Center for Mathematical Economics
Working Papers

643

February 2021

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bielefeld.de/zwe/imw/research/working-papers/
ISSN: 0931-6558

TWO-SIDED SINGULAR CONTROL OF AN INVENTORY WITH UNKNOWN DEMAND TREND

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Abstract: We study the problem of optimally managing an inventory with unknown demand trend. Our formulation leads to a stochastic control problem under partial observation, in which a Brownian motion with non-observable drift can be singularly controlled in both an upward and downward direction. We first derive the equivalent separated problem under full information, with state-space components given by the Brownian motion and the filtering estimate of its unknown drift, and we then completely solve this latter problem. Our approach uses the transition amongst three different but equivalent problem formulations, links between two-dimensional bounded-variation stochastic control problems and games of optimal stopping, and probabilistic methods in combination with refined viscosity theory arguments. We show substantial regularity of (a transformed version of) the value function, we construct an optimal control rule, and we show that the free boundaries delineating (transformed) action and inaction regions are bounded globally Lipschitz continuous functions. To our knowledge this is the first time that such a problem has been solved in the literature.

MSC2020 subject classifications: Primary 93E20, 93E11, 91A55, 49J40, 90B05.
Keywords and phrases: bounded-variation stochastic control, partial observation, inventory management, Dynkin games, free boundaries.

1. Introduction

In real-world situations, decision makers are usually faced with the uncertainty of noise or volatility in the dynamics of an underlying stochastic process. However, in many occasions they are also faced with uncertainty in their estimation of the drift of this monitored stochastic process. In other words, decision makers might not know the exact growth characteristics of the future value of the underlying process. They may find themselves observing the evolution of its value, but cannot perfectly distinguish whether the cause of its variations is due to the drift or the stochastic driver of the process. Through their observations, they can update their beliefs about the drift, however due to the aforementioned inability of distinguishing the cause of variations, the information acquired by observations is inevitably noisy. Such an uncertainty about the drift therefore adds a structural risk component to decision making, in addition to the noise from the stochastic driver of the underlying process. Such scenaria have already received attention in the mathematical economic/financial literature, such as [13] for investment timing, [8] for asset trading, [18] for optimal liquidation, [16] for contract theory, and [12] and [14] for dividend payments.

In this paper, we consider the optimal management of inventory when the demand is stochastic and partially observed. There exists an enormous literature on optimal inventory management (see, e.g. [40] for an overview and the significance of inventory control in operations and profitability of companies). The optimal singular/impulsive control literature of stochastic inventory systems has so far assumed that the dynamics of the inventory is fully known to decision makers, see e.g. [1], [6], [7], [22], [23], [24] [37], [38] [39], amongst many others. Some of the most celebrated results are the optimality of (constant) threshold strategies determining (a) base-stock policies – maintaining inventory above a fixed shortage level – and (b) restrictions on the size of inventory, in order to manage storage-related costs. In this paper, we generalise the existing literature on the singular control of inventories by assuming that the demand rate or the mean of the random demand for the product is unknown to decision makers. This can be relevant to companies operating in newly established markets or producing a novel good, for which there is limited knowledge about the demand trend. In particular, we will show in this paper how the aforementioned optimal strategies are no longer triggered by constant thresholds, but by functions of the decision maker's learning process of the unknown demand rate. We further note that our analysis and results in this paper can also contribute to applications way beyond the inventory management literature; for instance, to cash balance management problems (see, e.g. [19]), when the drift of the cash process is unknown to managers.

The model and general results. We consider decision makers who can observe in real time the evolution of the (random) inventory level $S_t = x + \mu t + \eta B_t$, which represents the production minus the stochastic demand for the product at time t (see [22], [37] for the first such models, and e.g. [39] for a detailed description of Brownian inventory systems). The inventory has a deterministic "net demand" rate μ, which is *unknown* to decision makers, and a stochastic part modelling the volatility associated to demand via a standard one-dimensional Brownian motion B and a constant volatility parameter $\eta > 0$. The decision makers can control the inventory via a bounded-variation process $P_t = P_t^+ - P_t^-$, where $P_t^\pm$ are increasing processes that provide the minimal decomposition of P and define the total amount of increase/decrease of the inventory process up to time t. The controlled inventory level is therefore given by

$$X_t = S_t + P_t = x + \mu t + \eta B_t + P_t^+ - P_t^- \quad \text{for all } t \geq 0\,.$$

Note that, a positive value of X naturally models the current excess inventory level, while the absolute value of a negative X models the backlog in production.

Both levels of excess inventory and backorder bare (non-necessarily symmetric) holding and shortage costs per unit of time, modelled via a suitable convex function $C(X)$ which is based on the level of X. On one hand, if the holding costs and expenses/investments into more storage space $C(X)$ to accommodate an increasing inventory X become too costly, the decision maker can unload part of the excess inventory in various ways (e.g. start promotions, send to outlets, donate, ship to another facility, or destroy) at a cost K^- proportional to the inventory volume that is unloaded. On the other hand, when shortage costs, loss of dissatisfied customers and penalties for delayed shipments $C(X)$ due to undesirable levels of backlog X, become too costly, the decision maker can place an inventory replenishment order to raise the inventory level. This would come at a cost K^+ proportional to the inventory volume that is ordered.

Overall, the aforementioned holding and shortage costs $C(X)$ need to be controlled but the proportional costs $K^\pm$ of controlling the inventory create a trade off. The decision maker thus needs to find the right balance between letting the storage system evolve freely according to the realised demand and the timings of controlling it, so that the overall cost is minimised. The question we therefore study in the sequel is *"What is the optimal inventory management strategy that minimises the total expected (discounted) future holding, shortage and control costs, when the demand rate is unknown?"*.

As in most of the aforementioned literature, we allow the rate of reduction $\mathrm{d}P^-$ and increase $\mathrm{d}P^+$ to be unbounded and allow them to reduce or increase, respectively, the level of X instantaneously. In mathematical terms, the aforementioned question is formulated as a bounded-variation stochastic control problem of a linearly controlled one-dimensional diffusion with the novelty of a random (non-observable) drift μ. To the best of our knowledge, this is the first time that the complete solution to a bounded-variation problem under partial observation is derived. Given that the drift of X is unknown to the decision maker, the analysis of this question becomes considerably harder than in standard versions of the aforementioned problem with full information (see, e.g. [22]). In order to model this additional uncertainty, we assume that the random variable $\mu \in \{\mu_0, \mu_1\}$, for some $\mu_0, \mu_1 \in \mathbb{R}$ such that $\mu_0 < \mu_1$. The decision makers can only observe the overall evolution of S, whose natural filtration modelling the information available to them up to time t, is denoted by $\mathcal{F}_t^S$, while they just have a prior belief $\pi := \mathsf{P}(\mu = \mu_1) \in (0, 1)$ on the value of μ at time $t = 0$. Their belief on the drift is however continuously updated as new information is revealed and their belief process takes the form $\Pi_t := \mathbb{P}(\mu = \mu_1 \,|\, \mathcal{F}_t^S)$, according to standard filtering techniques (for a survey, see e.g. [32]). Naturally, the decisions whether to act/control the system or not, are not based solely on the position of the Brownian (inventory) system X, as in standard problems where the drift is known (see, e.g. [22]). These decisions are now adapted dynamically according to the current belief on the drift μ of the system, thus they depend strongly on the learning process Π of the decision maker. However, under this filtering estimate of the drift, the dynamics of the problem becomes essentially two-dimensional and diffusive, which results in an associated variational formulation with partial differential equations (PDEs). Therefore, obtaining explicit solutions is not possible in general. Nevertheless, using our methodology that combines various different techniques (as we outline later), we manage to solve the problem and provide the complete characterisation of the optimal control strategy.

Given the convexity of C, when the (inventory) level X is relatively high (resp., low) resulting in a large holding (resp., shortage) marginal cost $C'(X)$, the decision maker has an incentive to exert control P^- (resp.,

P^+) to decrease (resp., increase) the level of X. The decision maker must find an optimal control strategy $P^{\star+}$ and $P^{\star-}$ that minimises the overall expected future holding and shortage costs counterbalanced with the proportional costs $K^{\pm}$ per unit of control exerted. Indeed, we successfully prove in this paper, that such an optimal strategy $P^{\star+}$ and $P^{\star-}$ exists and is explicitly characterised by two boundaries, each one associated with one of the control processes $P^{\star\pm}$. These boundaries then split the space in three distinct but connected regions: (a) An action region that is divided into two parts, namely the areas above or below these boundaries, prescribing that when X is either relatively large or small, the decision maker should intervene by decreasing or increasing X, respectively, and bring X inside the area which is between the two boundaries; and (b) an intermediate waiting (inaction) region for relatively intermediate values of X, which is precisely the aforementioned area between the two boundaries.

To the best of our knowledge, the study and complete characterisation of these boundaries which define the solution of a bounded-variation stochastic control problem under partial information on the dynamics of the underlying diffusion, has also never been addressed in the literature. We prove that the aforementioned boundaries triggered by X are monotone functions of the belief process Π and can be completely characterised in terms of monotone Lipschitz continuous curves solving a system of nonlinear integral equations. The dependence of the optimal boundaries on the belief variable Π is in contrast to the full information cases, where the decision makers must intervene whenever X breaches some constant thresholds, irrespective of its past evolution (see, e.g. [22]). In fact, we also prove that our boundaries are bounded by these (constant) thresholds of the full information cases. This further shows that our model extends and complements the existing literature on bounded-variation stochastic control problems in the case when there is uncertainty about the drift of the underlying process.

Our contributions, approach and an overview of the mathematical analysis. Our contribution in this paper is twofold. From the point of view of its application, even though the literature on the optimal management of inventory is extremely rich (see, e.g. papers cited before), there is no model where the demand is assumed to be partially observed and lump-sum as well as singularly continuous actions on the inventory are allowed. To the best of our knowledge, this makes our paper a pioneer in this class of problems, which is our first main contribution. From the mathematical theory perspective, the development of methods to tackle optimal control problems with absolutely continuous (regular) controls and partial observation has an extensive history, see e.g. [2], [27], [28], and [30]. However, the literature on the characterisation of the optimal policy in singular stochastic control problems with partial observation is limited, and actually deals only with monotone controls. We firstly refer to [33] that studies singular control problems with partial information via the study of their associated backward stochastic differential equations (BSDEs) leading to general maximum principles; [12] that solves the optimal dividend problem under partial information on the drift of the revenue process of a firm that can default, creating also an absorption state; [14] that studies a dynamic model of a firm whose shareholders learn about its profitability, face costs of external financing and costs of holding cash; and [4] that considers the debt-reduction problem of a government that has partial information on the underlying business conditions. Contrary to the aforementioned papers with monotone controllers, we allow the decision maker to both decrease and increase the underlying process by using controls of bounded-variation. Thus, our paper is expanding the traditional bounded-variation control theory towards the direction of partial information, by providing a methodology for dealing with such problems, achieving the complete characterisation of the free boundaries that define the optimal control, and achieving also notable value function regularity properties. This is our second main contribution, on which we elaborate in the remaining of this section.

By relying on classical filtering theory (see [32]) we first determine an equivalent problem under full information, the so-called "separated problem". This is a genuine two-dimensional bounded-variation singular stochastic control problem, with state-space described by the level of the inventory and the decision maker's belief on the demand rate. Given the two-dimensional nature of the problem, the traditional "guess and verify" approach is not effective. Indeed, this would require at first the construction of an explicit solution to a PDE with (gradient) boundary conditions, which in general cannot be obtained.

We instead use a more direct approach that allows for a thorough study of the regularity and structure of the problem's value function V, and eventually leads to the complete characterisation of the optimal control strategy. To be more precise, we begin with connecting our two-dimensional bounded-variation stochastic control problem to a suitable zero-sum optimal stopping game (Dynkin game), such that $V_x = v$ where v denotes the value of the game with underlying two-dimensional, uncontrolled, degenerate diffusion (S, Π) taking values in $\mathbb{R} \times (0, 1)$. The players in this game can be thought of as the two forces who wish to either increase or decrease marginally the level of the process X. By studying the game, we are able to characterise the optimal stopping strategy of each player via two free boundary functions $a_\pm(\pi)$ for $\pi \in (0,1)$, which are monotone and bounded.

Then, via a change of measure, the original two-dimensional controlled process (X, Π) is transformed into (X, Φ) with decoupled dynamics that takes values in $\mathbb{R} \times (0,\infty)$. Under these new (x,φ)–coordinates, we show that the transformed control value function $\overline{V}(x,\varphi)$, game value function $\overline{v}(x,\varphi)$, and associated free boundary functions $b_\pm(\varphi)$ inherit all properties proved for $V(x,\pi)$, $v(x,\pi)$ and $a_\pm(\pi)$. Using these properties, and proving local semiconcavity of $\overline{V}$, allow us to show via fine techniques from viscosity theory that $\overline{V} \in C^1(\mathbb{R} \times (0,\infty))$. Because of the degeneracy of the process (X,Φ) (in which X and Φ are driven by the same Brownian motion), in order to derive further regularity of the control problem's value function it is useful to derive the intrinsic parabolic formulation of the problem (see also [25] and [12]). This is achieved by passing yet to another transformation (X,Y) of our state process taking values in $\mathbb{R}^2$. In these new coordinates we prove that the transformed control value function $\widehat{V}(x,y)$ is also continuously differentiable and it is furthermore such that $\widehat{V}_{xx}$ admits a continuous extension to the closure of the associated inaction region (where a linear parabolic PDE holds). This regularity is then employed in order to prove a verification theorem identifying an optimal control rule. This keeps for almost all times the diffusion (X,Φ) within the closure of the inaction region $\{(x,\varphi) : b_+(\varphi) < x < b_-(\varphi)\}$, according to a Skorokhod reflection.

In order to obtain finer regularity and a characterisation of the free boundaries triggering the optimal control rule, we continue our analysis in the (x,y)–coordinates. Here, by introducing a new transformed Dynkin game with value $\widehat{v}(x,y)$, we are able to show that the (x,φ)-inaction region transforms into an open set of $\mathbb{R}^2$ which is delineated by two strictly increasing curves $x = c_\pm(y)$. By exploiting the structure of transformation linking the (x,φ)-plane to the (x,y)-plane, we then obtain an easy proof of the fact that $c_\pm$ are Lipschitz-continuous functions, with Lipschitz constant $L = 1$. Such a result is of particular independent interest, given the importance of Lipschitz regularity in obstacle problems (see the introduction of [10] for a detailed account on this and its related literature). Moreover, we believe that the simple argument of our proof can be applied also to other singular control/optimal stopping problems with partial observation, thus providing an alternative – to the more technical approach developed in [10] – for obtaining the Lipschitz regularity of the optimal stopping boundaries. The Lipschitz property of $c_\pm$ is then employed to show via probabilistic techniques à la [11] that the Dynkin game's value function is continuously differentiable in $\mathbb{R}^2$; that is, a global smooth-fit property holds. The latter fact is finally useful in proving that $\widehat{v}_{xx} \in L^\infty_{\text{loc}}(\mathbb{R}^2)$ and in obtaining a system of nonlinear integral equations solved by $c_\pm$.

Overall, notwithstanding the degeneracy of the associated PDE in the variational formulation of the original control problem, by using our probabilistic methodology in combination with viscosity theory arguments and switching between three equivalent formulations (under change of variables): (a) we achieve a notable global regularity of the value function $\overline{V}$, namely $\overline{V} \in C^1(\mathbb{R} \times (0,\infty))$, and we deduce that its transformed version $\widehat{V}$ is actually $C^{2,1}$ in the closure of its inaction region; (b) we use these properties in order to construct an optimal control strategy in terms of the belief-dependent process $t \mapsto b_\pm(\Phi_t)$; (c) we obtain global Lipschitz continuity of the free boundaries $c_\pm$ arising in the transformed problem $\widehat{V}$, which are then characterised via nonlinear integral equations.

Note that, using our methodology as described above, we manage to obtain the minimal (necessary) regularity in order to construct an optimal control strategy and verify its optimality. As in multi-dimensional settings proving regularity properties of the control value function can be very challenging, having a methodology that takes a different route can be very helpful in studying similar problems with singular controls under partial observation. Moreover, it is worth observing that backtracking all the involved change of variables, the characterisation of $c_\pm$ effectively turns into a characterisation of the free boundaries $b_\pm$ and consequently of $a_\pm$ in the original (x,π)–coordinates.

Structure of the paper. The rest of this paper is organised as follows. In Section 2, we present the model, formulate the control problem, and then derive the separated problem V. The first related optimal stopping game is derived in Section 3, while Section 4 introduces the first useful change of coordinates. Section 5 then studies the regularity of the (transformed) control problem's value function $\overline{V}$, and Section 6 presents the verification theorem and the construction of an optimal control. Finally, in Section 7: we introduce the last change of variables; we obtain the Lipschitz-continuity of the corresponding free boundaries $c_\pm$; we prove the smooth-fit property of the transformed Dynkin game's value function $\widehat{v}$; and we derive the integral equations for $c_\pm$.

2. Problem Formulation and the Separated Problem

On a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$, we define a one-dimensional Brownian motion $(B_t)_{t\geq 0}$ whose P-augmented natural filtration is denoted by $(\mathcal{F}^B_t)_{t\geq 0}$. Moreover, we define a random variable μ which is independent of the Brownian motion B and can take two possible real values, namely $\mu \in \{\mu_0, \mu_1\}$, where $\mu_0, \mu_1 \in \mathbb{R}$. Without loss of generality, we assume henceforth that $\mu_1 > \mu_0$ and that

$$\pi := \mathsf{P}(\mu = \mu_1) \in (0, 1).$$

In absence of any intervention, the underlying (stochastic inventory) process S_t as observed by the decision maker, follows the dynamics

$$\mathrm{d}S_t = \mu \mathrm{d}t + \eta \mathrm{d}B_t, \quad S_0 = x \in \mathbb{R},$$

for some $\eta > 0$. Recall that the drift μ of the process S is not observable by the decision maker, who can only monitor the evolution of the process S itself. In light of this observation, the decision maker select their control strategy P based solely on their observation of the process S. By denoting the natural filtration of any process Y by $\mathbb{F}^Y := (\mathcal{F}^Y_t)_{t\geq 0}$, we can therefore define the set of admissible controls

$$\begin{aligned} \mathcal{A} \;:=\; & \{P : \Omega \times \mathbb{R}^+ \to \mathbb{R} \text{ such that } t \mapsto P_t \text{ is right-continuous, (locally) of bounded variation} \\ & \text{and } P \text{ is } \mathbb{F}^S - \text{adapted}\}. \end{aligned}$$

To be more precise, we consider the minimal decomposition of the bounded-variation control $P \in \mathcal{A}$ to be

$$P_t = P^+_t - P^-_t,$$

where P^+ and P^- are then nondecreasing, right-continuous $\mathbb{F}^S$–adapted processes. From now on, we set $P^{\pm}_{0-} = 0$ a.s. for any $P \in \mathcal{A}$. Hence, the reference (controlled inventory) process is given by

$$X^P_t := S_t + P_t, \tag{2.1}$$

where $P \in \mathcal{A}$, and such that $X^P_{0-} = x$. Note that, when $P \equiv 0$, the inventory process is uncontrolled and takes the form $X^0 = S$.

Given the aforementioned setting, the decision maker's goal is to minimise the overall (discounted) cost of holding, shortage and controlling the inventory process. In mathematical terms, the bounded-variation control problem of the decision maker is given by

$$\inf_{P\in\mathcal{A}} \mathsf{E}\left[\int_0^\infty e^{-\rho t}\left(C(X^P_t)\mathrm{d}t + K^+\mathrm{d}P^+_t + K^-\mathrm{d}P^-_t\right)\right], \tag{2.2}$$

where E denotes the expectation under the probability measure P, $\rho > 0$ is the decision maker's discount rate of future costs, $K^+, K^- > 0$ are the marginal costs per unit of control exerted on X^P, and $C : \mathbb{R} \to \mathbb{R}^+$ is a holding and shortage cost function which satisfies the following **standing assumption**.

