Stochastic partial differential equations arising in self-organized criticality

Dissertation

zur Erlangung des akademischen Grades
Doktor der Mathematik (Dr. math.)

Eingereicht von

Marius Neuß

Fakultät für Mathematik
Universität Bielefeld
Dezember 2020

Summary

This thesis is motivated by stochastic particle systems arising in self-organized criticality, which are also known as “sandpile models”. As observed by Bak, Tang and Wiesenfeld [5], these systems stand out due to the fact that they converge without specific external tuning to a state in which power-law distributed intermittent events occur.
The present thesis aims to contribute to the mathematical understanding of this behaviour and of the underlying models in general by making them accessible to analytical tools. To this end, it is rigorously shown that under a suitable rescaling, modified sandpile models on finer and finer one-dimensional grids converge to the solutions of a stochastic partial differential equation (SPDE) with a singular-degenerate drift, driven by space-time white noise. Furthermore, the well-posedness of more general SPDEs of similar type is proved. Finally, the long time behaviour of solutions to the continuum limit SPDE is addressed by proving that the corresponding Markov process possesses a unique invariant measure.

Zusammenfassung

Diese Dissertation ist durch stochastische Teilchensysteme (sogenannte „Sandhaufenmodelle“) motiviert, die in Zusammenhang mit dem physikalischen Phänomen der selbstorganisierten Kritikalität prominent auftauchen. Wie von Bak, Tang und Wiesenfeld [5] beobachtet, konvergieren diese Modelle ohne spezifische externe Einflussnahme gegen einen Zustand, in dem stoßweise Ereignisse auftreten, deren Größe nach einem Potenzgesetz verteilt ist.
Diese Arbeit zielt darauf ab, zum mathematischen Verständnis dieses statistischen Verhaltens und allgemein der zugrundeliegenden Prozesse beizutragen, indem diese analytischen Methoden zugänglich gemacht werden. Dazu wird gezeigt, dass Sandhaufenmodelle auf feiner werdenden eindimensionalen Gittern unter Verwendung einer geeigneten Skalierung in einem bestimmten Sinne gegen die Lösung einer stochastischen partiellen Differentialgleichung (SPDG) mit singulär-degeneriertem Drift konvergieren. Anschließend werden allgemeinere SPDGs ähnlichen Typs auf ihre Wohlgestelltheit untersucht. Schließlich wird das Langzeitverhalten der Lösungen der Grenzgleichung untersucht, indem gezeigt wird, dass der entsprechende Markowprozess ein eindeutiges invariantes Maß besitzt.

Contents

Chapter 1

Introduction

The concept of self-organised criticality (SOC) has been introduced by Bak, Tang and Wiesenfeld in the seminal article [5]. In a very general formulation, it describes the behaviour of randomly driven processes which possess a “critical” non-equilibrium statistical invariant state, but the precise definition of SOC is still disputed (see e. g. [122, Section 7]). In the following, we will introduce the concept of SOC, which provides the physical motivation of the present thesis, based on the particle model which Bak, Tang and Wiesenfeld used in [5]. Led by heuristics (see e. g. the presentation in [6]), this type of model is referred to as “sandpile model”, and has become paradigmatic for the illustration of SOC, which is why we begin by briefly describing its setting.

We consider a rectangular spatial grid Λ of size $Z \in \mathbb{N}$ in d dimensions, i. e. $\Lambda = \{0, \dots, Z\}^d$, on which grid functions evolve in time steps $\{0, 1, \dots, N\}$, $N \in \mathbb{N}$. The resulting process will be denoted by $(X_{i,j})_{i=0,\dots,N;j\in\Lambda} \subset \mathbb{R}$, where j is a d-dimensional multi-index. Since we are going to prescribe zero-Dirichlet boundary conditions, we will from now on only take care of the bulk part

$$(X_{i,j})_{i=0,\dots,N;j\in\Lambda'}$$

where $\Lambda' = \{1, \dots, Z-1\}^d$. As a convenient notation, we will also use for $i \in \{0, \dots, N\}$

$$X_i := X_{i,\cdot} := (X_{i,j})_{j\in\Lambda'}.$$

Next, we introduce two related classical SOC models, where we gently amended the model in a way that it is easy to simulate and still displays the interesting effects explained below. The first one, which we will call **BTW model**, goes back to [5] and obeys the following dynamics for $d = 2$. It starts with the zero configuration, i. e. $X_{0,j} = 0$ for all $j \in \Lambda'$. As long as the process is subcritical, e. g. $X_{i,j} \leq K$ for all $j \in \Lambda'$, a particle of value 1 is added to a randomly chosen site $j \in \Lambda'$. Formally, this corresponds to the transition

$$X_{i+1} = X_i + (\delta_{j,s_i})_{j\in\Lambda'}, \tag{1.0.1}$$

where $(s_i)_{i=0}^{N-1}$, $s_i \sim \mathrm{Uni}(\Lambda')$, are independent identically distributed random variables. As soon as the process becomes supercritical, i. e. $X_{i,j} > K$ for some $j \in \Lambda'$, the site j becomes unstable, which means that it distributes one particle to each of its $2d$ direct neighbours. We will refer to this effect as **toppling**. Particles moved on a boundary site will just leave the system. This leads to the transition

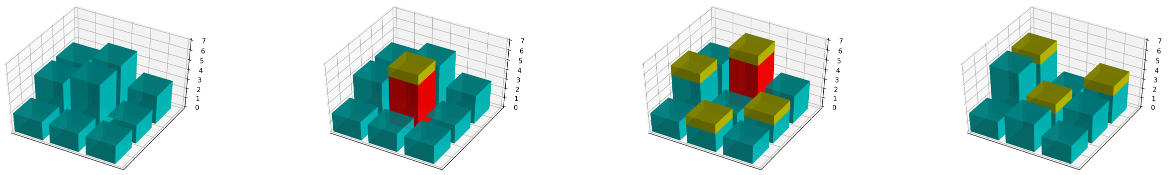
$$X_{i+1,j} = X_{i,j} + D \sum_{j'\sim j} \left(\phi(X_{i,j'}) - \phi(X_{i,j})\right), \tag{1.0.2}$$

where $D \in \left(0, \frac{1}{2d}\right]$, $j' \sim j$ if and only if j and j' are direct neighbours, and

$$\phi : \mathbb{R} \to \mathbb{R}, \quad \phi(x) = K \left(\mathbf{1}_{(K,\infty)}(x) - \mathbf{1}_{(-\infty,-K)}(x)\right), \tag{1.0.3}$$

where the negative part of ϕ will only become relevant later on. The parameter D has been included for the sake of completeness; in [5] only the case $D = 0.25$ was considered.

In [125], this model is slightly modified in the following ways, yielding the **Zhang model**: First, the amount being added in the subcritical regime is allowed to be a random real value between 0 and 1,



(a) initial state (b) one particle added (c) after first toppling (d) after second toppling

Figure 1.1: An example of an avalanche of length 2 in a two-dimensional BTW model. Increments compared to the previous state are coloured yellow, the red bar indicates a supercritical site.

which we will not implement in the following. Second, in the supercritical regime, a fixed proportion of the quantity is removed from the critical site and is equally distributed to its $2d$ direct neighbours. As a result, (1.0.2) stays unchanged and (1.0.3) is replaced by

$$\phi : \mathbb{R} \to \mathbb{R}, \quad \phi(x) = x\left(\mathbf{1}_{(K,\infty)}(x) - \mathbf{1}_{(-\infty,-K)}(x)\right), \tag{1.0.4}$$

where again only the positive part of ϕ is used for now. In the original article [125], only the case $D = 0.25$ was considered, which corresponds to always distributing the whole quantity from an active site in a toppling step.

In both of these models, the toppling events can in principle leave behind another supercritical state, which induces another toppling event. A number m of toppling events in a row will be called an **avalanche of size** m (see Figure 1.1). Avalanche sizes are the key observables, because they give rise to the following statistical effect. If the system is run for a large number of time steps, the frequency of the observed avalanche sizes will approach a **power law**, i. e.

$$\text{frequency (avalanche of size } m) \sim m^{-\alpha}$$

for some $\alpha > 1$, on a large range of possible avalanche sizes. This corresponds to lines in the plot of the logarithm of the frequency against the logarithm of the avalanche size (see Figure 1.2). Due to the discrete structure of the dynamics, the smallest measurable avalanche size is obviously 1, which explains the limited extension of the line to the left. The quick decay for large avalanche sizes is believed to be a finite-size effect. While emerging power laws are typical of systems for which a parameter is tuned to a critical value, typically at a phase transition, the systems introduced above apparently drive themselves into such a "critical" state without external tuning. This is why the described phenomenon is referred to as **self-organized criticality**. It raised a considerable interest in statistical physics, since it might explain why power laws arise in many contexts in nature without an obvious phase transition. For example, we mention the famous Gutenberg-Richter law for the strength of earthquakes, first published in [81]; for a large choice of similar observations, we refer to [4] and [116].

The statistical behaviour described above has been observed in a number of further theoretical particle models, see e. g. [104, 55], which share the properties of being infinitesimally slowly driven towards an unstable state and sudden diffusive relaxation events until a stable state is retained. Moreover, these features have also been collected in [48, Section III.1] as characteristics of systems displaying SOC. Following [48, Section III.2], it is possible to replace the global dependence of the drive by a local, continuous drive with infinitesimally small rate. This can be implemented by modifying the original model described by (1.0.1) and (1.0.2) into

$$X_{i+1,j} = X_{i,j} + D \sum_{j' \sim j} \left(\phi(X_{i,j'}) - \phi(X_{i,j})\right) + \mu + \xi_{i,j}, \tag{1.0.5}$$

where $(\xi_{i,j})_{i=0,\ldots,N; j\in\Lambda'}$ are independent random variables identically distributed with $\mathbb{E}\xi_{i,j} = 0$ and $\mathbb{E}\xi_{i,j}^2 = \sigma^2 < \infty$, $\mathbb{E}\xi_{i,j}^6 < \infty$, and $\mu > 0$. For $\xi_{i,j} \sim \mathcal{N}(0, \sigma^2)$ with μ and σ being chosen small enough, we were able to reproduce power laws looking very similar to the original sandpile models (see Figure 1.3).

The motivation for this work is given by three natural ambitions. First, as mentioned above, the discrete structure of the model and the finite system size present extrinsic bounds for the size of avalanches. As a consequence, the abovementioned power law can only be observed on a limited region of sizes, which

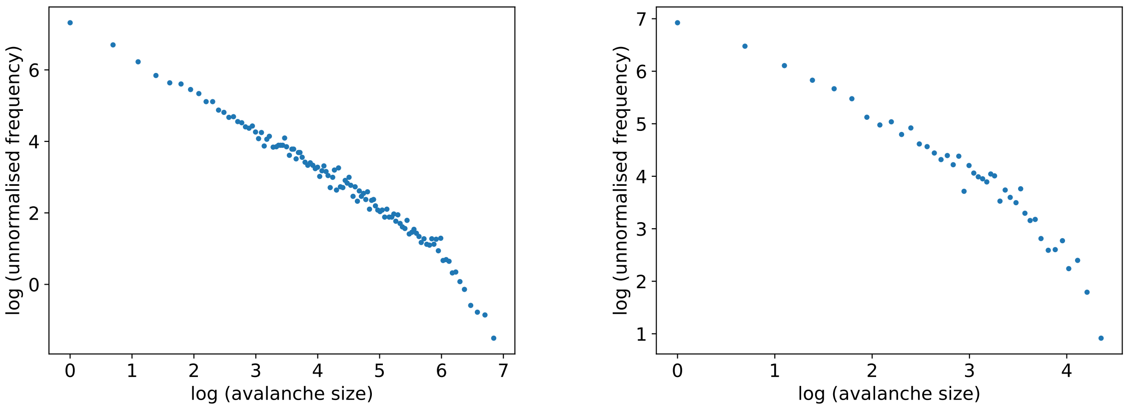


Figure 1.2: Frequency of avalanche sizes for the BTW model (left graph; $K = 10, D = 0.1$) and the Zhang model (right graph; $K = 10, D = 0.25$). The corresponding simulations have been carried out on a 2-dimensional 30×30 grid over $3 \cdot 10^4$ time steps. In order to smoothen out statistical fluctuations, each data point cumulates the information of at least 40 avalanches.

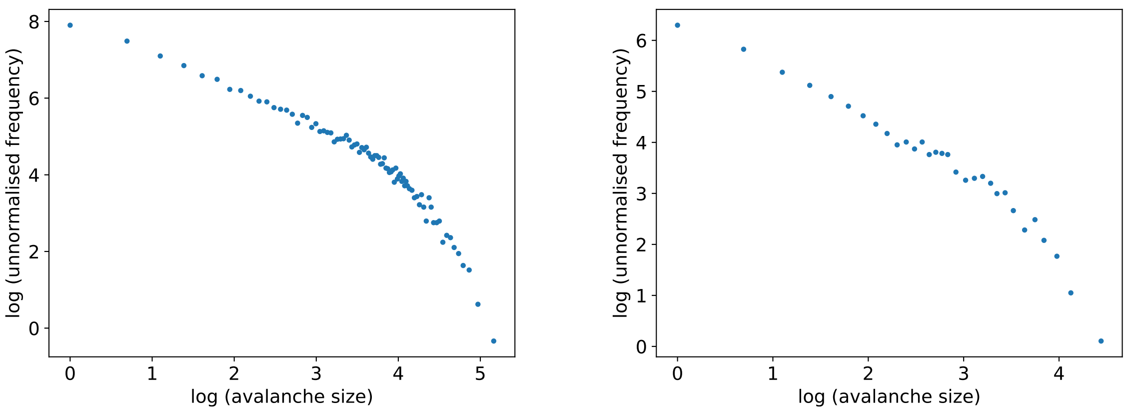


Figure 1.3: Frequency of avalanche sizes for the BTW model (left graph) and the Zhang model (right graph) with the same choice of parameters as above and with Gaussian forcing ($\mu = 0.0001$, $\sigma = 0.01$). The corresponding simulations have been carried out on a 2-dimensional 30×30 grid over 10^5 time steps. In order to smoothen out statistical fluctuations, each data point cumulates the information of at least 40 avalanches. The right picture becomes slightly clearer for different coefficients of the noise, but for the sake of comparability, we used the same parameters for both simulations.

might be extended by considering increasingly large lattices. Second, the discrete processes introduced so far contain many degrees of freedom, most notably the model parameters, but also "hidden" features such as the lattice structure, while it is not clear in which way these parameters are relevant for the overall statistical behaviour. Hence, we aim for a universal object which can be viewed as a scaling limit of a larger class of particle models. Finally, despite the ubiquity of power laws in nature, their occurrence in the context of SOC is not yet explained. To contribute to this question, it is interesting to work out a model on a continuous state space which is accessible to more analytical tools and can still be related to the sandpile models displaying SOC.

To this end, we introduce the following framework. As a first theoretical step, we increase the size of the space-time grid on which the discrete model as described above is run. We then reinterpret these larger lattices as finer and finer subdivisions of a fixed and bounded space-time domain $[0,T] \times [0,1]^d$. Noting that the spatial lattice size and the number N of time steps can be chosen independently from each other, we may write τ for the resulting time step size and h for the resulting spatial lattice constant. We then rewrite equation (1.0.5) in the form of a rescaled toppling mechanism at a grid point $x_i \in [0,1]^d$ and at time $n\tau$ as

$$\begin{aligned} X_{h,\tau}((n+1)\tau, x_j) =& X_{h,\tau}(n\tau, x_j) + \frac{\tau}{h^2} \sum_{j' \sim j} \left(\phi(X_{h,\tau}(n\tau, x_{j'})) - \phi(X_{h,\tau}(n\tau, x_j))\right) \\ &+ \mu\tau + \sqrt{\frac{\tau}{h^d}}\, \xi_{h,\tau}^{n,j} \quad \text{for } n \in \{0,\dots,N-1\}, j \in \Lambda', \\ X_{h,\tau}(0, x_j) =& x_j^* \quad \text{for all } j \in \Lambda', \\ X_{h,\tau}(n\tau, x_j) =& 0 \quad \text{for all } n \in \{0,\dots,N\}, j \in \Lambda \setminus \Lambda', \end{aligned} \tag{1.0.6}$$

where $\mu \geq 0$, D and σ^2 are now replaced by $\frac{\tau}{h^2}$ and $\sqrt{\frac{\tau}{h^d}}$, respectively, and $(x_j^*)_{j=1}^{Z-1} \subset \mathbb{R}$ allow for more general initial values. Furthermore, $(\xi_{h,\tau}^{n,i})_{i=1,\dots,Z-1; n \in \mathbb{N}\cup\{0\}}$ are centered, $\mathbb{R}$-valued, independent identically distributed random variables with unit variance and finite sixth moments, and $j' \sim j$ denotes all indices encoding direct neighbouring grid points of x_i. The non-positivity of the forcing term leads to the necessity of a two-sided nonlinearity, which is why ϕ is defined as in (1.0.3) or (1.0.4). We observe that the sum in (1.0.6) formally represents a discrete Laplacian, while the stochastic part is a discrete version of space-time white noise. Hence, (1.0.6) can be formally considered as a finite difference scheme for the (a priori ill-posed) generalized stochastic porous medium equation

$$\begin{aligned} \mathrm{d}X(t) &= \Delta\phi(X(t))\mathrm{d}t + \mu \mathrm{d}t + \mathrm{d}W(t) \quad \text{on } (0,T] \times (0,1), \\ X(0) &= x_0 \end{aligned} \tag{1.0.7}$$

with zero Dirichlet boundary conditions, where $x_0 \in L^2([0,1])$ is a "suitable" initial state and W denotes a cylindrical Id-Wiener process on $L^2([0,1])$. This fits to heuristic statements in [8, 49, 46, 109], according to which (S)PDEs similar to (1.0.7) are "continuous versions" of the previously introduced sandpile models.

It will be one main result of this thesis to make this correspondence rigorous. For the sake of simplicity, we will chose $K = 1$ and $\mu = 0$. For technical reasons, the stochastic model will be treated for the Zhang nonlinearity (1.0.4) in the case $d = 1$, while for the BTW nonlinearity (1.0.3), we present some results in a deterministic framework. We stress that there remains a considerable freedom to choose the way of rescaling space and time if $\tau, h \to 0$. For our subsequent analysis, we will need the relation $\tau \in o(h^2)$, which corresponds to decreasing the toppling proportion D on the discrete level when increasing the lattice size. Although this seems unjustified a priori, it can be viewed in the context of weak universality, where scaling limits of processes with decaying parameters are frequently considered (see e. g. [31, Definition 2.1]).

At this stage, three types of questions arise and will be addressed in the three chapters of this thesis.

1. The heuristic arguments above allude that the formal SPDE in (1.0.7) is a continuous analogue of the family of processes in (1.0.6). Do the processes in (1.0.6) actually converge in some sense to solutions of (1.0.7)? How strong is the convergence and under which conditions does it hold?

2. Is it possible to give a meaning to the general type of equations that are associated to the discrete processes, either as proven limits or as candidates by more heuristic arguments?

3. As mentioned above, the main reason to consider the original discrete model is its observed statistical behaviour. We address this area by analyzing the long time behaviour of solutions to one of the equations in continuous space and time identified before. More precisely, we prove that the solutions to this equation possess a unique statistical invariant state.

We briefly comment on these questions and thereby give an overview over this thesis. In Chapter 2, we introduce an embedding of $X_{h,\tau}$ as defined in (1.0.6) into a continuous space. After having defined a suitable notion of solution to (1.0.7) and showing uniqueness of solutions in this sense, we prove that $(X_{h,\tau})_{h,\tau>0}$ converges in law to a solution of this equation under a suitable scaling. For linearly growing ϕ as in the Zhang model, we treat the whole process in (1.0.6), while for bounded ϕ as in the BTW model, only the deterministic dynamics can be treated rigorously.

In Chapter 3, we analyze more general singular-degenerate stochastic porous media equations of the type as arisen above. We formulate a very weak notion of solution, the so-called SVI solution, which is easy to work with due to few assumptions on the regularity of the solutions are being made. The solution theory includes both linearly growing and bounded nonlinearities containing discontinuities in arbitrary space dimensions, but needs more regularity for the noise. For SVI solutions of the SPDEs under consideration, we prove existence, uniqueness and stability in the initial value.

In Chapter 4, we restrict to the one-dimensional version of the equation in Chapter 3 with linearly growing nonlinearity ϕ. Under an additional assumption on the noise, we prove the existence of a unique invariant measure for the corresponding solution process, using an abstract result on ergodicity for processes taking values in Polish phase spaces (cf. [91]).

Last but not least, it should be mentioned that generalized stochastic porous media equations with multi-valued coefficients are an emerging field of research, such that the classical questions of well-posedness and ergodicity also carry a considerable intrinsic mathematical interest. For more details in this direction, we refer to the mathematical expositions of each individual chapter.

1.1 Acknowledgements

I would like to express my gratitude to my advisor Prof. Dr. Benjamin Gess for introducing me to the realm of stochastic partial differential equations and their connection to statistical physics. Without his ongoing support, helpful advice and broad knowledge, this work would not have been possible. His many lectures shaped my mathematical knowledge significantly. Besides, I highly appreciate his generosity in the approval of conference participation. These events have greatly enhanced my scientific education.

Furthermore, I am indebted to Prof. Dr. Felix Otto, in whose group I was working during my PhD studies, for broad mathematical inspiration through his lectures and through occasional informal discussions during my everyday working life. Many thanks for fruitful mathematical and non-mathematical discussions go to my fellow PhD students Florian Kunick, who also helped me a lot by reading and commenting on my preprints, Pablo Linares, Orlando Marigliano, Tim Seynnaeve, Markus Tempelmayr and to my former colleagues Claudia Raithel and Theresa Simon. I would also like to thank the working group as a whole for providing a stimulating and lively atmosphere for my research.

I would like to thank the International Max Planck Research School in Leipzig for its financial support and several motivating events, such as the IMPRS retreats. Especially, I would like to highlight the commitment of Prof. Dr. Stephan Luckhaus and Prof. Dr. Stefan Hollands in the organization of these kinds of interdisciplinary activities. Moreover, financial and individual support by the "Cusanuswerk – Bischöfliche Studienförderung" is gratefully acknowledged.

Finally, many thanks to all my friends, my Ultimate Frisbee club and my family for the social backing during my time as a PhD student, and especially to Janna, whose firm support throughout the past several years and whose love I am deeply grateful to enjoy.

Chapter 2

Continuum limits of cellular automata related to self-organized criticality

2.1 Introduction

It is important to realize that there are two principally different scopes which lead to the issue of continuum limits of processes defined on a discrete space. The first one is to solve a problem posed in a continuous setting (e. g. a partial differential equation) constructively, which will be called the "numerics approach" in the following. In this case, the precise design of the discrete approximation scheme may vary, as long as it verifiably converges to a solution of the original problem. The second scope is to find a continuous simplification of a discrete model which provides a good description on large scales; this will be called the "scaling limit approach". In this case, the limiting equation is not known per se, but is a consequence of the behaviour of the original discrete model. In contrast to the numerics approach, any change in the setting of the discrete process would considerably affect the research goal.

As it has been discussed in Chapter 1, this work is supposed to provide a scaling limit for a certain class of discrete models arising in self-organized criticality, i. e. it follows the second approach. Since both perspectives lead to the same type of proof, most concepts we use stem from numerical analysis. However, the rigidity of the discrete scheme leads to a number of difficulties, which we aim to explain after introducing the basic objects.

For the first main result of this chapter, we consider the up to now formal SPDE

$$\begin{aligned} \mathrm{d}X(t) &\in \Delta(\phi(X(t)))\mathrm{d}t + \mathrm{d}W(t), \\ X(0) &= x_0 \end{aligned} \tag{2.1.1}$$

on the interval $(0,1) \subset \mathbb{R}$ with zero Dirichlet boundary conditions and $x_0 \in L^2 := L^2((0,1))$. In this setting, W is a cylindrical Id-Wiener process in L^2 and the nonlinearity $\phi : \mathbb{R} \to 2^{\mathbb{R}}$ is the maximal monotone extension of

$$\tilde{\phi} : \mathbb{R} \ni x \mapsto x\,\mathbf{1}_{|x|>1}(x), \tag{2.1.2}$$

which is the nonlinearity corresponding to the Zhang model in the sense described in Chapter 1.

For the second main result, we consider the singular-degenerate partial differential equation

$$\begin{aligned} \partial_t u(t) &\in \Delta(\phi_1(u(t))\mathrm{d}t, \\ u(0) &= u_0 \end{aligned} \tag{2.1.3}$$

on a bounded interval $(0,1) \subset \mathbb{R}$ with zero Dirichlet boundary conditions and $u_0 \in L^2$. In this case, $\phi_1 : \mathbb{R} \to 2^{\mathbb{R}}$ is the maximal monotone extension of

$$\tilde{\phi_1} : \mathbb{R} \ni x \mapsto \operatorname{sgn}(x)\,\mathbf{1}_{|x|>1}(x), \tag{2.1.4}$$

which is the nonlinearity corresponding to the BTW model in the sense described in Chapter 1.

Furthermore, on an equidistant space-time grid over $[0,T]\times[0,1]$ with N time steps of step size τ and $Z+1$ spatial nodes with distance h (including the boundary), such that $N\tau=T$ and $Zh=1$, we consider the process $(X^n_{h,\tau})_{n=0}^N\subset\mathbb{R}^{Z-1}$ given by

$$\begin{aligned} X^{n+1}_{h,\tau} &= X^n_{h,\tau}+\tau\Delta_h\phi(X^n_{h,\tau})+\sqrt{\frac{\tau}{h}}\xi^n_{h,\tau} \quad \text{for } n=0,\dots,N-1,\\ X^0_{h,\tau} &= x^0_h, \end{aligned} \tag{2.1.5}$$

where $x^0_h\in\mathbb{R}^{Z-1}$ is a suitably chosen initial configuration, $(\xi^{n,i}_{h,\tau})_{i=1}^{Z-1}$ are centered, $\mathbb{R}$-valued, independent identically distributed random variables with unit variance and finite sixth moments, Δ_h denotes the discrete Laplacian which is rigorously defined in section 2.1.2 below, and $X^{n,k}_{h,\tau}$ denotes the value of the approximating process at the point $(n\tau,kh)\in[0,T]\times[0,1]$. Note that by the assumed Dirichlet boundary conditions, it is sufficient to consider the bulk part of the approximating process, setting

$$X^{n,0}_{h,\tau}=X^{n,Z}_{h,\tau}=0 \quad \text{for } n=0,\dots,N-1$$

if required. Similarly, we define

$$\begin{aligned} u^{n+1}_{h,\tau} &= u^n_{h,\tau}+\tau\Delta_h\phi_1(u^n_{h,\tau}) \quad \text{for } n=0,\dots,N-1,\\ u^0_{h,\tau} &= u^*_h, \end{aligned} \tag{2.1.6}$$

where $u^*_h\in\mathbb{R}^{Z-1}$ is a suitably chosen initial configuration.

The main results of this chapter, Theorem 2.2.5 and 2.2.8 below, provide that the (S)PDEs (2.1.1) and (2.1.3) are, in a weak sense, the scaling limits of the discrete processes (2.1.5) and (2.1.6), respectively. The key difficulties which have to be overcome result from the rigid discretization scheme which is determined by the design of the discrete SOC models, which render most established numerical methods unusable. In the following, we illustrate this effect in terms of the discretization via finite differences and the singular-degenerate nonlinearity.

Finite difference schemes for classical porous media equations usually strongly rely on the regularity of the nonlinear diffusion coefficient. The most general work in this direction is [44], in which the convergence of explicit finite difference schemes of generalized porous medium equations with Lipschitz nonlinearities is proved. However, the proof relies on a comparison principle on the level of the discrete scheme leading to an L^1 bound, which is closely connected to the CFL-type condition

$$\frac{\tau}{h^2}\le\frac{1}{2d\,\mathrm{Lip}_\phi},$$

where Lip_ϕ is the Lipschitz constant of the nonlinearity ϕ (see [44, p. 2272]). It is obvious that none of the nonlinearities in (1.0.3) and (1.0.4) is Lipschitz-continuous, such that this condition can only be satisfied in a limiting sense. In view of this, the necessity of the technical assumption $\tau\in o(h^2)$, which will be crucial in order to establish an adequate convergence result in the present work, becomes plausible. Hence, in the present situation of discontinuous drift terms, we need different concepts to ensure convergence of the numerical approximations to a solution, which are mainly based on the maximal monotonicity of the nonlinearity ϕ, in case of the Zhang model in combination with its coercivity providing L^2 bounds. As another indication that the L^1 framework in [54, 44] is not applicable in the present setting, we remark that the compactness of the sequence of discrete process in $\mathcal{C}([0,T];L^1)$, which is exploited there, strongly relies either on the driving process being integral-preserving or in L^1 for almost all times. Both properties are not met by space-time white noise, which does not have higher spatial regularity than $\mathcal{C}^{-\frac{1}{2}}$.

Since the scaling limit will be taken in distribution with respect to weak topologies, the proof structure is flexible regarding the form of the discrete noise, especially allowing for non-Gaussian discrete input. This is necessary to treat numerical schemes which are as similar as possible to the original sandpile models. Later on, almost sure weak convergence will be recovered by passing to another stochastic basis, using a Skorohod-type result. As a consequence, the solution theory is posed in a probabilistically weak sense, which entails more work to ensure uniqueness.

Finally, we also mention inherent challenges that space-time white noise presents in the context of the continuous equation. In most of the literature, either the drift term is more regular, such as in the case of semi-linear SPDE, or the noise is restricted to be in L^2, such as in [14, 12, 103]. This gap has been closed by [100], where the well-posedness of multivalued SPDE driven by Levy noise has been proved by techniques which are also used in the present work to identify scaling limits.

The structure of this chapter is as follows: We first give an overview on the mathematical and physical literature in Section 2.1.1 and introduce some general notational conventions in Section 2.1.2. After stating the main results in Section 2.2, we prove uniqueness of weak solutions in Section 2.3. The proofs of the main theorems are given in Section 2.4 and 2.5 for the stochastic Zhang model and the deterministic BTW model, respectively.

A publication of the results of this chapter is in preparation.

2.1.1 Literature

Explicit finite difference discretizations of porous media equations have been subject to a lively research activity in numerical mathematics, advancing from the classical power functions (e. g. [47]) via differentiable nonlinearities ([54]) to merely Lipschitz nonlinearities ([44]). As related results, we mention convergence results for implicit finite difference schemes of degenerate porous media equations ([53, 44]) and a finite-difference discretization of a fractional porous medium equation ([43]).

For discretizations of stochastic porous media equations, we refer to [112] and [80], where a finite element approach is applied in order to construct and analyze solutions. In [64, 82], linear SPDE with multiplicative noise are discretized using finite difference approximations in space, while [101] considers space-time finite difference approximations of linear parabolic SPDE with additive noise. To the best of the author's knowledge, the present work is the first time that finite difference approximations of stochastic porous media equations are rigorously analyzed.

Concerning the underlying techniques the main arguments of this article rely on, we mention the following sources of theory and inspiration. For Yamada-Watanabe type results, we refer to [123] for the foundational work and to [94, 112] for applications to SPDE. The meanwhile classical weak convergence approach has been used before e. g. by [58, 25, 69], relying on a Skorohod-type result by Jakubowski [87]. For the identification of the limit of the discrete approximations as a solution, we use the theory of maximal monotone operators given in [7] in a similar way as [100].

Finally, we mention some further attempts to approach SOC in a continuous setting. Related to the scaling limit approach, one strategy consists in considering cellular automata resulting from a reformulation and modification of the original sandpile models, as proceeded in [28], in order to obtain a problem which is more accessible for analysis. For one of these models, a hydrodynamic limit PDE has been rigorously obtained in [29]. For the existence of a scaling limit for deterministic sandpiles started from specific initial configurations, we refer to [108]. In [111, 24, 85], systems of PDEs are analyzed as ad-hoc models for natural processes displaying power-law statistics.

2.1.2 Notation

We begin with a quick recap of frequently used concepts, most of which can be found in [112].

Let $\mathcal{O} \subset \mathbb{R}$ be an open and bounded interval. For $k \geq 0$, let $\mathcal{C}^k(\mathcal{O})$ ($\mathcal{C}^k_c(\mathcal{O})$) be the space of k times differentiable real-valued functions (with compact support). Let $L^2 := L^2(\mathcal{O})$ be the Lebesgue space of square integrable functions, endowed with the norm $\|\cdot\|_{L^2}$. Let $H^1_0 := H^1_0(\mathcal{O})$ be the Sobolev space of weakly differentiable functions with zero trace, endowed with the norm $\|u\|_{H^1_0} = \|\nabla u\|_{L^2}$, and let H^{-1} be its topological dual space. Recall the canonical continuous embedding $I : H^1_0 \to L^2$ provided by the Poincaré inequality, and define its dual map $I' : L^2 \to H^{-1}$ by

$$\langle I'u, v\rangle_{H^{-1}\times H^1_0} = \langle u, Iv\rangle_{L^2}\,.$$

for $u \in H^{-1}$, $v \in H^1_0$. Its dual map $I'' : H^{-1} \to (L^2)'$ is defined analogously. We will also use the adjoint operator $(I')^* : H^{-1} \to L^2$ of I', defined by

$$\langle (I')^* u, v\rangle_{L^2} = \langle u, I'v\rangle_{H^{-1}}$$

for $u \in H^{-1}$, $v \in L^2$. If there is no risk of misunderstanding, we will not mention the use of the embeddings I, I' and I''.

Recall that the negative Laplace operator $-\Delta : H_0^1 \to H^{-1}$ is defined by

$$\langle -\Delta u, v \rangle_{H^{-1} \times H_0^1} = \langle \nabla u, \nabla v \rangle_{L^2}.$$

We note that $I'' \circ (-\Delta) : H_0^1 \to (L^2)'$ is continuous with respect to the L^2 norm. Hence, it can be extended linearly and continuously to the whole of L^2. The resulting operator $L^2 \to (L^2)'$ will also be denoted by $-\Delta$. As pointed out in [112, Remark 4.1.14], we note that this operator is a surjective isometry, and we stress that it does not coincide with the Riesz isomorphism for the classical dualization of L^2 with itself.

For two separable Hilbert spaces H_1, H_2, we write $L_2(H_1, H_2)$ for the space of all Hilbert-Schmidt operators from H_1 to H_2.

For a Banach space B and $k \geq 0$, let $\mathcal{C}^k([0,T]; V)$ be the space of k times continuously differentiable curves in V parametrized by $t \in [0,T]$. For a measurable space $(S, \mathcal{A})$, we denote the Lebesgue-Bochner space of measurable, square integrable B-valued functions by $L^2(S; B)$, which is defined e. g. in [86, Definition 1.2.15]. If S is the product two Banach spaces $S_1 \times S_2$, we will use $L^2(S; B)$ and $L^2(S_1; L^2(S_2; B))$ interchangeably, see Lemma 2.B.3 for a justification. Let $f \in L^2(\Omega \times [0,T]; B)$ be a Banach-space valued random function. Then, f is called progressively measurable with respect to a filtration $(\mathcal{F}_t)_{t \in [0,T]}$, if $f|_{[0,t]}$ is measurable with respect to $\mathcal{F}_t \otimes \mathcal{B}([0,t]) - \mathcal{B}(B)$ for all $t \in [0,T]$.

For the product $V = V_1 \times \cdots \times V_n$ of topological spaces, where $n \in \mathbb{N}$, we define the i-th projection Π_i by $\Pi_i(v) = v_i$ for $i \in \{1, \ldots, n\}$, which is a continuous map by the definition of the product space. We will use Π_i for any such projection, regardless of the respective underlying spaces.

Let $T > 0$ and consider a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t \in [0,T]}$, where $\mathcal{F}_t \subset \mathcal{F}$ for all $t \in [0,T]$. Expected values with respect to $\mathbb{P}$ will be denoted by $\mathbb{E}$. The filtration is called normal, if it is complete, i. e. $\mathcal{F}_t$ contains all $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$ for all $t \in [0,T]$, and right-continuous, i. e.

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s \quad \text{for all } t \in [0,T].$$

Each filtration $(\mathcal{F}_t)_{t \in [0,T]}$ can be augmented to a normal filtration $(\mathcal{F}_t^*)_{t \in [0,T]}$ by defining

$$\begin{aligned} \mathcal{F}_T^* &= \sigma(\mathcal{F}_T \cup \mathcal{N}), \\ \mathcal{F}_t^* &= \bigcap_{s>t} \sigma(\mathcal{F}_s \cup \mathcal{N}) \quad \text{for all } t \in [0,T), \end{aligned}$$

where $\mathcal{N}$ denotes the collection of all $\mathbb{P}$-zero sets. We refer to [114, p. 45] for details.

We now turn to the finite-dimensional structures which we will use to formulate numerical convergence results. From now on, we fix

$$\mathcal{O} = [0,1] \subset \mathbb{R}.$$

Consider an equidistant grid on the unit interval with grid points $(x_i)_{i=0}^Z$ with $h = \frac{1}{Z}$, $Z \in \mathbb{N}$ and $x_i = ih$. For $i = 0, \ldots, Z-1$ let $y_i = \left(i + \frac{1}{2}\right) h$. Consider the sets of intervals $(K_i)_{i=0,\ldots,Z}$ and $(J_i)_{i=0,\ldots,Z-1}$ given by

$$\begin{aligned} &K_0 = [x_0, y_0),\ K_Z = [y_{Z-1}, x_Z],\ K_i = [y_{i-1}, y_i) \text{ for } i = 1, \ldots, Z-1, \\ &J_i = [x_i, x_{i+1}) \text{ for } i = 0, \ldots, Z-1. \end{aligned} \tag{2.1.7}$$

We consider the space of grid functions on $(x_i)_{i=0}^Z$ with zero boundary conditions, which is isomorphic to $\mathbb{R}^{Z-1}$, and we define the following prolongations (see Figure 2.1).

Definition 2.1.1. Let $u_h \in \mathbb{R}^{Z-1}$ and $v_h \in \mathbb{R}^Z$. We then define the piecewise linear prolongation with respect to the grid $(x_i)_{i=0,\ldots,Z}$ with zero-boundary conditions by

$$I_h^{\text{plx}} : \mathbb{R}^{Z-1} \hookrightarrow H_0^1,\ u_h \mapsto u_h^{\text{plx}} := \sum_{i=0}^{Z-1} \left[u_{h,i} + \frac{u_{h,i+1} - u_{h,i}}{h} (\cdot - x_i) \right] \mathbf{1}_{J_i},$$

and the piecewise constant prolongation by

$$I_h^{\text{pcx}} : \mathbb{R}^{Z-1} \hookrightarrow L^2,\ u_h \mapsto u^{\text{pcx}} := \sum_{i=1}^{Z-1} u_{h,i} \mathbf{1}_{K_i},$$

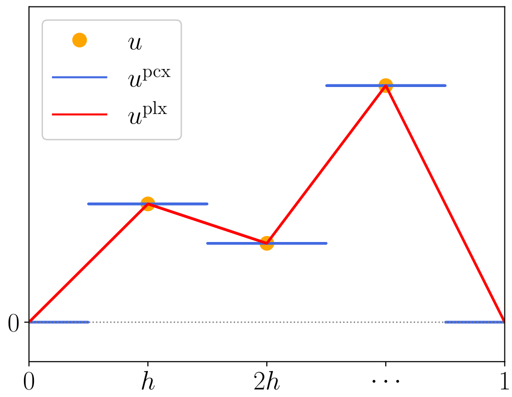


Figure 2.1: Different prolongations of a spatial grid function

with the convention $u_{h,0} = u_{h,Z} = 0$. The image of I_h^{pcx}, i. e. the space of piecewise constant functions on the partition $(K_i)_{i=0}^{Z}$ with zero Dirichlet boundary conditions, will be denoted by S_h^{pcx}. The L^2-orthogonal projection to this space will be denoted by Π_h^{pcx}. Note that $I_h^{\mathrm{pcx}} : \mathbb{R}^{Z-1} \to S_h^{\mathrm{pcx}}$ is bijective.

Lemma 2.1.2. *Let $\eta \in L^2$. Then, $\Pi_h^{pcx}\eta \to \eta$ in L^2 for $h \to 0$.*

Proof. The proof is a simpler version of the proof of Lemma 2.5.6 below, which is why it is omitted here. □

Let $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{l^2}$ denote the inner product arising from the Euclidean norm $\|\cdot\| := \|\cdot\|_{l^2}$ on $\mathbb{R}^{Z-1}$. For a matrix $A \in \mathbb{R}^{(Z-1)\times(Z-1)}$, $\|A\|$ denotes the matrix norm induced by $\|\cdot\|$, i. e.

$$\|A\| := \sup_{x\in\mathbb{R}^{Z-1}\setminus\{0\}} \frac{\|Ax\|}{\|x\|}. \tag{2.1.8}$$

Let $\Delta_h \in \mathbb{R}^{(Z-1)\times(Z-1)}$ be the matrix corresponding to the finite difference Laplacian on grid functions on $(x_i)_{i=0}^{Z}$ with zero Dirichlet boundary conditions, i. e.

$$\Delta_h = -\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & \text{\Large 0} & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & \text{\Large 0} & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}. \tag{2.1.9}$$

Recall that $-\Delta_h$ is symmetric and positive definite (for a formal argument, see Lemma 2.4.1 below). Hence, the following definition is admissible.

Definition 2.1.3. On $\mathbb{R}^{Z-1}$, we define the inner products $\langle\cdot,\cdot\rangle_0$, $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_{-1}$ by

$$\begin{aligned} \langle u,v\rangle_0 &= h\,\langle u,v\rangle \\ \langle u,v\rangle_1 &= \langle -\Delta_h u, v\rangle_0 \\ \langle u,v\rangle_{-1} &= \left\langle (-\Delta_h)^{-1}u, v\right\rangle_0 \end{aligned}$$

for $u, v \in \mathbb{R}^{Z-1}$.

Remark 2.1.4. The inner product $\langle\cdot,\cdot\rangle_0$ in Definition 2.1.3 corresponds to the L^2 norm on $\mathcal{O}$ by the fact that

$$I_h^{\mathrm{pcx}} : \left(\mathbb{R}^{Z-1}, \|\cdot\|_0\right) \to \left(S_h^{\mathrm{pcx}}, \|\cdot\|_{L^2}\right)$$

is an isometry, i. e.

$$\langle u,v\rangle_0 = \langle I_h^{\mathrm{pcx}} u, I_h^{\mathrm{pcx}} v\rangle_{L^2} \quad \text{for } u, v \in \mathbb{R}^{Z-1}.$$

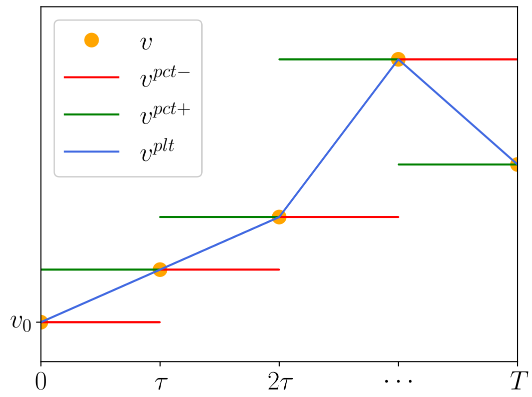


Figure 2.2: Different prolongations of a time grid function

Furthermore, Definition 2.1.3 suggests to view $\langle u, v \rangle_1$ and $\langle u, v \rangle_{-1}$ as discrete analogues of the H_0^1 and H^{-1} norm on $\mathcal{O}$, respectively. These connections are more subtle and will be made more precise in Lemma 2.4.7, Lemma 2.4.9 and Proposition 2.4.29 below.

Next, we consider a lattice for the time interval $[0, T]$, $T > 0$. For $\tau > 0$ such that $T = N\tau$, $N \in \mathbb{N}$, consider the equidistant grid $(0, \tau, 2\tau, \dots, N\tau)$. We then define the following prolongations of grid functions (see Figure 2.2).

Definition 2.1.5. Let $(v_k)_{k=0}^N \subseteq \mathbb{R}$ be a grid function on the previously described grid of length τ. Then we define the piecewise linear prolongation $v^{\text{plt}} : [0, T] \to \mathbb{R}$, the left-sided piecewise constant prolongation $v^{\text{pct-}} : [0, T] \to \mathbb{R}$ and the right-sided piecewise constant prolongation $v^{\text{pct+}} : [0, T] \to \mathbb{R}$ by

$$\begin{aligned} v^{\text{plt}}(t) &= \frac{t - t_\tau}{\tau} v_{\lfloor t/\tau \rfloor + 1} + \frac{t_\tau + \tau - t}{\tau} v_{\lfloor t/\tau \rfloor}, \\ v^{\text{pct-}}(t) &= v_{\lfloor t/\tau \rfloor}, \\ v^{\text{pct+}}(t) &= v_{\lfloor t/\tau \rfloor + 1}. \end{aligned}$$

Definition 2.1.6. Let $N, Z \in \mathbb{N}$ and $(u_{k,l})_{k=0,\dots,N; l=1,\dots,Z-1} \subset \mathbb{R}$ be a function on the space-time grid covering $[0, T] \times [0, 1]$, with time grid length τ and space grid length h, such that $\tau N = T$ and $Zh = 1$. Committing a slight abuse of notation, we define the componentwise time prolongations $u^{\text{plt}}, u^{\text{pct-}}, u^{\text{pct+}} : [0, T] \to \mathbb{R}^{Z-1}$ by

$$\begin{aligned} u^{\text{plt}}(t) &:= \left((u_{\cdot,l})^{\text{plt}}(t)\right)_{l=1}^{Z-1} := \left(\left((u_{k,l})_{k=0}^N\right)^{\text{plt}}(t)\right)_{l=1}^{Z-1}, \\ u^{\text{pct-}}(t) &:= \left((u_{\cdot,l})^{\text{pct-}}(t)\right)_{l=1}^{Z-1} := \left(\left((u_{k,l})_{k=0}^N\right)^{\text{pct-}}(t)\right)_{l=1}^{Z-1}, \\ u^{\text{pct+}}(t) &:= \left((u_{\cdot,l})^{\text{pct+}}(t)\right)_{l=1}^{Z-1} := \left(\left((u_{k,l})_{k=0}^N\right)^{\text{pct+}}(t)\right)_{l=1}^{Z-1}, \end{aligned}$$

and the componentwise spatial piecewise constant prolongation $u^{\text{pcx}} : [0, 1] \to \mathbb{R}^{N+1}$ by

$$u^{\text{pcx}}(x) := (u_k^{\text{pcx}}(x))_{k=0}^N := \left(\left((u_{k,l})_{l=1}^{Z-1}\right)^{\text{pcx}}(x)\right)_{k=0}^N,$$

where we used the extensions from Definition 2.1.1 and 2.1.5. Finally, we define the full prolongations

$$u^{\text{plt,pcx}}, u^{\text{pct-pcx}}, u^{\text{pct+pcx}} : [0, T] \times [0, 1] \to \mathbb{R}$$

by

$$\begin{aligned} u^{\text{plt,pcx}}(t, x) &= \left(u^{\text{plt}}(t)\right)^{\text{pcx}}(x) = (u^{\text{pcx}}(x))^{\text{plt}}(t), \\ u^{\text{pct-pcx}}(t, x) &= \left(u^{\text{pct-}}(t)\right)^{\text{pcx}}(x) = (u^{\text{pcx}}(x))^{\text{pct-}}(t), \\ \text{and} \quad u^{\text{pct+pcx}}(t, x) &= \left(u^{\text{pct+}}(t)\right)^{\text{pcx}}(x) = (u^{\text{pcx}}(x))^{\text{pct+}}(t). \end{aligned}$$

2.2 Setting and main results

We set the stage for the following analysis by defining a notion of solution to (2.1.1) in a probabilistically weak sense, which means that the solution is not bound to a specific stochastic basis, but that the stochastic basis is part of the solution.

Definition 2.2.1. Let $x_0 \in H^{-1}$. A triple $\Big((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in[0,T]}, \tilde{\mathbb{P}}), \tilde{X}, \tilde{W}\Big)$, where $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in[0,T]}, \tilde{\mathbb{P}})$ is a complete probability space endowed with a normal filtration,

$$\tilde{X} \in L^2(\tilde{\Omega} \times [0,T]; L^2) \cap L^2(\tilde{\Omega}; L^\infty([0,T]; H^{-1}))$$

is an $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$-progressively measurable process and $\tilde{W}$ is a cylindrical Id-Wiener process with respect to $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$ in L^2, is a weak solution to (2.1.1), if there exists an $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$-progressively measurable process $\tilde{Y} \in L^2(\tilde{\Omega} \times [0,T]; L^2)$ such that

$$\tilde{X}(t) = x_0 + \int_0^t \Delta \tilde{Y}(r)\,\mathrm{d}r + \tilde{W}(t) \tag{2.2.1}$$

is satisfied in $L^2(\tilde{\Omega} \times [0,T]; (L^2)')$, and

$$\tilde{Y}(t) \in \phi(\tilde{X}(t)) \quad (\mathrm{d}t \otimes \mathrm{d}x)\text{-almost everywhere } \tilde{\mathbb{P}}\text{-almost surely.} \tag{2.2.2}$$

Remark 2.2.2. If $\tilde{W}$ is a cylindrical Id-Wiener process, then $I'\tilde{W}$ is a classical Wiener process in H^{-1} with covariance operator $I'(I')^*$, which is trace class. We will frequently identify $\tilde{W}$ and $I'\tilde{W}$.

The following theorem is a preparatory result and will be proved at the end of section 2.3.

Theorem 2.2.3. *The processes $(\tilde{X}, \tilde{W})$ of every weak solution to* (2.1.1) *have the same law with respect to the Borel σ-algebra of $L^2([0,T]; L^2) \times \mathcal{C}^0([0,T]; H^{-1})$.*

We make the following central assumption for the rest of this article.

Assumptions 2.2.4. Let $T > 0$. Consider a sequence $(h_m)_{m\in\mathbb{N}} \subset (0,1)$, $(Z_m)_{m\in\mathbb{N}} \subset \mathbb{N}$ with

$$h_m \to 0 \text{ for } m \to \infty \quad \text{and} \quad h_m Z_m = 1 \quad \text{for all } m \in \mathbb{N}.$$

For each $m \in \mathbb{N}$, choose $\tau_m > 0, N_m \in \mathbb{N}$ in such a way that

$$\tau_m N_m = T \text{ for all } m \in \mathbb{N} \quad \text{and} \quad \frac{\tau_m}{h_m^2} \to 0 \quad \text{for } m \to \infty \tag{CFL}$$

is satisfied, which presents a strengthened Courant-Friedrichs-Lewy-type condition.

Motivated by the discrete Zhang model (cf. Chapter 1), we construct a family of time-discrete evolution processes on $\mathbb{R}^{Z_m-1}$ as follows. For each $m \in \mathbb{N}$, we define $(X^n_{h_m})_{n\in\{0,1,\dots,N_m+1\}} \subset \mathbb{R}^{Z_m-1}$ iteratively by

$$\begin{aligned} X^{n+1}_{h_m} &= X^n_{h_m} + \tau_m \Delta_{h_m} \tilde{\phi}(X^n_{h_m}) + \sqrt{\frac{\tau_m}{h_m}}\xi^n_{h_m}, \\ X^0_{h_m} &= x^0_{h_m}, \end{aligned} \tag{2.2.3}$$

where $(x^0_{h_m})_{m\in\mathbb{N}} \subset \mathbb{R}^{Z_m-1}$ such that $(x^0_{h_m})^{\mathrm{pcx}} \to x_0$ in L^2, and $(\xi^{n,l}_{h_m})_{n=0,\dots,N_m;l=1,\dots,Z_m-1}$ are centred independent random variables identically distributed on a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\mathbb{E}(\xi^{0,1}_{h_1})^2 = 1$ and that $\mathbb{E}(\xi^{0,1}_{h_1})^6$ is finite. Furthermore, let $(\mathcal{F}^n_{h_m})^{N_m}_{n=0}$, $\mathcal{F}^n_{h_m} \subseteq \mathcal{F}$, the filtration generated by $(\xi^k_{h_m})^{N_m}_{k=0}$, i.e.

$$\mathcal{F}^n_{h_m} = \sigma\left(\xi^k_{h_m} : k \in \{0,\dots,n-1\}\right) \quad \text{for } n \in \{0,\dots,N_m\}. \tag{2.2.4}$$

In order to obtain more complete estimates, we include $N_m + 1$ time steps instead of N_m. For this numerical scheme, we have the following main result, which will be proved at the end of Section 2.4.

Theorem 2.2.5. *Recall the notation from Section 2.1.2, let Assumption 2.2.4 be satisfied and, for $m \in \mathbb{N}$, consider the process $(X^n_{h_m})_{n=0}^{N_m}$ given by (2.2.3). Then, for $m \to \infty$, $X^{plt,pcx}_{h_m}$ converges in distribution to a probability measure μ on $L^2([0,T];L^2) \cap L^\infty([0,T];H^{-1})$ both on $L^2([0,T];L^2)$ endowed with the weak topology and on $L^\infty([0,T];H^{-1})$ endowed with the weak* topology. The measure μ is the law of a stochastic process*

$$\tilde{X} \in L^2(\tilde{\Omega} \times [0,T];L^2) \cap L^2(\tilde{\Omega};L^\infty([0,T];H^{-1})),$$

on a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, such that $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}},\tilde{X})$ can be extended to a weak solution

$$\left((\tilde{\Omega},\tilde{\mathcal{F}},(\tilde{\mathcal{F}}_t)_{t\in[0,T]},\tilde{\mathbb{P}}),\tilde{X},\tilde{W}\right)$$

of (2.1.1).

We give a brief overview of the proof structure. The uniqueness theorem is proven by applying a Yamada-Watanabe type result by Kurtz [94] using the monotonicity of the drift operator to obtain pathwise uniqueness.

In order to be in a setting in which the discrete process in (2.1.5) can be shown to approximate a solution to (2.1.1), we first need to embed it into a suitable state space. This can be realized in different ways, such as interpolating piecewise linearly or using piecewise constant extensions. Carefully chosen combinations of these embeddings will be used to meet different requirements, and it will be crucial to show that all of them converge in law to the same limit.

As a next step, we need a priori bounds for the embedded discrete processes $(X^{\mathrm{emb}}_{h_m})_{m\in\mathbb{N}}$, which are satisfied uniformly in the grid size. To this end, we need to define several norms on the vector space of grid functions and to analyze how they are related to their continuous counterparts. Afterwards, the resulting bounds are used to apply a Skorohod-type theorem by Jakubowski [87] in order to obtain a nonrelabeled subsequence $(X^{\mathrm{emb}}_{h_m})_{m\in\mathbb{N}}$, a different probability space and random variables $\tilde{X}$, $(\tilde{X}^{\mathrm{emb}}_{h_m})_{m\in\mathbb{N}}$ on this probability space such that

$$\begin{aligned} \mathcal{L}(X^{\mathrm{emb}}_{h_m}) &= \mathcal{L}(\tilde{X}^{\mathrm{emb}}_{h_m}) \quad \text{for all } m \in \mathbb{N} \\ \text{and } \tilde{X}^{\mathrm{emb}}_{h_m} &\to \tilde{X} \quad \text{for } m \to \infty \text{ almost surely.} \end{aligned}$$

This ansatz is often referred to as “weak convergence approach” or “compactness method”.

As a last step, we identify $\tilde{X}$ as a solution to (2.1.1) as required, using classical Taylor expansion arguments to identify the limit in the finite difference Laplacian, the discrete energy estimate, the maximal monotonicity of the drift operator in the H^{-1} setting and lower-semicontinuity of the norm together with the almost sure convergence mentioned above. The uniqueness of solutions allows to conclude that the whole sequence $(X^{\mathrm{emb}}_{h_m})_{m\in\mathbb{N}}$ converges in law to $\tilde{X}$ for $h \to 0$.

The previously mentioned a priori estimates yield a bound for $(X^{\mathrm{emb}}_{h_m})_{m\in\mathbb{N}}$ in $L^p(\Omega \times [0,T];L^p((0,1)))$, where Ω is the underlying probability space and p depends on the growth of the linearity. Unfortunately, this bound is much harder to use for compactness arguments once $p = 1$, which is the case for the BTW nonlinearity ϕ_1 (cf. (2.1.4)). As a result, the construction of a solution candidate, as in the approach described above, fails. This is why we work with the notion of (S)VI solutions as introduced in Chapter 3 below, which does not include space-time white noise as stochastic input. Hence, we restrict our analysis to the deterministic dynamics. The energy estimates are very similar to the Zhang model case. For the subsequent identification of the limit, we make use of the fact that the variational inequality, which is a defining property of VI solutions, does not require strong regularity properties.

To this end, we define

$$\psi : \mathbb{R} \to [0,\infty), \quad \psi(x) = \int_0^x \phi_1(y)\mathrm{d}y = \mathbf{1}_{\mathbb{R}\setminus[-1,1]}(x)(|x-1|), \tag{2.2.5}$$

and $\varphi : H^{-1} \to [0,\infty)$,

$$\varphi(u) = \begin{cases} \|\psi(u)\|_{TV}, & \text{if } u \in \mathcal{M} \cap H^{-1}, \\ +\infty, & \text{else,} \end{cases} \tag{2.2.6}$$

as in Chapter 3 below, where the precise definition of the convex functional of a measure is given in Definition 3.3.3. Furthermore, recall from Section 2.1 the partial differential equation (2.1.3) and the discrete process (2.1.6). We then define the following notion of solution, which is a special case of a stochastic variational inequality (SVI) solution (cf. Chapter 3 for a more detailed analysis).

Definition 2.2.6 (VI solution)**.** Let $u_0 \in H^{-1}$, $T > 0$. We say that $u \in \mathcal{C}([0,T]; H^{-1})$ is a VI solution to (2.1.3) if the following conditions are satisfied:

(i) (Regularity)
$$\varphi(X) \in L^1([0,T]).$$

(ii) (Variational inequality) For each $G \in L^2([0,T]; H^{-1})$, and $Z \in L^2([0,T]; L^2) \cap \mathcal{C}([0,T]; H^{-1})$ solving the equation
$$Z(t) - Z(0) = \int_0^t G(s)\,\mathrm{d}s \quad \text{for all } t \in [0,T],$$
we have
$$\begin{aligned}
&\|u(t) - Z(t)\|_{H^{-1}}^2 + 2\int_0^t \varphi(u(r))\mathrm{d}r \\
&\leq \|u_0 - Z(0)\|_{H^{-1}}^2 + 2\int_0^t \varphi(Z(r))\mathrm{d}r \\
&\quad - 2\int_0^t \langle G(r), u(r) - Z(r)\rangle_{H^{-1}}\,\mathrm{d}r
\end{aligned} \tag{2.2.7}$$
for almost all $t \in [0,T]$.

