Doctoral Thesis

Conformists and Anti-conformists in Opinion Formation and Diffusion

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Decision Making, Game Theory, Opinion Dynamics in Networks, Anti-conformism

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Master thesis: Global Sensitivity Analysis of Marine Ecosystem Dynamics Models based on the Morris Method and the Sobol Method

2009-2013 **Ocean University of China (OUC)**, Qingdao, Shandong, China
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Conferences, Workshops, Summer Schools

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- 12/2019 International workshop on "Network approach to opinion formation and interaction", Paris, France. Presentation entitled *The Transmission of Continuous Cultural Traits in Endogenous Social Networks*.
- 11/2019 Ordered Structures in Games and Decisions 13th seminar day, Paris.
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- 07/2019 30th International Conference on Game Theory at Stony Brook, U.S. Presentation entitled *Anti-conformism in the threshold model of collective behavior*.
- 07/2019 Summer School "Equilibria in Financial Markets: General Equilibrium and Game Theoretic Perspective" at Bielefeld University, Bielefeld, Germany.
- 04/2019 Network Science in Economics Conference at Indiana University-Bloomington, U.S. Presentation entitled *Anti-conformism in the threshold model of collective behavior*.
- 10/2018 The Lisbon Meetings in Game Theory and Applications. Presentation entitled *Opinion Dynamics with Anti-conformist agents*.
- 07/2018 The 14th European (formerly Spain-Italy-Netherlands) Meeting on Game Theory (SING14), Bayreuth, Germany. Presentation entitled *Opinion Dynamics with Anti-conformist agents*.
- 05/2018 Deliberation, Belief Aggregation, and Epistemic Democracy, Paris, France.
- 04/2018 23rd Coalition Theory Network Workshop, Maastricht, Netherlands.
- 11/2017 Ordered Structures in Games and Decision, Paris, France.
- 12/2017 Journées Internes Aussois, Aussois, France. Poster entitled *A Pólya Urn Model of Opinion Formation*.
- 07/2015 Summer school of the 3rd International Marine Frontier Research under funding support
- 09/2014 The 4th Australia - China Ocean Science & Technology Symposium (ACOST4). Presentation entitled *Distribution of Phytoplankton Biodiversity in relations to environmental variables in South-five island of Miaodao Archipelago*.
- 11/2013 The 2013 national doctoral student academic forum. Submitted the abstract of *Laizhou bay regulate control and optimize of pollution*

Papers Published

[1] Grabisch, M., Li, F. *Anti-conformism in the Threshold Model of Collective Behavior*. Dyn Games Appl 10, 444 - 477 (2020). https://doi.org/10.1007/s13235-019-00332-0.

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[7] Shi H, Shen C, Li F, Wang Y. *Jiaozhou Bay biology - Sensitivity analysis on physics coupling model parameter*. Acta Ecologica Sinica, 2014, 34(1): 41-49. (in Chinese)

Awards and honors

Academic

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2015 **2nd prize on academic forum of school of mathematical science of OUC.**
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2014 **Academic Scholarship of OUC**
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2014 **Outstanding graduate student of OUC**
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Abstract

The works of this thesis fall into the broad, highly interdisciplinary field of research on opinion dynamics and social networks, which has been studied from different perspectives by sociologists, social psychologists, economists, politicians, cybernetists, mathematicians, computer scientists and even theoretical physicists, for many decades. So far, most models make the basic assumption that agents tend to follow the trend (they are conformist) and that nobody has a kind of opposite behavior (anti-conformism). Recently, a few studies consider such kind of opposite behavior, under the name of *anti-conformists*, *hipsters*, *contrarians*, *anti-coordination*, etc. Unlike the well-developed theory of opinion dynamics with conformists, the studies with anti-conformists are taking their first steps. This Ph.D thesis aims to answer the following questions: *Given a society of agents in a (fixed or endogenous) network, given a mechanism of influence for each agent, how the behavior/opinion of the agents will evolve with time, and in particular can it be expected that it converges to some stable situation, and in this case, which one?* The main objective is to obtain the conditions (both on the agent- and network-level) required to generate specific network level phenomena, e.g., reaching a consensus/polarization.

The first work provides a detailed study of the threshold model, where both conformist and anti-conformist agents coexist. The study bears essentially on the convergence of the opinion dynamics in the society of agents, i.e., finding absorbing classes, cycles, etc. Also, we are interested in the existence of cascade effects, as this may constitute an undesirable phenomenon in collective behavior. The study is divided into two parts. In the first one, the threshold model is studied by supposing a fixed complete network, where every one is connected to every one, like in the seminal work of Granovetter. The cases of a uniform distribution of the threshold, of a Gaussian distribution are studied, and finally a result for arbitrary distributions is given, supposing there is one type of anti-conformist. In a second part, the graph is no more complete and we suppose that the neighborhood of an agent is random, drawn at each time step from a distribution. Two cases are distinguished where the degree (number of links) of an agent is fixed, and where there is an arbitrary degree distribution. We show the existence of cascades and that for most societies, the opinion converges to a chaotic situation.

The second work studies the dynamics of continuous cultural traits (as a specific type of continuous opinions) in an OLG (overlapping generation) structure and in an endogenous

social network, where the network changes are inherited. Children learn their cultural trait from their parents and their social environment modelled by network. Parents want their children to adopt a cultural trait that is similar to their own and engage in the socialization process of their children by forming new links or deleting connections. Changing links from the inherited network is costly, but having many links is beneficial. We propose three ways to endogenize the process of network formation. In the first one, the network is supposed to be directed and each dynasty can either form or delete a directed link unilaterally with another dynasty. Therefore, at each period, each family faces a utility opitimization problem where a trade-off between own utility losses and the improvements of child's cultural trait. We have shown that if the cost of network changes is greater than the cost of child care, extremists will never add links, and in the case of sufficiently low cost, extremists may cut all ties with the society. In the second and third models, the network is supposed to be undirected. In the second model, we assume that after each period, a pairwise stable network (PS network for short) is reached. In this case, there always exist sufficiently small cost parameters such that the empty network is the unique PS network. In the third model, we assume that after each period, a pairwise stable network with transfers (PST network for short) is reached. We have shown the existence of the PST network for each period, however, it is not necessary to be unique. Moreover, a necessary and sufficient condition is given such that a network is PST for given $V(t)$ and $G(t)$. The convergence of cultural traits in this case is guaranteed. Regarding the efficiency of the network, we show that there always exist sufficiently small cost parameters such that the empty network is the unique efficient network, and sufficiently large costs of child care such that the complete network is the unique efficient network. Moreover, more detailed dynamics of cultural traits are studied when the costs of network changes and benefits from integration are low, intermediate, and large, respectively.

The third work proposes an appropriate updating rule of continuous opinions for modeling anti-conformity behavior, defined according to the repelling function, which gives the shift of the opinion based on the current opinion and the reference opinion for an agent. Two models of continuous opinion dynamics (with opinion value on a continuous scale [0,1]) are studied in undirected networks, by introducing the heterogeneity in the sense of conformity and anti-conformity behavior either in nodes or in links. In the first one, the society is composed of both conformist and anti-conformist agents. Conformist agents update their opinions following the DeGroot rule with equal weights, however, anti-conformist agents would like to repel from others, and the repelling level is negatively related to the opinion distance between the anti-conformist and her reference point. No consensus will be reached for any connected network in the presence of anti-conformist agent. Instead, opinions converge to a disagreement or oscillate over time. In the second part, by supposing a signed graph where agents have positive links $(+1)$ with their friends and negative links (-1) with their enemies, agents update their opinion as the sum of the averaged opinion of their friends and repelling value from their enemies. When the network is balanced, i.e., there are two communitarian groups, and each sub-network corresponding to each group is connected and the initial opinion ranges of the two groups are disjoint, the consensus within each group is guaranteed. Both synchronous and asynchronous updating models are discussed in these two parts.

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Contents

Chapter 1

Introduction

1.1 Opinion Dynamics and Social Networks

The works of this thesis fall into the broad, highly interdisciplinary field of research on opinion dynamics and social networks which has been studied for many decades, in different communities related to social and behavioral sciences, economics, politics, physics, control theory, mathematics, computer science, etc. Opinions, beliefs, actions, decisions, attitudes, and social norms are intimately intertwined with each other. We form our opinions on every aspect of our life, from personal interests (e.g., favorite colors/foods), to social norms (e.g., the acceptable behavior in certain circumstances), to economic decisions (e.g., consumption budget, tax rate), and even to political attitudes, etc. How do we form our opinions? On the one hand, we acquire our beliefs and opinions through learning from our parents or guardians, as we are taught the basic values and beliefs since childhood (e.g. Bisin and Verdier (2000), Bisin and Verdier (2001)). On the other hand, to a large extent, we learn from our social environment and update our opinions and beliefs instantly based on our experience and the information obtained by observing the behaviors of others, communicating with people directly or indirectly via different means of communicational devices, surfing in social media and so on (Acemoglu and Ozdaglar (2011)). Social networks play a crucial role in modeling opinion dynamics as people are constantly interacting and influencing each other. Experimental evidences provided by Galton (1907), Lorge et al. (1958), Hommes et al. (2005) and Yaniv and Milyavsky (2007) demonstrate that the aggregate (such as median and averaged) estimates of a group are very close to the true value. This *wisdom of crowd effect* can somehow show the importance of social influence. As the social aspect of opinion dynamics, social learning focuses on studying how dispersed information are gathered and how accurate it is in the presence of misinformation and manipulation. When it refers to *true value* or *accurate* *estimation*, it is related to well-defined questions such as the weight of an ox (Galton (1907)). However, we also form our opinions on questions to which there is no correct and clear answer.

In social science, as described by Lewin (1947) and Proskurnikov and Tempo (2017), there was a shift from focusing on individual approaches to interests on social structures and social dynamics during 20th century. As the quantitative method *socialmetry* and graphical tool *sociogram* are introduced into social analysis (Moreno (1934), Moreno (1951)), it attracts more and more attention, not only from sociologists, but also from economists, cybernetists, politicians, physicists, computer scientists and so on (Proskurnikov and Tempo (2017)). At the same time, it also promotes the development of studies on networks and network dynamics. Tremendous amount of research from all fields focused on studying the convergence to agreement (consensus) and its conditions. For example, researchers working on control theory are interested in the global consensus protocols caused by local interactions. The interested readers are referred to Jackson (2010), Acemoglu and Ozdaglar (2011) and Proskurnikov and Tempo (2017) for detailed overview surveys, and Bullo (2019) for a thorough reference book on Network Systems.

In general, models of opinion dynamics study how agents' opinion on one topic or multiple topics are evolving and what are the convergence and consensus conditions on both individual level and network level. The opinion of agents' opinion on one topic is usually represented as an integer for discrete opinions (e.g., yes or no) or a real number for continuous opinions (e.g., tax rate). Compared to the large amount of research on opinions on single topic, few studies also consider opinions on multiple topics (see Axelrod (1997), Ye et al. (2020)), with the help of the *belief system* which is usually an interdependence matrix. The dynamic interacting process takes place either in a continuous time interval or in each discrete time instant. Agents communicate and influence within their neighborhood during the interactions and thereafter update their opinions.

As mentioned by Proskurnikov and Tempo (2017), the earliest models of opinion dynamics date back to 1930s from a macroscopic and statistical perspective (see Rashevsky (1939), Rashevsky (1947)), which are mainly for large-scale communities. One of the earliest and widely used agent-based model of opinion dynamics is the French-Harary-DeGroot model (French Jr (1956), Harary (1959), Harary et al. (1965), DeGroot (1974)) (which is also called DeGroot model in the literature), which can describe both small and large groups. As a sociologist and psychologist, French Jr. proposed the model of opinion dynamics in French Jr (1956) to study the social power of each agent, i.e., the ability of each agent to influence the final consensus opinion, which is also the *centrality* of each agent (node) in a social network (graph) from a network theoretical perspective. Later on,

Frank Harary (Harary (1959), Harary et al. (1965)) and Morris DeGroot (DeGroot (1974)) generalized this model and provide the network conditions to guarantee the consensus of opinions. The DeGroot model is also called "iterative opinion pooling", since agents update their opinions iteratively as the weighted average of the opinions of their neighbors. The weights are usually placed in a row-stochastic influence matrix W, where the element w_{ij} is referred to as the level of trust that agent i holds on agent j. Essentially, the DeGroot model is a discrete-time and synchronous model since all agents update their opinions *simultaneously* at each time instant. Extensions and variations of the DeGroot model have been proposed to generalize it to a continuous-time framework, e.g., the Abelson model (Abelson (1964)), and to a asynchronous framework, e.g., the gossip model (Ravazzi et al. (2014), Liu et al. (2011)). The typical behavior of the DeGroot model is the presence of consensus, while in real life disagreement is also ubiquitous (Abelson (1964)). Out of this consideration, different kinds of variations of the DeGroot model have been proposed. For example, some variations introduce stubborn agent whose opinion remains unchanged during the iterative pooling process (see Hegselmann and Krause (2015), Masuda (2015)), by introducing an attachment of each agent to its initial opinion (stubborn agents are also called by physicists as *independent agents (Sznajd-Weron et al. (2011), Sznajd-Weron et al. (2014)), inflexibles (Galam and Jacobs (2007)), zealots (Mobilia (2003)));* some other variations introduce negative influences, i.e., the element w_{ij} of the weight matrix W can be positive or negative, thus W is no more row-stochastic; some variations considered that agents are only interacting with those who hold opinions close enough to them by introducing confidence bounds (Hegselmann et al. (2002), Weisbuch (2004), see also the survey on continuous opinion dynamics with bounded confidence Lorenz (2007)).
Studies on the DeGoot model of continuous opinion dynamics are well-developed due to its technical tractability and simplicity. Another simple and tractable model of discrete (usually binary) opinion dynamics is the threshold model, in which a special case is the majority rule model (Galam (2002)) when the threshold is equal to 0.5, introduced by Granovetter (1978), Schelling (2006), among others. Suppose that each agent holds one opinion or action from the set $\{0, 1\}$, where action 1 refers to "yes, active, etc.," while action 0 refers to "no, inactive, etc." This model simply says that an agent takes action 1 if sufficiently many people in his neighborhood takes action 1. The simplicity of the model allows for a deep analysis (see the surveys by Mossel and Tamuz (2017) and Castellano et al. (2009a)), and one remarkable result already observed in the pioneering work of Granovetter (1978) was that a cascade effect occurs, supposing that the population of agents starts from an initial state where nobody is active, and that the distribution of the threshold value is uniform over the population. Then, after a finite number of steps, all

agents become active. Interestingly, the latter study was done in the context of a mob, where the available actions were "to riot" (action 1) or to be inactive (action 0). Then, agents with threshold 0 were called "instigators'" as they start to riot alone, and this indeed forms the seed of the cascade effect, ending in a mob rioting. This topic has been very much studied, as demonstrated by a recent monograph on mob control (Breer et al. (2017)), written by researchers in control theory.

Another stretch of modeling binary opinion dynamics (especially popular in the fields of physics and sociophysics) is the Sznajd model, the voter model and their extensions. Sznajd-Weron proposed the Sznajd model in Sznajd-Weron and Sznajd (2000), by adopting the Ising spin system (Galam (2004b)) which is widely used and applied in sociology, economy and statistical physics (Sznajd-Weron (2005)). The agent who holds binary opinions is called a *spin* or *spinson* (as a combination of **spin** and **person**), and it is imaged as ($\uparrow$ or $\downarrow$) in the Ising spin system. To describe the influence of local interaction on the global social phenomenon, the Sznajd model introduced a new concept of spin dynamics as follows. Suppose n agents are ordered in a line $A_1, A_2, \ldots, A_n$ such that A_i and A_{i+1} are neighbors, $\forall i = 1, \ldots, n-1$. As the first step, a pair of spins A_i and A_{i+1} are chosen to influence A_{i-1} and A_{i+2}. As the second step, if $A_i = A_{i+1}$, then $A_{i-1} = A_i$ and $A_{i+2} = A_{i+1}$, i.e., their nearest neighbors will copy their opinions if they agree (this is called *social validation*); if $A_i \neq A_{i+1}$, then $A_{i-1} = A_{i+1}$ and $A_{i+2} = A_i$. Slanina et al. (2008) modified this model such that A_i and A_{i+1} stay unchanged in case of $A_i \neq A_{i+1}$, and this coincides with a special case of the q-voter model which will be described below. The original voter model was proposed by Clifford and Sudbury (1973), where agents are situated in a static graph. At each time instant, an agent is chosen at random to be active and copies the opinion of a random neighbor. Castellano et al. (2009b) generalized this model to the q-voter model in the lattice network where the active agent is influenced by its q randomly picked neighbors (with possible repetitions). The active agent, say i, will copy the opinion of the q neighbors if they agree, otherwise agent i will flip with probability ϵ. The case when $q = 2$ and $\epsilon = 0$ coincides with the modified Sznajd model mentioned above (Jędrzejewski et al. (2016)). Many works are conducted based on these two models, e.g., Jędrzejewski et al. (2016) generalized the q-voter model to complex networks, Przybyła et al. (2011) extended the Sznajd model by incorporating also the idea of the q-voter model, and so on. Both the classical Sznajd model and the q-voter model imply the idea of conformity (or repetition, i.e., the copy of the opinion of others), while some latest research also considers the non-conformity (independent or anti-conformity) behavior. As described by Nyczka and Sznajd-Weron (2013b), the Ising model of social influence has three main components: *topology* consisting of a set of finite nodes and a set of finite links (e.g., regular lattice, complete graph); dynamical binary opinions; internal interactions including conformity and anti-conformity and the external interactions caused by some external force such as a strong leader.

Even though the amount of models of opinion dynamics is huge, they can be classified and applied differently from various perspectives. This will be discussed in the next section (Section 1.2).

1.2 Different Aspects of Modeling Opinion Dynamics

1.2.1 Continuous opinion and discrete opinion

The opinion dynamics models can be divided into continuous opinion dynamics models and discrete opinion dynamics models, while in the latter group of models, binary opinion dynamics are more often studied.

The group of models of continuous opinion dynamics deals with problems in which the opinion of people can be expressed as real numbers (e.g., tax rates, prices, quantitative predictions). As discussed in the previous section (Section 1.1), the most frequently used and the fundamental model of continuous opinion dynamics is the DeGroot model, based on which a large amount of generalizations and extensions are conducted. A remarkable modification of the DeGroot model is the Friedkin-Johnsen model (Friedkin and Johnsen (1990), Friedkin and Johnsen (1999)), which is validated for small and medium-size groups with experimental evidence (Childress and Friedkin (2012), Friedkin et al. (2016a), Proskurnikov and Tempo (2017)). The Friedkin-Johnsen model supposes that agents are continuously influenced by their initial opinion over the dynamic process (i.e., agents are attached to their initial opinions), by introducing a *diagonal* matrix into the DeGroot model, with the elements of the diagonal matrix indicating the *susceptibility* of agents to social influence (Proskurnikov and Tempo (2017)). Due to this introduction of agents' stubbornness, this model is able to explain the presence of *strong diversity*, i.e., the distribution of opinions is not concentrated into sharp clusters (Ye (2019)). These models are comparatively simple since they are all **linear** models so that they can be analyzed by powerful linear techniques such as graph theory, Markov chain theory and matrix theory.

The continuous-time counterpart of the DeGroot model was proposed by Abelson (Abelson (1964)). Meanwhile, Abelson also extended it to a **nonlinear** form and formulated the *community cleavage problem* (Friedkin (2015)) or *Abelson's diversity puzzle* (Kurahashi-

Nakamura et al. (2016)) as: "*Since universal ultimate agreement is an ubiquitous outcome of a very broad class of mathematical models, we are naturally led to inquire what on earth one must assume in order to generate the bimodal outcome of community cleavage studies.*" In general, *consensus* is too strong to be realistic (Abelson (1967)), *disagreement* such as *clusters* and *cleavage* are to be identified. One way is to introduce the role of stubborn agents (or partially stubborn agents) as in the Friedkin-Johnsen model. The other ways leading to community cleavage include introducing time-varying weight matrices (Lorenz (2005b)), considering bounded confidence (Hegselmann et al. (2002)), and incorporating negative influences (e.g., Altafini (2012a), Altafini (2012b) for continuous opinions and Grabisch et al. (2019) for binary opinions). The models of continuous opinion dynamic under bounded confidence, proposed by Krause (2000) and Deffuant et al. (2000) independently, are also nonlinear in the sense that the model changes with the opinion of agents. Therefore, the above-mentioned powerful mathematical tools such as Markov chains and matrix theory are not applicable and thus rigorous analysis is hard to obtain (Hegselmann et al. (2002)). Instead, some results are based on simulations on computers.

Next, the group of models of discrete opinion dynamics is mainly applied to cases when there is no compromises in between any two opinions, actions or decisions. For example, the Ising model was used to model the behavior of labors ("go on the strike" or not) during the strike in Galam et al. (1982); the voter model is a useful framework for the study on innovation diffusion, decisions and epidemics (Yildiz et al. (2011)); the Sznajd model and its modifications are able to be applied in politics, marketing and finance; Grabisch and Rusinowska (2013) proposed the model of influence based on aggregation functions for yes-no opinion dynamics, provided a detailed analysis of the convergence, and gave all the absorbing classes and states. While both continuous opinions and discrete opinions are common in life, one natural idea is that agents express internal continuous opinions (preferences on two alternatives) as discrete actions. Based on this idea, the CODA (*Continuous Opinions and Discrete Actions*) updating rule was proposed by Martins (2008) and applied to both voter model and the Sznajd model, which can explain the appearance of extremists.

1.2.2 Continuous-time and discrete-time

The opinion dynamics models can be divided into continuous-time and discrete-time opinion dynamics models. The DeGroot model is a discrete-time opinion dynamics model. For averaging systems similar to the DeGroot model, mathematical techniques can be used to find the conditions of convergence and consensus. By supposing that the time interval between two steps is very small, Abelson (1964) extended the DeGroot model to its continuous-time counterpart. The linear form of the Abelson model is represented by the *Laplacian flow*. Some results on ordinary differential equations and matrix exponential of Laplacian matrices are applied to find the equilibria and convergence of the Laplacian flow dynamics (Bullo (2019)).

Taylor extended the linear Abelson model (Taylor (1968)), by introducing a certain number (say, m) of *communication sources* such as mass media with static opinions (s_i, $i = 1, \dots, m$) that can constantly influence agents. Indeed, the Taylor model is equivalent to the Abelson model with extra m stubborn agents with their opinions s_i unchanged during the process. Alternatively, the Taylor model can also be transformed into the opinion dynamic model where agents are *prejudiced*, i.e., agents hold some internal opinion, formed by personal experience, mass media, etc. The multidimensional extension of the Taylor model is also applied to containment control problems in multi-agent systems (Cao et al. (2012)). The discrete-time counterpart of the Taylor model – the Friedkin-Johnsen model which is equivalent to the DeGroot model with stubborn agents, can also be considered as a containment control algorithm in discrete time process.

The models of opinion dynamics are not limited to the classifications mentioned in this section. Some other classification criteria also exist such as the dimension of opinions. Most of the models in this section are opinion dynamics on a single topic, however, recently some researchers are also working on opinion dynamics on multidimensional topics (Parsegov et al. (2016), Friedkin et al. (2016b), and Ye et al. (2020) for a continuous-time counterpart to the former two works).

1.2.3 Bayesian models and non-Bayesian models

In the opinion formation process, agents are initially attached to some initial opinions (i.e., *priors*) which will be updated based on a certain *updating rule*, after agents communicate with their friends, obtain some information from mass media, or observe the behavior of their neighbors (i.e., *information acquisition*). The models of opinion dynamics can be divided into Bayesian models and non-Bayesian models, according to whether agents use Bayes rule (given by the formula 1.1) as the opinions updating rule or not. Given two probabilistic events A and B, the Bayes rule states that the conditional probability that A is true given B being true, i.e., $\mathbb{P}(A \mid B)$, is given by the product of the conditional probability that B is true given that A is true and the unconditional probability of A, divided by the unconditional probability of B.

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)} \tag{1.1}$$

The Bayesian approach assumes that agents are Bayesian and thus update their opinion optimally given the information they have obtained. It applies to problems where the questions on which agents form opinions are well-defined, i.e., there is a clear answer to the question, and we call this clear answer *true value of the world*. Bayesian agents have a reliable model of the world such that they have a clear idea of the (subjective) probabilities or likelihood of all possible events (i.e., priors), and such that they can obtain information from the actions of others (Acemoglu and Ozdaglar (2011)). Motivated by the *Condorcet's Jury Theorem* stating that if agents report their information truthfully, it is sufficient to aggregate the dispersed information to reveal the true value of the world (Condorcet (1976)), Acemoglu et al. (2011), Bikhchandani et al. (1992) show that instead of information aggregation, herding effect and information cascades can occur in the context of Bayesian learning, when agents observe all the previous actions of the other agents.

All the models mentioned before this section (including the DeGroot model and its variations, the threshold model, the voter model, etc.) are non-Bayesian models, which will be discussed in detailed in Chapter 2. Compared to the challenging and restrictive perspective of the Bayesian models, non-Bayesian models seem to provide a more natural staring point to study the spread of misinformation (Acemoglu et al. (2010), Molavi et al. (2018)). However, Bayesian models are still useful benchmarks to evaluate non-Bayesian models. Mueller-Frank (2014) showed that in a strongly connected network consisting only of non-Bayesian agents, the presence of at least one Bayesian agent is sufficient for each agent to aggregate information perfectly. Molavi et al. (2017) provided a foundation of the non-Bayesian models of opinion dynamics in a general framework, where agents also receive private signals in addition to the iterated averaging, and furthermore identified the forces that lead to information aggregation in social networks.

1.3 Synchronous and Asynchronous

Different activation regimes (such as synchronization or synchronization of agents' activation, different interaction size at each time step and so on) can produce different results in opinion dynamic models (Alizadeh et al. (2015)). It may happen that some interesting phenomena exhibited in the synchronous updating model disappear in the asynchronous setting, and therein stability appears instead of striking spatial chaos (Huberman and Glance (1993), Nowak and May (1992)). By describing the order of updates as a sequence of subsets of the population N, Bredereck and Elkind (2017) defined the synchronous updating accordingly, as the updating sequence $(N, N, \ldots, N)$ and defined the asynchronous updating as that with each subset being a singleton. This captures the idea that only one agent is active at each time. The active agent can either meet another agent with a certain probability to exchange opinions or observe the opinions of all her neighbors, and thereafter has her own opinion updated.

In control theory, models of opinion dynamics are given either in the discrete time version or in the continuous time version, and they are all synchronous, such as the DeGroot model and its continuous-time counterpart, i.e., the linear Abelson's model (see Section 2.2). There is no relation between the continuous/discrete aspect and the synchronous/asynchronous aspect. However, in many models in physics, asynchronous models tend to continuous models as the time between two steps of iteration tends to 0.

Acemoglu and Ozdaglar (2011) modeled an asynchronous updating process by supposing that at each time, agent i is chosen to be active with probability $1/n, \forall i \in N$ and in case of agent i being active, agent i will meet agent j and exchange opinions with probability $p_{ij} \geq 0$, where $\sum_{j=1}^{n} p_{ij} = 1, \forall i \in N$. Moreover, for a better approximation of many real situations, some researchers also consider the opinion dynamics in a random neighborhood setting. For example, Grabisch and Li (2020) studied the synchronous opinion dynamics for binary opinions in a random neighborhood setting in which a random neighborhood is realized in each period. Nyczka and Sznajd-Weron (2013a) studied the asynchronous q-voter model and assumed that both the voter and the group that can influence the voter are randomly chosen (random active agent and random neighborhood). Ramazi et al. (2016) showed that for threshold-based dynamics, the equilibrium can be reached in both the synchronous and asynchronous setting, and it can also be almost surely reached in partial synchronous setting [1]. These results reveal that the asynchrony does not lead to cycles or non-convergence, neither does the irregular network topology. Instead, the coexistence of heterogenous behavior (such as conformity and anti-conformity behavior) play a role in the presence of cycles or non-convergence (Ramazi et al. (2016), Grabisch and Li (2020)).

1.4 Conformity and Anti-conformity Behavior

Before the 21st century, most of the models of opinion dynamics made the basic assumption that agents tend to follow the trend (i.e., they are conformist), and the existence of opposite behavior (anti-conformity or counter-conformity) was neglected. Even in

[1] In partial synchronous updating setting, a random number of agents update opinions simultaneously.

the field of psychology, as Jahoda (1959) criticized, conformity was over-emphasized in the psychological literature, and the emphasis obscured the reality of non-conformity or anti-conformity (Hornsey et al. (2003)). The famous experimental study by Asch (1955) showed that agents tend to conform to the wrong judgement of their predecessors even if some of them know already that the judgement was wrong. A follow-up study (Deutsch and Gerard (1955)) distinguished two forms of social influence that lead to the wrong judgement. While normative social influence drives some agents to behave like majority in order to avoid social censure, informational social influence explains the conformity behavior in the sense that agents are uncertain about the answer, so they might rely on the judgement of the majority of the society (Hornsey et al. (2003)). This was later supported by Frideres et al. (1971), Terry et al. (2000), Zafar (2011).
Motivated by this idea, Buechel et al. (2015) modeled the continuous opinion dynamics by allowing agents to misrepresent opinions in a conforming or anti-conforming way, and furthermore showed that agents' social power is decreasing in the degree of conformity. The other branch of study on continuous opinion dynamics with anti-conformity behavior (or negative social influence) is based on the notion of *coopetition*, which was first introduced by Carfì and Schilirò (2012) in the study of the Green Economy and then applied to opinion dynamics with negative influences for a better understanding and explaining the disagreement of opinions. In "coopetitive" networks agents can both cooperate and compete, corresponding to the positive and negative influences among agents, respectively (Proskurnikov and Tempo (2018)), i.e., agents are situated in a signed graph. Altafini proposed a model of influence with antagonistic interactions based on the theory of structurally balanced network (Altafini (2012b), Altafini (2012a), Harary et al. (1953)). The idea of structural balancedness can be interpreted as the ancient proverb *the friend of my enemy is my enemy, the enemy of my enemy is my friend* (Schwartz (2010)). The original Altafini model is coincident with the Abelson model with an influence matrix that can have both positive and negative elements. By doing gauge transformation, the structurally balanced network can be transformed into the corresponding nonnegative network sharing the same convergence properties. It was shown that in case of a structurally balanced network (without self-loops), the bipartite consensus can be achieved. However, for a structurally unbalanced and strongly connected network, the consensus value is always the origin, regardless of the initial conditions (Altafini (2012a), Meng et al. (2016)). In a recent paper coauthored by Altafini (Shi et al. (2019)), the authors defined two rules for negative influences: the opposing rule where the opinion of an agent is attracted by the opposite of the opinion of her neighbor via negative links, and the repelling rule where the two agents repel each other instead of being attracted via negative links.

It also attracts many researchers to study binary opinion dynamics with non-conformity behavior in recent years. In sociophysics, the first idea about anti-conformist agents seems to have been introduced by Galam (2004a) under the name of *contrarians*. Later works include those of Borghesi (Borghesi and Galam (2006)), Sznajd-Weron (Nyczka and Sznajd-Weron (2013b)) and also Juul and Porter (Juul and Porter (2019)). In Nyczka and Sznajd-Weron (2013b), the q-voter model is studied, where it is supposed that agents may adopt with some probability an anti-conformist attitude, while the threshold model is considered under this assumption in Nowak and Sznajd-Weron (2019). Close to this model is the recent study of Juul and Porter (2019) about the spreading of two competing products, say A and B, where anti-conformist agents are called *hipsters* (see Touboul (2014) where this terminology has been introduced). In Juul and Porter (2019), starting from a network with all nodes inactive, a single node is uniformly chosen at random to adopt one product, say A, which buries the seed for the spreading process. They assume the threshold of a player (who can be a conformist or a hipster) is the minimum proportion of their active neighbors such that this player becomes active, and the transition from active to inactive is a one-way process. Once the player becomes active, they must adopt one product according to the following rules: if he is a conformist, he will adopt the most popular product over his neighborhood; if he is a hipster, he will adopt the less popular product over the whole population. Under this assumption, they found that even a small proportion of hipsters can lead to a reversal of the popularity of two competing products. Javarone (2014) provided a computational study of the non-conformity behavior in both local (neighborhood) and global (population) perspective, based on the majority rule voting. Ramazi et al. (2016), Nowak and Sznajd-Weron (2019) and Grabisch and Li (2020) studied the linear threshold-based dynamics with anti-conformity behavior. In Nowak and Sznajd-Weron (2019), agents are selected at random for updating and their threshold is the same for all agents, however, an agent is not *a priori* conformist or anti-conformist, but is one or the other with some probability, which are different in Ramazi et al. (2016) and Grabisch and Li (2020). Both Ramazi et al. (2016) and Grabisch and Li (2020) have shown that the equilibrium can be reached in the society of exclusive conformist agents or in the society of exclusive anti-conformist agents, and that coexistence of coordinating and anti-coordinating agents leads to cycles or non-convergence in the binary opinion dynamics.

1.5 Network Theory and Network Formation

As network structure is crucial for the convergence of opinion dynamics and the social power of agents, it is important to incorporate network theory and network formation process into the study of modeling opinion dynamics.

One famous example in the field of network study is "Florentine Marriages", describing the Medici family, which did not stand out with respect to wealth and political clout, rose in power and eclipse those with both greater wealth and political power, due to its high *betweenness* (i.e., the number of the shortest paths that pass through the Medici divided by the number of all the shortest paths between any two distinct nodes, see Freeman (1977)) in the network of marriages between some key families in Florence (Padgett and Ansell (1993)). This example is always used as an example to show that the structure of social network is important beyond a simple comparison of degrees (i.e., how many social ties or links each agent has), and it further-on motivates the study of network formation, e.g., how did this marriage network form and was it optimal (Jackson (2010))?

Different measures of centrality have been proposed and developed to capture different aspects of the position that a node is placed in a certain network. The simplest measure of centrality of a given node, say i, is to count how many social ties it has, say d_i, and dividing this number by $n-1$ gives its *degree centrality*[2] , where n is the total number of nodes (Freeman (1978)), i.e., $C_d(i) = \frac{d_i}{n-1}$. Define the distance between two nodes i and j as the length of (i.e., the number of links in) a shortest path between two nodes, denoted by d_{ij} [3]. *Closeness centrality* measures how close the given node i is to all other nodes, i.e., how easily the given node can contact all other nodes, defined as the inverse of the average distance between one node and any other node $C_{cl}(i) = \frac{n-1}{\sum_{j \neq i} d_{ij}}$. Some other variants have been proposed for unconnected networks (Newman (2003), Csardi et al. (2006)). A particular variant is the *decay centrality* (Jackson and Wolinsky (1996), Jackson (2010)), defined as $C_{decay}(i) = \sum_{j \neq i} \delta^{d_{ij}}$, where $0 < \delta < 1$ is a decay parameter. As mentioned in the previous paragraph, the *betweenness centrality* of a node i is defined as

$$C_{be}(i) = \frac{2 \sum_{k \notin \{j,i\}} \sum_{j \neq i} \frac{g_{jk}(i)}{g_{jk}}}{(n-1)(n-2)},$$

where g_{jk} is the number of shortest paths (geodesics) from j to k, and $g_{jk}(i)$ is the number of shortest paths from j to k containing i (Freeman (1977)). Another widely-used branch of measure of centrality is eigenvector based centrality measures, among which *Bonacich*

[2] A node with degree $n-1$ is considered to be fully connected.

[3] $d_{ij} = \infty$, if i and j are not connected.

centrality became very popular. The Bonacich centrality is given by the eigenvector of the network g, i.e., $\lambda C_{bo}(g) = gC_{bo}(g)$, where λ is the eigenvalue corresponding to $C_{bo}(g)$ (Bonacich (1972)). Moreover, in opinion dynamics models, the Bonacich centrality can also characterize opinion leadership or social power (Friedkin (1991), Buechel et al. (2015)), i.e., the level of influence on the final consensus opinion.

One important question we need to answer by modeling network formation models is why are certain network structures formed instead of any other structure. Two concepts are proposed: one is *stability* or *equilibrium* of the network dynamics, based on maximizing individual incentives; the other is *efficiency* of the network, measuring the overall societal welfare (Jackson (2010)). *Pairwise stability* is a simple stability concept proposed by Jackson and Wolinsky (1996) to capture the mutual consent required for forming a link between two agents, while Nash equilibrium based solutions fail to capture this point. A pairwise stable (PS for short) network requires that no agents wants to delete a link unilaterally and no two unconnected agents both want to form a link, considering one link at a time. Later on, some refinements of the pairwise stability were proposed, such as *strong stability* (Dutta and Mutuswami (1997), Jackson and Van den Nouweland (2005)), allowing larger coalitions to deviate than just pairs of agents, *pairwise Nash stability* (Bloch and Jackson (2006), Calvó-Armengol and İlkılıç (2009)), considering multiple link deletion. An *strong efficient* network is the one maximizing the total societal utility, while an *efficient* network is the one that no other network can have larger total societal utility (Jackson and Wolinsky (1996)). The other standard notion of efficiency is *Pareto efficiency*, requiring that the Pareto efficient network is not Pareto dominated by any other network (Pareto (1964)). In general there is an incompatibility of stability and efficiency, thus the transfers among agents were introduced to ensure that at least one efficient network is PS (Jackson and Wolinsky (1996), Jackson (2010)). The notion of *pairwise stable network with transfers* (PST network for short) was proposed, assuming that transfers among agents are allowed (Bloch and Jackson (2007)). A PST network requires that no pair of agents can jointly benefit by forming or deleting a link. Jackson and Watts (2001), Goyal and Joshi (2006), Sarangi et al. (2011), Hellmann (2013) provided sufficient conditions for the existence and uniqueness of PS networks. Interesting readers should consult Jackson (2010) for a thorough reference book on Networks, and Jackson (2005) for a survey on network formation models.

Chapter 2

Preliminaries

2.1 Non-negative Matrices

This section heavily borrows from material contained in Dym (2013), Seneta (2006), Proskurnikov and Tempo (2017) and the compendium of Michel Grabisch. By convention, the transpose of a matrix is denoted by $'$. A matrix $T = (t_{ij})$ is *nonnegative* (*positive*) if all of its elements t_{ij} are nonnegative (positive). A nonnegative $n \times n$ matrix T is *row-stochastic* if $\sum_{j=1}^{n} t_{ij} = 1$, $\forall i = 1, \ldots, n$. Let $N := \{1, 2, \ldots, n\}$. The kth power of $T = [t_{ij}]$ is denoted by $T^k = [t_{ij}^{(k)}]$. To any nonnegative matrix $T = [T_{ij}]$ we associate a directed graph Γ with set of nodes N, and the set of arcs $\{(i, j) \mid i, j \in N, t_{ij} > 0\}$.

Definition 2.1 (walk). *A walk of length k from node i to node j (denoted by $i \to j$[1]) is a sequence of nodes $i = i_0, i_1, \ldots, i_k = j$ such that (i_{l-1}, i_l) is an arc in Γ for $l = 1, \ldots, k$[2].*

Definition 2.2 (cycle). *A cycle around i is defined as a walk from i to i which does not pass through i between the starting and the ending points.*

Definition 2.3 (component). *A (connected) component is a set of nodes C such that either C is a singleton or $i \leftrightarrow j$ for every distinct $i, j \in C$.*

Definition 2.4 (class, strongly connected component). *A class or strongly connected component is a set of nodes C such that either C is a singleton or $i \leftrightarrow j$ for every distinct $i, j \in C$, and any $C' \subset C$ does not fulfill the latter property.*

Definition 2.5 (essential, inessential). *A class is essential if no arc is going out of it, otherwise it is inessential.*

[1] $i \nrightarrow j$ refers to that there is no walk from node i to j and $i \leftrightarrow j$ means that both the walk from i to j and the walk from j to i exist.

[2] Remark that the existence of such a walk is equivalent to $t_{ij}^{(k)} > 0$.

The **canonical form** of a matrix T with q essential classes and w inessential classes is

$$T = \left[\begin{array}{cccc|c} T_1 & 0 & \dots & 0 & 0 \\ 0 & T_2 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & & 0 \\ 0 & \dots & 0 & T_q & 0 \\ \hline \multicolumn{4}{c|}{R} & Q \end{array}\right] \tag{2.1}$$

with

$$Q = \begin{bmatrix} Q_1 & 0 & \dots & 0 \\ & Q_2 & & \vdots \\ \vdots & & \ddots & \\ S & \dots & & Q_w \end{bmatrix}$$

where elements in N have been ordered so that essential classes come first (in any order), then inessential classes, so that if for any i and j in two distinct inessential classes, i is ranked before j, we have $i \nrightarrow j$. We have

$$T^k = \left[\begin{array}{cccc|c} T_1^k & 0 & \dots & 0 & 0 \\ 0 & T_2^k & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & & 0 \\ 0 & \dots & 0 & T_q^k & 0 \\ \hline \multicolumn{4}{c|}{R_k} & Q_k \end{array}\right] \tag{2.2}$$

with

$$Q_k = \begin{bmatrix} Q_1^k & 0 & \dots & 0 \\ & Q_2^k & & \vdots \\ \vdots & & \ddots & \\ S_k & \dots & & Q_w^k \end{bmatrix},$$

where S_k and R_k are in general difficult to compute.

Fact 2.1. $\lim\limits_{k\to\infty} Q_k = \mathbf{0}$.

Definition 2.6 (regular)**.** *T is regular if the limit $T^\infty = \lim_{k\to\infty} T^k$ exists.*

Definition 2.7 (fully regular)**.** *T is fully regular if $T^\infty = \mathbb{1}_n p'_\infty$ for some $p_\infty \in \mathbb{R}^n$, i.e., T^∞ has identical rows.*

Definition 2.8 (primitive)**.** *T is primitive if $T^k > 0$ for some integer k, i.e., each entry is positive.*

Definition 2.9 (irreducible). *T is irreducible if for every $i, j \in N$, $\exists$ an integer $m(i,j)$ such that $t_{ij}^{m(i,j)} > 0$.*

Definition 2.10 (period). *The period $\tau(i)$ of node $i \in N$ such that $i \to i$ is the greatest common divisor of those k satisfying $t_{ii}^k > 0$, i.e., k is the length of a walk from i to i.*

If $i \leftrightarrow j$, then $\tau(i) = \tau(j)$. So if i and j are from the same class, then they have the same period, and this is said to be the period of the class.

Definition 2.11 (aperiodic). *A class C is aperiodic if $\tau(i) = 1$ for some i in the class.*

Fact 2.2. *T is primitive if T is irreducible and aperiodic.*

Definition 2.12 (spectrum). *The spectrum of T is the set of its eigenvalues.*

Definition 2.13 (spectral radius). *The spectral radius of T is the largest absolute value of its eigenvalues (i.e., supremum among the absolute values of the elements in its spectrum), denoted by $\rho(T)$.*

Definition 2.14 (characteristic equation/polynomial, eigenvalue). *The characteristic equation or the characteristic polynomial of T is $\det(T - \lambda I) = 0$ (i.e., $Tu = \lambda u$ has a nonzero solution), with I the identity matrix. The solutions λ of this equation are called eigenvalues of T.*

Theorem 2.1 (Perron-Frobenius for primitive matrices). *If T is primitive, then $\exists$ an eigenvalue r such that:*

(i) *r is real and $r > 0$;*

(ii) *With r can be associated a left and right eigenvector with positive components;*

(iii) *$r > |\lambda|$ for every eigenvalue λ different from r;*

(iv) *There is a unique eigenvector associated to r, up to a multiplicative constant;*

(v) *If $\mathbf{0} \leq B \leq T$ and β is an eigenvalue of B, then $|\beta| \leq r$. Moreover, $|\beta| = r$ implies $B = T$;*

(vi) *r is a simple root of the characteristic equation of T.*

Fact 2.3. *If in addition T is row-stochastic, then $r = 1$.*

Fact 2.4. *If T is primitive, then $\exists v$ which is a left eigenvector of T, such that $\lim_{k\to\infty} T^k = \mathbf{1} \cdot v'$.*

Lemma 2.1. *Let T be a finite $n \times n$ matrix such that $\lim_{k\to\infty} T^k = 0$. Then $[I-T]^{-1}$ exists and $[I-T]^{-1} = \sum_{k=0}^{\infty} T^k$ with $T^0 = I$.*

Definition 2.15 (M-matrix)**.** *A square matrix Z is an M-matrix if it admits a decomposition $Z = sI - A$, with $s \geq \rho(A)$ and A is nonnegative, where $\rho(A)$ is the spectral radius of A.*

Lemma 2.2. *$Z = (z_{ij})$ is an M-matrix if $z_{ij} \leq 0$ when $i \neq j$ and $z_{ii} \geq \sum_{j\neq i} |z_{ij}|$.*

Definition 2.16 (Laplacian matrix)**.** *Given a weighted graph $\mathcal{G} = (V, E, A)$ where A is a weight matrix, its Laplacian matrix is defined by*

$$L[A] = (l_{ij})_{i,j\in V}, \text{ where } l_{ij} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{k\neq i} a_{ik}, & i = j. \end{cases} \tag{2.3}$$

Fact 2.5. *The Laplacian matrix is an M-matrix and 0 is an eigenvalue since $L[A]\cdot\mathbf{1} = \mathbf{0}$.*

Definition 2.17 (Hurwitz matrix)**.** *A square matrix T is Hurwitz if all of its eigenvalues have a strictly negative real part. Then the differential equation $\dot{x} = Ax$ is asymptotically stable, i.e., $x(t) \to 0$ when $t \to \infty$.*

Definition 2.18 (Schur (stable) matrix)**.** *A square matrix T is a Schur (stable) matrix if $\rho(T) \leq 1$, i.e., the spectral radius of T is strictly less than one. That is, all eigenvalues of T lie inside the unit circle. In this case, the dynamical system $x(t+1) = Ax(t)$ is stable.*

2.2 The DeGroot Model and The Abelson's Model

2.2.1 The DeGroot model

Consider a society of n agents, whose opinions are denoted as $x_1, \ldots, x_n$, situated in a weighted graph[3], associated with the influence matrix W. The element of the influence matrix w_{ij} refers to the weight that agent i put on the opinion of agent j, i.e., w_{ij} measures to what extent agent j influences agent i. W is nonnegative and row-stochastic. Agents exchange opinions and thereafter update opinions in each discrete time slot. Mathematically, at time $t = 0, 1, 2, \ldots$, agent i's opinion $x_i(t)$ evolves as follows:

$$x_i(t+1) = \sum_{j=1}^{n} w_{ij} x_j(t). \tag{2.4}$$

[3]The terms *graph* and *network* will be used interchangeably in this thesis.

Denote the vector of opinions of all agents at time t as $x(t) = (x_1(t), \dots, x_n(t))^T$, then the opinion dynamical system is given by

$$x(t+1) = Wx(t). \tag{2.5}$$

Definition 2.19. *The model (2.5) is convergent if the limit* $x(\infty) = \lim_{k\to\infty} x(k) = \lim_{k\to\infty} W^k x(0)$ *exists,* $\forall x(0) \in \mathbb{R}^n$*.*

Definition 2.20. *A convergent model (2.5) is said to reach a consensus if* $x_1(\infty) = \ldots = x_n(\infty)$*,* $\forall x(0) \in \mathbb{R}^n$*.*

Convergence and consensus conditions The conditions of convergence and consensus are direct consequences of the results on nonnegative matrices presented in Section 2.1.

Fact 2.6. *If W is regular, then the model (2.5) is convergent; if W is fully regular, then the model (2.5) reaches a consensus.*

Lemma 2.3. *If W is irreducible, then the model (2.5) is convergent if and only if W is primitive. In this case the model (2.5) also reaches a consensus.*

Obtaining the steady-state opinion The argument t is dropped from $x(t)$ and $x_i(t)$ if there is no risk of confusion. The influence matrix W can be put under the canonical form 2.1, after identification of the essential and inessential classes. Partition x into the inessential and essential classes, i.e., $x = [x_E, x_I] = [x_{E_1}, \cdots, x_{E_q}, x_{I_1}, \cdots, x_{I_w}]$. Denote the steady-state opinion vector as $\bar{x} = [\bar{x}_E, \bar{x}_I]$, where $\bar{x} = \lim_{t\to\infty} x(t)$. The model (2.5) can be written as:

$$\begin{cases} x_E(t+1) = T_i x_E(t) \\ x_I(t+1) = Rx_E(t) + Qx_I(t). \end{cases}$$

For essential classes, $\bar{x}_{E_i} = T_i^\infty x_{E_i}(0)$, $i = 1, \dots, q$. According to fact 2.4, for each primitive T_i, $\exists v(i)$ which is a left eigenvector of T_i, such that $T_i^\infty = \mathbf{1} \cdot v'(i)$. Therefore,

$$\bar{x}_{E_i} = \mathbf{1} \cdot v'(i) x_{E_i}(0), \tag{2.6}$$

i.e., each agent in E_i converges to consensus $v'(i)x_{E_i}(0)$. On the other hand, for inessential classes, the steady-state vector must satisfy

$$\bar{x}_I = R\bar{x}_E + Q\bar{x}_I, \tag{2.7}$$

i.e., $\bar{x}_I = (I - Q)^{-1} R \bar{x}_E$, if $I - Q$ is invertible. By fact 2.1 and lemma 2.1, $I - Q$ is always invertible. Thus the steady-state opinion always exists, obtained by 2.6 and 2.7.

Social power Suppose W is primitive, then all agents converge to consensus $\bar{x}$, obtained by $\bar{x} = v'x(0)$, where v is the left eigenvector of W. As the final consensus is a weighted sum of the initial opinion, the ith element of v is called the *social power* of agent i in the final consensus.

Stubborn agents in the DeGroot model

Definition 2.21. *An agent i is said to be stubborn if $x_i(t) = x_i(0), \forall t \in \mathbb{N}$, i.e., $w_{ii} = 1$.*

Fact 2.7. *If more than one stubborn agent exist, then consensus is impossible to be reached.*

Corollary 2.1. *Proskurnikov and Tempo (2017) If there are $s \geq 1$ stubborn agents, who are connected by walks to all other agents, then the model (2.5) is convergent. The final opinion $x\infty$ is a convex combination of the opinions of stubborn agents.*

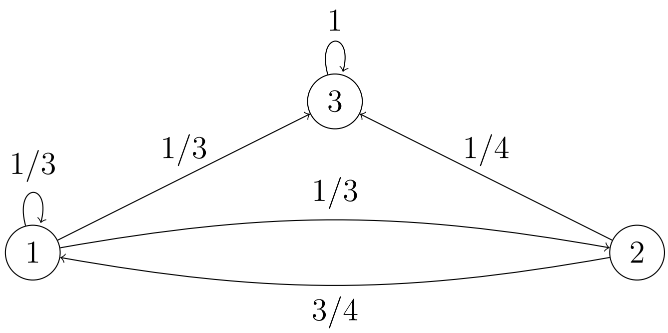
Example 2.1. *Consider the DeGroot model with $n = 3$ agents, corresponding to the graph in Figure 2.1. Agent 1 has equal weights on all agents. Agent 2 has no self-confidence since $w_{22} = 0$ and trust more agent 3 than agent 1. Agent 3 is a stubborn agent since $w_{33} = 1$ and will keep the initial opinion unchanged. Obviously $x_3(t) = 9, \forall t$. Rearranging the weight matrix such that the essential class (i.e., agent 3) comes first and then the inessential class (agents 2 and agent 1), we obtain:*

$$W_{canonical} = \begin{bmatrix} 1 & 0 & 0 \\ 3/4 & 0 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

By $\bar{x}_E = \bar{x}_3 = 9$ and 2.7, we have

$$\begin{bmatrix} \bar{x}_2 \\ \bar{x}_1 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/3 \end{bmatrix} 9 + \begin{bmatrix} 0 & 1/4 \\ 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} \bar{x}_2 \\ \bar{x}_1 \end{bmatrix} \tag{2.8}$$

which gives the solution $\bar{x}_2 = \bar{x}_1 = 9$. Remark that this is in accordance with Corollary 2.1.



(a) The network structure

$$W = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 0 & 3/4 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) The influence matrix W

Figure 2.1: Example 2.1 of the DeGroot model with $n = 3$ agents, among which agent 3 is stubborn. The steady-state opinion is fully determined by the initial opinion of the stubborn agent: $x_1(\infty) = x_2(\infty) = x_3(\infty) = 9$.

2.2.2 The linear Abelson's model

The linear Abelson's model is the continuous-time version of the DeGroot model (2.5) obtained by supposing the time between two steps of iteration is sufficiently small:

$$\dot{x}_i(t) = \sum_{j \neq i} a_{ij}(x_j(t) - x_i(t)), i = 1, \ldots, n \tag{2.9}$$

where $A = (a_{ij})$ is nonnegative but not necessarily row-stochastic. Its equivalent matrix form is

$$\dot{x}_i(t) = -L[A]x(t), \tag{2.10}$$

where $L[A]$ is the Laplacian matrix (2.3) of the weighted graph corresponding to A.

Converge and consensus conditions

Fact 2.8. *The model (2.10) is always convergent.*

Fact 2.9. *For any $A \geq 0$, $P^\infty = \lim_{t \to \infty} e^{-L[A]t}$ exists and the opinion vector in 2.10 converges to $x^\infty = P^\infty x(0)$.*

Fact 2.10. *Consensus is reached in model (2.10) if $\mathcal{G}[A]$ is strongly connected, and in this case the consensus vector is $x_1^\infty = \cdots = x_n^\infty = p'_\infty x(0)$ with p_∞ uniquely given by $p'_\infty \mathbf{1} = 1$ and $p'_\infty L[A] = 0$.*

Social power Similar to the DeGroot model, if consensus is reached in model (2.10), then the ith element of p_∞ is the social power of agent i in the final consensus.

2.3 The Friedkin-Johnsen Model and the Taylor's Model

2.3.1 The Friedkin-Johnsen Model

The Friedkin-Johnsen Model (F-J model for short) is given by

$$x(t+1) = \Lambda W x(t) + (I - \Lambda)u \tag{2.11}$$

where W is a nonnegative and row-stochastic weight matrix, $\Lambda = diag(\lambda_1, \dots, \lambda_n)$ is a *diagonal* matrix with $\lambda_i \in [0, 1]$ the *susceptibility* of agent i to social influence, and u is a constant vector of agents' *prejudices.*

Remark 2.1.
1) If $\Lambda = I$, then the F-J model recovers the DeGroot model.
2)The traditional F-J model supposed that $u = x(0)$, i.e., prejudices of agents are their initial opinions (Friedkin and Johnsen (1999)).

Definition 2.22 (prejudiced agent)**.** *Agent i is prejudiced if $\lambda_i < 1$.*

Definition 2.23 (P-(in)dependent)**.** *Agent i (situated in a graph $\mathcal{G}(A)$) is $P-dependent$ (prejudice-dependent) if i is prejudiced or there exists a walk from some prejudiced agent j to i in the graph $\mathcal{G}(A)$. Otherwise, agent i is P-independent.*

Converge conditions Writing $x = [x^1 x^2]$ with x^1, x^2 the vectors pertaining to the P-dependent and P-independent agents, respectively, the model (2.11) can be decomposed as

$$x^1(t+1) = \Lambda^{11}[W^{11}x^1(t) + W^{12}x^2(t)] + (I - \Lambda^{11})u^1 \tag{2.12}$$

$$x^2(t+1) = W^{22}x^2(t).\text{[4]} \tag{2.13}$$

Fact 2.11.
1) The system 2.12 is asymptotically stable, i.e., $\Lambda^{11}W^{11}$ is Schur stable.
2) The model (2.11) is convergent if and only if all agents are P-dependent or the model 2.13 is convergent (i.e., W^{22} is regular).

Steady-state opinions If the model 2.13 is convergent, then the steady-state opinion vector $x^2(\infty)$ can be obtained by applying the results on the Degroot model (Section 2.2). Then

$$x^1(\infty) = V \begin{bmatrix} u^1 \\ x^2(\infty) \end{bmatrix}$$

[4] Remark that W^{22} is row-stochastic, so the P-independent agents obey the DeGroot model.

where $V = (I - V^{11}W^{1}1)^{-1}[I - \Lambda^{11}\Lambda^{11}W^{12}]$.[5]

Social power Recall that the social power of agent i in the Degroot model is defined as the ratio of the initial opinion $x_i(0)$ to the final consensus opinion. This notion is extended to the F-J model when $u = x(0)$, as the mean weight of $x_i(0)$ in determining the final opinions (Friedkin (1991), Friedkin (2015)). The *Friedkin's influence centrality* vector is defined as $c := \frac{1}{n}V'\mathbb{1}_n$, which is obtained from

$$\frac{1}{n}\sum_{i=1}^{n} x_i(\infty) = \frac{1}{n}V'\mathbb{1}_n x(0) = c'x(0).$$

2.3.2 The Taylor's model

Introduced by Taylor (1968), the Taylor's model is an extension of the linear Abelson's model as well as the continuous-time counterpart of the the F-J model. Additional to the opinion vector $x(t)$, it is also assumed that there are m communication sources providing static opinions $s_1, \ldots, s_m$. The model is given by

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)) + \sum_{k=1}^{m} b_{ik}(s_k - x_i(t)), \tag{2.14}$$

where B is the matrix of persuability constants. This is equivalent to the Abelson's model with k stubborn agents. Mathematically, it is equivalent to the following model:

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)) + \gamma_i(u_i - x_i(t)). \tag{2.15}$$

Similar to the F-J model, u_i is the prejudice of agent i, and agent i is prejudiced if $\gamma_i > 0$.[6] 2.15 can be put in matrix form

$$\dot{x}_i(t) = -(L[A] + \Gamma)x(t) + \Gamma u, \tag{2.16}$$

where $\Gamma = diag(\gamma_1, \ldots, \gamma_n)$. P-(in)dependent agents are defined in the same way as in the F-J model. Similarly, writing $x = [x^1 x^2]$ with x^1, x^2 the vectors pertaining to the P-dependent and P-independent agents, respectively, the model (2.16) can be decomposed as

$$\dot{x}^1(t) = -(L^{11} + \Gamma^{11})x^1(t) - L^{12}x^2(t) + \Gamma^{11}u^1 \tag{2.17}$$

[5] If all agents are P-independent, then V reduces to $(I - \Lambda W)^{-1}(I - \Lambda)$.
[6] Recall that in the F-J model, Agent i is prejudiced if $\lambda_i < 1$ (Def. 2.22).

$$\dot{x}^2(t) = -L^{22}x^2(t). \tag{2.18}$$

Steady-state opinions

Fact 2.12.
1) The model (2.17) is asymptotically stable, i.e., $-(L^{11}+\Gamma^{11})$ is Hurwitz.
2) L^{22} is Laplacian, so the P-independent agents obey the Abelson's model, i.e., the model 2.18 is always convergent. Then the steady-state opinion vector $x^2(\infty)$ can be obtained by applying the results on the Abelson's model (Section 2.2.2).
3) The steady-state opinion vector of the P-dependent agents is given by

$$x^1(\infty) = M \begin{bmatrix} u^1 \\ x^2(\infty) \end{bmatrix}, \text{ with } M = (L^{11}+\Gamma^{11})^{-1}[\Gamma^{11} - L^{12}]). \tag{2.19}$$

4) The matrix M in 2.19 is stochastic, so $x_i(\infty)$ is a convex combination of u^1 and $x^2(\infty), \forall i \in N$.

