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Markov perfect equilibria in multi-mode differential games with endogenous timing of mode transitions*

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Abstract

In this paper we study Markov-perfect equilibria (MPE) of two-player multi-mode differential games with controlled state dynamics, where one player controls the transition between modes. Different types of MPE are characterized distinguishing between *delay equilbria*, inducing for some initial conditions mode switches after a positive finite delay, and *now or never equilbria*, under which, depending on the initial condition, a mode switch occurs immediately or never. These results are applied to analyze the MPE of a game capturing the dynamic interaction between two incumbent firms among which one has to decide when to extend its product range by introducing a new product. The market appeal of the new product can be (positively or negatively) influenced over time by the competing firms through costly investments. It is shown that under a wide range of market introduction costs a now or never equilibrium co-exists with a continuum of delay equilibria, with each of them inducing a different time of product introduction.

JEL classification: C73; L13; O31

Keywords: multi-mode differential games, Markov-perfect-equilibrium, product innovation, optimal timing

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1 Introduction

The main agenda of this paper is to improve our understanding of strategic effects arising in dynamic economic and managerial settings characterized by potential structural breaks, which induce jumps in the payoff functions of the economic actors, in the law governing the dynamics of relevant state variables, or both. Areas of application where such structural breaks, which we will refer to as mode changes, arise include environmental economics dealing with potential catastrophic transitions (e.g. Haurie and Roche (1994)), financial portfolio and real investment problems incorporating potential crashes and macroeconomic regime shifts (e.g. Liu and Loewenstein (2013); Guo et al. (2005)) or analyses of innovation dynamics capturing abrupt changes in the market structure due to the adoption of new technologies or the introduction of new products by some market participants (Chronopoulos and Lumbreras (2017); Dawid et al. (2015, 2017)). Whereas all these different problems can be formulated as multi-mode models, crucial differences arise with respect to the way mode transitions are triggered. In particular, the timing and the type of mode transition might be deterministic or stochastic and might by purely exogenous or directly respectively indirectly (through the state dynamics) controlled by some economic agent(s).

In this paper we focus on multi-mode settings with deterministic, controlled mode transitions and strategic interaction. In particular, we are interested in characterizing Markov-perfect-equilibria (MPE) in multi-mode differential games with a finite number of continuously evolving states, the dynamics of which are controlled by the actions of all players, and a set of modes, where the time of the transition between the modes is determined by one of the players, for easier exposition we assume this is player 1. Problems of this kind arise for example in dynamic competition models, like capital accumulation games or dynamic models of reputation formation, where one of the competitors through the introduction of new products or technologies to the market can change the demand structure. Whereas the literature on MPE in timing games with stochastic un-controlled state dynamics is large and well-established (see e.g. Hoppe and Lehmann-Grube (2005); Huisman and Kort (2015); Steg (2018)), there is considerably less work dealing with situations with controlled mode transitions where the players can also influence the state dynamics.[1] Intuitively, characterizing MPE in such a setting generates intricate strategic

[1] Multi-mode games with controlled state dynamics, where however the mode transition is not directly controlled by players but determined by state or time constraints have been analyzed in Reddy et al.

effects. The Markovian strategy determining the time of the mode switch induces a split of the state space in a given mode in regions with and without an immediate jump to an alternative mode.[2] In a setting where all players through their controls can influence the state dynamics this implies that each player might influence the timing of the mode switch and this has to be taken into account when determining the optimal strategies. At the same time, player 1, when determining the optimal mode switching strategy, has to take into account the state dynamics under the equilibrium strategies of all players.

Few contributions have addressed these issues. Closest to our paper is Long et al. (2017), where in a differential game model with multiple regimes, the concept of piecewise-closed loop Nash equilibria (PCNE) is introduced. They consider a two-player multi-mode differential where both players can induce a change of the regime of the game and study piecewise-closed loop Nash equilibria (PCNE). Under this equilibrium concept the state at which a player carries out a mode switch is derived from the condition, that it is optimal for the corresponding player to switch at that point, and the timing of the mode-switches is determined as the point in time when the state variable under the equilibrium controls arrives at that switching state. However, in their setting, it is assumed that firms commit to their switching time in the sense, that they do not alter that time even if the other firm would deviate from its equilibrium control path. Hence, the considered equilibrium is not fully Markov perfect with respect to the timing decision. Also in Dawid and Gezer (2021) a multi-mode differential game with controlled state dynamics and mode switches is considered, but again the strategic interactions arising under Markovian strategies are not fully captured. In particular, the authors consider a strategy space where the players' controls influencing the state dynamics are determined by Markovian strategies, whereas the time of the regime switches is determined with full commitment at time zero and hence can be seen as following an open-loop strategy. To our best knowledge so far no characterizations of full MPE in a setting similar to that considered in this paper are available in the literature.

To obtain first insights into the structure of MPE in multi-mode games with controlled states and mode switches we restrict attention in this paper to a relative simple setting with two players, two modes and a one-dimensional state space. We believe that main

(2015) and Gromov and Gromova (2017).

[2]Such a split is well known from the optimal stopping or the real-options literature, where typically 'continuation regions' and 'stopping regions' are distinguished.

strategic effects occurring in such a setting and the types of equilibria arising can already be seen in such a setup. We derive a set of sufficient conditions for a strategy profile to be a MPE of the game and, based on these conditions, identify different possible types of equilibria. There are *delay equilibria*, with the property that for some initial conditions the mode switch occurs after some finite delay, and *now or never equilibria*, under which the state space is partitioned in two areas with the property that for initial conditions in one area the game stays in the first mode forever, whereas for initial conditions in the other area player 1 already at $t = 0$ induces the switch to the second mode. Furthermore, we show that among the delay equilibria only for a special class of MPE, labeled as *maximum delay equilibria*, a standard smooth pasting condition holds for player 1. Such equilibria always exist if the game has any delay equilibria, but generically in addition to a maximum delay equilibrium there also exists a continuum of delay equilibria in which the mode transition is triggered earlier compared to the maximum delay equilibrium. For these equilibria no smooth pasting conditions holds at the mode switching threshold. Intuitively, the mode switch by player 1 is triggered by the fact that, if the state would cross the switching threshold while the game is in the initial mode, the action of player 2 would jump to an action with strong adversarial effects for player 1. Since under such an equilibrium strategy profile the game never stays for a positive amount of time at a state above the threshold in the first mode, any action of player 2 in this part of the state space and the first mode can be supported in a MPE. These arguments show that in the considered settings the fact that a strategy profile constitutes a Markov-perfect equilibrium does not prevent the occurrence of what could be described as 'incredible threats' in the sense that a player's strategy prescribes an action which would be suboptimal for the player to implement if forced to do so for a time interval with positive measure.

We illustrate these findings by studying the optimal timing of new product introduction for a producer (firm 1) facing on an established market a competitor (firm 2), which has not developed a new product yet and therefore does not have the option of a new product introduction. The new product introduction corresponds to a mode change and the state variable of the considered differential game is the appeal of the new product with consumers, which is influenced by the two firms through advertising (firm 1) and negative campaigning (firm 2).[3] Furthermore, the market introduction of the new product is

[3]Our modeling approach is embedded in a large literature using differential game models to study

associated with lump-sum costs. We show that for a wide range of values of the market introduction costs all three types of equilibria mentioned above co-exist, and therefore for a substantial subset of initial values of the new product's market appeal no clear cut prediction about the time of market introduction of the new product is possible. Furthermore, we determine a threshold such that for values of the market introduction costs above this value only now or never equilibria exist. Overall, this analysis highlights that multi-mode timing games of this kind almost generically give rise to multiplicity of Markov-perfect-equilibria. A potential implication of this insight is that more refined equilibrium concepts than MPE should be considered to analyze such games.

The remainder of the paper is organized as follows. In Section 2 we introduce the type of multi-mode games we are considering and in Section 3 derive necessary conditions for MPE in such a setting and also some additional results on equilibrium profiles. In Section 4 we study the optimal timing of new product introduction, thereby illustrating our general findings. A discussion of our results and conclusions are provided in Section 5. The Appendix contains all proofs and some additional analysis.

2 The Model

We consider a differential game between two players $i = 1, 2$, in which each player intends to maximize an infinite horizon discounted payoff stream of the form

$$J_i = \int_0^\infty e^{-rt} F_i(x(t), u(t), m(t))dt - \mathbb{1}_{[i=1]} e^{-r\tau} \kappa, \tag{1}$$

where $x \in \mathbb{R}$ is a 1-dimensional state and $m(t) \in \{m_1, m_2\}$ is the mode of the game. The interval $X = [x_l, x_u]$ is the state-space of the game. Here, $\mathbb{1}$ denotes the indicator function and $\tau = \inf\{t \geq 0 | m(t) = m_2\}$ is the point in time in which the mode process moves from m_1 to m_2. In case no transition to m_2 occurs we set $\tau = \infty$. At time τ transition costs of $\kappa \geq 0$ arise for player 1. The vector $u = (u_1, u_2)$ denotes the controls of both players with $u_i \in \mathcal{U}_i \subseteq \mathbb{R}^n_i$. We assume that $F_i(x, u, m)$ is continuous and differentiable with respect to x and u for each $m \in \{m_1, m_2\}$. The state evolves according to

$$\dot{x} = f(x, u, m(t)), \quad x(0) = x_{ini} \in X \tag{2}$$

where $f(x, u, m)$ is Lipschitz continuous and differentiable with respect to x and u for all $m \in \{m_1, m_2\}$. The mode process $m(t)$ initially is in $m(0) = m_1$ and is controlled

advertising under dynamic competition, see Jorgensen and Zaccour (2004) for a survey.

by player 1. More precisely, player 1 can determine at which point in time the process jumps from m_1 to m_2. Once $m(t) = m_2$ no additional mode transitions are possible.

In what follows we restrict attention to time-homogeneous Markovian strategies. More precisely, we consider strategy profiles $((\Phi_1(x, m), \Psi(x)), \Phi_2(x, m))$ with $\Phi_i : \mathbb{R} \times \{m_1, m_2\} \to \mathbb{R}^{n_i}$ and $\Psi : \mathbb{R} \to \{0, 1\}$ such that $u_i(t) = \Phi_i(x(t), m(t)), i = 1, 2$ for all t and the process $m(\cdot)$ jumps from m_1 to m_2 at t if and only if $m(\tau) = m_1 \ \forall \tau < t$ and $\Psi(x(t)) = 1$. A Markov Perfect Equilibrium of the game is a strategy profile such that each player solves the dynamic optimization problem defined by (1) and (2) given that the other player determines her control using her equilibrium strategy.

3 Markov Perfect Equilibria

In this section we derive necessary conditions to be satisfied by a Markov Perfect equilibrium profile of the problem described in Section 2. The fact that one of the players controls the transition from mode m_1 to m_2 implies that in addition to standard conditions characterizing the value functions and optimal strategies in each of the two modes the effect of the endogenous timing of the mode transition has to be taken into account. Intuitively, this means that on the one hand the time of the mode transition has to be optimal for player 1, thereby fulfilling standard conditions for optimal stopping problems. On the other hand, for a given mode switching strategy Ψ of player 1, the opponent player 2 can also influence the dynamics of x and thereby the time of the mode switch, by choosing her controls in mode m_1. The interplay of these effects has to be taken into account when characterizing a MPE profile. In what follows we focus on equilibria in which the mode switching strategy Ψ is of threshold type, i.e. there exists a threshold $\bar{x} \in X$ such that $\Psi(x) = 1$ if and only if $x \geq \bar{x}$.[4] We denote such a threshold strategy by $\Psi_{\bar{x}}(x)$. The following set of sufficient conditions characterizes MPE of such threshold type (the proof of this and all following propositions is given in Appendix A).

