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ON AN IRREVERSIBLE INVESTMENT PROBLEM WITH TWO-FACTOR UNCERTAINTY

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Abstract. We consider a real options model for the optimal irreversible investment problem of a profit maximizing company. The company has the opportunity to invest into a production plant capable of producing two products, of which the prices follow two independent geometric Brownian motions. After paying a constant sunk investment cost, the company sells the products on the market and thus receives a continuous stochastic revenue-flow. This investment problem is set as a two-dimensional optimal stopping problem. We find that the optimal investment decision is triggered by a convex curve, which we characterize as the unique continuous solution to a nonlinear integral equation. Furthermore, we provide analytical and numerical comparative statics results of the dependency of the project's value and investment decision with respect to the model's parameters.

Keywords: Real Options; Irreversible Investment; Optimal Stopping; Nonlinear Integral Equation; Comparative Statics
JEL Classification: G11, C61, D25

1. Introduction

In this paper, we study a real options model of a company facing an irreversible investment decision in the presence of two sources of uncertainty. By paying a fixed sunk cost the company generates a continuous stochastic cash-flow, which results from selling two products on the market. In this framework, the company aims at maximizing its total expected profit arising from this investment and seeks to find a decision rule, which determines the optimal time to undertake this expenditure. We will see that this amounts in solving a two-dimensional optimal stopping problem of the form

$$V(x,y) = \sup_{\tau} \mathbb{E}[e^{-r\tau} F(X_\tau^x, Y_\tau^y)], \tag{1.1}$$

where the supremum is taken over the set of stopping times and the function F represents the value of the investment, dependent on two Itô-diffusions X and Y modelling the prices of the two products on the market (cf. (2.5) below).

Dating back to the seminal works of Myers [27] and McDonald and Siegel [26], the real options approach to irreversible investment decisions has received much attention in economics and finance with various settings regarding the dimensionality and characteristics of the underlying stochastic process (cf. Dixit [13], Pindyck [32, 33] or Alvarez [2], Battauz et al. [3], Luo et al. [25] for more recent contributions). In the simplest form, where the underlying economic shock process is one-dimensional and the investment option gives rise to a perpetual payoff stream, explicit solutions are often feasible (cf. Dixit and Pindyck [14], Stokey [36], Trigeorgis [38] for a survey). On the other hand, there are

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still only few examples of solvable multidimensional optimal stopping problems, despite the fact that real options models naturally deal with multiple sources of uncertainty.

In some models, the dimensionality of the problem can be effectively reduced to one. For instance, McDonald and Siegel [26] derive the optimal solution for the ratio of investment value and investment cost, and thus trace the problem back to a one-dimensional problem, for which an explicit solution could be found. This operative method was used and improved by a number of authors such as Gerber and Shiu [19], Shepp and Shiryaev [35] as well as Thijssen [37] (see also Christensen et al. [9] ad references therein). Nevertheless, in presence of a constant sunk cost of investment, a reduction of dimension à la McDonald and Siegel [26] is typically not feasible, as the problem's value function fails to be homogeneous of degree one.

Characterizing the solution in optimal stopping/real options models where the state space cannot be reduced is a challenging task. Hu and Øksendal [21] as well as Olsen and Stensland [28] consider an investment problem involving a multidimensional geometric Brownian motion, but their given conjecture regarding the shape of the stopping region only holds true in trivial cases, as pointed out by Christensen and Irle [7]. Adkins and Paxson [1] proposed a quasi-analytical approach, which results in solving a set of simultaneous equations, but their methodology seems to trigger sub-optimal solutions (see also Compernolle et al. [10], Lange et al. [24]). There are, however, some recent contributions in which a complete characterization of the solution to truly multidimensional irreversible investment problems was derived. De Angelis et al. [11] study a singular stochastic control problem and the associated two-dimensional optimal stopping problem, for which they characterize the optimal boundary as the unique solution to a nonlinear integral equation. Christensen and Salminen [8] propose a solution method relying on the Riesz representation of excessive functions and study a classical investment problem, for which they derive an integral equation of similar structure. In both references, the uniqueness of the representation is established by relying on arguments first presented in Peskir [30].

In this paper, we consider and solve optimal investment problem (1.1), which was first introduced by Compernolle et al. [10]. In that work, the authors derived some important preliminary results regarding the value function as well as the corresponding optimal boundary, but did not achieve a complete characterization of the latter. In this work, we push the analysis of Compernolle et al. [10] much further. Borrowing arguments from De Angelis et al. [11], we determine an integral equation for the optimal investment boundary (cf. Theorem 5.10). Moreover, we provide an analytical rigorous study of the dependency of the optimal boundary on some model's parameters. To our knowledge, such a result appears here for the first time. As a matter of fact, the analytical approach to comparative statics in Olsen and Stensland [28] (also employed by Compernolle et al. [10]) seems to overlook the delicate issue of the regularity of the value function. We are able to fix this issue by providing the proper regularity property, that in turn allows for a rigorous proof of the claimed monotonicity results and for additional findings (cf. Section 6). Finally, inspired by the numerical analysis in Detemple and Kitapbayev [12] and Christensen and Salminen [8], we propose a probabilistic numerical approach for the determination of the optimal boundary through the derived integral equation. We provide details about the algorithm, with the aim of making a service to other studies dealing with related questions. It is worth noticing that the proposed probabilistic numerical method employs a Monte Carlo simulation, and as such it does not face the curse of dimensionality, which is typical of analytical methods in large dimensions.

Overall, we believe that our main contributions are the following. From a mathematical point of view, given the limited amount of solvable multidimensional optimal stopping problems, we believe that our detailed study nicely complements the existing literature on optimal stopping as well as real options theory. Moreover, we suggest that our approach also has a methodological value for other real options problems. In fact, it defines an operative recipe for the determination of the optimal investment trigger analytically (by an integral equation) and numerically (by an approximation scheme), which can be easily adapted to different settings as well.

The paper is organized as follows. In Section 2 we introduce the optimal investment problem. In Section 3 we consider two benchmark problems, before we continue by characterizing the value function and the related optimal boundary in Sections 4 and 5. Analytical and numerical comparative statics results are then obtained in Section 6. Finally, some technical proofs and results are collected in the Appendices.

2. The Irreversible Investment Problem

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space, with the filtration $\mathbb{F}$ generated by a two-dimensional Brownian motion $W = (W_t^X, W_t^Y)_{t\geq 0}$ and augmented with $\mathbb{P}$-null sets. We consider a profit-maximizing and risk-neutral company, which has the opportunity to invest into a production plant by paying a constant investment cost I. The production plant is capable of producing two goods in given quantities Q_1 and Q_2 and we assume that the prices of the two goods evolve stochastically according to the dynamics

$$\begin{cases} dX_t^x = \alpha_1 X_t^x dt + \sigma_1 X_t^x dW_t^X, & X_0^x = x > 0, \\ dY_t^y = \alpha_2 Y_t^y dt + \sigma_2 Y_t^y dW_t^Y, & Y_0^y = y > 0, \end{cases} \tag{2.1}$$

for some constants $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$. We assume that after the company has made the investment, it is able to sell the goods in their given quantities instantaneously and over an infinite time horizon on the market. If the investment is performed at initial time, its value for given price levels x and y is then obtained through the discounted perpetual revenue flow, net of the investment cost; that is,

$$\mathbb{E}\Big[\int_0^\infty e^{-rt}\pi(X_t^x, Y_t^y)dt \;-\; I\Big] =: F(x,y). \tag{2.2}$$

Here $\pi(x,y) := Q_1 x + Q_2 y$ denotes the profit function and $r > 0$ is a discount factor. In order to guarantee finite integrals, we make the following **standing assumption**.

Assumption 2.1. *We have $r > \alpha_1 \vee \alpha_2$.*

Clearly, an investment at initial time is not necessarily optimal. Hence, setting

$$\mathcal{T} := \{\tau : \;\; \tau \text{ are } \mathbb{F}\text{-stopping times}\}, \tag{2.3}$$

the company aims at determining the entry rule $\tau^* \in \mathcal{T}$ that maximizes its net total expected profits from τ^* on. That is, for any initial price levels $(x,y) \in \mathbb{R}_+$, it seeks to determine $\tau^* \in \mathcal{T}$ such that

$$V(x,y) := \mathcal{J}(x,y,\tau^*) = \max_{\tau\in\mathcal{T}} \mathcal{J}(x,y,\tau), \tag{2.4}$$

where

$$\mathcal{J}(x,y,\tau) := \mathbb{E}\Big[e^{-r\tau}F(X_\tau^x, Y_\tau^y)\Big] = \mathbb{E}\Big[e^{-r\tau}\Big(\frac{Q_1 X_\tau^x}{\delta_1} + \frac{Q_2 Y_\tau^y}{\delta_2} - I\Big)\Big] \tag{2.5}$$

for $\delta_i = r - \alpha_i$, $i = 1, 2$. The last equality in (2.5) follows by straightforward calculations upon using Assumption 2.1. Throughout this paper, we will refer to (2.4) as to *the optimal investment problem.*

Remark 2.2. *Assumption 2.1 guarantees $\mathbb{E}[\sup_{t\geq 0} e^{-rt}X_t^x] < +\infty$ and $\mathbb{E}[\sup_{t\geq 0} e^{-rt}Y_t^y] < +\infty$, standard technical assumptions in the theory of optimal stopping (cf. Karatzas and Shreve [22], p. 35). Amongst other things, these conditions imply that the families of random variables*

$$\{e^{-r\tau}X_\tau^x \mathbb{1}_{\{\tau<\infty\}},\, \tau \in \mathcal{T}\} \quad \text{and} \quad \{e^{-r\tau}Y_\tau^y \mathbb{1}_{\{\tau<\infty\}},\, \tau \in \mathcal{T}\} \tag{2.6}$$

are uniformly integrable. Moreover $\lim_{t\to\infty} e^{-rt}X_t^x = 0$ as well as $\lim_{t\to\infty} e^{-rt}Y_t^y = 0$, we thus adopt the convention

$$\begin{aligned} &e^{-r\tau}X_\tau^x \mathbb{1}_{\{\tau=\infty\}} := \lim_{t\to\infty} e^{-rt}X_t^x = 0, \quad \mathbb{P}\text{-a.s.} \\ \text{as well as} \quad &e^{-r\tau}\,Y_\tau^y \mathbb{1}_{\{\tau=\infty\}} := \lim_{t\to\infty} e^{-rt}\,Y_t^y = 0, \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{2.7}$$

and set

$$e^{-r\tau} \mid f(X^x_\tau, Y^y_\tau) \mid \mathbb{1}_{\{\tau=\infty\}} := \limsup_{t\to\infty} e^{-rt} \mid f(X^x_\tau, Y^y_\tau) \mid \quad \mathbb{P}\text{-}a.s. \tag{2.8}$$

for any Borel-measurable function f*.*

3. Two Benchmark Problems

Before we study the optimal entry problem introduced in the previous section, it is useful to focus on two related classical real options problems. Notice that the values $x = 0$ and $y = 0$ are absorbing boundaries for the processes X^x_t and Y^y_t. In particular, when $X^0_0 = 0$ (resp. $Y^0_0 = 0$) then $X^0_t = 0$ (resp. $Y^0_t = 0$) for all $t \geq 0$ $\mathbb{P}$-a.s. Therefore, we can naturally associate to (2.4) the two one-dimensional optimal stopping problems

$$v_1(x) := \sup_{\tau\in\mathcal{T}} \mathbb{E}\Big[e^{-r\tau}\Big(\frac{Q1X^x_\tau}{\delta_1} - I\Big)\Big] \quad \text{and} \quad v_2(y) := \sup_{\tau\in\mathcal{T}} \mathbb{E}\Big[e^{-r\tau}\Big(\frac{Q2Y^y_\tau}{\delta_2} - I\Big)\Big] \tag{3.1}$$

