Center for Mathematical Economics
Working Papers

645

March 2021

Multidimensional Singular Control and Related Skorokhod Problem: Sufficient Conditions for the Characterization of Optimal Controls

Jodi Dianetti and Giorgio Ferrari

Center for Mathematical Economics (IMW)
Bielefeld University
Universitätsstraße 25
D-33615 Bielefeld · Germany

e-mail: imw@uni-bielefeld.de
bielefeld.de/zwe/imw/research/working-papers/
ISSN: 0931-6558

MULTIDIMENSIONAL SINGULAR CONTROL AND RELATED SKOROKHOD PROBLEM: SUFFICIENT CONDITIONS FOR THE CHARACTERIZATION OF OPTIMAL CONTROLS

JODI DIANETTI AND GIORGIO FERRARI

Abstract. We characterize the optimal control for a class of singular stochastic control problems as the unique solution to a related Skorokhod reflection problem. The considered optimization problems concern the minimization of a discounted cost functional over an infinite time-horizon through a process of bounded variation affecting an Itô-diffusion. The setting is multidimensional, the dynamics of the state and the costs are convex, the volatility matrix can be constant or linear in the state. We prove that the optimal control acts only when the underlying diffusion attempts to exit the so-called waiting region, and that the direction of this action is prescribed by the derivative of the value function. Our approach is based on the study of a suitable monotonicity property of the derivative of the value function through its interpretation as the value of an optimal stopping game. Such a monotonicity allows to construct nearly optimal policies which reflect the underlying diffusion at the boundary of approximating waiting regions. The limit of this approximation scheme then provides the desired characterization. Our result applies to a relevant class of linear-quadratic models, among others. Furthermore, it allows to construct the optimal control in degenerate and non degenerate settings considered in the literature, where this important aspect was only partially addressed.

Keywords: Dynkin games, reflected diffusion, singular stochastic control, Skorokhod problem, variational inequalities.

AMS subject classification: 93E20, 60G17, 91A55, 49J40.

1. Introduction

This paper considers the problem of characterizing optimal policies for singular stochastic control problems in multidimensional settings. More precisely, we consider the problem of controlling, through a one-dimensional càdlàg (i.e., right-continuous with left limits) process v with locally bounded variation, the first component of a multidimensional diffusion with initial condition x. Namely, the controller can affect a state process $X^{x;v}$ which evolves according to the equation

$$dX_t^{x;v} = b(X_t^{x;v})dt + \sigma(X_t^{x;v})dW_t + e_1 dv_t,\ t \geq 0, \quad X_{0-}^{x;v} = x, \tag{1.1}$$

for a multidimensional Brownian motion W, a suitable convex Lipschitz function b, and a volatility matrix σ, which is either constant or linear in the state. The aim of the controller is to minimize the expected discounted cost

$$J(x;v) := \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X_t^{x;v})dt + \int_{[0,\infty)} e^{-\rho t} d|v|_t\bigg], \tag{1.2}$$

for a given convex function h and a suitable discount factor $\rho > 0$. Here, $|v|$ denotes the total variation of the process v. The value function V of the problem is defined, at any given initial condition x, as the minimum of $J(x;v)$ over the choice of controls v. Also, a control $\bar{v}$ is said to be optimal for x if $J(x;\bar{v}) = V(x)$. Existence of optimal controls can be proved

Date: March 15, 2021.

in very general frameworks using different probabilistic compactification methods (see, e.g., [11, 18, 34, 47, 52]).

Natural questions that immediately arise are whether it is possible to characterize V, and how one should act on the system in order to obtain the minimal cost V. As a matter of fact, the Markovian nature of the problem together with mild regularity and growth conditions on b and h, allows to employ the dynamic-programming approach. This leads to the characterization of the value function as a solution (in a suitable sense) to the Hamilton-Jacobi-Bellman equation

$$\max\{\rho V - bDV - \operatorname{tr}(\sigma\sigma^\top D^2 V)/2 - h, |V_{x_1}| - 1\} = 0. \tag{1.3}$$

This equation provides key insights on the way the controller should act on the system in order to minimize the cost of her actions. Indeed, when V is sufficiently regular, an application of Itô's formula suggests that the controller should make the state process not leaving the set $\mathcal{W} := \{|V_{x_1}| < 1\}$, usually referred to as the waiting region. In fact, in many examples (see, e.g., [23, 32, 44, 45, 50, 57, 62], among others) it is possible to construct the optimal control as the solution to a related Skorokhod reflection problem; that is, the optimal control can be characterized as that process $\bar{v}$, with minimal total variation, which is able to keep the process $X^{x;\bar{v}}$ inside the closure of the waiting region $\mathcal{W}$, by reflecting it in a direction prescribed by the gradient of the value function. However, in multidimensional settings, such a characterization often remains a conjecture (see the discussion in Chapter 6 in [58], Remark 5.2 in [8], and also [15, 16, 26, 27]), and many questions about the properties of optimal controls remain open, representing a strong limitation to the theory.

We now discuss more in detail the problem of the characterization of optimal rules. When the state process is one dimensional, optimal controls can be explicitly constructed as Skorokhod reflections in a general class of models in [1, 22, 38, 39, 50, 61], among others. Also, in the (non necessarily Markovian) one dimensional case, a similar characterization of optimal controls has been achieved in [2, 3, 4], without relying on the dynamic-programming approach. When the dimension of the problem becomes larger than one, the difficulty of characterizing optimal controls drastically increases. Indeed, classical results on the existence of solutions to the Skorokhod reflection problem in the multidimensional domain $\mathcal{W}$ require some regularity of the boundary of $\mathcal{W}$ and of the direction of reflection, which are, in most of the cases, unknown. When the value function V is convex, this difficulty is overcome in some specific settings. A celebrated example is presented in [57], where the problem of controlling a two-dimensional Brownian motion with a two-dimensional process of bounded variation is considered. There, the authors show that the boundary of the waiting region (the so-called free boundary) is of class C^2, and they are therefore able to construct the optimal policy as a solution to the associated Skorokhod problem. The problem of the characterization is also encountered in [15, 16, 26, 27], where the construction of the optimal control can be provided only by requiring additional properties on the boundary of the waiting region. Another example is exhibited in [22], in which the case of controlling a multidimensional Brownian motion with a multidimensional control is considered in the case of a radial running cost $h(x) = |x|^2$. We also refer to [44], where the construction of the optimal policy is provided in a two-dimensional context in which the drift is non-zero. To the best of our knowledge, in the case of a convex V, the most general multidimensional setting in which this characterization is shown is presented in [45], and in its finite time-horizon counterpart [9]. There, the problem of controlling a multidimensional Brownian motion with a multidimensional control is considered for a convex running cost. Remarkably, in [45] (and in [9]) the author presents an approach which allows to construct the unique optimal policy as a solution to the related Skorokhod problem bypassing the problems related to the regularity of the free boundary. In non-convex settings, the number of contributions are even rarer. The suitable regularity of the boundary of $\mathcal{W}$ is shown, in two-dimensional settings, in [32] and in [23], while a multidimensional case is considered in [62], via a connection with Dynkin games. We also mention that the construction of multidimensional reflected diffusions in polyhedral domains has been recently studied in [19, 31, 33], in the context of games with singular controls. To conclude, despite many decades of research in the field, the nature of optimal controls is, in general, far from being completely understood, and this motivates our study.

In this paper, we provide sufficient conditions for the characterization of the optimal policy of the singular control problem specified by (1.1) and (1.2) as the solution to the related Skorokhod reflection problem. Despite in our setting the control is one dimensional, the multidimensional nature of the problem arises from the fact that the components of the state process are interconnected; in particular, the action of the controller on the first component of the state process can affect all the other components. We will show the claimed characterization under two main classes of assumptions in which the volatility matrix is constant or linearly dependent on the state. In both cases additional monotonicity assumptions are enforced to the running cost h and to the drift b. These structural conditions are satisfied in a relevant class of linear-quadratic models, and in some specific settings considered in the literature for which the problem of constructing the optimal control remained partially open (see [15, 16, 26, 27]). The strategy of our proof is inspired by [45] and can be resumed in three main steps.

(1) We first derive important monotonicity properties on V_{x_1}. This is done by identifying V_{x_1} as the value of a related Dynkin game, through a variational formulation in the spirit of [15].
(2) We construct solutions $\bar{v}^\varepsilon$ to a family of Skorokhod problems in domains $\mathcal{W}_\varepsilon$ *approximating* $\mathcal{W}$. Here the monotonicity of V_{x_1} plays a crucial role in order to show the regularity of $\mathcal{W}_\varepsilon$. The controls $\bar{v}^\varepsilon$ are ε-optimal for (1.2); i.e. $J(x;\bar{v}^\varepsilon) \leq V(x) + \varepsilon$.
(3) We find a control $\bar{v}$ such that $\bar{v}^\varepsilon \to \bar{v}$, as $\varepsilon \to 0$. This implies that $\bar{v}$ is optimal for x, and, thanks to the properties of $\bar{v}^\varepsilon$, that $\bar{v}$ solves the Skorokhod problem on the original domain $\mathcal{W}$. This then provides the desired characterization of the optimal policy $\bar{v}$.

As a consequence of our result, some works (in particular [15] and [62]) in the literature on singular control can be revisited, and the characterization of optimal controls can provided under slightly different assumptions. Also, our approach allows to treat the singular control problems with degenerate diffusion matrix studied in [26, 27]. The results apply to problems with monotone controls, and to the case in which increasing the underlying diffusion has a different cost than decreasing it. The approach presented in this paper seems to be suitable to treat also singular control problems in the finite time-horizon.

Clearly, our results relate to stochastic differential equations (SDEs, in short) with reflecting boundary conditions, also known as Skorokhod reflection problems for SDEs. In this field, existence and uniqueness of strong solutions to reflected SDEs in convex time-independent domains with normal reflection was first shown in the seminal [60]. These results were then generalized to non-convex smooth domains with smooth oblique reflection in [48], and subsequently refined in [55]. Existence of strong solutions in a class of non-smooth domains has been proved in [24], and therefore generalized to the time-dependent case in [49]. This list is, however, far from being exhaustive, and we therefore refer the interested reader to [12, 13, 20, 21, 53, 59] and to the references therein. From this point of view, our results provide existence and uniqueness of a (strong) solution to a Skorokhod problem in which the domain is given by the noncoincidence set $\mathcal{W}$ of the solution of the variational inequality with gradient constraint (1.3), and in which the reflection direction is prescribed its gradient.

An essential tool for our analysis is the connection between optimal stopping and singular stochastic control theory. This connection is known since the seminal [5], where the authors observed that the derivative of the value function of a singular control problem identifies with the value of an optimal stopping problem. Since then, this connections has been elaborated through different approaches (see [6, 8, 42], among others), until the more recent interpretation given in [47]. When the control is assumed to be of locally bounded variation, and the system has dynamics with independent components, with one of them being controlled, the space derivative of the value function of the control problem coincides with the value of a zero-sum game of stopping; i.e., a Dynkin game (cf. [7, 15, 16, 32, 43]). This connection was described in a multi-dimensional setting with interconnected dynamics in [15] and [16] by employing a variational formulation of the problem. In this paper, we employ essentially the formulation and the techniques elaborated in [15], however extending their arguments to fit our convex setting.

The rest of this paper is organised as follows. In Section 2 we formulate the problem, we enforce some structural conditions, and we state the main result of this paper. The proof of the main result for a constant volatility is presented in Section 3, while the proof for a linear volatility is discussed in Section 4. Extensions and examples are provided in Section 5, while Appendix A and Appendix B are devoted to some auxiliary technical results.

1.1. **Notation.** For $d \in \mathbb{N}$ with $d \geq 1$, an open set $B \subset \mathbb{R}^d$, $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ and a funciton $f : B \to \mathbb{R}$, we denote by $D^\alpha f := D_1^{\alpha_1} ... D_d^{\alpha_d} f$ the weak derivative of f, where $D_i f := f_{x_i} := \partial f / \partial x_i$, and we set $|\alpha| := \alpha_1 + ... + \alpha_d$. For $\ell \in \mathbb{N}$, $q \in [1, \infty]$, and a measure space $(E, \mathcal{E}, m)$, we define the spaces:

- $L^q(E) := \{\text{measurable } f : E \to \mathbb{R} \text{ s.t. } \|f\|_{L^q(E)} < \infty\}$, where $\|f\|_{L^q(E)} := \int_E |f|^q dm$ if $q < \infty$, and $\|f\|_{L^\infty(E)} := \operatorname{ess\,sup}_E f$ for $q = \infty$;
- $C^\ell(B) := \{f : B \to \mathbb{R} \text{ with continuous } \ell\text{-order derivatives}\}$ and $C_c^\infty(B) := \{f : B \to \mathbb{R} \text{ with compact support, s.t. } f \in C^\ell(B) \text{ for each } \ell \in \mathbb{N}\}$;
- $C^{\ell;1}(B) := \{f : B \to \mathbb{R} \text{ with } \|f\|_{C^{\ell;1}(B)} < \infty\}$, where $\|f\|_{C^0(B)} := \sup_{x \in B} |f(x)|$, $\|f\|_{\text{Lip}\,(B)} := \sup_{x,y \in B} |f(y) - f(x)|/|y - x|$, and $\|f\|_{C^{\ell;1}(B)} := \sum_{|\alpha| \leq \ell} \|D^\alpha f\|_{C^0(B)} + \sum_{|\alpha| = \ell} \|D^\alpha f\|_{\text{Lip}\,(B)}$;
- $W^{\ell;q}(B) := \{f \in L^q(B) \text{ with } \|f\|_{W^{\ell;q}(B)} < \infty\}$, $W^{\ell;q}_{loc}(B) := \{f \,|\, f \in W^{\ell;q}(D) \text{ for each bounded open set } D \subset B\}$, and $W^{\ell;q}_0(B)$ as the closure of $C_c^\infty(B)$ in the norm $\|\cdot\|_{W^{\ell;q}(B)}$, where $\|f\|_{W^{\ell;q}(B)} := \sum_{|\alpha| \leq \ell} \|D^\alpha f\|_{L^q(B)}$.

For $x \in \mathbb{R}^d$ we denote by $x^\top$ the transpose of x. The vector $e_i \in \mathbb{R}^d$ indicates the i-th element of the canonical basis of $\mathbb{R}^d$ and, for $x \in \mathbb{R}^2$ and $R > 0$, set $B_R(x) := \{y \in \mathbb{R}^d \,|\, |y - x| < R\}$. Finally, in this paper C indicates a generic positive constant, which may change from line to line.

2. Problem formulation and main result

2.1. **Singular control and Skorokhod problem.** Fix $d \in \mathbb{N}$, $d \geq 2$, and a d-dimensional Brownian motion $W = (W^1, ..., W^d)$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions. For each $x = (x_1, ..., x_d) \in \mathbb{R}^d$, let the process $X^x = (X^{1,x}, ..., X^{d,x})$ denote the solution to the stochastic differential equation (SDE, in short)

$$(2.1) \qquad \begin{cases} dX_t^{1,x} = (a_1 + b_1^1 X_t^{1,x})dt + \bar{\sigma}(X_t^{1,x})dW_t^1, & t \geq 0, \quad X_{0-}^{1,x} = x_1, \\ dX_t^{i,x} = b^i(X_t^{1,x}, X_t^{i,x})dt + \bar{\sigma}(X_t^{i,x})dW_t^i, & t \geq 0, \quad X_{0-}^{i,x} = x_i, \quad i = 2, ..., d. \end{cases}$$

Here a_1, b_1^1 are constants, while the coefficients $b^i \in C(\mathbb{R}^2)$ and $\bar{\sigma} \in C(\mathbb{R})$ are deterministic Lipschitz continuous functions. The drift $\bar{b}(x) := (a_1 + b_1^1 x_1, b^2(x_1, x_2), .., b^d(x_1, x_d))^\top$ and the function $\bar{\sigma}$ satisfy Assumption 2.1 below. Next, introduce the set of *admissible controls* as

$$\mathcal{V} := \{\mathbb{R}\text{-valued } \mathbb{F}\text{-adapted and càdlàg processes with locally bounded variation}\},$$

and, for each $v \in \mathcal{V}$ and $x \in \mathbb{R}^d$, let the process $X^{x;v} = (X^{1,x;v}, ..., X^{d,x;v})$ denote the unique strong solution to the controlled stochastic differential equation

$$\begin{cases} dX_t^{1,x;v} = (a_1 + b_1^1 X_t^{1,x;v})dt + \bar{\sigma}(X_t^{1,x;v})dW_t^1 + dv_t, & t \geq 0,\ X_{0-}^{1,x;v} = x_1, \\ dX_t^{i,x;v} = b^i(X_t^{1,x;v}, X_t^{i,x;v})dt + \bar{\sigma}(X_t^{i,x;v})dW_t^i, & t \geq 0,\ X_{0-}^{i,x;v} = x_i,\ i = 2, ..., d. \end{cases} \tag{2.2}$$

For any given initial condition $x \in \mathbb{R}^d$, consider the problem of minimizing the expected discounted cost

$$J(x; v) := \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X_t^{x;v})dt + \int_{[0,\infty)} e^{-\rho t} d|v|_t \bigg], \quad v \in \mathcal{V}, \tag{2.3}$$

where $|v|$ denotes the total variation of the process v, $h : \mathbb{R}^d \to \mathbb{R}$ is a continuous function, and $\rho > 0$ is a constant discount factor. We will say that the control $\bar{v} \in \mathcal{V}$ is optimal if

$$V(x) := \inf_{v \in \mathcal{V}} J(x; v) = J(x; \bar{v}), \tag{2.4}$$

and, in the following, we will refer to the function V as to the value function of the problem, and to $X^{x;\bar{v}}$ as to the optimal trajectory.

The second integral appearing in (2.3) has to be understood in the Lebesgue-Stieltjes sense, and it is defined as

$$\int_{[0,\infty)} e^{-\rho t} d|v|_t := |v|_0 + \int_0^\infty e^{-\rho t} d|v|_t,$$

in order to take into account possible jumps of the control at time zero. Moreover, for $v \in \mathcal{V}$ we will often write $dv = \gamma d|v|$ to denote the disintegration

$$v_t = \int_0^t \gamma_s d|v|_s, \quad \text{for each } t \geq 0,\ \mathbb{P}\text{-a.s.},$$

where $|v|$ denotes the total variation of the signed measure v, and the process γ is the Radon-Nikodym derivative of the signed measure v with respect to $|v|$. Also, for a control v, the nondecreasing càdlàg processes ξ^+, ξ^- will denote the minimal decomposition of the signed measure v; that is, $v = \xi^+ - \xi^-$, and $\xi^+ \leq \bar{\xi}^+$ and $\xi^- \leq \bar{\xi}^-$ for any other couple of nondecreasing càdlàg processes $\bar{\xi}^+$, $\bar{\xi}^-$ which satisfy $v = \bar{\xi}^+ - \bar{\xi}^-$.

Finally, recall from [45] the following notion of solution to the Skorokhod problem, which we adapt to our setting.

Definition 1. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$ with closure $\overline{\mathcal{O}}$, $x \in \overline{\mathcal{O}}$, and set $S := \partial\mathcal{O}$. Let $\bar{\nu}$ be a continuous vector field on S, with $\bar{\nu} = e_1\nu$ and $|\nu(y)| = 1$ for each $y \in S$.*

We say that the process $v \in \mathcal{V}$ is a solution to the modified Skorokhod problem for the SDE (2.2) in $\overline{\mathcal{O}}$ starting at x with reflection direction $\bar{\nu}$ if

(1) $\mathbb{P}[X_t^{x;v} \in \overline{\mathcal{O}}, \forall t \geq 0] = 1$;

(2) $\mathbb{P}$*-a.s., for each $t \geq 0$ one has $dv = \gamma d|v|$, with*

$$|v|_t = \int_0^t \mathbb{1}_{\{X_{s-}^{x;v} \in S,\, \nu(X_{s-}^{x;v}) = \gamma_s\}} d|v|_s;$$

(3) $\mathbb{P}$*-a.s., for each $t \geq 0$, a possible jump of the process $X^{x;v}$ at time t occurs on some interval $I \subset S$ parallel to the vector field $\bar{\nu}$; i.e., such that $\bar{\nu}(y)$ is parallel to I for each $y \in I$. If $X^{x;v}$ encounters such an interval I, it instantaneously jumps to its endpoint in the direction $\bar{\nu}$ on I.*

Moreover, if v is continuous , then we say that v is a solution to the (classical) Skorokhod problem for the SDE (2.2) in $\overline{\mathcal{O}}$ starting at x with reflection direction $\bar{\nu}$.

2.2. **Assumptions and main result.** The main objective of this paper is to characterize optimal control policies for Problem (2.4) as solutions to related Skorokhod problems.

We will prove our main result under the following structural conditions, which we enforce throughout the rest of this paper. We postpone the discussion of some generalizations to Section 5.

Assumption 2.1. *For $p \geq 2$ we have:*

(1) *The running cost h is $C^{2;1}(\mathbb{R}^d)$, convex, and, for suitable constants $K, \kappa_1, \kappa_2 > 0$, it satisfies, for each $x, y \in \mathbb{R}^d$ and for all $\lambda \in [0,1]$, the conditions*

$$\kappa_1 |x_1|^p - \kappa_2 \leq h(x) \leq K(1+|x|^p),$$
$$|h(y) - h(x)| \leq K(1+|x|^{p-1}+|y|^{p-1})|y-x|,$$
$$\lambda h(x) + (1-\lambda)h(x) - h(\lambda x + (1-\lambda)y) \leq K\lambda(1-\lambda)(1+|x|^{p-2}+|y|^{p-2})|x-y|^2,$$
$$0 < h_{x_1 x_1}(x).$$

(2) *There exists a constant $\bar{L} \geq 0$ such that, for each $x, y \in \mathbb{R}^d$, we have*

$$|\bar{b}(x)| \leq \bar{L}(1+|x|),$$
$$|\bar{b}(y) - \bar{b}(x)| \leq \bar{L}|y-x|.$$

The functions b^i are convex of class $C^{2;1}(\mathbb{R}^d)$. Furthermore, we assume that $h_{x_i} \geq 0$ and $b^i_{x_1}$, $b^i_{x_1 x_i}$, $h_{x_1 x_i} \leq 0$ for each $i = 2, ..., d$, and that $D\bar{b}$ is globally Lipschitz.

(3) *For $\rho^* := p(2p-1)$ and a constant $\sigma > 0$, either of the two conditions below is satisfied:*
   (a) *$\bar{\sigma}(y) = \sigma,\ y \in \mathbb{R}$, and the discount factor satisfies the relation $\rho > 3\rho^* \bar{L}$.*
   (b) *$\bar{\sigma}(y) = \sigma y,\ y \in \mathbb{R}$, and the discount factor satisfies the relation $\rho > 2\rho^*(\bar{L} + \sigma^2(\rho^* - 1))$. In this case, we also assume that there exists $x_1^* > 0$ such that $h_{x_1}(x) \leq \min\{0, -b_1^1\}$ for each x with $x_1 < 2x_1^*$, that $b^i(x_1, x_i) \geq 0$ for $x_1, x_i \geq 0$ for each $i = 2, ..., d$, and that $a_1 \geq 0$.*

Natural examples in which the conditions above are satisfied are given –after discussing generalizations of Assumption 2.1– in Section 5. These include a relevant class of *linear-quadratic* singular stochastic control problems (see Example 1 and Subsection 5.4 below). Notice that the nature of problem (2.4) is genuinely multidimensional, as the components of the dynamics (2.2) are interconnected.