Assumption 2.1. *There exists constants $p > 1$, $\alpha_0, \alpha_1, \alpha_2 > 0$ such that the following hold true:*

(i) for every $x \in \mathbb{R}$

$$0 \leq C(x) \leq \alpha_0(1 + |x|^p);$$

(ii) for every $x, x' \in \mathbb{R}$,

$$|C(x) - C(x')| \leq \alpha_1\big(1 + C(x) + C(x')\big)^{1-\frac{1}{p}}|x - x'|;$$

(iii) for every $x, x' \in \mathbb{R}$ and $\lambda \in (0, 1)$,

$$0 \leq \lambda C(x) + (1-\lambda)C(x') - C(\lambda x + (1-\lambda)x') \leq \alpha_2\lambda(1-\lambda)(1 + C(x) + C(x'))^{\left(1-\frac{2}{p}\right)^+}|x - x'|^2;$$

Notice that Assumption 2.1.*(iii)* above implies that C is convex and locally semiconcave. Hence, by [5, Corollary 3.3.8], we have that $C \in C^{1,\mathrm{Lip}}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^+)$. A classical quadratic holding cost $C(x) = (x - \overline{x})^2$, for some target level $\overline{x} \in \mathbb{R}$, clearly satisfies Assumption 2.1.

Given the feature of a non-observable μ, Problem (2.2) is not Markovian and cannot be therefore tackled via a dynamic programming approach. In the following, we will derive a new equivalent Markovian problem under full information, the so-called "separated problem". This will be then solved by exploiting its connection to a zero-sum game of optimal stopping and by a careful analysis of the regularity of its value function.

2.1. The separated problem

In order to derive the equivalent problem under full information, we use standard arguments from filtering theory (see, e.g., [32, Section 4.2]) and we define the "belief" process

$$\Pi_t := \mathbb{P}(\mu = \mu_1 \,|\, \mathcal{F}_t^S), \quad t \geq 0,$$

according to which, decision makers update their beliefs on the (true) value of the drift μ based on the arrival of new information via the observation of the process S. Then, the dynamics of X^P and Π can be written as

$$\begin{cases} \mathrm{d}X_t^P = (\mu_1\Pi_t + \mu_0(1-\Pi_t))\mathrm{d}t + \eta \mathrm{d}W_t + \mathrm{d}P_t, & X_{0^-}^P = x \in \mathbb{R}, \\ \mathrm{d}\Pi_t \;\; = \gamma \Pi_t(1-\Pi_t)\mathrm{d}W_t, & \Pi_0 = \pi \in (0,1), \end{cases} \tag{2.3}$$

where the innovation process W, given by

$$\mathrm{d}W_t = \frac{\mathrm{d}S_t}{\eta} - \left(\frac{\mu_0}{\eta} + \gamma\Pi_t\right)\mathrm{d}t, \quad \text{for all } t \geq 0,$$

is an $\mathbb{F}^S$-Brownian motion on $(\Omega, \mathcal{F}, \mathsf{P})$ according to Lévy's characterisation theorem (see, e.g., [32, Theorem 4.1]), and

$$\gamma := \frac{\mu_1 - \mu_0}{\eta} > 0.$$

It can be verified that the pair (X^P, Π) is an $\mathbb{F}^S$-adapted (time-homogeneous strong) Markov process on $(\Omega, \mathcal{F}, \mathsf{P})$ as a unique strong solution of the system of stochastic differential equations in (2.3) (see, e.g. [35, Chapter V]). In (2.3), the (unknown/non-observable) drift μ of X in the original model is replaced with its filtering estimate $\mathsf{E}[\mu \,|\, \mathcal{F}_t]$. Moreover, the belief (learning) process $\Pi = (\Pi_t)_{t\geq 0}$ involved in the filtering is a bounded martingale on $[0,1]$ such that $\Pi_\infty \in \{0,1\}$, due to the fact that all information eventually gets revealed at time $t = \infty$.

Then, for (X^P, Π) as in (2.3), with $(x, \pi) \in \mathcal{O} := \mathbb{R} \times (0,1)$, we define the full-information problem

$$V(x,\pi) := \inf_{P \in \mathcal{A}} \mathsf{E}\left[\int_0^\infty e^{-\rho t}\left(C(X_t^P)\mathrm{d}t + K^+\mathrm{d}P_t^+ + K^-\mathrm{d}P_t^-\right)\right], \tag{2.4}$$

where all the processes involved are now $\mathbb{F}^S$-adapted. Hence, Problem (2.4) is a two-dimensional Markovian singular stochastic control problem with controls of bounded variation. Moreover, by uniqueness of the strong solution to the belief equation, a control $P^\star$ is optimal for (2.2) if and only if it is optimal for (2.4), and the values in (2.2) and (2.4) coincide.

Note that, in light of the dynamics of (X^P, Π) in (2.3), a high value of Π close to 1 would imply that the decision maker has a strong belief in a high drift μ_1, while a low Π close to 0 would imply, on the contrary, a strong belief in a low drift μ_0 scenario.

Remark 2.2 (Full information cases)**.** *In the formulation (2.2), the case of prior belief $\pi := \mathsf{P}(\mu = \mu_1) \in \{0,1\}$ implies the certainty of the decision maker regarding whether $\mu = \mu_0$ or $\mu = \mu_1$. Hence, in this case, there is no uncertainty about the value of the drift μ, which is not a random variable any more. Respectively, in the formulation (2.4), the case of prior belief $\Pi_0 = \pi \in \{0,1\}$ yields that the belief process Π will actually remain constant through time, due to its dynamics which imply that $\Pi_t = \pi$ for all $t > 0$. Therefore, we equivalently have that such values of $\pi \in \{0,1\}$ correspond to the full information cases.*

In these cases, the optimal control problem becomes a standard one-dimensional bounded-variation stochastic control problem, for which an early study can be found in [22]. The optimal control strategy in such a case is triggered by two constant boundaries within which the process X^P is kept (via a Skorokhod reflection).

Given the convexity of C as in Assumption 2.1, and the linear structure of $P \mapsto X^P$ in (2.3), by following standard arguments based on Komlós' theorem (see, e.g., [20, Proposition 3.4]) the next result can be shown.

Proposition 2.3. *There exists an optimal control $P^\star$ for (2.4). Moreover, this is unique (up to indistinguishability) if C is strictly convex.*

3. The First Related Optimal Stopping Game

We now derive a zero-sum optimal stopping game (Dynkin game) related to V, and we provide preliminary properties of its value function and of the geometry of its state space. In this section, the uncontrolled process X^0 with $P_t \equiv 0$ for all $t \geq 0$ becomes involved in the analysis, so we recall from (2.3) that $(X^0_t, \Pi_t)_{t\geq 0} \equiv (S_t, \Pi_t)_{t\geq 0}$ is the two-dimensional strong Markov process solving

$$\begin{cases} \mathrm{d}X^0_t &= (\mu_1 \Pi_t + \mu_0(1-\Pi_t))\mathrm{d}t + \eta \mathrm{d}W_t, \quad X^0_0 = x \in \mathbb{R}, \\ \mathrm{d}\Pi_t &= \gamma \Pi_t(1-\Pi_t)\mathrm{d}W_t, \qquad\qquad\qquad\quad \Pi_0 = \pi \in (0,1), \end{cases} \tag{3.1}$$

Proposition 3.1. *Consider the process $(X^0_t, \Pi_t)_{t\geq 0}$ defined in (3.1) and define*

$$v(x,\pi) := \inf_\sigma \sup_\tau \mathsf{E}_{(x,\pi)}\bigg[\int_0^{\tau\wedge\sigma} e^{-\rho t} C'(X^0_t)\mathrm{d}t - K^+ e^{-\rho\tau}\mathbf{1}_{\{\tau<\sigma\}} + K^- e^{-\rho\sigma}\mathbf{1}_{\{\tau>\sigma\}}\bigg], \tag{3.2}$$

where the optimisation is taken over the set of $\mathbb{F}^W$-stopping times and $\mathsf{E}_{(x,\pi)}$ denotes the expectation conditioned on $(X^0_0, \Pi_0) = (x,\pi) \in \mathcal{O}$. Consider also the control value function $V(x,\pi)$ defined in (2.4). Then, we have the following properties:

(i) $x \mapsto V(x,\pi)$ is differentiable and $v(x,\pi) = V_x(x,\pi)$.
(ii) $x \mapsto V(x,\pi)$ is convex and therefore $x \mapsto v(x,\pi)$ is nondecreasing.
(iii) $\pi \mapsto v(x,\pi)$ is nondecreasing.
(iv) $(x,\pi) \mapsto v(x,\pi)$ is continuous on $\mathbb{R}\times(0,1)$.

Proof. In this proof, whenever we need to stress the dependence of the state process on its starting point, we denote by $(X^{0;(x',\pi')}, \Pi^{\pi'})$ the unique strong solution to (3.1) starting at $(x',\pi') \in \mathcal{O}$ at time zero. We prove separately the four parts.

Proof of (i). Thanks to Proposition 2.3, it suffices to apply [29, Theorem 3.2] upon setting $G \equiv 0$,

$$H(\omega,t,x) := e^{-\rho t} C\Big(x + \eta W_t(\omega) + \int_0^t \big(\mu_1 \Pi_s(\omega) + \mu_0(1-\Pi_s(\omega))\big)\mathrm{d}s\Big), \quad (\omega,t,x) \in \Omega\times\mathbb{R}_+\times\mathbb{R},$$

$$\gamma_t := e^{-\rho t}K^+, \qquad \nu_t := e^{-\rho t}K^-, \quad t \geq 0,$$

and noticing that the proof in [29] can be easily adapted to our infinite-time horizon discounted setting.

Proof of (ii). Denote by $(X^{P;(x,\pi)}, \Pi^\pi)$ the unique strong solution to (2.3) when $(X^P_{0-}, \Pi_0) = (x,\pi)$. The convexity of $V(x,\pi)$ with respect to x, can be easily shown by exploiting the convexity of $C(x)$ and the linear structure of $(x,P) \mapsto X^{P;(x,\pi)}$, for any $P \in \mathcal{A}$ and $(x,\pi) \in \mathcal{O}$. The nondecreasing property of $v(\cdot,\pi)$ then follows from the fact that $v = V_x$ from part (i).

Proof of (iii). Notice that

$$X^0_t = x + \eta W_t + \int_0^t \big(\mu_1\Pi_s + \mu_0(1-\Pi_s)\big)\mathrm{d}s, \quad t \geq 0, \tag{3.3}$$

and that $\pi \mapsto \Pi^\pi$ is nondecreasing due to standard comparison theorems for strong solutions to one-dimensional stochastic differential equations [26, Chapter 5.2]. Then, the claim follows from (3.2) and Assumption 2.1 according to which $x \mapsto C'(x)$ is nondecreasing.

Proof of (iv). By [29, Theorem 3.1] and Proposition 2.3 we know that, for any $(x,\pi) \in \mathcal{O}$, (3.2) admits a saddle point. Take $(x_n,\pi_n) \to (x,\pi)$ as $n \uparrow \infty$, and let $(\tau^\star,\sigma^\star)$ and $(\tau^\star_n,\sigma^\star_n)$ realize the saddle-points for (x,π) and (x_n,π_n), respectively. Then, we have

$$\begin{aligned} v(x,\pi) - v(x_n,\pi_n) \;&\leq\; \mathsf{E}\bigg[\int_0^{\tau^\star\wedge\sigma^\star_n} e^{-\rho t}\Big(C'(X^{0;(x,\pi)}_t) - C'(X^{0;(x_n,\pi_n)}_t)\Big)\mathrm{d}t\bigg] \\ &\leq\; \mathsf{E}\bigg[\int_0^\infty e^{-\rho t}\Big|C'(X^{0;(x,\pi)}_t) - C'(X^{0;(x_n,\pi_n)}_t)\Big|\mathrm{d}t\bigg]. \end{aligned}$$

Without loss of generality, we can take $(x_n, \pi_n) \subset (x-\varepsilon, x+\varepsilon) \times (\pi-\varepsilon, \pi+\varepsilon)$, for a suitable $\varepsilon > 0$ and for n sufficiently large. Then, by Assumption 2.1.(ii) and standard estimates using Assumption 2.1.(i), (3.3) and the fact that Π is bounded in $[0,1]$, we can invoke the dominated convergence theorem and obtain

$$\limsup_{n\to\infty} (v(x,\pi) - v(x_n,\pi_n)) \leq 0.$$

Arguing symmetrically, now with the couple of stopping times $(\tau_n^\star, \sigma^\star)$, we also find

$$\limsup_{n\to\infty} (v(x_n,\pi_n) - v(x,\pi)) \leq 0.$$

Combining the last two inequalities, we obtain the desired continuity claim. □

In the rest of this section, we focus on the study of the optimal stopping game v presented in (3.2), due to its connection to our stochastic control problem (cf. Proposition 3.1). To that end, we define below the so-called continuation (waiting) region

$$\mathcal{C}_1 := \big\{(x,\pi) \in \mathcal{O} : \ -K^+ < v(x,\pi) < K^-\big\}, \tag{3.4}$$

and the stopping region $\mathcal{S}_1 := {\mathcal{S}_1}^+ \cup {\mathcal{S}_1}^-$, whose components are given by

$${\mathcal{S}_1}^+ := \big\{(x,\pi) \in \mathcal{O} : \ v(x,\pi) \leq -K^+\big\}, \quad {\mathcal{S}_1}^- := \big\{(x,\pi) \in \mathcal{O} : \ v(x,\pi) \geq K^-\big\}. \tag{3.5}$$

In light of the continuity of v in Proposition 3.1.(iv), we conclude that the continuation region $\mathcal{C}_1$ is an open set, while the two components of the stopping regions ${\mathcal{S}_1}^\pm$ are both closed sets. We can therefore define the free boundaries

$$a_+(\pi) := \sup\big\{x \in \mathbb{R} : \ v(x,\pi) \leq -K^+\big\} \quad \text{and} \quad a_-(\pi) := \inf\big\{x \in \mathbb{R} : \ v(x,\pi) \geq K^-\big\}. \tag{3.6}$$

Here, and throughout the rest of this paper, we use the convention $\sup\emptyset = -\infty$ and $\inf\emptyset = +\infty$. Then, by using the fact that v is nondecreasing with respect to x (see Proposition 3.1.(ii)), we can obtain the structure of the continuation and stopping regions, which take the form

$$\mathcal{C}_1 = \big\{(x,\pi) \in \mathcal{O} : \ a_+(\pi) < x < a_-(\pi)\big\}, \tag{3.7}$$

$$\mathcal{S}_1^+ = \big\{(x,\pi) \in \mathcal{O} : \ x \leq a_+(\pi)\big\} \quad \text{and} \quad \mathcal{S}_1^- = \big\{(x,\pi) \in \mathcal{O} : \ x \geq a_-(\pi)\big\}. \tag{3.8}$$

Clearly, the continuity of v further implies that the free boundaries $a_\pm$ are strictly separated, namely

$$a_+(\pi) < a_-(\pi) \quad \text{for all } \pi \in (0,1).$$

We now prove some preliminary properties of the free boundaries $\pi \mapsto a_\pm(\pi)$.

Proposition 3.2. *The free boundaries $a_\pm$ defined in (3.6) satisfy the following properties:*

(i) *$a_\pm(\cdot)$ are nonincreasing on $(0,1)$.*
(ii) *$a_+(\cdot)$ is left-continuous and $a_-(\cdot)$ is right-continuous on $(0,1)$.*
(iii) *There exist constants $x^*_\pm \in \mathbb{R}$, such that*

$$x_+^* \leq a_+(\pi) < a_-(\pi) \leq x_-^*, \quad \forall\, \pi \in (0,1). \tag{3.9}$$

Moreover, letting $(C')^{-1}$ be the generalised inverse of C', we have $a_+(\pi) \leq (C')^{-1}(-\rho K^+)$ and $a_-(\pi) \geq (C')^{-1}(\rho K^-)$ for all $\pi \in (0,1)$.

Proof. We prove separately the four parts.

Proof of (i). This is a consequence of the definitions of $a_\pm(\cdot)$ in (3.6) and the fact that $v(x,\cdot)$ is nondecreasing for any $x \in \mathbb{R}$; cf. Proposition 3.1.(iii).

Proof of (ii). This follows from part (i) above and the closedness of the sets ${\mathcal{S}_1}^\pm$.

Proof of (iii). The fact that $a_+(\pi) \leq (C')^{-1}(-\rho K^+)$ and $a_-(\pi) \geq (C')^{-1}(\rho K^-)$ follows by noticing that $\mathcal{S}_1^+ \subseteq \{x \in \mathbb{R} : x \leq (C')^{-1}(-\rho K^+)\}$ and $\mathcal{S}_1^- \subseteq \{x \in \mathbb{R} : x \geq (C')^{-1}(\rho K^-)\}$.

In order to show the other bounds, we proceed as follows. Since $\mu_1 > \mu_0$ and $\Pi_t > 0$, we have $\mathsf{P}_{(x,\pi)}$-a.s., for any $t \geq 0$, that

$$X^0_t = x + \eta W_t + \int_0^t \big(\mu_1 \Pi_s + \mu_0(1-\Pi_s)\big)\mathrm{d}s = x + \eta W_t + \mu_0 t + \int_0^t (\mu_1 - \mu_0)\Pi_s \mathrm{d}s$$
$$\geq x + \eta W_t + \mu_0 t =: \underline{X}^0_t.$$

Similarly, using that $\Pi_t < 1$, we get that

$$X^0_t \leq x + \eta W_t + \mu_1 t =: \overline{X}^0_t.$$

Therefore, the latter two estimates yield that $\underline{X}^0_t \leq X^0_t \leq \overline{X}^0_t$ for all $t \geq 0$. Combining these inequalities with the fact that $C'(\cdot)$ is nondecreasing due to Assumption 2.1 and the definition (3.2) of the value function $v(x,\pi)$, we conclude that

$$v_0(x) \leq v(x,\pi) \leq v_1(x), \quad \text{for all} \quad (x,\pi) \in \mathcal{O}, \tag{3.10}$$

where we have introduced the one-dimensional optimal stopping games

$$v_0(x) := \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathsf{E}\Big[\int_0^{\tau\wedge\sigma} e^{-\rho t} C'(\underline{X}^0_t)\mathrm{d}t - K^+ e^{-\rho\tau}\mathbf{1}_{\{\tau<\sigma\}} + K^- e^{-\rho\sigma}\mathbf{1}_{\{\tau>\sigma\}}\Big]$$

and

$$v_1(x) := \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathsf{E}\Big[\int_0^{\tau\wedge\sigma} e^{-\rho t} C'(\overline{X}^0_t)\mathrm{d}t - K^+ e^{-\rho\tau}\mathbf{1}_{\{\tau<\sigma\}} + K^- e^{-\rho\sigma}\mathbf{1}_{\{\tau>\sigma\}}\Big].$$

Because both $v_0(\cdot)$ and $v_1(\cdot)$ are nondecreasing on $\mathbb{R}$, standard techniques allow to show that there exists finite $x^\star_-, x^\star_+$ such that

$$\{x \in \mathbb{R} :\ x \geq x^\star_-\} = \{x \in \mathbb{R} :\ v_0(x) \geq K^-\}$$

and

$$\{x \in \mathbb{R} :\ x \leq x^\star_+\} = \{x \in \mathbb{R} :\ v_1(x) \leq -K^+\}.$$

Hence, combining the latter two regions together with the inequalities in (3.10), we eventually get that

$$\{x \in \mathbb{R} :\ x \geq x^\star_-\} \subseteq \{(x,\pi) \in \mathcal{O} :\ v(x,\pi) \geq K^-\} = \mathcal{S}^-_1. \tag{3.11}$$

and

$$\{x \in \mathbb{R} :\ x \leq x^\star_+\} \subseteq \{(x,\pi) \in \mathcal{O} :\ v(x,\pi) \leq -K^+\} = \mathcal{S}^+_1, \tag{3.12}$$

Hence, $\mathcal{S}^\pm_1 \neq \emptyset$ and the claim follows from (3.11)-(3.12). □

Recall that, the higher the value of π, the stronger the decision makers' belief is about μ begin equal to μ_1, which is the highest possible value (recall that $\mu_1 > \mu_0$). Taking this into account, we notice from the monotonicity (nonincreasing) of the free boundary functions $a_\pm(\pi)$ in Proposition 3.2.(i) that: The more the decision maker's belief tends towards μ_1 (higher inventory level on average), the more cautious they need to be, thus they tend to intervene (by unloading part of excess inventory) more often to make sure that the inventory level X is kept below the optimal threshold $a_-(\pi)$, despite its strong tendency to go up, so that the overall (holding and control) costs are minimised. On the other hand, they are more willing to delay interventions (by placing replenishment orders) to increase the inventory level X, by optimally setting a lower "base-stock" level $a_+(\pi)$ as their belief grows towards μ_1 (higher inventory level on average). This reflects the fact that the inventory level X will not breach this lower boundary too often under their belief that $\mu = \mu_1$ and eventually achieves the minimisation of the overall (shortage and control) costs.