Proposition 1. *Consider a multi-mode differential game described in Section 2. If there exists a set of except at $(\bar{x}, m_1)$ everywhere continuous and continuously differentiable value functions $V_i : \mathbb{R} \times \{m_1, m_2\} \to \mathbb{R}$ and a profile of Markovian strategies $((\Phi_1(x, m), \Psi_{\bar{x}}(x)), \Phi_2(x, m))$ such that the conditions below are satisfied, then this profile*

[4] The fact that we assume that the mode switch happens for all states above the threshold is not restrictive, since our equilibrium characterization also applies after transforming the state from x to $-x$.

constitutes to a Markov Perfect Equilibrium of the game ($i = 1, 2$):

i)

$$V_i(x, m_2) = \frac{1}{r}\Big(F_i(x, (\Phi_1(x, m_2), \Phi_2(x, m_2)), m_2) + \\ + \frac{\partial V_i(x,m_2)}{\partial x} f(x, (\Phi_1(x, m_2), \Phi_2(x, m_2)), m_2)\Big) \qquad x \in X,$$

ii)

$$\Phi_i(x, m_2) \in argmax_{u_i \in \mathcal{U}_i}\Big(F_i(x, (u_i, \Phi_j(x, m_2)), m_2) + \\ + \frac{\partial V_i(x,m_2)}{\partial x} f(x, (u_i, \Phi_j(x, m_2)), m_2)\Big) \qquad x \in X, j \neq i,$$

iii)

$$\limsup_{t \to \infty} e^{-rt} V_i(x, m_2) \leq 0, i = 1, 2$$

iv)

$$V_i(x, m_1) = \begin{cases} \frac{1}{r}\Big(F_i(x, (\Phi_1(x, m_1), \Phi_2(x, m_1)), m_1) + \\ \quad + \frac{\partial V_i(x,m_1)}{\partial x} f(x, (\Phi_1(x, m_1), \Phi_2(x, m_1)), m_1)\Big) & x < \bar{x}, \\ V_i(x, m_2) - \mathbb{1}_{[i=1]}\kappa & x \geq \bar{x}, \end{cases}$$

v)

$$\begin{aligned} \lim_{x \to \bar{x}-} V_1(x, m_1) &= V_1(\bar{x}, m_1) \\ \lim_{x \to \bar{x}-} V_2(x, m_1) &\geq V_2(\bar{x}, m_1), \end{aligned}$$

where the inequality for player 2 has to hold as equality if there exists an $\epsilon > 0$ such that $f(x, (\Phi_1(x, m_1), \Phi_2(x, m_1)), m_1) > 0$ for all $x \in (\bar{x} - \epsilon, \bar{x})$.

vi)

$$V_1(x, m_2) - \kappa < V_1(x, m_1), \quad \forall x < \bar{x}$$

vii)

$$r(V_1(x, m_2) - \kappa) > \max_{u_1 \in \mathcal{U}_1}\Big[F_1(x, (u_1, \Phi_2(x, m_1)), m_1) + \\ + \frac{\partial V_1(x,m_2)}{\partial x} f(x, (u_1, \Phi_2(x, m_1)), m_1)\Big], \quad \forall x > \bar{x}$$

viii)

$$\Phi_i(x, m_1) \begin{cases} \in argmax_{u_i \in \mathcal{U}_i}\Big(F_i(x, (u_i, \Phi_j(x, m_1)), m_1) + \\ \quad + \frac{\partial V_i(x,m_1)}{\partial x} f(x, (u_i, \Phi_j(x, m_1)), m_1)\Big) & x < \bar{x},\ j \neq i, \\ = \tilde{\Phi}_i(x) & x \geq \bar{x} \end{cases}$$

for some functions $\tilde{\Phi}_i : [\bar{x}, \infty) \to \mathbb{R}^{n_i},\ i = 1, 2$.

In order to interpret the conditions listed in the proposition, we first observe that conditions (i) and (ii) are standard conditions for a MPE of the game in mode m_2. Since for any value of the state x above the threshold $\bar{x}$ player 1 immediately switches to mode m_2 the value functions in mode m_1 in this part of the state space coincide with that in mode m_2 net of the costs associated to the switch from m_1 to m_2. For values of x below $\bar{x}$ the value function and the equilibrium strategies are characterized by standard Hamilton-Jacobi-Bellman (HJB) equations (see (iv) and (viii))). The boundary condition for these HJB equations are given by the value matching condition in (v) which guarantees that the value function of player 1 is continuous at the threshold $\bar{x}$, whereas the value function of player 2 might exhibit a jump at the threshold $\bar{x}$ at which player 1 switches to mode m_2. However, as we will illustrate below, a non-continuous value function for player 2 can arise only if the the state dynamics in equilibrium is such that the switch to mode m_2 occurs either immediately or never, depending on the initial state, i.e. if the dynamics in mode m_1 leads the state away from the threshold $\bar{x}$ (cf. condition (v)). If the switch does not occur immediately, i.e. in an equilibrium where for some initial states $\tau > 0$, condition (vi) ensures that it is not optimal for player 1 to switch to mode m_2 at any $x \in [x_l, \bar{x})$ and (vii) guarantees that for any $x > \bar{x}$ it is optimal for player 1 to switch immediately to mode m_2.

Finally, we like to point out that the characterization of MPEs provided in Proposition 1 does impose hardly any restrictions on the strategies $\phi_i(x, m_1)$ for $x > \bar{x}$. This implies that in spite of the fact that we consider Markov perfect equilibria, which constitute equilibria of every subgame defined by the current state and mode, the players might use strategies which for $x > \bar{x}$ in mode m_1 induce actions that would not be optimal for that player if implemented for a positive amount of time. This is still optimal because for these states the game immediately switches to mode m_2. We will discuss this issue in more detail below and illustrate that this feature might give rise to a wide range of co-existing equilibria.

For our further discussion it is helpful to distinguish different types of equilibria that can arise in our setting. The main property to be considered is whether the state dynamics under the equilibrium strategies $(\phi_1(x), \phi_2(x))$ for $x < \bar{x}$ in the neighborhood of $\bar{x}$ points towards the threshold $\bar{x}$, i.e. whether $f(x, (\phi_1(x, m_1), \phi_2(x, m_1)), m_1) > 0$ for $x \in (\bar{x} - \epsilon, \bar{x})$. If this condition holds, then for $x^{ini} \in (\bar{x} - \epsilon, \bar{x})$ the game switches to m_2 with some positive delay. We refer to such equilibria as *delay equilibria*. On the other hand, if $f(x, (\phi_1(x, m_1), \phi_2(x, m_1)), m_1) < 0$ for all $x \in (\bar{x} - \epsilon, \bar{x})$ then the game either switches immediately to mode m_2, for $x(0) \geq \bar{x}$, or remains in mode m_1 forever (for $x(0) < \bar{x}$). We denote such equilibria as *now or never equilibria.* Considering delay equilibria, the fact that player 1 has incentives to delay the switch to mode m_2 for any $x \in (\bar{x} - \epsilon, \bar{x})$ induces that the inequality

$$\begin{aligned} r(V_1(x, m_2) - \kappa) \quad &< F_1(x, (\Phi_1(x, m_1), \Phi_2(x, m_1), m_1) + \\ &+ \tfrac{\partial V_1(x, m_2)}{\partial x} f(x, (\Phi_1(x, m_1), \Phi_2(x, m_1), m_1) \end{aligned} \tag{3}$$

holds for $x \in (\bar{x} - \epsilon, \bar{x})$. It should be noted that in general (3) might hold as a strict inequality even in the limit as x converges to $\bar{x}$. In an equilibrium with this property player 1 has strict incentives to delay the switch to mode m_2 for any value of the state x up to $\bar{x}$ and the switch to m_2 at $\bar{x}$ is then triggered by a jump the action of player 2 would exhibit if the game would remain in mode m_1. If player 2 does not trigger the transition to mode m_2 by such a jump of its action, player 1 delays the switch until a value of x is reached where (3) holds as equality. We refer to such an equilibrium as a *maximum delay equilibrium.* It follows directly from (3) and the HJB equation in mode m_1 that in such equilibria a standard smooth pasting condition for player 1 holds at $x = \bar{x}$.

Generally speaking, the actions of both players might exhibit jumps at the point in time when the game switches from mode m_1 to mode m_2. This raises the question under which conditions the controls of the players are continuous at the time of the switch. Given that we consider deterministic models, where for a given strategy profile both player can perfectly predict the time of the switch to mode m_1, intuitively one could think that in equilibrium controls are continuous as long as the marginal effects of the controls on the players instantaneous profits and on the state dynamics is the same in both modes.

More formally we say that a game has *mode independent control effects* if and only if

$$\frac{\partial F_i(\cdot, m_1)}{\partial u_i} = \frac{\partial F_i(\cdot, m_2)}{\partial u_i} \quad \text{and} \quad \frac{\partial f(\cdot, m_1)}{\partial u_i} = \frac{\partial f(\cdot, m_2)}{\partial u_i}, \; i = 1, 2. \tag{4}$$

As will become clear below, even in such games a mode change might induce a jump in the control of some player. In that respect it is important to distinguish between games, depending on whether jumps in the opponent's control directly affect the (current) incentives of the other player or not. In many types of games, including for example different variants of investment games, such a direct effect on the incentives of the other player does not exist. We say a game has *separable control effects* if

$$\frac{\partial F_i(x,(u_i,u_j),m)}{\partial u_i} \quad \text{and} \quad \frac{\partial f(x,(u_i,u_j),m)}{\partial u_i}$$

do not depend on u_j for all $x \in X$ and $m \in \{m_1, m_2\}$.

Even if we consider games that have both of these properties, in general equilibrium profiles, as characterized by Proposition 1, might exhibit jumps in their feedback strategies such that $\lim_{x \to \bar{x}-} \Phi_i(x, m_1) \neq \Phi_i(x, m_2)$. In case of now or never equilibria the trajectories of the actions of both players are nevertheless continuous over time, since no switch from m_1 to m_2 occurs for $t > 0$. However, in the case of delay equilibria, as will be illustrated below, in general the actions of both player jump at the point in time when the modes switches to m_2. The only exception in this respect is the maximum delay equilibrium, for which the following proposition shows that the action of player 1 is continuous over time although the mode changes at some positive t from m_1 to m_2.

Proposition 2. *If* $((\Phi_1(x,m), \Psi_{\bar{x}}(x)), \Phi_2(x,m))$ *is a maximum delay equilibrium profile of a game with mode independent and separable control effects such that in each mode* m *the strategies* $\Phi_i, i = 1, 2$ *are continuous with respect to the state* x *and the right hand side of condition (ii) in Proposition 1 has a unique maximizer at* $x = \bar{x}$. *Then,*

$$\lim_{x \to \bar{x}-} \Phi_1(x, m_1) = \Phi_1(x, m_2).$$

The intuition for this result is straight forward. For the maximal delay equilibrium the smooth pasting condition holds for player 1 at the threshold $\bar{x}$. This ensures that the slope with respect to the state of her value function is identical in modes m_1 and m_2 at this point. If the marginal effects of the own control on the instantaneous payoff and the state dynamics is not affected directly by the change in mode and by the potential change in the other player's control due to the mode switch, then the optimization problem which player 1 faces at state $\bar{x}$ is equivalent, no matter whether the firm is in mode m_1 or in m_2. Hence, its equilibrium feedback strategy is continuous at $\bar{x}$.

Concerning the investment of player 2, who cannot directly control the switch from mode m_1 to m_2, in general jumps in the control occur as the game moves from mode m_1 to mode m_2 in all types of delay equilibria. In equilibrium player 2 perfectly predicts the time of the mode switch and also the value of its investment in mode m_1 for her future profits in mode m_2. Therefore, at first sight the discontinuity of the investment of player 2 might be surprising, in particular in the maximum delay equilibrium in which the control of player 1 is continuous. The jump in the action of player 2 results from the fact that in mode m_1 player 2 can influence the time of the switch to mode m_2, since the choice of her control affects the dynamics of the state and thereby the time the state hits the threshold $\bar{x}$ determined by the equilibrium strategy of player 1. This effect, which influences the optimal choice of the control of player 2, immediately disappears, at the point in time when the mode changes to m_2. Hence, the equilibrium strategy of player 2 in general exhibits a jump at the state $\bar{x}$ even in games with mode independent and separable control effects.

In delay equilibria which are different from the maximal delay equilibrium, in general the controls of both players jump at the point in time when the game moves from m_1 to m_2. In such an equilibrium player 1 has strict incentives to delay the jump to mode m_2 for all values of the state below the threshold $\bar{x}$, and, differently to the maximal delay equilibrium, this incentive to delay does not converge to zero as x approaches $\bar{x}$. The reason for player 1 to switch to mode m_2 at $\bar{x}$ is that player 2 threatens to discontinuously change its action at $\bar{x}$ if the game would stay in mode m_1. Hence, for a given strategy of player 2, the threshold $\bar{x}$ at which player 1 switches to mode m_2 is de-facto given and, in order to influence the duration of the game in mode m_1, player 1 has to adjust its investment. Similarly to what we described for player 2 above, this effect disappears as soon as the mode is m_2 and hence also the action of player 1 in general exhibits a jump at the mode switch. More formally, in the absence of a smooth pasting condition the value function of player 1 exhibits a kink at $\bar{x}$, and therefore the marginal return of investment for player 1 jumps as the game switches from mode m_1 to m_2 at $\bar{x}$. In the following section we will illustrate these scenarios using a simple example analyzing the optimal timing of new product introduction in a duopoly.

4 An Illustrative Example: Optimal Timing of New Product Introduction

To illustrate our general findings we now consider the timing problem of a firm, denoted as firm 1, which has to decide when to introduce a new product, that it has developed. We assume that the firm is already active producing an established product and competes with a second firm (firm 2) on the market for the established product. Only firm 1 has developed the new product and therefore has the option to introduce that product at any point in time $t \geq 0$.