Remark 2.2.7. Existence and uniqueness of solutions to (2.1.3) in this sense is treated in Theorem 3.2.6 below with the choice $B \equiv 0$ in Assumption 3.2.1, (A1).

Still using Assumption 2.2.4, for each $m \in \mathbb{N}$, let $(u_{h_m}^n)_{n\in\{0,\dots,N_m-1\}} \subset \mathbb{R}^{Z_m-1}$ be defined iteratively by
$$\begin{aligned}
u_{h_m}^{n+1} &= u_{h_m}^n + \tau_m \Delta_{h_m} \tilde{\phi}_1(u_{h_m}^n) \\
u_{h_m}^0 &= u_{h_m}^*,
\end{aligned} \tag{2.2.8}$$
where $(u_{h_m}^*)_{m\in\mathbb{N}} \subset \mathbb{R}^{Z_m-1}$ such that $(u_{h_m}^*)^{\mathrm{pcx}} \to u_0$ in L^2. Then, we have the following result, which will be proved in Section 2.5.

Theorem 2.2.8. *Recall the notation from Section 2.1.2 and let Assumption 2.2.4 be satisfied. Then, the process* $u_{h_m}^{plt,pcx}$ *obtained from* (2.2.8) *converges weakly* to the VI solution of* (2.1.3) *in* $L^\infty([0,T]; H^{-1})$ *for* $m \to \infty$.

2.3 Uniqueness of laws of weak solutions

In order to apply the main result from [94] to obtain uniqueness of the law of weak solutions, we first establish some preparatory results and helpful notions.

Definition 2.3.1. We define a multivalued operator by its graph $\mathcal{A}_T \subset L^2([0,T]; L^2) \times L^2([0,T]; L^2)$, given by
$$(f,g) \in \mathcal{A}_T \quad \text{if and only if} \quad g \in \phi(f) \text{ for almost every } (t,x) \in [0,T] \times [0,1]. \tag{2.3.1}$$

Lemma 2.3.2. *The operator* $\mathcal{A}_T$ *is maximal monotone.*

Proof. By [7, Theorem 2.8], it is enough to show that $\mathcal{A}_T$ is the subdifferential of a convex, proper and lower-semicontinuous functional $\varphi : H \to [0,\infty]$ on a real Banach space H. To this end, define $\tilde{\psi} : \mathbb{R} \to [0,\infty)$ by
$$\tilde{\psi}(x) = \mathbf{1}_{\{|x|\geq 1\}}(x^2 - 1),$$
which is proper, convex and continuous, and for which we have $\partial\tilde{\psi} = \phi$. We note that $H := L^2([0,T]; L^2)$ is a Hilbert space. Defining
$$\varphi_T : H \to [0,\infty], \quad \varphi_T(u) = \int_0^T \int_0^1 \tilde{\psi}(u(t,x))\mathrm{d}x\mathrm{d}t, \tag{2.3.2}$$
we obtain by [19, Theorem 16.50] that φ_T is convex, proper and lower-semicontinuous and $\mathcal{A}_T = \partial\varphi_T$, as required. ☐

Lemma 2.3.3. *The graph $\mathcal{A}_T$ is a closed subset of $L^2([0,T];L^2) \times L^2([0,T];L^2)$ and thus measurable with respect to the Borel σ-algebra on $L^2([0,T];L^2)$.*

Proof. The first statement is true for any maximal monotone operator by [7, Proposition 2.1]. The measurability then follows by definition of the Borel σ-algebra. $\square$

We define two kinds of Sobolev spaces that we are going to use.

Definition 2.3.4. Let $V \subset H \subset V'$ a Gelfand triple and $T > 0$. We define

$$\begin{aligned} W^{1,2}([0,T];V') &:= \{u \in L^2([0,T];V') : u' \in L^2([0,T];V')\} \\ \text{and} \quad W^{1,2}([0,T];V,H) &:= \{u \in L^2([0,T],V) : u' \in L^2([0,T];V')\}, \end{aligned}$$

where u' is the weak derivative of u as defined e. g. in [86, Definition 2.5.1]. These spaces are Banach spaces with the norms

$$\begin{aligned} \|u\|_{W^{1,2}([0,T];V')} &= \left(\|u\|^2_{L^2([0,T];V')} + \|u'\|^2_{L^2([0,T];V')}\right)^{\frac{1}{2}} \\ \text{and} \quad \|u\|_{W^{1,2}([0,T];V,H)} &= \left(\|u\|^2_{L^2([0,T];V)} + \|u'\|^2_{L^2([0,T];V')}\right)^{\frac{1}{2}}, \end{aligned}$$

respectively. These norms are norm-equivalent to the ones given in [86, Section 2.5.b] and [124, Proposition 23.23], respectively, where also the Banach space property is proved.

We have the following measurability properties.

Lemma 2.3.5. *The subset*

$$M_1 := \left\{ \begin{matrix} (u,z) \in L^2([0,T];L^2) \times L^2([0,T];(L^2)') : \\ \exists v \in L^2([0,T];L^2) \text{ such that } z = \Delta v \ \mathrm{d}t\text{-almost everywhere and } (u,v) \in \mathcal{A}_T \end{matrix} \right\} \tag{2.3.3}$$

is Borel-measurable. The map $\partial_t : W^{1,2}([0,T];(L^2)') \to L^2([0,T];(L^2)')$ is continuous and

$$M_2 := (\Pi_1, \partial_t(\Pi_2))^{-1}(M_1) \subseteq L^2([0,T];L^2) \times W^{1,2}([0,T];(L^2)') \tag{2.3.4}$$

is Borel-measurable. The set M_2 is also Borel-measurable as a subset of $L^2([0,T];L^2) \times L^2([0,T];(L^2)')$. Finally, the canonical embedding $I_{xw} : L^2([0,T];L^2) \times \mathcal{C}([0,T];H^{-1}) \hookrightarrow \left(L^2([0,T];(L^2)')\right)^2$ is continuous, and the subset

$$M_3 := \{(\Pi_1, \Pi_1(I_{xw}) - \Pi_2(I_{xw})) \in M_2\} \subseteq L^2([0,T];L^2) \times \mathcal{C}([0,T];H^{-1})$$

is Borel-measurable.

Proof. We notice that M_1 is the image of the set $\mathcal{A}_T$, which is Borel-measurable by Lemma 2.3.3, under the isometry $(\Pi_1, \Delta \circ \Pi_2)$, and hence Borel-measurable by the Kuratowski theorem (cf. [107, Theorem 3.9]). The operator $\partial_t : W^{1,2}([0,T];(L^2)') \to L^2([0,T];(L^2)')$ is linear and bounded by the definition of the Sobolev space. Hence it is continuous, which implies Borel-measurability. Thus, also $(\Pi_1, \partial_t(\Pi_2))$ is continuous and Borel-measurable, which yields measurability of M_2 using the measurability of M_1. The set M_2, viewed as a subset of $L^2([0,T];L^2) \times L^2([0,T];(L^2)')$, is the image of the canonical embedding and thus Borel-measurable by the Kuratowski theorem. The embedding I_{xw} is linear and bounded, the latter of which can be seen by computing

$$\begin{aligned} \|I_{xw}(u,z)\|^2_{L^2([0,T];(L^2)') \times L^2([0,T];(L^2)')} &= \|u\|^2_{L^2([0,T];(L^2)')} + \|z\|^2_{L^2([0,T];(L^2)')} \\ &= \int_0^T \|u(t)\|^2_{(L^2)'} \,\mathrm{d}t + \int_0^T \|z(t)\|^2_{H^{-1}} \,\mathrm{d}t \\ &\leq C \int_0^T \|u(t)\|^2_{L^2} \,\mathrm{d}t + CT \sup_{t \in [0,T]} \|z(t)\|^2_{H^{-1}} \\ &\leq CT \,\|(u,z)\|^2_{L^2([0,T];L^2) \times \mathcal{C}([0,T],H^{-1})}. \end{aligned}$$

The Borel-measurability of M_3 follows. $\square$

The previous lemma alludes that (2.2.1) and (2.2.2) are actually distributional properties, which motivates the following definition.

Definition 2.3.6. We call a probability measure Q on the probability space $L^2([0,T];L^2)\times\mathcal{C}([0,T];H^{-1})$ endowed with its Borel σ-algebra a pre-solution to (2.1.1), if

$$Q(M_3) = 1, \tag{2.3.5}$$

where M_3 is defined as in Lemma 2.3.5.

Lemma 2.3.7. *The joint law of the processes (X,W) of each weak solution to* (2.1.1) *in the sense of Definition 2.2.1 is a pre-solution.*

Proof. Let $\big((\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P}),X,W\big)$ be a weak solution to (2.1.1) and $Y\in L^2(\tilde{\Omega}\times[0,T];L^2)$ the corresponding drift process as in Definition 2.2.1. Then, [86, Proposition 2.5.9], (2.2.1) and (2.2.2) yield

$$\partial_t\left(\Pi_1(I_{xw}(X,W))-\Pi_2(I_{xw}(X,W))\right)=\Delta Y \quad \mathbb{P}\text{-almost surely}$$

with $(X,Y)\in\mathcal{A}_T$. Hence, using the notation from Lemma 2.3.5,

$$(X,\Pi_1(I_{xw}(X,W))-\Pi_2(I_{xw}(X,W)))\in M_2 \quad \mathbb{P}\text{-almost surely},$$

which by construction is equivalent to $(X,W)\in M_3$ $\mathbb{P}$-almost surely. This finishes the proof. ☐

We cite the concept of pointwise uniqueness from [94, Definition 1.4].

Definition 2.3.8. Pointwise uniqueness holds for pre-solutions, if for any processes (X_1,X_2,W) defined on the same probability space with $\mathcal{L}((X_1,W))$ and $\mathcal{L}((X_2,W))$ being pre-solutions, $X_1=X_2$ almost surely.

From [124, Proposition 23.23], we obtain the following.

Lemma 2.3.9. *Let $V\subset H\subset V'$ be a Gelfand triple and let $u\in W^{1,2}([0,T];V,H)$. Then, there exists a uniquely determined continuous function $u_1:[0,T]\to H$, which coincides* dt*-almost everywhere with u. Furthermore, for $t\in[0,T]$ we have*

$$\|u_1(t)\|_H^2=\|u_1(0)\|_H^2+2\int_0^t\langle u'(s),u(s)\rangle_{V'\times V}\,\mathrm{d}s.$$

Lemma 2.3.10. *Pointwise uniqueness holds for pre-solutions to* (2.1.1).

Proof. Let (X^1,X^2,W) be defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, such that $\mathcal{L}((X^1,W))$ and $\mathcal{L}((X^2,W))$ are two pre-solutions to (2.1.1). Let M_3 be defined as in Lemma 2.3.5, and let

$$\tilde{M}_3:=(X^1,W)^{-1}(M_3)\cap(X^2,W)^{-1}(M_3),$$

which implies that $\mathbb{P}(\tilde{M}_3)=1$ by construction. From now on, we conduct all arguments pointwise for $\omega\in\tilde{M}_3$. We define $Y^i\in L^2([0,T];L^2)$ for $i=1,2$ by

$$Y^i=\Delta^{-1}(\partial_t(X^i-W)),$$

which is well-defined due to the construction of M_3. Moreover, it follows that (2.2.1) and (2.2.2) are satisfied for (X^i,Y^i,W) for $i=1,2$, which implies

$$X^1(t)-X^2(t)=\int_0^t\Delta(Y^1(r)-Y^2(r))\mathrm{d}r \quad \text{in } L^2([0,T];(L^2)'). \tag{2.3.6}$$

By [86, Proposition 2.5.9], (2.3.6) implies that X^1-X^2 is weakly differentiable with $(X^1-X^2)'=\Delta(Y^1-Y^2)$. Since $X^i,Y^i\in L^2([0,T];L^2)$ for $i=1,2$ by construction, $X^1-X^2\in W^{1,2}([0,T];L^2,H^{-1})$. Lemma 2.3.9 then yields that there exists a continuous H^{-1}-valued dt-version Z of X^1-X^2, for which we have

$$\begin{aligned}\|Z(t)\|_{H^{-1}}^2&=\int_0^t\left\langle\Delta(Y^1(r)-Y^2(r)),X^1(r)-X^2(r)\right\rangle_{(L^2)'\times L^2}\mathrm{d}r\\&=-\int_0^t\left\langle Y^1(r)-Y^2(r),X^1(r)-X^2(r)\right\rangle_{L^2}\mathrm{d}r\le 0,\end{aligned}$$

where the last step follows from (2.2.2). This implies that Z and consequently X^1-X^2 is zero dt-almost everywhere. Since this is true for every $\omega\in\tilde{M}_3$, $X^1=X^2$ $\mathbb{P}$-almost surely, as required. ☐

Corollary 2.3.11. *There exists at most one pre-solution to* (2.1.1).

Proof. This is part of the statement of [94, Theorem 1.5]. ☐

Proof of Theorem 2.2.3. The claim is a direct consequence of Lemma 2.3.7 and Corollary 2.3.11. ☐

2.4 Continuum limit of a modified Zhang model

Throughout this section, we will drop the index m of the discretization sequences

$$(h_m)_{m\in\mathbb{N}}, (Z_m)_{m\in\mathbb{N}}, (\tau_m)_{m\in\mathbb{N}}, (N_m)_{m\in\mathbb{N}},$$

writing instead $(h)_{h>0}$ etc. Moreover, convergence of sequences and usually nonrelabeled subsequences indexed by m for $m\to\infty$ will be denoted by $h\to 0$. Expressions like "for $h>0$" have to be understood in the sense "for all elements of $(h_m)_{m\in\mathbb{N}}$" or "for all elements of the subsequence at hand".

We begin with recalling an estimate controlling the operator norm of the discrete Laplacian.

Lemma 2.4.1. *Let $\Delta_h \in \mathbb{R}^{(Z-1)\times(Z-1)}$ be defined as in* (2.1.9) *and recall the matrix norm $\|\cdot\|$ in* (2.1.8). *Then, $-\Delta_h$ is positive definite and*

$$\|-\Delta_h\| \leq \frac{4}{h^2}.$$

Proof. From [96, Equation (2.23)], we obtain that the eigenvalues of $-\Delta_h$ are

$$\lambda_j = \frac{2}{h^2}(1-\cos(j\pi h)) \in \left(0, \frac{4}{h^2}\right), \quad j = 1,\dots,Z-1,$$

which implies that $-\Delta_h$ is positive definite. Equation (2.77) in [117] then yields the second claim. ☐

We have the following bounds on the discrete process X_h defined in (2.2.3).

Lemma 2.4.2. *Let $\tau, h>0, Z, N\in\mathbb{N}$ as in Assumption 2.2.4, where we choose h small enough for $\frac{\tau}{h^2}\leq\frac{1}{4}$ to be satisfied. Then, the discrete process in* (2.2.3) *satisfies*

$$\|X_h^n\|_{-1}^2 + \mathcal{S}_{n,h} \leq \left\|x_h^0\right\|_{-1}^2 + \sum_{k=0}^{n-1}\left(2\sqrt{\frac{\tau}{h}}\left\langle X_h^k, \xi_h^k\right\rangle_{-1} + 2\tau^{\frac{3}{2}}h^{-\frac{1}{2}}\left\langle \Delta_h\tilde{\phi}(X_h^k), \xi_h^k\right\rangle_{-1} + \frac{\tau}{h}\left\|\xi_h^k\right\|_{-1}^2\right) \tag{2.4.1}$$

and

$$\mathbb{E}\,\|X_h^n\|_{-1}^2 + \mathbb{E}\,\mathcal{S}_{n,h} \leq \mathbb{E}\left\|x_h^0\right\|_{-1}^2 + n\tau\,\mathrm{Tr}(-\Delta_h^{-1}) \tag{2.4.2}$$

for all $n\in\{1,\dots,N+1\}$, where

$$\mathcal{S}_{n,h} \in \left\{\tau\sum_{k=0}^{n-1}\left\langle X_h^k, \tilde{\phi}(X_h^k)\right\rangle_0, \tau\sum_{k=0}^{n-1}\left\|\tilde{\phi}(X_h^k)\right\|_0^2, \tau\sum_{k=0}^{n-1}\left\|X_h^k\right\|_0^2 - n\tau\right\}.$$

Moreover, we have for $n\in\{1,\dots,N\}$

$$\frac{1}{h}\mathbb{E}\,\|\xi_h^n\|_{-1}^2 = \mathrm{Tr}(-\Delta_h^{-1}) \tag{2.4.3}$$

and

$$\mathbb{E}\,\|X_h^n\|_{-1}^2 + 2\tau\mathbb{E}\sum_{k=0}^{n-1}\left\langle X_h^k, \tilde{\phi}(X_h^k)\right\rangle_0 \leq \left(1+\frac{4\tau}{h^2}\right)\left(\mathbb{E}\left\|x_h^0\right\|_{-1}^2 + n\tau\,\mathrm{Tr}(-\Delta_h^{-1})\right). \tag{2.4.4}$$

Proof. For $\frac{\tau}{h^2} \leq \frac{1}{4}$ and $n \in \{0, \dots, N\}$, we compute

$$
\begin{aligned}
\left\|X_h^{n+1}\right\|_{-1}^2 &= \left\|X_h^n + \tau\Delta_h\tilde{\phi}(X_h^n) + \sqrt{\frac{\tau}{h}}\xi_h^n\right\|_{-1}^2 \\
&= \|X_h^n\|_{-1}^2 + 2\tau\left\langle X_h^n, \Delta_h\tilde{\phi}(X_h^n)\right\rangle_{-1} + \tau^2\left\|\Delta_h\tilde{\phi}(X_h^n)\right\|_{-1}^2 \\
&\quad + 2\sqrt{\frac{\tau}{h}}\left\langle X_h^n, \xi_h^n\right\rangle_{-1} + 2\tau^{\frac{3}{2}}h^{-\frac{1}{2}}\left\langle \Delta_h\tilde{\phi}(X_h^n), \xi_h^n\right\rangle_{-1} + \frac{\tau}{h}\|\xi_h^n\|_{-1}^2 .
\end{aligned} \tag{2.4.5}
$$

Using Lemma 2.4.1, we compute for $u \in \mathbb{R}^{Z-1}$

$$
\|\Delta_h u\|_{-1}^2 = |\langle -\Delta_h u, u\rangle_0| = h\,|\langle -\Delta_h u, u\rangle| \leq h\,\|-\Delta_h u\|\,\|u\| \leq \|-\Delta_h\|\,h\,\|u\|^2 \leq \frac{4}{h^2}\|u\|_0^2 . \tag{2.4.6}
$$

Furthermore, note that by the definition of $\tilde{\phi}$, we have for all $x \in \mathbb{R}^{Z-1}$

$$
\left\langle x, \tilde{\phi}(x)\right\rangle_0 = \left\|\tilde{\phi}(x)\right\|_0^2 . \tag{2.4.7}
$$

Hence, for the first three terms, we obtain

$$
\begin{aligned}
&\|X_h^n\|_{-1}^2 + 2\tau\left\langle X_h^n, \Delta_h\tilde{\phi}(X_h^n)\right\rangle_{-1} + \tau^2\left\|\Delta_h\tilde{\phi}(X_h^n)\right\|_{-1}^2 \\
&= \|X_h^n\|_{-1}^2 - 2\tau\left\langle X_h^n, \tilde{\phi}(X_h^n)\right\rangle_0 + \tau^2\left\|\Delta_h\tilde{\phi}(X_h^n)\right\|_{-1}^2 \\
&\leq \|X_h^n\|_{-1}^2 - (\tau+\tau)\left\langle X_h^n, \tilde{\phi}(X_h^n)\right\rangle_0 + 4\tau\frac{\tau}{h^2}\left\|\tilde{\phi}(X_h^n)\right\|_0^2 \\
&= \|X_h^n\|_{-1}^2 - \tau\left\langle X_h^n, \tilde{\phi}(X_h^n)\right\rangle_0 - \tau\left(1 - 4\frac{\tau}{h^2}\right)\left\|\tilde{\phi}(X_h^n)\right\|_0^2 \\
&\leq \|X_h^n\|_{-1}^2 - \tau\left\langle X_h^n, \tilde{\phi}(X_h^n)\right\rangle_0 .
\end{aligned} \tag{2.4.8}
$$

This yields (2.4.1) with the first choice for $\mathcal{S}_{n,h}$ by induction.

Now taking expectation in (2.4.5), we treat the last terms as follows. Recall the definition of the filtration $(\mathcal{F}_h^n)_{n=0}^N$ in (2.2.4). Note that for $h > 0$ and $n \in \{0, \dots, N+1\}$, $\|X_h^n\|$ and hence $\left\|\tilde{\phi}(X_h^n)\right\|$ is integrable by induction, and that for $\mathcal{F}_h^n$-measurable and integrable random variables $z \in L^1(\Omega; \mathbb{R}^{Z-1})$ we have

$$
\begin{aligned}
\mathbb{E}\left\langle z, \xi_h^n\right\rangle_{-1} &= h\mathbb{E}\left\langle -\Delta_h^{-1}z, \xi_h^n\right\rangle \\
&= h\sum_{i=0}^{Z}\mathbb{E}\left((-\Delta_h^{-1}z)_i\,\xi_h^{n,i}\right) \\
&= h\sum_{i=0}^{Z}\mathbb{E}\,\mathbb{E}\left[(-\Delta_h^{-1}z)_i\,\xi_h^{n,i}|\mathcal{F}_h^n\right] \\
&= h\sum_{i=0}^{Z}\mathbb{E}(-\Delta_h^{-1}z)_i\,\mathbb{E}\xi_h^{n,i} = 0,
\end{aligned}
$$

using that ξ_h^n is independent of $\mathcal{F}_h^n$ by assumption. Thus, the two mixed terms in the last line of (2.4.5) vanish. For the last term, we notice that

$$
\frac{\tau}{h}\mathbb{E}\,\|\xi_h^n\|_{-1}^2 = \frac{\tau}{h}\mathbb{E}\left\langle -\Delta_h^{-1}\xi_h^n, h\xi_h^n\right\rangle = \tau\,\mathbb{E}\left\langle -\Delta_h^{-1}\xi_h^n, \xi_h^n\right\rangle = \tau\operatorname{Tr}(-\Delta_h^{-1}), \tag{2.4.9}
$$

since for any family $(\xi_i)_{i=1}^{Z-1}$ of random variables with $\mathbb{E}(\xi_i\xi_j) = \delta_{ij}$ and for any matrix $A \in \mathbb{R}^{(Z-1)\times(Z-1)}$, we have

$$
\mathbb{E}\left\langle A\xi, \xi\right\rangle = \mathbb{E}\sum_{i,j=1}^{Z-1}A_{ij}\xi_j\xi_i = \sum_{i,j=1}^{Z-1}A_{ij}\mathbb{E}(\xi_j\xi_i) = \sum_{i,j=1}^{Z-1}A_{ij}\delta_{ij} = \operatorname{Tr}(A). \tag{2.4.10}
$$

In particular, (2.4.3) follows. Collecting all estimates, we conclude by induction that

$$
\mathbb{E}\,\|X_h^n\|_{-1}^2 + \tau\sum_{k=0}^{n-1}\mathbb{E}\left\langle X_h^k, \tilde{\phi}(X_h^k)\right\rangle_0 \leq \mathbb{E}\left\|x_h^0\right\|_{-1}^2 + n\tau\operatorname{Tr}(-\Delta_h^{-1}) \tag{2.4.11}
$$

for $n \in \{0, \dots, N+1\}$, which proves (2.4.2) for the first choice of $\mathcal{S}_{n,h}$. In view of (2.4.7), this immediately yields (2.4.1) and (2.4.2) for the second choice of $\mathcal{S}_{n,h}$. Considering

$$\left|\tilde{\phi}(x)\right|^2 \geq |x|^2 - 1,$$

we may compute

$$\begin{aligned}\tau \sum_{k=0}^{n-1} \left\|X_h^k\right\|_0^2 - n\tau &= \tau \sum_{k=0}^{n-1} \sum_{i=1}^{Z-1} h \left(\left|X_h^{k,i}\right|^2 - 1 \right) \\ &\leq \tau \sum_{k=0}^{n-1} \sum_{i=1}^{Z-1} h \left|\tilde{\phi}\left(X_h^{k,i}\right)\right|^2 \leq \tau \sum_{k=0}^{n-1} \left\|\tilde{\phi}(X_h^k)\right\|_0^2,\end{aligned}$$

which extends these statements to the last choice of $\mathcal{S}_{n,h}$. Carrying out only the first two steps of (2.4.8) before taking expectation and using induction, we obtain

$$\mathbb{E}\left\|X_h^n\right\|_{-1}^2 + 2\tau\mathbb{E}\sum_{k=0}^{n-1}\left\langle X_h^k, \tilde{\phi}(X_h^k)\right\rangle_0 \leq \mathbb{E}\left\|x_h^0\right\|_{-1}^2 + n\tau \operatorname{Tr}(-\Delta_h^{-1}) + \frac{4\tau}{h^2}\tau\sum_{k=0}^{n-1}\left\|\tilde{\phi}(X_h^k)\right\|_0^2.$$

Finally, using (2.4.2) yields (2.4.4). □

Lemma 2.4.3. *Let $h > 0$ as in Assumption 2.2.4, and let $I' : L^2 \to H^{-1}$ be the canonical embedding. Then, $I' \in L_2(L^2, H^{-1})$ and*

$$\lim_{h\to 0} \operatorname{Tr}(-\Delta_h^{-1}) = \sum_{k=1}^{\infty} \frac{1}{\pi^2 k^2} = \left\|I'\right\|_{L_2(L^2,H^{-1})}^2. \tag{2.4.12}$$

Proof. For $k = 1, 2, \dots$, define $b_k(x) = \sqrt{2}\sin(\pi k\, x)$, such that $(b_k)_{k\in\mathbb{N}}$ is an $L^2([0,1])$-orthonormal basis of eigenvectors of the Laplacian. For the last equality in (2.4.12), we compute

$$\left\|I'\right\|_{L_2(L^2,H^{-1})}^2 = \sum_{k=1}^{\infty} \left\langle I'e_k, I'e_k\right\rangle_{H^{-1}} = \sum_{k=1}^{\infty}\left\langle -\Delta^{-1}e_k, e_k\right\rangle_{L^2} = \sum_{k=1}^{\infty}\frac{1}{\pi^2k^2} = \frac{1}{6},$$

which also proves that I' is Hilbert-Schmidt.

For the convergence of the trace of the inverse discrete Laplacian, we first recall from [96, Section 2.10] that the eigenvalues of $-\Delta_h$ are

$$\lambda_{Z,k} = \frac{2}{h^2}\left(1 - \cos(\pi k h)\right) \quad \text{for } k = 1, \dots, Z-1.$$

This can be reformulated, using trigonometric identities and $\frac{1}{h} = Z$, to

$$\lambda_{Z,k} = 4Z^2 \sin^2\left(\frac{\pi k}{2Z}\right).$$

In order to prove the statement, we divide the sums into

$$\left|\sum_{k=1}^{Z-1}\frac{1}{\lambda_{Z,k}} - \sum_{k=1}^{\infty}\frac{1}{\pi^2k^2}\right| \leq \sum_{k=1}^{Z-1}\left|\frac{1}{\lambda_{Z,k}} - \frac{1}{\pi^2k^2}\right| + \sum_{k=Z}^{\infty}\frac{1}{\pi^2k^2}$$

Note that the last sum converges to zero for $Z \to \infty$ by the finiteness of $\zeta(2)$. It remains to estimate the first sum on the right hand side. Carrying out a Taylor expansion of the sine function around zero for each $k \in \{1, \dots, Z-1\}$, $Z \in \mathbb{N}$, the Lagrange remainder formula yields $\xi_{Z,k} \in [0, \frac{\pi}{2}]$ such that

$$\lambda_{Z,k} = 4Z^2\left(\frac{\pi k}{2Z} - \frac{1}{6}\left(\frac{\pi k}{2Z}\right)^3\cos(\xi_{Z,k})\right)^2.$$

Hence, for $k \in \{1, \dots, Z-1\}$, $Z \in \mathbb{N}$, we have

$$
\begin{aligned}
\frac{1}{\lambda_{Z,k}} - \frac{1}{\pi^2 k^2} &= \frac{1}{4Z^2 \left(\frac{\pi k}{2Z} - \frac{1}{6}\left(\frac{\pi k}{2Z}\right)^3 \cos(\xi_{Z,k})\right)^2} - \frac{1}{\pi^2 k^2} \\
&= \frac{\pi^2 k^2 - 4Z^2 \left(\left(\frac{\pi k}{2Z}\right)^2 - \frac{1}{3}\left(\frac{\pi k}{2Z}\right)^4 \cos(\xi_{Z,k}) + \frac{1}{36}\left(\frac{\pi k}{2Z}\right)^6 \cos^2(\xi_{Z,k})\right)}{4Z^2\pi^2 k^2 \left(\left(\frac{\pi k}{2Z}\right)^2 - \frac{1}{3}\left(\frac{\pi k}{2Z}\right)^4 \cos(\xi_{Z,k}) + \frac{1}{36}\left(\frac{\pi k}{2Z}\right)^6 \cos^2(\xi_{Z,k})\right)} \\
&= \frac{\pi^2 k^2 - \pi^2 k^2 + \frac{4}{3} Z^2 \left(\frac{\pi k}{2Z}\right)^4 \cos(\xi_{Z,k}) - \frac{1}{9} Z^2 \left(\frac{\pi k}{2Z}\right)^6 \cos^2(\xi_{Z,k})}{\pi^4 k^4 \left(1 - \frac{1}{3}\left(\frac{\pi k}{2Z}\right)^2 \cos(\xi_{Z,k}) + \frac{1}{36}\left(\frac{\pi k}{2Z}\right)^4 \cos^2(\xi_{Z,k})\right)} \\
&= \frac{\pi^4 k^4 \left(\frac{4}{3}\frac{1}{2^4 Z^2} \cos(\xi_{Z,k}) - \frac{1}{9}\frac{\pi^2 k^2}{2^6 Z^4} \cos^2(\xi_{Z,k})\right)}{\pi^4 k^4 \left(1 - \frac{1}{3}\left(\frac{\pi k}{2Z}\right)^2 \cos(\xi_{Z,k}) + \frac{1}{36}\left(\frac{\pi k}{2Z}\right)^4 \cos^2(\xi_{Z,k})\right)}.
\end{aligned}
\tag{2.4.13}
$$

Using $|\cos(\xi_{Z,k})| \leq 1$, $\frac{k}{Z} \leq 1$ and $\pi^2 < 10$, we have

$$
\left(1 - \frac{1}{3}\left(\frac{\pi k}{2Z}\right)^2 \cos(\xi_{Z,k}) + \frac{1}{36}\left(\frac{\pi k}{2Z}\right)^4 \cos^2(\xi_{Z,k})\right) \geq 1 - \frac{1}{3}\frac{\pi^2}{4} \geq \frac{1}{6}.
$$

Together with (2.4.13), we obtain for $Z \to \infty$

$$
\sum_{k=1}^{Z-1} \left| \frac{1}{\lambda_{Z,k}} - \frac{1}{\pi^2 k^2} \right| \leq \sum_{k=1}^{Z-1} \frac{6}{Z^2} \left(\frac{1}{12} + \frac{10}{9 \cdot 2^6}\right) \leq C \frac{Z-1}{Z^2} \to 0,
$$

which completes the proof. □

Corollary 2.4.4. *Let $\tau, h > 0$ as in Assumption 2.2.4. Then, there exists a constant $C > 0$ which only depends on T and x_h^0, such that the discrete process in* (2.2.3) *satisfies*

$$
\mathbb{E} \left\| X_h^{n+1} - X_h^n \right\|_{-1}^2 \leq C \frac{\tau}{h^2} \quad \textit{for all } n \in \{0, \dots, N-1\}.
$$

Proof. We compute

$$
\begin{aligned}
\mathbb{E} \left\| X_h^{n+1} - X_h^n \right\|_{-1}^2 &= \mathbb{E} \left\| \tau \Delta_h \tilde{\phi}(X_h^n) + \sqrt{\frac{\tau}{h}} \xi_h^n \right\|_{-1}^2 \\
&= \mathbb{E} \left\| \tau \Delta_h \tilde{\phi}(X_h^n) \right\|_{-1}^2 + 2\mathbb{E} \left\langle \tau \Delta_h \tilde{\phi}(X_h^n), \sqrt{\frac{\tau}{h}} \xi_h^n \right\rangle_{-1} + \frac{\tau}{h} \mathbb{E} \left\| \xi_h^n \right\|_{-1}^2.
\end{aligned}
\tag{2.4.14}
$$

Arguing as in (2.4.6), we have

$$
\tau^2 \mathbb{E} \left\| \Delta_h \tilde{\phi}(X_h^n) \right\|_{-1}^2 \leq 4 \frac{\tau^2}{h^2} \mathbb{E} \left\| \tilde{\phi}(X_h^n) \right\|_0^2,
$$

such that we can use Lemma 2.4.2 to obtain

$$
\mathbb{E} \left\| \tau \Delta_h \tilde{\phi}(X_h^n) \right\|_{-1}^2 \leq 4 \frac{\tau}{h^2} \tau \sum_{n=0}^{N} \mathbb{E} \left\| \tilde{\phi}(X_h^n) \right\|_0^2 \leq 4 \frac{\tau}{h^2} \left(\mathbb{E} \left\| x_h^0 \right\|_{-1}^2 + T \operatorname{Tr}(-\Delta_h^{-1}) \right) \leq C \frac{\tau}{h^2}. \tag{2.4.15}
$$

Moreover, as in the proof of Lemma 2.4.2, we have that

$$
\mathbb{E} \left\langle \tau \Delta_h \phi(X_h^n), \sqrt{\frac{\tau}{h}} \xi_h^n \right\rangle_{-1} = 0 \tag{2.4.16}
$$

by the independence of ξ_h^n of $\mathcal{F}_h^n$, where $(\mathcal{F}_h^n)_{n=0}^N$ is given as in (2.2.4). Finally, by (2.4.3) and Lemma 2.4.3, one can choose C independent of h satisfying

$$
\frac{\tau}{h} \mathbb{E} \left\| \xi_h^n \right\|_{-1}^2 \leq \tau C, \tag{2.4.17}
$$

such that (2.4.14) together with (2.4.15), (2.4.16) and (2.4.17) yields the claim. □

Lemma 2.4.5. *Let $\tau, h > 0$ and $N, Z \in \mathbb{N}$ as in Assumption 2.2.4, and let $(X_h^n)_{n=0}^N$ be constructed as in (2.2.3). Then*

$$\mathbb{E} \max_{n=0,\dots,N} \|X_h^n\|_{-1}^2 \le C,$$

where C is independent of h.

Proof. We recall equation (2.4.1), which leads us to consider the stochastic processes $(M_n)_{n=0}^N, (\tilde{M}_n)_{n=0}^N$ defined by

$$M_n = 2\sqrt{\frac{\tau}{h}} \sum_{k=0}^{n-1} \langle X_h^k, \xi_h^k \rangle_{-1}, \quad \tilde{M}_n = 2\tau^{\frac{3}{2}} h^{-\frac{1}{2}} \sum_{k=0}^{n-1} \left\langle \Delta_h \tilde{\phi}(X_h^k), \xi_h^k \right\rangle_{-1} \quad \text{for } n \in \{0, \dots, N\}.$$

Furthermore, let $(\mathcal{F}_n)_{n=0}^N$, $\mathcal{F}_n \subseteq \mathcal{F}$ be given as in (2.2.4). Note that for all $n \in \{0, \dots, N\}$, both $M_{n+1} - M_n$ and $\tilde{M}_{n+1} - \tilde{M}_n$ depend linearly on ξ_h^n, which is stochastically independent of $\mathcal{F}_n$, with coefficients which are $\mathcal{F}_n$-measurable. Hence, both $(M_n)_{n=0}^N$ and $(\tilde{M}_n)_{n=0}^N$ are $(\mathcal{F}_n)_{n=0}^N$-martingales, which allows to apply the Burkholder-Davis-Gundy inequality in the form of [40, Theorem 1]. We note that this theorem is proved for $C = 130$, i. e. C is independent of the process to which the inequality is applied. For $(M_n)_{n=0}^N$, this yields

$$\begin{aligned}
\mathbb{E} \max_{n=1,\dots,N} |M_n| &\le C\, \mathbb{E} \left(\sum_{k=1}^N (M_n - M_{n-1})^2 \right)^{\frac{1}{2}} \\
&= C\, \mathbb{E} \left(\sum_{k=0}^{N-1} 4 \frac{\tau}{h} \langle X_h^k, \xi_h^k \rangle_{-1}^2 \right)^{\frac{1}{2}} \\
&\le 2C\, \mathbb{E} \left(\sum_{k=0}^{N-1} \frac{\tau}{h} \|X_h^k\|_{-1}^2 \|\xi_h^k\|_{-1}^2 \right)^{\frac{1}{2}} \\
&\le 2C\, \mathbb{E} \left[\max_{k=0,\dots,N-1} \left(\|X_h^k\|_{-1} \right) \left(\sum_{k=0}^{N-1} \frac{\tau}{h} \|\xi_h^k\|_{-1}^2 \right)^{\frac{1}{2}} \right] \\
&\le \frac{1}{2} \mathbb{E} \max_{k=0,\dots,N-1} \|X_h^k\|_{-1}^2 + 2C^2\, \mathbb{E} \sum_{k=0}^{N-1} \frac{\tau}{h} \|\xi_h^k\|_{-1}^2 \\
&= \frac{1}{2} \mathbb{E} \max_{k=0,\dots,N-1} \|X_h^k\|_{-1}^2 + 2C^2\, T \operatorname{Tr}(-\Delta_h^{-1}) \le \frac{1}{2} \mathbb{E} \max_{k=0,\dots,N-1} \|X_h^k\|_{-1}^2 + C,
\end{aligned}$$

where we used the weighted Young inequality in the fifth step, (2.4.3) in the sixth step and Lemma 2.4.3 in the last step. Similarly, we compute for $(\tilde{M}_n)_{n=0}^N$

$$\begin{aligned}
\mathbb{E} \max_{n=0,\dots,N} \left| \tilde{M}_n \right| &\le C\, \mathbb{E} \left(\sum_{k=0}^{N-1} 4 \frac{\tau^3}{h} \left\langle \Delta_h \tilde{\phi}(X_h^k), \xi_h^k \right\rangle_{-1}^2 \right)^{\frac{1}{2}} \\
&\le 2C\, \mathbb{E} \left(\sum_{k=0}^{N-1} \frac{\tau^3}{h} \left\| \Delta_h \tilde{\phi}(X_h^k) \right\|_{-1}^2 \|\xi_h^k\|_{-1}^2 \right)^{\frac{1}{2}} \\
&\le 2C\, \mathbb{E} \left[\tau \max_{k=0,\dots,N-1} \left(\left\| \Delta_h \tilde{\phi}(X_h^k) \right\|_{-1} \right) \left(\sum_{k=0}^{N-1} \frac{\tau}{h} \|\xi_h^k\|_{-1}^2 \right)^{\frac{1}{2}} \right] \\
&\le C\tau^2\, \mathbb{E} \max_{k=0,\dots,N-1} \left\| \Delta_h \tilde{\phi}(X_h^k) \right\|_{-1}^2 + C\, \mathbb{E} \sum_{k=0}^{N-1} \frac{\tau}{h} \|\xi_h^k\|_{-1}^2 \\
&\le C\tau\, \mathbb{E}\, \tau \sum_{k=0}^{N-1} \left\| \Delta_h \tilde{\phi}(X_h^k) \right\|_{-1}^2 + CT \operatorname{Tr}(-\Delta_h^{-1}) \\
&\le 4C\, \frac{\tau}{h^2} \mathbb{E}\, \tau \sum_{k=0}^{N-1} \left\| \tilde{\phi}(X_h^k) \right\|_0^2 + C \le C \left(\frac{\tau}{h^2} + 1 \right) \le 2C,
\end{aligned}$$

where, in addition to the arguments above, we used (2.4.6) in the sixth step, Lemma 2.4.2 in the seventh step and Assumption 2.2.4 in the last step.

Collecting all estimates, we obtain from (2.4.1)

$$\begin{aligned}\mathbb{E}\max_{n=0,\dots,N}\left\|X_h^k\right\|_{-1}^2 &\le \mathbb{E}\left\|x_h^0\right\|_{-1}^2+\mathbb{E}\max_{n=0,\dots,N-1}|M_n|+\mathbb{E}\max_{n=0,\dots,N-1}\left|\tilde{M}_n\right|+\mathbb{E}\max_{n=0,\dots,N}\sum_{k=0}^{n-1}\frac{\tau}{h}\left\|\xi_h^k\right\|_{-1}^2\\ &\le C+\frac{1}{2}\mathbb{E}\max_{n=0,\dots,N-1}\left\|X_h^k\right\|_{-1}^2+\mathbb{E}\sum_{k=0}^{N-1}\frac{\tau}{h}\left\|\xi_h^k\right\|_{-1}^2\\ &\le \frac{1}{2}\mathbb{E}\max_{n=0,\dots,N}\left\|X_h^k\right\|_{-1}^2+C,\end{aligned}$$

which yields

$$\mathbb{E}\max_{n=0,\dots,N}\left\|X_h^k\right\|_{-1}^2\le C,$$

as required. □

Lemma 2.4.6. *Let $T>0$, $N\in\mathbb{N}$ and $(u_k)_{k=0}^N\subseteq\mathbb{R}$. Let $\tau=\frac{T}{N}$ and recall the prolongations from Definition 2.1.5. Then, there exists $C>0$ independent of N, such that*

$$\max\left\{\int_0^T\left|u^{plt}(t)\right|^2\mathrm{d}t,\int_0^T\left|u^{pct\text{-}}(t)\right|^2\mathrm{d}t,\int_0^T\left|u^{pct+}(t)\right|^2\mathrm{d}t\right\}\le C\sum_{k=0}^N\tau u_k^2\mathrm{d}t.$$

Proof. For the last two terms, we have

$$\int_0^T\left|u^{\mathrm{pct\text{-}}}(t)\right|^2\mathrm{d}t=\sum_{k=0}^{N-1}\tau u_k^2\mathrm{d}t\le\sum_{k=0}^N\tau u_k^2\mathrm{d}t$$

and

$$\int_0^T\left|u^{\mathrm{pct+}}(t)\right|^2\mathrm{d}t=\sum_{k=1}^{N}\tau u_k^2\mathrm{d}t\le\sum_{k=0}^N\tau u_k^2\mathrm{d}t,$$

such that it only remains to include the first term. To this end, we define the extended piecewise constant prolongation $v^{\mathrm{pct0}}:[-\tau,T+\tau]\to\mathbb{R}$ using the grid points as midpoints, i. e.

$$v^{\mathrm{pct0}}(t)=v_{\lfloor t/\tau\rceil},$$

where we set $v_{-1}=v_{N+1}=0$, and the extended piecewise linear prolongation $v^{\mathrm{plt0}}:[-\tau,T+\tau]\to\mathbb{R}$ by

$$v^{\mathrm{plt0}}=\frac{t-t_\tau}{\tau}v_{\lfloor t/\tau\rfloor+1}+\frac{t_\tau+\tau-t}{\tau}v_{\lfloor t/\tau\rfloor},$$

again setting $v_{-1}=v_{N+1}=0$. We compute

$$\int_0^T\left|u^{\mathrm{plt}}(t)\right|^2\mathrm{d}t\le\int_{-\tau}^{T+\tau}\left|u^{\mathrm{plt0}}(t)\right|^2\mathrm{d}t\le C\int_{-\tau}^{T+\tau}\left|u^{\mathrm{pct0}}(t)\right|^2\mathrm{d}t=C\tau\sum_{k=0}^N u_k^2\mathrm{d}t,$$

where in the second step, we used the following reasoning. We substitute t by $(T+2\tau)s-\tau$ and we notice that

$$\begin{aligned}u^{\mathrm{plt0}}((T+2\tau)\cdot-\tau)&=\mathbb{P}_\gamma\left((u_{k-1})_{k=1}^{N+1}\right)\\ \text{and}\quad u^{\mathrm{pct0}}((T+2\tau)\cdot-\tau)&=\mathbb{Q}_\gamma\left((u_{k-1})_{k=1}^{N+1}\right)\end{aligned}$$

where $\gamma=(N+1+1)^{-1}$ and $\mathbb{P}_\gamma,\mathbb{Q}_\gamma$ are the piecewise linear (respectively piecewise constant) extensions from [52, Equations (3.6) and (3.7)], with h replaced by γ and N replaced by $N+1$. Keeping these conventions, we use [52, Propositions 3.1 and 3.2] to obtain

$$\begin{aligned}
\int_{-\tau}^{T+\tau} \left|u^{\mathrm{plt0}}(t)\right|^2 \mathrm{d}t &= (T+2\tau) \int_0^1 \left|u^{\mathrm{plt0}}(s(T+2\tau)-\tau)\right|^2 \mathrm{d}t \\
&= (T+2\tau) \left\|\mathbb{P}_\gamma \left((u_{k-1})_{k=1}^{N+1}\right)\right\|_{L^2([0,1])}^2 \\
&\le C(T+2\tau) \left\|\mathbb{Q}_\gamma \left((u_{k-1})_{k=1}^{N+1}\right)\right\|_{L^2([0,1])}^2 \\
&= C(T+2\tau) \int_0^1 \left|u^{\mathrm{pct0}}(s(T+2\tau)-\tau)\right|^2 \mathrm{d}t \\
&= C \int_{-\tau}^{T+\tau} \left|u^{\mathrm{pct0}}(t)\right|^2 \mathrm{d}t,
\end{aligned}$$

for $C > 0$ independent of τ, which finishes the proof. □

Recall the partitions $(K_i)_{i=0}^{Z}$ and $(J_i)_{i=0}^{Z-1}$ and the grids $(x_i)_{i=0}^{Z}$ and $(y_i)_{i=0}^{Z-1}$ as given in (2.1.7), and the definition of prolongations of functions on the grid $(x_i)_{i=0,\dots,Z}$ as given in Definition 2.1.1.

Lemma 2.4.7. *Let $u = (u_i)_{i=1}^{Z-1} \in \mathbb{R}^{Z-1}$ and $v = (v_i)_{i=0}^{Z-1} \in \mathbb{R}^{Z}$ and recall the convention $u_0 = u_Z = 0$. Define the piecewise constant prolongation with respect to the grid $(y_i)_{i=0,\dots,Z-1}$ by*

$$I_h^{\mathrm{pcy}} : \mathbb{R}^Z \to L^2, v \mapsto \sum_{i=0}^{Z-1} v_i \boldsymbol{1}_{J_i},$$

and the piecewise linear prolongation with zero-Neumann boundary conditions by

$$I_h^{\mathrm{ply}} : \mathbb{R}^Z \to H^1, v \mapsto v_0 \boldsymbol{1}_{K_0} + \sum_{i=1}^{Z-1} \left[v_{i-1} + \frac{v_i - v_{i-1}}{h}(\cdot - x_i)\right] \boldsymbol{1}_{K_i} + v_{Z-1} \boldsymbol{1}_{K_Z}.$$

1. *We have*
$$\left\|I_h^{\mathrm{ply}} v\right\|_{L^2} \le \left\|I_h^{\mathrm{pcy}} v\right\|_{L^2} \le 3 \left\|I_h^{\mathrm{ply}} v\right\|_{L^2}.$$
2. *We have $\int_0^1 I_h^{\mathrm{ply}} v \, \mathrm{d}x = \int_0^1 I_h^{\mathrm{pcy}} v \, \mathrm{d}x$.*
3. *For all $a \in \mathbb{R}$, we have*
$$I_h^{\mathrm{ply}} v - a = I_h^{\mathrm{ply}} (v-a) \quad \textit{and} \quad I_h^{\mathrm{pcy}} v - a = I_h^{\mathrm{pcy}} (v-a),$$
where the difference is understood pointwise or componentwise, respectively.
4. *We have*
$$\partial_x I_h^{\mathrm{pcy}} v = \sum_{i=1}^{Z-1} \delta_{x_i}(v_i - v_{i-1}),$$
where ∂_x is the distributional derivative and for $z \in [0,1]$, δ_z is a Dirac mass on z.
5. *For $i = 0, \dots Z-1$, let $v_i = \sum_{j=0}^{i} h u_j$. Then*
$$\partial_x (I_h^{\mathrm{ply}} v) = I_h^{\mathrm{pcx}} u.$$
6. *We have $\|u\|_1 = \left\|I_h^{\mathrm{plx}} u\right\|_{H_0^1}$.*
7. *We have*
$$\left\|\partial_{xx}(I_h^{\mathrm{plx}} u)\right\|_{H^{-1}} = \left\|\sum_{i=1}^{Z-1} \frac{1}{h}(-u_{i-1} + 2u_i - u_{i+1})\delta_{x_i}\right\|_{H^{-1}}$$
with the same conventions as in 4.

Proof. 1. We first note that for $i = 0, \dots, Z-2$

$$\begin{aligned}
\int_{y_i}^{y_{i+1}} (I_h^{\mathrm{ply}}\, v)^2(x)\mathrm{d}x &= \int_{y_i}^{y_{i+1}} \left(\frac{x-y_i}{h} v_{i+1} + \frac{y_{i+1}-x}{h} v_i \right)^2 \mathrm{d}x \\
&= \int_0^1 (v_{i+1}x + (1-x)v_i)^2 h \,\mathrm{d}x \\
&= h \int_0^1 (v_i + (v_{i+1}-v_i)x)^2 \mathrm{d}x \\
&= h \int_0^1 v_i^2 + 2v_i(v_{i+1}-v_i)x + (v_{i+1}-v_i)^2 x^2 \mathrm{d}x \\
&= h(v_i^2 + v_i(v_{i+1}-v_i) + \frac{1}{3}(v_{i+1}-v_i)^2) \\
&= \frac{h}{3}(v_i^2 + v_i v_{i+1} + v_{i+1}^2) \\
&= \frac{h}{2}(v_{i+1}^2 + v_i^2) - \frac{h}{6}(v_{i+1}-v_i)^2 \\
&\le \frac{h}{2}(v_{i+1}^2 + v_i^2) \\
&= \int_{y_i}^{x_{i+1}} v_i^2 \mathrm{d}x + \int_{x_{i+1}}^{y_{i+1}} v_{i+1}^2 \mathrm{d}x = \int_{y_i}^{y_{i+1}} (I_h^{\mathrm{pcy}}\, v)^2(x)\mathrm{d}x.
\end{aligned}$$

Carrying out the same first seven steps, one can also continue by

$$\begin{aligned}
3\int_{y_i}^{y_{i+1}} (I_h^{\mathrm{ply}}\, v)^2(x)\mathrm{d}x &= \frac{h}{2}(v_{i+1}^2 + v_i^2) + h(v_{i+1}^2 + v_i^2) - \frac{h}{2}(v_{i+1}-v_i)^2 \\
&= \int_{y_i}^{y_{i+1}} (I_h^{\mathrm{pcy}}\, v)^2(x)\mathrm{d}x + \frac{h}{2}(2v_{i+1}^2 + 2v_i^2 - v_{i+1}^2 + 2v_i v_{i+1} - v_i^2) \\
&= \int_{y_i}^{y_{i+1}} (I_h^{\mathrm{pcy}}\, v)^2(x)\mathrm{d}x + \frac{h}{2}(v_{i+1}+v_i)^2 \ge \int_{y_i}^{y_{i+1}} (I_h^{\mathrm{pcy}}\, v)^2(x)\mathrm{d}x.
\end{aligned}$$

This can be used to conclude

$$\begin{aligned}
\left\| I_h^{\mathrm{ply}}\, v \right\|_{L^2}^2 &= \frac{h}{2}v_0^2 + \sum_{i=0}^{Z-2} \int_{y_i}^{y_{i+1}} (I_h^{\mathrm{ply}}\, v)^2(x)\mathrm{d}x + \frac{h}{2}v_{Z-1}^2 \\
&\le \frac{h}{2}v_0^2 + \sum_{i=0}^{Z-2} \int_{y_i}^{y_{i+1}} (I_h^{\mathrm{pcy}}\, v)^2(x)\mathrm{d}x + \frac{h}{2}v_{Z-1}^2 = \|I_h^{\mathrm{pcy}}\, v\|_{L^2}^2 \\
&\le \frac{3h}{2}v_0^2 + \sum_{i=0}^{Z-2} 3\int_{y_i}^{y_{i+1}} (I_h^{\mathrm{ply}}\, v)^2(x)\mathrm{d}x + \frac{3h}{2}v_{Z-1}^2 = 3\left\| I_h^{\mathrm{ply}}\, v \right\|_{L^2}^2,
\end{aligned}$$

as required.

2. We first note that for $i = 0, \dots, Z-2$

$$\int_{y_i}^{y_{i+1}} (I_h^{\mathrm{ply}}\, v)(x)\ \mathrm{d}x = h\int_0^1 v_{i+1}x + (1-x)v_i\ \mathrm{d}x = \frac{h}{2}(v_{i+1}+v_i) = \int_{y_i}^{y_{i+1}} (I_h^{\mathrm{pcy}}\, v)(x)\ \mathrm{d}x,$$

which yields

$$\begin{aligned}
\int_0^1 (I_h^{\mathrm{ply}}\, v)(x)\ \mathrm{d}x &= \frac{h}{2}v_0 + \sum_{i=0}^{Z-2} \int_{y_i}^{y_{i+1}} (I_h^{\mathrm{ply}}\, v)(x)\ \mathrm{d}x + \frac{h}{2}v_Z \\
&= \frac{h}{2}v_0 + \sum_{i=0}^{Z-2} \int_{y_i}^{y_{i+1}} (I_h^{\mathrm{pcy}}\, v)(x)\ \mathrm{d}x + \frac{h}{2}v_Z = \int_0^1 (I_h^{\mathrm{pcy}}\, v)(x)\ \mathrm{d}x,
\end{aligned}$$

as required.

3. The statement is clear for I_h^{pcy} and for I_h^{ply} on K_0 and K_Z. For $y_i \leq x < y_{i+1}$, $i = 0, \dots, Z-2$, we have

$$\begin{aligned}(I_h^{\text{ply}}(v-a))(x) &= \frac{x-y_i}{h}(v_{i+1}-a) + \frac{y_{i+1}-x}{h}(v_i - a) \\ &= (I_h^{\text{ply}} v)(x) - \frac{x - y_i + y_{i+1} - x}{h} a = (I_h^{\text{ply}} v)(x) - a,\end{aligned}$$

as required.

4. Let $\eta \in C_c^\infty([0,1])$. Then

$$\begin{aligned}\int_0^1 (I_h^{\text{pcy}} v)\, \partial_x \eta \, \mathrm{d}x &= \sum_{i=0}^{Z-1} \int_{x_i}^{x_{i+1}} v_i \partial_x \eta \, \mathrm{d}x \\ &= \sum_{i=0}^{Z-1} v_i(\eta(x_{i+1}) - \eta(x_i)) \\ &= \sum_{i=1}^{Z-1} \eta(x_i)(v_{i-1} - v_i) = -\int_0^1 \eta \left(\sum_{i=1}^{Z-1} (v_i - v_{i-1}) \delta_{x_i} \right) (\mathrm{d}x),\end{aligned}$$

as required.

5. The equation is satisfied on K_0 and K_Z, since

$$(I_h^{\text{pcx}} u)|_{K_0} = (I_h^{\text{pcx}} u)|_{K_0} \equiv 0,$$

while $I_h^{\text{ply}} v$ is constant on K_0 and K_Z. For all $x \in K_i$, $i = 1, \dots, Z-1$, we have

$$\partial_x (I_h^{\text{ply}} v)(x) = \frac{v_i - v_{i-1}}{h} = u_i = (I_h^{\text{pcx}} u)(x),$$

as required.