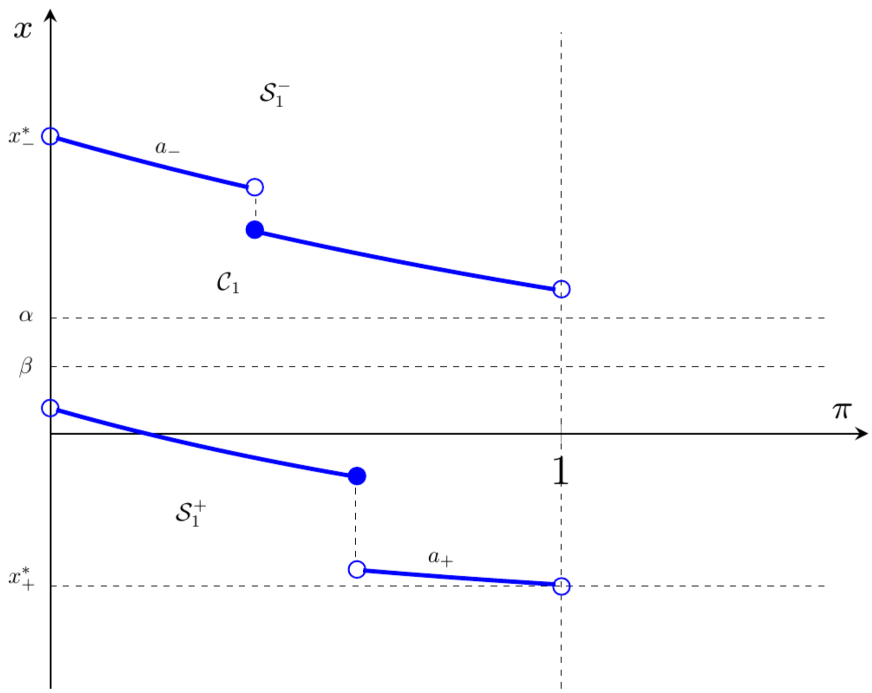

Figure 1: An illustrative drawing of the free boundaries a_+ and a_- satisfying Proposition 3.2. In the picture, $\alpha := (C')^{-1}(\rho K^-)$ and $\beta := (C')^{-1}(-\rho K^+)$.

4. A Decoupling Change of Measure

In order to provide further results about the optimal control problem (2.4) and the associated Dynkin game (3.2), it is convenient to decouple the dynamics of the controlled inventory process X^P and the belief process Π. This can be achieved via a transformation of state space and a change of measure, as we explain in the following subsections.

4.1. Transformation of process Π to Φ.

We first recall from (2.3) (see also (3.1)), that for any prior belief $\Pi_0 = \pi \in (0,1)$, we have $\Pi_t \in (0,1)$ for all $t \in (0,\infty)$. Hence, we define the process

$$\Phi_t := \frac{\Pi_t}{1-\Pi_t}, \quad t \geq 0,$$

whose dynamics are given via Itô's formula by

$$\mathrm{d}\Phi_t = \gamma^2 \Pi_t \Phi_t \mathrm{d}t + \gamma \Phi_t \mathrm{d}W_t = \gamma \Phi_t (\gamma \Pi_t \mathrm{d}t + \mathrm{d}W_t), \quad \Phi_0 = \varphi := \frac{\pi}{1-\pi}. \tag{4.1}$$

Note that, the process Φ is known as the "likelihood ratio process" in the literature of filtering theory (see, e.g. [25]).

4.2. Change of measure from P to Q_T, for some fixed $T > 0$.

We begin by defining the exponential martingale

$$\zeta_T := \exp\left\{ -\gamma \int_0^T \Pi_s \mathrm{d}W_s - \frac{1}{2} \int_0^T \gamma^2 \Pi_s^2 \mathrm{d}s \right\},$$

and the measure $\mathsf{Q}_T \sim \mathsf{P}$ on $(\Omega, \mathcal{F}_T)$ by

$$\frac{\mathrm{d}\mathsf{Q}_T}{\mathrm{d}\mathsf{P}} = \zeta_T.$$

Then, the process

$$W_t^* := W_s + \gamma \int_0^t \Pi_s \mathrm{d}s, \quad t \in [0,T],$$

is a Brownian motion in $[0,T]$ under Q_T, and the dynamics of Φ in (4.1) simplifies to

$$\mathrm{d}\Phi_t = \gamma \Phi_t \mathrm{d}W_t^*, \quad t \in (0,T], \quad \Phi_0 = \varphi, \tag{4.2}$$

hence Φ is an exponential martingale under Q_T.

Consequently, applying the same change of measure to the process X^P from (2.3), we get that

$$\mathrm{d}X_t^P = \mu_0 \mathrm{d}t + \eta \mathrm{d}W_t^* + \mathrm{d}P_t^+ - \mathrm{d}P_t^-, \quad t \in [0,T], \quad X_{0-}^P = x. \tag{4.3}$$

In order to change the measure also in the cost criterion of our value function in (2.4), we further define the process

$$Z_t := \frac{1+\Phi_t}{1+\varphi}, \quad t \in [0,T],$$

which can be verified via Itô's formula to satisfy

$$Z_t = 1/\zeta_t, \quad \text{for every } t \in [0,T].$$

Hence, denoting by $\mathsf{E}^{\mathsf{Q}_T}$ the expectation under Q_T, we have that

$$\begin{aligned} &\mathsf{E}\Big[\int_0^T e^{-\rho t} \left(C(X_t^P)\mathrm{d}t + K^+ \mathrm{d}P_t^+ + K^- \mathrm{d}P_t^- \right) \Big] \\ &= \frac{1}{1+\varphi} \mathsf{E}^{\mathsf{Q}_T} \Big[(1+\Phi_T) \int_0^T e^{-\rho t} \Big(C(X_t^P)\mathrm{d}t + K^+ \mathrm{d}P_t^+ + K^- \mathrm{d}P_t^- \Big) \Big]. \end{aligned} \tag{4.4}$$

Since the process $(1+\Phi_t)_{t\geq 0}$ defines a nonnegative martingale under Q_T, by [15, Theorem 57] (and the example after the theorem) we can write

$$\mathsf{E}^{\mathsf{Q}_T} \Big[(1+\Phi_T) \int_0^T e^{-\rho t} C(X_t^P)\mathrm{d}t \Big] = \mathsf{E}^{\mathsf{Q}_T} \Big[\int_0^T e^{-\rho t} (1+\Phi_t) C(X_t^P) \mathrm{d}t \Big],$$

as well as

$$\mathsf{E}^{\mathsf{Q}_T} \Big[(1+\Phi_T) \int_0^T e^{-\rho t} \mathrm{d}P_t^{\pm} \Big] = \mathsf{E}^{\mathsf{Q}_T} \Big[\int_0^T e^{-\rho t} (1+\Phi_t) \mathrm{d}P_t^{\pm} \Big].$$

Hence, combining together the above expressions of the expectations $\mathsf{E}^{\mathsf{Q}_T}$ we get that (4.4) can be expressed in the form of

$$\begin{aligned} &\mathsf{E}\Big[\int_0^T e^{-\rho t} \Big(C(X_t^P)\mathrm{d}t + K^+ \mathrm{d}P_t^+ + K^- \mathrm{d}P_t^- \Big) \Big] \\ &= \frac{1}{1+\varphi} \mathsf{E}^{\mathsf{Q}_T} \Big[\int_0^T e^{-\rho t} (1+\Phi_t) \Big(C(X_t^P)\mathrm{d}t + K^+ \mathrm{d}P_t^+ + K^- \mathrm{d}P_t^- \Big) \Big]. \end{aligned} \tag{4.5}$$

4.3. Passing to the limit as $T \to \infty$ and to the new measure Q.

We firstly notice that passing to the limit as $T \to \infty$ cannot be performed directly to the latter expression in (4.5), since the measure Q_T changes with T. Nevertheless, notice that the right-hand side of (4.5) only depends on the law of the processes involved. Given that we are only interested in the value function (2.4) and eventually in the optimal feedback control $P^\star$ (cf. Proposition 2.3) – which do not depend on the laws – we can introduce a new auxiliary problem.

To that end, we define a new filtered probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}}, \overline{\mathsf{Q}})$ supporting a Brownian motion $(\overline{W}_t)_{t\geq 0}$ and the strong solution to the controlled stochastic differential equation

$$\begin{cases} \mathrm{d}\overline{X}_t^{\overline{P}} = \mu_0 \mathrm{d}t + \eta \mathrm{d}\overline{W}_t + \mathrm{d}\overline{P}_t^+ - \mathrm{d}\overline{P}_t^-, & \overline{X}_{0-}^{\overline{P}} = x, \\ \mathrm{d}\overline{\Phi}_t = \gamma \overline{\Phi}_t \mathrm{d}\overline{W}_t, & \overline{\Phi}_0 = \varphi := \frac{\pi}{1-\pi}, \end{cases}$$

for $\overline{P} = \overline{P}^+ - \overline{P}^- \in \overline{\mathcal{A}}$, where

$$\begin{aligned} \overline{\mathcal{A}} \;:=\; & \big\{ \overline{P} : \Omega \times \mathbb{R}^+ \to \mathbb{R} \text{ such that } t \mapsto \overline{P}_t \text{ is right-continuous, (locally) of bounded variation} \\ & \text{and } \overline{P} \text{ is } \overline{\mathbb{F}} - \text{adapted} \big\}. \end{aligned}$$

Then, denoting by $\overline{\mathsf{E}}$ the expectation on $(\overline{\Omega}, \overline{\mathcal{F}})$ under $\overline{\mathsf{P}}$, we have for every $T > 0$, that

$$\begin{aligned} & \mathsf{E}^{\mathsf{Q}_T} \Big[\int_0^T e^{-\rho t} (1 + \Phi_t) \big(C(X_t^P) \mathrm{d}t + K^+ \mathrm{d}P_t^+ + K^- \mathrm{d}P_t^- \big) \Big] \\ & = \overline{\mathsf{E}} \Big[\int_0^T e^{-\rho t} (1 + \overline{\Phi}_t) \Big(C(\overline{X}_t^{\overline{P}}) \mathrm{d}t + K^+ \mathrm{d}\overline{P}_t^+ + K^- \mathrm{d}\overline{P}_t^- \Big) \Big], \end{aligned}$$

due to the equivalence in law of the process $(X_t^P, \Phi_t, W_t^*, P_t)_{t\geq 0}$ under Q_T and the process $(\overline{X}_t^{\overline{P}}, \overline{\Phi}_t, \overline{W}_t, \overline{P}_t)_{t\geq 0}$ under $\overline{\mathsf{Q}}$. Therefore, combining the above equality with (4.5), we eventually get

$$\begin{aligned} & \mathsf{E} \Big[\int_0^T e^{-\rho t} \Big(C(X_t^P) \mathrm{d}t + K^+ \mathrm{d}P_t^+ + K^- \mathrm{d}P_t^- \Big) \Big] \\ & = \frac{1}{1+\varphi} \overline{\mathsf{E}} \Big[\int_0^T e^{-\rho t} (1 + \overline{\Phi}_t) \Big(C(\overline{X}_t^{\overline{P}}) \mathrm{d}t + K^+ \mathrm{d}\overline{P}_t^+ + K^- \mathrm{d}\overline{P}_t^- \Big) \Big], \end{aligned} \tag{4.6}$$

Thanks to (4.6), we can now take limits as $T \to \infty$ and obtain, in view of the definitions (2.4) of the control value function and (4.1) of the starting value φ, that

$$V(x, \pi) = (1-\pi) \overline{V}\Big(x, \frac{\pi}{1-\pi}\Big), \quad \text{or equivalently} \quad \overline{V}(x, \varphi) = (1+\varphi) V\Big(x, \frac{\varphi}{1+\varphi}\Big), \tag{4.7}$$

where we define

$$\overline{V}\,(x, \varphi) := \inf_{\overline{P} \in \overline{\mathcal{A}}} \overline{\mathsf{E}} \Big[\int_0^\infty e^{-\rho t} (1 + \overline{\Phi}_t) \Big(C(\overline{X}_t^{\overline{P}}) \mathrm{d}t + K^+ \mathrm{d}\overline{P}_t^+ + K^- \mathrm{d}\overline{P}_t^- \Big) \Big].$$

Therefore, in order to obtain the value function $V(x, \pi)$ from (2.4), we could instead solve first the above problem to get $\overline{V}\,(x, \varphi)$ and then use the equality in (4.7). However, in order to simplify the notation, **from now on** in the study of $\overline{V}$ we will simply write $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{Q}, \mathsf{E}^{\mathsf{Q}}, W, X, \Phi, P, \mathcal{A})$ instead of $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}}, \overline{\mathsf{Q}}, \overline{\mathsf{E}}, \overline{W}, \overline{X}, \overline{\Phi}, \overline{P}, \overline{\mathcal{A}})$.

4.4. The optimal control problem with state-space process (X^P, Φ) *under the new measure* Q.

Summarising the results from Sections 4.1–4.3, we henceforth focus on the study of the following optimal control problem

$$\overline{V}\,(x, \varphi) := \inf_{P \in \mathcal{A}} \mathsf{E}^{\mathsf{Q}} \Big[\int_0^\infty e^{-\rho t} (1 + \Phi_t) \Big(C(X_t^P) \mathrm{d}t + K^+ \mathrm{d}P_t^+ + K^- \mathrm{d}P_t^- \Big) \Big] =: \inf_{P \in \mathcal{A}} \overline{\mathcal{J}}_{x,\varphi}(P). \tag{4.8}$$

under the dynamics

$$\begin{cases} \mathrm{d}X_t^P = \mu_0 \mathrm{d}t + \eta \mathrm{d}W_t + \mathrm{d}P_t^+ - \mathrm{d}P_t^-, & X_{0-}^P = x \in \mathbb{R}, \\ \mathrm{d}\Phi_t = \gamma \Phi_t \mathrm{d}W_t, & \Phi_0 = \varphi := \frac{\pi}{1-\pi} \in (0, \infty), \end{cases} \tag{4.9}$$

for a standard Brownian motion W. In light of the equality in (4.7), this will lead to the original value function $V(x, \pi)$ from (2.4). In the remaining of Section 4, we expand our study – beyond the values of the control problems – to the relationship between the free boundaries in the two formulations, since these boundaries will eventually define the optimal control strategy (see Section 6).

4.5. The optimal stopping game associated to (4.8)–(4.9) under the new measure Q.

The next result is concerned with properties of the value function defined in (4.8) and its connection to an associated optimal stopping game. The proof is omitted for brevity, since it can be proved by employing arguments similar to those used in the proof of Propositions 2.3 and 3.1 above.

Proposition 4.1. *Consider the problem defined in (4.8)–(4.9).*

(i) There exists an optimal control $P^\star$ solving (4.8). Moreover, $P^\star$ is unique (up to indistinguishability) if C is strictly convex.
(ii) $x \mapsto \overline{V}(x,\varphi)$ is convex and differentiable, such that $\overline{V}_x(x,\varphi) = \bar{v}(x,\varphi)$ on $\mathbb{R}\times(0,\infty)$, for

$$\bar{v}(x,\varphi) := \inf_{\sigma}\sup_{\tau} \mathsf{E}^{\mathsf{Q}}\bigg[\int_0^{\tau\wedge\sigma} e^{-\rho t}(1+\Phi_t)C'(X_t^0)\mathrm{d}t - K^+(1+\Phi_\tau)e^{-\rho\tau}\mathbf{1}_{\{\tau<\sigma\}} + K^-(1+\Phi_\sigma)e^{-\rho\sigma}\mathbf{1}_{\{\tau>\sigma\}}\bigg]. \tag{4.10}$$

Here, the optimisation is taken over the set of $\mathbb{F}^W$-stopping times and the state-space process is given by

$$\begin{cases} \mathrm{d}X_t^0 = \mu_0\mathrm{d}t + \eta\mathrm{d}W_t, & X_0^0 = x\in\mathbb{R}, \\ \mathrm{d}\Phi_t \;= \gamma\Phi_t\mathrm{d}W_t, & \Phi_0 = \varphi := \frac{\pi}{1-\pi}\in(0,\infty). \end{cases} \tag{4.11}$$

It further follows from the previous analysis, namely Sections 4.1–4.3, that the value function $v(x,\pi)$ of the optimal stopping game in (3.2) is connected to the value function $\bar{v}(x,\varphi)$ of the new game introduced above in (4.10), according to (see also (4.7) for the control value functions) the following equality

$$\bar{v}(x,\varphi) = (1+\varphi)\,v\Big(x, \frac{\varphi}{1+\varphi}\Big). \tag{4.12}$$

In view of the above relationship, the value function $\bar{v}(\cdot,\cdot)$ inherits important properties which have already been proved for $v(\cdot,\cdot)$ in Section 3. In particular, we have directly from Proposition 3.1.(ii) and (iv) the following result.

Proposition 4.2. *The value function $\bar{v}$ defined in (4.10) satisfies the following properties:*

(i) $(x,\varphi)\mapsto\bar{v}(x,\varphi)$ is continuous over $\mathbb{R}\times(0,\infty)$;
(ii) $x\mapsto\bar{v}(x,\varphi)$ is nondecreasing.

Following similar steps as in Section 3 to study the new game (4.10), we define below the so-called continuation (waiting) region

$$\mathcal{C}_2 := \big\{(x,\varphi)\in\mathbb{R}\times(0,\infty):\; -K^+(1+\varphi) < \bar{v}(x,\varphi) < K^-(1+\varphi)\big\}, \tag{4.13}$$

and the stopping region $\mathcal{S}_2 := {\mathcal{S}_2}^+\cup{\mathcal{S}_2}^-$, whose components are given by

$$\mathcal{S}_2^+ := \big\{(x,\varphi)\in\mathbb{R}\times(0,\infty):\; \bar{v}(x,\varphi)\le -K^+(1+\varphi)\big\}, \tag{4.14}$$
$$\mathcal{S}_2^- := \big\{(x,\varphi)\in\mathbb{R}\times(0,\infty):\; \bar{v}(x,\varphi)\ge K^-(1+\varphi)\big\}. \tag{4.15}$$

Moreover, in light of the continuity of $\bar{v}$ in Proposition 4.2.(i), we conclude that the continuation region $\mathcal{C}_2$ is an open set, while the two components of the stopping regions ${\mathcal{S}_2}^\pm$ are both closed sets. We can therefore define the free boundaries

$$b_+(\varphi) := \sup\big\{x\in\mathbb{R}:\; \bar{v}(x,\varphi)\le K^+(1+\varphi)\big\}, \tag{4.16}$$
$$b_-(\varphi) := \inf\{x\in\mathbb{R}:\; \bar{v}(x,\varphi)\ge K^-(1+\varphi)\}. \tag{4.17}$$

Then, by using the fact that $\bar{v}$ is nondecreasing with respect to x (see Proposition 4.2.(ii)), we can obtain the structure of the continuation and stopping regions, which take the form

$$\mathcal{C}_2 = \big\{(x,\varphi)\in\mathbb{R}\times(0,\infty):\; b_+(\varphi) < x < b_-(\varphi)\big\}, \tag{4.18}$$
$$\mathcal{S}_2^+ = \big\{(x,\varphi)\in\mathbb{R}\times(0,\infty):\; x\le b_+(\varphi)\big\} \quad\text{and}\quad \mathcal{S}_2^- = \big\{(x,\varphi)\in\mathbb{R}\times(0,\infty):\; b_-(\varphi)\le x\big\}. \tag{4.19}$$

Clearly, the continuity of $\bar{v}$ implies that these free boundaries $b_\pm$ are strictly separated, namely $b_+(\varphi) < b_-(\varphi)$ for all $\varphi \in (0,\infty)$.

