon^{\pm}(t,v) \\
&=: \sum_{j=1}^{3} \mathcal{N}_{1j}^{\pm}(t,v),
\end{aligned}
$$

where

$$
\Upsilon = \begin{pmatrix} \Upsilon^+ \\ \Upsilon^- \end{pmatrix}
$$

and consequently

$$
|\Upsilon|^2 = |\Upsilon^+|^2 + |\Upsilon^-|^2.
$$

Lemma 3.7 (Estimate for $\mathcal{N}_{11}^{\pm}$)**.** *For any $t \ge r$ and $v \in B_1(0)$, we have*

$$
|\mathcal{N}_{11}^{\pm}(t,v)| \lesssim \tilde{\delta}^3 t^{2\varepsilon - 5/4} \langle \xi_v \rangle^{-7/2}.
$$

Proof. The substitution $x = ty$ leads to

$$|\mathcal{N}_{11}^{\pm}(t,v)| \lesssim t^3 \int_{\mathbb{R}^3} \big|\big(V * |\Psi(t)|^2\big)(ty) - \big(V * |\Psi(t)|^2\big)(tv)\big| \cdot \big|\overline{\Omega_v^{\pm}}(t,ty)\Pi^{\pm}(\xi_v)\psi(t,ty)\big| \,\mathrm{d}y. \tag{187}$$

For the first factor in the integral, denote

$$\begin{aligned} &\big|\big(V * |\Psi(t)|^2\big)(ty) - \big(V * |\Psi(t)|^2\big)(tv)\big| \\ &\quad = \Big|\int_{\mathbb{R}^3} |u|^{-1}\Big(\big|\Psi(t,ty-u)\big|^2 - \big|\Psi(t,tv-u)\big|^2\Big)\,\mathrm{d}u\Big| \\ &\quad = t^2\Big|\int_{\mathbb{R}^3} |z|^{-1}\Big(\big|\Psi(t,t(y-z))\big|^2 - \big|\Psi(t,t(v-z))\big|^2\Big)\,\mathrm{d}z\Big| \\ &\quad =: t^2\Big|\int_{\mathbb{R}^3} I(t,v,y,z)\,\mathrm{d}z\Big|. \end{aligned} \tag{188}$$

Since

$$\begin{aligned} |\Psi|^2 &= |\psi^+|^2 + |\psi^-|^2, \\ |\psi^{\pm}| &= |w^{\pm}|, \end{aligned}$$

we get

$$\begin{aligned} &t^2\Big|\int_{B_2(0)} I(t,v,y,z)\,\mathrm{d}z\Big| \\ &\quad \lesssim t^2 \sum_{\pm_0\in\{+,-\}} \int_{B_2(0)} |z|^{-1}\big|w^{\pm_0}(t,t(y-z)) - w^{\pm_0}(t,t(v-z))\big|\,\big\|\psi^{\pm_0}(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}z \\ &\quad \lesssim \tilde{\delta} t^{1/2} \sum_{\pm_0\in\{+,-\}} \int_{B_2(0)} |z|^{-1}|y-v|^{1/2}\big\|w_{v-z}(t,t\cdot)\big\|_{\dot{H}^2(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}z \\ &\quad \lesssim \tilde{\delta}^2 t^{2\varepsilon-1}|y-v|^{1/2}. \end{aligned} \tag{189}$$

For $z \in B_2(0)^c$ and $v \in B_1(0)$, we have

$$\begin{aligned} |z| &\le |z-v| + |v| \\ &\le 2|z-v| \\ &\le 2\big(|z-y| + |y-v|\big). \end{aligned}$$

Therefore,

$$\begin{aligned} &t^2\Big|\int_{B_2(0)^c} I(t,v,y,z)\,\mathrm{d}z\Big| \\ &\qquad \lesssim t^2 \sum_{\pm_0\in\{+,-\}} \int_{B_2(0)^c} |z|^{-2}|z-v|\big|w^{\pm_0}(t,t(y-z)) - w^{\pm_0}(t,t(v-z))\big| \\ &\qquad\qquad \cdot \Big(\big|\psi^{\pm_0}(t,t(y-z))\big| + \big|\psi^{\pm_0}(t,t(v-z))\big|\Big)\,\mathrm{d}z. \end{aligned}$$

As before,

$$\big|w^{\pm_0}(t,t(y-z)) - w^{\pm_0}(t,t(v-z))\big| \lesssim \tilde{\delta} t^{2\varepsilon-3/2}|y-v|^{1/2}.$$

Hölder's inequality gives

$$\begin{aligned}
& t^2\Big|\int_{B_2(0)^c} I(t,v,y,z)\,\mathrm{d}z\Big| \\
&\quad \lesssim \tilde{\delta} t^{2\varepsilon+1/2}|y-v|^{1/2}\sum_{\pm_0\in\{+,-\}}\Big(\int_{B_2(0)^c}|z-v|^2 \\
&\qquad\qquad \cdot\Big(\big|\psi^{\pm_0}(t,t(y-z))\big|+\big|\psi^{\pm_0}(t,t(v-z))\big|\Big)^2\,\mathrm{d}z\Big)^{1/2} \\
&\quad \le \tilde{\delta} t^{2\varepsilon+1/2}|y-v|^{1/2}\sum_{\pm_0\in\{+,-\}}\Big(\int_{B_2(0)^c}|z-v|^2\big|\psi^{\pm_0}(t,t(v-z))\big|^2\,\mathrm{d}z\Big)^{1/2} \\
&\qquad + \tilde{\delta} t^{2\varepsilon+1/2}|y-v|^{1/2}\sum_{\pm_0\in\{+,-\}}\Big(\int_{B_2(0)^c}|z-y|^2\big|\psi^{\pm_0}(t,t(y-z))\big|^2\,\mathrm{d}z\Big)^{1/2} \\
&\qquad + \tilde{\delta} t^{2\varepsilon+1/2}|y-v|^{3/2}\sum_{\pm_0\in\{+,-\}}\Big(\int_{B_2(0)^c}\big|\psi^{\pm_0}(t,t(v-z))\big|^2\,\mathrm{d}z\Big)^{1/2} \\
&\quad =: \sum_{j=1}^{3} J_j(t,v,y). \qquad (190)
\end{aligned}$$

For J_1 and J_2, note that

$$\begin{aligned}
\max\{J_1(t,v,y),J_2(t,v,y)\} &\lesssim \tilde{\delta} t^{2\varepsilon+1/2}|y-v|^{1/2}\sum_{\pm_0\in\{+,-\}}\big\|y\psi^{\pm_0}(t,ty)\big\|_{L^2_y(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim \tilde{\delta} t^{2\varepsilon-2}|y-v|^{1/2}\sum_{\pm_0\in\{+,-\}}\big\|x\psi^{\pm_0}(t,x)\big\|_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim \tilde{\delta}^2 t^{2\varepsilon-1}|y-v|^{1/2}. \qquad (191)
\end{aligned}$$

For J_3, we have

$$\begin{aligned}
J_3(t,v,y) &\lesssim \tilde{\delta} t^{2\varepsilon+1/2}|y-v|^{3/2}\sum_{\pm_0\in\{+,-\}}\big\|\psi^{\pm_0}(t,ty)\big\|_{L^2_y(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim \tilde{\delta}^2 t^{2\varepsilon-1}|y-v|^{3/2}. \qquad (192)
\end{aligned}$$

For $k=1,3$, we obtain

$$\begin{aligned}
& t^3\int_{\mathbb{R}^3}\tilde{\delta}^2 t^{2\varepsilon-1}|y-v|^{k/2}\,\big|\overline{\Omega_v^{\pm}}(t,ty)\psi^{\pm}(t,ty)\big|\,\mathrm{d}y \\
&\lesssim \tilde{\delta}^2 t^{2\varepsilon+2}\big\|\psi^{\pm}(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\int_{\mathbb{R}^3}|y-v|^{k/2}\langle\xi_v\rangle^{7/2}\big|\mathcal{X}\big(t^{1/2}\langle\xi_v\rangle^2(y-v)\big)\big|\,\mathrm{d}y \\
&\lesssim \tilde{\delta}^3 t^{2\varepsilon+1/2}\langle\xi_v\rangle^{7/2}\big(t^{1/2}\langle\xi_v\rangle^2\big)^{-3-k/2} \\
&= \tilde{\delta}^3 t^{2\varepsilon-1-k/4}\langle\xi_v\rangle^{-5/2-k}. \qquad (193)
\end{aligned}$$

The desired estimate follows from (187)–(193). □

Lemma 3.8 (Estimate for $\mathcal{N}_{12}^{\pm}$)**.** *For any $t\ge r$ and $v\in B_1(0)$, we have*

$$|\mathcal{N}_{12}^{\pm}(t,v)|\lesssim\tilde{\delta}^3 t^{2\varepsilon-5/4}\langle\xi_v\rangle^{-7/2}.$$

Proof. Since $|\Psi|^2 = |\psi^+|^2 + |\psi^-|^2$ and $|\Upsilon|^2 = |\Upsilon^+|^2 + |\Upsilon^-|^2$, we get

$$\begin{aligned}
&|\mathcal{N}_{12}^{\pm}(t,v)| \\
&\quad\lesssim \sum_{\pm_0\in\{+,-\}} \Big|\big(V * |\psi^{\pm_0}(t)|^2\big)(tv) - t^{-1}\big(V * |\langle\xi_\bullet\rangle^{5/2}\Upsilon^{\pm_0}(t)|^2\big)(v)\Big| \cdot \tilde\delta\langle\xi_v\rangle^{-5/2} \\
&\quad\lesssim \tilde\delta\langle\xi_v\rangle^{-5/2} \sum_{\pm_0\in\{+,-\}} \Big|\int_{\mathbb{R}^3} |tv-x|^{-1}|\psi^{\pm_0}(t,x)|^2\,\mathrm{d}x \\
&\qquad\qquad - t^2\int_{\mathbb{R}^3} |v-x|^{-1}|t^{-3/2}\langle\xi_x\rangle^{5/2}\Upsilon^{\pm_0}(t,x)|^2\,\mathrm{d}x\Big| \\
&\quad\le \tilde\delta\langle\xi_v\rangle^{-5/2} \sum_{\pm_0\in\{+,-\}} t^2\int_{\mathbb{R}^3} |v-x|^{-1}\Big|\big|\psi^{\pm_0}(t,tx)\big|^2 - \big|t^{-3/2}\langle\xi_x\rangle^{5/2}\Upsilon^{\pm_0}(t,x)\big|^2\Big|\,\mathrm{d}x \\
&\quad\le \tilde\delta\langle\xi_v\rangle^{-5/2} \sum_{\pm_0\in\{+,-\}} t^2\int_{\mathbb{R}^3} |v-x|^{-1}|w^{\pm_0}(t,tx) - t^{-3/2}\langle\xi_x\rangle^{5/2}\Upsilon^{\pm_0}(t,x)| \\
&\qquad\qquad \cdot\Big(\big|\psi^{\pm_0}(t,tx)\big| + \big|t^{-3/2}\langle\xi_x\rangle^{5/2}\Upsilon^{\pm_0}(t,x)\big|\Big)\,\mathrm{d}x,
\end{aligned}$$

where we used (172) in the first step. Approximation (178) shows

$$|\mathcal{N}_{12}^{\pm}(t,v)| \lesssim \tilde\delta^2 t^{2\varepsilon+1/4}\langle\xi_v\rangle^{-7/2} \sum_{\pm_0\in\{+,-\}} \int_{\mathbb{R}^3} I^{\pm_0}(t,v,x)\,\mathrm{d}x, \tag{194}$$

where

$$I^{\pm_0}(t,v,x) = |v-x|^{-1}\Big(\big|\psi^{\pm_0}(t,tx)\big| + \big|t^{-3/2}\langle\xi_x\rangle^{5/2}\Upsilon^{\pm_0}(t,x)\big|\Big).$$

From estimates (70) and (172), we obtain

$$\int_{B_1(v)} I^{\pm_0}(t,v,x)\,\mathrm{d}x \lesssim \tilde\delta t^{-3/2}. \tag{195}$$

On the other hand, Hölder's inequality gives

$$\begin{aligned}
&\int_{B_1(v)^c} I^{\pm_0}(t,v,x)\,\mathrm{d}x \\
&\qquad\lesssim \bigg(\int_{B_1(v)^c} |v-x|^2\Big(\big|\psi^{\pm_0}(t,tx)\big| + \big|t^{-3/2}\langle\xi_x\rangle^{5/2}\Upsilon^{\pm_0}(t,x)\big|\Big)^2\,\mathrm{d}x\bigg)^{1/2}.
\end{aligned} \tag{196}$$

Since

$$|v-x|^2 \le \big(1+|x|\big)^2,$$

we get

$$\begin{aligned}
\int_{B_1(v)^c} I^{\pm_0}(t,v,x)\,\mathrm{d}x &\lesssim \Big(\big\|\langle x\rangle\psi^{\pm_0}(t,tx)\big\|_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)} + t^{-3/2}\big\|\langle x\rangle\langle\xi_x\rangle^{5/2}\Upsilon^{\pm_0}(t,x)\big\|_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)}\Big) \\
&\lesssim \tilde\delta t^{-3/2}
\end{aligned}$$

and the conclusion follows from estimates (194)–(196). □

For $\mathcal{N}_2^{\pm}$, we can no longer use that $|\psi^{\pm}| = |w^{\pm}|$. Instead, we apply estimate (178) to approximate $\langle \psi^+(t), \psi^-(t)\rangle_{\mathbb{C}^4}$ by

$$\langle \xi_\bullet \rangle^5 \big\langle e^{-i\Phi(t,t\cdot)}\Upsilon^+(t), e^{i\Phi(t,t\cdot)}\Upsilon^-(t)\big\rangle_{\mathbb{C}^4} = e^{-2i\Phi(t,t\cdot)}\langle \xi_\bullet \rangle^5 \big\langle \Upsilon^+(t), \Upsilon^-(t)\big\rangle_{\mathbb{C}^4}.$$

Essentially the same decomposition as for $\mathcal{N}_1^{\pm}$ leads to

$$\begin{aligned}
\mathcal{N}_2^{\pm}(t,v) = 2ic \int_{\mathbb{R}^3} \overline{\Omega_v^{\pm}}(t,x)\Big[&\big(V * \mathrm{Re}\big\langle \psi^+(t), \psi^-(t)\big\rangle_{\mathbb{C}^4}\big)(x) \\
&- \big(V * \mathrm{Re}\big\langle \psi^+(t), \psi^-(t)\big\rangle_{\mathbb{C}^4}\big)(tv)\Big]\psi^{\pm}(t,x)\,\mathrm{d}x \\
+ 2ic\Big[&\big(V * \mathrm{Re}\big\langle \psi^+(t), \psi^-(t)\big\rangle_{\mathbb{C}^4}\big)(tv) \\
&- t^{-1}\big(V * \mathrm{Re}\, e^{-2i\Phi(t,t\cdot)}\langle \xi_\bullet \rangle^5 \big\langle \Upsilon^+(t), \Upsilon^-(t)\big\rangle_{\mathbb{C}^4}\big)(v)\Big]\Upsilon^{\pm}(t,x) \\
+ 2ict^{-1}&\big(V * \mathrm{Re}\, e^{-2i\Phi(t,t\cdot)}\langle \xi_\bullet \rangle^5 \big\langle \Upsilon^+(t), \Upsilon^-(t)\big\rangle_{\mathbb{C}^4}\big)(v)\Upsilon^{\pm}(t,v) \\
=: 2c \sum_{j=1}^{3} \mathcal{N}_{2j}^{\pm}(t,v).&
\end{aligned}$$

Lemma 3.9 (Estimate for $\mathcal{N}_{21}^{\pm}$)**.** *For any $t \geq r$ and $v \in B_1(0)$, we have*

$$|\mathcal{N}_{21}^{\pm}(t,v)| \lesssim \tilde{\delta}^3 t^{-5/4}\langle \xi_v \rangle^{-7/2}.$$

Proof. The substitution $x = ty$ gives

$$\begin{aligned}
|\mathcal{N}_{21}^{\pm}(t,v)| \lesssim t^3 \int_{\mathbb{R}^3} &\big|\big(V * \mathrm{Re}\big\langle \psi^+(t), \psi^-(t)\big\rangle_{\mathbb{C}^4}\big)(ty) - \big(V * \mathrm{Re}\big\langle \psi^+(t), \psi^-(t)\big\rangle_{\mathbb{C}^4}\big)(tv)\big| \\
&\cdot \big|\overline{\Omega_v^{\pm}}(t,ty)\psi^{\pm}(t,ty)\big|\,\mathrm{d}y.
\end{aligned} \tag{197}$$

For the first factor in the integral, we write

$$\begin{aligned}
&\big|\big(V * \mathrm{Re}\big\langle \psi^+(t), \psi^-(t)\big\rangle_{\mathbb{C}^4}\big)(ty) - \big(V * \mathrm{Re}\big\langle \psi^+(t), \psi^-(t)\big\rangle_{\mathbb{C}^4}\big)(tv)\big| \\
&= \Big|\int_{\mathbb{R}^3} |u|^{-1}\Big(\mathrm{Re}\big\langle \psi^+(t,ty-u), \psi^-(t,ty-u)\big\rangle_{\mathbb{C}^4} \\
&\qquad\qquad - \mathrm{Re}\big\langle \psi^+(t,tv-u), \psi^-(t,tv-u)\big\rangle_{\mathbb{C}^4}\Big)\,\mathrm{d}u\Big| \\
&= t^2\Big|\int_{\mathbb{R}^3} |z|^{-1}\Big(\mathrm{Re}\big\langle \psi^+(t,t(y-z)), \psi^-(t,t(y-z))\big\rangle_{\mathbb{C}^4} \\
&\qquad\qquad - \mathrm{Re}\big\langle \psi^+(t,t(v-z)), \psi^-(t,t(v-z))\big\rangle_{\mathbb{C}^4}\Big)\,\mathrm{d}z\Big| \\
&=: t^2\Big|\int_{\mathbb{R}^3} I(t,v,y,z)\,\mathrm{d}z\Big|.
\end{aligned} \tag{198}$$

Since

$$\begin{aligned}
&\Big|\big\langle \psi^+(t,t(y-z)), \psi^-(t,t(y-z))\big\rangle_{\mathbb{C}^4} - \big\langle \psi^+(t,t(v-z)), \psi^-(t,t(v-z))\big\rangle_{\mathbb{C}^4}\Big| \\
&= \Big|\big\langle \psi^+(t,t(y-z)) - \psi^+(t,t(v-z)), \psi^-(t,t(y-z))\big\rangle_{\mathbb{C}^4} \\
&\qquad + \big\langle \psi^+(t,t(v-z)), \psi^-(t,t(y-z)) - \psi^-(t,t(v-z))\big\rangle_{\mathbb{C}^4}\Big| \\
&\leq \big|\psi^+(t,t(y-z)) - \psi^+(t,t(v-z))\big|\,\big\|\psi^-(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} \\
&\qquad + \big\|\psi^+(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\,\big|\psi^-(t,t(y-z)) - \psi^-(t,t(v-z))\big|,
\end{aligned}$$

estimate (176) and a priori assumption (44) yield

$$
\begin{aligned}
&\Big|\big\langle \psi^+(t,t(y-z)),\psi^-(t,t(y-z))\big\rangle_{\mathbb{C}^4}-\big\langle \psi^+(t,t(v-z)),\psi^-(t,t(v-z))\big\rangle_{\mathbb{C}^4}\Big|\\
&\qquad\lesssim \tilde{\delta}t^{-3/2}\Big(|y-v|^{1/2}\big\|\psi^+(t,t\cdot)\big\|_{\dot{H}^2(\mathbb{R}^3,\mathbb{C}^4)}+|y-v|^{1/2}\big\|\psi^-(t,t\cdot)\big\|_{\dot{H}^2(\mathbb{R}^3,\mathbb{C}^4)}\Big)\\
&\qquad\lesssim \tilde{\delta}t^{-3}|y-v|^{1/2}\Big(\big\|\psi^+(t,\cdot)\big\|_{\dot{H}^2(\mathbb{R}^3,\mathbb{C}^4)}+\big\|\psi^-(t,\cdot)\big\|_{\dot{H}^2(\mathbb{R}^3,\mathbb{C}^4)}\Big)\\
&\qquad\le \tilde{\delta}t^{-3}|y-v|^{1/2}\Big(\big\|\langle\xi\rangle^4\widehat{\psi^+(t)}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}+\big\|\langle\xi\rangle^4\widehat{\psi^-(t)}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\Big)\\
&\qquad\lesssim \tilde{\delta}^2t^{-3}|y-v|^{1/2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
t^2\Big|\int_{B_2(0)} I(t,v,y,z)\,\mathrm{d}z\Big| &\lesssim \tilde{\delta}^2t^{-1}|y-v|^{1/2}\int_{B_2(0)}|z|^{-1}\,\mathrm{d}z\\
&\lesssim \tilde{\delta}^2t^{-1}|y-v|^{1/2}. \qquad (199)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\Big|\big\langle \psi^+(t,t(y-z)),\psi^-(t,t(y-z))\big\rangle_{\mathbb{C}^4}-\big\langle \psi^+(t,t(v-z)),\psi^-(t,t(v-z))\big\rangle_{\mathbb{C}^4}\Big|\\
&\qquad\le \big|\psi^+(t,t(y-z))-\psi^+(t,t(v-z))\big|\,\big|\psi^-(t,t(y-z))\big|\\
&\qquad\qquad+\big|\psi^+(t,t(v-z))\big|\,\big|\psi^-(t,t(y-z))-\psi^-(t,t(v-z))\big|\\
&\qquad\lesssim \tilde{\delta}t^{-3/2}|y-v|^{1/2}\Big(\big|\psi^-(t,t(y-z))\big|+\big|\psi^+(t,t(v-z))\big|\Big).
\end{aligned}
$$

For $z\in B_2(0)^c$ and $v\in B_1(0)$, we have

$$
\begin{aligned}
|z| &\le |z-v|+|v|\\
&\le 2|z-v|\\
&\le 2\big(|z-y|+|y-v|\big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&t^2\Big|\int_{B_2(0)^c} I(t,v,y,z)\,\mathrm{d}z\Big|\\
&\qquad\lesssim \tilde{\delta}t^{1/2}|y-v|^{1/2}\int_{B_2(0)^c}|z|^{-2}|z-v|\Big(\big|\psi^-(t,t(y-z))\big|+\big|\psi^+(t,t(v-z))\big|\Big)\,\mathrm{d}z.
\end{aligned}
$$