6. We directly compute

$$\begin{aligned}\left\| I_h^{\text{plx}} u \right\|_{H_0^1}^2 = \left\| \partial_x (I_h^{\text{plx}} u) \right\|_{L^2}^2 &= \left\| I_h^{\text{pcy}} \left(\left(\frac{u_{i+1} - u_i}{h} \right)_{i=0}^{Z-1} \right) \right\|_{L^2}^2 \\ &= \sum_{i=0}^{Z-1} \left(\frac{u_{i+1} - u_i}{h} \right)^2 h \\ &= \sum_{i=0}^{Z-1} \frac{1}{h} (u_{i+1}^2 - 2u_{i+1} u_i + u_i^2) \\ &= \sum_{i=1}^{Z-1} \frac{2}{h} (u_i - u_{i+1}) u_i \\ &= \sum_{i=1}^{Z-1} \frac{1}{h} (-u_{i+1} + 2u_i - u_{i-1}) u_i \\ &= \sum_{i=1}^{Z-1} (-h \Delta_h u)_i \, u_i = \langle -\Delta_h u, u \rangle_0 = \|u\|_1^2 .\end{aligned}$$

7. Using 4., we compute

$$\begin{aligned}-\partial_{xx}(I_h^{\text{plx}} u) &= -\partial_x \left(I_h^{\text{pcy}} \left(\frac{u_{i+1} - u_i}{h} \right)_{i=0}^{Z-1} \right) \\ &= -\sum_{i=1}^{Z-1} \delta_{x_i} \left(\frac{u_{i+1} - u_i}{h} - \frac{u_i - u_{i-1}}{h} \right) \\ &= \sum_{i=1}^{Z-1} \frac{1}{h} (-u_{i+1} + 2u_i - u_{i-1}) \delta_{x_i}.\end{aligned}$$

Since $\|\cdot\|_{H^{-1}}$ is absolutely homogeneous, this finishes the proof. $\square$

Lemma 2.4.8. *Let $u \in H^{-1}$ and $v \in L^2$ such that $\partial_x v = u$ in the sense of distributions. Then*

$$\|u\|_{H^{-1}} = \left\| v - \int_0^1 v \mathrm{d}x \right\|_{L^2}.$$

In particular,

$$\|u\|_{H^{-1}} \leq C \|v\|_{L^2}.$$

Proof. Let $w : [0,1] \to \mathbb{R}$ be defined by

$$w(x) = \int_0^x v(y)\mathrm{d}y - x \int_0^1 v(y)\mathrm{d}y.$$

Obviously $w \in H_0^1$. We show that $\Delta w = u$ in the sense of distributions. To this end, let $\eta \in C_c^\infty$ and compute

$${}_{H^{-1}}\langle u, \eta \rangle_{H_0^1} = -\langle v, \partial_x \eta \rangle_{L^2} = -\langle \partial_x w, \partial_x \eta \rangle_{L^2} + \left\langle \int_0^1 v(y)\mathrm{d}y, \partial_x \eta \right\rangle_{L^2} = \langle w, \Delta \eta \rangle_{L^2},$$

where for the last step, we note that

$$\left\langle \int_0^1 v(y)\mathrm{d}y, \partial_x \eta \right\rangle_{L^2} = \int_0^1 v(y)\mathrm{d}y \int_0^1 \partial_x \eta \mathrm{d}x = \int_0^1 v(y)\mathrm{d}y(\eta(1) - \eta(0)) = 0.$$

Thus, we conclude

$$\|u\|_{H^{-1}} = \|w\|_{H_0^1} = \|\partial_x w\|_{L^2} = \left\| v - \int_0^1 v(y)\mathrm{d}y \right\|_{L^2}.$$

For the last statement, we compute

$$\begin{aligned}
\|u\|_{H^{-1}} &\leq \|v\|_{L^2} + \left\| \int_0^1 v \,\mathrm{d}x \right\|_{L^2} \\
&\leq \|v\|_{L^2} + \left(\int_0^1 \left| \int_0^1 v(x) \,\mathrm{d}x \right|^2 \mathrm{d}y \right)^{\frac{1}{2}} \\
&\leq \|v\|_{L^2} + \left(\int_0^1 \int_0^1 v(x)^2 \,\mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}} = 2\|v\|_{L^2},
\end{aligned}$$

as required. ☐

Lemma 2.4.9. *Let $u = (u_i)_{i=1}^{Z-1} \in \mathbb{R}^{Z-1}$, where Z is defined as in (2.1.7). Then*

$$\|I_h^{\mathrm{pcx}} u\|_{H^{-1}} \leq \|u\|_{-1} \leq 3 \|I_h^{\mathrm{pcx}} u\|_{H^{-1}}.$$

Proof. Let $v = (v_i)_{i=0}^{Z-1} \in \mathbb{R}^Z$ be defined by

$$v_i = \sum_{j=0}^{i} h u_j.$$

Then, using the convention $\left(\Delta_h^{-1} u\right)_0 = \left(\Delta_h^{-1} u\right)_Z = 0$ and the prolongations $I_h^{\mathrm{ply}}, I_h^{\mathrm{pcy}}$ from Lemma

2.4.7, we have

$$
\begin{aligned}
\|I_h^{\mathrm{pcx}}\,u\|_{H^{-1}} &= \left\|I_h^{\mathrm{ply}}\,v - \int_0^1 I_h^{\mathrm{ply}}\,v\;\mathrm{d}x\right\|_{L^2} \qquad \text{(by Lemma 2.4.7, 5., and Lemma 2.4.8)}\\
&= \left\|I_h^{\mathrm{ply}}\left(v - \int_0^1 I_h^{\mathrm{ply}}\,v\;\mathrm{d}x\right)\right\|_{L^2} \qquad \text{(by Lemma 2.4.7, 3.)}\\
&\leq \left\|I_h^{\mathrm{pcy}}\left(v - \int_0^1 I_h^{\mathrm{pcy}}\,v\;\mathrm{d}x\right)\right\|_{L^2} \qquad \text{(by Lemma 2.4.7, 1. and 2.)}\\
&= \|\partial_x(I_h^{\mathrm{pcy}}\,v)\|_{H^{-1}} \qquad \text{(by Lemma 2.4.7, 3. and Lemma 2.4.8)}\\
&= \left\|\sum_{i=1}^{Z-1}\delta_{x_i} h u_i\right\|_{H^{-1}} \qquad \text{(by Lemma 2.4.7, 4.)}\\
&= \left\|\sum_{i=1}^{Z-1}\delta_{x_i} h(\Delta_h\Delta_h^{-1}u)_i\right\|_{H^{-1}}\\
&= \left\|-\sum_{i=1}^{Z-1}\delta_{x_i}\frac{1}{h}\left(-\left(\Delta_h^{-1}u\right)_{i-1} + 2\left(\Delta_h^{-1}u\right)_i - \left(\Delta_h^{-1}u\right)_{i+1}\right)\right\|_{H^{-1}}\\
&= \left\|\partial_{xx}(I_h^{\mathrm{plx}}\,\Delta_h^{-1}u)\right\|_{H^{-1}} \qquad \text{(by Lemma 2.4.7, 7.)}\\
&= \left\|I_h^{\mathrm{plx}}\,\Delta_h^{-1}u\right\|_{H_0^1}\\
&= \left\|\Delta_h^{-1}u\right\|_1 \qquad \text{(by Lemma 2.4.7, 6.)}\\
&= \|u\|_{-1}\,,
\end{aligned}
$$

which yields the first inequality. The same calculation yields the second inequality if we start with $3\,\|I_h^{\mathrm{pcx}}\,u\|_{H^{-1}}$ and replace the third step by

$$
3\left\|I_h^{\mathrm{ply}}\left(v - \int_0^1 I_h^{\mathrm{ply}}\,v\;\mathrm{d}x\right)\right\|_{L^2} \geq \left\|I_h^{\mathrm{pcy}}\left(v - \int_0^1 I_h^{\mathrm{pcy}}\,v\;\mathrm{d}x\right)\right\|_{L^2},
$$

using the second part of Lemma 2.4.7, 1. □

We now use this correspondence between the discrete norm $\|\cdot\|_{-1}$ and the continuous H^{-1} norm to obtain estimates on the spatially embedded processes.

Corollary 2.4.10. *Let N, Z be given as in* (2.1.7)*, and let $X_h : \Omega \to \mathbb{R}^{N\times(Z-1)}$ be random variables (e. g. constructed as in* (2.2.3)*). Then*

$$
\begin{aligned}
&\max\left\{\sup_{t\in[0,T]}\mathbb{E}\left\|X_h^{plt,pcx}(t) - X_h^{pct\text{-}pcx}(t)\right\|_{H^{-1}}^2,\ \sup_{t\in[0,T]}\mathbb{E}\left\|X_h^{plt,pcx}(t) - X_h^{pct+pcx}(t)\right\|_{H^{-1}}^2\right\}\\
&\leq \max_{n\in\{0,\dots,N-1\}}\mathbb{E}\left\|X_h^{n+1} - X_h^n\right\|_{-1}^2.
\end{aligned}
$$

Proof. We compute, using Lemma 2.4.9 in the first step and the piecewise affine shape of $X_{h\tau}^{pl}$ in the second step,

$$
\begin{aligned}
\sup_{t\in[0,T]}\mathbb{E}\left\|X_h^{\mathrm{plt,pcx}}(t) - X_h^{\mathrm{pct\text{-}pcx}}(t)\right\|_{H^{-1}}^2 &\leq \sup_{t\in[0,T]}\mathbb{E}\left\|X_h^{\mathrm{plt}}(t) - X_h^{\mathrm{pct\text{-}}}(t)\right\|_{-1}^2\\
&= \max_{n\in\{0,\dots,N-1\}}\sup_{t\in[n\tau,(n+1)\tau)}\mathbb{E}\left\|\frac{t-n\tau}{\tau}\left(X_h^{n+1} - X_h^n\right)\right\|_{-1}^2\\
&\leq \max_{n\in\{0,\dots,N-1\}}\mathbb{E}\left\|X_h^{n+1} - X_h^n\right\|_{-1}^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\sup_{t\in[0,T]} \mathbb{E}\left\|X_h^{\mathrm{plt,pcx}}(t)-X_h^{\mathrm{pct+pcx}}(t)\right\|_{H^{-1}}^2 &\le \sup_{t\in[0,T]} \mathbb{E}\left\|X_h^{\mathrm{plt}}(t)-X_h^{\mathrm{pct+}}(t)\right\|_{-1}^2 \\
&= \max_{n\in\{0,\dots,N-1\}} \sup_{t\in[n\tau,(n+1)\tau)} \mathbb{E}\left\|\frac{(n+1)\tau-t}{\tau}\left(X_h^{n+1}-X_h^n\right)\right\|_{-1}^2 \\
&\le \max_{n\in\{0,\dots,N-1\}} \mathbb{E}\left\|X_h^{n+1}-X_h^n\right\|_{-1}^2,
\end{aligned}
$$

as required. □

Corollary 2.4.11. *Let X_h be constructed as in (2.2.3). Then, Assumption 2.2.4, Lemma 2.4.4 and Corollary 2.4.10 immediately yield*

$$
\max\left\{\sup_{t\in[0,T]} \mathbb{E}\left\|X_h^{plt,pcx}(t)-X_h^{pct\text{-}pcx}(t)\right\|_{H^{-1}}^2,\ \sup_{t\in[0,T]} \mathbb{E}\left\|X_h^{plt,pcx}(t)-X_h^{pct+pcx}(t)\right\|_{H^{-1}}^2\right\}\to 0 \quad (2.4.18)
$$

for $h\to 0$. Moreover, since $L^\infty([0,T];L^2(\Omega;H^{-1}))$ is continuously embedded into $L^2(\Omega\times[0,T];H^{-1})$, (2.4.18) *also implies*

$$
\mathbb{E}\int_0^T \left\|X_h^{plt,pcx}(t)-X_h^{pct\text{-}pcx}(t)\right\|_{H^{-1}}^2 \mathrm{d}t\to 0 \quad \textit{and} \quad \mathbb{E}\int_0^T \left\|X_h^{plt,pcx}(t)-X_h^{pct+pcx}(t)\right\|_{H^{-1}}^2 \mathrm{d}t\to 0.
$$

Lemma 2.4.12. *Let $\tau,h>0, N,Z\in\mathbb{N}$ as in Assumption 2.2.4, with h small enough for $\frac{\tau}{h^2}\le\frac{1}{4}$ to be satisfied, and let X_h be constructed as in* (2.2.3)*. Then, there exists $C>0$ only depending on T (in particular, independent of h), such that*

$$
\max\left\{\mathbb{E}\int_0^T \left\|X_h^{plt,pcx}\right\|_{L^2}^2 \mathrm{d}t,\ \mathbb{E}\int_0^T \left\|X_h^{pct\text{-}pcx}\right\|_{L^2}^2 \mathrm{d}t,\ \mathbb{E}\int_0^T \left\|X_h^{pct+pcx}\right\|_{L^2}^2 \mathrm{d}t,\right\}\le C, \quad (2.4.19)
$$

$$
\mathbb{E}\int_0^T \left\|\tilde\phi(X_h^{pct\text{-}pcx})\right\|_{L^2}^2 \mathrm{d}t\le C, \quad (2.4.20)
$$

$$
\textit{and}\quad \mathbb{E}\operatorname*{ess\,sup}_{t\in[0,T]} \left\|X_h^{plt,pcx}\right\|_{H^{-1}}^2\le C. \quad (2.4.21)
$$

Proof. We compute for the first term of (2.4.19), using Definition 2.1.6, Definition 2.1.5, Lemma 2.4.6 and Lemma 2.4.2,

$$
\begin{aligned}
\mathbb{E}\int_0^T \left\|X_h^{\mathrm{plt,pcx}}\right\|_{L^2}^2 \mathrm{d}t &= \mathbb{E}\int_0^T \left\|\left(X_h^{\mathrm{plt}}(t)\right)^{\mathrm{pcx}}\right\|_{L^2}^2 \mathrm{d}t \\
&= \mathbb{E}\int_0^T h\sum_{l=1}^{Z-1}\left(X_h^{\mathrm{plt}}(t)\right)_l^2 \mathrm{d}t \\
&= h\sum_{l=1}^{Z-1}\mathbb{E}\int_0^T \left|\left(X_h^{\cdot,l}\right)^{\mathrm{plt}}(t)\right|^2 \mathrm{d}t \\
&\le Ch\sum_{l=1}^{Z-1}\mathbb{E}\sum_{k=0}^N \tau\left(X_h^{k,l}\right)^2 \\
&= C\mathbb{E}\sum_{k=0}^N\tau\sum_{l=0}^{Z-1}h\left(X_h^{k,l}\right)^2 = C\mathbb{E}\sum_{k=0}^N\tau\left\|X_h^k\right\|_0^2\le C.
\end{aligned} \quad (2.4.22)
$$

The second and third term in (2.4.19) can be treated analogously. For (2.4.20), we note that

$$
\tilde\phi(X_h^{\mathrm{pct\text{-}pcx}}) = \left(\tilde\phi(X_h)\right)^{\mathrm{pct\text{-}pcx}},
$$

where on the right hand side, $\tilde\phi$ is applied component-wise. Then, an analogous computation to (2.4.22) applies. For (2.4.21), we first compute for arbitrary grid functions $(u_{k,l})_{k=1,\dots,N;l=1,\dots,Z-1}\subset\mathbb{R}$, using

Definition 2.1.6 and Lemma 2.4.9,

$$\begin{aligned}
\sup_{t\in[0,T]} \left\|u^{\mathrm{plt,pcx}}(t)\right\|_{H^{-1}} &= \sup_{t\in[0,T]} \left\|\left(u^{\mathrm{plt}}(t)\right)^{\mathrm{pcx}}\right\|_{H^{-1}} \\
&\le \sup_{t\in[0,T]} \left\|u^{\mathrm{plt}}(t)\right\|_{-1} \\
&= \sup_{t\in[0,T]} \left\|\frac{t-t_\tau}{\tau} u_{\lfloor t/\tau\rfloor+1,\cdot} + \frac{t_\tau+\tau-t}{\tau} u_{\lfloor t/\tau\rfloor,\cdot}\right\|_{-1} \\
&\le \sup_{t\in[0,T]} \left(\frac{t-t_\tau}{\tau} + \frac{t_\tau+\tau-t}{\tau}\right) \max_{n=0,\dots,N} \left\|u_{n,\cdot}\right\|_{-1} = \max_{n=0,\dots,N} \left\|u_{n,\cdot}\right\|_{-1}.
\end{aligned}$$

Hence, by Lemma 2.4.5, we obtain

$$\mathbb{E} \operatorname*{ess\,sup}_{t\in[0,T]} \left\|X_h^{\mathrm{plt,pcx}}\right\|_{H^{-1}}^2 \le \mathbb{E} \max_{n=0,\dots,N} \left\|X_h^n\right\|_{-1}^2 \le C,$$

as required. ☐

Definition 2.4.13. Let $(X_h^n)_{n=0}^N$ and $(\xi_h^n)_{n=0}^N$ be defined as in (2.2.3). We then define random variables $Y_h, W_h : \Omega \to \mathbb{R}^{(Z-1)(N+1)}$ by

$$Y_h = (\tilde{\phi}(X_h^n))_{n=0}^N \quad \text{and} \quad W_h = \left(\sum_{k=0}^{n-1} \sqrt{\frac{\tau}{h}} \xi_h^k\right)_{n=0}^{N+1}.$$

Furthermore, we define $F_h : \Omega \to \mathcal{C}([0,T]\times[0,1])$ to be the spatial antiderivative of $W_h^{\mathrm{plt,pcx}}$, i. e.

$$F_h(t,x) = \int_0^x W_h^{\mathrm{plt,pcx}}(t,x')\,\mathrm{d}x'. \tag{2.4.23}$$

Remark 2.4.14. Note that F_h is continuous in time by the continuity of the piecewise linear prolongation, and absolutely continuous in space, since $W_h^{\mathrm{plt,pcx}}(t,\cdot)$ is Lebesgue integrable at any time $t\in[0,T]$.

We proceed by showing that F_h converges in law to a Brownian sheet on $[0,T]\times[0,1]$ (for a Definition, see [88, p. 1]). To this end, we use the following formalism from [51, Definition 1.3], which we adapt to the special case we are going to use.

Definition 2.4.15. Let $h>0$, $N, Z \in \mathbb{N}$ be as in Assumption 2.2.4 and let $(\xi_h^{n,l})_{n=0,\dots,N;l=1,\dots,Z-1}$ be as in (2.2.3). For $n\in\{1,\dots,N\}, l\in\{1,\dots,Z-1\}$, we define rectangles

$$R_h^{n,l} = \left[\frac{n-1}{N}, \frac{n}{N}\right] \times \left[\frac{l-1}{Z-1}, \frac{l}{Z-1}\right].$$

We now define for $A \in \mathcal{B}([0,1]^2)$

$$F_h'(A) := \sum_{n=1}^{N} \sum_{l=1}^{Z-1} \left|R_h^{n,l}\right|^{-\frac{1}{2}} \left|R_h^{n,l}\cap A\right| \xi_h^{n-1,l}, \tag{2.4.24}$$

where $|\cdot|$ denotes the Lebesgue measure. Furthermore, we define a process $F_h^1 : \Omega\times[0,1]^2 \to \mathbb{R}$ by

$$F_h^1(t,x) = F_h'([0,t]\times[0,x]) \quad \text{for } (t,x)\in[0,1]^2.$$

Lemma 2.4.16. *Let F_h^1 be defined as in Definition 2.4.15. Then, F_h^1 is continuous $\mathbb{P}$-almost surely.*

Proof. We note that $\mathbb{P}$-almost surely,

$$M := \max\left\{\xi_h^{k,l} : k\in\{0,\dots,N-1\}, l\in\{1,\dots,Z-1\}\right\} < \infty,$$

and we have $\left|R_h^{n,l}\right| = (N(Z-1))^{-1}$ independent of k and l. Hence, for $(s,x),(t,y) \in [0,1]^2$, we have $\mathbb{P}$-almost surely

$$\begin{aligned}\left|F_h^1(s,x) - F_h^1(t,y)\right| &\le \left|F_h^1(s,x) - F_h^1(t,x)\right| + \left|F_h^1(t,x) - F_h^1(t,y)\right| \\ &\le M(N(Z-1))^{\frac{1}{2}} \sum_{n=1}^{N} \sum_{l=1}^{Z-1} \left|R_h^{n,l} \cap ([\min(s,t), \max(s,t)] \times [0,x])\right| \\ &\quad + M(N(Z-1))^{\frac{1}{2}} \sum_{n=1}^{N} \sum_{l=1}^{Z-1} \left|R_h^{n,l} \cap ([0,t] \times [\min(x,y), \max(x,y)])\right| \\ &\le M(N(Z-1))^{\frac{3}{2}} (|s-t| + |x-y|) \to 0\end{aligned}$$

for $(s,x) \to (t,y)$, as required. □

The following proposition corresponds to [51, Theorem 7.5 and Theorem 7.6].

Proposition 2.4.17. *For $h > 0$ as in Assumption 2.2.4, let F_h' be given as in Definition 2.4.15, and let $p > 2$. Then, there is a constant $K_p > 0$, such that for all admissible $h > 0$, $A, B \in \mathcal{B}([0,1]^2)$ we have*

$$\mathbb{E}\left|F_h'(A) - F_h'(B)\right|^p \le K_p\, \mathbb{E}\left|\xi_h^{1,1}\right|^p |A \Delta B|^{\frac{p}{2}},$$

where $A\Delta B$ denotes the symmetric difference of sets. Furthermore, F_h' converges in distribution to the Brownian sheet on $[0,1]^2$ as defined by [51, Definition 1.2] for $h \to 0$.

Corollary 2.4.18. *For $h > 0$ as in Assumption 2.2.4, let F_h^1 be given as in Definition 2.4.15, and let $p > 2$. Then, there is a constant $K_p > 0$, such that for all admissible $h > 0$, $(s,x),(t,y) \in [0,1]^2$, we have*

$$\mathbb{E}\left|F_h^1(s,x) - F_h^1(t,y)\right|^p \le K_p\, \mathbb{E}\left|\xi_h^{1,1}\right|^p \left(|s-t|^{\frac{p}{2}} + |x-y|^{\frac{p}{2}}\right). \tag{2.4.25}$$

Furthermore, for $h \to 0$, the finite-dimensional distributions of F_h^1 converge weakly to those of the Brownian sheet F^1 on $[0,1]^2$ as defined by [88]. Finally, $F_h^1 \to F^1$ in distribution with respect to $\mathcal{C}([0,1]^2)$ for $h \to 0$.

Proof. Inequality (2.4.25) and the convergence of finite-dimensional distributions immediately follow from Proposition 2.4.17 when setting $A = [0,s] \times [0,x]$ and $B = [0,t] \times [0,y]$ and observing that

$$|A \Delta B| \le |t-s| + |y-x|.$$

Since (2.4.25) together with $F_h^1(0,0) = 0$ $\mathbb{P}$-almost surely implies tightness of $(F_h^1)_{h>0}$ with respect to the strong topology on $\mathcal{C}([0,1]^2)$ by [93, Theorem 1.4.7], the convergence in distribution follows. □

We next relate the processes $(F_h^1)_{h>0}$ to $(F_h)_{h>0}$.

Lemma 2.4.19. *Let $h > 0$ as in Assumption 2.2.4. We define the affine transformation Q_h by*

$$Q_h : [0,1] \to [0,1]_h := \left[-\frac{h}{2-2h}, \frac{2-h}{2-2h}\right], \quad Q_h(x) = \frac{x - \frac{h}{2}}{1-h}.$$

Let $\bar{F}_h^1$ be the continuous extension of F_h^1 to $[0,1] \times [0,1]_h$ such that $\bar{F}_h^1$ is constant in space direction on $[0,1] \times ([0,1]_h \setminus (0,1))$. Then, for $(t,x) \in [0,T] \times [0,1]$, we have

$$F_h(t,x) = \sqrt{T}\sqrt{1-h}\,\bar{F}_h^1\left(\frac{t}{T}, Q_h(x)\right)$$

for all $h > 0$.

Proof. We first show the statement for

$$(t,x) = \left(n\tau, \left(l + \frac{1}{2}\right)h\right), \tag{2.4.26}$$

where $k \in \{0, \ldots, N\}$, $l \in \{0, \ldots, Z-1\}$. By Definition 2.4.13, we have

$$F_h(t,x) = \sum_{j=1}^{l} h\Big(W_h^{\cdot,j}\Big)^{\text{plt}}(t) = h \sum_{j=1}^{l} W_h^{n,j} = \sqrt{\tau} h \sum_{j=1}^{l}\sum_{k=0}^{n-1} \xi_h^{k,j}.$$

On the other hand, we have

$$R_h^{k,j} \cap \left(\left[0, \frac{n}{N}\right] \times \left[0, \frac{l}{Z-1}\right]\right) = R_h^{k,j}$$

if $k \leq n$ and $j \leq l$, otherwise the intersection is a Lebesgue zero set. Noting furthermore that $\left|R_h^{k,j}\right| = (N(Z-1))^{-1}$, we have

$$F_h^1\left(\frac{n}{N}, \frac{l}{Z-1}\right) = (N(Z-1))^{-\frac{1}{2}} \sum_{k=1}^{n}\sum_{j=1}^{l} \xi_h^{k-1,j} = T^{-\frac{1}{2}}\sqrt{\tau}\sqrt{\frac{Z}{Z-1}}\sqrt{h}\sum_{k=0}^{n-1}\sum_{j=1}^{l} \xi_h^{k,j},$$

where we recall that $Zh = 1$ and $N\tau = T$. Noticing that $Z(Z-1)^{-1} = (1-h)^{-1}$ and

$$Q_h\left(\left(l + \frac{1}{2}\right)h\right) = \frac{l}{Z-1} \quad \text{for } l \in \{0, \ldots, Z-1\}$$

yields the claim for grid points as set in (2.4.26). By definition, F_h is piecewise affine in space and time direction between those grid points. The same applies to $\sqrt{T(1-h)}\,\bar{F}_h^1(T^{-1}\cdot, Q_h(\cdot))$, since Q_h is linear and the image of $(T^{-1}\cdot, Q_h\cdot)$ of a rectangle formed by the grid points in (2.4.26) is one of the rectangles $R_h^{n,l}$. Hence, it suffices to show that F_h^1 is piecewise affine in each variable on each of those rectangles. In the time variable, this can be seen by writing for $(s,x),(t,x) \in R_h^{n,l}$ such that $s < t$, $l = \lfloor x/(Z-1)\rfloor + 1$, $x = \frac{l-1}{Z-1} + x'$,

$$F_h^1(t,x) - F_h^1(s,x) = \sqrt{N(Z-1)}(t-s)\left(\frac{1}{Z-1}\sum_{j=1}^{l-1}\xi_h^{n-1,j} + x'\xi_h^{n-1,l}\right).$$

An analogous calculation yields the claim in the space variable. Finally, both functions are piecewise constant in space direction for $x \leq \frac{h}{2}$ or $x \geq 1 - \frac{h}{2}$. Hence, both functions are uniquely defined by their values on the grid points in (2.4.26), which concludes the proof. □

Lemma 2.4.20. *The family $(F_h)_{h>0}$ from Definition 2.4.13 satisfies the conditions for [93, Theorem 1.4.7] with $\gamma = 6, d = 2$ and $\alpha_1 = \alpha_2 = 3$.*

Proof. Note that (2.4.25) is still true when replacing F_h^1 by $\bar{F}_h^1$, since for $t \in [0,1], x \in [0,1]_h \setminus [0,1]$, either $F_h^1(s,0)$ or $F_h^1(s,1)$ equals $\bar{F}_h^1(s,x)$ and can replace it yielding an even stronger inequality. Hence, we compute for $(s,x),(t,y) \in [0,T] \times [0,1]$

$$\begin{aligned}
\mathbb{E}\left(F_h(s,x) - F_h(t,y)\right)^6 &= T^3(1-h)^3\mathbb{E}\left(\bar{F}_h^1\left(\frac{s}{T}, Q_h(x)\right) - \bar{F}_h^1\left(\frac{t}{T}, Q_h(y)\right)\right) \\
&\leq T^3(1-h)^3 K_6\left(\left|\frac{s}{T} - \frac{t}{T}\right|^3 + |Q_h x - Q_h y|^3\right) \\
&\leq K_6\left(|s-t|^3 + T^3|x-y|^3\right).
\end{aligned}$$

Since $F_h(0,0) = 0$ $\mathbb{P}$-almost surely, this completes the proof. □

Corollary 2.4.21. *The family of the laws of $(F_h)_{h>0}$ is tight with respect to the strong topology in $\mathcal{C}([0,T] \times [0,1])$.*

Proof. Tightness with respect to the semi-weak topology on $\mathcal{C}([0,T] \times [0,1])$ follows from Lemma 2.4.20 and [93, Theorem 1.4.7]. Since Property (a) in [93, Theorem 1.4.6] is equivalent to Property (a') in [93, p. 38] in the case $S = \mathbb{R}$, relatively compact sets in the semi-weak topology are also relatively compact in the strong topology on $\mathcal{C}([0,T] \times [0,1])$. This implies tightness with respect to the strong topology. □

Lemma 2.4.22. *The finite-dimensional distributions of $(F_h)_{h>0}$ from Definition 2.4.13 converge weakly to those of a Brownian sheet F on $[0,T]\times[0,1]$ in the sense of [88, p. 1].*

Proof. In view of Corollary 2.4.18, we first note that the finite-dimensional distributions of $\sqrt{T}\,F_h^1\left(\frac{\cdot}{T},\cdot\right)$ converge weakly to the finite-dimensional distributions of $\sqrt{T}\,F^1\left(\frac{\cdot}{T},\cdot\right)$, where F^1 is a Brownian sheet on $[0,1]^2$. Moreover, we note that

$$\left(\sqrt{T}\,F^1\left(\frac{t}{T},x\right)\right)_{(t,x)\in[0,T]\times[0,1]}$$

is a centred Gaussian process, and that for $(s,x),(t,y)\in[0,T]\times[0,1]$, we have

$$\mathbb{E}\left[\sqrt{T}\,F^1\left(\frac{s}{T},x\right)\sqrt{T}\,F^1\left(\frac{t}{T},y\right)\right]=T\max\left\{\frac{s}{T},\frac{t}{T}\right\}\max\{x,y\}=\max\{s,t\}\max\{x,y\},$$

which implies that $\sqrt{T}\,F^1\left(\frac{\cdot}{T},\cdot\right)$ is a Brownian sheet on $[0,T]\times[0,1]$.

Furthermore, note that $\sqrt{1-h}\to 1$ stochastically for $h\to 0$, which implies that also

$$\sqrt{T}\,\sqrt{1-h}\,F_h^1\left(\frac{\cdot}{T},\cdot\right)\to\sqrt{T}\,F^1\left(\frac{\cdot}{T},\cdot\right)$$

in distribution for $h\to 0$ by [22, Theorem 4.4].

In order to apply Slutsky's theorem, we claim that the finite-dimensional distributions of

$$F_h-\sqrt{T}\,\sqrt{1-h}\,F_h^1\left(\frac{\cdot}{T},\cdot\right)$$

converge to zero stochastically for $h\to 0$, which we argue in the following. Since all norms on $\mathbb{R}^d$ are equivalent, we are free to choose the ℓ_1 norm. Using Lemma 2.4.19, we write for a finite family $((t_i,x_i))_{i=1}^M\subset[0,T]\times[0,1]$ and $\delta>0$

$$\begin{aligned}&\mathbb{P}\left(\sum_{i=1}^M\left|F_h(t_i,x_i)-\sqrt{T}\,\sqrt{1-h}\,F_h^1\left(\frac{t_i}{T},x_i\right)\right|>\delta\right)\\&\le\sum_{i=1}^M\mathbb{P}\left(\left|F_h(t_i,x_i)-F_h(t_i,Q_h^{-1}x_i)\right|>\frac{\delta}{M}\right),\end{aligned}\tag{2.4.27}$$

such that it is enough to show for $\delta>0$ and $(t,x)\in[0,T]\times[0,1]$

$$\mathbb{P}\left(\left|F_h(t,x)-F_h(t,Q_h^{-1}x)\right|>\delta\right)\to 0\quad\text{for } h\to 0.$$

To this end, we note that $\left|Q_h^{-1}x-x\right|<\frac{h}{2}$ for $x\in[0,1]$ and $Q_h^{-1}x\ge x$ if and only if $x\le\frac{1}{2}$. We restrict to this case since the case $x>\frac{1}{2}$ can be carried out analogously. Due to the construction of F_h, we then observe that

$$\left|F_h(t,x)-F_h(t,Q_h^{-1}x)\right|\le\frac{h}{2}\left(\left|\left(W_h^{\cdot,\lfloor x/h\rceil}\right)^{\text{plt}}(t)\right|+\left|\left(W_h^{\cdot,\lfloor x/h\rceil+1}\right)^{\text{plt}}(t)\right|\right),$$

such that, by the same argument as in (2.4.27), it is enough to show

$$\mathbb{P}\left(\frac{h}{2}\left|\left(W_h^{\cdot,\lfloor x/h\rceil}\right)^{\text{plt}}(t)\right|>\delta\right)\to 0\tag{2.4.28}$$

for $h\to 0$, where we omit the statement and argument for the second summand since it can be conducted analogously. Writing $t_\tau=\tau\lfloor t/\tau\rfloor$, we note that

$$\left(W_h^{\cdot,\lfloor x/h\rceil}\right)^{\text{plt}}(t)=\sum_{k=0}^{\lfloor t/\tau\rfloor-1}\sqrt{\frac{\tau}{h}}\xi_h^{k,\lfloor x/h\rceil}+\frac{t-t_\tau}{\tau}\sqrt{\frac{\tau}{h}}\xi_h^{\lfloor t/\tau\rfloor,\lfloor x/h\rceil},$$

such that we have by the space-time independence of the random input that

$$\operatorname{Var}\left(\frac{h}{2}\left(W_h^{\cdot,\lfloor x/h\rceil}\right)^{\text{plt}}(t)\right)=\frac{\tau h}{4}\left(\left\lfloor\frac{t}{\tau}\right\rfloor+\frac{(t-t_\tau)^2}{\tau^2}\right)\le\frac{Th}{4}.$$

Since moreover $\mathbb{E}\xi_h^{k,l} = 0$ for all $k \in \{0, \dots, N\}$, $l \in \{1, \dots, Z-1\}$, applying Chebyshev's inequality to (2.4.28) yields

$$\mathbb{P}\left(\frac{h}{2}\left|\left(W_h^{\cdot,\lfloor x/h \rceil}\right)^{\mathrm{plt}}(t)\right| > \delta\right) \le \frac{Th}{4\delta^2} \to 0$$

for $h \to 0$. This proves the previously claimed stochastic convergence. The proof is finished by applying Slutsky's theorem (cf. [90, Theorem 13.18]). □

Corollary 2.4.23. *The family $(F_h)_{h>0}$ converges in law to F, which is a Brownian sheet on $[0,T] \times [0,1]$.*

Proof. This is a consequence of Corollary 2.4.21 and Corollary 2.4.22. □

Lemma 2.4.24. *Let $(X_h)_{h>0}$ and $(Y_h)_{h>0}$ be defined as in (2.2.3) and Definition 2.4.13, respectively.*

1. *The family of the laws of $(X_h^{plt,pcx})_{h>0}$ is tight with respect to the weak* topology in $L^\infty([0,T]; H^{-1})$.*

2. *The families of the laws of $(X_h^{plt,pcx})_{h>0}, (X_h^{pct\text{-}pcx})_{h>0}, (X_h^{pct+pcx})_{h>0}$ and $(Y_h^{pct\text{-}pcx})_{h>0}$ are tight with respect to the weak topology in $L^2([0,T]; L^2)$, where $Y_h = \tilde{\phi}(X_h)$ as before.*

Proof. 1. By Lemma 2.4.12, we obtain

$$\mathbb{E} \operatorname*{ess\,sup}_{t\in[0,T]} \left\| X_h^{\mathrm{plt,pcx}}(t) \right\|_{H^{-1}} \le \left(\mathbb{E} \sup_{t\in[0,T]} \left\| X_h(t)^{\mathrm{plt,pcx}} \right\|_{H^{-1}}^2 \right)^{\frac{1}{2}} \le C \tag{2.4.29}$$

independently of h. Hence, by the Markov inequality, we have for $R \ge 1$

$$\mathbb{P}\left(\|X_h^{\mathrm{pcx}}\|_{L^\infty([0,T];H^{-1})} \ge R\right) \le \frac{C}{R},$$

which converges to 0 for $R \to \infty$ uniformly in h. Since bounded sets are compact in the weak* topology by the Banach-Alaoglu theorem, this yields the claim.

2. By Lemma 2.4.12, there exists $C > 0$ only depending on T, such that

$$\mathbb{E}\int_0^T \left\| X_h^{\mathrm{plt,pcx}} \right\|_{L^2}^2 \mathrm{d}t \le C$$

for all $h > 0$ as in Assumption 2.2.4. Using the Markov inequality, we obtain

$$\mathbb{P}\left(\left\| X_h^{\mathrm{plt,pcx}} \right\|_{L^2([0,T];L^2)} > R\right) \le \frac{C}{R},$$

which converges to 0 for $R \to \infty$ uniformly in h. Since $L^2([0,T]; L^2)$ is a Hilbert space and hence reflexive, we obtain that closed balls are weakly sequentially compact. This implies compactness with respect to the weak topology by the Eberlein-Smulian theorem. The remaining processes can be treated analogously. □

Lemma 2.4.25. *Let $(X_h)_{h>0}$, $(Y_h)_{h>0}$ and $(F_h)_{h>0}$ be defined as in (2.2.3) and Definition 2.4.13, respectively. Then, the family of the distributions of the tuples*

$$\left((X_h^{plt,pcx}, X_h^{plt,pcx}, X_h^{pct\text{-}pcx}, X_h^{pct+pcx}, Y_h^{pct\text{-}pcx}, F_h)\right)_{h>0}$$

is tight with respect to the product topology τ of $(\tau_w^*, \tau_w, \tau_w, \tau_w, \tau_w, \tau_C)$, where

$$\begin{aligned} &\tau_w^* \text{ is the weak* topology in } L^\infty([0,T]; H^{-1}), \\ &\tau_w \text{ is the weak topology in } L^2([0,T]; L^2), \text{ and} \\ &\tau_C \text{ is the strong topology in } \mathcal{C}([0,T] \times [0,1]; \mathbb{R}). \end{aligned} \tag{2.4.30}$$

Proof. Let $0 < \varepsilon < 1$. By Corollary 2.4.21 and Lemma 2.4.24, we obtain compact sets K_1, K_2, K_3, K_4, K_5 such that for all $h > 0$

$$\begin{aligned}
&K_0 \text{ is compact with respect to } \tau_w^* \quad \text{and} \quad \mathbb{P}(X_h^{\text{pcx}} \in K_1) \geq 1 - \frac{\varepsilon}{6}\\
&K_1 \text{ is compact with respect to } \tau_w \quad \text{and} \quad \mathbb{P}(X_h^{\text{plt,pcx}} \in K_1) \geq 1 - \frac{\varepsilon}{6}\\
&K_2 \text{ is compact with respect to } \tau_w \quad \text{and} \quad \mathbb{P}(X_h^{\text{pct-pcx}} \in K_2) \geq 1 - \frac{\varepsilon}{6}\\
&K_3 \text{ is compact with respect to } \tau_w \quad \text{and} \quad \mathbb{P}(X_h^{\text{pct+pcx}} \in K_3) \geq 1 - \frac{\varepsilon}{6}\\
&K_4 \text{ is compact with respect to } \tau_w \quad \text{and} \quad \mathbb{P}(Y_h^{\text{pct-pcx}} \in K_4) \geq 1 - \frac{\varepsilon}{6}\\
&K_5 \text{ is compact with respect to } \tau_C \quad \text{and} \quad \mathbb{P}(F_h \in K_5) \geq 1 - \frac{\varepsilon}{6}.
\end{aligned}$$

Then, $K := K_0 \times K_1 \times K_2 \times K_3 \times K_4 \times K_5$ is compact with respect to τ, and we have

$$\begin{aligned}
&\mathbb{P}\Big((X_h^{\text{plt,pcx}}, X_h^{\text{pct-pcx}}, X_h^{\text{pct+pcx}}, Y_h^{\text{pct-pcx}}, F_h) \in K\Big)\\
&= 1 - \mathbb{P}\Bigg(\{X_h^{\text{plt,pcx}} \notin K_0\} \cup \{X_h^{\text{plt,pcx}} \notin K_1\} \cup \{X_h^{\text{pct-pcx}} \notin K_2\}\\
&\qquad \cup \{X_h^{\text{pct+pcx}} \notin K_3\} \cup \{Y_h^{\text{pct-pcx}} \notin K_4\} \cup \{F_h \notin K_5\}\Bigg)\\
&\geq 1 - \mathbb{P}(X_h^{\text{plt,pcx}} \notin K_0) - \mathbb{P}(X_h^{\text{plt,pcx}} \notin K_1) - \mathbb{P}(X_h^{\text{pct-pcx}} \notin K_2)\\
&\qquad - \mathbb{P}(X_h^{\text{pct+pcx}} \notin K_3) - \mathbb{P}(Y_h^{\text{pct-pcx}} \notin K_4) - \mathbb{P}(F_h \notin K_5)\\
&\geq 1 - \varepsilon,
\end{aligned}$$

as required. □

Lemma 2.4.26. *Let $(X_h)_{h>0}$, $(Y_h)_{h>0}$ and $(W_h)_{h>0}$ be defined as in (2.2.3) and Definition 2.4.13, respectively. Then, there is a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, stochastic processes*

$$\begin{aligned}
&\tilde{X} \in L^2(\tilde{\Omega}; L^\infty([0,T]; H^{-1})) \cap L^2(\tilde{\Omega}; L^2([0,T]; L^2),\\
&\tilde{Y} \in L^2(\tilde{\Omega}; L^2([0,T]; L^2)),\\
&\tilde{W} \in L^2(\tilde{\Omega}; \mathcal{C}([0,T]; H^{-1})),
\end{aligned}$$

where $\tilde{W}$ is an $I'(I')^*$-Wiener process on H^{-1}, a nonrelabeled subsequence $h \to 0$ such that for each h in this subsequence, there are random variables $\tilde{X}_h, \tilde{Y}_h, \tilde{W}_h : \tilde{\Omega} \to \mathbb{R}^{(N+1)(Z-1)}$ satisfying the following properties. The processes

$$\tilde{X}_h^{plt,pcx} \in L^2(\tilde{\Omega}; L^\infty([0,T]; H^{-1})) \tag{2.4.31}$$

$$\tilde{X}_h^{plt,pcx}, \tilde{X}_h^{pct\text{-}pcx}, \tilde{X}_h^{pct+pcx}, \tilde{Y}_h^{pct\text{-}pcx} \in L^2(\tilde{\Omega}; L^2([0,T]; L^2), \tag{2.4.32}$$

$$\text{and} \quad \tilde{W}_h^{plt,pcx} \in L^2(\tilde{\Omega}; \mathcal{C}([0,T]; H^{-1})) \tag{2.4.33}$$

are bounded in the respective space uniformly in h. For each h in this subsequence,

$$\mathcal{L}\left((\tilde{X}_h, \tilde{Y}_h, \tilde{W}_h)\right) = \mathcal{L}\left((X_h, Y_h, W_h)\right) \tag{2.4.34}$$

and

$$\begin{aligned}
&\mathcal{L}\left((\tilde{X}_h^{plt,pcx}, \tilde{X}_h^{plt,pcx}, \tilde{X}_h^{pct\text{-}pcx}, \tilde{X}_h^{pct+pcx}, \tilde{Y}_h^{pct\text{-}pcx}, \tilde{W}_h^{plt,pcx})\right)\\
&= \mathcal{L}\left((X_h^{plt,pcx}, X_h^{plt,pcx}, X_h^{pct\text{-}pcx}, X_h^{pct+pcx}, Y_h^{pct\text{-}pcx}, W_h^{plt,pcx})\right)
\end{aligned} \tag{2.4.35}$$

with respect to the product topology of $(\tau_w^*, \tau_w, \tau_w, \tau_w, \tau_w, \tilde{\tau}_C)$, where τ_w^* and τ_w are defined in (2.4.30) and $\tilde{\tau}_C$ denotes the strong topology on $\mathcal{C}([0,T]; H^{-1})$. Finally, $\tilde{\mathbb{P}}$-almost surely, we have for $h \to 0$

$$\begin{aligned}
&\tilde{X}_h^{plt,pcx} \overset{*}{\rightharpoonup} \tilde{X} && \text{in } L^\infty([0,T]; H^{-1}),\\
&\tilde{X}_h^{plt,pcx} \rightharpoonup \tilde{X}, \tilde{X}_h^{pct\text{-}pcx} \rightharpoonup \tilde{X}, \tilde{X}_h^{pct+pcx} \rightharpoonup \tilde{X}, \tilde{Y}_h^{pct\text{-}pcx} \rightharpoonup \tilde{Y} && \text{in } L^2([0,T]; L^2),\\
\text{and} \quad &\tilde{W}_h^{plt,pcx} \to \tilde{W} && \text{in } \mathcal{C}([0,T]; H^{-1}).
\end{aligned}$$

Remark 2.4.27. Expected values with respect to $\tilde{\mathbb{P}}$ will be denoted by $\tilde{\mathbb{E}}$.

Proof of Lemma 2.4.26. Note that condition (2.C.1), which is the central requirement in the article [87], is satisfied for each of the topological spaces

$$\left(L^\infty([0,T];H^{-1}),\tau_w^*\right),\ \left(L^2([0,T];L^2),\tau_w\right),\ \text{and}\ \left(\mathcal{C}([0,T]\times[0,1]),\tau_C\right)$$

separately by Lemma 2.C.1. Hence,

$$\bigcup_{i=1}^{6}\{f\circ\Pi_i : f\in T_i\}$$

where Π_i is the canonical projection and T_i the separating class for the respective factor space, serves as a countable family satisfying condition (2.C.1) on the product space. Let $(F_h)_{h>0}$ be defined as in Definition 2.4.13. Lemma 2.4.25 and the generalized Skorohod-type theorem [87, Theorem 2] then yield for $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})=([0,1],\mathcal{B}([0,1]),\mathrm{d}x)$ the existence of a subsequence $h\to 0$ and random variables

$$\tilde{\tilde{X}}_h^{\text{plt,pcx}},\tilde{X}_0:\tilde{\Omega}\to L^\infty([0,T];H^{-1}), \tag{2.4.36}$$
$$\tilde{X}_h^{\text{plt,pcx}},\tilde{X}_1,\tilde{X}_h^{\text{pct-pcx}},\tilde{X}_2,\tilde{X}_h^{\text{pct+pcx}},\tilde{X}_3,\tilde{Y}_h^{\text{pct-pcx}},\tilde{Y}:\tilde{\Omega}\to L^2([0,T];L^2), \tag{2.4.37}$$
$$\tilde{F}_h,\tilde{F}:\tilde{\Omega}\to\mathcal{C}([0,T]\times[0,1]), \tag{2.4.38}$$

such that

$$\begin{aligned}&\mathcal{L}\left((\tilde{\tilde{X}}_h^{\text{plt,pcx}},\tilde{X}_h^{\text{plt,pcx}},\tilde{X}_h^{\text{pct-pcx}},\tilde{X}_h^{\text{pct+pcx}},\tilde{Y}_h^{\text{pct-pcx}},\tilde{F}_h)\right)\\&=\mathcal{L}\left((X_h^{\text{plt,pcx}},X_h^{\text{plt,pcx}},X_h^{\text{pct-pcx}},X_h^{\text{pct+pcx}},Y_h^{\text{pct-pcx}},F_h)\right)\end{aligned} \tag{2.4.39}$$

with respect to the topology τ from Lemma 2.4.25, and $\tilde{\mathbb{P}}$-almost surely

$$\tilde{\tilde{X}}_h^{\text{plt,pcx}}\overset{*}{\rightharpoonup}\tilde{X}_0 \qquad \text{in } L^\infty([0,T];H^{-1}), \tag{2.4.40}$$
$$\tilde{X}_h^{\text{plt,pcx}}\rightharpoonup\tilde{X}_1,\tilde{X}_h^{\text{pct-pcx}}\rightharpoonup\tilde{X}_2,\tilde{X}_h^{\text{pct+pcx}}\rightharpoonup\tilde{X}_3,\tilde{Y}_h^{\text{pct-pcx}}\rightharpoonup\tilde{Y} \qquad \text{in } L^2([0,T];L^2), \tag{2.4.41}$$
$$\text{and}\quad \tilde{F}_h\to\tilde{F} \qquad \text{in } \mathcal{C}([0,T]\times[0,1]), \tag{2.4.42}$$

where we recall that convergence in the product topology is equivalent to component-wise convergence, that convergence in the weak topology on a normed space is equivalent to weak convergence (cf. [61, Proposition A.49]) and that convergence in the weak* topology on the dual of a normed space is equivalent to weak* convergence (cf. [61, Proposition A.51]). Note that despite the suggestive notation, we have not yet shown at this stage that the approximating processes actually arise as the prolongations of corresponding $\mathbb{R}^{(N+1)(Z-1)}$-valued processes. This will be done at the end of the proof.

We next show that the newly defined processes have uniformly in h bounded second moments. Using Lemma 2.A.3 and Lemma 2.4.12, we obtain

$$\tilde{\mathbb{E}}\operatorname*{ess\,sup}_{t\in[0,T]}\left\|\tilde{\tilde{X}}_h^{\text{plt,pcx}}\right\|_{H^{-1}}^2=\mathbb{E}\operatorname*{ess\,sup}_{t\in[0,T]}\left\|X_h^{\text{plt,pcx}}\right\|_{H^{-1}}^2\le C$$

independent of h, and further, with (2.4.40), Fatou's lemma and the weak* lower-semicontinuity of the norm,

$$\begin{aligned}\tilde{\mathbb{E}}\operatorname*{ess\,sup}_{t\in[0,T]}\left\|\tilde{X}_0\right\|_{H^{-1}}^2&\le\tilde{\mathbb{E}}\liminf_{h\to0}\operatorname*{ess\,sup}_{t\in[0,T]}\left\|\tilde{\tilde{X}}_h^{\text{plt,pcx}}\right\|_{H^{-1}}^2\\&\le\liminf_{h\to0}\mathbb{E}\operatorname*{ess\,sup}_{t\in[0,T]}\left\|\tilde{\tilde{X}}_h^{\text{plt,pcx}}\right\|_{H^{-1}}^2\le C.\end{aligned}$$

Using Lemma 2.A.2 and Lemma 2.4.12, we obtain

$$\tilde{\mathbb{E}}\int_0^T\left\|\tilde{X}_h^{\text{plt,pcx}}\right\|_{L^2}^2\mathrm{d}t=\mathbb{E}\int_0^T\left\|X_h^{\text{plt,pcx}}\right\|_{L^2}^2\mathrm{d}t\le C$$

and further, with (2.4.41), Fatou's lemma and the weak lower-semicontinuity of the norm,

$$\begin{aligned}\tilde{\mathbb{E}}\int_0^T \left\|\tilde{X}\right\|_{L^2}^2 \mathrm{d}t &\leq \tilde{\mathbb{E}}\liminf_{h\to 0}\int_0^T \left\|\tilde{X}_h^{\mathrm{plt,pcx}}\right\|_{L^2}^2 \mathrm{d}t \\ &\leq \liminf_{h\to 0}\mathbb{E}\int_0^T \left\|\tilde{X}_h^{\mathrm{plt,pcx}}\right\|_{L^2}^2 \mathrm{d}t \leq C.\end{aligned}$$

The other processes in (2.4.37) can be treated analogously. To bound the moments of $\tilde{F}_h$, we note that $\mathbb{E}\sup_{t\in[0,T],x\in[0,1]}|F_h|^6 \leq C$ by [93, Theorem 1.4.1], where C is independent of h. For the latter statement, note that the moment $\mathbb{E}K^6$ of the random variable K used in inequality (2) in [93, Theorem 1.4.1] only depends on the parameters in [93, Theorem 1.4.1], which becomes clear from the proof of [93, Lema 1.4.3]. By the continuity of the norm in $\mathcal{C}([0,T]\times[0,1])$, this carries over to $\mathbb{E}\sup_{t\in[0,T],x\in[0,1]}|\tilde{F}_h|^6$. Again using Fatou's lemma and continuity of the norm, one shows that also $\tilde{F}$ has a finite second (even sixth) moment.

Next, we identify the processes X_0, X_1, X_2 and X_3. To this end, we first use Lemma 2.B.6 together with (2.4.41) and the previously shown moment bounds to obtain that the limits in (2.4.41) are also true as weak limits in $L^2(\tilde{\Omega};L^2([0,T];L^2))$ and thus as weak limits in $L^2(\tilde{\Omega};L^2([0,T];H^{-1}))$. Furthermore, we note that equality in $L^2(\tilde{\Omega};L^2([0,T];H^{-1}))$ implies equality in $L^2(\tilde{\Omega};L^2([0,T];L^2))$, which means that it is sufficient to show that the limits coincide in $L^2(\Omega;L^2([0,T];H^{-1}))$. We claim that Corollary 2.4.11 carries over to the processes $\tilde{X}_h^{\mathrm{plt,pcx}}, \tilde{X}_h^{\text{pct-pcx}}, \tilde{X}_h^{\text{pct+pcx}}$ for the subsequence chosen above, which we argue as follows. By (2.4.39), it is enough to show that

$$(x,y)\mapsto \int_0^T \|x-y\|_{H^{-1}}^2$$

is measurable with respect to the Borel σ-algebras of the weak topologies on $\left((L^2([0,T];L^2)\right)^2$ and $\left(L^2([0,T];H^{-1})\right)^2$. By continuity, this is clear if considering strong topologies, which immediately yields the claimed measurability by Lemma 2.A.2. Hence, with $\langle\cdot,\cdot\rangle_\sim$ denoting the inner product in $L^2(\tilde{\Omega};L^2([0,T];H^{-1}))$ and $\|\cdot\|_\sim$ the corresponding norm, we may compute for $Z\in L^2(\tilde{\Omega};L^2([0,T];H^{-1}))$

$$\begin{aligned}&\left|\left\langle \tilde{X}_h^{\mathrm{plt,pcx}} - \tilde{X}_2, Z(t)\right\rangle_\sim\right| \\ &\leq \left|\left\langle \tilde{X}_h^{\mathrm{plt,pcx}} - \tilde{X}_h^{\text{pct-pcx}}, Z\right\rangle_\sim\right| + \left|\left\langle \tilde{X}_h^{\text{pct-pcx}} - \tilde{X}_2, Z\right\rangle_\sim\right| \\ &\leq \left\|\tilde{X}_h^{\mathrm{plt,pcx}} - \tilde{X}_h^{\text{pct-pcx}}\right\|_\sim \|Z\|_\sim + \left|\left\langle \tilde{X}_h^{\text{pct-pcx}} - \tilde{X}_2, Z\right\rangle_\sim\right| \\ &\to 0\end{aligned} \tag{2.4.43}$$

for $h\to 0$, i. e. $\tilde{X}_1 = \tilde{X}_2$ by the uniqueness of weak limits. The identification with $\tilde{X}_3$ works analogously. Very similarly, we obtain that $\tilde{\tilde{X}}_h^{\mathrm{plt,pcx}} \rightharpoonup \tilde{X}_0$ in $L^2(\tilde{\Omega};L^2([0,T];H^{-1}))$ and $\tilde{\tilde{X}}_h^{\mathrm{plt,pcx}} = \tilde{X}_h^{\mathrm{plt,pcx}}$ $\tilde{\mathbb{P}}$-almost surely in $L^2([0,T];H^{-1})$ for all h in the subsequence chosen above, which especially implies

$$\tilde{X}_h^{\mathrm{plt,pcx}} \in L^2(\tilde{\Omega};L^2([0,T];L^2))\cap L^2(\tilde{\Omega};L^\infty([0,T];H^{-1})).$$

Passing to the limit $h\to 0$ yields

$$\tilde{X}_0 = \tilde{X}_1 \in L^2(\tilde{\Omega};L^2([0,T];L^2))\cap L^2(\tilde{\Omega};L^\infty([0,T];H^{-1})).$$

Hence, we may define

$$\tilde{X}_1 = \tilde{X}_2 = \tilde{X}_3 =: \tilde{X}. \tag{2.4.44}$$

Next, we pass from the auxiliary processes $\tilde{F}_h$ and $\tilde{F}$ to their spatial distributional derivative. Note that $\tilde{F}_h$ and $\tilde{F}$ can be continuously embedded into $\mathcal{C}([0,T];L^2)$, consequently (2.4.42) implies $\tilde{F}_h \to \tilde{F}$ in $\mathcal{C}([0,T];L^2)$. Defining

$$\tilde{W}_h^{\mathrm{plt,pcx}} := \partial_x\tilde{F}_h \quad\text{and}\quad \tilde{W} := \partial_x\tilde{F}, \tag{2.4.45}$$

we obtain that $(\tilde{W}_h^{\mathrm{plt,pcx}})_{h>0}$ and $\tilde{W}$ are uniformly bounded in $L^2(\tilde{\Omega};\mathcal{C}([0,T];H^{-1}))$ and $\tilde{W}_h^{\mathrm{plt,pcx}} \to \tilde{W}$ in $\mathcal{C}([0,T];H^{-1})$ $\tilde{\mathbb{P}}$-almost surely for $h\to 0$ in the subsequence provided above, using the fact that

$\partial_x : \mathcal{C}([0,T];L^2) \to \mathcal{C}([0,T];H^{-1})$ is a linear and bounded operator. We conclude that (2.4.35) is satisfied, which follows from (2.4.39), (2.4.44) and the continuity of the map

$$(\mathrm{Id},\mathrm{Id},\mathrm{Id},\mathrm{Id},\mathrm{Id},\partial_x).$$

Next, we show that $\tilde{W}$ is an $I'(I')^*$-Wiener process by checking the requirements in [112, Definition 2.1.9]. Using that $\tilde{F}$ is a Brownian sheet on $[0,T]\times[0,1]$, which follows from Corollary 2.4.23 and (2.4.39), we first note that $\tilde{F}(0,\cdot)=0$ $\tilde{\mathbb{P}}$-almost surely by definition. It follows that for $\eta \in H_0^1$, we have

$$\left\langle \tilde{W}(0,\cdot),\eta\right\rangle_{H^{-1}\times H_0^1} = -\left\langle \tilde{F}(0,\cdot),\partial_x\eta\right\rangle_{L^2} = 0$$

$\tilde{\mathbb{P}}$-almost surely. Furthermore we note that $\tilde{W}$ is $\tilde{\mathbb{P}}$-almost surely continuous in time by construction. For the independence of the increments, we note that the $\mathcal{C}([0,T]\times[0,1])$ norm is stronger than the $\mathcal{C}([0,T];L^2)$ norm, which implies

$$\mathcal{B}(\mathcal{C}([0,T];L^2)) \cap \mathcal{C}([0,T]\times[0,1]) \subseteq \mathcal{B}(\mathcal{C}([0,T]\times[0,1])).$$

Furthermore, we recall that for $n\in\mathbb{N}$, $t_1,\dots,t_n\in[0,T]$, $t_1<\cdots<t_n$

$$\tilde{F}(t_n,\cdot)-\tilde{F}(t_{n-1},\cdot),\dots,\tilde{F}(t_2,\cdot)-\tilde{F}(t_1,\cdot),\tilde{F}(t_1,\cdot)$$

are independent. Since $\partial_x : L^2 \to H^{-1}$ is continuous (see Lemma 2.4.8) and thus

$$A\in\mathcal{C}([0,T];H^{-1}) \quad \text{implies} \quad \partial_x^{-1}A \in \mathcal{C}([0,T];L^2),$$

we have for $A_1,\dots A_n \in \mathcal{C}([0,T];H^{-1})$

$$\begin{aligned}
&\tilde{\mathbb{P}}(\tilde{W}(t_n)-\tilde{W}(t_{n-1})\in A_n,\dots\tilde{W}(t_2)-\tilde{W}(t_1)\in A_2,\tilde{W}(t_1)\in A_1)\\
&=\tilde{\mathbb{P}}(\tilde{F}(t_n,\cdot)-\tilde{F}(t_{n-1},\cdot)\in\partial_x^{-1}(A_n),\dots,\tilde{F}(t_2,\cdot)-\tilde{F}(t_1,\cdot)\in\partial_x^{-1}(A_2),\tilde{F}(t_1,\cdot)\in\partial_x^{-1}(A_1))\\
&=\tilde{\mathbb{P}}(\tilde{F}(t_n,\cdot)-\tilde{F}(t_{n-1},\cdot)\in\partial_x^{-1}(A_n))\cdots\tilde{\mathbb{P}}(\tilde{F}(t_2,\cdot)-\tilde{F}(t_1,\cdot)\in\partial_x^{-1}(A_2))\,\tilde{\mathbb{P}}(\tilde{F}(t_1,\cdot)\in\partial_x^{-1}(A_1))\\
&=\tilde{\mathbb{P}}(\tilde{W}(t_n)-\tilde{W}(t_{n-1})\in A_n)\cdots\tilde{\mathbb{P}}(\tilde{W}(t_2)-\tilde{W}(t_1)\in A_2)\,\tilde{\mathbb{P}}(\tilde{W}(t_1)\in A_1).
\end{aligned}$$

Finally, we verify the distribution of $\tilde{W}(t)-\tilde{W}(s)$ $(0\le s\le t\le T)$. Comuputing for $\eta\in L^2$, $r\in[0,T]$

$$\left|\left\langle \tilde{F}(r,\cdot),\eta\right\rangle_{L^2}\right| = \left|\int_0^1 \tilde{F}(r,\cdot)\eta\mathrm{d}x\right| \le \sup_{x\in[0,1]}\left|\tilde{F}(r,x)\right| \|\eta\|_{L^1} \le C \sup_{(r',x)\in[0,T]\times[0,1]}\left|\tilde{F}(r',x)\right| \|\eta\|_{L^2},$$

we may conclude that $\tilde{W}(r)$ is Gaussian for all $r\in[0,T]$, since for $\eta\in H_0^1$, we have

$$\left\langle \tilde{W}(r,\cdot),\eta\right\rangle_{H^{-1}\times H_0^1} = -\left\langle \tilde{F}(r,\cdot),\partial_x\eta\right\rangle_{L^2}.$$

Hence, also $\tilde{W}(t)-\tilde{W}(s)$ is Gausssian. In order to establish the parameters of the distribution, we compute for $\eta,\xi\in H_0^1$

$$\tilde{\mathbb{E}}\left\langle \tilde{W}(t)-\tilde{W}(s),\eta\right\rangle_{H^{-1}\times H_0^1} = -\tilde{\mathbb{E}}\left\langle \tilde{F}(t,\cdot)-\tilde{F}(s,\cdot),\partial_x\eta\right\rangle_{L^2} = 0,$$

and

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\left\langle \tilde{W}(t)-\tilde{W}(s),\eta\right\rangle_{H^{-1}\times H_0^1}\left\langle \tilde{W}(t)-\tilde{W}(s),\xi\right\rangle_{H^{-1}\times H_0^1}\right]\\
&=\tilde{\mathbb{E}}\left[\left\langle \tilde{F}(t,\cdot)-\tilde{F}(s,\cdot),\partial_x\eta\right\rangle_{L^2}\left\langle \tilde{F}(t,\cdot)-\tilde{F}(s,\cdot),\partial_x\xi\right\rangle_{L^2}\right]\\
&=\tilde{\mathbb{E}}\left[\int_0^1(\tilde{F}(t,x)-\tilde{F}(s,x))\,\partial_x\eta(x)\,\mathrm{d}x\int_0^1(\tilde{F}(t,y)-\tilde{F}(s,y))\,\partial_x\xi(y)\,\mathrm{d}y\right]\\
&=\int_0^1\int_0^1\tilde{\mathbb{E}}\left[(\tilde{F}(t,x)-\tilde{F}(s,x))(\tilde{F}(t,y)-\tilde{F}(s,y))\right]\partial_x\eta(x)\,\partial_x\xi(y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{2.4.46}$$

where Fubini's theorem applies since $\tilde{F}$ has a finite second moment as shown above. By the definition of the Brownian sheet, we notice that

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[(\tilde{F}(t,x)-\tilde{F}(s,x))(\tilde{F}(t,y)-\tilde{F}(s,y))\right]\\
&=\tilde{\mathbb{E}}\left[\tilde{F}(t,x)\tilde{F}(t,y)\right]-\tilde{\mathbb{E}}\left[\tilde{F}(t,x)\tilde{F}(s,y)\right]-\tilde{\mathbb{E}}\left[\tilde{F}(s,x)\tilde{F}(t,y)\right]+\tilde{\mathbb{E}}\left[\tilde{F}(s,x)\tilde{F}(s,y)\right]\\
&=t\,(x\wedge y)-2s(x\wedge y)+s\,(x\wedge y)\\
&=(t-s)(x\wedge y).
\end{aligned}\tag{2.4.47}$$

Moreover, we compute

$$\begin{aligned}
\int_0^1\int_0^1 (x\wedge y)\,\partial_x\eta(x)\,\partial_x\xi(y)\,\mathrm{d}x\,\mathrm{d}y &= -\int_0^1 \partial_x\eta(x)\int_0^1 \xi(y)\mathbf{1}_{[0,x]}(y)\,\mathrm{d}y\,\mathrm{d}x\\
&= -\int_0^1 \xi(y)\int_0^1 \partial_x\eta(x)\mathbf{1}_{[y,1]}(x)\,\mathrm{d}x\,\mathrm{d}y\\
&= -\int_0^1 \xi(y)(\eta(1)-\eta(y))\mathrm{d}y = \langle\eta,\xi\rangle_{L^2}\,,
\end{aligned}\tag{2.4.48}$$

and further

$$\langle\eta,\xi\rangle_{L^2} = -\left\langle\Delta\eta, I'\xi\right\rangle_{H^{-1}} = -\left\langle (I')^*\Delta\eta,\xi\right\rangle_{L^2} = \left\langle I'(I')^*\Delta\eta,\Delta\xi\right\rangle_{H^{-1}}.\tag{2.4.49}$$

Combining (2.4.46) – (2.4.49), we obtain for $u,v\in H^{-1}$

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\left\langle \tilde{W}(t)-\tilde{W}(s),u\right\rangle_{H^{-1}}\left\langle \tilde{W}(t)-\tilde{W}(s),v\right\rangle_{H^{-1}}\right]\\
&=\tilde{\mathbb{E}}\left[\left\langle \tilde{W}(t)-\tilde{W}(s),\Delta^{-1}u\right\rangle_{H^{-1}\times H_0^1}\left\langle \tilde{W}(t)-\tilde{W}(s),\Delta^{-1}v\right\rangle_{H^{-1}\times H_0^1}\right]\\
&=(t-s)\left\langle I'(I')^*u,v\right\rangle_{H^{-1}}\,,
\end{aligned}$$

as required.