Moreover, observe that the relationship in (4.12) together with the definitions (3.4) and (4.13) of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, imply that the latter two regions are equal under the transformation from (x,π)- to (x,φ)-coordinates. To be more precise, for any $(x,\pi) \in \mathbb{R} \times (0,1)$, define the transformation

$$\overline{T} := (\overline{T}_1, \overline{T}_2) : \mathbb{R} \times (0,1) \to \mathbb{R} \times (0,\infty), \qquad (\overline{T}_1(x,\pi), \overline{T}_2(x,\pi)) = \Big(x, \frac{\pi}{1-\pi}\Big),$$

which is invertible and its inverse is given by

$$\overline{T}^{-1}(x,\varphi) = \Big(x, \frac{\varphi}{1+\varphi}\Big), \quad (x,\varphi) \in \mathbb{R} \times (0,\infty).$$

Hence, $\overline{T} : \mathbb{R} \times (0,1) \to \mathbb{R} \times (0,\infty)$ is a global diffeomorphism, which implies together with the expressions of (3.4)–(3.5) and (4.13)–(4.15) that

$$\mathcal{C}_2 = \overline{T}(\mathcal{C}_1) \quad \text{and} \quad \mathcal{S}_2^\pm = \overline{T}(\mathcal{S}_1^\pm).$$

Taking this into account together with the expressions (3.7)–(3.8) of $\mathcal{C}_1$ and $\mathcal{S}_1^\pm$, we can further conclude from the expressions (4.18)–(4.19) of $\mathcal{C}_2$ and $\mathcal{S}_2^\pm$ that

$$b_\pm(\varphi) = a_\pm\left(\frac{\varphi}{1+\varphi}\right). \tag{4.20}$$

Hence, in light of the previously proved results for $a_\pm$ in Proposition 3.2, we also obtain the following preliminary properties of the free boundaries $\varphi \mapsto b_\pm(\varphi)$.

Proposition 4.3. *The free boundaries $b_\pm$ defined in (4.16)–(4.17) satisfy the following properties:*

(i) $b_\pm(\cdot)$ *are nonincreasing on* $(0,\infty)$.
(ii) $b_+(\cdot)$ *is left-continuous and* $b_-(\cdot)$ *is right-continuous on* $(0,\infty)$.
(iii) $b_\pm(\cdot)$ *are bounded by* $x_\pm^*$ *as in Proposition 3.2:*

$$x_+^* \le b_+(\varphi) < b_-(\varphi) \le x_-^*, \quad \forall\, \varphi \in (0,\infty).$$

Moreover, we have $b_+(\varphi) \le (C')^{-1}(-\rho K^+)$ *and* $b_-(\varphi) \ge (C')^{-1}(\rho K^-)$ *for all* $\varphi \in (0,\infty)$.

Notice that the explicit relationship (4.20) between the free boundaries $a_\pm$ and $b_\pm$ that we proved above, is not only crucial for retrieving the original boundaries $a_\pm$ from $b_\pm$, but it is also particularly useful in the proof of Proposition 4.3.(*i*) and (*iii*). In fact, proving the monotonicity and boundedness of $b_\pm$ by directly working on the Dynkin game (4.10) is not a straightforward task.

Up this point, we managed to obtain the structure of the optimal stopping strategies and preliminary properties of the corresponding optimal stopping boundaries associated with these strategies, for both Dynkin games (3.2) and (4.10) connected to the optimal control problems (2.4) and (4.8), respectively. Moreover, we managed to obtain some regularity results for the value functions of the latter control problems (see Propositions 3.1, 4.1 and 4.2). In Sections 5 and 6 below, building on the aforementioned analysis, we show that the control value function $\overline{V}$ has the sufficient regularity needed to construct an optimal control strategy. This will involve the boundaries $b_\pm$.

5. HJB Equation and Regularity of $\overline{V}$

In this section, we introduce the Hamilton-Jacobi-Bellman (HJB) equation (variational inequality) associated to the control value function $\overline{V}$ defined in (4.8) and state-space process (X^P, Φ) given by (4.9). First, let $\mathcal{D} \subseteq \mathbb{R}^2$ be an open domain and define the space $C^{k,h}(\mathcal{D};\mathbb{R})$ as the space of functions $f : \mathcal{D} \to \mathbb{R}$ which are k-times continuously differentiable with respect to the first variable and h-times continuously differentiable with respect to the second variable. When $k = h$ we simply write C^h.

We begin our study with the following *ex ante* regularity result for $\overline{V}$. Its proof can be found in the Appendix.

Proposition 5.1. *The control value function $\overline{V}$ defined in (4.8) is locally semiconcave; that is, for every $R > 0$ there exists $L_R > 0$ such that for all $\lambda \in [0, 1]$ and all $(x, \varphi), (x', \varphi')$ such that $|(x, \varphi)| \le R$ and $|(x', \varphi')| \le R$, we have*

$$\lambda\overline{V}(x, \varphi) + (1 - \lambda)\overline{V}(x', \varphi') - \overline{V}(\lambda(x, \varphi) + (1 - \lambda)(x', \varphi')) \le L_R\lambda(1 - \lambda)|(x, \varphi) - (x', \varphi')|^2.$$

In particular, by [5, Theorem 2.17], we conclude that $\overline{V}$ is locally Lipschitz.

Given the locally Lipschitz continuity proved in the previous result, we now aim at employing the HJB equation to investigate further regularity of $\overline{V}$. To that end, we define on $f \in C^2(\mathbb{R} \times (0, \infty); \mathbb{R})$ the second order differential operator

$$\mathcal{L}f(x, \varphi) := \mu_0 f_x(x, \varphi) + \frac{1}{2}\left(\eta^2 f_{xx}(x, \varphi) + \gamma^2\varphi^2 f_{\varphi\varphi}(x, \varphi) + 2\gamma\eta\varphi f_{x\varphi}(x, \varphi)\right). \tag{5.1}$$

By the dynamic programming principle, we expect that $\overline{V}$ solves (in a suitable sense) the HJB equation (in the form of a variational inequality)

$$\max\Big\{(\rho - \mathcal{L})u(x, \varphi) - (1 + \varphi)C(x),\ -u_x(x, \varphi) - K^+(1 + \varphi),\ u_x(x, \varphi) - K^-(1 + \varphi)\Big\} = 0, \tag{5.2}$$

for $(x, \varphi) \in \mathbb{R} \times (0, \infty)$. In particular, we now first show that the value function $\overline{V}$ of the control problem defined in (4.8) is a viscosity solution to (5.2). We present the formal definition of the latter notion below.

Definition 5.2. *A function $u \in C^0(\mathbb{R} \times (0, \infty); \mathbb{R})$ is called a* viscosity solution *to* (5.2) *if it is both a viscosity subsolution and supersolution, where:*

(i) a function $u \in C^0(\mathbb{R} \times (0, \infty); \mathbb{R})$ is called a viscosity subsolution *to* (5.2) *if, for every $(x, \varphi) \in \mathbb{R} \times (0, \infty)$ and every $\beta \in C^2(\mathbb{R} \times (0, \infty); \mathbb{R})$ such that $u - \beta$ attains a local maximum at (x, φ), it holds*

$$\max\left\{(\rho - \mathcal{L})\beta(x, \varphi) - (1 + \varphi)C(x),\ -\beta_x(x, \varphi) - K^+(1 + \varphi),\ \beta_x(x, \varphi) - K^-(1 + \varphi)\right\} \le 0.$$

(ii) a function $u \in C^0(\mathbb{R} \times (0, \infty); \mathbb{R})$ is called a viscosity supersolution *to* (5.2) *if, for every $(x, \varphi) \in \mathbb{R} \times (0, \infty)$ and every $\beta \in C^2(\mathbb{R} \times (0, \infty); \mathbb{R})$ such that $u - \beta$ attains a local minimum at (x, φ), it holds*

$$\max\left\{(\rho - \mathcal{L})\beta(x, \varphi) - (1 + \varphi)C(x),\ -\beta_x(x, \varphi) - K^+(1 + \varphi),\ \beta_x(x, \varphi) - K^-(1 + \varphi)\right\} \ge 0.$$

Following the arguments developed in Theorem 5.1 in Section VIII.5 of [21], and using the a priori regularity obtained in Proposition 5.1, one can show the following classical result.

Proposition 5.3. *The value function $\overline{V}$ defined in (4.8) is a locally Lipschitz continuous viscosity solution to (5.2).*

Recall the definition (4.13) of the continuation region $\mathcal{C}_2$ of problem $\overline{v}(x, \varphi)$ in (4.10) and the relationship $\overline{V}_x(x, \varphi) = \overline{v}(x, \varphi)$ on $\mathbb{R} \times (0, \infty)$ from Proposition 4.1.(ii), to observe that

$$\mathcal{C}_2 = \big\{(x, \varphi) \in \mathbb{R} \times (0, \infty) :\ -K^+(1 + \varphi) < \overline{V}_x(x, \varphi) < K^-(1 + \varphi)\big\}, \tag{5.3}$$

This implies that $\mathcal{C}_2$ identifies also with the so-called "inaction region" of $\overline{V}$, as suggested also by the HJB equation (5.2). Combining the latter fact with Proposition 5.3 clearly implies the following result.

Corollary 5.4. *The value function $\overline{V}$ defined in (4.8) is a locally Lipschitz continuous viscosity solution to*

$$(\rho - \mathcal{L})u(x, \varphi) - (1 + \varphi)C(x) = 0, \quad \textit{for all } (x, \varphi) \in \mathcal{C}_2.$$

The result in Corollary 5.4 will be used in the forthcoming analysis to upgrade the regularity of the value function in the closure of its inaction region which is the main goal of Section 5. Before reaching this (final) step of our analysis in this section, we need to further prove that $\overline{V}$ is actually globally continuously differentiable. We present this result in the following proposition, which is proved by using once again Proposition 5.3 together with the properties of $\overline{V}$ proved in Proposition 5.1 and in Section 4.5.

Proposition 5.5. *The value function $\overline{V}$ defined in (4.8) satisfies $\overline{V} \in C^1(\mathbb{R} \times (0, \infty); \mathbb{R})$.*

Proof. In order to prove that $\overline{V} \in C^1(\mathbb{R}\times(0,\infty);\mathbb{R})$, we need to prove that both (classical) derivatives $\overline{V}_x(x,\varphi)$ and $\overline{V}_\varphi(x,\varphi)$ of $\overline{V}(x,\varphi)$ in the directions x and φ, respectively, are continuous on $\mathbb{R}\times(0,\infty)$. We therefore split the proof of the desired claim in the following two steps.

Step 1. Continuity of $\overline{V}_x$. We already know from Proposition 4.1.(ii) that $\overline{V}_x = \bar{v}$ exists and from Proposition 4.2.(i) that $(x,\varphi)\mapsto \bar{v}(x,\varphi)$ is continuous over $\mathbb{R}\times(0,\infty)$. Hence, we conclude that $(x,\varphi)\mapsto \overline{V}_x(x,\varphi)$ is continuous on $\mathbb{R}\times(0,\infty)$.

Step 2. Continuity of $\overline{V}_\varphi$. Let us now show that the (classical) derivative $\overline{V}_\varphi$ exists at each $(x_o,\varphi_o)\in\mathbb{R}\times(0,\infty)$.

We assume, without loss of generality[1], that $\overline{V}$ is actually concave in a neighborhood $\mathcal{I}$ of (x_o,φ_o). Then, by concavity of $\overline{V}$ in $\mathcal{I}$, the right- and left-derivatives of $\overline{V}$ exist in the φ-direction at (x_o,φ_o). We denote these derivatives by $\overline{V}^+_\varphi(x_o,\varphi_o)$ and $\overline{V}^-_\varphi(x_o,\varphi_o)$, respectively, and due to concavity they satisfy the inequality $\overline{V}^-_\varphi(x_o,\varphi_o)\geq \overline{V}^+_\varphi(x_o,\varphi_o)$. Then, in order to show that $\overline{V}_\varphi$ exists, it suffices to show that the strict inequality $\overline{V}^-_\varphi(x_o,\varphi_o) > \overline{V}^+_\varphi(x_o,\varphi_o)$ cannot hold. Aiming for a contradiction, we assume henceforth that $\overline{V}^-_\varphi(x_o,\varphi_o) > \overline{V}^+_\varphi(x_o,\varphi_o)$ does hold true.

It follows from [36, Theorem 23.4] and the fact that $\overline{V}_x$ exists and is continuous (cf. *Step 1* above) that there exist vectors

$$\zeta := (\overline{V}_x(x_o,\varphi_o),\zeta_\varphi),\ \eta := (\overline{V}_x(x_o,\varphi_o),\eta_\varphi)\ \in D^+\overline{V}(x_o,\varphi_o) \quad \text{such that} \quad \zeta_\varphi < \eta_\varphi\,,$$

where we denote by $D^+\overline{V}(x_o,\varphi_o)$ the superdifferential of $\overline{V}$ at (x_o,φ_o). For any $(x,\varphi)\in\mathcal{I}$, we then define

$$g(x,\varphi) := \overline{V}(x_o,\varphi_o) + \overline{V}_x(x_o,\varphi_o)(x-x_o) + \ \eta_\varphi(\varphi-\varphi_o)\wedge\zeta_\varphi(\varphi-\varphi_o)$$

and notice that $\overline{V}(x_o,\varphi_o) = g(x_o,\varphi_o)$, while we also get by concavity that

$$\overline{V}(x,\varphi)\leq g(x,\varphi), \quad \forall\,(x,\varphi)\in\mathcal{I}.$$

Next, we consider the sequence of functions $(f^n)_{n\in\mathbb{N}}\subset C^2(\mathbb{R}\times(0,\infty);\mathbb{R})$ defined by

$$f^n(x,\varphi) := g(x,\varphi_o) + \frac{1}{2}(\eta_\varphi+\zeta_\varphi)(\varphi-\varphi_o) - \frac{n}{2}(\varphi-\varphi_o)^2, \quad \forall\, n\in\mathbb{N}.$$

Such a sequence satisfies the following collection of properties:

$$\begin{cases} f^n(x_o,\varphi_o) = g(x_o,\varphi_o) = \overline{V}(x_o,\varphi_o), \quad \forall\, n\in\mathbb{N}, \\ f^n \geq \overline{V} \text{ in a neighborhood of } (x_o,\varphi_o), \quad \forall\, n\in\mathbb{N}, \\ f^n_x(x_o,\varphi_o) = \overline{V}_x(x_o,\varphi_o), \quad f^n_{xx}(x_o,\varphi_o) = 0 = f^n_{x\varphi}(x_o,\varphi_o), \quad f^n_{\varphi\varphi}(x_o,\varphi_o) = -n, \quad \forall\, n\in\mathbb{N}. \end{cases} \tag{5.4}$$

Then, using the viscosity subsolution property (cf. Definition 5.2.(i)) of $\overline{V}$ at (x_o,φ_o) yields

$$0 \geq (\rho-\mathcal{L})f^n(x_o,\varphi_o) - (1+\varphi_o)C(x_o) \overset{n\to\infty}{\longrightarrow} +\infty,$$

which gives the desired contradiction. Hence, by arbitrariness of (x_o,φ_o), we have that $\overline{V}$ is differentiable in the direction φ.

In view of the aforementioned differentiability in the direction φ and the semiconcavity of $\overline{V}$ (cf. Proposition 5.1) we conclude from [36, Theorem 25.5] that $\overline{V}_\varphi$ is continuous on $\mathbb{R}\times(0,\infty)$. □

We are now ready to show the final result of this section, namely to upgrade the regularity of the value function of the control problem to the minimal required regularity for constructing a candidate optimal control policy and verify its optimality in Section 6.

To this end, we define for any $(x,\varphi)\in\mathbb{R}\times(0,\infty)$ the transformation

$$T := (T_1,T_2):\mathbb{R}\times(0,\infty)\to\mathbb{R}^2, \qquad (T_1(x,\varphi),T_2(x,\varphi)) = \Big(x, x-\frac{\eta}{\gamma}\log(\varphi)\Big), \tag{5.5}$$

[1] This can be done by replacing the (locally) semiconcave $\overline{V}(x,\varphi)$ by $W(x,\varphi) := \overline{V}(x,\varphi) - C_0|(x-x_o,\varphi-\varphi_o)|^2$ for suitable $C_0>0$ in the subsequent argument.

which is invertible with inverse given by

$$T^{-1}(x,y) = \Big(x, e^{\frac{\gamma}{\eta}(x-y)}\Big), \quad (x,y) \in \mathbb{R}^2.$$

Using the latter inverse transformation, we introduce the *transformed* version $\widehat{V}(x,y)$ of the value function $\overline{V}(x,\varphi)$ defined in (4.8) by

$$\widehat{V}(x,y) := \overline{V}(x, e^{\frac{\gamma}{\eta}(x-y)}), \quad (x,y) \in \mathbb{R}^2. \tag{5.6}$$

Moreover, direct calculations yield that

$$\widehat{V}_x(x,y) + \widehat{V}_y(x,y) = \overline{V}_x(x, e^{\frac{\gamma}{\eta}(x-y)}), \quad (x,y) \in \mathbb{R}^2. \tag{5.7}$$

Given that $T : \mathbb{R} \times (0,\infty) \to \mathbb{R}^2$ is a global diffeomorphism, we have from (5.3) and (5.7) that the open set

$$\mathcal{C}_3 := \big\{(x,y) \in \mathbb{R}^2 :\ -K^+(1 + e^{\frac{\gamma}{\eta}(x-y)}) < \big(\widehat{V}_x + \widehat{V}_y\big)(x,y) < K^-(1 + e^{\frac{\gamma}{\eta}(x-y)})\big\} = T(\mathcal{C}_2). \tag{5.8}$$

Finally, define the second-order linear differential operator on $f \in C^{2,1}(\mathbb{R}^2;\mathbb{R})$ by

$$\mathcal{L}_{X,Y} f(x,y) := \frac{1}{2}\eta^2 f_{xx}(x,y) + \mu_0 f_x(x,y) + \frac{1}{2}(\mu_0 + \mu_1) f_y(x,y) \tag{5.9}$$

Proposition 5.6. *The transformed value function $\widehat{V}$ defined in (5.6) satisfies $\widehat{V} \in C^{2,1}(\overline{\mathcal{C}}_3;\mathbb{R})$, where $\overline{\mathcal{C}}_3$ denotes the closure of the open set $\mathcal{C}_3$ defined in (5.8). In addition, $\widehat{V}$ is a classical solution to*

$$\big(\rho - \mathcal{L}_{X,Y}\big)u(x,y) = C(x)(1 + e^{\frac{\gamma}{\eta}(x-y)}), \quad \text{for all } (x,y) \in \overline{\mathcal{C}}_3. \tag{5.10}$$

Proof. First of all, due to Corollary 5.4 and the expression of the transformed value function in (5.6), one can easily verify that $\widehat{V}$ is a viscosity solution to (5.10) on $\mathcal{C}_3$ due to (5.8). Then, in light of Proposition 5.5 and the above smooth transformation, we also obtain that $\widehat{V} \in C^1(\mathbb{R}^2;\mathbb{R})$.

By a standard localization argument based on the fact that $\widehat{V}$ is a continuously differentiable viscosity solution to (5.10) on $\mathcal{C}_3$ and results for Dirichlet boundary problems involving partial differential equations of parabolic type (see [31]), we have that actually $\widehat{V} \in C^{2,1}(\mathcal{C}_3;\mathbb{R})$ and solves (5.10) on $\mathcal{C}_3$ in a classical sense. Hence,

$$\frac{1}{2}\eta^2 \widehat{V}_{xx}(x,y) = -C(x)(1 + e^{\frac{\gamma}{\eta}(x-y)}) + \rho\widehat{V}(x,y) - \mu_0\widehat{V}_x(x,y) - \frac{1}{2}(\mu_0 + \mu_1)\widehat{V}_y(x,y), \tag{5.11}$$

for all $(x,y) \in \mathcal{C}_3$. However, since we know that $\widehat{V} \in C^1(\mathbb{R}^2;\mathbb{R})$ and since the right-hand side of (5.11) only involves functions that are continuous on $\mathbb{R}^2$, we conclude that $\widehat{V}_{xx}$ admits a continuous extension on $\overline{\mathcal{C}}_3$. This completes the proof of the claim. ☐

6. Verification Theorem and Optimal Control

Given the regularity of $\widehat{V}$ obtained in Proposition 5.6 and the relation (5.6) between $\widehat{V}$ with the value function $\overline{V}$ defined in (4.8), we are now able to prove a verification theorem. Namely, we provide in this section the optimal control for $\overline{V}$ in terms of the boundaries $b_\pm$ defined in (4.16)–(4.17). Before we commence the analysis, recall also the properties of the latter boundaries proved in Proposition 4.3.