Remark 2.2 (On the role of Assumption 2.1)**.** *We underline that the particular choice of $p \geq 2$ is motivated by quadratic running costs (cf. Example 1 in Section 5). From Condition 2 one can see that quite strong requirements are needed in order to treat models with a general b^i. However, when b^i has a simpler form, some conditions on the derivatives $b^i_{x_1}$, $b^i_{x_1 x_i}$, h_{x_i}, $h_{x_1 x_i}$ can be weakened (see Subsections 5.1.1 and 5.1.2). Also, the assumption on h_{x_1} in Condition 3b is to enforce that the optimal trajectories live in the set $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d |\, x_i > 0,\ i = 1, ..., d\}$, whenever the initial condition $x \in \mathbb{R}^d_+$ (cf. Lemma 4.1 below). This condition is a natural substitute, for minimization problems in dimension $d \geq 2$, of the classical Inada condition at 0 (see, e.g., equation (2.5) in [30]). The latter, is typically assumed in profit maximization problems, and it is satisfied by Cobb-Douglas production functions. Finally, the conditions on the discount factor ρ are in place in order to ensure a suitable "integrability" of the optimal trajectories, which allows to prove some semiconcavity estimates for the value function V (see steps 2 and 3 in the proof of Theorem A.1 in Appendix A).*

Observe that, when Condition 3a is in place, a generic controlled trajectory $X^{x;v}$, $v \in \mathcal{V}$, can reach the whole space with probability $\mathbb{P} > 0$. On the other hand, under Condition 3b, as mentioned in Remark 2.2, the natural domain for a controlled trajectory is $\mathbb{R}^d_+$. This suggest to define a domain D in the following way

$$D := \mathbb{R}^d \text{ if Condition 3a holds}, \quad D := \mathbb{R}^d_+ \text{ if Condition 3b holds}. \tag{2.5}$$

Indeed, it is possible to show that the value function V is finite and it is a convex solution in $W^{2;\infty}_{loc}(D)$ of the Hamilton-Jacobi-Bellman (HJB, in short) equation

$$\max\{\rho V - \mathcal{L}V - h, |V_{x_1}| - 1\} = 0, \quad \text{a.e. in } D, \tag{2.6}$$

where $\mathcal{L}V(x) := \bar{b}(x)DV(x) + \frac{1}{2}\sum_{i=1}^d \bar{\sigma}^2(x_i)V_{x_i x_i}(x)$, $x \in D$, is the generator of the uncontrolled SDE (2.1). For completeness, a proof of this result is provided in Appendix A (see Theorem A.1). During the proof of Theorem A.1, the convergence of a certain penalization method is studied: This convergence will be a useful tool in many of the proofs in this paper.

Define next the *waiting region* $\mathcal{W}$ as

$$\mathcal{W} := \{x \in D \,|\, |V_{x_1}(x)| < 1\}, \tag{2.7}$$

and notice that, by the $W^{2;\infty}_{loc}$-regularity of V, $\mathcal{W}$ is an open subset of D. Also, for each $z \in \mathbb{R}^{d-1}$, we define the sets

$$D_1(z) := \{y \in \mathbb{R} \,|\, (y,z) \in D\} \quad \text{and} \quad \mathcal{W}_1(z) := \{y \in \mathbb{R} \,|\, (y,z) \in \mathcal{W}\}.$$

In the sequel, the closure of $\mathcal{W}$ (resp. $\mathcal{W}_1(z)$) in D (resp. $D_1(z)$) will be denoted by $\overline{\mathcal{W}}$ (resp. $\overline{\mathcal{W}}_1(z)$). We state here a technical lemma, whose proof is given in Appendix B.

Lemma 2.3. *For any $x = (x_1, z) \in D$, with $z \in \mathbb{R}^{d-1}$, the set $\mathcal{W}_1(z)$ is a nonempty open interval; in particular, $\mathcal{W}$ is nonempty.*

Remark 2.4 (Existence and uniqueness of optimal controls)**.** *For each $\bar{x} \in D$, it is possible to show that, under Assumption 2.1, there exists a unique optimal control $\bar{v} \in \mathcal{V}$. This is a classical result when the drift is affine. In the case of a convex drift, it essentially follows from the convexity of J w.r.t. (x, v). The latter in turn follows from the convexity of the drift, the monotonicity of h, and a comparison principle for SDEs. The argument can be recovered from the proof of Lemma 3.7 below, which works for any sequence of controls minimizing the cost functional J. Finally, the uniqueness of the optimal control is a consequence of the strict convexity of h in the variable x_1.*

The following is the main result of our paper, characterizing the optimal policies in terms of the waiting region $\mathcal{W}$ and the derivative V_{x_1} in the sense of Definition 1.

Theorem 2.5. *Let $\bar{x} = (\bar{x}_1, \bar{z}) \in D$, with $\bar{z} \in \mathbb{R}^{d-1}$. The following statements hold true:*

(1) If $\bar{x} \in \overline{\mathcal{W}}$, then the optimal control $\bar{v}$ is the unique solution to the modified Skorokhod problem for the SDE (2.2) in $\overline{\mathcal{W}}$ starting at $\bar{x}$ with reflection direction $-V_{x_1}e_1$;

(2) If $\bar{x} \notin \overline{\mathcal{W}}$, then the optimal control $\bar{v}$ can be written as $\bar{v} = \bar{y}_1 - \bar{x}_1 + \bar{w}$, where $\bar{y}_1$ is the metric projection of $\bar{x}_1$ into the set $\overline{\mathcal{W}}_1(\bar{z})$, and $\bar{w}$ is the unique solution to the modified Skorokhod problem for the SDE (2.2) in $\overline{\mathcal{W}}$ starting at $\bar{y} := (\bar{y}_1, \bar{z})$ with reflection direction $-V_{x_1}e_1$.

In Section 3 we provide a proof of Theorem 2.5 under Condition 3a in Assumption 2.1. The strategy of the proof can be resumed in three main steps:

Step a. In Subsection 3.1 we study an important monotonicity property of V_{x_1}, through a connection with Dynkin games.

Step b. In Subsection 3.2, this property will allow us to construct ε-optimal policies as solutions to Skorokhod problems in domains $\overline{\mathcal{W}}_\varepsilon$ approximating $\overline{\mathcal{W}}$.

Step c. Finally, in Subsection 3.3 we prove that the ε-optimal policies approximate the optimal policy, and that the latter is a solution to the Skorokhod problem in the original domain $\overline{\mathcal{W}}$.

The proof of Theorem 2.5 under Condition 3b in Assumption 2.1 follows similar rationales, and it is discussed in Section 4. In particular, in Subsections 4.1 a preliminary lemma is proved, while in Subsection 4.2 we show how to use this lemma in order to repeat (with minor modifications) the arguments of Section 3.

3. Proof of Theorem 2.5 for constant volatility

In this section we assume that Condition 3a in Assumption 2.1 holds. To simplify the notation, the proof is given for $d = 2$, so that $D = \mathbb{R}^2$. The generalization to the case $d > 2$ is straightforward.

3.1. **Step a: A connection to Dynkin games and the monotonicity property.**

In this subsection we adopt an approach based on the variational formulation of the problem in order to show, in the spirit of [15], a connection between the singular control problem (2.4) and a Dynkin game. This connection will enable us to prove a monotonicity property of V_{x_1}, which will be then fundamental in order to construct ε-optimal controls.

3.1.1. *The related Dynkin game.* We begin by characterizing V_{x_1} as a $W^{2;\infty}_{loc}$-solution to a *two-obstacle problem.* The proof of the next result borrows arguments from [15] (see in particular Theorem 3.9, Proposition 3.10, and Theorem 3.11 therein). However, since in our case b can be convex, the techniques used in [15] needs to be refined, and used along with suitable estimates (described more in detail in the proof of Theorem A.1 in Appendix A) on a penalization method. We provide a detailed proof for the sake of completeness.

Theorem 3.1. *The function V_{x_1} is a $W^{2;\infty}_{loc}(\mathbb{R}^2)$-solution to the equation*

$$\max\{(\rho - b^1_1)V_{x_1} - \mathcal{L}V_{x_1} - \hat{h}, |V_{x_1}| - 1\} = 0, \quad \text{a.e. in } \mathbb{R}^2, \tag{3.1}$$

where $\hat{h} := h_{x_1} + b^2_{x_1}V_{x_2}$.

Proof. We organize the proof in two steps.

Step 1. In this step we show that the function V_{x_1} is a solution to a variational inequality with a local operator, and that $V_{x_1} \in W^{2;\infty}_{loc}(\mathbb{R}^2)$. Fix $B \subset \mathbb{R}^2$ open bounded and consider a nonnegative localizing function $\psi \in C^\infty_c(B)$. Define the sets

$$\mathcal{K} := \{U \in W^{1;2}_{loc}(\mathbb{R}^2) \,|\, |U| \leq 1 \text{ a.e.}\} \quad \text{and} \quad \mathcal{K}_\psi := \{\psi U \,|\, U \in \mathcal{K}\}.$$

We show in the sequel that the function $W := V_{x_1}\psi$ is a solution in $\mathcal{K}_\psi$ to the variational inequality

$$A_B(W, U - W) \geq \langle \hat{H}, U - W\rangle_B, \quad \text{for each } U \in \mathcal{K}_\psi, \tag{3.2}$$

where $\hat{H} := \hat{h}\psi - V_{x_1}\mathcal{L}\psi - DV_{x_1}D\psi$, the operator $A_B : W^{1;2}(B) \times W^{1;2}(B) \to \mathbb{R}$ is given by

$$A_B(\bar{U}, U) := \frac{\sigma^2}{2}\sum_{i=1}^{2}\langle \bar{U}_{x_i}, U_{x_i}\rangle_B - \langle \bar{b}D\bar{U}, U\rangle_B + (\rho - b^1_1)\langle \bar{U}, U\rangle_B \quad \text{for each } \bar{U}, U \in W^{1;2}(B),$$

and $\langle \cdot, \cdot\rangle_B$ denotes the scalar product in $L^2(B)$.

Let us begin by introducing a family of penalized versions of the HJB equation (2.6). Let $\beta \in C^\infty(\mathbb{R})$ be a convex nondecreasing function with $\beta(r) = 0$ if $r \leq 0$ and $\beta(r) = 2r - 1$ if

$r \geq 1$. For each $\varepsilon > 0$, let V^ε be defined as in (A.2). As in Step 1 in the proof of Theorem A.1 in Appendix A, V^ε is a C^2-solution to the partial differential equation

$$\rho V^\varepsilon - \mathcal{L}V^\varepsilon + \frac{1}{\varepsilon}\beta((V^\varepsilon_{x_1})^2 - 1) = h, \quad x \in \mathbb{R}^2. \tag{3.3}$$

It is possible to show (see Step 2 in the proof of Theorem A.1 in Appendix A) that, for each $R > 0$, there exists a constant C_R such that

$$\sup_{\varepsilon \in (0,1)} \|V^\varepsilon\|_{W^{2;\infty}(B_R)} \leq C_R. \tag{3.4}$$

Moreover (as in (A.18) in the proof of Theorem A.1), as $\varepsilon \to 0$, on each subsequence we have:

$$\begin{aligned} &(V^\varepsilon, DV^\varepsilon) \text{ converges to } (V, DV) \text{ uniformly in } B_R; \\ &D^2V^\varepsilon \text{ converges to } D^2V \text{ weakly in } L^2(B_R). \end{aligned} \tag{3.5}$$

We now show that $V_{x_1} \in \mathcal{K}$. Since the $W^{1;2}_{loc}$-regularity of V_{x_1} is already known (cf. Theorem A.1 in Appendix A), we only need to show that $|V_{x_1}| \leq 1$ in $\mathbb{R}^2$. To this end, take $R > 0$ and observe that, by (3.4) and (3.3), we have

$$\sup_{\varepsilon \in (0,1)} \|\beta((V^\varepsilon_{x_1})^2 - 1)\|_{L^2(B_R)} \leq C_R \varepsilon, \tag{3.6}$$

where the constant $C_R > 0$ does not depend on ε. Moreover, unless to consider a larger C_R, by the estimate (3.4) and the definition of β, we also have the pointwise estimate

$$|\beta((V^\varepsilon_{x_1})^2 - 1)| \leq 2((V^\varepsilon_{x_1})^2 + 1) \leq C_R, \quad \text{on } B_R, \text{ for each } \varepsilon \in (0,1). \tag{3.7}$$

Therefore, the limits in (3.5) and the estimates (3.7) allow to invoke the dominated convergence theorem to deduce, thanks to (3.6), that

$$\|\beta((V_{x_1})^2 - 1)\|_{L^2(B_R)} = \lim_{\varepsilon \to 0} \|\beta((V^\varepsilon_{x_1})^2 - 1)\|_{L^2(B_R)} = 0.$$

Since R is arbitrary, we conclude that $|V_{x_1}| \leq 1$ a.e. in $\mathbb{R}^2$, and therefore that $W \in \mathcal{K}_\psi$.

We continue by proving (3.2). Since V^ε is a solution to (3.3), a standard bootstrapping argument (using Theorem 6.17 at p. 109 in [29]) allows to improve the regularity of V^ε and to prove that $V^\varepsilon \in C^4$. Therefore, we can differentiate equation (3.3) with respect to x_1 in order to get an equation for $V^\varepsilon_{x_1}$. That is,

$$[(\rho - b^1_1) - \mathcal{L}]V^\varepsilon_{x_1} + \frac{2}{\varepsilon}\beta'((V^\varepsilon_{x_1})^2 - 1)V^\varepsilon_{x_1}V^\varepsilon_{x_1x_1} = \hat{h}^\varepsilon, \quad x \in \mathbb{R}^2, \tag{3.8}$$

where we have defined $\hat{h}^\varepsilon := h_{x_1} + b^2_{x_1}V^\varepsilon_{x_2}$. Moreover, by (3.8), the localized function $V^\varepsilon_\psi := V^\varepsilon_{x_1}\psi$ is a solution to the equation

$$[(\rho - b^1_1) - \mathcal{L}]V^\varepsilon_\psi + \frac{2}{\varepsilon}\beta'((V^\varepsilon_{x_1})^2 - 1)V^\varepsilon_\psi V^\varepsilon_{x_1x_1} = \hat{H}^\varepsilon, \quad x \in \mathbb{R}^2, \tag{3.9}$$

where $\hat{H}^\varepsilon := \hat{h}^\varepsilon\psi - V^\varepsilon_{x_1}\mathcal{L}\psi - DV^\varepsilon_{x_1}D\psi$.

Let now $U \in \mathcal{K}_\psi$. Taking the scalar product of (3.9) with $U - V^\varepsilon_\psi$, an integration by parts gives

$$A_B(V^\varepsilon_\psi, U - V^\varepsilon_\psi) + \frac{2}{\varepsilon}\langle \beta'((V^\varepsilon_{x_1})^2 - 1)V^\varepsilon_\psi V^\varepsilon_{x_1x_1}, U - V^\varepsilon_\psi\rangle_B = \langle \hat{H}^\varepsilon, U - V^\varepsilon_\psi\rangle_B. \tag{3.10}$$

Moreover, since $\sigma > 0$, the operator $\left(\frac{\sigma^2}{2}\sum_{i=1}^2 \langle U_{x_i}, U_{x_i}\rangle_B\right)^{1/2}$, $U \in W^{1;2}(B)$, defines a norm on $W^{1;2}_0(B)$, and it is therefore lower semi-continuous with respect to the weak convergence in $W_0^{1;2}(B)$. By the limits in (3.5), this implies that

$$\liminf_{\varepsilon\to 0} \frac{\sigma^2}{2}\sum_{i=1}^{2}\langle V^\varepsilon_{\psi x_i}, V^\varepsilon_{\psi x_i}\rangle_B \geq \frac{\sigma^2}{2}\sum_{i=1}^{2}\langle W_{x_i}, W_{x_i}\rangle_B. \tag{3.11}$$

Therefore exploiting the convergences in (3.5) and (3.11), taking the liminf as $\varepsilon \to 0$ in (3.10), we obtain

$$A_B(W, U-W) + \liminf_{\varepsilon\to 0}\frac{2}{\varepsilon}\langle \beta'((V^\varepsilon_{x_1})^2-1)V^\varepsilon_\psi V^\varepsilon_{x_1x_1}, U-V^\varepsilon_\psi\rangle_B \geq \langle \hat{H}, U-W\rangle_B. \tag{3.12}$$

In order to prove (3.2), it thus only remains to show that the scalar product in (3.12) involving β' is nonpositive. Write U as $U=\psi\bar{U}$, with $\bar{U}\in\mathcal{K}$. If $x\in\mathbb{R}^2$ is such that $(V^\varepsilon_{x_1}(x))^2 \leq (\bar{U}(x))^2$, then $\beta'((V^\varepsilon_{x_1}(x))^2-1)=0$ since $\bar{U}\in\mathcal{K}$. On the other hand, if $(V^\varepsilon_{x_1}(x))^2 > (\bar{U}(x))^2$ then we have $2V^\varepsilon_\psi(U-V^\varepsilon_\psi)\leq U^2-(V^\varepsilon_\psi)^2<0$. Hence, since V^ε is convex and β' nonnegative, in both cases we deduce that

$$\frac{2}{\varepsilon}\beta'((V^\varepsilon_{x_1})^2-1)V^\varepsilon_\psi V^\varepsilon_{x_1x_1}(U-V^\varepsilon_\psi)\leq 0.$$

Therefore, we conclude that W is a solution to the variational inequality (3.2).

Finally, since $\sigma>0$, the $W^{2;\infty}_{loc}$-regularity of V_{x_1} follows from Theorem 4.1 at p. 31 in [28], slightly modified in order to fit problem (3.2) (see Problem 1 at p. 44, combined with Problems 2 and 5 at p. 29 in [28]).

Step 2. We now prove that V_{x_1} is a pointwise solution to (3.1). For $B\subset\mathbb{R}^2$ open bounded and $\psi\in C^\infty_c(B)$, by Step 1 we have that $V_{x_1}\psi$ is a solution to the variational inequality (3.2). Moreover, thanks to the regularity of V_{x_1}, an integration by parts in (3.2) reveals that

$$\langle \hat{L}\psi, (U-V_{x_1})\psi\rangle_B \geq 0, \text{ for each } U\in\mathcal{K}, \tag{3.13}$$

where $\hat{L} := [(\rho-b^1_1)-\mathcal{L}]V_{x_1}-\hat{h}$. For every $\varepsilon>0$, define the sets $\widehat{\mathcal{W}_\varepsilon} := \{|V_{x_1}|<1-\varepsilon\}$ and, for $N>0$ and $0<\eta<\varepsilon/N$, set $\hat{\psi} := -\eta\hat{L}\mathbb{1}_{\widehat{\mathcal{W}_\varepsilon}}\mathbb{1}_{\{\hat{L}<N\}}$. Define next $U := V_{x_1}+\hat{\psi}$, and observe that $U\in\mathcal{K}$. With this choice of U, the inequality (3.13) rewrites as

$$0\leq\int_B \hat{L}(U-V_{x_1})\psi^2dx = -\eta\int_{\mathbb{R}^2}\hat{L}^2\psi^2\mathbb{1}_{\widehat{\mathcal{W}_\varepsilon}}\mathbb{1}_{\{|\hat{L}|<N\}}dx,$$

which in turn implies that $\int_{\mathbb{R}^2}\hat{L}^2\psi^2\mathbb{1}_{\widehat{\mathcal{W}_\varepsilon}}\mathbb{1}_{\{|\hat{L}|<N\}}dx=0$. Taking limits as $N\to\infty$ and $\varepsilon\to 0$, by monotone convergence theorem, we conclude that $\int_{\mathcal{W}}\hat{L}^2\psi^2dx=0$; that is, $\hat{L}=0$ a.e. in $\mathcal{W}$.

Finally, defining the two regions

$$\mathcal{I}_- := \{x\in\mathbb{R}^2\,|\,V_{x_1}(x)=-1\} \quad\text{and}\quad \mathcal{I}_+ := \{x\in\mathbb{R}^2\,|\,V_{x_1}(x)=1\}, \tag{3.14}$$

we can repeat the arguments above with $\hat{\psi} := -\eta\hat{L}^+\mathbb{1}_{\mathcal{I}_+}\mathbb{1}_{\{|\hat{L}|<N\}}$ and $\hat{\psi} := -\eta\hat{L}^-\mathbb{1}_{\mathcal{I}_-}\mathbb{1}_{\{|\hat{L}|<N\}}$, in order to deduce that $\hat{L}\leq 0$ a.e. in $\mathcal{I}_+\cup\mathcal{I}_-$, and thus completing the proof of the theorem. □

Theorem 3.1 allows to provide a probabilistic representation of V_{x_1} in terms of a Dynkin game. Let $\mathcal{T}$ be the set of $\mathbb{F}$-stopping times, and, for $\tau_1,\tau_2\in\mathcal{T}$, define the functional

$$G(x;\tau_1,\tau_2) := \mathbb{E}\bigg[\int_0^{\tau_1\wedge\tau_2} e^{-\hat{\rho}t}\hat{h}(X^x_t)dt - e^{-\hat{\rho}\tau_1}\mathbb{1}_{\{\tau_1\leq\tau_2,\,\tau_1<\infty\}} + e^{-\hat{\rho}\tau_2}\mathbb{1}_{\{\tau_2<\tau_1\}}\bigg],$$

where $\hat{h}=h_{x_1}+b^2_{x_1}V_{x_2}$ (cf. Theorem 3.1), the process X^x denotes the solution to the uncontrolled SDE (2.1), and $\hat{\rho} := \rho-b^1_1$. Consider the 2-player stochastic differential game of optimal stopping in which Player 1 (resp. Player 2) is allowed to choose a stopping time τ_1 (resp. τ_2) in order to maximize (resp. minimize) the functional G.

Recalling the definitions of $\mathcal{I}_-$ and $\mathcal{I}_+$ given in (3.14), from Theorem 3.1 we obtain the following verification theorem. Its proof is based on a generalized version of Itô's formula (see Theorem 1 at p. 122 in [46]) which can be applied to the process $(e^{-\hat{\rho}t}V_{x_1}(X_t^x))_{t\geq 0}$ because $V_{x_1} \in W^{2;\infty}_{loc}(\mathbb{R}^2)$ by Theorem 3.1. Since these arguments are standard, we omit the details in the interest of length.

Theorem 3.2. *For each $x \in \mathbb{R}^2$, the profile strategy $(\bar{\tau}_1, \bar{\tau}_2)$ given by the stopping times*

$$\bar{\tau}_1 := \inf\{t \geq 0 \,|\, X_t^x \in \mathcal{I}_-\} \quad \textit{and} \quad \bar{\tau}_2 := \inf\{t \geq 0 \,|\, X_t^x \in \mathcal{I}_+\}$$

is a saddle point of the Dynkin game, and its corresponding value equals $V_{x_1}(x)$; that is,

$$G(x;\tau_1,\bar{\tau}_2) \leq V_{x_1}(x) = G(x;\bar{\tau}_1,\bar{\tau}_2) \leq G(x;\bar{\tau}_1,\tau_2), \quad \textit{for each } \tau_1,\tau_2 \in \mathcal{T}.$$

Moreover, we have

$$V_{x_1}(x) = \sup_{\tau_1}\inf_{\tau_2} G(x;\tau_1,\tau_2) = \inf_{\tau_2}\sup_{\tau_1} G(x;\tau_1,\tau_2). \tag{3.15}$$

3.1.2. *The monotonicity property.* We now show how Condition 2 in Assumption 2.1 together with Theorems 3.1 and 3.2 lead to an important monotonicity of V_{x_1}.