Due to the one-dimensional structure of this problem, their solution is standard and can be obtained by a *guess-and-verify approach* (cf. Dixit and Pindyck [14]).
Let us consider v_1, as analogous considerations can be made for v_2. It is reasonable to assume that the company invests into the production plant only when the current price of the product is large enough. We thus expect that the optimal stopping time for problem (3.1) is of the form

$$\tau^*_x := \inf\{t \geq 0 : \ X^x_t \geq x^*\},$$

where x^* denotes the critical price level, at which the company decides to invest. Accordingly, the candidate value function w should satisfy $(\mathcal{L}_X - r)w(x) = 0$ for all $x < x^*$, where $\mathcal{L}_X$ denotes the second-order differential operator (acting on twice-continuously differentiable functions) given by

$$\mathcal{L}_X := \frac{1}{2}\sigma_1^2 x^2 \frac{\partial}{\partial x^2} + \alpha_1 x \frac{\partial}{\partial x}. \tag{3.2}$$

It is well known that the equation $(\mathcal{L}_X - r)w(x) = 0$ admits two fundamental solutions $\psi(x) = x^{\beta_1}$ and $\varphi(x) = x^{\beta_2}$, where β_1 and β_2 are the positive and negative solutions to the equation

$$\frac{1}{2}\sigma_1^2\beta(\beta-1) + \alpha_1\beta - r = 0$$

and Assumption 2.1 guarantees $\beta_1 > 1$. Consequently, any of its solutions takes the form $w(x) = A\psi(x) + B\varphi(x)$ for $x < x^*$, where A and B are constants to be found. As $x \mapsto \varphi(x)$ diverges as $x \downarrow 0$, and it is easy to see that v_1 has instead sublinear growth, we guess $B = 0$. The candidate value function w thus can be written as

$$w(x) = \begin{cases} Ax^{\beta_1} & x < x^* \\ \frac{Q_1 x}{\delta_1} - I & x \geq x^* \end{cases}$$

for A and x^* to be derived. By employing the standard smooth-pasting and smooth-fit condition, it is straightforward to see that they are given by

$$x^* = \frac{\beta_1}{(\beta_1 - 1)Q_1}\delta_1 I \quad \text{and} \quad A = \frac{Q_1}{\beta_1\delta_1}x^{*1-\beta_1}. \tag{3.3}$$

The following proposition verifies that the candidate value function w constructed in this way indeed coincides with the value function v_1 of (3.1). Its proof is standard and we refer to the classical textbook of Peskir and Shiryaev [31] for techniques and results.

Proposition 3.1. *Recall* v_1 *from (3.1). Then we have*

$$v_1(x) = \begin{cases} Ax^{\beta_1} & 0 < x < x^*, \\ \frac{Q_1 x}{\delta_1} - I & x \geq x^*, \end{cases}$$

where A and x^* are given by (3.3). Also,

$$\tau_x^* := \inf\{t \geq 0 : \; X_t^x \geq x^*\}$$

is the optimal stopping time.

Analogously, we have the next result concerning v_2.

Proposition 3.2. *Recall v_2 from (3.1). Then*

$$v_2(y) = \begin{cases} Dy^{\eta_1} & 0 < y < y^*, \\ \frac{Q_2 y}{\delta_2} - I & y \geq y^*, \end{cases}$$

where the constant D and the investment threshold y^* are given by

$$y^* = \frac{\eta_1}{(\eta_1 - 1)Q_2}\delta_2 I \qquad \text{and} \qquad D = \frac{Q_2}{\eta_1 \delta_2} y^{*1-\eta_1}, \tag{3.4}$$

where $\eta_1 > 1$ denotes the positive root of the quadratic equation $\frac{1}{2}\sigma_2^2\eta(\eta-1)+\alpha_2\eta-r=0$. Moreover, the optimal stopping time is of the form

$$\tau_y^* := \inf\{t \geq 0 : \; Y_t^y \geq y^*\}$$

As expected, the optimal thresholds x^* and y^* will be shown in our subsequent analysis to identify the limits as $y \downarrow 0$ and $x \downarrow 0$, respectively, of the curve triggering the optimal investment rule in (2.4).

4. On the Value Function of the Optimal Investment Problem

Consistently with the two benchmark problems of last section, we can expect that also for Problem (2.4) it will be optimal to invest when the price processes X and Y are sufficiently large. However, differently to v_1 and v_2 as in (3.1), (2.4) defines a two-dimensional optimal stopping problem for which a *guess-and-verify approach* is not feasible. Hence, in the following we will perform a direct study of V. After deriving some preliminary results, we move on by defining the associated continuation and stopping regions. The main result is then stated in Theorem 4.2, where we borrow arguments from De Angelis et al. [11] in order to derive a probabilistic representation of V. The proof of the next proposition can be found in Appendix A.

Proposition 4.1. *Recall V from (2.4). There exists a constant $C > 0$ such that for all $(x, y) \in \mathbb{R}_+$*

$$\max\{0, F(x,y)\} \leq V(x,y) \leq C(x+y), \tag{4.1}$$

and the value function V is nondecreasing with respect to x and y. Moreover, V is continuous and convex on $\mathbb{R}_+^2$.

Continuation and Stopping Regions. As it is customary in optimal stopping, continuation and stopping regions of the optimal stopping problem (2.4) are given by

$$\mathcal{C} := \{(x,y) \in \mathbb{R}_+^2 : \; V(x,y) > F(x,y)\} \qquad \mathcal{S} := \{(x,y) \in \mathbb{R}_+^2 : \; V(x,y) = F(x,y)\}. \tag{4.2}$$

Notice that, since the value function V and the function F are continuous, the continuation region is open and the stopping region is closed (cf. Peskir and Shiryaev [31], p. 36). Moreover, the optimal stopping time is given by the first entry time of the process (X_t^x, Y_t^y) into the stopping region

$$\tau^* = \tau^*(x,y) := \inf\{t \geq 0 : \; (X_t^x, Y_t^y) \in \mathcal{S}\}, \tag{4.3}$$

whenever it is $\mathbb{P}$-a.s. finite (cf. Peskir and Shiryaev [31], p. 46).

Probabilistic Representation of the Value Function. We now provide a probabilistic representation of the value function V of the stopping problem (2.4). This representation is essential for the forthcoming characterization of the optimal boundary being the solution to an integral equation. Its technical proof employs an approximation argument as in De Angelis et al. [11] and it is postponed to Appendix B.

Theorem 4.2. *The value function V of the optimal investment problem (2.4) admits the representation*

$$V(x,y)=\mathbb{E}\Big[\int_0^\infty e^{-rt}(Q_1X_t^x+Q_2Y_t^y-rI)\mathbb{1}_{\{(X_t^x,Y_t^y)\in\mathcal{S}\}}dt\Big]. \tag{4.4}$$

for all $(x,y)\in\mathbb{R}_+^2$.

Remark 4.3. *Let $H(x,y):=(Q_1x+Q_2y-rI)\mathbb{1}_{\{(x,y)\in\mathcal{S}\}}$. The expression (4.4) can thus be formulated as*

$$V(x,y)=\mathbb{E}\Big[\int_0^\infty e^{-rt}H(X_t^x,Y_t^y)dt\Big]. \tag{4.5}$$

Notice that $|H(x,y)|\le Q_1x+Q_2y-rI$. Upon using Assumption 2.1, the strong Markov property and standard arguments on conditional expectation, we have

$$\mathbb{E}\Big[\int_0^\infty e^{-rt}H(X_t^x,Y_t^y)dt\Big|\mathcal{F}_\tau\Big]=\int_0^\tau e^{-rs}H(X_s^x,Y_s^y)ds+e^{-r\tau}V(X_\tau^x,Y_\tau^y). \tag{4.6}$$

Consequently, the process

$$\big\{e^{-rt}V(X_t^x,Y_t^y)+\int_0^t e^{-rs}H(X_s^x,Y_s^y)ds,\, t\ge 0\big\} \tag{4.7}$$

is an $(\mathcal{F}_t)$-martingale. Furthermore, equation (4.6) implies

$$|e^{-rt}V(X_t^x,Y_t^y)|\,\le\,\mathbb{E}\Big[\int_0^\infty e^{-rt}H(X_t^x,Y_t^y)dt\Big|\mathcal{F}_\tau\Big] \tag{4.8}$$

and it follows that the family $\{e^{-rt}V(X_\tau^x,Y_\tau^y),\tau\in\mathcal{T}\}$ is uniformly integrable.

Remark 4.4. *Notice that the results of Theorem 4.2 can be generalized to the case in which X and Y are general one-dimensional Itô-diffusions. Indeed, the arguments of the proof in Appendix B do not actually hinge on the particular form of the price processes.*

5. On The Optimal Boundary

In this section, we study the optimal price level triggering the investment in Problem (2.4). Some of the subsequent results have already been derived by Compernolle et al. [10], Theorem 1, and we repeat them briefly for the sake of completeness. The main novel result is then stated in Theorem 5.10, where we characterize the optimal trigger as the unique solution to a nonlinear integral equation in a certain functional class.

Define

$$b(x):=\sup\{y\in\mathbb{R}_+:\ \ V(x,y)>F(x,y)\},\qquad\qquad x\in\mathbb{R}_+, \tag{5.1}$$

with the convention $\sup\emptyset=0$. We state the following proposition.

Proposition 5.1. *The continuation region and stopping region of (4.2) can be written as*

$$\mathcal{C}=\{(x,y)\in\mathbb{R}_+^2:\ \ y<b(x)\},\qquad \mathcal{S}=\{(x,y)\in\mathbb{R}_+^2:\ \ y\ge b(x)\} \tag{5.2}$$

Proof. It is sufficient to prove that the continuation region is down-connected. Take $(x, y) \in \mathcal{C}$ and $\tau^*(x, y)$ of (4.3). We then have

$$V(x,y) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau} F(X^x_\tau, Y^y_\tau)] = \mathbb{E}[e^{-r\tau^*} F(X^x_{\tau^*}, Y^y_{\tau^*})] > F(x,y)$$

Let $\epsilon \in (0, y]$ and notice that $\tau^*(x, y)$ is a-priori suboptimal for the stopping problem with value function $V(x, y - \epsilon)$. It follows that

$$\begin{aligned} V(x,y-\epsilon) &\geq \mathbb{E}[e^{-r\tau^*} F(X^x_{\tau^*}, Y^{y-\epsilon}_{\tau^*})] = \mathbb{E}[e^{-r\tau^*} F(X^x_{\tau^*}, Y^y_{\tau^*})] - \frac{Q_2\epsilon}{\delta_2}\mathbb{E}[e^{-r\tau^*} Y^1_{\tau^*}] \\ &\geq V(x,y) - \frac{Q_2\epsilon}{\delta_2} > F(x,y) - \frac{Q_2\epsilon}{\delta_2} = F(x, y-\epsilon) \end{aligned}$$

where we used that $\{e^{-rt}Y^y_t,\ t \geq 0\}$ is a supermartingale due to (2.6) and Assumption 2.1. Hence, $(x, y - \epsilon) \in \mathcal{C}$ for every $\epsilon \in (0, y]$, which concludes our proof. □

The next proposition states some preliminary results of the boundary (5.1).