Before the new product is introduced (mode m_1) the inverse demand for the established product is given by

$$p_o = \alpha_o - (q_{1o} + q_{2o}),$$

where p_o denotes the price of established product and q_{io} the quantity of that product supplied by firm i, $i = 1, 2$. After the introduction of the new product (mode m_2) the inverse demand changes to

$$\begin{aligned} p_o(q_{1o} + q_{2o}, q_{1n}) &= \alpha_o - (q_{1o} + q_{2o}) - \eta q_{1n} \\ p_n(q_{1o} + q_{2o}, q_{1n}) &= \alpha_n^0 + \alpha_n - \eta(q_{1o} + q_{2o}) - q_{1n}. \end{aligned}$$

Here α_n^0 is a minimal value for the reservation price of the new product, whereas $\alpha_n \geq 0$ is a state variable capturing the effects of the efforts of the two competitors to influence the demand for the new product. More precisely, we assume that

$$\dot{\alpha}_n = f(\alpha_n, (u_1, u_2, m) := u_1 - \gamma u_2 - \delta \alpha_n, \; m \in \{m_1, m_2\}, \tag{5}$$

where u_i denotes the effort of firm i, $\gamma > 0$ is a parameter, and δ is the rate by which the effect of firms' effort on demand vanishes. We assume that both controls u_i are non-negative, which means that firm 1, as the (potential) innovator, tries to increase the demand of the new product, whereas the competitor firm 2 might invest in reducing this demand, e.g. by providing information to consumers about negative features of the new product or by negative campaigning. The costs of effort of both firms are given by $\xi_i(u_i) = \frac{c_i}{2} u_i^2$ with $c_i > 0$. Production costs for the established product are assumed to be symmetric across firms and given by $\frac{\nu_o}{2} q_{io}^2, i = 1, 2$ and analogously for the new product firm 1 has production costs $\frac{\nu_n}{2} q_{1n}^2$.

Firms maximize profits by choosing the production quantities as well as the effort u_i at every point in time t. In addition firm 1 decides about the time at which the new product is introduced. The introduction of the new product is associated with lump-sum costs of $\kappa > 0$.

Since the quantity choice in this setting does not have any intertemporal effects, firms at t choose quantities according to the Cournot equilibrium, which depends on the value of $\alpha_n(t)$ for all t after the introduction of the new product. Standard calculations show that before the introduction of the new product we have

$$q_{io}^{m_1} = \frac{\alpha_o}{3 + \nu_o} \quad i = 1, 2,$$

and the market profit of each firm is given by $\left(1+\frac{\nu_o}{2}\right)(q_{io}^{m_1})^2$. Hence, we obtain for the instantaneous profit in mode m_1

$$F_i(\alpha_n, u_i, m_1) = \left(1+\frac{\nu_o}{2}\right)(q_{io}^{m_1})^2 - \xi_i(u_i) \quad i = 1, 2. \tag{6}$$

In order to guarantee that after the introduction of the new product also a positive amount of this good is produced in equilibrium regardless of the value of α_n, we assume that $\alpha_n^0 > \frac{3\eta\alpha_o}{3+\nu_o}$. Furthermore, we restrict to attention scenarios where in equilibrium both firms also sell a positive quantity of the established product. This is true as long as $\alpha_n < \alpha_n^{UB} := \frac{(2+\nu_n)(1+\nu_o)+\eta^2}{\eta(3+2\nu_o)}\alpha_o - \alpha_n^0$.[5] For values of $\alpha_n \in [0, \alpha_n^{UB}]$ we obtain as the equilibrium quantities in mode m_2:

$$\begin{aligned} q_{1o}^{m_2}(\alpha_n) &= \frac{\alpha_o((1+\nu_o)(2+\nu_n)+\eta^2)-\eta(\alpha_n^0+\alpha_n)(3+2\nu_o)}{(1-\eta^2)(6+5\nu_o)+3(\nu_o+\nu_n)+(2+\nu_n)\nu_o^2+4\nu_o\nu_n} \\ q_{2o}^{m_2}(\alpha_n) &= \frac{\alpha_o((1+\nu_o)(2+\nu_n)-2\eta^2)-\eta(\alpha_n^0+\alpha_n)\nu_o}{(1-\eta^2)(6+5\nu_o)+3(\nu_o+\nu_n)+(2+\nu_n)\nu_o^2+4\nu_o\nu_n} \\ q_{1n}^{m_2}(\alpha_n) &= \frac{(1+\nu_o)((\alpha_n^0+\alpha_n)(3+\nu_o)-3\alpha_o\eta)}{(1-\eta^2)(6+5\nu_o)+3(\nu_o+\nu_n)+(2+\nu_n)\nu_o^2+4\nu_o\nu_n} \end{aligned} \tag{7}$$

The corresponding instantaneous profit in mode m_2 is

$$\begin{aligned} F_1(\alpha_n, u_1, m_2) &= \left(1+\frac{\nu_o^2}{2}\right)(q_{1o}^{m_2}(\alpha_n))^2 + \left(1+\frac{\nu_n^2}{2}\right)(q_{1n}^{m_2}(\alpha_n))^2 + 2\eta q_{1o}^{m_2}(\alpha_n)q_{1n}^{m_2}(\alpha_n) - \xi_1(u_1) \\ F_2(\alpha_n, u_2, m_2) &= \left(1+\frac{\nu_o^2}{2}\right)(q_{2o}^{m_2}(\alpha_n))^2 - \xi_2(u_2) \end{aligned} \tag{8}$$

Overall, the dynamic strategic interaction between the two firms constitutes a two-mode differential game of the form considered in Section 2 with the single state α_n evolving according to (5) and the instantaneous profits in mode m_1 given by (6) and those in mode m_2 given by (8). Both firms decide on their effort u_i and additionally firm 1 determines the timing of the switch from m_1 to m_2. In accordance with Section 3 we consider profiles of Markovian strategies of the form $((\Phi_1(\alpha_n, m), \Psi_{\bar{\alpha}_n}(\alpha_n)), \Phi_2(\alpha_n, m))$ and in what follows characterize Markov Perfect Equilibria of this game.

4.1 MPE in mode m_2

In order to characterize the different types of equilibria in our game we first consider the final mode m_2. Since the instantaneous profits given in (8) are quadratic functions of state and controls and the state dynamics (5) is linear, the game in mode m_2 is of

[5] This condition is obtained by considering the condition $p_o(q_{2o}^*, q_{1n}^*) > \eta q_{1n}^*$ and inserting the optimal quantities q_{2o}^*, q_{1n}^* under the assumption that $q_{1o} = 0$.

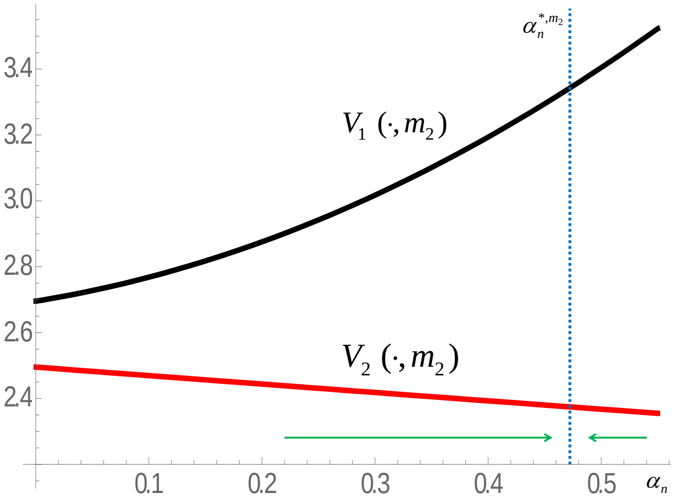


Figure 1: Value functions in mode m_2.

linear quadratic structure. Following a wide range of literature about such games (see e.g. Dockner et al. (2000)) we assume that the game has a MPE with linear feedback strategies[6], giving rise to quadratic value functions of both players. The value and feedback functions can then be determined by a guess and verify approach using the first order conditions and Hamilton-Jacobi-Bellman equations, see Appendix B for details.

Figure 1 shows the value functions of both players in the MPE for our benchmark parameter setting.[7] The green arrows indicate the direction of the state dynamics under the equilibrium feedback strategies. There is a unique stable steady state α_n^{*,m_2} and it is easy to verify that $\alpha_n^{*,m_2} < \alpha_n^{UB}$ under our parameter setting such that both firms sell positive quantities of the old product in the steady state. Figure 1 clearly shows that, as expected, the value of firm 1 increases with α_n, whereas firm 2 is hurt by an increase of the attractiveness of the new product.

4.2 Different types of MPEs in mode m_1

In order to characterize MPE strategies in mode m_1, we first determine the value functions of both players under the extreme scenarios in which the new product, regardless

[6]It should be noted that the game might as well have additional MPEs with non-linear feedback strategies.

[7]Since the purpose of the analysis of this model is to illustrate our theoretical findings we abstain from carrying out a systematic calibration exercise for our model. The parameters have been chosen in a way that they give rise to the different types of equilibria discussed in the previous Section. The values are $\eta = 0.5,\ c_1 = c_2 = 45,\ r = 0.04,\ \gamma = 0.5,\ \alpha_o = 1,\ \alpha_n^0 = 0.5,\ \delta = 0.1,\ \nu_o = 0.4,\ \nu_n = 0.2$.

of the initial condition, is either immediately or never introduced. We denote the value function of firm i in the former case of immediate introduction by $V_i^0(\alpha_n, m_1)$. Clearly we have

$$V_i^0(\alpha_n, m_1) = V_i(\alpha_n, m_2) - \mathbb{1}_{[i=1]}\kappa,$$

where $\mathbb{1}$ denotes the indicator function. If the new product is never introduced, positive investment is never optimal for either firm, i.e. $u_i(t) = 0,\ i = 1, 2,\ t \geq 0$, and hence the value functions read

$$V_i^\infty(\alpha_n, m_1) = \left(1 + \frac{\nu_o}{2}\right) \frac{(q_{io}^{m_1})^2}{r},\ i = 1, 2.$$

Note that V_i^∞ is constant with respect to α_n because the attractiveness of the new product is irrelevant if this product is never introduced. Given that $V_1^0(\alpha_n, m_1)$ increases with α_n, it follows that $V_1^0(0, m_1) > V_1^\infty(0, m_1)$ implies $V_1^0(\alpha_n, m_1) > V_1^\infty(\alpha_n, m_1)\ \forall \alpha \in [0, \alpha_n^{UB}]$ and it is optimal for firm 1 to introduce the new product at some finite point in time t. If this inequality is violated, then in general it might depend on the initial value $\alpha_n(0)$ whether the new product is introduced. In order to gain a better understanding of the potential structure of equilibria under which the new product introduction depends on the initial state, we first characterize the properties of equilibrium state dynamics on $(0, \bar{\alpha}_n)$.

The following proposition shows that under an MPE profile the state α_n is either monotonously increasing or decreasing on the entire interval below the threshold value at which firm 1 introduces the new product (i.e. for $\alpha \in (0, \bar{\alpha}_n)$).

Proposition 3. *If $((\Phi_1(\alpha_n, m), \Psi_{\bar{\alpha}_n}(\alpha_n)), \Phi_2(\alpha_n, m))$ is a MPE profile of the game, then it holds for all $\alpha_n^1, \alpha_n^2 \in (0, \bar{\alpha}_n)$ that*

$$sgn\left[f(\alpha_n^1, (\Phi_1(\alpha_n^1, m_1), \Phi_2(\alpha_n^1, m_1)))\right] = sgn\left[f(\alpha_n^2, (\Phi_1(\alpha_n^2, m_1), \Phi_2(\alpha_n^2, m_1)))\right].$$

The observation that in equilibrium the state α_n is either strictly increasing or strictly decreasing on the entire interval $(0, \bar{\alpha}_n)$ has several important implications. If in a given equilibrium the new product is introduced after a positive and finite delay for some initial value $\alpha_n(0)$, then under this equilibrium profile the product is introduced after finite time regardless of the initial state $\alpha_n(0)$. Conversely, if in a given equilibrium the product is never introduced for some initial value of the new market size, then under this equilibrium a positive finite delay in product introduction can never occur, regardless of $\alpha_n(0)$. Relating to the different types of equilibria introduced in Section 3, the Proposition shows that the properties 'delay equilibrium' respectively 'now or never equilibrium', which were defined locally around the threshold $\bar{\alpha}_n$, are actually global properties in the sense that in any delay equilibrium the switch to m_2 occurs after a positive finite delay for all $\alpha_n \in [0, \bar{\alpha}_n)$, whereas in any now or never equilibrium the switch never occurs for any initial value of the state below the threshold.

In what follows we illustrate our general results from Section 3 by showing that, depending on the size of the new product introduction costs κ, qualitatively different types of MPE constellations exist in in our model.