Proposition 3.3. *We have $b^2_{x_1}V_{x_1x_2} \geq 0$ in $\mathbb{R}^2$.*

Proof. Since $b^2_{x_1} \leq 0$ by Condition 2 in Assumption 2.1, it is enough to show that $V_{x_1x_2} \leq 0$. Fix an initial condition $x \in \mathbb{R}^2$, take $r > 0$, and define a new initial condition $x^r \in \mathbb{R}^2$ by setting $x^r := x + re_2$. Let $X^{x^r} = (X^{1,x^r}, X^{2,x^r})$ be the solution to the uncontrolled dynamics (2.1), with initial condition x^r. By the structure we assumed on the drift, this perturbation of the initial condition will affect only the second component of X^{x^r}. Indeed, since $x_2^r \geq x_2$, a standard comparison principle for SDE (see [37]) gives $X_t^{2,x^r} - X_t^{2,x} \geq 0$ for each $t \geq 0$, $\mathbb{P}$-a.s., while $X^{1,x^r} = X^{1,x}$. Hence, since $h_{x_1x_2} \leq 0$, we have

$$h_{x_1}(X_t^{x^r}) \leq h_{x_1}(X_t^x), \quad \text{for each } t \geq 0, \ \mathbb{P}\text{-a.s.} \tag{3.16}$$

Moreover, since $b^2_{x_1} \leq 0$, we can exploit the convexity of V to obtain

$$\begin{aligned} & b^2_{x_1}(X_t^{x^r})(V_{x_2}(X_t^{x^r}) - V_{x_2}(X_t^x)) \\ & \quad = b^2_{x_1}(X_t^{x^r})(X_t^{2,x^r} - X_r^{2,x}) \int_0^1 V_{x_2x_2}(X_t^x + s(X_t^{x^r} - X_r^x))ds \\ & \quad \leq 0, \quad \text{for each } t \geq 0, \ \mathbb{P}\text{-a.s.} \end{aligned} \tag{3.17}$$

Let us now prove that $V_{x_2}(y) \geq 0$, for each $y \in \mathbb{R}^2$. Fix $y \in \mathbb{R}^2$ and let v be an optimal control for y. Observe that, for each $\delta > 0$ we can still employ a comparison principle to deduce that $X_t^{1,y;v} - X_t^{1,y-\delta e_2;v} = 0$, and $X_t^{2,y;v} - X_t^{2,y-\delta e_2;v} \geq 0$, for each $t \geq 0$, $\mathbb{P}$-a.s. This, since $h_{x_2} \geq 0$ and $V \in C^1(\mathbb{R}^2)$, in turn implies that

$$\begin{aligned} V_{x_2}(y) &= \lim_{\delta\to 0} \frac{V(y) - V(y-\delta e_2)}{\delta} \\ &\geq \lim_{\delta\to 0} \frac{J(y;v) - J(y-\delta e_2;v)}{\delta} \\ &\geq \lim_{\delta\to 0} \frac{1}{\delta}\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}(h(X_t^{y;v}) - h(X_t^{y-\delta e_2;v}))dt\bigg] \geq 0, \end{aligned} \tag{3.18}$$

where we have used that the control v is suboptimal for the initial condition $y - \delta e_2$. Hence, since $b^2_{x_1x_2} \leq 0$, we obtain that

$$(b^2_{x_1}(X_t^{x^r}) - b^2_{x_1}(X_t^x))V_{x_2}(X_t^x) \leq 0, \quad \text{for each } t \geq 0, \ \mathbb{P}\text{-a.s.} \tag{3.19}$$

Summing now the inequalities (3.16), (3.17) and (3.19), we find

$$h_{x_1}(X_t^{x^r}) + b^2_{x_1}(X_t^{x^r})V_{x_2}(X_t^{x^r}) \leq h_{x_1}(X_t^{x}) + b^2_{x_1}(X_t^{x})V_{x_2}(X_t^{x}), \quad \text{for each } t \geq 0,\ \mathbb{P}\text{-a.s.}; \tag{3.20}$$

that is, $\hat{h}(X^{x^r}) \leq \hat{h}(X^x)$. Therefore, for each stopping time τ_1, $\tau_2 \in \mathcal{T}$, we deduce that

$$G(x^r; \tau_1, \tau_2) \leq G(x; \tau_1, \tau_2).$$

Taking the supremum over $\tau_1 \in \mathcal{T}$ and the infimum over $\tau_2 \in \mathcal{T}$ in the latter inequality, we deduce, in light of (3.15) in Theorem 3.2, that $V_{x_1}(x^r) \leq V_{x_1}(x)$. Hence, we conclude that $V_{x_1x_2} \leq 0$ in $\mathbb{R}^2$, which completes the proof of the proposition. □

3.2. **Step b: Construction of ε-optimal policies.** For every $\varepsilon > 0$ define the sets

$$\mathcal{W}_\varepsilon := \{x \in \mathbb{R}^2 \,|\, V^2_{x_1}(x) < 1 - \varepsilon\}, \quad S_\varepsilon := \partial \mathcal{W}_\varepsilon.$$

The proof of the following lemma is obtained combining arguments from [45] together with the monotonicity property shown in Proposition 3.3.

Lemma 3.4. *For each $\varepsilon > 0$ such that $\bar{x} \in \mathcal{W}_\varepsilon$, there exists a solution $v^\varepsilon \in \mathcal{V}$ to the (classical) Skorokhod problem for the SDE (2.2) in $\overline{\mathcal{W}}_\varepsilon$ starting at $\bar{x}$ with reflection direction $-V_{x_1}/|V_{x_1}|e_1$.*

Proof. Fix $\varepsilon > 0$ such that $\bar{x} \in \mathcal{W}_\varepsilon$. In order to employ the results of [48] to construct v^ε as the solution of the Skorokhod problem with reflection along S_ε, we first show that S_ε is a C^3 hypersurface.

To this end, we begin the proof by showing that

$$V_{x_1x_1}(x) > 0, \quad \text{for each } x \in \mathcal{W}. \tag{3.21}$$

Take indeed $x \in \mathcal{W}$ and $\delta > 0$ such that $B_\delta(x) \subset \mathcal{W}$. Since V solves the linear equation $\rho V - \mathcal{L}V = h$ in $\mathcal{W}$, from Theorem 6.17 at p. 109 in [29] it follows that $V \in C^4(\mathcal{W})$. Therefore, we can differentiate two times with respect to x_1 the HJB equation (2.6), and obtain an equation for $V_{x_1x_1}$

$$(\rho - 2b^1_1)V_{x_1x_1} - \mathcal{L}V_{x_1x_1} = h_{x_1x_1} + 2b^2_{x_1}V_{x_1x_2} + b^2_{x_1x_1}V_{x_2}, \quad \text{in } B_\delta(x). \tag{3.22}$$

Since by assumption $h_{x_1x_1} > 0$, thanks to Proposition 3.3 we have that $h_{x_1x_1} + 2b^2_{x_1}V_{x_1x_2} > 0$. By the inequality (3.18) in the proof of Proposition 3.3, and the fact that b^2 is convex, we deduce that $b^2_{x_1x_1}V_{x_2} \geq 0$. Therefore, the right hand side of (3.22) is positive. Next, by the strong maximum principle (see Theorem 3.5 at p. 35 in [29]), $V_{x_1x_1}$ cannot achieve a nonpositive local minimum in $B_\delta(x)$, unless it is constant. If $V_{x_1x_1}$ is constant in $B_\delta(x)$, then by (3.22) we obtain $V_{x_1x_1} > 0$ as desired. If $V_{x_1x_1}$ attains its minimum at the boundary $\partial B_\delta(x)$, then by convexity of V we still have

$$V_{x_1x_1}(y) > \min_{\partial B_\delta(x)} V_{x_1x_1} \geq 0, \quad \text{for each } y \in B_\delta(x),$$

which also proves (3.21)

Next, define $\bar{\nu}(x) := V_{x_1}(x)/|V_{x_1}(x)|e_1$ for each $x \in S_\varepsilon$, and $w(y) := |V_{x_1}(y)|^2$ for each $y \in \mathcal{W}$. Notice that $\sqrt{w(y)} = |\partial_{\bar{\nu}}V(y)|$. For $R > 0$, by compactness of $\overline{\mathcal{W}}^R_{\varepsilon/2} := \overline{\mathcal{W}}_{\varepsilon/2} \cap \overline{B}_R$, in light of (3.21) we can find a constant $c^R_\varepsilon > 0$ such that

$$\inf_{x \in \overline{\mathcal{W}}^R_{\varepsilon/2}} V_{x_1x_1}(x) \geq c^R_\varepsilon > 0. \tag{3.23}$$

Therefore, for $x \in S_\varepsilon$ and R large enough, by (3.23), we have

$$\sqrt{w(x + \lambda\bar{\nu})} = \partial_{\bar{\nu}}V(x + \lambda\bar{\nu}) \geq \partial_{\bar{\nu}}V(x) + \lambda c^R_\varepsilon/2 = \sqrt{w(x)} + \lambda c^R_\varepsilon/2,$$

and hence

$$\partial_{\bar{\nu}} \sqrt{w(x)} \geq c_\varepsilon^R/2. \tag{3.24}$$

It thus follows that $\partial_{\bar{\nu}} w \neq 0$ on S_ε. This implies, by the implicit function theorem, that S_ε is a C^3-hypersurface.

Now, by (3.24), arguing as in Lemma 2.7 in [45] it is possible to show that the vector $-\bar{\nu}$ is not tangential to S_ε, and, by definition of $\mathcal{W}_\varepsilon$ and of $\bar{\nu}$, we observe that the vector $-\bar{\nu}$ points inside $\mathcal{W}_\varepsilon$. Therefore, we can employ a version of Theorem 4.4 in [48] for unbounded domains in order to find a solution $v^\varepsilon \in \mathcal{V}$ to the Skorokhod problem for the SDE (2.2) in $\overline{\mathcal{W}}_\varepsilon$ starting at $\bar{x}$, with reflection direction $-V_{x_1}/|V_{x_1}|e_1$. □

We conclude this section with the following lemma. We omit its proof since this can be established as in the proof of Lemma 2.8 in [45].

Lemma 3.5. *For each $\bar{x} \in \mathcal{W}$ and $\varepsilon > 0$ such that $\bar{x} \in \mathcal{W}_\varepsilon$, let the control v^ε be as in Lemma 3.4. Then $J(\bar{x}; v^\varepsilon) \to V(\bar{x})$ as $\varepsilon \to 0$.*

3.3. **Step c: Characterization of the optimal control.** Thanks to the results of Subsections 3.1 and 3.2 we can now prove Theorem 2.5. We provide a separate proof for each of the two claims.

3.3.1. *Proof of Claim 1.* We will first prove Claim 1 for $\bar{x} \in \mathcal{W}$, and then, at the end of this subsection, we will give a proof for a general $\bar{x} \in \overline{\mathcal{W}}$. Fix $\bar{x} \in \mathcal{W}$ and a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to zero. To simplify the notation, according to Lemma 3.4 we define the processes

$$X^n := X^{\bar{x};v^{\varepsilon_n}}, \ v^n := v^{\varepsilon_n}, \ \xi^n := |v^{\varepsilon_n}|, \quad \text{for each } n \in \mathbb{N}.$$

Bear in mind that the processes v^n, γ^n and ξ^n depend on the initial condition $\bar{x}$, and that, according to Lemma 3.5, the sequence of controls $(v^n)_{n\in\mathbb{N}}$ is a minimizing sequence; that is, $J(\bar{x}; v^n) \to V(\bar{x})$ as $n \to \infty$.

We begin with the following estimate.

Lemma 3.6. *Let $p' := (2p-1)/2$. We have*

$$\sup_n \int_0^\infty e^{-\rho t}(\mathbb{E}[|X_t^{1,n}|^p] + \mathbb{E}[|X_t^n|^{p'}])dt \leq C(1+|\bar{x}|^p). \tag{3.25}$$

Proof. Denoting by $X^{\bar{x}}$ the solution to (2.1), a standard use of Grönwall's inequality and of Burkhölder-Davis-Gundy's inequality leads to the classical estimate

$$\mathbb{E}[|X_t^{\bar{x}}|^p] \leq Ce^{p\bar{L}t}(1+|\bar{x}|^p) \quad \text{for each } t \geq 0,$$

where $\bar{L}$ is the Lipschitz constant of $\bar{b}$ and $C > 0$ is a generic constant. Therefore, since the control constantly equal to zero is not necessarily optimal for $\bar{x}$, from the latter estimate and the growth rate of h we obtain

$$\begin{aligned} V(\bar{x}) \leq \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X_t^{\bar{x}})dt\bigg] &\leq C \int_0^\infty e^{-\rho t}(1+\mathbb{E}[|X_t^{\bar{x}}|^p])dt \\ &\leq C \int_0^\infty e^{-(\rho - p\bar{L})t}(1+|\bar{x}|^p)dt \leq C(1+|\bar{x}|^p), \end{aligned}$$

where we have used that, by Condition 3a in Assumption 2.1, $\rho > p\,\bar{L}$. Therefore, since v^n is a minimizing sequence, for all n big enough we find the estimate

$$\kappa_1 \int_0^\infty e^{-\rho t}\mathbb{E}[|X_t^{1,n}|^p]dt - \kappa_2 \leq J(\bar{x}; v^n) \leq C(1+|\bar{x}|^p),$$

from which

$$\int_0^\infty e^{-\rho t}\mathbb{E}[|X_t^{1,n}|^p]dt \le C(1+|\bar{x}|^p). \tag{3.26}$$

Next, using again Grönwall's inequality and Burkhölder-Davis-Gundy's inequality, we find

$$\mathbb{E}[|X_t^{2,n}|^{p'}] \le Ce^{p'\bar{L}t}\Big(1+|\bar{x}|^{p'}+p_t+p_t\int_0^t \mathbb{E}[|X_s^{1,n}|^{p'}]ds\Big), \quad \text{for each } t\ge 0,$$

where p_t is a suitable (deterministic) polynomial in t, not depending on n. Therefore

$$\begin{aligned}\int_0^\infty e^{-\rho t}\mathbb{E}[|X_t^{2,n}|^{p'}]dt &\le C\int_0^\infty e^{(p'\bar{L}-\rho)t}(1+|\bar{x}|^{p'}+p_t)dt \\ &\quad + C\int_0^\infty e^{[p'\bar{L}-\rho(1-p'/p)]t}p_t\int_0^t e^{-\rho(p'/p)s}\mathbb{E}[|X_s^{1,n}|^{p'}]dsdt \\ &\le C\int_0^\infty e^{(p'\bar{L}-\rho)t}(1+|\bar{x}|^{p'}+p_t)dt \\ &\quad + C\int_0^\infty e^{[p'\bar{L}-\rho(1-p'/p)]t}p_t\Big(\int_0^\infty e^{-\rho s}\mathbb{E}[|X_s^{1,n}|^{p}]ds\Big)^{\frac{p'}{p}}dt.\end{aligned} \tag{3.27}$$

After noticing that Condition 3a in Assumption 2.1 implies $p'\bar{L}-\rho<0$ and $p'\bar{L}-\rho(1-p'/p)<0$, using (3.26) in (3.27), we conclude that

$$\sup_n \int_0^\infty e^{-\rho t}\mathbb{E}[|X_t^{2,n}|^{p'}]dt \le C(1+|\bar{x}|^p),$$

which, together with (3.26), completes the proof of the lemma. □

Lemma 3.7. *Let $\bar{v}\in\mathcal{V}$ be the unique optimal control for $\bar{x}$. We have that*

$$X_t^n \to X_t^{\bar{x};\bar{v}} \quad \textit{and} \quad v^n\to\bar{v}, \quad \mathbb{P}\otimes dt\textit{-a.e. in } \Omega\times[0,\infty), \quad \textit{as } n\to\infty.$$

Proof. The proof employs arguments as those in the proof of Theorem 8 in [52], that however need to be suitably adapted in order to accommodate our more general convex setting.

We organize the proof in two steps.

Step 1. Arguing by contradiction, in this step we prove that the sequence X^n is Cauchy w.r.t. the convergence in the measure $\mathbb{P}\otimes e^{-\rho t}dt$; that is, for each $\delta>0$ we have

$$\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\mathbb{1}_{\{|X_t^n-X_t^m|>\delta\}}dt\Big]\to 0, \quad \text{as } n,m\to\infty. \tag{3.28}$$

Indeed, suppose that for a subsequence (not relabelled), one has

$$\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\mathbb{1}_{\{|X_t^n-X_t^m|>\delta\}}dt\Big]\ge\delta_0>0, \quad \text{for each } n,m\in\mathbb{N}, \tag{3.29}$$

for a certain constant $\delta_0>0$.

Fix $\lambda\in(0,1)$. We begin by defining the processes

$$Y^{n,m}:=X^{\bar{x};\lambda v^n+(1-\lambda)v^m} \quad \text{and} \quad Z^{n,m}:=\lambda X^n+(1-\lambda)X^m, \quad \text{for each } n,m\in\mathbb{N}.$$

We first need to show that

$$Y_t^{n,m}\le Z_t^{n,m}, \quad \text{for each } t\ge 0,\ \mathbb{P}\text{-a.s.} \tag{3.30}$$

Since the drift $\bar{b}^1$ is affine, we have $Y^{1;n,m} = Z^{1;n,m}$. Moreover, since b^2 is convex, we find

$$
\begin{aligned}
Z_t^{2;n,m} &= \bar{x}_2 + \int_0^t (\lambda b^2(X_s^n) + (1-\lambda) b^2(X_s^m)) dt + \sigma W_t^2 \\
&\geq \bar{x}_2 + \int_0^t b^2(Z_s^{1;n,m}, Z_s^{2;n,m}) dt + \sigma W_t^2 \\
&= \bar{x}_2 + \int_0^t b^2(Y_s^{1;n,m}, Z_s^{2;n,m}) dt + \sigma W_t^2,
\end{aligned} \tag{3.31}
$$

while $Y_t^{2;n,m} = \bar{x}_2 + \int_0^t b^2(Y_s^{1;n,m}, Y_s^{2;n,m}) ds + \sigma W_t^2$. This, by the comparison principle for SDE (see [37]), implies that $Y_t^{2;n,m} \leq Z_t^{2;n,m}$, for each $t \geq 0$, $\mathbb{P}$-a.s., and (3.30) follows.

Next, in light of (3.30), by the monotonicity of h in x_2 we find

$$
\begin{aligned}
&\lambda J(\bar{x}; v^n) + (1-\lambda) J(\bar{x}; v^m) - J(\bar{x}; \lambda v^n + (1-\lambda) v^m) \\
&\quad = \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} (\lambda h(X_t^n) + (1-\lambda) h(X_t^m) - h(Y_t^{n,m})) dt \\
&\qquad\qquad + \int_{[0,\infty)} e^{-\rho t} (\lambda d\xi_t^n + (1-\lambda) d\xi_t^m - d|\lambda v^n + (1-\lambda) v^m|_t) \bigg] \\
&\quad \geq \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} (\lambda h(X_t^n) + (1-\lambda) h(X_t^m) - h(Z_t^{n,m})) dt \bigg],
\end{aligned} \tag{3.32}
$$

as we have that $|\lambda v^n + (1-\lambda) v^m|_t \leq \lambda \xi_t^n + (1-\lambda) \xi_t^m$, and that $e^{-\rho t}$ is positive and decreasing.

Then, using (3.29), for $M > 0$ we observe that

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \mathbb{1}_{\{|X_t^n - X_t^m| > \delta\}} \mathbb{1}_{\{|X_t^n| \leq M,\, |X_t^m| \leq M\}} dt \bigg] \\
&\quad \geq \delta_0 - \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \mathbb{1}_{\{|X_t^n| > M\}} dt \bigg] - \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \mathbb{1}_{\{|X_t^m| > M\}} dt \bigg].
\end{aligned}
$$

Moreover, the estimate in Lemma 3.6 and an application of Chebyshev's inequality yield

$$
\mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \mathbb{1}_{\{|X_t^n| > M\}} dt \bigg] \leq \frac{C(1 + |\bar{x}|^p)}{M^{p'}}, \quad \text{for each } n \in \mathbb{N},
$$

so that we can find M big enough such that

$$
\mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \mathbb{1}_{\{|X_t^n - X_t^m| > \delta\}} \mathbb{1}_{\{|X_t^n| \leq M,\, |X_t^m| \leq M\}} dt \bigg] \geq \frac{\delta_0}{2}, \quad \text{for each } n, m \in \mathbb{N}.
$$

Combining the latter inequality with (3.32), we obtain

$$
\begin{aligned}
&\lambda J(\bar{x}; v^n) + (1-\lambda) J(\bar{x}; v^m) - J(\bar{x}; \lambda v^n + (1-\lambda) v^m) \\
&\quad \geq \delta_{(\delta_0, M)} \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \mathbb{1}_{\{|X_t^n - X_t^m| > \delta\}} \mathbb{1}_{\{|X_t^n| \leq M,\, |X_t^m| \leq M\}} dt \bigg] \\
&\quad \geq \delta_{(\delta_0, M)} \frac{\delta_0}{2},
\end{aligned} \tag{3.33}
$$

where, by strict convexity of h in the variable x_1, we have

$$
\delta_{(\delta_0, M)} := \inf \big\{ \lambda h(x) + (1-\lambda) h(y) - h(\lambda x + (1-\lambda) y) \big| \, |x - y| > \delta_0, |x|, |y| \leq M \big\} > 0.
$$

On the other hand, by Lemma 3.5, $J(\bar{x}; v^n)$ converges to $V(\bar{x})$ as $n \to \infty$. Therefore, from (3.33), we can find $\bar{n} \in \mathbb{N}$ such that

$$
V(\bar{x}) \geq \delta_{(\delta_0, M)} \frac{\delta_0}{4} + J(\bar{x}; \lambda v^n + (1-\lambda) v^m), \quad \text{for each } n, m \geq \bar{n},
$$

which contradicts the definition of V, completing the proof of (3.28).