Hölder's inequality gives

$$
\begin{aligned}
&t^2\Big|\int_{B_2(0)^c} I(t,v,y,z)\,\mathrm{d}z\Big|\\
&\lesssim \tilde{\delta}t^{1/2}|y-v|^{1/2}\Big(\int_{B_2(0)^c}|z-v|^2\Big(\big|\psi^-(t,t(y-z))\big|+\big|\psi^+(t,t(v-z))\big|\Big)^2\,\mathrm{d}z\Big)^{1/2}\\
&\le \tilde{\delta}t^{1/2}|y-v|^{1/2}\Big(\int_{B_2(0)^c}|z-v|^2\big|\psi^+(t,t(v-z))\big|^2\,\mathrm{d}z\Big)^{1/2}\\
&\qquad+\tilde{\delta}t^{1/2}|y-v|^{1/2}\Big(\int_{B_2(0)^c}|z-y|^2\big|\psi^-(t,t(y-z))\big|^2\,\mathrm{d}z\Big)^{1/2}\\
&\qquad+\tilde{\delta}t^{1/2}|y-v|^{3/2}\Big(\int_{B_2(0)^c}\big|\psi^-(t,t(y-z))\big|^2\,\mathrm{d}z\Big)^{1/2}\\
&=: J_1(t,v,y)+J_2(t,v,y)+J_3(t,v,y). \qquad (200)
\end{aligned}
$$

For J_j, $j \in \{1,2\}$, note that

$$\begin{aligned} J_j(t,v,y) &\lesssim \tilde{\delta} t^{1/2} |y-v|^{1/2} \sum_{\pm_0 \in \{+,-\}} \big\| y\psi^{\pm_0}(t,ty)\big\|_{L^2_y(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \tilde{\delta} t^{-2} |y-v|^{1/2} \sum_{\pm_0 \in \{+,-\}} \big\| x\psi^{\pm_0}(t,x)\big\|_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \tilde{\delta}^2 t^{2\varepsilon-1} |y-v|^{1/2}. \end{aligned} \tag{201}$$

For J_3, we have

$$\begin{aligned} J_3(t,v,y) &\lesssim \tilde{\delta} t^{1/2} |y-v|^{3/2} \big\| \psi^-(t,ty)\big\|_{L^2_y(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \tilde{\delta}^2 t^{-1} |y-v|^{3/2}. \end{aligned} \tag{202}$$

For $k \in \{1,3\}$, we obtain

$$\begin{aligned} &t^3 \int_{\mathbb{R}^3} \tilde{\delta}^2 t^{-1} |y-v|^{k/2} \left|\overline{\Omega_v^{\pm}}(t,ty)\psi^{\pm}(t,ty)\right| \mathrm{d}y \\ &\quad\lesssim \tilde{\delta}^2 t^2 \big\|\psi(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} \int_{\mathbb{R}^3} |y-v|^{k/2} \langle \xi_v\rangle^{7/2} \big|\mathcal{X}\big(t^{1/2}\langle\xi_v\rangle^2 (y-v)\big)\big| \,\mathrm{d}y \\ &\quad\lesssim \tilde{\delta}^3 t^{1/2} \langle\xi_v\rangle^{7/2} \big(t^{1/2}\langle\xi_v\rangle^2\big)^{-3-k/2} \\ &\quad= \tilde{\delta}^3 t^{2\varepsilon-1-k/4} \langle\xi_v\rangle^{-5/2-k}. \end{aligned} \tag{203}$$

The desired estimate follows from (197)–(203). □

Lemma 3.10 (Estimate for $\mathcal{N}_{22}^{\pm}$)**.** *Let $t \geq r$ and $v \in B_1(0)$. Then,*

$$|\mathcal{N}_{22}^{\pm}(t,v)| \lesssim \tilde{\delta}^3 t^{2\varepsilon-5/4} \langle\xi_v\rangle^{-7/2}.$$

Proof. Estimate (172) yields

$$\begin{aligned} &|\mathcal{N}_{22}^{\pm}(t,v)| \\ &\quad\lesssim \tilde{\delta}\langle\xi_v\rangle^{-5/2} \bigg| \int_{\mathbb{R}^3} |tv-z|^{-1} \mathrm{Re}\,\big\langle \psi^+(t,z), \psi^-(t,z)\big\rangle_{\mathbb{C}^4} \,\mathrm{d}z \\ &\qquad - t^2 \int_{\mathbb{R}^3} |v-u|^{-1} \mathrm{Re}\,\big[e^{-2i\Phi(t,tu)} \big\langle t^{-3/2}\langle\xi_u\rangle^{5/2}\Upsilon^+(t,u), t^{-3/2}\langle\xi_u\rangle^{5/2}\Upsilon^-(t,u)\big\rangle_{\mathbb{C}^4}\big] \,\mathrm{d}u \bigg| \end{aligned}$$

and therefore,

$$\begin{aligned} &|\mathcal{N}_{22}^{\pm}(t,v)| \\ &\lesssim \tilde{\delta}\langle\xi_v\rangle^{-5/2} t^2 \int_{\mathbb{R}^3} |v-u|^{-1} \bigg| \big\langle \psi^+(t,tu) - e^{-i\Phi(t,tu)} t^{-3/2}\langle\xi_u\rangle^{5/2}\Upsilon^+(t,u), \psi^-(t,tu)\big\rangle_{\mathbb{C}^4} \\ &\qquad + \big\langle e^{-i\Phi(t,tu)} t^{-3/2}\langle\xi_u\rangle^{5/2}\Upsilon^+(t,u), \psi^-(t,tu) - e^{i\Phi(t,tu)} t^{-3/2}\langle\xi_u\rangle^{5/2}\Upsilon^-(t,u)\big\rangle_{\mathbb{C}^4} \bigg| \,\mathrm{d}u \\ &\leq \tilde{\delta}\langle\xi_v\rangle^{-5/2} t^2 \int_{\mathbb{R}^3} |v-u|^{-1} \Big(\big|w^+(t,tu) - t^{-3/2}\langle\xi_u\rangle^{5/2}\Upsilon^+(t,x)\big| \, \big|\psi^-(t,tu)\big| \\ &\qquad + t^{-3/2}\langle\xi_u\rangle^{5/2} \big|\Upsilon^+(t,u)\big| \, \big|w^-(t,tu) - t^{-3/2}\langle\xi_u\rangle^{5/2}\Upsilon^-(t,u)\big| \Big) \,\mathrm{d}u. \end{aligned}$$

Approximation (178) leads to

$$|\mathcal{N}_{22}^{\pm}(t,v)| \lesssim \tilde{\delta}^2 t^{2\varepsilon+1/4}\langle\xi_v\rangle^{-7/2}\int_{\mathbb{R}^3} I(t,v,u)\,\mathrm{d}u, \tag{204}$$

where

$$I(t,v,u) = |v-u|^{-1}\Big(\big|\psi^-(t,tu)\big| + \big|t^{-3/2}\langle\xi_u\rangle^{5/2}\Upsilon^+(t,u)\big|\Big).$$

From estimates (70) and (172), we obtain

$$\int_{B_1(v)} I(t,v,u)\,\mathrm{d}u \lesssim \tilde{\delta}t^{-3/2}. \tag{205}$$

On the other hand, Hölder's inequality gives

$$\begin{aligned}\int_{B_1(v)^c} & I(t,v,u)\,\mathrm{d}u \\ &\lesssim \bigg(\int_{B_1(v)^c}|v-u|^2\Big(\big|\psi^-(t,tu)\big| + \big|t^{-3/2}\langle\xi_u\rangle^{5/2}\Upsilon^+(t,u)\big|\Big)^2\mathrm{d}u\bigg)^{1/2}.\end{aligned} \tag{206}$$

Since

$$|v-x|^2 \le \big(1+|x|\big)^2,$$

we get

$$\begin{aligned}\int_{B_1(v)^c} I^{\pm_0}(t,v,u)\,\mathrm{d}u &\lesssim \Big(\big\|\langle u\rangle\psi^{\pm_0}(t,tu)\big\|_{L^2_u(\mathbb{R}^3,\mathbb{C}^4)} + t^{-3/2}\big\|\langle u\rangle\langle\xi_u\rangle^{5/2}\Upsilon^{\pm_0}(t,u)\big\|_{L^2_u(\mathbb{R}^3,\mathbb{C}^4)}\Big) \\ &\lesssim \tilde{\delta}t^{-3/2}\end{aligned}$$

and the conclusion follows from estimates (204)–(206). □

Lemma 3.11 (Estimate for $\mathcal{N}_3^{\pm}$). *For any $t \ge r$ and $v \in B_1(0)$, we have*

$$|\mathcal{N}_3^{\pm}(t,v)| \lesssim \tilde{\delta}^3 t^{\varepsilon-5/4}\langle\xi_v\rangle^{-3}. \tag{207}$$

Proof. Let

$$m^{\pm}_{l,l_{12}}(\xi,\eta) = \tilde{\chi}_l(\xi)\tilde{\chi}_{l_{12}}(\eta)|\eta|^{-2}\big(\Pi^{\pm}(\xi)-\Pi^{\pm}(\xi-\eta)\big).$$

Note that

$$\pm 2\big[\Pi^{\pm}(\xi)-\Pi^{\pm}(\xi-\eta)\big] = \big(\langle\xi\rangle^{-1}-\langle\xi-\eta\rangle^{-1}\big)\gamma^0 + \gamma^0\gamma^j\Big(\frac{\xi_j}{\langle\xi\rangle}-\frac{\xi_j-\eta_j}{\langle\xi-\eta\rangle}\Big) \tag{208}$$

and

$$\big|\partial_\xi^\alpha\partial_\eta^\beta m^{\pm}_{l,l_{12}}(\xi,\eta)\big| \lesssim_{\alpha,\beta} 2^{-l^+}2^{-l_{12}}\tilde{\chi}_l(\xi)\tilde{\chi}_{l_{12}}(\eta)2^{-l|\alpha|}2^{-l_{12}|\beta|},$$

cf. also (49). Parseval's identity and Hölder's inequality lead to

$$\begin{aligned}\big|\mathcal{N}_3^{\pm}(t,v)\big| &= (2\pi)^{-2}\sum_{l,l_{12},l_3\in\mathbb{Z}}\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta)\overline{\widehat{\Omega_v(t)}}(\xi)\widehat{|\psi(t)|^2}(\eta) \\ &\qquad\qquad \cdot m^{\pm}_{l,l_{12}}(\xi,\eta)\widehat{\psi(t)}(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\Big| \\ &\le (2\pi)^{-2}t^{3/4}\langle\xi_v\rangle^{-3}\sum_{l,l_{12},l_3\in\mathbb{Z}}\Big\|\int_{\mathbb{R}^3}\chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta)\widehat{|\psi(t)|^2}(\eta) \\ &\qquad\qquad \cdot m^{\pm}_{l,l_{12}}(\xi,\eta)\widehat{\psi(t)}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}.\end{aligned}$$

It remains to show that

$$\sum_{l,l_{12},l_3\in\mathbb{Z}} \Big\| \int_{\mathbb{R}^3} \chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta)\widehat{|\psi(t)|^2}(\eta) m^{\pm}_{l,l_{12}}(\xi,\eta)\widehat{\psi(t)}(\xi-\eta)\,\mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}^3 t^{\varepsilon-2}.$$

a) First, consider $l \sim l_3 \succ l_{12}$. Lemmas 2.10, 2.9 and Hölder's and Bernstein's inequality give

$$\begin{aligned} \Big\| \int_{\mathbb{R}^3} \chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta)\widehat{|\psi(t)|^2}(\eta) m^{\pm}_{l,l_{12}}(\xi,\eta)\widehat{\psi(t)}(\xi-\eta)\,\mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\ \lesssim 2^{-l^+}2^{-l_{12}} \big\| P_{l_{12}}|\psi(t)|^2 \big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \big\| P_{l_3}\psi(t) \big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ \lesssim \tilde{\delta}^3 2^{-l^+}2^{-l_{12}} \min\{2^{3l_{12}}, 2^{-2l_{12}^+}\langle t\rangle^{-3}\} 2^{3l_3/2}2^{-8l_3^+}. \end{aligned} \tag{209}$$

Hence,

$$\begin{aligned} \sum_{l\sim l_3\succ l_{12}} \Big\| \int_{\mathbb{R}^3} \chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta)\widehat{|\psi(t)|^2}(\eta) m^{\pm}_{l,l_{12}}(\xi,\eta)\widehat{\psi(t)}(\xi-\eta)\,\mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\ \lesssim \tilde{\delta}^3 \Big(\sum_{2^{l_{12}}\le\langle t\rangle^{-1}} 2^{2l_{12}} + \sum_{2^{l_{12}}\ge\langle t\rangle^{-1}} 2^{-l_{12}}\langle t\rangle^{-3} \Big) \\ \lesssim \tilde{\delta}^3\langle t\rangle^{-2}. \end{aligned}$$

b) In the case $l \ll l_3 \sim l_{12}$, $2^l \le \langle t\rangle^{-2}$, the inequalities of Hölder, Hausdorff–Young and Bernstein yield

$$\begin{aligned} \Big\| \int_{\mathbb{R}^3} \chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta)\widehat{|\psi(t)|^2}(\eta)\, m_{l,l_{12}}(\xi,\eta)\widehat{\psi(t)}(\xi-\eta)\,\mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\ \lesssim 2^{l^+}2^{-l_{12}} \big\|\chi_l\big\|_{L^2(\mathbb{R}^3,\mathbb{C})} \big\| P_{l_{12}}|\psi(t)|^2 \big\|_{L^2(\mathbb{R}^3,\mathbb{C})} \big\| P_{l_3}\psi(t) \big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ \lesssim \tilde{\delta}^3 2^{-l^+} 2^{3l/2}2^{l_{12}/2}2^{3l_3/2}2^{-8l_3^+} \end{aligned}$$

and therefore,

$$\begin{aligned} \sum_{l\ll l_3\sim l_{12},\, 2^l\le\langle t\rangle^{-2}} \Big\| \int_{\mathbb{R}^3} \chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta)\widehat{|\psi(t)|^2}(\eta) \\ \cdot m^{\pm}_{l,l_{12}}(\xi,\eta)\widehat{\psi(t)}(\xi-\eta)\,\mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\ \lesssim \tilde{\delta}^3 \sum_{2^l\le\langle t\rangle^{-2}} 2^{3l/2} \\ \lesssim \tilde{\delta}^3\langle t\rangle^{-3}. \end{aligned}$$

c) In the case $l \ll l_3 \sim l_{12}$, $2^l \ge \langle t\rangle^{-2}$, we use again (209) to obtain

$$\begin{aligned} \sum_{l\ll l_3\sim l_{12},\, 2^l\le\langle t\rangle^{-2}} \Big\| \int_{\mathbb{R}^3} \chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta)\widehat{|\psi(t)|^2}(\eta) \\ \cdot m^{\pm}_{l,l_{12}}(\xi,\eta)\widehat{\psi(t)}(\xi-\eta)\,\mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\ \lesssim \tilde{\delta}^3 \sum_{2^{l_{12}}\le\langle t\rangle^{-1}} 2^{2l_{12}} \sum_{\langle t\rangle^{-2}\le 2^l\le\langle t\rangle^{-1}} 1 + \tilde{\delta}^3 \sum_{2^{l_{12}}\ge\langle t\rangle^{-1}} 2^{-l_{12}}\langle t\rangle^{-3} \sum_{2^l\ge\langle t\rangle^{-2}} 2^{-l^+} \\ \lesssim \tilde{\delta}^3\langle t\rangle^{-2+\varepsilon}. \end{aligned}$$

d) Finally, for the case $l_{12} \sim l \gg l_3$, estimate (209) gives

$$\begin{aligned}
&\sum_{l_{12}\sim l\gg l_3} \Big\| \int_{\mathbb{R}^3} \chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta)\widehat{|\psi(t)|^2}(\eta)\, m^\pm_{l,l_{12}}(\xi,\eta)\widehat{\psi(t)}(\xi-\eta)\,\mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\qquad \lesssim \tilde{\delta}^3 \Big(\sum_{2^{l_{12}}\le\langle t\rangle^{-1}} 2^{2l_{12}} + \sum_{2^{l_{12}}\ge\langle t\rangle^{-1}} 2^{-l_{12}}\langle t\rangle^{-3} \Big) \sum_{l_3} 2^{3l_3/2} 2^{-8l_3^+} \\
&\qquad \lesssim \tilde{\delta}^3 \langle t\rangle^{-2}.
\end{aligned}$$

□

Remark. In the previous proof, we essentially used the fact that

$$|\Pi^\pm(\xi) - \Pi^\pm(\xi-\eta)| \lesssim |\eta|$$

cancels part of the singularity $|\eta|^{-2}$ coming from the Fourier transform of the Coulomb potential. Another way to argue that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \overline{\Omega^\pm_v}(t,x)\Pi^\pm(D)\big[\big(V * |\psi|^2\big)(x)\psi(t,x)\big]\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} \overline{\Omega^\pm_v}(t,x)\big(V * |\psi|^2\big)(x)\big[\Pi^\pm(D)\psi^\pm(t)\big](x)\,\mathrm{d}x + \mathcal{O}(\langle t\rangle^{\varepsilon-5/4}\langle\xi_v\rangle^{-3})
\end{aligned}$$

is replacing $\Pi^\pm(D)$ with $\Pi^\pm(\xi_v)$ by using the localization of $\Omega^\pm_v(t)$ on the Fourier side.

3.3 Control of $\langle\xi\rangle^8\widehat{\psi^\pm(t)}(\xi)$ and proof of the main result

In the previous section, we have shown that

$$\begin{aligned}
\partial_t\Upsilon^\pm(t,v) &= c\mathcal{N}^\pm_{13}(t,v) + 2c\mathcal{N}^\pm_{23}(t,v) \\
&\qquad + \mathcal{O}\big((\delta+\tilde{\delta}^3)\min\big\{\langle t\rangle^{2\varepsilon-5/4}\langle\xi_v\rangle^{-1}, \langle t\rangle^{2\varepsilon-1}\langle\xi_v\rangle^{-2}\big\}\big). \qquad (210)
\end{aligned}$$

The error term is uniformly integrable in time leading to an $\mathcal{O}\big((\delta+\tilde{\delta})^3\langle t\rangle^{2\varepsilon-1/4}\langle\xi_v\rangle^{-1}\big)$-error for $\Upsilon^\pm(t,v)$. In contrast to $\mathcal{N}^\pm_{13}(t,v)$, the contribution $\mathcal{N}^\pm_{23}(t,v)$ will not affect the modified scattering behavior because of the oscillatory phase term $e^{-2i\Phi(t,t\cdot)}$. More precisely, we have:

Lemma 3.12 (Estimate for $\mathcal{N}^\pm_{23}$)**.** *Let $t \ge r$ and $v \in B_1(0)$. Then,*

$$\Big| \int_t^\infty \mathcal{N}^\pm_{23}(t',v)\,\mathrm{d}t' \Big| \lesssim (\delta+\tilde{\delta}^3)\langle t\rangle^{2\varepsilon-3/8}\langle\xi_v\rangle^{-5/2}. \qquad (211)$$

Proof. We have

$$\begin{aligned}
&\int_t^\infty \mathcal{N}^\pm_{23}(t',v)\,\mathrm{d}t' \\
&\quad = i\int_t^\infty (t')^{-1}\Big(V * \mathrm{Re}\big[e^{-2i\Phi(t',t'\cdot)}\langle\xi_\bullet\rangle^5\big\langle\Upsilon^+(t'),\Upsilon^-(t')\big\rangle_{\mathbb{C}^4}\big]\Big)(v)\Upsilon^\pm(t',v)\,\mathrm{d}t'.
\end{aligned}$$

Since

$$\Phi(t,tu) = -\frac{3\pi}{4} - t\langle\xi_u\rangle^{-1},$$

we get

$$e^{-2i\Phi(t,tu)} = \frac{\langle \xi_u \rangle}{2i} \frac{\mathrm{d}}{\mathrm{d}t} e^{-2i\Phi(t,tu)}.$$

Therefore, we integrate by parts to obtain

$$\begin{aligned}
&\int_t^\infty \mathcal{N}_{23}^\pm(t',v)\,\mathrm{d}t' \\
&\quad = -it^{-1}\Big(V * \mathrm{Re}\big[e^{-2i\Phi(t,t\cdot)}\langle \xi_\bullet \rangle^6 \big\langle \Upsilon^+(t), \Upsilon^-(t)\big\rangle_{\mathbb{C}^4}\big]\Big)(v)\,\Upsilon^\pm(t,v) \\
&\qquad - i\int_t^\infty (t')^{-2}\Big(V * \mathrm{Re}\big[e^{-2i\Phi(t',t'\cdot)}\langle \xi_\bullet \rangle^6 \big\langle \Upsilon^+(t'), \Upsilon^-(t')\big\rangle_{\mathbb{C}^4}\big]\Big)(v)\,\Upsilon^\pm(t',v)\,\mathrm{d}t' \\
&\qquad + i\int_t^\infty (t')^{-1}\Big(V * \mathrm{Re}\big[e^{-2i\Phi(t',t'\cdot)}\langle \xi_\bullet \rangle^6 \big\langle \partial_{t'}\Upsilon^+(t'), \Upsilon^-(t')\big\rangle_{\mathbb{C}^4}\big]\Big)(v)\,\Upsilon^\pm(t',v)\,\mathrm{d}t' \\
&\qquad + i\int_t^\infty (t')^{-1}\Big(V * \mathrm{Re}\big[e^{-2i\Phi(t',t'\cdot)}\langle \xi_\bullet \rangle^6 \big\langle \Upsilon^+(t'), \partial_t'\Upsilon^-(t')\big\rangle_{\mathbb{C}^4}\big]\Big)(v)\,\Upsilon^\pm(t',v)\,\mathrm{d}t' \\
&\qquad + i\int_t^\infty (t')^{-1}\Big(V * \mathrm{Re}\big[e^{-2i\Phi(t',t'\cdot)}\langle \xi_\bullet \rangle^6 \big\langle \Upsilon^+(t'), \Upsilon^-(t')\big\rangle_{\mathbb{C}^4}\big]\Big)(v)\,\partial_{t'}\Upsilon^\pm(t',v)\,\mathrm{d}t' \\
&\quad =: I_1^\pm(t,v) + \sum_{k=2}^5 \int_t^\infty I_k^\pm(t',v)\,\mathrm{d}t'.
\end{aligned}$$