2.4 The Bounded Confidence Model

The models of continuous opinion dynamic under bounded confidence, proposed by Krause (2000) and Deffuant et al. (2000) independently, with the latter being a gossip-based counterpart of the former model. Gossiping is one of approaches to model asynchronous interactions, assuming that agents interact in pairs instead of simultaneous interaction.

2.4.1 The Hegselmann-Krause model

The HK model is considered as time-varying extension of the DeGroot model, in the sense that the weight matrix $W(t)$ is changing over time, since agents only interact with those in their *confidence interval* which is state-dependent (i.e., related to the current opinion value). The *confidence interval* of agent i is defined as $[x_i - d, x_i + d] \subset \mathbb{R}$ with $d > 0$ the *range of confidence*. The set of agents in i's confidence interval is denoted as $CI_i(x) = \{j : |x_j - x_i| \le d\}$. Agents in the set $CI_i(x)$ are *trusted* by agent i.
$\forall i \in N$, agent i's opinion evolves as follows:

$$x_i(t+1) = \frac{\sum_{j \in CI_i(x)} x_j(t)}{|CI_i(x(t))|}, i = 1, \ldots, n \tag{2.20}$$

Definition 2.24 (*d-chain* and *maximal d-chain*)**.** *The opinions* $(x_i, \dots, x_j)$ *constitute a d-chain if the distances between two consecutive opinions* $x_{m+1} - x_m, \forall m = 1, \dots, j-1$ *are* $\leq d$*. A d-chain is maximal if it is not contained by any longer d-chain.*

Lemma 2.4. *Two different maximal d-chains cannot merge.*

Lemma 2.5. Krause (2000) *The model (2.20) preserves the order of* $x_1, \dots, x_n$*, i.e., if* $x_{i_1}(t) \leq \dots \leq x_{i_n}(t)$*, then* $x_{i_1}(t+1) \leq \dots \leq x_{i_n}(t+1)$*.*

Fact 2.13. Dittmer (2001) $\forall x(0)$*, the model 2.20) terminates in finite time steps. The final opinion* $\bar{x}$ *and the termination time depend on* $x(0)$ *and* d*.* $\forall i, j \in N$*, either* $\bar{x}_i = \bar{x}_j$ *or* $|\bar{x}_i - \bar{x}_j| > d$ *is true.*

2.4.2 The Deffuant-Weisbuch model

At each stage, a pair of agents, say i and j, is chosen at random to exchange their opinions. i and j interact if and only if they have close opinions. Denote the indicator function of an event A as $\mathbb{I}(A)$. The DW model is given by:

$$
\begin{aligned}
x_i(t+1) &= x_i(t) + \mu(x_j(t) - x_i(t))\mathbb{I}_k \\
x_j(t+1) &= x_j(t) + \mu(x_i(t) - x_j(t))\mathbb{I}_k \\
x_k(t+1) &= x_k(t), \forall k \neq i, j \\
\mathbb{I}(t) &= \mathbb{I}(|x_j(t) - x_i(t)| \leq d),
\end{aligned}
\tag{2.21}
$$

where μ is the *convergence parameter* which refers to the attraction between opinions.

Fact 2.14. *The model (2.21) is convergent almost surely, i.e.,* $\forall i \in N$*, the probability that* $\lim_{t\to\infty} x_i(t)$ *exists is 1. Denote that* $\bar{x}_i = \lim_{t\to\infty} x_i(t)$*, then* $\forall i, j \in N$*, one almost surely has either* $\bar{x}_i = \bar{x}_j$ *or* $|\bar{x}_i - \bar{x}_j| > d$*.* [7]

2.5 The Threshold Model

Remark that from Section 2.2 to Section 2.4, models of continuous opinion dynamics are discussed, while in this section and the next section, models of binary opinion dynamics will be discussed.

The threshold model focuses on the dynamics of binary opinions, supposing that each agent $i \in N$ is associated with a private threshold $\theta_i \in [0, 1]$. Denote the neighborhood

[7] Remark that this is in accordance with the results of HK model.

of agent i as N_i which is defined by the network. At each time t, agent i holds opinion $x_i(t) \in \{0,1\}$. Then the threshold model (Granovetter (1978), Schelling (2006)) is defined by:

$$x_i(t+1) = \begin{cases} 1, & \text{if } \frac{\sum_{j \in N_i} x_j(t)}{|N_i|} \geq \theta_i \\ 0, & \text{otherwise.} \end{cases} \tag{2.22}$$

Agents display inertia in switching opinions, but once their thresholds have been reached, the action of even a single neighbor can tip them from one opinion to another. The following theorem says that the threshold model either converges to a constant vector $\bar{x}$ or enters a cycle of period 2 in a general form.

Theorem 2.2. *Let $W = (w_{ij})$ be a symmetric square matrix, i.e., $w_{ij} = w_{ji}, \forall i, j \in N$, where $w_{ij} \in \mathbb{R}$ and threshold values $\theta_i \in \mathbb{R}, i \in N$. If the opinion evolves as follows:*

$$x_i(t+1) = \begin{cases} 1, & \textit{if} \sum_{j=1}^{n} w_{ij} x_j(t) \geq \theta_i \\ 0, & \textit{otherwise,} \end{cases} \tag{2.23}$$

then, $\forall x(0) \in \{0,1\}^n$, $\exists t^* \in \mathbb{N}$, such that $\forall t > t^*$, it holds that $x(t+2) = x(t)$.

Remark 2.2.
1) This theorem only holds for a symmetric matrix W.
2) If $\theta_i = 1/2, \forall i \in N$, then the smallest t^* is $|E|$, where $G = (N, E)$ is the network.

Granovetter (1978) studied the threshold model in the context of a mob, where the available actions were "to riot" (action 1) or to be "inactive" (action 0). One remarkable result was that a cascade effect occurs, supposing that the population of agents situated in a complete network, i.e., $N_i = N, \forall i \in N$. Suppose that agents starts from an initial state where nobody is active, and that the distribution of the threshold value is uniform over the population, i.e., the fractions of agents with thresholds $0, 1/n, \ldots, (n-1)/n$ are all equal to $1/n$. Then, agents with threshold 0 are called "instigators" as they start to riot alone, which leads to a domino effect and forms the seed of the cascade effect, ending in a mob rioting, i.e., all agents become active. More generally, suppose F is the cumulative distribution function of the threshold. Then the set of fixed points of F coincide with the set of equilibrium average opinion of the society, i.e., depending on the starting point, the average opinion of N converges to one of these fixed points.
Remark that the original threshold model supposes that agents are conformist agents since they will take action 1 if sufficiently enough people takes action 1. Ramazi et al. (2016)

proposed a "network game" ($\Gamma := (\mathbb{G}, \tau, \{+ \text{ or } -\})$) such that the best-response dynamics are in the form of the threshold model, where Γ is the network, τ is the threshold vector and $+$ or $-$ correspond to the case of all conformist agents or all anti-conformist agents. Furthermore, they studied the cases of asynchronous updating where only one agent is chosen at random to update opinion, of synchronous updating where all agents update opinions simultaneously and of partial synchronous updating where several agents update opinions at each time step. Assuming that every asynchronous activation sequence driving the dynamics is *persistent*, i.e., $\forall t$, each agent is guaranteed to be active at some finite future time, the following holds.

Fact 2.15.
1) Every network of all anti-conformist agents (resp., all conformist agents) who play $\Gamma := (\mathbb{G}, \tau, \{-\})$ *(resp.,* $\Gamma := (\mathbb{G}, \tau, \{+\})$*) asynchronously will reach an equilibrium in finite time (regardless of the distribution of threshold).*
2) The network game $\Gamma := (\mathbb{G}, \tau, \{-\})$ *(resp.,* $\Gamma := (\mathbb{G}, \tau, \{+\})$ *) admits a pure Nash Equilibrium.*

Assuming that in partial synchronous updating case, for every agent, the inter-activation times are drawn from mutually independent probability distributions with support on $\mathbb{R}_{\geq 0}$, the following holds.

Fact 2.16.
1) Every network of all anti-conformist agents (resp., all conformist agents) who play $\Gamma := (\mathbb{G}, \tau, \{-\})$ *(resp.,* $\Gamma := (\mathbb{G}, \tau, \{+\})$*) with partially synchronous updates almost surely reach an equilibrium in finite time.*

One remarkable conclusion of these results is that synchrony, population heterogeneity and irregular network topology are not enough for the presence of cycles or non-convergence, while the other factors such as the coexistence of anti-conformist and conformist agents must play a role (Ramazi et al. (2016)). Later on, Vanelli et al. (2019) studied the network games where both anti-conformist and conformist agents coexist and provide a complete characterization of the set of Nash equilibria for the complete network.

2.6 The Sznajd model and the q-voter Model

2.6.1 The Sznajd model

The Ising spins chain is defined as n agents ordered in a line $A_1, A_2, \ldots, A_n$ such that A_i and A_{i+1} are neighbors, and $A_i = +1$ (imaged as $\uparrow$) or -1 (imaged as $\downarrow$), $\forall i = 1, \ldots, n-1$.

The original Sznajd model (Sznajd-Weron and Sznajd (2000)) used the Ising spins chain to study the dynamics of binary opinions (in a closed community) according to the following rules [8]:

(a) At each time, a pair of neighbored spins A_i, A_{i+1} is chosen at random, where $i = 1, 2, \ldots, n-1$;

(b) If $A_i A_{i+1} = 1$, then $A_{i-1} = A_{i+2} = A_i$;

(c) If $A_i A_{i+1} = -1$, then $A_{i-1} = A_{i+1}$ and $A_{i+2} = A_i$.

Fact 2.17. *The opinion dynamics obeying the above rules lead to three types of steady states: 1) ferromagnetic state $FS = [1, \ldots, 1]$; 2) antiferromagnetic state $AS = [-1, \ldots, -1]$; 3) mixed state $MS = [\ldots, 1, -1, 1, -1 \ldots]$.*

Suppose now a noise p (which is called "social temperature" in Sznajd-Weron and Sznajd (2000)) is introduce in the model, where p is the probability that one agent makes a random decision instead of following the above rules. Then there exists a threshold p^*, such that for all $p < p^*$, the opinion dynamics will reach one of the three steady states. A later modification on the Sznajd model in Slanina et al. (2008) replaced the rules (b) and (c) by:

(b') If $A_i A_{i+1} = 1$, then $A_{i-1} = A_i$ with probability $1/2$ or $A_{i+2} = A_i$ with probability $1/2$, i.e., if the central pair forms an agreement, only one nearest neighbor update opinion to A_i or A_{i+1};

(c') If $A_i A_{i+1} = -1$, then $A_{i-1} = A_{i-1}$ and $A_{i+2} = A_{i+2}$, i.e., the disagreement of the central pair has no influence on the opinions of its nearest neighbors.

Slanina et al. (2008) provided an analytical form of the exit probability (defined below) for this new dynamics.

Definition 2.25. *For a given opinion dynamics, the exit probability is defined as the probability of reaching consensus on $+1$.*

Denote the probability that one randomly chosen agent is in state $+1$ at the beginning as p. If the initial states of all agents are completely uncorrelated, then the exit probability P_+ is approximated to $p^2/(2p^2-2p+1)$.

[8] This kind of dynamics is called one dimensional outflow dynamics.

2.6.2 The q-voter model

The original voter model was proposed by Clifford and Sudbury (1973), where agents are situated in a static graph. At each time instant, an agent is chosen at random to be active and copies the opinion of a random neighbor. Castellano et al. (2009b) generalized this model to the q-voter model to study the dynamics of binary opinions (with values $+1$ or -1) in a lattice network. At each time t, for a given integer q, one agent is chosen at random, say i, to update opinion according to the following rule:

(i) q neighbors of i are chosen one by one at random where repetition is allowed, so q can be an arbitrary integer;

(ii) If the q neighbors are in the same states $s \in \{+1, -1\}$, then $x_i(t+1) = s$;

(iii) Otherwise, agent i flips with probability ϵ, i.e.,

$$\mathbb{P}(x_i(t+1) = -x_i(t) \mid q \text{ neighbors are in different states}) = \epsilon.$$

Fact 2.18 (Remark that (2), (3) and (4) hold under the assumption of a complete network (i.e., in mean field).)**.**
(1) Denote the fraction of disagreeing neighbors of agent i as p, then the probability that agent flips can be computed as a function of p and q given by:

$$f(p, q) = p^q + \epsilon[1 - p^q - (1-p)^q]. \tag{2.24}$$

(2) The q-voter model with $q = 1$ reduces to the standard voter model, with exit probability proportional to p.
(3) The q-voter model with $q = 2$ and $\epsilon = 0$ coincides with the modified Sznajd model with rules (a), (b) and (c'). The q-voter model with $q = 2$ and $\epsilon = 1/2$ coincides with the voter model, and exhibits a "generalized-voter transition" in the sense that for $\epsilon > 1/2$ the system is disordered (in a paramagnetic phase) and for $\epsilon < 1/2$ the system is ordered (in a ferromagnetic phase).
(4) The q-voter model with $q = 3$ has a voterlike transition at $\epsilon = 1/3$, and separated two ordered (for smaller ϵ) and disordered phases (for larger ϵ) as when $q = 2$.

2.7 Altafini's Model

Remark that from Section 2.2 to Section 2.4, the (continuous) opinion vector at each time t is defined in $\mathbb{R}^n$, while in Section 2.5 and Section 2.6, it is defined in $\{0, 1\}^n$ and

$\{+1,-1\}^n$, respectively. The original Altafini model (Altafini (2012a)) is coincident with the Abelson model with an influence matrix that can have both positive and negative elements, corresponding to a signed graph, also assuming that $\bar{x}(t) \in \mathbb{R}^n, \forall t$.

2.7.1 Results on balance theory

Definition 2.26 (signed graph). *A signed graph $\mathcal{G}$ is a triple $\mathcal{G} = \{N, E, A\}$ where (N, E) is a graph and $A = (a_{ij})$, $a_{ij} \in \mathbb{R}$ is a matrix such that $a_{ij} \neq 0$ if and only if $(j, i) \in E$.*[9]

The followings are assumed throughout this section.

Assumption 2.1.
(1) The graph A has no self-loops, i.e., $a_{ii} = 0$, $\forall i \in N$.
(2) A is symmetric, i.e., $a_{ij} = a_{ji}, \forall i, j \in N$.

Definition 2.27 (structural balancedness). *A signed graph $\mathcal{G} = \{N, E, A\}$ is structurally balanced if N can be partitioned into two disjoint subsets $N = N_1 \cup N_2$ and $\forall i, j \in N$, $i \neq j$, such that*

$$\begin{cases} a_{ij} \geq 0, & \text{if } i, j \in N_m \ (m \in \{1, 2\}) \\ a_{ij} \leq 0, & \text{if } i \in N_m, j \in N_n \ (m, n \in \{1, 2\} \text{ and } m \neq n). \end{cases} \tag{2.25}$$

Otherwise, $\mathcal{G}$ is structurally unbalanced.

Definition 2.28 (generalized Laplacian matrix). *The generalized Laplacian matrix of the signed graph $\mathcal{G} = \{N, E, A\}$ is defined by:*

$$L[A] = (l_{ij})_{i,j \in N}, \ \text{where } l_{ij} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{k \neq i} |a_{ik}|, & i = j. \end{cases} \tag{2.26}$$

Remark 2.3.
(1) $l_{ii} = \sum_{k \neq i} |a_{ik}| = \sum_{k=1}^{n} |a_{ik}|$ since $a_{ii} = 0$;
(2) If A nonnegative, 2.26 coincides with the original Laplacian matrix of a weighted graph 2.3

Definition 2.29 (gauge transformation). *A gauge transformation is a change of orthant order in $\mathbb{R}^n$ performed by a diagonal matrix $D = diag(\sigma)$, where $\sigma = [\sigma_1, \ldots, \sigma_n]$ and $\sigma_i \in \{\pm 1\}, \forall i = 1, \ldots, n$.*

[9]Remark that an arc (i, j) means that i has influence on j.

Denote by L_D as the gauge transformation of L by a diagonal matrix D, i.e., $L_D = DLD$. Then the following holds:

Proposition 2.1. *L and L_D are isospectral: $sp(L) = sp(LD)$, i.e., the gauge transformation preserves the spectrum.*

Lemma 2.6. *The following conditions are equivalent:*

(1) $\mathcal{G} = \{N, E, A\}$ is structurally balanced;

(2) 0 is an eigenvalue of L;

(3) $\exists$ a diagonal matrix D such that DAD has all nonnegative entries;

(4) all cycles (resp., directed cycles) of $\mathcal{G}$ are positive for undirected (resp., directed) $\mathcal{G}$. [10]

Lemma 2.6 says that by doing a gauge transformation, the structurally balanced network can be transformed into the corresponding nonnegative network sharing the same convergence properties (Altafini (2012a)).

Example 2.2. *Let*

$$A = \begin{bmatrix} 0 & 5 & -4 \\ 5 & 0 & -2 \\ -4 & -2 & 0 \end{bmatrix}$$

corresponding to a structurally balanced network. The gauge transformation which transforms A into a nonnegative matrix is $D = diag(1, 1, -1)$. Indeed, we obtain

$$DAD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 5 & -4 \\ 5 & 0 & -2 \\ -4 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 4 \\ 5 & 0 & 2 \\ 4 & 2 & 0 \end{bmatrix}$$

which is a nonnegative matrix isospectral with A.

2.7.2 Model of opinion dynamics

The Altafini model is given by

$$\dot{x}(t) = -L[A(t)]x(t), t \geq 0, \tag{2.27}$$

where $L[A(t)]$ is the generalized Laplacian matrix of $\mathcal{G}$, and $x \in \mathbb{R}^n$.

[10] A cycle is said positive if the product of all weights on the linked arcs are positive.

Convergence

Similar to the Abelson's model, the Altafini model is always convergent.

Proposition 2.2. $\forall A, \forall x(0) \in \mathbb{R}^n$, $\exists \bar{x}_i = \lim_{t\to\infty} x_i(t)$.

Any structurally balanced network can be gauge transformed into a nonnegative matrix with the same spectrum, then it reduces to the Abelson's model. Indeed, opinions will converge to a bipartite consensus defined below.

Definition 2.30. *The model 2.27 admits a bipartite consensus if* $\lim_{t\to\infty} |x_i(t)| = x^* > 0, \forall i \in 1, \ldots, n$.

Theorem 2.3. *If $\mathcal{G}$ is strongly connected and structurally balanced, then the model 2.27 admits a bipartite consensus with solution given by* $\bar{x} = \frac{1}{n}(\mathbf{1}'Dx(0))D\mathbf{1}$. *If $\mathcal{G}$ is structurally unbalanced, then* $\lim_{t\to\infty} x(t) = 0$.

2.7.3 The opposing rule and the repelling rule along negative links

In a recent paper coauthored by Altafini (Shi et al. (2019)), the authors defined two rules for negative links in a signed graph: the opposing rule and the repelling rule. They consider an undirected network $G = (N, E)$ where N is the set of n nodes (agents) and E can be partitioned into the set of positive links and that of negative links $E = E^+ \cup E^-$. Denote the subgraphs $G^+ = (N, E^+)$ and $G^- = (N, E^-)$. The degree of each agent i is denoted by d_i, $\forall i \in N$. The positive (resp., negative) neighborhood of each agent i is denoted as $N_i^+ := \{j, \{i,j\} \in E^+\}$ (resp., $N_i^- := \{j, \{i,j\} \in E^-\}$), with positive (resp.,negative) degree $d_i^+ = |N_i^+|$ (resp., $d_i^- = |N_i^-|$). Throughout this section, it is assumed that G is connected and there is at least one edge in G^-.
Considering any link $\{i,j\} \in E$, the interaction between i and j (isolating all other interactions) is decided by the following rules:

(1) If $\{i,j\} \in E^+$, then they update opinions by the DeGroot rule:

$$\begin{aligned} x_i(t+1) &= x_i(t) + \alpha(x_j(t) - x_i(t)) = (1-\alpha)x_i(t) + \alpha x_j(t), \\ x_j(t+1) &= x_j(t) + \alpha(x_i(t) - x_j(t)) = (1-\alpha)x_j(t) + \alpha x_i(t), \end{aligned} \tag{2.28}$$

with $\alpha \in (0,1)$.

(2) If $\{i,j\} \in E^-$, then they update opinions either by the Opposing rule:

$$\begin{aligned} x_i(t+1) &= x_i(t) + \beta(-x_j(t) - x_i(t)) = (1-\beta)x_i(t) - \beta x_j(t), \\ x_j(t+1) &= x_j(t) + \beta(-x_i(t) - x_j(t)) = (1-\beta)x_j(t) - \beta x_i(t), \end{aligned} \tag{2.29}$$

(2') or by the Repelling rule:

$$\begin{aligned} x_i(t+1) &= x_i(t) - \beta(x_j(t) - x_i(t)) = (1+\beta)x_i(t) - \beta x_j(t), \\ x_j(t+1) &= x_j(t) - \beta(x_i(t) - x_j(t)) = (1+\beta)x_j(t) - \beta x_i(t), \end{aligned} \tag{2.30}$$

with $\beta \in (0,1)$.

By following the opposing rule, the opinion of an agent is attracted by the opposite of the opinion of her neighbor via negative links, as the updating rule for $x_i(t+1)$ can be rewritten as the weighted average of $x_i(t)$ and $x_j(t)$, i.e.,$x_i(t+1) = (1-\beta)x_i(t) + \beta(-x_j(t))$. In this case, the opinion of x_i is not necessarily farther from x_j at $t+1$, when $\{i,j\} \in E^-$. For example, when $x_i(t) < x_j(t) < 0$, $x_i(t)$ (resp., $x_j(t)$) is attracted by $-x_j(t)$ (resp., $-x_i(t)$), both x_i and x_j will move to the right at time $t+1$. However, by following the repelling rule, the two agents with negative links repel each other instead of being attracted like in the DeGroot model. So opinions of two agents with negative links always become farther at the next time step. For example, when $\{i,j\} \in E^-$ and $x_i(t) < x_j(t)$, x_i will move to the left and x_j will move to the right at time $t+1$.

2.7.4 Opposing negative dynamics

With the Opposing negative dynamics defined by 2.28 and 2.29, $x_i(t)$ updates opinion according to

$$x_i(t+1) = (1 - \alpha d_i^+ - \beta d_i^-)x_i(t) + \alpha \sum_{j \in N_i^+} x_j(t) - \beta \sum_{j \in N_i^-} x_j(t). \tag{2.31}$$

Theorem 2.4. Shi et al. (2019)
For opposing negative dynamics with all agents following the updating rules 2.28 and 2.29, if $0 < \alpha + \beta < \frac{1}{\max_{i \in N} d_i}$, *then* $\forall \mathbf{x}(\mathbf{0}) \in \mathbb{R}^n$:

(i) If G is structurally balanced with respect to the partition $N = N_1 \cup N_2$, then

$$\lim_{t\to\infty} x_i(t) = \frac{\sum_{j\in N_1} x_j(0) - \sum_{j\in N_2} x_j(0)}{n}, \forall i \in N_1;$$
$$\lim_{t\to\infty} x_i(t) = \frac{\sum_{j\in N_2} x_j(0) - \sum_{j\in N_1} x_j(0)}{n}, \forall i \in N_2. \tag{2.32}$$

(ii) Otherwise, $\lim_{t\to\infty} x_i(t) = 0, \forall i \in N$.

2.7.5 Repelling negative dynamics

With the Opposing negative dynamics defined by 2.28 and 2.30, $x_i(t)$ updates opinion according to

$$x_i(t+1) = (1 - \alpha d_i^+ + \beta d_i^-)x_i(t) + \alpha \sum_{j\in N_i^+} x_j(t) - \beta \sum_{j\in N_i^-} x_j(t). \tag{2.33}$$

Theorem 2.5. Shi et al. (2019)
For repelling negative dynamics with all agents following the updating rules 2.28 and 2.30, if G^+ is connected, and $0 < \alpha < 1/(\max_{i\in N} d_i^+)$, then $\exists$ a threshold value $\beta^* > 0$ such that:

(i) If $\beta < \beta^*$, then the average consensus is reached, i.e.,

$$\lim_{t\to\infty} x_i(t) = \frac{\sum_{j=1}^{n} x_j(0)}{n}, \forall \mathbf{x}(0) \in \mathbb{R}^n.$$

(ii) If $\beta > \beta^*$, then $\lim_{t\to\infty} ||\mathbf{x}(t)|| = \infty$ for almost all initial values w.r.t. Lebesgue measure.

Remark 2.4. *Theorems 2.4 and 2.5 also hold for the dynamic model of their continuous-time counterparts.*

2.7.6 Directed networks

Suppose now the network $G = (N, E)$ is directed, with $(i, j) \in E$ is a link starting from i to j. The neighborhood of i is partitioned into the positive and negative neighborhoods, respectively, i.e., $N_i = N_i^+ \cup N_i^-$, where $N_i^+ = \{j : (j, i) \in E^+\}$ and $N_i^- = \{j : (j, i) \in$

$E^-\}$, with positive (resp.,negative) degree $d_i^+ = |N_i^+|$ (resp., $d_i^- = |N_i^-|$). Then Theorems 2.4 and 2.5 can be generalized to the following theorems for directed networks.

Theorem 2.6. *Shi et al. (2019)*

For opposing negative dynamics in a directed network G, if $0 < \alpha + \beta < \frac{1}{\max_{i \in N} d_i}$, and G is strongly connected, then $\forall \mathbf{x}(\mathbf{0}) \in \mathbb{R}^n$:

(i) If G is structurally balanced with respect to the partition $N = N_1 \cup N_2$, then $\exists w_1, \ldots, w_n > 0$ with $\sum_{i=1}^{n} w_i = 1$ such that

$$\lim_{t\to\infty} x_i(t) = \frac{\sum_{j\in N_1} w_j x_j(0) - \sum_{j\in N_2} w_j x_j(0)}{n}, \forall i \in N_1;$$
$$\lim_{t\to\infty} x_i(t) = \frac{\sum_{j\in N_2} w_j x_j(0) - \sum_{j\in N_1} w_j x_j(0)}{n}, \forall i \in N_2. \tag{2.34}$$

(ii) Otherwise, $\lim_{t\to\infty} x_i(t) = 0, \forall i \in N$.

Theorem 2.7. *Shi et al. (2019)*

For repelling negative dynamics in a directed network G, if G^+ is strongly connected, and $0 < \alpha < {}^1\!/(\max_{i\in N} d_i^+)$, then $\exists$ a threshold value $\beta^* > 0$ such that $\forall \beta < \beta^*$, $\exists m_1(\beta), \ldots, m_n(\beta) \in \mathbb{R}^+$ with $\sum_{i=1}^{n} m_i(\beta) = 1$ such that a consensus is reached at

$$\lim_{t\to\infty} x_i(t) = \sum_{j=1}^{n} m_j(\beta) x_j(0), \forall \mathbf{x}(0) \in \mathbb{R}^n.$$

For structurally balanced networks, we can think that there are two hostile camps. Agents from the same camp are friends, and those from different camps are enemies. Remark that in 2.28, α can be considered as the trust level of the agent to the opinion of his friends, while in 2.29 and 2.30, β can be considered as the opposing or repelling level of the agent to the opinion of his enemy. The opposing negative dynamics tells that under certain conditions, the structurally balanced networks reach a within-group consensus, while the structurally unbalanced networks reach a consensus on the origin value 0. The repelling negative dynamics tells that if the friendship network G^+ is (strongly) connected, and if both the trust level (α) and the repelling level (β) are sufficiently small, then a consensus will be reached. However, for undirected networks, if the repelling level is larger than the threshold, opinions of agents will tend to (plus or minus) infinity almost everywhere. In models of continuous opinion dynamics, $+\infty$ and $-\infty$ can be referred to as two opposite extreme opinions, such as the left-wing stance and right-wing stance in politics. Thus the results imply that if the repelling level is too large, then every one will hold one of the extreme opinion in the limit behavior, which is unrealistic. One reason leading to this unrealistic result is that the repelling level β is assumed to be homogeneous (the same for every agent) in this model, so one can generalize this model to have heterogeneous βs. On the other hand, based on the assumption that agents hold opinions on $\mathbb{R}$, agents try to repel from enemies, and thus tend to hold the extreme opinion in the limit. One can also generalize this model to different domains of opinions, such as $[0, 1]$ or $[-1, 1]$, where the bounds are two extreme opinions.

Chapter 3

Summary of this thesis and contributions

As described in Section 1.4, unlike the well-developed theory of opinion dynamics with conformity behavior, the studies with anti-conformity behavior are taking their first steps. The main aim of this Ph.D thesis is to study the models of opinion dynamics in a network with both conformity behavior and anti-conformity behavior. More precisely, it aims to answer the following questions: *Given a society of agents in a (fixed or endogenous) network, given a mechanism of influence for each agent, how the behavior/opinion of the agents will evolve with time, and in particular can it be expected that it converges to some stable situation, and in this case, which one? Moreover, what are the conditions (both on the agent- and network-level) required to generate specific network level phenomena, e.g., reaching a consensus/polarization? Moreover, what are the differences between synchronous and asynchronous modelling?*

In Chapter 4, a first study of the threshold model, where both conformist and anti-conformist agents coexist, is provided. The study bears essentially on the convergence of the opinion dynamics in the society of agents, i.e., finding absorbing classes, cycles, etc. Also, we are interested in the existence of cascade effects, as this may constitute a undesirable phenomenon in collective behavior. The study is divided into two parts. In the first one, the threshold model is studied supposing a fixed complete network, where every one is connected to every one, like in the seminal work of Granovetter (Granovetter (1978)). We study the case of a uniform distribution of the threshold, of a Gaussian distribution, and finally give a result for arbitrary distributions, supposing there is one type of anti-conformist. In a second part, the graph is no more complete and we suppose that the neighborhood of an agent is random, drawn at each time step from a distribution. We distinguish the case where the degree (number of links) of an agent is fixed, and where there is an arbitrary degree distribution. We show the existence of cascades and that for most societies, the opinion converges to a chaotic situation.

In the second chapter (5), we study the dynamics of continuous cultural traits (as a specific type of continuous opinions) in an OLG (overlapping generation) structure and in an endogenous social network, where the network changes are inherited. Children learn their cultural trait from their parents and their social environment modelled by network. Parents want their children to adopt a cultural trait that is similar to their own and engage in the socialization process of their children by forming new links or deleting connections. Changing links from the inherited network is costly, but having many links is beneficial. We propose three ways to endogenize the process of network formation. In the first one, the network is supposed to be directed and each dynasty can either form or delete a directed link unilaterally with another dynasty. Therefore, at each period, each family faces a utility opitimization problem where a trade-off between own utility losses and the improvements of child's cultural trait. We have shown that if the cost of network changes is greater than the cost of child care, extremists will never add links, and in the case of sufficiently low cost, extremists may cut all ties with the society. In the second and third models, the network is supposed to be undirected. In the second model, we assume that after each period, a pairwise stable network with transfers (PST network for short) is reached. We have shown the existence of the PST network for each period, however, it is not necessary to be unique. Moreover, a necessary and sufficient condition is given such that a network is PST for given $V(t)$ and $G(t)$. The convergence of cultural traits in this case is guaranteed. In the third model, we assume that after each period, a pairwise stable network (PS network for short) is reached. In this case, there always exist sufficiently small cost parameters such that the empty network is the unique PS network. Regarding the efficiency of the network, we show that there always exist sufficiently small cost parameters such that the empty network is the unique efficient network, and sufficiently large costs of child care such that the complete network is the unique efficient network. Moreover, more detailed dynamics of cultural traits are studied when the costs of network changes and benefits from integration are low, intermediate, and large, respectively.

In Chapter 6, an appropriate updating rule of continuous opinions for anti-conformity behavior is proposed, defined according to the repelling function, which gives the shift of the opinion based on the current opinion and the reference opinion for an agent. Two models of continuous opinion dynamics are studied in undirected networks, by introducing the heterogeneity in the sense of conformity and anti-conformity behavior either in nodes or in links. The study is divided into two parts. In the first one, the society is composed of both conformist and anti-conformist agents. Conformist agents update their opinions following the DeGroot rule with equal weights, however, anti-conformist agents would like to repel from others, and the repelling level is negatively related to the opinion distance between the anti-conformist and her reference point. No consensus will be reached for any connected network in the presence of anti-conformist agent. Instead, opinions converge to a disagreement or oscillate over time. In the second part, by supposing a signed graph where agents have positive links (+1) with their friends and negative links (-1) with their enemies, agents update their opinion as the sum of the averaged opinion of their friends and repelling value from their enemies. When the network is balanced, i.e., there are two communitarian groups, and each sub-network corresponding to each group is connected and the initial opinion ranges of the two group are disjoint, the consensus within each group is guaranteed. Both synchronous and asynchronous updating models are discussed in these two parts.

Chapter 4

Anti-conformism in the threshold model of collective behavior [1]

4.1 Introduction

Human behavior is governed by many aspects, related to social context, culture, law and other factors. Most of these aspects tend to indicate that our behavior is heavily influenced by the behavior of the other people with whom we are in contact, either directly or indirectly by means of communication devices, information media, etc. Behavior refers here to any kind of action, decision to be taken, or opinion to be held on a given topic. As our environment is constantly changing, behavior and opinion of people, including us, are evolving with time, which makes central the following question: *Given a society of agents in a network, given a mechanism of influence for each agent, how the behavior/opinion of the agents will evolve with time, and in particular can it be expected that it converges to some stable situation, and in this case, which one?*

Evidently the question has been studied by sociologists and psychologists, and a number of pioneering models of "opinion dynamics'" have been proposed by them, e.g., Granovetter (1978), Abelson and Conference (1964), French Jr (1956), Friedkin and Johnsen (1990), Taylor (1968), but it has also attracted the attention of many physicists, assimilating agents to particles (this field is usually called "sociophysics", after the work of Galam (2004b); see a survey in Castellano et al. (2009a)), economists (see, e.g., the monograph of Jackson (2010), and the survey by Acemoglu and Ozdaglar (2011)), computer scientists and probabilists (by analogy with (probabilistic) cellular automata, see, e.g., Gravner and

[1] This chapter is a joint work with Michel Grabisch (University of Paris 1 Panthéon-Sorbornne), published in the journal *Dynamic Games and Applications*.
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Dyn Games Appl 10, 444-477 (2020). https://doi.org/10.1007/s13235-019-00332-0.

Griffeath (1998) and the survey by Mossel and Tamuz (2017)), etc.

One of the simplest model of behavior/opinion dynamics when the opinion or behavior is binary (yes/no, active/inactive, action 1 or 0, etc.) is the threshold model, also called the majority rule model (Galam, 2002), proposed by Granovetter (1978), Schelling (2006), among others. This model simply says that an agent takes action 1 if sufficiently enough people in his neighborhood takes action 1. The simplicity of the model allows for a deep analysis (see the surveys by Mossel and Tamuz (2017) and Castellano et al. (2009a)), and one remarkable result already observed in the pioneering work of Granovetter (1978) was that a cascade effect occurs, supposing that the population of agents starts from an initial state where nobody is active, and that the distribution of the threshold value is uniform over the population. Then, after a finite number of steps, all agents become active. Interestingly, the latter study was done in the context of a mob, where the available actions were "to riot" (action 1) or to be inactive (action 0). Then, agents with threshold 0 were called "instigators" as they start to riot alone, and this indeed forms the seed of the cascade effect, ending in a mob rioting. This topic has been very much studied, as demonstrated by a recent monograph on mob control (Breer et al., 2017), written by researchers in control theory.

So far, most models make the basic assumption that agents tend to follow the trend (they are conformist) and that nobody will have a kind of opposite behavior (anti-conformism), choosing action 0 if too many people take action 1. Although the literature on opinion dynamics is vast, very few studies consider that agents may be anti-conformist. In game theory, such kind of opposite behavior has been studied however, in what is called anti-coordination games (see, e.g., Bramoullé et al. (2004), López-Pintado (2009)), congestion games Rosenthal (1973), and fashion games Cao et al. (2013). In sociophysics, the first idea about anti-conformist agents seems to have been introduced by Galam (2004a) under the name of *contrarians*. Later works include those of Sznajd-Weron and also Juul and Porter. In Nyczka and Sznajd-Weron (2013b), the q-voter model is studied, where it is supposed that agents may adopt with some probability an anticonformist attitude, while the threshold model is considered under this assumption in Nowak and Sznajd-Weron (2019). Close to this model is the recent study of Juul and Porter (2019) about the spreading of two competing products, say A and B, where anticonformist agents are called *hipsters* (see Touboul (2019) where this terminology has been introduced). In Juul and Porter (2019), starting from a network with all nodes inactive, a single node is uniformly chosen at random to adopt one product, say A, which buries the seed for the spreading process. They assume the threshold of a player (which can be a conformist or a hipster) as the minimum proportion of their active neighbors such that this player becomes active and the transition from active to inactive is a one-way process. Once the player becomes active, they must adopt one product according to the following rules: if he is a conformist, he will adopt the most popular product over his neighborhood; if he is a hipster, he will adopt the less popular product over the whole population. Under this assumption, they found that even a small proportion of hipsters can lead to a reversal of the popularity of two competing products. The model is similar in Nowak and Sznajd-Weron (2019), in the sense that agents are selected at random for updating and their threshold is the same for all agents, however, an agent is not *a priori* conformist or anti-conformist, but is one or the other with some probability.

The present paper also studies a threshold model where both conformist and anti-conformist agents coexist, but in a rather different setting compared to Juul and Porter (2019) and Nowak and Sznajd-Weron (2019). Firstly, we assume that updating is done at every period for all agents. Secondly, in our setting the thresholds are drawn from a distribution which means that they are random and different, in general. In addition, the two possible states of an agent are not treated symmetrically. These are the assumptions of the seminal paper of Granovetter (1978).

Our paper is in the line of a previous work by the first author Grabisch et al. (2019), whose results will be used at some point in the present paper. Our study bears essentially in answering the main question raised in the first paragraph, that is, on the convergence of the process, analyzing if absorbing states exist (stable state of the society) or if a cycle occurs, or even more chaotic situations. Also, we are interested by the existence of cascade effects, as this may constitute a undesirable phenomenon in collective behavior. We divide our study into two parts. In the first one, we basically study the threshold model supposing a fixed complete network, where every one is connected to every one, like in the work of Granovetter (1978) (Section 4.2). We begin by giving a game-theoretic foundation to this model, which aims to give a rational explanation to human behavior, by means of a mix of coordination and anti-coordination games. Then, we study the case of a uniform distribution of the threshold, of a Gaussian distribution, and finally give a result for arbitrary distributions, supposing there is one type of anti-conformist. In a second part (Section 4.3), the graph is no more complete and we suppose that the neighborhood of an agent is random, drawn at each time step from a distribution. We distinguish the case where the degree (number of links) of an agent is fixed, and where there is an arbitrary degree distribution. Most of the proofs can be found in the Appendix.

4.2 The deterministic threshold model with anti-conformists

4.2.1 The model

Let $N = \{1, \dots, n\}$ be the society of agents. We suppose the existence of an underlying (exogenous) network $G = (N, E)$ whose nodes are the agents and E is the set of (undirected) edges or links. Each agent i has a set of neighbors $\Gamma_i = \{j \in N : \{i, j\} \in E\}$, and $|\Gamma_i| =: d_i$ is the degree of agent i. We consider that $i \in \Gamma_i$ for every agent i.
Two actions (or opinions, states) are available to each agent at every stage: 1 (agree, adopt, join, be active, etc.) and 0 (disagree, refuse, disjoin, be inactive, etc.). The action taken by agent i at stage t is denoted by $a_i(t)$. For short, we will often use the term "active" for agents taking action 1, and "inactive" for agents taking action 0.
In the classical *threshold model* introduced by Granovetter (1978) and Schelling (2006) among others, agent i will take action 1 at next stage if the proportion of his neighbors taking action 1 exceeds some threshold $\mu_i \in [0, 1]$, otherwise action 0 is taken:

$$a_i(t+1) = \begin{cases} 1, & \text{if } \frac{1}{|\Gamma_i|} \sum_{j \in \Gamma_i} a_j(t) \geq \mu_i \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Note that unlike some threshold models, e.g., in Juul and Porter (2019), Watts (2002), an agent having adopted action 1 may return to action 0, because not enough neighbors take action 1.[2]
Such behavior exhibits a tendency to follow the trend, and we call this type of agent a *conformist*. The tendency to do the opposite of the trend is called *anti-conformism*, and can be modelled as follows:

$$a_i(t+1) = \begin{cases} 0, & \text{if } \frac{1}{|\Gamma_i|} \sum_{j \in \Gamma_i} a_j(t) \geq \mu_i \\ 1, & \text{otherwise.} \end{cases} \tag{4.2}$$

When too many people take action 1, then an anti-conformist agent takes action 0, and vice-versa. In the rest of the paper, we denote by N_a the set of anti-conformist agents, and by $N_c := N \setminus N_a$ the set of conformist agents.
Observe that thresholds 0 and 1 play a particular role. For a conformist agent (respectively, an anti-conformist agent), a threshold equal to 0 means that he takes always action 1 (respectively, 0), while a threshold strictly greater than 1 implies to always take action 0 (respectively, 1). We call these agents *constant 0-player* and *constant 1-player*.

[2] When adoption of action 1 stays for ever, one speaks of "switch".

Our aim is to study the evolution of the dynamics of actions taken by the agents. To this aim, we define the *state* of the society at stage t, as the set S (or $S(t)$) of agents taking action 1 at stage t. Depending which one is more convenient, a state is either denoted as a set $S \subseteq N$ or as its characteristic vector $\mathbb{1}_S$ in $\{0,1\}^N$. The process is deterministic and Markovian, i.e., transitions from S to T (denoted by $S \to T$) are with probability 1 and do not depend on states before S.

We are interested in finding absorbing states, i.e., such that $S(t) = S(t+1)$ for some value of t, and cycles, i.e., sequences of transitions $S_1 \to S_2 \to \cdots \to S_k$ where $S_k = S_1$.

4.2.2 A game-theoretic foundation of the threshold models

It is well-known that the classical threshold model can be explained by a local coordination game (see, e.g., Morris (2000)). We show that the anti-conformist threshold model can be explained in a similar way via a local anti-coordination game. We recall first the result for the classical model.

Consider two players (row, column) whose set of strategies is $\{0,1\}$ with the following payoff matrix:

	0	1
0	q, q	0,0
1	0,0	$1-q, 1-q$

with $0 < q < 1$. This is a *coordination game* since the two pure Nash equilibria arise when the players choose the same action. It can be checked that the best response of one player is 1 if he assigns a probability at least q that the other player chooses 1. Consider now our network $G = (N, E)$ and a given player i. Given that player $j \in \Gamma_i$ takes action a_j, player's i best response is to choose action a iff

$$\sum_{j\in\Gamma_i} u^c(a, a_j) \geq \sum_{j\in\Gamma_i} u^c(1-a, a_j)$$

where the utility $u^c(a, a_j)$ is given in the above table. Taking $a = 1$ we find, assuming that m players in Γ_i choose action 1:

$$\sum_{j\in\Gamma_i} u^c(1, a_j) = \sum_{\substack{j\in\Gamma_i \\ a_j=1}} u^c(1,1) + \sum_{\substack{j\in\Gamma_i \\ a_j=0}} u^c(1,0) = m(1-q)$$

$$\sum_{j\in\Gamma_i} u^c(0, a_j) = q(d_i - m).$$

Therefore, player i's best response is 1 iff $m \geq qd_i$, assuming that in case of tie, 1 is chosen. This is exactly (4.1) with $\mu_i = q$. This yields an interpretation for the threshold μ_i: it is the minimum probability that player i assigns to players in his neighborhood for choosing action 1.

We consider now the 2-players *anti-coordination game* given by the following matrix defining the utility u^a (2-dimensional with coordinates u_1^a, u_2^a for row and column players, respectively):

	0	1
0	$0, 0$	$1-q, q$
1	$q, 1-q$	$0, 0$

The game is symmetric in the sense that $u_1^a(a, b) = u_2^a(b, a)$ for all actions a, b, and there are two pure Nash equilibria arising when the players take different actions. It can be checked that the best response of one player is 1 if he assigns a probability *at most* q that the other player chooses 1. Assuming that in case of tie, action 0 is taken, player i's best response is 1 iff

$$\sum_{\substack{j\in\Gamma_i\\ a_j=1}} u_1^a(1,1) + \sum_{\substack{j\in\Gamma_i\\ a_j=0}} u_1^a(1,0) > \sum_{\substack{j\in\Gamma_i\\ a_j=1}} u_1^a(0,1) + \sum_{\substack{j\in\Gamma_i\\ a_j=0}} u_1^a(0,0)$$

which leads to the condition $m < qd_i$. We recover the anticonformist model (4.2) with the same threshold as above $\mu_i = q$. Now, the threshold of player i is the maximum probability that he assigns to players in his neighborhood for choosing action 1.

We now combine both types of players and consider the 2-players game with one conformist (row player) and one anti-conformist (column player) given by the following payoff matrix defining utility u^m (u^m is 2-dimensional, with coordinates u_1^m and u_2^m for the row and column players, respectively, but not symmetric):

	0	1
0	$q, 0$	$0, q$
1	$0, 1-q$	$1-q, 0$

In this game, there is no pure Nash equilibria, but one can check that the best response of the conformist player (respectively, the anti-conformist player) is 1 if he assigns a probability at least q (respectively, at most q) that the other player chooses 1. Consider now a given conformist player i in the network $G = (N, E)$. Assuming that in case of a tie action 1 is taken, player i's best response is 1 iff

$$\sum_{\substack{j\in\Gamma_i\cap N_c\\ a_j=1}} u^c(1,1) + \sum_{\substack{j\in\Gamma_i\cap N_a\\ a_j=1}} u_1^m(1,1) + \sum_{\substack{j\in\Gamma_i\cap N_c\\ a_j=0}} u^c(1,0) + \sum_{\substack{j\in\Gamma_i\cap N_a\\ a_j=0}} u_1^m(1,0) \geq$$
$$\sum_{\substack{j\in\Gamma_i\cap N_c\\ a_j=1}} u^c(0,1) + \sum_{\substack{j\in\Gamma_i\cap N_a\\ a_j=1}} u_1^m(0,1) + \sum_{\substack{j\in\Gamma_i\cap N_c\\ a_j=0}} u^c(0,0) + \sum_{\substack{j\in\Gamma_i\cap N_a\\ a_j=0}} u_1^m(0,0)$$

which leads to the condition $m \geq qd_i$, while if i is anti-conformist, best response is 1 iff

$$\sum_{\substack{j\in\Gamma_i\cap N_c\\ a_j=1}} u_2^m(1,1) + \sum_{\substack{j\in\Gamma_i\cap N_a\\ a_j=1}} u_2^a(1,1) + \sum_{\substack{j\in\Gamma_i\cap N_c\\ a_j=0}} u_2^m(0,1) + \sum_{\substack{j\in\Gamma_i\cap N_a\\ a_j=0}} u_2^a(0,1) >$$
$$\sum_{\substack{j\in\Gamma_i\cap N_c\\ a_j=1}} u_2^m(1,0) + \sum_{\substack{j\in\Gamma_i\cap N_a\\ a_j=1}} u_2^a(1,0) + \sum_{\substack{j\in\Gamma_i\cap N_c\\ a_j=0}} u_2^m(0,0) + \sum_{\substack{j\in\Gamma_i\cap N_a\\ a_j=0}} u_2^a(0,0)$$

assuming that action 0 is taken in case of tie. This leads to the condition $m < qd_i$. Therefore, we can recover the threshold model with the same thresholds as above $\mu_i = q$ for both conformists and anti-conformists. For conformists (respectively, anti-conformists), the threshold is the minimum (respectively, maximum) probability that player i assigns to players in his neighborhood for choosing action 1.

4.2.3 A general result on cycles

There is a well-known result on the threshold model saying that the state converges to either a fixed state (absorbing) or a cycle of length 2. The most general form of this result is provided by Goles and Olivos (1980), where the process has the form

$$a_i(t+1) = \begin{cases} 1, & \text{if } \sum_{j\in N} \alpha_{ij} a_j(t) \geq \theta_i \\ 0, & \text{otherwise,} \end{cases}$$

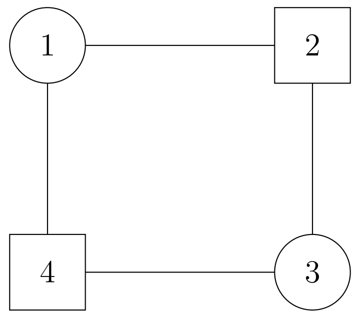
with $\alpha_{ij} = \alpha_{ji} \in \mathbb{R}$, $\theta_i \in \mathbb{R}$ for all i, j. Then there exists $t \in \mathbb{N}$ such that $a_i(t+2) = a_i(t)$. This general result applies to the case of a network of conformists, taking $\alpha_{ij} = 1$ if $\{i,j\} \in E$ and 0 otherwise, and $\theta_i = \mu_i|\Gamma_i|$, but it also applies to the case of a network where all agents are anti-conformist: just put $\alpha_{ij} = -1$ if $\{i,j\} \in E$ and 0 otherwise, and $\theta_i = -\mu_i|\Gamma_i| + 1$. Hence we have obtained:

Theorem 4.1. *Suppose $N_c = \emptyset$ or $N_a = \emptyset$. Then the process converges to either an absorbing state or to a cycle of length 2.*

The result is no more true if the network contains both conformists and anti-conformists, as the following example shows:

Example 4.1. *Consider a graph with $n = 4$, where agents 1 and 3 are conformist, while 2 and 4 are anticonformist, situated as in the figure below, and take $\mu_i = 1/2$ for all $i \in N$. Then we have the following cycle of length 4:*

$$(0,0,0,0) \to (0,1,0,1) \to (1,1,1,1) \to (1,0,1,0) \to (0,0,0,0).$$



4.2.4 Study of the complete network

We suppose in this section that the graph G is complete, i.e., every agent is connected to every other agent, so that the neighborhood Γ_i is N for every agent i.
We begin by recalling the classical result of Granovetter on absorbing states (Granovetter, 1978). Suppose $N = N_c$ and consider the cumulative distribution function F of the threshold of the agents:

$$F(x) = \frac{1}{n}|\{i \in N \, : \, \mu_i \leq x\}| = \frac{1}{n}\sum_{i \in N} \mathbb{1}_{\mu_i \leq x}.$$

This function is right-continuous, nondecreasing and has fixed points. It gives the proportion of agents whose threshold is below or equal to some quantity x, or put otherwise, the proportion of agents that will take action 1 when the current proportion of agents taking action 1 is x. As a consequence, if x^* is a fixed point of F, then $S^* := \{i \in N \, : \, \mu_i \leq x^*\}$ is an absorbing state, and conversely as well.
We generalize this result by incorporating anti-conformism as follows. We express by $G(x)$ the proportion of agents that will take action 1 when the current proportion of agents taking action 1 is x (hence $G(x) = F(x)$ is the cumulative distribution function of the threshold when there is no anti-conformists):

$$G(x) = \frac{1}{n}\Big(\sum_{i \in N_c} \mathbb{1}_{\mu_i \leq x} + \sum_{i \in N_a} \mathbb{1}_{\mu_i > x}\Big), \tag{4.3}$$

with $x \in [0,1]$. This function, which we call the *transition function,* is still right-continuous but is no more nondecreasing in general, and as a consequence, does not necessarily have fixed points (see Figure 4.3). Denote the list of all threshold values of conformist agents (respectively, anti-conformist agents) as $\mu_1^c, \mu_2^c, \ldots, \mu_\alpha^c$ (respectively, $\mu_1^a, \mu_2^a, \ldots, \mu_\beta^a$) in a strict increasing order with fractions $q_1^c, q_2^c, \ldots, q_\alpha^c$ (respectively, $q_1^a, q_2^a, \ldots, q_\beta^a$). The transition function $G(x)$ (Eq. 4.3) can also be written as

$$G(x) = \sum_{\mu_i^c \leq x} q_i^c + \sum_{\mu_j^a > x} q_j^a. \tag{4.4}$$

The following is an immediate generalization of Granovetter (1978). It establishes that the absorbing states of the process are the fixed points of G.

Theorem 4.2. *The process converges to either absorbing states or cycles. It has absorbing state only when G has fixed point(s). The absorbing states of the dynamic process coincide with the fixed points of G as follows: if x^* is a fixed point of G, then*

$$S^* = \{i \in N_c \,:\, \mu_i \leq x^*\} \cup \{i \in N_a \,:\, \mu_i > x^*\} \tag{4.5}$$

is an absorbing state, and vice versa each absorbing state S^* is associated with the fixed point $x^* = |S^*|/n$ of G. Which absorbing state can be reached is dependent on the initial state (when multiple absorbing states exist).

The theorem will be illustrated by several examples in the sequel. We begin our study by supposing that the distribution of the threshold is uniform, then the Gaussian case and the general case will be studied.

Uniform distribution

The case of a uniform distribution permits to get explicit results. It has been studied by Granovetter (1978), in order to explain riot phenomena (action 1: take part to a riot, action 0: be inactive). Supposing at the initial state that all agents are inactive, the presence of agents with threshold 0 (called “instigators” as they start rioting alone) initiates the phenomenon of rioting, which, by a domino or cascade effect, extends to the whole population if the distribution is uniform.

Specifically, we consider the thresholds are uniformly distributed over the set $\{0, {}^1\!/\!{}_n, {}^2\!/\!{}_n, \ldots, {}^{n-1}\!/\!{}_n\}$, as in Granovetter (1978),[3] and that w.l.o.g. we may consider that agent 1 has threshold 0, agent 2 has threshold ${}^1\!/\!{}_n$, etc., and agent n has threshold ${}^{n-1}\!/\!{}_n$.[4] We denote by $\mu_\ell = {}^\ell\!/\!{}_n$ the threshold of agent $\ell + 1$.

[3] We may adopt another definition where the thresholds value from ${}^1\!/\!{}_n$ to 1 (note that there is no constant player then). As we will see below, there is no fundamental change in the results, except for the case $N_a = \emptyset$, where the domino effect would not start and $\emptyset$ would be the only absorbing state.

[4] We consider for ease of notation that only one agent has a given value of threshold. We may

Consider first that $N_a = \emptyset$. As expected, function G (which is in this case the cumulative distribution function F) has only one fixed point, which is $x^* = 1$, corresponding to the absorbing state $S^* = N$ (see Figure 4.1).

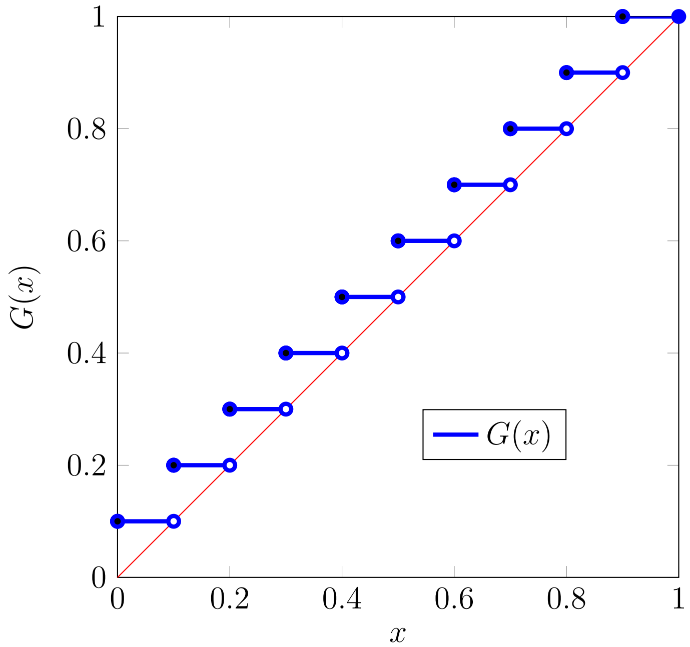


Figure 4.1: The transition function G for uniform distributed threshold model of conformists (i.e. $N_a = \emptyset$). In this example, $n = 10$. G has a unique fixed point, which is $x^* = 1$, corresponding to the absorbing state $S^* = N$.

Introducing one anti-conformist agent Imagine now the conformist agent $k+1$ with threshold k/n becoming anti-conformist with the same threshold, denoted by $\mu_a = k/n$. Suppose for example that $n = 10$ and $k = 3$, which makes agent 4 to be anti-conformist (see Figure 4.2). According to Theorem 4.2, the absorbing states correspond to the fixed points of G, which are, in set notation:

$$\{1, 2, 3\}, \ \{1, 2, 3, 5\}, \ \{1, 2, 3, 5, 6\}, \ldots, \{1, 2, 3, 5, 6, 7, 8, 9, 10\}.$$

Which absorbing state is reached depends on the initial condition. For example, starting from the state vector $(0, \ldots, 0)$, agent 1 and 4 become active, which makes agent 2 and 3 to become active in addition, then agent 4 becomes inactive and no more changes occurs: the absorbing state $\{1, 2, 3\}$ has been reached. If now we start from the state vector

consider a more general situation where several agents have the same threshold. This will be considered in Section 4.2.4 with arbitrary distribution, however, in the case of a uniform distribution, this has no interest as uniformity obliges to have for each value of the threshold the same number of agents, so that everything goes exactly the same as the case of one agent per threshold value.

$(1, \ldots, 1)$, agent 4 becomes inactive but all the other remain active, so that the absorbing state $\{1, 2, 3, 5, 6, 7, 8, 9, 10\}$ is reached.