6.1. Construction of control $\widehat{P}$ for state-space process $(X^{\widehat{P}}, \Phi)$.

For any given $(x,\varphi) \in \mathbb{R} \times (0,\infty)$, we define the admissible control strategy $\widehat{P} := \widehat{P}^+ - \widehat{P}^-$ such that the following couple of properties hold true:

$$\begin{cases} b_+(\Phi_t) \le X_t^{\widehat{P}} \le b_-(\Phi_t), \qquad \mathsf{Q} \otimes \mathrm{d}t - \text{a.e.}; \\ \widehat{P}_t^+ = \int_{[0,t]} \mathbb{1}_{\{X_s^{\widehat{P}} \le b_+(\Phi_s)\}} \mathrm{d}\widehat{P}_s^+ \quad \text{and} \quad \widehat{P}_t^- = \int_{[0,t]} \mathbb{1}_{\{X_s^{\widehat{P}} \ge b_-(\Phi_s)\}} \mathrm{d}\widehat{P}_s^-, \qquad t \ge 0. \end{cases} \tag{6.1}$$

In practice, according to the aforementioned strategy, a lump-sum increase or decrease of the inventory process X may be required, whenever the inventory level X_{t-} happens to be either strictly below the boundary $b_+(\Phi_t)$ or above the boundary $b_-(\Phi_t)$, respectively. The purpose of these *jumps* of at most one of the controls $\widehat{P}^{\pm}_t$ at each such $t \geq 0$, of size either $(b_+(\Phi_t) - X^{\widehat{P}}_{t-})^+$ or $(X^{\widehat{P}}_{t-} - b_-(\Phi_t))^+$, is to bring immediately the inventory level X_t inside the interval $[b_+(\Phi_t), b_-(\Phi_t)]$. Mathematically, these are the actions caused at any time $t \geq 0$, by the jump parts $\Delta\widehat{P}^{\pm}_t := \widehat{P}^{\pm}_t - \widehat{P}^{\pm}_{t-}$ of the controls $\widehat{P}^{\pm}$. Then, the strategy prescribes taking action (increase or decrease the inventory) when the inventory process X_t approaches, at any time $t \geq 0$, either boundary $b_+(\Phi_t)$ from above or boundary $b_-(\Phi_t)$ from below. The purpose of these actions now is to make sure (with a minimal effort) that the inventory level X_t is kept inside the interval $[b_+(\Phi_t), b_-(\Phi_t)]$. Mathematically, these actions are caused by the continuous parts of the respective controls $\widehat{P}^{\pm}$ and are the so-called *Skorokhod reflection-type* policies.

Given that the dynamics of $X^{\widehat{P}}$ and Φ are decoupled (cf. (4.9)), the solution triplet $(X^{\widehat{P}}_t, \Phi_t, \widehat{P}_t)_{t\geq 0}$ to the Skorokhod reflection problem at the boundaries $b_\pm$ can be constructed as in [20, Section 4.3]. It further follows from (6.1) above together with the definitions (4.16)–(4.17) of boundaries $b_\pm$, the region $\mathcal{C}_2$ from (4.18) and the fact that $\bar{v} = \overline{V}_x$ from Proposition 4.1.(ii), that the nondecreasing processes $\widehat{P}^{\pm}$ are such that the state-space process $(X^{\widehat{P}}, \Phi)$ and the induced (random) measures $\mathrm{d}\widehat{P}^{\pm}$ on $\mathbb{R}^+$ satisfy:

$$\begin{cases} (X^{\widehat{P}}_t, \Phi_t) \in \overline{\mathcal{C}_2}, \quad \text{for } \mathsf{Q} \otimes \mathrm{d}t\text{-a.e., with } \mathcal{C}_2 \text{ as in (4.18);} \\ \mathrm{d}\widehat{P}^{+} \text{ has support on } \big\{t \geq 0 : \overline{V}_x(X^{\widehat{P}}_t, \Phi_t) \leq -K^+(1+\Phi_t)\big\}; \\ \mathrm{d}\widehat{P}^{-} \text{ has support on } \big\{t \geq 0 : \overline{V}_x(X^{\widehat{P}}_t, \Phi_t) \geq K^-(1+\Phi_t)\big\}. \end{cases} \tag{6.2}$$

6.2. Transformation of controlled process $(X^{\widehat{P}}, \Phi)$ to $(X^{\widehat{P}}, Y^{\widehat{P}})$.

We now use the transformation (5.5) from (x, φ)- to (x, y)-coordinates, in order to define the controlled process

$$Y^{\widehat{P}}_t := X^{\widehat{P}}_t - \frac{\eta}{\gamma}\log(\Phi_t), \quad t \geq 0, \tag{6.3}$$

whose dynamics are given via Itô-Meyer's formula by

$$\mathrm{d}Y^{\widehat{P}}_t = \frac{1}{2}(\mu_0 + \mu_1)\mathrm{d}t + \mathrm{d}\widehat{P}^{+}_t - \mathrm{d}\widehat{P}^{-}_t, \quad Y^{\widehat{P}}_{0-} = y := x - \frac{\eta}{\gamma}\log(\varphi)\,.$$

Recalling the transformed value function (5.6) and the relation in (5.7), we have

$$\widehat{V}(X^{\widehat{P}}_t, Y^{\widehat{P}}_t) := \overline{V}\big(X^{\widehat{P}}_t, e^{\frac{\gamma}{\eta}(X^{\widehat{P}}_t - Y^{\widehat{P}}_t)}\big) \text{ and } \widehat{V}_x(X^{\widehat{P}}_t, Y^{\widehat{P}}_t) + \widehat{V}_y(X^{\widehat{P}}_t, Y^{\widehat{P}}_t) = \overline{V}_x\big(X^{\widehat{P}}_t, e^{\frac{\gamma}{\eta}(X^{\widehat{P}}_t - Y^{\widehat{P}}_t)}\big), \tag{6.4}$$

under the dynamics

$$\begin{cases} \mathrm{d}X^{\widehat{P}}_t = \mu_0\mathrm{d}t + \eta\mathrm{d}W_t + \mathrm{d}\widehat{P}^{+}_t - \mathrm{d}\widehat{P}^{-}_t, \qquad X^{\widehat{P}}_{0-} = x \in \mathbb{R}, \\ \mathrm{d}Y^{\widehat{P}}_t = \frac{1}{2}(\mu_0 + \mu_1)\mathrm{d}t + \mathrm{d}\widehat{P}^{+}_t - \mathrm{d}\widehat{P}^{-}_t, \qquad Y^{\widehat{P}}_{0-} = y := x - \frac{\eta}{\gamma}\log(\varphi) \in \mathbb{R}. \end{cases} \tag{6.5}$$

Hence, in light of (6.4)–(6.5), we can express the control $\widehat{P}$ defined in Section 6.1 in terms of the state-space process $(X^{\widehat{P}}, Y^{\widehat{P}})$ via

$$\begin{cases} (X^{\widehat{P}}_t, Y^{\widehat{P}}_t) \in \overline{\mathcal{C}_3}, \quad \text{for } \mathsf{Q} \otimes \mathrm{d}t\text{-a.e., where } \mathcal{C}_3 \text{ is defined in (5.8);} \\ \mathrm{d}\widehat{P}^{+} \text{ has support on } \big\{t \geq 0 : \big(\widehat{V}_x + \widehat{V}_y\big)(X^{\widehat{P}}_t, Y^{\widehat{P}}_t) \leq -K^+\big(1 + e^{\frac{\gamma}{\eta}(X^{\widehat{P}}_t - Y^{\widehat{P}}_t)}\big)\big\}; \\ \mathrm{d}\widehat{P}^{-} \text{ has support on } \big\{t \geq 0 : \big(\widehat{V}_x + \widehat{V}_y\big)(X^{\widehat{P}}_t, Y^{\widehat{P}}_t) \geq K^-\big(1 + e^{\frac{\gamma}{\eta}(X^{\widehat{P}}_t - Y^{\widehat{P}}_t)}\big)\big)\big\}. \end{cases} \tag{6.6}$$

6.3. Optimality of control $\widehat{P}$.

In this section we prove the optimality of the control $\widehat{P}$ defined through (6.1), which is equivalently expressed by (6.2) in terms of the state-space process $(X^{\widehat{P}}, \Phi)$ and by (6.6) in terms of the state-space process $(X^{\widehat{P}}, Y^{\widehat{P}})$, see Sections 6.1–6.2.

Theorem 6.1 (Verification Theorem)**.** *The admissible control $\widehat{P} \in \mathcal{A}$ defined through (6.1) (see also (6.2) and (6.6)) is optimal for Problem (4.8). Actually, $\widehat{P}$ is the unique optimal control (up to indistinguishability) if C is strictly convex.*

Proof. Recall that $\widehat{V} \in C^{2,1}(\overline{\mathcal{C}}_3;\mathbb{R})$ by Proposition 5.6, where $\widehat{V}$ is the transformed value function in (5.6). By the Tietze extension theorem, it can be extended to a function $\widetilde{V} \in C^{2,1}(\mathbb{R}^2;\mathbb{R})$.

Let now $(X^{\widehat{P}}_{0-}, Y^{\widehat{P}}_{0-}) = (x,y) \equiv (x, x-\eta\log(\varphi)/\gamma) \in \overline{\mathcal{C}_3}$ be given and fixed, and define $\tau_n := \inf\{t \geq 0 : |(X^{\widehat{P}}_t, Y^{\widehat{P}}_t)| > n\} \wedge n$, for $n \in \mathbb{N}$ with state-space process $(X^{\widehat{P}}, Y^{\widehat{P}})$ as defined in (6.5). Then, noticing that $(X^{\widehat{P}}_t, Y^{\widehat{P}}_t) \in \overline{\mathcal{C}_3}$, Q-a.s. for all $t \geq 0$, and that $\widetilde{V} = \widehat{V}$ on $\overline{\mathcal{C}_3}$ we can apply Dynkin's formula to the process $e^{-\rho t}\widetilde{V}(X^{\widehat{P}}_t, Y^{\widehat{P}}_t)$ on the (random) time interval $[0,\tau_n]$, obtaining

$$\begin{aligned}\widehat{V}(x,y) &= \mathsf{E}^{\mathsf{Q}}\Big[e^{-\rho\tau_n}\widehat{V}(X^{\widehat{P}}_{\tau_n}, Y^{\widehat{P}}_{\tau_n})\Big] - \mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau_n} e^{-\rho s}\big(\mathcal{L}_{X,Y}-\rho\big)\widehat{V}(X^{\widehat{P}}_s, Y^{\widehat{P}}_s)\mathrm{d}s\Big] \\ &\quad - \mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau_n} e^{-\rho s}\big(\widehat{V}_x+\widehat{V}_y\big)(X^{\widehat{P}}_s, Y^{\widehat{P}}_s)\mathrm{d}\widehat{P}^c_s\Big] - \mathsf{E}^{\mathsf{Q}}\Big[\sum_{0\leq s\leq\tau_n} e^{-\rho s}\Big(\widehat{V}(X^{\widehat{P}}_s, Y^{\widehat{P}}_s) - \widehat{V}(X^{\widehat{P}}_{s-}, Y^{\widehat{P}}_{s-})\Big)\Big],\end{aligned} \tag{6.7}$$

where $\widehat{P}^c$ denotes the continuous part of $\widehat{P}$ and the final sum is non-zero only for (at most countably many) times s such that $\Delta\widehat{P}_s := \widehat{P}_s - \widehat{P}_{s-} \neq 0$. Clearly, $\Delta\widehat{P}_s = \Delta\widehat{P}^+_s - \Delta\widehat{P}^-_s$, where $\Delta\widehat{P}^\pm_s := \widehat{P}^\pm_s - \widehat{P}^\pm_{s-}$ and notice that

$$\begin{aligned}\sum_{0\leq s\leq\tau_n} e^{-\rho s}\big(\widehat{V}(X^{\widehat{P}}_s, Y^{\widehat{P}}_s) - \widehat{V}(X^{\widehat{P}}_{s-}, Y^{\widehat{P}}_{s-})\big) &= \sum_{0\leq s\leq\tau_n} e^{-\rho s}\int_0^{\Delta\widehat{P}^+_s}\big(\widehat{V}_x+\widehat{V}_y\big)(X^{\widehat{P}}_{s-}+u, Y^{\widehat{P}}_{s-}+u)\mathrm{d}u \\ &\quad - \sum_{0\leq s\leq\tau_n} e^{-\rho s}\int_0^{\Delta\widehat{P}^-_s}\big(\widehat{V}_x+\widehat{V}_y\big)(X^{\widehat{P}}_{s-}-u, Y^{\widehat{P}}_{s-}-u)\mathrm{d}u.\end{aligned} \tag{6.8}$$

Hence, plugging (6.8) into (6.7) and using (5.10), we obtain

$$\begin{aligned}\widehat{V}(x,y) &= \mathsf{E}^{\mathsf{Q}}\Big[e^{-\rho\tau_n}\widehat{V}(X^{\widehat{P}}_{\tau_n}, Y^{\widehat{P}}_{\tau_n})\Big] + \mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau_n} e^{-\rho s}\big(1+e^{\frac{\gamma}{\eta}(X^{\widehat{P}}_s - Y^{\widehat{P}}_s)}\big)C(X^{\widehat{P}}_s)\mathrm{d}s\Big] \\ &\quad - \mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau_n} e^{-\rho s}\Big(\widehat{V}_x+\widehat{V}_y\Big)(X^{\widehat{P}}_s, Y^{\widehat{P}}_s)\mathrm{d}\big(\widehat{P}^{+,c}_s - \widehat{P}^{-,c}_s\big)\Big] \\ &\quad - \mathsf{E}^{\mathsf{Q}}\Big[\sum_{0\leq s\leq\tau_n} e^{-\rho s}\int_0^{\Delta\widehat{P}^+_s}\big(\widehat{V}_x+\widehat{V}_y\big)(X^{\widehat{P}}_{s-}+u, Y^{\widehat{P}}_{s-}+u)\mathrm{d}u \\ &\qquad - \sum_{0\leq s\leq\tau_n} e^{-\rho s}\int_0^{\Delta\widehat{P}^-_s}\big(\widehat{V}_x+\widehat{V}_y\big)(X^{\widehat{P}}_{s-}-u, Y^{\widehat{P}}_{s-}-u)\mathrm{d}u\Big].\end{aligned} \tag{6.9}$$

Using now the nonnegativity of $\widehat{V}$ as well as the second and third property of control $\widehat{P}$ in (6.6), we see that (6.9) becomes

$$\begin{aligned}\widehat{V}(x,y) &\geq \mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau_n} e^{-\rho s}\big(1+e^{\frac{\gamma}{\eta}(X^{\widehat{P}}_s - Y^{\widehat{P}}_s)}\big)C(X^{\widehat{P}}_s)\mathrm{d}s\Big] \\ &\quad + \mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau_n} e^{-\rho s}K^+\big(1+e^{\frac{\gamma}{\eta}(X^{\widehat{P}}_s - Y^{\widehat{P}}_s)}\big)\mathrm{d}\widehat{P}^+_s + \int_0^{\tau_n} e^{-\rho s}K^-\big(1+e^{\frac{\gamma}{\eta}(X^{\widehat{P}}_s - Y^{\widehat{P}}_s)}\big)\mathrm{d}\widehat{P}^-_s\Big].\end{aligned}$$

Then, we take limits as $n\uparrow\infty$ and we invoke Fatou's lemma (given the nonnegativity of all the integrands above) to find that

$$\begin{aligned}\widehat{V}(x,y) &\geq \mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\infty} e^{-\rho s}\big(1+e^{\frac{\gamma}{\eta}(X^{\widehat{P}}_s - Y^{\widehat{P}}_s)}\big)C(X^{\widehat{P}}_s)\mathrm{d}s\Big] \\ &\quad + \mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\infty} e^{-\rho s}K^+\big(1+e^{\frac{\gamma}{\eta}(X^{\widehat{P}}_s - Y^{\widehat{P}}_s)}\big)\mathrm{d}\widehat{P}^+_s + \int_0^{\infty} e^{-\rho s}K^-\big(1+e^{\frac{\gamma}{\eta}(X^{\widehat{P}}_s - Y^{\widehat{P}}_s)}\big)\mathrm{d}\widehat{P}^-_s\Big].\end{aligned} \tag{6.10}$$

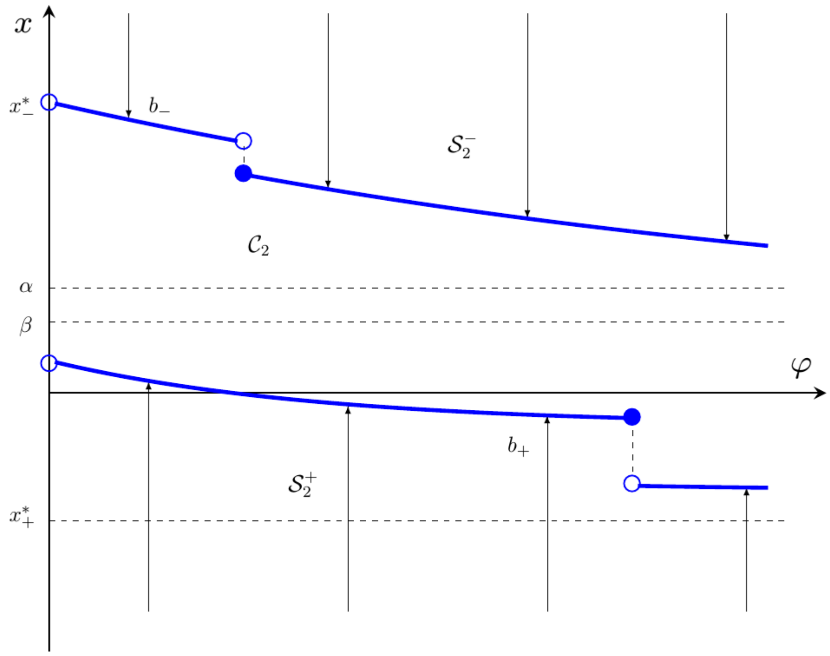

Figure 2: An illustrative drawing of the free boundaries b_+ and b_- satisfying Proposition 4.3. In the picture, $\alpha := (C')^{-1}(\rho K^-)$ and $\beta := (C')^{-1}(-\rho K^+)$. Moreover, the vertical arrows identify the directions of exercise of the optimal control $\widehat{P}$ defined through (6.1).

Given now that $X^{\widehat{P}} - Y^{\widehat{P}} = \eta \log(\Phi)/\gamma$ by definition (6.3), and that (5.6) yields $\widehat{V}(x,y) = \widehat{V}(x, x - \eta\log(\varphi)/\gamma) = \overline{V}(x,\varphi)$, we further conclude from (6.10) that for any $(x,\varphi) \in \overline{\mathcal{C}_2}$ (as we had assumed $(x,y) \equiv (x, x - \eta\log(\varphi)/\gamma) \in \overline{\mathcal{C}_3}$)

$$\overline{V}(x,\varphi) \geq \mathsf{E}^{\mathsf{Q}}\bigg[\int_0^\infty e^{-\rho s}\big(1+\Phi_s\big)C(X_s^{\widehat{P}})\mathrm{d}s + \int_0^\infty e^{-\rho s}\big(1+\Phi_s\big)\big(K^+\mathrm{d}\widehat{P}_s^+ + K^-\mathrm{d}\widehat{P}_s^-\big)\bigg] = \overline{\mathcal{J}}_{x,\varphi}(\widehat{P}). \tag{6.11}$$

Combining this inequality with definition (4.8), i.e. $\overline{V}(x,\varphi) \leq \overline{\mathcal{J}}_{x,\varphi}(\widehat{P})$, we prove that $\widehat{P}$ is an optimal control, for any $(x,\varphi) \in \overline{\mathcal{C}_2}$.