The Lorentz-type versions of Young's and Hölder's inequality yield

$$\begin{aligned}
&\big\|V * \mathrm{Re}\big[e^{-2i\Phi(t,t\cdot)}\langle \xi_\bullet \rangle^6 \big\langle \Upsilon^+(t), \Upsilon^-(t)\big\rangle_{\mathbb{C}^4}\big]\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \\
&\qquad \lesssim \big\|\langle \xi_\bullet \rangle^6 \big\langle \Upsilon^+(t), \Upsilon^-(t)\big\rangle_{\mathbb{C}^4}\big\|_{L^{3/2,1}(\mathbb{R}^3,\mathbb{C})} \\
&\qquad \lesssim \big\|\langle \xi_\bullet \rangle\big\|_{L^{12/7}(B_1(0),\mathbb{C})}\big\|\langle \xi_\bullet \rangle^5 \big\langle \Upsilon^+(t), \Upsilon^-(t)\big\rangle_{\mathbb{C}^4}\big\|_{L^{12,12/5}(\mathbb{R}^3,\mathbb{C})} \\
&\qquad \lesssim \big\|\langle \xi_\bullet \rangle^5 \big\langle \Upsilon^+(t), \Upsilon^-(t)\big\rangle_{\mathbb{C}^4}\big\|_{L^8(\mathbb{R}^3,\mathbb{C})}^{1/3}\big\|\langle \xi_\bullet \rangle^5 \big\langle \Upsilon^+(t), \Upsilon^-(t)\big\rangle_{\mathbb{C}^4}\big\|_{L^{16}(\mathbb{R}^3,\mathbb{C})}^{2/3} \\
&\qquad \lesssim \big\|\langle \xi_\bullet \rangle^{5/2}\Upsilon^+(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\big\|\langle \xi_\bullet \rangle^{5/2}\Upsilon^-(t)\big\|_{L^8(\mathbb{R}^3,\mathbb{C}^4)}^{1/3}\big\|\langle \xi_\bullet \rangle^{5/2}\Upsilon^-(t)\big\|_{L^{16}(\mathbb{R}^3,\mathbb{C}^4)}^{2/3} \\
&\qquad \lesssim \tilde{\delta}^2, \qquad (212)
\end{aligned}$$

where we used interpolation in the third step as well as (174), (172) (and interpolation) in the last step. For the fourth step, note that

$$\begin{aligned}
\big\|\langle \xi_\bullet \rangle\big\|_{L^{12/7}(B_1(0),\mathbb{C})}^{12/7} &= \int_0^1 \int_{\mathbb{S}^2} \big(1 - |r\omega|^2\big)^{-6/7}\,\mathrm{d}\sigma(\omega)\,r^2\,\mathrm{d}r \\
&\lesssim 1.
\end{aligned}$$

For $I_1^\pm$, estimates (212) and (172) imply immediately

$$|I_1^\pm(t,v)| \lesssim t^{-1}\tilde{\delta}^3\langle \xi_v \rangle^{-5/2}.$$

Similarly,

$$|I_2^\pm(t',v)| \lesssim (t')^{-2}\tilde{\delta}^3\langle \xi_v \rangle^{-5/2}$$

and therefore,

$$\Big|\int_t^\infty I_2^\pm(t',v)\,\mathrm{d}t'\Big| \lesssim t^{-1}\tilde{\delta}^3\langle \xi_v \rangle^{-5/2}.$$

For $I_3^\pm$, $I_4^\pm$ and $I_5^\pm$, it suffices to use the following weaker estimate for $\partial_{t'}\Upsilon^\pm(t',v)$: Lorentz-type Young's inequality together with estimates (174), (172) (and interpolation) gives

$$\begin{aligned}\big\|V * |\langle\xi_\bullet\rangle^{5/2}\Upsilon(t)|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} &\lesssim \big\|\langle\xi_\bullet\rangle^{5/2}\Upsilon(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\langle\xi_\bullet\rangle^{5/2}\Upsilon(t)\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde\delta^2\end{aligned}$$

and

$$\begin{aligned}&\big\|V * \mathrm{Re}\big[e^{-2i\Phi(t,t\cdot)}\langle\xi_\bullet\rangle^5\big\langle\Upsilon^+(t),\Upsilon^-(t)\big\rangle_{\mathbb{C}^4}\big]\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\\ &\quad\lesssim \sum_{\pm_1,\pm_2\in\{+,-\}}\big\|\langle\xi_\bullet\rangle^{5/2}\Upsilon^{\pm_1}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\langle\xi_\bullet\rangle^{5/2}\Upsilon^{\pm_2}(t)\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad\lesssim \tilde\delta^2.\end{aligned}$$

Applying again (172) leads to

$$\begin{aligned}|\partial_{t'}\Upsilon^\pm(t',v)| &\lesssim |\mathcal{N}_{13}^\pm(t',v)| + |\mathcal{N}_{23}^\pm(t,v)| + (\delta+\tilde\delta^3)t^{2\varepsilon-1}\langle\xi_v\rangle^{-2}\\ &\lesssim (\delta+\tilde\delta^3)t^{2\varepsilon-1}\langle\xi_v\rangle^{-2}.\end{aligned}\tag{213}$$

Hence,

$$\begin{aligned}\Big|\int_t^\infty I_5(t',v)\,\mathrm{d}t'\Big| &\lesssim \int_t^\infty (t')^{-1}(\delta+\tilde\delta^3)(t')^{2\varepsilon-1}\langle\xi_v\rangle^{-2}\,\mathrm{d}t'\\ &\lesssim t^{2\varepsilon-1}(\delta+\tilde\delta^3)\langle\xi_v\rangle^{-2}.\end{aligned}$$

For $I_3^\pm$ and $I_4^\pm$, we need to modify (212) slightly: We have

$$\begin{aligned}&\big\|V * \mathrm{Re}\big[e^{-2i\Phi(t,t\cdot)}\langle\xi_\bullet\rangle^6\big\langle\partial_t'\Upsilon^+(t),\Upsilon^-(t)\big\rangle_{\mathbb{C}^4}\big]\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\\ &\quad\lesssim \big\|\langle\xi_\bullet\rangle^{9/8}\big\|_{L^{12/7}(B_1(0),\mathbb{C})}\big\|\langle\xi_\bullet\rangle^{23/8}\Upsilon^-(t)\big\|^{1/3}_{L^8(\mathbb{R}^3,\mathbb{C}^4)}\big\|\langle\xi_\bullet\rangle^{23/8}\Upsilon^-(t)\big\|^{2/3}_{L^{16}(\mathbb{R}^3,\mathbb{C}^4)}\\ &\qquad\cdot\big\|\langle\xi_\bullet\rangle^2\partial_{t'}\Upsilon^+(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad\lesssim \tilde\delta^2 t^{2\varepsilon-3/8},\end{aligned}$$

where the second step follows from (213), interpolation of (174) and (173) as well as

$$\begin{aligned}\big\|\langle\xi_\bullet\rangle^{9/8}\big\|^{12/7}_{L^{12/7}(B_1(0),\mathbb{C})} &= \int_0^1\int_{\mathbb{S}^2}\big(1-|r\omega|^2\big)^{-13/14}\,\mathrm{d}\sigma(\omega)\,r^2\,\mathrm{d}r\\ &\lesssim 1.\end{aligned}$$

In the same way, we obtain

$$\big\|V * \mathrm{Re}\big[e^{-2i\Phi(t,t\cdot)}\langle\xi_\bullet\rangle^6\big\langle\Upsilon^+(t),\partial_t'\Upsilon^-(t)\big\rangle_{\mathbb{C}^4}\big]\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \lesssim \tilde\delta^2 t^{2\varepsilon-3/8}.$$

Hence,

$$\begin{aligned}\Big|\int_t^\infty I_k(t',v)\,\mathrm{d}t'\Big| &\lesssim \int_t^\infty (t')^{-1}(\delta+\tilde\delta^3)(t')^{2\varepsilon-3/8}\langle\xi_v\rangle^{-5/2}\,\mathrm{d}t'\\ &\lesssim t^{2\varepsilon-3/8}(\delta+\tilde\delta^3)\langle\xi_v\rangle^{-5/2}\end{aligned}$$

for $k=3,4$. □

Remark. Note that in the proof, we used the trivial estimate

$$|\mathcal{N}_{23}^{\pm}(t,v)| \lesssim (\delta + \tilde{\delta}^3) t^{-1} \langle \xi_v \rangle^{-5/2}$$

in order to derive the stronger estimate (211) via integration by parts.

From (210) and (211), we conclude that $\Upsilon^{\pm}$ solves the ODE

$$\begin{aligned}\partial_t \Upsilon^{\pm}(t,v) &= \mathcal{N}_{13}^{\pm}(t,v) + \mathcal{O}\big((\delta+\tilde{\delta}^3) t^{2\varepsilon-5/4} \langle \xi_v \rangle^{-1}\big) \\ &= ict^{-1}\big(V * |\langle \xi_\bullet \rangle^{5/2} \Upsilon(t)|^2\big)(v) \Upsilon^{\pm}(t,v) + \mathcal{O}\big((\delta+\tilde{\delta}^3) t^{2\varepsilon-5/4} \langle \xi_v \rangle^{-1}\big)\end{aligned}$$

which leads to

$$\Upsilon^{\pm}(t,v) = e^{ic\log(t)(V*|\langle \xi_\bullet \rangle^{5/2} \tilde{W}|^2)(v)} \tilde{W}^{\pm}(v) + \mathcal{O}\big((\delta+\tilde{\delta}^3) t^{2\varepsilon-1/4} \langle \xi_v \rangle^{-1}\big) \tag{214}$$

for some asymptotic state

$$\tilde{W} \colon \mathbb{R}^3 \to \mathbb{C}^4, \quad \tilde{W} = \begin{pmatrix} \tilde{W}^+ \\ \tilde{W}^- \end{pmatrix}.$$

Note that

$$\big|\tilde{W}^{\pm}(v)\big| = \big|\Upsilon^{\pm}(t,v)\big| + \mathcal{O}\big((\delta+\tilde{\delta}^3) t^{2\varepsilon-1/4} \langle \xi_v \rangle^{-1}\big)$$

which also implies

$$\tilde{W}(v) = 0 \quad \text{for } |v| \geq 1$$

and

$$\big\| \langle \xi_\bullet \rangle^{5/2} \tilde{W}^{\pm} \big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}.$$

Let

$$\begin{aligned} &W^{\pm} \colon \mathbb{R}^3 \to \mathbb{C}^4, \quad W^{\pm}(\xi) = \tilde{W}^{\pm}\big(\pm \langle \xi \rangle^{-1} \xi\big), \\ &W = \begin{pmatrix} W^+ \\ W^- \end{pmatrix}. \end{aligned}$$

Then,

$$\begin{aligned}\big(V * |\langle \xi_\bullet \rangle^{5/2} \tilde{W}|^2\big)\big(\pm \langle \xi \rangle^{-1} \xi\big) &= \int_{B_1(0)} \Big| \pm \frac{\xi}{\langle \xi \rangle} - x \Big|^{-1} \big|\langle \xi_x \rangle^{5/2} \tilde{W}(x)\big|^2 \,\mathrm{d}x \\ &= \int_{\mathbb{R}^3} \Big| \pm \frac{\xi}{\langle \xi \rangle} - \frac{\sigma}{\langle \sigma \rangle} \Big|^{-1} \big|\langle \sigma \rangle^{5/2} \tilde{W}\big(\langle \sigma \rangle^{-1} \sigma\big)\big|^2 \langle \sigma \rangle^{-5} \,\mathrm{d}\sigma \\ &= \int_{\mathbb{R}^3} \Big| \frac{\xi}{\langle \xi \rangle} \mp \frac{\sigma}{\langle \sigma \rangle} \Big|^{-1} \Big(\big|W^+(\sigma)\big|^2 + \big|W^-(-\sigma)\big|^2 \Big) \,\mathrm{d}\sigma.\end{aligned}$$

By estimates (177) and (214), we obtain

$$\begin{aligned}\widehat{\mathrm{w}^{\pm}(t)}(\xi) &= (2\pi)^{3/2} \Upsilon^{\pm}\big(t, \pm \langle \xi \rangle^{-1} \xi\big) + \mathcal{O}\big((\delta+\tilde{\delta}^3) t^{2\varepsilon-1/4} \langle \xi \rangle^{-1}\big) \\ &= (2\pi)^{3/2} e^{ic\log(t)\mathcal{B}^{\pm}(\xi)} W^{\pm}(\xi) + \mathcal{O}\big((\delta+\tilde{\delta}^3) t^{2\varepsilon-1/4} \langle \xi \rangle^{-1}\big),\end{aligned} \tag{215}$$

where $\mathcal{B}^{\pm}$ has been introduced in (35).

Remark. Note that

$$\begin{aligned}
\mathcal{N}^{\pm}_{13}(t,\pm\langle\xi\rangle^{-1}\xi) &= ict^{-1}\Big(\int_{\mathbb{R}^3}\Big|\pm\frac{\xi}{\langle\xi\rangle}-x\Big|^{-1}\big|\langle\xi_x\rangle^{5/2}\Upsilon(t,x)\big|^2\,\mathrm{d}x\Big)\Upsilon^{\pm}\big(t,\pm\langle\xi\rangle^{-1}\xi\big)\\
&= ict^{-1}\Big(\int_{\mathbb{R}^3}\Big|\pm\frac{\xi}{\langle\xi\rangle}-\frac{\sigma}{\langle\sigma\rangle}\Big|^{-1}\Big|\langle\sigma\rangle^{5/2}\Upsilon\big(t,\langle\sigma\rangle^{-1}\sigma\big)\Big|^2\langle\sigma\rangle^{-5}\,\mathrm{d}\sigma\Big)\Upsilon^{\pm}\big(t,\pm\langle\xi\rangle^{-1}\xi\big)\\
&= ict^{-1}\Big(\int_{\mathbb{R}^3}\Big|\frac{\xi}{\langle\xi\rangle}\mp\frac{\sigma}{\langle\sigma\rangle}\Big|^{-1}\Big|\Upsilon\big(t,\langle\sigma\rangle^{-1}\sigma\big)\Big|^2\,\mathrm{d}\sigma\Big)\Upsilon^{\pm}(t,\pm\langle\xi\rangle^{-1}\xi)\\
&= ict^{-1}\Big(\int_{\mathbb{R}^3}\Big|\frac{\xi}{\langle\xi\rangle}\mp\frac{\sigma}{\langle\sigma\rangle}\Big|^{-1}\Big(\big|\widehat{\psi^+(t)}(\sigma)\big|^2+\big|\widehat{\psi^-(t)}(-\sigma)\big|^2\Big)\,\mathrm{d}\sigma\Big)\Upsilon^{\pm}\big(t,\pm\langle\xi\rangle^{-1}\xi\big)\\
&\quad+\mathcal{O}(\tilde{\delta}^3t^{2\varepsilon-5/4}\langle\xi\rangle^{-1}),
\end{aligned}$$

where we used (177) in the last step. Solving the corresponding approximate ODE leads to

$$\begin{aligned}
&\widehat{\mathrm{w}^{\pm}(t)}(\xi)\\
&\quad= (2\pi)^{3/2}\exp\Big(ic\int_0^t(t')^{-1}\int_{\mathbb{R}^3}\Big|\frac{\xi}{\langle\xi\rangle}\mp\frac{\sigma}{\langle\sigma\rangle}\Big|^{-1}\Big(|\widehat{\psi^+(t)}(\sigma)|^2+|\widehat{\psi^-(t)}(-\sigma)|^2\Big)\,\mathrm{d}\sigma\,\mathrm{d}t'\Big)\\
&\qquad\cdot W^{\pm}(\xi)\\
&\qquad+\mathcal{O}\big((\delta+\tilde{\delta}^3)t^{2\varepsilon-1/4}\langle\xi\rangle^{-1}\big)
\end{aligned}$$

which coincides with Pusateri's result in [84, Thm. 1.1] for the scalar toy model.

In order to complete our bootstrap argument, we show:

Lemma 3.13. *Suppose that $\psi^\pm\in\mathcal{C}\big([0,T],H^N(\mathbb{R}^3)\big)$ satisfies a priori assumption (44) and let $t\in[r,T]$. Then,*

$$\big|\langle\xi\rangle^8\widehat{\mathrm{w}^{\pm}(t)}(\xi)\big|\lesssim\delta\langle t\rangle^{-1/56}\quad\textit{for }\langle\xi\rangle\geq\langle t\rangle^{1/56}\tag{216}$$

and

$$\big\|\langle\xi\rangle^8\widehat{\mathrm{w}^{\pm}(t)}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\lesssim\delta+\tilde{\delta}^3.\tag{217}$$

For arbitrary $t\geq r$, we have

$$\Big|\langle\xi\rangle^8\Big(\widehat{\mathrm{w}^{\pm}(t)}(\xi)-(2\pi)^{3/2}e^{ic\log(t)\mathcal{B}^{\pm}(\xi)}W^{\pm}(\xi)\Big)\Big|\lesssim\langle t\rangle^{2\varepsilon-1/8}\delta\quad\textit{for }\langle\xi\rangle\leq\langle t\rangle^{1/56}.\tag{218}$$

Proof. In Pusateri's paper [84], estimate (217) was essentially a consequence of his modified scattering result. Since we derive a stronger result by Ifrim and Tataru's method of testing solutions by wave packets, we present a modified proof of (217). Estimate (218) will be a consequence of (216) and (215).

Let $\langle\xi\rangle\geq\langle t\rangle^{1/56}$. Duhamel's equation (79) gives

$$\langle\xi\rangle^8\widehat{\mathrm{w}^{\pm_0}(t)}(\xi)=\langle\xi\rangle^8\widehat{\psi^{\pm_0}(t)}(\xi)+i4\pi c\sum_{\pm_1,\pm_2,\pm_3\in\{+,-\}}\int_0^t\langle\xi\rangle^8I^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t'.$$

Dyadic decomposition in η yields

$$\begin{aligned} I^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi) &= \Pi^{\pm_0}(\xi)e^{\pm_0 it\langle\xi\rangle}\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\langle\widehat{\psi^{\pm_1}(t')}(\sigma),\widehat{\psi^{\pm_2}(t')}(\sigma-\eta)\Big\rangle_{\mathbb{C}^4} \\ &\qquad\cdot\widehat{\psi^{\pm_3}(t')}(\xi-\eta)\chi_l(\eta)|\eta|^{-2}\,\mathrm{d}\sigma\,\mathrm{d}\eta \\ &=: \Pi^{\pm_0}(\xi)e^{\pm_0 it\langle\xi\rangle}\sum_{l\in\mathbb{Z}} I_l^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi). \end{aligned}$$

For $2^l \le \langle t\rangle^{-9/8}$, we obtain

$$\begin{aligned} &\Big|\int_0^t \langle\xi\rangle^8 I_l^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t'\Big| \\ &\quad\lesssim \Big|\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\big\langle\langle\sigma\rangle^8\widehat{\psi^{\pm_1}(t')}(\sigma),\langle\sigma-\eta\rangle^8\widehat{\psi^{\pm_2}(t')}(\sigma-\eta)\big\rangle_{\mathbb{C}^4} \\ &\qquad\qquad\cdot\langle\xi-\eta\rangle^8\widehat{\psi^{\pm_3}(t')}(\xi-\eta)\chi_l(\eta)|\eta|^{-2}\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}t'\Big| \\ &\quad\le \big\||\cdot|^{-2}\chi_l\big\|_{L^1(\mathbb{R}^3,\mathbb{C})} \\ &\qquad\cdot\int_0^t \big\|\big\langle\langle D\rangle^8\psi^{\pm_1}(t'),\langle D\rangle^8\psi^{\pm_2}(t')\big\rangle_{\mathbb{C}^4}\big\|_{L^1(\mathbb{R}^3,\mathbb{C})}\big\|\langle\xi\rangle^8\widehat{\psi^{\pm_3}(t')}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t' \\ &\quad\lesssim 2^l\int_0^t\big\|\psi^{\pm_1}(t')\big\|_{H^8(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi^{\pm_2}(t')\big\|_{H^8(\mathbb{R}^3,\mathbb{C}^4)}\tilde\delta\,\mathrm{d}t' \\ &\quad\lesssim 2^l\langle t\rangle^{1+2\varepsilon}\tilde\delta^3, \end{aligned} \tag{219}$$

where we used Hölder's inequality and Hausdorff–Young's inequality in the second step and a priori assumption (44) in the third and fourth step. Therefore,

$$\sum_{l\in\mathbb{Z},\,2^l\le\langle t\rangle^{-9/8}}\Big|\int_0^t\langle\xi\rangle^8 I_l^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t'\Big| \lesssim \langle t\rangle^{2\varepsilon-1/8}\tilde\delta^3. \tag{220}$$

For $2^l \ge \langle t\rangle^{-9/8}$, we have

$$\begin{aligned} &\Big|\int_0^t\langle\xi\rangle^8 I_l^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t'\Big| \\ &\quad\lesssim \langle\xi\rangle^{-95}\Big|\int_0^t\langle\xi\rangle^N I_l^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t'\Big| \\ &\quad\lesssim \langle\xi\rangle^{-95}\Big|\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\langle\xi\rangle^N\big\langle\widehat{\psi^{\pm_1}(t')}(\sigma),\widehat{\psi^{\pm_2}(t')}(\sigma-\eta)\big\rangle_{\mathbb{C}^4} \\ &\qquad\qquad\cdot\widehat{\psi^{\pm_3}(t')}(\xi-\eta)\chi_l(\eta)|\eta|^{-2}\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}t'\Big| \\ &\quad\le \langle\xi\rangle^{-95}\big\||\cdot|^{-2}\chi_l\big\|_{L^2(\mathbb{R}^3,\mathbb{C})}\int_0^t\big\|\psi(t')\big\|^2_{H^N(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t' \\ &\quad\lesssim \langle\xi\rangle^{-95}2^{-l/2}\langle t\rangle^{1+2\varepsilon}\tilde\delta^3 \\ &\quad\lesssim \langle t\rangle^{-95/56}2^{-l/2}\langle t\rangle^{1+2\varepsilon}\tilde\delta^3 \\ &\quad\lesssim 2^{-l/2}\langle t\rangle^{2\varepsilon-39/56}\tilde\delta^3, \end{aligned} \tag{221}$$

where we used Hölder's and Hausdorff–Young's inequality in the third step and a priori assumption (44) in the fourth and fifth step.