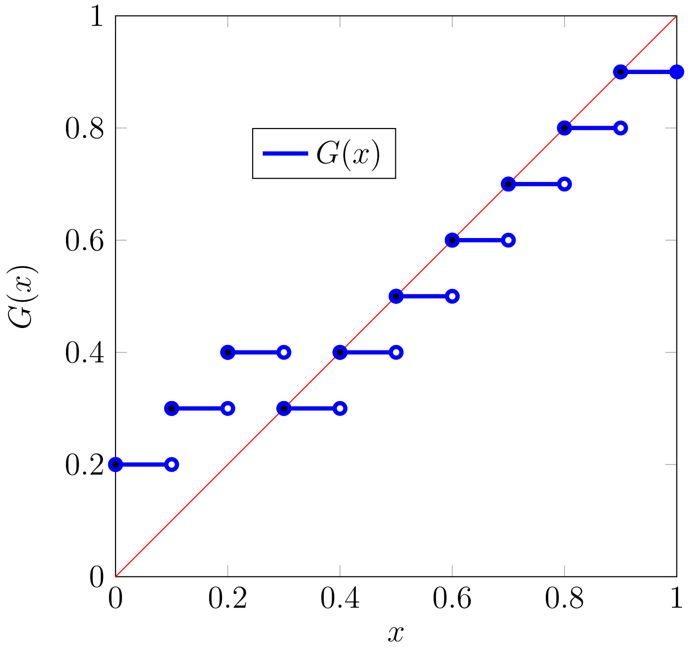


Figure 4.2: The transition function G for uniform distributed threshold model of conformists and anti-conformists when one anti-conformist agent is introduced. In this example, $k = 3$ (i,e., agent 4 is anti-conformist) and $n = 10$. There are seven fixed points of G, which is ${}^3/_{10}, {}^4/_{10}, {}^5/_{10}, \ldots, {}^9/_{10}$, corresponding to the absorbing states, in set notation:$\{1, 2, 3\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 3, 5, 6\}, \ldots, \{1, 2, 3, 5, 6, 7, 8, 9, 10\}$. Which absorbing state is reached depends on the initial condition.

The following proposition summarizes the uniform case with one anti-conformist.

Proposition 4.1. *Consider a group of agents whose thresholds follow a uniform distribution as described above, and suppose that there is only one anti-conformist, say agent $k + 1$, $k \in \{0, 1, \ldots, n - 1\}$, with threshold $\mu_a := {}^k/_n$. Then the opinion dynamic has $n(1 - \mu_a) = n - k$ absorbing states corresponding to the fixed points ${}^k/_n, \ldots, {}^{n-1}/_n$ of G, specifically:*

$$\{1, \ldots, k\}, \{1, \ldots, k, k + 2\}, \{1, \ldots, k, k + 2, k + 3\}, \ldots, N \setminus \{k + 1\}.$$

Moreover, starting from any initial state with the group opinion $x^* = {}^{k^*}/_n$, $k^* \in \{0, 1, \ldots, n\}$, if $k^* = n$, the reachable fixed point is ${}^{n-1}/_n$; if $k^* \geq k$ and $k^* \neq n$, the reachable fixed point is x^*; if $k^* < k$, then the reachable fixed point is ${}^k/_n$ or ${}^{k+1}/_n$ depending on which has the same parity as k^*.

Introducing two anti-conformist agents Imagine now that two conformist agents, say k_1+1, k_2+1 with thresholds $\mu_a^1 = {}^{k_1}\!/_n, \mu_a^2 = {}^{k_2}\!/_n$ become anti-conformist with the same thresholds. Assume w.l.o.g. that $k_1 < k_2$.

Suppose for example that $n = 10$ and $k_1 = 3$ and $k_2 = 5$, which makes agent 4 and agent 6 to be anti-conformists (see Figure 4.3). According to Theorem 4.2, the dynamic has no absorbing states since there is no fixed points of G. Instead, there will be a cycle $S_1 \to S_2 \to S_1$ with $S_1 = \{1,2,3,5\}, \quad S_2 = \{1,2,3,5,6\}$. For example, starting from the state vector $(0,\dots,0)$, agent 1, 4 and 6 become active, which makes agent 2 and 3 to become active in addition, then agent 4 becomes inactive, which will again activate agent 5 at the next stage i.e., the state $S_1 = \{1,2,3,5,6\}$ has reached with $x = {}^5\!/_{10}$. Thus agent 6 becomes inactive at the next stage with the state $S_2 = \{1,2,3,5\}$ and the cycle $S_1 \to S_2 \to S_1$ has been reached.

The following proposition summarizes the uniform case with two anti-conformists.

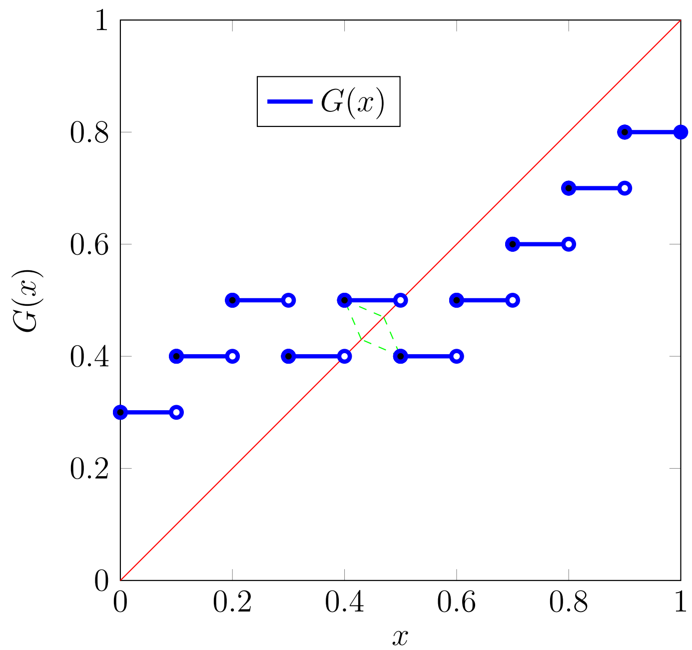


Figure 4.3: The transition function G for uniform distributed threshold model of conformists and anti-conformists when two anti-conformist agents are introduced. In this example, $n = 10$, $k_1 = 3$ and $k_2 = 5$ (i.e., agent 4 and agent 6 are anti-conformists). The dynamic has no absorbing states since there is no fixed points of G. Instead, there will be a cycle materialized in green, which is $S_1 \to S_2 \to S_1$ with $S_1 = \{1,2,3,5\}$ and $S_2 = \{1,2,3,5,6\}$.

Proposition 4.2. *Consider a group of agents whose thresholds follow a uniform distribution as described above, and suppose that there are only two anti-conformists, say agents*

k_1+1, k_2+1 with thresholds $\mu_a^1 = {}^{k_1}/_n$ and $\mu_a^2 = {}^{k_2}/_n$, respectively[5]*, and without loss of generality, assume that $k_1 < k_2$. Then there is no absorbing state but a cycle $S_1 \to S_2 \to S_1$ with*

$$S_1 = \{1, \dots, k_1, k_1+2, \dots, k_2\}, \quad S_2 = \{1, \dots, k_1, k_1+2, \dots, k_2+1\},$$

corresponding to group opinion ${}^{k_2}/_n \to {}^{(k_2-1)}/_n \to {}^{k_2}/_n$, regardless of the initial state.

The general case The previous results tend to indicate that with an odd number of types of anti-conformists, there are absorbing states, while there is no with an even number of types (only cycles occur). The main result of this section shows that this is indeed the case.

Theorem 4.3. *Consider a society N of agents whose thresholds follow a uniform distribution on $\{0, {}^1/_n, \dots, {}^{n-1}/_n\}$ and suppose that some agents are anti-conformists ($N_a \neq \emptyset$). The following holds.*

(i) Suppose that there are $2\ell+1$ ($\ell \geq 1$) anti-conformist agents with thresholds $\mu_a^1, \cdots, \mu_a^{2\ell+1}$, respectively, with $\mu_a^i = {}^{k_i}/_n, (i = 1, \dots, 2\ell+1)$ and $k_1 < k_2 < \cdots < k_{2\ell+1}$.[6] *Then G has fixed points ${}^{k_{\ell+1}}/_n, \cdots, {}^{(k_{\ell+2}-1)}/_n$, whose corresponding absorbing states are given by (4.5).*

(ii) Suppose there are 2ℓ ($\ell \geq 1$) anti-conformist agents with thresholds $\mu_a^1, \cdots, \mu_a^{2\ell}$, respectively, with $\mu_a^i = {}^{k_i}/_n, (i = 1, \dots, 2\ell)$ and $k_1 < k_2 < \cdots < k_{2\ell}$.[7] *Then there is no absorbing state, but there exist cycles of length 2, corresponding to the pairs of points $(x, G(x)), (y, G(y))$ such that $G(y) = x$ and $y = G(x)$, i.e., x is a fixed point of $G^{(2)} = G \circ G$. Moreover, there is no cycle of length greater than 2.*

When the uniform distribution is on $\{{}^1/_n, \dots, 1\}$, it is easy to see that the results are the same as in Theorem 4.3, except that the cases of odd and even numbers of anti-conformists are inverted: there are absorbing states when there is an even number of conformists, and no absorbing states but cycles otherwise. This is because in that case, the function G is shifted of ${}^1/_n$ to the right, and $G(0) = p$, the number of anti-conformists.
We illustrate the above result in the case of cycles with the following example.

Example 4.2. *$n = 10$, $\ell = 2$, $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $k_4 = 3$ (see* Figure *4.4). There is no absorbing state but cycles such as: $\{1,2,3\} \to \emptyset \to \{1,2,3\}$ with group opinion x: ${}^3/_{10} \to 0 \to {}^3/_{10}$. Other cycles might also exist, e.g., $\{3,4\} \to \{4\} \to \{3,4\}$ with group opinion x: ${}^1/_{10} \to {}^2/_{10} \to {}^1/_{10}$.*

[5] $k_1, k_2 \in \{0, 1, \dots, n-1\}$
[6] $k_1, k_2, \dots, k_{2l+1} \in \{0, 1, \dots, n-1\}$
[7] $k_1, k_2, \dots, k_{2l} \in \{0, 1, \dots, n-1\}$

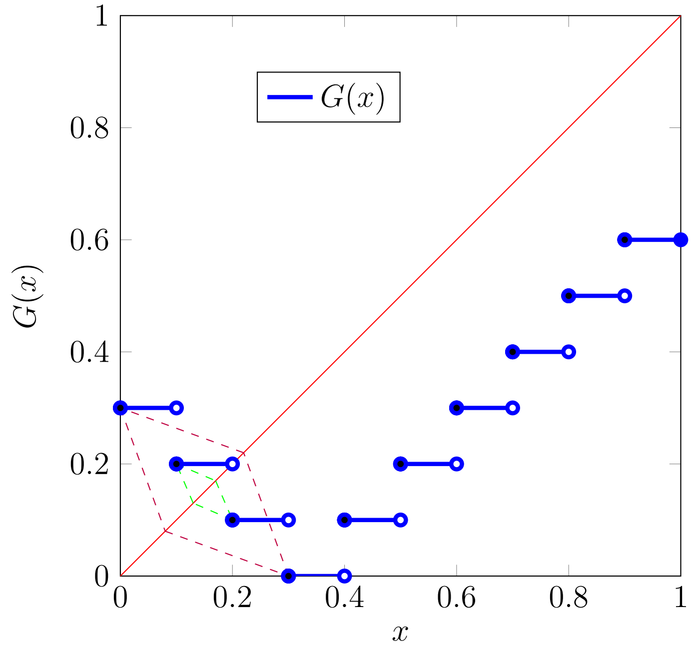


Figure 4.4: The transition function G for uniform distributed threshold model of conformists and anti-conformists when even number of anti-conformist agents are introduced. In this example 5.2, $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $k_4 = 3$ and $n = 10$. The cycles are materialized in red and green, which is $\{1, 2, 3\} \to \emptyset \to \{1, 2, 3\}$ and $\{3, 4\} \to \{4\} \to \{3, 4\}$, respectively.

Applying the results above, we get the case where the society is purely anti-conformist.

Corollary 4.1. *Consider a group of agents whose thresholds follow a uniform distribution. If all agents are anti-conformists, then there exist at most one absorbing state. The existence of absorbing state is decided by the parity of n. If n is even, then there is no absorbing state; if n is odd, then the absorbing state is the action profile associated to the fixed point $^{(n-1)}/_2$.*

Note that this is in accordance with the general result on cycles (Theorem 4.1).

Gaussian distribution

In this section we explore the dynamics of the process for a Gaussian distributed threshold in a complete network. Assume that the thresholds of conformists (respectively, anti-conformists), identically and independently distributed, follow the Gaussian distributions $\mathbf{N}(m_c, \sigma_c)$, $\mathbf{N}(m_a, \sigma_a)$ respectively, with the corresponding cumulative distribution functions F_c, F_a. Then

$$G(x) = qF_c(x) + (1 - q)(1 - F_a(x))$$

where q is the proportion of conformists, and

$$F_c(x) = \frac{1}{\sigma_c\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(t-m_c)^2}{2\sigma_c^2}\right) \mathrm{d}t = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x-m_c}{\sigma_c\sqrt{2}}\right)$$

and similarly for F_a, with $\mathrm{erf}(x)$ the error function defined by $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\mathrm{d}u$. Letting $m_c = m_a =: m$ and $\sigma_c = \sigma_a =: \sigma$ permit to derive some properties. We obtain for the transition function

$$G(x) = \left(q - \frac{1}{2}\right)\mathrm{erf}\left(\frac{x-m}{\sigma\sqrt{2}}\right) + \frac{1}{2}, \tag{4.6}$$

from which we deduce that we always have $G(m) = 1/2$. Hence $1/2$ is a fixed point and therefore an absorbing state when $m = 1/2$. We also notice that when $\frac{x-m}{\sigma\sqrt{2}}$ tends to infinity (that is, when x tends to 1 and σ tends to 0), $G(x)$ tends to q (all conformists take action 1). Similarly, when $\frac{x-m}{\sigma\sqrt{2}}$ tends to $-\infty$ (i.e., x and σ tend to 0), $G(x)$ tends to $1-q$.
Let us examine if other absorbing states exist. First and second derivatives of the transition function are

$$\begin{aligned} G'(x) &= \frac{2q-1}{2\sigma\sqrt{2}}\exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) \\ G''(x) &= -\frac{(2q-1)(x-m)}{2\sigma^3\sqrt{2}}\exp\left(-\frac{(x-m)^2}{2\sigma^2}\right). \end{aligned}$$

The function has an inflection point at $x = m$. If $q < 1/2$, it is decreasing, concave when $x \leq m$, and convex when $x \geq m$. If $q = 1/2$, $G(x)$ is constant and equal to $1/2$, while if $q > 1/2$, it is increasing, convex when $x \leq m$, and concave when $x \geq m$.

Let us first consider the case $q < 1/2$ (majority of anti-conformists). G being decreasing and passing through the value $1/2$, it has a single fixed point x_* solution of the equation

$$\mathrm{erf}\left(\frac{x-m}{\sigma\sqrt{2}}\right) = \frac{1-2x}{1-2q}. \tag{4.7}$$

We have $m < x_* < 1/2$ when $m < 1/2$, and $1/2 < x_* < m$ when $m > 1/2$. If the magnitude of the slope of G at x^* is greater than 1, x^* is an unstable fixed point, otherwise it is stable (this is a general observation). When it is unstable, the process reaches a limit cycle of length 2 formed by the points $(x_0, G(x_0)), (G(x_0), G(G(x_0)))$, where x_0 is a fixed point of $G \circ G$.

We consider now that $q > 1/2$ (majority of conformists). G being increasing makes the study more complex as several intersections with the diagonal may occur. As $\mathrm{erf}(x) \in$

$]-1,1[$, we have in general

$$1-q < G(0) < G(m) = \frac{1}{2} < G(1) < q.$$

Therefore, if $m < 1/2$, there is a fixed point $x_* > 1/2$ solution of (4.7), and possibly two other ones (only one when G is tangent to the diagonal) smaller than $1/2$ solution of the same equation. If $m > 1/2$, the situation is symmetric: there is a fixed point $x_* < 1/2$ and possibly two other ones greater than $1/2$, both solutions of (4.7). As a general observation, if there is a single fixed point, it is stable, while in case of three fixed points, the one in the middle is unstable, as a small negative (respectively, positive) variation makes it converge to the left one (respectively, the right one).
The case $m = 1/2$ is particular, because $1/2$ is a fixed point, and depending whether the tangent at $1/2$ is above or below the diagonal, there are two other fixed points, one greater than $1/2$ and the other one smaller, or no other fixed point. Observe that when $1/2$ is not the unique fixed point, it is unstable, as starting from state $x < 1/2$ (respectively, $> 1/2$) makes the process converge to the lower fixed point (respectively, the upper one). The condition for the tangent reads

$$G'(m) \geq 1 \Leftrightarrow \sigma \leq \frac{1}{\sqrt{2}}\left(q - \frac{1}{2}\right). \tag{4.8}$$

We summarize our findings in the next proposition.

Proposition 4.3. *Suppose that $m_a = m_c =: m$, $\sigma_a = \sigma_c =: \sigma$. Then*

- *If there are more conformists than anti-conformists ($q > 1/2$), there always exists a stable fixed point x_* (and possibly two other unstable fixed points) such that $x_* \geq m$ if $m \leq 1/2$ and $x_* \leq m$ if $m \geq 1/2$, being solution of (4.7). When $m = 1/2$, $1/2$ is a fixed point and two other fixed points exist, also solutions of (4.7), provided the variance is not greater than $\frac{1}{\sqrt{2}}\left(q - \frac{1}{2}\right)$. The fixed point $\frac{1}{2}$, when it is not unique, is unstable. No cycle can occur.*

- *If there are more anti-conformists than conformists ($q < 1/2$), there is a unique fixed point x_* given by solving (4.7). It is stable if $|G'(x_*)| \leq 1$, otherwise there exists a limit cycle of length 2.*

- *If there are exactly as many conformists as anti-conformists, then there is convergence in one shot from any state S to the absorbing state S^* corresponding to the fixed point $1/2$ by 4.5.*

Interestingly, cycles (always of length 2) occur only when there are more anti-conformists than conformists, and under the condition that the variance of the threshold distribution is small enough. Otherwise, there is always a stable absorbing state, and there is no cascade effect leading to N_c or N_a.

The following examples illustrate the above results.

Example 4.3. *$q = 0.9$; $m = 0.5$; $\sigma = 0.1, 0.2, \ldots, 0.9, 1$ (see Figure 4.5). This is the case of a majority of conformists and $m = 1/2$. One can observe the fixed point $1/2$ and the possible existence of 2 others. The limit value of σ for the tangent condition (4.8) is 0.283. Also, one can observe the asymptotic values q and $1 - q$ for $G(x)$.*

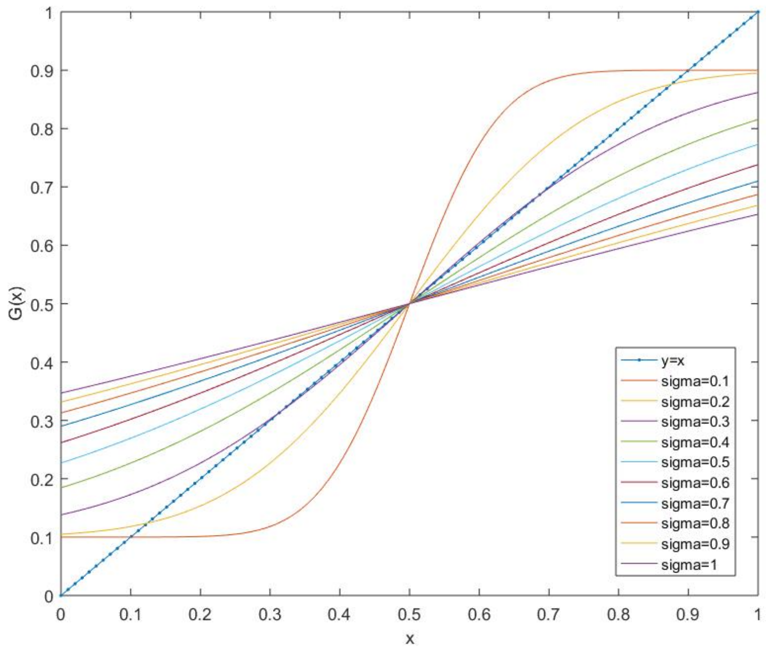


Figure 4.5: The transition functions for Gaussian distributed threshold model with a small proportion of anti-conformist agents ($q = 0.9$). In this example 4.3, $m = 0.5$, $\sigma = 0.1, 0.2, \ldots, 0.9, 1$.

Example 4.4. *$q = 0.95$; $m = 0.3$; $\sigma = 0.1, 0.2, \ldots, 0.9, 1$ (see Figure 4.6). This is the case of a majority of conformists and $m < 1/2$. There is fixed point greater than $1/2$, whose value is negatively related to the variance, and two other possible ones smaller than $1/2$.*

Example 4.5. *$q = 0.9$; $m = 0.8$; $\sigma = 0.1, 0.2, \ldots, 0.9, 1$ (see Figure 4.7): majority of conformists and $m > 1/2$. There is fixed point smaller than $1/2$, whose value is positively related to the value of the variance.*

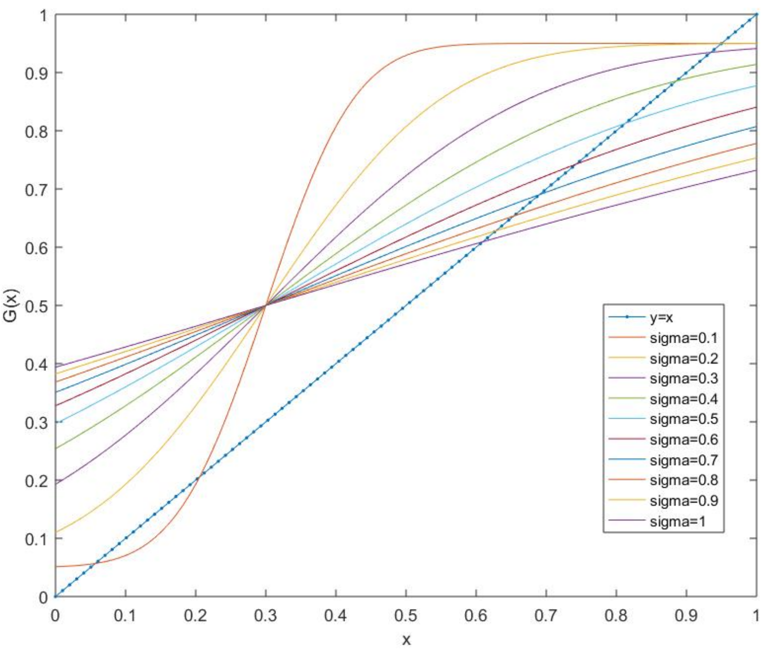


Figure 4.6: The transition functions for Gaussian distributed threshold model with a small proportion of anti-conformist agents ($q = 0.95$). In this example 4.4, $m = 0.3$, $\sigma = 0.1, 0.2, \ldots, 0.9, 1$.

Example 4.6. *$q = 0.1$; $m = 0.2$; $\sigma = 0.1, 0.2, \ldots, 0.9, 1$ (see Figure 4.8). This is the case of a majority of anti-conformists and $m < 1/2$. There is fixed point x_* smaller than $1/2$. Its value is positively related to the value of the variance. For $\sigma = 0.1$ and $\sigma = 0.2$, the slope at x_* begin greater than 1, the fixed point is unstable and there is a limit cycle corresponding to the fixed points of $G \circ G$. The plot of $G \circ G$ is shown on Figure 4.9. One can see that $G \circ G$ has 1 or 3 fixed points, one of them being the fixed point of G. When there are 3 fixed points (for $\sigma = 0.1$, 0.15 and 0.2), the extreme ones give the coordinates for the limit cycle. For $\sigma = 0.1$ the coordinates are (0.1, 0.7731) and (0.7731, 0.1), while for $\sigma = 0.2$ they are (0.1116, 0.6366) and (0.6366,0.1116). The cycles are materialized on Figure 4.8.*

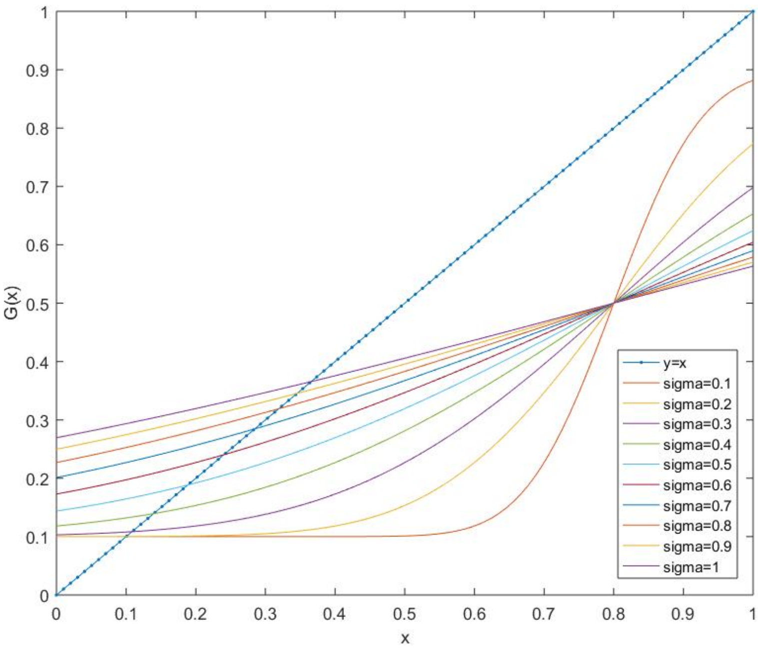


Figure 4.7: The transition functions for Gaussian distributed threshold model with a small proportion of anti-conformist agents ($q = 0.9$). In this example 6.11, $m = 0.8$, $\sigma = 0.1, 0.2, \ldots, 0.9, 1$.

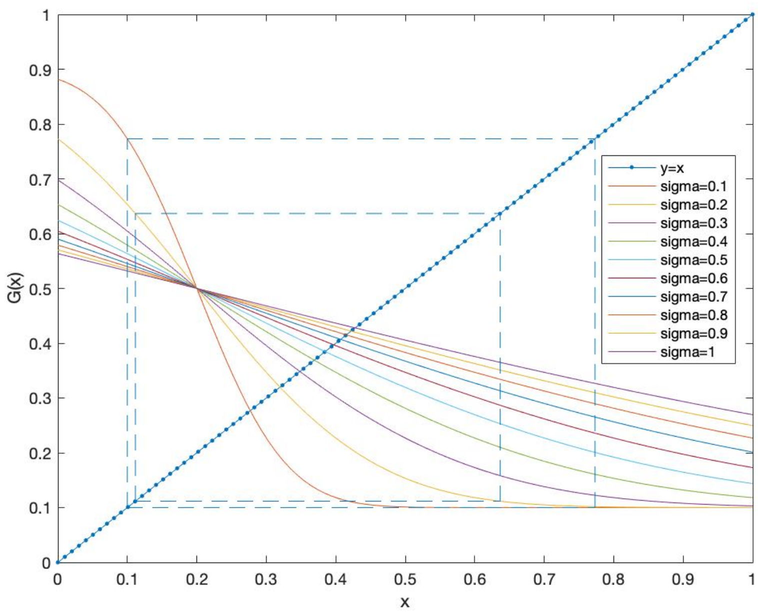


Figure 4.8: The transition functions for Gaussian distributed threshold model with a large proportion of anti-conformist agents ($q = 0.1$). In this example 4.6, $m = 0.2$, $\sigma = 0.1, 0.2, \ldots, 0.9, 1$.

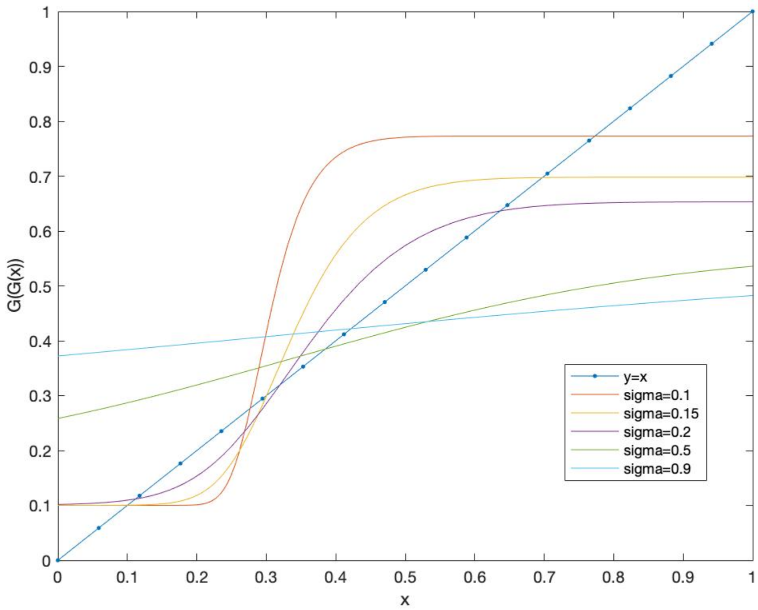


Figure 4.9: Plot of $G \circ G$ for Gaussian distributed threshold model with a large proportion of anti-conformist agents ($q = 0.1$). In this example 4.6, $m = 0.2$, $\sigma = 0.1, 0.2, \dots, 0.9, 1$.

General distribution

Recall that if $N_a \neq \emptyset$, the function G may not have fixed points because the presence of anticonformists makes it nonmonotonic. It is therefore difficult to get precise results in the general case. The next proposition elucidates the situation when there is only one type of anti-conformist agent.

Proposition 4.4. *Consider the following case where there is only one type of anti-conformists with threshold μ_a and assume its fraction among all players is δ_a. The list of all threshold values of conformists are $\mu_1, \mu_2, \ldots, \mu_p$ (in a strict increasing order) with fractions $q_1, q_2, \ldots, q_p$ respectively. Denote by k the largest number such that $\mu_k \leq \mu_a$ (see Table 4.1). The following holds.*

(i) If there is no agent i such that $\mu_i = \mu_a$, there exist absorbing states if and only if the thresholds and corresponding fractions violate one of the following inequalities.[8]

$$\begin{cases} \delta_a \geq \mu_1 & \\ \delta_a + \sum_{i=1}^{i_0} q_i \geq \mu_{i_0+1} & (i_0 = 1, 2, \ldots, k-1) \\ \delta_a + \sum_{i=1}^{k} q_i \geq \mu_a & \\ \sum_{i=1}^{k} q_i < \mu_a & \\ \sum_{i=1}^{i_0} q_i < \mu_{i_0} & (i_o = k+1, k+2, \ldots, p) \end{cases} \tag{4.9}$$

(ii) Otherwise, there exist absorbing states if and only if the thresholds and corresponding fractions violate one of the following inequalities.[9]

$$\begin{cases} \delta_a \geq \mu_1 & \\ \delta_a + \sum_{i=1}^{i_0} q_i \geq \mu_{i_0+1} & (i_0 = 1, 2, \ldots, k-1) \\ \sum_{i=1}^{i_0} q_i < \mu_{i_0} & (i_o = k, k+1, \ldots, p) \end{cases} \tag{4.10}$$

Proposition 4.4 gives the necessary and sufficient conditions by a system of inequalities for the existence of absorbing states for general distributions when there is only one type of

[8]Note that if $\mu_a < \mu_1$, we can think it as $k = 0$ and delete all the terms related to non positive indices. Thus inequalities (4.9) become

$$\begin{cases} \delta_a \geq \mu_a & \\ \sum_{i=1}^{i_0} q_i < \mu_{i_0} & (i_0 = 1, 2, \ldots, p) \end{cases}$$

[9]Note that if $\mu_a = \mu_1$, we can think it as $k = 1$ and delete all the terms related to non positive indices. Thus inequalities (4.10) will be

$$\begin{cases} \delta_a \geq \mu_1 & \\ \sum_{i=1}^{i_0} q_i < \mu_{i_0} & (i_0 = 1, 2, \ldots, p) \end{cases}$$

Table 4.1: Distribution of agents' thresholds with conformists and one type of anti-conformists

Proportion	Threshold	Behavior characteristics
q_1	μ_1	Conformism
$\cdots$	$\cdots$	$\cdots$
q_k	μ_k	Conformism
δ_a	μ_a	Anti-conformism
q_{k+1}	μ_{k+1}	Conformism
$\cdots$	$\cdots$	$\cdots$
q_p	μ_p	Conformism

anti-conformist agent. Figure 4.10 gives the possible absorbing states for different ranges of μ_a and δ_a based on Proposition 4.4 and its proofs. In the left area filled with only north west lines, i.e. $\mu_a \leq \sum_{i=1}^{k} q_i$, the process has only one absorbing state where only conformists with thresholds $\mu_1, \ldots, \mu_k$ saying yes, i.e., $x = \sum_{i=1}^{k} q_i$. In the right area filled with only north east lines, i.e., $\delta_a < \mu_a - \sum_{i=1}^{k} q_i$, the process has only one absorbing state where the anti-conformists and conformists with thresholds $\mu_1, \ldots, \mu_k$ saying yes, i.e., $x = \delta_a + \sum_{i=1}^{k} q_i$. In the bottom area filled with only horizontal lines, there are three possible absorbing states. The overlapped area means that absorbing states of both cases are possible. All the details can be found in Proof of Proposition 4.4.
The following example illustrates the case where there is no absorbing state.

Example 4.7. *We consider $n = 10$, with $N_c = \{1, 2, 3, 4, 5, 6\}$ and 4 anti-conformists. The parameters are $\mu_a = \delta_a = {}^4\!/_{10}$, $\mu_1 = q_1 = q_2 = q_3 = q_4 = q_5 = q_6 = {}^1\!/_{10}$, $\mu_2 = {}^2\!/_{10}$, $\mu_3 = {}^3\!/_{10}$, $\mu_4 = {}^5\!/_{10}$, $\mu_5 = {}^6\!/_{10}$, $\mu_6 = {}^7\!/_{10}$ (see* Figure *4.11). There is no absorbing state but a cycle: $\{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\} \to \{1, 2, 3\} \to \{1, 2, 3\} \cup N_a \to \{1, 2, 3, 4, 5, 6\}$ with group opinion x: ${}^6\!/_{10} \to {}^5\!/_{10} \to {}^4\!/_{10} \to {}^3\!/_{10} \to {}^7\!/_{10} \to {}^6\!/_{10}$.*

The previous example has shown the existence of cycles. The next theorem establishes that there could be at most one cycle, whose length has an upper bound.

Theorem 4.4. *Consider the same assumptions and notation as in Proposition 4.4. Then the process has either absorbing states or a unique cycle of length at most $m + 2$, where m is the number of values μ_i in the interval $\left]\sum_{i=1}^{k} q_i, \sum_{i=1}^{k} q_i + \delta_a\right]$. The upper bound of the length of the cycle, considering any possible values for the thresholds and fractions, is $n_a + 1$, where $n_a = n\delta_a$ is the number of anti-conformists.*

The next example illustrates the theorem and shows that the cycle can be shorter than $m + 2$ and that its length can be far below the upper bound $n_a + 1$. Note that in Example 4.7, this bound is attained.

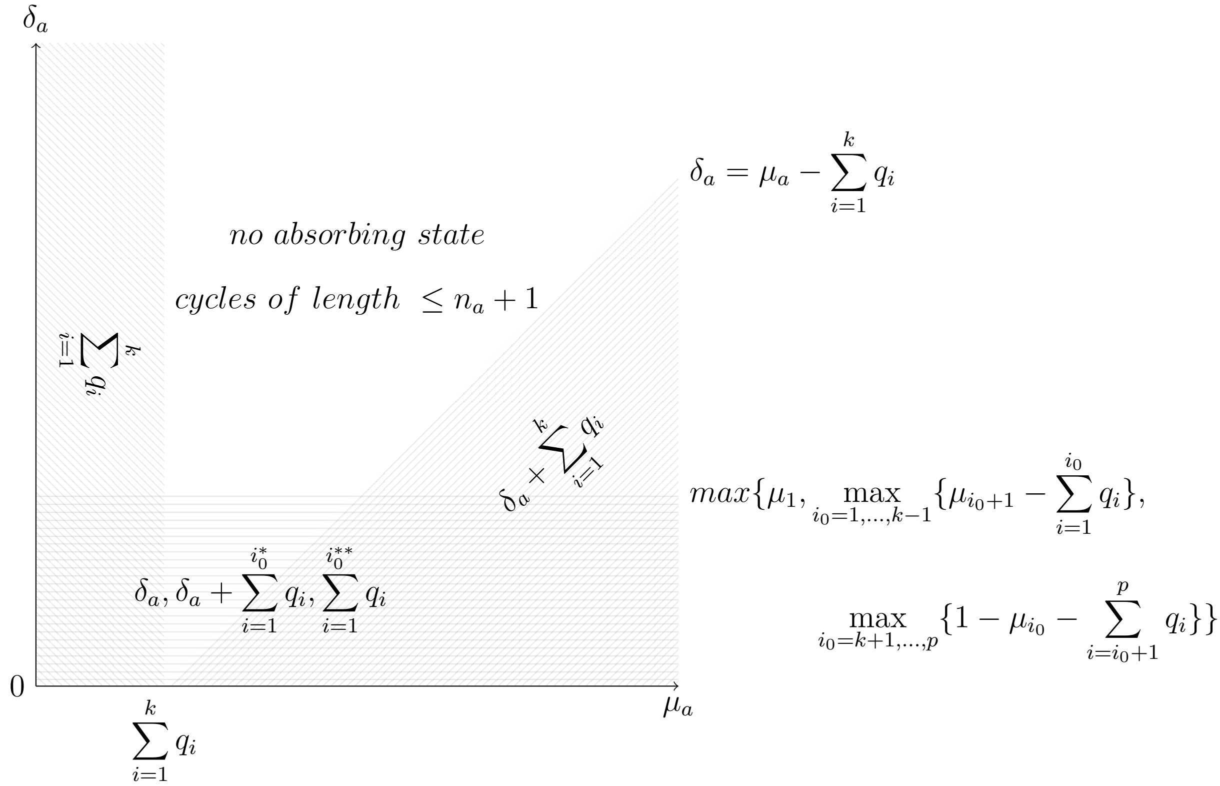


Figure 4.10: Existence of absorbing states for general distributions when there is only one type of anti-conformist agent with threshold μ_a and proportion δ_a (the variables in each area filled with lines indicates the values of the possible fixed points corresponding to each case.)

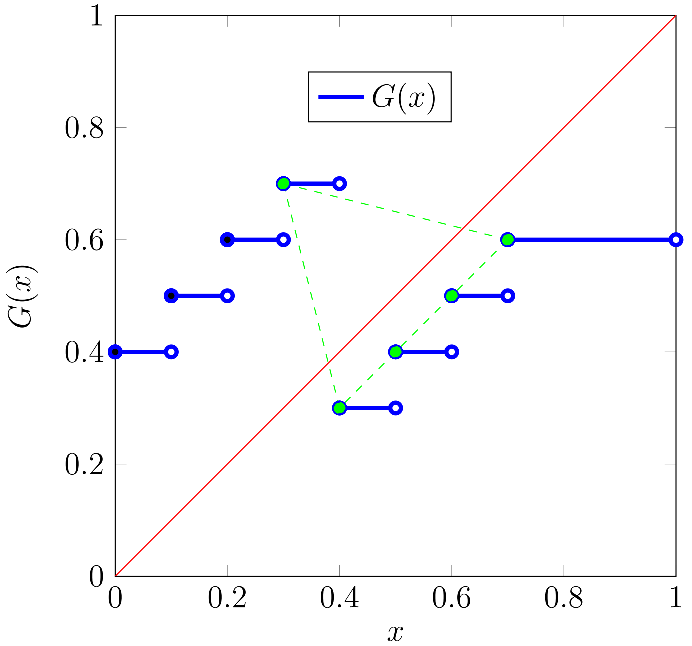


Figure 4.11: $G(x)$ of Example 4.7, with a cycle of length 5: $\{1,2,3,4,5,6\} \to \{1,2,3,4,5\} \to \{1,2,3,4\} \to \{1,2,3\} \to \{1,2,3\} \cup N_a \to \{1,2,3,4,5,6\}$, which is materialized in green.

Example 4.8. *We consider $n = 100$, with the following parameters: $\delta_a = 0.4$ (40 anti-conformists), $\mu_a = 0.5$, $\mu_1 = 0.2$, $q_1 = 0.1$, $\mu_2 = 0.3$, $q_2 = 0.15$, $\mu_3 = 0.7$, $q_3 = 0.3$, $\mu_4 = 0.8$, $q_4 = 0.05$. Then $k = 2$, and by the theorem, the cycle should be of length at most 3, while the upper bound $n_a + 1$ yields 41. One sees on* Figure *4.12 that the cycle (in green) has in fact length 2 and is formed of the two points* $(0.25, 0.5), (0.5, 0.25)$.

4.3 Random sampling models

The previous section considered a mechanism of diffusion with a complete and undirected network, where each agent was permanently in contact with all other agents. As this assumption may be unrealistic in some situations, we consider here a different mechanism where agents meet other agents at random (random neighborhood), with a certain size of the neighborhood to be either fixed or drawn from a distribution. When the size is fixed and identical for all agents, we speak of a *homogeneous network*. Note that in this model we implicitly assume that the network is directed, as if i selects j in its neighborhood, this does not necessarily imply that j has i in its neighborhood. Also, note that this model amounts to considering that at each time step a random graph on N realizes, where each agent has a random neighborhood (and therefore a random degree). We denote by $P(k)$ the distribution of the degree of each agent, who are supposed to have the same

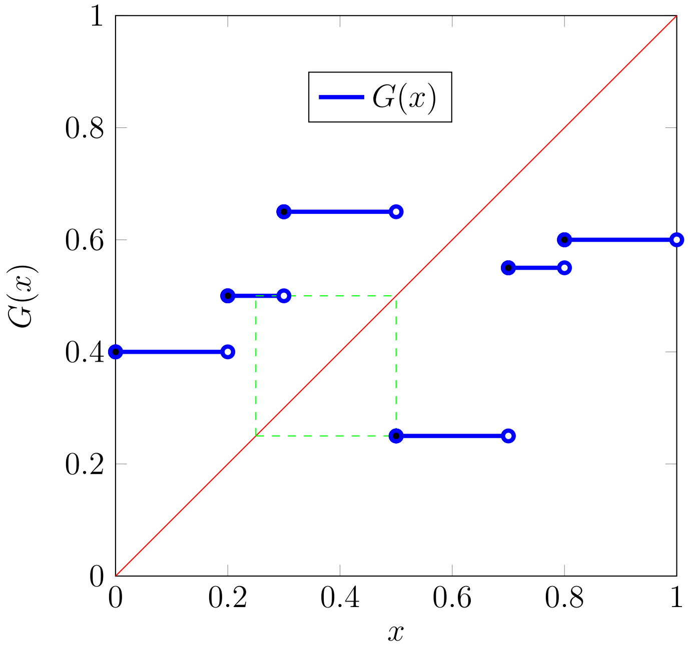


Figure 4.12: $G(x)$ of Example 4.8, with a cycle of length 2: $(0.25, 0.5) \to (0.5, 0.25) \to (0.25, 0.5)$, which is materialized in green.

distribution.

This can be interpreted in two ways. In the first one, at each time step, a random graph on N realizes, where directed links are picked at random so as to follow the specified degree distribution. In the second one, we consider the network to be the complete graph on N, meaning that any agent may potentially meet any other agent. Then, at each time step, for each agent a subset of agents is chosen randomly (i.e., links are drawn from the uniform distribution), so as to obey the specified degree distribution. We follow in the sequel the latter interpretation.

This model is a good approximation of many real situations, especially communication via online social media like Twitter, Sina weibo, etc. Take Twitter as an example, an individual receives at some time several tweets not only from his/her friends but also from strangers by checking the latest or hottest tweets. Moreover, he/she also can accessing information proactively on a certain topic by searching keywords or hashtags. At a different time, different users post some new tweets that attract this individual's attention, which can be seen as a random sample (neighborhood) whose size obeys some distribution. Another example is that we meet different people everyday and obtain information either by communicating directly with each other or observing their behavior.

To avoid intricacies, it is convenient to consider that the random neighborhood of agent i, knowing that its degree is d, is taken as a random subset of N, of size d. This means that sometimes i is in its neighborhood, sometimes it is not. Still, the agents are considered

to have a threshold, which can be drawn from a distribution or is fixed.
An important consequence of the model is that the process of updating of the opinion is no more deterministic, but still obeys a Markov chain. Its analysis is therefore much more complex, as not only absorbing states and cycles can exist but also aperiodic and periodic absorbing classes, where a class is a set of states such that a chain of transitions exists from any state to any other state in the set, and which is maximal for this property. It is easy to see that a state T different from $\emptyset$ and N cannot be absorbing anymore: this is because the neighborhood being random and smaller than N, it is not guaranteed that it will contain T at each period. However, $\emptyset$ and N can still be absorbing. The next lemma clarifies this point.

Lemma 4.1. *$\emptyset$ is absorbing (resp., N is absorbing) iff all anti-conformists are constant 0-players (resp., constant 1-players), while there is no constant conformist player (i.e., $0 < \mu_i \leq 1$ for all $i \in N_c$).*

Proof. Suppose $T = \emptyset$. By assumption, every conformist will take action 0 with certainty. Now, $i \in N_a$ takes action 0 iff $\mu_i = 0$. Hence, any anti-conformist must be a constant 0-player.
The argument for $T = N$ is much the same. □

The existence of non trivial absorbing classes will be shown in Section 4.3.1, where a complete analysis is done in a simple case (only two different thresholds, one for conformists and one for anti-conformists). The complexity of the results shows that it seems out of reach to get a complete study in more general cases. Nevertheless, general results, although not exhaustive, can be obtained (see Section 4.3.1).

We start by focusing on the case of fixed degree (homogeneous networks).

4.3.1 Homogeneous networks

We suppose in this section that the neighborhood of every agent has a fixed size d. A complete study of this case is possible when all conformist agents have the same threshold μ_c, and all the anti-conformist agents have threshold μ_a. Then we give a result in the general case. We begin by some general considerations.
Let us express the probability of transition $\mathbf{P}(S \to T)$ from one state S to another state T. We have by the independence assumption that $\mathbf{P}(S \to T) = \prod_{i \in T} p_i^1(S) \prod_{i \notin T} p_i^0(S)$, where $p_i^e(S)$ is the probability for agent i to take action $e \in \{0, 1\}$ knowing that the current state is S. To compute these probabilities, it is necessary to compute the distribution of the average opinion $\overline{a_i}$ in the neighborhood of i knowing the current state S. It is easy to

check that

$$\mathbf{P}(\overline{a_i} = \frac{k}{d} \mid S) = \begin{cases} \dfrac{\binom{s}{k}\binom{n-s}{d-x}}{\binom{n}{d}}, & \text{if } d-n+s \leq k \leq s \\ 0, \text{otherwise}, \end{cases} \tag{4.11}$$

for $k = 0, 1, \ldots, d$. Observe that these probabilities do not depend on i, therefore we can omit the subscript i and write $\overline{a}$, the average opinion in a neighborhood. Then,

$$\text{If } i \in N_c, \ \ p_i^1(S) = \mathbf{P}(\overline{a} \geq \mu_i \mid S) \text{ and } p_i^0(S) = 1 - p_i^1(S) \tag{4.12}$$

$$\text{If } i \in N_a, \ \ p_i^1(S) = \mathbf{P}(\overline{a} < \mu_i \mid S) \text{ and } p_i^0(S) = 1 - p_i^1(S). \tag{4.13}$$

As these probabilities depend only on the cardinality of S, we may write $p_i^1(s)$ for simplicity.

Case with two thresholds μ_a, μ_c

We assume here that there are two types of agents: anti-conformist agents with threshold μ_a and conformist agents with threshold μ_c, where $0 < \mu_a, \mu_c \leq 1$.
Observe that $p_i^1(s)$ depends only on whether i belongs to N_a or N_c. Specifically, for a conformist agent i, $p_i^1(S) = \mathbf{P}(\overline{a} \geq \mu_c \mid S)$ is a nondecreasing function of $s = |S| \in \{0, 1, \ldots, n\}$ to $[0, 1]$, depending only on μ_c, n and d. In addition, we have $p_i^1(0) = 0$ and $p_i^1(n) = 1$. Similarly, if i is anti-conformist, $p_i^1(s)$ is a nonincreasing function of s, starting at 1 with $s = 0$ and finishing at 0 with $s = n$. Thus, we fall into the framework studied in Grabisch et al. (2019) on an anonymous model of anti-conformism where each agent i has the probability $p_i(s)$ to take action 1 at next step knowing that the current state is S, with $s = |S|$, and $p_i(s)$ is a nondecreasing (respectively, nonincreasing) function reaching values 0 and 1 when i is conformist (respectively, anti-conformist).
In the model of Grabisch et al. (2019), all functions p_i can be different among the agents, but it is required that for all conformists, the functions p_i have the same domain where they take value 0 and 1, and similarly for the anti-conformists. These domains are characterized for the conformists by the quantities l^c (firing threshold) and $n - r^c$ (saturation threshold) which are the rightmost and leftmost values of s for which $p_i^1(s)$ is 0 and 1, respectively, given by

$$l^c := \min\{s \,:\, p_i(s) > 0\} - 1, \quad n - r^c = \min\{s \,:\, p_i(s) = 1\},$$

and similarly for the anti-conformists:

$$l^a := \min\{s \,:\, p_i(s) < 1\} - 1, \quad n - r^a = \min\{s \,:\, p_i(s) = 0\}.$$

Our case satisfies these requirements as $p_i^1(s)$ depends only on μ_a, μ_c and d. We easily obtain:

$$\begin{aligned}
l^c &= \max_{i \in N_c}\{s \mid \overline{a} < \mu_c\} = \lceil d\mu_c \rceil - 1 \\
n - r^c &= \min_{i \in N_c}\{s \mid \overline{a} \geq \mu_c\} = n - d + \lceil d\mu_c \rceil \\
l^a &= \max_{i \in N_a}\{s \mid \overline{a} < \mu_a\} = \lceil d\mu_a \rceil - 1 \\
n - r^a &= \max_{i \in N_a}\{s \mid \overline{a} \geq \mu_a\} = n - d + \lceil d\mu_a \rceil.
\end{aligned}$$

From Grabisch et al. (2019), we know that in full generality 20 possible absorbing classes can occur, depending on the values of l^c, r^c, l^a, r^a. Since it holds that in our case $l^c + r^c = l^a + r^a = d - 1 < n - 1$, 5 among the 20 are not possible. Denoting by n_c the number of conformist agents, we give below the list of the remaining 15 absorbing classes, put into categories.

Polarization: the society of agents is divided in two groups, one taking action 1, the other taking action 0.
(1) N_a if and only if $n_c \geq \max\{n - l^c, n - l^a\}$;
(2) N_c if and only if $n_c \geq \max\{n - r^c, n - r^a\}$;
Cycles: sequence of states made of the infinite repetition of a pattern.
(3) $N_a \xrightarrow{1} \emptyset \xrightarrow{1} N_a$ if and only if $n - l^c \leq n_c \leq r^a$;
(4) $N_c \xrightarrow{1} N \xrightarrow{1} N_c$ if and only if $n - r^c \leq n_c \leq l^a$;
(5) $N_a \xrightarrow{1} N_c \xrightarrow{1} N_a$ if and only if $n_c \leq \min\{l^c, l^a, r^c, r^a\}$;
(6) $\emptyset \xrightarrow{1} N_a \xrightarrow{1} N_c \xrightarrow{1} \emptyset$ if and only if $n_c \leq \min\{r^c, r^a, l^c\}$ and $n_c \geq n - r^a$;
(7) $N_a \xrightarrow{1} N \xrightarrow{1} N_c \xrightarrow{1} N_a$ if and only if $n_c \leq \min\{l^c, l^a, r^c\}$ and $n_c \geq n - l^a$;
Fuzzy cycles: the pattern contains states but also intervals of states. This means that there is no exact repetition of the same pattern, but at each repetition a state is picked at random in the interval.
(8) $N_a \xrightarrow{1} [\emptyset, N_c] \xrightarrow{1} N_a$ if and only if $n_c \leq \min\{l^c, l^a, r^a\}$ and $r^c < n_c < n - l^c$;
(9) $N_c \xrightarrow{1} [N_a, N] \xrightarrow{1} N_c$ if and only if $n_c \leq \min\{r^c, r^a, l^a\}$ and $l^c < n_c < n - r^c$;
(10) $[\emptyset, N_c] \xrightarrow{1} [N_a, N] \xrightarrow{1} [\emptyset, N_c]$ if and only if $\max\{r^c, l^c\} < n_c \leq \min\{r^a, l^a, n - l^c - 1, n - r^c - 1\}$;
Fuzzy polarization: the polarization is defined by an interval, which means that at each time step, a state is picked at random in the interval, representing the set of active agents.
(11) $[\emptyset, N_a]$ if and only if $\max\{n - l^c, r^a + 1\} \leq n_c < n - l^a$;
(12) $[N_c, N]$ if and only if $\max\{n - r^c, l^a + 1\} \leq n_c < n - r^a$;

Chaotic polarization: similar to the previous case but more complex as several intervals are involved.
(13) $[\emptyset, N_a] \cup [\emptyset, N_c]$ if and only if $l^c \geq n - r^a$ and $n_c \in (]r^c, n - l^c[\cap]l^a, n - r^c[) \cup ((]l^a, n - r^a[\cup]l^c, n - r^c[)\cap]0, r^c[)$;
(14) $[N_a, N] \cup [N_c, N]$ if and only if $l^a \geq n - r^c$ and $n_c \in (]l^c, n - r^c[\cap]r^a, n - l^c[) \cup ((]r^a, n - l^a[\cup]r^c, n - l^c[)\cap]0, l^c[)$;
Chaos: at each time step a state is picked at random among all possible states.
(15) 2^N otherwise.
These results are exhaustive and exact (no approximation), however it is difficult to get an intuitive idea of the behavior. More concrete results can be obtained by making the number of agents tend to infinity and by choosing special cases of parameters.

When n tends to infinity Assume that the number of agents tends to infinity. For simplicity, divide the previous parameters $n^a, n^c, l^a, l^c, r^a, r^c, d$ by n, keeping the same notation so that these parameters now take value in $[0, 1]$. Thus $l^c = d\mu_c$ and $r^c = d(1 - \mu_c)$, and smilarly for l^a, r^a.
We examine different typical situations for the value of the parameters, taking advantage of the study made in Grabisch et al. (2019) (full detail can be found in this reference).

- **Situation 1:** $l^a = l^c =: l$ and $r^a = r^c =: r$. This implies $\mu_c = \mu_a =: \mu$, i.e., all agents have the same threshold. Only the following four absorbing classes remain possible in this situation:

  - N^a iff $n^a \leq l$
  - N^c iff $n^a \leq d - l = d(1 - \mu)$
  - cycle $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ iff $n^a \geq 1 - d\mu$ and $n^a \geq 1 - d(1 - \mu)$
  - 2^N otherwise

  The general tendency is that as the proportion of anti-conformist agents increases, the society goes from consensus, to polarization or cascade, then to a chaos, finally to a cycle. A cascade effect (i.e., a convergence with probability 1 to N_c or with probability 1 to N_a, whatever the inital state) is likely to occur, all the more $l + r$ is close to 1 (i.e., the functions p_i^1 are close to threshold function). When l is smaller than $1/2$ and n^a is greater than l but sufficiently below $1 - l$, it will lead to a cascade with all conformist agents saying yes. When l is greater than $1/2$ and n_a between $1 - l$ and l, it will lead to a cascade with all anti-conformist agents saying yes.

- **Situation 2:** $l^a = l^c = r^a = r^c = \frac{d}{2}$. This implies $\mu_c = \mu_a = 1/2$. The three possible absorbing classes in this situation are:
  - N^a, N^c iff $n^a \leq d/2$ ("polarization")
  - cycle $N^a \xrightarrow{1} N^c \xrightarrow{1} N^a$ iff $n^a \geq 1 - d/2$ ("cycle")
  - 2^N otherwise ("chaos")

  The possible absorbing classes of "fuzzy cycle" and "fuzzy polarization" mentioned in Grabisch et al. (2019) become impossible since there is a constraint $l^a + r^a = l^c + r^c$ in this special context. Note that polarizations N^c and N^a always appear together, implying that there is no cascade effect.

- **Situation 3:** n^a tends to 0. Assume that $n^a = \epsilon > 0$ arbitrarily small, therefore $n^c = 1 - \epsilon$.

  Among the initial 15 possible absorbing classes, only 7 of them remain possible:
  - (1) N^a iff $\min(l^a, l^c) \geq \epsilon$;
  - (2) N^c iff $\min(r^c, r^a) \geq \epsilon$'
  - (3) $N^a \xrightarrow{1} \emptyset \xrightarrow{1} N^a$ iff $l^c \geq \epsilon$ and $r^a \geq 1 - \epsilon$;
  - (4) $N^c \xrightarrow{1} N \xrightarrow{1} N^c$ iff $r^c \geq \epsilon$ and $l^a \geq 1 - \epsilon$;
  - (11) $[\emptyset, N^a]$ iff $l^a < \epsilon$, $l^c \geq \epsilon$ and $r^a < 1 - \epsilon$;
  - (12) $[N^c, N]$ iff $r^a < \epsilon$, $r^c \geq \epsilon$ and $l^a < 1 - \epsilon$;
  - (15) 2^N otherwise.

  Again there is no cascade effect in this situation since two possible polarizations always appear together. When $l_c, r_c < \epsilon$, which means that d is very small, only chaos (2^N) appears.

General case

With more than two thresholds, the complexity of the previous study indicates that it seems to be hopeless to get exact and complete results. This negative conclusion is tempered by our next result, established with an arbitrary distribution of thresholds. It shows that in most cases, only chaos can occur, i.e., the only absorbing class is 2^N.

Theorem 4.5. *Suppose $n_a \geq d$, $n_c \geq d$ and suppose that there is no constant player (i.e., $0 < \mu_i \leq 1$ for every player i). Then 2^N is the only absorbing class, i.e., the transition matrix is irreducible.*

For example, if the distribution of thresholds has support $\{1/d, \dots, 1\}$ for the conformists and the anti-conformists. Then 2^N is the only absorbing class. Indeed, the assumption implies that there are at least d members in N_a, N_c.
This result is in accordance with those found in the previous section with two thresholds μ_a, μ_c. Indeed, one can check that under the condition $n_a, n_c \geq d$, none of the absorbing classes from (1) to (14) is possible. This is because we always have all four quantities l^c, l^a, r^c, r^a strictly smaller than d. Therefore $n_a \geq d$ implies that $n - n_a = n_c \geq n - l^c$ and $n_c \geq n - r^c$ are impossible (and similarly with $n_c \geq d$).

4.3.2 Arbitrary degree distribution

We suppose now that the degree of the neighborhood is not fixed but follows a distribution $P(d)$. We try to generalize the results of the homogeneous case.
The probabilities of taking action 1 or 0 for the conformist and anti-conformist agents given in (4.12) and (4.13) become:

$$\text{If } i \in N_c,\ \ p_i^1(S) = \sum_d \mathbf{P}(\overline{a}_i \geq \mu_i \mid S; d)P(d) \text{ and } p_i^0(S) = 1 - p_i^1(S) \tag{4.14}$$

$$\text{If } i \in N_a,\ \ p_i^1(S) = \sum_d \mathbf{P}(\overline{a}_i < \mu_i \mid S; d)P(d) \text{ and } p_i^0(S) = 1 - p_i^1(S), \tag{4.15}$$

where the summation over d is taken over the support of $P(d)$, and $\mathbf{P}(\overline{a}_i \geq \mu_i \mid S; d)$ is given by (4.11).