Suppose now that (x,φ) is such that $x < b_+(\varphi)$, so that $(x,\varphi) \in \mathcal{S}_2^+$. Then, according to (6.1) (see also (6.2)), and using (6.11), we have that

$$\overline{\mathcal{J}}_{x,\varphi}(\widehat{P}) = K^+(1+\varphi)\big(b_+(\varphi) - x\big) + \overline{\mathcal{J}}_{b_+(\varphi),\varphi}(\widehat{P}) \leq \overline{V}(b_+(\varphi),\varphi) - \int_x^{b_+(\varphi)} \overline{V}_x(z,\varphi) = \overline{V}(x,\varphi).$$

Proceeding similarly also for (x,φ) such that $x > b_-(\varphi)$, we conclude that $\widehat{P}$ is indeed optimal for any $(x,\varphi) \in \mathbb{R}^2$. □

7. Refined Regularity of the Free Boundaries and their Characterization

In this section we will obtain substantial regularity of the value $\bar{v}(x,\varphi)$ of the Dynkin game (4.10), as well as an analytical characterisation of its corresponding free boundaries $b_\pm$. Due to Theorem 6.1, this consequently leads to the complete knowledge of the optimal control rule $\widehat{P}$.

7.1. Parabolic formulation and Lipschitz continuity of the free boundaries

In view of a further change of variables, in line with (6.3), we define

$$Y_t^0 := X_t^0 - \frac{\eta}{\gamma}\log(\Phi_t), \quad t \geq 0, \tag{7.1}$$

with X^0 as in (4.11). Then, by Itô's formula, we have

$$\begin{cases} \mathrm{d}X_t^0 = \mu_0 \mathrm{d}t + \eta \mathrm{d}W_t, & X_0^0 = x \in \mathbb{R}, \\ \mathrm{d}Y_t^0 = \frac{1}{2}(\mu_0 + \mu_1)\mathrm{d}t, & Y_0^0 = y := x - \frac{\eta}{\gamma}\log(\varphi) \in \mathbb{R}, \end{cases} \tag{7.2}$$

and (4.10) rewrites in terms of the new coordinates $(x, y) = (X_0^0, Y_0^0)$ as

$$\begin{aligned} \widehat{v}(x,y) := \inf_{\sigma}\sup_{\tau} \mathsf{E}^{\mathsf{Q}}\bigg[\int_0^{\tau\wedge\sigma} e^{-\rho t}\Big(1 + e^{\frac{\gamma}{\eta}(X_t^0 - Y_t)}\Big)C'(X_t^0)\mathrm{d}t - K^+ e^{-\rho\tau}\Big(1 + e^{\frac{\gamma}{\eta}(X_\tau^0 - Y_\tau)}\Big)\mathbf{1}_{\{\tau<\sigma\}} \\ + K^- e^{-\rho\sigma}\Big(1 + e^{\frac{\gamma}{\eta}(X_\sigma^0 - Y_\sigma)}\Big)\mathbf{1}_{\{\tau>\sigma\}}\bigg] = \bar{v}\left(x, e^{\frac{\gamma}{\eta}(x-y)}\right), \quad (x,y) \in \mathbb{R}^2. \end{aligned} \tag{7.3}$$

In view of the relationship in (7.3), the value function $\widehat{v}(\cdot,\cdot)$ inherits important properties which have already been proved for $\bar{v}(\cdot,\cdot)$. To be more precise, we first conclude immediately from Proposition 4.2.(i) the following result.

Proposition 7.1. *The value function $(x,y) \mapsto \widehat{v}(x,y)$ defined in (7.3) is continuous over $\mathbb{R}^2$.*

Moreover, since $\bar{v}(x, \exp\{\gamma(x-y)/\eta\}) = \overline{V}_x(x, \exp\{\gamma(x-y)/\eta\})$ by Proposition 4.1.(ii), it follows from (5.7) that $\widehat{v}(x,y) = \widehat{V}_x(x,y) + \widehat{V}_y(x,y)$ for all $(x,y) \in \mathbb{R}^2$, and consequently the open set $\mathcal{C}_3$ defined in (5.8) takes the form

$$\mathcal{C}_3 = \big\{(x,y) \in \mathbb{R}^2 : \ -K^+\big(1 + e^{\frac{\gamma}{\eta}(x-y)}\big) < \widehat{v}(x,y) < K^-\big(1 + e^{\frac{\gamma}{\eta}(x-y)}\big)\big\} = T(\mathcal{C}_2). \tag{7.4}$$

Hence, by also defining the closed sets

$$\mathcal{S}_3^+ := \big\{(x,y) \in \mathbb{R}^2 : \ \widehat{v}(x,y) \leq -K^+\big(1 + e^{\frac{\gamma}{\eta}(x-y)}\big)\big\}, \tag{7.5}$$

$$\mathcal{S}_3^- := \big\{(x,y) \in \mathbb{R}^2 : \ \widehat{v}(x,y) \geq K^-\big(1 + e^{\frac{\gamma}{\eta}(x-y)}\big)\big\}, \tag{7.6}$$

the global diffeomorphism T from (5.5) implies that $\mathcal{S}_3^\pm = T(\mathcal{S}_2^\pm)$ as well, where $\mathcal{C}_2$ and $\mathcal{S}_2^\pm$ are the continuation and stopping regions (4.13)–(4.15) for the Dynkin game $\bar{v}$ in (4.10). Combining these relationships with the structure of the latter regions in (4.18)–(4.19) yields that $\mathcal{C}_3$ and ${\mathcal{S}_3}^\pm$ are connected.

In order to obtain the explicit structure of the regions $\mathcal{C}_3$ and ${\mathcal{S}_3}^\pm$, we now define the generalised inverses of the nonincreasing $b_\pm$ (cf. Proposition 4.3) by

$$b_+^{-1}(x) := \sup\{\varphi \in (0,\infty) : \ b_+(\varphi) \geq x\} \quad \text{and} \quad b_-^{-1}(x) := \inf\{\varphi \in (0,\infty) : \ b_-(\varphi) \leq x\}. \tag{7.7}$$

Since the map $\varphi \mapsto T_2(x,\varphi)$ in (5.5) is decreasing for any given $x \in \mathbb{R}$ (cf. the functions $b_\pm$ are nonincreasing due to Proposition 4.3.(i)), we have

$$\begin{aligned} (x,y) \in \mathcal{C}_3 \quad &\Leftrightarrow \quad \big(x, e^{\frac{\gamma}{\eta}(x-y)}\big) \in \mathcal{C}_2 \quad \Leftrightarrow \quad b_+^{-1}(x) < e^{\frac{\gamma}{\eta}(x-y)} < b_-^{-1}(x) \\ &\Leftrightarrow \quad x - \frac{\eta}{\gamma}\log(b_-^{-1}(x)) < y < x - \frac{\eta}{\gamma}\log(b_+^{-1}(x)), \end{aligned}$$

while similar relations hold true for the characterisation of $\mathcal{S}_3^\pm$. Then, by defining

$$c_\pm^{-1}(x) := x - \frac{\eta}{\gamma}\log(b_\pm^{-1}(x)), \tag{7.8}$$

we can obtain the structure of the continuation and stopping regions of $\widehat{v}$, which take the form

$$\mathcal{C}_3 = \{(x,y) \in \mathbb{R}^2 : \ c_-^{-1}(x) < y < c_+^{-1}(x)\}, \tag{7.9}$$

$$\mathcal{S}_3^+ = \{(x,y) \in \mathbb{R}^2 : \ y \geq c_+^{-1}(x)\} \quad \text{and} \quad \mathcal{S}_3^- = \{(x,y) \in \mathbb{R}^2 : \ y \leq c_-^{-1}(x)\}. \tag{7.10}$$

The proof of the next lemma can be found in the Appendix.

Lemma 7.2. *The functions $c_\pm^{-1}(\cdot)$ defined in (7.8) are strictly increasing, while $c_+^{-1}(\cdot)$ is left-continuous and $c_-^{-1}(\cdot)$ is right-continuous on $\mathbb{R}$.*

In light of Lemma 7.2, we may define the functions

$$c_+(y) := \inf\{x \in \mathbb{R} : \ y \le c_+^{-1}(x)\} \quad \text{and} \quad c_-(y) := \sup\{x \in \mathbb{R} : \ y \ge c_-^{-1}(x)\}, \quad y \in \mathbb{R}. \tag{7.11}$$

In the following result, we prove that $y \mapsto c_\pm(y)$ identify with the optimal free boundaries of the Dynkin game $\widehat{v}$ in (7.3) and provide some important properties such as their global Lipschitz continuity.

Proposition 7.3. *The free boundaries $c_\pm$ defined in (7.11) satisfy the following properties:*

(i) $c_\pm(\cdot)$ are strictly increasing on $\mathbb{R}$ and we have $x_+^* \le c_+(y) < c_-(y) \le x_-^*$ for all $y \in \mathbb{R}$ (with $x_\pm^*$ as in Proposition 3.2). Moreover, $c_+(y) \le (C')^{-1}(-\rho K^+)$ and $c_-(y) \ge (C')^{-1}(\rho K^-)$ for all $y \in \mathbb{R}$;

(ii) $c_\pm(\cdot)$ are Lipschitz-continuous on $\mathbb{R}$ with Lipschitz constant $L = 1$, namely

$$0 \le c_\pm(y) - c_\pm(y') \le y - y', \quad \forall\, y \ge y'.$$

(iii) The structure of the continuation and stopping regions for (7.3) take the form

$$\mathcal{C}_3 = \{(x,y) \in \mathbb{R}^2 : \ c_+(y) < x < c_-(y)\},$$
$$\mathcal{S}_3^+ = \{(x,y) \in \mathbb{R}^2 : \ x \le c_+(y)\} \quad \textit{and} \quad \mathcal{S}_3^- = \{(x,y) \in \mathbb{R}^2 : \ x \ge c_-(y)\}.$$

Proof. We prove separately the three parts.

Proof of (i). The first part of the claim follows from Lemma 7.2, together with the definition (7.11) of $c_\pm$. The second and third parts of the claim are due to the fact that T_1 as in (5.5) is the identity.

Proof of (ii). Using the definitions (7.8) of $c_\pm^{-1}$ and the monotonicity of $b_\pm^{-1}$ (see proof of Lemma 7.2) we get

$$c_\pm^{-1}(x) - c_\pm^{-1}(x') = \left(x - \frac{\eta}{\gamma}\log(b_\pm^{-1}(x))\right) - \left(x' - \frac{\eta}{\gamma}\log(b_\pm^{-1}(x'))\right) \ge x - x', \quad \forall\, x \ge x'. \tag{7.12}$$

Combining this with definitions (7.11) and part (i), we obtain the desired claim.

Proof of (iii). This is again due to the definitions (7.11) of $c_\pm$, their monotonicity from part (i) and the expressions of the sets in (7.9) and (7.10). ☐

7.2. Global C^1-regularity of $\widehat{v}$

For any $(x,y) \in \mathbb{R}^2$ given and fixed, we consider the strong solution to the dynamics in (7.2), denoted by

$$X_t^{0,x} = x + \mu_0 t + \eta W_t \quad \text{and} \quad Y_t^{0,y} = y + \frac{1}{2}(\mu_1 + \mu_0)t, \quad t \ge 0,$$

and we define

$$\tau^\star(x,y) := \inf\{t \ge 0 : \ (X_t^{0,x}, Y_t^{0,y}) \in \mathcal{S}_3^+\} \quad \text{and} \quad \sigma^\star(x,y) := \inf\{t \ge 0 : \ (X_t^{0,x}, Y_t^{0,y}) \in \mathcal{S}_3^-\}. \tag{7.13}$$

Notice that by [17] and [34], the pair $(\tau^\star, \sigma^\star)$ realises a saddle point for the Dynkin game with value $\widehat{v}$ in (7.3). In the sequel, we aim at deriving the global C^1-regularity of $\widehat{v}(\cdot,\cdot)$, following the arguments developed in [11]. In order to accomplish that, the next result about the regularity (in the probabilistic sense) of $(\tau^\star, \sigma^\star)$ is needed.

Lemma 7.4. *Suppose that $(x_n, y_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}_3$ is such that $(x_n, y_n) \to (x_o, y_o)$, where $y_o \in \mathbb{R}$ and $x_o := c_+(y_o)$ (resp., $x_o := c_-(y_o)$), then $\tau^\star(x_n, y_n) \to 0$ (resp., $\sigma^\star(x_n, y_n) \to 0$), Q-a.s..*

Proof. We prove the claim for $\tau^\star(x_n, y_n)$, since the proof for $\sigma^\star(x_n, y_n)$ can be performed analogously. Fix $\omega \in \Omega$ and assume (aiming for a contradiction) that

$$\limsup_{n\to\infty} \tau^\star(x_n, y_n)(\omega) =: \delta > 0.$$

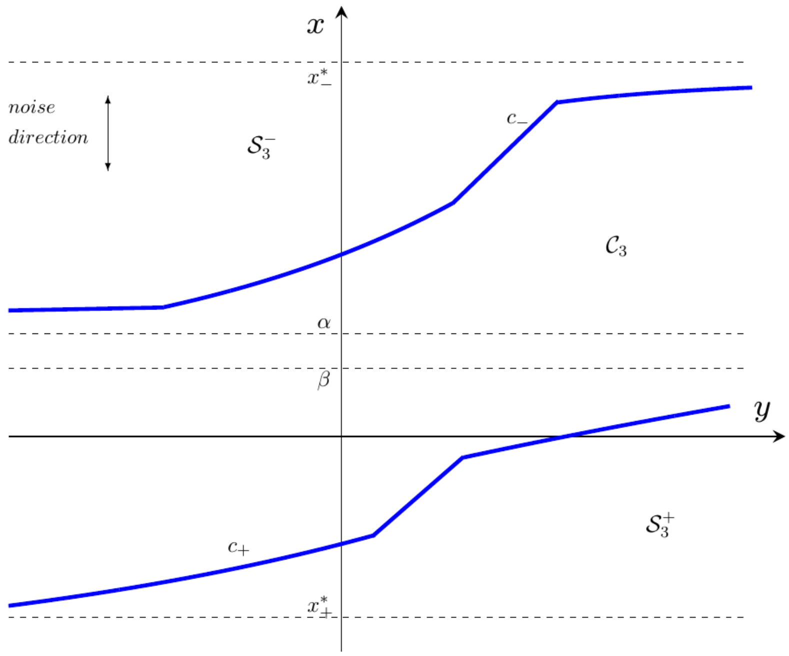

Figure 3: An illustrative drawing of the free boundaries c_+ and c_- satisfying Proposition 7.3. In the picture, $\alpha := (C')^{-1}(\rho K^-)$ and $\beta := (C')^{-1}(-\rho K^+)$.

This means that there exists a subsequence, still labelled by (x_n, y_n), such that

$$X_t^{0,x_n}(\omega) > c_+(Y_t^{0,y_n}) \qquad \forall\, n \in \mathbb{N}^*, \quad \forall\, t \in [0, \delta/2]; \tag{7.14}$$

that is,

$$x_n + \mu_0 t + \eta W_t(\omega) > c_+\Big(y_n + \frac{1}{2}(\mu_1 + \mu_0)t\Big) \qquad \forall\, n \in \mathbb{N}^*, \quad \forall\, t \in [0, \delta/2].$$

Hence, taking the limit as $n \to \infty$ and considering that c_+ is continuous (see Proposition 7.3.(ii)),

$$\eta W_t(\omega) \geq c_+\Big(y_o + \frac{1}{2}(\mu_1 + \mu_0)t\Big) - x_o - \mu_0 t, \quad \forall\, t \in [0, \delta/2].$$

Using now the Lipschitz continuity of c_+ (see again Proposition 7.3.(ii)), we further obtain

$$\eta W_t(\omega) \geq c_+(y_o) - \frac{1}{2}(\mu_1 + \mu_0)^- t - x_o - \mu_0 t \;=\; -\frac{1}{2}\big((\mu_1 + \mu_0)^- + \mu_0\big)t, \quad \forall\, n \in \mathbb{N}^*, \quad \forall\, t \in [0, \delta/2]. \tag{7.15}$$

However, by the law of iterated logarithm we know that for every $\varepsilon > 0$ there exists a sequence $(t_n)_{n\in\mathbb{N}}$ decreasing to 0 such that a.s. for any $n \in \mathbb{N}$ one has

$$W_{t_n} \leq -(1-\varepsilon)\sqrt{2t_n \log\big(\log\big(\frac{1}{t_n}\big)\big)}.$$

Hence, because $\sqrt{2t \log\big(\log\big(\frac{1}{t}\big)\big)}/t \to \infty$ as $t \downarrow 0$, we have that (7.15) can only happen for ω belonging to a Q-null set and the proof is complete. □

Remark 7.5. *From the previous proof one can easily observe that, by replacing the strict inequality with the large one in* (7.14)*, we can actually prove that* $\check{\tau}^\star(x_n, y_n) \to 0$ *and* $\check{\sigma}^\star(x_n, y_n) \to 0$, Q*-a.s., where*

$$\check{\tau}^\star(x,y) := \inf\{t \geq 0 : (X_t^{0,x}, Y_t^{0,y}) \in \mathrm{Int}(\mathcal{S}_3^+)\}, \quad \check{\sigma}^\star(x,y) := \inf\{t \geq 0 : (X_t^{0,x}, Y_t^{0,y}) \in \mathrm{Int}(\mathcal{S}_3^-)\}. \tag{7.16}$$

We now show that the value function $\widehat{v}(x,y)$ of the Dynkin game (7.3) is smooth across the topological boundary $\partial\mathcal{C}_3$ of the continuation region $\mathcal{C}_3$ from (7.4) in both directions x and y. The details of the proof of the following result can be found in the Appendix.

Proposition 7.6 (Smooth-fit)**.** *Let $y_o \in \mathbb{R}$ and set $x_o := c_\pm(y_o)$. Then the value function $\widehat{v}$ defined in (7.3) satisfies*

$$\lim_{\substack{(x,y)\to(x_o,y_o)\\(x,y)\in\mathcal{C}_3}} \widehat{v}_x(x,y) = \mp\frac{\gamma}{\eta}K^{\pm}e^{\frac{\gamma}{\eta}(x_o-y_o)} \quad \textit{and} \quad \lim_{\substack{(x,y)\to(x_o,y_o)\\(x,y)\in\mathcal{C}_3}} \widehat{v}_y(x,y) = \pm\frac{\gamma}{\eta}K^{\pm}e^{\frac{\gamma}{\eta}(x_o-y_o)}.$$

We are now ready to derive the global C^1-regularity of $\widehat{v}$ as well as the local boundedness of its second derivative in x.

Proposition 7.7. *The value function $\widehat{v}$ defined in (7.3) satisfies $\widehat{v} \in C^1(\mathbb{R}^2;\mathbb{R})$ and $\widehat{v}_{xx} \in L^\infty_{loc}(\mathbb{R}^2;\mathbb{R})$.*

Proof. By standard arguments based on the strong Markov property and Dirichlet boundary problems involving second-order partial differential equations of parabolic type, one can show that $\widehat{v}$ in (7.3) is a classical $C^{2,1}$-solution to

$$(\rho - \mathcal{L}_{X,Y})u(x,y) - \big(1 + e^{\frac{\gamma}{\eta}(x-y)}\big)C'(x) = 0, \quad \text{for all } (x,y) \in \mathcal{C}_3, \tag{7.17}$$

where $\mathcal{L}_{X,Y}$ is the second-order differential operator defined in (5.9) and $\mathcal{C}_3$ is given by (7.4) (see also Proposition 7.3.(iii)). Also, $\widehat{v} \in C^\infty$ in the interior of $\mathcal{S}_3^\pm$. Hence, by Proposition 7.6 we have that $\widehat{v} \in C^1(\mathbb{R}^2;\mathbb{R})$.