Therefore,

$$\sum_{l\in\mathbb{Z},\,2^l\geq\langle t\rangle^{-9/8}} \Big|\int_0^t \langle\xi\rangle^8 I_l^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t'\Big| \lesssim \langle t\rangle^{9/16}\langle t\rangle^{2\varepsilon-39/56}\tilde{\delta}^3$$
$$= \langle t\rangle^{2\varepsilon-15/112}\tilde{\delta}^3. \tag{222}$$

Hence, for $\langle\xi\rangle \geq \langle t\rangle^{1/56}$, we have

$$\begin{aligned}\big|\langle\xi\rangle^8\widehat{\mathrm{w}^\pm(t)}(\xi)\big| &\lesssim \langle\xi\rangle^{-1}\|\langle\cdot\rangle^9\widehat{\psi_0^\pm}\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} + \langle t\rangle^{2\varepsilon-1/8}\tilde{\delta}^3\\ &\lesssim \langle t\rangle^{-1/56}\delta + \langle t\rangle^{2\varepsilon-1/8}\tilde{\delta}^3.\end{aligned} \tag{223}$$

Now, let $\langle\xi\rangle \leq \langle t\rangle^{1/56}$. Our result (215) implies

$$\begin{aligned}\big|\langle\xi\rangle^8\big(\widehat{\mathrm{w}^\pm(t)}(\xi) - (2\pi)^{3/2}\Upsilon^\pm(t,\pm\langle\xi\rangle^{-1}\xi)\big)\big| &\lesssim \langle\xi\rangle^7\langle t\rangle^{2\varepsilon-1/4}(\delta+\tilde{\delta}^3)\\ &\lesssim \langle t\rangle^{2\varepsilon-1/8}(\delta+\tilde{\delta}^3).\end{aligned} \tag{224}$$

By (214),

$$\begin{aligned}\big|\Upsilon^\pm(t,\pm\langle\xi\rangle^{-1}\xi)\big| &\leq \big|W(v)\big| + C(\delta+\tilde{\delta}^3)t^{2\varepsilon-1/4}\langle\xi\rangle^{-1}\\ &\leq \big|\Upsilon^\pm(t/2,\pm\langle\xi\rangle^{-1}\xi)\big| + 2^{1/4-2\varepsilon}C(\delta+\tilde{\delta}^3)t^{2\varepsilon-1/4}\\ &\leq \big|\Upsilon^\pm(t/2,\pm\langle\xi\rangle^{-1}\xi)\big| + 2^{1/4-2\varepsilon}C(\delta+\tilde{\delta}^3)t^{2\varepsilon-1/8}\langle\xi\rangle^{-8}.\end{aligned}$$

Hence,

$$\begin{aligned}&\big|\Upsilon^\pm(2^n,\pm\langle\xi\rangle^{-1}\xi)\big|\\ &\quad\leq \big|\Upsilon^\pm(2^{n-1},\pm\langle\xi\rangle^{-1}\xi)\big| + 2^{1/4-2\varepsilon}C(\delta+\tilde{\delta}^3)2^{n(\varepsilon-1/8)}\langle\xi\rangle^{-8}\\ &\quad\leq \big|\Upsilon^\pm(2^{n-2},\pm\langle\xi\rangle^{-1}\xi)\big| + 2^{1/4-2\varepsilon}C(\delta+\tilde{\delta}^3)(2^{(n-1)(2\varepsilon-1/8)} + 2^{n(2\varepsilon-1/4)})\langle\xi\rangle^{-8}\end{aligned}$$

and iterating this argument leads to

$$\begin{aligned}\big|\Upsilon^\pm(2^n,\pm\langle\xi\rangle^{-1}\xi)\big| &\leq \big|\Upsilon^\pm(2^{-m},\pm\langle\xi\rangle^{-1}\xi)\big| + 2^{1/4-2\varepsilon}C(\delta+\tilde{\delta}^3)\sum_{k=0}^{n+m}2^{k(2\varepsilon-1/8)}\langle\xi\rangle^{-8}\\ &\leq C_1 2^{-3m/4}\langle\xi\rangle^{-3} + C_2(\delta+\tilde{\delta}^3)\langle\xi\rangle^{-8},\end{aligned}$$

where we used (173) in the last step and set

$$C_2 = C\sum_{k=0}^{\infty}2^{k(2\varepsilon-1/8)} \in (0,\infty)$$

which does not depend on n, m. Hence, we can choose $m\in\mathbb{N}$ arbitrarily large and conclude

$$\big|\Upsilon^\pm(t,\pm\langle\xi\rangle^{-1}\xi)\big| \leq C_2(\delta+\tilde{\delta}^3)\langle\xi\rangle^{-8} \tag{225}$$

which together with (224) finishes the bootstrap argument. Since we can choose

$$\tilde{\delta} = \delta^{1/2}$$

for sufficiently small $\delta > 0$, we may assume that

$$\tilde{\delta}^3 \leq \delta.$$

Finally, estimate (218) follows from (215) for arbitrarily large t. □

Remarks.

a) From the proof of (218), we obtain

$$W^\pm(\xi) = \lim_{t\to\infty} (2\pi)^{-3/2} e^{-ic\log(t)\mathcal{B}^\pm(\xi)} \widehat{\mathrm{w}^\pm(t)}(\xi) \mathbb{1}_{B_{\langle t\rangle^{1/56}}(0)}(\langle\xi\rangle),$$

where the limit is taken in $\langle\cdot\rangle^{-8} L^\infty(\mathbb{R}^3, \mathbb{C}^4)$ as well as

$$W^\pm(\xi) = \lim_{t\to\infty} (2\pi)^{-3/2} e^{-ic\log(t)\mathcal{B}^\pm(\xi)} \widehat{\mathrm{w}^\pm(t)}(\xi)$$

in $\langle\cdot\rangle^{-1} L^\infty(\mathbb{R}^3, \mathbb{C}^4)$. In particular,

$$\big\|\langle\xi\rangle^8 W^\pm(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)} = (2\pi)^{-3/2} \lim_{t\to\infty} \big\|\langle\xi\rangle^8 \widehat{\mathrm{w}^\pm(t)}(\xi) \mathbb{1}_{B_{\langle t\rangle^{1/56}}(0)}(\langle\xi\rangle)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}$$

and

$$\big\|\langle\xi\rangle W^\pm(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)} = (2\pi)^{-3/2} \lim_{t\to\infty} \big\|\langle\xi\rangle \widehat{\mathrm{w}^\pm(t)}(\xi)\big\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}$$

which implies (36). Since

$$\langle\cdot\rangle^{-8} \in L^2(\mathbb{R}^3, \mathbb{C}^4)$$

and

$$\big|e^{ic\log(t)\mathcal{B}^\pm(\xi)} \widehat{\mathrm{w}^\pm(t)}(\xi)\big| = \big|\widehat{\psi^\pm(t)}(\xi)\big|,$$

estimate (218) also shows (37) which finishes the proof of (32)–(34). Since estimate (31) is an immediate consequence of (178) and (214), the proof of Theorem 1.1 is complete.

b) $N = 103$ is the lowest integer such that – with our arguments – estimate (223) holds true with a $\langle t\rangle^{2\varepsilon-1/8}$-decay. One might derive a modified scattering result with weaker decay for $N \geq 52$. Indeed, the same arguments as in the proof of Lemma 3.13 lead to a $\langle t\rangle^{-\varepsilon}$-decay, $\varepsilon \in (0, 1/261]$, as follows: First, let $\langle\xi\rangle \geq \langle t\rangle^{1/29}$. By replacing $\langle t\rangle^{-9/8}$ with $\langle t\rangle^{-1-3\varepsilon}$ in (219)–(222) and $\langle\xi\rangle^{-95}$ in (221) with $\langle\xi\rangle^{-44}$, we get the following substitutes for (219)–(222):

$$\Big|\int_0^t \langle\xi\rangle^8 I_l^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t'\Big| \lesssim 2^l \langle t\rangle^{1+2\varepsilon}\tilde{\delta}^3, \quad 2^l \leq \langle t\rangle^{-1-3\varepsilon},$$

$$\sum_{l\in\mathbb{Z},\, 2^l \leq \langle t\rangle^{-1-3\varepsilon}} \Big|\int_0^t \langle\xi\rangle^8 I_l^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t'\Big| \lesssim \langle t\rangle^{-\varepsilon}\tilde{\delta}^3,$$

$$\begin{aligned}\Big|\int_0^t \langle\xi\rangle^8 I_l^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t'\Big| &\lesssim \langle t\rangle^{-44/29} 2^{-l/2} \langle t\rangle^{1+2\varepsilon}\tilde{\delta}^3 \\ &\lesssim 2^{-l/2}\langle t\rangle^{-15/29+2\varepsilon}, \quad 2^l \leq \langle t\rangle^{-1-3\varepsilon},\end{aligned}$$

$$\begin{aligned}\sum_{l\in\mathbb{Z},\, 2^l \geq \langle t\rangle^{-1-3\varepsilon}} \Big|\int_0^t \langle\xi\rangle^8 I_l^{\pm_0,\pm_1,\pm_2,\pm_3}(t',\xi)\,\mathrm{d}t'\Big| &\lesssim \langle t\rangle^{-1/58+7\varepsilon/2}\tilde{\delta}^3 \\ &\lesssim \langle t\rangle^{-\varepsilon}\tilde{\delta}^3,\end{aligned}$$

where we need $\varepsilon \in (0, 1/261]$ in the last step. Then, we have

$$\begin{aligned} \left|\langle\xi\rangle^8 \widehat{\mathrm{w}^\pm(t)}(\xi)\right| &\lesssim \langle t\rangle^{-1/29}\delta + \langle t\rangle^{-\varepsilon}\tilde{\delta}^3, \quad \langle\xi\rangle \geq \langle t\rangle^{1/29}, \\ \left|\langle\xi\rangle^8 \big(\widehat{\mathrm{w}^\pm(t)}(\xi) - \Upsilon^\pm(t, \pm\langle\xi\rangle^{-1}\xi)\big)\right| &\lesssim \langle\xi\rangle^7 \langle t\rangle^{2\varepsilon-1/4}(\delta + \tilde{\delta}^3) \\ &\lesssim \langle t\rangle^{2\varepsilon-1/116}(\delta + \tilde{\delta}^3) \\ &\lesssim \langle t\rangle^{-\varepsilon}, \quad \langle\xi\rangle \leq \langle t\rangle^{1/29}, \qquad (226) \\ \left|\Upsilon^\pm(t, \pm\langle\xi\rangle^{-1}\xi)\right| &\lesssim (\delta + \tilde{\delta}^3)\langle\xi\rangle^{-8}. \end{aligned}$$

Note that step (226) is true for $\varepsilon \in (0, 3/116]$ and in particular for $\varepsilon \in (0, 1/261]$.

c) For going below $N = 52$ and $\|\langle\xi\rangle^8 \widehat{\psi(t)}(\xi)\|_{L^\infty_\xi(\mathbb{R}^3,\mathbb{C}^4)}$, we would need an improvement of the refined decay estimate (69), cf. also (71).

d) Up to some slight modifications, our analysis carries over to the scalar toy model, so that we in fact improved Pusateri's result in [84]. Recall the nonlinearity

$$c\big(V * |u|^2\big)u$$

in the scalar toy model. With the notation of Section 2, this corresponds to $\pm_0 = \pm_1 = \pm_2 = \pm_3$, where we did not need to use the null structure of $\Pi^\pm(D)$. In the case $\pm_1 \neq \pm_2$, null structure of $\Pi^{\pm_1}\Pi^{\pm_2}$ was only needed when $\pm_0 \neq \pm_3$. This means, our method is also applicable to the scalar toy model with nonlinearities

$$\big(V * u^2\big)u, \quad \big(V * \overline{u}^2\big)u$$

whereas nonlinearities

$$\big(V * (u_1 u_2)\big)\overline{u}, \quad u_1, u_2 \in \{u, \overline{u}\}$$

would be problematic. One might try a more careful integration by parts to overcome the need of null-structure.

4 Asymptotic completeness

Given a suitable asymptotic state $W\colon \mathbb{R}^3 \to \mathbb{C}^4$, we want to find a unique solution to the Cauchy problem

$$\begin{cases} -i\partial_t\psi^\pm \pm \langle D\rangle\psi^\pm = c\Pi^\pm(D)\big[\big(V * |\psi|^2\big)\psi\big], \\ \lim\limits_{t\to\infty} \big\|\psi(t) - \psi_W(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} = 0, \end{cases} \qquad (227)$$

where

$$\begin{aligned} &\Pi^\pm(D)\big[\psi_W(t)\big](x) = t^{-3/2} e^{\mp i(t^2-|x|^2)^{1/2}} e^{ic\log(t)(V*|W|^2)(x/t)} W(x/t), \qquad (228) \\ &\psi_W = \Pi^+(D)\psi_W + \Pi^-(D)\psi_W. \end{aligned}$$

In the sequel, we write

$$\psi^\pm_W = \Pi^\pm(D)\psi_W.$$

We also denote

$$\psi^\pm_{\nabla W}(t, x) = t^{-3/2} e^{\mp i(t^2-|x|^2)^{1/2}} e^{ic\log(t)(V*|W|^2)(x/t)} \nabla W(x/t)$$

and analogously $\psi^\pm_{\Delta W}$ etc.

Theorem 4.1. *Let $W \in H^2(\mathbb{R}^3, \mathbb{C}^4)$ satisfy*

$$\operatorname{supp} W \subseteq B_1(0).$$

There is $C_W \in (0,1)$ such that if

$$\||\xi_v| W(v)\|_{L^2_v(\mathbb{R}^3,\mathbb{C}^4) \cap L^\infty_v(\mathbb{R}^3,\mathbb{C}^4)} + \|W\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \leq C_W$$

then (227) *admits a unique solution $\psi \in \mathcal{C}\big([0,\infty), H^1(\mathbb{R}^3, \mathbb{C}^4)\big)$.*

Remark. Our choice of ψ_W comes from the asymptotic result (31) on the spatial side. One might also think of using the corresponding result (32)–(34) on the Fourier side. However, while working with the Fourier transform is easier for the linear part, it would become very delicate for the nonlinear part, cf. Section 4.2.

Obviously,

$$\begin{aligned}\|\psi_W^\pm(t)\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} &= t^{-3/2}\|W\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim t^{-3/2}\|W\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim t^{-3/2} C_W\end{aligned}$$

and

$$\begin{aligned}\|\psi_W^\pm(t)\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &= t^{-3/2}\|W(x/t)\|_{L^2_x(\mathbb{R}^3,\mathbb{C}^4)} \\ &\leq C_W.\end{aligned}$$

Furthermore,

$$\begin{aligned}\nabla_x \psi_W^\pm(t,x) &= \pm i(t^2 - |x|^2)^{-1/2} x \psi_W^\pm(t,x) + t^{-1}\psi_{\nabla W}(t,x) \\ &\quad - ict^{-1}\log(t)\big(4\pi \nabla_x \Delta_x^{-2}|W(t)|^2\big)(x/t)\psi_W^\pm(t,x) \\ &= \pm i\Big(1 - \Big|\frac{x}{t}\Big|^2\Big)^{-1/2}\frac{x}{t}\psi_W^\pm(t,x) \\ &\quad + \mathcal{O}_{L^2(\mathbb{R}^3,\mathbb{C}^4)}(C_W t^{-1}\log(t)) \cap \mathcal{O}_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}(C_W t^{-5/2}\log(t)),\end{aligned}$$

where we used

$$\begin{aligned}\big\||D|^{-1}|W(t)|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} &= (2\pi^2)^{-1}\big\||\cdot|^{-2} * |W(t)|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \\ &\lesssim \big\||\cdot|^{-2}\big\|_{L^{3/2,\infty}(\mathbb{R}^3,\mathbb{C})}\big\||W(t)|^2\big\|_{L^{3,1}(\mathbb{R}^3,\mathbb{C})} \\ &\lesssim \|W(t)\|_{L^4(\mathbb{R}^3,\mathbb{C}^4)}\|W(t)\|_{L^{12}(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim C_W^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\nabla\psi_W^\pm(t)\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim t^{-3/2}\||\xi_v|W(v)\|_{L^\infty_v(\mathbb{R}^3,\mathbb{C}^4)} + C_W R t^{-5/2}\log(t) \\ &\lesssim C_W t^{-3/2}\end{aligned}$$

and

$$\begin{aligned}\|\nabla\psi_W^\pm(t)\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \||\xi_v|W(v)\|_{L^2_v(\mathbb{R}^3,\mathbb{C}^4)} \\ &\leq C_W\end{aligned}$$

which implies

$$\|\psi_W(t)\|_{W^{1,\infty}(\mathbb{R}^3,\mathbb{C}^4)} \lesssim C_W t^{-3/2}$$

and

$$\|\psi_W(t)\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \lesssim C_W.$$

By interpolation, we obtain

$$\|\psi_W(t)\|_{W^{1,p}(\mathbb{R}^3,\mathbb{C}^4)} \lesssim C_W t^{-3/2-3/p}, \quad p \in [2,\infty]. \tag{229}$$

We define $f(\psi_W) = f^+(\psi_W) + f^-(\psi_W)$ by

$$f^\pm(\psi_W) = -i\partial_t \psi_W^\pm \pm \langle D\rangle \psi_W^\pm - c\Pi^\pm(D)\big[\big(V * |\psi_W|^2\big)\psi_W\big]$$

and $\psi_{\text{dif}} = \psi_{\text{dif}}^+ + \psi_{\text{dif}}^-$ by

$$\psi_{\text{dif}}^\pm = \psi^\pm - \psi_W^\pm.$$

Then, Cauchy problem (227) is equivalent to

$$\begin{cases} -i\partial_t \psi_{\text{dif}}^\pm \pm \langle D\rangle \psi_{\text{dif}}^\pm = N^\pm(\psi_W, \psi_{dif}) - f^\pm(\psi_W), \\ \lim\limits_{t\to\infty} \big\|\psi_{\text{dif}}^\pm(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} = 0, \end{cases} \tag{230}$$

where

$$\begin{aligned} &N^\pm(\psi_W, \psi_{dif}) \\ &\quad = c\Pi^\pm(D)\big[\big(V * |\psi|^2\big)\psi\big] - c\Pi^\pm(D)\big[\big(V * |\psi_W|^2\big)\psi_W\big] \\ &\quad = c\Pi^\pm(D)\big[\big(V * |\psi_{\text{dif}}|^2\big)\psi_{\text{dif}} + \big(V * |\psi_{\text{W}}|^2\big)\psi_{\text{dif}} + 2\big(V * \text{Re}\langle \psi_{\text{dif}}, \psi_W\rangle_{\mathbb{C}^4}\big)\psi_{\text{dif}} \\ &\qquad\qquad + \big(V * |\psi_{\text{dif}}|^2\big)\psi_{\text{W}} + 2\big(V * \text{Re}\langle \psi_{\text{dif}}, \psi_W\big\rangle_{\mathbb{C}^4}\big)\psi_{\text{W}}\big]. \end{aligned} \tag{231}$$

In the sequel, we omit the dependence on the asymptotic data ψ_W and write

$$f^\pm = f^\pm(\psi_W), \quad N^\pm(\psi_{\text{dif}}) = N^\pm(\psi_W, \psi_{\text{dif}}).$$

We solve problem (230) by the standard contraction mapping principle in the Banach space Z induced via the norm

$$\|u\|_Z = \sup_{T\geq 0} \langle T\rangle^{1/2} \|u\|_{L_t^\infty H_x^1([T,2T]\times\mathbb{R}^3,\mathbb{C}^4)}.$$

More precisely, let $H = H^+ + H^-$ and

$$\begin{aligned} H^\pm(\psi_{\text{dif}})(t) &= \int_t^\infty e^{\mp i(t-t')\langle D\rangle} \big(N^\pm(\psi_{\text{dif}}(t')) + f^\pm(t')\big)\, \mathrm{d}t' \\ &=: H_N^\pm(\psi_{\text{dif}})(t) + H_{\text{inh}}^\pm(f^\pm)(t). \end{aligned}$$

We consider the closed ball

$$Z_R = \big\{u \in Z \colon \|u\|_Z \leq R\big\}$$

for some

$$R \in (C_W, 1)$$

to be chosen later. In order to show that H is a contraction mapping on Z_R and prove Theorem 4.1, we discuss the nonlinear part $N^\pm(\psi_{\text{dif}})$ and the inhomogeneous part $f^\pm$ separately.