Case with two thresholds μ_a, μ_c

The introduction of a distribution over the degree does not change the behavior of $p_i^1(S)$: there are still nonincreasing or nondecreasing functions of s taking boundary values 0 and 1. The identification of the absorbing classes depends only on the width of the domain where these functions take values 0 and 1, hence their exact form is unimportant for this purpose.
By (4.14) we see that $p_i^1(S) = 0$ for conformist agents iff every term in the summation is equal to 0. We have established in Section 4.3.1 that $\mathbf{P}(\overline{a}_i \geq \mu_i \mid S; d) = 0$ iff $s \leq l^c(d_i) := \lceil d_i\mu_c \rceil - 1$ and $\mathbf{P}(\overline{a}_i \geq \mu_i \mid S; d) = 1$ iff $s \geq n - r^c(d_i) := n - d_i + \lceil d_i\mu_c \rceil$. Introducing

$$l^c := \min\{l^c(d)\ :\ d \in \text{ support of } P(d)\}, r^c = \min\{r^c(d)\ :\ d \in \text{ support of } P(d)\}$$
$$l^a := \min\{l^a(d)\ :\ d \in \text{ support of } P(d)\}, r^a = \min\{r^a(d)\ :\ d \in \text{ support of } P(d)\},$$

the results of Section 4.3.1 can be readily extended to the general case by using the above quantities l^c, r^c in all the conditions of existence of the 15 absorbing classes.

An important consequence is the following: suppose that the distribution of d gives a positive probability to $d = 1$. Then we find $l^c = r^c = l^a = r^a = 0$. By inspection of the conditions of existence of the 15 absorbing classes, it follows that *only* the case of the chaos (2^N) remains possible. Note that this assumption is often satisfied (e.g., for the Poisson distribution, which arises when any pair of vertices is connected with a fixed probability).

General case

We suppose now that each agent has a fixed threshold but possibly different among agents. A generalization of Theorem 4.5 is possible: under mild assumptions, only chaos can occur. Let us denote by $\underline{d}, \overline{d}$ the lowest and greatest values of d with a positive probability, and by $\underline{\mu}, \overline{\mu}$ the lowest and highest threshold values among the agents.

Theorem 4.6. *Suppose that $\underline{\mu} > 0$ (no constant player) and that the number of conformist and anti-conformist agents satisfy*

$$\overline{\mu}\underline{d} \leq n_a, \;\; n_c < n - \underline{d}(1 - \underline{\mu})$$

Then 2^N is the only absorbing class, i.e., the transition matrix is irreducible.

Note that the conditions on n_a, n_c can be written equivalently as $n_a \geq \overline{\mu}\underline{d}$ and $n_a > \underline{d}(1-\underline{\mu})$ (same for n_c). Again, observe that if $\underline{d} = 1$, these conditions are *always* satisfied.

4.4 Concluding remarks

We have performed in this paper a detailed study of convergence of the threshold model incorporating anti-conformist agents. Two models were considered: a deterministic model supposing a complete graph, and a random neighborhood model, both corresponding to useful real situations. The first one represents a connected society where every agent is informed about the number of agents being in state 1 or 0 (active or inactive) at the present time, through media, etc. It is to be noted that no other information about the society is possessed by an agent, e.g., if there are anti-conformists and how many. The second model represents a society communicating via social networks like Facebook or Twitter, receiving randomly messages from other agents indicating their states. Here also, a given agent has no information on the type of his neighbor (conformist or anti-conformist). We have given a game-theoretic foundation of the threshold mechanism with anti-conformists, using coordination and anti-coordination games.

For the deterministic model, we have found that, generally speaking, the presence of anti-conformists causes the appearance of much more absorbing states, and cycles of length greater than 2 (when only (anti-)conformist agents are present cycles can only be of length 2). We have performed a complete and exact study when the distribution of threshold is uniform, generalizing the results of Granovetter (1978). We have also studied the case of a Gaussian distribution, where we showed the existence of unstable fixed points and limit cycles of length 2, and the case of an arbitrary distribution, where it is possible to find cycles of length greater than 2.

Based on a previous study, we have performed a complete and exact analysis of the random model when there are only two thresholds, one for the conformists, and another for the anti-conformists. The introduction of randomness causes a variety of absorbing classes to appear: polarization, periodic classes of more or less complex structure, and chaos, i.e., any state of the society can be reached. When thresholds are randomly distributed, such an analysis is no more possible, however, we have shown that in most cases, only chaos occurs.

The initial aim of the paper was to analyze the effect of the presence of anti-conformists in a society regarding the convergence of the state or opinion of the agents in the long run. The most remarkable findings in this respect are:

- The presence of anti-conformists introduces instability in the process, causing a multiplicity of absorbing states and a variety of cycles, periodic classes and chaos. Also, small variations in the parameters defining the society may induce important changes in the convergence: the model is highly sensitive, e.g., in the number of anti-conformists, the threshold values, etc. For example, it has been seen in the case of a uniform threshold distribution that introducing or deleting only one anti-conformist agent changes the convergence from a stable state to a cyclic behavior or vice versa.

- In the case of a random neighborhood, the process converges to chaos (every state is possible) for most values of the parameters defining the society (it suffices that there are more conformists and more anti-conformists than the size of a smallest neighborhood). Otherwise, cascades may occur: we have proved their existence in the case of fixed thresholds for conformists and anti-conformists. This shows that introducing a small proportion of anti-conformists in a society may lead, not only to chaotic situations, but also to *permanent* opinion reversal.

4.5 Appendix

4.5.1 Proof of Theorem 4.2

The opinion dynamic in a complete network is deterministic since the probability of a transition from one state to another is either 1 or 0. Note also that the state space is finite, which means that the elements of absorbing states can only be absorbing states or cycles. It remains to prove the correspondence between fixed points of $G(x)$ and absorbing states.

$\Rightarrow$) If x^* is a fixed point of $G(x)$, then assign actions to players according to the following rule: assign to the conformists whose tipping values are smaller than x^* the action 1 while to those whose tipping values are greater than x^* the action 0; assign to the anti-conformists whose tipping values are smaller than x^* the action 0 while to those whose tipping values are greater than x^* the action 1. Obviously this action profile corresponds to one absorbing state since nobody would like to change actions next period.

$\Leftarrow$) If $x^* = (x_1, x_2, \ldots, x_n)$ is an absorbing state, then $\overline{x} = {}^1\!/_n \sum_{i=1}^{p} x_i$ is a fixed point of $G(x)$. By contradiction, if $\overline{x} > G(\overline{x})$, there will be some players playing action 0 at the present period who would like to play action 1 in the next period (e.g. conformists $i \in N_c$ with $\mu_i < \overline{x}$ or anticonformists $j \in N_a$ with $\mu_j > \overline{x}$). It is similar for the case $\overline{x} < G(\overline{x})$. Thus $\overline{x} = G(\overline{x})$.

4.5.2 Proof of Proposition 4.1

Note that $x \in \{0, {}^1\!/_n, \ldots, {}^{n-1}\!/_n, 1\}$.

Fix $x \geq \mu_a$. All conformist agents with threshold less than or equal to x would like to take action "1" when observing x (with proportion x). The anticonformist agents with threshold μ_a as well as all conformist agents with threshold strictly greater than x would like to take action "0" when observing x. Thus $G(x) = x$.

Fix $x < \mu_a$. All conformist agents with threshold less than or equal to x as well as the anticonformist agents with threshold $\mu_a > x$ would like to take action "1" when observing x (with proportion $x + {}^2\!/_n$). All conformist agents with threshold strictly greater than x would like to take action "0" when observing x. Thus $G(x) = x + {}^2\!/_n > x$. As a conclusion, $\mathcal{F} = \{{}^k\!/_n, \ldots, {}^{n-1}\!/_n\}$ is the set of fixed points of the function G. By Theorem 4.2, it is also the set of absorbing states of the opinion dynamics. Obviously, $|\mathcal{F}| = n(1 - \mu_a) = n - k$. Starting from any initial state with the group opinion $x^* = {}^{k^*}\!/_n$, if $xk^* \geq k$ and $k^* \neq n$, then $k^* \in \mathcal{F}$. Thus x^* is a reachable absorbing state. If $k^* < k$, then $G(x) = x + {}^2\!/_n$. It means that two more conformist agents will be activated at the next stage. This "domino" effect stops till $x \geq \mu_a$ with $x = {}^{k}/_{n}$ or $x = {}^{k+1}/_{n}$ depending on which one has the same parity as k^*.

4.5.3 Proof of Proposition 4.2

Fix $x \in \{0, {}^{1}/_{n}, \ldots, {}^{n-1}/_{n}, 1\}$. If $x < \mu_a^1$, only the conformist agents with threshold less than or equal to x as well as the two anticonformist agents would take action "1" at the next stage (with proportion $x + {}^{3}/_{n}$ in total). Thus $G(x) = x + {}^{3}/_{n} > x$. Similarly, if $x \in [\mu_a^1, \mu_a^2[$, $G(x) = x + {}^{1}/_{n} > x$; if $x \in [\mu_a^2, 1[$, $G(x) = x - {}^{1}/_{n} < x$.
Obviously, $G(1) = {}^{n-2}/_{n} < 1$. Therefore, $\forall x \in S$, $G(x) \neq x$. By Theorem 4.2, there is no absorbing state.
To show that this dynamic end up with a cycle regardless of the initial state, let us distinguish the following cases.
Assume that the dynamic start with the state $x = {}^{k_2}/_{n}$. Then all conformist agents with threshold less than x (with proportion ${}^{k_2-1}/_{n}$) would take action "1". All conformist agents with threshold greater than x as well as the two anticonformist agents would take action "0" at the next stage (v_2 with $x = {}^{k_2-1}/_{n}$). Then, observing $x < \mu_a^2$, the anticonformist agents with threshold μ_a^2 would change her action into "1" at the following stage (v_1 with $x = {}^{k_2}/_{n}$). Similar analysis is applied to the initial state $x = {}^{k_2-1}/_{n}$.
Assume the initial state satisfies $x < {}^{k_2-1}/_{n}$. If $x < \mu_a^1$, then $G(x) = x + {}^{3}/_{n}$. After every stage, there will be 3 more types of conformist agents would like to take action "1". This activation process stops till $x \in [\mu_a^1, \mu_a^2]$, then $G(x) = x + {}^{1}/_{n}$. After every stage, there will be one more types of conformist agents would like to take action "1". This activation process stops till $x = \mu_a^2 = {}^{k_2}/_{n}$. Then it goes back to the first case and forms a cycle $v_2 \to v_1 \to v_2$.
Assume the initial state satisfies $x \in [{}^{k_2}/_{n}, 1[$, then $G(x) = x - {}^{1}/_{n}$. This desactivation process stops till $x = {}^{k_2}/_{n}$. Then it goes back to the first case and forms a cycle $v_2 \to v_1 \to v_2$.

4.5.4 Proof of Theorem 4.3

Fix $x \in \{0, {}^{1}/_{n}, \ldots, {}^{n-1}/_{n}, 1\}$.
(i) If $k_1 \neq 0$, and if $s \in [0, {}^{k_1}/_{n}[$, $G(x) = x + {}^{2\ell+2}/_{n} > s$. If $x \in [{}^{k_{2\ell+1}}/_{n}, 1[$, $G(x) = x - {}^{2\ell}/_{n} < x$.
In general, if $x \in [{}^{k_i}/_{n}, {}^{k_{i+1}}/_{n}[$ $(i = 1, \ldots, 2\ell)$, only the anti-conformist agents with threshold strictly greater than ${}^{k_i}/_{n}$ (with proportion ${}^{2\ell+1-i}/_{n}$) and the conformist agents with threshold less than or equal to x (with proportion $x + {}^{1}/_{n} - {}^{i}/_{n}$) would like to take action

1. That is, $G(x) = x + {}^{2\ell+2-2i}/_{n}$.

By $G(x) = x$, we get $i = k+1$. Therefore, $\forall x \in [{}^{k_{k+1}}/_{n}, {}^{k_{k+2}}/_{n}[$, $G(x) = x$. Thus the set of fixed points of $G(x)$ is $\mathcal{F} = \{{}^{k_{k+1}}/_{n}, \dots, {}^{k_{k+2}-1}/_{n}\}$. By Theorem 4.2, the absorbing states are the action profiles associated to $\mathcal{F}$.

(ii) If $k_1 \neq 0$, and if $x \in [0, {}^{k_1}/_{n}[$, $G(x) = x + {}^{2\ell+1}/_{n} > s$. If $x \in [{}^{k_{2\ell}}/_{n}, 1[$, $G(x) = x - {}^{2\ell-1}/_{n} < x$.

In general, if $x \in [{}^{k_i}/_{n}, {}^{k_{i+1}}/_{n}[$ $(i = 1, \dots, 2\ell - 1)$, only the anti-conformist agents with threshold strictly greater than ${}^{k_i}/_{n}$ (with proportion ${}^{2\ell-i}/_{n}$) and the conformist agents with threshold less than or equal to x (with proportion $x + {}^{1}/_{n} - {}^{i}/_{n}$) would like to take action 1. That is, $G(x) = x + {}^{2\ell+1-2i}/_{n}$.

By $G(x) = x$, we get $i = k + {}^{1}/_{2}$. But i should be an integer. Thus $G(x)$ has no fixed point (i.e., $\mathcal{F} = \emptyset$). By Theorem 4.2, there is no absorbing state but cycles.

It remains to prove the statement on cycles. For this, we observe the following property of the function $G(x)$: when x changes from ${}^{i}/_{n}$ to ${}^{(i+1)}/_{n}$, the value of G is increased or decreased by one unit, depending whether agent i is conformist or anti-conformist. Hence for G the variation in ordinate cannot be greater than the variation in abscissa. We proceed in three steps.

1. The sequence of points $(x, (Gx)), (y, G(y)), (x, G(x))$ is a cycle iff it corresponds to a sequence of transitions, i.e., $y = G(x)$, $x = G(y) = G^{(2)}(x)$.

2. We show that a cycle of length 3 cannot exist. Let $(x, G(x)) \rightarrow (G(x), G^{(2)}(x)) \rightarrow (G^{(2)}(x), G^{(3)}(x)) \rightarrow (x, G(x))$ be a cycle. This implies $G^{(3)}(x) = x$. As the origin of the cycle is unimportant, suppose that $(x, G(x))$ is the leftmost point, i.e., $x < \min(G(x), G^{(2)}(x))$. Observe that this entails that this point is above the diagonal ($x < G(x)$). We may suppose that the second point $(G(x), G^{(2)}(x))$ is also above the diagonal, so that we have $x < G(x) < G^{(2)}(x)$, which entails that the 3d point is below the diagonal since its ordinate is x. By the above observation on the function G, the jump in ordinate cannot exceed the jump in abscissa. In particular, concerning the jump between the 2nd and 3d point, we obtain $G^{(2)}(x) - x \leq G^{(2)}(x) - G(x)$, which cannot be true as $x < G(x)$. Suppose now that the second point is below the diagonal. As the 1st point is the leftmost point, we necessarily have $x < G^{(2)}(x) < G(x)$. The condition on the jump between the 1st point and the 3d point yields $G(x) - x \leq G^{(2)}(x) - x$, a contradiction with $G^{(2)}(x) < G(x)$. The case where the first point is under the diagonal works similarly.

3. We show that no cycle of length greater than 3 can exist. Simply observe that selecting the leftmost point and 2 other points in a sequence of more than 3 points amounts to the case of a cycle of length 3 as the jump conditions will impose the same contradictions.

4.5.5 Proof of Proposition 4.4

(1) Consider the case that $\nexists i \in N$ s.t. $\mu_i = \mu_a$.

This is equivalent to show that there is no absorbing state if and only if all the inequalities in 4.9 are satisfied.

$\Leftarrow$ **(Sufficiency)** Fix $x \in \{0, 1/n, \cdots, n-1/n, 1\}$.

If $x \in [0, \mu_1[$, only anti-conformist agents would like to take action 1. That is, $G(x) = \delta_a$. But by the first inequality of 4.9, $G(x) > x$.

If $x \in [\mu_{i_0}, \mu_{i_0+1}[$, $(i_0 = 1, \ldots, k-1)$, only anti-conformist agents and the conformists agents with threshold less than or equal to μ_{i_0} would like to take action 1. That is, $G(x) = \delta_a + \sum_{i=1}^{i_0} q_i$. By the second inequality of 4.9, $G(x) > x$.

Similarly, if $x \in [\mu_k, \mu_a[$, $G(x) = \delta_a + \sum_{i=1}^{k} q_i > x$; if $x \in [\mu_a, \mu_{k+1}[$, $G(x) = \sum_{i=1}^{k} q_i < x$; if $x \in [\mu_{i_0}, \mu_{i_0+1}[$, $(i_0 = k+1, \ldots, p-1)$, $G(x) = \sum_{i=1}^{i_0} q_i < x$; if $x \in [\mu_p, 1[$, $G(x) = \sum_{i=1}^{p} q_i < x$. As a conclusion, there is no absorbing state.

$\Rightarrow$ **(Necessity)** We provide a *reductio ad absurdum* proof. Suppose the thresholds and corresponding fractions do not satisfy inequalities 4.9. Then distinguish the following cases.

- $\delta_a < \mu_1$

  In this case, $x = \delta_a$ is a fixed point of $G(x)$ since given the group opinion δ_a, only the anti-conformist agent with threshold strictly greater than μ_1 would take action 1 (with proportion δ_a). Then the group opinion at the next stage is still δ_a.

- $\exists$ some $i_0 \in \{1, 2, \cdots, k-1\}$, such that $\delta_a + \sum_{i=1}^{i_0} q_i < \mu_{i_0+1}$.

  Assume that i_0^* is the smallest number satisfying this condition. That is, $\delta_a + \sum_{i=1}^{i_0^*} q_i < \mu_{i_0^*+1}$ and $\forall i_0 < i_0^*$, $\delta_a + \sum_{i=1}^{i_0} q_i \geq \mu_{i_0+1}$. Thus $\delta_a + \sum_{i=1}^{i_0^*} q_i < \mu_{i_0^*+1} < \mu_a < \mu_{k+1} < \cdots < \mu_p$ and $\mu_1 < \cdots < \mu_{i_o^*} < \delta_a + \sum_{i=1}^{i_0^*} q_i$. In this case, given the group opinion $s = \delta_a + \sum_{i=1}^{i_0^*} q_i$, only the conformist agent $i \leq i_0^*$ as well as the anti-conformist agent will take action "1" (with proportion $\sum_{i=1}^{i_0^*} q_i + \delta_a$ in total) at the next stage. So $\delta_a + \sum_{i=1}^{i_0^*} q_i$ is a fixed point of $G(x)$.

- $\delta_a + \sum_{i=1}^{k} q_i < \mu_a$

  Then $\delta_a + \sum_{i=1}^{k} q_i < \mu_a < \mu_{k+1} < \cdots < \mu_p$. On the other hand, $\forall i_0 = 1, \ldots, k-1$, $\delta_a + \sum_{i=1}^{i_0} q_i \geq \mu_{i_0+1}$ (Otherwise it coincides with the previous case), then $\forall i_0 = 1, \ldots, k-1$, $\mu_{i_0+1} \leq \delta_a + \sum_{i=1}^{i_0} q_i < \delta_a + \sum_{i=1}^{k} q_i$ (since $i_0 < k$). In this case, given the group opinion $x = \delta_a + \sum_{i=1}^{k} q_i$, only the anti-conformist agent and the conformist agent $i = 1, \ldots, k$ will take action 1 at the next stage with proportion $\delta_a + \sum_{i=1}^{k} q_i$ in total. Thus $\delta_a + \sum_{i=1}^{k} q_i$ is a fixed point of $G(x)$.

- $\exists$ some $i_0 \in \{k+1, k+2, \cdots, p\}$, such that $\sum_{i=1}^{i_0} q_i \geq \mu_{i_0}$.

  Assume that i_0^{**} is the largest number satisfying this condition. That is, $\sum_{i=1}^{i_0^{**}} q_i \geq \mu_{i_0^{**}}$ and $\forall i_0 > i_0^{**}$, $\sum_{i=1}^{i_0} q_i < \mu_{i_0}$. Thus $\sum_{i=1}^{i_0^{**}} q_i \geq \mu_{i_0^{**}} > \mu_{i_0^{**}-1} > \cdots > \mu_a > \cdots > \mu_1$ and $\forall i_0 > i_0^{**}$, $\sum_{i=1}^{i_0^{**}} q_i < \sum_{i=1}^{i_0} q_i < \mu_{i_0}$. In this case, given the group opinion $x = \sum_{i=1}^{i_0^{**}} q_i$, only the conformist agent $i < i_0^{**}$ will take the action 1 at the next stage with proportion $\sum_{i=1}^{i_0^{**}} q_i$ in total. So $\sum_{i=1}^{i_0^{**}} q_i$ is a fixed point of $G(x)$.

- $\sum_{i=1}^{k} q_i \geq \mu_a$

  Then $\sum_{i=1}^{k} q_i \geq \mu_a > \mu_k > \mu_{k-1} > \cdots, \mu_1$. On the other hand, $\forall i_0 \in \{k+1, k+2, \cdots, p\}$, $\mu_{i_0} > \sum_{i=1}^{i_0} q_i > \sum_{i=1}^{k} q_i$. In this case, given the group opinion $x = \sum_{i=1}^{k} q_i$, only the conformist agent $i = 1, \ldots, k$ will take action 1 at the next stage with proportion $\sum_{i=1}^{k} q_i$. Thus $s = \sum_{i=1}^{k} q_i$ is a fixed point of $G(x)$.

As a conclusion, the fixed point of $G(x)$ always exists. By Theorem 4.2, the absorbing state always exists which leads to a contradiction.

(2) It is analogous for the case that $\exists i \in N$ s.t. $\mu_i = \mu_a$.

4.5.6 Proof of Theorem 4.4

Suppose that there is no absorbing state, hence we are under the conditions of the system of inequalities (4.9) or (4.10). It remains to prove the statement on the cycle. The transition function G has the following behavior: considering that x grows from 0 to 1, it starts from the value $G(0) = \delta_a$, then increases at each point $x = \mu_i$ by the quantity q_i, and decreases by the quantity δ_a at point μ_a. Therefore, $G(\mu_k) = \delta_a + \sum_{i=1}^k q_i$ and $G(\mu_a) = \sum_{i=1}^k q_i$. Then $G(x)$ continues to increase at each value μ_i when x goes from μ_a to 1. It follows that the inequalities (4.9) (or (4.10)) imply the following properties of the transition function:

(i) The three first inequalities imply that $G(x) > x$ for all $x < \mu_a$. Hence the part of the transition function to the left of μ_a is strictly above the diagonal and nondecreasing;

(ii) The 4th inequality implies that $G(\mu_a) < \mu_a$;

(iii) The last inequality implies that $G(x) < x$ for all $x > \mu_a$, hence the part of the transition function to the right of μ_a including this point is strictly below the diagonal and nondecreasing.

We consider the square delimited by the diagonal points $(\sum_{i=1}^k q_i, \sum_{i=1}^k q_i)$ and $(\sum_{i=1}^k q_i + \delta_a, \sum_{i=1}^k q_i + \delta_a)$. We claim that if any point $(x, G(x))$ is chosen inside this square, the "next" point $(G(x), G(G(x)))$ is still inside the square. First observe that $\sum_{i=1}^k q_i$ and $\sum_{i=1}^k q_i + \delta_a$ are respectively the minimum value and the maximum value achieved by G in the interval $[\sum_{i=1}^k q_i, \sum_{i=1}^k q_i + \delta_a]$, hence if x lies in this interval, $(x, G(x))$ is in the square. Now, taking x to the right of μ_a, we have that $\sum_{i=1}^k q_i \leq G(x) < x \leq \sum_{i=1}^k q_i + \delta_a$, so by the previous observation $(G(x), G(G(x)))$ lies in the square. Similarly, if x is on the left of μ_a, then $\sum_{i=1}^k q_i \leq x < G(x) \leq \sum_{i=1}^k q_i + \delta_a$, so that again the image of the point by G is still in the square.
The number of different values (levels) of G in the square is the number of values μ_i in the interval $\left]\sum_{i=1}^k q_i, \sum_{i=1}^k q_i + \delta_a\right]$ plus one (taking into account right-continuity) and plus one corresponding to μ_a. As this number is finite, the successive points $(x, G(x))$, $(G(x), G(G(x))), \ldots$ must from some step form a cycle.
We prove the claim on the upper bound. The number of levels of G is maximal when the gap q_i at each μ_i is minimal. The minimal value of q_i is $1/n$, which yields a total number of levels to be $n\delta_a + 1$ because the total gap is δ_a.

4.5.7 Proof of Theorem 4.5

1. Suppose that the state is T with $t \geq d$ and $n - t \geq d$. This is possible because by the assumptions $n_a \geq d$, $n_c \geq d$, we have $d \leq n/2$. We claim that a transition to any $Q \in 2^N$ is possible. Indeed, we can have any type of neighborhood, so that the average opinion $\overline{a}(T)$ in a neighborhood can be any value in $\{0, 1/d, \dots, 1\}$. It follows that $p_i^0(T)$ and $p_i^1(T)$ can be positive for all conformists and all anti-conformists.
2. Consider that either $t < d$ or $n - t < d$. It suffices to prove that a transition to some set Q such that $q \geq d$ and $n - q \geq d$ is possible to conclude the proof. Suppose $t < d$ (the other case is similar). Then $n - t > d$, so that the 0-neighborhood has a positive probability. Supposing that all players take the 0-neighborhood, the next action will be 0 for the conformists and 1 for the anti-conformists. Hence $Q = N_a$, which does the job as $|N_a| \geq d$ and $|N \setminus N_a| = |N_c| \geq d$.

4.5.8 Proof of Theorem 4.6

Recall that at each time step, a random neighborhood of random size is drawn, for each agent. Each agent has a different threshold but it is fixed (distribution is known)
1. Consider a state T. In order to have for every conformist and anti-conformist a possibility of choosing action 1 and 0, we must have for conformist choosing action 1 $P(\overline{a}_i \geq \mu_i : S; d) > 0$ for at least one d and choosing action 0 $P(\overline{a}_i \geq \mu_i : S; d) < 1$ for at least one d in the support. For action 1 we must have T with $t \geq \overline{\mu}\underline{d}$ (must work for all agents) and for action 0 we must have $N \setminus T$ with $n - t > \underline{d}(1 - \underline{\mu})$. The conditions are inverted for anti-conformists. Then a transition to any Q is possible at next step.
2. Suppose now that T is such that $t < \overline{\mu}\underline{d}$. Suppose $n_a \geq \overline{\mu}\underline{d}$ and $n_a < n - \underline{d}(1 - \underline{\mu})$. (same conditions for n_c)(hence equivalently, $n_a > \underline{d}(1 - \underline{\mu})$ and idem for n_c). It follows that $\overline{\mu}\underline{d} \leq n/2$ and $\underline{d}(1 - \underline{\mu}) < n/2$. Observe that we have then

$$t < \overline{\mu}\underline{d} \leq {}^{n}\!/_{2} < n - \underline{d}(1 - \underline{\mu}).$$

Therefore every conformist can choose action 0 and every anti-conformist can choose action 1, so that a transition to N_a is possible. As by assumption $n_a \geq \overline{\mu}\underline{d}$ and $n_a < n - \underline{d}(1 - \underline{\mu})$, we are back to Step 1 and a transition to any state Q is possible.
The case where $t \geq n - \underline{d}(1 - \underline{\mu})$ works similarly as we have

$$t \geq n - \underline{d}(1 - \underline{\mu}) \geq {}^{n}\!/_{2} \geq \overline{\mu}\underline{d}.$$

Then a transition to N_c is possible, which allows then a transition to any state Q.

4.5.9 Further discussion

In this paper, we assume that agents update their opinions synchronously in discrete time and we adopt the basic assumption that agents are memoryless. The model can not include all situations in real world which is much more complicated. However, the current paper gives a baseline study of the threshold model with anti-conformist agent and possible future directions of research are discussed below.

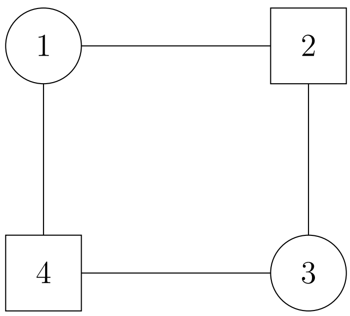
Firstly, the model is in a synchronous framework and we show that in the presence of anti-conformist agents cycles could be reached. However, in case of asynchronous updating, even if it is rare, such cycles are also possible to be reached, as the following simple example shows.

Example 4.9. *Consider a graph with $n = 4$, where agents 1 and 3 are conformist, while 2 and 4 are anticonformist, situated as in the figure below, and take $\mu_i = 1/2$ for all $i \in N$. Agents update opinions asynchronously in the order of agent 1, agent 2, agent 3, agent 4, agent 1, ... Assume the initial opinion is $S(0) = (0, 0, 0, 0)$. At time 1, conformist agent 1 will look at the opinions of her neighbor and update her opinion, thus we denote $S(0)$ as $(0^*, 0, 0, 0)$, where the location of $*$ refers to the agent that will update her opinion at next time period.*

Then we have the following cycle of length 9:

$$(0, 0^*, 0, 0) \to (0, 1, 0^*, 0) \to (0, 1, 0, 0^*) \to (0^*, 1, 0, 1) \to (1, 1^*, 0, 1) \to (1, 0, 0^*, 1)$$

$$\to (1, 0, 0, 1^*) \to (1^*, 0, 0, 0) \to (0, 0^*, 0, 0).$$



Therefore, the synchrony is not the main reason of the presence of cycles, instead, the introduction of anti-conformist agents play an important role.

In the current paper we compared the threshold model with and without anti-conformist agents, and we adopted the basic assumption that agents are completely memoryless such that they can switch between the two states. Indeed, the situation in real life is much more complicated. Agents can be memoryless or have a limited or unlimited memory. Introducing memory of agents is certainly one of the future directions we intend to explore.

In the first part of the paper, we studied the fixed complete graph. However, the findings also hold for fixed regular networks where the neighborhood of each agent is of the fixed size $d < N$, i.e., the proportion of anti-conformist agents and the values of threshold will together determine whether there is an equilibrium point or a cycle. The explicit study of this point is also among our future directions of research.

In the second part of the paper, we showed that for the random sampling model with two thresholds, 15 possible absorbing classes could appear for different conditions on the number of different types of agents and different values of threshold. One can relate these classes with real-life situations. For example, in a market with competitions, $N_a \xrightarrow{1} \emptyset \xrightarrow{1} N_a$ can explain the situation where N_a are the companies that adopt some special techniques, and N_c are the companies for which the special techniques are not available. With the special techniques, companies in N_a are able to produce some special products A and benefit. But once all companies in N_a are producing A such that there is no room for benefiting and thus all companies in N_a will stop producing A. Note here that we model it in a synchronous way, of course in real life, companies often are taking decisions asynchronously. More concretely, we studied the presence of cascade effect when the population of the society tends to infinity. When all agents have the same threshold, a cascade effect is likely to occur, which means that starting from a state of the society where a large majority of agents has the correct opinion (supposing that a ground truth exists), as misinformation spread, society converges to the wrong opinion. On the contrary, when all agents have the same threshold 1/2, or when the proportion of anti-conformist agents is sufficiently small, there is no cascade effect.

Chapter 5

The Dynamics of Cultural Traits in Inherited Endogenous Social Networks[1]

5.1 Introduction

Together with opinions and beliefs, cultural traits are part of factors that govern human behavior and are transmitted through generations. Transmission of cultural traits, such as trust, altruism, morality, risk aversion, persistence, etc., play an important role in shaping economic and social outcomes(Tabellini (2008), Bisin and Verdier (2011)). For instance, trust and risk preferences are important determinants of economic development. Hence, how the cultural traits are formed and evolved are of central interest.

The first theoretical model of cultural transmission was provided by Cavalli-Sforza and Feldman (1981) and Rindos et al. (1985), who also proposed a clear terminology that was widely-adopted by subsequent literature (Bisin and Verdier (2011)). Cavalli-Sforza and Feldman (1981) explained how the custom spread, and showed that if cultural traits were passed down solely from one generation to next generation, some demographic change could not exist, i.e., purely *vertical transmission* is not enough. It required also the consideration of the influence by parents' peers and social environment, i.e., some mixture of *horizontal* and *oblique transmission* (Sober (1992)). Evidently, a large amount of empirical research has also shown that cultural traits of children are shaped by the observable cultural traits of both their parents and social environment (e.g. Dohmen et al. (2012), Tabellini (2008)). Moreover, cultural traits are observed to be persistent among generations (see, e.g., Guiso et al. (2008), Nunn and Wantchekon (2011), Algan and Cahuc

[1]This chapter is a joint work with Tim Hellmann (University of Southampton).

(2010), Voigtländer and Voth (2012)). Thus, social networks play a central role in the process of the formation and evolution of cultural traits.

The seminal work by Bisin and Verdier (2001) provided a model of intergenerational cultural transmission for binary cultural traits and explain its global persistence theoretically, based on the assumption of *imperfect empathy* that parents are willing to use costly controls to increase the probability that their children adopt the same cultural trait as them. In case that a different cultural trait from their parents is adopted, children would learn the trait from a random individual among the population (i.e., *oblique socialization*). Imperfect empathy implies a cultural substitution effect which drives persistence of heterogeneous traits in the long run.

Buechel et al. (2014) and Panebianco (2014) extended this approach by modeling cultural traits as continuous variables and by introducing local interactions represented by a *social network*, considering the network as exogenously given. The role of endogenous networks is introduced in a recent research by Hellmann and Panebianco (2018), but network changes are not inherited by supposing a fixed underlying network. The aim of this paper is to study the dynamic of continuous cultural traits in an endogenous network where network changes are inherited. Parents are able to influence their children's networks to prevent undesirable peer effects or encourage desirable peer effects, by means of school choice, sports clubs, intervene in children's friendships, etc. This makes central the following questions: *How do parents bias children's network optimally? How do cultural traits evolve under this presumption on optimal networks? Under which conditions do heterogeneous and homogeneous societies emerge?*

These questions are closely related to whether the network is directed or not. Directed networks are good approximation to the situation where parents can guide (resp., prevent) their children to learn from some role families from public resources. For undirected networks, *Pairwise stability* is a simple stability concept proposed by Jackson and Wolinsky (1996) to capture the mutual consent required for forming a link between two agents, while Nash equilibrium based solutions fail to capture this point. It supposes that any individual can delete a link unilaterally, while adding a link requires the agreement of both involved individuals. Motivated by this idea of *pairwise stability*, some other notions of *stability* in network formation were proposed, e.g., pairwise Nash stability (Bloch and Jackson (2006)), pairwise stability with transfers (PST) where the transfers among individuals are allowed (Bloch and Jackson (2007)), strong stability (Dutta and Mutuswami (1997)), bilateral stability (Goyal and Vega-Redondo (2007)) and so on.

We endogenize the network formation process in the following three different frameworks: 1) the network is directed, and at each time, any adult can form or delete any link unilaterally with the other adults, by maximizing his or her own utility; 2) the network is undirected, and at each stage $t+1$, a PS network is reached based on $V(t)$ and $G(t)$; 3) the network is undirected, and at each stage $t+1$, a PST network is reached based on $V(t)$ and $G(t)$, i.e., the cultural traits and network of the previous stage. The first framework applies to the situations where cultural traits can be learnt by observing the behavior of others. In this case, parents can reduce or prevent their children from learning the undesirable traits. The second and third framework apply to those situations where cultural traits are influenced through bilateral contact, e.g., trust, altruism, etc. The difference is whether transfers among dynasties are allowed in the process of network formation. PS is a very standard concept for modeling formation of undirected networks, while visible and invisible transfers are also very common when people build their connections in real life. So both notions are adopted here to study the dynamics of cultural traits.

We emphasize the role of two degrees of imperfect empathy relative to (i) cost of network changes and (ii) a desire to be integrated in the society. Moreover, we consider that a dynasty always weights more on own cultural trait than that of any other neighbor's due to the consideration that cultural traits of children are more likely to be influenced vertically by their parents than by their neighbors (peer effects) horizontally.

This paper is structured as follows. In Section 5.2, we introduced the model with notions, assumptions on the networks and the utility function. How the process of network formation is endogenized (in three ways as mentioned previously) is explained in detail in Section 5.3, together with some remarkable results on network changes for one period. In Section 5.4, the dynamics (convergence, limit behavior) of these three models are discussed in detail, in both theory and simulations. In Section 5.5, we discuss the efficiency of networks and Section 5.6 concludes this paper.

5.2 The model

We employ the model by Buechel et al. (2014) and Panebianco (2014) for the transmission of continuous cultural traits on social networks. Consider an overlapping generations society populated by the adults of a finite set of dynasties $N = \{1, \dots, n\}$. At the beginning of any period $t \in \mathbb{N}$, each adult has one offspring. Adults of period $t \in \mathbb{N}$ are characterized by a *cultural trait* $V_i(t) \in I$. As we consider continuous traits, $I \subseteq \mathbb{R}$ is assumed to be compact and convex. Following empirical evidence, children learn their cultural trait from their parents and their social environment which is determined by a social network represented by a $n \times n$ row-stochastic matrix $G = (g_{ij})_{i,j \in N}$ (i.e., $g_{ij} \geq 0$, such that $\sum_{j \in N} g_{ij} = 1$).

We assume that there is a network $G(t)$ (either directed or undirected) in each period $t \in N$. Let $N_i(t) = \{j \in N : (i,j) \in G(t)\}$ be the neighbors of dynasty i in period t, i.e. the dynasties to whom dynasty i has links in period t and denote by $\eta_i(t) = |N_i(t)|$ the (out-) degree of dynasty i in period t.
We then get the influence network by

$$g_{ij}(t) = \begin{cases} \frac{1}{n}, & \text{if } j \in N_i(t); \\ \frac{n-\eta_i(t)}{n}, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

In each period dynasty i can form new links or delete current links. The changes then carry over to the next generation. Obviously, $G(t)$ is always row stochastic for any t. For dynasty i, parents can invest into peer socialization by adding set of links $l_i^+ \subset N - (N_i(t) + \{i\})$ as well as vertical socialization by deleting any set of existing links $l_i^- \subset N_i(t)$. We denote the total investment as $X_i = l_i^+ \cup l_i^-$. This results in a new neighborhood of children denoted by $N_i(t+1) := (N_i(t) \setminus l_i^-) \cup l_i^+$. The detailed network formation process will be explained in Section 5.3).
Children then adopt the cultural trait according to

$$V_i(t+1) = \sum_{j \in N_i(t+1)} g_{ij}(t+1) V_j(t), \tag{5.1}$$

which is formed according to the influences they are exposed to:

$$g_{ij}(t+1) = \begin{cases} \frac{1}{n}, & \text{if } j \in (N_i(t) \setminus l_i^-) \cup l_i^+; \\ \frac{n-\eta_i(t)-\lambda_i^+ + \lambda_i^-}{n}, & \text{if } j = i; \\ 0, & \text{otherwise.} \end{cases}$$

where $\lambda_i^+ := |l_i^+|$ and $\lambda_i^- := |l_i^-|$. At the end of any period $t \in \mathbb{N}$, the adults die, the children become adults in period $t+1$, and carry over the adopted trait into their adult period.

Utility of each dynasty

We assume imperfect empathy: parents want their children to adopt the same cultural trait as they carry themselves (see for a motivation e.g. Bisin and Verdier, 2010). Moreover, as both parental socialization and network changes are costly, we assume the following functional form of utility

$$U_i(t+1) = -[V_i(t+1) - V_i(t)]^2 - c_i^{\Delta}(\lambda_i^+ + \lambda_i^-)^2 - c_i^{\eta}(n - \eta_i(t+1)), \tag{5.2}$$

where $c_i^{\Delta}, c_i^{\eta} > 0$ for all $i \in N$.

The first term $-[V_i(t+1) - V_i(t)]^2$ is the intergenerational utility component which is decreasing in the distance between parent's and child's trait and therefore reflects imperfect empathy. The remaining term is composed of the cost of network intervention and the cost of parental socialization. Cost of network intervention depends on the number of altered links between $G_i(t+1)$ and the previous original influence $G_i(t)$ weighted by the dynasty dependent cost factor c^{Δ}. Cost of parental socialization is created by the amount of time parents spend with the children $\frac{n-\eta_i(t+1)}{n}$ weighted with a cost term nc^{η}. This cost term can also be interpreted as a (positive) benefit term from interaction with others.

5.3 Endogenize network formation $(G(t) \to G(t+1))$

In this section, we propose three ways to endogenize the network formation process, based on whether the network is directed or not and stability of the network dynamics. In Section 5.3.1, the network is supposed to be directed and thus dynasty can form or delete any link unilaterally. In Section 5.3.3 and Section 5.3.2, the network is supposed to be undirected. We endogenize the undirected network formation by adopting the notions of PST and PS networks in Section 5.3.3 and Section 5.3.2, respectively.

5.3.1 Directed network with optimal network changes

Optimization problem of purposeful socialization

Suppose that $G(t)$ is directed and thus dynasties face a trade-off between own utility losses (due to costs of child care and network changes) and eventual improvements in locations of child's trait relative to peak. Thus the optimization problem of adult i in period $t+1$ is:

$$\begin{aligned}
&\max_{l_i^+, l_i^-} -(V_i(t+1) - V_i(t))^2 - c_i^{\Delta}(\lambda_i^+ + \lambda_i^-)^2 - c_i^{\eta}(n - \eta_i(t) - \lambda_i^+ + \lambda_i^-) \\
&\text{s.t.} \quad V_i(t+1) = \sum_{j \in N} g_{ij}(t+1) V_j(t) \\
&\qquad\qquad\quad = V_i(t) + \frac{1}{n} \sum_{j \in (N_i(t) \setminus l_i^-) \cup l_i^+} (V_j(t) - V_i(t)) \,.
\end{aligned}$$

Optimal network changes in each period

Consider the change of network for dynasty $i \in N$. Adult i's utility then changes according to

$$\begin{aligned}&\Big[U_i(g(t+1)) - c_i^{\Delta}(g(t+1) \mid g(t)) - c_i^{\eta}(g(t+1))\Big] - [U_i(g(t)) - c_i^{\eta}(g(t))]\\ =& -\frac{1}{n^2}\left[d_i^{l_i^+} - d_i^{l_i^-}\right]\left[d_i^{(N_i+l_i^+-l_i^-)} + d_i^{N_i}\right] - (\lambda_i^- - \lambda_i^+)c_i^{\eta} - (\lambda_i^+ + \lambda_i^-)^2 c_i^{\Delta},\end{aligned}$$

where for $L \subseteq N \setminus \{i\}$ we have set $d_i^L := \sum_{j\in L}(V_i(t) - V_j(t))$, and the set operations union and subtraction are denoted as $+$ and $-$ among sets.
Denote by $V_i^<(t) := \{j \in N : V_j(t) < V_i(t)\}$ and by $V_i^>(t) := \{j \in N : V_j(t) > V_i(t)\}$. A necessary condition for optimal links is given by the following Proposition.

Proposition 5.1. *Suppose $c^{\eta} \le c^{\Delta}$. The necessary conditions for a set of links L_i^+ added to network $g(t)$ and a set of links L_i^- deleted from network $g(t)$ by player $i \in N$ to be optimal are that:*

A) for all $j \in L_i^+ \cup V_i^<(t)$ it holds that $d_i^{N+L_i^+-L_i^--j}(t) < 0$,

B) for all $j \in L_i^+ \cup V_i^>(t)$ it holds that $d_i^{N+L_i^+-L_i^--j}(t) > 0$,

C) for all $j \in L_i^- \cup V_i^<(t)$ it holds that $d_i^{N+L_i^+-L_i^-}(t) > 0$,

D) for all $j \in L_i^- \cup V_i^>(t)$ it holds that $d_i^{N+L_i^+-L_i^-}(t) < 0$.

Proof. We only show the first point, the remainder is completely analogous. Suppose that contrarily to the assertion we have $j \in L_i^+ \cup V_i^<(t)$ but $d_i^{N+L_i^+-L_i^--j}(t) > 0$. Since $j \in V_i^<(t)$ we have $d_i^j(t) = (V_i(t) - V_j(t)) > 0$. We get that

$$\begin{aligned}d_i^{N+L_i^+-L_i^-}(t) &= \sum_{k\in N+L_i^+-L_i^-}(V_i(t) - V_k(t))\\ &= \sum_{k\in N+L_i^+-L_i^--j}(V_i(t) - V_k(t)) + (V_i(t) - V_k(t))\\ &= d_i^{N+L_i^+-L_i^--j}(t) + d_i^j(t) > 0.\end{aligned}$$

For the marginal utility of the link ij we then get:

$$\begin{aligned}&u_i(g(t) + L_i^+ - L_i^-) - u_i(g(t) + L_i^+ - L_i^- - j)\\ &= -\frac{1}{n^2}\left(d_i^j(t)\right)\left(d_i^{N_i(t)+L_i^+-L_i^-}(t) + d_i^{N_i(t)+L_i^+-L_i^--j}(t)\right) + c^{\eta} - c^{\Delta}(2(l_i^+ + l_i^-) + 1)\\ &< c^{\eta} - c^{\Delta}(2(l_i^+ + l_i^-) + 1) \le 0.\end{aligned}$$

The last inequality is due to the fact that $c^\eta \leq c^\Delta$. □

Note that condition C and condition D are mutually exclusive. Thus either $L_i^-(t) \subset V_i^<(t)$ or $L_i^-(t) \subset V_i^>(t)$. From equation (5.1), we get that

$$V_i(t+1) = V_i(t) - \frac{1}{n} d_i^{N_i(t+1)}. \tag{5.3}$$

Then it is obvious that if condition C holds, $V_i(t+1) < V_i(t)$; if condition D (thus D') holds, $V_i(t+1) > V_i(t)$. It means that if agent cuts a link to an agent with greater cultural trait in period t, her cultural trait in period $t+1$ will become greater; *vice versa*.

Single link changes The following corollary shows that link changes are only optimal if these countervail the peer infuence.

Corollary 5.1. *Suppose $c^\eta \leq c^\Delta$ and let $|l_i^+ \cup l_i^-| \leq 1$ for some $i \in N$. Then this can be optimal only if:*

(i) If $V_i(t) < \frac{1}{\eta_i(t)} \sum_{j \in N_i(t)} V_j(t)$, then $l_i^+ \subset V_i^<(t)$ and $l_i^- \subset V_i^>(t)$.

(ii) If $V_i(t) > \frac{1}{\eta_i(t)} \sum_{j \in N_i(t)} V_j(t)$, then $l_i^- \subset V_i^<(t)$ and $l_i^+ \subset V_i^>(t)$.

(iii) If $V_i(t) = \frac{1}{\eta_i(t)} \sum_{j \in N_i(t)} V_j(t)$, then $l_i^- = l_i^+ = \emptyset$.

Extremists' behavior We call the adults who hold the lowest or the greatest cultural traits as the extremists. The following corollaries show that extremists never add links and in the case of sufficiently low cost, extremists may cut all ties with the society.

Corollary 5.2. *Suppose $c^\eta \leq c^\Delta$ and let $i \in \arg\min_{k \in N} V_k(t)$ and $j \in \arg\max_{k \in N} V_k(t)$. Then, $l_i^+ = l_j^+ = \emptyset$.*

Proof. For all $i \in \min_{k \in N} V_k(t)$ it is $d^{N_i}(t) \leq 0$ for all $N_i \subset N \setminus \{i\}$ and $V_i^<(t) = \emptyset$. Thus the first two conditions of Proposition 5.1 cannot be satisfied implying that $L_i^+ = \emptyset$. Similarly for all $j \in \max_{k \in N} V_k(t)$ it is $d^{N_j}(t) \geq 0$ for all $N_j \subset N \setminus \{j\}$ and $V_j^>(t) = \emptyset$. Thus, again, the first two conditions of Proposition 5.1 cannot be satisfied implying that $L_j^+ = \emptyset$. □

Corollary 5.3. *Let $i \in \arg\min_{k \in N} V_k(t)$ and $j \in \arg\max_{k \in N} V_k(t)$. There exists $\epsilon > 0$ such that for all $0 \leq c^\eta \leq c^\Delta < \epsilon$, we get $l_i^- = V_i^> \cap N_i(t)$ and $l_j^- = V_i^< \cap N_j(t)$.*

The following example illustrates the network changes in one period.

Example 5.1. *Consider 10 dynasties situated in an initial network given by Figure 5.1 with its incidence matrix being*

$$\begin{array}{cc} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} & \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10, \end{matrix} \end{array}$$

and the costs of child care and network changes are $c^\eta = 0$ and $c^\Delta = 0.0045$, respectively. The initial state of cultural traits is $\mathbf{V}(0) = (1,2,3,3,4,4,5,7,9,11)$. To illustrate the differences of network changes between extremists and integrated families, take $n = 1,7,10$ as an example. See Figure 5.2 for the network structure, restricted to dynasties 1, 7 and 10. By solving the corresponding optimization problems, dynasty 1 will delete links with 7 and 9, dynasty 10 will delete links with 2 and 3, and dynasty 7 would rather add links to dynasty10 (see Figure 5.3). The resulting network at time 1 is shown in Figure 5.4 with its incidence matrix to be

$$\begin{array}{cc} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} & \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10. \end{matrix} \end{array}$$

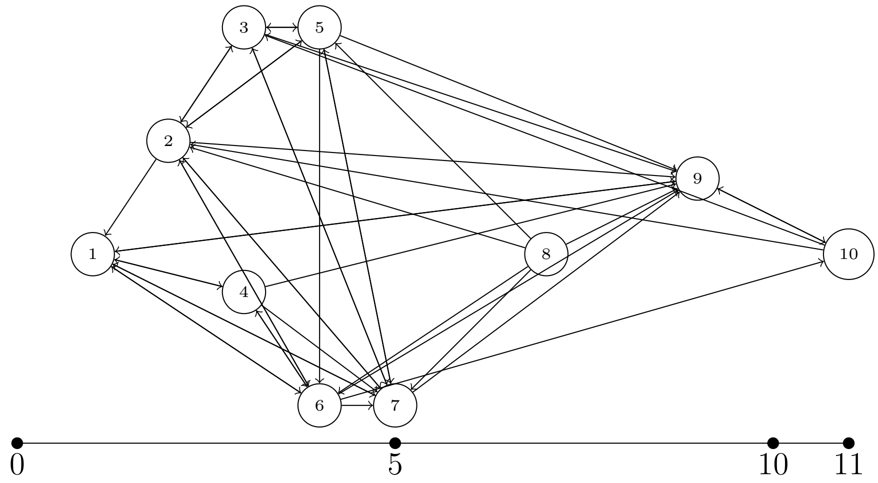


Figure 5.1: Initial network.

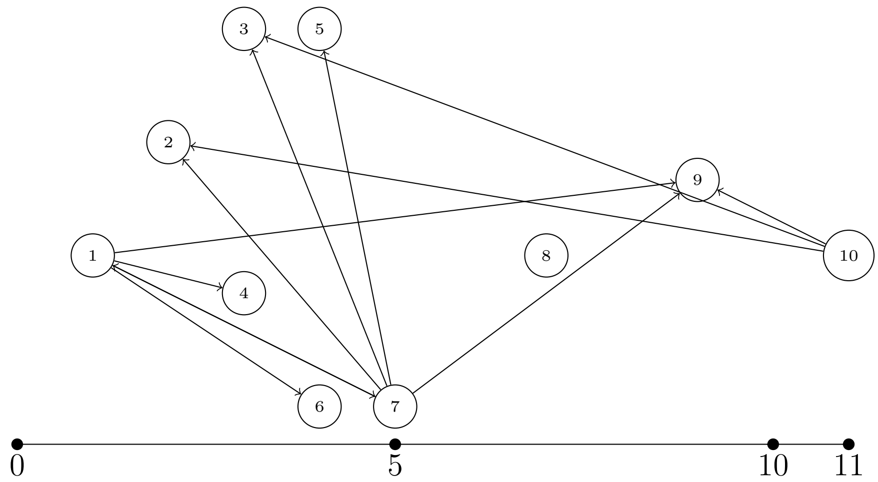


Figure 5.2: Initial network, restricted to dynasties 1,7 and 10.

5.3.2 Undirected network with PS (Pairwise Stable) networks

Suppose that $G(t)$ is undirected. The PS network is defined as follows.

Definition 5.1 (Pairwise Stable)**.** *For all $V(t), g(t)$, a network g^* is Pairwise Stable (PS for short) if*

(i) $\forall ij \in g^*$,

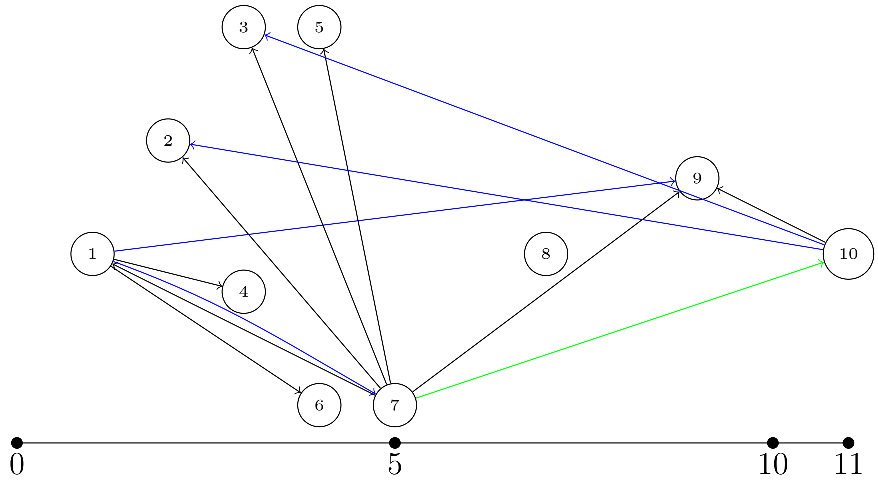


Figure 5.3: Network changes of 1,7 and 10, with deleted links colored in blue and added links colored in green.

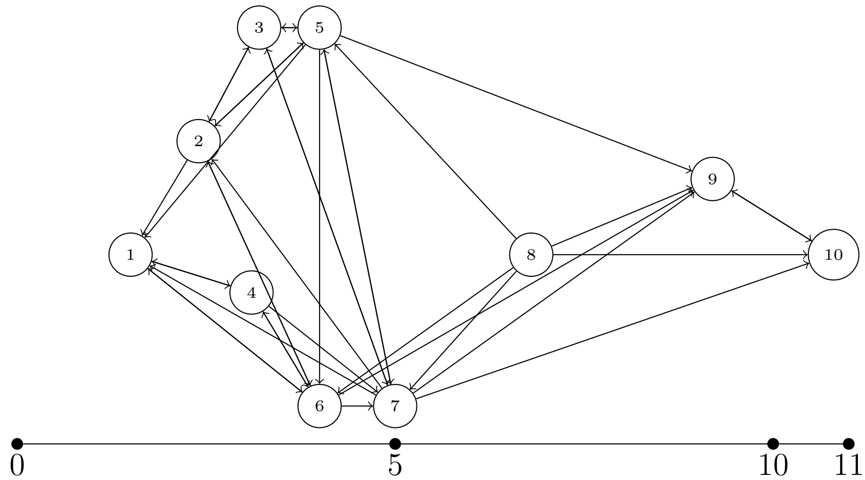


Figure 5.4: Resulting network at time 1.

$$u_i(g^* - ij|g(t), V(t)) - u_i(g^*|g(t), V(t)) \leq 0,$$ [2]

$$\textit{and } u_j(g^* - ij|g(t), V(t)) - u_j(g^*|g(t), V(t)) \leq 0;$$

(ii) $\forall ij \notin g^*$,

$$u_i(g^*+ij|g(t), V(t)) > u_i(g^*|g(t), V(t)) \implies u_j(g^*+ij|g(t), V(t)) \leq u_j(g^*|g(t), V(t)).$$

Pairwise stable network requires that no adult wants to delete a link unilaterally and no two unconnected adults both want to form a link, considering one link at a time. It requires the mutual agreement to form a link between two adults.

The following proposition gives the sufficient condition such that the empty network is Pairwise Stable.

[2] $u_i(g^* - ij|g(t), V(t))$ refers to the utility of agent i of deleting the link ij from g^* given the current network $g(t)$ and current cultural trait vector $V(t)$. It is analogous for $u_i(g^* + ij|g(t), V(t))$.

Proposition 5.2. *For any $V(t)$ and $g(t)$, a sufficient condition such that the empty network is Pairwise Stable for given $V(t), g(t)$ is that $\frac{c^\eta}{c^\Delta} \leq \min_{i \in N}\{2x_i + 1\}$, where x_i and x_j are the numbers of links agent i and j have in $g(t)$, respectively.*

Proof. Suppose that in $g(t)$, agent i has x_i links, $\forall i \in N$. Then

$$\begin{aligned} & mu_i(g^\emptyset + ij, ij|V(t), g(t)) \\ = & -\frac{1}{n^2}(V_i - V_j)^2 - c^\Delta(2x_i + 1) + c^\eta. \end{aligned}$$

Thus $\frac{c^\eta}{c^\Delta} \leq 2x_i + 1 \implies mu_i(g^\emptyset + ij, ij|V(t), g(t)) \leq 0$.
So

$$\begin{aligned} & \frac{c^\eta}{c^\Delta} \leq \min_{i \in N}\{2x_i + 1\} \\ \implies & mu_i(g^\emptyset + ij, ij|V(t), g(t)) \leq 0, \forall i \in N \\ \implies & g^\emptyset \text{ is PS.} \end{aligned}$$

□

Remark 5.1. *When $c^\eta = c^\Delta = 0$, $g^\emptyset$ is PS.*

Proposition 5.2 says that if $\frac{c^\eta}{c^\Delta}$ is sufficiently small, the empty network will be pairwise stable. Furthermore, if the two cost parameters are sufficiently small, the empty network will be the unique pairwise stable network.