Moreover, we have from (7.17) that

$$\frac{1}{2}\eta^2\widehat{v}_{xx}(x,y) = \rho\widehat{v}(x,y) - \frac{1}{2}(\mu_1+\mu_0)\widehat{v}_y(x,y) - \mu_0\widehat{v}_x(x,y) - \big(1 + e^{\frac{\gamma}{\eta}(x-y)}\big)C'(x), \quad \forall\,(x,y) \in \mathcal{C}_3.$$

Given that $\widehat{v} \in C^1(\mathbb{R}^2;\mathbb{R})$, the right-hand side of the latter equation only involves functions that are continuous on $\mathbb{R}^2$, hence $\widehat{v}_{xx}$ admits a continuous extension on the closure of $\mathcal{C}_3$, and it is therefore bounded therein. Therefore, for $y \in \mathbb{R}$, we have that $\widehat{v}_x(\cdot,y)$ is Lipschitz continuous on $[c_+(y), c_-(y)]$, with Lipschitz constant $K(y)$ which is locally bounded on $\mathbb{R}$. Combining this with the fact that $\widehat{v}_x(\cdot,y)$ is infinitely many times continuously differentiable, and therefore locally bounded, in the stopping regions $\mathcal{S}_3^\pm$, we conclude that $\widehat{v}_{xx} \in L^\infty_{\text{loc}}(\mathbb{R}^2)$. □

7.3. Integral equations for the free boundaries

By Proposition 7.7, and by using standard arguments based on the strong Markov property (cf. [17] and [34]), we have that the value function $\widehat{v}$ defined in (7.3) and the free boundaries c_+ and c_- solve the free-boundary problem

$$\begin{cases} \big(\mathcal{L}_{X,Y}-\rho\big)\widehat{v}(x,y) = -(1+e^{\frac{\gamma}{\eta}(x-y)})C'(x), & c_+(y) < x < c_-(y),\ y \in \mathbb{R} \\ \big(\mathcal{L}_{X,Y}-\rho\big)\widehat{v}(x,y) \le -(1+e^{\frac{\gamma}{\eta}(x-y)})C'(x), & x < c_+(y),\ y \in \mathbb{R} \\ \big(\mathcal{L}_{X,Y}-\rho\big)\widehat{v}(x,y) \ge -(1+e^{\frac{\gamma}{\eta}(x-y)})C'(x), & x > c_-(y),\ y \in \mathbb{R} \\ -K^+(1+e^{\frac{\gamma}{\eta}(x-y)}) \le \widehat{v}(x,y) \le K^+(1+e^{\frac{\gamma}{\eta}(x-y)}), & (x,y) \in \mathbb{R}^2 \\ \widehat{v}(x,y) = -K^+(1+e^{\frac{\gamma}{\eta}(x-y)}), & x \le c_+(y),\ y \in \mathbb{R} \\ \widehat{v}(x,y) = K^-(1+e^{\frac{\gamma}{\eta}(x-y)}), & x \ge c_-(y),\ y \in \mathbb{R} \\ \widehat{v}_x(x,y) = \mp\frac{\gamma}{\eta}K^\pm e^{\frac{\gamma}{\eta}(x-y)}, & x = c_\pm(y),\ y \in \mathbb{R} \\ \widehat{v}_y(x,y) = \pm\frac{\gamma}{\eta}K^\pm e^{\frac{\gamma}{\eta}(x-y)}, & x = c_\pm(y),\ y \in \mathbb{R}. \end{cases} \tag{7.18}$$

Here $\mathcal{L}_{X,Y}$ is the second-order differential operator defined in (5.9) and $\widehat{v} \in C^{2,1}$ inside $\mathcal{C}_3$ (cf. Proposition 7.3.(iii)). Hence, via the above results and a suitable application of (a week version of) Itô's lemma, we firstly aim at obtaining an integral representation of $\widehat{v}$. This will then lead to a system of coupled integral equations solved by the free boundaries $c_\pm$ defined in (7.11) (see also Proposition 7.3 for their properties).

Proposition 7.8. *Consider the free boundaries $c_\pm$ defined in (7.11). Then, for any $(x,y) \in \mathbb{R}^2$, the value function $\widehat{v}$ of (7.3) can be written as*

$$\widehat{v}(x,y) = \mathsf{E}^{\mathsf{Q}}_{(x,y)}\bigg[\int_0^\infty e^{-\rho s}\big(1+e^{\frac{\gamma}{\eta}(X^0_s - Y^0_s)}\big)C'(X^0_s)\mathbb{1}_{\{c_+(Y^0_s)<X^0_s<c_-(Y^0_s)\}}\mathrm{d}s\bigg] + \mathsf{E}^{\mathsf{Q}}_{(x,y)}\bigg[\int_0^\infty e^{-\rho s}\rho\big(1+e^{\frac{\gamma}{\eta}(X^0_s - Y^0_s)}\big)\big(K^-\mathbb{1}_{\{X^0_s\geq c_-(Y^0_s)\}} - K^+\mathbb{1}_{\{X^0_s\leq c_+(Y^0_s)\}}\big)\mathrm{d}s\bigg], \tag{7.19}$$

where $\mathsf{E}^{\mathsf{Q}}_{(x,y)}$ is the expectation under $\mathbb{P}_{(x,y)}$ such that (X^0, Y^0) from (7.2) starts at $(x,y) \in \mathbb{R}^2$.

The previous representation of $\widehat{v}$ in (7.19), which is proved in the Appendix, allows us to determine a system of integral equations for $c_\pm$, which is the main aim of this section. Before we present this result, we denote by

$$G(z;m,\nu) := \frac{1}{\sqrt{2\pi\nu^2}}e^{-\frac{(z-m)^2}{2\nu^2}}, \quad z\in\mathbb{R},\ m\in\mathbb{R},\ \nu>0,$$

the density function of a Gaussian random variable with mean m and variance ν^2.

Proposition 7.9. *The free boundaries $c_\pm$ defined in (7.11) solve the system of integral equations*

$$\mp K^\pm q(c_\pm(y),y) = \int_0^\infty e^{-\rho s}\bigg(\int_{\mathbb{R}}\big(1+e^{\frac{\gamma}{\eta}(z-Y^0_s)}\big)C'(z)\mathbb{1}_{\{c_+(Y^0_s)<z<c_-(Y^0_s)\}}G(z;c_\pm(y)+\mu_0 s,\eta^2 s)\mathrm{d}z\bigg)\mathrm{d}s + \int_0^\infty e^{-\rho s}\bigg(\int_{\mathbb{R}}\rho\big(1+e^{\frac{\gamma}{\eta}(z-Y^0_s)}\big)\big(K^-\mathbb{1}_{\{z\geq c_-(Y^0_s)\}} - K^+\mathbb{1}_{\{z\leq c_+(Y^0_s)\}}\big)G(z;c_\pm(y)+\mu_0 s,\eta^2 s)\mathrm{d}z\bigg)\mathrm{d}s.$$

Moreover, (c_+,c_-) is the unique solution pair belonging to the set $\mathcal{D}_+\times\mathcal{D}_-$, where

$$\mathcal{D}_+ := \big\{g:\mathbb{R}\to\mathbb{R}:\ g \text{ is continuous, strictly increasing, s.t. } x^*_+ \leq g(y) \leq (C')^{-1}(-\rho K^+)\big\}$$
$$\mathcal{D}_- := \big\{g:\mathbb{R}\to\mathbb{R}:\ g \text{ is continuous, strictly increasing, s.t. } (C')^{-1}(\rho K^-) \leq g(y) \leq x^*_-\big\}.$$

Proof. Taking $x = c_\pm(y)$ in Proposition 7.8, and employing the value function's continuity (i.e. $\widehat{v}(c_\pm(y),y) = \mp K^\pm\big(1+\exp\{\gamma(c_\pm(y)-y)/\eta\}\big)$, for any $y\in\mathbb{R}$), we find that

$$\mp K^\pm q(c_\pm(y),y) = \mathsf{E}^{\mathsf{Q}}_{(c_\pm(y),y)}\bigg[\int_0^\infty e^{-\rho s}\big(1+e^{\frac{\gamma}{\eta}(X^0_s - Y^0_s)}\big)C'(X^0_s)\mathbb{1}_{\{c_+(Y^0_s)<X^0_s<c_-(Y^0_s)\}}\mathrm{d}s\bigg] + \mathsf{E}^{\mathsf{Q}}_{(c_\pm(y),y)}\bigg[\int_0^\infty e^{-\rho s}\rho\big(1+e^{\frac{\gamma}{\eta}(X^0_s - Y^0_s)}\big)\big(K^-\mathbb{1}_{\{X^0_s\geq c_-(Y^0_s)\}} - K^+\mathbb{1}_{\{X^0_s\leq c_+(Y^0_s)\}}\big)\mathrm{d}s\bigg]. \tag{7.20}$$

Hence, by noticing that Y^0 is a deterministic process and that $X^{0,c_\pm(y)}_s$ is Gaussian under Q with mean $c_\pm(y)+\mu_0 s$ and variance $\eta^2 s$, we easily obtain from (7.20) the desired equations.

The fact that $c_\pm$ belong to the classes $\mathcal{D}_\pm$ follows from their continuity, monotonicity, and boundedness in Proposition 7.3.

Finally, in order to prove the uniqueness claim one can proceed as in [9] (see, in particular Lemma 3.15, Lemma 3.16, Proposition 3.17 and Theorem 3.18 therein). Given that the present setting creates no additional difficulties, we omit further details of this verification, hence this completes the proof. ☐

Remark 7.10. *The complete characterisation of the boundaries $c_\pm$ provided by Proposition 7.9 together with (7.8), yield a complete description of the free boundaries $b_\pm$, at which the optimal control rule $\widehat{P}$ constructed in (6.1)–(6.2) (see Section 6.1 for details) commands the process $(X^{\widehat{P}}_t, \Phi_t)_{t\geq 0}$ to be reflected.*

Indeed, once $c_\pm$ are determined by solving (numerically) the system (7.20), we can use (7.8) to obtain $b^{-1}_\pm$, and consequently determine $b_\pm$ by inverting (7.7). However, since a numerical treatment of (7.20) is non trivial and outside the scopes of the present work, we do not address it in this paper.

Appendix A

A.1. Proof of Proposition 5.1

It follows from (4.9), that $\Phi_t = \varphi \mathcal{M}_t$, where $\mathcal{M}_t := \exp\{\gamma W_t - \gamma^2 t/2\}$, for any $t \geq 0$ and $\varphi > 0$.

For any $(x, \varphi) \in \mathbb{R} \times (0, \infty)$ given and fixed, one clearly has $\overline{V}(x, \varphi) \leq \overline{\mathcal{J}}_{x,\varphi}(0)$. Hence, without loss of generality, we can restrict the attention to all those controls $P \in \mathcal{A}$ such that, for some constant $\kappa_o > 0$, we have

$$\mathsf{E}^{\mathsf{Q}}\bigg[\int_0^\infty e^{-\rho t}\big(1 + \varphi\mathcal{M}_t\big)C(X_t^{x;P})\mathrm{d}t\bigg] \leq \overline{\mathcal{J}}_{x,\varphi}(P) \leq \overline{\mathcal{J}}_{x,\varphi}(0) = \mathsf{E}^{\mathsf{Q}}\bigg[\int_0^\infty e^{-\rho t}\big(1 + \varphi\mathcal{M}_t\big)C(X_t^{x;0})\mathrm{d}t\bigg]$$
$$= (1+\varphi)\mathsf{E}\bigg[\int_0^\infty e^{-\rho t}C(X_t^{x;0})\mathrm{d}t\bigg] \leq \kappa_o(1+\varphi)(1+|x|^p). \tag{A-1}$$

Here, the second equality follows from a change of measure as in Section 4, $X^{x;0}$ in the second expectation evolves as in (4.11), while in the third expectation it evolves as in (3.1), and the last step is due to Assumption 2.1.(i) and standard estimates. In the rest of this proof, we denote by $\mathcal{A}_o$ the class of admissible controls P for which (A-1) holds true.

Then, let $(x, \varphi), (x', \varphi')$ such that $|(x, \varphi)| \leq R$, $|(x', \varphi')| \leq R$ be given and fixed, and take $\lambda \in [0, 1]$. Observe that, by using the definition (4.8) of $\overline{V}$ (and restricting to the class $\mathcal{A}_o$) we get

$$\begin{aligned}
&\lambda\overline{V}(x,\varphi)+(1-\lambda)\overline{V}(x',\varphi') - \overline{V}(\lambda(x,\varphi)+(1-\lambda)(x',\varphi'))\\
&\leq \sup_{P\in\mathcal{A}_o} \mathsf{E}^{\mathsf{Q}}\bigg[\int_0^\infty e^{-\rho t}\Big[\lambda(1+\varphi\mathcal{M}_t)C(X_t^{x;P}) + (1-\lambda)(1+\varphi'\mathcal{M}_t)C(X_t^{x';P})\\
&\qquad - \big(1+(\lambda\varphi+(1-\lambda)\varphi')\mathcal{M}_t\big)C(X_t^{\lambda x+(1-\lambda)x';P})\Big]\mathrm{d}t\\
&\qquad + \int_0^\infty e^{-\rho t}K^+\Big[\lambda(1+\varphi\mathcal{M}_t) + (1-\lambda)(1+\varphi'\mathcal{M}_t) - \big(1+(\lambda\varphi+(1-\lambda)\varphi')\mathcal{M}_t\big)\Big]\mathrm{d}P_t^+\\
&\qquad + \int_0^\infty e^{-\rho t}K^-\Big[\lambda(1+\varphi\mathcal{M}_t) + (1-\lambda)(1+\varphi'\mathcal{M}_t) - \big(1+(\lambda\varphi+(1-\lambda)\varphi')\mathcal{M}_t\big)\Big]\mathrm{d}P_t^-\bigg].
\end{aligned}$$

By adding and subtracting $(1-\lambda)\varphi\mathcal{M}\big(C(X^{x';P}) + C(\lambda X_t^{x;P} + (1-\lambda)X_t^{x';P})\big)$ in the dt-integral appearing in the last equation, using the semiconcavity property of C in Assumption 2.1.(iii) together with the solution $X^{x;P}$ of (4.9), as well as the fact that $\sup(f+g) \leq \sup(f) + \sup(g)$, we obtain

$$\begin{aligned}
&\lambda\overline{V}(x,\varphi)+(1-\lambda)\overline{V}(x',\varphi') - \overline{V}(\lambda(x,\varphi)+(1-\lambda)(x',\varphi'))\\
&\leq \sup_{P\in\mathcal{A}_o} \mathsf{E}^{\mathsf{Q}}\bigg[\int_0^\infty e^{-\rho t}\alpha_2\lambda(1-\lambda)\Big(1+C(X_t^{x;P})+C(X_t^{x';P})\Big)^{(1-\frac{2}{p})^+}|x-x'|^2\mathrm{d}t\bigg]\\
&+ \sup_{P\in\mathcal{A}_o} \mathsf{E}^{\mathsf{Q}}\bigg[\int_0^\infty e^{-\rho t}\varphi\mathcal{M}_t\Big(\lambda C(X_t^{x;P}) + (1-\lambda)C(X_t^{x';P}) - C(\lambda X_t^{x;P}+(1-\lambda)X_t^{x';P})\Big)\mathrm{d}t\\
&\qquad + \int_0^\infty e^{-\rho t}(1-\lambda)(\varphi-\varphi')\mathcal{M}_t\Big(C(\lambda X_t^{x;P}+(1-\lambda)X_t^{x';P}) - C(X_t^{x';P})\Big)\mathrm{d}t\bigg].
\end{aligned}$$

Using again the assumed semiconcavity of C and Hölder's inequality, we further conclude that

$$\begin{aligned}
&\lambda\overline{V}(x,\varphi)+(1-\lambda)\overline{V}(x',\varphi') - \overline{V}(\lambda(x,\varphi)+(1-\lambda)(x',\varphi'))\\
&\leq \alpha_2\lambda(1-\lambda)|x-x'|^2\mathsf{E}^{\mathsf{Q}}\bigg[\int_0^\infty e^{-\rho t}\mathrm{d}t\bigg]^{\frac{2}{p}} \sup_{P\in\mathcal{A}_o} \mathsf{E}^{\mathsf{Q}}\bigg[\int_0^\infty e^{-\rho t}\Big(1+C(X_t^{x;P})+C(X_t^{x';P})\Big)\mathrm{d}t\bigg]^{(1-\frac{2}{p})^+}\\
&+ \alpha_2\lambda(1-\lambda)|x-x'|^2 \sup_{P\in\mathcal{A}_o} \mathsf{E}^{\mathsf{Q}}\bigg[\int_0^\infty e^{-\rho t}\varphi\mathcal{M}_t\Big(1+C(X_t^{x;P})+C(X_t^{x';P})\Big)^{(1-\frac{2}{p})^+}\mathrm{d}t\bigg]\\
&+ \alpha_1\lambda(1-\lambda)|\varphi-\varphi'||x-x'| \sup_{P\in\mathcal{A}_o} \mathsf{E}^{\mathsf{Q}}\bigg[\int_0^\infty e^{-\rho t}\mathcal{M}_t\Big(1+C(X_t^{x;P})+C(X_t^{x';P})\Big)^{(1-\frac{1}{p})}\mathrm{d}t\bigg].
\end{aligned}$$

We now distinguish the cases $p \in (1,2]$ and $p > 2$. If $p \in (1,2]$, using that $\mathsf{E}^{\mathsf{Q}}[\int_0^\infty e^{-\rho t}\mathcal{M}_t \mathrm{d}t] = 1/\rho$ and Hölder's inequality, we further obtain that

$$\begin{aligned}&\lambda\overline{V}(x,\varphi) + (1-\lambda)\overline{V}(x',\varphi') - \overline{V}(\lambda(x,\varphi)+(1-\lambda)(x',\varphi'))\\&\le \alpha_2\lambda(1-\lambda)|x-x'|^2\Big(\rho^{-\frac{2}{p}}+\rho^{-1}\Big) + \alpha_1\lambda(1-\lambda)|\varphi-\varphi'||x-x'|\mathsf{E}^{\mathsf{Q}}\Big[\int_0^\infty e^{-\rho t}\mathcal{M}_t\mathrm{d}t\Big]^{\frac{1}{p}}\\&\qquad\qquad \times \sup_{P\in\mathcal{A}_o}\mathsf{E}^{\mathsf{Q}}\Big[\int_0^\infty e^{-\rho t}\mathcal{M}_t\Big(1+C(X_t^{x;P})+C(X_t^{x';P})\Big)\mathrm{d}t\Big]^{1-\frac{1}{p}}. \qquad \text{(A-2)}\end{aligned}$$

Hence, employing the estimate (A-1) in (A-2), we find for some $\kappa > 0$ that

$$\begin{aligned}&\lambda\overline{V}(x,\varphi) + (1-\lambda)\overline{V}(x',\varphi') - \overline{V}(\lambda(x,\varphi)+(1-\lambda)(x',\varphi'))\\&\le \kappa\lambda(1-\lambda)\Big(|x-x'|^2 + |\varphi-\varphi'||x-x'|(1+|x|+|x'|)^{p-1}\Big),\end{aligned}$$

which gives the claimed semiconcavity. The case $p > 2$ can be actually proved using similar arguments as for $p \in (1,2]$, therefore its precise proof in omitted for brevity. □

A.2. Proof of Lemma 7.2

Due to (7.7) and Proposition 4.3, we have that $b_\pm^{-1}$ are nonincreasing, with b_+^{-1} left-continuous and b_-^{-1} right-continuous. Combining the aforementioned properties together with the definition (7.8) yields the desired properties. □

A.3. Proof of Proposition 7.6

We focus on proving the continuity of $\widehat{v}_x$ across c_+, since the other claims can be obtained similarly. To that end, we firstly simplify the notation by defining (cf. (7.4)–(7.5))

$$q(x,y) := 1 + e^{\frac{\gamma}{\eta}(x-y)} \quad \text{and} \quad \widehat{w}(x,y) := \widehat{v}(x,y) + K^+ q(x,y)\begin{cases} > 0, & \text{for all } (x,y)\in\mathbb{R}^2\setminus\mathcal{S}_3^+,\\ = 0, & \text{for all } (x,y)\in\mathcal{S}_3^+,\end{cases} \qquad \text{(A-3)}$$

and notice that, for every $(x,y)\in\mathbb{R}^2$ we have

$$\begin{aligned}&\widehat{w}(x,y)\\&= \sup_{\tau\in\mathcal{T}}\inf_{\sigma\in\mathcal{T}}\mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau\wedge\sigma} e^{-\rho t}q(X_t^{0,x},Y_t^{0,y})\big(C'(X_t^{0,x})+\rho K^+\big)\mathrm{d}t + (K^+ + K^-)e^{-\rho\sigma}q(X_\sigma^{0,x},Y_\sigma^{0,y})\mathbf{1}_{\{\sigma<\tau\}}\Big].\end{aligned}$$

Then, the desired continuity of $\widehat{v}_x$ across c_+ is equivalent to

$$\lim_{\mathcal{C}_3\ni(x,y)\to(x_o,y_o)}\widehat{w}_x(x,y) = 0, \quad \text{for} \quad x_o := c_+(y_o) \quad \text{and} \quad y_o\in\mathbb{R}. \qquad \text{(A-4)}$$

In the remaining of the proof, we therefore focus on proving (A-4).