4.2.1 Small costs of market introduction

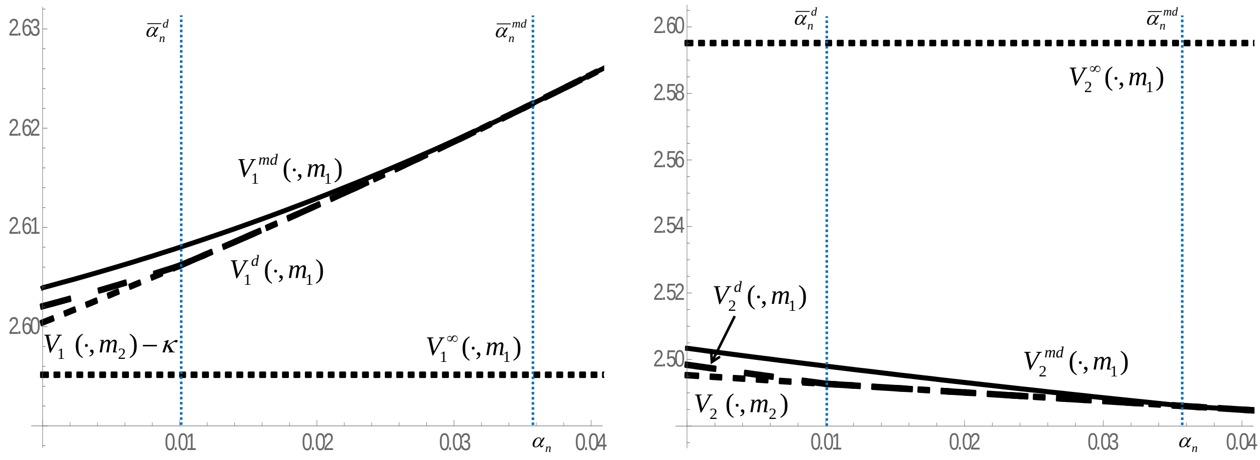
If κ is sufficiently small, i.e. $\kappa < \underline{\kappa} := V_1(0, m_2) - V_1^{\infty}(0, m_1)$ then it follows directly that immediate introduction is more profitable for Firm 1 than no introduction regardless of the value of α_n.[8] Hence, in equilibrium the new product is always introduced. Concentrating first on a maximum delay equilibrium, we need to characterize the threshold value of $\bar{\alpha}_n$, above which the product is introduced. Denoting the threshold in the maximum delay equilibrium by $\bar{\alpha}_n^{md}$, we obtain that the following condition have to be satisfied for some positive value of $u_2^{m_1}$ (see Appendix C):

$$\begin{aligned} r\left(V_1(\bar{\alpha}_n^{md}, m_2) - \kappa\right) &= F_1(\bar{\alpha}_n^{md}, \Phi_1(\bar{\alpha}_n^{md}, m_2), m_1) + \frac{\partial V_1(\bar{\alpha}_n^{md}, m_2)}{\partial \alpha_n} f(\bar{\alpha}_n^{md}, \Phi_1(\bar{\alpha}_n^{md}, m_2), u_2^{m_1}), \\ rV_2(\bar{\alpha}_n^{md}, m_2) &= F_2(\bar{\alpha}_n^{md}, u_2^{m_1}, m_1) - \frac{c_2 u_2^{m_1}}{\gamma} f(\bar{\alpha}_n^{md}, \Phi_1(\bar{\alpha}_n^{md}, m_2), u_2^{m_1}) \end{aligned} \quad (9)$$

Whereas the first equation captures the smooth pasting condition, the second is the HJB equation of firm 2 at α_n^{md}. To obtain these equations we have used the fact that the equilibrium value function of firm 2 has to coincide between the two modes for all $\alpha_n \geq \bar{\alpha}_n$ (see Proposition 1) and that the control of firm 1 is identical in both modes for $\alpha_n = \bar{\alpha}_n^{md}$ since all conditions of Proposition 2 are satisfied in our model.

For our benchmark parametrization together with market introduction costs of $\kappa = 0.095$ the system (9) has a unique positive solution given by $\bar{\alpha}_n^{md} = 0.036$ and $u_2^{m_1} = 0.0047$. The corresponding maximum delay equilibrium is illustrated in Figures 2 and 3(a). The solid lines in the two panels of Figure 2 show the value functions of the two firms in mode m_1 under such an equilibrium. More precisely, these functions have been determined as the solutions of the HJB equations for mode m_1 on $[0, \bar{\alpha}_n^{md}]$ under

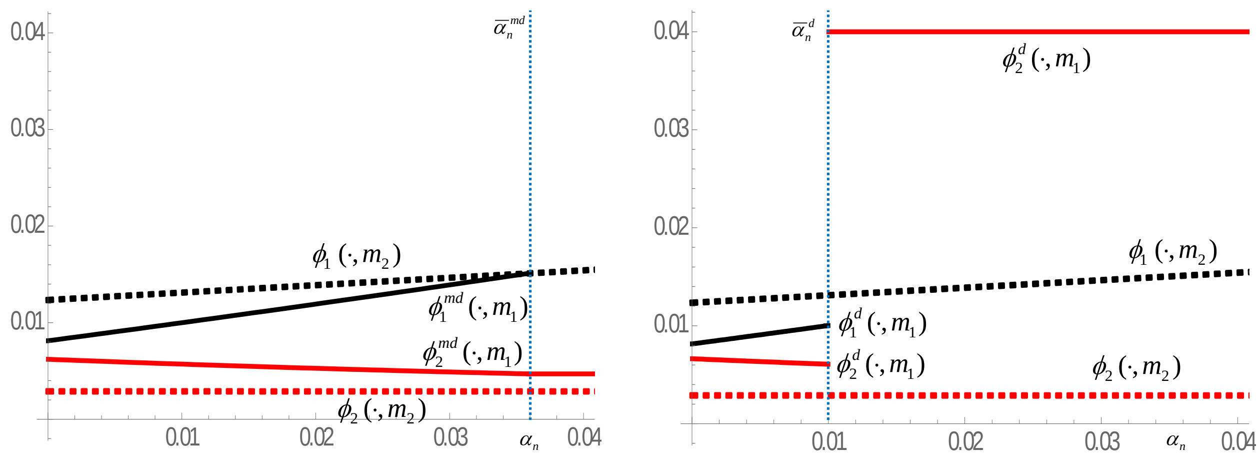
[8] For our benchmark parameter constellation, we have $V_1(0, m_2) = 2.7$ and $V_1^{\infty}(0, m_1) = 2.6$ such that for all values $\kappa < 0.1$ this condition is fulfilled.



(a) Value functions of firm 1 (b) Value functions of firm 2

Figure 2: Value functions of firm 1 (left panel) and firm 2 (right panel) under the maximum delay equilibrium (solid line), a delay equilibrium with $\bar{\alpha}_n = \bar{\alpha}_n^d = 0.01$ (coarsely dashed line) and immediate introduction of the new product (dashed line). The dotted line indicates the value function if the new product is never introduced ($\kappa = 0.095$).

the boundary conditions that $V_i(\bar{\alpha}_n^{md}, m_1) = V_i(\bar{\alpha}_n^{md}, m_2) - \mathbb{I}_{[i=1]}\kappa, i = 1, 2$. It can be easily checked that these value functions in combination with feedback functions derived from the maximization of the right hand side of the HJB equation on $[0, \bar{\alpha}_n^{md}]$ satisfy all conditions of Proposition 1 for any profile (Φ_1, Φ_2) satisfying $\Phi_2(\alpha_n, m_1) \geq \Phi_2(\bar{\alpha}_n^{md}, m_1)$ for all $\alpha_n \geq \bar{\alpha}_n^{md}$ (in order to ensure condition (vii) of Proposition 1). The equilibrium investment functions under the maximum delay equilibrium are shown in Figure 3(a). It can be clearly seen that the investment of firm 1 in mode m_1 (solid black line) intersects with the equilibrium investment in mode m_2 (black dashed line) exactly at the threshold $\bar{\alpha}_n^{md}$ above which firm 1 immediately introduces the new product. We do not show any values for the function $\Phi_1(\alpha_n, m_1)$ for $\alpha_n \geq \bar{\alpha}_n^{md}$ since any choice of the function on this interval is compatible with equilibrium. The figure also illustrates the downward jump of the investment of firm 2 at the threshold $\alpha_n = \bar{\alpha}_n^{md}$ where the mode switches from m_1 to m_2. In mode m_1, there is an additional incentive for player 2 to invest. Such investment decreases the speed of the increase of α_n and delays the point in time when the state variable arrives at $\bar{\alpha}_n^{md}$ and the new product is introduced. Firm 2 has an incentive to delay the new product introduction, which provides additional investment incentives. Once the new product is introduced, this additional incentive vanishes which



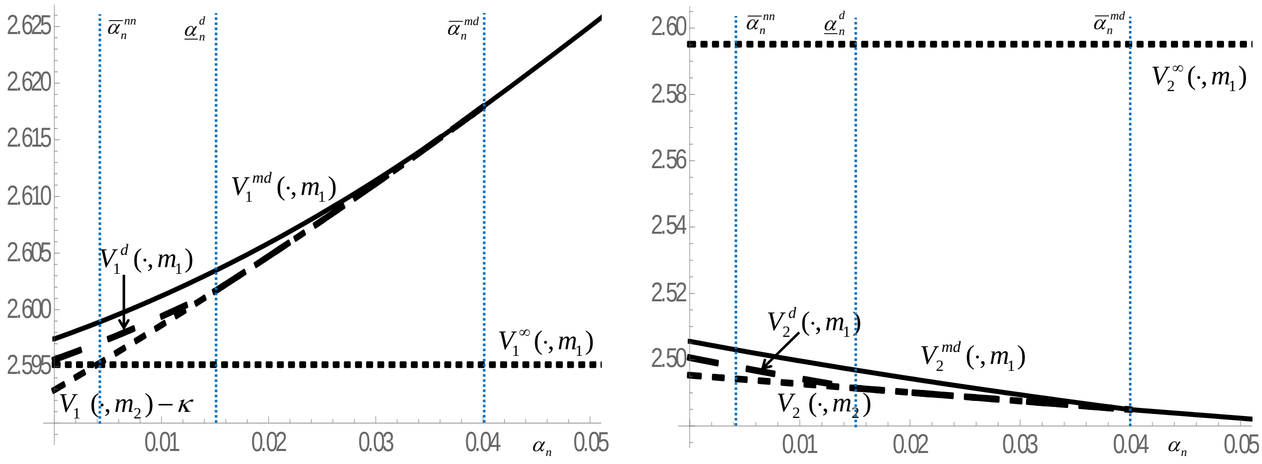
(a) Maximum delay equilibrium

(b) Delay equilibrium

Figure 3: Investment strategies of firm 1 (black) and firm 2 (red) in the maximum delay equilibrium and the a delay equilibrium with $\bar{\alpha}_n = \bar{\alpha}_n^d = 0.01$ ($\kappa = 0.095$). Strategies in mode m_1 are indicated by solid lines, those in mode m_2 by dashed lines.

results in a downward jump in player 2's investment.

However, the maximum dealy equilibrium is not the only MPE in our setting. Actually, for any threshold $\bar{\alpha}_n^d \in [0, \bar{\alpha}_n^{md}]$ there is an MPE such that firm 1 introduces the new product immediately for all $\alpha_n \geq \bar{\alpha}_n^d$. In Figure 2 the value functions corresponding to such an equilibrium with $\bar{\alpha}_n^d = 0.01$ are illustrated with coarsely dashed lines. It can be clearly seen that both value functions have a kink at $\alpha_n = \bar{\alpha}_n^d$ such that also for firm 1 the smooth pasting condition does not hold at this threshold where the firm introduces the new product. The equilibrium feedback function corresponding to this MPE are shown in Figure 3(b).The figure illustrates that the investments of both firms jump at the point in time when the new product is introduced, where the jump is upwards for firm 1 and downwards for firm 2. Whereas the intuition for the downward jump of firm 2 is analogous to that developed for the maximum delay equilibrium, the upward jump for firm 1 is due to the fact that after the introduction of the new product an increase of α_n has a positive impact on the instantaneous profit of firm 1, whereas in mode m_1 such an increase only has impact on the remaining time till the new product is introduced. If the new product is introduced at a level of α_n where the value function in mode m_2 is still strictly steeper than that in mode m_1 the investment incentives jump upwards at the point of market introduction. A crucial feature of this equilibrium profile is that $\Phi_2(\alpha_n, m_1)$ for $\alpha_n \geq \bar{\alpha}_n^d$ is sufficiently large such that it is optimal for firm



(a) Value functions for player 1

(b) Value functions for player 2

Figure 4: Value functions of firm 1 (left panel) and firm 2 (right panel) under the maximum delay equilibrium (solid line), the equilibrium with minimal delay (coarsely dashed line) and immediate introduction of the new product (dashed line). The dotted line indicates the value function if the new product is never introduced ($\kappa = 0.1025$).

1 to introduce the new product immediately at $\bar{\alpha}_n^d$. In our example this is guaranteed by setting $\Phi_2(\bar{\alpha}_n^d, m_1) > \frac{1}{\gamma}\Phi_1(\bar{\alpha}_n^d, m_1)$, which implies that under optimal investment by firm 1 the state variable α_n does not move above $\bar{\alpha}_n^d$. As discussed in Section 3, although such high investments by firm 2 would not be optimal, if it were to be carried out for a time interval with positive measure, in equilibrium the firm is never required to carry out such investment, regardless of the value of $\alpha_n(0)$.