Proposition 5.3. *There always exist sufficiently small c^η and c^Δ such that the empty network is the unique PS network for given $g(t)$ and $V(t)$.*

Proof. Given $g(t)$ and $V(t)$, Proposition 5.2 guarantees the existence of c^Δ and c^η such that the empty network is PS. It suffices to prove that $\forall g \neq g^\emptyset$, $\exists c^\Delta > 0$ and $c^\eta > 0$ such that g is not PS.
We will show that for any nonempty network is not PS. Fix any $g \neq g^\emptyset$, among all the agents having at least one link in g, we can always find the agent holding either the largest or the smallest cultural trait, say i, and i wants to delete the links. Suppose w.l.o.g. i is connected to those holding larger culture traits, i.e., $V_j > V_i$, $\forall j \in N_i$, where N_i is the neighborhood of agent i in g. Denote y_i as the number of links that agent i need to change from $g(t)$ to g. Depending on whether $ij \in G(t)$ or not, the number of links that agent i need to change from $G(t)$ to $g - ij$ can be computed as $y_i + 1$ (for the case of

$ij \in G(t)$) or $y_i - 1$ (for the case of $ij \notin G(t)$). Denote N_i as the neighborhood of agent i in g. The marginal utility of deleting any link ij from g for agent i is

$$mu_i(g, ij|V(t), G(t)) = \frac{1}{n^2}[(d_i^{N_i-\{j\}})^2 - (d_i^{N_i})^2] + c^\Delta(1 \pm 2y_i) + c^\eta.$$

When

$$c^\Delta(1 \pm 2y_i) + c^\eta < \frac{1}{n^2}[(d_i^{N_i})^2 - (d_i^{N_i-\{j\}})^2], \tag{5.4}$$

$mu_i(g, ij|V(t), G(t)) < 0$, i.e., agent i would like to delete the link ij. Since $\forall j \in N_i$, $V_j > V_i$, $(d_i^{N_i-\{j\}})^2 - (d_i^{N_i})^2 > 0$. Thus we can always find small enough c^Δ and c^η such that inequality 5.4 holds, so g is not PS.

Above all, $g^\emptyset$ is the unique PS network.

□

5.3.3 Undirected network with PST (Pairwise Stable with Transfers) networks

In this section and the next section, we suppose that $G(t)$ is undirected. The PST network is defined as follows.

Definition 5.2 (Pairwise Stable with Transfers). *For all $V(t), g(t)$, a network g^* is Pairwise Stable with Transfers (PST) if*

(i) $\forall ij \in g^*$,

$U_i(g^* \mid g(t), V(t)) + u_j(g^* \mid g(t), V(t)) \geq U_i(g^* - ij \mid g(t), V(t)) + u_j(g^* - ij \mid g(t), V(t))$;

(ii) $\forall ij \notin g^*$,

$U_i(g^* \mid g(t), V(t)) + u_j(g^* \mid g(t), V(t)) \geq U_i(g^* + ij \mid g(t), V(t)) + u_j(g^* + ij \mid g(t), V(t))$.

Pairwise Stable network with Transfers requires that no pair of agents can jointly benefit by forming or deleting a link. In other words, it allows the utility exchange between two agents.

Existence of PST networks at each period

Jackson and Watts (2002) showed that there exist at least one pairwise stable network or closed cycle of networks. Following the same reasoning, it also holds for PST networks.

Lemma 5.1. *For any network society, there exists at least one PST network or a closed cycle of networks.*

Theorem 5.1. *For all $V(t)$ and $g(t)$, there exist at least one PST network.*

Proof. We show here there is no closed cycle. For any given $g(t)$, assume there is a closed cycle $g^1, g^2, \dots, g^m$, where $g^1 = g^m = g(t)$. Note here that $u_i(g) = u_i(g')$ if $N_i(g) = N_i(g')$. Therefore the sum of the utilities is strictly increasing along the improving path, i.e., $\sum_{i=1}^n u_i(g^k) < \sum_{i=1}^n u_i(g^{k+1}), \forall k \in \{1, 2, \dots, m-1\}$. Thus $\sum_{i=1}^n u_i(g^1) < \sum_{i=1}^n u_i(g^m)$ leading to a contradiction to $\sum_{i=1}^n u_i(g^1) = \sum_{i=1}^n u_i(g^m) = \sum_{i=1}^n u_i(g(t)))$. So there is no closed improving cycle but at lease one pairwise stable network with transfers by Lemma 5.1.

□

Network changes in one period

We suppose that in each period, a PST network $G(t+1)$ is reached given $V(t)$ and $G(t)$[3]. The following example illustrates the network changes in one period.

Example 5.2. *Consider 10 dynasties situated in an initial network given by Figure 5.5 with its incidence matrix being*

$$
\begin{array}{cc}
\begin{array}{cccccccccc} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} & \mathit{7} & \mathit{8} & \mathit{9} & \mathit{10} \end{array} & \\
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
&
\begin{array}{c} \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \\ \mathit{7} \\ \mathit{8} \\ \mathit{9} \\ \mathit{10}, \end{array}
\end{array}
$$

and the costs of child care and network changes are $c^\eta = 0.01$ and $c^\Delta = 0.002$, respectively. The initial state of cultural traits is $\mathbf{V}(0) = (1, 2, 3, 3, 4, 4, 5, 7, 9, 11)$. To illustrate the differences of network changes between extremists and integrated families, take $n = 1, 3, 10$

[3] In case of the existence of multiple PST networks, a random PST network will be reached.

as an example. See Figure 5.6 for the network structure, restricted to dynasties 1, 6 and 10. By solving the corresponding optimization problems, dynasty 1 will delete links with 6, 7 and 9, dynasty 10 will delete links with 6, and dynasty 3 would delete link with 9, and at the same time add link with 4 (see Figure 5.7). The resulting network at time 1 *is shown in Figure 5.8 with its incidence matrix to be*

$$\begin{array}{cc} \begin{array}{cccccccccc} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} & \mathit{7} & \mathit{8} & \mathit{9} & \mathit{10} \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} & \begin{array}{c} \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \\ \mathit{7} \\ \mathit{8} \\ \mathit{9} \\ \mathit{10.} \end{array} \end{array}$$

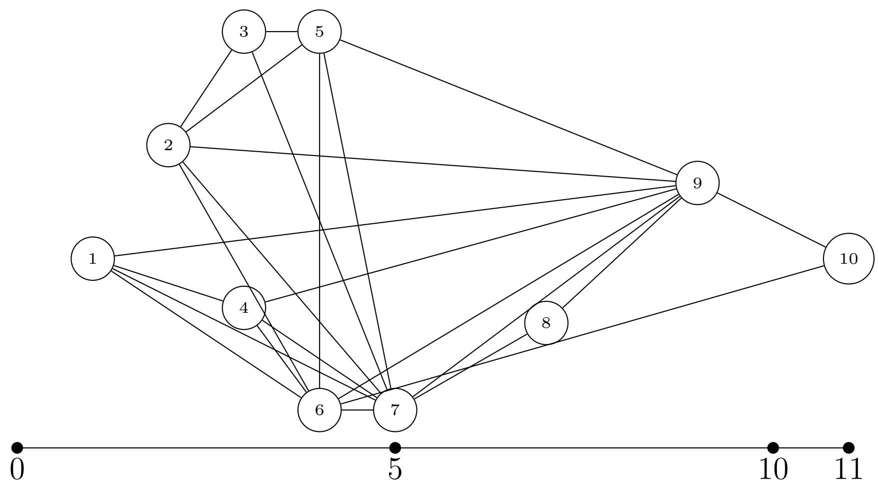


Figure 5.5: Initial network.

In case that a PST network is reached in each period, extremists tend to cut links and intermediate agents tend to add links.

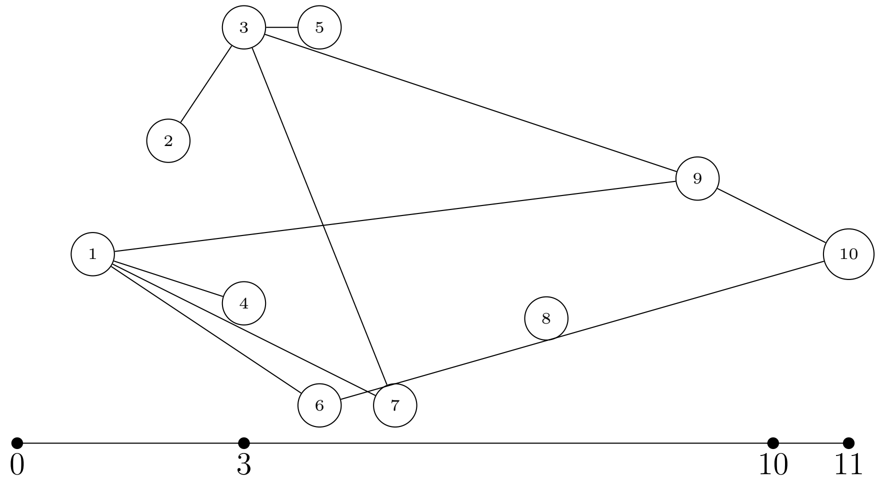


Figure 5.6: Initial network, restricted to dynasties 1, 3 and 10.

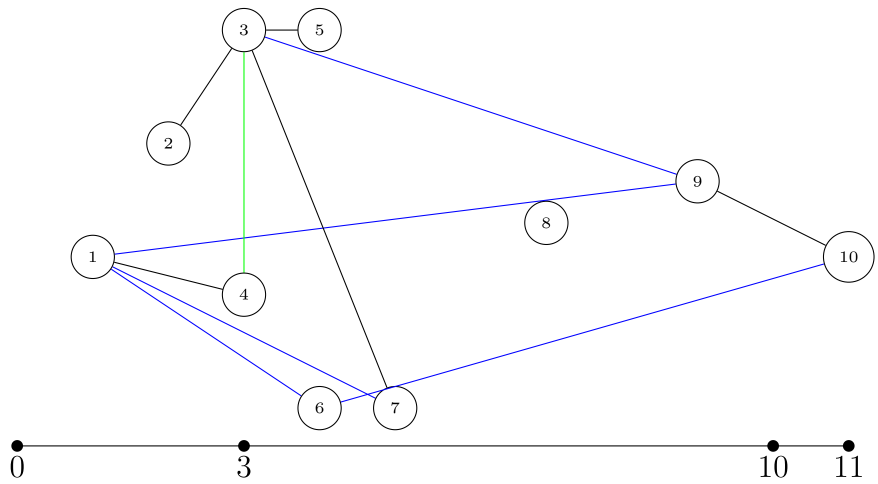


Figure 5.7: Network changes of 1,3 and 10, with deleted links colored in blue and added links colored in green.

Necessary and sufficient conditions for a network to be PST

Given $V(t)$ and $G(t)$, we define the sum of marginal utilities of i and j for adding a link to g^* or cutting the link ij from g^* as follows.

Definition 5.3 (Sum of marginal utilities of i and j)**.**

The sum of marginal utilities of i and j for adding a link to g^* is defined as

$$\begin{aligned}&mu_{i+j}(g^*+ij, ij|g(t), V(t))\\:=&u_i(g^*+ij|g(t), V(t)) - u_i(g^*|g(t), V(t)) + u_j(g^*+ij|g(t), V(t)) - u_j(g^*|g(t), V(t)).\end{aligned}$$

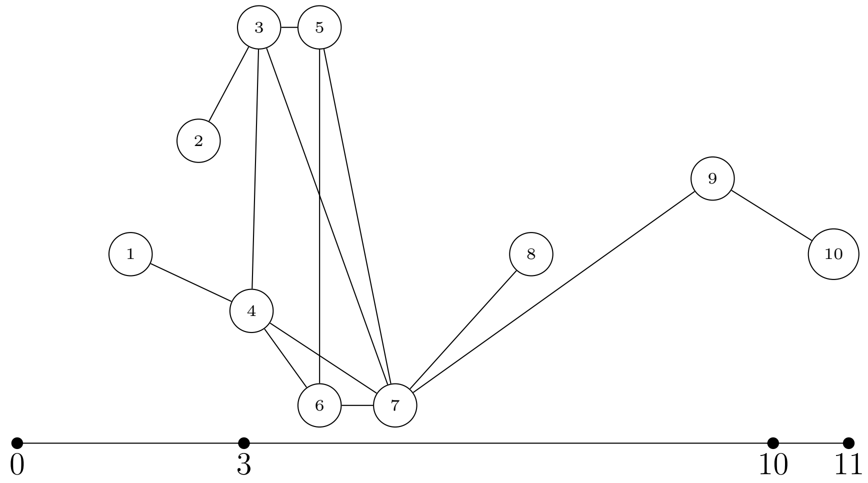


Figure 5.8: Resulting network at time 1

The sum of marginal utilities of i and j for cutting the link ij from g^* is defined as

$$\begin{aligned} &mu_{i+j}(g^*, ij|g(t), V(t)) \\ :=&u_i(g^*|g(t), V(t)) - u_i(g^* - ij|g(t), V(t)) + u_j(g^*|g(t), V(t)) - u_j(g^* - ij|g(t), V(t)). \end{aligned}$$

The following proposition provides the necessary and sufficient condition such that the empty network is PST network.

Proposition 5.4. *$\forall i, j \in N$, suppose that in $g(t)$, i has x_i links and j has x_j links. For any $V(t), g(t)$, the necessary and sufficient condition such that the empty network is PST network is that*

$$\begin{aligned} &-\frac{(v_i - v_j)^2}{n^2} \leq c^\Delta - c^\eta - c^\Delta(x_i + x_j), \forall ij \in g(t), \text{ and} \\ &-\frac{(v_i - v_j)^2}{n^2} \leq c^\Delta - c^\eta + c^\Delta(x_i + x_j), \forall ij \notin g(t) \end{aligned}$$

Note that when $c^\eta = c^\Delta = 0$, $g^\emptyset$ is always a PST network.

Proof. The sum of the marginal utilities of adding any link ij to $g^\emptyset$ is

$$\begin{aligned}
&u_i(g^{\emptyset}+ij|g(t),V(t))+u_j(g^{\emptyset}+ij|g(t),V(t))-u_i(g^{\emptyset}|g(t),V(t))-u_j(g^{\emptyset}|g(t),V(t)\\
=&-\frac{2(v_i-v_j)^2}{n^2}-c^{\Delta}[(x_i-1)^2-x_i^2+(x_j-1)^2-x_j^2]-2c^{\eta}(n-1-n)\\
=&-\frac{2(v_i-v_j)^2}{n^2}-2c^{\Delta}+2c^{\eta}+2c^{\Delta}(x_i+x_j),\ \text{if } ij\in g(t);
\end{aligned}$$

$$\begin{aligned}
=&-\frac{2(v_i-v_j)^2}{n^2}-c^{\Delta}[(x_i+1)^2-x_i^2+(x_j+1)^2-x_j^2]-c^{\eta}(n-1-n)\\
=&-\frac{2(v_i-v_j)^2}{n^2}-2c^{\Delta}+2c^{\eta}-2c^{\Delta}(x_i+x_j),\ \text{if } ij\notin g(t).
\end{aligned}$$

When $ij \in g(t)$, a sufficient condition such that $(mu_i + mu_j)(g^{\emptyset} + ij, ij) < 0$ is that $2c^{\Delta} - 2c^{\eta} - 2c^{\Delta}(x_i + x_j) > 0$, i.e. $\frac{c^{\eta}}{c^{\Delta}} < 1 - (x_i + x_j)$; (So there could be the case that the. empty network is not PST).
When $ij \notin g(t)$, a sufficient condition such that $(mu_i + mu_j)(g^{\emptyset} + ij, ij) < 0$ is that $2c^{\Delta} - 2c^{\eta} + 2c^{\Delta}(x_i + x_j) > 0$, i.e. $\frac{c^{\eta}}{c^{\Delta}} < 1 + (x_i + x_j)$.

$$\begin{aligned}
g^{\emptyset} \text{ is PST } &\Leftrightarrow (mu_i+mu_j)(g^{\emptyset}+ij,ij)\le 0\\
&\Leftrightarrow -\frac{(v_i-v_j)^2}{n^2}\le c^{\Delta}-c^{\eta}-c^{\Delta}(x_i+x_j),\forall ij\in g(t)\\
&\text{and } -\frac{(v_i-v_j)^2}{n^2}\le c^{\Delta}-c^{\eta}+c^{\Delta}(x_i+x_j),\forall ij\notin g(t).
\end{aligned}$$

□

More generally, the following proposition gives the necessary and sufficient condition such that any g^* is PST network.

Proposition 5.5. *For any $V(t), g(t), g^*$, suppose that $\forall i \in N$, i has x_i links changed from $g(t)$ to g^*. The neighbourhood of player i in networks $g(t), g^*, g^* + ij$ are denoted as $N_i, N_i^*, N_i^* + \{j\}$, with $|N_i| = \eta_i$, $|N_i^*| = \eta_i^*$ and $|N_i^* + \{j\}| = \eta_i^* + 1$ respectively. For any $V(t), g(t)$, the necessary and sufficient condition such that g^* is PST network is that*
$\forall ij \notin g^*$, $\frac{1}{n^2}[(V_i - V_j)(d_i^{N_i^*} - d_j^{N_j^*} + V_i - V_j)] + c^{\Delta}(x_i + x_j - 1) + c^{\eta} \le 0$, if $ij \in g(t)$, $\frac{1}{n^2}[(V_i - V_j)(d_i^{N_i^*} - d_j^{N_j^*} + V_i - V_j)] - c^{\Delta}(x_i + x_j - 1) + c^{\eta} \le 0$, if $ij \notin g(t)$;
$\forall ij \in g^*$, $\frac{1}{n^2}[(V_i - V_j)(d_j^{N_j^*} - d_i^{N_i^*} + V_i - V_j)] - c^{\Delta}(x_i + x_j - 1) + c^{\eta} \le 0$, if $ij \in g(t)$, $\frac{1}{n^2}[(V_i - V_j)(d_j^{N_j^*} - d_i^{N_i^*} + V_i - V_j)] + c^{\Delta}(x_i + x_j - 1) + c^{\eta} \le 0$, if $ij \notin g(t)$.

Proof.

$$\begin{aligned}
&g^* \text{ is a PST network given } V(t) \text{ and } g(t)\\
\Leftrightarrow\ & mu_{i+j}(g^*+ij, g^*|V(t), g(t)) \leq 0, \forall ij \notin g^*;\\
&\text{and } mu_{i+j}(g\text{'}g^* - ij|V(t), g(t)) \geq 0, \forall ij \in g^*.
\end{aligned}$$

We distinguish the following cases:

- $ij \notin g^*$

  - $ij \in g(t)$

$$\begin{aligned}
&mu_{i+j}(g^*+ij, g^*|V(t), g(t))\\
=&-(-\frac{1}{n}d_i^{N_i^*+\{j\}})^2 - c^{\Delta}(x_i-1)^2 - c^{\eta}(n-\eta_i^*-1)\\
&-(-\frac{1}{n}d_j^{N_j^*+\{i\}})^2 - c^{\Delta}(x_j-1)^2 - c^{\eta}(n-\eta_j^*-1)\\
&+(-\frac{1}{n}d_i^{N_i^*})^2 - c^{\Delta}(x_i)^2 - c^{\eta}(n-\eta_i^*)\\
&+(-\frac{1}{n}d_j^{N_j^*})^2 - c^{\Delta}(x_j)^2 - c^{\eta}(n-\eta_j^*)\\
=&\frac{2}{n^2}[(V_i-V_j)(d_i^{N_i^*} - d_j^{N_j^*} + V_i - V_j)] + 2c^{\Delta}(x_i+x_j-1) + 2c^{\eta};
\end{aligned}$$

  - $ij \notin g(t)$

$$\begin{aligned}
&mu_{i+j}(g^*+ij, g^*|V(t), g(t))\\
=&-(-\frac{1}{n}d_i^{N_i^*+\{j\}})^2 - c^{\Delta}(x_i+1)^2 - c^{\eta}(n-\eta_i^*-1)\\
&-(-\frac{1}{n}d_j^{N_j^*+\{i\}})^2 - c^{\Delta}(x_j+1)^2 - c^{\eta}(n-\eta_j^*-1)\\
&+(-\frac{1}{n}d_i^{N_i^*})^2 - c^{\Delta}(x_i)^2 - c^{\eta}(n-\eta_i^*)\\
&+(-\frac{1}{n}d_j^{N_j^*})^2 - c^{\Delta}(x_j)^2 - c^{\eta}(n-\eta_j^*)\\
=&\frac{2}{n^2}[(V_i-V_j)(d_i^{N_i^*} - d_j^{N_j^*} + V_i - V_j)] - 2c^{\Delta}(x_i+x_j-1) + 2c^{\eta};
\end{aligned}$$

- $ij \in g^*$

– $ij \in g(t)$

$$
\begin{aligned}
&- mu_{i+j}(g^*, ij|V(t), g(t))\\
=&- (-\frac{1}{n}d_i^{N_i^*-\{j\}})^2 - c^{\Delta}(x_i+1)^2 - c^{\eta}(n-\eta_i^*-1)\\
&- (-\frac{1}{n}d_j^{N_j^*-\{i\}})^2 - c^{\Delta}(x_j+1)^2 - c^{\eta}(n-\eta_j^*-1)\\
&+ (-\frac{1}{n}d_i^{N_i^*})^2 - c^{\Delta}(x_i)^2 - c^{\eta}(n-\eta_i^*)\\
&+ (-\frac{1}{n}d_j^{N_j^*})^2 - c^{\Delta}(x_j)^2 - c^{\eta}(n-\eta_j^*)\\
=&\frac{2}{n^2}[(V_i-V_j)(d_j^{N_j^*} - d_i^{N_i^*} + V_i - V_j)] - 2c^{\Delta}(x_i+x_j-1) + 2c^{\eta};
\end{aligned}
$$

– $ij \notin g(t)$

$$
\begin{aligned}
&- mu_{i+j}(g^*, ij|V(t), g(t))\\
=&- (-\frac{1}{n}d_i^{N_i^*-\{j\}})^2 - c^{\Delta}(x_i-1)^2 - c^{\eta}(n-\eta_i^*-1)\\
&- (-\frac{1}{n}d_j^{N_j^*-\{i\}})^2 - c^{\Delta}(x_j-1)^2 - c^{\eta}(n-\eta_j^*-1)\\
&+ (-\frac{1}{n}d_i^{N_i^*})^2 - c^{\Delta}(x_i)^2 - c^{\eta}(n-\eta_i^*)\\
&+ (-\frac{1}{n}d_j^{N_j^*})^2 - c^{\Delta}(x_j)^2 - c^{\eta}(n-\eta_j^*)\\
=&\frac{2}{n^2}[(V_i-V_j)(d_j^{N_j^*} - d_i^{N_i^*} + V_i - V_j)] + 2c^{\Delta}(x_i+x_j-1) + 2c^{\eta}.
\end{aligned}
$$

□

5.4 Dynamics

For simplicity, we denote the dynamic model of cultural traits in directed networks (described in Section 5.3.1), in undirected PST networks (described in Section 5.3.3) and in undirected PS networks (described in Section 5.3.2) as the DN model, the UPST model and the UPS model, respectively.

5.4.1 The DN model

The optimization Problem for adult $i \in N$ at time t is:

$$\begin{aligned}&\max_{l_i^+, l_i^-} -(V_i(t+1) - V_i(t))^2 - c^{\Delta}(\lambda_i^+ + \lambda_i^-)^2 - c^{\eta}(n - \eta(t) - \lambda_i^+ + \lambda_i^-)\\ =&\max_{l_i^+, l_i^-} -\frac{1}{n^2}(d_i^{N_i(t+1)})^2 - c^{\Delta}(\lambda_i^+ + \lambda_i^-)^2 - c^{\eta}(n - \eta(t) - \lambda_i^+ + \lambda_i^-).\end{aligned}$$

A steady state is such that either all traits are homogenous or the extremists are disconnected from all others.

Definition 5.4. *A state $V(t)$ is called a steady state if for all $t' > t$, $V_i(t') = V_i(t)$, for all $i \in N$.*

General results

Assume that each dynasty applies optimal network changes in each period. If at some time t, a dynasty i deleted all the links with the others, i..e, $N_i(t) = \emptyset$, then the dynasty will never add links.

Proposition 5.6. *Assume that $c^{\eta} < c^{\Delta}$. $\forall i \in N$, if $\exists t$, such that $N_i(t) = \emptyset$, then $N_i(t') = \emptyset, \forall t' > t$. As a result, $V_i(t') = V_i(t), \forall t' > t$.*

Proof. The utility of adding any number of links is decreasing for every dynasty since the intergenerational utility would be non-increasing by linking to other dynasties and the sum of the cost of network intervention and the cost of parental socialization would be decreasing due to the assumption that $c^{\eta} < c^{\Delta}$. □

The following proposition gives the necessary conditions for agents with only one link to add one more link, or to add one more link and to delete the existing link at the same time, respectively.

Proposition 5.7. *For any $n \in \mathbb{N}$, assume adult i has one link to adult j at time t, i.e., $N_i(t) = \{j\}$. Then $N_i(t+1) \neq \{k\}$ if*

$$c^{\Delta} > \frac{1}{4n^2}((d_i^j(t))^2 - (d_i^k(t))^2);$$

$N_i(t+1) \neq \{j, k\}$ if

$$c^{\Delta} - c^{\eta} > -\frac{1}{n^2}((d_i^{\{j,k\}}(t))^2 - (d_i^{\{j\}}(t))^2).$$

Proof.

$$U_i(N_i(t+1)=\{j\}|N_i(t)=\{j\}) = -\frac{1}{n^2}((d_i^j(t))^2 - (n-1)c^\eta.$$

$$U_i(N_i(t+1)=\{j,k\}|N_i(t)=\{j\}) = -\frac{1}{n^2}(d_i^{\{j,k\}})^2 - c^\eta - (n-2)c^\Delta.$$

$$U_i(N_i(t+1)=\{k\}|N_i(t)=\{j\}) = -\frac{1}{n^2}(d_i^k(t))^2 - (n-1)c^\eta - 4c^\Delta.$$

Adult i will keep the link with adult j and form a link with adult k only if $U_i(N_i(t+1)=\{j,k\}|N_i(t)=\{j\}) > U_i(N_i(t+1)=\{j\}|N_i(t)=\{j\})$ which leads to $c^\Delta - c^\eta < -\frac{1}{n^2}((d_i^{\{j,k\}}(t))^2 - (d_i^{\{j\}}(t))^2)$.
Adult i will deltete the link with adult j and form a link with adult k only if $U_i(N_i(t+1)=\{k\}|N_i(t)=\{j\}) > U_i(N_i(t+1)=\{j\}|N_i(t)=\{j\})$ which leads to $c^\Delta < \frac{1}{4n^2}((d_i^j(t))^2 - (d_i^k(t))^2)$.

□

Proposition 5.7 tells that adult would not form other links if the current cultural distance with the neighbourhood is sufficiently small.

Corollary 5.4. *Assume $n=3$ and adult i has one link to adult j at time t, i.e., $N_i(t)=\{j\}$. Then $N_i(t+1)=\{j,k\}$ only if*

$$c^\Delta - c^\eta < -\frac{1}{9}((V_j(t)+V_k(t)-2V_i(t))^2) - (V_j(t)-V_i(t))^2$$

which also can be written as

$$c^\Delta - c^\eta < -\frac{1}{9}((d_i^{\{j,k\}}(t))^2 - (d_i^{\{j\}}(t))^2);$$

$N_i(t+1)=\{k\}$ only if

$$c^\Delta < \frac{1}{36}((d_i^j(t))^2 - (d_i^k(t))^2).$$

This is equivalent to say that $N_i(t+1)\neq\{j,k\}$ if

$$c^\Delta - c^\eta > -\frac{1}{9}((d_i^{\{j,k\}}(t))^2 - (d_i^{\{j\}}(t))^2);$$

$N_i(t+1)\neq\{k\}$ if

$$c^\Delta > \frac{1}{36}((d_i^j(t))^2 - (d_i^k(t))^2).$$

Proof.

$$U_i(N_i(t+1)=\{j\}|N_i(t)=\{j\}) = -\frac{1}{9}(V_j(t)-V_i(t))^2 - 2c^\eta.$$

$$U_i(N_i(t+1) = \{j,k\}|N_i(t) = \{j\}) = -\frac{1}{9}(V_j(t) + V_k(t) - 2V_i(t))^2 - c^\eta - c^\Delta.$$

$$U_i(N_i(t+1) = \{k\}|N_i(t) = \{j\}) = -\frac{1}{9}(V_k(t) - V_i(t))^2 - 2c^\eta - 4c^\Delta.$$

Adult i will keep the link with adult j and form a link with adult k only if $U_i(N_i(t+1) = \{j,k\}|N_i(t) = \{j\}) > U_i(N_i(t+1) = \{j\}|N_i(t) = \{j\})$ which leads to $c^\Delta - c^\eta < -\frac{1}{9}((V_j(t) + V_k - 2V_i(t))^2) - (V_j(t) - V_i)(t)^2$.

Adult i will deltete the link with adult j and form a link with adult k only if $U_i(N_i(t+1) = \{k\}|N_i(t) = \{j\}) > U_i(N_i(t+1) = \{j\}|N_i(t) = \{j\})$ which leads to $c^\Delta < \frac{1}{36}((d_i^j)(t)^2 - (d_i^k(t))^2)$. □

Small costs

Assume the initial state is heterogeneous, i.e., there exist i and j such that $V_i(0) \neq V_j(0)$. If both costs of changing the network and cost of child care are small (relative to the degree of imperfect empathy), then the extremists (possibly groups) will disconnect and there will be long term heterogeneity.

Proposition 5.8. *Let $c^\eta < c^\Delta$ small enough such that*

$$c^\Delta < \min_{i \in N} \frac{\min_{j \in N}(V_i(0) - V_j(0))^2}{n^2((n-1)^2 + \eta_i(0) + 1)}.$$

Then, for any $\epsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that for $\underline{V}(t) := \min_{k \in N} V_k(t)$ and $\bar{V}(t) := \max_{k \in N} V_k(t)$ we get that there exists $\underline{V}, \bar{V} \in \mathbb{R}$ such that $|\underline{V} - \underline{V}(t)| < \epsilon$ and $|\bar{V} - \bar{V}(t)| < \epsilon$ for all $t \geq t_0$ and $\bar{V} - \underline{V} > 0$.

Proof. We will show that if $c^\Delta < \frac{\min_{i,j \in N}(V_i(0) - V_j(0))^2}{n^2((n-1)^2 + \eta_i(0) + 1)}$, the optimal network at time 1 will be the empty network and stay empty after time 1.

For any $i \in N$, if he deletes all his links to others, his utility of deleting all links will be $u_i^{g^\emptyset}(1) = -c^\Delta(n-1)^2 - c^\eta(\eta_i + 1)$. The utility of any other network changes $u_i^{g'}(1) < -\frac{1}{n^2} \min_{j \in N}(V_i(0) - V_j(0))^2$, for any $g' \neq g^\emptyset$. Thus $u_i^{g^\emptyset}(1) > u_i^{g'}(1), \forall g' \neq g^\emptyset$. This holds for any $i \in N$, so the network will be $g^\emptyset$ after time 1. □

Example 5.3. *Consider the same initial network structure and cultural traits with Example 5.1 and $c^\eta = 0.001$, $c^\Delta = 0.003$. The dynamic of cultural traits is shown in Figure 5.9. The extremists disconnect and it leads to a long term heterogeneity.*

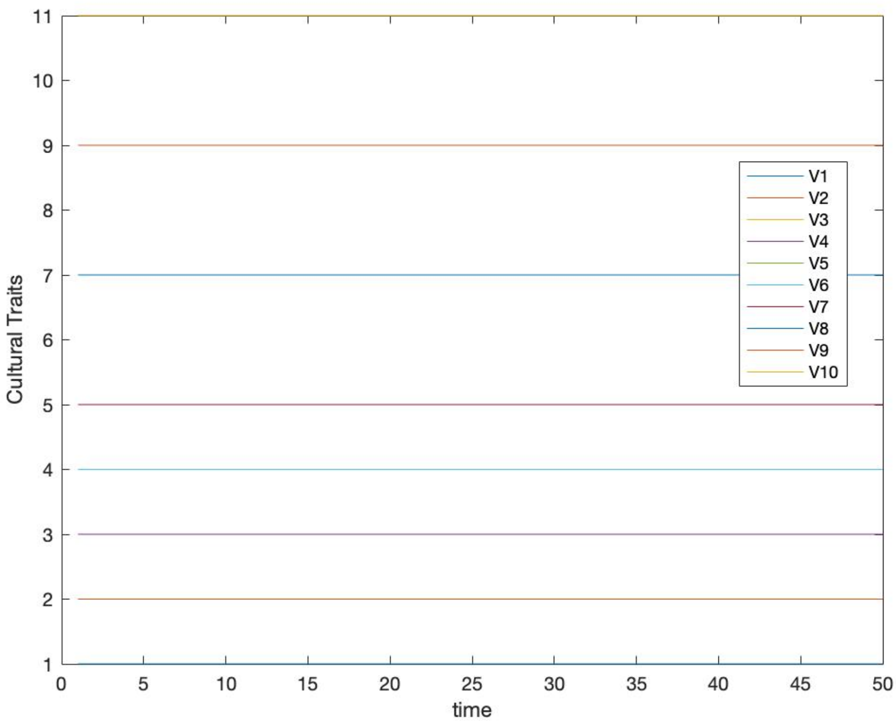


Figure 5.9: Dynamic of Cultural Traits for $c^\eta = 0.001$, $c^\Delta = 0.003$.

Intermediate costs

We can almost always find (intermediate) cost values (relative to degree of imperfect empathy) such that the traits of the whole society converge to that of an extremist subgroup.

Conjecture 5.1. *Suppose the initial traits are randomly distributed according to the uniform distribution on some interval $I \subset \mathbb{R}$ and suppose the initial network is a Bernoulli random network. Then, for almost all initial cultural traits $\mathbf{V}(0)$ and for all initial networks $g(0)$, there exists costs $0 \leq c^\eta \leq c^\Delta$ and a (strict) subset of players $E \subset N$ such that for all $\epsilon > 0$ there exists a $t_0 \in \mathbb{N}$:*

$$g_{ij}(t) = 0 \; \forall i \in E, j \in N \setminus E \quad \textit{and} \quad |V_i(t) - V_j(t)| < \epsilon.$$

Example 5.4. *Consider the same initial network structure and cultural traits with Example 5.1 and $c^\eta = 0.009$, $c^\Delta = 0.023$. The dynamic of cultural traits is shown in Figure 5.10. The traits of the whole society converge to the lowest extremist.*

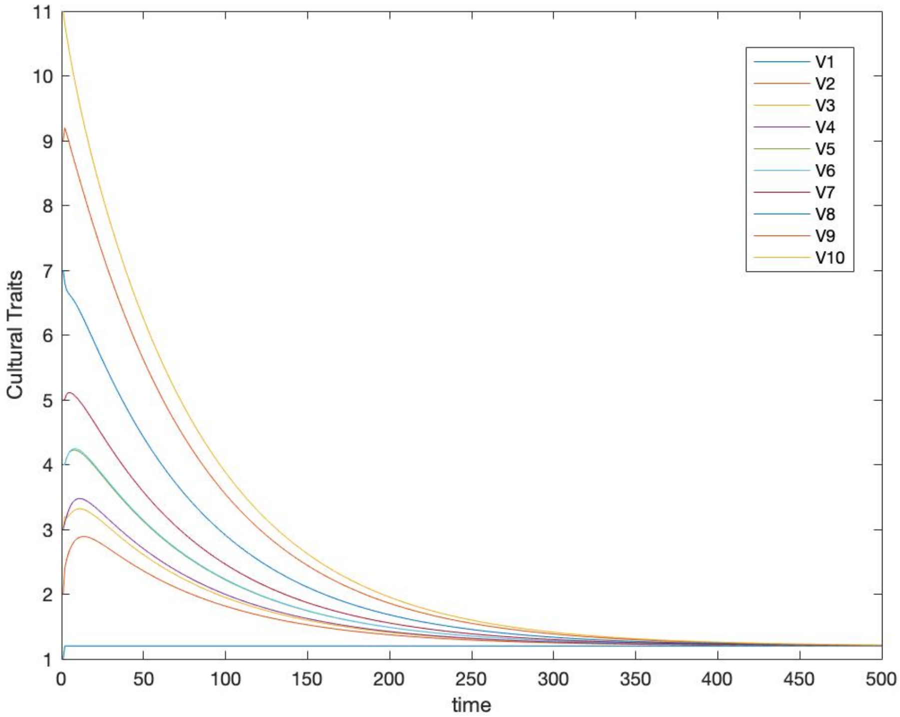


Figure 5.10: Dynamic of Cultural Traits for $c^{\eta} = 0.009$, $c^{\Delta} = 0.023$.

Large costs

Large costs of either child care or cost of network change (relative to the degree of imperfect empathy) imply convergence to a homogenous society.

Proposition 5.9. *Let c^{η} or c^{Δ} be large enough and suppose for all $i, j \in N$ there exists a directed path from i to j in $g(0)$. Then, for all $\epsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that for all $i, j \in N : |V_i(t) - V_j(t)| < \epsilon$ for all $t \geq t_0$ and $g(t)$ is connected.*

Example 5.5. *Consider the same initial network structure and cultural traits with Example 5.1 and $c^{\eta} = 0.001$, $c^{\Delta} = 0.5$. The dynamic of cultural traits is shown in Figure 5.11.*

Indeed, assume the cost of network changes c^{Δ} is sufficiently large, then no agent wants to change links. So the network stays unchanged and cultural traits form a consensus in the limit.

Example 5.6. *Consider the same initial network structure and cultural traits with Example 5.1 and $c^{\eta} = 0.5$, $c^{\Delta} = 0.001$. The dynamic of cultural traits is shown in Figure 5.12.*

Instead, assume the cost of child care c^{Δ} is sufficiently large, it means that dynasties

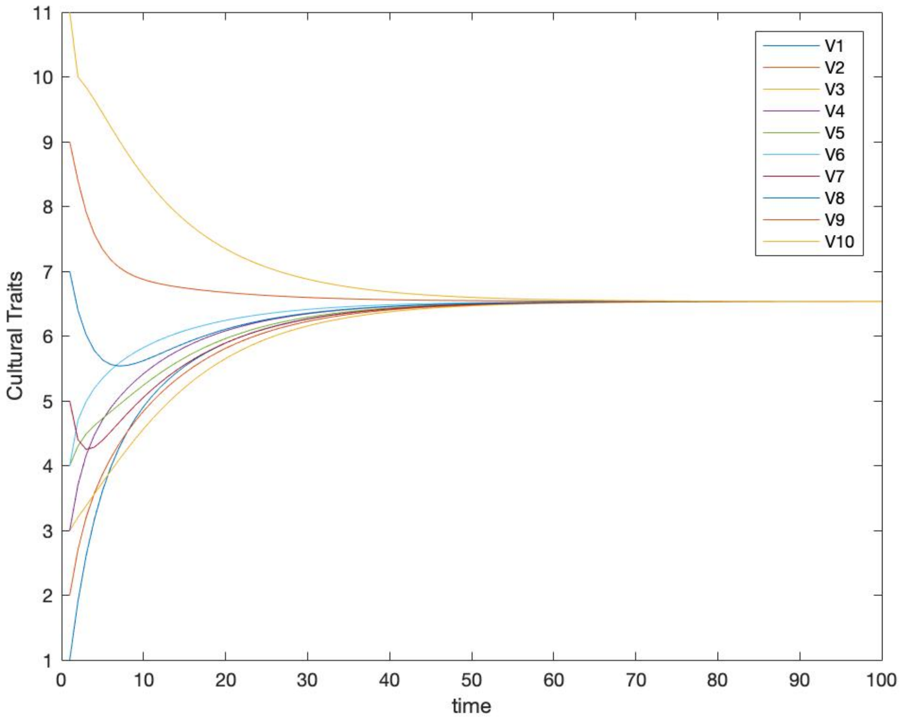


Figure 5.11: Dynamic of Cultural Traits for $c^{\eta} = 0.001$, $c^{\Delta} = 0.5$.

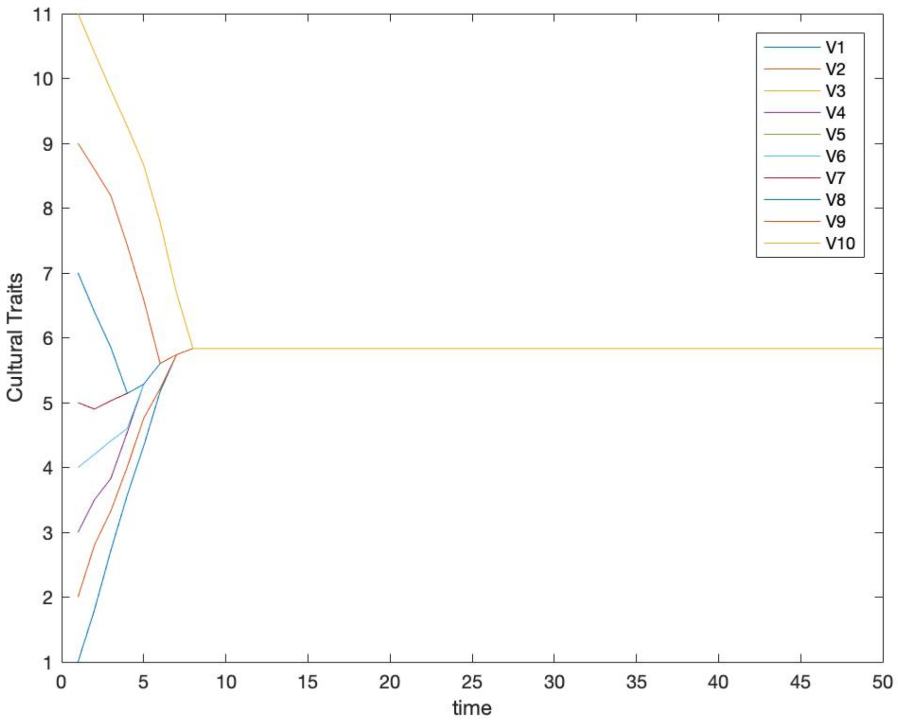


Figure 5.12: Dynamic of Cultural Traits for $c^{\eta} = 0.5$, $c^{\Delta} = 0.001$.

will benefit a lot from integration, then all dynasties want to add more links. So it will converge to a complete network and cultural traits also form a consensus quickly.

5.4.2 The UPST model

Assume that a pairwise stable network with transfers (PST) is reached in each period, i.e., $g(t+1)$ is a PST network with respect to $g(t)$ and $V(t)$ by definition 5.2 [4].

General results on convergence

Theorem 5.2. *(Convergence) For any given $V(0)$ and $g(0)$, for all $i \in N$, all $\epsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that $|V_i(t) - V_i(t')| < \epsilon$ for all $t, t' \geq t_0$.*

Proof. Note here the series of the network matrices $g(0), g(1), \ldots$. satisfies the following properties:

(i) $g_{ii}(t) = \frac{n-\eta_i(t)}{n} > 0, \forall i \in N$;

(ii) $g(t)$ is symmetric for all t;

(iii) $\min^+_{i,j\in N}\{g_{ij}(t)\} \geq \frac{1}{n}$, where $\min^+_{i,j\in N}\{g_{ij}(t)\}$ stands for the minimum that is taken over all positive entries among $g_{ij}(t)$.

Then by the Stabilisation Theorem proposed in Lorenz (2005a), $\exists t_0$ and pairwise disjoint classes of agents $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \ldots \cup \mathcal{I}_p = N$ such that

$$A(\infty)\cdots A(1)A(0) = \begin{bmatrix} K_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & K_p \end{bmatrix} A(t_0)\cdots A(1)A(0),$$

where $K_1, \cdots, K_p$ are quadratic consensus matrices in the sizes of $\mathcal{I}_1, \cdots, \mathcal{I}_p$. Thus

$$V(\infty) = A(\infty)\cdots A(1)A(0)V(0) = \begin{bmatrix} K_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & K_p \end{bmatrix} A(t_0)\cdots A(1)A(0)V(0).$$

The convergence of the cultural traits for all agents is guaranteed.

□

[4] In case of the existence of multiple PST networks, a random PST network will be reached.

Simulation results on the dynamics with PST networks

The following example shows that the PST network is not unique in each period.

Example 5.7. *Consider the same initial network structure and cultural traits with Example 5.2 and $c^\eta = 0$, $c^\Delta = 0$. The dynamic of cultural traits is shown at Figure 5.13. As time goes by, dynasty 1 disconnect with all other dynasties. The steady-state cultural traits are $\bar{V}_1 = 1, \bar{V}_i = 5.3, i = 2, \ldots, 10$. $\mathbf{V}(0) = (1,2,3,3,4,4,5,7,9,11)$.*

$$g(0) = \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10, \end{array}$$

$$g(1) = \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10. \end{array}$$

Given $V(0)$ and $g(0)$, both $g^\emptyset$ and $g(1)$ are PST networks.
Let us check for $23 \in g(1)$, would 2 and 3 delete the link. The sum of the marginal utilities

of deleting the link 23 of $g(1)$ is

$$
\begin{aligned}
&(mu_{2+3})(g(1), 23|V(0), g(0)) \\
=& -\frac{(V_2 - V_3)^2}{n^2} - \frac{(2V_3 - V_2 - V_7)^2}{n^2} + \frac{(V_3 - V_7)^2}{n^2} \\
=& \frac{1}{50} > 0.
\end{aligned}
$$

Therefore 2 and 3 would not delete the link 23. All the links can be checked likewise to show that $g(1)$ is a PST network.

Remark 5.2. *When $c^\eta = c^\Delta = 0$, g(1) only depends on $V(0)$.*

Remark 5.3. *PST network exists but is not unique at each period.*

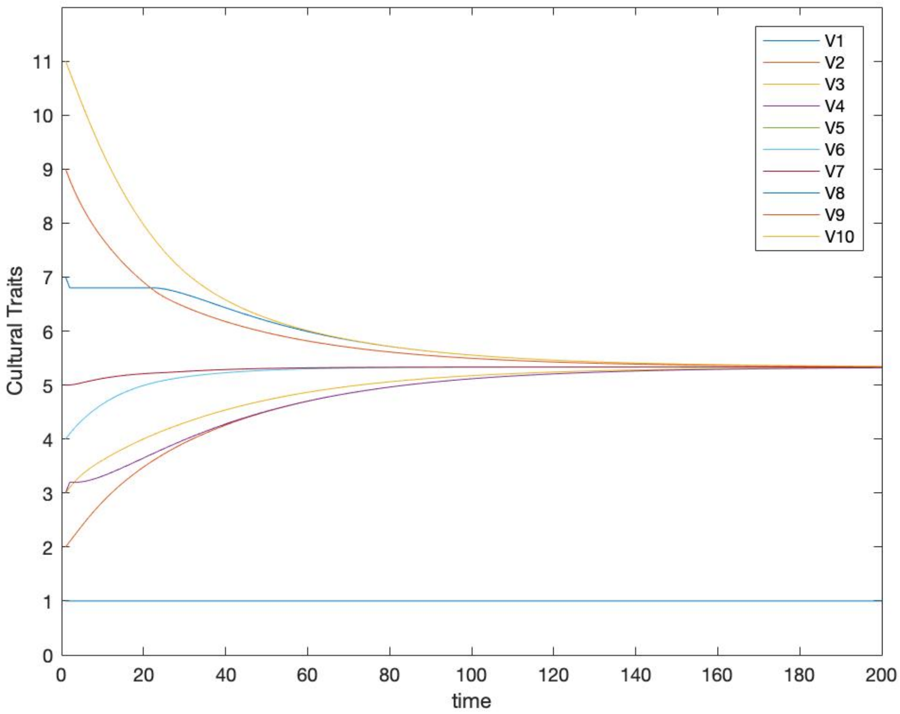


Figure 5.13: Dynamic of Cultural Traits for $c^\eta = 0$, $c^\Delta = 0$.

As either c^η or c^Δ increases enough, the network will converge to a connected (but may be not complete) network, thus all dynasties reach a consensus on cultural traits.

Example 5.8. *Consider the same initial network structure and cultural traits with Example 5.2 and $c^\eta = 0.1$, $c^\Delta = 0.5$. The dynamic of cultural traits is shown at Figure 5.14. The network converges to a connected network, and all dynasties reach a consensus on cultural traits with the steady-state cultural traits being $\bar{V}_i = 4.9, i = 1, \ldots, 10$.*

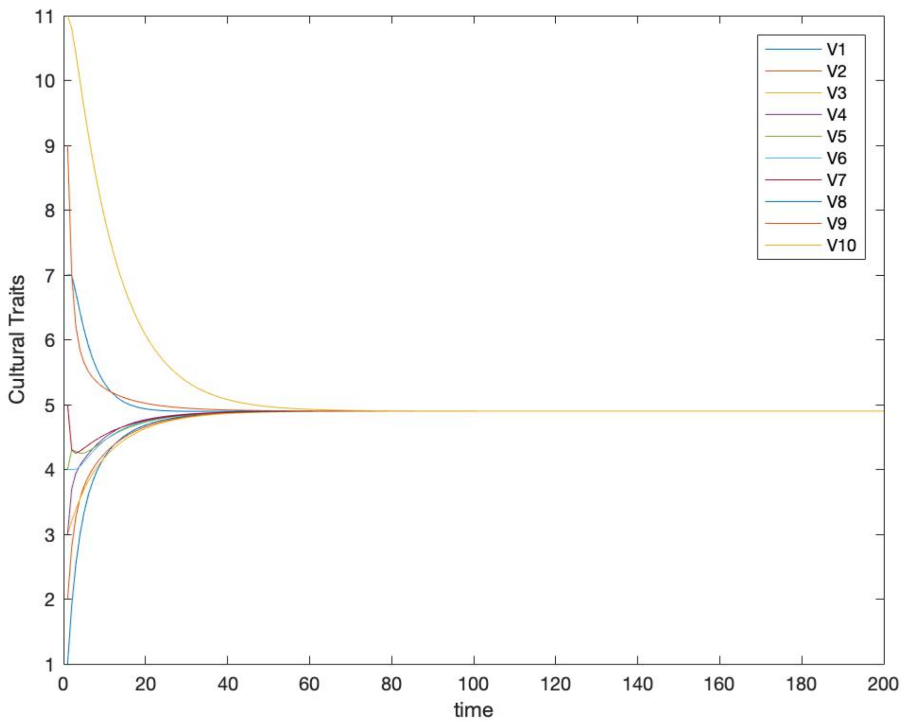


Figure 5.14: Dynamic of Cultural Traits for $c^\eta = 0.1$, $c^\Delta = 0.5$.

Let c^η be large enough, the network will converge to a connected network, thus all dynasties reach a consensus on cultural traits. Let c^Δ be large enough, the network will stay unchanged as time goes on.

Example 5.9. *Consider the same initial network structure and cultural traits with Example 5.2 and* $c^\eta = 1$, $c^\Delta = 0.5$. *The dynamic of cultural traits is shown at Figure 5.15. The network converges to a complete network, and all dynasties reach a consensus on cultural traits with the steady-state cultural traits being* $\bar{V}_i = 4.9, i = 1, \ldots, 10$.

Example 5.10. *Consider the same initial network structure and cultural traits with Example 5.2 and* $c^\eta = 0.01$, $c^\Delta = 5$. *The dynamic of cultural traits is shown at Figure 5.16. The network stays unchanged and connected, and all dynasties reach a consensus on cultural traits with the steady-state cultural traits being* $\bar{V}_i = 4.9, i = 1, \ldots, 10$.

5.4.3 The UPS model

Assume the initial state is heterogeneous, i.e., there exist i and j such that $V_i(0) \neq V_j(0)$. If both costs of changing the network and cost of child care are small (relative to the degree of imperfect empathy), then there will be long term heterogeneity.

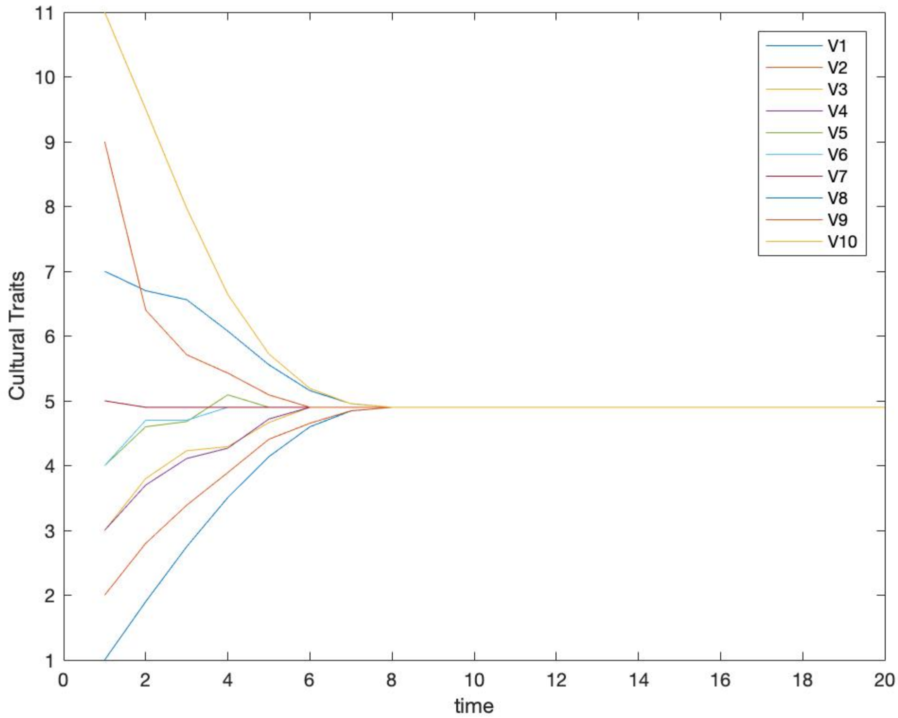


Figure 5.15: Dynamic of Cultural Traits for $c^{\eta} = 1$, $c^{\Delta} = 0.5$.

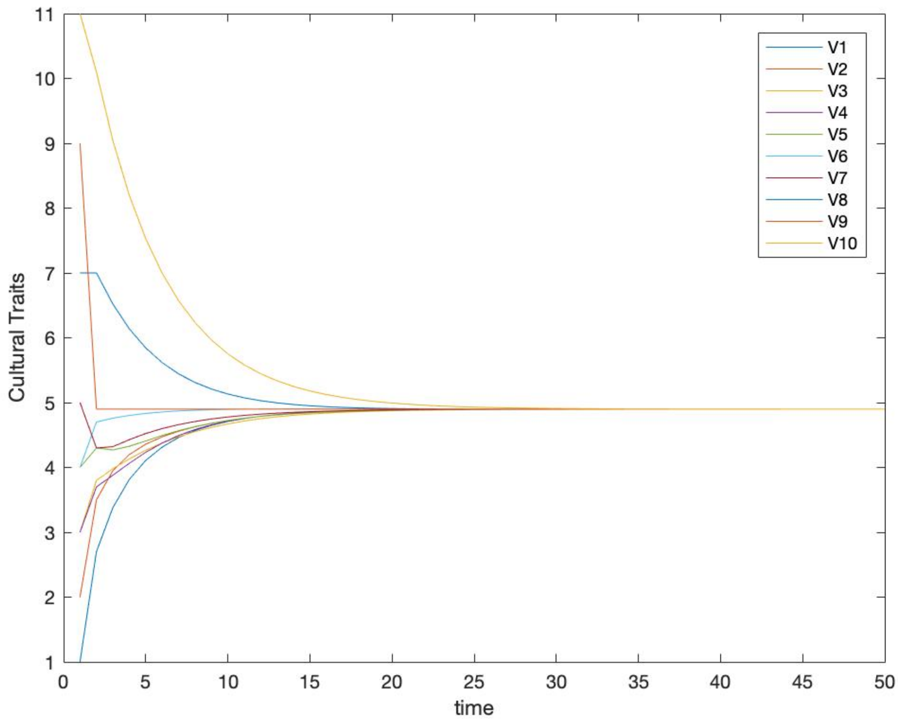


Figure 5.16: $c^{\eta} = 0.01$, $c^{\Delta} = 5$.

Proposition 5.10. *There exist sufficiently small c^{Δ} and c^{η} such that for any $\epsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that for $\underline{V}(t) := \min_{k \in N} V_k(t)$ and $\bar{V}(t) := \max_{k \in N} V_k(t)$, there exists $\underline{V}, \bar{V} \in \mathbb{R}$ such that $|\underline{V} - \underline{V}(t)| < \epsilon$ and $|\bar{V} - \bar{V}(t)| < \epsilon$ for all $t \geq t_0$ and $\bar{V} - \underline{V} > 0$.*

Proof. Denote x_i and x_j as the number of links agent i and j have in $g(0)$, respectively. By Proposition 5.3, there exist sufficiently small c^{Δ} and c^{η} such that the empty network is the unique PS network. Thus $V_i(t) = V_i(0)$, $\forall i \in N$. □

As both c^{η} and c^{Δ} increases, more and more dynasties reach a consensus, as shown in Figure 5.19. Once c^{η} or c^{Δ} increases over a threshold, the network will converge to a connected (but may not complete) network, thus all dynasties reach a consensus on cultural traits.

Example 5.11. *Consider the same initial network structure with Example 5.2. The initial cultural traits are $V(0) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$.*

Small costs

When $c^{\eta} = 0.001$, $c^{\Delta} = 0.001$, the dynamic of cultural traits is shown at Figure 5.17. The network converges to the empty network, and the steady-state cultural traits are $\bar{V}_i = V_i(0), i = 1, \ldots, 10$.

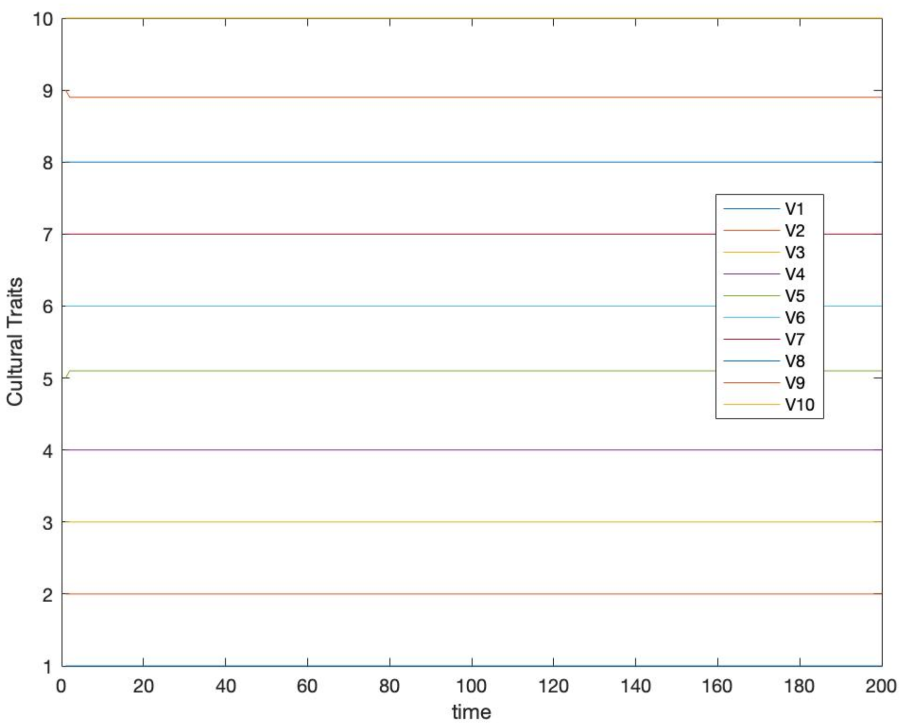


Figure 5.17: Dynamic of Cultural Traits for $c^{\eta} = 0.001$, $c^{\Delta} = 0.001$.