Fix $(x,y)\in\mathcal{C}_3$ and let $\varepsilon > 0$ be such that $(x+\varepsilon,y)\in\mathcal{C}_3$. Denote by $\tau^\star \equiv \tau^\star(x,y)$ and $\check{\tau}^\star \equiv \check{\tau}^\star(x,y)$ from (7.13) and (7.16), respectively. Then, define $\tau_\varepsilon^\star := \tau^\star(x+\varepsilon,y)$ according to (7.13) and $\check{\tau}_\varepsilon^\star := \check{\tau}^\star(x+\varepsilon,y)$ according to (7.16). In view of Proposition 7.3.(iii), these take the form

$$\begin{aligned}\tau_\varepsilon^\star &= \inf\{t\ge 0:\ X_t^{0,x+\varepsilon} \le c_+(Y_t^{0,y})\}, \quad \check{\tau}_\varepsilon^\star = \inf\{t\ge 0:\ X_t^{0,x+\varepsilon} < c_+(Y_t^{0,y})\},\\ \tau^\star &= \inf\{t\ge 0:\ X_t^{0,x}\le c_+(Y_t^{0,y})\} \quad \text{and} \quad \check{\tau}^\star = \inf\{t\ge 0:\ X_t^{0,x} < c_+(Y_t^{0,y})\}.\end{aligned}$$

By the regularity of the Brownian motion, we have $\tau_\varepsilon^\star = \check{\tau}_\varepsilon^\star$ and $\tau^\star = \check{\tau}^\star$, and by the continuity of trajectories of the Brownian motion, we have

$$\lim_{\varepsilon\downarrow 0}\check{\tau}_\varepsilon^\star \to \check{\tau}^\star \quad \text{which eventually yields that} \quad \lim_{\varepsilon\downarrow 0}\tau_\varepsilon^\star \to \tau^\star. \qquad \text{(A-5)}$$

Moreover, Proposition 7.3.(iii) further implies that $\sigma^\star \equiv \sigma^\star(x,y)$ from (7.13) takes the form

$$\sigma^\star = \inf\{t \geq 0 : \ X_t^{0,x} \geq c_-(Y_t^{0,y})\}.$$

Then, we have

$$\begin{aligned}\frac{\widehat{w}(x+\varepsilon,y)-\widehat{w}(x,y)}{\varepsilon} &\leq \frac{1}{\varepsilon}\,\mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau_\varepsilon^\star\wedge\sigma^\star} e^{-\rho t}\big(q(X_t^{0,x+\varepsilon},Y_t^{0,y})-q(X_t^x,Y_t)\big)\big(C'(X_t^{0,x+\varepsilon}+\rho K^+\big)\mathrm{d}t\Big] \\ &+\frac{1}{\varepsilon}\,\mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau_\varepsilon^\star\wedge\sigma^\star} e^{-\rho t}q(X_t^{0,x},Y_t^{0,y})\big(C'(X_t^{0,x+\varepsilon})-C'(X_t^{0,x})\big)\mathrm{d}t\Big] \\ &+\frac{1}{\varepsilon}\,\mathsf{E}^{\mathsf{Q}}\Big[e^{-\rho\sigma^\star}\mathbf{1}_{\{\tau_\varepsilon^\star>\sigma^\star\}}(K^++K^-)\big(q(X_{\sigma^\star}^{0,x+\varepsilon},Y_{\sigma^*}^{0,y})-q(X_{\sigma^\star}^{0,x},Y_{\sigma^*}^{0,y})\big)\Big].\end{aligned}$$

Using the Mean-Value Theorem, the above inequality becomes

$$\begin{aligned}\frac{\widehat{w}(x+\varepsilon,y)-\widehat{w}(x,y)}{\varepsilon} &\leq \mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau_\varepsilon^\star\wedge\sigma^\star} e^{-\rho t}q_x(\Lambda_t^\varepsilon,Y_t^{0,y})\big(C'(X_t^{0,x+\varepsilon})+\rho K^+\big)\mathrm{d}t\Big] \\ &+\mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau_\varepsilon^\star\wedge\sigma^\star} e^{-\rho t}q(X_t^{0,x},Y_t^{0,y})C''(\Xi_t^\varepsilon)\mathrm{d}t+e^{-\rho\sigma^\star}\mathbf{1}_{\{\tau_\varepsilon^\star>\sigma^\star\}}(K^++K^-)q_x(\Theta_{\sigma^\star}^\varepsilon)\Big].\end{aligned} \tag{A-6}$$

where $\Lambda_t^\varepsilon, \Xi_t^\varepsilon \in (X_t^{0,x}, X_t^{0,x+\varepsilon})$ and $\Theta_{\sigma^\star}^\varepsilon \in (X_{\sigma^\star}^{0,x}, X_{\sigma^\star}^{0,x+\varepsilon})$. If now the dominated convergence theorem can be applied, by taking limits and using (A-5) in (A-6), we get

$$\begin{aligned}\limsup_{\varepsilon\downarrow 0}\frac{\widehat{w}(x+\varepsilon,y)-\widehat{w}(x,y)}{\varepsilon} &\leq \mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau^\star\wedge\sigma^\star} e^{-\rho t}q_x(X_t^{0,x},Y_t^{0,y})\big(C'(X_t^{0,x})+\rho K^+\big)\mathrm{d}t\Big] \\ &+\mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau^\star\wedge\sigma^\star} e^{-\rho t}q(X_t^{0,x},Y_t^{0,y})C''(X_t^{0,x})\mathrm{d}t+e^{-\rho\sigma^\star}\mathbf{1}_{\{\tau^\star\geq\sigma^\star\}}(K^++K^-)q_x(X_{\sigma^\star}^{0,x})\Big].\end{aligned}$$

With similar estimates, we can also get

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0}\frac{\widehat{w}(x+\varepsilon,y)-\widehat{w}(x,y)}{\varepsilon} &\geq \mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau^\star\wedge\sigma^\star} e^{-\rho t}q_x(X_t^{0,x},Y_t^{0,y})\big(C'(X_t^{0,x})+\rho K^+\big)\mathrm{d}t\Big] \\ &+\mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau^\star\wedge\sigma^\star} e^{-\rho t}q(X_t^{0,x},Y_t^{0,y})C''(X_t^{0,x})\mathrm{d}t+e^{-\rho\sigma^\star}\mathbf{1}_{\{\tau^\star\geq\sigma^\star\}}(K^++K^-)q_x(X_{\sigma^\star}^{0,x})\Big].\end{aligned}$$

Hence,

$$\begin{aligned}\widehat{w}_x(x,y) &= \mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau^\star\wedge\sigma^\star} e^{-\rho t}q_x(X_t^{0,x},Y_t^{0,y})\big(C'(X_t^{0,x})+\rho K^+\big)\mathrm{d}t\Big] \\ &+\mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau^\star\wedge\sigma^\star} e^{-\rho t}q(X_t^{0,x},Y_t^{0,y})C''(X_t^{0,x})\mathrm{d}t+e^{-\rho\sigma^\star}\mathbf{1}_{\{\tau^\star\geq\sigma^\star\}}(K^++K^-)q_x(X_{\sigma^\star}^{0,x})\Big].\end{aligned}$$

Then, we obtain (A-4) by taking the limit as $(x,y) \to (x_0,y_0)$, using Lemma 7.4 and noticing that clearly $\liminf_{(x,y)\to(x_0,y_0)} \sigma^\star(x,y) > 0$ (cf. (7.5)–(7.6)).

In order to complete the proof, it remains to show that the dominated convergence theorem can be indeed invoked when taking limits in (A-6). We show this only for the term

$$\mathsf{E}^{\mathsf{Q}}\Big[\int_0^{\tau_\varepsilon^\star\wedge\sigma^\star} e^{-\rho t}q_x(\Lambda_t^\varepsilon,Y_t^{0,y})\big(C'(X_t^{0,x+\varepsilon})+\rho K^+\big)\mathrm{d}t\Big]$$

in (A-6), as the others can be treated similarly.

Notice that, since $q_x(\cdot, y)$ is positive and increasing, $C'(\cdot)$ is nondecreasing, and $\Lambda_t^\varepsilon \leq X_t^{0,x+\varepsilon} \leq X_t^{0,x+1}$ (for any $\varepsilon < 1$, without loss of generality), we can write

$$\begin{aligned}&\int_0^{\tau_\varepsilon^\star \wedge \sigma^\star} e^{-\rho t} q_x(\Lambda_t^\varepsilon, Y_t^{0,y})\big(C'(X_t^{0,x+\varepsilon}) + \rho K^+\big)\mathrm{d}t \\ &\leq \int_0^{\tau_\varepsilon^\star \wedge \sigma^\star} e^{-\rho t} q_x(X_t^{0,x+1}, Y_t^{0,y})\big(C'(X_t^{0,x+1}) + \rho K^+\big)\mathrm{d}t \\ &\leq \frac{\gamma}{\eta}\int_0^\infty e^{-\rho t}\big(q(X_t^{0,x+1}, Y_t^{0,y}) + 1\big)\big(\big|C'(X_t^{0,x+1})\big| + \rho K^+\big)\mathrm{d}t\end{aligned}$$

Now, on one hand, $\mathsf{E}^{\mathsf{Q}}[\int_0^\infty e^{-\rho t}|C'(X_t^{0,x+1})|\mathrm{d}t] < \infty$ due to Assumption 2.1 and standard estimates on the Brownian motion. On the other hand, by using the definition of $q(\cdot,\cdot)$ and (7.1), one has $q(X_t^{0,x+1}, Y_t^{0,y}) = 1 + \Phi_t^\varphi$, with $\varphi \equiv e^{\frac{\gamma}{\eta}(x+1-y)}$. Hence,

$$\begin{aligned}&\mathsf{E}^{\mathsf{Q}}\Big[\int_0^\infty e^{-\rho t} q(X_t^{0,x+1}, Y_t^{0,y})\left(\big|C'(X_t^{0,x+1})\big| + \rho K^+\right)\mathrm{d}t\Big] \\ &= \mathsf{E}^{\mathsf{Q}}\Big[\int_0^\infty e^{-\rho t}\big(1 + \Phi_t^\varphi\big)\big(\big|C'(X_t^{0,x+1})\big| + \rho K^+\big)\mathrm{d}t\Big]\Big|_{\varphi = e^{\frac{\gamma}{\eta}(x+1-y)}} \\ &= \left(1 + e^{\frac{\gamma}{\eta}(x+1-y)}\right)\mathsf{E}\Big[\int_0^\infty e^{-\rho t}\big(\big|C'(X_t^{0,x+1})\big| + \rho K^+\big)\mathrm{d}t\Big],\end{aligned} \tag{A-7}$$

where the last equality is due to a change of measure as in Section 4 and X^0 in the last expectation evolves as in (3.1). But then, standard estimates together with the growth requirements on C in Assumption 2.1 ensure that the last expectation in (A-7) is finite, thus completing the proof. □

A.4. Proof of Proposition 7.8

In this proof, we recall the notation $q(x,y) := 1 + e^{\gamma(x-y)/\eta}$, which will be used in the following four steps.

Step 1. Let $R > 0$ and define $\tau_R := \inf\{t \geq 0 : |X_t^0| \geq R \text{ or } |Y_t^0| \geq R\}$ under $\mathbb{P}_{(x,y)}$. Since $\widehat{v} \in C^1(\mathbb{R}^2;\mathbb{R})$ and $\widehat{v}_{xx} \in L^\infty_{\text{loc}}(\mathbb{R}^2;\mathbb{R})$ (cf. Proposition 7.7), we can apply a weak version of Itô's lemma (see, e.g., [3], Lemma 8.1 and Theorem 8.5, pp. 183–186) up to the stopping time $\tau_R \wedge T$, for some $T > 0$, to obtain

$$\widehat{v}(x,y) = \mathsf{E}^{\mathsf{Q}}_{(x,y)}\Big[e^{-\rho(\tau_R \wedge T)}\widehat{v}(X^0_{\tau_R\wedge T}, Y^0_{\tau_R \wedge T}) - \int_0^{\tau_R \wedge T} e^{-\rho s}\big(\mathcal{L}_{X,Y^0} - \rho\big)\widehat{v}(X_s^0, Y_s^0)\mathrm{d}s\Big]. \tag{A-8}$$

The right-hand side of (A-8) is well defined, since Y^0 is a deterministic process, the transition probability of X^0 is absolutely continuous with respect to the Lebesgue measure and $(\mathbb{L}_{X,Y} - \rho)u$ is defined up to a set of zero Lebesgue measure.

Since $\widehat{v}$ solves the free-boundary problem (7.18), we have for almost all $(x,y) \in \mathbb{R}^2$, that

$$\big(\mathcal{L}_{X,Y} - \rho\big)\widehat{v}(x,y) = -q(x,y)C'(x)\mathbb{1}_{\{c_+(y)<x<c_-(y)\}} + \rho K^+ q(x,y)\mathbb{1}_{\{x \leq c_+(y)\}} - \rho K^- q(x,y)\mathbb{1}_{\{x \geq c_-(y)\}},$$

and using again that the transition probability of X^0 is absolutely continuous with respect to the Lebesgue measure, the equation in (A-8) becomes

$$\begin{aligned}\widehat{v}(x,y) = {}&\mathsf{E}^{\mathsf{Q}}_{(x,y)}\Big[e^{-\rho(\tau_R\wedge T)}\widehat{v}(X^0_{\tau_R\wedge T}, Y^0_{\tau_R\wedge T}) + \int_0^{\tau_R\wedge T} e^{-\rho s} q(X_s^0, Y_s^0) C'(X_s^0)\mathbb{1}_{\{c_+(Y_s^0) < X_s^0 < c_-(Y_s^0)\}}\mathrm{d}s\Big] \\ &- \mathsf{E}^{\mathsf{Q}}_{(x,y)}\Big[\int_0^{\tau_R\wedge T} e^{-\rho s}\rho K^+ q(X_s^0, Y_s^0)\mathbb{1}_{\{X_s^0 \leq c_+(Y_s^0)\}}\mathrm{d}s - \int_0^{\tau_R\wedge T} e^{-\rho s}\rho K^- q(X_s^0, Y_s^0)\mathbb{1}_{\{X_s^0 \geq c_+(Y_s^0)\}}\mathrm{d}s\Big].\end{aligned} \tag{A-9}$$

Step 2. Using the relationship (7.3) between $\widehat{v}$ and $\overline{v}$ and the definition (7.2) of (X^0, Y^0), we obtain

$$\begin{aligned}&\mathsf{E}^{\mathsf{Q}}_{(x,y)}\Big[e^{-\rho(\tau_R\wedge T)}|\widehat{v}(X^0_{\tau_R\wedge T}, Y^0_{\tau_R\wedge T})|\Big] = \mathsf{E}^{\mathsf{Q}}_{(x,y)}\Big[e^{-\rho(\tau_R\wedge T)}\Big|\overline{v}\Big(X^0_{\tau_R\wedge T}, e^{\frac{\gamma}{\eta}(X^0_{\tau_R\wedge T} - Y^0_{\tau_R\wedge T})}\Big)\Big|\Big] \\ &\leq (K^+ \vee K^-)\mathsf{E}^{\mathsf{Q}}_{(x,\exp\{\frac{\gamma}{\eta}(x-y)\})}\Big[e^{-\rho(\tau_R\wedge T)}\Big(1 + \Phi_{\tau_R\wedge T}\Big)\Big] = \big(1 + e^{\frac{\gamma}{\eta}(x-y)}\big)\mathsf{E}_{(x,\pi)}\Big[e^{-\rho(\tau_R\wedge T)}\Big],\end{aligned} \tag{A-10}$$

for $\pi := e^{\gamma(x-y)/\eta}/(1+e^{\gamma(x-y)/\eta})$, and where the last step can be justified by performing a change of measure in the same spirit of Section 4. Clearly, taking limits as $R \uparrow \infty$ and $T \uparrow \infty$ in (A-10) yields

$$\lim_{T\uparrow\infty}\lim_{R\uparrow\infty} \mathsf{E}^{\mathsf{Q}}_{(x,y)}\Big[e^{-\rho(\tau_R\wedge T)}\widehat{v}(X^0_{\tau_R\wedge T}, Y^0_{\tau_R\wedge T})\Big] = 0. \tag{A-11}$$

Step 3. On one hand, notice that using the strong solution (X^0, Y^0) to (7.2), we get

$$\begin{aligned}
&\mathsf{E}^{\mathsf{Q}}_{(x,y)}\Big[\int_0^{\tau_R\wedge T} e^{-\rho s} q(X^0_s, Y^0_s)\mathbb{1}_{\{X^0_s\le c_+(Y^0_s)\}}\mathrm{d}s\Big] \le \mathsf{E}^{\mathsf{Q}}_{(x,y)}\Big[\int_0^{\infty} e^{-\rho s} q(X^0_s, Y^0_s)\mathrm{d}s\Big] \\
&= \int_0^\infty e^{-\rho s}\Big(1+\mathsf{E}^{\mathsf{Q}}_{(x,y)}\Big[e^{\frac{\gamma}{\eta}(X^0_s - Y^0_s)}\Big]\Big)\mathrm{d}s = \int_0^\infty e^{-\rho s}\Big(1+e^{\frac{\gamma}{\eta}(x-y)}\mathsf{E}^{\mathsf{Q}}\Big[e^{\gamma W_s - \frac{\gamma^2}{2}s}\Big]\Big)\mathrm{d}s < \infty,
\end{aligned}$$

since W is a Q-Brownian motion, thus the latter expectation is equal to 1. This clearly implies the finiteness of the latter expectation in (A-9). On the other hand, by a change of measure as that of Section 4 and Assumption 2.1, we also have

$$\begin{aligned}
&\mathsf{E}^{\mathsf{Q}}_{(x,y)}\Big[\int_0^{\tau_R\wedge T} e^{-\rho s} q(X^0_s, Y^0_s) C'(X^0_s)\mathbb{1}_{\{c_+(Y^0_s)<X^0_s<c_-(Y^0_s)\}}\mathrm{d}s\Big] \\
&\le \mathsf{E}^{\mathsf{Q}}_{(x,y)}\Big[\int_0^{\infty} e^{-\rho s} q(X^0_s, Y^0_s)|C'(X^0_s)|\mathrm{d}s\Big] = (1+e^{\frac{\gamma}{\eta}(x-y)})\mathsf{E}_{(x,\pi)}\Big[\int_0^{\infty} e^{-\rho s}|C'(X^0_s)|\mathrm{d}s\Big] \\
&\le \Big(1+e^{\frac{\gamma}{\eta}(x-y)}\Big)\mathsf{E}_{(x,\pi)}\Big[\int_0^{\infty} e^{-\rho s}\alpha_0(1+|X^0_s|^p)\mathrm{d}s\Big] < \infty,
\end{aligned}$$

for $\pi := e^{\frac{\gamma}{\eta}(x-y)}/(1+e^{\frac{\gamma}{\eta}(x-y)})$, where the finiteness of this expectation in the last step follows from standard estimates on the Brownian motion.

Step 4. Finally, given the finiteness of all the expectations of integrals appearing in (A-9) due to *Step 3*, we can apply the monotone convergence theorem to interchange limits as $R \uparrow \infty$ and $T \uparrow \infty$ with these expectations in (A-9). Therefore, using this fact together with *Step 2* we obtain (7.19), which completes the proof. □

Acknowledgments. Financial support by the German Research Foundation (DFG) through the Collaborative Research Centre 1283 is gratefully acknowledged by the second author.

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