Proposition 5.2. *The function b of (5.1) inherits the following properties.*

(i) $x \mapsto b(x)$ is nonincreasing on $\mathbb{R}_+$,
(ii) $x \mapsto b(x)$ is right continuous on $\mathbb{R}_+$.

Proof. (i) The proof follows in the same spirit as the proof of Proposition 5.1, with the roles of x and y reversed.

(ii) The functions V and F are continuous on $\mathbb{R}^2_+$, consequently b is lower-semicontinuous. Since it is nonincreasing by point *(i)*, the claim follows. □

The results stated in Propositions 5.1 and 5.2 guarantee that the continuation region $\mathcal{C}$ and the stopping region $\mathcal{S}$ are connected. Moreover, we can rewrite the optimal stopping time (4.3) due to (5.2) and obtain

$$\tau^* = \tau^*(x,y) := \inf\{t \geq 0 \mid Y^y_t \geq b(X^x_t)\} \tag{5.3}$$

for any $(x, y) \in \mathbb{R}^2_+$. Furthermore, the probabilistic representation (4.4) rewrites as

$$V(x,y) = \mathbb{E}\Big[\int_0^\infty e^{-rt}\Big(Q_1X^x_t + Q_2Y^y_t - rI\Big)\mathbb{1}_{\{Y^y_t \geq b(X^x_t))\}}dt\Big]. \tag{5.4}$$

for any $(x, y) \in \mathbb{R}^2_+$. In the next step, we prove $V \in C^1(\mathbb{R}^2_+)$. As a by-product, we obtain the well known *smooth-fit* condition across the free-boundary, which states the continuity of V_x as well as V_y at $\partial\mathcal{C}$. To this end, it is important to bear in mind the following well known fact.

Lemma 5.3. *The processes X^x and Y^y are given by two independent geometric Brownian motions, hence they have a log-normal distribution with transition densities*

$$\rho_1(t,x,\psi) = \frac{1}{\sigma_1\psi\sqrt{2\pi t}}\exp\Big(-\frac{(\log\psi - \log x - (\alpha_1 - \frac{1}{2}\sigma_1^2)t)^2}{2\sigma_1^2 t}\Big)$$
$$\rho_2(t,y,\eta) = \frac{1}{\sigma_2\eta\sqrt{2\pi t}}\exp\Big(-\frac{(\log\eta - \log y - (\alpha_2 - \frac{1}{2}\sigma_2^2)t)^2}{2\sigma_2^2 t}\Big)$$

for every $(t, x, \psi), (t, y, \eta) \in (0, \infty) \times \mathbb{R}_+ \times \mathbb{R}_+$. Moreover,

i) $(t, \zeta, \xi) \mapsto \rho_i(t, \zeta, \xi)$ is continuous on $(0, \infty) \times \mathbb{R}_+ \times \mathbb{R}_+$ for i=1,2;
ii) Let $\mathcal{K} \subset \mathbb{R}^2_+$ be a compact set. Then there exists some $q > 1$, which is possibly depending on $\mathcal{K}$, such that

$$\int_0^\infty e^{-rt}\Big(\int_{\mathcal{K}} \mid \rho_1(t,x,\psi)\rho_2(t,y,\eta) \mid^q d\psi d\eta\Big)^{1/q} dt < +\infty$$

for all $(x, y) \in \mathcal{K}$.

Proposition 5.4. *The value function of (2.4) is such that $V \in C^1(\mathbb{R}^2_+)$.*

Proof. We can rewrite (4.4) with Lemma 5.3 and obtain

$$V(x,y) = \int_0^\infty e^{-rt} \int_0^\infty \rho_1(t,x,\psi) \int_{b(\psi)}^\infty (Q_1\psi + Q_2\eta - rI)\rho_2(t,y,\eta)\, d\eta\, d\psi\, dt.$$

Due to (2.6) and Lemma 5.3, we are able to take derivates with respect to x and y. Standard dominated convergence arguments then show that $\partial_x V(x,y)$ as well as $\partial_y V(x,y)$ are continuous for all $(x,y) \in \mathbb{R}^2_+$. □

Continuity of the Optimal Boundary. In order to derive the continuity of the optimal boundary over the whole state-space, it would be sufficient to prove the left-continuity, as we already established the right-continuity of b in Proposition 5.2. Nevertheless, we follow the arguments of Compernolle et al. [10] relying on the convexity of the optimal boundary.

Proposition 5.5. *The stopping region $\mathcal{S}$ of (4.2), (equivalently of (5.2)) is convex on $\mathbb{R}^2_+$.*

Proof. Assume there exist $(x_1,y_1),(x_2,y_2) \in \mathcal{S}$ and $\lambda \in (0,1)$ such that $(x,y) := \lambda(x_1,y_1) + (1-\lambda)(x_2,y_2) \in \mathcal{C}$. Thus we must have $V(x,y) > F(x,y)$ as well as $V(x_i,y_i) = F(x_i,y_i)$ for $i = 1,2$. It follows that

$$V(x,y) \;>\; F(x,y) \;=\; \lambda F(x_1,y_1) + (1-\lambda)F(x_2,y_2) \;=\; \lambda V(x_1,y_1) + (1-\lambda)V(x_2,y_2),$$

which contradicts the convexity of V, as seen in Proposition 4.1. □

Proposition 5.6. *The optimal boundary b of (5.1) is convex on $\mathbb{R}_+$.*

Proof. Notice that the stopping region $\mathcal{S} = \{(x,y) \in \mathbb{R}^2_+ : \quad y \geq b(x)\}$ is the epigraph of b. Due to Proposition 5.5 it follows from standard results (see for example Borwein and Lewis [5], p. 43) that the boundary is convex on $\mathbb{R}_+$. □

Proposition 5.7. *The optimal boundary b of (5.1) is continuous on $\mathbb{R}_+$.*

Proof. The continuity of b on $(0,\infty)$ follows from Proposition 5.6, as b is convex on an open set. It remains to show that the boundary is continuous in $x = 0$. Assume that $b(0) \neq b(0+)$. While $b(0) < b(0+)$ is a contradiction to Proposition 5.2 as b is nonincreasing, supposing that $b(0) > b(0+)$ contradicts the closedness of the stopping region $\mathcal{S}$. The boundary b is thus continuous on $\mathbb{R}_+$. □

Remark 5.8. *Alternatively, for the proof of Proposition 5.7 we could rely on a result by Peskir [29], which connects the principle of smooth-fit with the continuity of the boundary for models with two-dimensional diffusions.*

It now becomes clear to what extent the optimal investment problem (2.4) is related to the benchmark problems we studied in Section 3. Since the optimal boundary is continuous, convex and nonincreasing on $\mathbb{R}_+$, it follows that $b(0) = y^*$ and $b(x) < y^*$ for all $x > 0$. Furthermore, due to the fact that x^* is the solution to the optimal stopping problem on $\mathbb{R}_+ \times \{0\}$, the boundary (5.1) is such that $b(x) = 0$ for $x \geq x^*$. The solutions to the benchmark problems therefore give the investment thresholds for the company at the x- and y-axis.

An Integral Equation for the Optimal Boundary. In this section, we aim at characterizing the optimal boundary b as the unique solution to an integral equation in a certain functional class. For that purpose, we make use of the probabilistic representation of the value function V developed in Theorem 4.2. As a first step, we derive a lower bound for b of (5.1). Notice that Dynkin's formula implies

$$\mathbb{E}[e^{-r\tau}F(X^x_\tau, Y^y_\tau)] = F(x,y) + \mathbb{E}\Big[\int_0^\tau e^{-rs}(\mathcal{L}-r)F(X^x_s,Y^y_s)ds\Big] \tag{5.5}$$

for any bounded stopping time τ. By localization arguments and (2.8), we conclude that (5.5) holds for any $\tau \in \mathcal{T}$ and we can thus rewrite V as

$$V(x,y) = F(x,y) + \sup_{\tau\in\mathcal{T}} \mathbb{E}\Big[\int_0^\tau e^{-rs}(\mathcal{L}-r)F(X_s^x, Y_s^y)ds\Big].$$

Observe that it is never optimal to stop whenever $(\mathcal{L}-r)F(x,y) > 0$, consequently we have

$$\{(x,y)\in\mathbb{R}_+^2 : \ (\mathcal{L}-r)F(x,y) > 0\} \ \subseteq \ \{(x,y)\in\mathbb{R}_+^2 : \ V(x,y) > F(x,y)\} \ = \ \mathcal{C}. \tag{5.6}$$

Define

$$h(x) := \sup\{y\in\mathbb{R}_+ : \quad (\mathcal{L}-r)F(x,y) > 0\}. \tag{5.7}$$

We state the following result.

Lemma 5.9. *The function h of (5.7) is nonincreasing, continuous and it is given by the unique solution to the equation $(\mathcal{L}-r)F(x,\cdot) = 0$. In particular,*

$$\{(x,y)\in\mathbb{R}_+^2 : \quad y > h(x)\} \ = \ \{(x,y)\in\mathbb{R}_+^2 : \quad rI - Q_1x - Q_2y < 0\}. \tag{5.8}$$

Proof. We set $g(x,y) := (\mathcal{L}-r)F(x,y) = rI - Q_1x - Q_2y$. Notice that g is strictly decreasing and continuous in x and y. Hence, for $x_2 > x_1$ we have $g(x_1, h(x_2)) \geq g(x_2, h(x_2)) \geq 0$, where the latter inequality is due to (5.7), and it follows that $h(x_1) \geq h(x_2)$. The continuity of g and (5.7) guarantee that h solves $g(x,\cdot) = 0$. Furthermore, as $g(x,\cdot)$ is strictly decreasing, h the unique solution. Consequently, it admits the representation

$$h(x) = \frac{1}{Q_2}(rI - Q_1x).$$

It is evident that h is continuous on $\mathbb{R}_+$ and (5.8) follows from the above results. □

Consider the class of functions

$$\mathcal{M} := \{f : \mathbb{R} \mapsto \mathbb{R}, \text{ continuous, decreasing and s.t. } f(x) \geq h(x)\}$$

and notice that $\mathcal{M}$ is nonempty as $h \in \mathcal{M}$ due to Lemma 5.9.