4.1 Nonlinear estimates

Lemma 4.2. *Let* $u_1, u_2 \in Z_R$ *and* $H_N = H_N^+ + H_N^-$. *Then,*

$$\|H_N(u_1)\|_Z \leq \frac{R}{2}, \tag{232}$$

$$\|H_N(u_1) - H_N(u_2)\|_Z \leq \frac{1}{2}\|u_1 - u_2\|_Z. \tag{233}$$

Proof. We treat the terms of (231) separately. For the last term, note that

$$\begin{aligned}
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^\pm(D)\big[\big(V * \mathrm{Re}\langle u_1(t'), \psi_W(t')\rangle_{\mathbb{C}^4}\big)\psi_W(t')\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \big\|V * \mathrm{Re}\langle u_1(t'), \psi_W(t')\rangle_{\mathbb{C}^4}\big\|_{W^{1,6}(\mathbb{R}^3,\mathbb{C})}\big\|\psi_W(t')\big\|_{W^{1,3}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \big\|\mathrm{Re}\langle u_1(t'), \psi_W(t')\rangle_{\mathbb{C}^4}\big\|_{W^{1,6/5}(\mathbb{R}^3,\mathbb{C})}\big\|\psi_W(t')\big\|_{W^{1,3}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|^2_{W^{1,3}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \langle t'\rangle^{-3/2} R C_W^2,
\end{aligned}$$

where we used weak Young's inequality in the second step and estimate (229) in the last step. Similarly,

$$\begin{aligned}
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^\pm(D)\big[\big(V * |u_1(t')|^2\big)\psi_W(t')\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \big\|V * |u_1(t')|^2\big\|_{W^{1,6}(\mathbb{R}^3,\mathbb{C})}\big\|\psi_W(t')\big\|_{W^{1,3}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \big\||u_1(t')|^2\big\|_{W^{1,6/5}(\mathbb{R}^3,\mathbb{C})}\big\|\psi_W(t')\big\|_{W^{1,3}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_1(t')\big\|_{L^3(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|_{W^{1,3}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \big\|u_1(t')\big\|^2_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|_{W^{1,3}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \langle t'\rangle^{-3/2} R^2 C_W,
\end{aligned}$$

where we used Sobolev's embedding $H^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ in the fourth step, and

$$\begin{aligned}
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^\pm(D)\big[\big(V * |\psi_W(t')|^2\big)u_1(t')\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \big\|V * |\psi_W(t')|^2\big\|_{W^{1,\infty}(\mathbb{R}^3,\mathbb{C})}\big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \big\|\psi_W(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|_{W^{1,6}(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \langle t'\rangle^{-3/2} R C_W^2.
\end{aligned}$$

For the first term of (231), we have

$$\begin{aligned}
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^\pm(D)\big[\big(V * |u_1(t')|^2\big)u_1(t')\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \Big\|\langle\xi\rangle \int_{\mathbb{R}^3} |\eta|^{-2}\widehat{|u_1(t')|^2}(\eta)\widehat{u_1(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \Big\|\int_{\mathbb{R}^3} \langle\eta\rangle^{-1}\widehat{|u_1(t')|^2}(\eta)\widehat{u_1(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\qquad + \Big\|\int_{\mathbb{R}^3} |\eta|^{-2}\widehat{|u_1(t')|^2}(\eta)\langle\xi-\eta\rangle\widehat{u_1(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \big\|\langle\xi\rangle^{-1}\widehat{|u_1(t')|^2}(\xi)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\big\|u_1(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\
&\qquad + \Big\|\int_{\mathbb{R}^3} |\xi|^{-2}\widehat{|u_1(t')|^2}(\xi)\Big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^{\pm}(D)\big[\big(V * |u_1(t')|^2\big)u_1(t')\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \big\|\langle\xi\rangle\widehat{|u_1(t')|^2}(\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C})}\big\|u_1(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\|u_1(t')\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)\cap L^4(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \big\|u_1(t')\big\|^3_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\lesssim \langle t'\rangle^{-3/2}R^3,
\end{aligned}$$

where we used Sobolev's embedding $H^{3/4}(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3,\mathbb{C}^4)$ in the second step. Similarly,

$$\begin{aligned}
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^{\pm}(D)\big[\big(V * \mathrm{Re}\langle u_1(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big)u_1(t')\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \Big\|\langle\xi\rangle\int_{\mathbb{R}^3}|\eta|^{-2}\big[\mathrm{Re}\langle u_1(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big]\widehat{\ }(\eta)\widehat{u_1(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \Big\|\int_{\mathbb{R}^3}\langle\eta\rangle^{-1}\big[\mathrm{Re}\langle u_1(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big]\widehat{\ }(\eta)\widehat{u_1(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad+\Big\|\int_{\mathbb{R}^3}|\eta|^{-2}\big[\mathrm{Re}\langle u_1(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big]\widehat{\ }(\eta)\langle\xi-\eta\rangle\widehat{u_1(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \big\|\langle\xi\rangle^{-1}\big[\mathrm{Re}\langle u_1(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big]\widehat{\ }(\xi)\big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\big\|u_1(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad+\Big\|\int_{\mathbb{R}^3}|\xi|^{-2}\big[\mathrm{Re}\langle u_1(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big]\widehat{\ }(\xi)\Big\|_{L^1_\xi(\mathbb{R}^3,\mathbb{C})}\big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}
\end{aligned}$$

which leads to

$$\begin{aligned}
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^{\pm}(D)\big[\big(V * \mathrm{Re}\langle u_1(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big)u_1(t')\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim_\beta \big\|\langle\xi\rangle\big[\langle u_1(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big]\widehat{\ }(\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C})}\big\|u_1(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad+\Big(\int_{|\xi|\ge 1}|\xi|^{-4}\,\mathrm{d}\xi\Big)^{1/2}\big\|\big[\langle u_1(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big]\widehat{\ }(\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C})}\big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad+\Big(\int_{|\xi|<1}|\xi|^{-6/(2+\beta)}\,\mathrm{d}\xi\Big)^{(2+\beta)/3}\big\|\big[\langle u_1(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big]\widehat{\ }(\xi)\big\|_{L_\xi^{3/(1-\beta)}(\mathbb{R}^3,\mathbb{C})}\big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim_\beta \big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|_{W^{1,\infty}(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_1(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad+\big\|u_1(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad+\big\|\langle u_1(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big\|_{L^{3/(2+\beta)}(\mathbb{R}^3,\mathbb{C})}\big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim_\beta \big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|_{W^{1,\infty}(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_1(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad+\big\|u_1(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|_{L^{6/(1+2\beta)}(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim_\beta \langle t'\rangle^{-2+\beta}R^2C_W,
\end{aligned}$$

where we can choose $\beta \in (0,1]$ (here it suffices to take $\beta = 1/2$). By assumption, $C_W \le R$. Therefore,

$$\begin{aligned}
\big\|H_N(u_1)\big\|_Z &\le C\sup_{T\ge 0}\langle T\rangle^{1/2}\Big\|\int_t^\infty \langle t'\rangle^{-3/2}R^3\,\mathrm{d}t'\Big\|_{L_t^\infty[T,2T]}\\
&\le CR^3.
\end{aligned}$$

Choosing $R \in (0,1)$ (and $C_W \in (0,1)$) such that $CR^2 \le 1/2$ leads to (232). For estimate (233), we first note that

$$H^{\pm}(u_1)(t) - H^{\pm}(u_2)(t) = \int_t^{\infty} e^{i(t-t')\langle D\rangle}\big(N^{\pm}(u_1(t')) - N^{\pm}(u_2(t'))\big)\,\mathrm{d}t'$$

and

$$\begin{aligned}
N^{\pm}(u_1(t')) - N^{\pm}(u_2(t')) = c\Pi^{\pm}(D)\Big[&\big(V * |u_1(t')|^2\big)\big(u_1(t') - u_2(t')\big)\\
&+ \big(V * \big(|u_1(t')|^2 - |u_2(t')|^2\big)\big)u_2(t')\\
&+ \big(V * |\psi_W(t')|^2\big)\big(u_1(t') - u_2(t')\big)\\
&+ 2\big(V * \mathrm{Re}\langle u_1(t'), \psi_W(t')\rangle_{\mathbb{C}^4}\big)\big(u_1(t') - u_2(t')\big)\\
&+ 2\big(V * \mathrm{Re}\langle u_1(t') - u_2(t'), \psi_W(t')\rangle_{\mathbb{C}^4}\big)u_2(t')\\
&+ \big(V * \big(|u_1(t')|^2 - |u_2(t')|^2\big)\big)\psi_W(t')\\
&+ 2\big(V * \mathrm{Re}\langle u_1(t') - u_2(t'), \psi_W(t')\rangle_{\mathbb{C}^4}\big)\psi_W(t')\Big]
\end{aligned}$$

as well as

$$|u_1(t')|^2 - |u_2(t')|^2 = \big\langle u_1(t') - u_2(t')), u_1(t')\big\rangle_{\mathbb{C}^4} + \big\langle u_2(t'), u_1(t') - u_2(t')\big\rangle_{\mathbb{C}^4}.$$

Hence, the same arguments as for (232) show that

$$\begin{aligned}
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^{\pm}(D)\big[\big(V * \mathrm{Re}\langle u_1(t') - u_2(t'), \psi_W(t')\rangle_{\mathbb{C}^4}\big)\psi_W(t')\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \big\|u_1(t') - u_2(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|^2_{W^{1,3}(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \langle t'\rangle^{-3/2}C_W^2\|u_1 - u_2\|_Z,\\
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^{\pm}(D)\big[\big(V * \big(|u_1(t')|^2 - u_2(t')|^2\big)\big)\psi_W(t')\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \big\|u_1(t') - u_2(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\sum_{j=1}^{2}\big\|u_j(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|_{W^{1,3}(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \langle t'\rangle^{-3/2}RC_W\|u_1 - u_2\|_Z,\\
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^{\pm}(D)\big[\big(V * |\psi_W(t')|^2\big)\big(u_1(t') - u_2(t')\big)\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \big\|\psi_W(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|_{W^{1,6}(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_1(t') - u_2(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \langle t'\rangle^{-3/2}C_W^2\|u_1 - u_2\|_Z,\\
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^{\pm}(D)\big[\big(V * |u_1(t')|^2\big)\big(u_1(t') - u_2(t')\big)\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \big\|\langle\xi\rangle\widehat{|u_1(t')|^2}(\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C})}\big\|u_1(t') - u_2(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad + \big\|u_1(t')\big\|^2_{L^2(\mathbb{R}^3,\mathbb{C}^4)\cap L^4(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_1(t') - u_2(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \big\|u_1(t')\big\|^2_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_1(t') - u_2(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \langle t'\rangle^{-3/2}R^2\|u_1 - u_2\|_Z,
\end{aligned}$$

$$
\begin{aligned}
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^{\pm}(D)\big[\big(V*\big(|u_1(t')|^2-|u_2(t')|^2\big)\big)u_2(t')\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \big\|\langle\xi\rangle\big[\big\langle u_1(t')-u_2(t'),u_1(t')\big\rangle_{\mathbb{C}^4}+\big\langle u_2(t'),u_1(t')-u_2(t')\big\rangle_{\mathbb{C}^4}\big]\widehat{\ }(\xi)\big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C})}\\
&\qquad\qquad\cdot\big\|u_2(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad+\big\|u_1(t')-u_2(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)\cap L^4(\mathbb{R}^3,\mathbb{C}^4)}\sum_{j=1}^{2}\big\|u_j(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)\cap L^4(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_2(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \Big(\big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}+\big\|u_2(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\Big)^2\big\|u_1(t')-u_2(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \langle t'\rangle^{-3/2}R^2\|u_1-u_2\|_Z,\\
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^{\pm}(D)\big[\big(V*\mathrm{Re}\langle u_1(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big)\big(u_1(t')-u_2(t')\big)\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \big\|u_1(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C})}\big\|\psi_W(t')\big\|_{W^{1,\infty}(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_1(t')-u_2(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad+\big\|u_1(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|_{L^3(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_1(t')-u_2(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \langle t'\rangle^{-3/2}RC_W\|u_1-u_2\|_Z,\\
&\big\|e^{\mp i(t-t')\langle D\rangle}\Pi^{\pm}(D)\big[\big(V*\mathrm{Re}\langle u_1(t')-u_2(t'),\psi_W(t')\rangle_{\mathbb{C}^4}\big)u_2(t')\big]\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \big\|u_1(t')-u_2(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C})}\big\|\psi_W(t')\big\|_{W^{1,\infty}(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_2(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad+\big\|u_1(t')-u_2(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi_W(t')\big\|_{L^3(\mathbb{R}^3,\mathbb{C}^4)}\big\|u_2(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim \langle t'\rangle^{-3/2}RC_W\|u_1-u_2\|_Z.
\end{aligned}
$$

Therefore,

$$\big\|H_N(u_1)-H_N(u_2)\big\|_Z\le CR^2\|u_1-u_2\|_Z$$

and choosing $R\in(0,1)$ such that $CR^2\le 1/2$ leads to (233). □

4.2 Inhomogeneous estimates

We shall first replace

$$\Pi^{\pm}(D)\big[\big(V*|\psi_W|^2\big)\psi_W\big]$$

with

$$\big(V*|\psi_W|^2\big)\psi_W^{\pm}.$$

One might try to exploit the identity

$$
\begin{aligned}
f&=f^++f^-\\
&=-i\partial_t\psi_W^++\langle D\rangle\psi_W^+-i\partial_t\psi_W^--\langle D\rangle\psi_W^--c\big(V*|\psi_W|^2\big)\psi_W\\
&=-i\partial_t\psi_W^++\langle D\rangle\psi_W^+-c\big(V*|\psi_W|^2\big)\psi_W^+\\
&\quad-i\partial_t\psi_W^--\langle D\rangle\psi_W^--c\big(V*|\psi_W|^2\big)\psi_W^-,
\end{aligned}
$$

where we used $\Pi^+(D)+\Pi^-(D)=I_4$ in the second step. However, we need to consider f^+ and f^- separately since the linear propagator $e^{-i(t-t')\langle D\rangle}$ hits f^+ while the linear propagator $e^{i(t-t')\langle D\rangle}$ hits f^-. Let

$$\psi_{\mathrm{err}}^{\pm}=\Pi^{\pm}(D)\big[\big(V*|\psi_W|^2\big)\psi_W\big]-\big(V*|\psi_W|^2\big)\psi_W^{\pm}.$$

We use the same idea as in the proof of Lemma 3.11.

Lemma 4.3. *Let $\varrho > 0$. For any $t' \geq 0$, it holds that*

$$\|\psi_{\mathrm{err}}^{\pm}(t')\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \lesssim_\varrho C_W^3 \langle t' \rangle^{\varrho-2}. \tag{234}$$

Proof. It suffices to consider $\varrho \in (0,3)$. We have

$$\begin{aligned}&\|\psi_{\mathrm{err}}^{\pm}(t')\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad= 4\pi \Big\| \langle \xi \rangle^2 \int_{\mathbb{R}^3} |\eta|^{-2} \big[|\psi_W(t')|^2\big]\widehat{\ }(\eta) \big(\Pi^{\pm}(\xi) - \Pi^{\pm}(\xi-\eta)\big) \widehat{\psi_W(t')}(\xi-\eta)\,\mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}.\end{aligned}$$

Let

$$m_{l,l_{12}}^{\pm}(\xi,\eta) = \tilde{\chi}_l(\xi)\tilde{\chi}_{l_{12}}(\eta)\langle \xi \rangle |\eta|^{-2}\big(\Pi^{\pm}(\xi) - \Pi^{\pm}(\xi-\eta)\big).$$

Then,

$$\big|\partial_\xi^\alpha \partial_\eta^\beta m_{l,l_{12}}^{\pm}(\xi,\eta)\big| \lesssim_{\alpha,\beta} 2^{-l_{12}} \tilde{\chi}_l(\xi)\tilde{\chi}_{l_{12}}(\eta) 2^{-l|\alpha|} 2^{-l_{12}|\beta|}$$

and

$$\begin{aligned}\|\psi_{\mathrm{err}}^{\pm}(t')\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \sum_{l,l_{12},l_3 \in \mathbb{Z}} \Big\| \int_{\mathbb{R}^3} \chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta) \big[|\psi_W(t')|^2\big]\widehat{\ }(\eta)&\\ \cdot\, m_{l,l_{12}}^{\pm}(\xi,\eta) \widehat{\psi_W(t')}(\xi-\eta)\,\mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}&.\end{aligned}$$

a) First, consider $l \sim l_3 \succ l_{12}$. Lemmas 2.10, 2.9 and Bernstein's estimate together with (229) give

$$\begin{aligned}&\Big\| \int_{\mathbb{R}^3} \chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta) \big[|\psi_W(t')|^2\big]\widehat{\ }(\eta) m_{l,l_{12}}^{\pm}(\xi,\eta) \widehat{\psi_W(t')}(\xi-\eta)\,\mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad\lesssim 2^{-l_{12}} \big\|P_{l_{12}}|\psi_W(t')|^2\big\|_{L^{12/\varrho}(\mathbb{R}^3,\mathbb{C})} \big\|P_{l_3}\psi_W(t')\big\|_{L^{12/(6-\varrho)}(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad\lesssim C_W^3 2^{-l_{12}} \min\{2^{(3-\varrho/4)l_{12}}, 2^{-l_{12}^+}\langle t' \rangle^{\varrho/2-3}\} 2^{\varrho l_3/4} 2^{-l_3^+}.\end{aligned} \tag{235}$$

Hence,

$$\begin{aligned}&\sum_{l\sim l_3 \succ l_{12}} \Big\| \int_{\mathbb{R}^3} \chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta) \big[|\psi_W(t')|^2\big]\widehat{\ }(\eta)\, m_{l,l_{12}}^{\pm}(\xi,\eta) \widehat{\psi_W(t')}(\xi-\eta)\,\mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad\lesssim_\varrho C_W^3 \Big(\sum_{2^{l_{12}} \leq \langle t' \rangle^{-1}} 2^{(2-\varrho/4)l_{12}} + \sum_{2^{l_{12}} \geq \langle t' \rangle^{-1}} 2^{-l_{12}} \langle t' \rangle^{\varrho/2-3} \Big)\\ &\quad\lesssim C_W^3 \langle t' \rangle^{\varrho/2-2}.\end{aligned}$$

b) In the case $l \ll l_3 \sim l_{12}$, $2^l \leq \langle t' \rangle^{-2}$, the inequalities of Hölder and Bernstein in combination with (229) yield

$$\begin{aligned}&\Big\| \int_{\mathbb{R}^3} \chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta) \big[|\psi_W(t')|^2\big]\widehat{\ }(\eta)\, m_{l,l_{12}}^{\pm}(\xi,\eta) \widehat{\psi_W(t')}(\xi-\eta)\,\mathrm{d}\eta \Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad\lesssim 2^{-l_{12}} \|\chi_l\|_{L^2(\mathbb{R}^3,\mathbb{C})} \big\|P_{l_{12}}|\psi_W(t')|^2\big\|_{L^2(\mathbb{R}^3,\mathbb{C})} \big\|P_{l_3}\psi_W(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad\lesssim C_W^3 2^{3l/2} 2^{l_{12}/2} 2^{-l_3^+}\end{aligned}$$

and therefore,

$$\sum_{l\ll l_3\sim l_{12},\,2^l\leq\langle t'\rangle^{-2}}\Big\|\int_{\mathbb{R}^3}\chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta)\big[|\psi_W(t')|^2\big]\widehat{\ }(\eta)\cdot m^{\pm}_{l,l_{12}}(\xi,\eta)\widehat{\psi_W(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}$$
$$\lesssim C_W^3\sum_{2^l\leq\langle t'\rangle^{-2}}2^{3l/2}$$
$$\lesssim C_W^3\langle t'\rangle^{-3}.$$

c) In the case $l\ll l_3\sim l_{12}$, $2^l\geq\langle t'\rangle^{-2}$, we use again (235) to obtain

$$\sum_{l\ll l_3\sim l_{12},\,2^l\leq\langle t'\rangle^{-2}}\Big\|\int_{\mathbb{R}^3}\chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta)\big[|\psi_W(t')|^2\big]\widehat{\ }(\eta)\cdot m^{\pm}_{l,l_{12}}(\xi,\eta)\widehat{\psi_W(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}$$
$$\lesssim_\varrho C_W^3\sum_{2^{l_{12}}\leq\langle t'\rangle^{-1}}2^{(2-\varrho/4)l_{12}}\sum_{\langle t'\rangle^{-2}\leq 2^l\leq\langle t'\rangle^{-1}}1+C_W^3\sum_{2^{l_{12}}\geq\langle t'\rangle^{-1}}2^{-l_{12}}\langle t'\rangle^{\varrho/2-3}\sum_{2^l\geq\langle t'\rangle^{-2}}2^{-l^+}$$
$$\lesssim C_W^3\langle t'\rangle^{\varrho-2}.$$

d) Finally, for the case $l_{12}\sim l\gg l_3$, estimate (235) leads to

$$\sum_{l_{12}\sim l\gg l_3}\Big\|\int_{\mathbb{R}^3}\chi_l(\xi)\chi_{l_{12}}(\xi-\eta)\chi_{l_3}(\eta)\big[|\psi_W(t')|^2\big]\widehat{\ }(\eta)\,m^{\pm}_{l,l_{12}}(\xi,\eta)\widehat{\psi_W(t')}(\xi-\eta)\,\mathrm{d}\eta\Big\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)}$$
$$\lesssim C_W^3\Big(\sum_{2^{l_{12}}\leq\langle t'\rangle^{-1}}2^{(2-\varrho/4)l_{12}}+\sum_{2^{l_{12}}\geq\langle t'\rangle^{-1}}2^{-l_{12}}\langle t'\rangle^{\varrho/2-3}\Big)\sum_{l_3}2^{\varrho l_3/4}2^{-l_3^+}$$
$$\lesssim_\varrho C_W^3\langle t'\rangle^{\varrho/2-2}.$$

□

For the next step, we calculate

$$\begin{aligned}
&-i\partial_t\psi_W^{\pm}(t,tv)\\
&\quad=-i\frac{\mathrm{d}}{\mathrm{d}t}\psi_W^{\pm}(t,tv)+it^{-1}\nabla_v\psi_W^{\pm}(t,tv)\cdot v\\
&\quad=\Big(\frac{3i}{2}t^{-1}\mp(1-|v|^2)^{1/2}+ct^{-1}\big(V*|\langle\xi_\bullet\rangle^{5/2}W|^2\big)(v)+it^{-1}v\cdot\nabla_v\Big)\psi_W^{\pm}(t,tv)
\end{aligned}\tag{236}$$

which has to coincide with $\pm\langle D\rangle\psi^{\pm}(t,tv)$ up to $\mathcal{O}_{H^1(\mathbb{R}^3,\mathbb{C}^4)}(C_Wt^{-3/2})$-error terms. For the nonlocal operator $\langle D\rangle$, we perform a Taylor approximation on the Fourier side as follows:

$$\begin{aligned}
\langle\xi\rangle&=\langle\xi_v\rangle\pm\langle\xi_v\rangle^{-1}\xi_v\cdot(\xi\mp\xi_v)+\mathcal{R}^{\pm}(v,\xi)\\
&=(1-|v|^2)^{-1/2}\pm v\cdot(\xi\mp(1-|v|^2)^{-1/2}v)+\mathcal{R}^{\pm}(v,\xi)\\
&=(1-|v|^2)^{-1/2}\pm\frac{1}{2}v\cdot\xi\pm\frac{1}{2}\xi\cdot v-(1-|v|^2)^{-1/2}|v|^2+\mathcal{R}^{\pm}(v,\xi)\\
&=(1-|v|^2)^{1/2}\pm\frac{1}{2}v\cdot\xi\pm\frac{1}{2}\xi\cdot v+\mathcal{R}^{\pm}(v,\xi)\\
&=:\mathcal{M}^{\pm}(v,\xi)+\mathcal{R}^{\pm}(v,\xi),
\end{aligned}$$

where the remainder is

$$\mathcal{R}^{\pm}(v,\xi) = \frac{1}{2}(\xi \mp \xi_v)^T \mathrm{Hess}\,\langle\cdot\rangle\,(\pm\xi_v + \vartheta^{\pm}(\xi \mp \xi_v))(\xi \mp \xi_v) \tag{237}$$

for some $\vartheta^{\pm} \in [0,1]$. Note that

$$\partial^{\alpha}\langle\cdot\rangle(\eta) \lesssim \langle\eta\rangle^{1-|\alpha|}. \tag{238}$$