Step 2. By the previous step, there exists a limit process $\hat{X}$ and, unless to consider a subsequence, we can assume that

$$X_t^n \to \hat{X}_t \quad \mathbb{P} \otimes dt\text{-a.e. in } \Omega \times [0, \infty), \quad \text{as } n \to \infty. \tag{3.34}$$

Next, defining the process $v_t := \hat{X}_t^1 - \bar{x}^1 - \int_0^t \bar{b}^1(\hat{X}_s^1)ds - \sigma W_t^1$, using the estimate from Lemma 3.6 and (3.34) we find

$$|v_t^n - v_t| \leq |X_t^{1,n} - \hat{X}_t^1| + \bar{L} \int_0^t |X_s^{1,n} - \hat{X}_s^1| ds \to 0 \quad \mathbb{P} \otimes dt\text{-a.e. in } \Omega \times [0, \infty),$$

which implies that

$$v_t^n \to v_t \quad \mathbb{P} \otimes dt\text{-a.e. in } \Omega \times [0, \infty), \text{ as } n \to \infty. \tag{3.35}$$

We also observe that, by using Lemma 3.5 in [40], we can assume the processes $\hat{X}^1$ and v to be right-continuous. Also, denoting with ξ the total variation of v, from (3.35) we easily find

$$\xi_t \leq \liminf_n \xi_t^n \quad \text{for each } t \geq 0. \tag{3.36}$$

Next, exploiting the limits in (3.34), the Lipschitz continuity of b^2 and the estimate from Lemma 3.6, we can see that the process $\hat{X}^2$ is continuous and it solves the SDE $d\hat{X}_t^2 = b^2(\hat{X}_t^1, \hat{X}_t^2)dt + \sigma dW_t^2, \ t \geq 0, \ \hat{X}_{0-}^2 = \bar{x}_2$. This, together with the definition of v, implies that

$$\hat{X} = X^{\bar{x};v}. \tag{3.37}$$

Finally, thanks to the limits in (3.34), (3.35) and (3.36), to the identity (3.37), and to the continuity of h, we invoke Fatou's lemma and, with an integration by parts (see, e.g., Corollary 2 at p. 68 in [54]), we find

$$\begin{aligned} J(\bar{x}; v) &= \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X_t^{\bar{x};v})dt + \rho \int_0^\infty e^{-\rho t} \xi_t dt\bigg] \\ &\leq \liminf_n \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X_t^n)dt + \rho \int_0^\infty e^{-\rho t} \xi_t^n dt\bigg] = \liminf_n J(\bar{x}; v^n) = V(\bar{x}), \end{aligned} \tag{3.38}$$

where we have used that the sequence $(v^n)_{n\in\mathbb{N}}$ is minimizing for $\bar{x}$, according to Lemma 3.5. Thus, the process v has locally bounded variation, and $v \in \mathcal{V}$. Also, from (3.38) we deduce that the control v is optimal for $\bar{x}$, and, by uniqueness of optimal controls (see Remark 2.4), we conclude that $v = \bar{v}$ and $\hat{X} = X^{\bar{x};\bar{v}}$, completing the proof of the lemma. □

The proofs of the next two propositions follow by employing arguments similar to those employed in Sections 2.3 and 2.4 in [45] (we provide details here in order to recall these arguments in the sequel).

Proposition 3.8. *We have that* $\mathbb{P}[X_t^{\bar{x};\bar{v}} \in \overline{\mathcal{W}}, \forall t \geq 0] = 1$.

Proof. By Lemma 3.7, $X_t^n \to X_t^{\bar{x};\bar{v}}$, $\mathbb{P} \otimes dt$-a.e. in $\Omega \times [0, \infty)$, and, by Lemma 3.4, $\mathbb{P}[X_t^n \in \overline{\mathcal{W}}, t \geq 0] = 1$, as $\overline{\mathcal{W}}_{\varepsilon_n} \subset \overline{\mathcal{W}}$ for each $n \in \mathbb{N}$. Therefore, it is clear that $X_t^{\bar{x};\bar{v}} \in \overline{\mathcal{W}}$, $\mathbb{P} \otimes dt$-a.e. in $\Omega \times [0, \infty)$, which, by right-continuity, implies that $\mathbb{P}[X_t^{\bar{x};\bar{v}} \in \overline{\mathcal{W}}, t \geq 0] = 1$. □

Proposition 3.9. *We have* $d\bar{v} = \bar{\gamma} d|\bar{v}|$ *with*

$$|\bar{v}|_t = \int_0^t \mathbb{1}_{\{X_{s-}^{\bar{x};\bar{v}} \in S, \, -V_{x_1}(X_{s-}^{\bar{x};\bar{v}}) = \bar{\gamma}_s\}} d|\bar{v}|_s, \quad \textit{for each } t \geq 0, \ \mathbb{P}\textit{-a.s.}$$

Proof. Take $R > 0$ such that $\bar{x} \in B_R$ and define $\tau_R := \inf\{t \in [0, \infty) | X_s^{\bar{x};\bar{v}} \notin B_R\}$. For each $\varepsilon > 0$, let V^ε be as in (A.2). As in the Step 1 in the proof of Theorem A.1 in Appendix A, V^ε is a convex C^2-solution to (A.3). By Itô's formula for semimartingales (see, e.g., Theorem

33 at p. 81 in [54]), applied on the process $(e^{-\rho t}V^{\varepsilon}(X_t^{\bar{x};\bar{v}}))_{t\geq 0}$ on the time interval $[0,\tau_R]$, we find

$$\mathbb{E}[e^{-\rho\tau_R}V^{\varepsilon}(X_{\tau_R}^{\bar{x};\bar{v}})] = V^{\varepsilon}(\bar{x})+\mathbb{E}\bigg[\int_0^{\tau_R} e^{-\rho t}(\mathcal{L}V^{\varepsilon}-\rho V^{\varepsilon})(X_t^{\bar{x};\bar{v}})dt + \int_{[0,\tau_R)} e^{-\rho t}V_{x_1}^{\varepsilon}(X_{t-}^{\bar{x};\bar{v}})\bar{\gamma}_t d|\bar{v}|_t$$
$$+\sum_{0\leq t\leq \tau_R} e^{-\rho t}(V^{\varepsilon}(X_t^{\bar{x};\bar{v}}) - V^{\varepsilon}(X_{t-}^{\bar{x};\bar{v}}) - V_{x_1}^{\varepsilon}(X_{t-}^{\bar{x};\bar{v}})\bar{\gamma}_t(|\bar{v}|_t - |\bar{v}|_{t-}))\bigg].$$

By the convexity of V^{ε}, the last sum above is nonnegative. Also, since the function β in (A.3) in nonnegative, we have $\rho V^{\varepsilon} - \mathcal{L}V^{\varepsilon} \leq h$ a.e. in $\mathbb{R}^2$. Hence from the latter equality we

$$V^{\varepsilon}(\bar{x}) \leq \mathbb{E}\bigg[\int_0^{\tau_R} e^{-\rho t}h(X_t^{\bar{x};\bar{v}})dt - \int_{[0,\tau_R)} e^{-\rho t}V_{x_1}^{\varepsilon}(X_{t-}^{\bar{x};\bar{v}})\bar{\gamma}_t d|\bar{v}|_t\bigg]. \tag{3.39}$$

Therefore, taking first limits in (3.39) as $\varepsilon \to 0$ (using (A.18) and the dominated convergence theorem), and then letting $R\to\infty$ (using the monotone convergence theorem and the dominated convergence theorem), we obtain

$$V(\bar{x}) \leq \mathbb{E}\bigg[\int_0^{\infty} e^{-\rho t}h(X_t^{\bar{x};\bar{v}})dt - \int_{[0,\infty)} e^{-\rho t}V_{x_1}(X_{t-}^{\bar{x};\bar{v}})\bar{\gamma}_t d|\bar{v}|_t\bigg]. \tag{3.40}$$

Next, by the optimality of $\bar{v}$, we have that $V(\bar{x}) = J(\bar{x};\bar{v})$, and, from (3.40), it follows that

$$\mathbb{E}\bigg[\int_{[0,\infty)} e^{-\rho t}(1+V_{x_1}(X_{t-}^{\bar{x};\bar{v}})\bar{\gamma}_t)d|\bar{v}|_t\bigg] \leq 0. \tag{3.41}$$

This in turn implies, using $0\leq 1-|V_{x_1}|\leq 1+V_{x_1}\gamma$ for all $\gamma\in\mathbb{R}$ with $|\gamma|=1$, that

$$0\leq \mathbb{E}\bigg[\int_{[0,\infty)} e^{-\rho t}(1-|V_{x_1}(X_{t-}^{\bar{x};\bar{v}})|)d|\bar{v}|_t\bigg] \leq \mathbb{E}\bigg[\int_{[0,\infty)} e^{-\rho t}(1+V_{x_1}(X_{t-}^{\bar{x};\bar{v}})\bar{\gamma}_t)d|\bar{v}|_t\bigg] \leq 0.$$

From the latter chain of inequalities we deduce that the support of the random measure $d|\bar{v}|$ is $\mathbb{P}$-a.s. contained in the set $\{(\omega,t)\in\Omega\times[0,\infty)\,|\,X_{t-}^{\bar{x};\bar{v}}(\omega)\in\partial\mathcal{W}, \bar{\gamma}_t(\omega) = -V_{x_1}(X_{t-}^{\bar{x};\bar{v}}(\omega))\}$, which completes the proof of the proposition. □

The proof of the next proposition also follows by employing the arguments in [45]. Details are provided in Appendix B for the sake of completeness.

Proposition 3.10. *We have that, $\mathbb{P}$-a.s., a possible jump of the process $X^{\bar{x};\bar{v}}$ at time $t\geq 0$ occurs on some interval $I\subset\partial\mathcal{W}$ parallel to the vector field $-V_{x_1}e_1$, i.e., such that $-V_{x_1}(x)e_1$ is parallel to I for each $x\in I$. If $X^{\bar{x};\bar{v}}$ encounters such an interval I, it instantaneously jumps to its endpoint in the direction $-V_{x_1}e_1$ on I.*

Combining then the Propositions 3.8, 3.9 and 3.10, we see that, for $\bar{x}\in\mathcal{W}$, the optimal control $\bar{v}\in\mathcal{V}$ is a solution to the modified Skorokhod problem for the SDE (2.2) in $\overline{\mathcal{W}}$ starting at $\bar{x}$ with reflection direction $-V_{x_1}e_1$.

Take next $\bar{x}\in\overline{\mathcal{W}}$. By definition, there exists a sequence $(x^k)_{k\in\mathbb{N}}\subset\mathcal{W}$ such that $x^k\to\bar{x}$ as $k\to\infty$. For each k, let w^k be the optimal control for x^k, and consider the controls $x^k-\bar{x}+w^k$, which consist in following the policy w^k after an initial jump from $\bar{x}$ to x^k. Using the fact that $x^k\in\mathcal{W}$, from Proposition 3.8 we have that $\mathbb{P}[X_t^{x^k;w^k}\in\overline{\mathcal{W}}, t\geq 0]=1$. Observe, moreover, that $X^{x^k;w^k} = X^{\bar{x};x^k-\bar{x}+w^k}$, and that $|J(\bar{x};x^k-\bar{x}+w^k) - J(x^k;w^k)| = |\bar{x}-x^k|$. By the continuity of V, we now see that

$$V(\bar{x}) = \lim_k V(x^k) = \lim_k J(x^k;w^k) = \lim_k J(\bar{x};x^k-\bar{x}+w^k).$$

Therefore, the sequence of controls $(x^k-\bar{x}+w^k)_{k\in\mathbb{N}}$ is a minimizing sequence for the initial condition $\bar{x}$. Repeating the proof of Lemma 3.7 with the sequence of controls $(x^k-\bar{x}+w^k)_{k\in\mathbb{N}}$, we see that $X_t^{x^k;w^k} \to X_t^{\bar{x};\bar{v}}$, $\mathbb{P}\otimes dt$-a.e. in $\Omega\times[0,\infty)$. This allows to repeat the arguments in the proofs of Propositions 3.8, 3.9 and 3.10 in order to conclude that, also for $\bar{x}\in\overline{\mathcal{W}}$, the optimal control $\bar{v}\in\mathcal{V}$ is a solution to the modified Skorokhod problem for the SDE (2.2) in $\overline{\mathcal{W}}$ starting at $\bar{x}$ with reflection direction $-V_{x_1}e_1$.

Finally, through a verification theorem (which can be proved by using Itô's formula as in the proof of Proposition 3.9), it is easy to show that any solution to the modified Skorokhod problem for the SDE (2.2) in $\overline{\mathcal{W}}$ starting at $\bar{x}$ with reflection direction $-V_{x_1}e_1$ is an optimal control. This, by uniqueness of the optimal control (see Remark 2.4) implies that such a solution is unique, completing the proof of Claim 1 of Theorem 2.5.

3.3.2. *Proof of Claim 2.* Fix $\bar{x}=(\bar{x}_1,\bar{z})\notin\overline{\mathcal{W}}$ and denote again by $\bar{v}$ the optimal control for $\bar{x}$. Let $\bar{y}_1\in\mathbb{R}$ be the metric projection of $\bar{x}_1$ into the set $\overline{\mathcal{W}}_1(\bar{z})$. The set $\overline{\mathcal{W}}_1(\bar{z})$ is a closed interval (cf. Lemma 2.3), hence the point $\bar{y}_1$ is uniquely determined. Set then $\bar{y}:=(\bar{y}_1,\bar{z})$ and observe that $\bar{y}\in\partial\mathcal{W}$. Let $\bar{w}$ be the optimal control for $\bar{y}$. Notice that, since V_{x_1} is pointing outside $\overline{\mathcal{W}}_1(\bar{z})$, we have $V_{x_1}(\bar{y})(\bar{x}_1-\bar{y}_1)=|\bar{x}_1-\bar{y}_1|$. Therefore, since $(\bar{y}_1+\lambda(\bar{x}_1-\bar{y}_1),\bar{z})\notin\mathcal{W}$ for each $\lambda\in(0,1)$, we get

$$V(\bar{x})=V(\bar{y}_1,\bar{z})+\int_0^1 V_{x_1}(\bar{y}_1+\lambda(\bar{x}_1-\bar{y}_1),\bar{z})(\bar{x}_1-\bar{y}_1)d\lambda=V(\bar{y})+|\bar{x}_1-\bar{y}_1|.$$

This means that $V(\bar{x})=J(\bar{y};\bar{w})+|\bar{x}_1-\bar{y}_1|=J(\bar{x};\bar{x}_1-\bar{y}_1+\bar{w})$, which, by uniqueness of the optimal control, implies that $\bar{v}=\bar{x}_1-\bar{z}_1+\bar{w}$. Moreover, since $\bar{y}\in\overline{\mathcal{W}}$ and $\bar{w}$ is optimal for $\bar{y}$, by Claim 1 we have that $\bar{w}$ is the unique solution to the modified Skorokhod problem for the SDE (2.2) in $\overline{\mathcal{W}}$ starting at $\bar{y}$ with reflection direction $-V_{x_1}e_1$. This completes the proof of Claim 2 and therefore also of Theorem 2.5.

4. On the proof of Theorem 2.5 for linear volatility

In this section we assume that Condition 3b in Assumption 2.1 holds. To simplify the notation, also this proof is given for $d=2$, so that $D=\mathbb{R}^2_+=\{x\in\mathbb{R}^2\,|\,x_1,\,x_2>0\}$. The generalization to the case $d>2$ is straightforward.

4.1. **A preliminary lemma.** We first state the following technical result. Define the set

$$\mathcal{V}^x_+:=\{v\in\mathcal{V}\,|\,X_t^{1,x;v},\,X_t^{2,x;v}>0 \text{ for each } t\geq 0,\ \mathbb{P}\text{-a.s.}\}.$$

Lemma 4.1. *We have* $V(x)=\min_{v\in\mathcal{V}^x_+}J(x;v)$, *for each* $x\in\mathbb{R}^2_+$.

Proof. Let $v\in\mathcal{V}$ be an optimal control for $x\in\mathbb{R}^2_+$, and denote by (ξ^+,ξ^-) its minimal decomposition. In order to simplify the notation, set $X:=X^{x;v}$. Assuming that $v_s=0$ for each $s<0$, define the family of random variables

$$\tau_k:=\inf\{t\geq 0\,|\,(X_t^1,\xi^-_{t+1/k}-\xi^-_{t-1/k})\in(-\infty,x_1^*)\times(0,\infty)\},\quad k\in\mathbb{N}.$$

Define the filtration $\mathbb{F}^k:=(\mathcal{F}_{t+1/k})_{t\geq 0}$ and notice that, for each $k\geq 1$, τ_k is an $\mathbb{F}^k$-stopping time. Set $\tau:=\sup_k\tau_k$, and observe that, for $k\leq\bar{k}$, we have $\tau_k\leq\tau_{\bar{k}}$. This implies that, for each $\bar{k}\geq 1$, $\tau=\sup_{k\geq\bar{k}}\tau_k$, so that τ is an $\mathbb{F}^{\bar{k}}$-stopping time, and, by right-continuity of the filtration $\mathbb{F}$, we deduce that τ is an $\mathbb{F}$-stopping time. Also, such a definition of τ is such that the negative part ξ^- of v acts at time τ; that is, τ is in the support of the measure ξ^-.

If $\mathbb{P}[\tau<\infty]=0$, then the control ξ^- never acts when the state process X^1 lies in the region $(-\infty,x_1^*)$. Since $a_1\geq 0$ and $b^2\geq 0$, this is enough to ensure that $X_t^{1,x;v}$, $X_t^{2,x;v}>0$ for each $t\geq 0$, $\mathbb{P}$-a.s., which in turn implies that $v\in\mathcal{V}^x_+$.

Arguing by contradiction, suppose that $\mathbb{P}[\tau<\infty]>0$. Define the control $\tilde{v}_t:=\mathbb{1}_{\{t<\tau\}}v_t+\mathbb{1}_{\{t\geq\tau\}}(\xi_t^+ +\min\{x_1^*-X^1_{\tau-},0\}\mathbb{1}_{\{\Delta\xi^-_\tau>0\}})$, and the process $\tilde{X}:=X^{x;\tilde{v}}$. Define next the stopping time $\bar{\tau} := \inf\{t \geq \tau \,|\, \tilde{X}^1_t \geq 2x^*_1\}$, the control $\bar{v}_t := \mathbb{1}_{\{t<\bar{\tau}\}}\tilde{v} + \mathbb{1}_{\{t\geq\bar{\tau}\}}(X^1_{\bar{\tau}} - \tilde{X}^1_{\bar{\tau}-} + v_t - v_{\bar{\tau}})$ and the process $\bar{X} := X^{x;\bar{v}}$. Since at time τ only the negative part ξ^- of v acts, on $\{\tau < \infty\}$ we have $\tau < \bar{\tau}$. Also, by the definition of $\bar{v}$, for k such that $\tau + 1/k < \bar{\tau}$, on $\{\tau < \infty\}$ we have

$$v_{\tau+1/k} - \bar{v}_{\tau+1/k} \geq \xi^-_{\tau+1/k} \geq \xi^-_{\tau_k+1/k} > \xi^-_{\tau_k-1/k} \geq 0,$$

so that the processes v and $\bar{v}$ are not indistinguishable. Moreover, v and $\bar{v}$ are such that, on $\{\tau < \infty\}$, we have

$$\begin{cases} X_t = \bar{X}_t \text{ for } t \in [0, \tau) \cup [\bar{\tau}, \infty), \\ X_t \leq \bar{X}_t \text{ for } t \in [\tau, \bar{\tau}). \end{cases} \tag{4.1}$$

After some manipulations, from (4.1) we deduce that

$$\begin{aligned} J(x; v) = J(x; \bar{v}) &+ \mathbb{E}\bigg[\int_{(\tau,\bar{\tau})} e^{-\rho t} d\xi^-_t + \int_\tau^{\bar{\tau}} e^{-\rho t} Dh(\hat{X}_t)(X_t - \bar{X}_t)dt\bigg] \\ &+ \mathbb{E}[e^{-\rho\tau}(|X^1_\tau - X^1_{\tau-}| - |\bar{X}^1_\tau - \bar{X}^1_{\tau-}|) + e^{-\rho\bar{\tau}}(|X^1_{\bar{\tau}} - X^1_{\bar{\tau}-}| - |\bar{X}^1_{\bar{\tau}} - \bar{X}^1_{\bar{\tau}-}|)], \end{aligned} \tag{4.2}$$

for $\hat{X}_t = \lambda_t \bar{X}_t + (1 - \lambda_t)X^{x;v}_t \in (-\infty, 2x^*_1) \times \mathbb{R}$, and suitable choice of $\lambda_t(\omega) \in [0, 1]$. Since ξ^- acts at time τ, we have $X^1_\tau - X^1_{\tau-} \leq 0$. Also, at time τ the control $\bar{v}$ can only jump to the left, giving $\bar{X}^1_\tau - \bar{X}^1_{\tau-} \leq 0$. Hence, using $\bar{X}^1_\tau - X^1_\tau \geq 0$, we obtain

$$e^{-\rho\tau}(|X^1_\tau - X^1_{\tau-}| - |\bar{X}^1_\tau - \bar{X}^1_{\tau-}|) = e^{-\rho\bar{\tau}}(\bar{X}^1_\tau - X^1_\tau) \geq 0. \tag{4.3}$$

Now, if $\bar{X}^1_{\bar{\tau}} \geq \bar{X}^1_{\bar{\tau}-}$, then using (4.1) we find

$$|X^1_{\bar{\tau}} - X^1_{\bar{\tau}-}| - |\bar{X}^1_{\bar{\tau}} - \bar{X}^1_{\bar{\tau}-}| \geq \bar{X}^1_{\bar{\tau}-} - X^1_{\bar{\tau}-} \geq 0. \tag{4.4}$$

Therefore, plugging (4.3) and (4.4) into (4.2), we obtain the inequality

$$J(x; v) - J(x; \bar{v}) \geq \mathbb{E}\bigg[\int_\tau^{\bar{\tau}} e^{-\rho t}[h_{x_1}(\hat{X}_t)(X^1_t - \bar{X}^1_t) + h_{x_2}(\hat{X}_t)(X^2_t - \bar{X}^2_t)]dt\bigg] \geq 0, \tag{4.5}$$

where we have also used (4.1), Condition 3b in Assumption 2.1, and that, due to the monotonicity of b^2 in the variable x_1, via a comparison principle we have $X^2_t - \bar{X}^2_t \leq 0$ for $t \in (\tau, \bar{\tau})$. On the other hand, if $\bar{X}^1_{\bar{\tau}} \leq \bar{X}^1_{\bar{\tau}-}$, from (4.1) we obtain

$$\begin{aligned} |X^1_{\bar{\tau}} - X^1_{\bar{\tau}-}| - |\bar{X}^1_{\bar{\tau}} - \bar{X}^1_{\bar{\tau}-}| &\geq X^1_{\bar{\tau}-} - \bar{X}^1_{\bar{\tau}-} \\ &= X^1_\tau - \bar{X}^1_\tau + \int_\tau^{\bar{\tau}-} b^1_1(X^1_t - \bar{X}^1_t)dt + \int_\tau^{\bar{\tau}-} \sigma(X^1_t - \bar{X}^1_t)dW^1_t - \int_{(\tau,\bar{\tau})} d\xi^-_t. \end{aligned} \tag{4.6}$$

If $b^1_1 \leq 0$ substituting (4.3) and (4.6) into (4.2), we obtain again (4.5). Similarly, for $b^1_1 \geq 0$ we find

$$J(x; v) - J(x; \bar{v}) \geq \mathbb{E}\bigg[\int_\tau^{\bar{\tau}} [(h_{x_1}(\hat{X}_t) - b^1_1)(X^1_t - \bar{X}^1_t) + h_{x_2}(\hat{X}_t)(X^2_t - \bar{X}^2_t)]dt\bigg] \geq 0. \tag{4.7}$$

However, both (4.5) and (4.7) contradict the uniqueness of the optimal control v, completing the proof of the lemma. □

4.2. **Sketch of the proof of Theorem 2.5.** Since we are interested in characterizing the optimal control for any given $\bar{x} \in \mathbb{R}^2_+$, thanks to Lemma 4.1 we can restrict the domain of the HJB equation to the set $\mathbb{R}^2_+$. We observe that, upon exploiting the ellipticity of the operator $\mathcal{L}$ in the domain $\mathbb{R}^2_+$ (and, in particular, the uniform ellipticity of $\mathcal{L}$ on each ball $B \subset \mathbb{R}^2_+$), all the results from Sections 3.1 and 3.2 can be recovered, with minimal adjustments of the arguments therein.