Theorem 5.10. *The optimal boundary b of (5.1) is the unique function $y \in \mathcal{M}$ such that*

$$F(x, y(x)) = \mathbb{E}\Big[\int_0^\infty e^{-rt}\Big(Q_1X_t^x + Q_2Y_t^{y(x)} - rI\Big)\mathbb{1}_{\{Y_t^{y(x)} \geq y(X_t^x)\}}dt\Big], \quad x > 0. \tag{5.9}$$

Equivalently, with regards to Lemma 5.3, one has

$$\frac{Q_1x}{\delta_1} + \frac{Q_2y(x)}{\delta_2} - I = \int_0^\infty e^{-rt}\Big(\int_0^\infty \rho_1(t,x,\psi)\Big(\int_{y(\psi)}^\infty (Q_1\psi + Q_2\eta - rI)\rho_2(t,y(x),\eta)d\eta\Big)d\psi\Big)dt. \tag{5.10}$$

Proof. As for the existence, it is sufficient to show that b of (5.1) solves the equation. Notice that $b \in \mathcal{M}$ due to Proposition 5.2, Proposition 5.7 and simple comparison arguments resulting from (5.6). Furthermore, by evaluating both sides of the probabilistic representation (4.4) of V at points $y = b(x)$, one finds (5.9), upon using $V(x, b(x)) = F(x, b(x))$.
In order to show that b is the unique solution to (5.9) in $\mathcal{M}$, one can adopt the four-step procedure in De Angelis et al. [11], extending and refining the original probabilistic arguments from Peskir [30]. □

Remark 5.11. *The equation (5.10) can be reformulated in the canonical Fredholm form. Define*

$$K(x,\psi,\alpha,\beta) = \int_0^\infty e^{-rt}\rho_1(t,x,\psi)\int_\beta^\infty \Big(Q_1\psi + Q_2\eta - rI\Big)\rho_2(t,\alpha,\eta)d\eta dt$$

and after applying Fubini's theorem the equation (5.10) can be written as

$$\frac{Q_1 x}{\delta_1} + \frac{Q_2 b(x)}{\delta_2} - I = \int_0^\infty K(x,\psi,b(x),b(\psi))d\psi.$$

We thus obtain the representation

$$b(x) = f(x) + \underbrace{\lambda \int_0^\infty K(x,\psi,b(x),b(\psi))d\psi}_{:=G}, \tag{5.11}$$

where we set

$$\lambda := \frac{\delta_2}{Q_2} \qquad \text{and} \qquad f(x) := \frac{\delta_2}{Q_2}\Big(I - \frac{Q_1 x}{\delta_1}\Big).$$

Following Press and Teukolsky [34], (5.11) is a nonlinear, inhomogeneous Fredholm integral equation of second kind.
It is interesting to notice that $b(x) \geq f(x)$ for all $x \in \mathbb{R}_+$, where $f(x)$ represents the price of the second product that makes the company indifferent between investing and passing up on the investment opportunity. However, as the company wants to maximize its expected profit, it aims to invest at a larger price level of the second product. Consequently, it adds the quantity G, which is strictly positive due to (5.6).

6. Comparative Statics Analysis

In this section we perform some comparative statics analysis of the value function V and the optimal boundary b of (5.1). Differently to the majority of the contributions on real options problems, we are able to propose rigorous analytical proofs of the dependency of the value function V on σ_i, $i = 1,2$ and α_i, $i = 1,2$. Moreover, we implement a recursive numerical method to investigate the sensitivity of the optimal boundary with respect to the model's parameters.
The next important technical proposition will be used in Propositions 6.2 and 6.3. Its proof can be found in Appendix C.

Proposition 6.1. *The value function V of (2.4) is such that $V \in C^1(\mathbb{R}^2_+) \cap \mathcal{W}^{2,2}_{loc}(\mathbb{R}^2_+)$ and satisfies the variational inequality*

$$\max\{(\mathcal{L} - r)v(x,y),\, F(x,y) - v(x,y)\} = 0$$

for a.e. $(x,y) \in \mathbb{R}^2_+$.

The next result exploits the convexity and regularity of V in order to prove monotonicity of V with respect to σ_1. The same rationale has already been employed in Olsen and Stensland [28], where, however, the delicate issue of the regularity of V seems to be overlooked (as a matter of fact, the optimal value in Olsen and Stensland [28] is implicitly assumed to be of class C^2).

Proposition 6.2. *Let $\hat{\sigma}_1 > \sigma_1$ and let $\hat{X}^x_t$ denote the unique solution to (2.1) when volatility is $\hat{\sigma}_1$. Moreover, let $\hat{V}$ denote the value function of (2.4) with underlying state $(\hat{X}^x_t, Y^y_t)$. It follows that*

$$\hat{V}(x,y) \geq V(x,y),$$

for all $(x,y) \in \mathbb{R}^2_+$.

Proof. Recall $\mathcal{L}$ as in (B.10) and let $\hat{\mathcal{L}}$ denote the infinitesimal generator when volatility is $\hat{\sigma_1}$. Notice that Proposition 6.1 implies $(\mathcal{L} - r)V(x,y) \leq 0$ as well as $(\hat{\mathcal{L}} - r)\hat{V}(x,y) \leq 0$ for a.e. $(x,y) \in \mathbb{R}^2_+$. Moreover, the fact that $V \in \mathcal{W}^{2,2}(\mathbb{R}^2_+)$ by Proposition 6.1 implies that we can argue as in the proof of Proposition B.2 in order to apply Dynkin's formula and obtain

$$\mathbb{E}[e^{-r\tau^*}\hat{V}(X^x_{\tau^*}, Y^y_{\tau^*})] = \hat{V}(x,y) + \mathbb{E}\Big[\int_0^{\tau^*} e^{-rs}(\mathcal{L} - r)\hat{V}(X^x_s, Y^y_s)ds\Big], \tag{6.1}$$

where $\tau^* = \tau^*(x,y)$ denotes the optimal stopping time for the stopping problem with value function $V(x,y)$. From (6.1) we obtain by simple manipulation that

$$\begin{aligned}
&\mathbb{E}[e^{-r\tau^*}\hat{V}(X^x_{\tau^*},Y^y_{\tau^*})]\\
&\qquad = \hat{V}(x,y) + \mathbb{E}\Big[\int_0^{\tau^*} e^{-rs}\Big((\hat{\mathcal{L}}-r)\hat{V}(X^x_s,Y^y_s) + (\mathcal{L}-\hat{\mathcal{L}})\hat{V}(X^x_s,Y^y_s)\Big)ds\Big]\\
&\qquad \le \hat{V}(x,y) + \mathbb{E}\Big[\int_0^{\tau^*} e^{-rs}(\mathcal{L}-\hat{\mathcal{L}})\hat{V}(X^x_s,Y^y_s)ds\Big]\\
&\qquad = \hat{V}(x,y) + \mathbb{E}\Big[\int_0^{\tau^*} e^{-rs}\Big(\frac{1}{2}(\sigma_1^2-\hat{\sigma}_1^2)x^2\hat{V}_{xx}(X^x_s,Y^y_s)\Big)ds\Big]\\
&\qquad \le \hat{V}(x,y)
\end{aligned}$$

where the latter inequality follows from the convexity of V. By noticing that Proposition 6.1 implies $\hat{V} \ge F$ on $\mathbb{R}^2_+$, we then obtain

$$\hat{V}(x,y) \ge \mathbb{E}[e^{-r\tau^*}\hat{V}(X^x_{\tau^*},Y^y_{\tau^*})] \ge \mathbb{E}[e^{-r\tau^*}F(X^x_{\tau^*},Y^y_{\tau^*})] = V(x,y),$$

which concludes our claim. □

Proposition 6.3. *Let $\hat{\alpha}_1 > \alpha_1$ and let $\hat{X}^x_t$ denote the unique solution to (2.1) when the drift is $\hat{\alpha}_1$. Furthermore, let $\hat{V}$ denote the value function of (2.4) when the underlying state is $(\hat{X}^x_t, Y^y_t)$. We have*

$$\hat{V}(x,y) \ge V(x,y)$$

for all $(x,y) \in \mathbb{R}^2_+$.

Proof. We can argue similarly as in Proposition 6.2. Let $\hat{\mathcal{L}}$ be as in (B.10) but with drift coefficient $\hat{\alpha}_1$. Proposition 6.1 again implies $(\mathcal{L}-r)V(x,y) \le 0$ as well as $(\hat{\mathcal{L}}-r)\hat{V}(x,y) \le 0$ for a.e. $(x,y) \in \mathbb{R}^2_+$. Moreover, due to Proposition 6.1 we can argue as in the proof of Proposition B.2 in order to apply Dynkin's formula and obtain

$$\mathbb{E}[e^{-r\tau^*}\hat{V}(X^x_{\tau^*},Y^y_{\tau^*})] = \hat{V}(x,y) + \mathbb{E}\Big[\int_0^{\tau^*} e^{-rs}(\mathcal{L}-r)\hat{V}(X^x_s,Y^y_s)ds\Big], \tag{6.2}$$

where $\tau^* = \tau^*(x,y)$ denotes the optimal stopping time for the stopping problem with value function $V(x,y)$. It follows that

$$\begin{aligned}
&\mathbb{E}[e^{-r\tau^*}\hat{V}(X^x_{\tau^*},Y^y_{\tau^*})]\\
&\qquad = \hat{V}(x,y) + \mathbb{E}\Big[\int_0^{\tau^*} e^{-rs}\Big((\hat{\mathcal{L}}-r)\hat{V}(X^x_s,Y^y_s) + (\mathcal{L}-\hat{\mathcal{L}})\hat{V}(X^x_s,Y^y_s)\Big)ds\Big]\\
&\qquad \le \hat{V}(x,y) + \mathbb{E}\Big[\int_0^{\tau^*} e^{-rs}(\mathcal{L}-\hat{\mathcal{L}})\hat{V}(X^x_s,Y^y_s)ds\Big]\\
&\qquad = \hat{V}(x,y) + \mathbb{E}\Big[\int_0^{\tau^*} e^{-rs}\Big((\alpha_1-\hat{\alpha}_1)x\hat{V}_x(X^x_s,Y^y_s)\Big)ds\Big]\\
&\qquad \le \hat{V}(x,y),
\end{aligned}$$

for all $(x,y) \in \mathbb{R}^2_+$, upon using that $\hat{V}$ is nondecreasing by Proposition 4.1. We now write $F(x,y;\alpha_1)$ in order to emphasize the dependency of F on the drift coefficient α_1. Notice that $F(\cdot\,;\hat{\alpha}_1) \ge F(\cdot\,;\alpha_1)$. Repeating arguments as in the proof of Proposition 6.2, we obtain

$$\hat{V}(x,y) \ge \mathbb{E}[e^{-r\tau^*}\hat{V}(X^x_{\tau^*},Y^y_{\tau^*})] \ge \mathbb{E}[e^{-r\tau^*}F(X^x_{\tau^*},Y^y_{\tau^*};\hat{\alpha}_1)] \ge \mathbb{E}[e^{-r\tau^*}F(X^x_{\tau^*},Y^y_{\tau^*};\alpha_1)] = V(x,y),$$

where $\tau^* = \tau^*(x,y)$ again denotes the optimal stopping time for the stopping problem with value function $V(x,y)$. □

The results of Propositions 6.2 and 6.3 remain valid for a change in the coefficients α_2 and σ_2.