It remains to show that the processes in (2.4.31) – (2.4.33) are actually images of $\mathbb{R}^{(N+1)(Z-1)}$-valued random variables $(\tilde{X}_h,\tilde{Y}_h,\tilde{W}_h)$ under the respective prolongations. To this end, for $k\in\{0,\ldots,N\}$ and $l\in\{1,\ldots,Z-1\}$, let

$$e_h^{k,l} = (\delta_{k'k}\delta_{l'l})_{k'=0,\ldots,N;l'=1,\ldots,Z-1}\,.$$

Then, we define linear subspaces $S_h^{\text{pct-pcx}}, S_h^{\text{pct+pcx}}$ of $L^2([0,T];L^2)$, and $S_h^{\text{plt,pcx}}$, which can be interpreted as a subspace of both $L^2([0,T];L^2)$ and $\mathcal{C}([0,T];H^{-1})$, by

$$\begin{aligned}
S_h^{\text{plt,pcx}} &= \operatorname{span}\left\{\left(e_h^{k,l}\right)^{\text{plt,pcx}} : k\in\{0,\ldots,N\}, l\in\{1,\ldots,Z-1\right\},\\
S_h^{\text{pct-pcx}} &= \operatorname{span}\left\{\left(e_h^{k,l}\right)^{\text{pct-pcx}} : k\in\{0,\ldots,N\}, l\in\{1,\ldots,Z-1\right\},
\end{aligned}$$

noting that both generating systems are linearly independent. Since these subspaces are finite-dimensional, there exist continuous projections

$$\begin{aligned}
\Pi_h^{\text{plt,pcx}} &: L^2([0,T];L^2)\to S_h^{\text{plt,pcx}},\\
\Xi_h^{\text{plt,pcx}} &: \mathcal{C}([0,T];H^{-1})\to S_h^{\text{plt,pcx}},\\
\Pi_h^{\text{pct-pcx}} &: L^2([0,T];L^2)\to S_h^{\text{pct-pcx}},
\end{aligned}$$

(see e. g. [1, 7.2(1), 7.15]), and bounded linear coordinate functions

$$\begin{aligned}
\Gamma_h^{\text{plt,pcx}} &: S_h^{\text{plt,pcx}}\to\mathbb{R}^{(N+1)(Z-1)},\\
\Gamma_h^{\text{pct-pcx}} &: S_h^{\text{pct-pcx}}\to\mathbb{R}^{(N+1)(Z-1)}.
\end{aligned}$$

It follows from the definition of the projection and the injectivity of all prolongations involved that for $u\in S_h^{\text{plt,pcx}}$ we have

$$\left(\left(\Gamma_h^{\text{plt,pcx}}\circ\Pi_h^{\text{plt,pcx}}\right)(u)\right)^{\text{plt,pcx}} = u,\tag{2.4.50}$$

with analogous statements for $\Gamma_h^{\text{plt,pcx}} \circ \Xi_h^{\text{plt,pcx}}$ and $\Gamma_h^{\text{pct-pcx}} \circ \Pi_h^{\text{pct-pcx}}$. We now set

$$\begin{aligned}
\tilde{X}_h &= \Gamma_h^{\text{plt,pcx}}\left(\Pi_h^{\text{plt,pcx}}\left(\tilde{X}_h^{\text{plt,pcx}}\right)\right),\\
\tilde{Y}_h &= \Gamma_h^{\text{pct+pcx}}\left(\Pi_h^{\text{pct-pcx}}\left(\tilde{Y}_h^{\text{pct-pcx}}\right)\right),\\
\tilde{W}_h &= \Gamma_h^{\text{plt,pcx}}\left(\Xi_h^{\text{plt,pcx}}\left(\tilde{W}_h^{\text{plt,pcx}}\right)\right),
\end{aligned}$$

which is compatible with the previously defined processes in (2.4.36) – (2.4.38) due to (2.4.50) and thus does not cause notational ambiguities. The equality of the laws in (2.4.34) follows by (2.4.35) and the measurability of

$$\left(\Gamma_h^{\text{plt,pcx}} \circ \Pi_h^{\text{plt,pcx}}, \Gamma_h^{\text{pct-pcx}} \circ \Pi_h^{\text{pct-pcx}}, \Gamma_h^{\text{plt,pcx}} \circ \Xi_h^{\text{plt,pcx}}\right),$$

which completes the proof. □

We now turn to identify the limiting processes belong to a weak solution, starting by providing a stochastic basis and proving that $\tilde{W}$ is a Wiener process with respect to this basis in the sense of [112, Definition 2.1.12].

Lemma 2.4.28. *In the setting of Lemma 2.4.26, $\tilde{W}$ is an $I'(I')^*$-Wiener process on H^{-1} with respect to the augmented filtration $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$ of $(\tilde{\mathcal{F}}'_t)_{t\in[0,T]}$ in the sense of [112, Defintion 2.1.12], where*

$$\tilde{\mathcal{F}}'_t := \sigma\left(\tilde{X}|_{\tilde{\Omega}\times[0,t]}, \tilde{Y}|_{\tilde{\Omega}\times[0,t]}, \tilde{W}|_{\tilde{\Omega}\times[0,t]}\right).$$

Proof. It has already been shown in Lemma 2.4.26 that $\tilde{W}$ is an $I'(I')^*$-Wiener process. Furthermore, $\tilde{W}$ is adapted to $(\tilde{\mathcal{F}}'_t)_{t\in[0,T]}$ by construction and hence also to its augmentation. Thus, it remains to show that for $0 \le s < t \le T$

$$\tilde{W}(t) - \tilde{W}(s) \text{ is independent of } \tilde{\mathcal{F}}_s. \tag{2.4.51}$$

We begin with proving (2.4.51) for $\tilde{\mathcal{F}}_s$ replaced by $\tilde{\mathcal{F}}'_s$. To this end, let

$$\Phi \in \mathcal{C}\left(L^2([0,s];L^2) \times L^2([0,s];L^2) \times \mathcal{C}([0,s];H^{-1});[0,1]\right) \quad \text{and} \quad \Psi \in \mathcal{C}\left(H^{-1};[0,1]\right),$$

where $L^2([0,s];L^2)$ is endowed with the weak topology and $\mathcal{C}([0,s];H^{-1})$ with the strong topology. We note that for $0 < s \le T$ the operator $P_s : f \mapsto f|_{[0,s]}$ is continuous both as a map

$$L^2([0,T];L^2) \to L^2([0,s];L^2)$$

with respect to the weak topologies and as a map

$$\mathcal{C}([0,T];H^{-1}) \to \mathcal{C}([0,s];H^{-1})$$

with respect to the strong topologies. Let $(s_n)_{n\in\mathbb{N}} \subset (s,T]$ such that $s_n \searrow s$ for $n \to \infty$. Using the continuity of $\tilde{W}$, Lemma 2.4.26, dominated convergence due to the boundedness and continuity of Φ and Ψ, and independence on the discrete level by construction, we obtain

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\Psi\left(\tilde{W}(t) - \tilde{W}(s)\right)\Phi\left(\tilde{X}|_{[0,s]}, \tilde{Y}_{[0,s]}, \tilde{W}|_{[0,s]}\right)\right]\\
&= \lim_{n\to\infty} \tilde{\mathbb{E}}\left[\Psi\left(\tilde{W}(t) - \tilde{W}(s_n)\right)\Phi\left(\tilde{X}|_{[0,s]}, \tilde{Y}_{[0,s]}, \tilde{W}|_{[0,s]}\right)\right]\\
&= \lim_{n\to\infty}\lim_{h\to 0} \tilde{\mathbb{E}}\left[\Psi\left((\tilde{W}_h^{\text{plt,pcx}}(t) - \tilde{W}_h^{\text{plt,pcx}}(s_n))\right)\Phi\left(\tilde{X}_h^{\text{plt,pcx}}|_{[0,s]}, \tilde{Y}_h^{\text{pct-pcx}}|_{[0,s]}, \tilde{W}_h^{\text{plt,pcx}}|_{[0,s]}\right)\right]\\
&= \lim_{n\to\infty}\lim_{h\to 0} \mathbb{E}\left[\Psi\left((W_h^{\text{plt,pcx}}(t) - W_h^{\text{plt,pcx}}(s_n))\right)\Phi\left(X_h^{\text{plt,pcx}}|_{[0,s]}, Y_h^{\text{pct-pcx}}|_{[0,s]}, W_h^{\text{plt,pcx}}|_{[0,s]}\right)\right]\\
&= \lim_{n\to\infty}\lim_{h\to 0}\left(\mathbb{E}\left[\Psi\left((W_h^{\text{plt,pcx}}(t) - W_h^{\text{plt,pcx}}(s_n))\right)\right]\right.\\
&\qquad\qquad \left.\times\,\mathbb{E}\left[\Phi\left(X_h^{\text{plt,pcx}}|_{[0,s]}, Y_h^{\text{pct-pcx}}|_{[0,s]}, W_h^{\text{plt,pcx}}|_{[0,s]}\right)\right]\right)\\
&= \lim_{n\to\infty}\lim_{h\to 0}\left(\tilde{\mathbb{E}}\left[\Psi\left((\tilde{W}_h^{\text{plt,pcx}}(t) - \tilde{W}_h^{\text{plt,pcx}}(s_n))\right)\right]\right.\\
&\qquad\qquad \left.\times\,\tilde{\mathbb{E}}\left[\Phi\left(\tilde{X}_h^{\text{plt,pcx}}|_{[0,s]}, \tilde{Y}_h^{\text{pct-pcx}}|_{[0,s]}, \tilde{W}_h^{\text{plt,pcx}}|_{[0,s]}\right)\right]\right)\\
&= \lim_{n\to\infty}\left(\tilde{\mathbb{E}}\left[\Psi\left(\tilde{W}(t) - \tilde{W}(s_n)\right)\right]\tilde{\mathbb{E}}\left[\Phi\left(\tilde{X}|_{[0,s]}, \tilde{Y}|_{[0,s]}), \tilde{W}|_{[0,s]}\right)\right]\right)\\
&= \tilde{\mathbb{E}}\left[\Psi\left(\tilde{W}(t) - \tilde{W}(s)\right)\right]\tilde{\mathbb{E}}\left[\Phi\left(\tilde{X}|_{[0,s]}, \tilde{Y}|_{[0,s]}, \tilde{W}|_{[0,s]}\right)\right].
\end{aligned}$$

This proves that $\tilde{W}(t) - \tilde{W}(s)$ is independent of $\tilde{\mathcal{F}}'_s$.

Next, we note that the family

$$\mathcal{A} := \{B \cup N : B \in \tilde{\mathcal{F}}'_s, N \in \mathcal{N}\},$$

where we recall that $\mathcal{N}$ is the collection of all $\tilde{\mathbb{P}}$-zero sets, is stable under intersections, since for $B_1, B_2 \in \tilde{\mathcal{F}}'_s$, $N_1, N_2 \in \mathcal{N}$ we have

$$(B_1 \cup N_1) \cap (B_2 \cup N_2) = (B_1 \cap B_2) \cup (B_1 \cap N_2) \cup (B_2 \cap N_1) \cup (N_1 \cap N_2),$$

where $B_1 \cap B_2 \in \tilde{\mathcal{F}}'_s$ and the last three sets are $\tilde{\mathbb{P}}$-zero sets. Using the independence of $\tilde{W}(t) - \tilde{W}(s)$ of $\tilde{\mathcal{F}}'_s$, we then compute for $A \in \sigma(\tilde{W}(t) - \tilde{W}(s))$, $B \in \tilde{\mathcal{F}}'_s$ and $N \in \mathcal{N}$

$$\tilde{\mathbb{P}}(A \cap (B \cup N)) = \tilde{\mathbb{P}}(A \cap B) + \tilde{\mathbb{P}}(A \cap (N \cap B^c)) = \tilde{\mathbb{P}}(A)\mathbb{P}(B) = \tilde{\mathbb{P}}(A)\mathbb{P}(B \cup N),$$

which proves that $\tilde{W}(t) - \tilde{W}(s)$ is independent of

$$\tilde{\mathcal{F}}^0_t := \sigma(\tilde{\mathcal{F}}'_t \cup \mathcal{N})$$

by [90, Theorem 2.13]. Finally, if $B \in \tilde{\mathcal{F}}_s$, then, by construction of the augmentation, $B \in \tilde{\mathcal{F}}^0_{s_n}$ for all $n \in \mathbb{N}$, where $(s_n)_{n\in\mathbb{N}} \subset (s, T]$ is an arbitrary subsequence for which $s_n \to s$ for $n \to \infty$. Hence, by almost sure continuity of $\tilde{W}$ and dominated convergence, we obtain

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\Psi\left(\tilde{W}(t) - \tilde{W}(s)\right)\mathbf{1}_B\right] &= \lim_{n\to\infty} \tilde{\mathbb{E}}\left[\Psi\left(\tilde{W}(t) - \tilde{W}(s_n)\right)\mathbf{1}_B\right] \\
&= \lim_{n\to\infty} \tilde{\mathbb{E}}\left[\Psi\left(\tilde{W}(t) - \tilde{W}(s_n)\right)\right]\tilde{\mathbb{P}}(B) \\
&= \tilde{\mathbb{E}}\left[\Psi\left(\tilde{W}(t) - \tilde{W}(s)\right)\right]\tilde{\mathbb{P}}(B),
\end{aligned}$$

which proves (2.4.51), as required. □

Proposition 2.4.29. *Let $h > 0$ denote a sequence converging to 0, $u \in L^2([0,T]; H^{-1})$, $\eta \in L^2([0,T]; L^2)$, and for all h in this sequence, $t \in [0,T]$ let $u_h(t), \eta_h(t) \in \mathbb{R}^{Z-1}$, such that $u_h^{pcx} \in L^2([0,T]; H^{-1})$ with $u_h^{pcx} \rightharpoonup u$ in $L^2([0,T]; H^{-1})$, and $\eta_h^{pcx} \in L^2([0,T]; L^2)$ with $\eta_h^{pcx} \to \eta$ in $L^2([0,T]; L^2)$. Then, for $h \to 0$,*

$$\int_0^T \langle \eta_h(t), u_h(t)\rangle_{-1}\,\mathrm{d}t \to \int_0^T \langle \eta(t), u(t)\rangle_{H^{-1}}\,\mathrm{d}t.$$

Proof. First note that $u_h^{\mathrm{pcx}} \rightharpoonup u$ in $L^2([0,T]; H^{-1})$ implies that $(u_h^{\mathrm{pcx}})_{h>0}$ is bounded in $L^2([0,T]; H^{-1})$ (see e. g. [1, Bemerkungen 6.3, (5)]). Thus, $\int_0^T \|u_h(t)\|_{-1}^2$ is uniformly bounded by Lemma 2.4.9. Furthermore, we have a Poincaré inequality for the discrete norms by

$$\|v_h\|_0^2 = \|v_h^{\mathrm{pcx}}\|_{L^2}^2 \le C\left\|v_h^{\mathrm{plx}}\right\|_{L^2}^2 \le C\left\|\nabla v_h^{\mathrm{plx}}\right\|_{L^2}^2 = C\|v_h\|_1^2 \tag{2.4.52}$$

for C independent of h and grid functions $v \in \mathbb{R}^{Z-1}$, where the first inequality can be obtained by connecting [52, Propositions 3.1 and 3.2], and the last equality is the statement in Lemma 2.4.7, 6. Recalling that $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^{Z-1}$, this leads to

$$\begin{aligned}
\int_0^T \left\|\left(-\Delta_h^{-1}u_h(t)\right)^{\mathrm{pcx}}\right\|_{L^2}^2 \mathrm{d}t &= \int_0^T \left\|-\Delta_h^{-1}u_h(t)\right\|_0^2 \mathrm{d}t \\
&\le C\int_0^T \left\|-\Delta_h^{-1}u_h(t)\right\|_1^2 \mathrm{d}t \\
&= C\int_0^T \left\langle -\Delta_h^{-1}u_h(t), u_h(t)\right\rangle_0 \mathrm{d}t = C\int_0^T \|u_h(t)\|_{-1}^2\,\mathrm{d}t,
\end{aligned}$$

which is uniformly bounded in h as argued above, such that we can extract a subsequence of $\left(-\Delta_h^{-1}u_h\right)^{\mathrm{pcx}}$ weakly converging to some $f \in L^2([0,T]; L^2)$.

For $\eta \in L^\infty([0,1])$, let $D_h^-\eta, D_h^+\eta \in L^\infty([0,1])$ be the h-difference quotients to the left and right, i. e.

$$D_h^-\eta(x) = \frac{\eta_{\mathrm{ext}}(x-h) - \eta(x)}{h} \quad \text{and} \quad D_h^+\eta(x) = \frac{\eta_{\mathrm{ext}}(x+h) - \eta(x)}{h},$$

where η_{ext} is the extension of η by zero to a function on $\mathbb{R}$.

Let $\xi \in \mathcal{C}_c^\infty([0,T]\times[0,1])$. Then, for h chosen small enough such that

$$\operatorname{supp}(\xi) \subseteq [0,T]\times(3h, 1-3h),$$

we have

$$\xi(t,x)(-\Delta_h u_h(t))^{\text{pcx}}(x) = -\xi(t,x)\left(D_h^- D_h^+ u_h^{\text{pcx}}(t)\right)(x) \quad \text{for almost all } t\in[0,T], x\in[0,1]. \tag{2.4.53}$$

Hence, using discrete integration by parts and considering that ξ has compact support, we compute

$$\begin{aligned}\int_0^T \langle u,\xi\rangle_{H^{-1}\times H_0^1}\,\mathrm{d}t &= \lim_{h\to 0}\int_0^T \langle \bar{u}_h^{\text{pcx}},\xi\rangle_{L^2}\,\mathrm{d}t\\ &= \lim_{h\to 0}\int_0^T \left\langle \left(\Delta_h\Delta_h^{-1}u_h\right)^{\text{pcx}},\ \xi\right\rangle_{L^2}\mathrm{d}t\\ &= \lim_{h\to 0}\int_0^T \left\langle -D_h^- D_h^+\left(-\Delta_h^{-1}u_h\right)^{\text{pcx}},\ \xi\right\rangle_{L^2}\mathrm{d}t\\ &= \lim_{h\to 0}\int_0^T \left\langle \left(-\Delta_h^{-1}u_h\right)^{\text{pcx}}, -D_h^- D_h^+\xi\right\rangle_{L^2}\mathrm{d}t = \int_0^T \langle f,-\Delta\xi\rangle_{L^2}\,\mathrm{d}t.\end{aligned} \tag{2.4.54}$$

To justify the last step, we note that ξ is smooth, hence

$$\sup_{(t,x)\in\mathbb{R}^2}|\partial_{xxx}\xi_{\text{ext}}(t,x)| = \sup_{(t,x)\in[0,T]\times[0,1]}|\partial_{xxx}\xi(t,x)| \le C_\xi,$$

which allows to use Taylor's formula and the Lagrange form of the remainder to obtain $x_1, x_2 \in [-h, 1+h]$, such that for all $(t,x)\in[0,T]\times[0,1]$

$$\begin{aligned}\left|D_h^- D_h^+\xi(t,x) - \partial_{xx}\xi(t,x)\right| \le& \frac{1}{h^2}\left(\xi_{\text{ext}}(t,x+h) - 2\xi_{\text{ext}}(t,x) + \xi_{\text{ext}}(t,x-h)\right) - \partial_{xx}\xi(t,x)\\ =& \frac{1}{h^2}\big(\xi(t,x) + h\partial_x\xi(t,x) + \frac{h^2}{2}\partial_{xx}\xi(t,x) + \frac{h^3}{3}\partial_{xxx}\xi_{\text{ext}}(t,x_1) - 2\xi(t,x)\\ &+ \xi(t,x) - h\partial_x\xi(t,x) + \frac{h^2}{2}\partial_{xx}\xi(t,x) - \frac{h^3}{3}\partial_{xxx}\xi_{\text{ext}}(t,x_2)\big) - \partial_{xx}\xi(t,x)\\ \le& \frac{h}{3}C_\xi\end{aligned}$$

and hence $D_h^- D_h^+\xi \to \partial_{xx}\xi$ in $L^2([0,T]\times[0,1])$ for $h\to 0$.

Next, we show that (2.4.54) implies $f = -\Delta^{-1}u$ for almost every $t\in[0,T]$ in the sense of distributions. Since $\mathcal{C}^2((0,1))$ is separable, so is $(C_c^\infty((0,1)), \|\cdot\|_{\mathcal{C}^2})$, such that there is a countable dense set $(\eta_i)_{i\in\mathbb{N}}$ in this space. For all $\theta\in\mathcal{C}_c^\infty((0,T))$ and $i\in\mathbb{N}$, we have that $\theta(t)\eta_i(x)\in\mathcal{C}_c^\infty((0,T)\times(0,1))$, such that (2.4.54) yields

$$\int_0^T \langle u(t),\eta_i\rangle_{H^{-1}\times H_0^1}\theta(t)\mathrm{d}t = \int_0^T \langle f(t),-\partial_{xx}\eta_i\rangle_{L^2}\theta(t)\mathrm{d}t.$$

By the fundamental lemma of the calculus of variations, there exists a zero set $N_i\subset[0,T]$ such that

$$\langle u(t),\eta_i\rangle_{H^{-1}\times H_0^1} = \langle f(t),-\partial_{xx}\eta_i\rangle_{L^2} \quad \text{for all } t\in[0,T]\setminus N_i. \tag{2.4.55}$$

Defining $N := \cup_{i\in\mathbb{N}}N_i$, which is again a zero set, we obtain by linearity that (2.4.55) is true for η_i replaced by any function in the linear span $\operatorname{span}(\eta_i; i\in\mathbb{N})$ for each $t\in[0,T]\setminus N$. Let finally $\eta\in\mathcal{C}_c^\infty((0,1))$ be chosen arbitrarily. Since $(\eta_i)_{i\in\mathbb{N}}$ was chosen to be a dense subset, there is a sequence $(\eta_k)_{k\in\mathbb{N}}\subset\operatorname{span}(\eta_i; i\in\mathbb{N})$ such that

$$\eta_k\to\eta \quad \text{in } \mathcal{C}^2((0,1)).$$

Hence, we obtain for $t\in[0,T]\setminus N$

$$\langle u(t),\eta\rangle_{H^{-1}\times H_0^1} = \lim_{k\to\infty}\langle u(t),\eta_k\rangle_{H^{-1}\times H_0^1} = \lim_{k\to\infty}\langle f(t),-\partial_{xx}\eta_k\rangle_{L^2} = \langle f(t),-\partial_{xx}\eta\rangle_{L^2},$$

as required. The proof can then be finished by computing

$$\int_0^T \langle u_h, \eta_h\rangle_{-1}\,\mathrm{d}t = \int_0^T \left\langle -\Delta_h^{-1} u_h, \eta_h\right\rangle_0 \mathrm{d}t$$
$$= \int_0^T \left\langle \left(-\Delta_h^{-1} u_h\right)^{\mathrm{pcx}}, \eta_h^{\mathrm{pcx}}\right\rangle_{L^2} \mathrm{d}t \to \int_0^T \left\langle -\Delta^{-1} u, \eta\right\rangle_{L^2} \mathrm{d}t = \int_0^T \langle u, \eta\rangle_{H^{-1}}\,\mathrm{d}t.$$

□

Lemma 2.4.30. *Let $(\tilde{X}_h, \tilde{Y}_h, \tilde{W}_h)$ be the processes from Lemma 2.4.26. Then,*

$$\tilde{X}_h^{plt}(t) = x_h^0 + \int_0^t \Delta_h \tilde{Y}_h^{pct\text{-}}(r)\,\mathrm{d}r + \tilde{W}_h^{plt}(t) \tag{2.4.56}$$

in $L^2([0,T];\mathbb{R}^{Z-1})$ $\tilde{\mathbb{P}}$-almost surely, and the limits in Lemma 2.4.26 satisfy

$$\tilde{X}(t) = x_0 + \int_0^t \Delta \tilde{Y}(r)\mathrm{d}r + \tilde{W}(t) \tag{2.4.57}$$

in $L^2([0,T];(L^2)')$ $\tilde{\mathbb{P}}$-almost surely.

Proof. Step 1: We first prove (2.4.56). We note that by construction of the prolongations in use here, (2.4.56) is equivalent to

$$\tilde{X}_h^n = x_h^0 + \tau \sum_{k=0}^{n-1} \Delta_h \tilde{Y}_h^k + \tilde{W}_h^n$$

for all $n \in \{0, \ldots, N\}$ $\tilde{\mathbb{P}}$-almost surely. This is given by the construction of (X_h, Y_h, W_h) in (2.2.3) and Definition 2.4.13, and by the equality of laws in (2.4.34).

Step 2: We need to show that $\tilde{\mathbb{P}}$-almost surely, for every $\zeta \in L^2([0,T];L^2)$

$$\int_0^T \left\langle \tilde{X}(t), \zeta(t)\right\rangle_{(L^2)'\times L^2} \mathrm{d}t = \int_0^T \left\langle x_0 + \int_0^t \Delta\tilde{Y}(r)\mathrm{d}r + \tilde{W}(t), \zeta(t)\right\rangle_{(L^2)'\times L^2} \mathrm{d}t, \tag{2.4.58}$$

where $\langle u, v\rangle_{(L^2)'\times L^2} = \left\langle -\Delta^{-1} u, v\right\rangle_{L^2}$. In this step, we first show (2.4.58) for a test function ζ of the type

$$\zeta = \theta(t)\eta, \tag{2.4.59}$$

where $\eta \in L^2$ and $\theta \in L^\infty([0,T])$. For $h > 0$, $Z \in \mathbb{N}$ with $hZ = 1$, let $\Pi_h^{\mathrm{pcx}}\eta$ be the L^2-orthogonal projection of η to the space

$$S_h^{\mathrm{pcx}} := \operatorname{span}\left(\{I_h^{\mathrm{pcx}}\, e_i : i = 1, \ldots, Z-1\}\right)$$

and $\eta_h \in \mathbb{R}^{Z-1}$ the corresponding coefficients, i. e. $\eta_h^{\mathrm{pcx}} = \Pi_h^{\mathrm{pcx}}\eta$. Then

$$\eta_h^{\mathrm{pcx}} \to \eta \quad \text{in } L^2 \text{ for } h \to 0$$

by Lemma 2.1.2, which implies

$$\theta\eta_h^{\mathrm{pcx}} \to \theta\eta \text{ in } L^2([0,T];L^2). \tag{2.4.60}$$

From now on, we consider a subsequence $h \to 0$ realizing the limits in Lemma 2.4.26. Using (2.4.56), we obtain

$$\begin{aligned}&\tilde{\mathbb{P}}\left(\int_0^T \left\langle \tilde{X}_h^{\mathrm{plt}}(t), \theta(t)\eta_h\right\rangle_{-1} \mathrm{d}t = \int_0^T \left\langle x_h^0 + \int_0^t \Delta_h \tilde{Y}_h^{\mathrm{pct\text{-}}}(r)\mathrm{d}r + \tilde{W}_h^{\mathrm{plt}}(t), \theta(t)\eta_h\right\rangle_{-1} \mathrm{d}t\right)\\ &\geq \tilde{\mathbb{P}}\left(\tilde{X}_h^{\mathrm{plt}}(t) = x_h^0 + \int_0^t \Delta_h \tilde{Y}_h^{\mathrm{pct\text{-}}}(r)\mathrm{d}r + \tilde{W}_h^{\mathrm{plt}}(t) \text{ in } L^2([0,T];\mathbb{R}^{Z-1})\right) = 1.\end{aligned} \tag{2.4.61}$$

Moreover, we have $\tilde{X}_h^{\mathrm{plt,pcx}} \rightharpoonup \tilde{X}$ in $L^2([0,T];L^2)$ and thus in $L^2([0,T];H^{-1})$ $\tilde{\mathbb{P}}$-almost surely by Lemma 2.4.26, and $\tilde{X}_h^{\mathrm{plt,pcx}} = I_h^{\mathrm{pcx}}\,\tilde{X}_h^{\mathrm{plt}}$ $\tilde{\mathbb{P}} \otimes \mathrm{d}t$-almost surely by construction. Hence, Proposition 2.4.29 applies and yields

$$\int_0^T \left\langle \tilde{X}_h^{\mathrm{plt}}(t), \theta(t)\eta_h\right\rangle_{-1} \mathrm{d}t \to \int_0^T \left\langle \tilde{X}, \zeta\right\rangle_{H^{-1}} \mathrm{d}t = \int_0^T \left\langle \tilde{X}, \zeta\right\rangle_{(L^2)'\times L^2} \mathrm{d}t \quad \tilde{\mathbb{P}}\text{-almost surely.} \tag{2.4.62}$$

Furthermore, we notice that $\tilde{Y}_h^{\text{pct-pcx}} \rightharpoonup \tilde{Y}$ in $L^2([0,T];L^2)$ $\tilde{\mathbb{P}}$-almost surely by Lemma 2.4.26 and $\eta_h^{\text{pcx}}\theta \to \eta\,\theta$ in $L^2([0,T];L^2)$ as in (2.4.60), where $\theta \in L^\infty([0,T])$ can be chosen as $\theta(r) = \mathbf{1}_{[0,t]}(r)$ for any $t \in [0,T]$. This allows to write for arbitrary $t \in [0,T], h \to 0$

$$\begin{aligned} \int_0^t \left\langle \tilde{Y}_h^{\text{pct-pcx}}(r), \eta_h^{\text{pcx}} \right\rangle_{L^2} \mathrm{d}r &= \int_0^T \left\langle \tilde{Y}_h^{\text{pct-pcx}}(r), \eta_h^{\text{pcx}}\,\mathbf{1}_{[0,t]}(r) \right\rangle_{L^2} \mathrm{d}r \\ &\to \int_0^T \left\langle \tilde{Y}(r), \eta\,\mathbf{1}_{[0,t]}(r) \right\rangle_{L^2} \mathrm{d}r = \left\langle \int_0^t \tilde{Y}(r)\mathrm{d}r, \eta \right\rangle_{L^2}, \end{aligned} \tag{2.4.63}$$

where for the last step we used the compatibility of Bochner integrals with bounded linear operators (see e.g. [112, Proposition A.2.2]). Furthermore, we have for all $t \in [0,T]$

$$\begin{aligned} \left| \left\langle \int_0^t \tilde{Y}_h^{\text{pct-pcx}}(r)\mathrm{d}r, \eta_h^{\text{pcx}} \right\rangle_{L^2} \right| &\le \int_0^t \left| \left\langle \tilde{Y}_h^{\text{pct-pcx}}(r), \eta_h^{\text{pcx}} \right\rangle_{L^2} \right| \mathrm{d}r \\ &\le \left(\int_0^T \left\| \tilde{Y}_h^{\text{pct-pcx}}(r) \right\|_{L^2}^2 \mathrm{d}r \right)^{\frac{1}{2}} \sqrt{T}\, \|\eta\|_{L^2} < \infty \end{aligned} \tag{2.4.64}$$

uniformly in h by Lemma 2.4.26. Thus, we may use (2.4.63), (2.4.64) and dominated convergence to obtain for $h \to 0$

$$\begin{aligned} \int_0^T \left\langle \int_0^t \Delta_h \tilde{Y}_h^{\text{pct-}}(r)\mathrm{d}r, \theta(t)\eta_h \right\rangle_{-1} \mathrm{d}t &= \int_0^T \int_0^t \left\langle \tilde{Y}_h^{\text{pct-}}(r), \theta(t)\eta_h \right\rangle_0 \mathrm{d}r\mathrm{d}t \\ &= \int_0^T \int_0^t \left\langle \tilde{Y}_h^{\text{pct-pcx}}(r), \theta(t)\eta_h^{\text{pcx}} \right\rangle_{L^2} \mathrm{d}r\mathrm{d}t \\ &\to \int_0^T \left\langle \int_0^t \tilde{Y}(r)\mathrm{d}r, \theta(t)\eta \right\rangle_{L^2} \mathrm{d}t \\ &= \int_0^T \left\langle \int_0^t \Delta\tilde{Y}(r)\mathrm{d}r, \theta(t)\eta \right\rangle_{(L^2)'\times L^2} \mathrm{d}t. \end{aligned} \tag{2.4.65}$$

For the remaining terms, note that Corollary 2.4.26 yields that $\tilde{W}^{\text{plt,pcx}} \to \tilde{W}$ in $C([0,T];H^{-1})$ $\tilde{\mathbb{P}}$-almost surely, which is stronger than weak convergence in $L^2([0,T];H^{-1})$, and $(x_h^0)^{\text{pcx}} \to x_0$ in L^2 and thus in H^{-1} by assumption, which implies weak convergence in $L^2([0,T];H^{-1})$ if x_h^0 and x_0 are interpreted as constant functions in time. Moreover, $\tilde{W}_h^{\text{plt,pcx}} = I_h^{\text{pcx}}\,\tilde{W}_h^{\text{plt}}$ $\tilde{\mathbb{P}} \otimes \mathrm{d}t$-almost everywhere by construction, which allows to apply Proposition 2.4.29 to obtain $\tilde{\mathbb{P}}$-almost surely

$$\begin{aligned} \int_0^T \left\langle x_h^0 + \tilde{W}_h^{\text{plt}}(t), \theta(t)\eta_h \right\rangle_{-1} \mathrm{d}t &\to \int_0^T \left\langle x_0 + \tilde{W}(t), \theta(t)\eta \right\rangle_{H^{-1}} \mathrm{d}t \\ &= \int_0^T \left\langle x_0 + \tilde{W}(t), \theta(t)\eta \right\rangle_{(L^2)'\times L^2} \mathrm{d}t. \end{aligned} \tag{2.4.66}$$

Using (2.4.62), (2.4.65) and (2.4.66), Equation (2.4.58) follows by taking limits in (2.4.61) $\tilde{\mathbb{P}}$-almost surely.

Step 3: By linearity, (2.4.58) is also true for linear combinations of test functions ζ of type (2.4.59) and thus for every polynomial. Thus, we obtain a $\tilde{\mathbb{P}}$-zero set outside of which (2.4.58) is satisfied for any polynomial. Using the density of polynomials in $L^2([0,T]\times[0,1])$ given by the Stone-Weierstrass theorem, outside this zero set the full statement (2.4.58) is satisfied by passing to the limit of approximating sequences. □

Lemma 2.4.31. *Let $\tilde{Y}$ and $\tilde{W}$ be constructed as in Lemma 2.4.26 and define the continuous $(L^2)'$-valued process*

$$\tilde{Z}(t) := x_0 + \int_0^t \Delta\tilde{Y}(r)\mathrm{d}t + \tilde{W}(t)$$

for $t \in [0,T]$. Then, we have

$$\tilde{\mathbb{E}}\left(\sup_{t\in[0,T]} \left\| \tilde{Z}(t) \right\|_{H^{-1}}^2 \right) < \infty$$

and

$$\tilde{\mathbb{E}}\left\|\tilde{Z}(t)\right\|_{H^{-1}}^{2}+2\,\tilde{\mathbb{E}}\int_{0}^{t}\left\langle\tilde{Z}(r),\tilde{Y}(r)\right\rangle_{L^{2}}\mathrm{d}r=\|x_{0}\|_{H^{-1}}^{2}+t\,\|I'\|_{L_{2}(L^{2},H^{-1})}^{2}. \tag{2.4.67}$$

Proof. By Lemma 2.4.30, we have that $\tilde{X}$ and $\tilde{Z}$ are in the same $\tilde{\mathbb{P}}\otimes\mathrm{d}t$-equivalence class, and by the construction in Lemma 2.4.26 we know that $\tilde{X}\in L^{2}(\tilde{\Omega}\times[0,T];L^{2})$. Moreover, $\tilde{Y}\in L^{2}(\tilde{\Omega};L^{2}([0,T];L^{2}))$ and progressively measurable with respect to $(\tilde{\mathcal{F}}_{t})_{t\in[0,T]}$ by construction (cf. Corollary 2.B.5). Thus, Ito's formula from [112, Theorem 4.2.5] applies, which yields both claims.. □

Remark 2.4.32. By (2.4.57) and the definition of $\tilde{Z}$ above, we have $\tilde{Z}=\tilde{X}$ in $L^{2}(\tilde{\Omega}\times[0,T];(L^{2})')$. Furthermore, we have $\tilde{X}\in L^{2}(\tilde{\Omega}\times[0,T];L^{2})$, such that the injectivity of the embedding $I''I':L^{2}\hookrightarrow(L^{2})'$, which carries over to an embedding

$$L^{2}(\tilde{\Omega}\times[0,T];L^{2})\hookrightarrow L^{2}(\tilde{\Omega}\times[0,T];(L^{2})'),$$

implies that $\tilde{Z}\in L^{2}(\tilde{\Omega}\times[0,T];L^{2})$ and $\tilde{Z}=\tilde{X}$ in $L^{2}(\tilde{\Omega}\times[0,T];L^{2})$.

In view of (2.2.2), it remains to inspect the relation of $\tilde{X}$ and $\tilde{Y}$. To this end, we aim to use (2.4.67) for $\tilde{Z}$ replaced by $\tilde{X}$. Since this is only possible $\mathrm{d}t$-almost surely, we need to use an integrated version of (2.4.67). The resulting double integral in the second term leads to the following definition, which will be useful in the proof of Lemma 2.4.36 below.

Definition 2.4.33. We define the measure μ on $[0,T]$ as the measure with density

$$[0,T]\ni t\mapsto T-t,$$

with respect to $\mathrm{d}t$ and we write $[0,T]_{\mu}$ for the measure space $([0,T],\mu)$. Let $\mathcal{A}\subset L^{2}(\tilde{\Omega}\times[0,T]_{\mu};L^{2})\times L^{2}(\tilde{\Omega}\times[0,T]_{\mu};L^{2})$ be a multivalued operator (which we identify with its graph by a slight abuse of notation) defined by

$$(X,Y)\in\mathcal{A}\quad\text{if and only if}\quad Y\in\phi(X)\text{ for almost every }(\tilde{\omega},t,x)\in\tilde{\Omega}\times[0,T]\times[0,1]. \tag{2.4.68}$$

Remark 2.4.34. We note that $\tilde{\mathbb{P}}\otimes\mu\otimes\mathrm{d}x$ (resp. $\mu\otimes\mathrm{d}x$) and $\tilde{\mathbb{P}}\otimes\mathrm{d}t\otimes\mathrm{d}x$ (resp. $\mathrm{d}t\otimes\mathrm{d}x$) are equivalent (i. e. mutually absolutely continuous with respect to each other) by the fact that μ and $\mathrm{d}t$ are equivalent. Moreover, since $(T-t)$ is bounded, we have

$$L^{2}(\tilde{\Omega}\times[0,T];L^{2})\subseteq L^{2}(\tilde{\Omega}\times[0,T]_{\mu};L^{2}).$$

Lemma 2.4.35. *The operator $\mathcal{A}$ is maximal monotone.*

Proof. The proof is identical to the proof of Lemma 2.3.2 with $H:=L^{2}(\tilde{\Omega}\times[0,T]_{\mu};L^{2})$ and

$$\varphi:H\to[0,\infty],\quad\varphi(u)=\mathbb{E}\int_{0}^{T}(T-t)\int_{0}^{1}\tilde{\psi}(u(t,x))\mathrm{d}x\mathrm{d}t.$$

□

Lemma 2.4.36. *Let $h>0$, $(\tilde{X}_{h}^{pct\text{-}pcx})_{h>0},(\tilde{Y}_{h}^{pct\text{-}pcx})_{h>0},\tilde{X},\tilde{Y}$ be as in Lemma 2.4.26. Then,*

$$\limsup_{h\to0}\tilde{\mathbb{E}}\int_{0}^{T}(T-t)\left\langle\tilde{X}_{h}^{pct\text{-}pcx}(t),\tilde{Y}_{h}^{pct\text{-}pcx}(t)\right\rangle_{L^{2}}\mathrm{d}t\leq\tilde{\mathbb{E}}\int_{0}^{T}(T-t)\left\langle\tilde{X}(t),\tilde{Y}(t)\right\rangle_{L^{2}}\mathrm{d}t.$$

Proof. We notice that for $f\in L^{1}([0,T];\mathbb{R})$ or measurable $f\geq0$, we have by Fubini's (resp. Tonelli's) theorem

$$\int_{0}^{T}\int_{0}^{t}f(r)\mathrm{d}r\mathrm{d}t=\int_{0}^{T}\int_{0}^{T}\mathbf{1}_{[0,t]}(r)f(r)\mathrm{d}r\mathrm{d}t=\int_{0}^{T}f(r)\int_{0}^{T}\mathbf{1}_{[r,T]}(t)\mathrm{d}t\mathrm{d}r=\int_{0}^{T}(T-r)f(r)\mathrm{d}r. \tag{2.4.69}$$

Furthermore, we note that, due to Lemma 2.4.26, $(\tilde{X}_{h}^{\mathrm{pct+pcx}})_{h>0}$ is bounded in $L^{2}(\tilde{\Omega};L^{2}([0,T];L^{2}))$ uniformly in h, and

$$\tilde{X}_{h}^{\mathrm{pct+pcx}}\rightharpoonup\tilde{X}\quad\tilde{\mathbb{P}}\text{-almost surely in }L^{2}([0,T];L^{2}).$$

Hence, Lemma 2.B.6 yields that

$$\tilde{X}_h^{\text{pct+pcx}} \rightharpoonup \tilde{X} \quad \text{in } L^2(\tilde{\Omega}; L^2([0,T]; L^2))$$

for $h \to 0$. Since weak convergence in $L^2(\tilde{\Omega}; L^2(\times[0,T]; L^2))$ is stronger than weak convergence in $L^2(\tilde{\Omega}; L^2([0,T]; H^{-1}))$, we have by weak lower-semicontinuity of the norm that

$$\tilde{\mathbb{E}} \int_0^T \left\| \tilde{X}(t) \right\|_{H^{-1}}^2 \mathrm{d}t \le \liminf_{h\to 0} \tilde{\mathbb{E}} \int_0^T \left\| \tilde{X}_h^{\text{pct+pcx}}(t) \right\|_{H^{-1}}^2 \mathrm{d}t,$$

or, equivalently,

$$-\tilde{\mathbb{E}} \int_0^T \left\| \tilde{X}(t) \right\|_{H^{-1}}^2 \mathrm{d}t \ge \limsup_{h\to 0} \left(-\tilde{\mathbb{E}} \int_0^T \left\| \tilde{X}_h^{\text{pct+pcx}}(t) \right\|_{H^{-1}}^2 \mathrm{d}t \right). \tag{2.4.70}$$

Furthermore, by the same arguments as in the proof of Proposition 2.4.29, we obtain

$$\left(-\Delta_h^{-1} x_h^0)\right)^{\text{pcx}} \rightharpoonup -\Delta^{-1} x_0 \quad \text{in } L^2 \text{ for } h \to 0,$$

which allows to compute

$$\begin{aligned} \lim_{h\to 0} \left\| x_h^0 \right\|_{-1}^2 = \lim_{h\to 0} \left\langle -\Delta_h^{-1} x_h^0, x_h^0 \right\rangle_0 &= \lim_{h\to 0} \left\langle (-\Delta_h^{-1} x_h^0)^{\text{pcx}}, (x_h^0)^{\text{pcx}} \right\rangle_{L^2} \\ &= \left\langle -\Delta^{-1} x_0, x_0 \right\rangle_{L^2} = \left\| x_0 \right\|_{H^{-1}}^2 . \end{aligned} \tag{2.4.71}$$

For each $h > 0$ in the subsequence of Lemma 2.4.26, consider $\tilde{X}_h$ and $\tilde{Y}_h$ as constructed in Lemma 2.4.26. Then, by (2.4.69) and Remark 2.1.4, we obtain

$$\begin{aligned} &\limsup_{h\to 0} \tilde{\mathbb{E}} \int_0^T (T-t) \left\langle \tilde{X}_h^{\text{pct-pcx}}(t), \tilde{Y}_h^{\text{pct-pcx}}(t) \right\rangle_{L^2} \mathrm{d}t \\ &= \limsup_{h\to 0} \int_0^T \tilde{\mathbb{E}} \int_0^t \left\langle \tilde{X}_h^{\text{pct-pcx}}(r), \tilde{Y}_h(r)^{\text{pct-pcx}} \right\rangle_{L^2} \mathrm{d}r \, \mathrm{d}t \\ &= \limsup_{h\to 0} \int_0^T \tilde{\mathbb{E}} \int_0^t \left\langle \tilde{X}_h^{\text{pct-}}(s), \tilde{Y}_h^{\text{pct-}}(s) \right\rangle_0 \mathrm{d}s \, \mathrm{d}t. \end{aligned} \tag{2.4.72}$$

Writing $t_\tau = \lfloor t/\tau \rfloor \tau$ and using the definition of the left-sided piecewise constant embedding embedding, the positive sign of $\left\langle \tilde{X}_h^{\text{pct-}}, \tilde{Y}_h^{\text{pct-}} \right\rangle_0$ $\tilde{\mathbb{P}} \otimes \mathrm{d}t$-almost everywhere and Lemma 2.4.2, we continue by

$$\begin{aligned} (2.4.72) &= \limsup_{h\to 0} \int_0^T \tilde{\mathbb{E}} \left[\sum_{n=0}^{\lfloor t/\tau \rfloor} \tau \left\langle \tilde{X}_h^n, \tilde{Y}_h^n \right\rangle_0 - \int_t^{t_\tau + \tau} \left\langle \tilde{X}_h^{\text{pct-}}(s), \tilde{Y}_h^{\text{pct-}}(s) \right\rangle_0 \mathrm{d}s \right] \mathrm{d}t \\ &\le \frac{1}{2} \limsup_{h\to 0} \left(-\int_0^T \tilde{\mathbb{E}} \left\| \tilde{X}_h^{\lfloor t/\tau \rfloor + 1} \right\|_{-1}^2 \mathrm{d}t \right) \\ &\quad + \frac{1}{2} \lim_{h\to 0} \left(\left(1 + \frac{4\tau}{h^2} \right) \left(\int_0^T \left\| x_h^0 \right\|_{-1}^2 \mathrm{d}t + \int_0^T (t_\tau + \tau) \mathrm{Tr}(-\Delta_h^{-1}) \, \mathrm{d}t \right) \right). \end{aligned} \tag{2.4.73}$$

Using Assumption 2.2.4, the definition of the right-sided piecewise constant embedding, (2.4.71), Lemma 2.4.3 and Lemma 2.4.9, we obtain

$$\begin{aligned} (2.4.73) &= \limsup_{h\to 0} \left(-\frac{1}{2} \int_0^T \tilde{\mathbb{E}} \left\| \tilde{X}_h^{\text{pct+}}(t) \right\|_{-1}^2 \mathrm{d}t \right) + \frac{1}{2} \int_0^T \left\| x_0 \right\|_{H^{-1}}^2 \mathrm{d}t + \frac{1}{2} \int_0^T t \left\| I' \right\|_{L_2(L^2, H^{-1})}^2 \mathrm{d}t \\ &\le \limsup_{h\to 0} \left(-\frac{1}{2} \int_0^T \tilde{\mathbb{E}} \left\| \tilde{X}_h^{\text{pct+pcx}}(t) \right\|_{H^{-1}}^2 \mathrm{d}t \right) + \frac{1}{2} \int_0^T \left\| x_0 \right\|_{H^{-1}}^2 \mathrm{d}t + \frac{1}{2} \int_0^T t \left\| I' \right\|_{L_2(L^2, H^{-1})}^2 \mathrm{d}t. \end{aligned} \tag{2.4.74}$$

Using (2.4.70) and Remark 2.4.32, we obtain

$$
\begin{aligned}
(2.4.74) &\leq -\frac{1}{2}\int_0^T \tilde{\mathbb{E}} \left\|\tilde{X}(t)\right\|_{H^{-1}}^2 \mathrm{d}t + \frac{1}{2}\int_0^T \|x_0\|_{H^{-1}}^2 \mathrm{d}t + \frac{1}{2}\int_0^T t\,\|I'\|_{L_2(L^2,H^{-1})}^2 \mathrm{d}t \\
&= -\frac{1}{2}\int_0^T \tilde{\mathbb{E}} \left\|\tilde{Z}(t)\right\|_{H^{-1}}^2 \mathrm{d}t + \frac{1}{2}\int_0^T \|x_0\|_{H^{-1}}^2 \mathrm{d}t + \frac{1}{2}\int_0^T t\,\|I'\|_{L_2(L^2,H^{-1})}^2 \mathrm{d}t. \qquad (2.4.75)
\end{aligned}
$$

Finally, we obtain by Lemma 2.4.31, Remark 2.4.32 and (2.4.69) combined with the integrability in Lemma 2.4.26

$$
\begin{aligned}
(2.4.75) &= \int_0^T \tilde{\mathbb{E}} \int_0^t \left\langle \tilde{Z}(r), \tilde{Y}(r)\right\rangle_{L^2} \mathrm{d}r\,\mathrm{d}t \\
&= \int_0^T \tilde{\mathbb{E}} \int_0^t \left\langle \tilde{X}(r), \tilde{Y}(r)\right\rangle_{L^2} \mathrm{d}r\,\mathrm{d}t = \tilde{\mathbb{E}}\int_0^T (T-t)\left\langle \tilde{X}(t), \tilde{Y}(t)\right\rangle_{L^2} \mathrm{d}t,
\end{aligned}
$$

which finishes the proof. □

Proof of Theorem 2.2.5. By Lemma 2.4.26, we have that a (nonrelabeled) subsequence of $\left(\tilde{X}_h^{\mathrm{plt,pcx}}\right)_{h>0}$ converges to $\tilde{X}$ weakly in $L^2([0,T];L^2)$ and weakly* in $L^\infty([0,T];H^{-1})$ $\tilde{\mathbb{P}}$-almost surely, which implies by the Slutsky theorem (cf. [90, Theorem 13.18]) that

$$
\mathcal{L}\left(\tilde{X}_h^{\mathrm{plt,pcx}}\right) \to \mathcal{L}(\tilde{X})
$$

with respect to the weak topology in $L^2([0,T];L^2)$ and the weak* topology in $L^\infty([0,T];H^{-1})$. Since we also have by Lemma 2.4.26 that $\mathcal{L}\left(\tilde{X}_h^{\mathrm{plt,pcx}}\right) = \mathcal{L}\left(X_h^{\mathrm{plt,pcx}}\right)$ in both spaces, these convergence results transfer to $\mathcal{L}\left(X_h^{\mathrm{plt,pcx}}\right)$.

We next show that $\left((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in[0,T]}, \tilde{\mathbb{P}}), \tilde{X}, \tilde{W}\right)$ as constructed in Lemma 2.4.26 and Lemma 2.4.28, is a weak solution to (2.1.1) in the sense of Definition 2.2.1 belonging to the process $\tilde{Y}$ given in Lemma 2.4.26. Considering the definition of the filtration $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$, progressive measurability of $\tilde{X}$ and $\tilde{Y}$ is clear by Corollary 2.B.5. In Lemma 2.4.28, it is shown that $\tilde{W}$ is an $I'(I')^*$-Wiener process in H^{-1} and hence a cylindrical Id-Wiener process in L^2 with respect to $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$. Equality (2.2.1) is proved in Lemma 2.4.30. Hence, it only remains to show (2.2.2), or, equivalently, $(\tilde{X}, \tilde{Y}) \in \mathcal{A}$, which can be done by proving

$$
\left(\tilde{X}_h^{\text{pct-pcx}}, \tilde{Y}_h^{\text{pct-pcx}}\right) \in \mathcal{A} \quad \text{for all } h \in (0,1), \qquad (2.4.76)
$$

$$
\begin{cases} \tilde{X}_h^{\text{pct-pcx}} \rightharpoonup \tilde{X} & \text{in } L^2(\tilde{\Omega}\times[0,T]_\mu; L^2), \\ \tilde{Y}_h^{\text{pct-pcx}} \rightharpoonup \tilde{Y} & \text{in } L^2(\tilde{\Omega}\times[0,T]_\mu; L^2), \end{cases} \qquad (2.4.77)
$$

$$
\text{and} \quad \limsup_{h\to 0} \tilde{\mathbb{E}}\int_0^T (T-t)\left\langle \tilde{X}_h^{\text{pct-pcx}}, \tilde{Y}_h^{\text{pct-pcx}}\right\rangle_{L^2} \mathrm{d}t \leq \tilde{\mathbb{E}}\int_0^T (T-t)\left\langle \tilde{X}, \tilde{Y}\right\rangle_{L^2} \mathrm{d}t \qquad (2.4.78)
$$

according to [7, Corollary 2.4].

Ad (2.4.76)*:* We notice that by Lemma 2.4.26 and Definition 2.4.13, we have $\tilde{\mathbb{P}}$-almost surely

$$
\tilde{Y}_h = \tilde{\phi}(\tilde{X}_h)
$$

and hence

$$
\tilde{Y}_h^{\text{pct-pcx}} = \tilde{\phi}\left(\tilde{X}_h^{\text{pct-pcx}}\right) \in \phi\left(\tilde{X}_h^{\text{pct-pcx}}\right) \qquad (2.4.79)
$$

$\tilde{\mathbb{P}}$-almost surely in $L^2([0,T];L^2)$. By [50, Korollar V.1.6], this implies that (2.4.79) is satisfied for almost every $(\omega,t,x) \in \tilde{\Omega}\times[0,T]\times[0,1]$, which is equivalent to 2.4.76.

Ad (2.4.77)*:* As in the proof of Lemma 2.4.36, we use Lemma 2.4.26, Lemma 2.B.3 and Lemma 2.B.6 to obtain

$$
\tilde{X}_h^{\text{pct-pcx}} \rightharpoonup \tilde{X} \quad \text{and} \quad \tilde{Y}_h^{\text{pct-pcx}} \rightharpoonup \tilde{Y} \quad \text{in } L^2(\tilde{\Omega}; L^2([0,T];L^2)) \qquad (2.4.80)
$$

for $h \to 0$.