For $\bar{x} \in \mathcal{W}$ we can consider the processes $X^n := X^{\bar{x};v^n}$, v^n for $n \in \mathbb{N}$, with $(v^n)_{n\in\mathbb{N}}$ minimizing sequence of solutions to the Skorkokhod problems on domains $\overline{\mathcal{W}}_n$, according to Lemma 3.5 (here $\overline{\mathcal{W}}_n$ denotes the closure of $\mathcal{W}_n$ in $\mathbb{R}^2_+$).

Estimates as those of Lemma 3.6 can now be proved as follows. Denoting by $X^{\bar{x}}$ the solution to (2.1), by standard results (see, e.g., Theorem 4.1 at p. 59 in [51]) we have $\mathbb{E}[|X_t^{\bar{x}}|^p] \leq Ce^{p(2\bar{L}+\sigma^2(p-1))t}(1+|\bar{x}|^p)$ for each $t \geq 0$. Hence, arguing as in the proof of Lemma 3.6 and using the requirement on ρ from Condition 3b in Assumption 2.1, we find

$$\sup_n \int_0^\infty e^{-\rho t}\mathbb{E}[|X_t^{1,n}|^p]dt \leq C(1+|\bar{x}|^p). \tag{4.8}$$

Next, for $p' := (2p-1)/2$, we use (4.8) to estimate $|X^{2,n}|^{p'}$. We underline that, since $\overline{\mathcal{W}}_n \subset \mathcal{W}$, we have $X_t^n > 0$ $\mathbb{P}\otimes dt$-a.e. in $\Omega\times[0,\infty)$. For each $n \in \mathbb{N}$, define the process Λ^n as the solution to the SDE

$$d\Lambda_t^n = \bar{L}(1+|X_t^{1,n}|+\Lambda_t^n)dt + \sigma\Lambda_t^n dW_t^2,\ t \geq 0, \quad \Lambda_0^n = \bar{x}_2.$$

Since $X_t^{2,n} \leq \bar{x}_2 + \int_0^t \bar{L}(1+|X_s^{1,n}|+|X_s^{2,n}|)ds + \sigma\int_0^t X_s^{2,n}dW_s^2$, by a comparison principle we obtain $X^{2,n} \leq \Lambda^n$. Therefore, using that $\Lambda_t^n = \hat{E}_t[\bar{x}_2 + \int_0^t \bar{L}(1+|X_s^{1,n}|)\hat{E}_s^{-1}ds]$, with $\hat{E}_t := \exp[(\bar{L}-\sigma^2/2)t+\sigma W_t]$, we find

(4.9)
$$\begin{aligned}
\int_0^\infty e^{-\rho t}\mathbb{E}[|X_t^{2,n}|^{p'}]dt &\leq \int_0^\infty e^{-\rho t}\mathbb{E}[|\Lambda_t^n|^{p'}]dt \\
&\leq C\int_0^\infty e^{-\rho t}\mathbb{E}\bigg[\hat{E}_t^{p'}\bar{x}_2^{p'} + p_t\int_0^t \hat{E}_t^{p'}\hat{E}_s^{-p'}ds\bigg] \\
&\quad + C\int_0^\infty \bigg(\int_0^t e^{-\rho s}\mathbb{E}[|X_s^{1,n}|^p]ds\bigg)^{\frac{1}{q}} e^{-\rho(1-\frac{1}{q})t}\bigg(\int_0^t \mathbb{E}[(\hat{E}_t/\hat{E}_s)^{p'q^*}]ds\bigg)^{\frac{1}{q^*}}dt,
\end{aligned}$$

where we have also used Hölder's inequality with exponent $q = p/p'$, q^* denoting the conjugate of q. Exploiting the requirement on ρ made in Condition 3b in Assumption 2.1, after elementary computations one can see that

$$\int_0^\infty e^{-\rho(1-\frac{1}{q})t}\bigg(\int_0^t \mathbb{E}[(\hat{E}_t/\hat{E}_s)^{p'q^*}]ds\bigg)^{\frac{1}{q^*}}dt < \infty. \tag{4.10}$$

Finally, substituting (4.8) and (4.10) in (4.9), we conclude that

$$\sup_n \int_0^\infty e^{-\rho t}\mathbb{E}[|X_t^{2,n}|^{p'}]dt \leq C(1+|\bar{x}|^p),$$

which, combined with (4.8), gives

$$\sup_n \int_0^\infty e^{-\rho t}(\mathbb{E}[|X_t^{1,n}|^p] + \mathbb{E}[|X_t^n|^{p'}])dt \leq C(1+|\bar{x}|^p). \tag{4.11}$$

Thanks to the estimate (4.11), the arguments of Step 1 in the proof of Lemma 3.7 can be recovered, so that (up to a subsequence)

$$X_t^n \to \hat{X}_t \quad \mathbb{P}\otimes dt\text{-a.e. in } \Omega\times[0,\infty), \quad \text{as } n\to\infty, \tag{4.12}$$

for an adapted process $\hat{X}$. Using again (4.11) and the assumption $p \geq 2$, a standard use of Banach-Saks' theorem allows to find a subsequence of indexes $(n_j)_{j\in\mathbb{N}}$ such that the Cesàro means of $(X^{1,n_j})_{j\in\mathbb{N}}$ converge in $\mathbb{L}^2$ to the process $\hat{X}^1$; that is,

$$\bar{X}^{1,m} := \frac{1}{m}\sum_{j=1}^m X^{1,n_j} \to \hat{X}^1,\ \text{as } m\to\infty, \quad \text{in } \mathbb{L}^2(\Omega\times[0,T];\mathbb{P}\otimes dt),\ \text{for each } T>0. \tag{4.13}$$

Next, defining the process $v_t := \hat{X}^1_t - \bar{x}_1 - \int_0^t \hat{X}^1_s ds - \int_0^t \hat{X}^1_s dW_s$, and exploiting the $\mathbb{L}^2$ convergence in (4.13) and the linearity of the dynamics for the first component, we deduce that

$$\bar{v}^m := \frac{1}{m}\sum_{j=1}^m v^{n_j} \to v, \text{ as } m \to \infty, \quad \text{in } \mathbb{L}^2(\Omega \times [0,T]; \mathbb{P} \otimes dt), \text{ for each } T > 0. \tag{4.14}$$

Again, by using Lemma 3.5 in [40], we can assume the processes $\hat{X}^1$ and v to be right-continuous. Next, observe that the processes $X^{2,n}$ can be expressed as

$$X^{2,n}_t = E_t\big[\bar{x}_2 + \textstyle\int_0^t b^2(X^n_s)/E_s ds\big], \quad \text{with} \quad E_t := \exp\big(\sigma W^2_t - \frac{\sigma^2}{2}t\big), \quad t \geq 0.$$

Hence, taking limits as $n \to \infty$ in the latter equality (exploiting (4.12) and the uniform integrability deriving from (4.11)), we deduce that

$$\hat{X}^2_t = E_t\big[\bar{x}_2 + \textstyle\int_0^t b^2(\hat{X}_s)/E_s ds\big], \quad t \geq 0,$$

so that, thanks also to the very definition of v, we have $\hat{X} = X^{\bar{x};v}$. Overall, from (4.12), (4.14) and the latter equality, we have

$$\bar{X}^m := \frac{1}{m}\sum_{j=1}^m X^{n_j} \to X^{\bar{x};v}, \text{ and } \bar{v}^m \to v, \quad \mathbb{P}\otimes dt\text{-a.e. in } \Omega \times [0,\infty), \text{ as } m \to \infty. \tag{4.15}$$

It is however worth noticing that $\bar{X}^m$ is not the solution of the SDE controlled by $\bar{v}^m$, unless b^2 is affine. Similarly to (3.36), using the fact that the sequence of controls v^n is minimizing, and exploiting the limits in (4.15) and the convexity of h, we find

$$\begin{aligned}
J(\bar{x}; v) &= \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X^{\bar{x};v}_t) dt + \rho \int_0^\infty e^{-\rho t} |v|_t dt\bigg] \\
&\leq \liminf_m \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(\bar{X}^m_t) dt + \rho \int_0^\infty e^{-\rho t} |\bar{v}^m|_t dt\bigg] \\
&\leq \liminf_m \frac{1}{m}\sum_{j=1}^m \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X^{n_j}_t) dt + \rho \int_0^\infty e^{-\rho t} |v^{n_j}|_t dt\bigg] = V(\bar{x}),
\end{aligned}$$

so that the control v has locally bounded variation and it is optimal. By uniqueness of the optimal control, we deduce that $\bar{v} = v$ and $\hat{X} = X^{\bar{x};\bar{v}}$.

Finally, thanks to the properties of (X^n, v^n), by repeating the arguments leading to Propositions 3.8, 3.9 and 3.10 (see Appendix B), the optimal control $\bar{v}$ for $\bar{x} \in \mathcal{W}$ can be characterized as the unique solution to the modified Skorokhod problem for the SDE (2.2) in $\overline{\mathcal{W}}$ starting at $\bar{x}$ with reflection direction $-V_{x_1}e_1$. On the other hand, for $\bar{x} \in \overline{\mathcal{W}}$, we can repeat the rationale at the end of Subsection 3.3.1, which yields that the optimal control can be characterized also for $\bar{x} \in \overline{\mathcal{W}}$, completing the proof of Claim 1 of Theorem 2.5.

When $\bar{x} \notin \overline{\mathcal{W}}$, following the arguments of Subsection 3.3.2, one can characterize the initial jump of $\bar{v}$. This completes the proof of Theorem 2.5 under Condition 3b in Assumption 2.1.

5. Comments, extensions and examples

5.1. **Refinements of Assumption 2.1.** Assumption 2.1 can be improved as follows.

5.1.1. *Affine drift.* If $\bar{\sigma}$ is constant, Theorem 2.5 holds also for a drift $\bar{b}(x) := a + bx$, for a vector $a \in \mathbb{R}^d$ and a matrix $b \in \mathbb{R}^{d\times d}$ such that the vector $\beta := (0, b^2_1, ..., b^d_1)^\top \in \mathbb{R}^d$ is an eigenvector of b and $h_{x_1\beta} \geq 0$. Here the vector $(0, b^2_1, ..., b^d_1)^\top$ is the first column of b, with b^1_1 replaced by 0, while $h_{x_1\beta}$ denotes the β-directional derivative of h_{x_1}. In this case, for $x \in \mathbb{R}^d$, $r > 0$ and $x^r := x + r\beta$, the solution X^{x^r} of (2.1) writes (see, e.g., p. 99 in [51]) as $X^{x^r}_t = e^{bt}x^r + P_t$, where P_t does not depend on x^r. Hence, since the vector β is by assumption an eigenvector of the matrix b with eigenvalue λ, we find $X_t^{x^r} - X_t^x = r\, e^{tb}\beta = re^{t\lambda}\beta$, for each $t \geq 0$, $\mathbb{P}$-a.s. This easily allows to repeat the arguments in the proof of Proposition 3.3, so that $V_{x_1\beta} \geq 0$, while all of the other results in this paper still hold (often with less technical proofs). Also, in this case, for $p = 2$ it is sufficient to require that

$$\rho > 2\Lambda(b), \quad \Lambda(b) := \max\{\mathrm{Re}(\lambda) \,|\, \lambda \text{ eigenvalue of } b\}.$$

We refer to Lemma 2.2 and Theorem 2.3 in [15] for more details. Finally, all the results in this paper apply for a constant volatility matrix $\bar\sigma$ such that $\bar\sigma\bar\sigma^\top$ is positive definite, $\bar\sigma^\top$ denoting the transpose of $\bar\sigma$.

5.1.2. *On Condition 2.* A careful look into the proofs of Proposition 3.3 and of Lemma 3.7 reveals that the results in this paper remain valid if the drift coefficients b^i in Condition 2 in Assumption 2.1 satisfy one of the following more general requirements.

(1) Under Condition 3a, for $i = 2, ..., d$, either of the following is satisfied:
   (a) b^i is convex, $h_{x_i} \geq 0$, and either $b^i_{x_1}$, $b^i_{x_1x_i}$, $h_{x_1x_i} \leq 0$ or $b^i_{x_1}$, $b^i_{x_1x_i}$, $h_{x_1x_i} \geq 0$;
   (b) b^i is concave, $h_{x_i} \leq 0$, and either $b^i_{x_1}$, $-b^i_{x_1x_i}$, $h_{x_1x_i} \leq 0$ or $b^i_{x_1}$, $-b^i_{x_1x_i}$, $h_{x_1x_i} \geq 0$.
(2) Under Condition 3b, for $i = 2, ..., d$, either of the following is satisfied:
   (a) b^i is convex, $h_{x_i} \geq 0$, and $b^i_{x_1}$, $b^i_{x_1x_i}$, $h_{x_1x_i} \leq 0$;
   (b) b^i is concave, $h_{x_i} \leq 0$, and $b^i_{x_1}$, $-b^i_{x_1x_i}$, $h_{x_1x_i} \leq 0$.

We point out that the conditions to deal with a linear volatility need to be compatible with the arguments in the proof of Lemma 4.1 and are, for this reason, more restrictive.

5.1.3. *On the lower-growth of h.* We underline that the lower-growth requirement on h in Condition 1 can be improved in some particular settings: If the drift is affine and the volatility is constant, for $p \leq 2$ it is sufficient to assume $h \geq -\kappa_2$. Indeed, in this case, the proof of the estimate (A.5) in Step 2 in the proof of Theorem A.1 in Appendix A simplifies (in particular, in (A.6), $M_2 = 0$) and it can be provided without relying on Lemma 3.6. Also, for any $x \in \mathbb{R}^d$ and any sequence of minimizing controls $(v^n)_{n\in\mathbb{N}}$, we have the estimate

$$\sup_n \mathbb{E}\big[\textstyle\int_{[0,\infty)} e^{-\rho t} d|v^n|_t \big] \leq C(1 + |x|^p),$$

which, combined with $\mathbb{E}[|X_t^{x;v^n}|] \leq C(1 + |x|^p + \mathbb{E}[|v^n|_t])e^{\bar L t}$, gives

$$\sup_n \mathbb{E}\big[\textstyle\int_{[0,\infty)} e^{-(\rho+\bar L)t} |X_t^{x;v^n}| dt\big] \leq \sup_n C\Big(1 + |x|^p + \mathbb{E}\big[\textstyle\int_0^\infty e^{-\rho t} |v^n|_t dt\big]\Big) \leq C(1 + |x|^p).$$

Therefore, a limit process $\hat X$ such that $X_t^{x;v^n} \to \hat X_t$ $\mathbb{P}\otimes dt$-a.e. as $n \to \infty$ can be found, by adapting the reasoning in Step 1 in the proof of Lemma 3.7. Also, using Lemma 3.5 in [40], in the spirit of what has been done in Subsection 4.2, we can exploit the convexity of h and the fact that $\bar b$ is affine in order to prove that $\hat X = X^{x;v}$, with v optimal control for the given x. This allows to recover Lemma 3.7 and to characterize the optimal control v.

5.2. **Some remarks.** We provide here some extensions to the results contained in this paper.

Remark 5.1 (Asymmetric costs of action)**.** *Unless to slightly modify some of the arguments in this paper, Theorem 2.5 extends to the case in which increasing the first component of the state process has a different cost than decreasing it; that is, to the cost functional*

$$J_{\kappa_1,\kappa_2}(x; v) := \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X_t^{x;v}) dt + \kappa_1 \int_{[0,\infty)} e^{-\rho t} d\xi_t^+ + \kappa_2 \int_{[0,\infty)} e^{-\rho t} d\xi_t^- \bigg], \quad \kappa_1,\, \kappa_2 > 0.$$

In this case, the value function V solves the HJB equation

$$\max\{\rho V - \mathcal{L}V - h, -V_{x_1} - \kappa_1, V_{x_1} - \kappa_2\} = 0, \quad a.e. \text{ in } \mathbb{R}^2.$$

This can be shown by employing arguments similar to those in the proof of Theorem A.1 in Appendix A, by replacing the penalizing term in (A.3) with an "asymmetric" penalization $[\beta(-V_{x_1} - \kappa_1) + \beta(V_{x_1} - \kappa_2)]/\varepsilon$. *Most of the arguments in this paper remains essentially unchanged, and the optimal control can be characterized as the solution to a Skorokhod problem on the domain* $\mathcal{W}_{\kappa_1,\kappa_2} := \{y \in \mathbb{R}^d \,|\, \kappa_1 < V_{x_1}(y) < \kappa_2\}$.

Remark 5.2 (Monotone controls). *The approach in this paper allows also to characterize optimal controls for stochastic singular control problems where the minimization problem is formulated over the set of monotone controls; that is, when*

$$V(x) := \inf_{\xi \in \mathcal{V}_\uparrow} J(x;\xi) \quad \textit{with} \quad \mathcal{V}_\uparrow := \{\xi \in \mathcal{V},\, \xi \textit{ nondecreasing}\}.$$

In this case, V solves the HJB equation $\max\{\rho V - \mathcal{L}V - h, -V_{x_1} - 1\} = 0$, *a.e. in D, and its derivative* V_{x_1} *is the value function of an optimal stopping problem (rather than a Dynkin game). The arguments in this paper can be easily adapted, and the optimal control can be characterized as the solution to a Skorokhod problem on the domain* $\mathcal{W}_+ := \{y \in \mathbb{R}^d \,|\, 1 < V_{x_1}(y)\}$. *We stress that, in this case, the additional requirements on h and* $\bar{b}$ *in Condition 3b in Assumption 2.1 are not anymore needed (see Remark 2.2).*

Remark 5.3 (Finite time horizon). *A characterization result analogous to Theorem 2.5 could also be investigated for an optimal control problem over a finite time-horizon. For example, when* $d = 2$ *and b is affine, a connection with Dynkin games is known from [16]. Therefore, it seems possible to use this connection in order to investigate the monotonicity of the value of the game (as in Proposition 3.3), and to use this monotonicity in order to construct* ε*-optimal controls* v^ε. *In this case, building on the results in [9], one can try to study the limit as* $\varepsilon \to 0$ *of* $(v^\varepsilon)_{\varepsilon>0}$, *in order to provide a characterization of the optimal control.*

5.3. **Examples.** For the sake of illustration, we begin with the following:

Example 1. *For* $d = 2$, ρ *large enough, a convex nonincreasing function* ϕ *and a convex nondecreasing function f, in light of the discussion in Section 5.1 the optimal control can be characterized in the following settings:*

(1) $\bar{\sigma}$ *as in Condition 3a and*
   (a) $b^2(x) = a^2 + b_1^2 x_1 + b_2^2 x_2$, $h(x) = |x|^2$, $h(x) = (x_1 - x_2)^2$ *with* $b_1^2 \leq 0$, $h(x) = (x_1 + x_2)^2$ *with* $b_1^2 \geq 0$;
   (b) $b^2(x) = \phi(x_1) + b_2^2 x_2$, $h(x) = |x_1|^2 + f(x_2)$;
(2) $\bar{\sigma}$ *as in Condition 3b,* $x_1^* > 0$ *and*
   (a) $b^2(x) = a^2 + b_1^2 x_1 + b_2^2 x_2$, $h(x) = |x_1 - x_1^*|^2 + f(x_2)$;
   (b) $b^2(x) = \phi(x_1) + b_2^2 x_2$, $h(x) = |x_1 - x_1^*|^2 + f(x_2)$, $h(x) = |x_1 - x_1^*|^2 + f(x_2 - x_1)$.

In particular, Example 1a represents a relevant class of linear-quadratic *stochastic singular control problems, and it is the main example of this paper.*

Example 2. *Here we discuss a model of pollution control. In the sequel,* $x \in \mathbb{R}^2_+$ *is the given and fixed initial condition of the state variable. Consider a company that can increase via an irreversible investment plan* $\xi \in \mathcal{V}_\uparrow$ *(cf. Remark 5.2) its production capacity* $X^{1,x;\xi}$. *The latter depreciates at constant rate* $\delta > 0$ *and is randomly fluctuating, e.g. because of technological uncertainty. Production leads to emissions of pollutants and thus impacts the level of a state process* $X^{2,x;\xi}$ *which summarizes one or more stocks of environmental pollutants (such as the average concentration of CO2 in the atmosphere). We assume that such an externality of production on the stock of pollutants is measured by a positive, convex, increasing, Lipschitz continuous function* ϕ *that has bounded second order derivative. Overall, the dynamics of*

$X^{x;\xi}$ *is given by*

$$\begin{cases} dX_t^{1,x;\xi} = -\delta X_t^{1,x;\xi} dt + \sigma_1 X_t^{1,x;\xi} dW_t^1 + d\xi_t, \\ dX_t^{2,x;\xi} = (\phi(X_t^{1,x;\xi}) - X_2^{2,x;\xi}) dt + \sigma_2 X_t^{2,x;\xi} dW_t^2. \end{cases}$$

The company aims at choosing a production plan that minimizes the sum of different costs: the cost of not meeting a given production level θ; the penalty of leading to a level of pollution that exceeds some environmental target ϑ; the proportional costs of investment. That is,

$$V(x) = \inf_{\xi \in \mathcal{V}_\uparrow} \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \big((X_t^{1,x;\xi} - \theta)^2 + c(X_t^{2,x;\xi} - \vartheta) \big) dt + \int_{[0,\infty)} e^{-\rho t} d\xi_t \bigg].$$

Here, $c \in C^{2;1}(\mathbb{R})$ is a nonnegative, nondecreasing, convex, Lipschitz continuous function such that $c(y) = 0$ for $y \leq 0$, and with bounded second order derivative. In light of the discussion in Subsections 5.1 and 5.2, the optimal control for V can be characterized as the solution to its related Skorokhod problem.

We next turn our focus to examples of bounded-variation problems treated in the literature and for which our results apply.

Example 3. *We discuss the model studied in [15]. For $d = 2$, consider the singular control problem with running cost $h(x_1, x_2) = \nu x_1^2 + x_2^2$, for $\nu > 0$, and drift $\bar{b}(x) = a + bx$, for a constant vector $a \in \mathbb{R}^2$ and a matrix*

$$b = \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix} \in \mathbb{R}^{2\times 2},$$

Observe that the requirements discussed in Subsection 5.1.1, are satisfied by assuming $b_2^1 = 0$ and $\rho > 2\Lambda(b)$. Therefore, Theorem 2.5 gives the optimal control as the solution of the related Skorokhod problem. The same result was obtained in [15] only under the additional assumption of a global Lipschitz-continuous free boundary.

Example 4. *Another example of set up similar to ours has been studied in [62], where a multidimesional singular control problem with $d \geq 2$ and constant drift and volatility is considered. There, the author shows the C^2-regularity of the value function, allowing for the characterization of the optimal policy as a solution to the related Skorokhod problem (even in the case of a state dependent cost of intervention). It is easy to see that, when the drift $\bar{b}$ is assumed to be constant, no monotonicity of the running cost h is required in order to obtain our Theorem 2.5. In comparison with [62], our main result (cf. Theorem 2.5) allows to characterize the optimal policy even in cases in which the dynamics are interconnected (at the cost of additional structural conditions on the running cost h).*

5.4. **An example with degenerate dynamics.** A more involved discussion is required to treat the degenerate singular control problem studied in [27] (see also [26]).