Corollary 6.4. *The value function V of (2.4) is increasing in α_2 as well as σ_2.*

Numerical evaluation. In the following, we implement a numerical scheme in order to investigate the sensitivity of the optimal boundary b with respect to the parameters r and α_i as well as σ_i for $i = 1, 2$. Recall that b uniquely solves the integral equation (5.9). Let ζ be an auxiliary exponentially distributed random variable with parameter r that is independent of (W^X, W^Y). It follows that (5.9) can be reformulated as

$$\begin{aligned} b(x) &= f(x) + \lambda \mathbb{E}\Big[\int_0^\infty e^{-rt}\Big(Q_1 X_t^x + Q_2 Y_t^{b(x)} - rI\Big)\mathbb{1}_{\{Y_t^{b(x)} \geq b(X_t^x)\}} dt\Big] \\ &= f(x) + \lambda \frac{1}{r}\mathbb{E}\Big[\Big(Q_1 X_\zeta^x + Q_2 Y_\zeta^{b(x)} - rI\Big)\mathbb{1}_{\{Y_\zeta^{b(x)} \geq b(X_\zeta^x)\}}\Big]. \end{aligned} \tag{6.3}$$

The latter representation is useful, as it allows for an application of Monte-Carlo methods in order to estimate expectations. In the following, we apply an iterative procedure, inspired by the contributions of Christensen and Salminen [8] and Detemple and Kitapbayev [12]. For $(x, y) \in \mathbb{R}_+^2$ and a function $b : \mathbb{R}_+ \to \mathbb{R}_+$, we define the operator

$$\Psi(x, y; b) := f(x) + \lambda \frac{1}{r}\mathbb{E}\Big[\Big(Q_1 X_\zeta^x + Q_2 Y_\zeta^y - rI\Big)\mathbb{1}_{\{Y_\zeta^y \geq b(X_\zeta^x)\}}\Big], \tag{6.4}$$

It follows that the equation (6.3) rewrites as a fixed point problem

$$b(x) = \Psi(x, b(x); b), \quad x \in \mathbb{R}_+, \tag{6.5}$$

which we aim to solve by an iterative scheme. In order to do so, we define the sequence of boundaries

$$b^{(n)}(x) = \Psi(x, b^{(n-1)}(x); b^{(n-1)}), \quad x \in \mathbb{R}_+, \tag{6.6}$$

for $n \geq 1$ and choose the initial boundary $b^{(0)}$ such that $b^{(0)}(0) = y^*$, $b^{(0)}(x^*) = 0$, $b^{(0)}(x^*)$ is the vertex of a parabola and $b^{(0)}(x) = 0$ for all $x \geq x^*$. Moreover, for a given boundary $b^{(k)}$ we estimate the expectation in (6.4) by

$$\frac{1}{N}\sum_{i=1}^N \big(Q_1 X_{\zeta_i}^{i,x} + Q_2 Y_{\zeta_i}^{i,b^{(k)}(x)} - rI\big)\mathbb{1}_{\big\{Y_{\zeta_i}^{i,b^{(k)}(x)} \geq b^{(k)}(X_{\zeta_i}^{i,x})\big\}}, \tag{6.7}$$

where N is the total amount of implemented realizations of an exponential random variable with parameter r. Consequently, for each $i = 1, ..., N$, ζ_i denotes the value of time, while $X_{\zeta_i}^{i,x}$ and $Y_{\zeta_i}^{i,y}$ are the prices of the two products. Under the described procedure, the scheme (6.6) is then iterated until the variation between steps falls below a predetermined level.

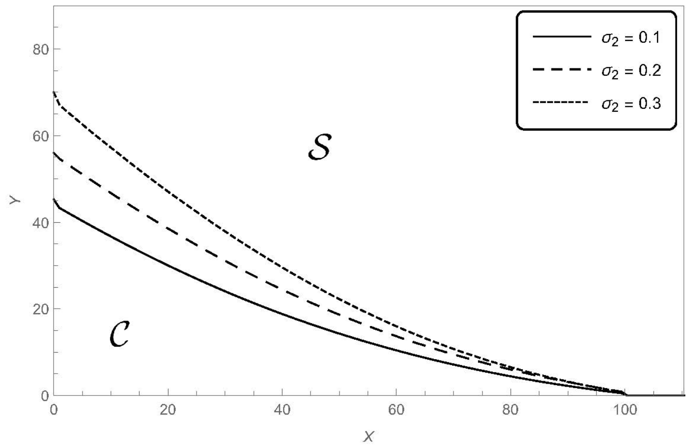

FIGURE 1. The optimal boundary for different values of σ_2 and following parameters: $r = 0.1$, $\alpha_1 = 0.03$, $\alpha_2 = 0.03$, $\sigma_1 = 0.15$, $Q_1 = 5$, $Q_2 = 10$, $I = 4000$

In Figure 1 we can observe the dependency of the optimal boundary b with respect to σ_2. It is evident, that the boundary increases with σ_2. A larger volatility coefficient may be interpreted as a higher level of uncertainty, which is equivalent to a higher price fluctuation in our model. The price thus has larger distortions in the downward direction - but also upwards. The firm exploits the latter fact and thus waits for higher prices to evolve. Furthermore, this effect also results in a larger expected profit of the firm, which also follows by Proposition 6.2. Notice that the threshold value x^* does not change in Figure 1, as it depends exclusively on the parameters Q_1, r, α_1 and σ_1. A change in the volatility coefficient σ_2 thus has no influence on the investment threshold on the x-axis.

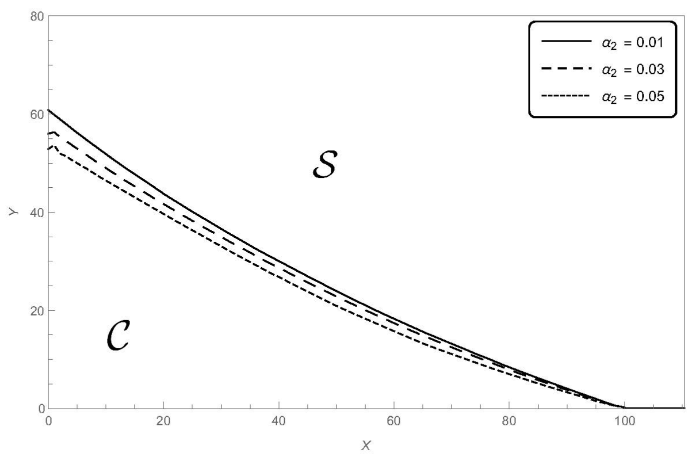

FIGURE 2. The optimal boundary for different values of α_2 and following parameters: $r = 0.1$, $\alpha_1 = 0.03$, $\sigma_1 = 0.15$, $\sigma_2 = 0.2$, $Q_1 = 5$, $Q_2 = 10$, $I = 4000$

Figure 2 shows the optimal boundary b for different values of α_2. We can see that, differently to what is happening for the volatility, the optimal boundary b is decreasing in α_2. A larger drift coefficient α_2 implies higher expected prices of the second product on the market and, as a result, the value of the investment increases. To understand the observed effect on the optimal boundary b we notice that the function F, which represents the value of exercising the investment option immediately, depends explicitely on α_2. Notice that F increases for larger values of α_2. The company thus has an incentive to invest earlier into the production plant and consequently, the boundary decreases.

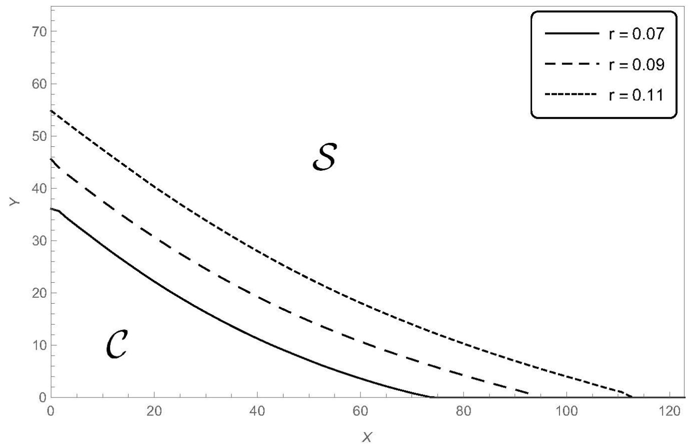

FIGURE 3. The optimal boundary for different values of r and following parameters: $\alpha_1 = 0.02$, $\alpha_2 = 0.03$, $\sigma_1 = 0.15$, $\sigma_2 = 0.15$, $Q_1 = 5$, $Q_2 = 10$, $I = 4000$

In Figure 3 we can observe the sensitivity of the optimal boundary b with respect to the discount factor r. We observe that the boundary b increases in the discount factor. As in the case of a change in the drift coeffiecient α_2, we notice that the function F depends explicitly on r. Since F decreases with r, the value of exercising the investment immediately decreases, so that the company prefers to delay the investment. Consequently, the boundary b increases. Notice that, differently to what we can observe in Figures 1 and 2, a change in r shifts the investment thresholds on both axes, as in fact r affects both x^* and y^*.

APPENDIX A. PROOF OF PROPOSITION 4.1

Lower and Upper Bounds. Observe that the first lower bound follows by taking the a-priori suboptimal stopping time $\tau = 0$. For the second lower bound, consider the stopping time $\sigma := \sigma(x,y) := \inf\{t \geq 0 : \ F(X^x_\tau, Y^y_\tau) > 0\}$ and notice that $e^{-r\sigma}F(X^x_\sigma, Y^y_\sigma)\mathbb{1}_{\{\sigma=\infty\}} = 0$ under the convention (2.7). It is evident that $\mathbb{P}(\sigma < +\infty) > 0$, and we thus have

$$V(x,y) \geq \mathbb{E}[e^{-r\sigma}F(X^x_\sigma, Y^y_\sigma)] > 0,$$

for all $(x,y) \in \mathbb{R}_+$. On the other hand, one obtains the upper bound by observing that

$$\begin{aligned} V(x,y) &= \sup_{\tau\in\mathcal{T}} \mathbb{E}\Big[e^{-r\tau}\Big(\frac{Q_1}{\delta_1}X^x_\tau + \frac{Q_2}{\delta_2}Y^y_\tau - I\Big)\Big] \leq \sup_{\tau\in\mathcal{T}}\Big[\frac{Q_1}{\delta_1}\mathbb{E}[e^{-r\tau}X^x_\tau] + \frac{Q_2}{\delta_2}\mathbb{E}[e^{-r\tau}Y^y_\tau]\Big] \\ &= \frac{Q_1}{\delta_1}x + \frac{Q_2}{\delta_2}y \ \leq \ C(x+y) \end{aligned}$$

upon setting $C := \max\{\frac{Q_1}{\delta_1}, \frac{Q_2}{\delta_2}\} > 0$, and using the uniform integrability stated in Remark 2.2.

Monotonicity. Let $(x,y) \in \mathbb{R}^2_+$ and $\epsilon > 0$. Consider an ϵ-optimal stopping time $\tau^\epsilon := \tau^\epsilon(x,y)$ for the optimal investment problem with value function $V(x,y)$. For any $\varphi > 0$, it follows that

(A.1)
$$V(x+\varphi,y) - V(x,y) + \epsilon \geq \mathbb{E}[e^{-r\tau^\epsilon}F(X^{x+\varphi}_{\tau^\epsilon}, Y^y_{\tau^\epsilon})] - \mathbb{E}[e^{-r\tau^\epsilon}F(X^x_{\tau^\epsilon}, Y^y_{\tau^\epsilon})] = \frac{Q_1\varphi}{\delta_1}\mathbb{E}[e^{-r\tau^\epsilon}X^1_{\tau^\epsilon}] \ \geq \ 0,$$

where the last inequality holds due to the nonnegativity of X^x_t and Assumption 2.1. Rearranging terms yields

$$V(x+\varphi,y) + \epsilon \geq V(x,y)$$

and V is thus nondecreasing in x by arbitrariness of $\epsilon > 0$. Moreover, by employing similar arguments, we obtain that V is nondecreasing in y.