Intermediate costs

When $c^\eta = 0.004$, $c^\Delta = 0.005$, *the dynamic of cultural traits is shown at Figure 5.18. Dynasties 1, 2, 3, 4, 10 are disconnected with the others, and the steady-state cultural traits are* $\bar{V}_1 = 1, \bar{V}_2 = 2.1, \bar{V}_3 = 3.1, \bar{V}_4 = 4, \bar{V}_{10} = 9.9, \bar{V}_i = 349/50, i = 5, \ldots, 9.$

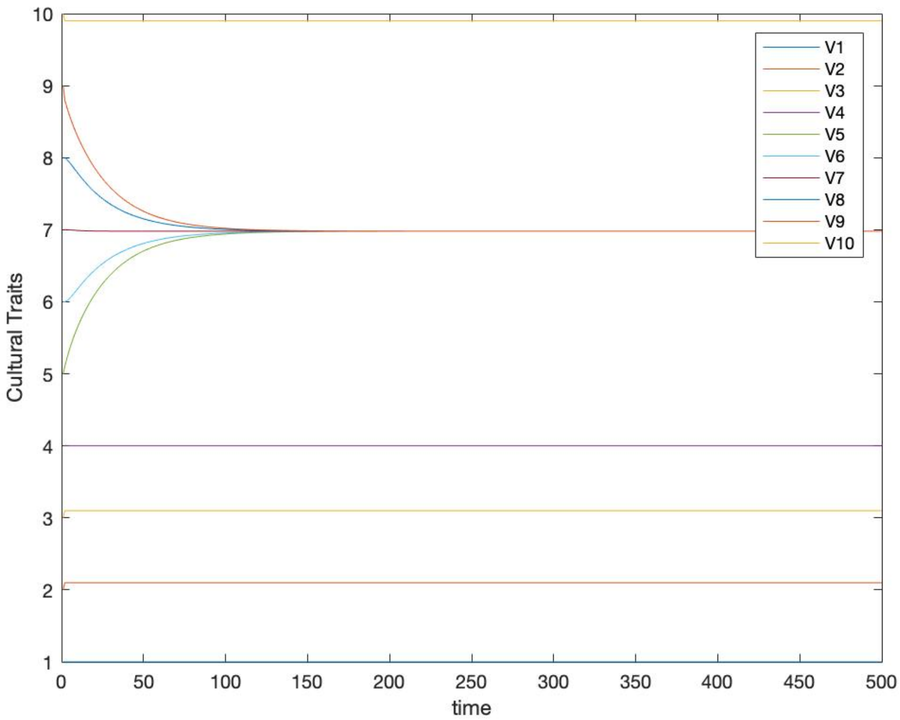


Figure 5.18: Dynamic of Cultural Traits for $c^\eta = 0.004$, $c^\Delta = 0.005$.

When $c^\eta = 0.005$, $c^\Delta = 0.006$, *the dynamic of cultural traits is shown at Figure 5.19. Dynasties 1, 4, 10 are disconnected with the others, and the steady-state cultural traits are* $\bar{V}_1 = 1, \bar{V}_4 = 4.2, \bar{V}_{10} = 9.9, \bar{V}_2 = \bar{V}_3 = \bar{V}_i = 5.7, i = 5, \ldots, 9.$

When $c^\eta = 0.01$, $c^\Delta = 0.02$, *the dynamic of cultural traits is shown at Figure 5.20. Dynasty 1 is disconnected with the others, and the steady-state cultural traits are* $\bar{V}_1 = 13/10, \bar{V}_i = 179/30, i = 2, \ldots, 10.$

Large costs

When $c^\eta = 0.04$, $c^\Delta = 0.05$, *the dynamic of cultural traits is shown at Figure 5.21. The network converges to the connected network, and the steady-state cultural traits are* $\bar{V}_i = 11/2, i = 1, \ldots, 10.$

Large c^Δ

When $c^\eta = 0.07$, $c^\Delta = 4.9$, *the dynamic of cultural traits is shown at Figure 5.22. The network remains unchanged, and the steady-state cultural traits are* $\bar{V}_i = 11/2, i = 1, \ldots, 10.$

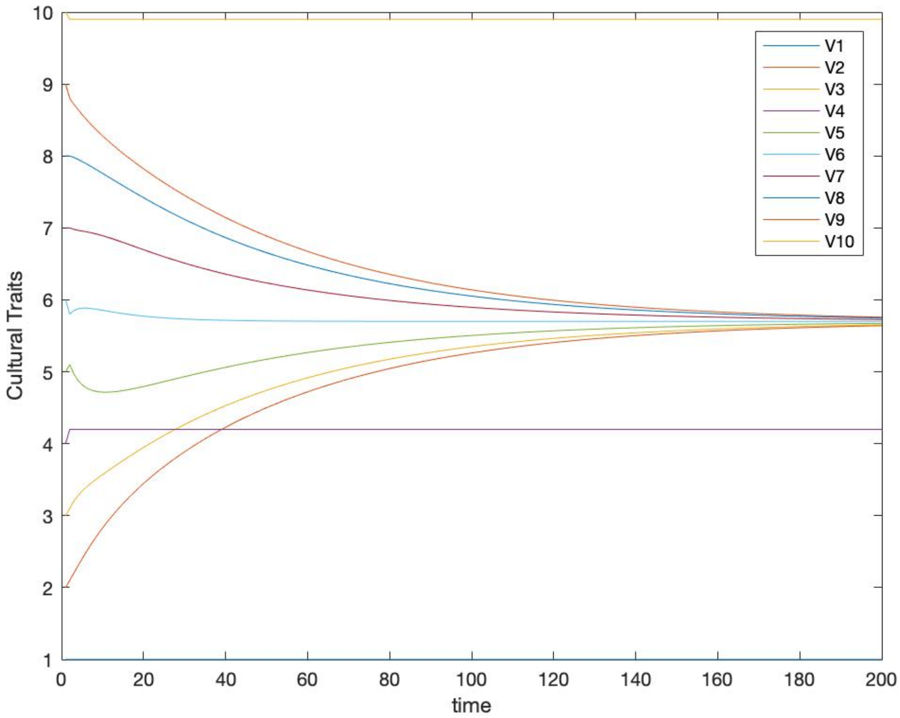


Figure 5.19: Dynamic of Cultural Traits for $c^{\eta} = 0.005$, $c^{\Delta} = 0.006$.

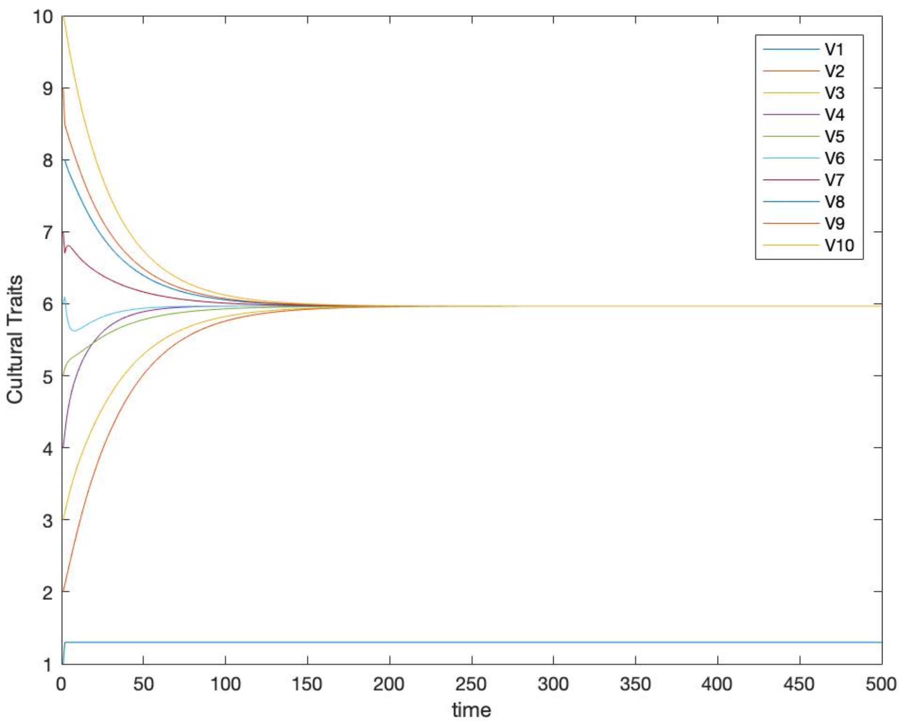


Figure 5.20: Dynamic of Cultural Traits for $c^{\eta} = 0.01$, $c^{\Delta} = 0.02$.

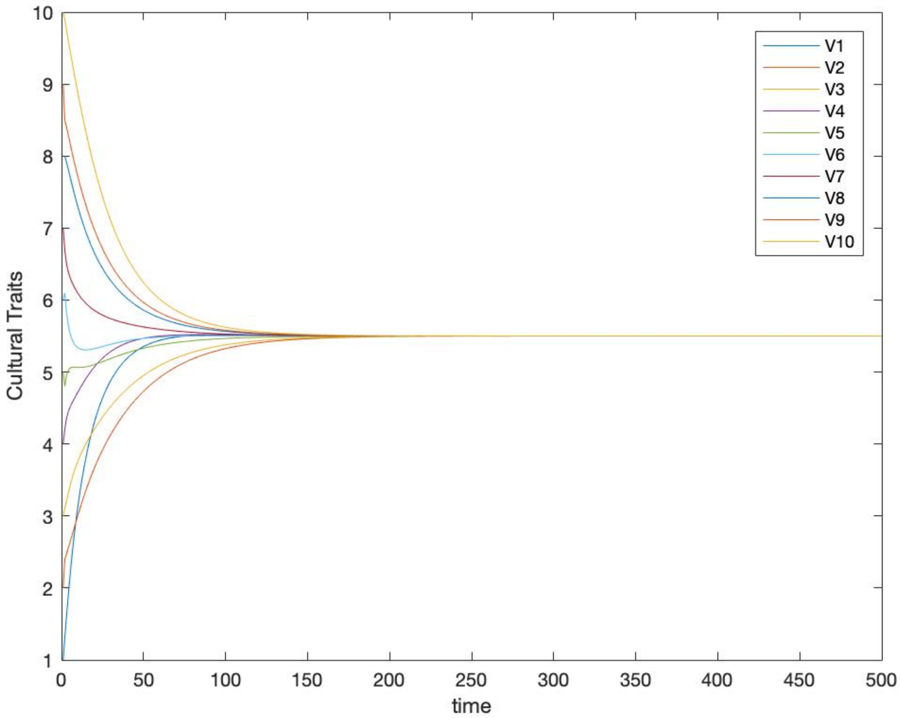


Figure 5.21: Dynamic of Cultural Traits for $c^{\eta} = 0.04$, $c^{\Delta} = 0.05$.

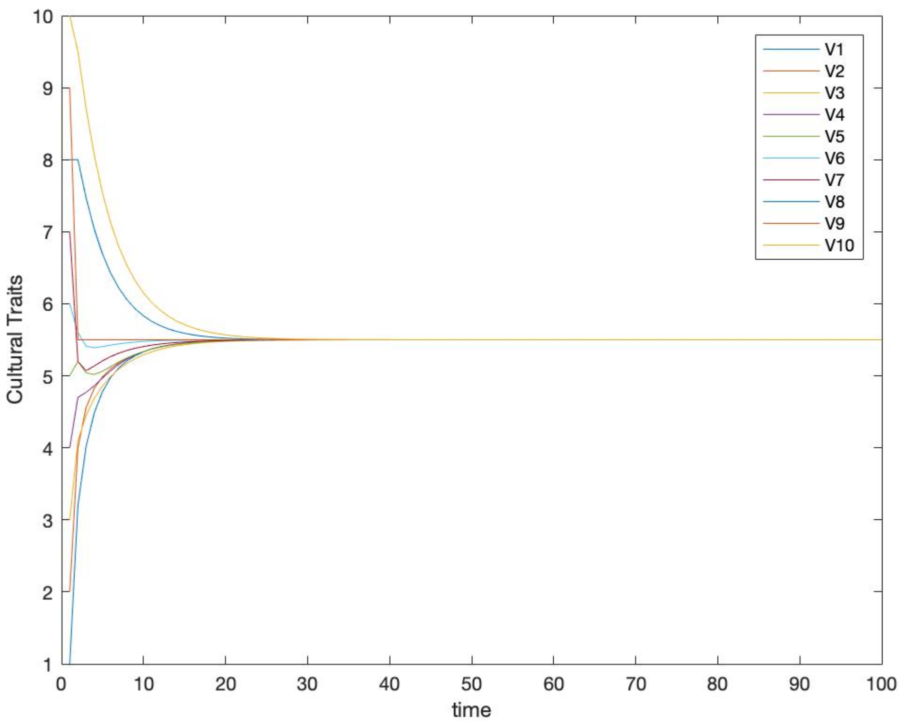


Figure 5.22: Dynamic of Cultural Traits for $c^{\eta} = 0.07$, $c^{\Delta} = 4.9$.

Large c^η

When $c^\eta = 0.1$, $c^\Delta = 0.05$, *the dynamic of cultural traits is shown at Figure 5.23. The network converges to the complete network, and the steady-state cultural traits are* $\bar{V}_i = 11/2, i = 1, \ldots, 10$.

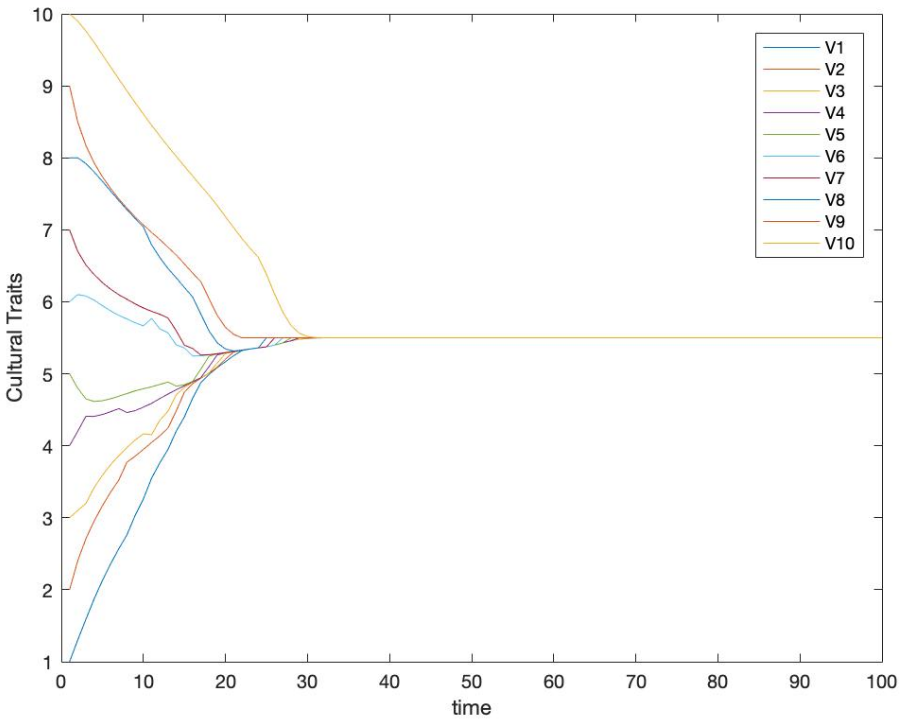


Figure 5.23: Dynamic of Cultural Traits for $c^\eta = 0.1$, $c^\Delta = 0.05$.

5.5 Efficiency of networks

All the simulation results of these models showed that for sufficiently small cost parameters, it converges to a heterogeneous society, while for large cost parameters, it converges to a homogeneous society. In this section, we consider the efficiency of networks such that all dynasties would not benefit from deviating to the other networks.

Definition 5.5. *A network g is efficient, if $\forall g' \in \mathcal{G}$, $\sum_{i=1}^{n} U_i(g \mid V(t), g(t)) \geq \sum_{i=1}^{n} U_i(g' \mid V(t), g(t))$.*

Definition 5.6. *A network g is strongly efficient, if $\forall g' \in \mathcal{G}$ and $\forall i \in N$, $U_i(g \mid V(t), g(t)) \geq U_i(g' \mid V(t), g(t))$.*

The following proposition further shows that the empty network $g^{\emptyset}$ is the only (strongly) efficient network for small costs and the complete network g^N is the only (strongly) efficient network for large costs.

Proposition 5.11. *Assume that the initial cultural traits are heterogenous, i.e., $V_i \neq V_j$, $\forall i \neq j$ and $i, j \in N$. There exist sufficiently small c^{η} and c^{Δ} such that the empty network $g^{\emptyset}$ is the only (strongly) efficient network, and sufficiently large c^{η} such that the complete network g^N is the only (strongly) efficient network.*

Proof. Remark that any strongly efficient network is also efficient, thus it suffices to show the argument on strong efficiency. First to show that there exist sufficiently small c^{η} and c^{Δ} such that the empty network $g^{\emptyset}$ is the only strongly efficient network. Fix any $g(t)$, $V(t)$ and $g \neq g^{\emptyset}$. It suffices to show that $\exists c^{\eta}, c^{\Delta}$, such that $U_i(g^{\emptyset} \mid V(t), g(t)) - U_i(g \mid V(t), g(t)) \geq 0$, $\forall i \in N$. Denote x_i as the number of links that agent i need to change from $g(t)$ to $g^{\emptyset}$ and y_i the number of links that agent i need to change from $g(t)$ to g. Denote the neighborhood of agent i in g as N_i and the degree of i as η_i. Then

$$
\begin{aligned}
& U_i(g^{\emptyset} \mid V(t), g(t)) - U_i(g \mid V(t), g(t)) \\
= & - c^{\Delta}(x_i)^2 - nc^{\eta} + \frac{1}{n^2}(d_i^{N_i})^2 + c^{\Delta}(y_i)^2 + c^{\eta}(n - \eta_i) \\
= & c^{\Delta}(y_i - x_i)^2 - c^{\eta}\eta_i + \frac{1}{n^2}(d_i^{N_i})^2 \geq 0 \\
\iff & c^{\eta}\eta_i + c^{\Delta}(x_i - y_i)^2 \leq \frac{1}{n^2}(d_i^{N_i})^2.
\end{aligned}
$$

Due to the assumption that $V_i \neq V_j$, $\forall i \neq j$ and $i, j \in N$, $d_i^{N_i} \neq 0$. Thus such cost parameters always exist to guarantee the empty network $g^{\emptyset}$ is the only strongly efficient network.

Then to show that there exist sufficiently large c^η such that the complete network g^N is the only (strongly) efficient network. Denote z_i as the number of links that agent i need to change from $g(t)$ to g^N. Then

$$\begin{aligned}
&U_i(g^N \mid V(t), g(t)) - U_i(g \mid V(t), g(t)) \\
=& -\frac{1}{n^2}(d_i^N)^2 - c^\Delta(z_i)^2 + \frac{1}{n^2}(d_i^{N_i})^2 + c^\Delta(y_i)^2 + c^\eta(n-\eta_i) \\
=& c^\Delta(y_i - z_i)^2 + \frac{1}{n^2}[(d_i^{N_i})^2 - (d_i^N)^2] + c^\eta(n-\eta_i) \geq 0 \\
\iff& c^\eta \geq \frac{1}{n-\eta_i}\Big[c^\Delta(z_i - y_i)^2 + \frac{1}{n^2}[(d_i^N)^2 - (d_i^{N_i})^2]\Big].
\end{aligned}$$

Thus such cost parameters always exist to guarantee the complete network g^N is the only strongly efficient network.

□

5.6 Conclusion

We studied the dynamics of intergenerational cultural transmission in endogenous networks where the network changes are inherited. We proposed three ways to endogenize the process of network formation. In the first one, the network is supposed to be directed and each dynasty can either form or delete a directed link unilaterally with another dynasty. Therefore, at each period, each family faces a utility optimization problem where a trade-off between own utility losses and the improvements of child's cultural trait. We have shown that if the cost of network changes is greater than the cost of child care, extremists will never add links, and in the case of sufficiently low cost, extremists may cut all ties with the society. In the second and third models, the network is supposed to be undirected. In the second model, we assume that after each period, a pairwise stable network with transfers (PST network for short) is reached, i.e., $\forall t \in \mathbb{N}$, $G(t+1)$ is a PST network for $G(t)$ and $V(t)$. We have shown the existence of the PST network for each period, however, it is not necessary to be unique, evidenced by a counter example. Moreover, a necessary and sufficient condition is given such that a network is PST for given $V(t)$ and $G(t)$. The convergence of cultural traits is guaranteed. In the third model, we assume that after each period, a pairwise stable network (PS network for short) is reached, i.e., $\forall t \in \mathbb{N}$, $G(t+1)$ is a PS network for $G(t)$ and $V(t)$. In this case, there always exist sufficiently small cost parameters such that the empty network is the unique PS network.

There always exist sufficiently small cost parameters such that the empty network is the unique efficient network, and sufficiently large costs of child care such that the complete network is the unique efficient network.

The dynamics of these three models are studied by both analytics and simulations. For sufficiently small costs of network changes and child care, extremists will disconnect from the other dynasties and there will be a long term heterogeneity of the society. Specially, in the first and the third model, we show that the network will converge to the empty network. While in the second model, the network might or might not converge to the empty network, since the PST network is not unique in each period. As costs of network changes and child care increase, more and more dynasties reach a consensus even though there are still some other dynasties disagree with this consensus. For large costs of network changes and child care, it converges to a homogeneous society such that all dynasties have the same cultural trait in the limit. This give us some insights on how to reduce extremism in our real life. For example, one can consider to foster the interaction of children with different cultural backgrounds such that the cost of network change is increased (extremists will less probably disconnect with others). Some work can also be done to increase value of integration (i.e., increase the benefits from relations). Extremists play an important role in the dynamical process, policy makers should take it into account and provide more opportunities for extremists to connect with others.

Chapter 6

Continuous opinion dynamics with anti-conformity behavior

6.1 Introduction

We form our opinions on every aspect of our life, from personal interests (e.g., favorite songs/foods), to social norms (e.g., the acceptable behavior in certain circumstances), to economic decisions (e.g., consumption budget, tax rate), and even to political attitudes, etc. Social networks play a crucial role in modeling opinion dynamics as people are constantly interacting and influencing each other. Experimental evidences provided by Galton (1907), Lorge et al. (1958), Hommes et al. (2005) and Yaniv and Milyavsky (2007) demonstrate that the aggregate (such as median and averaged) estimates of a group are very close to the true value.

Even though the amount of models of opinion dynamics is huge, they can be classified into continuous opinion models and discrete opinion models. The group of models of discrete opinion dynamics is mainly applied to cases when there is no compromises in between any two opinions, actions or decisions, while the group of models of continuous opinion dynamics deals with problems in which the opinion of people can be expressed as real numbers (e.g., tax rates, prices, quantitative predictions). The classical and widely used agent-based model of continuous opinion dynamics is the DeGroot model (French Jr (1956), Harary (1959), Harary et al. (1965), DeGroot (1974)), assuming agents update their opinions iteratively as the weighted average of the opinions of their neighbors. The typical behavior of the DeGroot model is the presence of consensus due to the implicit assumption of conformity, while in real life disagreement is also ubiquitous (Abelson (1964)). Out of this consideration, different kinds of variations of the DeGroot model have been proposed. For example, some variations introduce the stubborn agent[1] whose opinion remains unchanged during the iterative pooling process (see Friedkin and Johnsen (1990), Friedkin and Johnsen (1999), Hegselmann and Krause (2015), Masuda (2015)), by introducing an attachment of each agent to its initial opinion; some other variations consider time-varying weight matrices (Lorenz (2005b)); some variations considered that agents are only interacting with those who hold opinions close enough to them by introducing confidence bounds (Hegselmann et al. (2002), Weisbuch (2004), Krause (2000), Deffuant et al. (2000), see also the survey on continuous opinion dynamics with bounded confidence Lorenz (2007)); some other variations introduce negative influences, i.e., the element w_{ij} of the weight matrix W can be positive or negative, thus W is no more row-stochastic (Altafini (2012a), Altafini (2012b)). The DeGroot model implies the assumption of conformity since opinions of agents are attracting each other. Introducing negative influences provides a way to model the anti-conformity behavior.

Conformity V.S. Anti-conformity Before the 21st century, most of the models of opinion dynamics made the basic assumption that agents tend to follow the trend (i.e., they are conformist), and the existence of opposite behavior (anti-conformity or counter-conformity) was neglected. Even in the field of psychology, as Jahoda (1959) criticized, conformity was over-emphasized in the psychological literature, and the emphasis obscured the reality of non-conformity or anti-conformity (Hornsey et al. (2003)). The famous experimental study by Asch (1955) showed that agents tend to conform to the wrong judgement of their predecessors even if some of them know already that the judgement was wrong. A follow-up study (Deutsch and Gerard (1955)) distinguished two forms of social influence that lead to the wrong judgement. While normative social influence drives some agents to behave like majority in order to avoid "social censure", informational social influence explains the conformity behavior in the sense that agents are uncertain about the answer, so they might rely on the judgement of the majority of the society (Hornsey et al. (2003)). This was later supported by Frideres et al. (1971), Terry et al. (2000), Zafar (2011). Motivated by this idea, Buechel et al. (2015) modeled the continuous opinion dynamics by allowing agents to misrepresent opinions in a conforming or anti-conforming way, and furthermore showed that agents' social power is decreasing in the degree of conformity.

The other branch of study on continuous opinion dynamics with anti-conformity behavior (or negative social influence) is based on the notion of *coopetition*, which was introduced by Carfì and Schilirò (2012) in the study of the Green Economy and then applied to

[1] Stubborn agents are also called by physicists as *independent agents* (Sznajd-Weron et al. (2011), Sznajd-Weron et al. (2014)), *inflexibles* (Galam and Jacobs (2007)), *zealots* (Mobilia (2003)).

opinion dynamics with negative influences for a better understanding and explaining the disagreement of opinions. In "coopetitive" networks agents can both cooperate and compete, corresponding to the positive and negative influences among agents, respectively (Proskurnikov and Tempo (2018)), i.e., agents are situated in a signed graph where each edge of the graph is assigned a positive sign or a negative sign. Altafini proposed a model of influence with antagonistic interactions based on the theory of structurally balanced network (Altafini (2012b), Altafini (2012a), Harary et al. (1953)). The idea of structural balancedness can be interpreted as the ancient proverb *the friend of my enemy is my enemy, the enemy of my enemy is my friend* (Schwartz (2010)). The original Altafini model is coincident with the Abelson model[2] with an influence matrix that can have both positive and negative elements. By doing gauge transformation, the structurally balanced network can be transformed into the corresponding nonnegative network sharing the same convergence properties. It was shown that in case of a structurally balanced network (without self-loops), the bipartite consensus can be achieved. However, for a structurally unbalanced and strongly connected network, the consensus value is always the origin, regardless of the initial conditions (Altafini (2012a), Meng et al. (2016)). In a recent paper coauthored by Altafini (Shi et al. (2019)), the authors defined two rules for negative influences: the opposing rule where the opinion of an agent is attracted by the opposite of the opinion of her neighbor via negative links, and the repelling rule where the two agents repel each other instead of being attracted via negative links. However, none of these two rules is appropriate for modeling the anti-conformity behavior for the following reasons. By adopting the opposing rule, the agent is attracted by the opposite of the reference opinion, so whether the agent is conforming or anti-conforming depends on the relative position of the reference opinion to the origin. By adopting the repelling rule, the deviation of the opinion of one agent is decreasing as the opinion distance with her neighbor decreases, i.e., the opinion of an anti-conformist agent will stay unchanged if she has the same opinion value of her reference opinion which is counterintuitive. Moreover, since opinions are defined in $\mathbb{R}$, as the force of repelling increases, the norm of the opinions tends to infinity as t goes to infinity. If $+\infty$ and $-\infty$ are considered as the two extreme opinions in real life, it implies that the extreme opinions are never reached, which is also counterintuitive. Thus an appropriate updating rule of continuous opinions for anti-conformity behavior still needs to be developed, which is one of the aims of the current paper.

Social behavior is described by sociologists in the following three dimensions: (ir-)relevance, (in-)dependence and (anti-)conformity (Willis (1965)). The current paper focuses on the

[2]The Abelson model is the continuous counterpart of the DeGroot model.

third dimension, and aims to study the continuous opinion dynamics in undirected networks, considering both conformity and anti-conformity behaviors. As defined by Willis (1965), "conformity is behavior intended to fulfill normative group expectations as these expectations are perceived by the individual", while "anti-conformity behavior is directly antithetical to the norm prescription", in other words, anti-conformity behavior intends to get away from normative group expectations. Starting from these two definitions, we propose a new opinion updating rule for anti-conformity behavior which is defined by *the repelling function*, meanwhile we adopt the DeGroot updating for conformity behavior. The (anti-)conformity behavior is introduced either in nodes or in links, respectively. On the one hand, each agent is given a fixed behavioral characteristic, i.e., either conformist or anti-conformist, and they will treat all of their neighbors equally, i.e., the nodes are heterogenous and the links are all the same. On the other hand, all agents adopt the same opinion updating rules, but they divide their neighborhood into friends and enemies, i.e., the nodes are all the same and each link is associated with either a positive or a negative weight. Based on this idea, two models of continuous opinion dynamics are proposed. Opinions are assumed to be a real number in the interval $[0, 1]$. 0 and 1 can be considered as the two extreme opinions. The repelling function of an agent is a real-valued function of the current opinion and *the reference opinion* of the agent, which gives the deviation of the opinion for anti-conformity behavior. The reference opinion of an agent is a baseline that one agent would like to repel. It can be the average opinions of all her neighbors or the average opinions of all her enemies.

The paper is structured as follows. In section 6.2, two models of continuous opinion dynamics are introduced based on the repelling function, together with a description of synchronous setting and asynchronous setting for opinions updates. The model of opinion dynamics with conformist and anti-conformists is studied in Section 6.3, while the model of opinion dynamics over signed graphs is studied in Section 6.4. Both synchronous and asynchronous updating are studied for the two models. Section 6.5 concludes the paper with some remarks.

6.2 The model

6.2.1 Notation

Let $N = \{1, 2, \ldots, n\}$ be the society of agents situated in a fixed and undirected network $\mathbf{G} = (N, E)$ whose nodes are the agents and E is the set of edges or links. The neighborhood of agent i is denoted as $N_i = \{j \in N : \{i, j\} \in E\}$ with its cardinality $|N_i| =: \eta_i$

being the degree of agent i. We consider that $i \in N_i$ always holds for each agent $i \in N$. Opinion of agent i at time t is denoted by $x_i(t)$ which is a real number in [0,1]. In case that opinions converge, the steady state opinion vector is denoted as $\bar{x} = [\bar{x}_1, \ldots, \bar{x}_n]$. Opinions are bounded between 0 and 1, thus we introduce the notation $[x]_0^1$ to denote the truncated value of x, i.e.,

$$[x]_0^1 = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{otherwise.} \end{cases} \tag{6.1}$$

The (anti-)conformity behavior is introduced into either nodes or links, respectively, based on the repelling function. In the first model, the society is supposed to contain agents of two different types, i.e., conformist agents and anti-conformist agents. Conformists would like to hold opinions closer to the averaged social opinion, while anti-conformist would like to do the opposite (see detailed explanation in Section 6.2.3). In the second model, we suppose that homogeneous agents are situated in an undirected signed network $G = (N, E)$ where links are associated with a positive or negative sign. Agents linked by an edge with positive sign have opinions which are attracting each other (i.e., conforming influence), while agents linked by an edge with negative sign have opinions which are repelling each other (i.e., anti-conformity influence, see detailed explanation in Section 6.2.4). In the remaining part of the paper, CODA refers to continuous opinion dynamics with anti-conformity, and these two models will be called the CODA-node model and the CODA-link model for short, respectively.

6.2.2 The repelling function $f_i(x_i, r_i)$

To depict the anti-conformity behavior, for a given agent $i \in N$, the *repelling function* f_i is defined as a real-valued function of the current opinion x_i and the reference opinion r_i in $[0, 1] \times [0, 1]$. As described in Section 6.1, anti-conformity is behavior intended to get away from normative group expectations Willis (1965). Here, f_i is a deviation function referring to the shift of the opinion due to anti-conformity behavior, and the reference opinion r_i can be seen as the normative group expectations. The *reference opinion* of agent i gives the benchmark opinion that agent i would like to repel. For example, r_i can be the average opinion of neighbors of anti-conformist agent i in the CODA-node model (Section 6.2.3), while r_i can also be the average opinion of agents from different groups in the CODA-link model (Section 6.2.4).

We do not give an explicit form of the repelling function f_i, instead, a set of properties are provided as the requirements of f_i according to the definition of anti-conformity behavior.

If we position $x_i(t)$ and $r_i(t)$ on the [0,1] axis, and w.l.o.g. assume that $x_i(t) < r_i(t)$, as shown in Figure 6.1, the first intuition is that anti-conformity causes x_i to move to the left, i.e., f_i and $x_i - r_i$ have the same sign (property (c)). The second intuition is that as $r_i(t)$ becomes closer to $x_i(t)$, the force of anti-conformity is getting stronger, so $|f_i|$ is decreasing in $|x_i - r_i|$ (property (a)). The case of $x_i(t) > r_i(t)$ is symmetric (property (b)).

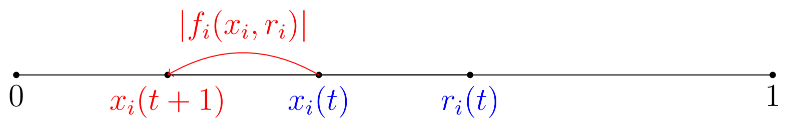


Figure 6.1: The repelling function $f_i(x_i, r_i)$

Denoting $d_i := x_i - r_i$, the opinion distance between agent i and the reference point is $|d_i|$. When $|d_i| \geq \epsilon$, f_i is a function of d_i, where $\epsilon > 0$ is a small number. Above all, the repelling function f_i should satisfy the following properties:

- When $d_i \in]-1, -\epsilon[\cup]\epsilon, 1[$, f_i can be written as $f_i(d_i)$, and the following holds:
  - (a) $|f_i(d_i)|$ is a decreasing function with respect to $|d_i|$;
  - (b) $f_i(d_i)$ is symmetric with respect to the origin, i.e., $f_i(-d_i) = -f_i(d_i)$;
  - (c) $f_i(d_i)$ and d_i have the same sign, i.e., $f_i(d_i) \cdot d_i \geq 0$;
  - (d) f_i is piecewise continuous in the intervals $[-1, -\epsilon[$ and $]\epsilon, 1]$.
- When $d_i \in [-\epsilon, \epsilon]$, i.e., $|x_i - r_i| \leq \epsilon$, the following cases are distinguished:
  - (e)
  $$f_i = \begin{cases} \delta_i, & \text{if } 0 \leq x_i < 0.5 - \epsilon, \\ \alpha, & \text{if } 0.5 - \epsilon \leq x_i \leq 0.5 + \epsilon, \\ -\delta_i, & \text{otherwise } 0.5 + \epsilon < x_i \leq 1, \end{cases}$$
  where α is a random variable taking values δ and $-\delta$ with equal probabilities $\frac{1}{2}$; [3]

When d_i is not very small, i.e., x_i is not very close to r_i: property (a) says that the magnitude of the deviation of agent i decays as the magnitude of d_i increases, and agent would deviate more as the reference opinion becomes closer; property (c) implies that f_i has the same sign as d_i. That is, opinion of agent i will move further away from r_i. Property (e) says that if x_i is very close to r_i, then the sign of f_i, i.e., the direction of the

[3] α can also take value δ_i or $-\delta_i$ to avoid the randomness.

deviation, will depend on the location of x_i, and x_i always moves in the direction such that there is more room for the deviation. Here, ϵ is introduced as an approximation parameter to avoid the unreachable consensus. Take a simple example of the CODA-node model where anti-conformist agent 1 and conformist agent 2 are connected. f_1 gives the shift of opinion for anti-conformist agent 1, i.e., $x_1(t+1) = x_1(t) + f_1(x_1, r_1)$. Assume that $x_1(0) < x_2(0)$, and they consider the opinion of each other as the referenced opinion, i.e., $r_1(t) = x_2(t)$, and $r_2(t) = x_1(t)$. Agents update opinions according to $x_1(t+1) = x_1(t) + f_1(x_1(t), r_1(t))$ and $x_2(t+1) = \frac{1}{2}(x_1(t) + x_2(t))$. As time goes by, $x_2(t)$ would approach $x_1(t)$ (i.e., shift to the left) gradually while $x_1(t)$ would deviate from $x_2(t)$ (i.e., also shift to the left). Without introducing this approximation parameter, one would obtain $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$ which is counterintuitive.
$(-\sigma_i, 0), (\sigma_i, 0)$ are the x-intercepts of $f_i(d_i)$, i.e., where f_i crosses x-axis (d_i-axis), with $\sigma_i > 0$ capturing the maximal opinion distance from the referenced point such that agent i is influenced. $[-\sigma_i, \sigma_i]$ is called *the repelling interval* of agent i. If f_i does not cross x-axis or $\sigma_i \geq 1$, we will adopt the convention that $\sigma_i = 1$. δ_i captures the maximum repelling level, i.e., how much is agent i influenced at most. Even though results in the current paper hold for all forms of f_i fulfilling the previous properties, $f_i(d)$ can be **convex**, **linear** or **concave** on $[\epsilon, \sigma_i]$ with respect to $d_i(t)$, depending on the context. Some corresponding examples are given below.

Example 6.1 (Linear $f_i(d_i)$ when $\epsilon < d_i < \sigma_i$)**.**
Consider the following piecewise linear form of $f_i(d_i)$:

$$f_i = \begin{cases} \max(0, \delta_i - \frac{\delta_i}{\sigma_i} d_i), & \text{if } d_i > \epsilon; \\ \min(0, -\delta_i - \frac{\delta_i}{\sigma_i} d_i), & \text{if } d_i < -\epsilon. \end{cases} \tag{6.2}$$

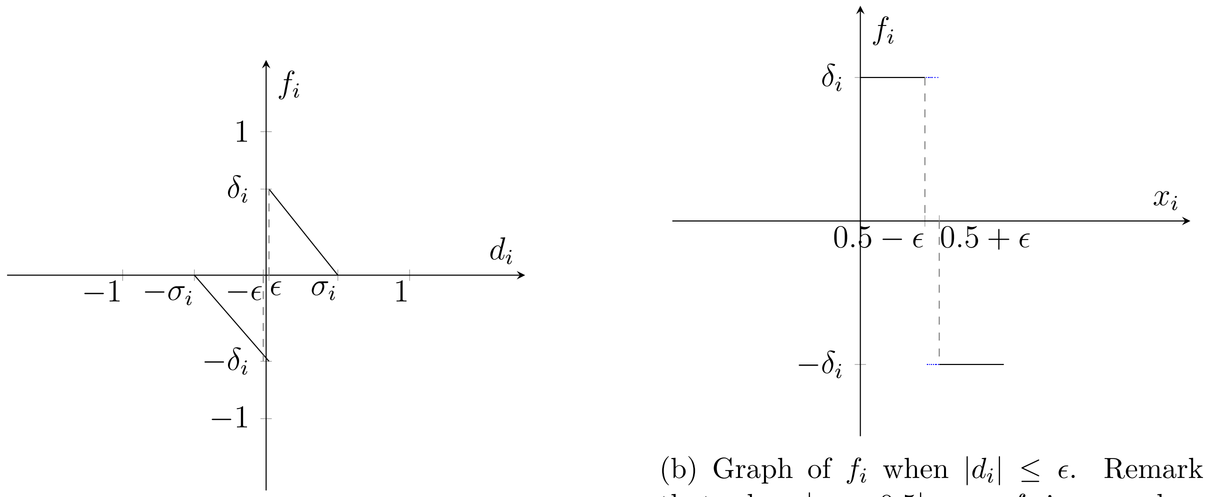
When $\epsilon \leq d_i < \sigma_i$, f_i is linear in d_i, i.e., the ratio of the changes in f_i and in d_i is fixed.

Example 6.2 (Convex $f_i(d_i)$ when $\epsilon < d_i < \sigma_i$)**.**
Consider the following convex form of $f_i(d_i)$ when $\epsilon < d_i < \sigma_i$:

$$f_i = \frac{1}{10 d_i}, \ d_i \in [-1, -\epsilon[\cup]\epsilon, 1]. \tag{6.3}$$

When $\epsilon < d_i < \sigma_i$, f_i is convex in d_i, i.e., the derivative of f_i w.r.t. d_i is increasing with d_i.

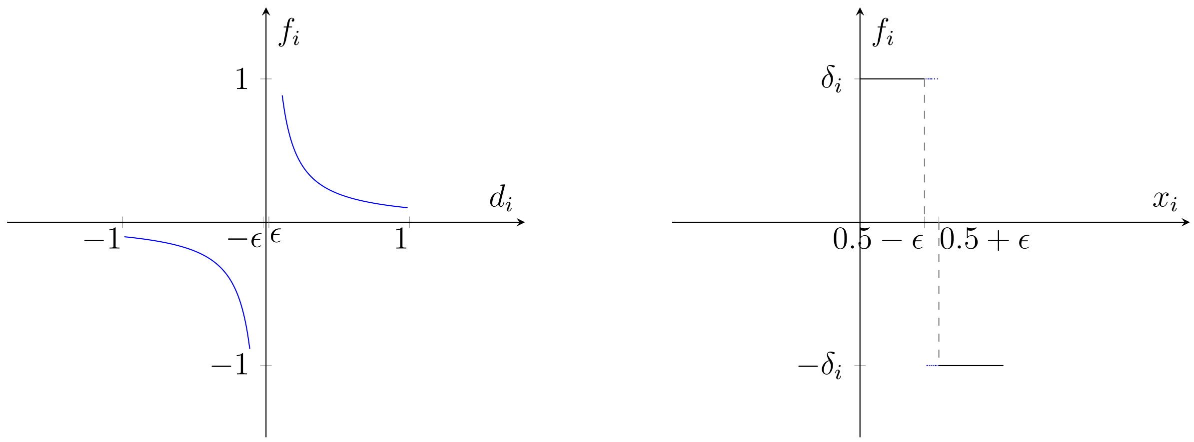
Example 6.3 (Concave $f_i(d_i)$ when $\epsilon < d_i < \sigma_i$)**.**



(a) Graph of f_i when $|d_i| > \epsilon$.

(b) Graph of f_i when $|d_i| \leq \epsilon$. Remark that when $|x_i - 0.5| < \epsilon$, f_i is a random variable taking values δ and $-\delta$ with equal probabilities (dotted in blue).

Figure 6.2: An example of linear f_i when $\epsilon < d_i < \sigma_i$



(a) Graph of f_i when $|d_i| > \epsilon$. In case that f_i does not cross x-axis (resp., y-axis), we adopt the convention that $\sigma_i = 1$ (resp., $\delta_i = 1$).

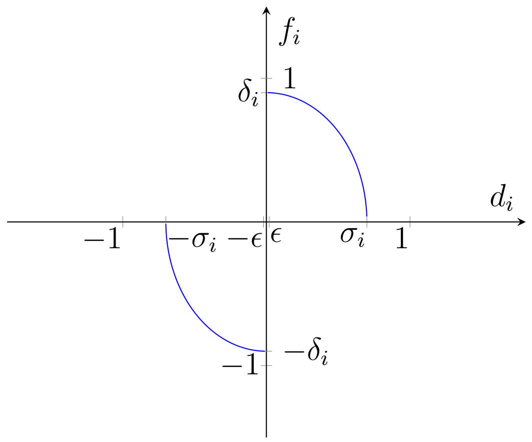
(b) Graph of f_i when $|d_i| \leq \epsilon$. Remark that when $|x_i - 0.5| < \epsilon$, f_i is a random variable taking values 1 and -1 with equal probabilities (dotted in blue).

Figure 6.3: An example of convex f_i when $\epsilon < d_i < \sigma_i$

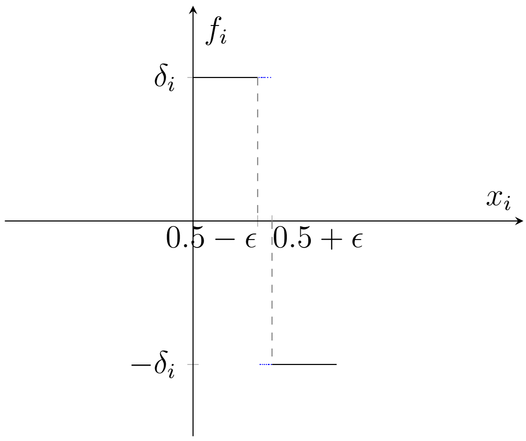
Consider the following concave form of $f_i(d_i)$ when $\epsilon < d_i < \sigma_i$:

$$f_i = \begin{cases} \frac{\delta_i}{\sigma_i}\sqrt{\sigma_i^2 - d_i^2}, & \text{if } d_i \in]\epsilon, 1], \\ -\frac{\delta_i}{\sigma_i}\sqrt{\sigma^2 - d_i^2}, & \text{if } d_i \in [-1, -\epsilon[. \end{cases} \tag{6.4}$$

When $\epsilon < d_i < \sigma_i$, f_i is concave in d_i, i.e., the derivative of f_i w.r.t. d_i is decreasing with d_i.



(a) Graph of f_i when $|d_i| > \epsilon$.



(b) Graph of f_i when $|d_i| \leq \epsilon$. Remark that when $|x_i - 0.5| < \epsilon$, f_i is a random variable taking values $-\delta_i$ and δ_i with equal probabilities (dotted in blue).

Figure 6.4: An example of concave f_i when $\epsilon < d_i < \sigma_i$

6.2.3 Heterogeneous nodes (conformists and anti-conformists)

In this section, we consider the model of opinion dynamics with heterogeneous nodes (agents), i.e., with both conformist and anti-conformist agents. Thus, the society N is partitioned into the set of conformist agents C and that of anti-conformist agents A, i.e., $N = C \cup A$. Opinion of agent i at time t is denoted by $x_i(t)$ which is a real number in [0,1]. Conformist agents update their opinions following the DeGroot rule with equal weights, i.e.,

$$x_i(t+1) = \frac{1}{\eta_i} \sum_{j \in N_i} x_j(t), \forall i \in C. \tag{6.5}$$

However, anti-conformist agents would like to deviate from others, i.e., from their reference opinions. The reference opinion of anti-conformist agent i can take different forms. For example, when considering only the synchronous updating of opinions, $r_i(t)$ is defined as the weighted average of opinions of agent i's neighbors, i.e.,

$$r_i(t) = r_i^S(t) := \frac{1}{\eta_i - 1} \sum_{j \in N_i \backslash \{i\}} x_j(t), \forall i \in A. \tag{6.6}$$

The shift of opinion for anti-conformist agent i is measured by the repelling function f_i defined in Section 6.2.2. Thus the updating rule followed by anti-conformist agents reads:

$$x_i(t+1) = \Big[x_i(t) + f_i\Big(x_i(t), r_i(t)\Big)\Big]_0^1, \forall i \in A. \tag{6.7}$$

Example 6.4. *Consider three agents situated in the following network (Figure 6.5) with agent 1 being conformist and agents 2, 3 being anti-conformists. The initial opinion vector* *is* $x^a(0) = [0.6, 0.4, 0.9]$, $\sigma_i = \delta_i = 1/2, i = 2, 3,$ *and* $\epsilon = 0.001$.

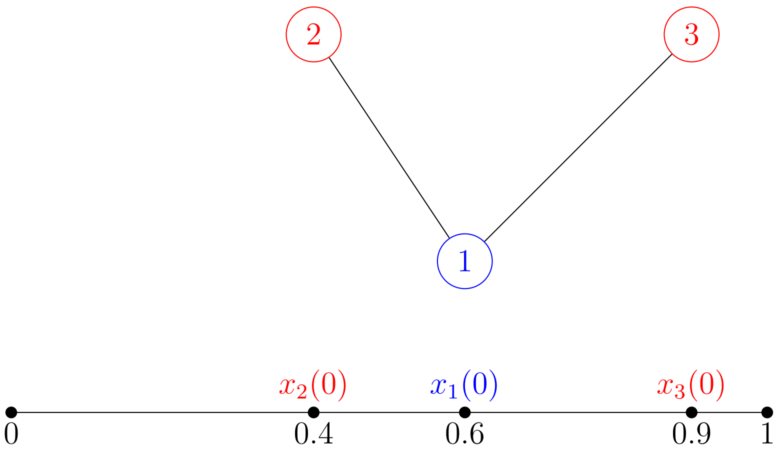


Figure 6.5: Network Structure with agent 1 being conformist and agent 2, 3 being anti-conformists. The real line at bottom refers to the value of initial opinions of agents.

Conformist agent 1 *updates opinion following the DeGroot rule:*

$$x_1(t+1) = \frac{1}{3}(x_1(t) + x_2(t) + x_3(t)).$$

For anti-conformist agents 2 *and* 3, $r_2^S(t) = r_3^S(t) = x_1(t)$, *resulting in* $d_2(t) = x_2(t) - x_1(t)$ *and* $d_3(t) = x_3(t) - x_1(t)$. *Agents* 2 *and* 3 *update opinions according to*

$$x_i(t+1) = [x_i(t) + f_i(t)]_0^1,$$

where $f_i(t)$ *is taking the linear form of the repelling function defined in Example 6.1 when* $i = 2, 3$.

Take agent 2 for example. At time 0, $d_2(0) = x_2(0) - x_1(0) = -0.2$. $f_2(0) = min(0, -0.5 + 0.2) = -0.3$. *Hence* $x_2(1) = x_2(0) + f_2(0) = 0.1$ *(see Figure 6.6). However, assuming a different initial opinion of agent 2, say* $x_2'(0) = 0.5$ *which is still lower than* $x_1(0)$*, this leads to the same opinion of agent 2 at time* 1*:* $x_2(1) = x_2(0) + min(0, -0.5 - (x_2(0) - x_1(0))) = 0.1$. *Indeed* $x_2(t+1)$ *only depends on the relative position of* $x_2(t)$ *with respect to* $x_1(t)$ *and on the value of* $x_1(t)$ *since* $x_2(t+1) = x_2(t) + min(0, -0.5 - (x_2(t) - x_1(t))) = min(x_2(t), x_1(t) - 0.5)$.

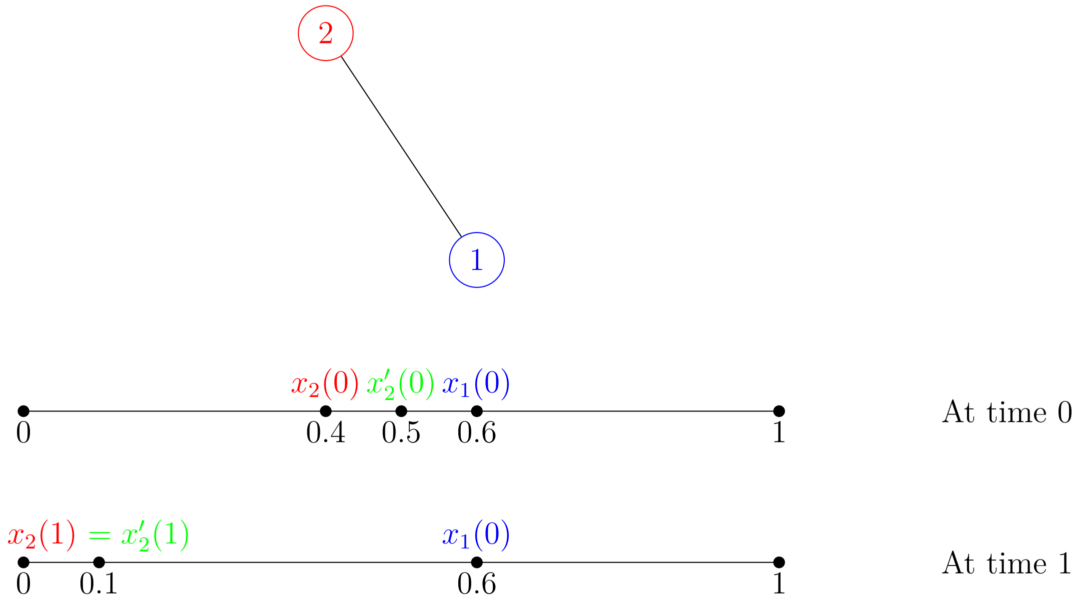


Figure 6.6: Evolution of x_2 for one period

In general, $\forall i \in A$,

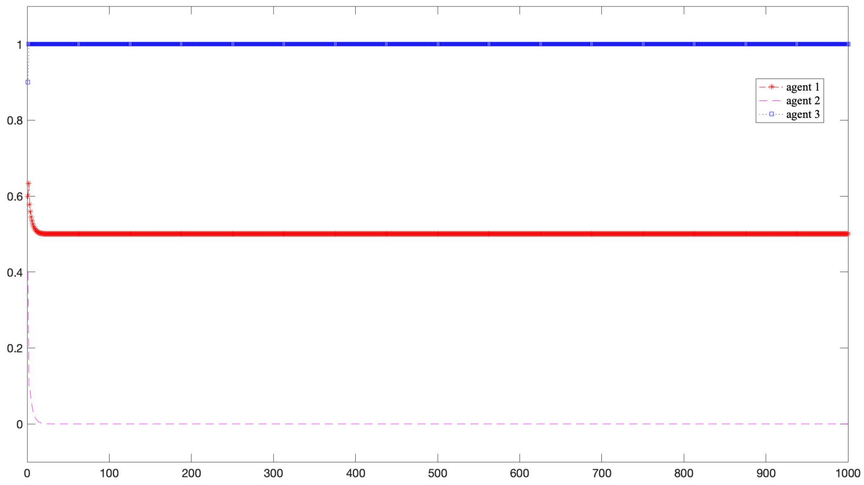
$$x_i(t{+}1) = \begin{cases} [\max(x_i(t), x_i(t) + \delta_i - \frac{\delta_i}{\sigma_i}(x_i(t) - r_i(t)|))]_0^1 & \textit{if } x_i(t) - r_i(t) > \epsilon, \\ [\min(x_i(t), x_i(t) - \delta_i - \frac{\delta_i}{\sigma_i}(x_i(t) - r_i(t)))]_0^1 & \textit{if } x_i(t) - r_i(t) < -\epsilon, \\ [x_i(t) + \delta_i]_0^1 & \textit{if } 0 \leq x_i(t) < 0.5 - \epsilon, x_i(t) \approx r_i(t), \\ [x_i(t) + \alpha]_0^1 & \textit{if } 0.5 - \epsilon \leq x_i(t) \leq 0.5 + \epsilon, x_i(t) \approx r_i(t), \\ [x_i(t) - \delta_i]_0^1 & \textit{otherwise.} \end{cases} \tag{6.8}$$

When $\delta_i = \sigma_i$, *it reduces to*

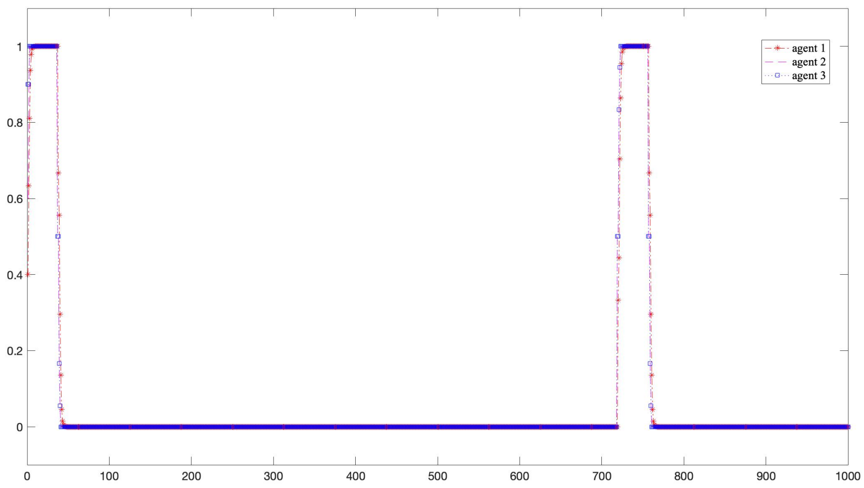
$$x_i(t+1) = \begin{cases} [\max(x_i(t), r_i(t) + \delta_i)]_0^1 & \textit{if } x_i(t) - r_i(t) > \epsilon, \\ [\min(x_i(t), r_i(t) - \delta_i)]_0^1 & \textit{if } x_i(t) - r_i(t) < -\epsilon, \\ [x_i(t) + \delta_i]_0^1 & \textit{if } 0 \leq x_i(t) < 0.5 - \epsilon, x_i(t) \approx r_i(t), \\ [x_i(t) + \alpha]_0^1 & \textit{if } 0.5 - \epsilon \leq x_i(t) \leq 0.5 + \epsilon, x_i(t) \approx r_i(t), \\ [x_i(t) - \delta_i]_0^1 & \textit{otherwise.} \end{cases} \tag{6.9}$$

The opinion value of an anti-conformist agent, say i*, at time* $t{+}1$ *only depends on the sign of the difference of the current opinion and the reference opinion (i.e.,* $sgn(x_i(t) - r_i(t))$*) and on the value of the reference opinion (i.e.,* $r_i(t)$*) rather than the value of its own current opinion (i.e.,* $x_i(t)$*) in case of* $\delta_i = \sigma_i$*. This is because* $\sigma_i = \delta_i$ *leads to the fact that the slope of* f_i *is* -1*, so each unit increase in* d_i *causes one unit decrease in* f_i*.*

The opinions converge to $[0, 0.5, 1]$ *as shown in Figure 6.7a, corresponding to the initial opinion* $x^a(0)$. *However, taking a different initial opinion vector equal to* $x^b(0) = [0.4, 0.6, 0.9]$, *the opinions oscillate as shown in Figure 6.7b. In this case, from time* 1 *on, anti-conformist agent 2 and agent 3 want to be away from agent 2, so they move to the right till reaching value* 1; *conformist agent* 1 *also moves to the right due to the conformity behavior. When* x_1 *becomes close enough to* 1, *i.e.,* $1 - x_1 < \epsilon$, x_2 *and* x_3 *will jump to the value of* $1/2$, *as shown in Figure 6.8a. Again* x_1 *will also gradually move to* $1/2$. *As* x_1 *becomes close enough to* $1/2$, x_2 *and* x_3 *will move to the left till reaching value* 0. *Under the same reasoning,* x_1 *will also gradually move to* 0 *and at some time,* x_2 *and* x_3 *will jump to* $1/2$, *as shown in Figure 6.8b.*

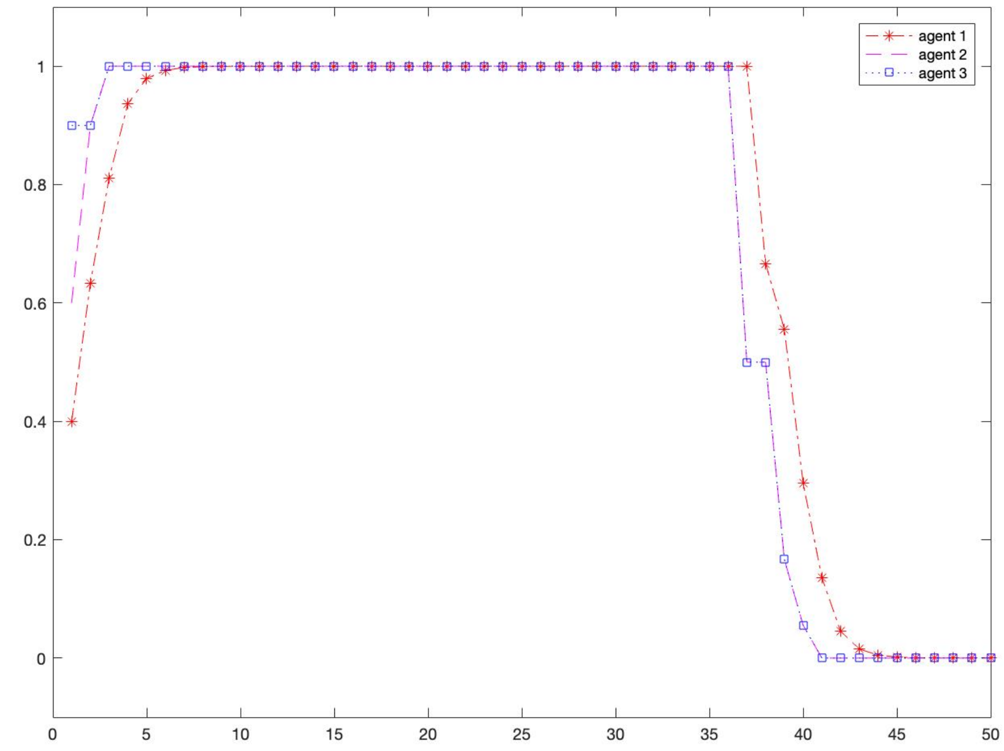


(a) Opinion dynamics when $\delta_i = \sigma_i = \frac{1}{2}$, $\epsilon = 0.001$, $x_1(0) = 0.6$, $x_2(0) = 0.4$ and $x_3(0) = 0.9$.