In this subsection, we take $d = 2$, h satisfying Condition 1 in Assumption 2.1, $\bar{b}(x) = (\bar{b}^1(x), \bar{b}^2(x))^\top = a + bx$, and

$$a = \begin{pmatrix} 0 \\ a^2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ b_1^2 & b_2^2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 \\ 0 & \eta \end{pmatrix}, \quad b_1^2,\, \eta,\, \rho > 0, \quad b_2^2 \leq 0, \quad h_{x_1 x_2} \geq 0. \tag{5.1}$$

In order to simplify the analysis of this example, assume $p = 2$ and $b_2^2 < 0$ and observe that, in this case, $\lambda(b) = 0$ (see the discussion in Subsection 5.1.1). The analysis of this subsection can be repeated also for $b_2^2 = 0$ and for a general $p \geq 1$.

Despite in this example the matrix $\sigma\sigma^\top$ is degenerate, the arguments in this paper can be employed in order to characterize the optimal control. However, some extra care is needed in order to prove the regularity of the value function inside the waiting region, which in fact follows from the properties of the free boundary proved in [27] and [26].

We begin the discussion by observing that results analogous to the ones contained in Appendix A hold. In particular, Theorem A.1 can be shown by using a suitable perturbation of the matrix σ (see the Appendix A in [27], for more details). The connection with Dynkin games holds as well (see Theorem 3.1 in [27]), so that the arguments leading to Proposition 3.3 (which make no use of the non-degeneracy of $\sigma\sigma^\top$) can be recovered.

5.4.1. *Regularity of V in $\mathcal{W}$.* We enforce an additional hypothesis, which is satisfied by $h(x) = |x|^2$ or $h(x) = (x_1 + x_2)^2$.

Assumption 5.4.

(1) $\lim_{x_2 \to \pm\infty} h_{x_2}(x_1, x_2) = \pm\infty$ for any $x_1 \in \mathbb{R}$;
(2) One of the following hold true:
  (a) $h_{x_1}(x_1, \cdot)$ is strictly increasing for any $x_1 \in \mathbb{R}$;
  (b) $h_{x_1 x_2} = 0$ and $h(x_1, \cdot)$ is strictly convex for any $x_1 \in \mathbb{R}$.

As in Proposition 5.8 in [26] (see otherwise Proposition 4.25 at p. 92 in [56]), under the additional Assumption 5.4, there exist two nonincreasing locally Lipschitz continuous functions g_1, $g_2 : \mathbb{R} \to \mathbb{R}$ such that

$$\mathcal{I}_- = \{x \in \mathbb{R}^2 \,|\, x_2 \le g_1(x_1)\} \quad \text{and} \quad \mathcal{I}_+ = \{x \in \mathbb{R}^2 \,|\, x_2 \ge g_2(x_1)\}. \tag{5.2}$$

For each $x \in \mathbb{R}^2$, recall the definition of $\bar\tau_1$, $\bar\tau_2$ given in Theorem 3.2 and define the stopping times

$$\bar\tau_1^\delta := \inf\{t \ge 0 \,|\, X_t^{x+\delta e_1} \in \mathcal{I}_-\}, \quad \bar\tau_2^\delta := \inf\{t \ge 0 \,|\, X_t^{x+\delta e_1} \in \mathcal{I}_+\}, \quad \delta \in \mathbb{R}. \tag{5.3}$$

The Lipschitz continuity of g_1 and of g_2 allows to prove the following lemma.

Lemma 5.5. *Under the additional Assumption 5.4, for $x \in \mathbb{R}^2$, we have*

$$\lim_{\delta \to 0} \bar\tau_1^\delta = \bar\tau_1, \quad \textit{and} \quad \lim_{\delta \to 0} \bar\tau_2^\delta = \bar\tau_2, \quad \mathbb{P}\text{-}a.s.$$

Proof. We only prove the first of the two limits for $\delta \to 0^+$, since the same limit for $\delta \to 0^-$ follows by identical arguments, and the second limit can be proved in the same way. We first observe that, since g_1 is finite, we have $\mathbb{P}[\bar\tau_1 < \infty] = 1$. Also, when $\delta > 0$, we have, by convexity of V and by Proposition 3.3, that $V_{x_1}(x_1 + \delta, X_t^{2,x+\delta e_1}) \ge V_{x_1}(x_1, X_t^{2,x+\delta e_1}) \ge V_{x_1}(x_1, X_t^{2,x})$, from which we deduce that

$$\bar\tau_1^\delta \ge \bar\tau_1, \quad \mathbb{P}\text{-a.s.} \tag{5.4}$$

We continue the proof arguing by contradiction. In light of (5.4), suppose that there exists $E \in \mathcal{F}$, with $\mathbb{P}[E] > 0$, such that for each $\omega \in E$ there exists $\varepsilon(\omega) > 0$ and a sequence $(\delta_j(\omega))_{j\in\mathbb{N}}$ with $\delta_j > 0$ and $\delta_j \to 0$ as $j \to \infty$, for which $\bar\tau_1^{\delta_j}(\omega) > \bar\tau_1(\omega) + \varepsilon(\omega)$ for each $j \in \mathbb{N}$. Using the representation in (5.2), (dropping the dependence on ω to simplify the notation) this is equivalent to

$$X_{\bar\tau_1}^{2,x} \le g_1(x_1) \text{ and } X_{\bar\tau_1+s}^{2,x+\delta_j e_1} > g_1(x_1 + \delta_j), \text{ for each } s \in [0, \varepsilon],\ j \in \mathbb{N}. \tag{5.5}$$

Notice that, due to the particular structure of the dynamics, we have

$$X_s^{2,x+\delta_j e_1} = X_s^{2,x} + \delta_j b_1^2 \big(e^{b_2^2 s} - 1\big)/b_2^2, \quad s \ge 0,\ j \in \mathbb{N}, \tag{5.6}$$

from which we can write

$$\begin{aligned} X_{\bar\tau_1}^{2,x} &= (X_{\bar\tau_1}^{2,x} - X_{\bar\tau_1+s}^{2,x}) + X_{\bar\tau_1+s}^{2,x} \\ &= -\int_0^s (a_2 + b_1^2 x_1 + b_2^2 X_{\bar\tau_1+r}^{2,x}) dr - \eta(W_{\bar\tau_1+s} - W_{\bar\tau_1}) + X_{\bar\tau_1+s}^{2,x+\delta_j e_1} - \delta_j b_1^2 \big(e^{b_2^2(\bar\tau_1+s)} - 1\big)/b_2^2, \end{aligned}$$

From the latter equality, using (5.5), by Lipschitz continuity of g_1, and pathwise boundedness of $X^{2,x}$ and of $\bar\tau_1$, we obtain

$$X^{2,x}_{\bar\tau_1} \geq -\delta_j C - sC + \eta(W_{\bar\tau_1+s} - W_{\bar\tau_1}) + g_1(x_1) - \delta_j, \text{ for each } s \in [0,\varepsilon],\ j \in \mathbb{N}, \tag{5.7}$$

where the constant C depends on $\bar\tau_1$ (which is finite, by assumption) and on $\sup_{r\in[0,\varepsilon]} X^{2,x}_{\bar\tau_1+r}$, but it is independent from s and j. Next, by the law of iterated logarithm (see, e.g., Theorem 9.23 at p. 112 in [41]) we find a sequence $(s_k)_{k\in\mathbb{N}}$ converging to zero and $\bar k \in \mathbb{N}$ (depending on ω) such that

$$(W_{\bar\tau_1+s_k} - W_{\bar\tau_1}) \geq \sqrt{s_k}\sqrt{\log\log(1/s_k)} \geq \sqrt{s_k}, \quad \text{for each } k \geq \bar k. \tag{5.8}$$

Finally, from (5.7) and (5.8), for suitable choice of δ_j and s_k, we conclude that

$$X^{2,x}_{\bar\tau_1} \geq -\delta_j(C+1) + \sqrt{s_k}(\eta - C\sqrt{s_k}) + g_1(x_1) > g_1(x_1),$$

which contradicts (5.5), and therefore completes the proof of the lemma. □

Lemma 5.6. *Under the additional Assumption 5.4, we have* $V \in C^2(\mathcal{W})$.

Proof. We split the proof in two steps.

Step 1. Take $z \in \mathcal{W}$ and $\varepsilon > 0$ such that $B^1_\varepsilon(z) \times B^2_\varepsilon(z) \subset \mathcal{W}$, where $B^1_\varepsilon(z) := \{x_1 \in \mathbb{R}\,|\,|z_1 - x_1| < \varepsilon\}$ and $B^2_\varepsilon(z) := \{x_2 \in \mathbb{R}\,|\,|z_2 - x_2| < \varepsilon\}$. We prove that $V_{x_2x_2}, V_{x_1x_2}$ are locally Lipschitz in $B^1_\varepsilon(z) \times B^2_\varepsilon(z)$ and that $V_{x_1x_1}(x_1,\cdot)$ is locally Lipschitz in $B^2_\varepsilon(z)$ for each $x_1 \in B^1_\varepsilon(z)$.

We begin by observing that, under (5.1), the HJB equation can be regarded a second order ordinary differential equation (ODE, in short) in the variable $x_2 \in \mathbb{R}$ depending on the parameter $x_1 \in \mathbb{R}$. In particular, V solves the equation

$$\rho V - \bar b^2 V_{x_2} - (\eta^2/2)V_{x_2x_2} = h, \quad \text{for a.a. } x_2 \in B^2_\varepsilon(z), \text{ for each fixed } x_1 \in B^1_\varepsilon(z). \tag{5.9}$$

Therefore we have $V(x_1,\cdot) \in C^{4;1}(B^2_\varepsilon(z))$, for each $x_1 \in B^1_\varepsilon(z)$. Next, for any $y_1, x_1 \in B^1_\varepsilon(z)$ we define the function $W(x_2) := V(y_1,x_2) - V(x_1,x_2),\ x_2 \in B^2_\varepsilon(z)$, which satisfies the ODE

$$\rho W - \bar b^2(y_1,\cdot)W_{x_2} - (\eta^2/2)W_{x_2x_2} = F, \quad x_2 \in B^2_\varepsilon(z),$$

where $F = h(y_1,\cdot) - h(x_1,\cdot) + b_1^2 V_{x_2}(x_1,\cdot)(y_1 - x_1)$. Therefore, by employing Schauder interior estimates (see Theorem 6.2 at p. 90 in [29]), we obtain

$$\|W\|_{C^{2;1}(B^2_{\varepsilon/2}(z))} \leq C(\|W\|_{C^0(B^2_\varepsilon(z))} + \|F\|_{C^{0;1}(B^2_\varepsilon(z))}).$$

Moreover, by the $W^{2;\infty}_{loc}$-regularity of V (cf. Theorem A.1 in Appendix A), the function F is Lipschitz in $B^1_\varepsilon(z) \times B^2_\varepsilon(z)$. Thus, the latter estimate implies that

$$\|V(y_1,\cdot) - V(x_1,\cdot)\|_{C^{2;1}(B^2_{\varepsilon/2}(z))} \leq C|y_1 - x_1|,$$

for a constant C which is independent from y_1 and x_1, as long as they are elements of $B^1_\varepsilon(z)$. Hence, the functions V, V_{x_2}, $V_{x_2x_2}$ are Lipschitz continuous in $B^1_\varepsilon(z) \times B^2_{\varepsilon/2}(z)$.

We can therefore compute the weak derivative of (5.9) with respect to x_1, obtaining, for each fixed $x_1 \in B^1_\varepsilon(z)$, the ODE

$$\rho V_{x_1} - b^2 V_{x_1x_2} - (\eta^2/2)V_{x_1x_2x_2} = h_{x_1} + b_1^2 V_{x_2}, \quad \text{for a.a. } x_2 \in B^2_{\varepsilon/2}(z). \tag{5.10}$$

Since $V_{x_2x_2}$ is Lipschitz, we have $V_{x_1}(x_1,\cdot) \in C^{3;1}(B^2_{\varepsilon/2}(z))$, for each $x_1 \in B^1_\varepsilon(z)$. Also, we can again define a function function $W^1(x_2) := V_{x_1}(y_1,x_2) - V_{x_1}(x_1,x_2),\ x_2 \in B^2_{\varepsilon/2}(z)$, which satisfies the elliptic equation

$$\rho W^1 - b^2(y_1,\cdot)W^1_{x_2} - (\eta^2/2)W^1_{x_2x_2} = F^1, \quad x_2 \in B^2_{\varepsilon/2}(z),$$

where $F^1 = h_{x_1}(y_1,\cdot) - h_{x_1}(x_1,\cdot) + b_1^2(V_{x_2}(y_1,\cdot) - V_{x_2}(x_1,\cdot)) + b_1^2 V_{x_1x_2}(x_1,\cdot)(y_1 - x_1)$. By employing again Schauder interior estimates, we obtain

$$\|W^1\|_{C^{2;1}(B^2_{\varepsilon/3}(z))} \leq C(\|W^1\|_{C^0(B^2_{\varepsilon/2}(z))} + \|F^1\|_{C^{0;1}(B^2_{\varepsilon/2}(z))}).$$

This, by the local Lipschitz continuity of V_{x_2} and $V_{x_1x_2}$ (since we have shown that $V_{x_1x_2x_2}$ exists bounded) in the variable x_2, implies that

$$\|V_{x_1}(y_1,\cdot) - V_{x_1}(x_1,\cdot)\|_{C^{2;1}(B^2_{\varepsilon/2}(z))} \leq C|y_1 - x_1|;$$

that is, the functions V_{x_1}, $V_{x_1x_2}$, $V_{x_1x_2x_2}$ are Lipschitz continuous in $B^1_\varepsilon \times B^2_{\varepsilon/3}(z)$.

This allows to compute once more the weak derivative w.r.t. x_1 in equation (5.10), obtaining for each fixed $x_1 \in B^1_\varepsilon(z)$, the ODE

$$\rho V_{x_1x_1} - b^2 V_{x_1x_1x_2} - (\eta^2/2)V_{x_1x_1x_2x_2} = h_{x_1x_1} + 2b_1^2 V_{x_1x_2}, \quad \text{for a.a. } x_2 \in B^2_{\varepsilon/3}(z). \tag{5.11}$$

Therefore, since we have shown that $V_{x_1x_2}$ is Lipschitz, after employing one more time Schauder interior estimates, we obtain

$$\|V_{x_1x_1}\|_{C^{2;1}(B^2_{\varepsilon/4}(z))} \leq C(\|V_{x_1x_1}\|_{C^0(B^2_{\varepsilon/3}(z))} + \|h_{x_1x_1} + 2b_1^2 V_{x_1x_2}\|_{C^{0;1}(B^2_{\varepsilon/3}(z))}) \leq C, \quad x_1 \in B^1_\varepsilon(z),$$

for C large enough, not depending on x_1. In particular we deduce that $V_{x_1x_1}(x_1,\cdot)$ is Lipschitz in $B^2_{\varepsilon/4}(z)$, with Lipschitz constant uniformly bounded for $x_1 \in B^1_\varepsilon(z)$.

Step 2. We now prove that $V_{x_1x_1}(\cdot, x_2)$ is continuous in $\mathcal{W}^1(x_2)$ (see Lemma 2.3), for each $x_2 \in \mathbb{R}$. This is done by employing a direct computation to find an expression for $V_{x_1x_1}$.

Fix $x \in \mathcal{W}$ and let $\hat{h}$ be as in Theorem 3.1. For $\delta > 0$, from (5.4) in the proof of Lemma 5.5, we have $\bar{\tau}_1^\delta \geq \bar{\tau}_1$. Then, from (5.6) and Theorem 3.2, we write

$$\begin{aligned}
\frac{V_{x_1}(x+\delta e_1) - V_{x_1}(x)}{\delta} &\leq \frac{G(x+\delta e_1; \bar{\tau}_1^\delta, \bar{\tau}_2) - G(x; \bar{\tau}_1^\delta, \bar{\tau}_2)}{\delta} \\
&= \mathbb{E}\Big[\int_0^{\bar{\tau}_1^\delta \wedge \bar{\tau}_2} e^{-\rho t}\Big(\frac{\hat{h}(X_t^{x+\delta e_1}) - \hat{h}(X_t^x)}{\delta}\Big) dt\Big] \\
&= \mathbb{E}\Big[\int_0^{\bar{\tau}_1 \wedge \bar{\tau}_2} \int_0^1 e^{-\rho t}\Big(\hat{h}_{x_1}(Z_t^{\delta,r}) + \hat{h}_{x_2}(Z_t^{\delta,r}) b_1^2 (e^{b_2^2 t} - 1)/b_2^2\Big) dr dt\Big] \\
&\quad + \mathbb{E}\Big[\int_{\bar{\tau}_1 \wedge \bar{\tau}_2}^{\bar{\tau}_1^\delta \wedge \bar{\tau}_2} \int_0^1 e^{-\rho t}\Big(\hat{h}_{x_1}(Z_t^{\delta,r}) + \hat{h}_{x_2}(Z_t^{\delta,r}) b_1^2 (e^{b_2^2 t} - 1)/b_2^2\Big) dr dt\Big] =: M_1^\delta + M_2^\delta,
\end{aligned} \tag{5.12}$$

where $Z_t^{\delta,r} := X_t^x + r(X_t^{x+\delta e_1} - X_t^x)$. Next, in order to study M_1^δ and M_2^δ, define

$$H(t,y) := \hat{h}_{x_1}(y) + \hat{h}_{x_2}(y) b_1^2 (e^{b_2^2 t} - 1)/b_2^2, \quad y \in \mathbb{R}^2. \tag{5.13}$$

Notice that, by (5.1), Proposition 3.3 (see the discussion in Subsection 5.1.1) and the convexity of V we have $h_{x_1x_1}$, $b_1^2 h_{x_1x_2}$, $b_1^2 V_{x_1x_2}$, $V_{x_2x_2} \geq 0$, and hence

$$H \geq 0. \tag{5.14}$$

Moreover, since $p = 2$, from Proposition 2.4 in [27], for each $\bar{y}$, $y \in \mathbb{R}^2$, and $\lambda \in [0,1]$, we have

$$\lambda V(\bar{y}) + (1-\lambda)V(y) - V(\lambda \bar{y} + (1-\lambda)y) \leq K\lambda(1-\lambda)|\bar{y} - y|^2, \tag{5.15}$$

for some $K > 0$. Hence, (5.14) and (5.15) together with Condition 1 in Assumption 2.1 give

$$0 \leq H(t,y) \leq C. \tag{5.16}$$

By Step 1, the function $H(t,\cdot)$ is continuous in $\mathcal{W}$. Moreover, since $Z^{\delta,r} \to X^x$ for $\mathbb{P}\otimes dt\otimes dr$-a.a. $(\omega,t,r)\in\Omega\times[0,\infty)\times(0,1)$, as $\delta\to 0$, we deduce that $H(t,Z_t^{\delta,r})\to H(t,X_t^x)$, $\mathbb{P}\otimes dt\otimes dr$-a.e. as $\delta\to 0$. Therefore, thanks to (5.16), by the dominated convergence theorem we have

$$\lim_{\delta\to 0^+} M_1^\delta = \mathbb{E}\bigg[\int_0^{\bar\tau_1\wedge\bar\tau_2} e^{-\rho t}\Big(\hat h_{x_1}(X_t^x) + \hat h_{x_2}(X_t^x) b_1^2 (e^{b_2^2 t}-1)/b_2^2\Big)dt\bigg]. \tag{5.17}$$

Also, by Lemma 5.5 we have $\mathbb{1}_{(\bar\tau_1\wedge\bar\tau_2,\bar\tau_1^\delta\wedge\bar\tau_2)}\to 0$, $\mathbb{P}$-a.s. as $\delta\to 0$. Therefore we can again employ (5.16) and the dominated convergence theorem to conclude that

$$\lim_{\delta\to 0} M_2^\delta = 0. \tag{5.18}$$

Hence, since we already know that $V_{x_1x_1}$ exists a.e., (5.12), (5.17) and (5.18) implies that

$$V_{x_1x_1}(x) \le \mathbb{E}\bigg[\int_0^{\bar\tau_1\wedge\bar\tau_2} e^{-\rho t}\Big(\hat h_{x_1}(X_t^x) + \hat h_{x_2}(X_t^x) b_1^2 (e^{b_2^2 t}-1)/b_2^2\Big)dt\bigg], \quad \text{a.e. in } \mathcal{W}. \tag{5.19}$$

Also, arguments similar to the one leading to (5.19), allow to estimate $V_{x_1x_1}$ from below, obtaining

$$V_{x_1x_1}(x) \ge \mathbb{E}\bigg[\int_0^{\bar\tau_1\wedge\bar\tau_2} e^{-\rho t}\Big(\hat h_{x_1}(X_t^x) + \hat h_{x_2}(X_t^x) b_1^2 (e^{b_2^2 t}-1)/b_2^2\Big)dt\bigg], \quad \text{a.e. in } \mathcal{W},$$

which, together with (5.19), implies that

$$V_{x_1x_1}(x) = \mathbb{E}\bigg[\int_0^{\bar\tau_1\wedge\bar\tau_2} e^{-\rho t}\Big(\hat h_{x_1}(X_t^x) + \hat h_{x_2}(X_t^x) b_1^2 (e^{b_2^2 t}-1)/b_2^2\Big)dt\bigg], \quad \text{a.e. in } \mathcal{W}. \tag{5.20}$$

We can finally study the continuity of $V_{x_1x_1}$ in the variable x_1. From (5.20) we have

$$\begin{aligned} |V_{x_1x_1}(x+\delta e_1) - V_{x_1x_1}(x)| &\le \bigg|\mathbb{E}\bigg[\int_0^{\bar\tau_1\wedge\bar\tau_2} e^{-\rho t}(H(t,X_t^{x+\delta e_1}) - H(t,X_t^x))dt\bigg]\bigg| \\ &\quad + \bigg|\mathbb{E}\bigg[\int_{\bar\tau_1\wedge\bar\tau_2}^{\bar\tau_1^\delta\wedge\bar\tau_2^\delta} e^{-\rho t} H(t,X_t^{x+\delta e_1})dt\bigg]\bigg| =: N_1^\delta + N_2^\delta, \end{aligned} \tag{5.21}$$

with H defined in (5.13). Following arguments similar to the ones leading to (5.17) and (5.18), we can show that $\lim_{\delta\to 0} N_1^\delta = 0$ and that $\lim_{\delta\to 0} N_2^\delta = 0$. Therefore, taking limits as $\delta\to 0$ in (5.21), we deduce that $V_{x_1x_1}$ is a.e. equal to a function which is continuous the variable x_1.