$\mathcal{M}^{\pm}(v,\xi)$ corresponds to the operator $\mathcal{M}^{\pm}(v,D)$ defined by

$$\begin{aligned}\mathcal{M}^{\pm}(v,D)\psi_W^{\pm}(t,tv) &= (1-|v|^2)^{1/2}\psi_W^{\pm}(t,tv) \mp \frac{i}{2}v\cdot\nabla_x\psi_W^{\pm}(t,tv)\\ &\quad \mp \frac{i}{2}\nabla_x\cdot\big(v\,\psi_W^{\pm}(t,tv)\big)\psi_W^{\pm}(t,tv)\\ &= \Big((1-|v|^2)^{1/2} \mp \frac{i}{2}t^{-1}v\cdot\nabla_v\\ &\quad \mp \frac{i}{2}t^{-1}(\nabla_v\cdot v) \mp \frac{i}{2}t^{-1}v\cdot\nabla_v\Big)\psi_W^{\pm}(t,tv)\\ &= \Big((1-|v|^2)^{1/2} \mp it^{-1}v\cdot\nabla_v \mp \frac{3}{2}it^{-1}\Big)\psi_W^{\pm}(t,tv)\end{aligned}$$

and from (236), we obtain

$$-i\partial_t\psi_W^{\pm}(t,tv) \pm \mathcal{M}^{\pm}(v,D)\psi^{\pm}(t,tv) = ct^{-1}\big(V*|\langle\xi_\bullet\rangle^{5/2}W|^2\big)(v)\psi_W^{\pm}(t,tv).$$

Since

$$\begin{aligned}\big(V*|\langle\xi_\bullet\rangle^{5/2}W|^2\big)(v) &= \int_{\mathbb{R}^3}|y|^{-1}\big|t^{3/2}\psi_W(t,v-ty)\big|^2\,\mathrm{d}y\\ &= t\int_{\mathbb{R}^3}|x|^{-1}|\psi_W(v-x)|^2\,\mathrm{d}x\\ &= t\big(V*|\psi_W(t)|^2\big)(v),\end{aligned}$$

we conclude that

$$-i\partial_t\psi_W^{\pm}(t,tv) \pm \mathcal{M}^{\pm}(v,D)\psi^{\pm}(t,tv) = c\big(V*|\psi_W(t)|^2\big)(v)\psi_W^{\pm}(t,tv). \tag{239}$$

Because of (234) and (239), it remains to show that $\mathcal{M}^{\pm}(v,D)$ coincides with $\langle D\rangle$ up to $\mathcal{O}_{H^1(\mathbb{R}^3,\mathbb{C}^4)}(C_W R t^{-3/2})$-expressions. Indeed,

$$\begin{aligned}&\big(\langle\xi\rangle - \mathcal{M}^{\pm}(v,\xi)\big)\widehat{\psi_W^{\pm}(t)}(\xi)\\ &\quad= \mathcal{R}^{\pm}(v,\xi)\widehat{\psi_W^{\pm}(t)}(\xi)\\ &\quad= t^{-3/2}\int_{B_t(0)} e^{-i(x\cdot\xi\pm(t^2-|x|^2)^{1/2})}\mathcal{R}^{\pm}(x/t,\xi)e^{ic\log(t)(V*|W|^2)(x/t)}W(x/t)\,\mathrm{d}x\\ &\quad= t^{3/2}\int_{B_1(0)} e^{-it(v\cdot\xi\pm(1-|v|^2)^{1/2})}\mathcal{R}^{\pm}(v,\xi)e^{ic\log(t)(V*|W|^2)(v)}W(v)\,\mathrm{d}v.\end{aligned}$$

For the phase

$$\phi(v,\xi) = v\cdot\xi \pm (1-|v|^2)^{1/2},$$

note that

$$\nabla_v \phi(v,\xi) = \xi \mp (1-|v|^2)^{-1/2} v$$
$$= \xi \mp \xi_v.$$

Because of (237) and (238), two integrations by parts lead to

$$\begin{aligned}
&\left\| \langle\xi\rangle \big(\langle\xi\rangle - M(v,\xi)\big) \widehat{\psi_W^{\pm}(t)}(\xi) \right\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim t^{-1/2} \left\| \langle\xi\rangle [\psi_{\Delta W}(t,t\cdot)]^{\frown}(t\xi) \right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} + t^{-1/2}\log(t)^2 \left\| \langle\xi\rangle [\psi_{|W|^2 W}(t,t\cdot)]^{\frown}(t\xi) \right\|_{L^2_\xi(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim t^{-2} \left\| \langle t^{-1} D_x\rangle \psi_{\Delta W}(t,tv) \right\|_{L^2_v(\mathbb{R}^3,\mathbb{C}^4)} + t^{-2}\log(t)^2 \left\| \langle t^{-1} D_x\rangle \psi_W(t,tv) \right\|_{L^2_v(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim t^{-2}\log(t)^2 C_W,
\end{aligned}$$

where we used $t^{-1}\nabla_x = \nabla_v$ in the last step. All in all, we have shown that

$$\left\| f^{\pm}(t') \right\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \lesssim C_W (t')^{-3/2}$$

which implies

$$\|H^{\pm}_{\mathrm{inh}}(f^{\pm})\|_Z \le C C_W.$$

Choosing $C_W \in (0,1)$ such that $CC_W \le R/4$ leads to

$$\|H_{\mathrm{inh}}(f)\|_Z \le \frac{R}{2}. \tag{240}$$

Let $u_1, u_2 \in Z$. Estimates (240) and (232) give

$$\|H(u_1)\|_Z \le R.$$

Since

$$H(u_1) - H(u_2) = H_N(u_1) - H_N(u_2),$$

we apply estimate (233) to obtain

$$\|H(u_1) - H(u_2)\|_Z \le \frac{R}{2}.$$

Therefore, H is a contraction mapping on Z_R which finishes the proof of Theorem 4.1.

5 Outlook: Nonzero magnetic field

In the previous sections, we derived modified scattering and asymptotic completeness for the Dirac–Maxwell system with zero magnetic field. The oscillating correction factor was determined by the electric potential $A_0(t) = -(4\pi)^{-1}\big(V * |\psi(t)|^2\big)$ whose spatial L^2-norm decays like t^{-1}. This decay of $A_0(t)$ also played a crucial role for the weighted energy estimates as a key ingredient in the bootstrap argument. This final section is devoted to presenting some ideas on how to treat the magnetic potential (A_1, A_2, A_3). We focus on estimates for the H^N-part of the X_T-norm, cf. (42). We also point out the necessity of the t^{-1}-decay of $\|A_j(t)\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}$ in order to control the H^N-norm for solutions of Dirac–Maxwell with nonzero magnetic field and present some attempts on how to get this desired decay of A_j. By exploiting a hidden null-structure which appears for the A_j but not for A_0, one might hope that, in contrast to A_0, the A_j do not affect the asymptotic behavior so that our modified scattering result would also hold for Dirac–Maxwell with nonzero magnetic field.

We recall that the Dirac–Maxwell system is equivalent to

$$\begin{aligned}
&(-i\partial_t \pm \langle D\rangle)\psi^\pm = \Pi^\pm(D)(A_0\psi + \gamma^0\gamma^j A_j\psi),\\
&\Delta A_0 = |\psi|^2,\\
&\Box A = -\operatorname{curl}\Delta^{-1}\operatorname{curl}\big((\overline{\psi}\gamma^j\psi)_{j=1,2,3}\big),\\
&\partial^j A_j = 0,
\end{aligned}$$

see (17)–(20). The Duhamel equation for $\psi^\pm$ reads

$$\psi^\pm(t) = e^{\mp it\langle D\rangle}\psi_0^\pm + i\int_0^t e^{\mp i(t-t')\langle D\rangle}\Pi^\pm(D)\Big[A_0(t')\psi(t') + \gamma^0\gamma^j A_j(t')\psi(t')\Big]\,\mathrm{d}t'.$$

We have

$$\begin{aligned}
A_0 &= \Delta^{-1}|\psi|^2\\
&= -(4\pi)^{-1}(V * |\psi|^2).
\end{aligned}$$

So for our bootstrap argument, it remains to control

$$\int_0^t e^{\mp i(t-t')\langle D\rangle}\Pi^\pm(D)\Big[\gamma^0\gamma^j A_j(t')\psi(t')\Big]\,\mathrm{d}t'$$

in $\|\cdot\|_{X_T}$. For the H^N-norm, we get

$$\begin{aligned}
&\Big\|\int_0^t e^{\mp i(t-t')\langle D\rangle}\Pi^\pm(D)\Big[\gamma^0\gamma^j A_j(t')\psi(t')\Big]\,\mathrm{d}t'\Big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}\\
&\qquad\lesssim \int_0^t \big\|A_j(t')\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t'\\
&\qquad\le \int_0^t \big\|A_j^{\mathrm{hom}}(t')\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C})} + \big\|A_j^{\mathrm{inh}}(t')\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t',
\end{aligned}$$

where the homogeneous part A_j^{hom} solves

$$\begin{cases}\Box A_j^{\mathrm{hom}} = 0,\\ A_j^{\mathrm{hom}}(0) = a_j, \quad \partial_t A_j^{\mathrm{hom}}(0) = \dot{a}_j,\end{cases}$$

and the inhomogeneous part A_j^{inh} solves

$$\begin{cases}\Box A^{\mathrm{inh}} = -\mathrm{curl}\Delta^{-1}\mathrm{curl}\big(\overline{\psi}\gamma^k\psi\big)_{k=1,2,3},\\ A^{\mathrm{inh}}(0) = 0, \quad \partial_t A^{\mathrm{inh}}(0) = 0.\end{cases}$$

5.1 H^N-estimate for A_j

For the homogeneous part, we have

$$\begin{aligned}\int_0^t \big\|A_j^{\text{hom}}(t')\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t' &\lesssim \int_0^t \big\|A_j^{\text{hom}}(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t' \\ &\quad + \int_0^t \big\|A_j^{\text{hom}}(t')\big\|_{H^{N,3}(\mathbb{R}^3,\mathbb{C})}\big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t' \\ &\lesssim \int_0^t \big\|A_j^{\text{hom}}(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\tilde{\delta}\langle t'\rangle^{\varepsilon}\,\mathrm{d}t' \\ &\quad + \int_0^t \big\|A_j^{\text{hom}}(t')\big\|_{H^{N,3}(\mathbb{R}^3,\mathbb{C}^4)}\tilde{\delta}\langle t'\rangle^{-1}\,\mathrm{d}t' \\ &=: I^{\text{hom}}(t) + II^{\text{hom}}(t).\end{aligned}$$

Provided initial data $a_j \in W^{2,1}(\mathbb{R}^3,\mathbb{R})$ and $\dot{a}_j \in W^{1,1}(\mathbb{R}^3,\mathbb{R})$ such that

$$\big\|a_j\big\|_{W^{2,1}(\mathbb{R}^3,\mathbb{C})} + \big\|\dot{a}_j\big\|_{W^{1,1}(\mathbb{R}^3,\mathbb{C})} \le \delta,$$

von Wahl's estimate (cf. [96]) shows

$$\big\|A_j^{\text{hom}}(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \lesssim \delta\langle t\rangle^{-1}.$$

Hence,

$$I^{\text{hom}}(t) \lesssim \delta\tilde{\delta}\langle t\rangle^{\varepsilon}.$$

For II^{hom}, the assumption

$$\big\|\langle D\rangle^N a_j\big\|_{\dot{H}^{1/3}(\mathbb{R}^3,\mathbb{C})}, \big\|\langle D\rangle^N \dot{a}_j\big\|_{\dot{H}^{-4/3}(\mathbb{R}^3,\mathbb{C})} \le \delta$$

would be sufficient to obtain

$$\begin{aligned} II^{\text{hom}}(t) &\lesssim \big\|A_j(t',x)\big\|_{L^6_{t'}H^{N,3}_x((0,t)\times\mathbb{R}^3,\mathbb{C})}\Big(\int_0^t \langle t'\rangle^{-6/5}\,\mathrm{d}t'\Big)^{5/6} \\ &\lesssim \delta,\end{aligned}$$

where we used the Strichartz estimate for the wave equation (cf. e.g. [90, Thm. 3]) in the second step.

For the inhomogeneous part, we denote

$$\begin{aligned}\int_0^t \big\|A_j^{\text{inh}}(t')\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t' &\lesssim \int_0^t \big\|A_j^{\text{inh}}(t')\big\|_{H^{N,4/(1-\varepsilon)}(\mathbb{R}^3,\mathbb{C})}\big\|\psi(t')\big\|_{L^{4/(1+\varepsilon)}(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t' \\ &\quad + \int_0^t \big\|A_j^{\text{inh}}(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t' \\ &=: I^{\text{inh}}(t) + II^{\text{inh}}(t).\end{aligned}$$

For I^{inh}, an application of Strichartz' estimate yields

$$\begin{aligned} I^{\text{inh}}(t) &\lesssim \big\|A_j^{\text{inh}}(t',x)\big\|_{L^{4/(1+\varepsilon)}_{t'}H^{N,4/(1-\varepsilon)}_x((0,t)\times\mathbb{R}^3,\mathbb{C})}\big\|\psi(t',x)\big\|_{L^{4/(3-\varepsilon)}_{t'}L^{4/(1+\varepsilon)}_x((0,t)\times\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \big\|\mathrm{curl}\Delta^{-1}\mathrm{curl}\big(\overline{\psi}(t')\gamma^k\psi(t')\big)_{k=1,2,3}(x)\big\|_{L^{4/(1+\varepsilon)}_{t'}H^{N,4/(1-\varepsilon)}_x((0,t)\times\mathbb{R}^3,\mathbb{C}^3)} \\ &\qquad\cdot\Big(\int_0^t \big(\tilde{\delta}\langle t'\rangle^{-3/2+3(1+\varepsilon)/4}\big)^{4/(3-\varepsilon)}\,\mathrm{d}t'\Big)^{(3-\varepsilon)/4}\end{aligned}$$

and therefore,

$$\begin{aligned} I^{\text{inh}}(t) &\lesssim \Big(\int_0^t \big(\|\psi(t')\|_{H_x^N(\mathbb{R}^3,\mathbb{C}^4)} \|\psi(t')\|_{L_x^{4/(1-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^4)} \big)^{4/(3+\varepsilon)} \, \mathrm{d}t' \Big)^{(3+\varepsilon)/4} \\ &\quad \cdot \tilde{\delta} \big(\langle t \rangle^{(-3+3\varepsilon)/(3-\varepsilon)+1} \big)^{(3-\varepsilon)/4} \\ &\lesssim \tilde{\delta}^3 \Big(\int_0^t \big(\langle t' \rangle^{\varepsilon - 3/2 + 3(1-\varepsilon)/4} \big)^{4/(3+\varepsilon)} \, \mathrm{d}t' \Big)^{(3+\varepsilon)/4} \langle t \rangle^{\varepsilon/2} \\ &\lesssim \tilde{\delta}^3 \big(\langle t \rangle^{(\varepsilon-3)/(3+\varepsilon)+1} \big)^{(3+\varepsilon)/4} \langle t \rangle^{\varepsilon/2} \\ &\lesssim \tilde{\delta}^3 \langle t \rangle^{\varepsilon}. \end{aligned}$$

It remains to control II^{inh}.

5.2 Approaches how to control II^{inh}

In the previous subsection, we were able to control the homogenous part of A for the H^N-estimate of ψ under sufficient regularity assumptions for the initial data $a, \dot{a}$. For closing the bootstrap argument of $\|\psi(t)\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}$, it remains to show

$$\|A_j^{\text{inh}}(t')\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \lesssim \tilde{\delta}^2 \langle t' \rangle^{-1}.$$

Then, we would get

$$\begin{aligned} II^{\text{inh}}(t) &\lesssim \tilde{\delta}^2 \int_0^t \langle t' \rangle^{-1+\varepsilon} \, \mathrm{d}t' \\ &\lesssim \langle t \rangle^{\varepsilon}. \end{aligned} \tag{241}$$

However, in contrast to the homogeneous part $A_j^{\text{hom}}(t')$ and to $A_0(t')$, estimate (241) for $A_j^{\text{inh}}(t')$ seems to be more delicate. In the following three subsections, we present different (incomplete) approaches, how one might try to obtain estimate (241).

5.2.1 Young's convolution inequality in Lorentz spaces

For $A_0(t)$, we established the required $\langle t \rangle^{-1}$-decay as follows: Since

$$\begin{aligned} A_0(t) &= \Delta^{-1} |\psi(t)|^2 \\ &= -(4\pi)^{-1} \big(V * |\psi(t)|^2 \big), \end{aligned}$$

the Lorentz space version of Young's convolution inequality and interpolation lead to

$$\begin{aligned} \big\|A_0(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} &\lesssim \big\|V\big\|_{L^{3,\infty}(\mathbb{R}^3,\mathbb{C})} \big\||\psi(t)|^2\big\|_{L^{3/2,1}(\mathbb{R}^3,\mathbb{C})} \\ &\lesssim \big\|\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \big\|\psi(t)\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \tilde{\delta}^2 \langle t \rangle^{-1}. \end{aligned}$$

Let us try to transfer this approach to A_j^{inh}. We have

$$\begin{aligned} A^{\text{inh}}(t) &= \square^{-1} \big(\mathrm{curl} \Delta^{-1} \mathrm{curl} \big(\overline{\psi}(t) \gamma^k \psi(t) \big)_{k=1,2,3} \big) \\ &= \int_0^t |D|^{-1} \sin \big((t-t')|D| \big) \mathrm{curl} \Delta^{-1} \mathrm{curl} \big(\overline{\psi}(t') \gamma^k \psi(t') \big)_{k=1,2,3} \, \mathrm{d}t'. \end{aligned}$$

For the "simplified expression"

$$A^{\mathrm{inh}}_{\mathrm{simp}}(t) := \int_{t/4}^{t} |D|^{-1}\big(\overline{\psi}(t)\gamma^k\psi(t)\big)_{k=1,2,3}\,\mathrm{d}t',$$

we would indeed obtain

$$\begin{aligned}
\big\|A^{\mathrm{inh}}_{\mathrm{simp}}(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^3)} &\lesssim \int_{t/4}^{t} \big\||\cdot|^{-2} * |\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\,\mathrm{d}t' \\
&\lesssim \int_{t/4}^{t} \big\||\cdot|^{-2}\big\|_{L^{3/2,\infty}(\mathbb{R}^3,\mathbb{C})}\big\||\psi(t')|^2\big\|_{L^{3,1}(\mathbb{R}^3,\mathbb{C})}\,\mathrm{d}t' \\
&\lesssim \int_{t/4}^{t} \big\|\psi(t')\big\|_{L^4(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^{12}(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t' \\
&\lesssim \tilde{\delta}^2 \int_{t/4}^{t} \langle t'\rangle^{-3/4-5/4}\,\mathrm{d}t' \\
&\lesssim \tilde{\delta}^2\langle t\rangle^{-1}. \qquad (242)
\end{aligned}$$

In fact, it suffices to consider $\int_{t/4}^{t}$ instead of $\int_0^t$ since

$$|A^{\mathrm{inh}}(t,x)| \lesssim \langle t\rangle^{-2+} \quad \text{for } x \notin [t/2, 2t].$$

Furthermore, the operator $\mathrm{curl}\Delta^{-1}\mathrm{curl}$ can essentially be treated like a Fourier multiplier of size 1. However, the argument for (242) does no longer hold true when the $\sin\big((t-t')|D|\big)$-multiplier is included. More precisely, let

$$m(t-t', D) := \chi_{\lesssim 1}(D)\sin\big((t-t')|D|\big), \qquad (243)$$

where $\chi_{\lesssim 1}$ is a smooth $[0,1]$-valued function supported in a ball $B_R(0)$ of radius $R \lesssim 1$ with $\chi = 1$ on $B_{R/2}(0)$. Then,

$$\begin{aligned}
&\big\|\chi_{\lesssim 1}A^{\mathrm{inh}}(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^3)} \\
&\quad\lesssim \int_{t/4}^{t} \big\|m(t-t',D)\mathrm{curl}\Delta^{-1}\mathrm{curl}|D|^{-1}\big(\overline{\psi(t')}\gamma^k\psi(t')\big)_{k=1,2,3}\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^3)}\,\mathrm{d}t' \\
&\quad\lesssim \int_{t/4}^{t} \big\|m(t-t',\cdot)\check{}\,\big\|_{L^1(\mathbb{R}^3,\mathbb{C})}\big\||D|^{-1}\big(\overline{\psi(t')}\gamma^k\psi(t')\big)_{k=1,2,3}\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^3)}\,\mathrm{d}t' \\
&\quad\lesssim \int_{t/4}^{t} \big\|m(t-t',\cdot)\check{}\,\big\|_{L^1(\mathbb{R}^3,\mathbb{C})}\tilde{\delta}^2\langle t'\rangle^{-2}\,\mathrm{d}t',
\end{aligned}$$

where the last step can be seen by the same arguments as in the integrals of (242). For the multiplier, we have

$$\big\|m(t-t',\cdot)\check{}\,\big\|_{L^1(\mathbb{R}^3,\mathbb{C})} = (2\pi)^{-3}\int_{\mathbb{R}^3}\Big|\int_{\mathbb{R}^3} e^{ix\cdot\xi}\chi_{\lesssim 1}(\xi)\sin\big((t-t')|\xi|\big)\,\mathrm{d}\xi\Big|\,\mathrm{d}x.$$

Obviously,

$$\begin{aligned}
\int_{|x|\le 1}\Big|\int_{\mathbb{R}^3} e^{ix\cdot\xi}\chi_{\lesssim 1}(\xi)\sin\big((t-t')|\xi|\big)\,\mathrm{d}\xi\Big|\,\mathrm{d}x &\le \int_{|x|\le 1}\int_{\mathbb{R}^3}\big|\chi_{\lesssim 1}(\xi)\big|\,\mathrm{d}\xi\,\mathrm{d}x \\
&\lesssim 1.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{|x|\geq 1}\Big|\int_{\mathbb{R}^3} e^{ix\cdot\xi}\chi_{\lesssim 1}(\xi)\sin\big((t-t')|\xi|\big)\,\mathrm{d}\xi\Big|\,\mathrm{d}x\\
&\quad=\int_{|x|\geq 1}\Big|\int_0^\infty\int_{\mathbb{S}^2} e^{irx\cdot\omega}\chi_{\lesssim 1}(r)\sin\big((t-t')r\big)\,\mathrm{d}\sigma(\omega)\,r^2\,\mathrm{d}r\Big|\,\mathrm{d}x\\
&\quad\lesssim\int_{|x|\geq 1}\Big|\int_0^C\int_{\mathbb{S}^2} e^{irx\cdot\omega}\,\mathrm{d}\sigma(\omega)\,\sin\big((t-t')r\big)\,r^2\,\mathrm{d}r\Big|\,\mathrm{d}x\\
&\quad\lesssim\int_{|x|\geq 1}\Big|\int_0^C\int_{-1}^1 e^{irs|x|}\,\mathrm{d}s\,\sin\big((t-t')r\big)\,r^2\,\mathrm{d}r\Big|\,\mathrm{d}x\\
&\quad\lesssim\int_{|x|\geq 1}\Big|\int_0^C r^{-1}|x|^{-1}\big(e^{ir|x|}-e^{-ir|x|}\big)\,\sin\big((t-t')r\big)\,r^2\,\mathrm{d}r\Big|\,\mathrm{d}x\\
&\quad\lesssim\sum_{\pm_1,\pm_2\in\{+,-\}}\int_{|x|\geq 1}\Big|\int_0^C |x|^{-1}e^{\pm_1 ir(|x|\pm_2(t-t'))}\,r\,\mathrm{d}r\Big|\,\mathrm{d}x.
\end{aligned}$$

The last expression remains bounded when we are able to improve the $|x|^{-1}$-decay to a $|x|^{-3-}$-decay which is fulfilled for

$$|x|\not\approx(t-t')$$

but not for

$$|x|\approx(t-t').$$

We also note that it might suffice to control $\chi_{\lesssim 1}(D)A^{\mathrm{inh}}$ since $\chi_{\gg 1}A^{\mathrm{inh}}$ could be controlled by Klainerman–Sobolev estimate, cf. Subsection 5.2.3.