4.2.2 Intermediate costs of market introduction

We now consider the case where $\kappa > \underline{\kappa}$, which implies that for $\alpha_n(0) = 0$ immediate introduction of the new product yields a lower value for firm 1 than never introducing the new product and abstaining from any investment into the build-up of α_n. Although immediate introduction of the new product is not optimal, an introduction with some delay might still be more profitable than no introduction. Taking into account that the largest value for firm 1 is obtained under the maximum delay equilibrium, $V_1^{md}(0, m_1) > V_1^{\infty}(0, m_1)$ is a necessary and sufficient condition for the existence of delay equilibria in our setting. It should be noted that $V_1^{md}(0, m_1)$ is a decreasing function of κ such that this condition implies an upper bound $\bar{\kappa}$ for the costs of market introduction such that for all $\kappa \leq \bar{\kappa}$ there exists an equilibrium such that for all $\alpha_n(0) \geq 0$ the new product is introduced to the market after finite time. For this subsection we assume that $\kappa \in [\underline{\kappa}, \bar{\kappa}]$, i.e. the market introduction costs are in an intermediate range such that for $\alpha_n(0) = 0$ immediate introduction is not optimal but there exists an equilibrium which induces new product introduction with some delay for this initial value.

In Figure 4 this case is illustrated by showing the value functions for immediate market introduction (dashed lines), the maximum delay equilibrium (solid line) and no market introduction (dotted line). We denote by $\bar{\alpha}_n^{nn}$ the largest value of α_n such that $V_1^\infty(\alpha_n, m_1) \geq V_1(\alpha_n, m_2) - \kappa$. Clearly, for all values of $\alpha_n < \bar{\alpha}_n^{nn}$ immediate market introduction is not optimal, therefore the range of threshold values $\bar{\alpha}_n^d$ for which delay equilibria can exist is restricted to $\bar{\alpha}_n^d \in [\bar{\alpha}_n^{nn}, \bar{\alpha}_n^{md}]$. However, for a delay equilibrium to exist, the associated value function must also satisfy $V_1^d(0, m_1; \bar{\alpha}_n^d) \geq V_1^\infty(0, m_1)$, where we denote by $V_1^d(0, m_1; \bar{\alpha}_n^d)$ the value function of firm 1 under a (candidate for a) delay equilibrium with threshold $\bar{\alpha}_n^d$. If this inequality does not hold condition (ii) in Proposition 1 would be violated at $\alpha_n = 0$. This can be seen by realizing that by choosing $u_i = 0$ for $\alpha_n = 0$ we have $\dot{\alpha} = 0$ and the value of the right hand side of the HJB equation becomes

$$\left(1 + \frac{\nu_o}{2}\right) (q_{io}^{m_1})^2 = rV_1^\infty(0, m_1) > rV_1^d(0, m_1; \bar{\alpha}_n^d).$$

Since under the equilibrium feedback functions in the delay equilibrium the right hand side of the HJB equation has to be equal to $rV_1^d(0, m_1; \bar{\alpha}_n^d)$ (see condition (i) in Proposition 1) the feedback function $\Phi^d(\alpha_n, m_1; \bar{\alpha}_n^d)$ in that equilibrium candidate does not maximize the right hand side of the HJB equation. This shows that no delay equilibrium with $V_1^d(0, m_1; \bar{\alpha}_n^d) < V_1^\infty(0, m_1)$ can exist with the property that the new product is introduced after a positive delay for $\alpha_n(0) = 0$. Furthermore, we know from Proposition 3 that in any equilibrium the sign of the state dynamics cannot change on the interval $[0, \bar{\alpha}_n]$. This rules out any equilibrium profile under which the new product is never introduced (and hence α_n decreases over time) for a small value of $\alpha_n(0)$, but is introduced after a delay for larger values of $\alpha_n(0)$. Put together these arguments establish that for any threshold $\bar{\alpha}_n^d$ with $V_1^d(0, m_1; \bar{\alpha}_n^d) < V_1^\infty(0, m_1)$ no delay equilibrium can exist. However, there exists a continuum of delay equilibria where at the market introduction threshold this inequality is violated. In particular, there exists a unique $\underline{\alpha_n^d} \in (\bar{\alpha}_n^{nn}, \bar{\alpha}_n^{md}]$ such that $V_1^d(0, m_1; \bar{\alpha}_n^d) \geq V_1^\infty(0, m_1)$ for all $\bar{\alpha}_n^d \in [\underline{\alpha}_n^d, \bar{\alpha}_n^{md}]$. To see this note that since the smooth pasting condition is satisfied at $\bar{\alpha}_n^{md}]$ for all $\alpha < \bar{\alpha}_n^{md}]$ a later

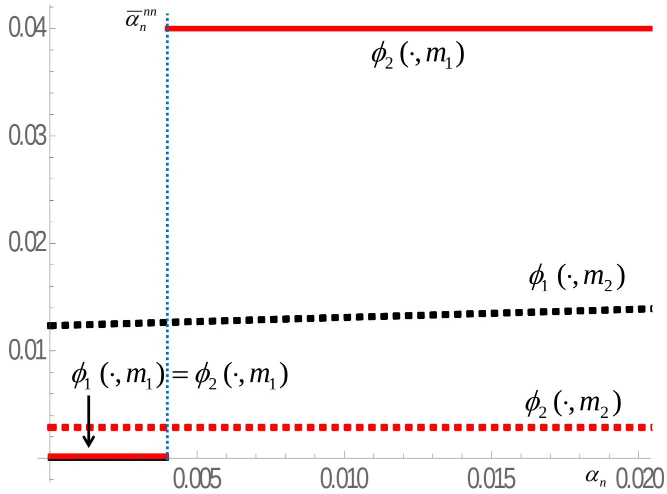


Figure 5: Equilibrium investment strategies under the MPE with 'now or never' product introduction ($\kappa = 0.1025$). The strategies of player 2 are depicted in red.

switch to m_2 increases the value for Firm 1. Hence, $V_1^d(0, m_1; \bar{\alpha}_n^d)$ is an increasing and continuous function of $\bar{\alpha}_n^d$. Furthermore, in light of the definition of $\bar{\alpha}_n^{nn}$ we have

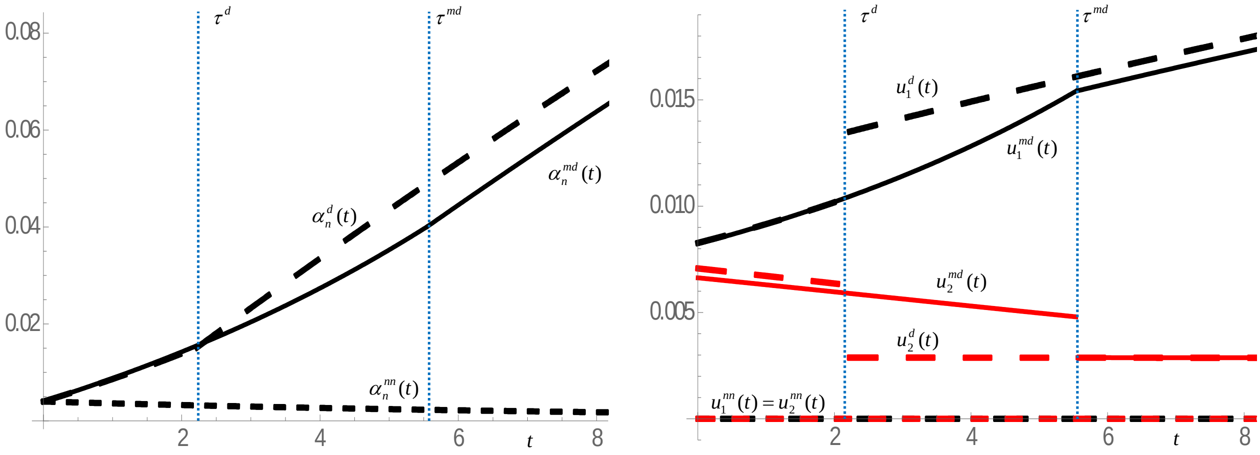
$$V_1^d(0, m_1; \bar{\alpha}_n^{nn}) < V_1^d(\bar{\alpha}_n^{nn}, m_1; \bar{\alpha}_n^{nn}) = V_1(\bar{\alpha}_n^{nn}, m_2) - \kappa = V_1^\infty(\bar{\alpha}_n^{nn}, m_1)$$

Since without market introduction of the new product the value of firm 1 does not depend on α_n, we have $V_1^\infty(0, m_1) = V_1^\infty(\bar{\alpha}_n^{nn}, m_1) > V_1^d(0, m_1; \bar{\alpha}_n^{nn})$. Since $\kappa < \bar{\kappa}$ we also have $V_1^\infty(0, m_1) < V_1^d(0, m_1; \bar{\alpha}_n^{md})$ and the intermediate value theorem implies the existence of $\underline{\alpha}_n^d$ with $V_1^d(0, m_1; \underline{\alpha}_n^d \alpha_n^d) = V_1^\infty(0, m_1)$. In the two panels of Figure 4 we illustrate this observation by showing, as coarsely dashed lines, the value functions of the two firms under the delay equilibrium with threshold $\bar{\alpha}_n^d = \underline{\alpha}_n^d$. For any threshold value $\bar{\alpha}_n^d$ in the interval $[\underline{\alpha}_n^d, \bar{\alpha}_n^{md}]$ a delay equilibrium profile can be constructed in the same way as discussed in the previous subsection.

However, in the case of intermediate values of market introduction costs considered here, there also exists a now or never equilibrium, as discussed in Section 3. Similarly to the delay equilibria, also this equilibrium is characterized by a threat of a strong investment of player 2 in m_1 as soon as the state variable α_n is larger or equal than $\bar{\alpha}_n^{nn}$. Hence, firm 1 introduces the product immediately if $\alpha_n \geq \bar{\alpha}_n^{nn}$. As discussed above, under a potential delay equilibrium in which firm 1 invests positive amounts we would have

$$V_1^d(\alpha_n, m_1; \bar{\alpha}_n^{nn}) < V_1^d(\bar{\alpha}_n^{nn}, m_1; \bar{\alpha}_n^{nn}) = V_1^\infty(\bar{\alpha}_n^{nn}, m_1) = V_1^\infty(\alpha_n, m_1)$$

for all $\alpha_n < \bar{\alpha}_n^{nn}$ and therefore setting $\Phi_1(\alpha_n, m_1) = 0$ is the optimal strategy for firm 1 on



(a) Dynamics of α_n (b) Dynamics of Investments

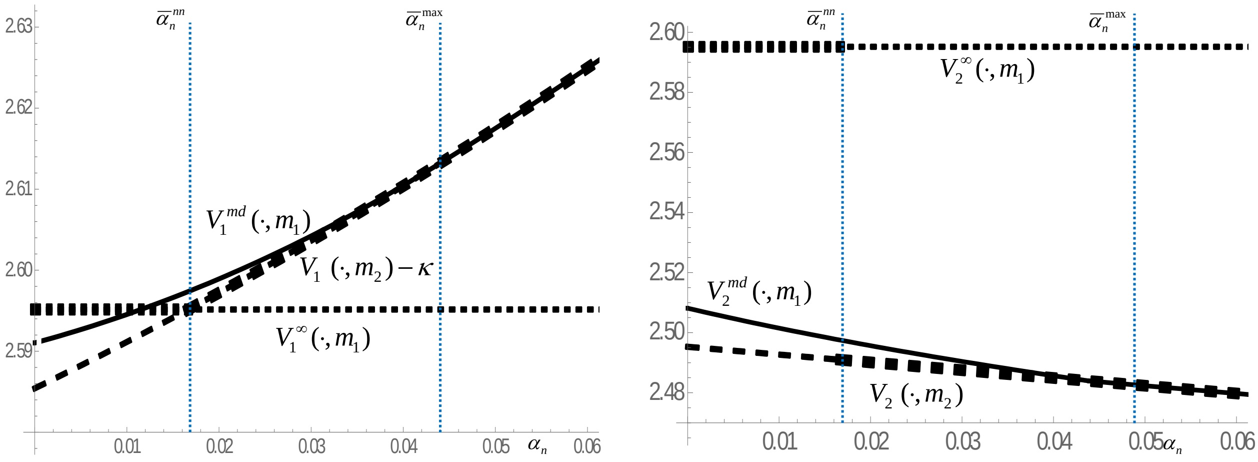
Figure 6: Dynamic of the attractiveness of the new product (a) and the investments of both firms (b) for $\alpha_n(0) = \bar{\alpha}_n^{nn} = 0.004$ under the maximum delay equilibrium (solid line), the equilibrium with minimal delay ($\bar{\alpha}_n^d = 0.015$, dashed line) and the now or never equilibrium (dotted line) for $\kappa = 0.1025$.

this interval. Hence, in equilibrium we also must have $\Phi_2(\alpha_n, m_1) = 0$ for all $\alpha_n < \bar{\alpha}_n^{nn}$. We obtain a MPE profile under which for all $\alpha_n(0) < \bar{\alpha}_n^{nn}$ both firms invest nothing and the new product is never introduced, whereas for $\alpha_n(0) \geq \bar{\alpha}_n^{nn}$ the product is introduced at $t = 0$ and both firms afterwards invest according to the MPE strategies in mode m_2. The equilibrium feedback functions underlying this equilibrium are illustrated in Figure 5, where, as in Figure 3, we do not show any values of $\Phi_1(\alpha_n, m_1)$ for $\alpha_n \geq \bar{\alpha}_n^{nn}$ since any choice is compatible with an equilibrium profile.