For $\zeta \in L^2(\tilde{\Omega} \times [0,T]_\mu; L^2)$, we note that

$$\tilde{\mathbb{E}} \int_0^T \|(T-t)\zeta\|_{L^2}^2 \,\mathrm{d}t \leq T\,\tilde{\mathbb{E}} \int_0^T (T-t)\,\|\zeta\|_{L^2}^2 \,\mathrm{d}t = T\,\|\zeta\|_{L^2(\tilde{\Omega}\times[0,T]_\mu;L^2)}^2 ,$$

which yields that $(T-t)\zeta \in L^2(\tilde{\Omega} \times [0,T]; L^2)$. Thus, for $h \to 0$, we have

$$\begin{aligned}\tilde{\mathbb{E}} \int_0^T \left\langle \tilde{X}_h^{\text{pct-pcx}}(t), \zeta(t) \right\rangle_{L^2} \mu(\mathrm{d}t) &= \tilde{\mathbb{E}} \int_0^T \left\langle \tilde{X}_h^{\text{pct-pcx}}(t), (T-t)\zeta(t) \right\rangle_{L^2} \mathrm{d}t \\ &\to \tilde{\mathbb{E}} \int_0^T \left\langle \tilde{X}(t), (T-t)\zeta(t) \right\rangle_{L^2} \mathrm{d}t = \tilde{\mathbb{E}} \int_0^T \left\langle \tilde{X}(t), \zeta(t) \right\rangle_{L^2} \mu(\mathrm{d}t),\end{aligned}$$

as required. For $\tilde{Y}$, an analogous calculation applies.

Ad (2.4.78)*:* This is proven in Lemma 2.4.36.

The same course of arguments also applies to any subsequence of $(h_m)_{m\in\mathbb{N}}$, which means that each subsequence of $(X_h^{\text{plt,pcx}})_{h>0}$ contains a subsubsequence converging in law to a weak solution of (2.1.1). Since every weak solution to (2.1.1) is distributed according to the same law by Theorem 2.2.3, each of these subsubsequences converges in law to the same limit, which implies convergence in law of the whole sequence. This completes the proof. □

2.5 Continuum limit for the deterministic BTW model

We keep the convention of dropping the index m of the discretization sequences

$$(h_m)_{m\in\mathbb{N}}, (Z_m)_{m\in\mathbb{N}}, (\tau_m)_{m\in\mathbb{N}}, (N_m)_{m\in\mathbb{N}},$$

writing instead $(h)_{h>0}$ etc. Moreover, convergence of sequences and usually nonrelabeled subsequences indexed by h_m for $m \to \infty$ will be denoted by $h \to 0$. Finally, we will drop the index ϕ_1 to indicate the different nonlinearity. Of course, $\tilde{\phi}$ (ϕ) denote the (maximally monotone extended) BTW nonlinearity.

In oder to obtain convergent subsequences by compactness arguments, we use a very similar strategy as in Section 2.4. Hence, we will often refer to the proofs of the corresponding lemmas.

Lemma 2.5.1. *Let $\tau, h > 0$ and $Z, N \in \mathbb{N}$ as in Assumption 2.2.4, where we choose h small enough for $\frac{\tau}{h^2} \leq \frac{1}{4}$ to be satisfied. Let $(u_h)_{h\geq 0}$ be the discrete process defined in* (2.2.8)*. Then,*

$$\max_{n\in\{0,\ldots,N+1\}} \|u_h^n\|_{-1}^2 \leq \|u_h^*\|_{-1}^2 .$$

The proof of Lemma 2.5.1 is conducted by the same arguments as the proof of Lemma 2.4.2, using

$$\left\langle x, \tilde{\phi}(x) \right\rangle_0 \geq \left\| \tilde{\phi}(x) \right\|_0^2 \quad \text{for } x \in \mathbb{R}^{Z-1}$$

instead of (2.4.7).

We have the following stronger version of Corollary 2.4.4 due to the boundedness of the BTW nonlinearity.

Corollary 2.5.2. *Let $\tau, h > 0$ as in Assumption 2.2.4. Then, the discrete process in (2.2.8) satisfies*

$$\left\|u_h^{n+1} - u_h^n\right\|_{-1}^2 \leq 4\frac{\tau^2}{h^2} \quad \text{for all } n \in \{0, \ldots, N-1\}.$$

Proof. Using Lemma 2.4.1, we compute for $n \in \{0, \ldots, N-1\}$

$$\left\|u_h^{n+1} - u_h^n\right\|_{-1}^2 = \left\|\tau\Delta_h \tilde{\phi}(X_h^n)\right\|_{-1}^2 \leq \tau^2 \left\|-\Delta_h\right\| \mathbb{E} \left\|\tilde{\phi}(X_h^n)\right\|_0^2 \leq 4\frac{\tau^2}{h^2},$$

using the boundedness of $\tilde{\phi}$ in the last step. □

Lemma 2.5.3. *Let $\tau, h > 0$ and $Z, N \in \mathbb{N}$ as in Assumption 2.2.4, where we choose h small enough for $\frac{\tau}{h^2} \leq \frac{1}{4}$ to be satisfied. Let $(u_h)_{h\geq 0}$ be the discrete process defined in* (2.2.8)*. Then, for $h > 0$*

$$\max\left\{ \operatorname*{ess\,sup}_{t\in[0,T]} \left\| u_h^{plt,pcx}(t) \right\|_{H^{-1}}^2, \operatorname*{ess\,sup}_{t\in[0,T]} \left\| u_h^{pct\text{-}pcx}(t) \right\|_{H^{-1}}^2 \right\} \leq \|u_h^*\|_{-1} \leq C \tag{2.5.1}$$

for a positive constant C independent of h. Moreover,

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left\| u_h^{plt,pcx}(t) - u_h^{pct\text{-}pcx}(t) \right\|_{H^{-1}}^2 \leq C \frac{\tau^2}{h^2} \tag{2.5.2}$$

for $h > 0$.

Proof. The bound of the piecewise linear in time interpolated extension of u_h in (2.5.1) is shown analogous to the last part of the proof of Lemma 2.4.12, using Lemma 2.5.1. For the piecewise constant in time extension of u_h, we see, using Definition 2.1.6, Lemma 2.4.9 and Lemma 2.5.1,

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left\| u_h^{\text{pct-pcx}}(t) \right\|_{H^{-1}}^2 \leq \operatorname*{ess\,sup}_{t\in[0,T]} \left\| u_h^{\text{pct-}}(t) \right\|_{-1}^2 = \max_{n\in\{0,\ldots,N-1\}} \|u_h^n\|_{-1}^2 \leq \|u_h^*\|_{-1}^2 .$$

A uniform bound C exists, because $(u_h^*)^{\text{pcx}} \to u_0$ in L^2 by construction. Hence, $(u_h^*)^{\text{pcx}}$ is bounded in L^2 and thus in H^{-1}, which by Lemma 2.4.9 yields

$$\|u_h^*\|_{-1} \leq 3 \left\|(u_h^*)^{\text{pcx}}\right\|_{H^{-1}} \leq C.$$

The last statement is proved analogously to Lemma 2.4.10, using Corollary 2.5.1. $\square$

Lemma 2.5.4. *Let $\tau, h > 0$ and $Z, N \in \mathbb{N}$ as in Assumption 2.2.4, and let $(u_h)_{h\geq 0}$ be the discrete process defined in* (2.2.8)*. Then, there exists $u \in L^\infty([0,T]; H^{-1})$ and a nonrelabeled subsequence such that*

$$u_h^{plt,pcx} \overset{*}{\rightharpoonup} u \quad \text{and} \quad u_h^{pct\text{-}pcx} \overset{*}{\rightharpoonup} u$$

for $h \to 0$.

Proof. The existence of $u \in H^{-1}$ and a nonrelabeled subsequence such that $u_h^{\text{plt,pcx}} \overset{*}{\rightharpoonup} u$ for $h \to 0$ follows by the Banach-Alaoglu theorem and the fact that convergence with respect to the weak* topology on the dual of a normed space is equivalent to weak* convergence (cf. [61, Proposition A.51]). From this subsequence, the same argument allows to extract another subsequence such that $u_h^{\text{pct-pcx}} \overset{*}{\rightharpoonup} \tilde{u}$ for some $\tilde{u} \in L^\infty([0,T]; H^{-1})$. Using (2.5.2), a similar argument as in (2.4.43) yields that $u = \tilde{u}$, which finishes the proof. $\square$

Lemma 2.5.5. *Let $\mathcal{O} \subset \mathbb{R}^d$ a domain and $v \in \mathcal{C}_c(\mathcal{O})$. For $h > 0$, let $\mathcal{P}_h$ be a partition of $\mathcal{O}$ such that*

$$\max\{\operatorname{diam}(P) : P \in \mathcal{P}_h\} \to 0$$

for $h \to 0$. For $P \in \mathcal{P}_h$, let

$$v_h^P := \begin{cases} \frac{1}{|P|} \int_P v \, \mathrm{d}x, & \text{if } |P| > 0, \\ 0, & \text{else.} \end{cases} \tag{2.5.3}$$

Then,

$$\sum_{P\in\mathcal{P}_h} v_h^P \boldsymbol{1}_P \to v \quad \text{for } h \to 0 \text{ in } L^2([0,T] \times [0,1]).$$

Proof. Note that v is equicontinuous by the fact that it is continuous and has compact support, i. e.

$$\sup\left\{|v(x) - v(y)| : x, y \in \mathcal{O}, \|x - y\|_{\mathbb{R}^d} < h\right\} \to 0$$

for $h \to 0$. Furthermore, for $x \in P \subset \mathcal{O}$, we have

$$\left| v(x) - \frac{1}{|P|} \int_P v(y) \, \mathrm{d}y \right| \leq \frac{1}{|P|} \int_P |v(x) - v(y)| \, \mathrm{d}y \leq \sup\{|v(x) - v(y)| : \|x - y\|_{\mathbb{R}^d} < \operatorname{diam}(P)\}.$$

Hence,

$$
\begin{aligned}
\left\| \sum_{P\in\mathcal{P}_h} v_h^P \mathbf{1}_P - v \right\|^2_{L^2([0,T]\times[0,1])} &= \int_0^T \int_0^1 \left| \sum_{P\in\mathcal{P}_h} v_h^P \mathbf{1}_P(x,t) - v(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\le \int_0^T \int_0^1 \sum_{P\in\mathcal{P}_h} \mathbf{1}_P \sup\{|v(x)-v(y)|^2 : \|x-y\|_{\mathbb{R}^d} < \operatorname{diam}(P)\} \mathrm{d}x\,\mathrm{d}t \\
&\le T\left(\sup\{|v(x)-v(y)| : \|x-y\|_{\mathbb{R}^d} < \max\{\operatorname{diam}(P) : P\in\mathcal{P}_h\}\}\right)^2 \\
&\to 0,
\end{aligned}
$$

for $h\to 0$, as required. ☐

Recall the notation in Section 2.1.2. For $v\in L^2$, let $\Pi_h v$ be the coordinates of $\Pi_h^{\mathrm{pcx}} v$ with respect to the basis

$$
\{e_i^{\mathrm{pcx}} : i\in\{1,\dots,Z-1\}\} \subset S_h^{\mathrm{pcx}},
$$

i. e. $\Pi_h v := (I_h^{\mathrm{pcx}})^{-1}\Pi_h^{\mathrm{pcx}} v$.

Lemma 2.5.6. *Let $\tau, h>0$ and $N,Z\in\mathbb{N}$ be as in Assumption 2.2.4 and $v\in\mathcal{C}^1([0,T];L^2)$. Consider the subsequence which realizes the convergence in Lemma 2.5.4. Then, there exists another nonrelabeled subsequence $h\to 0$ and, for each h in this subsequence, there exists $v_h\in\mathcal{C}([0,T];\mathbb{R}^{Z-1})$, such that v_h is differentiable in time almost everywhere and for $h\to 0$*

$$
v_h^{pcx} \to v \quad \text{and} \quad (\partial_t v_h)^{pcx} \to \partial_t v \quad \text{in } L^2([0,T];L^2). \tag{2.5.4}
$$

Proof. Fix $m\in\mathbb{N}$ and choose $w_m\in\mathcal{C}_c^0([0,T]\times[0,1])$, such that

$$
\|\partial_t v - w_m\|_{L^2([0,T]\times[0,1])} < \frac{1}{2m}.
$$

Hence, there exists $h_m^*>0$ such that for all $h<h_m^*$

$$
\operatorname{supp}(w_m) \subseteq [0,T]\times\left[\frac{h}{2}, 1-\frac{h}{2}\right]. \tag{2.5.5}
$$

We now choose $h<h_m^*$ in the original sequence, and we consider the space-time partition

$$
\begin{aligned}
\mathcal{P}_h &= \mathcal{P}_h^{\mathrm{bulk}} \cup \mathcal{P}_h^{\mathrm{bdry}}, \\
\mathcal{P}_h^{\mathrm{bulk}} &= \left\{ \left(\left(j-\frac{1}{2}\right)h, \left(j+\frac{1}{2}\right)h\right) \times (n\tau,(n+1)\tau) : j=1,\dots,Z-1; n=0,\dots,N-1 \right\}, \\
\mathcal{P}_h^{\mathrm{bdry}} &= \left\{ \left(0,\frac{h}{2}\right) \times (n\tau,(n+1)\tau) : n=0,\dots,N-1 \right\} \\
&\quad \cup \left\{ \left(1-\frac{h}{2},1\right) \times (n\tau,(n+1)\tau) : n=0,\dots,N-1 \right\}.
\end{aligned}
$$

of $[0,T]\times[0,1]$, where we neglect Lebesgue-zero sets. Recall the notation in (2.5.3) and note that for $h<h_n^*$, we have by (2.5.5)

$$
(w_m)_h^P = 0 \tag{2.5.6}
$$

for $P\in\mathcal{P}_h^{\mathrm{bdry}}$. Using Lemma 2.5.5, there exists $h_m<h_m^*$ in the original sequence, such that for all $h\le h_m$, we have

$$
\left\| \sum_{P\in\mathcal{P}_h} (w_m)_h^P \mathbf{1}_P - w_m \right\|_{L^2([0,T]\times[0,1])} < \frac{1}{2m}.
$$

For $n\in\{0,\dots,N-1\}$ and $j\in\{1,\dots,Z-1\}$ we then define

$$
w_{h_m}^{n,j} := (w_m)_{h_m}^P, \quad \text{where } P = \left(\left(j-\frac{1}{2}\right)h, \left(j+\frac{1}{2}\right)h\right) \times (n\tau,(n+1)\tau) \in \mathcal{P}_h^{\mathrm{bulk}},
$$

so that we may compute

$$\begin{aligned}\left\|w_{h_m}^{\text{pct-pcx}} - \partial_t v\right\|_{L^2([0,T]\times[0,1])} &\le \left\|w_{h_m}^{\text{pct-pcx}} - w_m\right\|_{L^2([0,T]\times[0,1])} + \|w_m - \partial_t v\|_{L^2([0,T]\times[0,1])} \\ &\le \left\| \sum_{P\in\mathcal{P}_{h_m}} (w_m)_{h_m}^P \mathbf{1}_P - w_m \right\|_{L^2([0,T]\times[0,1])} + \frac{1}{2m} \le \frac{1}{m},\end{aligned} \tag{2.5.7}$$

where we used the construction of w_m and (2.5.6) in the second step.

For $m \in \mathbb{N}$, define $v_{h_m} \in \mathcal{C}([0,T];\mathbb{R}^{Z-1})$ by

$$v_{h_m}(t) := \Pi_{h_m} v(0) + \int_0^t w_{h_m}^{\text{pct-}}(r)\,\mathrm{d}r,$$

which is clearly differentiable in time everywhere except at the time grid points. Using the linearity of the integral to interchange the time integral with the spatial embedding in the first step and Jensen's inequality in the last step, we compute

$$\begin{aligned}\int_0^T \left\|v_{h_m}^{\text{pcx}}(t) - v(t)\right\|_{L^2}^2 \mathrm{d}t &= \int_0^T \left\|\Pi_{h_m}^{\text{pcx}} v(0) + \int_0^t w_{h_m}^{\text{pct-pcx}}(r)\,\mathrm{d}r - v(0) - \int_0^t \partial_t v(r)\,\mathrm{d}r\right\|_{L^2}^2 \mathrm{d}t \\ &\le 2T \left\|\Pi_{h_m}^{\text{pcx}} v(0) - v(0)\right\|_{L^2}^2 + 2\int_0^T \left\|\int_0^t w_{h_m}^{\text{pct-pcx}}(r) - \partial_t v(r)\,\mathrm{d}r\right\|_{L^2}^2 \mathrm{d}t \\ &\le 2T \left\|\Pi_{h_m}^{\text{pcx}} v(0) - v(0)\right\|_{L^2}^2 + 2\int_0^T \left(\int_0^T \left\|w_{h_m}^{\text{pct-pcx}}(r) - \partial_t v(r)\right\|_{L^2} \mathrm{d}r\right)^2 \mathrm{d}t \\ &\le 2T \left\|\Pi_{h_m}^{\text{pcx}} v(0) - v(0)\right\|_{L^2}^2 + 2T^2 \left\|w_{h_m}^{\text{pct-pcx}} - \partial_t v(r,x)\right\|_{L^2([0,T]\times[0,1])}^2,\end{aligned}$$

which converges to 0 for $m \to \infty$ by (2.5.7), where similar techniques as used for this approximation result lead to the convergence of the projected initial value $v(0)$. Hence, the subsequence $(h_m)_{m\in\mathbb{N}}$ and the approximating functions v_{h_m} and $w_{h_m} = \partial_t v_{h_m}$ satisfy the requirements of (2.5.4). □

Definition 2.5.7. Let $h > 0$ and $Z \in \mathbb{N}$ as in Assumption 2.2.4. We then define the functional $\varphi_h : \mathbb{R}^{Z-1} \to [0,\infty)$ by

$$\varphi_h(w_h) = \sum_{i=1}^{Z-1} h\,\psi(w_{h,i}),$$

where $\partial\psi = \phi$ as defined in (2.2.5).

Remark 2.5.8. We note that $\varphi_h(w_h) = \varphi(w_h^{\text{pcx}})$, where φ is defined as in (2.2.6).

Lemma 2.5.9. *Let $h > 0$ and $Z \in \mathbb{N}$ as in Assumption 2.2.4 and let $w_h \in \mathbb{R}^{Z-1}$. Then*

$$-\Delta_h \tilde{\phi}(w_h) \in \partial_{-1}\varphi_h(w_h),$$

where ∂_{-1} denotes the subdifferential with respect to the inner product $\langle\cdot,\cdot\rangle_{-1}$.

Proof. Note that we have for $w_h, v_h \in \mathbb{R}^{Z-1}$

$$\begin{aligned}\varphi_h(v_h) - \varphi_h(w_h) &= h\sum_{i=1}^{Z-1} (\psi(v_{h,i}) - \psi(w_{h,i})) \\ &\ge h\sum_{i=1}^{Z-1} \tilde{\phi}(w_{h,i})(v_{h,i} - w_{h,i}) \\ &= \left\langle \Delta_h^{-1}\Delta_h\tilde{\phi}(w_h), v_h - w_h\right\rangle_0 = \left\langle -\Delta_h\tilde{\phi}(w_h), v_h - u_h\right\rangle_{-1},\end{aligned}$$

as required. □

Lemma 2.5.10. *Let $h > 0$ and $Z \in \mathbb{N}$ as in Assumption 2.2.4. Let $v_h \in \mathcal{C}([0,T];\mathbb{R}^{Z-1})$ be almost everywhere differentiable, $\partial_t v_h \in L^2([0,T];\mathbb{R}^{Z-1})$ and u_h be defined as in (2.2.8). For all $t \in [0,T]$, we then have*

$$\begin{aligned}\left\|v_h(t) - u_h^{plt}(t)\right\|_{-1}^2 \leq \|v_h(0) - u_h^*\|_{-1}^2 &+ 2\int_0^t \varphi_h(v_h(r))\mathrm{d}r - 2\int_0^t \varphi_h(u_h^{pct\text{-}}(r))\mathrm{d}r \\ &+ 2\int_0^t \left\langle v_h(r) - u_h^{plt}(r), \partial_t v_h(r)\right\rangle_{-1} \mathrm{d}r \\ &+ 2\int_0^t \left\langle u_h^{pct\text{-}}(r) - u_h^{plt}(r), -\Delta_h\tilde{\phi}(u_h^{pct\text{-}}(r))\right\rangle_{-1} \mathrm{d}r.\end{aligned} \tag{2.5.8}$$

Proof. By construction of u_h and the chain rule, we obtain

$$\begin{aligned}&\left\|v_h(t) - u_h^{\mathrm{plt}}(t)\right\|_{-1}^2 - \left\|v_h(0) - u_h^{\mathrm{plt}}(0)\right\|_{-1}^2 \\ &= \int_0^t 2\left\langle v_h(r) - u_h^{\mathrm{plt}}(r), \partial_t v_h(r) - \Delta_h\tilde{\phi}(u_h^{\mathrm{pct\text{-}}}(r))\right\rangle_{-1} \mathrm{d}r \\ &= \int_0^t 2\left\langle v_h(r) - u_h^{\mathrm{pct\text{-}}}(r), -\Delta_h\tilde{\phi}(u_h^{\mathrm{pct\text{-}}}(r))\right\rangle_{-1} \\ &\quad + 2\left\langle u_h^{\mathrm{pct\text{-}}}(r) - u_h^{\mathrm{plt}}(r), -\Delta_h\tilde{\phi}(u_h^{\mathrm{pct\text{-}}}(r))\right\rangle_{-1} \\ &\quad + 2\left\langle v_h(r) - u_h^{\mathrm{plt}}(r), \partial_t v_h(r)\right\rangle_{-1} \mathrm{d}r \\ &\leq 2\int_0^t \varphi_h(v_h(r))\mathrm{d}r - 2\int_0^t \varphi_h(u_h^{\mathrm{pct\text{-}}}(r))\mathrm{d}r + 2\int_0^t \left\langle v_h(r) - u_h^{\mathrm{plt}}(r), \partial_t v_h(r)\right\rangle_{-1} \mathrm{d}r \\ &\quad + 2\int_0^t \left\langle u_h^{\mathrm{pct\text{-}}}(r) - u_h^{\mathrm{plt}}(r), -\Delta_h\tilde{\phi}(u_h^{\mathrm{pct\text{-}}}(r))\right\rangle_{-1} \mathrm{d}r,\end{aligned}$$

as required. ☐

Proposition 2.5.11. *Let $v \in \mathcal{C}^1([0,T];L^2)$ and $u \in L^\infty([0,T];H^{-1})$ be the limit process of $(u_h)_{h>0}$ as in Lemma 2.5.4. Then*

$$\begin{aligned}\|v(t) - u(t)\|_{H^{-1}}^2 + 2\int_0^t \varphi(u(r))\mathrm{d}r \leq \|v(0) - u(0)\|_{H^{-1}}^2 &+ 2\int_0^t \varphi(v(r))\mathrm{d}r \\ &+ 2\int_0^t \langle v(r) - u(r), \partial_t v(r)\rangle_{H^{-1}} \mathrm{d}r\end{aligned} \tag{2.5.9}$$

for almost all $t \in [0,T]$.

Proof. Let Assumption 2.2.4 be satisfied. To show (2.5.9), we aim to pass to the limit in (2.5.8), where we use the sequence $(v_h)_{h>0}$ constructed in Lemma 2.5.6 for the function v. Note that Lemma 2.5.10 applies, since (2.5.4) implies that $(\partial_t v_h)^{\mathrm{pcx}}$ is bounded in $L^2([0,T];L^2)$ and hence

$$\int_0^T \|\partial_t v_h\|_0^2 \,\mathrm{d}t = \int_0^T \|(\partial_t v_h)^{\mathrm{pcx}}\|_{L^2}^2 \,\mathrm{d}t < \infty$$

by the isometry in Remark 2.1.4.

To this end, let $\gamma \in L^\infty([0,T])$ satisfy $\gamma \geq 0$ $\mathrm{d}t$-almost everywhere, consider the subsequence $(h_m)_{m\in\mathbb{N}}$ constructed in Lemma 2.5.6, which we will denote by h, and let $(v_h)_{h>0}$ be defined as in Lemma 2.5.6.

Integrating against γ in (2.5.8) yields

$$\begin{aligned}
&\int_0^T \gamma(t) \left\| v_h(t) - u_h^{\text{plt}}(t) \right\|_{-1}^2 \mathrm{d}t + 2\int_0^T \gamma(t) \int_0^t \varphi_h(u_h^{\text{pct-}}(r)) \,\mathrm{d}r \,\mathrm{d}t \\
&\le \int_0^T \gamma(t) \left\| v_h(0) - u_h^* \right\|_{-1}^2 \mathrm{d}t + 2\int_0^T \gamma(t) \int_0^t \varphi_h(v_h(r)) \,\mathrm{d}r \,\mathrm{d}t \\
&\quad + 2\int_0^T \gamma(t) \int_0^t \left\langle v_h(r) - u_h^{\text{plt}}(r), \partial_t v_h(r) \right\rangle_{-1} \mathrm{d}r \,\mathrm{d}t \\
&\quad + 2\int_0^T \gamma(t) \int_0^t \left\langle u_h^{\text{pct-}}(r) - u_h^{\text{plt}}(r), -\Delta_h \phi(u_h^{\text{pct-}}(r)) \right\rangle_{-1} \mathrm{d}r \,\mathrm{d}t.
\end{aligned} \tag{2.5.10}$$

We treat each term in (2.5.10) separately. For the first term, we use Lemma 2.4.9 and the weak lower-semicontinuity of the norm in $L^2(([0,T],\gamma\,\mathrm{d}t);H^{-1})$ to obtain

$$\begin{aligned}
\liminf_{h\to 0} \int_0^T \gamma(t) \left\| v_h(t) - u_h^{\text{plt}}(t) \right\|_{-1}^2 \mathrm{d}t &\ge \liminf_{h\to 0} \int_0^T \gamma(t) \left\| v_h^{\text{pcx}}(t) - u_h^{\text{plt,pcx}}(t) \right\|_{H^{-1}}^2 \mathrm{d}t \\
&\ge \int_0^T \gamma(t) \left\| v(t) - u(t) \right\|_{H^{-1}}^2 \mathrm{d}t,
\end{aligned}$$

where the weak convergence necessary for the last step is due to Lemma 2.5.4 and Lemma 2.5.6. Note that as in the proof of (2.4.77), we have that

$$u_h^{\text{plt,pcx}} \rightharpoonup u \text{ in } L^2([0,T];H^{-1}) \quad \text{implies} \quad u_h^{\text{plt,pcx}} \to u \text{ in } L^2(([0,T],\gamma\,\mathrm{d}t);H^{-1}), \tag{2.5.11}$$

For the second term, we first use Fubini's theorem to rewrite for any measurable function $f:[0,T]\to\mathbb{R}$

$$\begin{aligned}
\int_0^T \gamma(t) \int_0^t f(r)\mathrm{d}r\,\mathrm{d}t &= \int_0^T \int_0^T \gamma(t) \mathbf{1}_{[0,t]}(r) f(r)\,\mathrm{d}r\,\mathrm{d}t \\
&= \int_0^T f(r) \int_0^T \mathbf{1}_{[r,T]}(t)\gamma(t)\,\mathrm{d}t\,\mathrm{d}r \\
&= \int_0^T f(r)\tilde{\gamma}(r)\mathrm{d}r,
\end{aligned}$$

where $\tilde{\gamma}:[0,T]\to\mathbb{R}$ with

$$\tilde{\gamma}(r) = \int_r^T \gamma(t)\mathrm{d}t.$$

Since $\varphi: H^{-1}\to[0,\infty]$ is convex and lower-semicontinuous, as proved in Section 3.3, the map

$$L^2(([0,T],\tilde{\gamma}\mathrm{d}t);H^{-1}) \ni u \mapsto \int_0^T \varphi(u(t))\tilde{\gamma}(t)\mathrm{d}t$$

is convex and lower-semicontinuous by [19, Proposition 16.50] and hence, by [19, Theorem 9.1], weakly sequentially lower-semicontinuous. Since (2.5.11) is also satisfied wit γ replaced by $\tilde{\gamma}$ and $u_h^{\text{plt,pcx}}$ replaced by $u_h^{\text{pct-pcx}}$, we obtain

$$\begin{aligned}
\liminf_{h\to 0} \int_0^T \gamma(t) \int_0^t \varphi_h(u_h^{\text{pct-}}(r))\,\mathrm{d}r\,\mathrm{d}t &= \liminf_{h\to 0} \int_0^T \varphi(u_h^{\text{pct-pcx}}(r))\tilde{\gamma}(r)\,\mathrm{d}r \\
&\ge \int_0^T \varphi(u(r))\tilde{\gamma}(r)\,\mathrm{d}r \\
&= \int_0^T \gamma(t) \int_0^t \varphi(u(r))\,\mathrm{d}r\,\mathrm{d}t,
\end{aligned}$$

where we used Remark 2.5.8 in the first step and Lemma 2.5.4 in the second step.

For the third term, recall from Lemma 2.1.2 that

$$(v_h(0))^{\text{pcx}} = \Pi_h^{\text{pcx}} v(0) \to v(0) \quad \text{in } L^2 \text{ for } h\to 0$$

and $(u_h^*)^{\mathrm{pcx}} \to u_0$ in L^2 for $h \to 0$ by assumption. By the same arguments as in the proof of Proposition 2.4.29, we then have

$$\left(-\Delta_h^{-1}(v_h(0) - u_h^*)\right)^{\mathrm{pcx}} \rightharpoonup -\Delta^{-1}(v(0) - u_0) \quad \text{in } L^2 \text{ for } h \to 0,$$

which allows to compute

$$\begin{aligned}
\lim_{h\to 0} \int_0^T \gamma(t)\, \|v_h(0) - u_h^*\|_{-1}^2 \,\mathrm{d}t &= \lim_{h\to 0} \left\langle -\Delta_h^{-1}(v_h(0) - u_h^*), v_h(0) - u_h^* \right\rangle_0 \int_0^T \gamma(t)\,\mathrm{d}t \\
&= \lim_{h\to 0} \left\langle (-\Delta_h^{-1}(v_h(0) - u_h^*))^{\mathrm{pcx}}, (v_h(0) - u_h^*)^{\mathrm{pcx}} \right\rangle_{L^2} \int_0^T \gamma(t)\,\mathrm{d}t \\
&= \left\langle -\Delta^{-1}(v(0) - u_0), v(0) - u_0 \right\rangle_{L^2} \int_0^T \gamma(t)\,\mathrm{d}t \\
&= \int_0^T \gamma(t)\, \|v(0) - u_0\|_{H^{-1}}^2 \,\mathrm{d}t.
\end{aligned}$$

For the fourth term, we use that ψ defined in (2.2.5) is Lipschitz continuous with Lipschitz constant 1, Remark 2.5.8 and the fact that $v(t), v_h^{\mathrm{pcx}}(t) \in L^1$ for all $t \in [0,T]$ to obtain

$$\begin{aligned}
&\left| \int_0^T \gamma(t) \int_0^t \varphi(v(r))\,\mathrm{d}r\,\mathrm{d}t - \int_0^T \gamma(t) \int_0^t \varphi_h(v_h(r))\,\mathrm{d}r\,\mathrm{d}t \right| \\
&\le \int_0^T \gamma(t) \left| \int_0^t \varphi(v(r)) - \varphi(v_h^{\mathrm{pcx}}(r))\,\mathrm{d}r \right| \mathrm{d}t \\
&\le \int_0^T \gamma(t) \int_0^t \int_0^1 |\psi(v(r,x)) - \psi(v_h^{\mathrm{pcx}}(r,x))|\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}t \\
&\le \int_0^T \gamma(t)\mathrm{d}t \int_0^T \int_0^1 |v(r,x) - v_h^{\mathrm{pcx}}(r,x)|\,\mathrm{d}x\,\mathrm{d}r \\
&\le \sqrt{T} \int_0^T \gamma(t)\mathrm{d}t\, \|v - v_h^{\mathrm{pcx}}\|_{L^2([0,T];L^2)} \to 0
\end{aligned} \tag{2.5.12}$$

for $h \to 0$ by the construction of $(v_h)_{h>0}$ in Lemma 2.5.6.

We notice that the integrand of the fifth term is uniformly bounded in t due to

$$\begin{aligned}
&\gamma(t) \int_0^t \left\langle v_h(r) - u_h^{\mathrm{plt}}(r), \partial_t v_h(r) \right\rangle_{-1} \mathrm{d}r \\
&\le \|\gamma\|_{L^\infty([0,T])} \int_0^T \left\| v_h(r) - u_h^{\mathrm{plt}}(r) \right\|_{-1} \|\partial_t v_h(r)\|_{-1}\,\mathrm{d}r \\
&\le \|\gamma\|_{L^\infty([0,T])} \left(\int_0^T \left\| v_h(r) - u_h^{\mathrm{plt}}(r) \right\|_{-1}^2 \mathrm{d}r \right)^{\frac{1}{2}} \left(\int_0^T \|\partial_t v_h(r)\|_{-1}^2\,\mathrm{d}r \right)^{\frac{1}{2}} \\
&\le 9\,\|\gamma\|_{L^\infty([0,T])} \left(\int_0^T \left\| v_h^{\mathrm{pcx}}(r) - u_h^{\mathrm{plt,pcx}}(r) \right\|_{H^{-1}}^2 \mathrm{d}r \right)^{\frac{1}{2}} \left(\int_0^T \|(\partial_t v_h(r))^{\mathrm{pcx}}\|_{H^{-1}}^2\,\mathrm{d}r \right)^{\frac{1}{2}},
\end{aligned} \tag{2.5.13}$$

where we used Lemma 2.4.9 in the last step. Since $(v_h^{\mathrm{pcx}})_{h>0}$ and $((\partial_t v_h)^{\mathrm{pcx}})_{h>0}$ are weakly convergent in $L^2([0,T];H^{-1})$ by Lemma 2.5.6 and $\left(u_h^{\mathrm{plt,pcx}}\right)_{h>0}$ is weakly* convergent in $L^\infty([0,T];H^{-1})$ by Lemma 2.5.4 and hence also weakly convergent in $L^2([0,T];H^{-1})$, these sequences are bounded, i. e.

$$(2.5.13) \le C$$

independently of t. For each t, we note that

$$\partial_t v_h^{\mathrm{pcx}} \to \partial_t v \text{ in } L^2([0,T];L^2) \quad \text{implies} \quad \mathbf{1}_{[0,t]} \partial_t v_h^{\mathrm{pcx}} \to \mathbf{1}_{[0,t]} \partial_t v \text{ in } L^2([0,T];L^2),$$

such that Proposition 2.4.29 applies and yields

$$\begin{aligned}\int_0^t \left\langle v_h(r) - u_h^{\mathrm{plt}}(r), \partial_t v_h(r)\right\rangle_{-1} \mathrm{d}r &= \int_0^T \left\langle v_h(r) - u_h^{\mathrm{plt}}(r), \mathbf{1}_{[0,t]}(r)\partial_t v_h(r)\right\rangle_{-1} \mathrm{d}r \\ &\to \int_0^T \left\langle v(r) - u(r), \mathbf{1}_{[0,t]}(r)\partial_t v(r)\right\rangle_{H^{-1}} \mathrm{d}r \\ &= \int_0^t \left\langle v(r) - u(r), \partial_t v(r)\right\rangle_{H^{-1}} \mathrm{d}r\end{aligned}$$

for $h \to 0$. Hence, by dominated convergence,

$$\int_0^T \gamma(t) \int_0^t \left\langle v_h(r) - u_h^{\mathrm{plt}}(r), \partial_t v_h(r)\right\rangle_{-1} \mathrm{d}r\, \mathrm{d}t \to \int_0^T \gamma(t) \int_0^t \left\langle v(r) - u(r), \partial_t v(r)\right\rangle_{H^{-1}} \mathrm{d}r\, \mathrm{d}t$$

for $h \to 0$.

For the last term, we note that

$$\begin{aligned}\sup_{t\in[0,T]} \left\| u_h^{\mathrm{plt}}(t) - u_h^{\mathrm{pct\text{-}}}(t)\right\|_{-1}^2 &= \max_{n\in\{0,\dots,N-1\}} \sup_{t\in[n\tau,(n+1)\tau)} \left\| \frac{t-n\tau}{\tau}\left(u_h^{n+1} - u_h^n\right)\right\|_{-1}^2 \\ &\leq \max_{n\in\{0,\dots,N-1\}} \left\| u_h^{n+1} - u_h^n\right\|_{-1}^2 \leq 4\frac{\tau^2}{h^2},\end{aligned}$$

using Corollary 2.5.2. Using Estimate (2.4.6) and the boundedness of $\tilde{\phi}$, this leads to

$$\begin{aligned}&\left| \int_0^T \gamma(t) \int_0^t \left\langle u_h^{\mathrm{pct\text{-}}}(r) - u_h^{\mathrm{plt}}(r), -\Delta_h \tilde{\phi}(u_h^{\mathrm{pct\text{-}}}(r))\right\rangle_{-1} \mathrm{d}r\, \mathrm{d}t\right| \\ &\leq \int_0^T \gamma(t)\, \mathrm{d}t \sup_{t\in[0,T]} \left\| u_h^{\mathrm{plt}}(t) - u_h^{\mathrm{pct\text{-}}}(t)\right\|_{-1} \int_0^T \left\| -\Delta_h \tilde{\phi}(u_h^{\mathrm{pct\text{-}}}(r))\right\|_{-1} \mathrm{d}r \\ &\leq \int_0^T \gamma(t)\, \mathrm{d}t\, 2\frac{\tau}{h} \int_0^T \frac{2}{h} \left\| \tilde{\phi}(u_h^{\mathrm{pct\text{-}}}(r))\right\|_0 \mathrm{d}r \\ &\leq \int_0^T \gamma(t)\, \mathrm{d}t\, 4T\frac{\tau}{h^2} \to 0\end{aligned} \tag{2.5.14}$$

for $h \to 0$. Hence, taking $\liminf_{h\to 0}$ in (2.5.10), we obtain

$$\begin{aligned}&\int_0^T \gamma(t) \left\| v(t) - u(t)\right\|_{H^{-1}} \mathrm{d}t + 2\int_0^T \gamma(t) \int_0^t \varphi(u(r))\, \mathrm{d}r\, \mathrm{d}t \\ &\leq \int_0^T \gamma(t) \left\| v(0) - u(0)\right\|_{H^{-1}} \mathrm{d}t + 2\int_0^T \gamma(t) \int_0^t \varphi(v(r))\, \mathrm{d}r\, \mathrm{d}t \\ &\quad + 2\int_0^T \gamma(t) \int_0^t \left\langle v(r) - u(r), \partial_t v(r)\right\rangle_{H^{-1}} \mathrm{d}r\, \mathrm{d}t,\end{aligned} \tag{2.5.15}$$

and since $\gamma \in L^\infty([0,T])$, $\gamma \geq 0$, was chosen arbitrarily, (2.5.9) follows. □

Proposition 2.5.12. *Let*

$$v \in W^{1,2}(0,T;L^2,H^{-1}) := \left\{ v \in L^2([0,T];L^2) | \partial_t v \in L^2([0,T];H^{-1})\right\}$$

and $u \in L^\infty([0,T];H^{-1})$ be the limit process of $(u_h)_{h>0}$ as in Lemma 2.5.4. Then,

$$\begin{aligned}\left\| v(t) - u(t)\right\|_{H^{-1}}^2 + 2\int_0^t \varphi(u(r))\mathrm{d}r &\leq \left\| v(0) - u(0)\right\|_{H^{-1}}^2 + 2\int_0^t \varphi(v(r))\mathrm{d}r \\ &\quad + 2\int_0^t \left\langle v(r) - u(r), \partial_t v(r)\right\rangle_{H^{-1}} \mathrm{d}r\end{aligned} \tag{2.5.16}$$

for almost all $t \in [0,T]$.

Proof. By [97, Theorem 2.1], the space $\mathcal{C}^1([0,T];L^2)$ is dense in $W^{1,2}(0,T;L^2,H^{-1})$ with respect to the norm

$$\|u\|^2_{W^{1,2}(0,T;L^2,H^{-1})} = \|u\|^2_{L^2([0,T];L^2)} + \|\partial_t u\|^2_{L^2([0,T];H^{-1})}.$$

Hence, there is a sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{C}^1([0,T];L^2)$ such that

$$v_n \to v \text{ in } L^2([0,T];L^2) \quad \text{and} \quad \partial_t v_n \to \partial_t v \text{ in } L^2([0,T];H^{-1}). \tag{2.5.17}$$

By the continuous embedding

$$W^{1,2}(0,T;L^2,H^{-1}) \hookrightarrow \mathcal{C}([0,T];H^{-1})$$

(see e. g. [97, Theorem 3.1]), we also have

$$v_n(t) \to v(t) \quad \text{in } H^{-1} \text{ for } n \to \infty \tag{2.5.18}$$

for all $t \in [0,T]$. By Proposition 2.5.11, (2.5.16) is satisfied for v replaced by v_n for all $n \in \mathbb{N}$. Then, the convergence properties (2.5.17) and (2.5.18) are sufficient to pass to the limit, where the limit

$$\int_0^t \varphi(v_n(r))\,\mathrm{d}r \to \int_0^t \varphi(v(r))\,\mathrm{d}r \quad \text{for } n \to \infty$$

is obtained as in (2.5.12). This finishes the proof. □

Proof of Theorem 2.2.8. For each sequence $(h_m)_{m\in\mathbb{N}}$ satisfying Assumption 2.2.4, Lemma 2.5.4 provides a subsequence denoted by $h \to 0$ and $u \in L^\infty([0,T];H^{-1})$, such that $u_h \overset{*}{\rightharpoonup} u$ in $L^\infty([0,T];H^{-1})$ for $h \to 0$. In order to show that the limit u of any such subsequence is a dt-version of the unique VI solution to (2.1.3), we will apply a procedure similar to the uniqueness argument in the proof of Theorem 3.2.6, to which we refer for details. Let $(v_0^n)_{n\in\mathbb{N}} \subset L^2$ satisfying

$$v_0^n \to u_0 \quad \text{in } H^{-1} \text{ for } n \to \infty \tag{2.5.19}$$

and let $(v^{\varepsilon,n})_{\varepsilon>0,n\in\mathbb{N}}$ be the solutions to

$$\begin{aligned} \mathrm{d}v^{\varepsilon,n}(t) &= \varepsilon\Delta v^{\varepsilon,n}(t)\,\mathrm{d}t + \Delta\phi^\varepsilon(v^{\varepsilon,n}(t))\,\mathrm{d}t \\ v^{\varepsilon,n}(0) &= v_0^n, \end{aligned} \tag{2.5.20}$$

which is an admissible choice for v in Proposition 2.5.12. It is known that

$$\sup_{t\in[0,T]} \|v^{\varepsilon,n}(t)\|^2_{L^2} + \varepsilon \int_0^T \|v^{\varepsilon,n}(t)\|^2_{H_0^1}\,\mathrm{d}t \le C \tag{2.5.21}$$

for some $C > 0$ independent of ε and n, and

$$v^{\varepsilon,n} \to v \quad \text{in } \mathcal{C}([0,T];H^{-1}) \tag{2.5.22}$$

for $\varepsilon \to 0$ and then $n \to \infty$, where v is the unique VI solution to (2.1.3). For almost all $t \in [0,T]$, Proposition 2.5.12 yields

$$\begin{aligned} &\|u(t) - v^{\varepsilon,n}(t)\|^2_{H^{-1}} + 2\int_0^t \varphi(u(r))\,\mathrm{d}r \\ &\le \|u_0 - v_0^n\|^2_{H^{-1}} + 2\int_0^t \varphi(v^{\varepsilon,n}(r))\,\mathrm{d}r \\ &\quad - 2\int_0^t \langle \varepsilon\Delta v^{\varepsilon,n}(r) + \Delta\phi^\varepsilon(v^{\varepsilon,n}(r)), u(r) - v^{\varepsilon,n}(r)\rangle_{H^{-1}}\,\mathrm{d}r. \end{aligned} \tag{2.5.23}$$

Using

$$\langle -\Delta\phi^\varepsilon(v^{\varepsilon,n}), u - v^{\varepsilon,n}\rangle_{H^{-1}} + \varphi(v^{\varepsilon,n}) \le \varphi(u) + C\varepsilon\left(1 + \|v^{\varepsilon,n}\|^2_{L^2}\right) \quad \mathrm{d}t\text{-almost everywhere}$$

and the weighted Young inequality, (2.5.23) turns into

$$\begin{aligned}\|u(t)-v^{\varepsilon,n}(t)\|_{H^{-1}}^2 \leq\ & \|u_0-v_0^n\|_{H^{-1}}^2\\ &+2\int_0^t \varepsilon^{\frac{4}{3}}\,\|\Delta v^{\varepsilon,n}(r)\|_{H^{-1}}^2\,\mathrm{d}r+\varepsilon^{\frac{2}{3}}\,\|u(r)-v^{\varepsilon,n}(r)\|_{H^{-1}}^2\,\mathrm{d}r\\ &+C\varepsilon\int_0^t\Big(1+\|v^{\varepsilon,n}(r)\|_{L^2}^2\Big)\,\mathrm{d}r.\end{aligned}$$

Passing to the limit $\varepsilon \to 0$ and then $n \to \infty$, using (2.5.19), (2.5.21) and (2.5.22), we obtain

$$\|u(t)-v(t)\|_{H^{-1}}^2 \leq 0 \quad \text{for almost all } t \in [0,T],$$

which means that u is uniquely determined in $L^\infty([0,T];H^{-1})$ as the dt-equivalence class belonging to the VI solution to (2.1.3). Since Assumption 2.2.4 is stable under taking subsequences, this means that each subsequence of $(h_m)_{m\in\mathbb{N}}$ contains a subsequence $h \to 0$ such that $u_h \overset{*}{\rightharpoonup} v$ for $h \to 0$ in $L^\infty([0,T];H^{-1})$. This proves the theorem. ☐

2.A Measurability with respect to the weak(*) topology

Lemma 2.A.1. *Let H be a separable metric space. Then, each open nonempty set is an at most countable union of open balls.*

Proof. Let $Y \subset H$ a countable set such that $\overline{Y} = H$ and let $A \subset H$ open. We claim that

$$A = A' := \bigcup\{B_r(y) \subseteq A : y \in Y, r \in \mathbb{Q}_{\geq 0}\}, \tag{2.A.1}$$

where $B_r(y)$ denotes the open ball around y with radius r. By construction, $A' \subseteq A$. To see the reverse inclusion, let $x \in A$ and choose $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$. Then, choose $y \in Y$ such that $d(x,y) < \frac{\varepsilon}{2}$ and $r \in \mathbb{Q}_{\geq 0}$ such that $d(x,y) < r < \frac{\varepsilon}{2}$. Then

$$x \in B_r(y) \subset B_\varepsilon(x) \subset A',$$

which proves (2.A.1) and thus completes the proof. ☐

Lemma 2.A.2. *Let H be a separable Hilbert space, τ the strong topology and τ_w the weak topology. For any topology ρ, let $\mathcal{B}(\rho)$ be the σ-algebra generated by ρ. Then,*

$$\mathcal{B}(\tau) = \mathcal{B}(\tau_w).$$

Proof. Recall that each closed set with respect to τ_w is closed with respect to τ, hence $\tau_w \subset \tau$. Consequently, we have $\mathcal{B}(\tau_w) \subseteq \mathcal{B}(\tau)$. It remains to show that each open set with respect to τ is contained in $\mathcal{B}(\tau_w)$. In view of Lemma 2.A.1, it is enough to show this for open balls $B_r(y) \subset H$, $r > 0, y \in H$.

To this end, choose an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of H, which exists according to [1, Satz 7.8]. Then, by definition of the weak topology, for each $n \in \mathbb{N}$ the function $f_n : H \to \mathbb{R}$ given by

$$f_n(x) = \sum_{i=1}^n |\langle x-y, e_i\rangle_H|^2$$

is continuous with respect to τ_w and thus $\mathcal{B}(\tau_w)$-measurable. Consequently, $f : H \to \mathbb{R}$ defined by

$$f(x) = \|x-y\|_H^2 = \lim_{n\to\infty} f_n(x)$$

is $\mathcal{B}(\tau_w)$-measurable as the pointwise limit of measurable functions. Thus,

$$B_r(y) = f^{-1}\big([0,r)\big) \in \mathcal{B}(\tau_w),$$

which completes the proof. ☐

Lemma 2.A.3. *Let V be a separable Banach space, V' its dual space and τ_w^* the weak* topology on V'. Then, $\|\cdot\|_{V'}$ is measurable with respect to $\mathcal{B}(\tau_w^*)$.*

Proof. First, notice that for $\phi \in V$ we have

$$V' \ni u \mapsto \frac{|{}_{V'}\langle u, \phi\rangle_V|}{\|\phi\|_V}$$

is continuous with respect to τ_w^* by definition and hence measurable with respect to $\mathcal{B}(\tau_w^*)$. Let D be a dense countable subset of V and B_1^V the open unit ball in V. Then, writing

$$\|u\|_{V'} = \sup_{\phi\in\overline{B_1^V}} \frac{|{}_{V'}\langle u, \phi\rangle_V|}{\|\phi\|_V} = \sup_{\phi\in B_1^V\cap D} \frac{|{}_{V'}\langle u, \phi\rangle_V|}{\|\phi\|_V}, \tag{2.A.2}$$

we observe that $\|\cdot\|_{V'}$ is the countable supremum of $\mathcal{B}(\tau_w^*)$-measurable functions and thus measurable. To justify the "$\leq$" direction in the last equality of (2.A.2), first note that $B_1^V \cap D$ is dense in B_1^V, such that for any $\phi \in \overline{B_1^V}$, we may choose a sequence $(\phi_n)_{n\in\mathbb{N}} \subset B_1^V \cap D$ such that $\phi_n \to \phi$ in V for $n \to \infty$. Then,

$$\frac{|{}_{V'}\langle u, \phi_n\rangle_V|}{\|\phi_n\|_V} \to \frac{|{}_{V'}\langle u, \phi\rangle_V|}{\|\phi\|_V} \quad \text{for } n \to \infty$$

by the fact that $u : V \to \mathbb{R}$ is continuous. This completes the proof. □

2.B Measurability and weak convergence in Bochner spaces

Definition 2.B.1 (Bochner space)**.** Let $1 \leq p < \infty$, V be a separable Banach space and $(\Omega, \mathcal{F}, \mathbb{P})$ a finite measure space. Then the space $L^p((\Omega, \mathcal{F}, \mathbb{P}); V)$, also denoted by $L^p(\Omega; V)$ is the space of functions $f : \Omega \to V$ which are strongly $\mathcal{F}$-$\mathcal{B}(V)$-measurable, i. e. there are measurable sets $(A_k)_{k\in\mathbb{N}} \subset \mathcal{F}$ and a sequence $(\eta_k)_{k\in\mathbb{N}} \subset V$ such that

$$f(\omega) = \lim_{N\to\infty} \sum_{k=1}^{N} \mathbf{1}_{A_k}(\omega)\eta_k \quad \text{for } \mathbb{P} - \text{almost all } \omega \in \Omega,$$

and

$$\int_\Omega \|f\|_V^p \,\mathrm{d}\mathbb{P} < \infty. \tag{2.B.1}$$

Remark 2.B.2. Note that for any strongly measurable function $f : \Omega \to V$, $\|f\|_V : \Omega \to \mathbb{R}$ is measurable, such that (2.B.1) is well-defined.

Lemma 2.B.3. *Let V be a separable Banach space and $(\Omega, \mathcal{F}, \mathbb{P})$, $(\Omega', \mathcal{F}', \mathbb{P}')$ finite measure spaces. Let*

$$f \in L^p\left((\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}'); V\right) \quad \textit{and} \quad g \in L^p((\Omega, \mathcal{F}, \mathbb{P}); L^p((\Omega', \mathcal{F}', \mathbb{P}'); V)).$$

Then $\tilde{f} \in L^p((\Omega, \mathcal{F}, \mathbb{P}); L^p((\Omega', \mathcal{F}', \mathbb{P}'); V))$ and $\tilde{g} \in L^p((\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}'); V)$, where

$$\tilde{f}(\omega) = f(\omega, \cdot) \quad \textit{and} \quad \tilde{g}(\omega, \omega') = g(\omega)(\omega').$$

The identification is linear and isometric, i. e.

$$\|f\|_{L^p((\Omega\times\Omega', \mathcal{F}\otimes\mathcal{F}', \mathbb{P}\otimes\mathbb{P}'); V)} = \|\tilde{f}\|_{L^p((\Omega, \mathcal{F}, \mathbb{P}); L^p((\Omega', \mathcal{F}', \mathbb{P}'); V))}.$$

Proof. See e. g. [86, Proposition 1.2.24]. □

Remark 2.B.4. We will mostly treat f and $\tilde{f}$ as equivalent.

Corollary 2.B.5. *Let V be a separable Banach space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space endowed with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Let*

$$f \in L^2((\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T]), \mathbb{P} \otimes \mathrm{d}t); V). \tag{2.B.2}$$

Then, f is progressively measurable with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ if and only if

$$\tilde{f}|_{[0,t]} \text{ is } \mathcal{F}_t - \mathcal{B}(L^2([0,t];V)) \text{-measurable for all } t \in [0,T], \tag{2.B.3}$$

where

$$\tilde{f} \in L^2((\Omega,\mathcal{F},\mathbb{P}); L^2(([0,T],\mathcal{B}([0,T]),\mathrm{d}t);V)), \quad \tilde{f}(\omega) = f(\omega,\cdot) \tag{2.B.4}$$

as in Lemma 2.B.3. Note that the isomorphism in Lemma 2.B.3 commutes with restriction of one of the two variables, which is why there is no need to reflect the order of these operations in the notation.

Proof. If f is progressively measurable, we have $f|_{[0,t]}$ is $\mathcal{F}_t \otimes \mathcal{B}([0,t])$-$\mathcal{B}(V)$-measurable. Since V is separable, this amounts to strong measurability, such that together with the integrability of f, we obtain $f|_{[0,t]} \in L^2((\Omega\times[0,t], \mathcal{F}_t\otimes\mathcal{B}([0,t]), \mathbb{P}\otimes \mathrm{d}t);V)$. Lemma 2.B.3 then yields that $\tilde{f}|_{[0,t]} \in L^2((\Omega,\mathcal{F}_t,\mathbb{P}); L^2(([0,t],\mathcal{B}([0,t],\mathrm{d}t);V))$, which includes $\mathcal{F}_t$-measurability.

For the reverse statement, note that $\tilde{f}|_{[0,t]}$ is separably valued, such that (2.B.3) implies strong measurability. Moreover, (2.B.4) provides the integrability requirement to conclude

$$\tilde{f}|_{[0,t]} \in L^2((\Omega,\mathcal{F}_t,\mathbb{P}); L^2(([0,t],\mathcal{B}([0,t]),\mathrm{d}t);V)).$$

Then, Lemma 2.B.3 yields $f|_{[0,t]} \in L^2((\Omega\times[0,t],\mathcal{F}_t\otimes\mathcal{B}([0,t]),\mathbb{P}\otimes\mathrm{d}t);V)$, which implies that f is progressively measurable with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, as required. □

Lemma 2.B.6. *Let H be a separable Hilbert space, and $(\Omega,\mathcal{F},\mathbb{P})$ a measurable space. Let $(f_n)_{n\in\mathbb{N}} \subset L^2(\Omega;H)$ be uniformly bounded and $f \in L^2(\Omega;H)$ such that $f_n \rightharpoonup f$ pointwise $\mathbb{P}$-almost everywhere. Then,*

$$f_n \rightharpoonup f \quad \text{in } L^2(\Omega,H).$$

Proof. We begin with showing that for $A\in\mathcal{F}$, $\eta\in H$, $n\to\infty$

$$\mathbb{E}\left[\langle f_n,\eta\rangle_H \mathbf{1}_A\right] \to \mathbb{E}\left[\langle f,\eta\rangle_H \mathbf{1}_A\right]. \tag{2.B.5}$$

To this end, we note that by assumption

$$\langle f_n,\eta\rangle_H \to \langle f,\eta\rangle_H \quad \mathbb{P}\text{-almost everywhere for } n\to\infty,$$

and

$$\mathbb{E}\langle f_n,\eta\rangle_H^2 \le \|\eta\|_H^2\,\mathbb{E}\|f_n\|_H^2 \le C_\eta,$$

where C_η is independent of n. Thus, [50, Satz VI.5.9] provides

$$\langle f_n,\eta\rangle_H \rightharpoonup \langle f,\eta\rangle_H \quad \text{in } L^2(\Omega;\mathbb{R}),$$

which yields (2.B.5). For a general function $\zeta \in L^2(\Omega;H)$, we use the density of simple functions (see e.g. [86, Lemma 1.2.19]) to choose $M\in\mathbb{N}$, $\{A_i\}_{i=1}^M \subset \mathcal{F}$, $\{\eta_i\}_{i=1}^M \subset H$ such that we have for a given $\varepsilon>0$

$$\zeta_M := \sum_{i=1}^M \mathbf{1}_{A_i}\eta_i, \tag{2.B.6}$$

$$S := \max\left\{\sup_{n\in\mathbb{N}}\left(\mathbb{E}\|f_n\|_H^2\right)^{\frac{1}{2}}, \left(\mathbb{E}\|f\|_H^2\right)^{\frac{1}{2}}\right\}, \tag{2.B.7}$$

$$\mathbb{E}\|\zeta_M-\zeta\|_H^2 \le \frac{\varepsilon}{3S}, \tag{2.B.8}$$

where $S<\infty$ by assumption. Then, using (2.B.5), we choose $N\in\mathbb{N}$ such that for $n\ge N$ we have

$$\left|\mathbb{E}\left[\langle f_n-f,\eta_i\rangle_H \mathbf{1}_{A_i}\right]\right| \le \frac{\varepsilon}{3M} \quad \text{for all } i\in\{1,\dots,M\}, \tag{2.B.9}$$

which allows to compute for $n\ge N$

$$\begin{aligned}
&\left|\mathbb{E}\langle f_n,\zeta\rangle_H - \mathbb{E}\langle f,\zeta\rangle_H\right| \\
&\le \left|\mathbb{E}\langle f_n,\zeta-\zeta_M\rangle_H\right| + \left|\mathbb{E}\langle f_n-f,\zeta_M\rangle_H\right| + \left|\mathbb{E}\langle f,\zeta_M-\zeta\rangle_H\right| \\
&\le \left(\left(\mathbb{E}\|f_n\|_H^2\right)^{\frac{1}{2}} + \left(\mathbb{E}\|f\|_H^2\right)^{\frac{1}{2}}\right)\left(\mathbb{E}\|\zeta-\zeta_M\|_H^2\right)^{\frac{1}{2}} + \sum_{i=1}^M \left|\mathbb{E}\left[\langle f_n-f,\eta_i\rangle_H \mathbf{1}_{A_i}\right]\right| \\
&\le \varepsilon,
\end{aligned}$$

using (2.B.8) and (2.B.9) in the last step. This proves the claim. □

2.C Separating families in topological spaces

In [87, Theorem 2], the following condition for a topological space $(\mathcal{X}, \mathcal{T})$ plays a key role.

$$\begin{aligned}&\text{There exists a countable family } \{f_i : \mathcal{X} \to [-1,1]\}_{i\in I}\\ &\text{of } \mathcal{T}\text{-continuous functions which separate points of } \mathcal{X}.\end{aligned} \tag{2.C.1}$$

Lemma 2.C.1. *Condition* (2.C.1) *is satisfied for separable Banach spaces endowed with the strong topology, for separable Hilbert spaces endowed with the weak topology and for duals of separable Banach spaces endowed with the weak* topology.*

Proof. Let $\xi : \mathbb{R} \to [-1,1]$ be continuous and injective, e. g. $\xi = 2\pi^{-1} \arctan$. Let X be a separable Banach space and let $\mathcal{S}_X \subset X$ be a countable dense subset. We choose $x_1, x_2 \in X$, $x_1 \neq x_2$ arbitrary, as well as a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{S}_X$ such that $x_n \to x_1$ for $n \to \infty$. Then there exists $N \in \mathbb{N}$ such that

$$\|x_N - x_1\|_X \leq \frac{1}{4} \|x_1 - x_2\|_X$$

and thus

$$\|x_N - x_2\|_X \geq \|x_1 - x_2\|_X - \|x_N - x_1\|_X \geq \frac{3}{4} \|x_1 - x_2\|_X > \|x_N - x_1\|_X .$$

Hence, $\xi \circ \|x_N - \cdot\|_X$ separates x_1 from x_2, takes values in $[-1,1]$ and is obviously continuous with respect to the strong topology. It follows that the family

$$\{\xi \circ \|x - \cdot\|_X\}_{x\in\mathcal{S}_X}$$

satisfies (2.C.1) in the first case.