5.2.2 Strichartz estimate

As discussed in the previous section, it suffices to consider $\int_{t/4}^t$ instead of $\int_0^t$. The approach

$$\begin{aligned}
&\int_{t/4}^t \big\|A_j^{\mathrm{inh}}(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\big\|\psi(t')\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t'\\
&\quad\lesssim\big\|A_j(t',x)\big\|_{L^2_{t'}L^\infty_x((t/4,t)\times\mathbb{R}^3,\mathbb{C})}\big\|\psi(t',x)\big\|_{L^2_{t'}H^N((t/4,t)\times\mathbb{R}^3,\mathbb{C}^4)}\\
&\quad\lesssim\big\|\big(\overline{\psi(t')}\gamma^k\psi(t')\big)_{k=1,2,3}\big\|_{L^1_{t'}L^2_x((t/4,t)\times\mathbb{R}^3,\mathbb{C}^3)}\Big(\int_{t/4}^t\big(\tilde\delta\langle t'\rangle^\varepsilon\big)^2\Big)^{1/2}\,\mathrm{d}t'\\
&\quad\lesssim\int_{t/4}^t\big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t'\,\tilde\delta\langle t\rangle^{\varepsilon+1/2}\\
&\quad\lesssim\tilde\delta^3\langle t\rangle^{\varepsilon+1/2}\int_{t/4}^t\langle t'\rangle^{-3/2}\,\mathrm{d}t'\\
&\quad\lesssim\tilde\delta^3\langle t\rangle^\varepsilon
\end{aligned}$$

fails since we used the endpoint Strichartz estimate in $L^2_tL^\infty_x$ in the second step which is not valid in three space dimensions (cf. e.g. Klainerman and Machedon [61]). This endpoint Strichartz estimate can be recovered by working with additional angular weights (cf. e.g. Sterbenz–Rodnianski [93] and Fang–Wang [38])

$$\Omega_{ij}=x_j\partial_i-x_i\partial_j,\quad 1\leq i<j\leq 3$$

and using

$$\begin{aligned}\big\|\Omega_{ij}\psi(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &= \big\|\Omega_{ij}\mathrm{w}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \big\|x\mathrm{w}(t)\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde\delta\langle t\rangle^{\varepsilon},\end{aligned}$$

see also Section 5.2.3 for a deeper discussion of Ω_{ij}. However, this additional factor $\langle t\rangle^{\varepsilon}$ already destroys our bootstrap argument.

As done in the estimate for I^{inh}, we could try to use a perturbed $L_t^4L_x^4$-Strichartz estimate. But while we have the estimates

$$\begin{aligned}\big\|\psi(t)\big\|_{W^{2,p}(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \tilde\delta\langle t\rangle^{-3/2+3/p}, \quad p\in[2,\infty],\\ \big\|\psi(t)\big\|_{H^N(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \tilde\delta\langle t\rangle^{\varepsilon},\end{aligned}$$

there is no estimate of the form

$$\big\|\psi(t)\big\|_{H^{N,p}(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde\delta\langle t\rangle^{\varepsilon-3/2+3/p} \tag{244}$$

when $p\in(2,\infty]$ and $N>2$. If estimate (244) was applicable, we would obtain

$$\begin{aligned}&II^{\text{inh}}(t)\\ &\lesssim \int_0^t \big\|A^{\text{inh}}(t')\big\|_{L^{4/(1+\varepsilon)}(\mathbb{R}^3,\mathbb{C}^3)}\big\|\psi(t')\big\|_{H^{N,4/(1-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^4)}\,\mathrm{d}t'\\ &\lesssim \big\|A^{\text{inh}}(t',x)\big\|_{L_{t'}^{4/(1-\varepsilon)}L_x^{4/(1+\varepsilon)}((0,t)\times\mathbb{R}^3,\mathbb{C}^3)}\big\|\psi(t',x)\big\|_{L_{t'}^{4/(3+\varepsilon)}H_x^{N,4/(1-\varepsilon)}((0,t)\times\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \big\|\big(\overline{\psi}(t')\gamma^k\psi(t')\big)_{k=1,2,3}\big\|_{L_{t'}^{4/(3-\varepsilon)}L_x^{4/(3+\varepsilon)}((0,t)\times\mathbb{R}^3,\mathbb{C}^3)}\\ &\qquad\cdot\Big(\int_0^t\big(\tilde\delta\langle t'\rangle^{\varepsilon-3/2+3(1-\varepsilon)/4}\big)^{4/(3+\varepsilon)}\,\mathrm{d}t'\Big)^{(3+\varepsilon)/4}\\ &\lesssim \Big(\int_0^t\big(\|\psi(t)\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\|\psi(t)\|_{L^{4/(3+\varepsilon)}(\mathbb{R}^3,\mathbb{C}^4)}\big)^{4/(3-\varepsilon)}\,\mathrm{d}t'\Big)^{(3-\varepsilon)/4}\tilde\delta\big(\langle t\rangle^{(\varepsilon-3)/(3+\varepsilon)+1}\big)^{(3+\varepsilon)/4}\\ &\lesssim \tilde\delta^3\Big(\int_0^t\langle t'\rangle^{-3/(3-\varepsilon)}\,\mathrm{d}t'\Big)^{(3-\varepsilon)/4}\langle t\rangle^{\varepsilon/2}\\ &\lesssim \tilde\delta^3\langle t\rangle^{\varepsilon},\end{aligned}$$

where we require estimate (244) with $p=4/(1-\varepsilon)$ in the third step.

5.2.3 Klainerman–Sobolev inequality

The Klainerman–Sobolev inequality (cf. e.g. Klainerman [60] and Selberg [88]) for $u\in\mathcal{S}([0,\infty)\times\mathbb{R}^3)$ reads

$$|u(t,x)|\lesssim\big(1+t+|x|\big)^{-1}\sum_{\alpha\in\mathbb{N}_0^{11}:\,|\alpha|\le 2}\big\|\Gamma^{\alpha}u(t)\big\|_{L^2(\mathbb{R}^3)},$$

where

$$\Gamma=\big(\Gamma_1,\dots,\Gamma_{11}\big)$$

is an enumeration of the vector fields

$$\begin{aligned}
&\partial_t, \partial_1, \partial_2, \partial_3,\\
&\Omega_{ij} = x_j\partial_i - x_i\partial_j, \quad 1 \le i < j \le 3,\\
&\Omega_{0j} = t\partial_j + x_j\partial_t, \quad j = 1, 2, 3,\\
&L_0 = t\partial_t + \sum_{i=1}^{3} x_i\partial_i
\end{aligned}$$

and

$$\Gamma^\alpha = \Gamma_1^{\alpha_1} \cdots \Gamma_{11}^{\alpha_{11}}, \quad \alpha = (\alpha_1, \dots, \alpha_{11}) \in \mathbb{N}_0^{11}.$$

We can avoid the presence of L_0 which does not behave well with the Dirac propagator $e^{\mp it\langle D\rangle}$ by allowing possible future times on the right-hand side (cf. again Klainerman [60]). More precisely,

$$\begin{aligned}
&|u(t,x)|\\
&\lesssim \langle t\rangle^{-1} \sup_{t' \in [0, 2(t^2 - |x|^2)^{1/2}]} \Big(\sum_{\alpha \in \mathbb{N}_0^{11}:\, |\alpha| \le 2} \big\|\Omega^\alpha u(t')\big\|_{L^2(\mathbb{R}^3)}^2 + \big\|\Omega^\alpha \nabla u(t')\big\|_{L^2(\mathbb{R}^3)}^2 \Big)^{1/2}
\end{aligned} \tag{245}$$

for any $u \in \mathcal{S}([0,\infty) \times \mathbb{R}^3)$, where

$$\begin{aligned}
\Omega &= \big(\Omega_1, \dots, \Omega_6\big)\\
&= \big(\Omega_{01}, \Omega_{02}, \Omega_{03}, \Omega_{12}, \Omega_{13}, \Omega_{23}\big)
\end{aligned}$$

and

$$\Omega^\alpha = \Omega_1^{\alpha_1} \dots \Omega_6^{\alpha_6}, \quad \alpha \in \mathbb{N}_0^6.$$

We establish the a priori assumption

$$\big\|A^{\mathrm{inh}}(t)\big\|_{W^{2,\infty}(\mathbb{R}^3,\mathbb{C}^3)} \lesssim \tilde{\delta}\langle t\rangle^{-1} \tag{246}$$

which we want to improve to

$$\big\|A^{\mathrm{inh}}(t)\big\|_{W^{2,\infty}(\mathbb{R}^3,\mathbb{C}^3)} \lesssim \tilde{\delta}^2\langle t\rangle^{-1}. \tag{247}$$

For any $1 \le i < j \le 3$, we have

$$\begin{aligned}
\Omega_{ij}A^{\mathrm{inh}}(t,x) = &\int_0^t x_j\big[|D|^{-1}\sin\big((t-t')|D|\big)\partial_i N(\psi(t'))\big](x)\,\mathrm{d}t'\\
&- \int_0^t x_i\big[|D|^{-1}\sin\big((t-t')|D|\big)\partial_j N(\psi(t'))\big](x)\,\mathrm{d}t',
\end{aligned}$$

where

$$N(\psi) = -\mathrm{curl}\Delta^{-1}\mathrm{curl}\big(\overline{\psi}\gamma^k\psi\big)_{k=1,2,3}.$$

Since

$$\begin{aligned}
x_j\big[|D|^{-1}\sin\big((t-t')|D|\big)\partial_i N(\psi(t'))\big] = &-|D|^{-3}\partial_j\sin\big((t-t')|D|\big)\partial_i N(\psi(t'))\\
&+ |D|^{-1}\cos\big((t-t')|D|\big)(t-t')|D|\partial_j\partial_i N(\psi(t'))\\
&+ |D|^{-1}\sin\big((t-t')|D|\big)x_j\partial_i N(\psi(t'))
\end{aligned}$$

and analogously for $x_i\big[|D|^{-1}\sin\big((t-t')|D|\big)\partial_j N(\psi(t'))\big]$, we obtain

$$\Omega_{ij}A^{\mathrm{inh}}(t,x)=\int_0^t |D|^{-1}\sin\big((t-t')|D|\big)\big[\Omega_{ij}N(\psi(t'))\big](x)\,\mathrm{d}t'.$$

For $j=1,2,3$, note that

$$\begin{aligned}\Omega_{0j}A^{\mathrm{inh}}(t,x)=&\int_0^t |D|^{-1}\sin\big((t-t')|D|\big)t\partial_j N(\psi(t'))\big](x)\,\mathrm{d}t'\\&+x_j\int_0^t \cos\big((t-t')|D|\big)N(\psi(t'))(x)\,\mathrm{d}t'\end{aligned}$$

and

$$\begin{aligned}x_j\big[\cos\big((t-t')|D|\big)N(\psi(t'))\big]=&-\sin\big((t-t')|D|\big)(t-t')|D|^{-1}\partial_j N(\psi(t'))\\&+\cos\big((t-t')|D|\big)x_j N(\psi(t')).\end{aligned}$$

Integration by parts leads to

$$\begin{aligned}\int_0^t \cos\big((t-t')|D|\big)x_j N(\psi(t'))(x)\,\mathrm{d}t'=&\int_0^t |D|^{-1}\sin\big((t-t')|D|\big)x_j\partial_{t'}N(\psi(t'))(x)\,\mathrm{d}t'\\&+|D|^{-1}\sin\big(t|D|\big)x_j N(\psi_0).\end{aligned}$$

Hence,

$$\begin{aligned}\Omega_{0j}A^{\mathrm{inh}}(t,x)=&\int_0^t |D|^{-1}\sin\big((t-t')|D|\big)\Omega_{0j}N(\psi(t'))(x)\,\mathrm{d}t'\\&+|D|^{-1}\sin\big(t|D|\big)x_j N(\psi_0).\end{aligned}$$

The vector fields Ω_{kj}, $1\le k<j\le 3$ commute with $e^{\mp it\langle D\rangle}$. Indeed,

$$\begin{aligned}\Omega_{kj}\psi^{\pm}(t)&=(x_j\partial_k-x_k\partial_j)\big(e^{\mp it\langle D\rangle}\mathrm{w}^{\pm}(t)\big)\\&=\mp it\langle D\rangle^{-1}\partial_j\partial_k e^{\mp it\langle D\rangle}\mathrm{w}^{\pm}(t)+e^{\mp it\langle D\rangle}x_j\partial_k\mathrm{w}^{\pm}(t)\\&\quad\pm it\langle D\rangle^{-1}\partial_k\partial_j e^{\mp it\langle D\rangle}\mathrm{w}^{\pm}(t)-e^{\mp it\langle D\rangle}x_k\partial_j\mathrm{w}^{\pm}(t)\\&=e^{\mp it\langle D\rangle}\Omega_{kj}\mathrm{w}^{\pm}(t)\end{aligned}\tag{248}$$

and analogously

$$\Omega_{k'j'}\Omega_{kj}\psi^{\pm}(t)=e^{\mp it\langle D\rangle}\Omega_{k'j'}\Omega_{kj}\mathrm{w}^{\pm}(t).\tag{249}$$

For the vector fields Ω_{0j}, $j=1,2,3$, we have

$$\begin{aligned}\Omega_{0j}\psi^{\pm}(t)&=(t\partial_j+x_j\partial_t)\big(e^{\mp it\langle D\rangle}\mathrm{w}^{\pm}(t)\big)\\&=e^{\mp it\langle D\rangle}t\partial_j\mathrm{w}^{\pm}(t)+x_j\big(\mp i\langle D\rangle e^{\mp it\langle D\rangle}\mathrm{w}^{\pm}(t)\big)+x_j\big(e^{\mp it\langle D\rangle}\partial_t\mathrm{w}^{\pm}(t)\big)\\&=e^{\mp it\langle D\rangle}\big(\mp i\langle D\rangle^{-1}\partial_j\mp i\langle D\rangle x_j\mp i\langle D\rangle^{-1}\partial_j t\partial_t+x_j\partial_t\big)\mathrm{w}^{\pm}(t)\\&=e^{\mp it\langle D\rangle}\big(\mp i\langle D\rangle^{-1}\partial_j\mp i\langle D\rangle x_j+L_j^{\mp}\partial_t\big)\mathrm{w}^{\pm}(t)\end{aligned}\tag{250}$$

and

$$\begin{aligned}\Omega_{0j'}\Omega_{0j}\psi^{\pm}(t)=e^{\mp it\langle D\rangle}&\big(\mp i\langle D\rangle^{-1}\partial_{j'}\mp i\langle D\rangle x_{j'}+\big(L_{j'}^{\mp}\big)\partial_t\big)\\&\big(\mp i\langle D\rangle^{-1}\partial_j\mp i\langle D\rangle x_j+L_j^{\mp}\partial_t\big)\mathrm{w}^{\pm}(t).\end{aligned}\tag{251}$$

For interactions of Ω_{kj} and $\Omega_{0j'}$, note that

$$\big[\Omega_{\alpha\beta},\Omega_{\gamma\delta}\big] = g^{\alpha\gamma}\Omega_{\beta\delta} + g^{\beta\delta}\Omega_{\alpha\gamma} - g^{\beta\gamma}\Omega_{\alpha\delta} - g^{\beta\delta}\Omega_{\beta\gamma} \tag{252}$$

and

$$\Omega_{0j'}\Omega_{kj}\psi^{\pm}(t) = e^{\mp it\langle D\rangle}\big(\mp i\langle D\rangle^{-1}\partial_{j'} \mp i\langle D\rangle x_{j'} + L^{\mp}_{j'}\partial_t\big)\Omega_{kj}\mathrm{w}^{\pm}(t). \tag{253}$$

We now consider the contributions

$$\big\|\Omega^{\alpha}\big(\overline{\psi}(t')\gamma^k\psi(t')\big)\big\|_{L^2(\mathbb{R}^3,\mathbb{C})},\quad k=1,2,3,\ \alpha\in\mathbb{N}_0^6,\ |\alpha|\le 2.$$

For $|\alpha|=0$, we obtain immediately

$$\begin{aligned}\big\|\overline{\psi}(t')\gamma^k\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C})} &\le \big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\\ &\le \tilde{\delta}^2\langle t'\rangle^{-3/2}.\end{aligned} \tag{254}$$

Let $|\alpha|=1$. Leibniz' rule yields

$$\begin{aligned}\Omega_{ij}\big(\overline{\psi}\gamma^k\psi\big) &= \big(\Omega_{ij}\overline{\psi}\big)\gamma^k\psi - \overline{\psi}\gamma^k\big(\Omega_{ij}\psi\big),\quad 1\le i<j\le 3,\\ \Omega_{0j}\big(\overline{\psi}\gamma^k\psi\big) &= \big(\Omega_{ij}\overline{\psi}\big)\gamma^k\psi + \overline{\psi}\gamma^k\big(\Omega_{ij}\psi\big),\quad 1\le j\le 3.\end{aligned}$$

Therefore,

$$\begin{aligned}\big\|\Omega_{ij}\big(\overline{\psi}(t')\gamma^k\psi(t')\big)\big\|_{L^2(\mathbb{R}^3,\mathbb{C})} &\lesssim \big\|\Omega_{ij}\mathrm{w}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \big\|x\mathrm{w}(t')\big\|_{\dot{H}^1(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}^2\langle t'\rangle^{\varepsilon-3/2}\end{aligned}$$

and

$$\begin{aligned}\big\|\Omega_{0j}\big(\overline{\psi}(t')\gamma^k\psi(t')\big)\big\|_{L^2(\mathbb{R}^3,\mathbb{C})} &\lesssim \Big(\big\|\mathrm{w}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\|x_j\mathrm{w}(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\ &\qquad + \sum_{\pm_0\in\{+,-\}}\big\|L_j^{\mp_0}\partial_{t'}\mathrm{w}^{\pm_0}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\Big)\big\|\psi(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \Big(\tilde{\delta}\langle t'\rangle^{\varepsilon} + \sum_{\pm_0\in\{+,-\}}\big\|L_j^{\mp_0}\partial_{t'}\mathrm{w}^{\pm_0}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\Big)\tilde{\delta}\langle t'\rangle^{-3/2}.\end{aligned}$$

Since

$$\partial_t\mathrm{w}^{\pm_0}(t) = e^{\pm_0 it\langle D\rangle}\Pi^{\pm_0}(D)\big[\gamma^0\gamma^\mu A_\mu(t)\psi(t)\big], \tag{255}$$

we have

$$L_j^{\mp_0}\partial_t\mathrm{w}^{\pm_0}(t) = e^{\pm_0 it\langle D\rangle}x_j\Big(\Pi^{\pm_0}(D)\big[\gamma^0\gamma^\mu A_\mu(t)\psi(t)\big]\Big)$$

and

$$\begin{aligned}\big\|L_j^{\mp_0}\partial_{t'}\mathrm{w}^{\pm_0}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \big\|\gamma^0\gamma^\mu A_\mu(t')\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} + \big\|\gamma^0\gamma^\mu x_j\big(A_\mu(t')\psi(t')\big)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \langle t'\rangle\big\|\gamma^0\gamma^\mu A_\mu(t')\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \langle t'\rangle\Big(\big\|A_0(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} + \big\|\gamma^j A_j(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^3)}\Big)\big\|\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}\langle t'\rangle\Big(\tilde{\delta}^2\langle t'\rangle^{-1} + \big\|A^{\mathrm{inh}}(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^3)}\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\big\|\Omega_{0j}\big(\overline{\psi}(t')\gamma^k\psi(t')\big)\big\|_{L^2(\mathbb{R}^3,\mathbb{C})} &\lesssim \tilde{\delta}^2\langle t'\rangle^{\varepsilon-3/2}+\tilde{\delta}\langle t'\rangle^{-1/2}\big\|A^{\mathrm{inh}}(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^3)}\\ &\lesssim \tilde{\delta}^2\langle t'\rangle^{\varepsilon-3/2},\end{aligned}$$

where we used a priori assumption (246) in the second step. Therefore,

$$\sum_{\alpha\in\mathbb{N}_0^6\colon |\alpha|=1}\big\|\Omega^\alpha\big(\overline{\psi}(t')\gamma^k\psi(t')\big)\big\|_{L^2(\mathbb{R}^3,\mathbb{C})}\lesssim \tilde{\delta}^2\langle t'\rangle^{\varepsilon-3/2}. \tag{256}$$