Continuity. Let $\{(x_n, y_n),\ n \in \mathbb{N}\} \subset \mathbb{R}^2_+$ be a sequence converging to $(x, y) \in \mathbb{R}^2_+$. For $\epsilon > 0$, consider an ϵ-optimal stopping time $\tau^\epsilon := \tau^\epsilon(x, y)$ for the stopping problem with value function $V(x, y)$. It follows that

$$\begin{aligned} V(x,y) - V(x_n,y_n) \ &\leq \epsilon \ + \ \mathbb{E}[e^{-r\tau^\epsilon}(F(X^x_{\tau^\epsilon}, Y^y_{\tau^\epsilon}) \ - \ F(X^{x_n}_{\tau^\epsilon}, Y^{y_n}_{\tau^\epsilon}))] \\ &= \epsilon \ + \ (x - x_n)\frac{Q_1}{\delta_1}\mathbb{E}[e^{-r\tau^\epsilon}X^1_{\tau^\epsilon}] \ + \ (y - y_n)\frac{Q_2}{\delta_2}\mathbb{E}[e^{-r\tau^\epsilon}Y^1_{\tau^\epsilon}]. \end{aligned}$$

and by rearranging terms and letting $n \to \infty$ we obtain

$$\liminf_{n\to\infty} V(x_n, y_n) \ \geq \ V(x, y) \ - \ \epsilon. \tag{A.2}$$

On the other hand, consider an ϵ-optimal stopping time $\tau^\epsilon_n := \tau^\epsilon(x_n, y_n)$ for the stopping problem with value function $V(x_n, y_n)$. By noticing that

$$\mathbb{E}[e^{-r\tau}X^x_t] \ = \ x - \mathbb{E}\Big[\int_0^\tau e^{-rt}\delta_1 X^x_t dt\Big] \quad \text{and} \quad \mathbb{E}[e^{-r\tau}Y^y_t] \ = \ y - \mathbb{E}\Big[\int_0^\tau e^{-rt}\delta_2 Y^y_t dt\Big]. \tag{A.3}$$

for all stopping times $\tau \in \mathcal{T}$, we obtain

$$\begin{aligned} V(x_n,y_n) - V(x,y) \ &\leq \ \epsilon \ + \ \mathbb{E}[e^{-r\tau^\epsilon_n}(F(X^{x_n}_{\tau^\epsilon_n}, Y^{y_n}_{\tau^\epsilon_n}) - F(X^x_{\tau^\epsilon_n}, Y^y_{\tau^\epsilon_n}))] \\ &= \ \epsilon \ + \ \frac{Q_1}{\delta_1}\Big((x_n - x) \ + \ \mathbb{E}\Big[\int_0^{\tau^\epsilon_n} e^{-rt}\delta_1(X^x_t - X^{x_n}_t)dt\Big]\Big) \\ &\quad + \ \frac{Q_2}{\delta_2}\Big((y_n - y) \ + \ \mathbb{E}\Big[\int_0^{\tau^\epsilon_n} e^{-rt}\delta_2(Y^y_t \ - \ Y^{y_n}_t)dt\Big]\Big) \\ &\leq \ \epsilon \ + \ \frac{Q_1}{\delta_1}\Big((x_n - x) \ + \ |x_n - x|\delta_1\mathbb{E}\Big[\int_0^\infty e^{-rt}X^1_t dt\Big]\Big) \\ &\quad + \ \frac{Q_2}{\delta_2}\Big((y_n - y) \ + \ |y_n - y|\delta_2\mathbb{E}\Big[\int_0^\infty e^{-rt}Y^1_t dt\Big]\Big) \end{aligned}$$

and taking the limit as $n \to \infty$ this results to

$$\limsup_{n\to\infty} V(x_n, y_n) \ \leq \ \epsilon \ + \ V(x, y). \tag{A.4}$$

The continuity of V then follows from (A.2) and (A.4) by arbitrariness of $\epsilon > 0$.

Convexity. Take any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2_+$ and consider a convex combination $(x, y) := \lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2)$ for $\lambda \in (0, 1)$. We obtain

$$\begin{aligned} V(x,y) &= \sup_{\tau\in\mathcal{T}} \mathbb{E}\Big[e^{-r\tau}\Big(\frac{Q_1 x X^1_\tau}{\delta_1} + \frac{Q_2 y Y^1_\tau}{\delta_2} - I\Big)\Big] \\ &= \sup_{\tau\in\mathcal{T}} \mathbb{E}\Big[\lambda e^{-r\tau}\Big(\frac{Q_1 x_1 X^1_\tau}{\delta_1} + \frac{Q_2 y_1 Y^1_\tau}{\delta_2} - I\Big) \\ &\qquad\qquad + (1-\lambda)e^{-r\tau}\Big(\frac{Q_1 x_2 X^1_\tau}{\delta_1} + \frac{Q_2 y_2 Y^1_\tau}{\delta_2} - I\Big)\Big] \\ &\leq \lambda \sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r\tau}F(X^{x_1}_\tau, Y^{y_1}_\tau)] + (1-\lambda)\sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r\tau}F(X^{x_2}_\tau, Y^{y_2}_\tau)] \\ &= \lambda V(x_1, y_1) + (1-\lambda)V(x_2, y_2), \end{aligned}$$

and the claim follows.

Appendix B. Proof of Theorem 4.2

We argue by adopting arguments presented in Section 3.1 in De Angelis et al. [11]. At this point it would be convenient for us to study the variational inequality associated with the optimal stopping problem, but since the coefficients of the stochastic differential equations in (2.1) are unbounded on the state space $\mathbb{R}^2_+$, classical results from the PDE literature are not directly applicable. Instead we will approximate the optimal stopping problem (2.4) by a sequence of problems on bounded domains. To this end, define a sequence $\{Q_n, n \in \mathbb{N}\}$ of sets satisfying the following conditions.

(B.1) i) Q_n is open, bounded and connected for all $n \in \mathbb{N}$,

(B.2) ii) $Q_n \subset Q_{n+1}$ for all $n \in \mathbb{N}$,

(B.3) iii) $\lim_{n\to\infty} Q_n := \cup_{n\geq 0} Q_n = \mathbb{R}^2_+$,

(B.4) iv) $\partial Q_n \in C^{2+\alpha_n}$ for some $\alpha_n > 0$.

Note that it is always possible to find such a sequence. Furthermore, we define

$$\sigma_n = \sigma_n(x,y) := \inf\{t \geq 0 : \ (X^x_t, Y^y_t) \notin Q_n\} \tag{B.5}$$

and state the following remark.

Remark B.1. *The condition (B.2) implies that the sequence $\{\sigma_n, n \in \mathbb{N}\}$ is strictly increasing as $n \to \infty$ with limit*

$$\sigma_n \uparrow \sigma_\infty = \sigma_\infty(x,y) := \inf\{t \geq 0 : \ (X^x_t, Y^y_t) \notin \mathbb{R}^2_+\}. \tag{B.6}$$

The boundaries 0 as well as $+\infty$ of the processes X^x_t and Y^y_t are natural, meaning they are unattainable whenever the processes are started in the interior of the state space (cf. Borodin and Salminen [4], p. 136). For the stopping time $\sigma_\infty(x,y)$ specified in (B.6) it thus follows that

$$\sigma_\infty = \sigma_\infty(x,y) = \infty \quad \mathbb{P}\text{-}a.s. \tag{B.7}$$

for every $(x,y) \in \mathbb{R}^2_+$.

Upon using the stopping time $\sigma_n(x,y)$ of (B.5) we localize the optimal stopping problem (2.4) by setting

$$V_n(x,y) := \sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r(\tau\wedge\sigma_n)} F(X^x_{\tau\wedge\sigma_n}, Y^y_{\tau\wedge\sigma_n})], \qquad (x,y) \in \mathbb{R}^2_+. \tag{B.8}$$

The continuation and stopping regions of this stopping problem are given by

$$\mathcal{C}_n := \{(x,y) \in \mathbb{R}^2_+ \ | \ V_n(x,y) > F(x,y)\} \qquad \mathcal{S}_n := \{(x,y) \in \mathbb{R}^2_+ \ | \ V_n(x,y) = F(x,y)\}, \tag{B.9}$$

respectively. Furthermore, we note that the second-order elliptic differential operator associated with the two-dimensional diffusion (X^x_t, Y^y_t) is given by $\mathcal{L} := \mathcal{L}_X + \mathcal{L}_Y$, where

$$\mathcal{L}_X := \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2}{\partial x^2} + \alpha_1 x \frac{\partial}{\partial x} \qquad \mathcal{L}_Y := \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2}{\partial y^2} + \alpha_2 y \frac{\partial}{\partial y}, \tag{B.10}$$

since we are dealing with two *uncorrelated* geometric brownian motions (cf. Borodin and Salminen [4], p. 136). Moreover, by employing standard arguments (cf. Peskir and Shiryaev [31], p. 49), we can associate the function $V_n\,|_{Q_n}$ to the variational inequality

$$\max\{(\mathcal{L} - r)u(x,y), -u(x,y) + F(x,y)\} = 0, \quad (x,y) \in Q_n \tag{B.11}$$

with the boundary condition

$$u(x,y) = F(x,y), \quad (x,y) \in \partial Q_n. \tag{B.12}$$

The following Proposition verifies that the function of (B.8) indeed solves the system of equations stated in (B.11) and (B.12) above.

Proposition B.2. *The function V_n of (B.8) uniquely solves the variational inequality (B.11) a.e. in Q_n with boundary condition (B.12) and we have $V_n \in \mathcal{W}^{2,p}(Q_n)$ for $1 \leq p < \infty$, where $\mathcal{W}^{2,p}(Q_n)$ denotes the Sobolev space of order 2 (cf. Brezis [6], Chapter 8.2). Furthermore, the stopping time*

$$\tau_n^* := \inf\{t \geq 0 : \ (X_t^x, Y_t^y) \notin \mathcal{C}_n\}, \qquad (x,y) \in \mathbb{R}_+^2 \tag{B.13}$$

is optimal for the problem (B.8).

Proof. (i) The existence and uniqueness of a function $u_n \in \mathcal{W}^{2,p}(Q_n)$ for all $p \in [1,\infty)$ solving the variational inequality (B.11) with boundary condition (B.12) is guaranteed by the results derived in Friedman [18], since the coefficients of the dynamics (2.1) are continuous as well as bounded on $\bar{Q}_n$ and we have made sufficient assumptions regarding the set Q_n and their boundary in (B.1) and (B.4) (cf. Friedman [18], Theorem 3.2 & 3.4). Furthermore, we are able to continuously extend the function u_n outside of Q_n by setting

$$u_n(x,y) = F(x,y), \quad (x,y) \in \mathbb{R}_+^2 \setminus Q_n. \tag{B.14}$$

In the following we refer to this extension and denote it, with a slight abuse of notation, again by u_n.