Figure 6 illustrates the dynamics emerging under the different types of equilibria that co-exist for an intermediate level of the market introduction costs. In particular, the trajectories of the market attractiveness (α_n) and of the investments of both firms (u_1, u_2) is depicted for a small initial value of the market attractiveness. The dotted lines, corresponding to the now or never equilibrium shows that in such an equilibrium, due to absence of any investments the attractiveness of the new product decreases towards zero and the product is never introduced. The dashed and the solid lines correspond to the dynamics under the delay equilibrium with the minimal possible threshold, i.e. $\bar{\alpha}_n^d = \underline{\alpha_n^d}$, and the maximum delay equilibrium. The points in time at which the new product is introduced under the delay and the maximum delay equilibrium are denoted by τ^d and τ^{md}, where $\tau^d < \tau^{md}$. The figure illustrates that under both equilibria the state α_n increases over time. In the time interval $[0, \tau^d]$ the increase is slower under the delay equilibrium than under the maximum delay equilibrium, because the investments of firm 2, which decrease the speed of growth of α_n, are larger under the delay than under the maximum delay equilibrium. The investments of firm 1, fostering the growth of α_n, are virtually identical under both equilibria. Intuitively, firm 2 invests more under the delay equilibrium, because the downward jump of its instantaneous profit associated with the switch to mode m_2 is much closer time-wise in the delay equilibrium than in the maximum delay equilibrium and therefore less heavily discounted. Hence, the incentive to invest in delaying the switch is stronger in the delay equilibrium. For firm 1, no such effect occurs and the incentive to invest during mode m_1 is hardly affected by the type of the delay equilibrium. Under the delay equilibrium, both controls exhibit a jump at τ^d with the investment of firm 1 jumping upwards and those of firm 2 jumping downwards. As a result, α_n grows faster after τ^d compared to the time before the new product introduction and we observe a higher attractiveness of the new market under the delay equilibrium than under the maximum delay equilibrium. It should however be noticed that under both equilibria the state α_n converges to the steady state in mode m_2 (α_n^{*,m_2}) and therefore this difference between the two equilibria disappears in the long run. Finally, Figure 6(b) illustrates again that in the maximum delay equilibrium the investment of firm 1 is continuous throughout the entire trajectory, i.e. also at period τ^{md} at which the mode switches from m_1 to m_2.

4.2.3 Large costs of market introduction

If market introduction costs are large, i.e. $\kappa > \bar{\kappa}$ then we have $V_1^{md}(0, m_1) < V_1^{\infty}(0, m_1)$, which means that any candidate for a delay equilibrium yields a smaller value for firm 1 at $\alpha_n(0) = 0$ than not investing in the build-up of α_n and never introducing the new product. We illustrate the value functions of both firms for this case in Figure 7. Due to $V_1^{md}(0, m_1) < V_1^{\infty}(0, m_1)$ there can be no equilibria where for the initial condition $\alpha_n(0) = 0$ firm 1 invests in the build-up of α_n and eventually introduces the new product. Using the same arguments as developed in the previous subsection, this rules out the existence of any MPE under which the product is introduced with delay for some initial value $\alpha_n(0)$. Hence, the only equilibrium that still exists in this scenario is the now or never equilibrium. The value functions of both firms corresponding to this equilibrium are indicated in bold in Figure 7. It should be noted that under this equilibrium value



(a) Value function for player 1

(b) Value function for player 2

Figure 7: Value functions of firm 1 (left panel) and firm 2 (right panel) under the candidate for a maximum delay equilibrium (solid line), the immediate introduction of the new product (dashed line) and if the new product is never introduced. The value functions under the now or never equilibrium are indicated in bold ($\kappa = 0.11$).

of the game for firm 2 exhibits a downward jump as the initial state $\alpha_n(0)$ crosses the threshold $\bar{\alpha}_n^{nn}$ (see panel (b)).

5 Discussion and Conclusions

Our analysis of a simple model of dynamic competition under potential new product introduction illustrates the properties of all three types of equilibria identified in Section 3 and also shows that delay equilibria, maximum delay equilibra and now or never equilibria, might co-exist for some parameter constellations. More precisely, we have demonstrated that the following three scenarios arise:

(i) $\kappa \leq \underline{\kappa}$: There exists a maximum delay equilibrium with product introduction at $\alpha = \alpha_n^{md}$ and a continuous investment path of player 1 across both modes of the game. Furthermore, for any $\tilde{\alpha}_n \in [0, \alpha_n^{md})$ there exists a MPE with new product introduction at $\tilde{\alpha}_n$. In each of these equilibria the investments of both players jump as the mode switches from m_1 to m_2.

(ii) $\underline{\kappa} < \kappa \leq \bar{\kappa}$: There exists a maximum delay equilibrium plus for each $\bar{\alpha}_n^d \in [\underline{\alpha}_n^d, \bar{\alpha}_n^{md})$ there is also a delay equilibrium. Furthermore, a now or never MPE exists.

(iii) $\kappa > \bar{\kappa}$: Whereas strategy profiles inducing market introduction with delay for some initial value of α_n do not constitute a MPE, the now or never equilibrium still exists.

In scenario (iii) for each initial condition a unique prediction about the occurrence and timing of the new product introduction can be made if we restrict attention to the MPE analyzed here. This is not true in the first two cases. In case (ii) there is a set of initial conditions $\alpha_n(0) \in [0, \bar{\alpha}_n^{nn}]$ such that MPEs under which the new product is eventually introduced co-exist with the now or never equilibrium under which no introduction of the new product occurs. For $\alpha_n(0) > \bar{\alpha}_n^{nn}$ the new product is introduced under all existing equilibria although the time of the introduction varies across equilibria. Similarly, under scenario (i) the product is eventually introduced for all initial conditions, where the time of the introduction depends on the chosen equilibrium.

Whereas, all equilibria considered here are Markov perfect, intuitively the delay equilibria seem less plausible compared to the maximum delay and the now or never equilibrium. As discussed above, the equilibrium profiles underlying these equilibria include strategies for firm 2, under which in mode m_1 for $\alpha_n \geq \bar{\alpha}_n^d$ investments are chosen which would not be optimal if the game would stay in mode m_1 for an amount of time with positive measure even after the state α_n has crossed the threshold $\bar{\alpha}_n^d$. Since in equilibrium the state is never larger or equal than α_n^d in mode m_1 for a positive amount of time, this feature does not contradict the optimality of the strategy of firm 2. However, it nevertheless constitutes an 'incredible threat' in the sense that it would not be optimal for firm 2 to stick to this investment strategy in case firm 1 would deviate from its own equilibrium strategy by increasing the threshold at which it switches to mode m_2.

A main insight of our analysis is that a strategy profile might be Markov perfect even though the strategy of some players induces actions in some parts of the state space which would not be optimal for the player if carried out for a positive amount of time. Although this phenomenon has been demonstrated in this paper only for multi-mode games, it seems that it might arise also in other classes of differential games. For example, in games in which the state dynamics is controlled by potentially singular controls of both players MPEs might exist in which the strategy of one player in a certain region of the state space make it optimal for some other player to induce a jump of the state out of that region by means of a singular control. Similar to our setup, Markov perfection does not put any restrictions on the regular control of that player in the region in which the other player chooses a singular control.[9] Intuitively, under such a MPE profile the state always immediately jumps out of that singular control region such that the amount of time for which the actions induced by the players' strategies in that region are actually implemented has measure zero. Hence, also in the framework of such differential games incredible threats, in the sense that actions are chosen which would be suboptimal if implemented for a positive amount of time, can occur as part of Markov perfect equilibrium profiles. An interesting question for future research might be to explore whether such Markov perfect equilibria with 'incredible threats' in multi-mode or singular control games can be eliminated by appropriate equilibrium refinements. Developing such a refinement would also provide a valuable theoretical basis for alleviating the problem of non-uniqueness of predictions about equilibrium timing of regime switches in applications like the dynamic innovation model considered in this paper.

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[9] This can for example be seen in Theorem 1 of Kwon (2021), characterizing best responses in a dynamic common goods game in continuous time.

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Appendix A: Proofs

Proof of Proposition 1

In order to prove that the considered profile is an equilibrium, we first observe that conditions (i) to (iii) are standard conditions implying that the profile in mode m_2 is a Markov Perfect Equilibrium of the infinite horizon game in this mode (see Theorem 4.4 in Dockner et al. (2000)).

Consider now the optimization problem of player 2 in mode m_1 for a given strategy $\Phi_1(x, m_1), \Psi_{\bar{x}}(x)$ of player 1. Given that player 1 switches to mode m_2 instantaneously for $x > \bar{x}$ the choice of $\Phi_2(x, m_1)$ for all $x > \bar{x}$ does not affect the objective function of firm 2. Hence any choice for $\tilde{\Phi}_2(x)$ is optimal.

We now turn to the problem of player 2 for an arbitrary initial state $x^{ini} \in [x_l, \bar{x})$. Define an auxiliary value function $\hat{V}_2(x)$ on $[x_l, \bar{x}]$ as follows

$$\hat{V}_2(x) = \begin{cases} V_2(x, m_1) & x \in [x_l, \hat{x}), \\ \lim_{x \to \bar{x}-} V_2(x, m_1) & x = \bar{x}. \end{cases}$$

By definition this auxiliary value function is continuous and differentiable on $[x_l, \bar{x}]$ and due to (iv) satisfies the Hamilton Jacobi Bellman equation for the optimal control problem of player 2 on that interval. Standard results show that this is therefore the value function of the auxiliary control problem of player 2, in which player 2 receives the payoff $\lim_{x \to \bar{x}-} V_2(x, m_1)$ once the state hits $\bar{x}$. Furthermore, because of condition (viii), $\Phi_2(x, m_1)$ is the optimal feedback function for player 2 with respect to this auxiliary problem. It should be noted that any control path under which the state does not hit $\bar{x}$ yields the same value for player 2 under the auxiliary and the original problem. Any control path under which the state hits $\bar{x}$, due to condition (v) yields a larger or equal value for player 2 under the auxiliary problem than under the original problem. In particular this implies that, if a path under which $\bar{x}$ is not hit is optimal under the auxiliary problem, it is also optimal under the original problem.

Consider now an arbitrary $x^{ini} \in [x_l, \bar{x})$. If the optimal path under the auxiliary problem, induced by $\Phi_2(x, m_1)$ does not hit $\bar{x}$ then it is also optimal under the original problem. If the optimal path under the auxiliary problem, induced by $\Phi_2(x, m_1)$ hits $\bar{x}$, then we must have $f(x, (\Phi_1(x, m_1), \Phi_2(x, m_1)), m_1) > 0$ for all $x \in (\bar{x} - \epsilon, \bar{x})$ for some $\epsilon > 0$. By condition (v) this implies that $\hat{V}_2(\bar{x}) = V_2(\bar{x}, m_1)$ and therefore the auxiliary problem coincides with the original problem. Hence $\Phi_2(x^{ini}, m_1)$ is optimal for player 2 under the original problem for any $x^{ini} \in [x_l, \bar{x})$.

We now consider the optimal control problem of player 1 in mode m_1 for a given strategy $\Phi_2(x, m_1)$ of player 2. Considering the switching strategy $\Psi_{\bar{x}}(x)$ as given, the same arguments as just applied to player 2 establish the optimality of $\Phi_1(x, m_1)$. Hence, what remains to be shown is that switching to mode m_2 for all $x \geq \bar{x}$ is optimal for player 1 given $\Phi_2(x, m_1)$. It follows from (iv) and (v) and by continuity that any path from the initial state $x^{ini} = \bar{x}$ which stays in the interval $[x_l, \bar{x}]$ yields a value for player 1 that is not larger than $\lim_{x \to \bar{x}-} V_1(x, m_1) = V_1(\bar{x}, m_2) - \kappa$. Hence, switching to mode m_2 at $\bar{x}$ yields at least the same value as any such path staying in $[x_l, \bar{x}]$.

Consider now a potentially optimal path from $\bar{x}$ which stays in mode m_1 and has $x(t) > \bar{x}$ for all $t > 0$. Taking into account the compactness of the state space and the monotonicity of any optimal path there must exist a steady state $\tilde{x} > \bar{x}$, associated with a steady state control $\tilde{u}_1$, of such a path and we must have

$$\frac{F_1(\tilde{x}, (\tilde{u}_1, \Phi_2(\tilde{x}, m_1)))}{r} \geq V(\tilde{x}, m_2) - \kappa,$$

since otherwise it would be optimal for player 1 to switch to mode m_2 at $\tilde{x}$. Taking into account that $f(\tilde{x}, (\tilde{u}_1, \Phi_2(\tilde{x}, m_1))) = 0$ we obtain

$$F_1(\tilde{x}, (\tilde{u}_1, \Phi_2(\tilde{x}, m_1))) + \frac{\partial V_1(\tilde{x}, m_2)}{\partial x} f(\tilde{x}, (\tilde{u}_1, \Phi_2(\tilde{x}, m_1))) \geq r(V(\tilde{x}, m_2) - \kappa),$$

which contradicts condition (vii). This shows that under the given conditions no path from $\bar{x}$ which never jumps to mode m_2 can be strictly better than switching to mode m_2 at $\bar{x}$. The same arguments establish that this holds also for any $x^{ini} > \bar{x}$.