By Step 1, the functions $V_{x_1x_1}(x_1,\cdot)$ are locally Lipschitz continuous, uniformly in x_1. Thus, by the continuity of $V_{x_1x_1}(\cdot,x_2)$, we conclude that the function $V_{x_1x_1}$ is jointly continuous in both variables in $\mathcal{W}$. This completes the proof of the lemma. □

5.4.2. *Characterization of the optimal control.* In light of Lemma 5.6, under the additional Assumption 5.4, we can construct the ε-optimal policies. Indeed, by employing the comparison principle to the second order ODE (5.11) (regarded as an equation in the variable x_2, depending on the parameter x_1), one still obtains that $V_{x_1x_1} > 0$ in $\mathcal{W}$. This, together with the fact that $V_{x_1}\in C^1(\mathcal{W})$ (by Lemma 5.6), allows to show that S_ε is a C^1 curve in $\mathbb{R}^2$ and that the vector field $-e_1 V_{x_1}/|V_{x_1}|$ is C^1 on S_ε, and nontangential to S_ε. All the assumptions in CASE 2 at p. 557 in [24] (up to the boundedness of $\mathcal{W}$) are then satisfied, and we can therefore employ (a suitable extension to unbounded domains of) Theorem 5.1 at p. 572 in [24] in order to find the ε-optimal controls as in Lemma 3.4. Finally, all the arguments in Section 3.3 can be repeated in the case in which $\sigma\sigma^\top$ is degenerate. Overall, we have proved the following result.

Theorem 5.7. *Consider the degenerate singular control problem described in* (5.1)*, with h satisfying Condition 1 in Assumption 2.1 and Assumption 5.4. Then, the thesis of Theorem 2.5 holds.*

Concluding, with respect to [27], we require in addition that $h_{x_1x_1} > 0$ and that Assumption 5.4 is satisfied. In this case, Theorem 5.7 applies, and the construction of the optimal control discussed in Section 7 in [27] can be provided. We underline that in [27] a construction of an optimal control is given in weak formulation, under a quite strong requirement on the running cost h. We refer to Proposition 7.3 in [27] for more details.

Appendix A. On the HJB equation

In this subsection we prove that V is a solution (in the a.e. sense) to the related HJB equation. The argument of the proof exploits the penalization method introduced in [25] for bounded domains (see also [36] and the references therein), which we extend to D thanks to suitable semiconcavity estimates, in the spirit of [10]. Although this result is somehow classical, we have not been able to find versions that exactly fit our setting, and we therefore provide its proofs in the following.

Theorem A.1. *The value function V is a $W^{2;\infty}_{loc}(D)$-solution to the equation*

$$\max\{\rho V - \mathcal{L}V - h, |V_{x_1}| - 1\} = 0, \quad a.e. \text{ in } D. \tag{A.1}$$

Proof. We divide the proof in four steps.

Step 1. Let us start by introducing a family of penalized versions of the HJB equation (A.1). Let $\beta \in C^\infty(\mathbb{R})$ be a convex nondecreasing function with $\beta(r) = 0$ if $r \le 0$ and $\beta(r) = 2r - 1$ if $r \ge 1$. For each $\varepsilon > 0$, let V^ε be the the value function of the penalized control problem

$$V^\varepsilon(x) := \inf_{\alpha \in \mathcal{U}_\varepsilon} J_\varepsilon(x;\alpha) := \inf_{\alpha \in \mathcal{U}_\varepsilon} \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}(h(X_t^{x;\alpha}) + |\alpha_t^1| + \alpha_t^2)dt\bigg], \quad x \in D, \tag{A.2}$$

where $\mathcal{U}_\varepsilon$ is the set of E_ε-valued $\mathbb{F}$-progressively measurable processes, wtih $E_\varepsilon := \{\alpha = (\alpha^1,\alpha^2) \in \mathbb{R} \times [0,\infty) \,|\, |\alpha^1|r - \frac{1}{\varepsilon}\beta(r(r+2)) \le \alpha^2 \le \frac{1}{\varepsilon},\ \forall r > 0\}$. Here, with a slight abuse of notation, $X^{x;\alpha}$ denotes the solution to $dX_t^{x;\alpha} = (b(X_t^{x;\alpha}) + e_1\alpha_t^1)dt + \sigma dW_t,\ t \ge 0,\ X_0^{x;\alpha} = x$. We point out that, under Condition 3b in Assumption 2.1, a result analogus to Lemma 4.1 holds. Arguing as in [36] (throught a localization argument), it is possible to show that V^ε is a $C^2(D)$ solution to the partial differential equation

$$\rho V^\varepsilon - \mathcal{L}V^\varepsilon + \frac{1}{\varepsilon}\beta((V^\varepsilon_{x_1})^2 - 1) = h, \quad \text{in } D. \tag{A.3}$$

Moreover, the family $(V^\varepsilon)_{\varepsilon \in (0,1)}$ provides an approximation of V; that is,

$$\lim_{\varepsilon \to 0} V^\varepsilon(x) = V(x), \quad \text{for each } x \in D. \tag{A.4}$$

Take indeed $x \in D$. Observe that, for each $\varepsilon > 0$, we have $V^\varepsilon(x) \ge V(x)$, as $\alpha^2 \ge 0$. Moreover, as in Theorem 2.2. in [17], one can show that for each $\delta > 0$ there exists a Lipschitz admissible process $w \in \mathcal{V}$ such that $J(x;w) \le V(x) + \delta$. Since w is Lipschitz, we have $dw_t = \alpha_t^1 dt$, for some bounded progressively measurable process α^1. Then, defining $\alpha_t^2 = \rho\delta/2$, we can find $\bar\varepsilon > 0$ such that $\alpha := (\alpha^1,\alpha^2) \in \mathcal{U}_\varepsilon$ for each $\varepsilon \in (0,\bar\varepsilon)$. Moreover, with this choice of α, we have that $J_\varepsilon(x;\alpha) \le J(x;w) + \delta/2 \le V(x) + \delta$, for each $\varepsilon \in (0,\bar\varepsilon)$, completing the proof of (A.4).

Step 2. In this step we show that, under Condition 3a in Assumption 2.1, for each $R > 0$, there exists a constant C_R such that

$$0 \le \lambda V^\varepsilon(\bar x) + (1-\lambda)V^\varepsilon(x) - V^\varepsilon(\lambda\bar x + (1-\lambda)x) \le C_R\lambda(1-\lambda)|\bar x - x|^2, \tag{A.5}$$

for each $\lambda \in [0,1]$, $\bar x, x \in B_R$ and $\varepsilon > 0$. By the same arguments leading the convexity of V (cf. Remark 2.4), it is possible to show that, for each $\varepsilon > 0$, the function V^ε is convex. Therefore, we only need to prove the last inequality in (A.5). Take $\bar x$, $x \in B_R$, $\lambda \in [0,1]$ and set $x^\lambda := \lambda\bar x + (1-\lambda)x$. Fix $\varepsilon > 0$, an arbitrary $\delta > 0$, and let $\alpha \in \mathcal{U}_\varepsilon$ be a δ-optimal control for the problem (A.2) with initial condition x^λ; that is, $J_\varepsilon(x^\lambda;\alpha) \le V^\varepsilon(x^\lambda) + \delta$. Since α is not necessarily optimal for x or $\bar x$, we have

$$\begin{aligned}
&\lambda V^\varepsilon(\bar x) + (1-\lambda)V^\varepsilon(x) - V^\varepsilon(x^\lambda) - \delta \\
&\qquad \le \lambda J_\varepsilon(\bar x;\alpha) + (1-\lambda)J_\varepsilon(x;\alpha) - J_\varepsilon(x^\lambda;\alpha) \\
&\qquad \le \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\big(\lambda h(X_t^{\bar x;\alpha}) + (1-\lambda)h(X_t^{x;\alpha}) - h(X_t^{x^\lambda;\alpha})\big)dt\bigg].
\end{aligned}$$

Setting $Z_t := \lambda X_t^{\bar x;\alpha} + (1-\lambda)X_t^{x;\alpha}$, using Condition 1 in Assumption 2.1 , we continue the latter chain of estimates to find

$$\begin{aligned}
&\lambda V^\varepsilon(\bar x) + (1-\lambda)V^\varepsilon(x) - V^\varepsilon(x^\lambda) - \delta \\
&\quad \le \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\big(\lambda h(X_t^{\bar x;\alpha}) + (1-\lambda)h(X_t^{x;\alpha}) - h(Z_t)\big)dt\bigg] \\
&\qquad + \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\big(h(Z_t) - h(X_t^{x^\lambda;\alpha})\big)dt\bigg] \\
&\quad \le C\lambda(1-\lambda)\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\big(1 + |X_t^{x;\alpha}|^{p-2} + |X_t^{\bar x;\alpha}|^{p-2}\big)|X_t^{\bar x;\alpha} - X_t^{x;\alpha}|^2 dt\bigg] \\
&\qquad + C\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\big(1 + |Z_t|^{p-1} + |X_t^{x^\lambda;\alpha}|^{p-1}\big)|Z_t - X_t^{x^\lambda;\alpha}|dt\bigg] \\
&\quad =: M_1 + M_2.
\end{aligned} \tag{A.6}$$

We will now estimate M_1 and M_2 separately.

First of all, by a standard use of Grönwall's inequality, we find

$$\big|X_t^{\bar x;\alpha} - X_t^{x;\alpha}\big| \le Ce^{\bar L t}|\bar x - x|. \tag{A.7}$$

When $p = 2$, from (A.7) and our assumptions on ρ, we immediately deduce that

$$M_1 \le C_R\lambda(1-\lambda)|\bar x - x|^2, \tag{A.8}$$

as desired. On the other hand, if $p > 2$, set $p' := (2p-1)/2$. Defining $q := p'/(p-2)$ and denoting by q^* its conjugate, we can employ Hölder's inequality and obtain

$$\begin{aligned}
M_1 &\le C\lambda(1-\lambda)|\bar x - x|^2\bigg(\mathbb{E}\bigg[\int_0^\infty e^{(2\bar L - \rho(1-\frac{1}{q}))q^* t}dt\bigg]\bigg)^{\frac{1}{q^*}}\bigg(\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\big(|X_t^{x;\alpha}|^{p'} + |X_t^{\bar x;\alpha}|^{p'}\big)dt\bigg]\bigg)^{\frac{1}{q}} \\
&\le C\lambda(1-\lambda)(1 + |x|^p + |\bar x|^p)^{\frac{1}{q}}|\bar x - x|^2 \le C_R\lambda(1-\lambda)|\bar x - x|^2,
\end{aligned}$$

where we have used the requirements on ρ in Condition 3a in Assumption 2.1, and the estimate (3.26), which holds also for the penalized problem.

We next estimate M_2. Since the gradient Db is Lipschitz we have the estimate (see, e.g., Proposition 1.1.3 at p. 2 in [14])

$$|\lambda b(\bar y) + (1-\lambda)b(y) - b(\lambda\bar y + (1-\lambda)y)| \le C\lambda(1-\lambda)|\bar y - y|^2, \quad \text{for each } \bar y,\, y \in \mathbb{R}^2.$$

This, together with the Lipschitz proprerty of b, allows to obtain

$$\begin{aligned}
\text{(A.9)} \qquad \big|X_t^{x^\lambda;\alpha} - Z_t\big| &\leq \int_0^t \big|b(X_s^{x^\lambda;\alpha}) - \lambda b(X_s^{\bar{x};\alpha}) - (1-\lambda) b(X_s^{x;\alpha})\big| ds \\
&\leq \bar{L} \int_0^t \big(\big|X_s^{x^\lambda;\alpha} - Z_s\big| + \lambda(1-\lambda)\big|X_s^{\bar{x};\alpha} - X_s^{x;\alpha}\big|^2\big) ds, \\
&\leq \bar{L} \int_0^t \big(\big|X_s^{x^\lambda;\alpha} - Z_s\big| + \lambda(1-\lambda)|\bar{x} - x|^2 e^{2\bar{L}s}\big) ds, \\
&\leq C\lambda(1-\lambda)|\bar{x} - x|^2 e^{2\bar{L}t} + \bar{L} \int_0^t \big|X_s^{x^\lambda;\alpha} - Z_s\big| ds.
\end{aligned}$$

The latter estimate, after employing Grönwall's inequality, leads to

$$\text{(A.10)} \qquad \big|X_t^{x^\lambda;\alpha} - Z_t\big| \leq C\lambda(1-\lambda) e^{3\bar{L}t} |\bar{x} - x|^2.$$

Defining $q := p'/(p-1)$ and denoting by q^* is conjugate, we can again employ Hölder's inequality and (A.10) in order to obtain

$$\begin{aligned}
M_2 &\leq C\lambda(1-\lambda)|\bar{x} - x|^2 \mathbb{E}\Big[\int_0^\infty e^{(3\bar{L}-\rho)t}\big(1 + \big|Z_t\big|^{p-1} + \big|X_t^{x^\lambda;\alpha}\big|^{p-1}\big) dt\Big] \\
&\leq C\lambda(1-\lambda)|\bar{x} - x|^2 \Big(\mathbb{E}\Big[\int_0^\infty e^{(3\bar{L} - \rho(1-\frac{1}{q}))q^* t} dt\Big]\Big)^{\frac{1}{q^*}} \Big(\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\big(\big|X_t^{x^\lambda;\alpha}\big|^{p'} + \big|Z_t\big|^{p'}\big) dt\Big]\Big)^{\frac{1}{q}} \\
&\leq C\lambda(1-\lambda)(1 + |x|^p + |\bar{x}|^p)^{\frac{1}{q}} |\bar{x} - x|^2 \leq C_R \lambda(1-\lambda)|\bar{x} - x|^2,
\end{aligned}$$

where we have used the estimate (3.26) and the requirements on ρ in Condition 3a in Assumption 2.1. This, together with (A.8) and (A.6), thanks again to the arbitrariness of δ, completes the proof of (A.5).

Step 3. We now prove the estimate (A.5) under Condition 3b in Assumption 2.1. To simplify the notation, we assume $d = 2$, the generalization to $d > 2$ being straightforward. We proceed from (A.6), and we estimate M_1 and M_2 from above. To this end, define the processes

$$E_t := \exp[(b_1^1 - \sigma^2/2)t + W_t^1] \quad \text{and} \quad \hat{E}_t := \exp[(\bar{L} - \sigma^2/2)t + \sigma W_t^2].$$

We first estimate M_1. Observe that

$$\text{(A.11)} \qquad |X_t^{1,\bar{x};\alpha} - X_t^{1,x;\alpha}| = |\bar{x}_1 - x_1| E_t,$$

which we will use to estimate $|X_t^{2,\bar{x};\alpha} - X_t^{2,x;\alpha}|$. Define the process Δ as the solution to the SDE

$$d\Delta_t = \bar{L}(|X_t^{1,\bar{x};\alpha} - X_t^{1,x;\alpha}| + \Delta_t)dt + \sigma\Delta_t dW_t^2, \quad t \geq 0, \quad \Delta_0 = |\bar{x}_2 - x_2|.$$

Through a comparison principle, it is easy to check that $|X_t^{2,\bar{x};\alpha} - X_t^{2,x;\alpha}| \leq \Delta_t$, so that, using (A.11) and the explicit expression for Δ, we get

$$\text{(A.12)} \qquad |X_t^{2,\bar{x};\alpha} - X_t^{2,x;\alpha}| \leq C|\bar{x} - x| \hat{E}_t \big[1 + \textstyle\int_0^t E_s/\hat{E}_s ds\big] =: C|\bar{x} - x| P_t.$$

When $p = 2$, the estimate of M_1 can be easily deduced from (A.11) and (A.12). For $p > 2$, by employing Hölder's inequality with exponent $q = p'/(p-2)$, we find

(A.13)

$$\begin{aligned}&\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\big(1+\big|X_t^{x;\alpha}\big|^{p-2}+\big|X_t^{\bar{x};\alpha}\big|^{p-2}\big)(E_t^2+P_t^2)dt\Big]\\&\leq C\Big(\int_0^\infty e^{-\rho t}\mathbb{E}\big[1+\big|X_t^{x;\alpha}\big|^{p'}+\big|X_t^{\bar{x};\alpha}\big|^{p'}\big]dt\Big)^{\frac{1}{q}}\Big(\int_0^\infty e^{-\rho(1-\frac{1}{q})q^*t}\mathbb{E}\big[E_t^{2q*}+P_t^{2q*}\big]dt\Big)^{\frac{1}{q^*}}\\&\leq C(1+|x|^p)^{\frac{1}{q}}\Big(\int_0^\infty e^{-\rho(1-\frac{1}{q})q^*t}\mathbb{E}\big[E_t^{2q*}+P_t^{2q*}\big]dt\Big)^{\frac{1}{q^*}}\leq C_R<\infty.\end{aligned}$$

Here, we have also used (4.11), while the finiteness of the latter integral follows, after some elementary computations, from the requirements on ρ in Condition 3b in Assumption 2.1. Finally, by (A.11), (A.12) and (A.13), we obtain

$$M_1 \leq C_R\lambda(1-\lambda)|\bar{x}-x|^2. \tag{A.14}$$

We next estimate M_2. Since $\bar{b}^1$ is affine, we have $Z^1 - X^{1,x^\lambda;\alpha}$. Similarly to (A.9), one has

$$Z_t^2 - X_t^{2,x^\lambda;\alpha} \leq \int_0^t (C\lambda(1-\lambda)|X_s^{2,\bar{x};\alpha}-X_s^{2,x;\alpha}|^2 + \bar{L}\big|X_s^{x^\lambda;\alpha}-Z_s\big|)ds + \sigma\int_0^t (Z_s - X_s^{x^\lambda;\alpha})dW_s^2.$$

Therefore, employing again a comparison principle and using (A.12), we see that

$$|Z_t^2 - X_t^{2,x^\lambda;\alpha}| \leq C\lambda(1-\lambda)\hat{E}_t\int_0^t \frac{|X_s^{2,\bar{x};\alpha}-X_s^{2,x;\alpha}|^2}{\hat{E}_s}ds \leq C\lambda(1-\lambda)|\bar{x}-x|^2\int_0^t \frac{\hat{E}_t}{\hat{E}_s}P_s^2 ds. \tag{A.15}$$

Also, Hölder's inequality with exponent $q = p'/(p-1)$ yields

(A.16)

$$\begin{aligned}&\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\big(1+\big|Z_t\big|^{p-1}+\big|X_t^{x^\lambda;\alpha}\big|^{p-1}\big)\int_0^t\frac{\hat{E}_t}{\hat{E}_s}P_s^2ds\,dt\Big]\\&\leq C\Big(\int_0^\infty e^{-\rho t}\mathbb{E}\big[1+\big|X_t^{x;\alpha}\big|^{p'}+\big|X_t^{\bar{x};\alpha}\big|^{p'}\big]dt\Big)^{\frac{1}{q}}\Big(\mathbb{E}\Big[\int_0^\infty e^{-\rho(1-\frac{1}{q})q^*t}\Big(\int_0^t\frac{\hat{E}_t}{\hat{E}_s}P_s^2ds\Big)^{q*}dt\Big]\Big)^{\frac{1}{q^*}}\\&\leq C(1+|x|^p)^{\frac{1}{q}}\Big(\mathbb{E}\Big[\int_0^\infty e^{-\rho(1-\frac{1}{q})q^*t}\Big(\int_0^t\frac{\hat{E}_t}{\hat{E}_s}P_s^2ds\Big)^{q*}dt\Big]\Big)^{\frac{1}{q^*}}\leq C_R<\infty,\end{aligned}$$

Again, here we have also employed (4.11), while the finiteness of the latter integral follows, after some elementary computations, from the requirements on ρ in Condition 3b in Assumption 2.1. Finally, combining (A.15) and (A.16), we obtain $M_2 \leq C_R\lambda(1-\lambda)|\bar{x}-x|^2$, which, together with (A.14) and (A.6), implies (A.5).

Step 4. From (A.5) we deduce that, for each bounded open set $B \subset D$, there exists a constant $C_B > 0$ such that

$$\sup_{\varepsilon\in(0,1)}\|V^\varepsilon\|_{W^{2;\infty}(B)} \leq C_B. \tag{A.17}$$

This estimate allows, by mean of classical arguments (exploiting Sobolev compact embedding theorem of $W^{2;q}(B)$ into $C^1(B)$ for $q > 2+d$ and the weak compactness of the closed unit ball in $W^{2;2}(B)$) to improve the convergence in (A.4). Indeed (on each subsequence) we now have:

$$\begin{aligned}&(V^\varepsilon, DV^\varepsilon) \text{ converges to } (V, DV) \text{ uniformly in } B;\\&D^2V^\varepsilon \text{ converges to } D^2V \text{ weakly in } L^2(B).\end{aligned} \tag{A.18}$$

Let us now prove that V solves the HJB equation (A.1). First of all observe that, from (A.3) and (A.17), (unless to take a larger C_B) we have

$$\frac{1}{\varepsilon}\beta((V^{\varepsilon}_{x_1})^2-1)\le C_B, \quad \text{in } B. \tag{A.19}$$

Hence, taking pointwise limits in (A.3) and (A.19), we obtain

$$\rho V-\mathcal{L}V-h\le 0, \quad \text{and} \quad |V_{x_1}|-1\le 0 \quad \text{a.e. in } D.$$

Suppose now that the inequality $|V_{x_1}|-1\le 0$ is strict in $\bar{x}\in D$. By continuity of V_{x_1}, there exist $\eta>0$ and a neighborhood N of $\bar{x}$ such that $|V_{x_1}(x)|-1\le -\eta$ for each $x\in N$. Therefore, by uniform convergence in N, for each ε small enough we have $|V^{\varepsilon}_{x_1}(x)|-1\le -\eta/2$, and therefore, by (A.3), that $V^{\varepsilon}-\mathcal{L}V^{\varepsilon}-h=0$ in N. Passing again to the limit, this in turn implies that $\rho V-\mathcal{L}V-h=0$ in N, completing the proof of the theorem.

□

Appendix B. Proof of Lemma 2.3 and of Proposition 3.10

B.1. **Proof of Lemma 2.3.** We give a proof for $d=2$, the case $d>2$ is analogous. The set $\mathcal{W}_1(z)$ is an open interval, since, by convexity of V, the function $V_{x_1}(\cdot,z)$ is nondecreasing. We therefore show that the set $\mathcal{W}_1(z)$ is nonempty. Suppose that Condition 3a in Assumption 2.1 is in place. Arguing by contradiction, if $\mathcal{W}_1(z)=\emptyset$, then, by the continuity of V_{x_1}, we have $V_{x_1}(\cdot,z)=1$ or $V_{x_1}(\cdot,z)=-1$. If $V_{x_1}(\cdot,z)=1$, we have $V(x_1,z)+\kappa_2\ge V(x_1,z)-V(y,z)=\int_y^{x_1}V_{x_1}(r,z)dr=x_1-y\to\infty$ as $y\to-\infty$. Therefore $V(x_1,z)=\infty$, contradicting the finiteness of V (see Theorem A.1 in Appendix A). In the same way, we can not have that $V_{x_1}(\cdot,z)=-1$, which implies $\mathcal{W}_1(z)\neq\emptyset$.

On the other hand, suppose that Condition 3b in Assumption 2.1. Arguing by contradiction, we assume that $\mathcal{W}_1(z)$ is empty. From the continuity of V_{x_1}, we have $V_{x_1}(\cdot,z)=1$ or $V_{x_1}(\cdot,z)=-1$. If $V_{x_1}(\cdot,z)=-1$, then we have $V(x_1,z)+\kappa_2\ge V(x_1,z)-V(y,z)=-\int_{x_1}^{y}V_{x_1}(r,z)dr=y-x_1\to\infty$ as $y\to\infty$. Therefore $V(x_1,z)=\infty$, contradicting the finiteness of V. We therefore assume that $V_{x_1}(\cdot,z)=1$ and we show that this leads anyway to a contradiction.