(ii) It is therefore left to check that the value function V_n of (B.8) in fact coincides with the unique solution u_n introduced in point (i) over $\mathbb{R}_+^2$ as well as that the stopping time (B.13) is optimal for the problem stated in (B.8). We first treat the case of $(x,y) \in \mathbb{R}_+^2 \setminus Q_n$, for which the definition of σ_n in (B.5) evidently yields $\sigma_n(x,y) = 0$. Together with (B.8) it consequently follows

$$V_n(x,y) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r(\tau\wedge\sigma_n)} F(X_{\tau\wedge\sigma_n}^x, Y_{\tau\wedge\sigma_n}^y)] = F(x,y).$$

Upon using (B.14), we therefore obtain

$$V_n(x,y) = F(x,y) = u_n(x,y),$$

and the claim follows for any $(x,y) \in \mathbb{R}_+^2 \setminus Q_n$. We now let $(x,y) \in Q_n$. In this case we obtain the proof by showing both inequalities $V_n \geq u_n$ as well as $V_n \leq u_n$. The theorem of Meyers-Serrin (cf. Gilbarg and Trudinger [20], Theorem 7.9) implies there exists a sequence of smooth functions $\{u_n^k(\cdot), \ k \in \mathbb{N}\} \subset C^\infty(Q_n)$ such that

$$u_n^k \to u_n, \qquad \text{as} \quad k \to \infty \tag{B.15}$$

in $\mathcal{W}^{2,p}(Q_n)$ for $1 \leq p < \infty$. Since the function u_n is continuous and $\bar{Q}_n$ is compact, the convergence in (B.15) is actually uniform on $\bar{Q}_n$ (cf. Gilbarg and Trudinger [20], Lemma 7.1). We have enough regularity for the functions u_n^k to apply Dynkin's formula and obtain

$$u_n^k(x,y) = \mathbb{E}\Big[e^{-r(\tau\wedge\sigma_n)} u_n^k(X_{\tau\wedge\sigma_n}^x, Y_{\tau\wedge\sigma_n}^y) - \int_0^{\tau\wedge\sigma_n} e^{-rt}(\mathcal{L}-r)u_n^k(X_t^x, Y_t^y)dt\Big] \tag{B.16}$$

for any bounded stopping time τ. Using standard localization arguments as well as (2.8), one can check that this equality holds true for all stopping times $\tau \in \mathcal{T}$.

In the next step we will study this equation in the limit as $k \to \infty$. The term on the left-hand side of (B.16) converges pointwisely by (B.15), whereas the uniform convergence on $\bar{Q}_n$ guarantees

$$\lim_{k\to\infty} \mathbb{E}\Big[e^{-r(\tau\wedge\sigma_n)} u_n^k(X_{\tau\wedge\sigma_n}^x, Y_{\tau\wedge\sigma_n}^y)\Big] = \mathbb{E}\Big[e^{-r(\tau\wedge\sigma_n)} u_n(X_{\tau\wedge\sigma_n}^x, Y_{\tau\wedge\sigma_n}^y)\Big].$$

It is therefore left to check that the integral term in (B.16) converges, and we need to show

$$\lim_{k\to\infty} \mathbb{E}\Big[\int_0^{\tau\wedge\sigma_n} e^{-rt}(\mathcal{L}-r)u_n^k(X_t^x, Y_t^y)dt\Big] = \mathbb{E}\Big[\int_0^{\tau\wedge\sigma_n} e^{-rt}(\mathcal{L}-r)u_n(X_t^x, Y_t^y)dt\Big].$$

We recall Lemma 5.3 (ii) and take $q > 1$ suitable for $\bar{Q}_n$ and p such that $\frac{1}{q} + \frac{1}{p} = 1$. For a multi-index α we specify the following norm on the Sobolev space $\mathcal{W}^{2,p}(Q_n)$

$$||u||_{\mathcal{W}^{2,p}(Q_n)} := \sum_{|\alpha|\leq 2} ||D^\alpha u||_{L^p(Q_n)} = \sum_{|\alpha|\leq 2} \Big(\int_{Q_n} |D^\alpha u(\xi,\zeta)|^p d\xi d\zeta \Big)^{\frac{1}{p}}$$

and note that $\mathcal{W}^{2,p}(Q_n)$ is a Banach space equipped with this norm. Hölder's inequality for p and q as defined above then yields

$$\Big|\mathbb{E}\Big[\int_0^{\tau\wedge\sigma_n} e^{-rt}(\mathcal{L}-r)(u_n^k - u_n)(X_t^x, Y_t^y)dt\Big]\Big| \leq C||u_n^k - u_n||_{\mathcal{W}^{2,p}(Q_n)}, \tag{B.17}$$

where $C > 0$ denotes a positive constant. The right-hand side of (B.17) vanishes as $k \to \infty$ since (B.15) holds true and we finally obtain

$$u_n(x,y) = \mathbb{E}\Big[e^{-r(\tau\wedge\sigma_n)} u_n(X^x_{\tau\wedge\sigma_n}, Y^y_{\tau\wedge\sigma_n}) - \int_0^{\tau\wedge\sigma_n} e^{-rt}(\mathcal{L}-r)u_n(X_t^x, Y_t^y)dt\Big], \tag{B.18}$$

for all $\tau \in \mathcal{T}$. As this is well-defined, since $(\mathcal{L}-r)u_n$ is defined up to a null set of Lebesgue measure, the variational inequality (B.11) implies on the one hand that

$$u_n(x,y) \geq \mathbb{E}[e^{-r(\tau\wedge\sigma_n)} u_n(X^x_{\tau\wedge\sigma_n}, Y^y_{\tau\wedge\sigma_n})], \qquad \forall \tau \in \mathcal{T}$$

and furthermore

$$u_n(x,y) \geq \mathbb{E}[e^{-r(\tau\wedge\sigma_n)} F(X^x_{\tau\wedge\sigma_n}, Y^y_{\tau\wedge\sigma_n})], \qquad \forall \tau \in \mathcal{T}.$$

By the arbitrariness of $\tau \in \mathcal{T}$ we have

$$u_n(x,y) \geq \sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r(\tau\wedge\sigma_n)} F(X^x_{\tau\wedge\sigma_n}, Y^y_{\tau\wedge\sigma_n})] = v_n(x,y) \tag{B.19}$$

and this concludes the first part of the proof. To obtain the reverse, we consider the stopping time

$$\hat{\tau} = \hat{\tau}(x,y) := \inf\{t \geq 0 : \ u_n(X_t^x, Y_t^y) = F(X_t^x, Y_t^y)\}$$

and recall $u_n = F$ on $\mathbb{R}^2_+ \setminus Q_n$. Since u_n is continuous on the bounded set $\bar{Q}_n$, it is bounded as well and with the convention (2.7) we have

$$e^{-r(\hat{\tau}\wedge\sigma_n)} u_n(X^x_{\hat{\tau}\wedge\sigma_n}, Y^y_{\hat{\tau}\wedge\sigma_n}) \mathbb{1}_{\{\hat{\tau}\wedge\sigma_n = \infty\}} = \limsup_{t\to\infty} e^{-rt} u_n(X_t^x, Y_t^y) = 0.$$

Consequently, we obtain

$$\begin{aligned} e^{r(\hat{\tau}\wedge\sigma_n)} u_n(X^x_{\hat{\tau}\wedge\sigma_n}, Y^y_{\hat{\tau}\wedge\sigma_n}) &= e^{r(\hat{\tau}\wedge\sigma_n)} u_n(X^x_{\hat{\tau}\wedge\sigma_n}, Y^y_{\hat{\tau}\wedge\sigma_n}) \mathbb{1}_{\{\hat{\tau}\wedge\sigma_n < \infty\}} \\ &= e^{-r(\hat{\tau}\wedge\sigma_n)} F(X^x_{\hat{\tau}\wedge\sigma_n}, Y^y_{\hat{\tau}\wedge\sigma_n}) \mathbb{1}_{\{\hat{\tau}\wedge\sigma_n < \infty\}} \\ &= e^{-r(\hat{\tau}\wedge\sigma_n)} F(X^x_{\hat{\tau}\wedge\sigma_n}, Y^y_{\hat{\tau}\wedge\sigma_n}), \end{aligned}$$

where the last equality follows from Remark 2.2. Hence, upon using the fact that $(\mathcal{L}-r)u_n = 0$ on the set $\{(x,y) \in Q_n : \ u_n(x,y) > F(x,y)\}$ by (B.11), we finally have

$$\begin{aligned} u_n(x,y) &= \mathbb{E}\Big[e^{-r(\hat{\tau}\wedge\sigma_n)} u_n(X^x_{\hat{\tau}\wedge\sigma_n}, Y^y_{\hat{\tau}\wedge\sigma_n}) - \int_0^{\hat{\tau}\wedge\sigma_n} e^{-rt}(\mathcal{L}-r)u_n(X_t^x, Y_t^y)dt\Big] \\ &= \mathbb{E}[e^{-r(\hat{\tau}\wedge\sigma_n)} F(X^x_{\hat{\tau}\wedge\sigma_n}, Y^y_{\hat{\tau}\wedge\sigma_n})] \leq \sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r(\tau\wedge\sigma_n)} F(X^x_{\tau\wedge\sigma_n}, Y^y_{\tau\wedge\sigma_n})] \\ &= V_n(x,y). \end{aligned} \tag{B.20}$$

Combining (B.19) and (B.20) we conclude that $u_n = V_n$ on $\mathbb{R}^2_+$. Moreover, as the inequality in (B.20) becomes an equality, the stopping time $\hat{\tau}$ is optimal for the problem (B.8) and coincides with the stopping time τ_n^* defined in (B.13). □

Remark B.3. *In the following, we will refer to the unique C^1 representative of the elements in the class $\mathcal{W}^{2,p}(Q_n)$, as the Sobolev inclusions (cf. Brezis [6], Corollary 9.13 & 9.15) guarantee a continuous embedding of $\mathcal{W}^{2,p}(Q_n)$ into $C^1(\bar{Q}_n)$ for $p \in (2,\infty)$, and the boundary condition (B.12) is thus well posed for such functions.*

In the next Proposition we derive a probabilistic representation for the value function of (B.8). The proof follows, apart from a small technicality, by our results stated before in this section.

Proposition B.4. *The function V_n of (B.8) admits the representation*

$$V_n(x,y) = \mathbb{E}\Big[e^{-r\sigma_n} F(X^x_{\sigma_n}, Y^y_{\sigma_n}) - \int_0^{\sigma_n} e^{-rt}(rI - Q_1 X^x_t - Q_2 Y^y_t)\mathbb{1}_{\{(X^x_t, Y^y_t)\in\mathcal{S}\}} dt\Big] \tag{B.21}$$

for all $(x,y) \in \mathbb{R}^2_+$.

Proof. The proof follows by adapting arguments presented in the proof of Proposition B.2, that is finding a sequence (B.15), applying Dynkins formula as in (B.16) and taking the limit as $k \to \infty$, we then obtain the representation

$$V_n(x,y) = \mathbb{E}\Big[e^{-r\sigma_n} F(X^x_{\sigma_n}, Y^y_{\sigma_n}) - \int_0^{\sigma_n} e^{-rt}(\mathcal{L} - r)V_n(X^x_t, Y^y_t) dt\Big], \tag{B.22}$$

by recalling that V_n solves the boundary condition (B.12). Moreover, due to Proposition B.2 and arguing as in Lemma B.1 in De Angelis et al. [11], we have

$$(\mathcal{L} - r)V_n(x,y) = (\mathcal{L} - r)F(x,y)\mathbb{1}_{\{(x,y)\in\mathcal{S}_n\}} = (rI - Q_1 x - Q_2 y)\mathbb{1}_{\{(x,y)\in\mathcal{S}_n\}} \tag{B.23}$$

for a.e. $(x,y) \in Q_n$. Due to Lemma 5.3 we can use (B.23) in (B.22) and the claim follows. ☐

In the forthcoming Proposition we explore some properties of the *sequence* of functions V_n, most importantly its behaviour in the limit as $n \to \infty$. This is essential for the proof of Theorem 4.2, as we aim at studying the limit of (B.21) for $n \to \infty$.