Let H be a separable Hilbert space and let $\mathcal{S}_H \subset H$ be a countable dense subset. We observe that $\langle x, \cdot\rangle_H$ is by definition continuous with respect to the weak topology for every $x \in \mathcal{S}_H$. Now choose $y_1, y_2 \in H$, $y_1 \neq y_2$ arbitrary and note that

$$\langle y_1 - y_2, y_1\rangle_H \neq \langle y_1 - y_2, y_2\rangle_H ,$$

since otherwise $\|y_1 - y_2\|^2 = 0$ in contradiction to the assumption. Moreover, we choose a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{S}_H$ such that $x_n \to y_1 - y_2$ for $n \to \infty$, which means that there exists $N \in \mathbb{N}$ such that for $n \geq N$

$$\|x_n - (y_1 - y_2)\|_H \leq \min\left\{ \frac{\|y_1 - y_2\|_H^2}{4\|y_1\|_H}, \frac{\|y_1 - y_2\|_H^2}{4\|y_2\|_H} \right\}.$$

Thus,

$$\begin{aligned}|\langle x_N, y_1\rangle_H - \langle x_N, y_2\rangle_H| &= |\langle x_N - (y_1 - y_2) + (y_1 - y_2), y_1 - y_2\rangle_H| \\ &= \left|\langle x_N - (y_1 - y_2), y_1 - y_2\rangle_H + \|y_1 - y_2\|_H^2\right| \\ &\geq \|y_1 - y_2\|_H^2 - |\langle x_N - (y_1 - y_2), y_1\rangle_H| - |\langle x_N - (y_1 - y_2), y_2\rangle_H| \\ &\geq \frac{1}{2} \|y_1 - y_2\|_H^2 > 0.\end{aligned}$$

Hence, $\xi \circ \langle x_N, \cdot\rangle_H$ separates y_1 from y_2, takes values in $[-1,1]$ and is continuous with respect to the weak topology. It follows that the family

$$\{\xi \circ \langle x, \cdot\rangle_H\}_{x\in\mathcal{S}_H}$$

satisfies (2.C.1) in the second case.

Finally, let B' be the dual space of a separable Banach space B and let $\mathcal{S}_B \subset B$ be a countable and dense subset. We observe that $\langle \cdot, y\rangle_{B'\times B}$ is by definition continuous with respect to the weak* topology for every $y \in B$. Now choose $x_1, x_2 \in B'$, $x_1 \neq x_2$ arbitrary, which means that there exists $y \in B$, $y \neq 0$ such that

$$\langle x_1, y\rangle_{B'\times B} \neq \langle x_2, y\rangle_{B'\times B} .$$

For this y, choose a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathcal{S}_B$ such that $y_n \to y$ for $n \to \infty$, which means that there exists $N \in \mathbb{N}$ such that

$$\|y_N - y\|_B \leq \min\left\{ \frac{\left|\langle x_1 - x_2, y\rangle_{B'\times B}\right|}{4\,\|x_1\|_{B'}}, \frac{\left|\langle x_1 - x_2, y\rangle_{B'\times B}\right|}{4\,\|x_2\|_{B'}} \right\}.$$

Thus,

$$\begin{aligned}
\left|\langle x_1, y_N\rangle_H - \langle x_2, y_N\rangle_{B'\times B}\right| &= \left|\langle x_1 - x_2, y_N - y + y\rangle_{B'\times B}\right| \\
&\geq \left|\langle x_1 - x_2, y\rangle_{B'\times B}\right| - \left|\langle x_1, y_N - y\rangle_{B'\times B}\right| - \left|\langle x_2, y_N - y\rangle_{B'\times B}\right| \\
&\geq \frac{1}{2}\left|\langle x_1 - x_2, y\rangle_{B'\times B}\right| > 0.
\end{aligned}$$

Hence, $\xi \circ \langle \cdot, y_N\rangle_{B'\times B}$ separates x_1 from x_2, takes values in $[-1, 1]$ and is continuous with respect to the weak topology. It follows that the family

$$\{\xi \circ \langle \cdot, y\rangle_{B'\times B}\}_{y\in\mathcal{S}_B}$$

satisfies (2.C.1) in the last case, as required. $\square$

Chapter 3

Well-posedness of SVI solutions to singular-degenerate stochastic porous media equations arising in self-organized criticality

3.1 Introduction

We consider a class of singular-degenerate generalized stochastic porous media equations

$$
\begin{aligned}
\mathrm{d}X_t \in&\ \Delta\left(\phi(X_t)\right)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t,\\
X_0 =&\ x_0,
\end{aligned}
\tag{3.1.1}
$$

on a bounded, smooth domain $\mathcal{O} \subseteq \mathbb{R}^d$ with zero Dirichlet boundary conditions and $x_0 \in H^{-1}$, where H^{-1} is the dual of $H_0^1(\mathcal{O})$. In the following, W is a cylindrical Wiener process on some separable Hilbert space U, and the diffusion coefficients $B : [0,T] \times H^{-1} \times \Omega \to L_2(U, H^{-1})$ take values in the space of Hilbert-Schmidt operators $L_2(U, H^{-1})$. The nonlinearity $\phi : \mathbb{R} \to 2^{\mathbb{R}}$ is the subdifferential of a convex lower-semicontinuous symmetric function $\psi : \mathbb{R} \to \mathbb{R}$ (sometimes called "potential"), which grows at least linearly and at most quadratically for $|x| \to \infty$. As paradigmatic examples, we mention the maximal monotone extensions of

$$
\phi_1(x) = \operatorname{sgn}(x)\left(1 - \mathbf{1}_{(-1,1)}(x)\right) \quad \text{and} \quad \phi_2(x) = x\left(1 - \mathbf{1}_{(-1,1)}(x)\right), \tag{3.1.2}
$$

which are encountered in the context of self-organized criticality, as elaborated in Chapter 1.

The main merits of this article are as follows. First, we give a meaning to (3.1.1) with nonlinearities which are general enough to include ϕ_1 and ϕ_2 in (3.1.2), by defining a suitable notion of solution and proving the existence and uniqueness of such solutions. Second, we extend the applicability of the framework of SVI solutions, which features several properties which are desirable independently of the specific equation presented above. For instance, it applies to stochastic partial differential equations (SPDE) with a very general nonlinear drift term, which is exploited here by relatively lose conditions on the potential ψ. Moreover, solutions for general initial data can be identified by means of the equation and not only in a limiting sense.

We briefly outline the strategy that we are going to apply. First, we rewrite (3.1.1) into the form

$$
\mathrm{d}X_t \in -\partial\varphi(X_t)\,\mathrm{d}t + B(t, X_t)\,\mathrm{d}W_t, \tag{3.1.3}
$$

which incorporates the multivalued function ϕ into an energy functional $\varphi : H^{-1} \to [0, \infty]$. For example, in case of the nonlinearity ϕ_1 in (3.1.2), we define

$$
\varphi(u) = \begin{cases} \|\psi(u)\|_{TV}, & \text{if } u \text{ is a finite Radon measure on } \mathcal{O},\\ +\infty, & \text{else,} \end{cases} \tag{3.1.4}
$$

where ψ is the anti-derivative of ϕ, i. e. $\partial\psi = \phi$, with $\psi(0) = 0$. For the precise definition of a convex function of a measure, we refer to Section 3.3 below. We then derive a stochastic variational inequality (SVI) from (3.1.3) and define a corresponding notion of solution, see Definition 3.2.4 below. In order to construct such a solution we first show that φ as defined above is lower-semicontinuous, which then allows to show the convergence of an approximating sequence gained by a Yosida approximation of the nonlinearity and the addition of a viscosity term. Furthermore, in the proof of uniqueness, it is crucial to show that φ can be well approximated by its values on L^2, which we ensure by showing that it coincides with the lower-semicontinuous hull of $\varphi|_{L^2}$ in H^{-1}. To this end, we will construct approximating sequences by an interplay of mollification and shifts, inspired by the construction of Lemma A6.7 in [1]. This constitutes one technical focus of this work.

The structure of this chapter is as follows. In the subsequent sections of the introduction, we will give a brief overview on the mathematical literature concerning the solution theory of generalized stochastic porous media equations, and we will point out how equation (3.1.1) is motivated by the physics literature. In Section 3.2 we state the precise assumptions and formulate the first main result of this article, in which the well-posedness of Equation (3.1.1) is established (see Theorem 3.2.6 below). We prove the lower-semicontinuity of the abovementioned energy functional φ and the property of φ being the lower-semicontinuous hull of $\varphi|_{L^2}$ in H^{-1} in Section 3.3, the latter of which is the second main result (see Theorem 3.3.8 below). In Section 3.4, the well-posedness result will be proved, following the arguments of Section 2 in [74].

The results of this chapter are accepted for publication, see [103].

3.1.1 Mathematical Literature

In the recent decades, stochastic porous media equations have been very present in the mathematical literature. For the original case

$$\mathrm{d}X_t = \Delta\phi(X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t, \tag{3.1.5}$$

where $\phi(r) = r^{[m]} := |r|^{m-1} r$ for $r \in \mathbb{R}$ and $m \geq 1$ ($m = 1$ representing the stochastic heat equation), a concisely summarized well-posedness analysis can be found in [112], which goes back to the work of Krylov and Rozovskii [92] and Pardoux [106]. In [113], the theory is extended to the fast diffusion case $m \in (0, 1)$, and other nonlinear functions ϕ are considered. A setting with a more general monotone and differentiable nonlinearity is considered in [13].

A severe additional difficulty arises when one considers the limit case $m = 0$, in which ϕ becomes multivalued. The first articles treating this type of porous medium equations, [14] and [12], either require ϕ to be surjective or more restrictions on the initial state or the noise. In [78], the $m = 0$ limit of (3.1.5) can be treated, but one has to restrict to more regular initial data or to the concept of limiting solutions. For general initial conditions, this notion of solution contains no characterization in terms of the equation, which is often necessary for further work such as stability results (see e. g. [76]).

In [11] and later in [17, 73], the concept of stochastic variational inequalities (SVIs) and a corresponding notion of solution have been used to overcome these issues. We note that in [73], an identification of a functional as a lower-semicontinuous hull was needed in the context of p-Laplace type equations with a C^2 potential, going back to results from [2, 57]. In [74], the existence and uniqueness of SVI solutions was proven for the $m = 0$ limit of (3.1.5), for which a refinement of previous methods became necessary, because the naive choice for the energy functional does not lead to an energy space with adequate compactness properties. The arising difficulties when setting up the energy functional are similar to the ones mentioned above for φ from (3.1.4). They have been overcome in [74] by using the specific shape of the nonlinearity, which allows to set the energy functional to

$$\varphi(u) = \begin{cases} \|u\|_{TV}, & \text{if } u \text{ is a finite Radon measure on } \mathcal{O}, \\ +\infty, & \text{else} \end{cases}$$

for $u \in H^{-1}$, which then allows to use structural properties of the TV norm. With more regularity or structural assumptions on the noise and/or the initial state, more regularity for SVI solutions or the existence of strong solutions can be proved, as e. g. in [74, 73, 17, 65]. For the regularization by noise of quasi-linear SPDE with possibly singular drift terms, we also mention the works [63, 83].

We next mention several different approaches to stochastic porous media equations. The article [18] considers the equation on an unbounded domain, the works [9, 35] use an approach via Kolmogorov equations. In [16], an operatorial approach to SPDE is introduced which can be applied to generalized stochastic porous media equations with continuous nonlinearities. In [70, 41] and [38], stochastic porous media equations are solved in the sense of kinetic or entropy solutions, respectively. Previous works in those directions are, e. g., [20, 42] and [23, 56, 89]. [75] makes use of a rough path approach leading to pathwise rough kinetic/entropy solutions and including regularity results, with [62, 98] as some of the related preceding works.

Regarding the construction and analysis of the energy functional arising in the context of SVIs, we rely on techniques from [45, 118] on convex functionals of Radon measures. For the deterministic theory on porous medium equations, we refer to [105] and [119]. Regarding results on the long-time behaviour of singular-degenerate SPDE we refer to the literature exposition in Chapter 4.

3.1.2 Notation

Unless specified differently, function or measure spaces will be understood to be defined on a smooth, bounded domain $\mathcal{O} \subset \mathbb{R}^d, d \in \mathbb{N}$. We write $L^p = L^p(\mathcal{O})$ for the usual Lebesgue spaces with norm $\|\cdot\|_{L^p}$ and scalar product $\langle \cdot, \cdot \rangle_{L^2}$ if $p = 2$. The Lebesgue measure is denoted by $\mathrm{d}x$, and a measure with density $h \in L^1$ with respect to $\mathrm{d}x$ is denoted by $h\,\mathrm{d}x$. Furthermore, $H_0^1 = H_0^1(\mathcal{O})$ denotes the Sobolev space of L^2 functions whose first-order weak derivatives exist and are in L^2, and which have zero trace, with norm $\|u\|_{H_0^1} = \|\nabla u\|_{L^2}$. The full space analogues $L^2(\mathbb{R}^d)$, $H^1(\mathbb{R}^d)$ are defined correspondingly. Furthermore, let H^{-1} denote the topological dual of H_0^1. We use $-\Delta$ to denote the corresponding Riesz isomorphism, which gives rise to the inner product

$$\langle u, v \rangle_{H^{-1}} = {}_{H^{-1}}\langle u, (-\Delta)^{-1} v \rangle_{H_0^1} \quad \text{for all } u, v \in H^{-1},$$

where the notation ${}_{V'}\langle u, v \rangle_V = {}_V\langle v, u \rangle_{V'}$ denotes evaluating a functional u belonging to the dual space V' of a Banach space V at a vector $v \in V$.

Moreover, we let $\mathcal{C}_0^0 = \mathcal{C}_0^0(\mathcal{O})$ denote the set of all continuous functions on $\mathcal{O}$ vanishing at the boundary, while we write $\mathcal{C}_c^0 = \mathcal{C}_c^0(\mathcal{O})$ for continuous functions with compact support. The same notation applies to spaces $\mathcal{C}^k$ of k times continuously differentiable functions.

For $m \in [0, 1]$ we define the set

$$L^{m+1} \cap H^{-1} := \left\{ v \in L^{m+1} : \exists\, C \geq 0 \text{ s. t.} \int v\eta \,\mathrm{d}x \leq C \|\eta\|_{H_0^1} \text{ for all } \eta \in C_c^1 \right\}.$$

Note that $L^2 = L^2 \cap H^{-1}$ by the Cauchy-Schwarz and Poincaré inequalities. To each $v \in L^{m+1} \cap H^{-1}$ one can injectively assign a map

$$\mathcal{C}_c^1 \ni \eta \mapsto \int v\eta \,\mathrm{d}x. \tag{3.1.6}$$

By continuity, (3.1.6) can be injectively extended to a bounded linear functional on H_0^1, which we call $\iota_m(v)$. The resulting map $\iota_m : L^{m+1} \cap H^{-1} \to H^{-1}$ is thus injective, which allows to identify v with $\iota_m(v)$.

Let $\mathcal{M} = \mathcal{M}(\mathcal{O})$ be the space of all signed Radon measures on $\mathcal{O}$ with finite total variation, which is isomorphic to the dual space $\left(\mathcal{C}_0^0\right)'$ via

$$\mathcal{M} \ni \mu \mapsto \tilde{\mu} \in \left(\mathcal{C}_0^0\right)', \quad \tilde{\mu}(f) = \int f \mathrm{d}\mu. \tag{3.1.7}$$

This allows us to use $\left(\mathcal{C}_0^0\right)'$ and $\mathcal{M}$, as well as $\tilde{\mu}$ and μ interchangeably. The variation measure of $\mu \in \mathcal{M}$ is denoted by $|\mu| := \mu_+ + \mu_-$ and the total variation of μ is given by

$$\|\mu\|_{TV} = |\mu|(\mathcal{O}).$$

Note that the total variation is also the operator norm if the measure is interpreted as an element of $\left(\mathcal{C}_0^0\right)'$ by the Riesz-Markov representation theorem (see e. g. Theorem 1.200 in [61]). We define the space of *measures of bounded energy* by

$$\mathcal{M} \cap H^{-1} := \left\{ \mu \in \mathcal{M} : \exists\, C \geq 0 \text{ s. t.} \int \eta(x) \,\mathrm{d}\mu(x) \leq C \|\eta\|_{H_0^1} \text{ for all } \eta \in \mathcal{C}_c^1(\mathcal{O}) \right\}.$$

By a density argument, restricting a measure $\mu \in \mathcal{M} \cap H^{-1}$ to a function on $\mathcal{C}_c^1$ is an injective operation. Moreover, by continuity $\mu|_{\mathcal{C}_c^1}$ can be injectively extended to a bounded linear functional on H_0^1, which we call $\iota(\mu)$. The resulting map $\iota : \mathcal{M} \cap H^{-1} \to H^{-1}$ is thus injective, which allows to identify μ with $\iota(\mu)$.

In general, constants may vary from line to line, but are always positive and finite.

3.2 Assumptions and main result

Assumptions 3.2.1. We require the following assumptions throughout this article.

(A1) W is a cylindrical Id-Wiener process in some separable Hilbert space U defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $(\mathcal{F}_t)_{t\geq 0}$, which means the following: There is a Hilbert-Schmidt embedding J from U to another Hilbert space U_1, which can be chosen to be bijective (see e. g. Remark 2.5.1 in [112]). Defining $Q_1 := JJ^*$, Q_1 is linear, bounded, non-negative definite, symmetric and has finite trace, so that we obtain a classical Q_1-Wiener process $\tilde{W}$ on U_1. Moreover, for an operator $\tilde{B} : U \to H^{-1}$ we have

$$\tilde{B} \in L_2(U, H^{-1}) \Leftrightarrow \tilde{B} \circ J^{-1} \in L_2\left(Q_1^{\frac{1}{2}}(U_1), H^{-1}\right), \tag{3.2.1}$$

such that if (3.2.1) is satisfied, we can define

$$\int_0^T \tilde{B}\, \mathrm{d}W_t := \int_0^T \tilde{B} \circ J^{-1} \mathrm{d}\tilde{W}_t.$$

(A2) The diffusion coefficients $B : [0,T] \times H^{-1} \times \Omega \to L_2(U, H^{-1})$ take values in the space of Hilbert-Schmidt operators, are progressively measurable and satisfy

$$\|B(t,v) - B(t,w)\|_{L_2(U,H^{-1})}^2 \leq C \|v - w\|_{H^{-1}}^2 \qquad \text{for all } v, w \in H^{-1}, \tag{3.2.2}$$

$$\|B(t,v)\|_{L_2(U,L^2)}^2 \leq C(1 + \|v\|_{L^2}^2) \qquad \text{for all } v \in L^2, \tag{3.2.3}$$

$$\|B(t,0)\|_{L_2(U,H^{-1})}^2 \leq C, \tag{3.2.4}$$

for some constant $C > 0$ and all $(t, \omega) \in [0,T] \times \Omega$.

(A3) The so-called *potential* $\psi : \mathbb{R} \to [0, \infty)$ is convex and lower-semicontinuous, and we assume $\psi(0) = 0$, which then implies $0 \in \partial\psi(0)$. For simplicity, we furthermore impose the symmetry assumption $\psi(x) = \psi(-x)$ for all $x \in \mathbb{R}$.

(A4) Define $\phi = \partial\psi : \mathbb{R} \to 2^{\mathbb{R}}$, the subdifferential of ψ, and assume for all $r \in \mathbb{R}$

$$\inf\{|\eta|^2 : \eta \in \phi(r)\} \leq C(1 + |r|^2). \tag{3.2.5}$$

In case that

$$\lim_{|x|\to\infty} \frac{\psi(x)}{|x|} \to \infty, \tag{3.2.6}$$

i. e. ψ is superlinear, we require

(A5) There exists $m \in (0,1]$, such that $\psi(v) \in L^1(\mathcal{O})$ if and only if $v \in L^{m+1}(\mathcal{O})$.

In case that the potential is sublinear, i. e. that there exists a constant $C > 0$ such that

$$\psi(x) \leq C(1 + |x|) \quad \text{for all } x \in \mathbb{R}, \tag{3.2.7}$$

we require

(A5') There exists $y > 0$ such that $\psi(y) > 0$.

Note that by convexity, Assumption (A5') implies that

$$\psi(x) \geq \frac{\psi(y)}{y} |x| - \psi(y) \quad \text{for all } x \in \mathbb{R}.$$

Next, we define the energy functional for the notion of solution we are going to consider.

Definition 3.2.2. Let Assumptions 3.2.1 be satisfied.

(i) In the case of a superlinear potential, i. e. if (3.2.6) is satisfied, we define for $u \in H^{-1}$ the functional

$$\varphi(u) = \begin{cases} \int \psi(u) \, \mathrm{d}x, & \text{if } u \in L^{m+1} \cap H^{-1}, \\ +\infty, & \text{else,} \end{cases} \tag{3.2.8}$$

where m is the exponent from (A5).

(ii) In the case of a sublinear potential, i. e. if (3.2.7) is satisfied, we define for $u \in H^{-1}$ the functional

$$\varphi(u) = \begin{cases} \|\psi(u)\|_{TV}, & \text{if } u \in \mathcal{M} \cap H^{-1}, \\ +\infty, & \text{else,} \end{cases} \tag{3.2.9}$$

where the construction of a nonlinear functional of a measure, which is needed in (3.2.9), is given in Definition 3.3.3 below.

Remark 3.2.3. The choice of the energy functional in Definition 3.2.2 allows us to reformulate (3.1.1) as a gradient flow, i. e. to rewrite it in the form

$$\begin{aligned} \mathrm{d}X_t &\in -\partial\varphi(X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t, \\ X_0 &= x_0, \end{aligned} \tag{3.2.10}$$

where the subdifferential is well-defined due to Proposition 3.3.7 below. More precisely, let a "classical" solution to (3.1.1) with $x_0 \in H^{-1}$ be defined as an $(\mathcal{F}_t)_{t\geq 0}$-adapted process $X \in L^2(\Omega; \mathcal{C}([0,T]; H^{-1})$ with the following properties: $\mathbb{P}$-almost surely, for all $t \in [0,T]$ we have $X_t \in L^2$, there is a choice $v_t \in \phi(X_t)$ such that $v_t \in H_0^1$, and

$$X_t = x_0 + \int_0^t \Delta v_r \, \mathrm{d}r + \int_0^t B(r, X_r) \, \mathrm{d}W_r.$$

Furthermore, we impose $\Delta v \in L^2([0,T] \times \Omega; H^{-1})$. If X is a classical solution in this sense, then $\Delta v \in -\partial\varphi(X)$ $\mathbb{P} \otimes \mathrm{d}t$-almost everywhere, which means that $(X, \Delta v)$ is a strong solution to (3.2.10) in the sense of Definition 3.C.1.

Proof. We only need to show that $\Delta v_t \in -\partial\varphi(X_t)$ $\mathbb{P}$-almost surely for all $t \in [0,T]$, which is done by verifying the subdifferential inequality

$$\varphi(u) \geq \varphi(X_t) + {}_{H^{-1}}\langle u - X_t, -\Delta v_t \rangle_{H^{-1}}$$

for arbitrary $u \in H^{-1}$ and for $(t, \omega) \in [0,T] \times \Omega$, for which the abovestated properties of classical solutions are satisfied. For $\varphi(u) = \infty$, there is nothing to show. For the superlinear case with Assumption 3.2.1 (A5) satisfied for $m \in (0,1]$, we consider $u \in L^{m+1} \cap H^{-1}$, which is equivalent to $\varphi(u) < \infty$. Since $X_t \in L^2 \subseteq L^{m+1} \cap H^{-1}$ by assumption, we have $\varphi(X_t) < \infty$, such that we can subtract the term and obtain, using $v_t \in \partial\psi(X_t)$,

$$\begin{aligned} \varphi(u) - \varphi(X_t) &= \int_{\mathcal{O}} \psi(u) - \psi(X_t) \, \mathrm{d}x \\ &\geq \int_{\mathcal{O}} v_t(u - X_t) \, \mathrm{d}x \\ &= {}_{H^{-1}}\langle u - X_t, v_t \rangle_{H_0^1} = \langle u - X_t, (-\Delta)v_t \rangle_{H^{-1}}. \end{aligned} \tag{3.2.11}$$

In the sublinear case, i.e. (3.2.7) is satisfied, let $u \in \mathcal{M} \cap H^{-1}$. Let $(u_n)_{n\in\mathbb{N}}$ be the approximating sequence for u given by Theorem 3.3.8 below. Then, using Theorem 3.3.8 below, we compute

$$\begin{aligned}
\varphi(u) - \varphi(X_t) &= \lim_{n\to\infty} \varphi(u_n) - \varphi(X_t) \\
&= \lim_{n\to\infty} \int_{\mathcal{O}} \psi(u_n) - \psi(X_t)\mathrm{d}x \\
&\geq \limsup_{n\to\infty} \int v_t(u_n - X_t)\mathrm{d}x \\
&= \limsup_{n\to\infty} {}_{H^{-1}}\langle u_n - X_t, v_t\rangle_{H_0^1} \\
&= \lim_{n\to\infty} \langle u_n - X_t, (-\Delta)v_t\rangle_{H^{-1}} = \langle u - X_t, (-\Delta)v_t\rangle_{H^{-1}},
\end{aligned}$$

as required. □

Now we are in the position to formulate the notion of solution we will consider.

Definition 3.2.4 (SVI solution)**.** Given Assumptions 3.2.1, let $x_0 \in L^2(\Omega, \mathcal{F}_0; H^{-1})$, $T > 0$ and φ be defined as in Definition 3.2.2. We say that an $\mathcal{F}_t$-adapted process $X \in L^2(\Omega; \mathcal{C}([0,T]; H^{-1}))$ is an SVI solution to (3.1.1) if the following conditions are satisfied:

(i) (Regularity)

$$\varphi(X) \in L^1([0,T] \times \Omega).$$

(ii) (Variational inequality) For each $\mathcal{F}_t$-progressively measurable process $G \in L^2([0,T]\times\Omega; H^{-1})$, and each $\mathcal{F}_t$-adapted process $Z \in L^2(\Omega; \mathcal{C}([0,T]; H^{-1})) \cap L^2([0,T] \times \Omega; L^2)$ solving the equation

$$Z_t - Z_0 = \int_0^t G_s \,\mathrm{d}s + \int_0^t B(s, Z_s)\,\mathrm{d}W_s \quad \text{for all } t \in [0,T],$$

we have

$$\begin{aligned}
&\mathbb{E}\,\|X_t - Z_t\|_{H^{-1}}^2 + 2\mathbb{E}\int_0^t \varphi(X_r)\mathrm{d}r \\
&\leq \mathbb{E}\,\|x_0 - Z_0\|_{H^{-1}}^2 + 2\mathbb{E}\int_0^t \varphi(Z_r)\mathrm{d}r \\
&\quad - 2\,\mathbb{E}\int_0^t \langle G_r, X_r - Z_r\rangle_{H^{-1}}\,\mathrm{d}r \\
&\quad + C\,\mathbb{E}\int_0^t \|X_r - Z_r\|_{H^{-1}}^2\,\mathrm{d}r \quad \text{for all } t \in [0,T]
\end{aligned} \tag{3.2.12}$$

for some $C > 0$.

Remark 3.2.5. If (X,η) is a strong solution to (3.2.10) in H^{-1} according to Definition 3.C.1, then X is an SVI solution to (3.1.1).

Proof. For (i) from Definition 3.2.4, we first note that $\varphi(0) = 0$ and for $s \in [0,T]$

$$0 \leq \varphi(X_s) \leq \varphi(0) + \langle \eta_s, 0 - X_s\rangle_{H^{-1}} = -\langle \eta_s, X_s\rangle_{H^{-1}}$$

by the subdifferential inequality. Hence, using the assumptions on (X,η), we can compute

$$\begin{aligned}
\mathbb{E}\int_0^T |\varphi(X_s)|\,\mathrm{d}s = \mathbb{E}\int_0^T \varphi(X_s)\,\mathrm{d}s &\leq \mathbb{E}\int_0^T \|\eta_s\|_{H^{-1}}\|X_s\|_{H^{-1}}\,\mathrm{d}s \\
&\leq \frac{1}{2}\mathbb{E}\int_0^T \|\eta_s\|_{H^{-1}}^2\,\mathrm{d}s + \frac{1}{2}\mathbb{E}\int_0^T \|X_s\|_{H^{-1}}^2\,\mathrm{d}s \\
&\leq \frac{1}{2}\mathbb{E}\int_0^T \|\eta_s\|_{H^{-1}}^2\,\mathrm{d}s + \frac{T}{2}\mathbb{E}\left(\sup_{s\in[0,T]}\|X_s\|_{H^{-1}}\right)^2 < \infty,
\end{aligned}$$

as required. For (ii), let G and Z be given as in 3.2.4. Then Ito's formula (e. g. [112, Theorem 4.2.5]) implies for all $t \in [0,T]$

$$\begin{aligned}\mathbb{E}\,\|X_t - Z_t\|_{H^{-1}}^2 &= \mathbb{E}\,\|x_0 - Z_0\|_{H^{-1}}^2 + 2\mathbb{E}\int_0^t \langle \eta_r - G_r, X_r - Z_r\rangle_{H^{-1}}\,\mathrm{d}r \\ &\quad + \mathbb{E}\int_0^t \|B(r,X_r) - B(r,Z_r)\|_{L_2(U,H^{-1})}^2\,\mathrm{d}r.\end{aligned}$$

Since $\eta_r \in -\partial\varphi(X_r)$ $(\mathbb{P}\otimes \mathrm{d}t)$-almost everywhere, we have

$$\langle \eta_r, X_r - Z_r\rangle_{H^{-1}} \le \varphi(Z_r) - \varphi(X_r) \quad \mathrm{d}t\otimes \mathrm{d}\mathbb{P}\text{-a.e..}$$

Using moreover the Lipschitz condition (3.2.2) on B, we obtain for all $t \in [0,T]$

$$\begin{aligned}\mathbb{E}\,\|X_t - Z_t\|_{H^{-1}}^2 \le \mathbb{E}\,\|x_0 - Z_0\|_{H^{-1}}^2 &+ 2\mathbb{E}\int_0^t \varphi(Z_r) - \varphi(X_r)\mathrm{d}r \\ &- 2\mathbb{E}\int_0^t \langle G_r, X_r - Z_r\rangle_{H^{-1}}\,\mathrm{d}r \\ &+ \mathbb{E}\int_0^t C\,\|X_r - Z_r\|_{H^{-1}}^2\,\mathrm{d}r,\end{aligned}$$

which is equivalent to (3.2.12). □

The main result of this article is as follows.

Theorem 3.2.6. *Given Assumptions 3.2.1, let $x_0 \in L^2(\Omega,\mathcal{F}_0;H^{-1})$ and $T>0$. Then there is a unique SVI solution X to* (3.1.1). *For two SVI solutions X,Y with initial conditions $x_0,y_0 \in L^2(\Omega,\mathcal{F}_0;H^{-1})$, we have*

$$\sup_{t\in[0,T]} \mathbb{E}\,\|X_t - Y_t\|_{H^{-1}}^2 \le C\,\mathbb{E}\,\|x_0 - y_0\|_{H^{-1}}^2\,. \tag{3.2.13}$$

The proof of this theorem will be given in Section 3.4 below.

3.3 Properties of the energy functional

The aim of this section is to make Definition 3.2.2 rigorous by recalling the concept of convex functionals on measures, and to prove certain properties of the energy functional defined in Definition 3.2.2, which are needed for the proof of the main theorem. We start with some basic concepts concerning convex functions.

Definition 3.3.1. Let $f:\mathbb{R}\to[0,\infty]$ be a convex and lower-semicontinuous function with $f(0)=0$. We then define its convex conjugate $f^*:\mathbb{R}\to[0,\infty]$ by

$$f^*(x) = \sup_{y\in\mathbb{R}}(xy - f(y)), \tag{3.3.1}$$

and its recession function $f_\infty:\mathbb{R}\to[0,\infty]$ by

$$f_\infty(x) = \lim_{t\to\infty}\frac{f(tx)}{t}. \tag{3.3.2}$$

Remark 3.3.2. Note that f_∞ and f^* are convex. If f is symmetric, so are f_∞ and f^*. Moreover, f_∞ is positively homogeneous.

For the notion of solution that we are aiming at, we need the concept of a convex function of a measure, which has been developed in [45].

Definition 3.3.3. Let ψ satisfy (3.2.7) as well as Assumptions 3.2.1 (A3), (A5'). Define the set

$$\mathcal{D}_\psi = \{v \in \mathcal{C}_c^0(\mathcal{O}) : \psi^*(v) \in L^1(\mathcal{O})\}$$

and let $\mu \in \mathcal{M}(\mathcal{O})$. We then define the positive measure $\psi(\mu) \in \mathcal{M}(\mathcal{O})$ by

$$\begin{aligned}\int_{\mathcal{O}} \eta\, \psi(\mu) &:= {}_{\mathcal{M}(\mathcal{O})}\langle \psi(\mu), \eta \rangle_{\mathcal{C}_0^0(\mathcal{O})} \\ &:= \sup \left\{ \int_{\mathcal{O}} v\eta \,\mathrm{d}\mu - \int_{\mathcal{O}} \psi^*(v)\eta \,\mathrm{d}x : v \in \mathcal{D}_\psi \right\}\end{aligned} \tag{3.3.3}$$

for $\eta \in \mathcal{C}_0^0(\mathcal{O}), \eta \geq 0$, and for general $\eta \in \mathcal{C}_0^0(\mathcal{O})$ we set

$$ {}_{\mathcal{M}(\mathcal{O})}\langle \psi(\mu), \eta \rangle_{\mathcal{C}_0^0(\mathcal{O})} = {}_{\mathcal{M}(\mathcal{O})}\langle \psi(\mu), \eta \vee 0 \rangle_{\mathcal{C}_0^0(\mathcal{O})} - {}_{\mathcal{M}(\mathcal{O})}\langle \psi(\mu), (-\eta) \vee 0 \rangle_{\mathcal{C}_0^0(\mathcal{O})},$$

according to Theorem 1.1 in [45].

Remark 3.3.4. As argued in Lemma 1.1 in [45], one can write for $\mu \in \mathcal{M}(\mathcal{O})$

$$\int_{\mathcal{O}} \psi(\mu) = \|\psi(\mu)\|_{TV} = \sup \left\{ \int_{\mathcal{O}} v \,\mathrm{d}\mu - \int_{\mathcal{O}} \psi^*(v) \,\mathrm{d}x : v \in \mathcal{D}_\psi \right\}.$$

Remark 3.3.5. Let $\mu \in \mathcal{M}(\mathcal{O})$ with Lebesgue decomposition $\mu^a + \mu^s$, where μ^a has the density $h \in L^1(\mathcal{O})$ with respect to the Lebesgue measure. Then, by Theorem 1.1 in [45], we have

$$\int_{\mathcal{O}} \eta\ \psi(\mu) = \int_{\mathcal{O}} \eta(x)\psi(h(x))\mathrm{d}x + \int_{\mathcal{O}} \eta\ \psi_\infty(\mu^s), \tag{3.3.4}$$

where the recession function ψ_∞ is defined as in (3.3.2). In particular, this formulation shows the useful fact that

$$\psi(\mu) = \psi(\mu^a) + \psi(\mu^s). \tag{3.3.5}$$

Our next aim is to prove the lower-semicontinuity of the energy functional defined in Definition 3.2.2 and Definition 3.3.3. First, we show that the Radon measure $\psi(\mu)$ constructed in Definition 3.3.3 controls the norm of its original measure μ in the following way.

Lemma 3.3.6. *Let ψ satisfy* (3.2.7) *as well as Assumptions 3.2.1 (A3), (A5'). Let $\mu \in \mathcal{M}(\mathcal{O})$ and let $y > 0$ such that $\psi(y) > 0$ as demanded in Assumption 3.2.1 (A5'). Then*

$$\|\psi(\mu)\|_{TV} \geq \frac{\psi(y)}{y} \|\mu\|_{TV} - \psi(y) |\mathcal{O}|.$$

Proof. For $\mu \in \mathcal{M}(\mathcal{O})$, denote by $\mu = \mu^a + \mu^s$ the Lebesgue decomposition of μ with respect to Lebesgue measure, and let $h = \frac{\mathrm{d}\mu^a}{\mathrm{d}x}$ be the Radon-Nikodym derivative of μ^a. As $\psi_\infty(\mu^s)$ is singular by Theorem 4.2 in [118], we can use the decomposition (3.3.4) to obtain

$$\|\psi(\mu)\|_{TV} = \int_{\mathcal{O}} \psi(h) \,\mathrm{d}x + \|\psi_\infty(\mu^s)\|_{TV}. \tag{3.3.6}$$

We now estimate the summands separately. For the absolutely continuous part we obtain using Assumption 3.2.1 (A5')

$$\int_{\mathcal{O}} \psi(h) \,\mathrm{d}x \geq \frac{\psi(y)}{y} \int_{\mathcal{O}} |h| \,\mathrm{d}x - \psi(y) |\mathcal{O}| = \frac{\psi(y)}{y} \|\mu^a\|_{TV} - \psi(y) |\mathcal{O}|.$$

For the singular part, we note by Lemma 3.A.5 that for $v \in \mathcal{C}_c^0(\mathcal{O})$ being in $\mathcal{D}_{\psi_\infty}$ is equivalent to $-\psi_\infty(1) \leq v \leq \psi_\infty(1)$, and for such v, $\psi_\infty^*(v) \equiv 0$. Thus, we get with Corollary 3.A.4 with $k := \frac{\psi(y)}{y}$

$$\int_{\mathcal{O}} \psi_\infty(\mu^s) = \sup_{v \in \mathcal{D}_{\psi_\infty}} \left(\int_{\mathcal{O}} v \ \mathrm{d}\mu^s - \int \psi_\infty^*(v) \mathrm{d}x \right) \geq \sup_{\substack{v \in \mathcal{C}_c^0(\mathcal{O}) \\ -k \leq v \leq k}} \int_{\mathcal{O}} v \ \mathrm{d}\mu^s = k \|\mu^s\|_{TV}.$$

Thus, we can continue (3.3.6) by

$$\|\psi(\mu)\|_{TV} \geq \frac{\psi(y)}{y} \|\mu^a\|_{TV} + k \|\mu^s\|_{TV} - \psi(y) |\mathcal{O}| = \frac{\psi(y)}{y} \|\mu\|_{TV} - \psi(y) |\mathcal{O}|,$$

as required. ☐

Proposition 3.3.7. *In both settings of Definition 3.2.2, $\varphi : H^{-1} \to [0, \infty]$ is convex and lower-semicontinuous.*

Proof. In the superlinear case, i. e. Definition 3.2.2 (i) applies, convexity and lower-semicontinuity of φ are proved on p. 68 in [7]. In the sublinear case, i. e. Definition 3.2.2 (ii) applies, convexity becomes clear by Remark 3.3.4. It remains to prove lower-semicontinuity in the sublinear case.

Step 1: As a preparatory step, we establish weak* lower-semicontinuity of the functional $\tilde{\varphi} : \mathcal{M}(\mathcal{O}) \to [0, \infty)$,

$$\tilde{\varphi}(\mu) = \|\psi(\mu)\|_{TV},$$

for which we have

$$\tilde{\varphi}|_{\mathcal{M}(\mathcal{O}) \cap H^{-1}} = \varphi.$$

Consider $\mu_n \to \mu$ weakly* for $n \to \infty$. We can assume that $\psi(\mu_n)$ contains a subsequence which is bounded in TV norm (otherwise there is nothing to show). Then we select a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ such that $\|\psi(\mu_{n_k})\|_{TV} \to \liminf_{n\to\infty} \|\psi(\mu_n)\|_{TV}$ for $k \to \infty$, from which we can choose a nonrelabeled subsequence $(\psi(\mu_{n_k}))_{k\in\mathbb{N}}$ which converges weakly* to some $\nu \in \mathcal{M}(\mathcal{O})$ (e. g. by Satz 6.5 in [1]). By Lemma 2.1 in [45], we get that

$$\begin{aligned} {}_{\mathcal{M}(\mathcal{O})}\langle \psi(\mu), \eta \rangle_{\mathcal{C}^0_0(\mathcal{O})} &\leq {}_{\mathcal{M}(\mathcal{O})}\langle \nu, \eta \rangle_{\mathcal{C}^0_0(\mathcal{O})} \\ &= \lim_{k\to\infty} {}_{\mathcal{M}(\mathcal{O})}\langle \psi(\mu_{n_k}), \eta \rangle_{\mathcal{C}^0_0(\mathcal{O})} \\ &\leq \lim_{k\to\infty} \|\psi(\mu_{n_k})\|_{TV} \|\eta\|_{\mathcal{C}^0_0(\mathcal{O})} \end{aligned}$$

for $\eta \in \mathcal{C}^0_c(\mathcal{O})$, $\eta \geq 0$. Now, using that $\psi(\rho)$ is a positive measure for any $\rho \in \mathcal{M}(\mathcal{O})$ by (3.3.3), we obtain

$$\begin{aligned} \|\psi(\mu)\|_{TV} &= \sup_{\substack{\eta\in\mathcal{C}^0_c(\mathcal{O}) \\ \eta\in[0,1]}} {}_{\mathcal{M}(\mathcal{O})}\langle \psi(\mu), \eta \rangle_{\mathcal{C}^0_0(\mathcal{O})} \\ &\leq \sup_{\substack{\eta\in\mathcal{C}^0_c(\mathcal{O}) \\ \eta\in[0,1]}} \lim_{k\to\infty} {}_{\mathcal{M}(\mathcal{O})}\langle \psi(\mu_{n_k}), \eta \rangle_{\mathcal{C}^0_0(\mathcal{O})} \\ &\leq \sup_{\substack{\eta\in\mathcal{C}^0_c(\mathcal{O}) \\ \eta\in[0,1]}} \lim_{k\to\infty} \|\psi(\mu_{n_k})\|_{TV} = \liminf_{n\to\infty} \|\psi(\mu_n)\|_{TV}, \end{aligned}$$

as required.

Step 2: Assume now that $(u_n)_{n\in\mathbb{N}} \subset H^{-1}$, $u \in H^{-1}$, and $u_n \to u$ for $n \to \infty$. Being the only non-trivial case, we can assume that $(u_n)_{n\in\mathbb{N}}$ contains a subsequence (which we call again (u_n)) for which $(\varphi(u_n))_{n\in\mathbb{N}}$ is bounded. Thus, there are measures $\mu_n \in \mathcal{M}(\mathcal{O}) \cap H^{-1}$ such that

$$u_n(\eta) = \int_{\mathcal{O}} \eta \, \mathrm{d}\mu_n \quad \text{for all } \eta \in \mathcal{C}^1_c(\mathcal{O}).$$

By definition of φ, $\varphi(u_n) = \|\psi(\mu_n)\|_{TV}$, such that Lemma 3.3.6 implies that $\|\mu_n\|_{TV}$ is bounded. Thus, there is $\tilde{\mu} \in \mathcal{M}(\mathcal{O})$ and an again nonrelabeled subsubsequence $(\mu_n)_{n\in\mathbb{N}}$ such that $\mu_n \overset{*}{\rightharpoonup} \tilde{\mu}$. For $\eta \in \mathcal{C}^1_c(\mathcal{O}) \subseteq \mathcal{C}^0_c(\mathcal{O})$ we have

$$\int_{\mathcal{O}} \eta \, \mathrm{d}\tilde{\mu} = \lim_{n\to\infty} \int_{\mathcal{O}} \eta \, \mathrm{d}\mu_n = \lim_{n\to\infty} u_n(\eta) = u(\eta) \leq \|u\|_{H^{-1}} \|\eta\|_{H^1_0(\mathcal{O})},$$

so $\tilde{\mu} \in \mathcal{M}(\mathcal{O}) \cap H^{-1}$ and $u = \tilde{\mu}$. Using the weak* lower-semicontinuity of $\tilde{\varphi}$ from Step 1, we get

$$\varphi(u) = \tilde{\varphi}(\tilde{\mu}) \leq \liminf_{n\to\infty} \tilde{\varphi}(\mu_n) = \liminf_{n\to\infty} \varphi(u_n). \tag{3.3.7}$$

As this argument works for any bounded subsequence of $(u_n)_{n\in\mathbb{N}}$, (3.3.7) is also true for the original sequence $(u_n)_{n\in\mathbb{N}}$. □

As one can see from the definition of the energy functional φ in the second part of Definition 3.2.2, it has an explicit representation on $H^{-1} \setminus \mathcal{M}(\mathcal{O})$, where it is ∞, and on $L^1(\mathcal{O}) \cap H^{-1}$, where it is an integral. However, whenever we evaluate φ for general measures in $\mathcal{M}(\mathcal{O}) \cap H^{-1}$, e. g. in the uniqueness part of the proof of Theorem 3.2.6, we need an approximation reducing it to evaluations on $L^1(\mathcal{O})$ functions. This will be made precise in the following theorem, the proof of which will take the rest of this section.

Theorem 3.3.8. *Assume that ψ satisfies* (3.2.7) *as well as Assumptions 3.2.1 (A3), (A5'). Let φ be defined as in Definition 3.2.2 (ii) and $u \in \mathcal{M}(\mathcal{O}) \cap H^{-1}$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset L^2(\mathcal{O})$ such that*

$$u_n \rightharpoonup u \quad \text{in } H^{-1}, \text{ and} \tag{3.3.8}$$

$$\varphi(u_n) \to \varphi(u) \tag{3.3.9}$$

for $n \to \infty$.

Corollary 3.3.9. *Since convex functions on a real Hilbert space are lower-semicontinuous if and only if they are weakly sequentially lower-semicontinuous (see e. g. Theorem 9.1 in [19]), Theorem 3.3.8 implies that φ is the lower-semicontinuous hull of $\varphi|_{L^2(\mathcal{O})}$ in H^{-1}, which means that*

$$\varphi = \sup \left\{ \begin{matrix} \beta : H^{-1} \to [0,\infty] : \\ \beta \text{ convex and lower-semicontinuous, } \beta|_{L^2(\mathcal{O})} \le \varphi|_{L^2(\mathcal{O})} \end{matrix} \right\}, \tag{3.3.10}$$

where sup *denotes the pointwise supremum.*

We will approach Theorem 3.3.8 by giving an explicit construction for the sequence $(u_n)_{n\in\mathbb{N}}$, inspired by the construction in Lemma A6.7 in [1]. It will rely on applying the original functional to modified functions, which is why we first introduce several modifications to functions on $\mathcal{O}$.

We next introduce further notation and recall some concepts relying on the regularity of the boundary.

Notations 3.3.10. Since the domain $\mathcal{O}$ is bounded and smooth, its boundary is locally the graph of a smooth function. More precisely, we recall from Section A6.2 in [1] that for each $y \in \partial\mathcal{O}$ there is a neighbourhood $\tilde{U} \subset \mathbb{R}^d$, an orthonormal system $e_1, \dots, e_d$ of $\mathbb{R}^d$, $r, h \in \mathbb{R}$ with $r > h > 0$, and a smooth bounded function $g : \mathbb{R}^{d-1} \to \mathbb{R}$, such that with the notation

$$x_{,d} := (x_1, \dots, x_{d-1}), \quad \text{for } x = \sum_{i=1}^d x_i e_i,$$

we have

$$\tilde{U} = \left\{ x \in \mathbb{R}^d : |x_{,d} - y| < r \text{ and } |x_d - g(x_{,d})| < h \right\},$$

and for $x \in \tilde{U}$

$$\begin{aligned} x_d = g(x_{,d}) \quad & \text{if and only if } x \in \partial\mathcal{O}, \\ x_d \in (g(x_{,d}), g(x_{,d}) + h) \quad & \text{if and only if } x \in \mathcal{O}, \text{ and} \\ x_d \in (g(x_{,d}) - h, g(x_{,d})) \quad & \text{if and only if } x \notin \mathcal{O}. \end{aligned}$$

For technical reasons we set

$$U = \left\{ x \in \tilde{U} : |x_{,d} - y| < \frac{r}{2} \text{ and } |x_d - g(x_{,d})| < \frac{h}{2} \right\}. \tag{3.3.11}$$

The boundary $\partial\mathcal{O}$ is covered by those open sets U belonging to all possible reference points y. As $\partial\mathcal{O}$ is compact, we can choose a finite subcovering $(U^j)_{j=1}^l$, and for each U^j, we denote the elements belonging to it by a superindex j, e. g. $y^j, e_d^j, g^j, h^j, \tilde{U}^j$. At last, we fix an open set U^0 with $\overline{U^0} \subset \mathcal{O}$, such that $\overline{\mathcal{O}} \subset \cup_{j=0}^l U^j$ and we set $e_d^0 := 0$.

Subordinate to the covering $\cup_{j=0}^l U^j$, let now $\zeta^0, \dots, \zeta^l$ be a partition of unity on $\overline{\mathcal{O}}$, i. e. $0 \le \zeta^j \le 1, \zeta^j \in C_c^\infty(\mathbb{R}^d), \operatorname{supp}(\zeta^j) \subseteq U^j$ for all $j = 0, \dots, l$, and

$$\sum_{j=0}^l \zeta^j = 1 \quad \text{on } \overline{\mathcal{O}}.$$

For $\eta : \mathcal{O} \to \mathbb{R}$ and $\mu \in \mathcal{M}(\mathcal{O})$, we define $\eta_{\text{ext}} : \mathbb{R}^d \to \mathbb{R}$ and $\mu_{\text{ext}} \in \mathcal{M}(\mathbb{R}^d)$ as the extended function (resp. measure) by zero. Finally, we define for $\rho \in C_c^\infty(\mathbb{R}^d)$ with

$$\operatorname{supp}(\rho) \subseteq B_1(0), \ \int_{\mathcal{O}} \rho \, \mathrm{d}x = 1, \ \rho(x) = \rho(-x) \tag{3.3.12}$$

a Dirac sequence $(\rho_\delta)_{\delta>0} \subset C_c^\infty(\mathbb{R}^d)$ of mollifiers by

$$\rho_\delta(x) = \frac{1}{\delta^d}\,\rho\left(\frac{x}{\delta}\right). \tag{3.3.13}$$

For $\eta \in L^2(\mathbb{R}^d), \mu \in \mathcal{M}(\mathbb{R}^d)$, we then define functions $\rho_\delta * \eta$, $\rho_\delta * \mu \in C^\infty(\mathbb{R}^d)$ by

$$\rho_\delta * \eta(x) = \int_{\mathbb{R}^d} \rho_\delta(x-y)\eta(y)\,\mathrm{d}y \quad \text{and} \quad \rho_\delta * \mu(x) = \int_{\mathbb{R}^d} \rho_\delta(x-y)\mathrm{d}\mu(y).$$

For brevity, we write $\rho_\delta * \eta := \rho_\delta * \eta_{\mathrm{ext}}$ for $\eta \in L^2(\mathcal{O})$.

The following construction allows to shift a function "away from the boundary".

Definition 3.3.11. Let $\varepsilon > 0$ and $\eta : \mathcal{O} \to \mathbb{R}$. Then we define $\eta_\varepsilon : \mathcal{O} \to \mathbb{R}$ by

$$\eta_\varepsilon(x) = \sum_{j=0}^{l} \zeta^j(x)\eta_{\mathrm{ext}}(x - \varepsilon e_d^j), \tag{3.3.14}$$

where we recall that e_d^0 is set to 0.

Remark 3.3.12. By this construction, we achieve that $\eta_\varepsilon = 0$ on a $w(\varepsilon)$-neighbourhood of $\partial\mathcal{O}$ with

$$w(\varepsilon) := \min\left\{\mathrm{dist}(U^0, \mathcal{O}^c), \min_{j=1,\dots,l}\left(\min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2L^j}, \frac{h^j}{4}, \frac{h^j}{4L^j}\right\}\right)\right\} > 0, \tag{3.3.15}$$

where L^j denotes the Lipschitz constant of g^j defined in Notations 3.3.10.

Proof. The number $w(\varepsilon)$ is obviously strictly positive by the construction of the covering $(U^j)_{j=0}^l$. To show the support property, let $j \in \{0, 1, \dots, l\}$ and $U_\varepsilon^j := U^j \cap ((U^j \cap \mathcal{O}) + \varepsilon e_d^j)$. By definition, $\eta_{\mathrm{ext}}(x - \varepsilon e_d^j) = 0$ if $x \in U^j \setminus U_\varepsilon^j$. By the definition of ζ^j, we furthermore conclude that $\zeta^j(x)\eta_{\mathrm{ext}}(x - \varepsilon e_d^j) = 0$ for $x \notin U_\varepsilon^j$. Consequently,

$$\eta_\varepsilon : x \mapsto \sum_{j=0}^{l} \zeta^j(x)\eta_{\mathrm{ext}}(x - \varepsilon e_d^j)$$

is supported on

$$U_\varepsilon := \bigcup_{j=0}^{l} U_\varepsilon^j,$$

such that it remains to show that $\mathrm{dist}(U_\varepsilon, \mathcal{O}^c) \geq w(\varepsilon)$, or equivalently, that $\mathrm{dist}(U_\varepsilon^j, \mathcal{O}^c) \geq w(\varepsilon)$ for all $j \in \{0, \dots, l\}$.

For $j = 0$, this is trivial by construction of $U_\varepsilon^0 = U^0$ and $w(\varepsilon)$. For $j = 1, \dots, l$, using the coordinate system $(x_{,d}^j, x_d^j)$ we can rewrite

$$U_\varepsilon^j = \{x \in U^j : x_d^j > g^j(x_{,d}^j) + \varepsilon\}.$$

Hence, we can compute for any $x \in U_\varepsilon^j$, i. e. $x = \left(x_{,d}^j\,,\, g^j(x_{,d}^j) + \varepsilon'\right)$ for some $\varepsilon' \in (\varepsilon, \frac{h^j}{2})$, and $y \in \partial\mathcal{O} \cap \tilde{U}^j$

$$\begin{aligned}\|x - y\|^2 &= \|x_{,d} - y_{,d}\|^2 + |g(x_{,d}) + \varepsilon' - g(y_{,d})|^2 \\ &\geq \|x_{,d} - y_{,d}\|^2 + (\varepsilon' - |g(x_{,d}) - g(y_{,d})|)^2,\end{aligned}$$

where $\|\cdot\|$ denotes the Euclidean norm both in $\mathbb{R}^d$ and in $\mathbb{R}^{d-1}$. Letting L^j be the Lipschitz constant of g^j, we can then argue that either $\|x_{,d} - y_{,d}\| > \frac{\varepsilon}{2L^j}$ or

$$|g(x_{,d}) - g(y_{,d})| \leq L^j \frac{\varepsilon}{2L^j} = \frac{\varepsilon}{2},$$

such that $\mathrm{dist}(U_\varepsilon^j, \partial\mathcal{O} \cap \tilde{U}^j)$ is at least $\min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2L^j}\right\}$. By similar arguments, we can obtain from the construction of U^j in (3.3.11) (note that $r^j > h^j$ by construction) that

$$\mathrm{dist}(U_\varepsilon^j, (\tilde{U}^j)^c) \geq \min\left\{\frac{h^j}{4}, \frac{h^j}{4L^j}\right\},$$

such that we conclude

$$\begin{aligned}\operatorname{dist}(U^j_\varepsilon,\partial\mathcal{O}) &= \min\{\operatorname{dist}(U^j_\varepsilon,\partial\mathcal{O}\cap\tilde U^j),\operatorname{dist}(U^j_\varepsilon,\partial\mathcal{O}\cap(\tilde U^j)^c)\}\\ &\geq \min\{\operatorname{dist}(U^j_\varepsilon,\partial\mathcal{O}\cap\tilde U^j),\operatorname{dist}(U^j_\varepsilon,(\tilde U^j)^c)\}\\ &\geq \min\left\{\frac{\varepsilon}{2},\frac{\varepsilon}{2L^j},\frac{h^j}{4},\frac{h^j}{4L^j}\right\}\geq w(\varepsilon).\end{aligned}$$

□

This allows to define the following approximating objects for $u\in\mathcal{M}(\mathcal{O})\cap H^{-1}$.