For a generic $x_1\in\mathbb{R}$ with $0<x_1<x_1^*$, let $v\in\mathcal{V}$ be optimal for the initial condition $x:=(x_1,z)$, with $dv=\gamma d|v|$. By repeating the arguments leading to (3.41) in the proof of Proposition 3.9, an application of Itô's formula leads to

$$\mathbb{E}\bigg[\int_{[0,\infty)}e^{-\rho t}(1+V_{x_1}(X^{x;v}_{t-})\gamma_t)d|v|_t\bigg]\le 0.$$

This in turn implies, using $0\le 1-|V_{x_1}|\le 1+V_{x_1}u$ for all $u\in\mathbb{R}$ with $|u|=1$, that

$$\mathbb{E}[|v|_0(1+\gamma_0)]=\mathbb{E}[|v|_0(1+\gamma_0V_{x_1}(X^{x;v}_{0-})]\le\mathbb{E}\bigg[\int_{[0,\infty)}e^{-\rho t}(1+V_{x_1}(X^{x;v}_{t-})\gamma_t)d|v|_t\bigg]\le 0,$$

where the first equality follows from the assumption $V_{x_1}(\cdot,z)=1$. Also, since $|\gamma_0|=1$, $\mathbb{E}[|v|_0(1+\gamma_0)]\ge 0$, which combined with the latter inequality gives $\mathbb{E}[|v|_0(1+\gamma_0)]=0$. In other words, a possible jump a time zero must be of negative size. Therefore, since $x_1<x_1^*$, as in the proof of Lemma 4.1, we deduce that v has no jumps at time zero; that is,

$$\mathbb{P}[|v|_0>0]=0. \tag{B.1}$$

Next, fix $0<x_1<y_1<x_1^*$ and set $x=(x_1,z)$ and $y=(y_1,z)$. Since we are assuming that $V_{x_1}(\cdot,z)=1$, we have

$$V(y)-V(x)=\int_{x_1}^{y_1}V_{x_1}(r,z)dr=y_1-x_1. \tag{B.2}$$

Next, denote by v and w the optimal control for the initial conditions x and y, respectively. By (B.1), neither v or w has a jump a time zero, so that, using (B.2), we find

$$J(y; v + x_1 - y_1) = J(x; v) + |x_1 - y_1| = V(x) + y_1 - x_1 = V(y).$$

This, by uniqueness of the optimal control implies that $w = v + x_1 - y_1$, so that, since $x_1 < y_1$, the control w has a negative jump at time zero, contradicting (B.1).

Therefore also the assumption $V_{x_1}(\cdot, z) = 1$ leads to a contradiction, completing the proof of Lemma 2.3 under Condition 3b in Assumption 2.1.

B.2. **Proof of Proposition 3.10.** We split the proof in three steps.

Step 1. Let $x \in \partial\mathcal{W}$ be such that $x \in I$ for some interval $I \subset \mathbb{R}^2$, with $I \subset \partial\mathcal{W}$ and of the form

$$I = I_{a,c} := \{a + r\eta \,|\, r \in [0, c]\},$$

for some $a \in \mathbb{R}^2$, with $\eta = V_{x_1}(y)e_1$, for each $y \in I \setminus \{a\}$. Denote by $\mathcal{H}$ the set of all such x. Furthermore, assume that I in the above definition is maximal, in the sense that $a - r\eta \notin \partial\mathcal{W}$, for every $r > 0$.

Observe that, since $\partial_\eta V(\cdot) = \eta DV = |V_{x_1}(\cdot)|^2 = 1$, then

$$V(a + r\eta) = V(a) + r, \quad \text{for each } r \in [0, c]. \tag{B.3}$$

We have that

$$\mathcal{H} = \bigcup_{i=1}^{\infty} \big\{y \in \partial\mathcal{W} \,|\, V(y) - V(y - V_{x_1}(y)/i) = 1/i\big\}.$$

Suppose now that $\bar{x} \in \mathcal{H}$. Then there exists $a \in \mathbb{R}^2$ and $c > 0$ such that $x \in I_{a,c}$. Let $v^a \in \mathcal{V}$ be an optimal control for a. By (B.3), we find

$$J(\bar{x}; a - \bar{x} + v^a) = J(a; v^a) + |a - x| = V(a) + |a - x| = V(x),$$

which, by the uniqueness of the optimal control, implies that $\bar{v}_t = a - \bar{x} + v^a_t$, for any $t \geq 0$. This means exactly that the optimally controlled state starting from $\bar{x}$ jumps immediately to a.

Step 2. Let now $\bar{x} \in \overline{\mathcal{W}}$ be generic. We want to prove that $X^{\bar{x};\bar{v}}$ jumps only at those times t for which $X^{\bar{x};\bar{v}}_{t-} \in \mathcal{H}$. We argue by contradiction, and suppose that

$$\mathbb{P}[\,\omega \in \Omega \text{ s.t. there exists } t \geq 0 \text{ s.t. } X^{\bar{x};\bar{v}}_{t-}(\omega) \notin \mathcal{H} \text{ and } |X^{\bar{x};\bar{v}}_{t}(\omega) - X^{\bar{x};\bar{v}}_{t-}(\omega)| > 0] > 0.$$

For each $\varepsilon > 0$, let

$$\tau_\varepsilon := \inf\{t \geq 0 \,|\, X^{\bar{x};\bar{v}}_{t-} \notin \mathcal{H},\ |X^{\bar{x};\bar{v}}_{t} - X^{\bar{x};\bar{v}}_{t-}| \geq \varepsilon\}. \tag{B.4}$$

Take $\varepsilon > 0$ small enough such that $\mathbb{P}[\tau_\varepsilon < \infty] > 0$. Consider a sequence $(\bar{\tau}_k)_{k\in\mathbb{N}}$ of stopping times exhausting the jumps of $X^{\bar{x};\bar{v}}$ (see, e.g., Proposition 2.26 at p. 10 in [41], for a construction of such a sequence), so that

$$\tau_\varepsilon := \inf\{\bar{\tau}_k \,|\, k \in \mathbb{N},\ X^{\bar{x};\bar{v}}_{\bar{\tau}_k-} \notin \mathcal{H},\ |X^{\bar{x};\bar{v}}_{\bar{\tau}_k} - X^{\bar{x};\bar{v}}_{\bar{\tau}_k-}| \geq \varepsilon\}. \tag{B.5}$$

Since the jumps of $\bar{v}$ coincides with the jumps of $X^{\bar{x};\bar{v}}$, if $X^{\bar{x};\bar{v}}$ would have an infinite number of jumps of size grater than ε on some interval $[0, T]$ with $T \in (0, \infty)$, then $\bar{v}$ would not be of bounded variation on the interval $[0, T]$. Thus $X^{\bar{x};\bar{v}}$ has only a finite number of jumps of size grater than ε on each interval $[0, T]$. This reveals that τ_ε in (B.5) is actually the minimum of a finite number of stopping times, which implies that τ_ε is itself a stopping time.

Next, on $\{\tau_\varepsilon < \infty\}$, we find

$$V(X^{\bar{x};\bar{v}}_{\tau_\varepsilon}) - V(X^{\bar{x};\bar{v}}_{\tau_\varepsilon-}) = \int_0^1 DV(\tau_\varepsilon, X^{\bar{x};\bar{v}}_{\tau_\varepsilon-} + \lambda(X^{\bar{x};\bar{v}}_{\tau_\varepsilon} - X^{\bar{x};\bar{v}}_{\tau_\varepsilon-}))(X^{\bar{x};\bar{v}}_{\tau_\varepsilon} - X^{\bar{x};\bar{v}}_{\tau_\varepsilon-})d\lambda$$
$$= \int_0^1 V_{x_1}(\tau_\varepsilon, X^{\bar{x};\bar{v}}_{\tau_\varepsilon-} + \lambda(X^{\bar{x};\bar{v}}_{\tau_\varepsilon} - X^{\bar{x};\bar{v}}_{\tau_\varepsilon-}))\bar{\gamma}_{\tau_\varepsilon}(|\bar{v}|_{\tau_\varepsilon} - |\bar{v}|_{\tau_\varepsilon-})d\lambda$$
$$> -|X^{\bar{x};\bar{v}}_{\tau_\varepsilon} - X^{\bar{x};\bar{v}}_{\tau_\varepsilon-}|, \tag{B.6}$$

where the strict inequality follows from the fact that, by Proposition 3.8, $X^{\bar{x};\bar{v}}_{\tau_\varepsilon} \in \overline{\mathcal{W}}$ but τ_ε is such that $X^{\bar{x};\bar{v}}_{\tau_\varepsilon-} \notin \mathcal{H}$. Recalling that τ_ε is a stopping time, define the sequence of stopping times $\tau_k := (\tau_\varepsilon + \frac{1}{k}) \wedge T$. By the dynamic programming principle (see, e.g., [35]) we have, for each k

$$V(\bar{x}) = \mathbb{E}\bigg[\int_0^{\tau_k} e^{-\rho t} h(X^{\bar{x};\bar{v}}_t)dt + \int_{[0,\tau_k)} e^{-\rho t} d|\bar{v}|_t + e^{-\rho\tau_k} V(X^{\bar{x};\bar{v}}_{\tau_k-})\bigg]. \tag{B.7}$$

Therefore, taking limits as $k \to \infty$ in (B.7), using (B.6) we find

$$V(\bar{x}) = \mathbb{E}\bigg[\int_0^{\tau_\varepsilon} e^{-\rho t} h(X^{\bar{x};\bar{v}}_t)dt + \int_{[0,\tau_\varepsilon]} e^{-\rho t} d|\bar{v}|_t + e^{-\rho\tau_\varepsilon} V(X^{\bar{x};\bar{v}}_{\tau_\varepsilon})\bigg]$$
$$= \mathbb{E}\bigg[\int_0^{\tau_\varepsilon} e^{-\rho t} h(X^{\bar{x};\bar{v}}_t)dt + \int_{[0,\tau_\varepsilon)} e^{-\rho t} d|\bar{v}|_t + e^{-\rho\tau_\varepsilon}|X^{\bar{x};\bar{v}}_{\tau_\varepsilon} - X^{\bar{x};\bar{v}}_{\tau_\varepsilon-}| + e^{-\rho\tau_\varepsilon} V(X^{\bar{x};\bar{v}}_{\tau_\varepsilon-})\bigg]$$
$$> \mathbb{E}\bigg[\int_0^{\tau_\varepsilon} e^{-\rho t} h(X^{\bar{x};\bar{v}}_t)dt + \int_{[0,\tau_\varepsilon)} e^{-\rho t} d|\bar{v}|_t + e^{-\rho\tau_\varepsilon} V(X^{\bar{x};\bar{v}}_{\tau_\varepsilon-})\bigg] = V(\bar{x}),$$

which is a contradiction, hence $X^{\bar{x};\bar{v}}$ jumps only at times t such that $X^{\bar{x};\bar{v}}_{t-} \in \mathcal{H}$.

Step 3. Suppose now that $X^{\bar{x};\bar{v}}_{t-} \in \mathcal{H}$ for some $t > 0$. It remains to prove that, also in this case, $\mathbb{P}$-a.s. the process $X^{\bar{x};\bar{v}}$ jumps at time t to the endpoint of the interval I. Now, for any $\mathbb{F}$-stopping time τ, for $\mathbb{P} \circ (X^{\bar{x};\bar{v}}_\tau)^{-1}$-a.a. $x \in \mathbb{R}^2$, we have that the control

$$\bar{v}^\tau_t := \bar{v}_{\tau+t} - \bar{v}_{\tau-}, \quad t \geq 0, \tag{B.8}$$

is optimal for the initial condition $X^{\bar{x};\bar{v}}_{\tau-}$ (see Lemma 2.11 and the discussion at p. 1616 in [45]). Let now τ^1 be the first time at which the optimally controlled process $X^{\bar{x};\bar{v}}$ enters the set $\mathcal{H}$. Combining (B.8) together with Step 1, we obtain that $X^{\bar{x};\bar{v}}$ jumps to the endpoint of I. By constructing an increasing sequence τ_k of hitting times of the set $\mathcal{H}$, which exhausts the set in which $X^{\bar{x};\bar{v}} \in \mathcal{H}$, we conclude that $\mathbb{P}$-a.s. the process $X^{\bar{x};\bar{v}}$ jumps at time t to the endpoint of the interval I.

Acknowledgements. Financial support by the German Research Foundation (DFG) through the Collaborative Research Centre 1283 is gratefully acknowledged.

References

[1] L. H. Alvarez, *Singular stochastic control, linear diffusions, and optimal stopping: A class of solvable problems*, SIAM Journal on Control and Optimization, 39 (2001), pp. 1697–1710.
[2] P. Bank, *Optimal control under a dynamic fuel constraint*, SIAM Journal on Control and Optimization, 44 (2005), pp. 1529–1541.
[3] P. Bank and N. El Karoui, *A stochastic representation theorem with applications to optimization and obstacle problems*, Annals of Probability, 32 (2004), pp. 1030–1067.
[4] P. Bank and F. Riedel, *Optimal consumption choice with intertemporal substitution*, Annals of Applied Probability, 11 (2001), pp. 750–788.
[5] J. Bather and H. Chernoff, *Sequential decisions in the control of a spaceship*, in Fifth Berkeley Symposium on Mathematical Statistics and Probability, vol. 3, 1967, pp. 181–207.
[6] F. E. Benth and K. Reikvam, *A connection between singular stochastic control and optimal stopping*, Applied Mathematics and Optimization, 49 (2004), pp. 27–41.

[7] F. Boetius, *Bounded variation singular stochastic control and Dynkin game*, SIAM Journal on Control and Optimization, 44 (2005), pp. 1289–1321.
[8] F. Boetius and M. Kohlmann, *Connections between optimal stopping and singular stochastic control*, Stochastic Process. Appl., 77 (1998), pp. 253–281.
[9] M. Boryc and Ł. Kruk, *Characterization of the optimal policy for a multidimensional parabolic singular stochastic control problem*, SIAM Journal on Control and Optimization, 54 (2016), pp. 1657–1677.
[10] R. Buckdahn, P. Cannarsa, and M. Quincampoix, *Lipschitz continuity and semiconcavity properties of the value function of a stochastic control problem*, Nonlinear Differential Equations and Applications, 17 (2010), pp. 715–728.
[11] A. Budhiraja and K. Ross, *Existence of optimal controls for singular control problems with state constraints*, Annals of Applied Probability, 16 (2006), pp. 2235–2255.
[12] K. Burdzy, Z.-Q. Chen, and J. Sylvester, *The heat equation and reflected Brownian motion in time-dependent domains*, Annals of Probability, 32 (2004), pp. 775–804.
[13] K. Burdzy, W. Kang, and K. Ramanan, *The Skorokhod problem in a time-dependent interval*, Stochastic Processes and their Applications, 119 (2009), pp. 428–452.
[14] P. Cannarsa and C. Sinestrari, *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control*, vol. 58, Springer Science & Business Media, 2004.
[15] M. Chiarolla and U. Haussmann, *Controlling inflation: the infinite horizon case*, Applied Mathematics and Optimization, 41 (2000), pp. 25–50.
[16] M. B. Chiarolla and U. G. Haussmann, *Optimal control of inflation: a central bank problem*, SIAM Journal on Control and Optimization, 36 (1998), pp. 1099–1132.
[17] P. L. Chow, J. L. Menaldi, and M. Robin, *Additive control of stochastic linear systems with finite horizon*, SIAM Journal on Control and Optimization, 23 (1985), pp. 858–899.
[18] A. Cohen, *On singular control problems, the time-stretching method, and the weak-M1 topology*, SIAM Journal on Control and Optimization, 59 (2021), pp. 50–77.
[19] R. Cont, X. Guo, and R. Xu, *Pareto optima for a class of singular control games*, preprint arXiv:2005.05766, (2020).
[20] C. Costantini, *The Skorohod oblique reflection problem in domains with corners and application to stochastic differential equations*, Probability Theory and Related Fields, 91 (1992), pp. 43–70.
[21] C. Costantini, E. Gobet, and N. El Karoui, *Boundary sensitivities for diffusion processes in time dependent domains*, Applied Mathematics and Optimization, 54 (2006), pp. 159–187.
[22] M. H. Davis and M. Zervos, *A pair of explicitly solvable singular stochastic control problems*, Applied Mathematics and Optimization, 38 (1998), pp. 327–352.
[23] T. De Angelis, G. Ferrari, and J. Moriarty, *A solvable two-dimensional degenerate singular stochastic control problem with nonconvex costs*, Mathematics of Operations Research, 44 (2019), pp. 512–531.
[24] P. Dupuis and H. Ishii, *SDEs with oblique reflection on nonsmooth domains*, Annals of Probability, (1993), pp. 554–580.
[25] L. C. Evans, *A second order elliptic equation with gradient constraint*, Communications in Partial Differential Equations, 4 (1979), pp. 555–572.
[26] S. Federico, G. Ferrari, and P. Schuhmann, *Singular control of the drift of a Brownian system*, Center for Mathematical Economics Working Papers, (2020).
[27] ———, *A singular stochastic control problem with interconnected dynamics*, SIAM Journal on Control and Optimization, 58 (2020), pp. 2821–2853.
[28] A. Friedman, *Variational Principles and Free-Boundary Problems*, Dover Books on Mathematics, Dover Publications, Mineola, NY, 2010.
[29] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, 1431-0821, Springer Berlin Heidelberg, Berlin, Heidelberg, 2nd ed., 2001.
[30] X. Guo and H. Pham, *Optimal partially reversible investment with entry decision and general production function*, Stochastic Processes and their Applications, 115 (2005), pp. 705–736.
[31] X. Guo, W. Tang, and R. Xu, *A class of stochastic games and moving free boundary problems*, preprint arXiv:1809.03459, (2018).
[32] X. Guo and P. Tomecek, *A class of singular control problems and the smooth fit principle*, SIAM Journal on Control and Optimization, 47 (2009), pp. 3076–3099.
[33] X. Guo and R. Xu, *Stochastic games for fuel follower problem: N versus mean field game*, SIAM Journal on Control and Optimization, 57 (2019), pp. 659–692.
[34] U. G. Haussmann and W. Suo, *Singular optimal stochastic controls I: Existence*, SIAM Journal on Control and Optimization, 33 (1995), pp. 916–936.
[35] ———, *Singular optimal stochastic controls II: Dynamic programming*, SIAM Journal on Control and Optimization, 33 (1995), pp. 937–959.

[36] R. Hynd, *Analysis of Hamilton-Jacobi-Bellman equations arising in stochastic singular control*, ESAIM: Control, Optimisation and Calculus of Variations, 19 (2013), pp. 112–128.

[37] N. Ikeda and S. Watanabe, *A comparison theorem for solutions of stochastic differential equations and its applications*, Osaka Journal of Mathematics, 14 (1977), pp. 619–633.

[38] A. Jack, T. C. Johnson, and M. Zervos, *A singular control model with application to the goodwill problem*, Stochastic Processes and their Applications, 118 (2008), pp. 2098–2124.

[39] A. Jack and M. Zervos, *A singular control problem with an expected and a pathwise ergodic performance criterion*, International Journal of Stochastic Analysis, 2006 (2006), pp. 1–19.

[40] Y. M. Kabanov, *Hedging and liquidation under transaction costs in currency markets*, Finance Stoch., 3 (1999), pp. 237–248.

[41] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, vol. 113, Springer, 1998.

[42] I. Karatzas and S. E. Shreve, *Connections between optimal stopping and singular stochastic control I. Monotone follower problems*, SIAM Journal on Control and Optimization, 22 (1984), pp. 856–877.

[43] I. Karatzas and H. Wang, *Connections between bounded variation control and Dynkin games*, Optimal Control and Partial Differential Equations, 363 (2001).

[44] T. Koch and T. Vargiolu, *Optimal installation of solar panels with price impact: a solvable singular stochastic control problem*, arXiv preprint:1911.04223, (2019).

[45] L. Kruk, *Optimal policies for n-dimensional singular stochastic control problems part I: The Skorokhod problem*, SIAM Journal on Control and Optimization, 38 (2000), pp. 1603–1622.

[46] N. V. Krylov, *Controlled Diffusion Processes*, vol. 14, Springer Science & Business Media, 2008.

[47] J. Li and G. Zitkovic, *Existence, characterization, and approximation in the generalized monotone-follower problem*, SIAM Journal on Control and Optimization, 55 (2017), pp. 94–118.

[48] P. L. Lions and A. S. Sznitman, *Stochastic differential equations with reflecting boundary conditions*, Communications on Pure and Applied Mathematics, 37 (1984), pp. 511–537.

[49] N. L. Lundström and T. Önskog, *Stochastic and partial differential equations on non-smooth time-dependent domains*, Stochastic Processes and their Applications, 129 (2019), pp. 1097–1131.

[50] J. Ma, *On the principle of smooth fit for a class of singular stochastic control problems for diffusions*, SIAM Journal on Control and Optimization, 30 (1992), pp. 975–999.

[51] X. Mao, *Stochastic Differential Equations and Applications*, Horwood Pub., second ed., 2008.

[52] J. L. Menaldi and M. I. Taksar, *Optimal correction problem of a multidimensional stochastic system*, Automatica J. IFAC, 25 (1989), pp. 223–232.

[53] K. Nyström, T. Önskog, et al., *The Skorohod oblique reflection problem in time-dependent domains*, Annals of Probability, 38 (2010), pp. 2170–2223.

[54] P. E. Protter, *Stochastic Integration and Differential Equations*, vol. 21 of Stochastic Modelling and Applied Probability, Springer-Verlag, Berlin, 2nd ed., 2005.

[55] Y. Saisho, *Stochastic differential equations for multi-dimensional domain with reflecting boundary*, Probability Theory and Related Fields, 74 (1987), pp. 455–477.

[56] P. Schuhmann, *On some Two-Dimensional Singular Stochastic Control Problems and their Free-Boundary Analysis*, PhD thesis, Bielefeld University, 2021.

[57] H. M. Soner and S. E. Shreve, *Regularity of the value function for a two-dimensional singular stochastic control problem*, SIAM Journal on Control and Optimization, 27 (1989), pp. 876–907.

[58] W. Suo, *The Existence of Optimal Singular Controls for Stochastic Differential Equations*, PhD thesis, University of British Columbia, 1994.

[59] M. Taksar, *Skorohod problems with nonsmooth boundary conditions*, Journal of Computational and Applied Mathematics, 40 (1992), pp. 233–251.

[60] H. Tanaka, *Stochastic differential equations with reflecting boundary condition in convex regions*, Hiroshima Math. J., 9 (1979), pp. 163–177.

[61] A. Weerasinghe, *A bounded variation control problem for diffusion processes*, SIAM Journal on Control and Optimization, 44 (2005), pp. 389–417.

[62] Y. Yang, *A multidimensional stochastic singular control problem via Dynkin game and Dirichlet form*, SIAM Journal on Control and Optimization, 52 (2014), pp. 3807–3832.

J. Dianetti: Center for Mathematical Economics (IMW), Bielefeld University, Universitätsstrasse 25, 33615, Bielefeld, Germany

Email address: jodi.dianetti@uni-bielefeld.de

G. Ferrari: Center for Mathematical Economics (IMW), Bielefeld University, Universitätsstrasse 25, 33615, Bielefeld, Germany

Email address: giorgio.ferrari@uni-bielefeld.de