Proposition B.5. *The sequence $\{V_n(\cdot),\ n \in \mathbb{N}\}$ is ascending and such that $V_n \le V$ on $\mathbb{R}^2_+$ for all $n \in \mathbb{N}$. Moreover, it converges pointwisely to the value function V of the stopping problem (2.4).*

Proof. The first two claims follow by recalling (B.6) and simple comparison arguments. In order to check the convergence of the sequence, we consider an ϵ-optimal stopping time $\tau^\epsilon = \tau^\epsilon(x,y)$ for the stopping problem with value function $V(x,y)$. We obtain

$$\begin{aligned}0 \le V(x,y) - V_n(x,y) &\le \mathbb{E}[e^{-r\tau^\epsilon} F(X^x_{\tau^\epsilon}, Y^y_{\tau^\epsilon})] - \mathbb{E}[e^{-r(\tau^\epsilon\wedge\sigma_n)} F(X^x_{\tau^\epsilon\wedge\sigma_n}, Y^y_{\tau^\epsilon\wedge\sigma_n})] + \epsilon \\ &= \mathbb{E}[(e^{-r\tau^\epsilon} F(X^x_{\tau^\epsilon}, Y^y_{\tau^\epsilon}) - e^{-r\sigma_n} F(X^x_{\sigma_n}, Y^y_{\sigma_n}))\mathbb{1}_{\{\sigma_n < \tau^\epsilon\}}] + \epsilon. \end{aligned} \tag{B.24}$$

and due to Assumption 2.1 and (2.6), the sequence of random variables

$$W_n = \Big(e^{-r\tau^\epsilon} F(X^x_{\tau^\epsilon}, Y^y_{\tau^\epsilon}) - e^{-r\sigma_n} F(X^x_{\sigma_n}, Y^y_{\sigma_n})\Big)\mathbb{1}_{\{\sigma_n < \tau^\epsilon\}}$$

is uniformly integrable. Moreover, the sequence converges in measure and we have $\lim_{n\to\infty} W_n = 0$ $\mathbb{P}$-a.s. The convergence theorem of Vitali (cf. Folland [16], p. 187) then implies

$$\lim_{n\to\infty} \mathbb{E}[(e^{-r\tau^\epsilon} F(X^x_{\tau^\epsilon}, Y^y_{\tau^\epsilon}) - e^{-r\sigma_n} F(X^x_{\sigma_n}, Y^y_{\sigma_n}))\mathbb{1}_{\{\sigma_n < \tau^\epsilon\}}] = 0$$

and the claim follows by the arbitrariness of $\epsilon > 0$ in (B.24). ☐

We now state the proof of Theorem 4.2, in which we derive the probabilistic representation (4.4) of the value function V of the optimal stopping problem (2.4).

Proof of Theorem 4.2. We study the representation (B.21) in the limit as $n \to \infty$. Notice that the left-hand side converges pointwisely to the function V, due to Proposition B.5. It is therefore left to check that the following equality holds true

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\Big[e^{-r\sigma_n}F(X^x_{\sigma_n},Y^y_{\sigma_n}) - \int_0^{\sigma_n} e^{-rt}(rI - Q_1X^x_t - Q_2Y^y_t)\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{S}\}}dt\Big] \\ = \mathbb{E}\Big[\int_0^\infty e^{-rt}(Q_1X^x_t + Q_2Y^y_t - rI)\,\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{S}\}}dt\Big].\end{aligned} \tag{B.25}$$

Note that since

$$\mathbb{E}[e^{-r\sigma_n}F(X^x_{\sigma_n},Y^y_{\sigma_n})] = \frac{Q_1}{\delta_1}\mathbb{E}[e^{-r\sigma_n}X^x_{\sigma_n}] + \frac{Q_2}{\delta_2}\mathbb{E}[e^{-r\sigma_n}Y^y_{\sigma_n}] - e^{-r\sigma_n}I$$

and $\sigma_n \uparrow \infty$, Remark 2.2 together with Vitali's convergence theorem yields

$$\lim_{n\to\infty}\mathbb{E}[e^{-r\sigma_n}F(X^x_{\sigma_n},Y^y_{\sigma_n})] = 0. \tag{B.26}$$

We now seek to study the limit of the integral term of (B.25). Observe that $V_n \le V_{n+1} \le V$ implies $\mathcal{S} \subset \mathcal{S}_{n+1} \subset \mathcal{S}_n$ for all $n \in \mathbb{N}$, while the pointwise convergence of $V_n \uparrow V$ implies that $\lim_{n\to\infty}\mathcal{S}_n := \bigcap_{n\ge 0}\mathcal{S}_n = \mathcal{S}$. We therefore have

$$\lim_{n\to\infty}\mathbb{1}_{[0,\sigma_n]}(t)e^{-rt}(rI - Q_1X^x_t - Q_2Y^y_t)\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{S}_n\}} \;=\; e^{-rt}(rI - Q_1X^x_t - Q_2Y^y_t)\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{S}\}}$$

for a.e. $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^2_+$. Moreover, notice that for a constant C depending on Q_1, Q_2, I and r, we have

$$\begin{aligned}|e^{-rt}(rI - Q_1X^x_t - Q_2Y^y_t)\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{S}_n\}}| &\le e^{-rt}|rI - Q_1X^x_t - Q_2Y^y_t)| \\ &\le e^{-rt}C(1 + X^x_t + Y^y_t),\end{aligned} \tag{B.27}$$

where the last term is integrable due to Assumption 2.1. Applying dominated convergence then yields

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\Big[\int_0^{\sigma_n} e^{-rt}(rI - Q_1X^x_t - Q_2Y^y_t)\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{S}_n\}}dt\Big] \\ = \mathbb{E}\Big[\int_0^\infty e^{-rt}(rI - Q_1X^x_t - Q_2Y^y_t)\mathbb{1}_{\{(X^x_t,Y^y_t)\in\mathcal{S}\}}dt\Big],\end{aligned} \tag{B.28}$$

and the claim follows by (B.26) and (B.28).

Appendix C. Proof of Proposition 6.1

By taking the stopping time $\tau = 0$, it immediately follows that $V(x,y) \ge F(x,y)$ for all $(x,y) \in \mathbb{R}^2_+$. It is thus left to prove that $(\mathcal{L} - r)V(x,y) \le 0$ a.e. on $\mathbb{R}^2_+$. Since the function F is continuous on $\mathbb{R}^2_+$, standard results from optimal stopping theory (cf. Peskir and Shiryaev [31], Chapter 3, 7.1) and PDE theory for elliptic equations imply that the value function is such that $V \in C^{2,2}(\mathcal{C})$ and it solves

$$(\mathcal{L} - r)V = 0 \quad \text{on } \mathcal{C}. \tag{C.1}$$

As V solves C.1, rearranging terms upon using Proposition 4.1 implies

$$0 \;\le\; \frac{1}{2}\sigma_1^2x^2V_{xx} \;=\; -\frac{1}{2}\sigma_2^2y^2V_{yy} - \alpha_1xV_x - \alpha_2yV_y + rV \;\le\; rV - \alpha_1xV_x - \alpha_2yV_y, \quad \text{on } \mathcal{C},$$

which is equivalent to

$$0 \;\le\; V_{xx} \;\le\; \frac{2}{\sigma_1^2x^2}\Big[rV - \alpha_1xV_x - \alpha_2yV_y\Big], \quad \text{on } \mathcal{C}.$$

But the right-hand side defines a continuous function on $\mathbb{R}^2_+$ by Proposition 5.4, and therefore there exists finite

$$\lim_{\mathcal{C}\ni(x_n,y_n)\to(x,y)\in\partial\mathcal{C}} V_{xx}(x,y).$$

Similarly, one is able to prove existence of the second derivative with respect to y at $\partial\mathcal{C}$. Hence, V_x as well as V_y are therefore locally Lipschitz on $\bar{\mathcal{C}}$, as well as on $\mathrm{int}(\mathcal{S})$, where $V = F$. We now show that V_x and V_y are locally Lipschitz continuous on $\mathbb{R}^2_+$. To this end, set $b^{-1}(y) := \inf\{y \in \mathbb{R}_+ :\ y > b(x)\}$ and $g(x,y) := -\frac{2}{\sigma_1^2}(\alpha_1 x V_x(x,y) + \alpha_2 V_y(x,y) + rV(x,y)) \in C(\mathbb{R}^2_+)$. Then, for $x < b^{-1}(y)$ and $x' > b^{-1}(y)$ we obtain by convexity of V that

$$\begin{aligned}0 \le V_x(x',y) - V_x(x,y) &= \int_x^{b^{-1}(y)} \big(g(z,y) - \frac{\sigma_2^2}{\sigma_1^2} y^2 V_{yy}(z,y)\big)\frac{1}{z^2}dz + \int_{b^{-1}(y)}^{x'} F_{xx}(z,y)dz \\ &\le \int_x^{b^{-1}(y)} g(z,y)\frac{1}{z^2}dz \le K(x,y)(b^{-1}(y)-x) \le K(x,y)|x'-x|,\end{aligned}$$

where we used that V solves (C.1). Here, the constant $K > 0$ is such that $K := K(x,y) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2_+)$. Therefore, $V_x(\cdot,\, y)$ is locally Lipschitz on $\mathbb{R}^2_+$; i.e. $V_{xx} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2_+)$. Analogously, one obtains the same result for $y \mapsto V_y(x,y)$. We thus have $V(\cdot,y), V(x,\cdot) \in \mathcal{W}^{2,2}_{\mathrm{loc}}(\mathbb{R}^2_+)$ (cf. Evans and Gariepy [15], p. 164, Theorem 2(ii)), and finally a result by S. Bernstein (cf. Krantz [23], Theorem 3) yields $V \in \mathcal{W}^{2,2}_{\mathrm{loc}}(\mathbb{R}^2_+)$.

It remains to check that $(\mathcal{L}-r)V \le 0$ a.e. in $\mathcal{S}$. For that, we notice $(\mathcal{L}-r)V = (\mathcal{L}-r)F$ for a.e. $(x,y) \in \mathcal{S}$, which can in fact be proved by arguing as in De Angelis et al. [11], Lemma B.1, due to the fact that $V \in \mathcal{W}^{2,2}_{\mathrm{loc}}(\mathbb{R}^2_+)$. Moreover, we have

$$\mathcal{S} = \{(x,y) \in \mathbb{R}^2_+ :\quad V(x,y) = F(x,y)\} \quad\subseteq\quad \{(x,y) \in \mathbb{R}^2_+ :\quad (\mathcal{L}-r)F(x,y) \le 0\}$$

which follows from (5.6). We thus obtain $(\mathcal{L}-r)V(x,y) \le 0$ for a.e. $(x,y) \in \mathcal{S}$, which then completes the proof.

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