Focusing on paths on which player 1 switches to mode m_2, it can be shown that (vii) implies that by for $x > \bar{x}$ switching immediately is strictly better than marginally delaying the switch to mode m_2.

To see this, consider for some $x(t) > \bar{x}$ the value in mode m_1 of delaying the switch to mode m_2 from t to $t + \epsilon$ which is denoted by $V_1^\epsilon(x(t), m_1)$:

$$V_1^\epsilon(x(t), m_1) = \max_{u_1 \in \mathcal{U}_1} \int_t^{t+\epsilon} F_1(x(s), (u_1, m_1), \Phi_2(x(s), m_1)), m_1) ds + e^{-r\epsilon}(V_1(x(t+\epsilon), m_2) - \kappa).$$

We need to show that $V_1^\epsilon(x(t), m_1) < V_1(x(t), m_2) - \kappa$ for all (small) ϵ. Direct calculations yield

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{V_1^{\epsilon}(x,m_1)-(V_1(x,m_2)-\kappa)}{\epsilon}\\
&\quad=\max_{u_1\in\mathcal{U}_1}\left[F_1(x,(u_1,\Phi_2(x,m_1)),m_1)+\frac{\partial V_1(x,m_2)}{\partial x}f(x,(u_1,\Phi_2(x,m_1)),m_1)\right]-r(V_1(x,m_2)-\kappa)\\
&\quad<0,
\end{aligned}
$$

where the last inequality follows from condition (vii). Hence, for all $x>\bar{x}$ it is optimal for player 1 to immediately switch to mode m_2.

Consider now a value $x^{ini}<\bar{x}$. Applying $\Phi_1(x,m_1)$ and switching to mode m_2 once the generated path hits $\bar{x}$ yields a value given by $V_1(x^{ini},m_1)$. Due to (vi) we have ¡ $V_1(x^{ini},m_1)<V_1(x^{ini},m_2)-\kappa$ and therefore switching to mode m_2 at x^{ini} is not optimal. This shows that the switching strategy $\Psi_{\bar{x}}(x)$ is indeed optimal for player 1. □

Proof of Proposition 2

Considering conditions (ii) and (viii) of Proposition 1 we have to show that

$$
\lim_{x\to\bar{x}-}\arg\max_{u_1\in\mathcal{U}_1} g_1(x,u_1)=\arg\max_{u_1\in\mathcal{U}_1} g_2(u_1) \tag{10}
$$

with

$$
\begin{aligned}
g_1(x,u_1)&=F_1(x,(u_1,\Phi_2(x,m_1)),m_1)+\frac{\partial V_1(x,m_1)}{\partial x}f(x,(u_1,\Phi_2(x,m_1)),m_1) &(11)\\
g_2(u_1)&=F_1(\bar{x},(u_1,\Phi_2(\bar{x},m_2)),m_2)+\frac{\partial V_1(\bar{x},m_2)}{\partial x}f(\bar{x},(u_1,\Phi_2(\bar{x},m_2)),m_2). &(12)
\end{aligned}
$$

The derivatives with respect to u_1 for the expressions to be maximized in the two modes read:

$$
\begin{aligned}
&\frac{\partial F_1(x,(u_1,\Phi_2(x,m_1)),m_1)}{\partial u_1}+\frac{\partial V_1(x,m_1)}{\partial x}\frac{\partial f(x,(u_1,\Phi_2(x,m_1)),m_1)}{\partial u_1}\\
&\frac{\partial F_1(\bar{x},(u_1,\Phi_2(\bar{x},m_2)),m_2)}{\partial u_1}+\frac{\partial V_1(\bar{x},m_2)}{\partial x}\frac{\partial f(\bar{x},(u_1,\Phi_2(\bar{x},m_2)),m_2)}{\partial u_1}.
\end{aligned}
$$

Due to the assumption of mode independent and separable control effects we have

$$
\frac{\partial F_1(x,(u_1,\Phi_2(\bar{x},m_1)),m_1)}{\partial u_1}=\frac{\partial F_1(x,(u_1,\Phi_2(\bar{x},m_2)),m_2)}{\partial u_1}
$$

and

$$
\frac{\partial f(x,(u_1,\Phi_2(\bar{x},m_1)),m_1)}{\partial u_1}=\frac{\partial f(x,(u_1,\Phi_2(\bar{x},m_2)),m_2)}{\partial u_1}.
$$

for all $u_1 \in \mathcal{U}_1$. Furthermore, since we consider a maximum delay equilibrium, at $x = \bar{x}$ the following has to hold:

$$\begin{aligned} rV_1(\bar{x}, m_1) =& F_1(\bar{x}, (\Phi_1(\bar{x}, m_1), \Phi_2(\bar{x}, m_1)), m_1) + \\ &+ \frac{\partial V_1(\bar{x}, m_2)}{\partial x} f(\bar{x}, (\Phi_1(\bar{x}, m_1), \Phi_2(\bar{x}, m_1)), m_1). \end{aligned} \tag{13}$$

The HJB equation (iv) in mode m_1 for $x < \bar{x}$ implies

$$\begin{aligned} rV_1(x, m_1) =& F_1(\bar{x}, (\Phi_1(x, m_1), \Phi_2(x, m_1)), m_1) + \\ &+ \frac{\partial V_1(x, m_1)}{\partial x} f(x, (\Phi_1(x, m_1), \Phi_2(x, m_1)), m_1). \end{aligned} \tag{14}$$

Taking into account the continuity of $F_1(\cdot, m_1), f(\cdot, m_1), \Phi_1(\cdot, m_1)$ and $\Phi_2(\cdot, m_1)$ with respect to x equations (13) and (14) together imply that the following smooth pasting condition:

$$\lim_{x \to \bar{x}-} \frac{\partial V_1(x, m_1)}{\partial x} = \frac{\partial V_1(\bar{x}, m_2)}{\partial x}. \tag{15}$$

Denote by

$$g_3(u_1) = \Big(F_1(\bar{x}, (u_1, \hat{u}_2), m_1) + \frac{\partial V_1(\bar{x}, m_2)}{\partial x} f(\bar{x}, (u_1, \hat{u}_2), m_1) \Big)$$

with $\hat{u}_2 = \lim_{x \to \bar{x}-} \Phi_2(x, m_1)$. Recalling g_1 and g_2 from (11) and (12) it follows directly from our arguments above that $g_2'(u_1) = g_3'(u_1)$ for all $u_1 \in \mathcal{U}_1$. Therefore,

$$\arg\max_{u_1 \in \mathcal{U}_1} g_2(u_1) = \arg\max_{u_1 \in \mathcal{U}_1} g_3(u_1).$$

The assumption that the right hand side of condition (ii) in Proposition 1 has a single maximizer implies that the argmax of g_2 is unique, which means that also the argmax of g_3 has only a single element. Furthermore, using (15) we obtain

$$\begin{aligned} &\lim_{x \to \bar{x}-} g_1(x, u_1) \\ =\;& F_1(\bar{x}, (u_1, \lim_{x \to \bar{x}-} \Phi_2(x, m_1)), m_1) + \lim_{x \to \bar{x}-} \frac{\partial V_1(x, m_1)}{\partial x} f(\bar{x}, (u_1, \lim_{x \to \bar{x}-} \Phi_2(x, m_1)), m_1) \\ =\;& F_1(\bar{x}, (u_1, \hat{u}_2), m_1) + \frac{\partial V_1(\bar{x}, m_2)}{\partial x} f(\bar{x}, (u_1, \hat{u}_2), m_1) \\ =\;& g_3(u_1). \end{aligned}$$

Since both g_1 and g_3 are continuous with respect to x and u_1 it follows that

$$\lim_{x \to \bar{x}-} \arg\max_{u_1 \in \mathcal{U}_1} g_1(x, u_1) = \arg\max_{u_1 \in \mathcal{U}_1} g_3(u_1),$$

where we have used that the argmax of g_3 has only one element. Hence

$$\lim_{x \to \bar{x}-} \arg\max_{u_1 \in \mathcal{U}_1} g_1(x, u_1) = \arg\max_{u_1 \in \mathcal{U}_1} g_2(u_1),$$

and since $\phi_1(x, m_1)$ is a maximizer of g_1 and $\phi_1(\bar{x}, m_2)$ is the maximizer of g_2, this proves the proposition. □

Proof of Proposition 3

We define as $g(\alpha_n) = f(\alpha_n, (\Phi_1(\alpha_n, m_1), \Phi_2(\alpha_n, m_1)))$ the right hand side of the state dynamics under the equilibrium profile. Assume first that there exists a state $\alpha_n^* \in (0, \bar{\alpha}_n)$ such that in a neighborhood around α_n^* we have $g(\alpha_n) > 0$ for all $\alpha_n < \alpha_n^*$ and $g(\alpha_n) < 0$ for all $\alpha_n > \alpha_n^*$. Then this neighborhood is invariant under the state dynamics and for initial values $\alpha_n(0)$ in this neighborhood the threshold $\bar{\alpha}_n$ is never reached and therefore the game never switches to mode m_2. Given that m_2 is never reached any positive investment $u_1 > 0$ is clearly suboptimal for player 1, which contradicts $g(\alpha_n) > 0$ for $\alpha_n < \alpha_n^*$.

Therefore, the only remaining possibility for a scenario in which the direction of the state dynamics change in the interval $[0, \bar{\alpha}_n)$ is that there exists a unique point $\alpha_n^* \in (0, \bar{\alpha}_n)$ such that $g(\alpha_n) > 0 \ \forall \alpha_n > \alpha_n^*$ and $g(\alpha_n) < 0 \ \forall \alpha_n < \alpha_n^*$. Assume that such an MPE exists and $(\tilde{\Phi}_1(\alpha_n^1, m_1), \tilde{\Phi}_2(\alpha_n^1, m_1)))$ is the strategy profile giving rise to this pattern. We denote the value functions of the two firms corresponding to this profile by $\tilde{V}_i(\alpha_n, m_1), i = 1, 2$.

The proof now proceeds by showing first that $\lim_{\epsilon \to 0} \tilde{V}_2(\alpha_n^* - \epsilon, m_1) > \lim_{\epsilon \to 0} \tilde{V}_2(\alpha_n^* + \epsilon, m_1)$ and, second, that in light of this inequality the optimal strategy of player 2 at a state $\alpha_n^* + \epsilon$ for sufficiently small ϵ has to be such that $g(\alpha_n^* + \epsilon) < 0$, which contradicts our assumption and therefore rules out the existence of a state α_n^* with the properties given above.

To show that

$$\lim_{\epsilon \to 0} \tilde{V}_2(\alpha_n^* - \epsilon, m_1) > \lim_{\epsilon \to 0} \tilde{V}_2(\alpha_n^* + \epsilon, m_1) \tag{16}$$

we first observe that in light of $g(\alpha_n) < 0$ for all $\alpha_n \in [0, \alpha_n^*)$ and $\alpha_n^* < \bar{\alpha}_n$ the new product is never introduced under this investment profile. Therefore, any optimal investment strategy for firm 2 must have $\tilde{\Phi}_2(\alpha_n, m_1) = 0$ for all $\alpha_n \in [0, \alpha_n^*)$ and therefore for any $\alpha_n < \alpha_n^*$ we must have

$$\tilde{V}_2(\alpha_n, m_1) = V_2^{\infty}(\alpha_n, m_1) = \left(1 + \frac{\nu_o}{2}\right) \frac{(q_{2o}^{m_1})^2}{r}.$$

Hence,

$$\lim_{\epsilon\to 0}\tilde{V}_2(\alpha_n^*-\epsilon,m_1)=\left(1+\frac{\nu_o}{2}\right)\frac{(q_{2o}^{m_1})^2}{r}.$$

Furthermore, taking into account that for $\alpha_n(0)>\alpha_n^*$, the threshold $\bar{\alpha}_n$ is reached in finite time under the strategies $(\tilde{\Phi}_1(\alpha_n,m_1),\tilde{\Phi}_2(\alpha_n,m_1))$, at which point the game switches to mode m_2. Hence, we obtain

$$\begin{aligned}\lim_{\epsilon\to 0}\tilde{V}_2(\alpha_n^*+\epsilon) &= \int_{t=0}^{\tau} e^{-rt}\left(1+\frac{\nu_o}{2}\right)(q_{2o}^{m_1})^2-\xi_2(\Phi_2(\alpha_n,m_1))dt\\ &\quad+\int_{t=\tau}^{\infty} e^{-rt}\left(1+\frac{\nu_o}{2}\right)q_{2o}^{m_2}(\alpha_n)^2-\xi_2(\Phi_2(\alpha_n,m_2))dt,\end{aligned}$$

where τ is the point in time when the mode switches from m_1 to m_2. Taking into account that $\xi_2(u)\geq 0\ \forall u\geq 0$, it is sufficient for proving (16) to show that $q_{2o}^{m_2}(\alpha_n)<q_{2o}^{m_1}$ for all $\alpha_n\geq 0$. To see this, we first observe that inserting $\alpha_n=0$ and $\alpha_n^0=\frac{3\eta\alpha_o}{3+\nu_o}$ into (7) gives $q_{2o}^{m_2}(0)=q_{2o}^{m_1}$. Furthermore,

$$\frac{\partial q_{2o}^{m_2}}{\partial\alpha_n}=\frac{\partial q_{2o}^{m_2}}{\partial\alpha_n^0}=-\frac{\eta\nu_o}{(1-\eta^2)(6+5\nu_o)+3(\nu_o+\nu_n)+(2+\nu_n)\nu_o^2+4\nu_o\nu_n}<0,$$

and therefore, due to our assumption that $\alpha_n^0>\frac{3\eta\alpha_o}{3+\nu_o}$ we have $q_{2o}^{m_2}(\alpha_n)<q_{2o}^{m_1}$ for all $\alpha_n\geq 0$. This establishes that the inequality (16) holds.