Finally, consider $|\alpha|=2$. Similarly to (256), we would like to obtain

$$\sum_{\alpha\in\mathbb{N}_0^6\colon |\alpha|=2}\big\|\Omega^\alpha\big(\overline{\psi}(t')\gamma^k\psi(t')\big)\big\|_{L^2(\mathbb{R}^3,\mathbb{C})}\lesssim \tilde{\delta}^2\langle t'\rangle^{2\varepsilon-3/2}. \tag{257}$$

Then, we would be able to conclude

$$\begin{aligned}&\sum_{\alpha\in\mathbb{N}_0^6\colon |\alpha|\le 2}\big\|\chi_{\gg 1}(D)\Omega^\alpha A^{\mathrm{inh}}(t)\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^3)}\\ &\qquad\lesssim \int_0^t\big\|\chi_{\gg 1}(D)|D|^{-1}\sin\big((t-t')|D|\big)\Omega^\alpha N(\psi(t'))\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^3)}\,\mathrm{d}t'\\ &\qquad\lesssim \int_0^t\tilde{\delta}^2\langle t'\rangle^{2\varepsilon-3/2}\,\mathrm{d}t'\\ &\qquad\lesssim \tilde{\delta}^2,\end{aligned} \tag{258}$$

where $\chi_{\gg 1}=1-\chi_{\lesssim 1}$ and $\chi_{\lesssim 1}$ comes from (243). However, the contributions in (251) and (253) turn out to be more delicate and we will only present a sketch of proof how one might try to handle these terms. We need the restriction on large frequencies for two reasons. On the one hand, the contribution $|D|^{-1}$ would lead to problems for small frequencies. On the other hand, we want to use in (262) that $\langle D\rangle$ behaves like $|D|$ for large frequencies. From (249), we get

$$\begin{aligned}\big\|\Omega_{kj}\Omega_{k'j'}\psi^\pm(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &= \big\|\Omega_{kj}\Omega_{k'j'}\mathrm{w}^\pm(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \big\|\langle x\rangle^2\mathrm{w}^\pm(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\le \tilde{\delta}\langle t'\rangle^{2\varepsilon}.\end{aligned} \tag{259}$$

By (253), we have

$$\begin{aligned}\big\|\Omega_{0j}\Omega_{0j'}\psi^\pm(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \big\|\langle x\rangle^2\mathrm{w}^\pm(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} &+ \big\|\langle x\rangle L_j^\mp\partial_t\mathrm{w}^\pm(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\ &+\big\|L_{j'}^\mp\partial_t L_j^\mp\partial_t\mathrm{w}^\pm(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}.\end{aligned} \tag{260}$$

The first term in (260) is obviously bounded by

$$\big\|\langle x\rangle^2\mathrm{w}^\pm(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\lesssim\tilde{\delta}\langle t'\rangle^{2\varepsilon}.$$

For the second term in (260), note that

$$L_j^\mp\big[\gamma^0\gamma^\mu A_\mu(t')\psi(t')\big]=\gamma^0\gamma^\mu A_\mu(t')L_j^\mp\psi(t')\mp it'\gamma^0\gamma^\mu\partial_j A_\mu(t')\,\langle D\rangle^{-1}\psi(t')$$

and

$$L_{j'}^{\mp} x_j = x_j L_{j'}^{\mp} \mp it' \langle D \rangle^{-4} \partial_{j'} \partial_j.$$

Hence,

$$\begin{aligned}
&\big\| x L_j^{\mp} \partial_t \mathrm{w}^{\pm}(t') \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad = \big\| L_j^{\mp} x \big(\Pi^{\pm_0}(D) [\gamma^0 \gamma^\mu A_\mu(t') \psi(t')] \big) \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad \lesssim \big\| x \gamma^0 \gamma^\mu A_\mu(t') L_j^{\mp} \psi(t') \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\
&\qquad + \big\| t' \gamma^0 \gamma^\mu \partial_j A_\mu(t') \langle D \rangle^{-1} \psi(t')] \big) \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\
&\qquad + \langle t' \rangle \big\| \gamma^0 \gamma^\mu A_\mu(t') \psi(t') \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad \lesssim \langle t' \rangle \Big(\big\| A_0(t') \big\|_{W^{1,\infty}(\mathbb{R}^3,\mathbb{C})} + \big\| \gamma^0 \gamma^j A_j(t') \big\|_{W^{1,\infty}(\mathbb{R}^3,\mathbb{C}^4)} \Big) \big\| x \mathrm{w}(t') \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \\
&\qquad + \langle t' \rangle \Big(\big\| A_0(t') \big\|_{W^{2,6/(2-\varepsilon)}(\mathbb{R}^3,\mathbb{C})} + \big\| \gamma^0 \gamma^j A_j(t') \big\|_{W^{2,6/(2-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^4)} \Big) \big\| \psi(t') \big\|_{L^{6/(1+\varepsilon)}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\qquad + \langle t' \rangle \Big(\big\| A_0(t') \big\|_{W^{1,\infty}(\mathbb{R}^3,\mathbb{C})} + \big\| \gamma^0 \gamma^j A_j(t') \big\|_{W^{1,\infty}(\mathbb{R}^3,\mathbb{C}^4)} \Big) \big\| \psi(t') \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}.
\end{aligned}$$

As before,

$$\begin{aligned}
\big\| A_0(t') \big\|_{W^{1,\infty}(\mathbb{R}^3,\mathbb{C})} &\lesssim \big\| \psi(t') \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \big\| \psi(t') \big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim \tilde{\delta}^2 \langle t' \rangle^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\big\| A_0(t') \big\|_{W^{2,6/(2-\varepsilon)}(\mathbb{R}^3,\mathbb{C})} &\lesssim \big\| V \big\|_{L^{3,1}(\mathbb{R}^3,\mathbb{R})} \big\| |\psi(t')|^2 \big\|_{W^{2,6/(6-\varepsilon),\infty}(\mathbb{R}^3,\mathbb{C})} \\
&\lesssim \big\| \psi(t') \big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)} \big\| \psi(t') \big\|_{L^{6/(3-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\lesssim \tilde{\delta}^2 \langle t' \rangle^{-\varepsilon/2}.
\end{aligned}$$

Since we also know that

$$\begin{aligned}
&\big\| x \mathrm{w}(t') \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta} \langle t' \rangle^{\varepsilon}, \\
&\big\| \psi(t') \big\|_{L^{6/(1+\varepsilon)}(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta} \langle t' \rangle^{-1+\varepsilon/2}, \\
&\big\| \psi(t') \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta},
\end{aligned}$$

we end up with

$$\begin{aligned}
\big\| x L_j^{\mp} \partial_t \mathrm{w}^{\pm}(t') \big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \tilde{\delta}^2 \langle t' \rangle^{\varepsilon} + \tilde{\delta} \langle t' \rangle^{1+\varepsilon} \big\| A^{\mathrm{inh}}(t') \big\|_{W^{1,\infty}(\mathbb{R}^3,\mathbb{C}^3)} \\
&\qquad + \tilde{\delta} \langle t' \rangle^{\varepsilon/2} \big\| A^{\mathrm{inh}}(t') \big\|_{W^{2,6/(2-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^3)} \\
&\lesssim \tilde{\delta}^2 \langle t' \rangle^{\varepsilon} + \tilde{\delta} \langle t' \rangle^{\varepsilon/2} \big\| A^{\mathrm{inh}}(t') \big\|_{W^{2,6/(2-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^3)} \qquad (261)
\end{aligned}$$

using a priori assumption (246) in the second step. The second term in (261) will be controlled by applying Strichartz' estimate. For the last term in (260), note that

$$\partial_t L_j^{\mp} = L_j^{\mp} \partial_t \mp \langle D \rangle^{-1} \partial_j$$

and

$$\partial_t^2 \mathrm{w}^{\pm}(t) = e^{\pm it\langle D \rangle} \Pi^{\pm}(D) \gamma^0 \gamma^\mu \Big(\pm i \langle D \rangle \big[A_\mu(t) \psi(t) \big] + \gamma^\mu A_\mu(t) \partial_t \psi(t) + \gamma^\mu \partial_t A_\mu(t) \psi(t) \Big).$$

Provided sufficiently large frequencies, one might try to replace $\langle D\rangle$ with $|D|$ (up to acceptable error terms). Then, we could use

$$\begin{aligned} A^{\mathrm{inh}}(t) &\approx \frac{1}{2i}\int_0^t |D|^{-1}e^{i(t-t')|D|}N(\psi)(t')\,\mathrm{d}t' - \frac{1}{2i}\int_0^t |D|^{-1}e^{-i(t-t')|D|}N(\psi)(t')\,\mathrm{d}t' \\ &=: \frac{1}{2i}A^{\mathrm{inh},+}(t) - \frac{1}{2i}A^{\mathrm{inh},-}(t), \end{aligned}$$

where $A^{\mathrm{inh},\pm}$ solves the half-wave equation

$$-i\partial_t A^{\mathrm{inh},\pm} \pm |D|A^{\mathrm{inh},\pm} = -i|D|^{-1}N(\psi),$$

and therefore,

$$\begin{aligned} &\|L^{\mp}_{j'}\partial_t L^{\mp}_j\partial_t \mathrm{w}^{\pm}(t')\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad\lesssim \big\|L^{\mp}_{j'}L^{\mp}_j(-i\partial_t \pm \langle D\rangle)A_0(t')\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + \big\|L^{\mp}_{j'}L^{\mp}_j\gamma^0\gamma^j(-i\partial_t \pm |D|)A_j(t')\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + \big\|L^{\mp}_{j'}L^{\mp}_j\gamma^0\gamma^\mu A_\mu(t')(-i\partial_t \pm \langle D\rangle)\psi^{\pm}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + \big\|\gamma^0\gamma^\mu A_\mu(t')\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\quad\lesssim \langle t'\rangle^2\big\|V*\big[\big(\gamma^0\gamma^\mu A_\mu(t')\psi(t')\big)\overline{\psi}(t')\big]\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + \langle t'\rangle^2\big\|\gamma^0\gamma^j|D|^{-1}N(\psi)(t')\,\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + \langle t'\rangle^2\big\|\gamma^0\gamma^\mu A_\mu(t')\,\gamma^0\gamma^\mu A_\mu(t')\psi^{\pm}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + \tilde\delta\Big(\tilde\delta^2\langle t'\rangle^{-1} + \big\|\gamma^j A_j(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\Big) \\ &\quad\lesssim \tilde\delta\langle t'\rangle^2\big\|\gamma^0\gamma^\mu A_\mu(t')\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + \tilde\delta\langle t'\rangle^2\big\||D|^{-1}N(\psi)(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^3)} \\ &\qquad + \tilde\delta\langle t'\rangle^2\big\|\gamma^0\gamma^\mu A_\mu(t')\big\|^2_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} \\ &\qquad + \tilde\delta\Big(\tilde\delta^2\langle t'\rangle^{-1} + \big\|\gamma^j A_j(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\Big) \\ &\quad\lesssim \tilde\delta\langle t'\rangle^2\big\|\gamma^j A_j(t')\big\|^2_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} + \tilde\delta\langle t'\rangle\big\|\gamma^j A_j(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} + \tilde\delta^3, \end{aligned} \tag{262}$$

where the last step follows from

$$\big\|A_0(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} \lesssim \tilde\delta^2\langle t'\rangle^{-1}$$

and

$$\begin{aligned} \big\||D|^{-1}N(\psi)(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^3)} &\lesssim \big\||\cdot|^{-2}\big\|_{L^{3/2,\infty}(\mathbb{R}^3,\mathbb{C})}\big\|N(\psi)(t')\big\|_{L^{3,1}(\mathbb{R}^3,\mathbb{C}^3)} \\ &\lesssim \big\|\psi(t')\big\|_{L^4(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t')\big\|_{L^{12}(\mathbb{R}^3,\mathbb{C}^4)} \\ &\lesssim \tilde\delta^2\langle t'\rangle^{-2}. \end{aligned}$$

Applying a priori assumption (246) in (262) leads to

$$\|L^{\mp}_{j'}\partial_t L^{\mp}_j\partial_t \mathrm{w}^{\pm}(t')\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde\delta^2.$$

For the contributions $\Omega_{0j'}\Omega_{kj}$, we first note that

$$\begin{aligned}\big\|\big(\langle D\rangle^{-1}\partial_j + \langle D\rangle x_j\big)\Omega_{kj}\mathrm{w}^{\pm}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \big\|\langle x\rangle^2\mathrm{w}^{\pm}(t')\big\|_{H^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}\langle t'\rangle^{\varepsilon}.\end{aligned}$$

Because of (253), it remains to consider $\|L_j^{\mp}\partial_t\Omega_{kj}\mathrm{w}^{\pm}(t')\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$. We have

$$\begin{aligned}\big\|L_j^{\mp}\partial_t\Omega_{kj}\mathrm{w}^{\pm}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &= \big\|L_j^{\mp}\Omega_{kj}\partial_t\mathrm{w}^{\pm}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &= \big\|L_j^{\mp}e^{\pm it\langle D\rangle}\Omega_{kj}\Pi^{\pm}(D)\gamma^0\gamma^{\mu}A_{\mu}(t')\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \big\|\gamma^0\gamma^{\mu}x_j\Omega_{kj}A_{\mu}(t')\,\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad + \big\|\gamma^0\gamma^{\mu}x_jA_{\mu}(t')\,\Omega_{kj}\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \langle t'\rangle\big\||D|^{-3}\partial_k\partial_j|\psi(t')|^2\,\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad + \langle t'\rangle\big\|V*(\overline{\psi}(t')\Omega_{kj}\psi(t'))\,\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad + \langle t'\rangle\big\|\gamma^0\gamma^j\Omega_{kj}A_j(t')\,\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\quad + \langle t'\rangle\big\|\gamma^0\gamma^{\mu}A_{\mu}(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)}\big\|\Omega_{kj}\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}.\end{aligned}$$

Since

$$\begin{aligned}\big\|\Omega_{kj}\psi(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &= \big\|\Omega_{kj}\mathrm{w}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \big\|x\mathrm{w}(t')\big\|_{H^1(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}\langle t'\rangle^{\varepsilon}\end{aligned}$$

and

$$\begin{aligned}\big\||D|^{-3}\partial_k\partial_j|\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} &\lesssim \big\||D|^{-1}|\psi(t')|^2\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})}\\ &\lesssim \big\|\psi(t')\big\|_{L^4(\mathbb{R}^3,\mathbb{C}^4)}\|\psi(t')\|_{L^{12}(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}^2\langle t'\rangle^{-2}\end{aligned}$$

as well as

$$\begin{aligned}\big\|V*(\overline{\psi}(t')\Omega_{kj}\psi(t'))\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C})} &\lesssim \big\|\psi(t')\big\|_{L^6(\mathbb{R}^3,\mathbb{C}^4)}\|\Omega_{kj}\psi(t')\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}\\ &\lesssim \tilde{\delta}^2\langle t'\rangle^{\varepsilon-1},\end{aligned}$$

we obtain

$$\begin{aligned}\big\|L_j^{\mp}\partial_t\Omega_{kj}\mathrm{w}^{\pm}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} &\lesssim \tilde{\delta}^2\langle t'\rangle^{\varepsilon} + \tilde{\delta}\langle t'\rangle^{-1/2}\big\|\Omega_{kj}A^{\mathrm{inh}}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^3)}\\ &\quad + \tilde{\delta}\langle t'\rangle\big\|A^{\mathrm{inh}}(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^3)}.\end{aligned}$$

A priori assumption (246) and estimate (256) for the case $|\alpha| = 1$ imply

$$\big\|L_j^{\mp}\partial_t\Omega_{kj}\mathrm{w}^{\pm}(t')\big\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}^2\langle t'\rangle^{\varepsilon}.$$

We can skip the contributions $\Omega_{kj}\Omega_{0j'}$ because they coincide with $\Omega_{0j'}\Omega_{kj}$ up to the correction terms of (252) which have already been treated in the case $|\alpha| = 1$. So, we can conclude that

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{N}_0^6:\,|\alpha|=2} \big\|\Omega^\alpha\big(\overline{\psi}(t')\gamma^k\psi(t')\big)\big\|_{L^2(\mathbb{R}^3,\mathbb{C})} \\
&\qquad\lesssim \Big(\tilde{\delta}\langle t'\rangle^{2\varepsilon} + \tilde{\delta}\langle t'\rangle^{\varepsilon/2}\big\|A^{\mathrm{inh}}(t')\big\|_{W^{1,6/(2-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^3)}\Big)\big\|\psi(t')\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} \\
&\qquad\lesssim \tilde{\delta}^2\langle t'\rangle^{2\varepsilon-3/2} + \tilde{\delta}^2\langle t'\rangle^{-(3-\varepsilon)/2}\big\|A^{\mathrm{inh}}(t')\big\|_{W^{2,6/(2-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^3)}. \qquad (263)
\end{aligned}$$

For the first term of (263), we can follow estimate (258). For the second term in (263), note that

$$\begin{aligned}
&\int_{t/4}^t \tilde{\delta}^2\langle t'\rangle^{-(3-\varepsilon)/2}\big\|A^{\mathrm{inh}}(t')\big\|_{W^{2,6/(2-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^3)}\,\mathrm{d}t' \\
&\quad\lesssim \tilde{\delta}^2\Big(\int_{t/4}^t \langle t'\rangle^{-3(3-\varepsilon)/(5-\varepsilon)}\,\mathrm{d}t'\Big)^{(5-\varepsilon)/6}\big\|A^{\mathrm{inh}}(t',x)\big\|_{L_{t'}^{6/(1+\varepsilon)}(t/4,t)W_x^{2,6/(2-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^3)} \\
&\quad\lesssim \tilde{\delta}^2\langle t\rangle^{(-2+\varepsilon)/3}\|N(\psi)(t',x)\|_{L_{t'}^{6/(4+\varepsilon)}(t/4,t)W_x^{2,6/(5-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^3)} \\
&\quad\lesssim \tilde{\delta}^2\langle t\rangle^{(-2+\varepsilon)/3}\big\|\psi(t',x)\big\|_{L_{t'}^\infty(t/4,t)H_x^2(\mathbb{R}^3,\mathbb{C}^4)}\big\|\psi(t',x)\big\|_{L_{t'}^{6/(4+\varepsilon)}(t/4,t)L_x^{6/(2-\varepsilon)}(\mathbb{R}^3,\mathbb{C}^4)} \\
&\quad\lesssim \tilde{\delta}^3\langle t\rangle^{(-2+\varepsilon)/3}\Big(\int_{t/4}^t \big(\langle t'\rangle^{-1/2-\varepsilon/2}\big)^{6/(4+\varepsilon)}\,\mathrm{d}t'\Big)^{(4+\varepsilon)/6} \\
&\quad\lesssim \tilde{\delta}^3\langle t\rangle^{(-2+\varepsilon)/3}\langle t\rangle^{(1-2\varepsilon)/6} \\
&\quad= \tilde{\delta}^3\langle t\rangle^{-1/2}. \qquad (264)
\end{aligned}$$

From estimates (254), (256), (263), (264) and (245), we would be able to conclude

$$\big\|\chi_{\gg 1}A^{\mathrm{inh}}(t)\big\|_{L^\infty(\mathbb{R}^3,\mathbb{C}^4)} \lesssim \tilde{\delta}^2\langle t\rangle^{-1}$$

and similarly for $|D|A^{\mathrm{inh}}$ and $|D|^2A^{\mathrm{inh}}$ (in fact derivatives on $A^{\mathrm{inh}}(t)$ even cancel the problematic $|D|^{-1}$-contribution, which was one reason to restrict on large frequencies). Therefore,

$$\big\|\chi_{\gg 1}A^{\mathrm{inh}}(t)\big\|_{W^{2,\infty}(\mathbb{R}^3,\mathbb{C}^3)} \lesssim \tilde{\delta}^2\langle t\rangle^{-1}.$$

Remark. We should point out that the calculations in the case $|\alpha| = 2$ have not been completely rigorous, especially in (262) concerning the error terms arising from replacing $\langle D\rangle$ with $|D|$ for large frequencies. However, even if we were able to restrict on $|\alpha| \leq 1$, the operator $|D|^{-1}$ would lead to problems for small frequencies. On the other hand, we also recall that small frequencies might be controlled by our approach in Subsection 5.2.1.

5.3 Conclusion

The three approaches in Section 5.2 are not yet sufficient to get the desired H^N-estimate for II^{inh}. We should also emphasize that the other parts of the X_T-norm still need to be controlled for the A_j, as well. After having shown the $\langle t\rangle^{-1}$-decay for $A_j(t)$, a similar proof as done in Section 2 for A_0 might produce the corresponding estimates for the A_j. After closing the bootstrap argument for A_j, we would be able to conclude global well-posedness. It also remains open whether the A_j do not affect the oscillating correction term in contrast to A_0. One might try to exploit the hidden null-structure in

$$\overline{\psi}\gamma^j\psi$$

which is not present in the $|\psi|^2$-contribution of A_0, see e.g. D'Ancona–Foschi–Selberg [25, Section 4] for details.

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Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel verwendet habe. Die Stellen der Arbeit, die dem Wortlaut oder dem Sinn nach anderen Werken entnommen sind, wurden unter Angabe der Quelle kenntlich gemacht.

Bielefeld, 08.12.2020	C. Cloos
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