Definition 3.3.13. Let $\varepsilon>0$ and $0<\delta\leq\frac{w(\varepsilon)}{2}$. For $u\in H^{-1}$, we define for $\eta\in H^1_0(\mathcal{O})$

$$\begin{aligned}u_\varepsilon(\eta) &= {}_{H^{-1}}\langle u,\eta_\varepsilon\rangle_{H^1_0(\mathcal{O})}\\ \text{and}\quad u_{\varepsilon,\delta}(\eta) &= {}_{H^{-1}}\langle u,\rho_\delta*\eta_\varepsilon\rangle_{H^1_0(\mathcal{O})}.\end{aligned}\tag{3.3.16}$$

These functionals are in H^{-1} by Lemma 3.3.14 and Lemma 3.3.15 below. For $u\in\mathcal{M}(\mathcal{O})$, we define for $\eta\in\mathcal{C}^0_0(\mathcal{O})$

$$\begin{aligned}\tilde u_\varepsilon(\eta) &= {}_{\mathcal{M}(\mathcal{O})}\langle u,\eta_\varepsilon\rangle_{\mathcal{C}^0_0(\mathcal{O})}\\ \text{and}\quad \tilde u_{\varepsilon,\delta}(\eta) &= {}_{\mathcal{M}(\mathcal{O})}\langle u,\rho_\delta*\eta_\varepsilon\rangle_{\mathcal{C}^0_0(\mathcal{O})}.\end{aligned}\tag{3.3.17}$$

These functionals are in $\mathcal{M}(\mathcal{O})$ by Lemma 3.3.16 below. If $u\in\mathcal{M}(\mathcal{O})\cap H^{-1}$, the uniqueness of the linear continuation allows to conclude that

$$u_\varepsilon,u_{\varepsilon,\delta}\in\mathcal{M}\cap H^{-1},\quad\text{as well as}\quad u_\varepsilon=\tilde u_\varepsilon\text{ and }u_{\varepsilon,\delta}=\tilde u_{\varepsilon,\delta}.$$

Lemma 3.3.14. *Let $\varepsilon>0$ and $\eta\in H^1_0(\mathcal{O})$. Then the map $H^1_0(\mathcal{O})\ni\eta\mapsto\eta_\varepsilon\in H^1_0(\mathcal{O})$ is linear, and*

$$\|\eta_\varepsilon\|_{H^1_0(\mathcal{O})}\leq C\|\eta\|_{H^1_0(\mathcal{O})},$$

where C only depends on the localizing functions $(\zeta^j)_{j=0}^l$, the number of covering sets l, the Poincaré constant of the domain $\mathcal{O}$ and the spatial dimension d.

Proof. The proof of the linearity claim is straightforward and therefore skipped. In order to prove boundedness, let $V^j=U^j\cap\mathcal{O}$ and $U^j_\varepsilon:=U^j\cap((U^j\cap\mathcal{O})+\varepsilon e^j_d)$ as before. We first note

$$\|\eta_\varepsilon\|_{H^1_0(\mathcal{O})}=\left\|\sum_{j=0}^l\zeta^j\eta^j_\varepsilon\right\|_{H^1_0(\mathcal{O})}\leq\sum_{j=0}^l\left\|\zeta^j\eta^j_\varepsilon\right\|_{H^1_0(\mathcal{O})},\tag{3.3.18}$$

where we have written

$$\eta^j_\varepsilon\in H^1(\mathbb{R}^d),\quad\eta^j_\varepsilon(x)=\eta_{\text{ext}}(x-\varepsilon e^j_d).$$

We now analyze the summands separately, where we make use of the fact that for all $j\in\{1,\dots,l\}$, $\zeta^j\in\mathcal{C}^\infty_c(U^j)$ and $\zeta^j\eta^j_\varepsilon$ is supported on V^j. In the following, $(\partial_i)_{i=1}^d$ represent the weak partial derivatives of first order. We then compute for $i\in\{1,\dots,d\}$

$$\begin{aligned}\left\|\partial_i(\zeta^j\eta^j_\varepsilon)\right\|_{L^2(\mathcal{O})}&=\left\|\partial_i(\zeta^j\eta^j_\varepsilon)\right\|_{L^2(V^j)}\leq\left\|(\partial_i\zeta^j)\eta^j_\varepsilon\right\|_{L^2(V^j)}+\left\|\zeta^j\partial_i\eta^j_\varepsilon\right\|_{L^2(V^j)}\\ &\leq C\left\|\eta^j_\varepsilon\right\|_{L^2(V^j)}+\left(\int_{V^j}\left|\partial_i(\eta_{\text{ext}}(x-\varepsilon e^j_d))\right|^2\mathrm{d}x\right)^{\frac12}\\ &\leq C\|\eta\|_{L^2(\mathcal{O})}+\left(\int_{U^j_\varepsilon}\left|(\partial_i\eta)(x-\varepsilon e^j_d)\right|^2\mathrm{d}x\right)^{\frac12}\\ &\leq C\|\eta\|_{L^2(\mathcal{O})}+\|\partial_i\eta\|_{L^2(\mathcal{O})}.\end{aligned}$$

This yields

$$\begin{aligned}\left\|\zeta^j\eta_\varepsilon^j\right\|^2_{H^1_0(\mathcal{O})} &= \sum_{i=1}^d \left\|\partial_i(\zeta^j\eta_\varepsilon^j)\right\|^2_{L^2(\mathcal{O})} \le \sum_{i=1}^d \Big(C\left\|\eta\right\|_{L^2(\mathcal{O})} + \left\|\partial_i\eta\right\|_{L^2(\mathcal{O})}\Big)^2 \\ &\le C\left\|\eta\right\|^2_{H^1_0(\mathcal{O})} + 2\sum_{i=1}^d \left\|\partial_i\eta\right\|^2_{L^2(\mathcal{O})} \le C\left\|\eta\right\|^2_{H^1_0(\mathcal{O})},\end{aligned}$$

where C may depend on $d, \mathcal{O}$ (through the Poincare constant) and ζ^j. Thus, we can continue (3.3.18) by

$$\left\|\eta_\varepsilon\right\|_{H^1_0(\mathcal{O})} \le \sum_{j=0}^l \left\|\zeta^j\eta_\varepsilon^j\right\|_{H^1_0(\mathcal{O})} \le (l+1)C\left\|\eta\right\|_{H^1_0(\mathcal{O})},$$

as required. □

Concerning the mollification step, we note that by Remark 3.3.12, $\rho_\delta * \eta_\varepsilon(x) = 0$ if $\operatorname{dist}(x, \partial\mathcal{O}) \le \frac{w(\varepsilon)}{2}$ and $0 < \delta \le \frac{w(\varepsilon)}{2}$, so that in this case we can restrict $\rho_\delta * \eta_\varepsilon$ to $\mathcal{O}$ to get a $\mathcal{C}^1_c(\mathcal{O})$ function. By a slight abuse of notation, we then write

$$(\rho_\delta * \eta_\varepsilon)|_{\mathcal{O}} = \rho_\delta * \eta_\varepsilon \in \mathcal{C}^1_c(\mathcal{O}) \subseteq H^1_0(\mathcal{O}) \cap \mathcal{C}^0_0(\mathcal{O}). \tag{3.3.19}$$

Also for this step, we have to ensure linearity, which is clear, and an estimate on the $H^1_0(\mathcal{O})$ norm, which is done in the following lemma.

Lemma 3.3.15. *Let $\varepsilon > 0$ and $0 < \delta \le \frac{w(\varepsilon)}{2}$. Then the map $H^1_0(\mathcal{O}) \ni \eta \mapsto \eta_{\varepsilon,\delta} \in H^1_0(\mathcal{O})$ is linear, and*

$$\left\|\rho_\delta * \eta_\varepsilon\right\|_{H^1_0(\mathcal{O})} \le C\left\|\eta\right\|_{H^1_0(\mathcal{O})} \quad \text{for all } \eta \in H^1_0(\mathcal{O}),$$

where C is the constant from Lemma 3.3.14.

Proof. The proof of linearity is straightforward. In order to show boundedness, for any $g \in L^2(\mathcal{O})$ such that $\rho_\delta * g = 0$ on $\mathcal{O}^c$ we can compute

$$\begin{aligned}\left\|\rho_\delta * g\right\|^2_{L^2(\mathcal{O})} &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\rho_\delta(x-y)g_{\text{ext}}(y)\,\mathrm{d}y\right)^2\mathrm{d}x \\ &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_\delta(x-y)\,(g_{\text{ext}}(y))^2\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_\delta(x-y)\mathrm{d}x\,(g_{\text{ext}}(y))^2\,\mathrm{d}y \\ &= \left\|g_{\text{ext}}\right\|^2_{L^2(\mathbb{R}^d)} = \left\|g\right\|^2_{L^2(\mathcal{O})},\end{aligned} \tag{3.3.20}$$

where in the second step we could apply Jensen's inequality since $\rho_\delta(x-y)\,\mathrm{d}y$ is a probability measure for each $x \in \mathbb{R}^d$. By Remark 3.3.12 for all $i \in \{1, \ldots, d\}$, $\rho_\delta * (\partial_i\eta_\varepsilon)$ vanishes outside of $\mathcal{O}$ if $0 < \delta \le \frac{w(\varepsilon)}{2}$. Hence g in (3.3.20) can be replaced by each partial derivative $\partial_i\eta_\varepsilon$ which yields

$$\begin{aligned}\left\|\rho_\delta * \eta_\varepsilon\right\|^2_{H^1_0(\mathcal{O})} &= \sum_{i=1}^d \left\|\partial_i(\rho_\delta * \eta_\varepsilon)\right\|^2_{L^2(\mathcal{O})} = \sum_{i=1}^d \left\|\rho_\delta * \partial_i(\eta_\varepsilon)\right\|^2_{L^2(\mathcal{O})} \\ &\le \sum_{i=1}^d \left\|\partial_i\eta_\varepsilon\right\|^2_{L^2(\mathcal{O})} = \left\|\eta_\varepsilon\right\|^2_{H^1_0(\mathcal{O})} \le C\left\|\eta\right\|^2_{H^1_0(\mathcal{O})},\end{aligned}$$

where the second equality can be found e. g. in Section 2.23 in [1] and the last inequality is the statement of Lemma 3.3.14. □

Lemma 3.3.16. *Let $\varepsilon > 0$, $0 < \delta \le \frac{w(\varepsilon)}{2}$ and $\eta \in \mathcal{C}^0_c(\mathcal{O})$. Then, the map*

$$\mathcal{C}^0_c(\mathcal{O}) \ni \eta \mapsto (\eta_\varepsilon, \rho_\delta * \eta_\varepsilon) \in (\mathcal{C}^0_c(\mathcal{O}))^2$$

is linear. Furthermore, we have

$$\left\|\rho_\delta * \eta_\varepsilon\right\|_\infty \le \left\|\eta_\varepsilon\right\|_\infty \le \left\|\eta\right\|_\infty, \tag{3.3.21}$$

where $\left\|\cdot\right\|_\infty$ denotes the supremum norm.

Proof. The proof of the linearity claim is straightforward. In order to show boundedness, we first note that for $0 < \delta \leq \frac{w(\varepsilon)}{2}$, $\rho_\delta * \eta_\varepsilon \in \mathcal{C}_c^0(\mathcal{O})$ by construction and Remark 3.3.12. To obtain (3.3.21), we estimate for arbitrary $x \in \mathcal{O}$

$$|\eta_\varepsilon(x)| \leq \sum_{j=0}^{l} \zeta^j(x) \left|\eta_{\mathrm{ext}}(x - \varepsilon e_d^j)\right| \leq \sum_{j=0}^{l} \zeta^j(x) \|\eta\|_\infty = \|\eta\|_\infty ,$$

which yields the second relation. The first one can be seen by

$$|\rho_\delta * \eta_\varepsilon(x)| \leq \int_{\mathbb{R}^d} \rho_\delta(x-y) \|\eta_\varepsilon\|_\infty \, \mathrm{d}x = \|\eta_\varepsilon\|_\infty ,$$

which concludes the proof. □

We next analyze how φ as given in Definition 3.2.2 (ii) acts on the approximating measures from Definition 3.3.13. First, we state that if μ is absolutely continuous with respect to the Lebesgue measure, so is μ_ε, which we show by giving its density.

Lemma 3.3.17. *Let $\varepsilon > 0$, $h \in L^1(\mathcal{O})$ and $\mu := h \, \mathrm{d}x \in \mathcal{M}(\mathcal{O})$. Then μ_ε has the density*

$$\mathcal{O} \ni x \mapsto \sum_{j=0}^{l} \zeta^j(x + \varepsilon e_d^j) h_{\mathrm{ext}}(x + \varepsilon e_d^j)$$

with respect to the Lebesgue measure.

Proof. For $\eta \in \mathcal{C}_c^0(\mathcal{O})$, we compute

$$\begin{aligned}
\int_{\mathcal{O}} \eta \, \mathrm{d}\mu_\varepsilon &= \int_{\mathcal{O}} \left(\sum_{j=0}^{l} \zeta^j(x) \eta_{\mathrm{ext}}(x - \varepsilon e_d^j) \right) \mu(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} \left(\sum_{j=0}^{l} \zeta^j(x) \eta_{\mathrm{ext}}(x - \varepsilon e_d^j) \right) h_{\mathrm{ext}}(x) \, \mathrm{d}x \\
&= \sum_{j=0}^{l} \int_{\mathbb{R}^d} \zeta^j(x + \varepsilon e_d^j) \eta_{\mathrm{ext}}(x) h_{\mathrm{ext}}(x + \varepsilon e_d^j) \, \mathrm{d}x \\
&= \int_{\mathcal{O}} \eta(x) \sum_{j=0}^{l} \zeta^j(x + \varepsilon e_d^j) h_{\mathrm{ext}}(x + \varepsilon e_d^j) \, \mathrm{d}x,
\end{aligned}$$

as required. The switching of integration domains is possible as the integrands are supported on $\mathcal{O}$ by Remark 3.3.12 or by assumption, respectively. □

A more direct construction of $\mu_{\varepsilon,\delta}$ is given by the following lemma.

Lemma 3.3.18. *Let $\varepsilon > 0$, $0 < \delta \leq \frac{w(\varepsilon)}{2}$ and $\mu \in \mathcal{M}(\mathcal{O})$. Then, the measure*

$$\tilde{\mu}_{\varepsilon,\delta} := \left((\rho_\delta * \mu_{\mathrm{ext}})|_{\mathcal{O}} \, \mathrm{d}x \right)_\varepsilon \in \mathcal{M}(\mathcal{O}) \tag{3.3.22}$$

coincides with $\mu_{\varepsilon,\delta}$.

Proof. We apply $\tilde{\mu}_{\varepsilon,\delta}$ to $\eta \in \mathcal{C}_c^0(\mathcal{O})$ and obtain

$$\begin{aligned}
\int_{\mathcal{O}} \eta \, \mathrm{d}\tilde{\mu}_{\varepsilon,\delta} &= \int_{\mathcal{O}} \eta \, \mathrm{d}((\rho_\delta * \mu_{\mathrm{ext}})|_{\mathcal{O}} \, \mathrm{d}x)_\varepsilon \\
&= \int_{\mathcal{O}} \eta_\varepsilon \, (\rho_\delta * \mu_{\mathrm{ext}})|_{\mathcal{O}} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} (\eta_\varepsilon)_{\mathrm{ext}} \, (\rho_\delta * \mu_{\mathrm{ext}}) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \rho_\delta * (\eta_\varepsilon)_{\mathrm{ext}} \, \mathrm{d}\mu_{\mathrm{ext}} \\
&= \int_{\mathcal{O}} \rho_\delta * \eta_\varepsilon \, \mathrm{d}\mu,
\end{aligned}$$

where for the last step, we used Remark 3.3.12 to extend the integration domain. We conclude by noticing that the last term is precisely the definition of ${}_{\mathcal{M}(\mathcal{O})}\langle \mu_{\varepsilon,\delta}, \eta\rangle_{\mathcal{C}_c^0(\mathcal{O})}$. ☐

In the rest of this section, we will argue that the sequence

$$\left(\mu_{\frac{1}{n},\frac{1}{2}w(\frac{1}{n})}\right)_{n\in\mathbb{N}}$$

is an approximation of $\mu \in \mathcal{M}(\mathcal{O}) \cap H^{-1}$ in the sense of Theorem 3.3.8. First we address the regularity of $\mu_{\varepsilon,\delta}$, where $\varepsilon > 0$ and $0 < \delta \leq \frac{w(\varepsilon)}{2}$.

Lemma 3.3.19. *Let $\varepsilon > 0$, $0 < \delta \leq \frac{w(\varepsilon)}{2}$ and $\mu \in \mathcal{M}(\mathcal{O})$. Then, the measure $\mu_{\varepsilon,\delta}$ has a bounded density with respect to Lebesgue measure.*

Proof. The fact that $\mu_{\varepsilon,\delta}$ has a density with respect to Lebesgue measure follows from its characterization in Lemma 3.3.18 and Lemma 3.3.17. This density is bounded in space since

$$\left|\sum_{j=0}^{l} \zeta^j(x+\varepsilon e_d^j)(\rho_\delta * \mu_{\text{ext}})(x+\varepsilon e_d^j)\right| \leq (l+1) \sup_{x\in\mathbb{R}^d} |\rho_\delta(x)| \, \|\mu\|_{TV}\,.$$

☐

The first part of the following proposition allows to deduce property (3.3.8), while the second part is needed for the further proof of (3.3.9).

Proposition 3.3.20. *Let ρ be as in (3.3.13), and for each $\varepsilon > 0$, let $0 < \delta_\varepsilon \leq \frac{w(\varepsilon)}{2}$.*

1. *For $\eta \in H_0^1(\mathcal{O})$, we have*
$$\rho_{\delta_\varepsilon} * \eta_\varepsilon \to \eta \quad \text{for } \varepsilon \searrow 0 \text{ in } H_0^1(\mathcal{O}). \tag{3.3.23}$$
2. *For $\eta \in \mathcal{C}_c^0(\mathcal{O})$, we have*
$$\rho_{\delta_\varepsilon} * \eta_\varepsilon \to \eta \quad \text{for } \varepsilon \searrow 0 \text{ in } \mathcal{C}_c^0(\mathcal{O}). \tag{3.3.24}$$

Proof. Throughout this proof, we will write δ instead of δ_ε, always assuming that $0 < \delta \leq \frac{w(\varepsilon)}{2}$.

Proof of part 1: It is enough to show that for all $i \in \{1, \dots, d\}$

$$\|\partial_i(\rho_\delta * \eta_\varepsilon) - \partial_i \eta\|_{L^2(\mathcal{O})} \to 0 \quad \text{for } \varepsilon \to 0. \tag{3.3.25}$$

By the density of $\mathcal{C}_0^\infty(\mathcal{O})$ in $H_0^1(\mathcal{O})$, for any $\beta > 0$ we can choose $\varphi \in \mathcal{C}_0^\infty(\mathcal{O})$ such that

$$\max\left\{\|\varphi - \eta\|_{L^2(\mathcal{O})}, \|\partial_i\varphi - \partial_i\eta\|_{L^2(\mathcal{O})}\right\} \leq \frac{\beta}{6(l+1)\tilde{C}}, \tag{3.3.26}$$

where

$$\tilde{C} := \max\left\{\max_{j=1,\dots,l} (\sup_{\mathbb{R}^d} |\partial_i \zeta^j|), 1\right\}.$$

As $\varphi_{\text{ext}}, \zeta^j \in \mathcal{C}_b^1(\mathcal{O})$ for each $j \in \{1, \dots, l\}$, we can choose $\varepsilon_0 > 0$ small enough, such that for all $x \in \mathbb{R}^d$ and $y, z \in B_{\varepsilon_0}(x)$

$$\left|\partial_i\zeta^j(y)\varphi_{\text{ext}}(z) - \partial_i\zeta^j(x)\varphi_{\text{ext}}(x)\right| \leq \frac{\beta}{6(l+1)\,|\mathcal{O}|^{\frac{1}{2}}} \tag{3.3.27}$$

and

$$\left|\zeta^j(y)\partial_i\varphi_{\text{ext}}(z) - \zeta^j(x)\partial_i\varphi_{\text{ext}}(x)\right| \leq \frac{\beta}{6(l+1)\,|\mathcal{O}|^{\frac{1}{2}}}. \tag{3.3.28}$$

We approach (3.3.25) by splitting the term under consideration into the more convenient pieces

$$\begin{aligned}
&\|\partial_i(\rho_\delta * \eta_\varepsilon) - \partial_i\eta\|_{L^2(\mathcal{O})} = \|\rho_\delta * \partial_i\eta_\varepsilon - \partial_i\eta_{\text{ext}}\|_{L^2(\mathbb{R}^d)}\\
&= \|\rho_\delta * \partial_i(\eta_\varepsilon - \varphi_\varepsilon) + \rho_\delta * \partial_i\varphi_\varepsilon - \partial_i\varphi_{\text{ext}} + \partial_i\varphi_{\text{ext}} - \partial_i\eta_{\text{ext}}\|_{L^2(\mathbb{R}^d)}\\
&\leq \|\rho_\delta * \partial_i(\eta_\varepsilon - \varphi_\varepsilon)\|_{L^2(\mathbb{R}^d)} + \|\rho_\delta * \partial_i\varphi_\varepsilon - \partial_i\varphi_{\text{ext}}\|_{L^2(\mathbb{R}^d)} + \|\partial_i\varphi_{\text{ext}} - \partial_i\eta_{\text{ext}}\|_{L^2(\mathbb{R}^d)}\\
&= \text{(I)} + \text{(II)} + \text{(III)}.
\end{aligned}$$

We estimate the summands separately. For the first one we get with the convolution estimate (e. g. Section 2.13 in [1])

$$
\begin{aligned}
(\mathrm{I}) &\leq \|\partial_i(\eta_\varepsilon - \varphi_\varepsilon)\|_{L^2(\mathbb{R}^d)} \\
&= \left\| \sum_{j=0}^{l} \partial_i [\zeta^j (\eta_{\mathrm{ext}}(\cdot - \varepsilon e_d^j) - \varphi_{\mathrm{ext}}(\cdot - \varepsilon e_d^j))] \right\|_{L^2(\mathbb{R}^d)} \\
&\leq \sum_{j=0}^{l} \left\| \partial_i \zeta^j \left(\eta_{\mathrm{ext}}(\cdot - \varepsilon e_d^j) - \varphi_{\mathrm{ext}}(\cdot - \varepsilon e_d^j) \right) \right\|_{L^2(\mathbb{R}^d)} \\
&\qquad + \sum_{j=0}^{l} \left\| \zeta^j \left(\partial_i \eta_{\mathrm{ext}}(\cdot - \varepsilon e_d^j) - \partial_i \varphi_{\mathrm{ext}}(\cdot - \varepsilon e_d^j) \right) \right\|_{L^2(\mathbb{R}^d)} \\
&\leq \sum_{j=0}^{l} \left(\sup_{\mathbb{R}^d} \left|\partial_i \zeta^j\right| \|\eta_{\mathrm{ext}} - \varphi_{\mathrm{ext}}\|_{L^2(\mathbb{R}^d)} + \|\partial_i \eta_{\mathrm{ext}} - \partial_i \varphi_{\mathrm{ext}}\|_{L^2(\mathbb{R}^d)} \right) \leq \frac{\beta}{3},
\end{aligned}
\tag{3.3.29}
$$

where we used (3.3.26) in the last step. For the second term, we recall that $(\zeta^j)_{j=0}^l$ is a partition of unity on the support of φ. Thus, we can compute

$$
\begin{aligned}
(\mathrm{II}) &\leq \sum_{j=0}^{l} \left\| \rho_\delta * \partial_i \left(\zeta^j \varphi_{\mathrm{ext}}(\cdot - \varepsilon e_d^j) \right) - \partial_i \left(\zeta^j \varphi_{\mathrm{ext}} \right) \right\|_{L^2(\mathbb{R}^d)} \\
&= \sum_{j=0}^{l} \left\| \rho_\delta * \left(\partial_i \zeta^j \varphi_{\mathrm{ext}}(\cdot - \varepsilon e_d^j) + \zeta^j \partial_i \varphi_{\mathrm{ext}}(\cdot - \varepsilon e_d^j) \right) - \partial_i \zeta^j \varphi_{\mathrm{ext}} - \zeta^j \partial_i \varphi_{\mathrm{ext}} \right\|_{L^2(\mathbb{R}^d)} \\
&\leq \sum_{j=0}^{l} \left\| \rho_\delta * (\partial_i \zeta^j \varphi_{\mathrm{ext}}(\cdot - \varepsilon e_d^j)) - \partial_i \zeta^j \varphi_{\mathrm{ext}} \right\|_{L^2(\mathbb{R}^d)} \\
&\quad + \sum_{j=0}^{l} \left\| \rho_\delta * (\zeta^j \partial_i \varphi_{\mathrm{ext}}(\cdot - \varepsilon e_d^j)) - \zeta^j \partial_i \varphi_{\mathrm{ext}} \right\|_{L^2(\mathbb{R}^d)} \\
&=: \sum_{j=0}^{l} (\mathrm{IV})_j + \sum_{j=0}^{l} (\mathrm{V})_j.
\end{aligned}
$$

$(\mathrm{IV})_j$ and $(\mathrm{V})_j$ are treated analogously, so we only show the estimate for $(\mathrm{V})_j$, where we choose $\varepsilon < \frac{\varepsilon_0}{2}$ with ε_0 as for (3.3.27). Noting that ρ_δ integrates to 1 for any $\delta > 0$ and using Jensen's inequality in the second step, we obtain

$$
\begin{aligned}
(\mathrm{V})_j^2 &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \rho_\delta(x-y) \left(\zeta^j(y) \partial_i \varphi_{\mathrm{ext}}(y - \varepsilon e_d^j) - \zeta^j(x) \partial_i \varphi_{\mathrm{ext}}(x) \right) \mathrm{d}y \right|^2 \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} \int_{B_\delta(x)} \rho_\delta(x-y) \left| \zeta^j(y) \partial_i \varphi_{\mathrm{ext}}(y - \varepsilon e_d^j) - \zeta^j(x) \partial_i \varphi_{\mathrm{ext}}(x) \right|^2 \mathrm{d}y \, \mathrm{d}x.
\end{aligned}
\tag{3.3.30}
$$

As $\partial_i \varphi_{\mathrm{ext}}$ is supported on $\mathcal{O}$ and, for the analogous step for (IV), so is φ_{ext}, we can argue as in the proof of Remark 3.3.12 to see that the integrand of the outer integral is supported on $\mathcal{O}$. Thus, we can restrict the integration domain to obtain

$$
\begin{aligned}
(3.3.30) &= \int_{\mathcal{O}} \int_{B_\delta(x)} \rho_\delta(x-y) \left| \zeta^j(y) \partial_i \varphi_{\mathrm{ext}}(y - \varepsilon e_d^j) - \zeta^j(x) \partial_i \varphi_{\mathrm{ext}}(x) \right|^2 \mathrm{d}y \, \mathrm{d}x \\
&\leq \int_{\mathcal{O}} \frac{\beta^2}{36\,(l+1)^2\,|\mathcal{O}|} \int_{\mathbb{R}^d} \rho_\delta(x-y) \, \mathrm{d}y \, \mathrm{d}x = \left(\frac{\beta}{6(l+1)} \right)^2.
\end{aligned}
$$

While we have used (3.3.28) in the second step, the estimate for $(\mathrm{IV})_j$ uses (3.3.27) instead and gets the same result. We conclude

$$
(\mathrm{II}) = \sum_{j=0}^{l} \left((\mathrm{IV})_j + (\mathrm{V})_j \right) \leq \frac{\beta}{3}. \tag{3.3.31}
$$

Finally the estimate

$$(\mathrm{III}) \leq \frac{\beta}{3} \tag{3.3.32}$$

is obvious by property (3.3.26). Collecting (3.3.29), (3.3.31), and (3.3.32), we obtain

$$\|\partial_i(\rho_\delta * \eta_\varepsilon - \eta)\|_{L^2(\mathcal{O})} \leq \beta$$

only by choosing ε small enough and adapting $0 < \delta \leq \frac{w(\varepsilon)}{2}$, which proves (3.3.23).

Proof of part 2: Since η is now assumed to be continuous and to have compact support, it is uniformly continuous. For arbitrary $\beta > 0$, we can thus fix $\varepsilon_0 > 0$ such that for all $x, y \in \mathbb{R}^d$

$$|x - y| \leq \varepsilon_0 \quad \text{implies} \quad |\eta_{\text{ext}}(x) - \eta_{\text{ext}}(y)| \leq \frac{\beta}{l+1}.$$

For $\varepsilon \leq \frac{1}{2}\varepsilon_0$, we use $\delta \leq \frac{w(\varepsilon)}{2} \leq \varepsilon$ by (3.3.15) to calculate for $x \in \mathcal{O}$

$$\begin{aligned}
|\rho_\delta * \eta_\varepsilon(x) - \eta(x)| &= \left| \int_{B_\delta(x)} \rho_\delta(x-y) \left(\sum_{j=0}^{l} \zeta^j(y)(\eta_{\text{ext}}(y - \varepsilon e_d^j) - \eta(x)) \right) \mathrm{d}y \right| \\
&\leq \int_{B_\delta(x)} \rho_\delta(x-y) \sum_{j=0}^{l} \left| \eta_{\text{ext}}(y - \varepsilon e_d^j) - \eta_{\text{ext}}(x) \right| \mathrm{d}y \\
&\leq \int_{B_\delta(x)} \rho_\delta(x-y) \sum_{j=0}^{l} \frac{\beta}{l+1} \mathrm{d}y = \beta,
\end{aligned}$$

where for the second step we observe that for $y \in B_\delta(x)$, we have

$$\left| (y - \varepsilon e_d^j) - x \right| \leq \delta + \varepsilon \leq 2\varepsilon \leq \varepsilon_0.$$

This proves (3.3.24). □

We now turn to prove Property (3.3.9). Recall the definition of a convex function of a measure from Definition 3.3.3. We need some more lemmas on measures obtained by this technique, the first of which can be found in Equation (2.11) in [45].

Lemma 3.3.21. *Let ψ satisfy* (3.2.7) *as well as Assumptions 3.2.1 (A3),(A5'). Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and let $(\rho_\delta)_{\delta>0}$ be a family of mollifying kernels as specified in* (3.3.12) *and* (3.3.13)*. Then*

$$\int_{\mathbb{R}^d} \psi(\rho_\delta * \mu)\, \mathrm{d}x \leq \int_{\mathbb{R}^d} \psi(\mu) \quad \textit{for all } \delta > 0. \tag{3.3.33}$$

Remark 3.3.22. Given the assumptions on ψ, the theory of Definition 3.3.3 indeed also applies to finite measures on $\mathbb{R}^d$ (cf. p. 202 in [118]).

Lemma 3.3.23. *Let ψ satisfy* (3.2.7) *as well as conditions Assumptions 3.2.1 (A3),(A5'). For $\mu \in \mathcal{M}(\mathcal{O})$ we have*

$$\int_{\mathbb{R}^d} \psi(\mu_{\text{ext}}) = \int_{\mathcal{O}} \psi(\mu). \tag{3.3.34}$$

Proof. We define

$$\mathcal{D}_1 := \left\{ \int_{\mathcal{O}} v \,\mathrm{d}\mu - \int_{\mathcal{O}} \psi^*(v)\, \mathrm{d}x : v \in L^1(\mu), \psi^*(v) \in L^1(\mathcal{O}) \right\}$$

and

$$\mathcal{D}_2 := \left\{ \int_{\mathbb{R}^d} v \,\mathrm{d}\mu_{\text{ext}} - \int_{\mathbb{R}^d} \psi^*(v)\, \mathrm{d}x : v \in L^1(\mu_{\text{ext}}), \psi^*(v) \in L^1(\mathbb{R}^d) \right\},$$

which allows us to write

$$\int_{\mathcal{O}} \psi(\mu) = \sup \mathcal{D}_1 \quad \text{and} \quad \int_{\mathbb{R}^d} \psi(\mu_{\text{ext}}) = \sup \mathcal{D}_2.$$

We note that for v satisfying the conditions of $\mathcal{D}_1$, v_{ext} satisfies the conditions of $\mathcal{D}_2$, while the involved integrals agree due to the definition of μ_{ext} and $\psi^*(0) = 0$. This yields "$\geq$".

Conversely, for v satisfying the conditions of $\mathcal{D}_2$ we can define $\tilde{v} = v|_{\mathcal{O}}$. $\tilde{v}$ satisfies the conditions of $\mathcal{D}_1$. Furthermore, we have

$$\int_{\mathcal{O}} \tilde{v}\,\mathrm{d}\mu = \int_{\mathbb{R}^d} v\,\mathrm{d}\mu_{\text{ext}} \quad \text{and}$$
$$\int_{\mathcal{O}} \psi^*(\tilde{v})\,\mathrm{d}x \leq \int_{\mathbb{R}^d} \psi^*(v)\,\mathrm{d}x \quad \text{due to } \psi^* \geq 0.$$

Thus, we have found an element in $\mathcal{D}_1$ being larger than or equal to

$$\int_{\mathbb{R}^d} v\,\mathrm{d}\mu_{\text{ext}} - \int_{\mathbb{R}^d} \psi^*(v)\,\mathrm{d}x,$$

which yields "$\leq$", completing the proof. □

The key tool to prove the approximation property (3.3.9) is the following proposition.

Proposition 3.3.24. *Let $\varepsilon > 0$, $0 < \delta \leq \frac{w(\varepsilon)}{2}$ and $\mu \in \mathcal{M}(\mathcal{O})$. Let ψ satisfy (3.2.7) as well as Assumptions 3.2.1 (A3),(A5'). Then,*

$$\|\psi(\mu_{\varepsilon,\delta})\|_{TV} \leq \|\psi(\mu)\|_{TV}\,. \tag{3.3.35}$$

Proof. Recall Notations 3.3.10 and let $V^j = U^j \cap \mathcal{O}$. Let $(\xi_\alpha)_{\alpha>0} \subset C_c^0(\mathbb{R}^d)$ be a sequence of non-negative cut-off functions compactly supported in $\mathcal{O}$, which converge to 1 pointwise in $\mathcal{O}$ for $\alpha \to 0$, and each of which is monotonically increasing on each V^j in e_d^j direction.

Let $h \in L^1(\mathcal{O})$ and $\mu = h\,\mathrm{d}x$. In the following argument, we will need $\xi_\alpha(x) \geq \xi_\alpha(x - \varepsilon e_d^j)$ for $x \in V^j$, where $x - \varepsilon e_d^j$ is not a priori in $\mathcal{O}$. However, since $\xi_\alpha = 0$ outside of $\mathcal{O}$, it is clear that the statement is valid even if $x - \varepsilon e_d^j \notin \mathcal{O}$. By the convexity of ψ, the construction of $(\zeta^j)_{j=0}^l$ and Lemma 3.3.17, we then estimate

$$\begin{aligned}
\int_{\mathcal{O}} \xi_\alpha \psi(\mu_\varepsilon) &= \int_{\mathcal{O}} \xi_\alpha(x)\,\psi\left(\sum_{j=0}^{l} \zeta^j(x + \varepsilon e_d^j) h_{\text{ext}}(x + \varepsilon e_d^j)\right) \mathrm{d}x \\
&\leq \int_{\mathcal{O}} \xi_\alpha(x) \sum_{j=0}^{l} \zeta^j(x + \varepsilon e_d^j) \psi(h_{\text{ext}}(x + \varepsilon e_d^j))\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \xi_\alpha(x) \sum_{j=0}^{l} \zeta^j(x + \varepsilon e_d^j) \psi(h_{\text{ext}}(x + \varepsilon e_d^j))\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \psi(h_{\text{ext}}(x)) \sum_{j=0}^{l} \xi_\alpha(x - \varepsilon e_d^j) \zeta^j(x)\mathrm{d}x.
\end{aligned} \tag{3.3.36}$$

We note that $\sum_{j=0}^{l} \xi_\alpha(x - \varepsilon e_d^j)\zeta^j(x)$ is supported on $\mathcal{O}$ by Remark 3.3.12. Furthermore, by the construction of ξ_α, we have

$$\xi_\alpha(x - \varepsilon e_d^j) \leq \xi_\alpha(x)$$

for all $x \in V^j$, so this holds especially for $x \in \mathcal{O}$ for which $\zeta^j(x) > 0$. Thus, we can continue

$$\begin{aligned}
(3.3.36) &= \sum_{j=0}^{l} \int_{\mathcal{O}} \xi_\alpha(x - \varepsilon e_d^j) \zeta^j(x) \psi(h(x))\,\mathrm{d}x \\
&\leq \int_{\mathcal{O}} \sum_{j=0}^{l} \zeta^j(x) \xi_\alpha(x) \psi(h(x))\,\mathrm{d}x \\
&= \int_{\mathcal{O}} \xi_\alpha(x) \psi(h(x))\,\mathrm{d}x = \int_{\mathcal{O}} \xi_\alpha\,\psi(\mu).
\end{aligned} \tag{3.3.37}$$

For a positive Radon measure μ, we have $\mu(\mathcal{O}) = \sup\{\mu(K) : K \subseteq \mathcal{O} \text{ compact}\}$. Since any such K is included in

$$K_\alpha := \{x \in \mathcal{O} : \operatorname{dist}(x, \mathcal{O}^c) \geq \alpha\}$$

for α small enough, we can as well write $\mu(\mathcal{O}) = \lim_{\alpha\to 0} \mu(K_\alpha)$. Then, noting that $\xi_\alpha \geq \mathbf{1}_{K_\alpha}$, we can argue by definition of the Radon measure of compact sets that

$$\mu(\mathcal{O}) \geq \int_{\mathcal{O}} \xi_\alpha \mathrm{d}\mu \geq \mu(K_\alpha) \overset{\alpha\to 0}{\longrightarrow} \mu(\mathcal{O}),$$

thus $\mu(\mathcal{O}) = \lim_{\alpha\to 0} \int_{\mathcal{O}} \xi_\alpha \mathrm{d}\mu$.

Hence, we conclude by (3.3.37) for $\mu = h\,\mathrm{d}x$, $h \in L^1(\mathcal{O})$, that

$$\int_{\mathcal{O}} \psi(\mu_\varepsilon) = \lim_{\alpha\to 0} \int_{\mathcal{O}} \xi_\alpha \psi(\mu_\varepsilon) \leq \lim_{\alpha\to 0} \int_{\mathcal{O}} \xi_\alpha \psi(\mu) = \int_{\mathcal{O}} \psi(\mu). \tag{3.3.38}$$

Using (3.3.38), Lemma 3.3.21 and Lemma 3.3.23, we then obtain for $0 < \delta \leq \frac{w(\varepsilon)}{2}$

$$\begin{aligned}
\int_{\mathcal{O}} \psi(\mu_{\varepsilon,\delta}) &= \int_{\mathcal{O}} \psi(((\rho_\delta * \mu_{\mathrm{ext}})|_{\mathcal{O}}\,\mathrm{d}x)_\varepsilon) \\
&\leq \int_{\mathcal{O}} \psi((\rho_\delta * \mu_{\mathrm{ext}})|_{\mathcal{O}})\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \psi(\rho_\delta * \mu_{\mathrm{ext}})\mathbf{1}_{\mathcal{O}}\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} \psi(\rho_\delta * \mu_{\mathrm{ext}})\,\mathrm{d}x \leq \int_{\mathbb{R}^d} \psi(\mu_{\mathrm{ext}}) = \int_{\mathcal{O}} \psi(\mu),
\end{aligned}$$

which finishes the proof. $\square$

Corollary 3.3.25. *Together with Remark 3.3.4, Proposition 3.3.24 immediately implies*

$$\limsup_{\varepsilon\searrow 0} \int_{\mathcal{O}} \psi(\mu_{\varepsilon,\delta_\varepsilon}) \leq \int_{\mathcal{O}} \psi(\mu),$$

where $\mu \in \mathcal{M}(\mathcal{O})$ and $0 < \delta_\varepsilon \leq \frac{w(\varepsilon)}{2}$ for each $\varepsilon > 0$.

Proof of Theorem 3.3.8. For u as in Theorem 3.3.8, we show that the sequence

$$(u_n)_{n\in\mathbb{N}} := \left(u_{\frac{1}{n},\frac{1}{2}w(\frac{1}{n})}\right)_{n\in\mathbb{N}},$$

where w was defined in Remark 3.3.12, meets all requirements.

By construction, $u_n \in \mathcal{M}(\mathcal{O}) \cap H^{-1}$ for all $n \in \mathbb{N}$, and by Lemma 3.3.19, the density of u_n is bounded and thus in $L^2(\mathcal{O})$. Property (3.3.8) is proved in the first part of Proposition 3.3.20. For Property (3.3.9), note that Corollary 3.3.25 especially shows that $(\psi(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in the TV norm, which means that it contains a subsequence that converges weakly* to $\psi(u)$ by Proposition 3.3.20, Corollary 3.3.25 and Lemma 2.1 in [45]. Since this argument can be carried out for any subsequence, we get weak* convergence for the whole sequence and, also by Lemma 2.1 in [45],

$$\left\|\psi\left(u_{\frac{1}{n},\frac{1}{2}w(\frac{1}{n})}\right)\right\|_{TV} = \int_{\mathcal{O}} \psi\left(u_{\frac{1}{n},\frac{1}{2}w(\frac{1}{n})}\right) \to \int_{\mathcal{O}} \psi(u) = \|\psi(u)\|_{TV} \quad \text{as } n \to \infty.$$

This yields (3.3.9) and thereby concludes the proof. $\square$

3.4 Proof of the main result

Throughout this section, we work under Assumptions 3.2.1.

We first solve a modified SPDE by the variational approach, which will yield ε-approximate solutions. Moreover, we show improved regularity for those approximations, which is used later to prove their convergence to a limit in $L^2(\Omega; \mathcal{C}([0,T]; H^{-1}))$ for $\varepsilon \to 0$.

We consider the SPDE

$$\begin{aligned} \mathrm{d}X_t^\varepsilon &= \varepsilon\Delta X_t^\varepsilon \mathrm{d}t + \Delta\phi^\varepsilon(X_t^\varepsilon)\mathrm{d}t + B(t, X_t^\varepsilon)\mathrm{d}W_t, \\ X_0^\varepsilon &= x_0, \end{aligned} \tag{3.4.1}$$

where we use the notation for the Yosida approximation of Appendix 3.D and assume $x_0 \in L^2(\Omega, \mathcal{F}_0; L^2)$. Now and in the following we omit the domain $\mathcal{O}$ when using Lebesgue and Sobolev spaces as well as spaces of continuous or continuously differentiable functions, as introduced in Section 3.1.2.

Lemma 3.4.1. *For all $T > 0$, Problem* (3.4.1) *gives rise to a solution in sense of Definition 3.B.1 with respect to the Gelfand triple $V := L^2 \hookrightarrow H^{-1} \hookrightarrow (L^2)' = V'$.*

Proof. We prove that (3.4.1) fits into the framework of Appendix 3.B with the operator

$$A(u) = \Delta(\varepsilon u + \phi^\varepsilon(u)) \quad \text{for } u \in L^2.$$

In [112, Example 4.1.11], it is shown that an operator A of the form $u \mapsto \Delta(\Psi(u))$ satisfies the four properties of Appendix 3.B with respect to the Gelfand triple $L^p \hookrightarrow H^{-1} \hookrightarrow (L^p)'$, if the following conditions are satisfied.

(Ψ1) Ψ is continuous.

(Ψ2) For all $s, t \in \mathbb{R}$ we have

$$(t - s)(\Psi(t) - \Psi(s)) \geq 0.$$

(Ψ3) There exist $p \in [2, \infty), a \in (0, \infty), c \in [0, \infty)$ such that for all $s \in \mathbb{R}$ we have

$$s\Psi(s) \geq a\,|s|^p - c.$$

(Ψ4) There exist $c_3, c_4 \in (0, \infty)$ such that for all $s \in \mathbb{R}$

$$|\Psi(s)| \leq c_4 + c_3\,|s|^{p-1},$$

where p is as in (Ψ3).

We briefly check (Ψ1) – (Ψ4) for $\Psi := \varepsilon\,\mathrm{Id}_\mathbb{R} + \phi^\varepsilon$. The first condition is satisfied by Lemma 3.D.2, the second one by the maximal monotonicity of ϕ^ε, together with [7, Corollary 2.1]. Using $\phi^\varepsilon(0) = 0$ and again the monotonicity of ϕ^ε, we obtain $s\phi^\varepsilon(s) \geq 0$ and thereby

$$s\Psi(s) \geq s\varepsilon\,\mathrm{Id}_\mathbb{R}(s) = \varepsilon\,|s|^2.$$

Thus, (Ψ3) is satisfied for $p = 2, a = \varepsilon$ and $c = 0$. (Ψ4) is then clear by Lemma 3.D.2. Thus, Theorem 3.B.2 is applicable as required. $\square$

The following lemma provides an important estimate on the regularity of these approximate solutions and corresponds to [74, Lemma B.1]:

Lemma 3.4.2. *Let $\varepsilon > 0$, $x_0 \in L^2(\Omega, \mathcal{F}_0; L^2)$ and $T > 0$. Then for the solution $(X_t^\varepsilon)_{t\in[0,T]}$ to* (3.4.1) *we have*

$$\mathbb{E}\sup_{t\in[0,T]} \|X_t^\varepsilon\|_2^2 + \varepsilon\mathbb{E}\int_0^T \|X_r^\varepsilon\|_{H_0^1}^2\,\mathrm{d}r \leq C(\mathbb{E}\,\|x_0\|_2^2 + 1)$$

with a constant $C > 0$ independent of ε.

Proof. Let $(e_i)_{i\in\mathbb{N}} \subset C_0^2$ be a sequence of smooth eigenvectors to $-\Delta$, i.e. $-\Delta e_i = \lambda_i e_i$ for some $(\lambda_i)_{i\in\mathbb{N}} \subset (0, \infty)$, such that $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis in H^{-1}. Such a sequence can be obtained by first choosing an L^2-orthonormal basis of $(-\Delta)$-eigenvectors $(\tilde{e}_i)_{i\in\mathbb{N}} \subset C_0^2 \subset L^2$, where

$$-\Delta\tilde{e}_i = \lambda_i\tilde{e}_i \quad \text{for some } \lambda_i > 0. \tag{3.4.2}$$

Then, setting

$$e_i = \sqrt{\lambda_i}\,\tilde{e}_i \quad \text{for } i \in \mathbb{N}$$

keeps (3.4.2) true for $\tilde{e}_i$ replaced by e_i and makes $(e_i)_{i\in\mathbb{N}}$ an orthonormal basis in H^{-1} as required. The latter can be seen by computing for $i, j \in \mathbb{N}$

$$\langle e_i, e_j\rangle_{H^{-1}} = \sqrt{\lambda_i\lambda_j}\left\langle -\Delta^{-1}\tilde{e}_i, \tilde{e}_j\right\rangle_{L^2} = \frac{\sqrt{\lambda_i\lambda_j}}{\lambda_i}\left\langle \tilde{e}_i, \tilde{e}_j\right\rangle_{L^2} = \delta_{ij}.$$

We further let $P^n : H^{-1} \to H_n := \operatorname{span}\{e_1, \dots, e_n\}$ be the H^{-1}-orthogonal projection onto the span of the first n eigenvectors, i. e.

$$P^n(y) = \sum_{i=1}^n \langle y, e_i\rangle_{H^{-1}} e_i.$$

Recall that the unique variational solution X^ε to (3.4.1) is constructed in [112, Section 4.2] as a (weak) limit in $L^2([0,T]\times\Omega; L^2)$ of the solutions to the Galerkin approximation

$$\begin{aligned} \mathrm{d}X_t^n &= \varepsilon P^n\Delta X_t^n \mathrm{d}t + P^n\Delta\phi^\varepsilon(X_t^n)\mathrm{d}t + P^nB(t, X_t^n)\mathrm{d}W_t^n \\ X_0^n &= P^n x_0, \end{aligned}$$

in H_n, where for simplicity we omit the ε-dependence of X^n, and for an orthonormal basis $(g_i)_{i\in\mathbb{N}}$ of U (as defined in Assumption 3.2.1 (A1)) we let

$$W_t^n = \sum_{i=1}^n \left\langle J^{-1}(W_t), g_i\right\rangle_U g_i.$$

We first note that for $x \in H_n$ we have $\Delta x \in H_n \subset L^2$ and thus $P_n(-\Delta x) = -\Delta x$. Using $X_t^n \in H_n$ for all $t \in [0,T]$, we have

$$\langle X_t^n, P^n(-\Delta X_t^n)\rangle_{L^2} = \|X^n\|_{H_0^1}^2.$$

We note by Lemma 3.D.3 and (3.2.5) that

$$|\phi^\varepsilon(X^n)|^2 \le C(1 + (X^n)^2),$$

so $\phi^\varepsilon(X^n) \in L^2$ since $X^n \in H_n \subseteq L^2$. Thus, $\phi^\varepsilon(X^n) \in H_0^1$ by [126, Theorem 2.1.11], and we can compute

$$\begin{aligned} \langle X^n, P^n(\Delta\phi^\varepsilon(X^n))\rangle_{L^2} &= \left\langle X^n, \sum_{i=1}^n \langle \Delta\phi^\varepsilon(X^n), e_i\rangle_{H^{-1}} e_i \right\rangle_{L^2} \\ &= \left\langle \Delta\phi^\varepsilon(X^n), \sum_{i=1}^n \langle X^n, e_i\rangle_{L^2} e_i \right\rangle_{H^{-1}} \\ &= \left\langle \Delta\phi^\varepsilon(X^n), \sum_{i=1}^n \langle -\Delta X^n, e_i\rangle_{H^{-1}} e_i \right\rangle_{H^{-1}} \\ &= \langle \Delta\phi^\varepsilon(X^n), -\Delta X^n\rangle_{H^{-1}} \\ &= \langle \Delta\phi^\varepsilon(X^n), X^n\rangle_{H^{-1}\times H_0^1}. \end{aligned}$$

Again by [126, Theorem 2.1.11], we obtain for all $r \in [0,T]$

$$\langle \Delta\phi^\varepsilon(X_r^n), X_r^n\rangle_{H^{-1}\times H_0^1} = -\langle \nabla X_r^n, \nabla\phi^\varepsilon(X_r^n)\rangle_{L^2} = -\left(\phi^\varepsilon\right)'(X_r^n)\,\|X_r^n\|_{H_0^1}^2 \le 0,$$

where we used that $(\phi^\varepsilon)'(X_r^n) \ge 0$ almost everywhere by the monotonicity of ϕ^ε. Along with the finite-dimensional Ito formula, this can be used to estimate

$$\begin{aligned} e^{-Kt}\left\|X_t^n\right\|_{L^2}^2 &= \left\|P^n x_0\right\|_{L^2}^2 + 2\int_0^t e^{-Kr}\left\langle X_r^n, \varepsilon P^n(\Delta X_r^n) + P^n(\Delta\phi^\varepsilon(X_r^n))\right\rangle_{L^2} \mathrm{d}r \\ &\quad +2\int_0^t e^{-Kr}\left\langle X_r^n, P^n B(r, X_r^n)\,\mathrm{d}W_r^n\right\rangle_{L^2} \\ &\quad +\int_0^t e^{-Kr}\left\|P^n B(r, X_r^n)\right\|_{L_2(U,L^2)}^2 \mathrm{d}r - K\int_0^t e^{-Kr}\left\|X_r^n\right\|_{L^2}^2 \mathrm{d}r \\ &\leq \left\|P^n x_0\right\|_{L^2}^2 - 2\varepsilon\int_0^t e^{-Kr}\left\|X_r^n\right\|_{H_0^1}^2 \mathrm{d}r \\ &\quad +2\int_0^t e^{-Kr}\left\langle X_r^n, P^n B(r, X_r^n)\,\mathrm{d}W_r^n\right\rangle_{L^2} \\ &\quad +\int_0^t e^{-Kr}\left\|P^n B(r, X_r^n)\right\|_{L_2(U,L^2)}^2 \mathrm{d}r - K\int_0^t e^{-Kr}\left\|X_r^n\right\|_{L^2}^2 \mathrm{d}r. \end{aligned} \tag{3.4.3}$$

Using lemma 3.E.1, the Burkholder-Davis-Gundy inequality (see e. g. [112, Appendix D]) and (3.2.3), we get for the stochastic integral term in (3.4.3)

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \left\langle e^{-\frac{Kr}{2}}X_r^n, e^{-\frac{Kr}{2}}P^n B(r, X_r^n)\,\mathrm{d}W_r^n\right\rangle_{L^2}\right| \\ &\leq 3\,\mathbb{E}\left\langle \int_0^\cdot \left\langle e^{-\frac{Kr}{2}}X_r^n, e^{-\frac{Kr}{2}}P^n B(r, X_r^n)\,\mathrm{d}W_r^n\right\rangle_{L^2}\right\rangle_T^{\frac{1}{2}} \\ &\leq 3\,\mathbb{E}\left(\int_0^T \left\|e^{-\frac{Kr}{2}}X_r^n\right\|_{L^2}^2 \left\|e^{-\frac{Kr}{2}}P^n B(r, X_r^n)\right\|_{L_2(U,L^2)}^2 \mathrm{d}r\right)^{\frac{1}{2}} \\ &\leq 3\,\mathbb{E}\left(\int_0^T \left\|e^{-\frac{Kr}{2}}X_r^n\right\|_{L^2}^2 \left\|e^{-\frac{Kr}{2}}B(r, X_r^n)\right\|_{L_2(U,L^2)}^2 \mathrm{d}r\right)^{\frac{1}{2}} \\ &\leq 3\,\mathbb{E}\left[\sup_{r\in[0,T]}\left(e^{-Kr}\left\|X_r^n\right\|_{L^2}^2\right)^{\frac{1}{2}}\left(\int_0^T \left\|e^{-\frac{Kr}{2}}B(r, X_r^n)\right\|_{L_2(U,L^2)}^2\right)^{\frac{1}{2}}\right] \\ &\leq 3\,\mathbb{E}\left[\frac{1}{12}\sup_{r\in[0,T]}\left(e^{-Kr}\left\|X_r^n\right\|_{L^2}^2\right) + \frac{6}{2}\int_0^T e^{-Kr}\left\|B(r, X_r^n)\right\|_{L^2(U,L^2)}^2 \mathrm{d}r\right] \\ &\leq \frac{1}{4}\mathbb{E}\sup_{r\in[0,T]}\left(e^{-Kr}\left\|X_r^n\right\|_{L^2}^2\right) + 9\mathbb{E}\int_0^T e^{-Kr}C(1+\left\|X_r^n\right\|_{L^2}^2)\,\mathrm{d}r \\ &= \frac{1}{4}\mathbb{E}\sup_{r\in[0,T]}\left(e^{-Kr}\left\|X_r^n\right\|_{L^2}^2\right) + 9C\mathbb{E}\int_0^T e^{-Kr}\left\|X_r^n\right\|_{L^2}^2\,\mathrm{d}r + \tilde{C} \end{aligned} \tag{3.4.4}$$

We can now estimate from (3.4.3) and the previous calculation that

$$\begin{aligned} &\mathbb{E}\sup_{r\in[0,T]}\left(e^{-Kr}\left\|X_r^n\right\|_{L^2}^2\right) + K\mathbb{E}\int_0^T e^{-Kr}\left\|X_r^n\right\|_{L^2}^2 \mathrm{d}r + 2\varepsilon\int_0^T e^{-Kr}\left\|X_r^n\right\|_{H_0^1}^2 \mathrm{d}r \\ &\leq 3\,\mathbb{E}\sup_{t\in[0,T]}\left(e^{-Kr}\left\|X_r^n\right\|_{L^2}^2 + K\mathbb{E}\int_0^t e^{-Kr}\left\|X_r^n\right\|_{L^2}^2 \mathrm{d}r + 2\varepsilon\int_0^t e^{-Kr}\left\|X_r^n\right\|_{H_0^1}^2 \mathrm{d}r\right) \\ &\leq 3\left(\mathbb{E}\left\|x_0\right\|_{L^2}^2 + \frac{1}{4}\mathbb{E}\sup_{r\in[0,T]}\left(e^{-Kr}\left\|X_r^n\right\|_{L^2}^2\right) + C\mathbb{E}\int_0^T e^{-Kr}\left\|X_r^n\right\|_{L^2}^2\,\mathrm{d}r + \tilde{C}\right), \end{aligned}$$

where we absorbed the second-to-last term in (3.4.3) into the terms with the constants C and $\tilde{C}$. Thus, we get

$$\mathbb{E}\sup_{r\in[0,T]}\left(e^{-Kr}\left\|X_r^n\right\|_{L^2}^2\right)+\varepsilon\int_0^T e^{-Kr}\left\|X_r^n\right\|_{H_0^1}^2\mathrm{d}r$$
$$\le 4\mathbb{E}\left\|x_0\right\|_{L^2}^2+4(C-K)\mathbb{E}\int_0^T e^{-Kr}\left\|X_r^n\right\|_{L^2}^2\mathrm{d}r+\tilde{C},$$

and by choosing K large enough and multiplying by e^{KT}, which we absorb in the constant, we obtain

$$\mathbb{E}\sup_{r\in[0,T]}\left\|X_r^n\right\|_{L^2}^2+\varepsilon\,\mathbb{E}\int_0^T\left\|X_r^n\right\|_{H_0^1}^2\mathrm{d}r\le C(\mathbb{E}\left\|x_0\right\|_{L^2}^2+1).$$

Thus, $(X^n)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega;L^\infty([0,T];L^2))$ and in $L^2(\Omega\times[0,T];H_0^1)$. The latter is a Hilbert space, thus we can extract a weakly converging subsequence whose limit can be identified with the unique weak $L^2(\Omega\times[0,T];L^2)$ limit X^ε. Furthermore, we can interpret the former as the dual space of $L^2(\Omega;L^1([0,T];L^2))$ which is separable. Thus, we can extract a weak* converging subsequence whose limit can again be identified with X^ε. By weak (respectively weak*) lower-semicontinuity of the norms, we can thus pass to the limit $n\to\infty$ to obtain the required inequality. □

Proof of Theorem 3.2.6. The proof will be carried out in three steps. We first construct a solution candidate as a limit of solutions to (3.4.1). Then we show that this limit indeed is an SVI solution and we conclude by showing uniqueness, which relies on the same construction which was already used to show the existence of a solution.