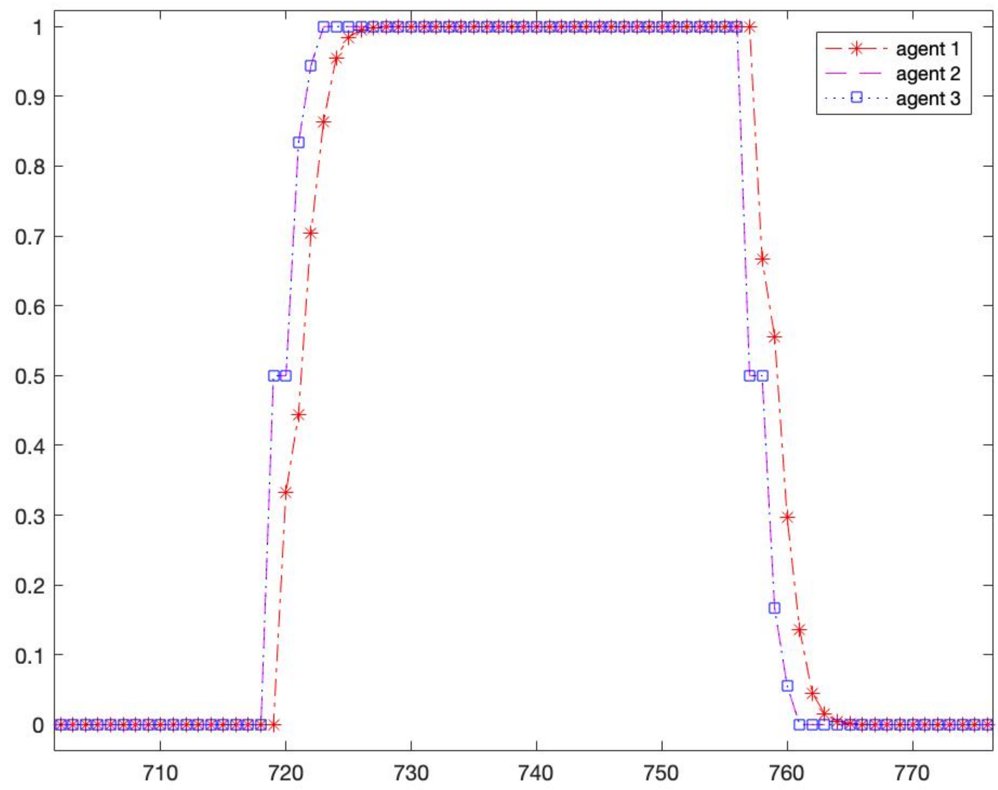


(b) Opinion dynamics when $\delta_i = \sigma_i = \frac{1}{2}$, $\epsilon = 0.001$, $x_1(0) = 0.4$, $x_2(0) = 0.6$ and $x_3(0) = 0.9$.

Figure 6.7: Opinion dynamics with anti-conformism in Example 6.4.



(a) Opinion dynamics in 50 time steps.



(b) Opinion dynamics between time steps 700 and 800.

Figure 6.8: Opinion dynamics with anti-conformism in Example 6.4 when $\delta_i = \sigma_i = \frac{1}{2}$, $\epsilon = 0.001$, $x_1(0) = 0.4$, $x_2(0) = 0.6$ and $x_3(0) = 0.9$.

6.2.4 Heterogeneous links (signed graph)

In this section, we consider the model of opinion dynamics with heterogeneous links (relations), i.e., with both positive and negative links. Thus, $G = (N, E)$ is a signed graph, and the set of edges E is partitioned into the set of positive edges E^+ and the set of negative edges E^-, i.e., $E = E^+ \cup E^-$. Then the network G is decomposed into two subnetworks $G^+ = (N, E^+)$ and $G^- = (N, E^-)$. The neighborhood of agent i is partitioned into the set of her friends and the set of her enemies, i.e., $N_i = N_i^+ \cup N_i^-$, with $N_i^+ := \{j \in N_i : g_{ij} > 0\}$ and $N_i^- := \{j \in N_i : g_{ij} < 0\}$. The opinion of a given agent is updated as the truncated sum of the average opinion of her friends and the deviation from her enemies, i.e.,

$$x_i(t+1) = \left[\frac{1}{\eta_i^+} \sum_{j \in N_i^+} x_j(t) + f\Big(x_i(t), \frac{1}{\eta_i^-} \sum_{j \in N_i^-} x_j(t)\Big)\right]_0^1 \tag{6.10}$$

where $f\Big(x_i(t), \frac{1}{\eta_i^-} \sum_{j \in N_i^-} x_j(t)\Big)$ is the repelling function defined in section 6.2.2 with $d_i(t) = x_i(t) - \frac{1}{\eta_i^-} \sum_{j \in N_i^-} x_j(t))$. We adopt the convention that $f\Big(x_i(t), \frac{1}{\eta_i^-} \sum_{j \in N_i^-} x_j(t)\Big) := 0$ when $\eta_i^- = 0$.

Example 6.5. *Consider 3 agents situated in the following network (see Figure 6.9). Agent 2 and agent 3 are friends and they are enemies to agent 1.*

$$G = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{l} 1 \\ 2 \\ 3. \end{array}$$

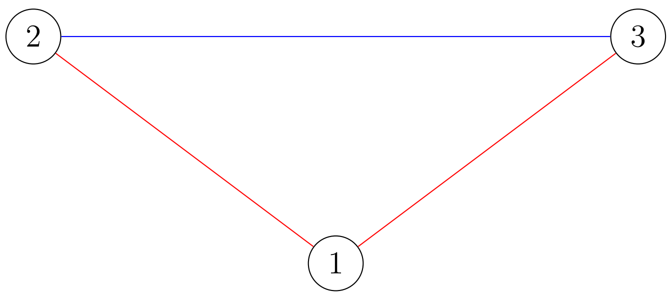


Figure 6.9: A signed graph. Blue edge represents positive influence and red edges represent negative influences.

Agents update opinions according to the following rules:

Agent 1:

$$x_1(t+1) = x_1(t) + f\Big(x_1(t), \frac{1}{2}(x_2(t) + x_3(t))\Big);$$

Agent 2:

$$x_2(t+1) = \frac{1}{2}(x_2(t) + x_3(t)) + f\Big(x_2(t), x_1(t)\Big);$$

Agent 3:

$$x_3(t+1) = \frac{1}{2}(x_2(t) + x_3(t)) + f\Big(x_3(t), x_1(t)\Big).$$

Taking the linear form of the repelling function as in Example 6.1, and supposing two different initial opinions $x^a(0) = [0.4, 0.6, 0.9]$ *and* $x^b(0) = [0.6, 0.4, 0.9]$*, the opinions converge as shown in figure 6.10a and figure 6.10b.*

Let us compare our model to the model of Shi et al. (2019), presented in Section 2.7.3. The formulas for the opposing and repelling rules are as follows.

Opposing rule:

Agent 1:

$$x_1(t+1) = x_1(t) - \beta(x_2(t) + x_3(t)) - 2\beta x_1(t);$$

Agent 2:

$$x_2(t+1) = x_2(t) + \alpha(x_3(t) - x_2(t)) - \beta(x_1(t) + x_2(t));$$

Agent 3:

$$x_3(t+1) = x_3(t) + \alpha(x_2(t) - x_3(t) - \beta(x_1(t) + x_3(t))).$$

Repelling rule:

Agent 1:

$$x_1(t+1) = x_1(t) - \beta(x_2(t) + x_3(t)) + 2\beta x_1(t);$$

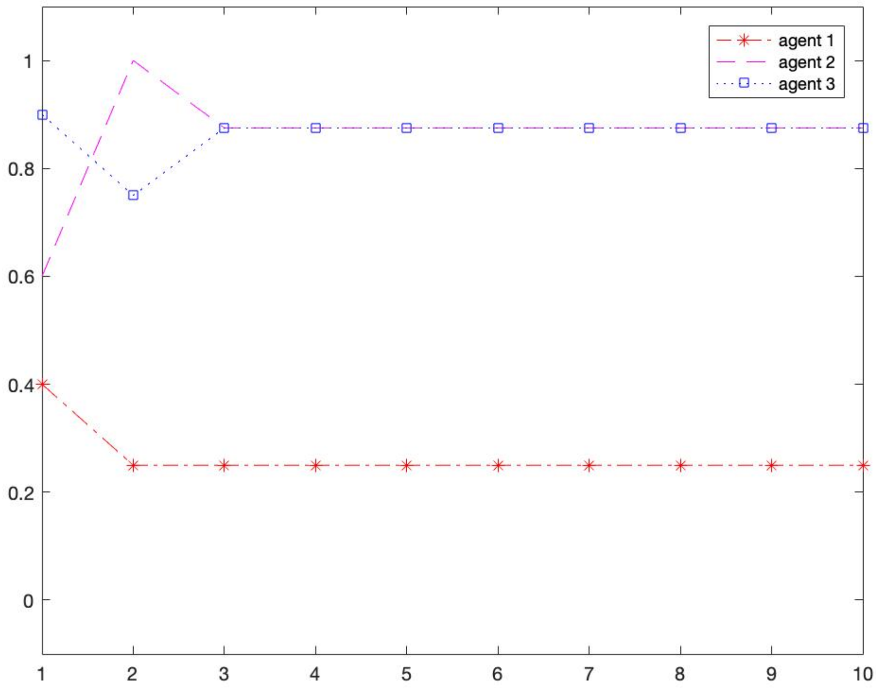
Agent 2:

$$x_2(t+1) = x_2(t) + \alpha(x_3(t) - x_2(t)) - \beta(x_1(t) - x_2(t));$$

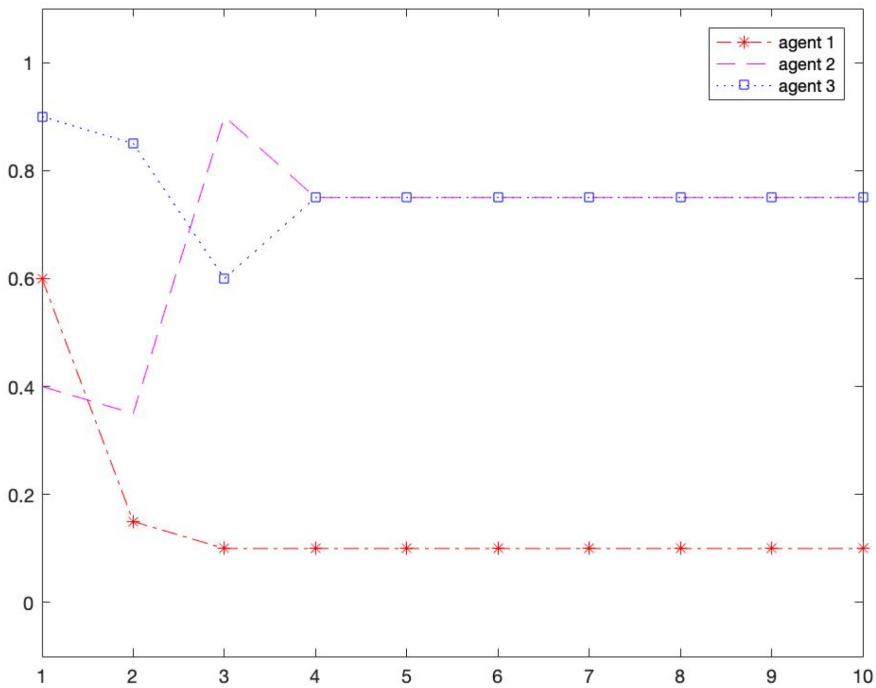
Agent 3:

$$x_3(t+1) = x_3(t) + \alpha(x_2(t) - x_3(t) - \beta(x_1(t) - x_3(t))).$$

The corresponding graphs of opinion dynamics are shown in Figure 6.11a and Figure 6.11b. By following the opposing rule, agents are attracted by the opinions of their friends and the opposite opinions of their enemies. However, during the second period of Example 6.5, agent 1 seems to be attracted by her enemies since $x_1(1) < -x_2(1)$ *and* $x_1(1) < -x_3(1)$ *(see Figure 6.11a). Indeed,* $\forall i, j \in N$*, agent* i *will be attracted by* x_j *as long as* $x_j - x_i$ *and* $-x_j - x_i$ *have the same sign, which is counterintuitive. On the other hand, by following the* *repelling rule, agents are attracted by the opinions of their friends and repel the opinions of their enemies. As a result, opinion of agent 1 tends to* $-\infty$ *and opinions of agent* 2 *and agent* 3 *tend to* ∞ *as* $t \to \infty$*(see Figure 6.11b). None of these rules are applicable for modelling anti-conformity behavior. However, our updating rule based on the repelling functions is able to capture anti-conformity behavior via negative links, in the sense that agents are attracted by opinions of their friends and repel the opinions of their enemies, and the repelling level is related to the distance between her own opinion and her reference opinion (the average opinion of her neighbors). The repelling level is decreasing as the distance increases. In Example 6.5, agents form consensus within each group (see Figure 6.10).*

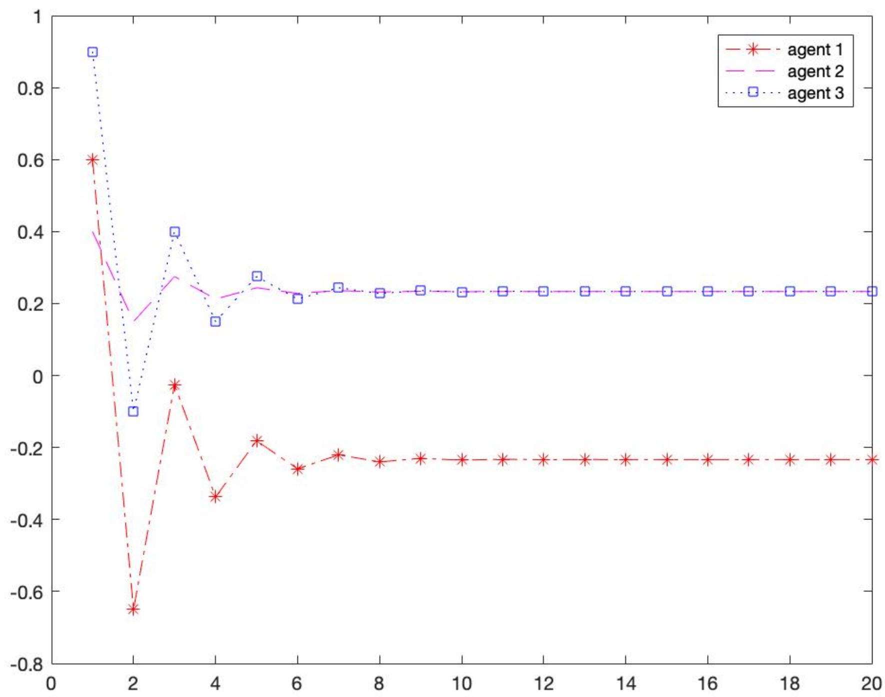


(a) $\delta_i = \sigma_i = \frac{1}{2}$, $\epsilon = 0.001$, $x_1(0) = 0.4$, $x_2(0) = 0.6$ and $x_3(0) = 0.9$.

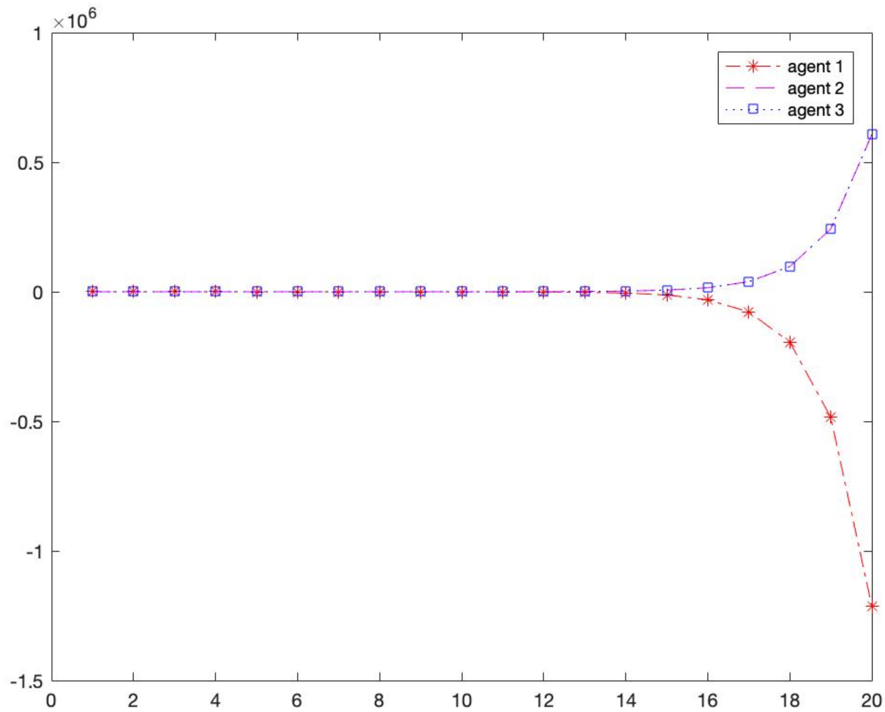


(b) $\delta_i = \sigma_i = \frac{1}{2}$, $\epsilon = 0.001$, $x_1(0) = 0.6$, $x_2(0) = 0.4$ and $x_3(0) = 0.9$.

Figure 6.10: Opinion dynamics of Example 6.5 over signed graphs.



(a) Opinion dynamics with opposing rule when $\alpha = \beta = \frac{1}{2}$, $x_1(0) = 0.6$, $x_2(0) = 0.4$ and $x_3(0) = 0.9$.



(b) Opinion dynamics with repelling rule when $\alpha = \beta = \frac{1}{2}$, $x_1(0) = 0.6$, $x_2(0) = 0.4$ and $x_3(0) = 0.9$.

Figure 6.11: Opinion dynamics of Example 6.5 over signed graphs with opposing rule and repelling rule.

6.2.5 Synchronous updating and Asynchronous updating

Different activation regimes (such as synchronization or synchronization of agents' activation, different interaction size at each time step and so on) can produce different results in opinion dynamic models (Alizadeh et al. (2015)). It may happen that some interesting phenomena exhibited in the synchronous updating model disappear in the asynchronous setting, and therein stability appears instead of striking spatial chaos (Huberman and Glance (1993), Nowak and May (1992)). By describing the order of updates as a sequence of subsets of the population N, Bredereck and Elkind (2017) defined the synchronous updating accordingly, as the updating sequence $(N, N, \ldots, N)$ and defined the asynchronous updating as that with each subset being a singleton. This captures the idea that only one agent is active at each time. The active agent can either meet another agent with a certain probability to exchange opinions or observe the opinions of all her neighbors, and thereafter has her own opinion updated.

Acemoglu and Ozdaglar (2011) modeled an asynchronous updating process by supposing that at each time, agent i is chosen to be active with probability $1/n, \forall i \in N$ and in case of agent i being active, agent i will meet agent j and exchange opinions with probability $p_{ij} \geq 0$, where $\sum_{j=1}^{n} p_{ij} = 1, \forall i \in N$. Moreover, for a better approximation of many real situations, some researchers also consider the opinion dynamics in a random neighborhood setting. For example, Grabisch and Li (2020) studied the synchronous opinion dynamics for binary opinions in a random neighborhood setting in which a random neighborhood is realized in each period. Nyczka and Sznajd-Weron (2013a) studied the asynchronous q-voter model and assumed that both the voter and the group that can influence the voter are randomly chosen (random active agent and random neighborhood). Ramazi et al. (2016) showed that for threshold-based dynamics, the equilibrium can be reached in both the synchronous and asynchronous setting, and it can also be almost surely reached in partial synchronous setting [4]. These results reveal that the asynchrony does not lead to cycles or non-convergence, neither does the irregular network topology. Instead, the coexistence of heterogenous behavior (such as conformity and anti-conformity behavior) play a role in the presence of cycles or non-convergence (Ramazi et al. (2016), Grabisch and Li (2020)).

For a given network structure $G = (N, E)$, it is natural to think the following ways to modeling asynchronous updating process. One is to choose an agent at random to be active with probability $^1/_n$ and the active agent will meet one of her neighbor at random with probability $^1/_{\eta_i}$. Then the pair of agents ij will exchange their opinions. This is also called randomised gossip model in Boyd et al. (2006), used by Shi et al. (2019) to

[4] In partial synchronous updating setting, a random number of agents update opinions simultaneously

describe asynchronous random interactions. An alternative way is to choose an agent at random (with probability $1/n$) to be active and the active agent will update her opinion. Take the CODA-node model for example, and assume that agent i is active at time t. If i is conformist she will update her opinion as the average opinion of her neighbors, i.e., according to 6.5, while if she is anti-conformist, she will update her opinion according to 6.7, taking the average opinion of her neighbors as the referenced opinion, i.e.,

$$r_i^A(t) := \frac{1}{\eta_i - 1} \sum_{j \in N_i \setminus \{i\}} x_j(t), \forall i \in A.^5 \tag{6.11}$$

This paper aims to study the anti-conformity behavior, which is related to the response to the average behavior of the society or a group. Therefore, we adopt the latter form of asynchrony where the active agent is able to observe the behavior of the local neighborhood, based on which she will update opinions.

6.3 Opinion dynamics with conformists and anti-conformists

6.3.1 Synchronous updating model

In this section, we consider the synchronous updating where agents update opinions simultaneously following rules 6.5 and 6.7. We are interested in whether opinions converge, and if so, whether agents will form a consensus in the long run. Define *convergence* and *consensus* as follows.

Definition 6.1 (Opinion convergence)**.** *Opinions of agents in set N are convergent if $\forall i \in N, \exists x_i^* \in [0,1]$, such that $\lim_{t\to\infty} x_i(t) = x_i^*$.*

Definition 6.2 (Consensus)**.** *The society is said to reach a consensus if there exists $x^* \in [0,1]$, such that $\lim_{t\to\infty} x_i(t) = x^*, \forall i \in N$.*

Different from the classic model of opinion dynamics with only conformist agents, introducing anti-conformist agents into any connected network of conformist agents makes the consensus impossible.

Fact 6.1. *There is no consensus for any connected network with $A \neq \emptyset$.*

Proof. By contradiction, suppose the society will reach a consensus, then $\exists t \in \mathbb{N}$, such that $x_i(t) = x^*, \forall i \in N$. Then for any anti-conformist agent i, $r_i(t) = x^*$.

[5] Remark here that $r_i^A(t) = r_i^S(t)$. Other forms of the reference opinion can be adopted, depending on the context.

By property (e) of the repelling function, $x_i(t+1)$ is either $x^*+\delta$ or $x^*-\delta$ which contradicts the definition of consensus. $\square$

Consider two connected conformist agents with any initial opinions in $[0,1]^2$, they will form a consensus on the average of their initial opinions since time 2. However, for the society of two connected anti-conformist agents, the existence of the steady state depends on the value of initial opinions, and the opinions may form a disagreement or oscillations.

Proposition 6.1. *Assume that $\sigma_i = \sigma, \forall i \in N$. Consider two connected anti-conformist agents with initial opinions $x_1(0)$ and $x_2(0)$. Recall that the steady state opinion vector is denoted as $\bar{x}$.*

(i) If $|x_1(0) - x_2(0)| \geq \sigma$, then $\bar{x} = [x_1(0), x_2(0)]$ (independence and disagreement);

(ii) If $\epsilon < |x_1(0) - x_2(0)| < \sigma$, then $\bar{x} = [x^*, x^{**}]$ and $x^* \neq x^{**}$ (disagreement);

(iii) If $|x_1(0) - x_2(0)| < \epsilon$, $|x_1(0) - 0.5| \geq \epsilon$ and $|x_2(0) - 0.5| \geq \epsilon$, then there is no steady state but an oscillation with period 2, i.e., $\exists t^* \in \mathbb{N}$, such that $\forall t > t^*$, $x_1(t) = x_2(t) = x_1(t+2)$;

(iv) Otherwise, opinions will almost surely converge to a disagreement.

Proof. $(\sigma_i, 0)$ is the x-intercept of f_i, and by properties (a) and (c) of f_i, we have $f_i(d_i) = 0, \forall d_i \geq \sigma_i$. (It is analogous for $d_i \leq -\sigma_i$.) Then it is easy to check case by case according to the updating rule 6.7 and the properties of f_i. $\square$

Example 6.6. *Consider two connected anti-conformist agents with two different initial opinion vectors $x^a(0) = [0.4, 0.4]$ and $x^b(0) = [0.3, 0.5]$. Taking the linear form of the repelling function as in Example 6.1, the opinions oscillate for the former case and converge to different values for the latter case, as shown in Figure 6.12 and 6.13.*

If there is one anti-conformist agent connected with one conformist agent, then for any initial opinions in $[0,1]^2$, the opinions do not converge but oscillate.

Proposition 6.2.

Consider the society consisting of two connected agents, with agent 1 conformist and agent 2 anti-conformist. Then there will be oscillations instead of a convergence of opinions. And for x_2, the oscillations are between 0 and 1.

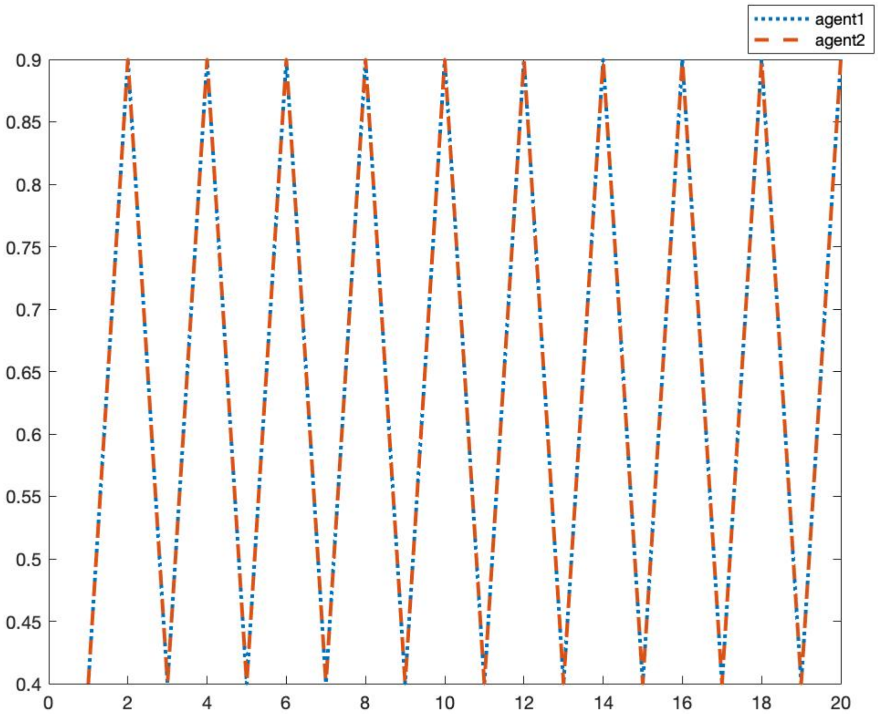


Figure 6.12: Opinion dynamics with two anti-conformists in Example 6.6. $x_1(0) = x_2(0) = 0.4$, $\epsilon = 0.001$ and $\delta = \sigma = 0.5$.

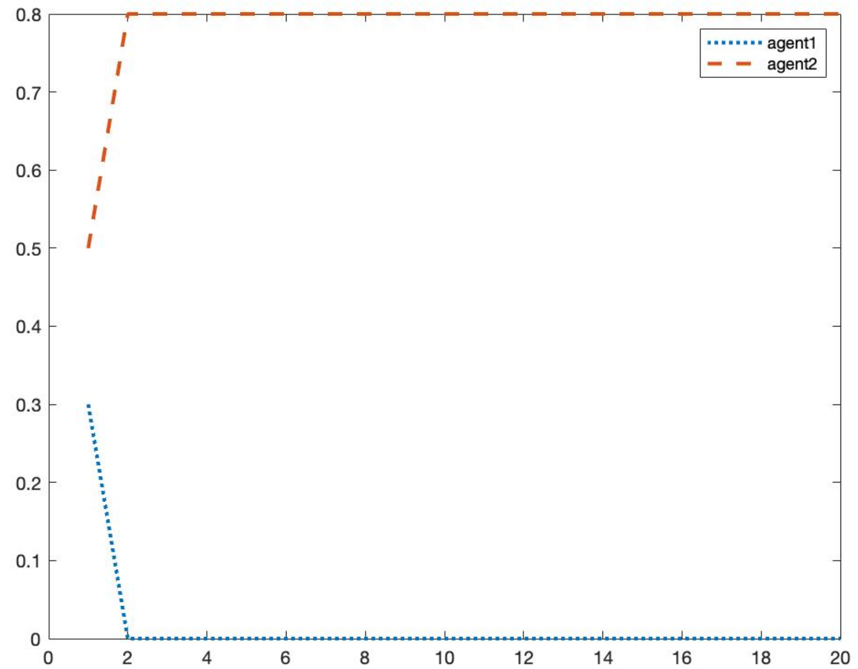


Figure 6.13: Opinion dynamics with two anti-conformists in Example 6.6. $x_1(0) = 0.3, x_2(0) = 0.5$, $\epsilon = 0.001$ and $\delta = \sigma = 0.5$.

Proof. We show first there is no convergence by contradiction. Suppose the opinions converge, i.e., $\exists \bar{x} = [\bar{x}_1, \bar{x}_2]$ where $\bar{x}_1, \bar{x}_2 \in [0,1]$ and a time t_0, such that for each $t' > t_0$, $x_1(t') = \bar{x}_1$ and $x_2(t') = \bar{x}_2$. Fix a $t' > t_0$, then:
1) if $\bar{x}_1 = \bar{x}_2 < 1/2$, then $x_2(t'+1) = \min\{1, x_2 + \delta_2\}$ contradicting $x_2(t'+1) = x_2(t')$;
2) if $\bar{x}_1 = \bar{x}_2 > 1/2$, then $x_2(t'+1) = \max\{0, x_2 - \delta_2\}$ contradicting $x_2(t'+1) = x_2(t')$;
3) if $\bar{x}_1 \neq \bar{x}_2$, then $x_1(t'+1) = \frac{(\bar{x}_1+\bar{x}_2)}{2}$ contradicting $x_1(t'+1) = x_1(t')$.
By contradiction, there is no convergence. Then it suffices to show opinions oscillate.
Supposing that at some time t, $x_1(t) \leq x_2(t)$, distinguish the following cases:
1) $x_2(t) - x_1(t) < \epsilon$ and $x_2 \leq 1/2 - \epsilon$, then x_2 will move to the right, and x_1 will also move to the right due to its conformity, till $x_2 = 1$;
2) $x_2(t) - x_1(t) < \epsilon$ and $x_2 \geq 1/2 + \epsilon$, then x_2 will move to the left, and x_1 will also move to the left due to its conformity, till $x_2 = 0$. As x_1 becomes close enough to 0, x_2 will jump to the right, followed by x_1 moving to the right, till $x_2 = 1$;
3) $x_2(t) - x_1(t) < \epsilon$ and $|x_2 - 1/2| < \epsilon$, then x_2 will move either to the right or to the left, and for both cases x_2 will eventually reach the value 1;
4) $\epsilon \leq x_2(t) - x_1(t) < \sigma_2$, then x_2 will move to the right, and x_1 will also move to the right due to its conformity, till $x_2 = 1$;
5) $x_2(t) - x_1(t) \geq \sigma_2$, then $x_2(t+1) = x_2(t)$. However, x_1 will be closer to x_2 as time goes on. Thus there must be a time such that it reduces to case 4).
Above all, there exists a time, such that x_2 reaches value 1. As x_1 will move closer to x_2, there must be a time, such that the difference between x_1 and x_2 is less than ϵ. Denote by t_1 the smallest time such that $x_2(t_1) = 1$ and $x_2(t_1) - x_1(t_1) < \epsilon$.
Analogously, assuming that at some time t, $x_1(t) \geq x_2(t)$, distinguish the following cases:
1') $x_1(t) - x_2(t) < \epsilon$ and $x_2 \geq 1/2 + \epsilon$, then x_2 will move to the left, and x_1 will also move to the left due to its conformity, till $x_2 = 0$;
2') $x_1(t) - x_2(t) < \epsilon$ and $x_2 \leq 1/2 - \epsilon$, then x_2 will move to the right, and x_1 will also move to the right due to its conformity, till $x_2 = 1$. As x_1 becomes close enough to 1, x_2 will jump to the left, followed by x_1 moving to the left, till $x_2 = 0$;
3') $x_1(t) - x_2(t) < \epsilon$ and $|x_2 - 1/2| < \epsilon$, then x_2 will move either to the right or to the left, and for both cases x_2 will reach the value 0;
4') $\epsilon \leq x_1(t) - x_2(t) < \sigma_2$, then x_2 will move to the left, and x_1 will also move to the left due to its conformity, till $x_2 = 0$;
5') $x_1(t) - x_2(t) \geq \sigma_2$, then $x_2(t+1) = x_2(t)$. However, x_1 will be closer to x_2 as time goes on. Thus there must be a time such that it reduces to case 4).
Above all, there exists a time, such that x_2 reaches value 0. As x_1 will move closer to x_2, there must be a time, such that the difference between x_1 and x_2 is less than ϵ. Denote

by t_1' the smallest time such that $x_2(t_1') = 0$ and $x_1(t_1') - x_2(t_1') < \epsilon$.
Let us take the assumption again that $x_1(t) \leq x_2(t)$. $x_2(t_1+1) = 1 - \delta_2$, $x_1(t_1+1) = \frac{x_1(t_1)}{2} + 1/2$. x_1 will move to the left, again trying to be closer to x_2. This is back to the case that $x_1 \geq x_2$. Denote by t_2 the smallest time such that $x_2(t_2) = 0$ and $x_1(t_1) - x_2(t_2) < \epsilon$. Then $x_2(t_2+1) = \delta_2$ and $x_1(t_2+1) = \frac{x_2(t)}{2} + 1/2$. This is back to the case that $x_1 \leq x_2$, so $\exists t_3$, such that $x_2(t_3) = 1$ and $x_2(t_3) - x_1(t_3) < \epsilon$. This process will be repeated over time. Thus opinions oscillate and there are oscillations between 0 and 1 for x_2. It also holds for the case of $x_1(t) \geq x_2(t)$ by the same reasoning.

□

Example 6.7. *Consider two connected agents with agent 1 being anti-conformist and agent 2 being conformist. The initial opinion vector is* $x(0) = [0.4, 0.8]$. *Taking the linear form of the repelling function as in Example 6.1, opinions oscillate as shown in Figure 6.15.*

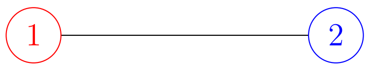


Figure 6.14: Network structure of Example 6.7

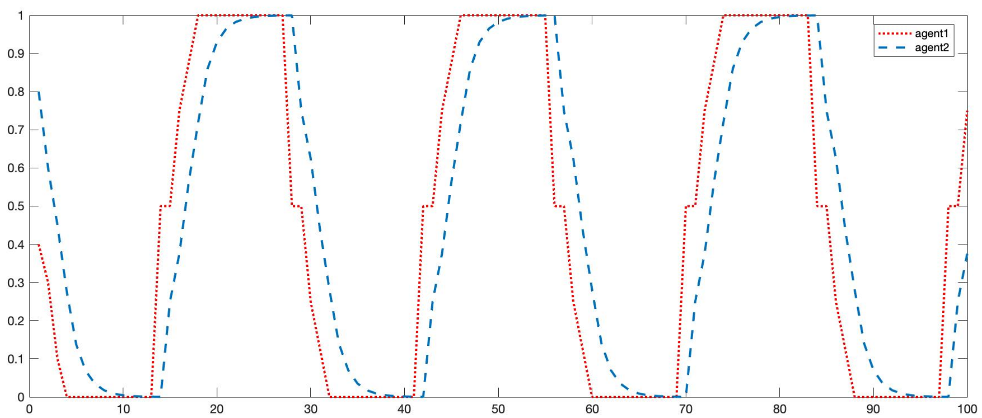


Figure 6.15: Opinion dynamics with one conformist agent and one anti-conformist agent of Example 6.7. $x_1(0) = 0.4$, $x_2(0) = 0.8$, $\delta = \sigma = 0.5$ and $\epsilon = 0.001$.

As a consequence, if $\exists i \in C$ and $j \in A$ such that $N_i = N_j = \{i, j\}$, then there will be oscillations instead of a convergence of opinions.

Fact 6.2. *Any connected component of conformist agents will form a consensus.* [6]

[6] Remark that the notion of a *strongly connected component* is equivalent to a *class* (see Definition 2.4)

Now let us consider the case where one anti-conformist agent is connected to a set of connected conformist agents. The following proposition shows that the opinion dynamics do not converge but oscillate.

Proposition 6.3. *If there is only one anti-conformist agent, and this agent has at least one link to a set of connected conformist agents, i.e.,* $A = \{i\}$, *and* $N_i \cap C \neq \emptyset$, *then opinions do not converge.*

Proof. W.l.o.g., assume that $i = 1$, so $A = \{1\}$ and $C = \{2, \ldots, n\}$.
We show first there is no convergence by contradiction. Suppose the opinions converge, i.e., $\exists \bar{x} = [\bar{x}_1, \ldots, \bar{x}_n]$ where $\bar{x}_1, \ldots, \bar{x}_n \in [0, 1]$ and a time t_0, such that for each $t' > t_0$, $x_i(t') = \bar{x}_i, \forall i \in N$. Since $N_i \cap C \neq \emptyset$, we can assume w.l.o.g. that $2 \in N_i \cap C$, while $N_i \cap C$ may also contain other agents. Fix $t' > t_0$, then distinguish the following cases.
1) If $\bar{x}_1 = \bar{x}_2 < 1/2$, then $x_1(t'+1) = \min\{1, x_1 + \delta_1\}$ contradicting $x_1(t'+1) = x_1(t')$.
2) If $\bar{x}_1 = \bar{x}_2 > 1/2$, then $x_1(t'+1) = \max\{0, x_1 - \delta_1\}$ contradicting $x_1(t'+1) = x_1(t')$.
3) If $\bar{x}_1 < \bar{x}_2$, then

$$x_2(t'+1) = \frac{\sum_{j \in N_2}(\bar{x}_j)}{\eta_2} = x_2(t') = \bar{x}_2.$$

There must exist a $j_1 \in N_2$ and $j_1 \neq 1$, such that $\bar{x}_2 < \bar{x}_{j_1}$ due to the fact that agent 1 is linked to agent 2 and $x_1 < x_2$. Again,

$$x_{j_1}(t'+1) = \frac{\sum_{j \in N_{j_1}}(\bar{x}_j)}{\eta_{j_1}} = x_{j_1}(t') = \bar{x}_{j_1}.$$

There must exist a $j_2 \in N_{j_1}$, such that $\bar{x}_{j_1} < \bar{x}_{j_2}$. So there is a infinite series of $j_1, j_2, \ldots$ such that $\bar{x}_{j_1} < \bar{x}_{j_2} < \ldots$, contradicting the assumption of a finite number of agents.
4) if $\bar{x}_1 > \bar{x}_2$, it is analogous to case 3) to get the contradiction. [7]
By contradiction, there is no convergence. □

For a society of connected conformist agents, introducing only one anti-conformist agent will break the consensus. Furthermore, if the society is fully connected, there will be oscillations between 0 and 1.

Proposition 6.4. *If the network is complete and there is only one anti-conformist agent, say, agent 1, then there will be oscillations instead of a convergence of opinions. And for* x_1, *the oscillations are between* 0 *and* 1.

Proof. By Proposition 6.3, opinions do not converge. Then it suffices to show opinions oscillate. W.l.o.g., assume that $A = \{1\}$ and $C = \{2, \ldots, n\}$. Since the network is

[7] Indeed, 3) and 4) imply that in the presence of only one anti-conformist agent, if the opinions converge, then the set of connected conformist agents form a consensus. Otherwise, a contradiction will happen in the same way.

complete, $x_i(1) = \sum_{j=1}^{n} x_j(0)$, $\forall i \in C$, i.e., from time 1, all conformist agents form a consensus. Then all conformist agents will have the same behavior after time 1, so we can treat them as one conformist agent. Thus by Proposition 6.2, there is no convergence but oscillations of opinions. And for x_1, i.e., the opinion of the anti-conformist agent, the oscillations are between 0 and 1.

□

Example 6.8. *Consider one anti-conformist agent* 1 *and a set of connected conformist agents* $2, \ldots, 6$, *situated in the society with correponding network structure as shown in Figure 6.16. The initial opinion vector is* $x(0) = [0.4, 0.7, 0.1, 0.16, 0.9, 0.4]$. *Take the linear form of the repelling function as in Example 6.1. As shown in Figure 6.17, along*

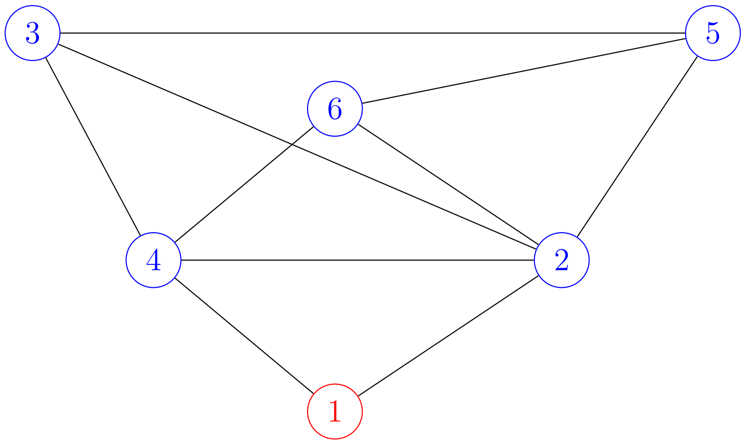


Figure 6.16: Network structure of Example 6.8

the dynamics, anti-conformist agent 1 reaches value 0 *during first several steps. And this causes agent 2 and agent 4 who are neighbors of agent 1 to decrease their opinion, which will again influence the other conformist agents to decrease their opinions. As opinions of agent 2 and agent 4 become close enough to* 0 *so that* $\frac{(x_2+x_4)}{2} \leq \epsilon$, x_1 *will jump to* δ_1, *which will again cause the other conformist agents to increase their opinions. Opinions oscillate instead of converging, and oscillations are between 0 and 1 for anti-conformist agent* 1.

However, if more than one anti-conformist agents are introduced into the model, even though the consensus is impossible, the convergence of opinion can still be reached under certain conditions. Consider a common situation where anti-conformist agents hold relatively extreme opinions and conformist agents hold relatively mild opinions, Theorem 6.1 gives the convergence conditions on the initial opinions and the number of neighbours from each group.

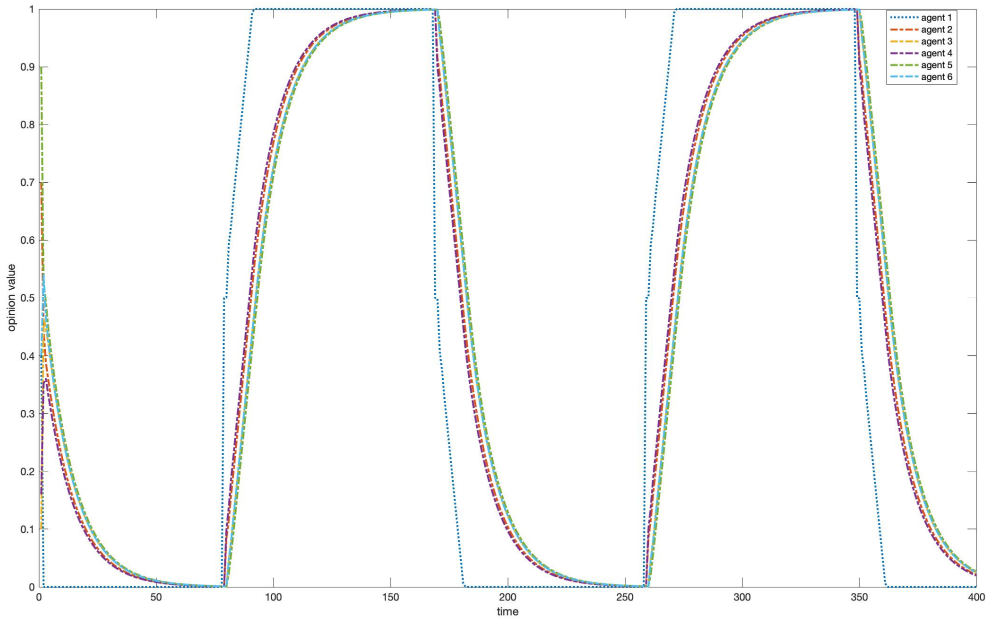


Figure 6.17: Opinion dynamics with one conformist agent and one anti-conformist agent of Example 6.8. $x(0) = [0.4, 0.7, 0.1, 0.16, 0.9, 0.4]$, $\delta = \sigma = 0.5$ and $\epsilon = 0.001$.

Definition 6.3 (connected sets)**.** *Two disjoint sets of agents B and D are said to be connected if each node of one set has at least one neighbor in the other group.*

Theorem 6.1 (anti-conformists being extremist and conformists being moderate)**.** *Suppose that $A = A_1 \cup A_2$, where A_1, A_2, C are non-empty and pairwise connected. Furthermore, $x_i(0) < x' \le x_j(0) \le x'' < x_k(0)$ holds for $\forall i \in A_1, \forall j \in C$ and $\forall k \in A_2$. Let $\eta_{i,A_1}, \eta_{i,A_2}$ and $\eta_{i,C}$ be the number of neighbors of agent i that belong to sets A_1, A_2 and C, respectively, and assume that $\eta_{i,A_1} > 0, \eta_{i,A_2} > 0, \eta_{i,C} > 0, \forall i \in N$. If the following inequalities are satisfied for all $i \in N$:*

$$\frac{x'}{x''} \le \frac{\eta_{i,A_2}}{\eta_{i,A_1} + \eta_{i,A_2}} \tag{6.12}$$

$$\frac{1 - x'}{1 - x''} \ge \frac{\eta_{i,A_2}}{\eta_{i,A_1}} + 1, \tag{6.13}$$

the following will hold:

Ordering consistency $\forall t, \forall i \in A_1, \forall j \in C, \forall k \in A_2$, $x_i(t) \le x_j(t) \le x_k(t)$.

Opinion convergence $\forall i \in N, \exists x_i^* \in [0, 1]$, *such that* $\lim_{t\to\infty} x_i(t) = x_i^*$.

Steady-state opinions *Denote by* $\bar{x} = [\bar{x}_{A_1}, \bar{x}_{A_2}, \bar{x}_C] \in [0, 1]^N$ *the steady-state opinion vector. If* $\sigma_i = 1, \forall i \in N$, *then* $\bar{x}_{A_1} = (0, \ldots, 0)$, $\bar{x}_{A_2} = (1, \ldots, 1)$, $\bar{x}_C = [I -$

$Q]^{-1}R_{A_2}$. The matrix Q and R_{A_2} can be obtained as follows: define a weight matrix as $W = (w_{ij})$ where

$$w_{ij} = \begin{cases} 1, & \textit{if } i,j \in A_1 \cup A_2 \textit{ and } i = j \\ \frac{1}{\eta_i}, & \textit{if } i \in C \textit{ and } g_{ij} = 1 \\ 0, & \textit{otherwise,} \end{cases}$$

and put W into the canonical form as

$$W = \begin{bmatrix} I_{|A_1|} & 0 & 0 \\ 0 & I_{|A_2|} & 0 \\ R_{A_1} & R_{A_2} & Q \end{bmatrix},$$

where I_k is the identity matrix of size k.

Proof. We will show first the ordering consistency, i.e., under conditions 6.12 and 6.13, $x_i(t) < x' \leq x_j(t) \leq x'' < x_k(t)$ holds for $\forall i \in A_1, \forall j \in C, \forall k \in A_2$, and for all t. It suffices to show this inequality holds for $t = 1$ under the given condition. For any conformist agent i, on one hand, $x_i(1) = \frac{1}{\eta_i}\sum_{j\in N_i} x_j(0) = \frac{1}{\eta_i}(\sum_{j\in N_{i,A_1}} x_j(0) + \sum_{j\in N_{i,C}} x_j(0) + \sum_{j\in N_{i,A_2}} x_j(0)) \geq \frac{1}{\eta_i}(\eta_{i,C}x' + \eta_{i,A_2}x'') \geq x'$ by inequality 6.12 and the fact that $\eta_i = \eta_{i,C} + \eta_{i,A_1} + \eta_{i,A_2}$. On the other hand, $x_i(1) \leq \frac{1}{\eta_i}(\eta_{i,C}x'' + \eta_{i,A_1}x' + \eta_{i,A_2}) \leq x''$. The last inequality is guaranteed by inequality 6.13 and the fact that $\eta_i = \eta_{i,C} + \eta_{i,A_1} + \eta_{i,A_2}$.
For any anti-conformist agent i in A_1, the reference opinion is

$$\begin{aligned} r_i(0) &= \frac{1}{\eta_i - 1}\sum_{j\neq i; j\in N_i} x_j(0) = \frac{1}{\eta_i - 1}(\sum_{j\in N_{i,A_1}} x_j(0) + \sum_{j\in N_{i,C}} x_j(0) + \sum_{j\in N_{i,A_2}} x_j(0)) \\ &> \frac{1}{\eta_i - 1}(\eta_{i,C}x' + \eta_{i,A_2}x'') > x' > x_i(0) \end{aligned}$$

by inequality 6.12. In case of $r_i(0) - x_i(0) < \sigma_i$, it will lead to $x_i(1) < x_i(0) < x'$, and otherwise $x_i(1) = x_i(0) < x'$.
For any anti-conformist agent i in A_2, the referenced opinion

$$\begin{aligned} r_i(0) &= \frac{1}{\eta_i - 1}\sum_{j\neq i; j\in N_i} x_j(0) = \frac{1}{\eta_i - 1}(\sum_{j\in N_{i,A_1}} x_j(0) + \sum_{j\in N_{i,C}} x_j(0) + \sum_{j\in N_{i,A_2}} x_j(0)) \\ &< \frac{1}{\eta_i - 1}(\eta_{i,A_1}x' + \eta_{i,C}x'' + \eta_{i,A_2} - 1) < x'' \end{aligned}$$

by inequality 6.13. In case of $x_i(0) - r_i(0) < \sigma_i$, it will lead to $x_i(1) > x_i(0) > x''$, and otherwise $x_i(1) = x_i(0) > x''$.

Above all, inequalities 6.12 and 6.13 are sufficient conditions such that $x_i(t) \leq x_j(t) \leq x_k(t)$ holds for $\forall i \in A_1, \forall j \in C, \forall k \in A_2, \forall t$.
The opinion values of agents in set A_1 (resp., A_2) is decreasing (resp., increasing) as time goes to infinity, thus the opinion convergence of anti-conformist agents is guaranteed due to the boundness of opinions.
$\sigma_i = 1, \forall i \in A$ means that the repelling area is $[0, 1)$ for all anti-conformist agents in A_1 and A_2. $\forall i \in A_1$, $x_i(t)$ is strictly decreasing until it reaches 0 and $\forall k \in A_2$, $x_k(t)$ is strictly increasing until it reaches 1. So $\lim_{t\to\infty} x_i(t) = 0$, and $\lim_{t\to\infty} x_k(t) = 1$. Then this model is equivalent to the DeGroot model with anti-conformist agents being stubborn agents, and the weight matrix is $W = (w_{ij})$ where

$$w_{ij} = \begin{cases} 1, & i, j \in A_1 \cup A_2 \text{ and } i = j, \\ \frac{1}{\eta_i}, & i \in C \text{ and } g_{ij} = 1, \\ 0, & \textit{otherwise}. \end{cases}$$

The weight matrix W can be written into the canonical form (see Section 2.2) with two set A_1 and A_2 being two essential classes and the set C being inessential classes as

$$W = \begin{bmatrix} I_{|A_1|} & 0 & 0 \\ 0 & I_{|A_2|} & 0 \\ R_{A_1} & R_{A_2} & Q \end{bmatrix}.$$

By Fact2.1 and Lemma 2.1 in Section 2.1, we have $[I-Q]^{-1}$ exists and $[I-Q]^{-1} = \sum_{k=0}^{\infty} Q^k$ with $Q^0 = I$.
So the steady-state opinion vector of conformist agents $\bar{x}_C$ must satisfy that $\bar{x}_C = R_{A_1}\bar{x}_{A_1} + R_{A_2}\bar{x}_{A_2} + Q\bar{x}_C$. By $\bar{x}_{A_1} = 0$ and $\bar{x}_{A_2} = 1$, we have $\bar{x}_C = [I - Q]^{-1} R_{A_2}$

□

Example 6.9. *Let $x' = 1/3$ and $x'' = 2/3$, then inequalities 6.12 and 6.13 imply that $\eta_{i,A_1} = \eta_{i,A_2}$. Every agent has exactly the same number of neighbors in A_1 and A_2.*

Example 6.10. *More generally, take $x' = \frac{1}{q}$ with $q \in \mathbb{N} \setminus \{1, 2\}$ and $x'' = 1 - x'$. The conditions being*

$$\frac{x'}{x''} \leq \frac{\eta_{i,A_2}}{\eta_{i,A_1} + \eta_{i,A_2}}, \quad \frac{1 - x'}{1 - x''} \geq \frac{\eta_{i,A_2}}{\eta_{i,A_1}} + 1,$$

they become

$$\eta_{i,A_1} \geq \frac{\eta_{i,A_2}}{q-2}, \quad \eta_{i,A_2} \geq \frac{\eta_{i,A_1}}{q-2}.$$

Supposing $\eta_{i,A_1} = k$, we must have $(q-2)k \geq \eta_{i,A_2}$, i.e., η_{i,A_2} has the form

$$\eta_{i,A_2} = (q-2)k - \ell, \quad \ell = 0, \dots, (q-2)k.$$

However, the second condition implies $(q-2)k - \ell \geq \frac{k}{q-2}$, which yields to:

$$\ell \leq \left\lfloor \frac{k(q-3)(q-1)}{q-2} \right\rfloor.$$

Remarking that

$$(q-2)k - \frac{k(q-3)(q-1)}{q-2} = \frac{k}{q-2},$$

the final result is: For any node i, its numbers of neighbors in A_1, A_2 must be of the form

$$\eta_{i,A_1} \in \mathbb{N}, \quad \eta_{i,A_2} = \left\lceil \frac{\eta_{i,A_1}}{q-2} \right\rceil, \dots, (q-2)\eta_{i,A_1}.$$

Example with $q = 4$: possible couples $(\eta_{i,A_1}, \eta_{i,A_2})$ are

$$(1,1), (1,2), (2,1), (2,2), (2,3), (2,4), \dots, (2n, n), \dots, (2n, 4n), \dots$$

In words, the number of neighbors in one group is at most twice and at least half the number of neighbors in the other group. As q increases, there are more and more possibilities.

Example 6.10 illustrates that as long as the influence from two extreme anti-conformist groups remain balanced, the ordering consistency will hold, such that extreme anti-conformist agents stay extreme and conformist agents remain moderate.