To complete the proof we show that there exists an alternative strategy $\hat{\Phi}_2(\alpha_n,m_1)$ such that for an initial value $\alpha_n(0)=\alpha_n^*+\hat{\epsilon}$ for sufficiently small $\hat{\epsilon}$ the generated value for firm 2 is larger than $\tilde{V}_2(\alpha_n^*+\hat{\epsilon})$ if firm 1 sticks to $\tilde{\Phi}_1(\alpha_n,m_1)$. In particular, we define

$$\hat{\Phi}_2(\alpha_n,m_1)=\begin{cases}\tilde{\Phi}_2(\alpha_n,m_1) & \alpha_n\notin[\alpha_n^*,\alpha_n^*+\hat{\epsilon}]\\ \frac{1}{\gamma}\tilde{\Phi}_1(\alpha_n,m_1) & \alpha_n\in[\alpha_n^*,\alpha_n^*+\hat{\epsilon}]\end{cases}$$

Under the strategy profile $(\tilde{\Phi}_1,\hat{\Phi}_2)$ we have $\dot{\alpha}_n=-\delta\alpha_n<0$ for $\alpha_n\in[\alpha_n^*,\alpha_n^*+\hat{\epsilon}]$. Hence for $\alpha_n(0)=\alpha_n^*+\hat{\epsilon}$ the state α_n^* is reached at $t=\tau(\hat{\epsilon}):=\frac{\ln(\alpha_n^*+\hat{\epsilon})-\alpha_n^*}{\delta}$. The value for firm 2 generated by this strategy for $\alpha_n(0)=\alpha_n^*+\hat{\epsilon}$ therefore reads

$$\hat{V}_2(\alpha_n^*+\hat{\epsilon})=\int_0^{\tau(\hat{\epsilon})} e^{-rt}\left(1+\frac{\nu_o}{2}\right)(q_{2o}^{m_1})^2-\xi_2(\hat{\Phi}_2(\alpha_n,m_1))dt+e^{-r\tau(\hat{\epsilon})}\hat{V}_2(\alpha_n^*,m_1)\quad(17)$$

Furthermore, since under this profile we have $\dot{\alpha}_n<0$ at $\alpha_n=\alpha_n^*$ and therefore $\hat{V}_2(\alpha_n^*,m_1)=\lim_{\epsilon\to 0}\tilde{V}_2(\alpha_n^*-\epsilon,m_1)$. Taking into account that $\lim_{\hat{\epsilon}\to 0}\tau(\hat{\epsilon})=0$ we therefore obtain from (16) and (17) that

$$\lim_{\hat{\epsilon}\to 0}\hat{V}_2(\alpha_n^*+\hat{\epsilon},m_1)=\lim_{\epsilon\to 0}\tilde{V}_2(\alpha_n^*-\epsilon,m_1)>\lim_{\epsilon\to 0}\tilde{V}_2(\alpha_n^*+\epsilon,m_1)$$

holds. Accordingly, for sufficiently small $\hat{\epsilon}$ we have $\hat{V}_2(\alpha_n^*+\hat{\epsilon},m_1)>\tilde{V}_2(\alpha_n^*+\hat{\epsilon},m_1)$, which contradicts our assumption that $\tilde{\Phi}_2(\alpha_n,m_1)$ is the optimal feedback strategy of firm 2. This completes the proof of the proposition. □

Appendix B: Analysis of the MPE in linear strategies in mode m_2

In mode m_2 the two firms interact through a linear quadratic differential game with a one-dimensional state. Standard arguments (see Dockner et al. (2000)) establish that a pair of functions $V_i(., m_2), i = 1, 2$ satisfying the Hamilton-Jacobi-Bellman equations

$$rV_i(\alpha_n, m_2) = \max_{u_i} \Big[F_i(\alpha_n, u_i, m_2) + \frac{\partial V_i}{\partial \alpha_n}(u_1 - \gamma u_2 - \delta\alpha_n))\Big], \; i = 1, 2 \tag{18}$$

and the transversality conditions

$$\lim_{t\to\infty} e^{-rt} V_i(\alpha_n) = 0, \; i = 1, 2 \tag{19}$$

constitute value function of a MPE. Maximizing the right hand side of the HJB-equations yields

$$u_i = \frac{1}{c_i}\frac{\partial V_i}{\partial \alpha_n}, \quad i = 1, 2. \tag{20}$$

Due to the linear-quadratic structure, the infinite time horizon and the time-autonomous nature of the game, we assume the following form for the value function:

$$V_i = C_i + D_i\alpha_n + E_i\alpha_n^2, \; i = 1, 2. \tag{21}$$

Comparison of coefficients yields the following system of 6 algebraic equations which are solved by standard numerical methods:

$$\begin{aligned}
rC_1 =& \frac{1}{2}\left(\frac{D_1^2}{c_1} + \frac{2D_1D_2\gamma^2}{c_2}\frac{\alpha_o^2(2+\nu_n+\eta^2(-2+\nu_o))(1+\nu_o)^2(-4\eta^2+(2+\nu_n)(2+\nu_o))}{K}\right)\\
rD_1 =& -D_1\delta + \frac{2D_1E_1}{c_1} + \frac{2(D_2E_1+D_1E_2)\gamma^2}{c_2} - \frac{\alpha_o\eta\nu_o(1+\nu_o)(2+\nu_o)(4\eta^2-(2+\nu_n)(2+\nu_o))}{K}\\
rE_1 =& \frac{2E_1^2}{c_1} + \frac{4E_1E_2\gamma^2}{c_2} - \frac{-(2+\nu_n)(1+\nu_o)^2(3+\nu_o)^2+\eta^2(3+2\nu_o)(6+\nu_o(9+2\nu_o))+4\delta E_1K}{2K}\\
rC_2 =& \frac{1}{2}\left(\frac{2D_1D_2}{c_1} + \frac{D_2^2\gamma^2}{c_2} + \frac{\alpha_o^2(2+\nu_o)\left((2+\nu_n)(1+\nu_o)-\eta^2(2+\nu_o)\right)^2}{K}\right)\\
rD_2 =& -D_2\delta + \frac{2(D_2E_1+D_1E_2)}{c_1} + \frac{2D_2E_2\gamma^2}{c_2} + \frac{\alpha_o\eta\nu_o(2+\nu_o)(-(2+\nu_n)(1+\nu_o)+\eta^2(2+\nu_o))}{K}\\
rE_2 =& 2E_2(-\delta + \frac{2E_1}{c_1} + \frac{E_2\gamma^2}{c_2}) + \frac{\eta^2\nu_o^2(2+\nu_o)}{2K}
\end{aligned} \tag{22}$$

where

$$K = \left((2+\nu_n)(1+\nu_o)(3+\nu_o) - \eta^2(6+5\nu_o)\right)^2 \tag{23}$$

The corresponding equilibrium feedback functions in mode m_2 are then given by

$$\Phi_i(\alpha_n, m_2) = \frac{1}{c_i}(D_i + 2E_i\alpha_n).$$

Appendix C: Characterization of the Maximum Delay Equilibrium

In the maximum delay equilibrium, the unknown variables to be determined are the threshold $\bar{\alpha}_n$ and the control of player 2 in mode m_1 for $\alpha_n = \bar{\alpha}_n$. In particular, we denote by $u_2^{m_1}$ equilibrium feedback of player 2 in mode m_1 at $\alpha_n = \bar{\alpha}_n^{md}$. Then, requiring that inequality (3) holds as an equality yields

$$\begin{aligned} r(V_1(\bar{\alpha}_n^{md}, m_2) - \kappa) &= \lim_{\epsilon \to 0} \Big[F_1\left(\bar{\alpha}_n^{md}, \Phi_1\left(\bar{\alpha}_n^{md} - \epsilon, m_1\right), m_1\right) + \\ &\quad + \frac{\partial V_1(\bar{\alpha}_n^{md}, m_2)}{\partial \alpha_n} f\left(\bar{\alpha}_n^{md}, \Phi_1\left(\bar{\alpha}_n^{md} - \epsilon, m_1\right), \Phi_2\left(\bar{\alpha}_n^{md} - \epsilon, m_1\right)\right)\Big] \\ &= F_1\left(\bar{\alpha}_n^{md}, \Phi_1\left(\bar{\alpha}_n^{md}, m_2\right), m_1\right) + \\ &\quad + \frac{\partial V_1(\bar{\alpha}_n^{md}, m_2)}{\partial \alpha_n} f(\bar{\alpha}_n^{md}, \Phi_1\left(\bar{\alpha}_n^{md}, m_2\right), u_2^{m_1}), \end{aligned}$$

where we have used that due to the smooth pasting condition the control of player 1 is continuous at $\alpha_n = \bar{\alpha}_n^{md}$.

Moreover, considering the limit of the HJB-equation of player 2 in mode m_1 for $\alpha_n \to \bar{\alpha}_n^{md}$ yields, again using the continuity of the control of player 1,

$$\begin{aligned} &\lim_{\epsilon \to 0} \Big[rV_2(\bar{\alpha}_n^{md} - \epsilon, m_1) - F_2(\bar{\alpha}_n^{md} - \epsilon, \Phi_2\left(\bar{\alpha}_n^{md} - \epsilon, m_1\right), m_1) \\ &\qquad - \frac{\partial V_2(\bar{\alpha}_n^{md} - \epsilon, m_1)}{\partial \alpha_n} f(\bar{\alpha}_n^{md} - \epsilon, \Phi_1\left(\bar{\alpha}_n^{md} - \epsilon, m_1\right), \Phi_2\left(\bar{\alpha}_n^{md} - \epsilon, m_1\right))\Big] \\ &= rV_2(\bar{\alpha}_n^{md}, m_2) - F_2(\bar{\alpha}_n^{md}, u_2^{m_1}, m_1) \\ &\qquad - \lambda(\bar{\alpha}_n^{md}, u_2^{m1}) f(\bar{\alpha}_n^{md}, \Phi_1\left(\bar{\alpha}_n^{md}, m_2\right), u_2^{m_1}) \\ &= 0 \end{aligned} \tag{24}$$

with $\lambda(\bar{\alpha}_n^{md}, u_2^{m1}) = \lim_{\epsilon \to 0} \frac{\partial V_2(\bar{\alpha}_n^{md} - \epsilon, m_1)}{\partial \alpha_n}$. To determine $\lambda(\bar{\alpha}_n^{md}, u_2^{m1})$ we use that the first order condition for the optimal control of player 2 in mode m_1 for $\alpha_n < \bar{\alpha}_n^{md}$ is given by

$$\frac{\partial F_2(\alpha_n, u_2, m_1)}{\partial u_2} + \frac{\partial V_2(\alpha_n, m_1)}{\partial \alpha_n} \frac{\partial f(\alpha_n, \Phi_1(\alpha_n, m_1), u_2)}{\partial u_2} = 0,$$

which yields $\lambda(\bar{\alpha}_n^{md}, u_2^{m1}) = -\left(\frac{\partial F_2(\bar{\alpha}_n^{md}, u_2^{m_1}, m_1)}{\partial u_2}\right) / \left(\frac{\partial f(\bar{\alpha}_n^{md}, \Phi_1(\bar{\alpha}_n^{md}, m_2), u_2^{m_1})}{\partial u_2}\right)$, where again we have used the equility of the control of player 1 across the two modes at $\alpha_n = \bar{\alpha}_n^{md}$. Taking into account the fucntional forms of F_2 and f in our model we obtain $\lambda(\bar{\alpha}_n^{md}, u_2^{m1}) = -\frac{c_2 u_2^{m_1}}{\gamma}$. Inserting this expression yields the second line in (9).