Example 6.11. *$A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, $C = \{5, 6\}$, $\delta_i = 0.5$, $\sigma_i = 1, \forall i \in A$ and $\epsilon = 0.001$. Consider 6 agents situated in the following network (see Figure 6.18) with the corresponding matrix equal to*

$$\begin{array}{cc} & \begin{array}{cccccc} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} \end{array} \\ G = & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{array}{c} \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6}. \end{array} \end{array}$$

The opinion dynamics are shown in Figure 6.19 with the limit opinion equal to $x(\infty) =$

$(0, 0, 1, 1, 4/11, 5/11)$. *The weight matrix of the equivalent DeGroot model is*

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 0 & 1/5 & 1/5 \\ 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

Thus

$$\begin{cases} x_5 = 1/5 + 1/5x_5 + 1/5x_6 \\ x_6 = 1/4 + 1/4x_5 + 1/4x_6 \end{cases}$$

which yields

$$\begin{cases} x_5 = 4/11 \\ x_6 = 5/11. \end{cases}$$

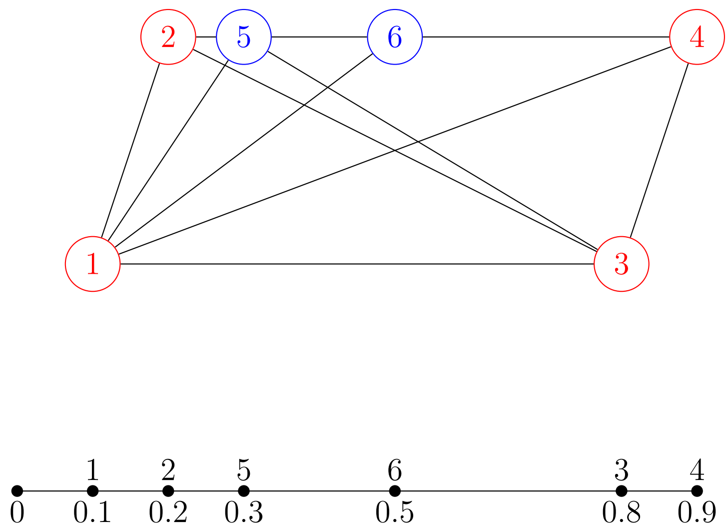


Figure 6.18: Network structure of Example 6.11

As one can see on Figure 6.19, the simulation confirms the theoretical result.

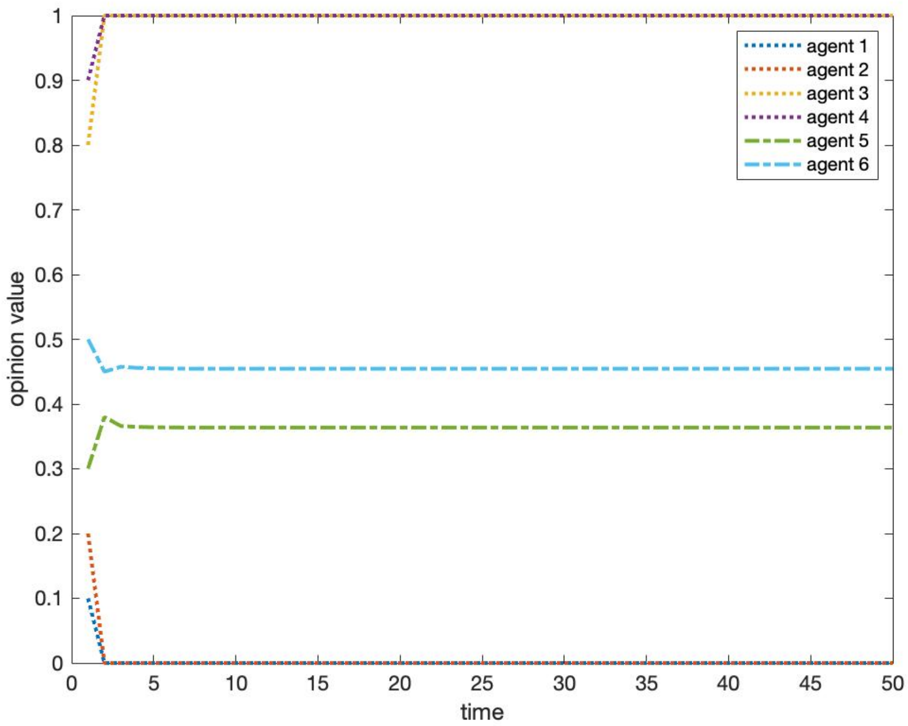


Figure 6.19: Opinion dynamics with anti-conformists of Example 6.11

6.3.2 Asynchronous updating model

In this section, at each time, one agent is chosen at random (with probability $1/n$) to update her opinion. Assume that agent i is active at time t. If i is conformist she will update her opinion as the average opinion of her neighbors, i.e., according to 6.5, while if she is anti-conformist, she will update her opinion according to 6.7, taking the average opinion of her neighbors as the referenced opinion, i.e.,

$$r_i^A(t) := \frac{1}{\eta_i - 1} \sum_{j \in N_i \setminus \{i\}} x_j(t), \forall i \in A. \tag{6.14}$$

Similar to the synchronous updating model, if there exist at least one anti-conformist agents in a connected network, then there is almost surely no consensus.

Fact 6.3. *There is almost surely no consensus for any connected network with $A \neq \emptyset$.*

Proof. By contradiction, suppose the society will reach a consensus, then $\exists t \in \mathbb{N}$, such that $x_i(t) = x^*, \forall i \in N$. Then for any anti-conformist agent i, $r_i(t) = x^*$.
Since each agent is chosen with probability $1/n > 0$, so almost surely there exist a time $t' > t$, such that an anti-conformist agent $i \in A$ is chosen to be active. By property (e) of the repelling function, $x_i(t+1)$ is either $x^* + \delta$ or $x^* - \delta$ which contradicts the definition of consensus.

□

Recall that in the synchronous updating model, if there are two connected anti-conformist agents $N = A = \{i, j\}$ holding similar opinions which are not around 0.5, then their opinions will oscillate with period 2; otherwise, they will form a disagreement. However, in the asynchronous updating model, the oscillation will not happen, and the two anti-conformist agents will for sure form a disagreement. Indeed, the oscillation in Proposition 6.5 is due to the synchronization of the opinions updates.

Proposition 6.5. *Assume that $\sigma_i = \sigma, \forall i \in N$. Consider two connected anti-conformist agents with initial opinions $x_1(0)$ and $x_2(0)$. Regardless of their initial opinions, they will form a disagreement in the end.*

Proof. It suffices to illustrate the case when $|x_1(0) - x_2(0)| < \epsilon$, $|x_1(0) - 0.5| \geq \epsilon$ and $|x_2(0) - 0.5| \geq \epsilon$, since for the other cases it will for sure go to a disagreement as in Proposition 6.1. Suppose agent $i, i = 1, 2$ is active at time 1, then $x_i(1)$ will jump with length δ, and $x_2(1) = x_2(0)$. Then $|x_1(1) - x_2(1)| > \epsilon$ falling into one of the other cases, and eventually a disagreement is reached.

□

If there is one anti-conformist agent connected with one conformist agent, then for any initial opinions in $[0, 1]^2$, the opinions do not converge but oscillate, which is in accordance with the synchronous model. Indeed, the asynchronization does not change the presence but the speed of the oscillations. Indeed, the oscillation in Proposition 6.2 is not caused by the synchronization of the opinions updates, but by the presence of both conformist and anti-conformist agents.

Proposition 6.6. *Consider the society consisting of two connected agents, with agent 1 conformist and agent 2 anti-conformist. Then there will be oscillations instead of a convergence of opinions. And for x_2, the oscillations are between 0 and 1.*

Proposition 6.7. *If there is only one anti-conformist agent, and this agent has at least one link to a set of connected conformist agents, i.e., $A = \{i\}$, and $N_i \cap C \neq \emptyset$, then opinions do not converge.*

Consider the case of anti-conformists being extremist and conformists being moderate, the same results will be obtained as in the synchronous model. The asynchronization does not change the limit behavior but makes the speed of the convergence slower.

Proposition 6.8. *Theorem 6.1 holds for asynchronous updating model with active agents observing opinion of all neighbors.*

Proof. The techniques used in the proof of Theorem 6.1 can also be applied to the case of the asynchronous model, since in both synchronous and asynchronous updating model, agents are following the same updating rules, and all the inequalities in the proof of Theorem 6.1 also hold here. □

6.4 Opinion dynamics over signed graphs

6.4.1 Synchronous updating model

In this section, we consider the synchronous updating where agents update opinions simultaneously following rules 6.10.

One widely-studied signed graph is the structurally balanced graph (see its general definition in Section 2.7). For any structurally balanced graphs, the agents can be partitioned into two groups, where agents connect to agents in the same group with positive links and connect to agents in the different group with negative links. So it can also be considered as two communitarian groups. We study first the opinion dynamics for two communitarian groups with disjoint initial opinions.

Definition 6.4. *We say that an undirected network $\mathcal{G} = (N, E)$ is connected, if $\forall i, j \in N$, there is a path from i to j, i.e., $i \leftrightarrow j$.*[8]

We are interested in under which conditions would opinions of agents converge if there are two communitarian groups with disjoint initial opinions, and will the order of the initial opinions between the two group be consistent over time.

Proposition 6.9 (Two communitarian groups with disjoint initial opinions)**.**
Consider a society $\mathcal{G} = (N, E)$ composed of two communitarian groups $N = G_1 \cup G_2$, that is, $g_{ij} \in \{1, 0\}$ if i and j are from the same group; $g_{ij} \in \{-1, 0\}$ otherwise. The sub-network (G_k, E_k) is a connected network where $E_k = \{\{i, j\} \in E \mid i \in G_k, j \in G_k\}, k = 1, 2$, i.e., $\forall i, j \in G_k, i \leftrightarrow j, k = 1, 2$. [9] *Assume that $\sigma_i = \sigma \in (0, 1), \forall i \in N$. If $x_i(0) + \epsilon < x^* < x_j(0) - \epsilon, \forall i \in G_1, \forall j \in G_2$, then the following will hold:*

Ordering consistency

$\forall t, \forall i \in G_1, \forall j \in G_2$, *it holds that* $x_i(t) < x_j(t)$.

Opinion convergence and consensus

$\forall i \in G_1, \forall j \in G_2$, $\exists x', x'' \in [0, 1]$ *and* $x' < x''$, *such that* $\lim_{t\to\infty} x_i(t) = x'$, $\lim_{t\to\infty} x_j(t) = x''$.

[8] Indeed, this is equivalent to the notion of strongly connectedness defined in Section 2.1 (see Definition 2.4).

[9] Note that we do not exclude the case where there is no link between G_1 and G_2.

Steady-state opinion

Moreover, if G_1 and G_2 are connected sets, then at least one of the following cases holds true:

(i) $x'' - x' \geq \sigma$;

(ii) $x' = 0$;

(iii) $x'' = 1$.

Proof. $\forall i \in N$, denote the neighborhood of agent i in G_k as $N_{i,k}$ and its cardinality as $\eta_{i,k}$, where $k = 1, 2$.
Fix any $i \in G_1$, then

$$x_i(1) = \left[\frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(0) + f\left(x_i(0), \frac{1}{\eta_{i,2}} \sum_{j \in N_{i,2}} x_j(0)\right)\right]_0^1.$$

Remark here that in case of $\eta_{i,2} = 0$, we adopt the convention that $f\left(x_i(0), \frac{1}{\eta_{i,2}} \sum_{j \in N_{i,2}} x_j(0)\right) = 0$, and that $\eta_{i,1} \geq 1$ since (G_1, E_1) is connected.
By $x_i(0) + \epsilon < x^* < x_j(0) - \epsilon, \forall i \in G_1, \forall j \in G_2$, $\frac{1}{\eta_{i,2}} \sum_{j \in N_{i,2}} x_j(0) > x_i(0) + \epsilon$, thus $f\left(x_i(0), \frac{1}{|g_2|} \sum_{j \in G_2} x_j(0)\right) \leq 0$. Moreover, $\frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(0) < x^* - \epsilon$. Therefore, $x_i(1) < x^* - \epsilon, \forall i \in G_1$.
Analogously, $x_j(1) > x^* + \epsilon, \forall j \in G_2$. So $x_i(t) + \epsilon < x^* < x_j(t) - \epsilon$ holds for any t, i.e., the ordering consistency is satisfied.

$$\forall i \in G_1, \forall t, r_i(t) = \frac{1}{\eta_{i,2}} \sum_{j \in N_{i,2}} x_j(t) > x^* + \epsilon > x_i(t) + \epsilon,$$

so $f(x_i(t), \frac{1}{\eta_{i,2}} \sum_{j \in N_{i,2}} x_j(t) \leq 0$ always holds. Thus

$$x_i(t+1) \leq \frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(t), \forall i \in G_1.$$

This implies $\frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(t+1) \leq \frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(t)$. Therefore the limit of the series $\{\frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(t)\}_t$ exists (i.e., $\exists x'_i$, such that $\lim\limits_{t \to \infty} \frac{1}{\eta_{i,1}} \sum\limits_{j \in N_{i,1}} x_j(t) = x'_i$) due to the weak monotonicity and boundedness. By $x_i(t+1) \leq \frac{1}{\eta_{i,1}} \sum_{j \in N_{i,1}} x_j(t), \forall i \in G_1$, it must hold that $\lim\limits_{t \to \infty} x_i(t) = x'_i, \forall i \in G_1$. Analogously, for all $i \in G_2$, $\exists x''_i$ such that $\lim\limits_{t \to \infty} x_i(t) = x''_i$ and $x'_i < x''_j, \forall i \in G_1, \forall j \in G_2$. Till now, the convergence of the opinion vector is guaranteed.
Denote the steady state opinions as x'_i and x''_j, $\forall i \in G_1, j \in G_2$, i.e., $\exists t^*$, such that $\forall t > t^*$, $x'_i(t) = x'_i$ and $x''_j(t) = x''_j$.

Then let us show the consensus is reached within each group. Due to the connectedness of each sub-network $(G_k, E_k), k = 1, 2$, it suffices to show that if any two agents i, j of the same group are directly connected, i.e., $i, j \in G_k, g_{ij} = 1, k = 1, 2$, then $x'_i = x'_j$. By contradiction, suppose that $x'_i \neq x'_j$, and w.l.o.g., assume that $x'_i < x'_j$, then $x'_j(t^* + 1) < x'_j(t^*)$ which leads to a contradiction. Hence the consensus is reached within each group. To complete the proof, it suffices to show that if G_1 and G_2 are connected, $x' \neq 0$ and $x'' \neq 1$, then $x'' - x' \geq \sigma$. By contradiction, suppose that $x'' - x' < \sigma$, then for any $i \in G_1$, $\exists j \in G_2$ such that i and j are connected, thus $x_i(t^* + 1)$ will be strictly less than $x'_i(t^*)$ which leads to a contradiction.

□

Example 6.12 (Signed graph with two communitarian groups)**.**
Consider two communitarian groups $G_1 = \{1, 2, 9, 10\}$ and $G_2 = \{3, 4, 5, 6, 7, 8\}$ situated in the following network (see Figure 6.20) with the corresponding matrix equal to

$$G = \begin{array}{c c} \begin{array}{cccccccccc} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} & \mathit{7} & \mathit{8} & \mathit{9} & \mathit{10} \end{array} & \\ \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix} & \begin{array}{c} \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \\ \mathit{7} \\ \mathit{8} \\ \mathit{9} \\ \mathit{10.} \end{array} \end{array} \tag{6.15}$$

$\delta = \sigma = 0.5$ and $\epsilon = 0.001$. The opinion dynamics are shown in Figure 6.21. Opinions within each group form a consensus very quickly and there is opinion ordering consistency in the group level. The steady state opinions are $^{11}/_{80}$ and $^{3}/_{4}$ for agents of G_1 and G_2, respectively.

However, all the properties that hold in Proposition 6.9 may fail when it is generalized to several communitarian groups with disjoint opinions. Let us consider a society $\mathcal{G} = (N, E)$ composed of several communitarian groups $N = G_1 \cup \ldots \cup G_k$, where $k > 2$ is an integer, that is, $g_{ij} \in \{1, 0\}$ if i and j are from the same group; $g_{ij} \in \{-1, 0\}$ otherwise. Let us

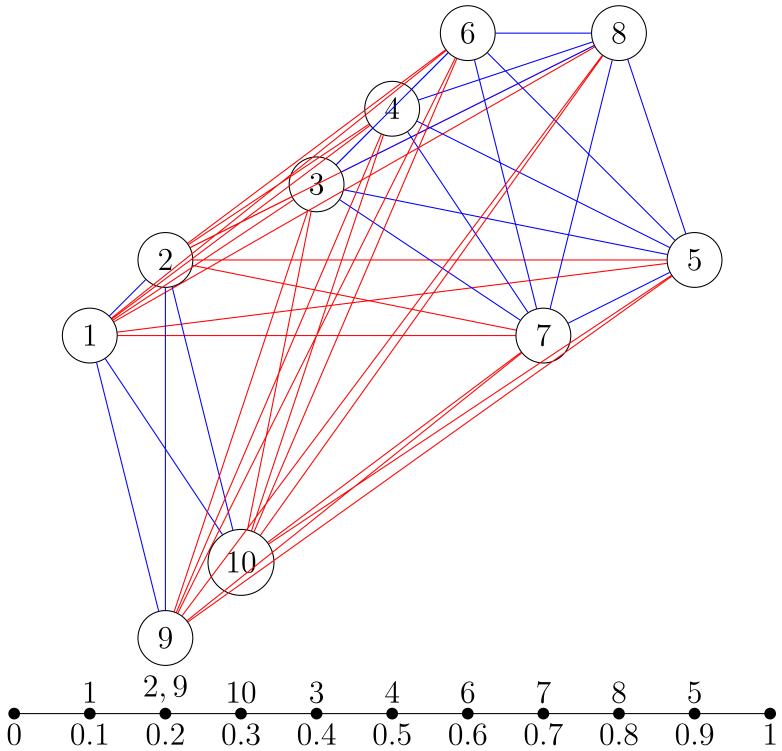


Figure 6.20: Network structure for two communitarian groups of Example 6.12

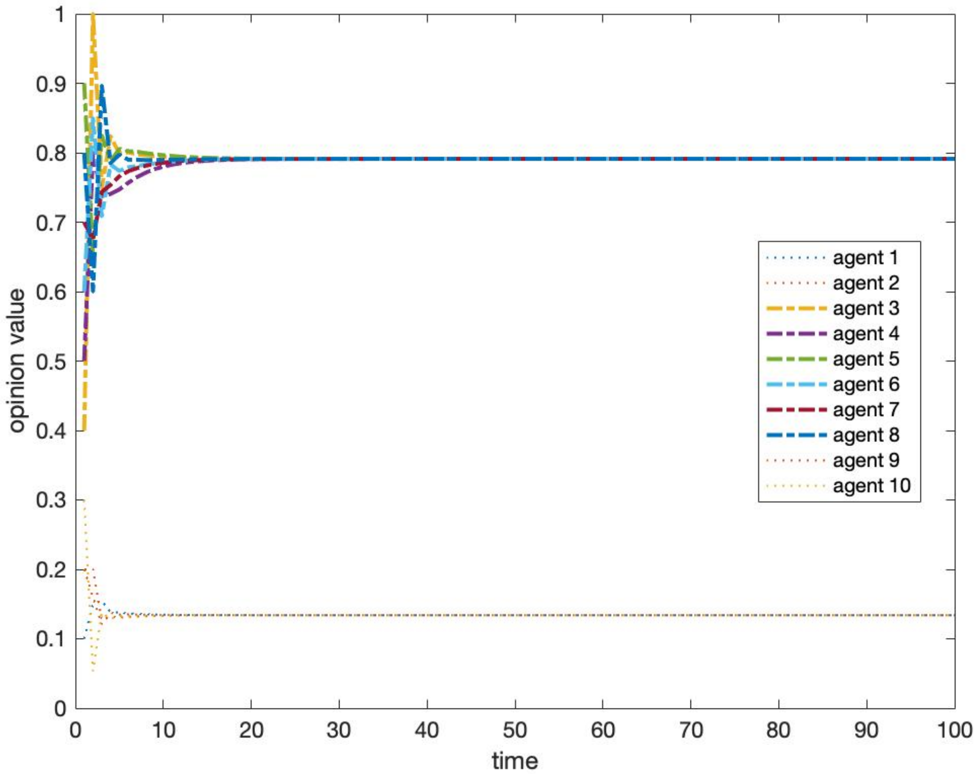


Figure 6.21: Opinion dynamics for two communitarian groups of Example 6.12

suppose that opinions of agents from distinct groups belong to disjoint opinion ranges, i.e., $\forall i \in G_p, \forall j \in G_{p+1}$, $x_i(0)+\epsilon < x_p^* < x_j(0)-\epsilon, p = 1, \ldots, k-1$. The following example will give some counter-evidence showing that the ordering consistency for all groups is no more true, and opinions of agents who belong to groups in the middle may oscillate or converge, depending on the values of σ and δ. A detailed study of all possible cases seems to be out of reach.

Example 6.13 (Signed graph with four communitarian groups).

Consider four communitarian groups $G_1 = \{1, 2\}$, $G_2 = \{3, 4\}$, $G_3 = \{5, 6\}$ *and* $G_4 = \{7, 8, 9, 10\}$ *situated in the following network (see Figure 6.22) with the corresponding matrix equal to*

$$G = \begin{array}{cl} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} & \\ \begin{bmatrix} 1 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 \\ -1 & 0 & 1 & 1 & -1 & 0 & 0 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & -1 \\ -1 & -1 & 0 & -1 & -1 & -1 & 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 1 & 1 \end{bmatrix} & \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10. \end{array} \end{array} \tag{6.16}$$

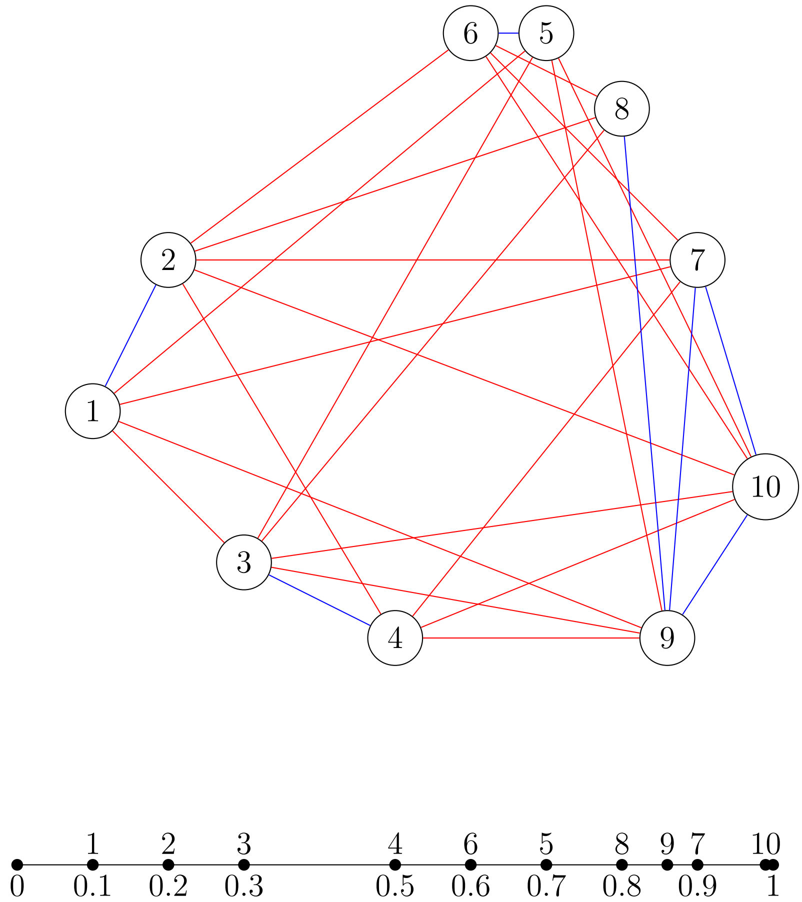


Figure 6.22: Network structure for two communitarian groups of Example 6.13

When $\delta = \sigma = 0.5$, the opinion dynamics are shown in Figure 6.23. Opinions of agents within groups G_1, G_2 and G_4 form a consensus very quickly, while opinions of agents in group G_3 oscillate.

If we fix $\delta = 0.5$, and decrease the value of σ to $\sigma = 0.2$, then opinions of agents within every group form a consensus, but the ordering consistency is no more true in this case since $x'_i > x'_j$, $\forall i \in G_3$, $\forall j \in G_2$ (see Figure 6.24). If we fix $\delta = 0.5$, and increase the value of σ to $\sigma = 0.7$, then opinions of agents within every group form a consensus, and the ordering consistency holds in this case (see Figure 6.25).

Instead, if we fix $\sigma = 0.5$, and increase the value of δ to $\delta = 0.9$, then similar to the case of $\delta = \sigma = 0.5$, opinions of agents within groups G_1, G_2 and G_4 form a consensus very quickly, but opinions of agents in group G_3 oscillate with a larger extent (see Figure 6.26).

If we fix $\sigma = 0.5$, and decrease the value of δ to $\delta = 0.3$, then again opinions of agents within every group form a consensus, but the ordering consistency fails in this case since $x'_i > x'_j$, $\forall i \in G_1$, $\forall j \in G_2$ (see Figure 6.27).

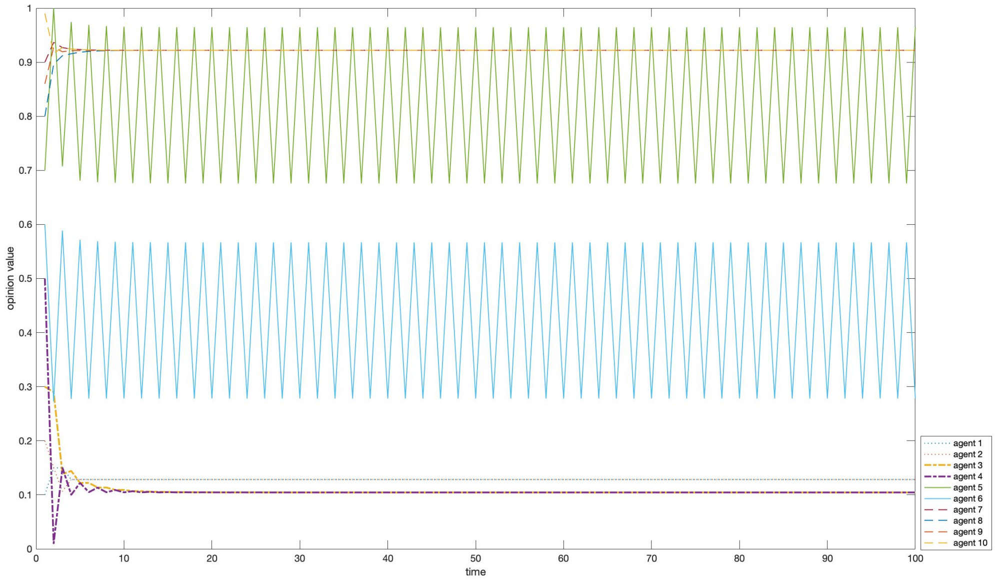


Figure 6.23: Opinion dynamics for two communitarian groups of Example 6.13, with $\delta = \sigma = 0.5$.

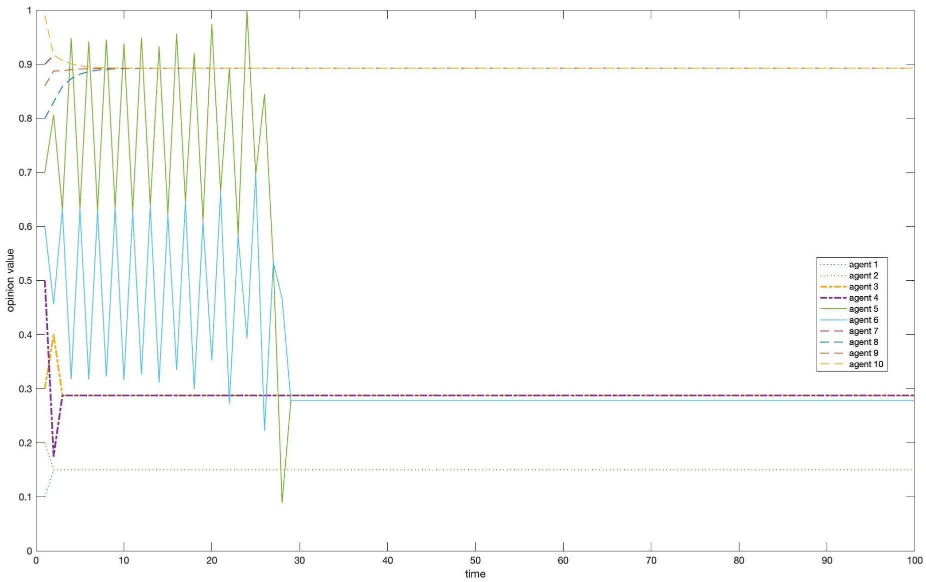


Figure 6.24: Opinion dynamics for two communitarian groups of Example 6.13, with $\delta = 0.5$ and $\sigma = 0.2$.

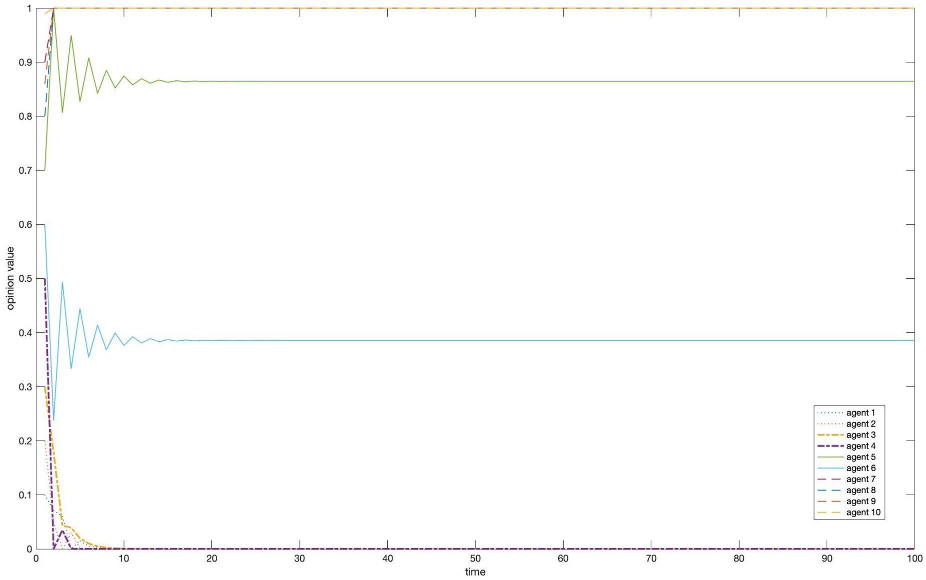


Figure 6.25: Opinion dynamics for two communitarian groups of Example 6.13, with $\delta = 0.5$ and $\sigma = 0.7$.

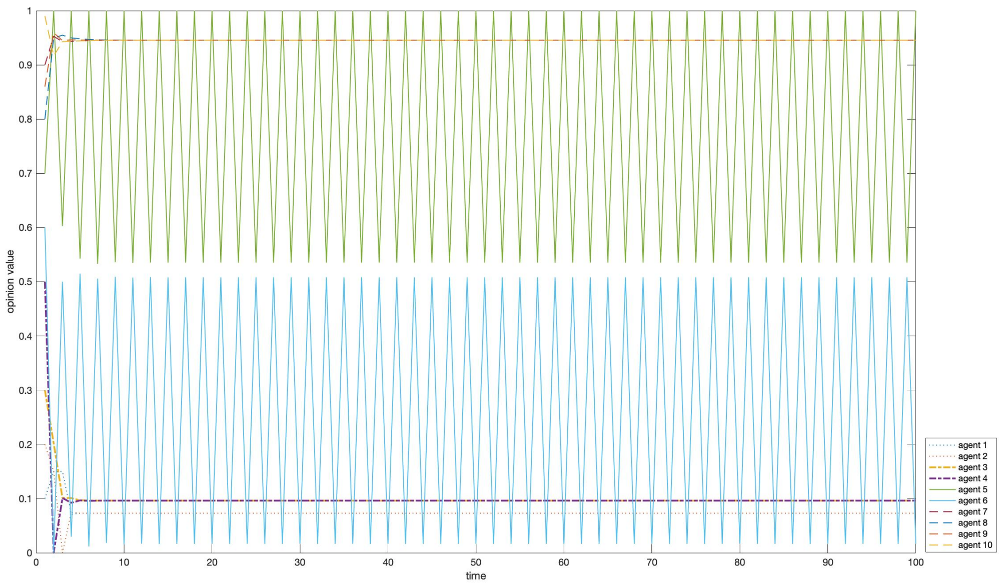


Figure 6.26: Opinion dynamics for two communitarian groups of Example 6.13, with $\delta = 0.9$ and $\sigma = 0.5$.

6.4.2 Asynchronous updating model

In this section, at each time, one agent is chosen at random (with probability $1/n$) to update her opinion according to equation 6.10, taking the average opinion of her neighbors as the

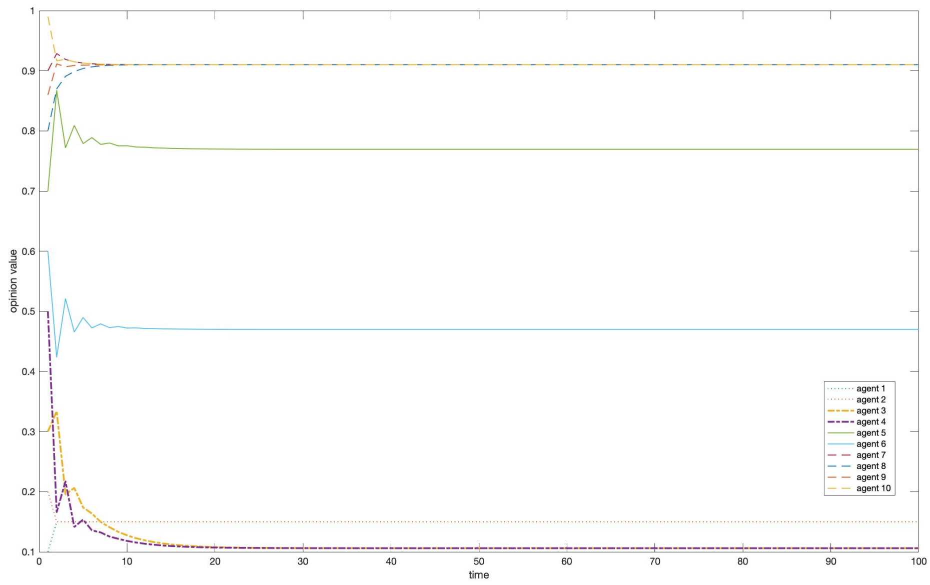


Figure 6.27: Opinion dynamics for two communitarian groups of Example 6.13, with $\delta = 0.3$ and $\sigma = 0.5$.

referenced opinion, i.e.,

$$r_i^A(t) := \frac{1}{\eta_i^-} \sum_{j \in N_i^-} x_j(t), \forall i \in A. \tag{6.17}$$

Consider the case of two communitarian groups with disjoint initial opinions.

Proposition 6.10. *Proposition 6.9 holds for asynchronous updating model with active agents observing opinion of all neighbors.*

Proof. The techniques used in the proof of Proposition 6.9 can also be applied to the case of the asynchronous model, since in both synchronous and asynchronous updating model, agents are following the same updating rules, and all the inequalities in the proof of Proposition 6.9 also hold here. □

Example 6.14.

Consider two communitarian groups $G_1 = \{1, 2, 9, 10\}$ and $G_2 = \{3, 4, 5, 6, 7, 8\}$ situated in the same network as in Example 6.12. $\delta = \sigma = 0.5$ and $\epsilon = 0.001$. Figures 6.28 and 6.29 are two realizations of the asynchronous opinion dynamics. Opinions within each group form a consensus very quickly and there is opinion ordering consistency in the group level. However, it is path-dependent and the value of the steady state opinion depends on the activation order of agents. The steady state opinions of agents in G_1 and G_2 are $[0.05, 0.61]$ (corresponding to Figures 6.28) and $0.2, 0.8$ (corresponding to Figures 6.29), respectively.

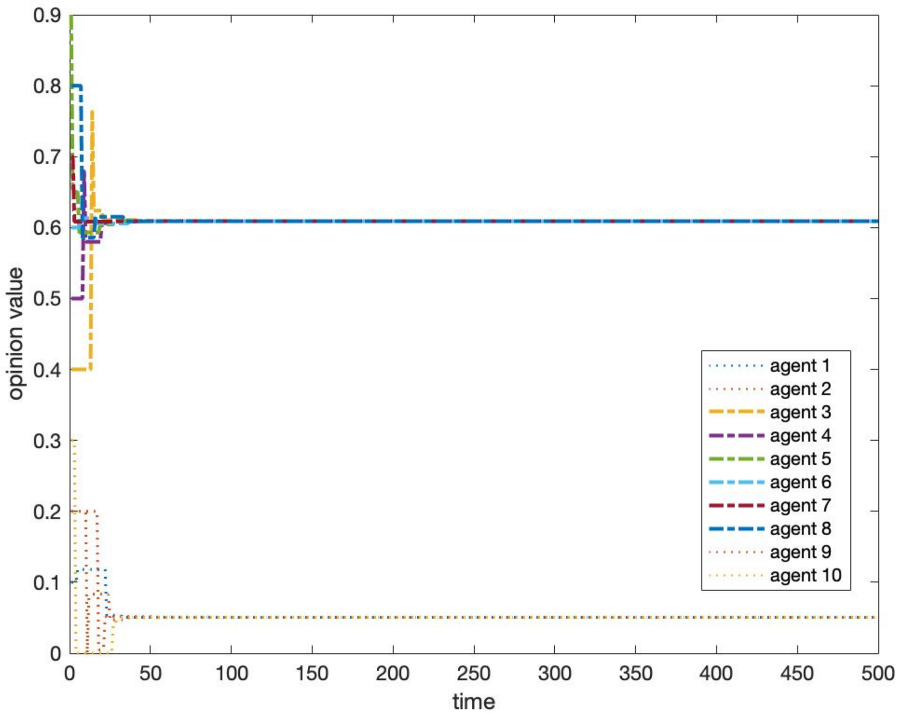


Figure 6.28: Realization 1 of the asynchronous opinion dynamics for two communitarian groups of Example 6.14. $\delta = \sigma = 0.5$ and $\epsilon = 0.001$.

Example 6.15. *Consider four communitarian groups* $G_1 = \{1, 2\}$, $G_2 = \{3, 4\}$, $G_3 = \{5, 6\}$ *and* $G_4 = \{7, 8, 9, 10\}$ *situated in the same network as in Example 6.13. Then the phenomenon of oscillations disappear in the asynchronous updating model. Instead, opinions always converge, regardless of the values of initial opinions and the values of* δ *and* σ*. The dynamic is path-dependent and the value of the steady state opinion depends on the activation order of agents.*

When $\delta = \sigma = 0.5$*, Figure 6.30 shows one realization of the opinion dynamics with the steady state opinion equal to* $[0.119, 0.119, 0.039, 0.039, 0.802, 0.411, 0.915, 0.915, 0.915, 0.915]$.

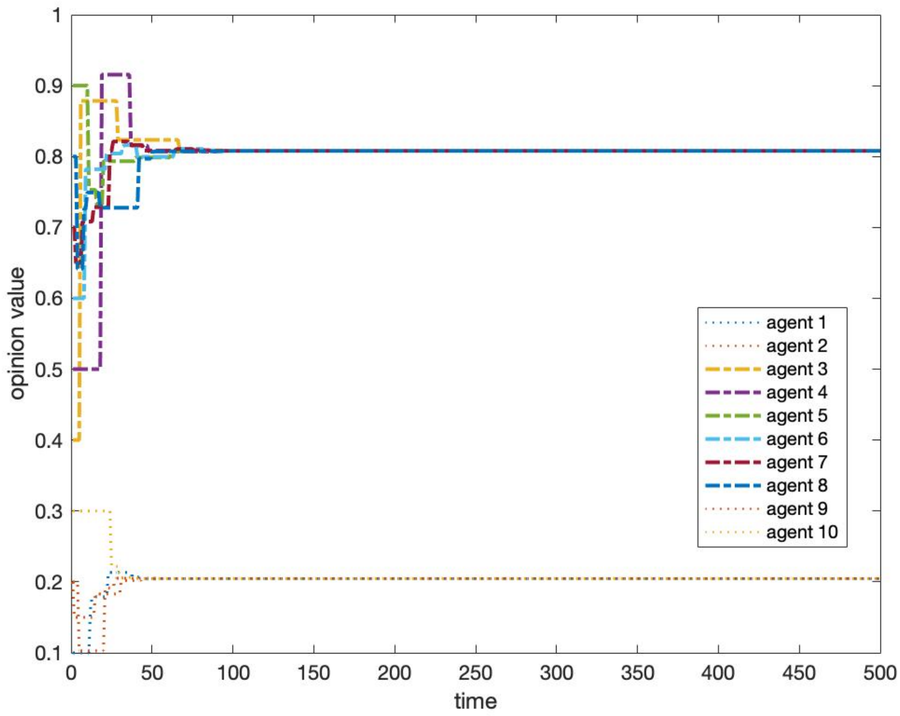


Figure 6.29: Realization 2 of the asynchronous opinion dynamics for two communitarian groups of Example 6.14. $\delta = \sigma = 0.5$ and $\epsilon = 0.001$.

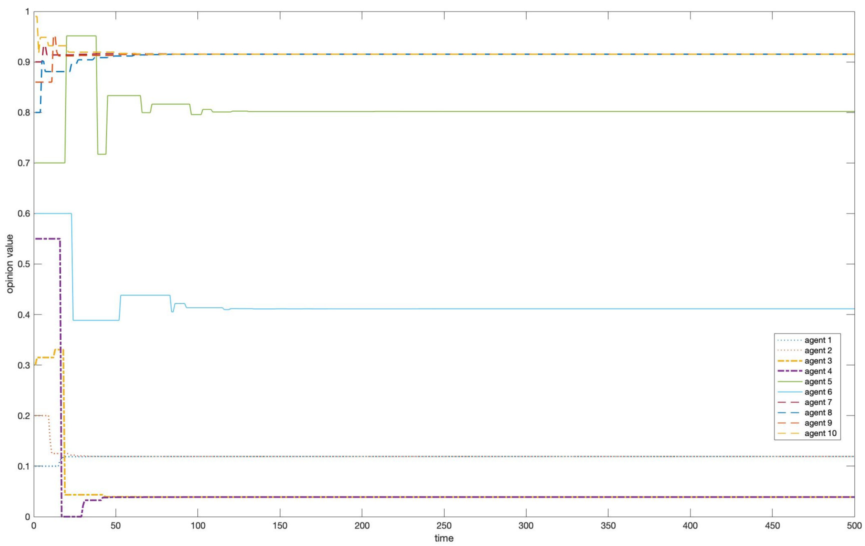


Figure 6.30: Opinion dynamics for two communitarian groups of Example 6.13, with $\delta = \sigma = 0.5$.

6.5 Concluding remarks

This paper proposes two models of continuous opinion dynamics in undirected networks, by introducing the heterogeneity either into nodes or into links in the sense of conformity and anti-conformity behavior. We propose an appropriate updating rule of continuous opinions for anti-conformity behavior, defined according to the repelling function, which gives the shift of the opinion based on the current opinion and the reference opinion for an agent. Both synchronous and asynchronous opinion updates are studied in these two models. In synchronous models, all agents are assumed to have the opinion updates simultaneously, while in asynchronous models, an agent is chosen at random to be active following the same updating rule as in synchronous models, and the opinions of other inactive agents stay unchanged.

In the first part of the paper, the model of opinion dynamics is studied with both conformist and anti-conformist agents. Conformist agents update opinions according to the DeGroot rule with equal weights on her neighbors, while anti-conformist agents deviate from the average opinions of her neighbors. For any connected network, consensus will never be reached as long as the set of anti-conformist agents is nonempty in both synchronous and asynchronous models. Instead, opinions of agents oscillate or converge to a disagreement, which is more common in real life. When one anti-conformist agent is connected to a set of connected conformist agents, then opinions do not converge regardless of the value of initial opinion. Instead, we see from the simulations that opinions are periodically oscillating between 0 and 1 for the anti-conformist agent, and between ϵ' and $1 - \epsilon'$ (where $\epsilon' < \epsilon$) for the conformist agents symmetrically in synchronous models, in the sense that the increasing process from 0 (resp., ϵ') to 1 (resp., $1 - \epsilon'$) of opinions is symmetric to the decreasing process from 1 (resp., $1 - \epsilon'$) to 0 (resp., ϵ'). Although the non-convergence and oscillations also hold for asynchronous models, the periodicity and symmetry disappear, since the dynamics of the asynchronous updates become path-dependent on the activation order of the agents. For the case of anti-conformist agents being extremist and conformist agents being moderate, under mild conditions on the initial opinions and the number of neighbors in each group (such that the influence from two extreme groups are balanced), the ordering consistency and opinion convergence hold, so anti-conformist agents remain extremist and conformist agents continue to be moderate. The exact value of the steady-state opinion is given when $\sigma = 1$, i.e., when the repelling interval is maximal. This result holds for both synchronous and asynchronous opinion updates.

In the second part of the paper, the model of opinion dynamics is studied over signed graphs with both positive and negative influence. We show that for two communitarian groups (i.e., structurally balanced graphs) with disjoint initial opinions, if each sub-network (i.e., each communitarian group) is connected, then the ordering consistency and opinion convergence holds, and opinions of agents from the same group form a consensus, for both synchronous and asynchronous opinion updates. In addition, the steady-state opinion is characterized for the case that two groups are connected. However, when considering more than two communitarian groups and synchronous updating, all the properties such as ordering consistency, opinions convergence may fail, and oscillation can sometimes happen, depending on the values of δ and σ. In the asynchronous updating model, ordering consistency may fail, but opinions always converge, regardless of the values of initial opinions and the values of δ and σ. But the steady-state opinion is not pre-determined, since the dynamics are also path-dependent and the activation order of agents matters.

If we exclude the case when the oscillations are only caused by the synchronization like in the example of two anti-conformist agents with the same opinions, in general, the asynchronization of opinions updates do not change the convergence behavior of the opinions but the speed of the dynamics. This is also in accordance with the results of Ramazi et al. (2016), emphasizing that the asynchrony does not lead to cycles or non-convergence. In other words, the asynchrony preserves the convergence property of the corresponding synchronous models, and sometimes even turns the oscillation behavior shown in the synchronous model into convergence.

The updating rule of continuous opinions proposed in this paper is flexible and appropriate for modeling anti-conformity behavior for the following reasons. First of all, the driving force of anti-conformity urges agents (anti-conformist agents or agents with negative influences) to repel from the reference opinion, which can take various forms depending on the context. For example, it is the average opinion of agents in one's neighborhood in the CODA-node model and the average opinion of one's enemies in the CODA-link model. It can also be the average opinion of agents from the entire society in the fashion context since everyone including the anti-conformist agent can easily obtain relevant information about fashion trend due to the popularity of the Internet. When modeling from a game-theoretic point of view, it can also take value of the average opinion of agents from a certain group. Secondly, the results presented in the current paper apply to different forms of the repelling functions such as, but not limited to, convex, concave or linear functions as shown in the examples. Thirdly, the deviation of the opinion for an agent is increasing as the reference opinion becomes closer to her current opinion, it implies the idea of always distancing the others. This sometimes causes oscillations such as in the very simple case of one anti-conformist agent linked to one (or a set of connected) conformist agent(s) in the CODA-node model, the anti-conformist agent playing the role of a leader (e.g., of a certain fashion), always followed by the conformist agent(s). Recall that opinions are defined in the close interval $[0, 1]$ where the two boundaries are referred to as the two extreme opinions. Once the anti-conformist agent reached some extreme opinion, she would lead the others back to some mild opinions, which can explain well the fashion fluctuations. Opinions can also reach equilibria. For example, in the case of two groups of anti-conformist agents holding relatively extreme opinions and conformist agents holding relatively moderate opinions, opinions converge when the influences from two different groups of anti-conformist agents are balanced. Moreover, as each anti-conformist agent has her own repelling interval $[-\sigma_i, \sigma_i]$ within which she is influenced, her new neighborhood at $t+1$ would be $\{j \in N : |x_j(t+1) - x_i(t+1)| < \sigma_i\}$. Thus the CODA-node model can also be seen as a coevolution model of networks and opinions.

Even though models of opinion dynamics with anti-conformity behavior is drawing attention in recent years, mainly on binary opinions, the study of continuous opinions with anti-conformity behavior still requires to be developed. To the best of the author's knowledge, there are two related works on continuous opinion dynamics. As mentioned in Section 6.1, Buechel et al. (2015) assumed that each agent is assigned with a level of conformity, measured by a parameter taking value in $[-1, 1]$, where -1 and 1 refer to full anti-conformity and full conformity, respectively. By contrast, in the current paper, we suppose no continuum in between conformity and anti-conformity. Another work is given by Altafini in several papers, who focused on the opinion dynamics with negative influences. As shown by Example 6.5, this model is not appropriate for describing anti-conformity behavior. Moreover, the study of opinion dynamics with anti-conformity behavior also requires the empirical and experimental support. Future studies include testing the new updating rule with some actual data, introducing strategic network formation into the model, and considering directed networks instead of a fixed undirected network supposed in the current paper.

Chapter 7

Conclusions and Future Work

7.1 Thesis Summary and Contributions

This thesis studied different models of opinion dynamics in networks, especially emphasizing the influence of introducing the role of anti-conformity behavior.

In Chapter 4, a detailed study of convergence of the threshold model is provided, in which both conformist and anti-conformist agents are included. Firstly, a deterministic threshold model is studied supposing a fixed complete graph, where every one is connected to every one, like in the seminal work of Granovetter (Granovetter (1978)). It represents a connected society where every agent is informed about the number of agents being in state 1 or 0 (active or inactive) at the present time, through media, etc. It is to be noted that no other information about the society is possessed by an agent, e.g., if there are anti-conformists and how many. A game-theoretic foundation has been given for the threshold mechanism with anti-conformists, using coordination and anti-coordination games. The remarkable result is that the presence of anti-conformists causes the appearance of much more absorbing states, and cycles of length greater than 2 (when only (anti-)conformist agents are present cycles can only be of length 2). In detail, a complete and exact study is performed when the distribution of threshold is uniform, generalizing the results of Granovetter (1978). The case of a Gaussian distribution has also been studied, in which we showed the existence of unstable fixed points and limit cycles of length 2. However, for the case of an arbitrary distribution, it is possible to find cycles of length greater than 2. Secondly, the graph is no more complete and we suppose that the neighborhood of an agent is random, drawn at each time step from a distribution. It represents a society communicating via social networks like Facebook or Twitter, receiving randomly messages from other agents indicating their states. Here also, a given agent has no information on the type of her neighbor (conformist or anti-conformist). The introduction of randomness

causes a variety of absorbing classes to appear: polarization, periodic classes of more or less complex structure, and chaos, i.e., any state of the society can be reached. When thresholds are randomly distributed, we have shown that for most societies, the opinion converges to a chaotic situation (every state is possible).
One most remarkable finding of Chapter 4 is that the presence of anti-conformists introduces instability in the process, causing a multiplicity of absorbing states and a variety of cycles, periodic classes and chaos. Also, the model is highly sensitive, e.g., in the number of anti-conformists, the threshold values, etc. For example, it has been seen in the case of a uniform threshold distribution that introducing or deleting only one anti-conformist agent changes the convergence from a stable state to a cyclic behavior or vice versa. Another most remarkable finding is that in the case of a random neighborhood, the process converges to chaos for most values of the parameters defining the society (e.g., if there are more conformists and more anti-conformists than the size of a smallest neighborhood). Otherwise, cascades may occur, e.g., in the case of fixed thresholds for conformists and anti-conformists. This shows that introducing a small proportion of anti-conformists in a society may lead, not only to chaotic situations, but also to *permanent* opinion reversal.

In Chapter 5, we study the dynamics of continuous cultural traits (as a specific type of continuous opinions) in an OLG (overlapping generation) structure and in an endogenous social network, where the network changes are inherited. Children learn their cultural trait from their parents and their social environment modelled by network. Parents want their children to adopt a cultural trait that is similar to their own and engage in the socialization process of their children by forming new links or deleting connections. However, changing links from the inherited network is costly, but having many links is beneficial. We proposed three ways to endogenize the process of network formation. In the first one, the network is supposed to be directed and each dynasty can either form or delete a directed link unilaterally with another dynasty. Therefore, at each period, each family faces a utility opitimization problem where a trade-off between own utility losses and the improvements of child's cultural trait. We have shown that if the cost of network changes is greater than the cost of child care, extremists will never add links, and in the case of sufficiently low cost, extremists may cut all ties with the society. In the second and third models, the network is supposed to be undirected. In the second model, we assume that after each period, a pairwise stable network with transfers (PST network for short) is reached, i.e., $\forall t \in \mathbb{N}$, $G(t+1)$ is a PST network for $G(t)$ and $V(t)$. We have shown the existence of the PST network for each period, however, it is not necessary to be unique, evidenced by a counterexample. Moreover, a necessary and sufficient condition is given such that a network is PST for given $V(t)$ and $G(t)$. The convergence of cultural traits is guaranteed. In the third model, we assume that after each period, a pairwise stable network (PS network for short) is reached, i.e., $\forall t \in \mathbb{N}$, $G(t+1)$ is a PS network for $G(t)$ and $V(t)$. In this case, there always exist sufficiently small cost parameters such that the empty network is the unique PS network. Regarding the efficiency of the network, we show that there always exist sufficiently small cost parameters such that the empty network is the unique efficient network, and sufficiently large costs of child care such that the complete network is the unique efficient network.

The dynamics of these three models are studied by both analytics and simulations. For sufficiently small costs of network changes and child care, extremists will disconnect from the other dynasties and there will be a long term heterogeneity of the society. Specially, in the first and the third model, we show that the network will converge to the empty network. While in the second model, the network might or might not converge to the empty network, since the PST network is not unique in each period. For large costs of network changes and child care, it converges to a homogeneous society such that all dynasties have the same cultural trait in the limit. This give us some insights on how to reduce extremism in our real life. For example, one can consider to foster the interaction of children with different cultural backgrounds such that the cost of network change is increased (extremists will less probably disconnect with others). Some work can also be done to increase value of integration (i.e., increase the benefits from relations). Extremists play an important role in the dynamical process, policy makers should take it into account and provide more opportunities for extremists to connect with others.

In Chapter 6, an appropriate updating rule of continuous opinions for anti-conformity behavior is proposed, defined according to the repelling function, which gives the shift of the opinion based on the current opinion and the reference opinion for an agent. Two models of continuous opinion dynamics are studied in undirected networks, by introducing the heterogeneity in the sense of conformity and anti-conformity behavior either in nodes or in links. Both synchronous and asynchronous opinion updates are studied in these two models. In synchronous models, all agents are assumed to have the opinion updates simultaneously, while in asynchronous models, an agent is chosen at random to be active following the same updating rule as in synchronous models, and the opinions of other inactive agents stay unchanged.

In the first part of the paper, the model of opinion dynamics is studied with both conformist and anti-conformist agents. Conformist agents update opinions according to the DeGroot rule with equal weights on her neighbors, while anti-conformist agents deviate from the average opinions of her neighbors, and the repelling level is negatively related to the opinion distance between the anti-conformist and her reference point. For any connected networks, the consensus will never be reached as long as the set of anti-conformist agents is nonempty in both synchronous and asynchronous models. Instead, opinions of agents oscillate or converge to a disagreement, which is more common in real life. For the case of anti-conformist agents being extremist and conformist agents being moderate, under mild conditions on the initial opinions and the number of neighbors in each group (such that the influence from two extreme groups are balanced), the ordering consistency and opinion convergence hold, so anti-conformist agents remain extremist and conformist agents continue to be moderate. The exact value of the steady-state opinion is given when $\sigma = 1$, i.e., when the repelling interval is maximal. This result holds for both synchronous and asynchronous opinion updates. In the second part of the paper, the model of opinion dynamics is studied over signed graphs where agents have positive links (+1) with their friends and negative links (-1) with their enemies. Agents update their opinion as the sum of the averaged opinion of their friends and repelling value from their enemies. When the network is balanced, i.e., there are two communitarian groups, and each sub-network corresponding to each group is connected and the initial opinion ranges of the two group are disjoint, the ordering consistency and opinion convergence holds, and opinions of agents from the same group form a consensus, for both synchronous and asynchronous opinion updates. In addition, the steady-state opinion is characterized for the case that two groups are connected.

The new updating rule of continuous opinions proposed in this paper is flexible and appropriate for modeling anti-conformity behavior due to the flexibility of choosing the reference opinion and the flexibility of the forms of the repelling function, and the good approximation of the anti-conformity behavior of distancing the aggregate opinions. Regarding the comparison of synchronization and asynchronization, if we exclude the case when the oscillations are only caused by the synchronization like in the example of two anti-conformist agents with the same opinions, in general, the asynchronization of opinions updates do not change the convergence behavior of the opinions but the speed of the dynamics. This is also in accordance with the results of Ramazi et al. (2016), emphasizing that the asynchrony does not lead to cycles or non-convergence. In other words, the asynchrony preserves the convergence property of the corresponding synchronous models, and sometimes even turns the oscillation behavior shown in the synchronous model into convergence. However, even though the behaviors of non-convergence and oscillations may be preserved for asynchronous models, the periodicity and symmetry may disappear, e.g., in the example where one anti-conformist agent is connected to a set of connected conformist agents, due to the path-dependence on the activation order of the agents in the asynchronous model. Moreover, although the behavior of convergence is preserved for asynchronous models, the value of the steady-state opinion might be unique (e.g., Theorem 6.1) or not (Proposition 6.9), also due to the the path-dependence on the activation order of the agents.

7.2 Future work

Future work on modeling opinion dynamics includes a number of different paths. Although studies on opinion dynamics are drawing more and more attention in recent years, research on modelling opinion dynamics with anti-conformity behavior is still taking the first step, and it requires very much the empirical and experimental support. Thus the first important future direction is to carry out some experiments to figure out what are the social response due to anti-conformity behavior and how it functions, excluding the influence of all the other factors.

The second is to study the opinion dynamics with anti-conformity behavior in directed networks, as in the current thesis (i.e., in Chapter 4 and Chapter 6) we study mainly on undirected networks. Some special networks such as star network, small-world network are also of great interest.

The third one is considering time-varying networks, as most of networks are changing over time in real life. We have studied convergence of cultural traits in the model with pairwise stable networks with transfers in Chapter 5, while for the other two models, it is quite challenging. What are the convergence conditions with time-varying networks is also an important question to be figured out in the field of opinion dynamics modelling. Future work is to find some general conditions on the networks to guarantee the convergence of opinions.

Introducing memory into the models of opinion dynamics is also of great interest. Memories influence agent's opinion and decisions (Kahneman (2003)). Recently, some researchers incorporated the notion of memory into binary opinion dynamics, mostly, into the voter model and its extensions, e.g., Stark et al. (2008), Xiong and Liu (2011), Woolcock et al. (2017). Jędrzejewski and Sznajd-Weron (2018) studied the impact of memory on the q-voter models of opinion dynamics, where agents can be conforming or independent, based on past awards memories. One possible generalization of the two models in Chapter 4 and Chapter 6 is assuming that agents are flexible to change their characteristics, between conforming and anti-conforming, and introducing memory on this basis.

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