Step 1: We begin by showing that the solutions $(X^\varepsilon)_{\varepsilon>0}$ to (3.4.1) for $\varepsilon\to0$ form a Cauchy sequence in $L^2(\Omega;\mathcal{C}([0,T];H^{-1}))$. To this end, we first consider two of those solutions $X^{\varepsilon_1},X^{\varepsilon_2}$ with respective initial condition $x_0^1,x_0^2\in L^2(\Omega,\mathcal{F}_0;L^2)$. By subsequently applying the Ito formula for the squared norm in Hilbert spaces (see e. g. [112, Theorem 4.2.5]) and the finite-dimensional Ito formula (see e. g. [114, IV.§3]), we have for $K>0$

$$\begin{aligned}e^{-Kt}\left\|X_t^{\varepsilon_1}-X_t^{\varepsilon_2}\right\|_{H^{-1}}^2=&\left\|x_0^1-x_0^2\right\|_{H^{-1}}^2\\&+2\int_0^t e^{-Kr}\langle\varepsilon_1\Delta X_r^{\varepsilon_1}-\varepsilon_2\Delta X_r^{\varepsilon_2},X_r^{\varepsilon_1}-X_r^{\varepsilon_2}\rangle_{H^{-1}}\mathrm{d}r\\&+2\int_0^t e^{-Kr}\langle\Delta\phi^{\varepsilon_1}(X_r^{\varepsilon_1})-\Delta\phi^{\varepsilon_2}(X_r^{\varepsilon_2}),X_r^{\varepsilon_1}-X_r^{\varepsilon_2}\rangle_{H^{-1}}\mathrm{d}r\\&+2\int_0^t e^{-Kr}\langle X_r^{\varepsilon_1}-X_r^{\varepsilon_2},B(r,X_r^{\varepsilon_1})-B(r,X_r^{\varepsilon_2})\mathrm{d}W_r\rangle_{H^{-1}}\\&+\int_0^t e^{-Kr}\left\|B(r,X_r^{\varepsilon_1})-B(r,X_r^{\varepsilon_2})\right\|_{L_2(U,H^{-1})}^2\mathrm{d}r\\&-K\int_0^t e^{-Kr}\left\|X_r^{\varepsilon_1}-X_r^{\varepsilon_2}\right\|_{H^{-1}}^2\mathrm{d}r.\end{aligned}\tag{3.4.5}$$

We note that

$$\begin{aligned}\langle\varepsilon_1\Delta X_r^{\varepsilon_1}-\varepsilon_2\Delta X_r^{\varepsilon_2},X_r^{\varepsilon_1}-X_r^{\varepsilon_2}\rangle_{H^{-1}}&=-\int_{\mathcal{O}}(\varepsilon_1X_r^{\varepsilon_1}-\varepsilon_2X_r^{\varepsilon_2})(X_r^{\varepsilon_1}-X_r^{\varepsilon_2})\mathrm{d}x\\&\le C(\varepsilon_1+\varepsilon_2)\left(\left\|X_r^{\varepsilon_1}\right\|_{L^2}^2+\left\|X_r^{\varepsilon_2}\right\|_{L^2}^2\right)\end{aligned}$$

and, using Corollary 3.D.9 for the second step,

$$\begin{aligned}&\langle\Delta\phi^{\varepsilon_1}(X_r^{\varepsilon_1})-\Delta\phi^{\varepsilon_2}(X_r^{\varepsilon_2}),X_r^{\varepsilon_1}-X_r^{\varepsilon_2}\rangle_{H^{-1}}=\\&\qquad=-\int_{\mathcal{O}}\left(\phi^{\varepsilon_1}(X_r^{\varepsilon_1})-\phi^{\varepsilon_2}(X_r^{\varepsilon_2})\right)(X_r^{\varepsilon_1}-X_r^{\varepsilon_2})\,\mathrm{d}x\\&\qquad\le C(\varepsilon_1+\varepsilon_2)\left(1+\left\|X_r^{\varepsilon_1}\right\|_{L^2}^2+\left\|X_r^{\varepsilon_2}\right\|_{L^2}^2\right)\end{aligned}$$

$\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-almost everywhere. Using this and the Lipschitz property (3.2.2) of B, we continue (3.4.5) by

$$
\begin{aligned}
e^{-Kt} \|X_t^{\varepsilon_1} - X_t^{\varepsilon_2}\|_{H^{-1}}^2 \leq \left\|x_0^1 - x_0^2\right\|_{H^{-1}}^2 & \\
&+ C(\varepsilon_1 + \varepsilon_2) \int_0^t e^{-Kr} \left(1 + \|X_r^{\varepsilon_1}\|_{L^2}^2 + \|X_r^{\varepsilon_2}\|_{L^2}^2\right) \mathrm{d}r \\
&+ 2 \int_0^t e^{-Kr} \langle X_r^{\varepsilon_1} - X_r^{\varepsilon_2}, B(r, X_r^{\varepsilon_1}) - B(r, X_r^{\varepsilon_2}) \mathrm{d}W_r \rangle_{H^{-1}} \mathrm{d}r \\
&+ C \int_0^t e^{-Kr} \|X_r^{\varepsilon_1} - X_r^{\varepsilon_2}\|_{H^{-1}}^2 \mathrm{d}r \\
&- K \int_0^t e^{-Kr} \|X_r^{\varepsilon_1} - X_r^{\varepsilon_2}\|_{H^{-1}}^2 \mathrm{d}r.
\end{aligned}
$$

With (3.2.2), Lemma 3.4.2 and, as in (3.4.4), the Burkholder-Davis-Gundy inequality, we obtain

$$
\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} \left(e^{-Kt} \|X_t^{\varepsilon_1} - X_t^{\varepsilon_2}\|_{H^{-1}}^2\right) \leq C\mathbb{E} \left\|x_0^1 - x_0^2\right\|_{H^{-1}}^2 & \\
&+ C(\varepsilon_1 + \varepsilon_2) \left(\mathbb{E} \left\|x_0^1\right\|_{L^2}^2 + \mathbb{E} \left\|x_0^2\right\|_{L^2}^2 + 1\right)
\end{aligned}
\tag{3.4.6}
$$

for K large enough, where we use the assumption that $x_0^1, x_0^2 \in L^2$. If we assume that $x_0^1 = x_0^2 =: x_0$, (3.4.6) implies

$$
\mathbb{E} \sup_{t\in[0,T]} \left(e^{-Kt} \|X_t^{\varepsilon_1} - X_t^{\varepsilon_2}\|_{H^{-1}}^2\right) \leq C(\varepsilon_1 + \varepsilon_2)(\mathbb{E} \|x_0\|_{L^2}^2 + 1),
$$

and thus, by completeness there exists a process $X \in L^2(\Omega; \mathcal{C}([0,T]; H^{-1}))$ satisfying

$$
\begin{cases}
\mathbb{E} \sup_{t\in[0,T]} \|X_t^\varepsilon - X_t\|_{H^{-1}}^2 \to 0 & \text{for } \varepsilon \to 0 \\
X_0 = x_0.
\end{cases}
$$

In particular, we have for each $t \in [0,T]$ that $X_t^\varepsilon \to X_t$ for $\varepsilon \to 0$ in $L^2(\Omega; H^{-1})$. Since X_t^ε is $\mathcal{F}_t$-measurable by construction (see Theorem 3.B.2), so is X_t, which makes X an adapted process. If the initial condition is indeed in L^2, this will be the candidate for an SVI solution.

It remains to construct a solution candidate if the initial state is not in L^2 but a general H^{-1} functional. To this end, we first notice that for two different initial conditions $x_0^1, x_0^2 \in L^2(\Omega, \mathcal{F}_0; L^2)$, we can construct the limit of the approximate solutions in $L^2(\Omega; \mathcal{C}([0,T]; H^{-1}))$ as before, call them X^1 and X^2, respectively, and take the limit $\varepsilon_1, \varepsilon_2 \to 0$ in (3.4.6) to obtain

$$
\mathbb{E} \sup_{t\in[0,T]} \left(e^{-Kt} \left\|X_t^1 - X_t^2\right\|_{H^{-1}}^2\right) \leq 2\mathbb{E} \left\|x_0^1 - x_0^2\right\|_{H^{-1}}^2. \tag{3.4.7}
$$

Let now $x_0 \in L^2(\Omega, \mathcal{F}_0; H^{-1})$ and select a sequence $(x_0^n)_{n\in\mathbb{N}} \subset L^2(\Omega, \mathcal{F}_0; L^2)$ such that $x_0^n \to x_0$ in $L^2(\Omega; H^{-1})$ for $n \to \infty$. Let $(X^{\varepsilon,n})_{\varepsilon>0,n\in\mathbb{N}}$ be the unique variational solutions to (3.4.1) with respective initial conditions $(x_0)_{n\in\mathbb{N}}$, for which Lemma 3.4.2 applies. We first construct the sequence $(X^n)_{n\in\mathbb{N}}$ as the unique limits in $L^2(\Omega; \mathcal{C}([0,T]; H^{-1}))$ obtained as in the argument above, and notice that it is a Cauchy sequence by (3.4.7). Thus, we obtain another limit $X \in L^2(\Omega; \mathcal{C}([0,T]; H^{-1}))$ which we identify as a solution to (3.1.1) in the sense of Definition 3.2.4 in the following step.

Step 2: We show that the limit process satisfies the properties of Definition 3.2.4. Let $\varepsilon > 0$, $x_0 \in L^2(\Omega, \mathcal{F}_0; H^{-1})$ and $(x_0^n)_{n\in\mathbb{N}} \subset L^2(\Omega, \mathcal{F}_0; L^2)$ such that $x_n \to x \in L^2(\Omega; H^{-1})$ for $n \to \infty$. Let $(X^{\varepsilon,n})_{\varepsilon>0,n\in\mathbb{N}}$ be the solutions to (3.4.1) with initial values x_0^n. For part (i) of Definition 3.2.4, we apply Ito's formula as in (3.4.5) to obtain for $t \in [0,T]$

$$\begin{aligned}
e^{-Kt}\left\|X_t^{\varepsilon,n}\right\|_{H^{-1}}^2 = \|x_0\|_{H^{-1}}^2 &+ 2\int_0^t e^{-Kr}\left\langle \varepsilon\Delta X_r^{\varepsilon,n}, X_r^{\varepsilon,n}\right\rangle_{H^{-1}} \mathrm{d}r \\
&+ 2\int_0^t e^{-Kr}\left\langle \Delta\phi^\varepsilon(X_r^{\varepsilon,n}), X_r^{\varepsilon,n}\right\rangle_{H^{-1}} \mathrm{d}r \\
&+ 2\int_0^t e^{-Kr}\left\langle X_r^{\varepsilon,n}, B(r,X_r^{\varepsilon,n})\mathrm{d}W_r\right\rangle_{H^{-1}} \mathrm{d}r \\
&+ \int_0^t e^{-Kr}\left\|B(r,X_r^{\varepsilon,n})\right\|_{L_2(U,H^{-1})}^2 \mathrm{d}r \\
&- K\int_0^t e^{-Kr}\left\|X_r^{\varepsilon,n}\right\|_{H^{-1}}^2 \mathrm{d}r.
\end{aligned} \tag{3.4.8}$$

Note that we have

$$\left\langle \varepsilon\Delta X_r^{\varepsilon,n}, X_r^{\varepsilon,n}\right\rangle_{H^{-1}} = -\varepsilon\left\|X_r^{\varepsilon,n}\right\|_{L^2} \le 0.$$

With the notation of Appendix 3.D and setting

$$\varphi^\varepsilon(v) = \begin{cases} \int_{\mathcal{O}} \psi^\varepsilon(v)\mathrm{d}x, & v \in L^{m+1}, \\ +\infty, & \text{otherwise}, \end{cases} \tag{3.4.9}$$

for $v \in H^{-1}, m \in (0,1]$ as in Assumption 3.2.1 (A5) in the superlinear case, i. e. if (3.2.6) is satisfied, and $m = 0$ in the sublinear case, i. e. if (3.2.7) is satisfied. We can use $\phi^\varepsilon = \partial\psi^\varepsilon$, the fact that $\phi^\varepsilon(X^{\varepsilon,n}) \in H_0^1$ $\mathrm{d}t \otimes \mathbb{P}$-almost everywhere by Lemma 3.4.2 and Lemma 3.D.2, and the chain rule for Sobolev functions (see e. g. [126, Theorem 2.1.11]), to obtain

$$\begin{aligned}
\left\langle \Delta\phi^\varepsilon(X_r^{\varepsilon,n}), X_r^{\varepsilon,n}\right\rangle_{H^{-1}} &= \left\langle -\Delta\phi^\varepsilon(X_r^{\varepsilon,n}), 0 - X_r^{\varepsilon,n}\right\rangle_{H^{-1}} \\
&\le \varphi^\varepsilon(0) - \varphi^\varepsilon(X_r^{\varepsilon,n}) = -\varphi^\varepsilon(X_r^{\varepsilon,n}).
\end{aligned} \tag{3.4.10}$$

Furthermore, we can use (3.2.2) and (3.2.4) to obtain

$$\begin{aligned}
\left\|B(t,X^{\varepsilon,n})\right\|_{L_2(U,H^{-1})}^2 &\le 2\left(\left\|B(t,X^{\varepsilon,n}) - B(t,0)\right\|_{L_2(U,H^{-1})}^2 + \left\|B(t,0)\right\|_{L_2(U,H^{-1})}^2\right) \\
&\le C(1 + \left\|X^{\varepsilon,n}\right\|_{H^{-1}}^2).
\end{aligned}$$

Thus, (3.4.8) implies

$$\begin{aligned}
\mathbb{E}\left(e^{-Kt}\left\|X_t^{\varepsilon,n}\right\|_{H^{-1}}^2\right) \le &\mathbb{E}\left\|x_0^n\right\|_{H^{-1}}^2 - 2\mathbb{E}\int_0^t e^{-Kr}\varphi^\varepsilon(X_r^{\varepsilon,n})\,\mathrm{d}r \\
&+ (C-K)\mathbb{E}\int_0^t e^{-Kr}\left\|X_r^{\varepsilon,n}\right\|_{H^{-1}}^2 \mathrm{d}r + \int_0^t Ce^{-Kr}\mathrm{d}s.
\end{aligned}$$

Choosing K large enough, we get

$$\mathbb{E}\left(e^{-Kt}\left\|X_t^{\varepsilon,n}\right\|_{H^{-1}}^2\right) \le \mathbb{E}\left\|x_0^n\right\|_{H^{-1}}^2 + C - 2e^{-Kt}\mathbb{E}\int_0^t \varphi^\varepsilon(X_r^{\varepsilon,n})\,\mathrm{d}r \tag{3.4.11}$$

and thus, by choosing $t = T$ and multiplying with $\frac{1}{2}e^{KT}$,

$$\mathbb{E}\int_0^T \varphi^\varepsilon(X_r^{\varepsilon,n})\,\mathrm{d}r \le C(1 + \mathbb{E}\left\|x_0^n\right\|_{H^{-1}}^2) \le \tilde{C} < \infty, \tag{3.4.12}$$

for some $C, \tilde{C} > 0$. Note that $\tilde{C}$ can be chosen independent of ε and n due to the convergence of $(x_0^n)_{n\in\mathbb{N}}$ to x_0. By Assumption 3.2.1 (A4) we can use Corollary 3.D.7 to obtain for $v \in L^2$

$$\begin{aligned}
\left|\varphi^\varepsilon(v) - \varphi(v)\right| &\le \int_{\mathcal{O}} \left|\psi^\varepsilon(v) - \psi(v)\right|\,\mathrm{d}x \\
&\le \int_{\mathcal{O}} C\varepsilon(1 + v^2)\,\mathrm{d}x \\
&= C\varepsilon(1 + \|v\|_{L^2}^2).
\end{aligned} \tag{3.4.13}$$

Since $X^{\varepsilon,n} \in L^2$ $\mathrm{d}t \otimes \mathbb{P}$-almost everywhere by Lemma 3.4.2, this leads to

$$\mathbb{E} \int_0^T \varphi^\varepsilon(X_r^{\varepsilon,n})\,\mathrm{d}r \geq \mathbb{E} \int_0^T \varphi(X_r^{\varepsilon,n})\,\mathrm{d}r - C\varepsilon\mathbb{E} \int_0^T 1 + \|X_r^{\varepsilon,n}\|_{L^2}^2 \,\mathrm{d}r. \tag{3.4.14}$$

With these statements about fixed values of ε, we can now consider the limit $\varepsilon \to 0$. Taking into account that φ is convex and lower-semicontinuous as shown in Section 3.3 and that $X^{\varepsilon,n} \to X^n$ in $L^2(\Omega; \mathcal{C}([0,T]; H^{-1}))$ and thus in $L^2(\Omega \times [0,T]; H^{-1})$, we can use [19, Proposition 16.50] and (3.4.14) to obtain

$$\begin{aligned}
\mathbb{E} \int_0^T \varphi(X_r^n)\,\mathrm{d}r &\leq \liminf_{\varepsilon\to 0} \mathbb{E} \int_0^T \varphi(X_r^{\varepsilon,n})\,\mathrm{d}r \\
&\leq \liminf_{\varepsilon\to 0} \left(\mathbb{E} \int_0^T \varphi^\varepsilon(X_r^{\varepsilon,n})\,\mathrm{d}r + C\varepsilon\mathbb{E} \int_0^T 1 + \|X_r^{\varepsilon,n}\|_{L^2}^2 \,\mathrm{d}r \right).
\end{aligned} \tag{3.4.15}$$

Since, by Lemma 3.4.2, the last term converges to 0 for $\varepsilon \to 0$ and $n \in \mathbb{N}$ fixed, we deduce that

$$\mathbb{E} \int_0^T \varphi(X_r^n)\,\mathrm{d}r \leq \liminf_{\varepsilon\to 0} \mathbb{E} \int_0^T \varphi^\varepsilon(X_r^{\varepsilon,n})\,\mathrm{d}r. \tag{3.4.16}$$

Thus, taking $\liminf_{\varepsilon\to 0}$ in (3.4.12) and then $\liminf_{n\to\infty}$, using lower-semicontinuity of φ as in (3.4.15), we obtain

$$\mathbb{E} \int_0^T \varphi(X_r)\,\mathrm{d}r \leq C(1 + \|x_0\|_{H^{-1}}^2) < \infty,$$

as required.

For the variational inequality part, let G, Z, t be as in Definition 3.2.4 (ii). Ito's formula (e.g. [112, Theorem 4.2.5]) then implies for all $t \in [0,T]$

$$\begin{aligned}
\mathbb{E}\,\|X_t^{\varepsilon,n} - Z_t\|_{H^{-1}}^2 = \mathbb{E}\,\|x_0^n - Z_0\|_{H^{-1}}^2 & \\
+ 2\mathbb{E} \int_0^t \langle \varepsilon\Delta X_r^{\varepsilon,n} + \Delta\phi^\varepsilon(X_r^{\varepsilon,n}) - G_r, X_r^{\varepsilon,n} - Z_r \rangle_{H^{-1}}\,\mathrm{d}r & \\
+ \mathbb{E} \int_0^t \|B(r, X_r^{\varepsilon,n}) - B(r, Z_r)\|_{L_2(U,H^{-1})}^2\,\mathrm{d}r. &
\end{aligned}$$

Analogous to (3.4.10), we have

$$\langle \Delta\phi^\varepsilon(X_r^{\varepsilon,n}), X_r^{\varepsilon,n} - Z_r \rangle_{H^{-1}} + \varphi^\varepsilon(X_r^{\varepsilon,n}) \leq \varphi^\varepsilon(Z_r) \tag{3.4.17}$$

$\mathrm{d}t \otimes \mathbb{P}$-almost everywhere, where we recall that both $X^{\varepsilon,n}$ and Z are in L^2 $\mathrm{d}t \otimes \mathbb{P}$-almost everywhere. Moreover, using the weighted Young inequality,

$$\begin{aligned}
\langle \varepsilon\Delta X_r^{\varepsilon,n}, X_r^{\varepsilon,n} - Z_r \rangle_{H^{-1}} &\leq \varepsilon\,\|\Delta X_r^{\varepsilon,n}\|_{H^{-1}}\,\|X_r^{\varepsilon,n} - Z_r\|_{H^{-1}} \\
&\leq \frac{1}{2}\varepsilon^{\frac{4}{3}}\,\|\Delta X_r^{\varepsilon,n}\|_{H^{-1}}^2 + \frac{1}{2}\varepsilon^{\frac{2}{3}}\,\|X_r^{\varepsilon,n} - Z_r\|_{H^{-1}}^2
\end{aligned} \tag{3.4.18}$$

$\mathrm{d}t \otimes \mathbb{P}$-almost everywhere. Hence, by (3.2.2), (3.4.17) and (3.4.18),

$$\begin{aligned}
&\mathbb{E}\,\|X_t^{\varepsilon,n} - Z_t\|_{H^{-1}} + 2\mathbb{E} \int_0^t \varphi^\varepsilon(X_r^{\varepsilon,n})\,\mathrm{d}r \\
&\leq \mathbb{E}\,\|x_0^n - Z_0\|_{H^{-1}}^2 + 2\mathbb{E} \int_0^t \varphi^\varepsilon(Z_r)\,\mathrm{d}r \\
&\quad - 2\mathbb{E} \int_0^t \langle G_r, X_r^{\varepsilon,n} - Z_r \rangle_{H^{-1}}\,\mathrm{d}r + C\mathbb{E} \int_0^t \|X_r^{\varepsilon,n} - Z_r\|_{H^{-1}}^2\,\mathrm{d}r \\
&\quad + 2\mathbb{E} \int_0^t \frac{1}{2}\varepsilon^{\frac{4}{3}}\,\|\Delta X_r^{\varepsilon,n}\|_{H^{-1}}^2 + \frac{1}{2}\varepsilon^{\frac{2}{3}}\,\|X_r^{\varepsilon,n} - Z_r\|_{H^{-1}}^2\,\mathrm{d}r.
\end{aligned} \tag{3.4.19}$$

As for (3.4.15), we have

$$\mathbb{E} \int_0^t \varphi(X_r^n)\,\mathrm{d}r \leq \liminf_{\varepsilon\to 0} \mathbb{E} \int_0^t \varphi^\varepsilon(X_r^{\varepsilon,n})\,\mathrm{d}r. \tag{3.4.20}$$

We notice that by $Z \in L^2$ $\mathrm{d}t \otimes \mathbb{P}$-almost everywhere, we have $\varphi^\varepsilon(Z_r) \leq \varphi(Z_r)$ due to Corollary 3.D.5. Moreover, any other term in (3.4.19) converges because $X^{\varepsilon,n} \to X^n$ in $L^2(\Omega; \mathcal{C}([0,T]; H^{-1}))$, the requirement of G belonging to $L^2(\Omega \times [0,T]; H^{-1})$ and Lemma 3.4.2. Thus, we can take $\liminf_{\varepsilon\to 0}$ in (3.4.19) to obtain

$$
\begin{aligned}
\mathbb{E}\int_0^t \varphi(X_r^n)\,\mathrm{d}r \leq &-\frac{1}{2}\mathbb{E}\,\|X_t^n - Z_t\|_{H^{-1}} + \frac{1}{2}\mathbb{E}\,\|x_0^n - Z_0\|_{H^{-1}}^2 + \mathbb{E}\int_0^t \varphi(Z_r)\,\mathrm{d}r \\
&- \mathbb{E}\int_0^t \langle G_r, X_r^n - Z_r\rangle_{H^{-1}}\,\mathrm{d}r + \frac{1}{2}C\mathbb{E}\int_0^t \|X_r^n - Z_r\|_{H^{-1}}^2\,\mathrm{d}r.
\end{aligned}
$$

Now taking $\liminf_{n\to\infty}$, using the lower-semicontinuity of φ and convergence of all the other terms, yields (3.2.12), as required.

Step 3: It remains to show that the solution constructed in the previous step is unique. To this end, let $x_0, y_0 \in L^2(\Omega, \mathcal{F}_0; H^{-1})$, $(y_0^n)_{n\in\mathbb{N}} \subset L^2(\Omega, \mathcal{F}_0; L^2)$ satisfying $y_0^n \to y_0$ in $L^2(\Omega; H^{-1})$ for $n \to \infty$. Let X be an arbitrary SVI solution to (3.1.1) with initial condition x_0 and let $(Y^{\varepsilon,n})_{\varepsilon>0,n\in\mathbb{N}}$ be the solutions to (3.4.1) with respective initial conditions $(y_0^n)_{n\in\mathbb{N}}$. We first check that

$$
Z = Y^{\varepsilon,n} \quad \text{and} \quad G = \varepsilon\Delta Y^{\varepsilon,n} + \Delta\phi^\varepsilon(Y^{\varepsilon,n}) \tag{3.4.21}
$$

are admissible choices for (3.2.12). First,

$$
Y^{\varepsilon,n} \in L^2(\Omega; \mathcal{C}([0,T]; H^{-1}))
$$

by construction and

$$
Y^{\varepsilon,n} \in L^2(\Omega \times [0,T]; H_0^1) \subset L^2(\Omega \times [0,T]; L^2)
$$

by Lemma 3.4.2 with norm bounded uniformly in ε. Also by Lemma 3.4.2, we have

$$
\mathbb{E}\int_0^T \|\varepsilon\Delta Y_t^{\varepsilon,n}\|_{H^{-1}}^2\,\mathrm{d}t = \varepsilon^2\,\mathbb{E}\int_0^T \|Y_t^{\varepsilon,n}\|_{H_0^1}^2\,\mathrm{d}t < \infty.
$$

Finally, for the nonlinear term, we have by the chain rule for the composition of Lipschitz functions with H_0^1 functions (e. g. [126, Theorem 2.1.11]) that almost everywhere in $\mathcal{O}$

$$
\nabla\phi^\varepsilon(Y_t^{\varepsilon,n}) = (\phi^\varepsilon)'(Y_t^{\varepsilon,n})\,\nabla Y_t^{\varepsilon,n},
$$

such that we can compute using Lemma 3.D.2

$$
\int_{\mathcal{O}} |\nabla\phi^\varepsilon(Y^{\varepsilon,n})|^2\,\mathrm{d}x = \int_{\mathcal{O}} \left|(\phi^\varepsilon)'(Y_t^{\varepsilon,n})\nabla Y_t^{\varepsilon,n}\right|^2\mathrm{d}x \leq \frac{1}{\varepsilon^2}\,\|\nabla Y_t^{\varepsilon,n}\|_{L^2}^2
$$

$\mathrm{d}t \otimes \mathbb{P}$-almost everywhere. Consequently,

$$
\|\phi^\varepsilon(Y_t^{\varepsilon,n})\|_{H_0^1}^2 \leq C(\varepsilon)\,\|Y_t^{\varepsilon,n}\|_{H_0^1}^2\,,
$$

such that we can conclude by Lemma 3.4.2

$$
\begin{aligned}
\mathbb{E}\int_0^T \|\Delta\phi^\varepsilon(Y_t^{\varepsilon,n})\|_{H^{-1}}^2\,\mathrm{d}t &= \mathbb{E}\int_0^T \|\phi^\varepsilon(Y_t^{\varepsilon,n})\|_{H_0^1}^2\,\mathrm{d}t \\
&\leq C(\varepsilon)\,\mathbb{E}\int_0^T \|Y_t^{\varepsilon,n}\|_{H_0^1}^2\,\mathrm{d}t \\
&\leq \tilde{C}(\varepsilon) < \infty,
\end{aligned}
$$

which yields that the choices in (3.4.21) were admissible.

As a consequence, (3.2.12) yields for $t \in [0,T]$

$$
\begin{aligned}
&\mathbb{E}\,\|X_t - Y_t^{\varepsilon,n}\|_{H^{-1}}^2 + 2\mathbb{E}\int_0^t \varphi(X_r)\,\mathrm{d}r \\
\leq &\mathbb{E}\,\|x_0 - y_0^n\|_{H^{-1}}^2 + 2\mathbb{E}\int_0^t \varphi(Y_r^{\varepsilon,n})\,\mathrm{d}r \\
&- 2\mathbb{E}\int_0^t \langle\varepsilon\Delta Y_r^{\varepsilon,n} + \Delta\phi^\varepsilon(Y_r^{\varepsilon,n}), X_r - Y_r^{\varepsilon,n}\rangle_{H^{-1}}\,\mathrm{d}r \\
&+ C\mathbb{E}\int_0^t \|X_r - Y_r^{\varepsilon,n}\|_{H^{-1}}^2\,\mathrm{d}r.
\end{aligned} \tag{3.4.22}
$$

For $u \in L^2$ and φ^ε as in (3.4.9), we have, as in (3.4.10) and (3.4.17),

$$\langle -\Delta\phi^\varepsilon(Y^{\varepsilon,n}), u - Y^{\varepsilon,n}\rangle_{H^{-1}} + \varphi^\varepsilon(Y^{\varepsilon,n}) \le \varphi^\varepsilon(u) \quad \mathrm{d}t \otimes \mathbb{P}\text{-a. e.} \tag{3.4.23}$$

Since $Y^{\varepsilon,n} \in H_0^1 \subset L^2$ $\mathrm{d}t \otimes \mathbb{P}$-a. e. we can use Corollary 3.D.7 as in (3.4.13) to obtain $\mathrm{d}t \otimes \mathbb{P}$-almost everywhere

$$|\varphi^\varepsilon(Y^{\varepsilon,n}) - \varphi(Y^{\varepsilon,n})| \le C\varepsilon \left(1 + \|Y^{\varepsilon,n}\|_{L^2}^2\right).$$

Thus, we can modify (3.4.23) and get

$$\langle -\Delta\phi^\varepsilon(Y^{\varepsilon,n}), u - Y^{\varepsilon,n}\rangle_{H^{-1}} + \varphi(Y^{\varepsilon,n}) \le \varphi(u) + C\varepsilon \left(1 + \|Y^{\varepsilon,n}\|_{L^2}^2\right) \quad \mathrm{d}t \otimes \mathbb{P}\text{-a. e..} \tag{3.4.24}$$

Note that (3.4.24) is trivial if $\varphi(u) = \infty$. Furthermore, (3.4.24) can be deduced analogously for $u \in \iota_m(L^{m+1} \cap H^{-1})$ in the superlinear setting, i. e. when φ is given by (3.2.8), with m as in Assumption 3.2.1 (A5). In the sublinear setting, i. e. φ is given by (3.2.9), and $u \in \mathcal{M}(\mathcal{O}) \cap H^{-1}$, we consider an approximating sequence $(\mu_j)_{j\in\mathbb{N}} \subset \mathcal{M} \cap H^{-1}$ with densities $(u_j)_{j\in\mathbb{N}} \subset L^2$ given by Theorem 3.3.8, such that (3.4.24) is satisfied for all $u_j, j \in \mathbb{N}$. We then pass to the limit $j \to \infty$ and notice that $(\mu_j)_{j\in\mathbb{N}}$ has been constructed in such a way that both $\varphi(u_j) \to \varphi(u)$ and

$$\begin{aligned}
&\langle -\Delta\phi^\varepsilon(Y^{\varepsilon,n}), u_j - Y^{\varepsilon,n}\rangle_{H^{-1}} \\
&= {}_{H_0^1}\langle \phi^\varepsilon(Y^{\varepsilon,n}), u_j - Y^{\varepsilon,n}\rangle_{H^{-1}} \quad \longrightarrow \quad {}_{H_0^1}\langle \phi^\varepsilon(Y^{\varepsilon,n}), u - Y^{\varepsilon,n}\rangle_{H^{-1}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = \langle -\Delta\phi^\varepsilon(Y^{\varepsilon,n}), u - Y^{\varepsilon,n}\rangle_{H^{-1}}.
\end{aligned}$$

Consequently, replacing u by X in (3.4.24), we have in any case

$$\langle -\Delta\phi^\varepsilon(Y^{\varepsilon,n}), X - Y^{\varepsilon,n}\rangle_{H^{-1}} + \varphi(Y^{\varepsilon,n}) \le \varphi(X) + C\varepsilon \left(1 + \|Y^{\varepsilon,n}\|_{L^2}^2\right) \quad \mathrm{d}t \otimes \mathbb{P}\text{-a. e..} \tag{3.4.25}$$

Using (3.4.25) and the same estimate as in (3.4.18), we can modify (3.4.22) to obtain for $t \in [0, T]$

$$\begin{aligned}
\mathbb{E}\,\|X_t - Y_t^{\varepsilon,n}\|_{H^{-1}}^2 \le{}& \mathbb{E}\,\|x_0 - y_0^n\|_{H^{-1}}^2 \\
&+ 2\mathbb{E}\int_0^t \frac{1}{2}\varepsilon^{\frac{4}{3}}\,\|\Delta Y_r^{\varepsilon,n}\|_{H^{-1}}^2\,\mathrm{d}r + \frac{1}{2}\varepsilon^{\frac{2}{3}}\,\|X_r - Y_r^{\varepsilon,n}\|_{H^{-1}}^2\,\mathrm{d}r \\
&+ C\mathbb{E}\int_0^t \|X_r - Y_r^{\varepsilon,n}\|_{H^{-1}}^2\,\mathrm{d}r + C\varepsilon\mathbb{E}\int_0^t \left(1 + \|Y_r^{\varepsilon,n}\|_{L^2}^2\right)\mathrm{d}r.
\end{aligned}$$

Taking $\varepsilon \to 0$ and then $n \to \infty$ yields

$$\mathbb{E}\,\|X_t - Y_t\|_{H^{-1}}^2 \le \mathbb{E}\,\|x_0 - y_0\|_{H^{-1}}^2 + C\mathbb{E}\int_0^t \|X_r - Y_r\|_{H^{-1}}^2\,\mathrm{d}r \quad \text{for } t \in [0, T], \tag{3.4.26}$$

where Y is the SVI solution which has been constructed from $(Y^{\varepsilon,n})$ in the limiting procedure of the first two steps of this proof. Gronwall's inequality then yields $X = Y$ if $x_0 = y_0$, and thus uniqueness of SVI solutions. Then, estimate (3.2.13) follows by applying Gronwall's inequality to (3.4.26) with different initial values, which concludes the proof. □

3.A Generalities on convex functions

We collect and prove some statements on convex functions defined on $\mathbb{R}$.

Lemma 3.A.1. *Let $f : \mathbb{R} \to [0, \infty)$ be convex with $f(0) = 0$ and $x, y \in \mathbb{R} \setminus \{0\}$ with $x < y$. Then*

$$\frac{f(x)}{x} \le \frac{f(y)}{y}. \tag{3.A.1}$$

In particular, for $x > 0$ this implies $f(x) \le f(y)$.

Proof. Note that by convexity, we have for $\lambda \in (0,1)$, $x \in \mathbb{R}$

$$f(\lambda x) = f(\lambda x + (1-\lambda)0) \leq \lambda f(x) + (1-\lambda) f(0) = \lambda f(x). \tag{3.A.2}$$

If $x < 0 < y$, the statement is obvious by the nonnegativity of f. If $0 < x < y$, we use (3.A.2) with $\lambda = \frac{x}{y}$ to get

$$\frac{f(x)}{x} = \frac{f(\lambda y)}{\lambda y} \leq \frac{\lambda f(y)}{\lambda y} = \frac{f(y)}{y},$$

while for $x < y < 0$ we use (3.A.2) with $\lambda := \frac{y}{x}$ to get

$$\frac{f(y)}{y} = \frac{f(\lambda x)}{\lambda x} \geq \frac{\lambda f(x)}{\lambda x} = \frac{f(x)}{x},$$

as required. □

Lemma 3.A.2. *Let ψ satisfy Assumptions 3.2.1 and $y > 0$. Then, if $\psi(y) > 0$, we have*

$$\psi^*(-x) = \psi^*(x) \leq \psi(y) \quad \text{for } x \in \left[0, \frac{\psi(y)}{y}\right],$$

where ψ^* is defined as in Definition 3.3.1.

Proof. By Remark 3.3.2, the last part of Lemma 3.A.1 and the nonnegativity of ψ^*, it is enough to show

$$\psi^* \left(\frac{\psi(y)}{y} \right) \leq \psi(y). \tag{3.A.3}$$

To verify (3.A.3), we distinguish three cases for $y' \in \mathbb{R}$. For $y' \geq y$ we have by Lemma 3.A.1

$$\frac{\psi(y)}{y} y' - \psi(y') = y' \left(\frac{\psi(y)}{y} - \frac{\psi(y')}{y'} \right) \leq 0,$$

for $y' \leq 0$ we have by the nonnegativity of ψ

$$\frac{\psi(y)}{y} y' - \psi(y') \leq 0,$$

and for $y' \in (0, y)$ we have

$$\frac{\psi(y)}{y} y' - \psi(y') \leq \frac{\psi(y)}{y} y = \psi(y),$$

which yields the claim. □

Lemma 3.A.3. *Let ψ satisfy Assumptions 3.2.1. For $K = dom(\psi^*) := \{x \in \mathbb{R} : \psi^*(x) < \infty\}$ we have*

$$\sup K = \lim_{t\to\infty} \frac{\psi(t)}{t} \quad \text{and} \quad \sup(-K) = \lim_{t\to\infty} \frac{\psi(-t)}{t}.$$

Proof. We only prove the first statement, the second one then becomes clear by symmetry. To this end, note first that the limit is actually a supremum, as $\frac{\psi(t)}{t}$ is increasing (by (3.A.1)). Let now $x \in K$, which means that $xt - \psi(t) \leq c_x < \infty$ and thus $\frac{\psi(t)}{t} \geq x - \frac{c_x}{t}$ for all $t \in [0, \infty)$, which yields "$\leq$" by letting $t \to \infty$.

Conversely, we have $\frac{\psi(t)}{t} \in K$ for $t > 0, \psi(t) > 0$ by by Lemma 3.A.2. As $\psi^*(0) = 0$, this is true also if $\psi(t) = 0$, thereby proving "$\geq$". □

Corollary 3.A.4. *Let ψ satisfy Assumptions 3.2.1. By Lemma 3.A.2 and Lemma 3.A.3, we have that*

$$\psi_\infty(1) = \psi_\infty(-1) \geq \frac{\psi(y)}{y}$$

for $y > 0$ with $\psi(y) > 0$, where ψ_∞ is defined as in Definition 3.3.1.

Lemma 3.A.5. *Let ψ satisfy Assumptions 3.2.1. For the convex conjugate of the recession function, we have*

$$\psi_\infty^*(x) := (\psi_\infty)^*(x) = \chi_{[-\psi_\infty(1),\psi_\infty(1)]}(x)$$

for $x \in \mathbb{R}$, where for an Interval I we have written

$$\chi_I(x) = \begin{cases} 0, & \text{if } x \in I \\ +\infty, & \text{else.} \end{cases}$$

Proof. In the superlinear case, i. e. (3.2.6) is satisfied, we have $\psi_\infty = \chi_{\{0\}}$ and thus $\psi_\infty^* \equiv 0$, as required. In the sublinear case, we first note that ψ_∞ is, by definition, positively homogeneous, which by symmetry amounts to absolute homogeneity. Thus

$$\psi_\infty(x) = \psi_\infty(1)\,|x|\,,$$

where $\psi_\infty(1) > 0$ by Corollary 3.A.4, which allows to conclude by the definition of the convex conjugate. □

3.B Variational solutions to nonlinear SPDE

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space, $V \subset H \subset V'$ a Gelfand triple, $(W_t)_{t\in[0,T]}$ a cylindrical Id-Wiener process taking values in another separable Hilbert space $(U, \langle,\rangle_U)$ with normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Let

$$A : [0,T] \times V \times \Omega \to V',\ \ B : [0,T] \times V \times \Omega \to L_2(U,H),$$

be progressively measurable and satisfy the following conditions:

(H1) (*Hemicontinuity*) For all $u, v, x \in V, \omega \in \Omega$ and $t \in [0,T]$, the map

$$\mathbb{R} \ni \lambda \mapsto {}_{V'}\langle A(t, u + \lambda v, \omega), x\rangle_V$$

is continuous.

(H2) (*Weak monotonicity*) There exists $c \in \mathbb{R}$, such that for all $u, v \in V$

$$2\ {}_{V'}\langle A(\cdot,u) - A(\cdot,v), u - v\rangle_V + \|B(\cdot,u) - B(\cdot,v)\|^2_{L_2(U,H)} \le c\,\|u - v\|^2_H$$

on $[0,T] \times \Omega$.

(H3) (*Coercivity*) There exist $\alpha \in (1,\infty), c_1 \in \mathbb{R}, c_2 \in (0,\infty)$ and an $(\mathcal{F}_t)$-adapted process $f \in L^1([0,T]\times \Omega, \mathrm{d}t \otimes P)$, such that for all $v \in V, t \in [0,T]$

$$2\ {}_{V'}\langle A(t,v), v\rangle_V + \|B(t,v)\|^2_{L_2(U,H)} \le c_1\,\|v\|^2_H - c_2\,\|v\|^\alpha_V + f(t) \quad \text{on } \Omega. \tag{3.B.1}$$

(H4) (*Boundedness*) There exist $c_3 \in [0,\infty)$ and an $(\mathcal{F}_t)$-adapted process

$$g \in L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega, \mathrm{d}t \otimes P),$$

such that for all $v \in V, t \in [0,T]$

$$\|A(t,v)\|_{V'} \le g(t) + c_3\,\|v\|^{\alpha-1}_V$$

on Ω, where α is as in (H3).

We then consider the stochastic partial differential equation

$$\mathrm{d}X_t = A(t,X_t)\mathrm{d}t + B(t,X_t)\,\mathrm{d}W_t, \tag{3.B.2}$$

for which we establish the following notion of solution:

Definition 3.B.1. A continuous H-valued $(\mathcal{F}_t)$-adapted process $(X_t)_{t\in[0,T]}$ is called a **(variational) solution of (3.B.2)**, if for its $\mathrm{d}t\otimes\mathbb{P}$-equivalence class $\hat{X}$ we have

$$\hat{X}\in L^\alpha([0,T]\times\Omega,\mathrm{d}t\otimes\mathbb{P};V)\cap L^2([0,T]\times\Omega,\mathrm{d}t\otimes\mathbb{P};H),$$

with α as in (3.B.1), and $\mathbb{P}$-a. s.

$$X_t=X_0+\int_0^t A(s,\bar{X}_s)\mathrm{d}s+\int_0^t B(s,\bar{X}_s)\,\mathrm{d}W_s,\quad t\in[0,T],$$

where $\bar{X}$ is any V-valued progressively measurable $\mathrm{d}t\otimes\mathbb{P}$-version of X.

We then have the following well-posedness result (see [112, Theorem 4.4], relying on [92]).

Theorem 3.B.2. *Let $X_0\in L^2(\Omega,\mathcal{F}_0,\mathbb{P};H)$. Then there exists a unique solution X to (3.B.2) in the sense of Definition 3.B.1.*

3.C Strong solutions to gradient-type SPDE

Let $\varphi: H\to\mathbb{R}$ be a proper, lower-semicontinuous, convex function on a separable real Hilbert space H. We consider an SPDE of the type

$$\begin{aligned}\mathrm{d}X_t&\in-\partial\varphi(X_t)\mathrm{d}t+B(t,X_t)\mathrm{d}W_t,\\ X_0&=x_0,\end{aligned}\tag{3.C.1}$$

where W is a cylindrical Wiener process in a separable Hilbert space U defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with normal filtration $(\mathcal{F}_t)_{t\geq 0}$ and $B:[0,T]\times H\times\Omega\to L_2(U,H)$ is Lipschitz continuous, i. e. for all $v,w\in H$

$$\|B(t,v)-B(t,w)\|^2_{L_2(U,H)}\leq C\,\|v-w\|^2_H\,,$$

and for all $(t,\omega)\in[0,T]\times\Omega$. Furthermore, we assume that

$$\|B(\cdot,0)\|_{L_2(U,H)}\in L^2([0,T]\times\Omega).$$

Definition 3.C.1. Let $x_0\in L^2(\Omega,\mathcal{F}_0;H)$. An H-continuous, $\mathcal{F}_t$-adapted process $X\in L^2(\Omega;\mathcal{C}([0,T];H))$ for which there exists a selection $\eta\in-\partial\varphi(X)$, $\mathrm{d}t\otimes\mathbb{P}$-a. e., is said to be a **strong solution to** (3.C.1) if

$$\eta\in L^2([0,T]\times\Omega;H)$$

and $\mathbb{P}$-a. s.

$$X_t=x_0+\int_0^t\eta_r\;\mathrm{d}r+\int_0^t B(r,X_r)\;\mathrm{d}W_r\quad\text{for all } t\in[0,T].$$

3.D Yosida approximation of multivalued operators

The theory of Yosida approximations can be applied to general maximal monotone operators from Banach spaces to their dual, see e. g. [7, Section 2]. However, we constrain ourselves to the case of the Hilbert space $\mathbb{R}$.

Fix $\varepsilon>0$. For a convex, lower-semicontinuous proper function $\psi:\mathbb{R}\to[0,\infty)$ we define its *Moreau-Yosida approximation* $\psi^\varepsilon:\mathbb{R}\to[0,\infty)$ by

$$\psi^\varepsilon(r)=\inf_{s\in\mathbb{R}}\left(\frac{|r-s|^2}{2\varepsilon}+\psi(s)\right).\tag{3.D.1}$$

Let $\phi=\partial\psi:\mathbb{R}\to 2^{\mathbb{R}}$ be the subdifferential of ψ. For each $r\in\mathbb{R}$, we define the resolvent $J^\varepsilon(r)$ as the unique solution s to

$$s+\varepsilon\phi(s)\ni r.$$

Hereby the resolvent is well-defined, since ϕ is maximal monotone as a subdifferential (see e. g. [7, Theorem 2.8]), which implies that $\mathrm{Id}_{\mathbb{R}} + \varepsilon\phi$ is bijective. We then define the *Yosida approximation* $\phi^\varepsilon : \mathbb{R} \to \mathbb{R}$ of ϕ by

$$\phi^\varepsilon(r) = \frac{1}{\varepsilon}(r - J^\varepsilon r). \tag{3.D.2}$$

We state and prove some properties of this approximation, most of which are true for general subpotential operators. The usage of additional assumptions will be highlighted.

Proposition 3.D.1. *We have*

$$\phi^\varepsilon(r) \in \phi(J^\varepsilon r). \tag{3.D.3}$$

Furthermore, ψ^ε is continuous, convex and Gateaux differentiable, and $\phi^\varepsilon = (\psi^\varepsilon)'$. In particular, ϕ^ε is also maximal monotone.

Proof. The first claim is clear by construction. The remaining statements are proved in [7, Theorem 2.9]. □

Lemma 3.D.2. *The Yosida approximation ϕ^ε is Lipschitz continuous with Lipschitz constant $\frac{1}{\varepsilon}$.*

Proof. Fix $x, y \in \mathbb{R}$. By definition of J^ε, we have

$$J^\varepsilon x - J^\varepsilon y + \varepsilon\left(\phi^\varepsilon(x) - \phi^\varepsilon(y)\right) = x - y.$$

By multiplying with $\phi^\varepsilon(x) - \phi^\varepsilon(y)$ and keeping (3.D.3) in mind, we obtain

$$\varepsilon(\phi^\varepsilon(x) - \phi^\varepsilon(y))^2 \leq |\phi^\varepsilon(x) - \phi^\varepsilon(y)| \; |x - y|,$$

which immediately yields the claim. □

Lemma 3.D.3. *Defining $|\phi(r)| := \inf\{|\eta| : \eta \in \phi(r)\}$, we have $|\phi^\varepsilon(r)| \leq |\phi(r)|$ for all $r \in \mathbb{R}$.*

Proof. By monotonicity of ϕ, we get for $\eta \in \phi(r)$

$$0 \leq (r - J^\varepsilon(r))(\eta - \phi^\varepsilon(r)).$$

Noting that $r - J^\varepsilon(r) = \varepsilon\phi^\varepsilon(r)$, we can simplify

$$0 \leq \varepsilon\, |\phi^\varepsilon(r)| \; |\eta| - \varepsilon(\phi^\varepsilon(r))^2$$

to obtain the estimate. □

The next lemma is proved in [7, Theorem 2.9]:

Lemma 3.D.4. *For each $r \in \mathbb{R}$, we have*

$$\psi^\varepsilon(r) = \frac{1}{2\varepsilon}\, |r - J^\varepsilon(r)|^2 + \psi(J^\varepsilon r),$$

in other words, the infimum in (3.D.1) is assumed at $J^\varepsilon r$.

As an immediate consequence, we get

Corollary 3.D.5. *For each $r \in \mathbb{R}$, we have*

$$\psi(J^\varepsilon r) \leq \psi^\varepsilon(r) \leq \psi(r).$$

Proof. The first inequality is clear by Lemma 3.D.4, the second one by setting $r = s$ in (3.D.1). □

Lemma 3.D.6. *For each $r \in \mathbb{R}$, we have*

$$|\psi(r) - \psi^\varepsilon(r)| \leq \varepsilon\, |\phi(r)|^2 \quad \textit{for all } r \in \mathbb{R}. \tag{3.D.4}$$

Proof. Fix an arbitrary $r \in \mathbb{R}$. For any $\eta \in \phi(r)$ we have, using Corollary 3.D.5 in the first step and the subdifferential inequality in the second step,

$$0 \leq \psi(r) - \psi(J^\varepsilon r) \leq -\eta(J^\varepsilon r - r) \leq |\eta|\, \varepsilon \, |\phi^\varepsilon(r)| \, .$$

Since $\eta \in \phi(r)$ was arbitrary, we can pass to its infimum. Using Lemma 3.D.3, we obtain (3.D.4). $\square$

Corollary 3.D.7. *With Lemma 3.D.6, under the additional assumption $|\phi(r)|^2 \leq C(1+|r|^2)$, we obtain*

$$|\psi(r) - \psi^\varepsilon(r)| \leq C\varepsilon(1 + r^2) \quad \text{for all } r \in \mathbb{R}.$$

Lemma 3.D.8. *We have for all $a, b \in \mathbb{R}, \varepsilon_1, \varepsilon_2 > 0$*

$$(\phi^{\varepsilon_1}(a) - \phi^{\varepsilon_2}(b))\,(a - b) \geq -C(\varepsilon_1 + \varepsilon_2)\left(|\phi^{\varepsilon_1}(a)|^2 + |\phi^{\varepsilon_2}(b)|^2\right).$$

Proof. We compute

$$\begin{aligned}
(\phi^{\varepsilon_1}(a) - \phi^{\varepsilon_2}(b))\,(a - b) &= (\phi^{\varepsilon_1}(a) - \phi^{\varepsilon_2}(b))(J^{\varepsilon_1} a - J^{\varepsilon_2} b) \\
&\quad + (\phi^{\varepsilon_1}(a) - \phi^{\varepsilon_2}(b))(a - J^{\varepsilon_1} a - (b - J^{\varepsilon_2} b)) \\
&\geq (\phi^{\varepsilon_1}(a) - \phi^{\varepsilon_2}(b))(\varepsilon_1 \phi^{\varepsilon_1}(a) - \varepsilon_2 \phi^{\varepsilon_2}(b)) \\
&\geq -\frac{1}{2}(\varepsilon_1 + \varepsilon_2)\left(|\phi^{\varepsilon_1}(a)|^2 + |\phi^{\varepsilon_2}(b)|^2\right),
\end{aligned}$$

where the second step uses (3.D.3) for the first summand to be positive and (3.D.2) for the second summand. In the last step, we neglect the squared terms and use Young's inequality for the mixed terms. $\square$

Corollary 3.D.9. *Under the additional assumption $|\phi(r)|^2 \leq C(1 + |r|^2)$, Lemma 3.D.3 immediately yields*

$$(\phi^{\varepsilon_1}(a) - \phi^{\varepsilon_2}(b))\,(a - b) \geq -C(\varepsilon_1 + \varepsilon_2)\left(1 + |a|^2 + |b|^2\right).$$

3.E Estimate on specific quadratic variations

Lemma 3.E.1. *Let U, H be Hilbert spaces, $Q : U \to U$ linear, bounded, non-negative definite and symmetric, W a (possibly cylindrical) Q-Wiener process on U defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and normal filtration $(\mathcal{F}_t)_{t \geq 0}$. Further let $B : \Omega \times [0, T] \to L_2\left(Q^{\frac{1}{2}}(U), H\right)$ such that B is predictable and*

$$\mathbb{P}\left(\int_0^T \|B(s)\|_{L_2\left(Q^{\frac{1}{2}}(U),H\right)} \mathrm{d}s < \infty\right) = 1,$$

and f an $(\mathcal{F}_t)$-adapted continuous H-valued process. Then, the quadratic variation of a stochastic integral on H of the form

$$M_t = \int_0^t \langle f_r, B_r \, \mathrm{d}W_r \rangle_H$$

can be estimated from above by

$$\langle M \rangle_t \leq \int_0^t \|f_r\|_H^2 \, \|B_r\|^2_{L_2\left(Q^{\frac{1}{2}}(U),H\right)} \mathrm{d}r.$$

Proof. If Q is of finite trace and thus W is a classical Wiener process, the statement follows from [112, Lemma 2.4.2] and [112, Lemma 2.4.3]. In case of a cylindrical Wiener process, we can compute, using the notation of Assumption 3.2.1 (A1),

$$\begin{aligned}
\langle M \rangle_t &= \left\langle \int_0^\cdot \left\langle f_r, B_r \circ J^{-1} \, \mathrm{d}\tilde{W}_r \right\rangle_H \right\rangle_t \\
&\leq \int_0^t \|f_r\|_H^2 \left\|B_r \circ J^{-1}\right\|^2_{L_2\left(Q_1^{\frac{1}{2}}(U_1),H\right)} \mathrm{d}r \\
&= \int_0^t \|f_r\|_H^2 \, \|B_r\|^2_{L_2\left(Q^{\frac{1}{2}}(U),H\right)} \mathrm{d}r,
\end{aligned}$$

where in the second step, we use the Lemma for the classical Q_1-Wiener process $\tilde{W}$ on U_1. The last step can be seen by the fact that for an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $Q^{\frac{1}{2}}(U)$, we have that $(Je_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L_2\big(Q_1^{\frac{1}{2}}(U_1), H\big)$; see [112, section 2.5.2] for details. □

Chapter 4

Ergodicity for singular-degenerate stochastic porous media equations

4.1 Introduction

We consider the singular-degenerate generalized stochastic porous medium equation

$$\begin{aligned} \mathrm{d}X_t \in&\ \Delta(\phi(X_t))\mathrm{d}t + B\mathrm{d}W_t, \\ X_0 =&\ x_0, \end{aligned} \tag{4.1.1}$$

on a bounded interval $\mathcal{O} \subseteq \mathbb{R}$ with zero Dirichlet boundary conditions. The multi-valued function ϕ is the maximal monotone extension of

$$\mathbb{R} \ni x \mapsto x\mathbf{1}_{\{|x|>1\}}, \tag{4.1.2}$$

W is a cylindrical Wiener process on some separable Hilbert space U, and the diffusion coefficient B is an $L^2(\mathcal{O})$-valued Hilbert-Schmidt operator satisfying a non-degeneracy condition (see (4.2.5) below). Equation (4.1.1) is understood as an evolution equation on H^{-1}, the dual of $H_0^1(\mathcal{O})$, where it can be solved uniquely in the sense of SVI solutions, as shown in Chapter 3. The main result of the present work is the existence and uniqueness of an invariant probability measure for solutions to (4.1.1).

The above form of stochastic porous media equations is motivated by the analysis of non-equilibrium systems, appearing in the context of self-organized criticality (for a survey, see e. g. [122]). Self-organized criticality is a statistical property of systems displaying intermittent events, such as earthquakes, which are activated when the underlying system locally exceeds a threshold. These dynamics are reflected by the discontinuity and degeneracy of the nonlinearity ϕ above. In order to get a better understanding of the long-time behaviour of these systems, we prove the existence of a unique non-equilibrium statistical invariant state for (4.1.1). Since this is the candidate to which the transition probabilities are expected to converge for long times, it is the key object for the statistical behaviour of the respective process.

A previous approach to the long-time behaviour of Markov processes stemming from monotone SPDEs with singular drift, by which the present article is inspired, is [77], which in turn uses the more abstract framework of [78]. In these works, the existence and uniqueness of invariant probability measures to stochastic local and non-local p-Laplace equations is proved, where the multivalued regime $p = 1$ is included. In one dimension, the paradigmatic case is the equation

$$\mathrm{d}X_t = \Delta(\mathrm{sgn}(X_t)) + \mathrm{d}W_t, \tag{4.1.3}$$

where sgn denotes the maximal monotone extension of the classical sign function. The proof relies on sufficient criteria from [91], where the so-called lower bound technique has been extended to Polish spaces which are not necessarily locally compact. This technique relies on the existence of a state being an accessible point for the time averages of the transition probabilities uniformly in time, and the so-called "e-property", which is a uniform continuity assumption on the Markov semigroup. To verify these criteria, the focus of [77] rests on energy estimates to first bound the mass of these averages to L^m balls for some suitably chosen $m \in (2, 3]$. As a next step, the convergence to a chosen accessible state with probability bounded below is shown, which is done by comparing the solution of (4.1.3) to a control process, which obeys the mere deterministic dynamics of (4.1.3), i. e.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} X_t &= \Delta(\operatorname{sgn}(X_t)), \\ X_0 &= y, \end{aligned} \tag{4.1.4}$$

for $y \in L^m, \|y\|_m \leq R$ for some $R > 0$. In this, simpler setting than (4.1.1), there is a unique limiting state to (4.1.4) which is a natural candidate for the aforementioned accessible point.

In the present article, we aim to prove the existence and uniqueness of an invariant probability measure by similar ideas. While energy estimates for (4.1.1) are easier to obtain due to the linear growth of ϕ (cf. (4.1.2)) at $\pm\infty$, the degenerate form of the nonlinearity destroys the convergence of the noise-free system to a unique fixed point. This is why we have to add a forcing term to the control process and rely on a more refined deterministic analysis of the resulting inhomogeneous monotone evolution equation. To guarantee the convergence of this modified control process, the forcing term has to be sufficiently non-degenerate, and as the connection of the solution to (4.1.1) to the control process only works if the noise is "close" to the deterministic forcing with non-zero probability, this relies on some non-degeneracy requirements on the noise. As in[77], it is important that the convergence of the deterministic process takes place uniformly for initial values in sets of bounded energy. We tackle this problem with the help of a comparison principle, which, however, only works if the energy actually controls the L^∞ norm. This leads to the restriction to one spatial dimension. Finally, most of the above-mentioned steps have to be argued on an approximate level due to the singularity of the drift, so that stability of the statements under these approximations also has to be ensured.

The structure of this chapter is as follows. After stating the exact setting in the first part of section 4.2, we state the main result of this article, Theorem 4.2.1 at the end of section 4.2. Section 4.3 then collects auxiliary results in the natural order of the argumentation, which finally allow to prove Theorem 4.2.1.

The results of this chapter are accepted for publication up to minor revisions, see [102].

4.1.1 Literature

We give a brief overview on the existing results on ergodicity of stochastic nonlinear diffusions, with a focus on approaches applicable to stochastic (generalized) porous media equations.

In the "classical" approach, e. g. in the monograph [37], the existence of invariant measures to semilinear SPDEs with non-degenerate noise is proven by bounds that imply the tightness of the averaged transition probabilities, allowing to use the Krylov-Bogoliubov theorem. Uniqueness is then relying on the Doob-Khasminskii theorem, using the regularity of the Markov semigroup which can be guaranteed by the strong Feller property and irreducibility. This technique has been considerably improved by [84], using smoothing in form of the asymptotic Feller property, though the scope was still on semilinear equations.

Invariant measures to quasilinear diffusions with additive noise have been initially studied in [36] and [35] on the level of Kolmogorov equations. In [110] (see also the monograph [10]), the strong monotonicity of the porous medium operator was exploited, which leads to the existence and uniqueness of invariant measures by strong dissipativity.

In the situation of weakly monotone drift operators, there have been several approaches to obtain contraction estimates which ensure ergodicity, e. g. via Harnack inequalities (cf. [121, 120]), weighted L^1 dissipativity (cf. [39]) or lower bound techniques (cf. [95], [91]). We note that the first approach also works for a partly multiplicative noise and the second one even for full multiplicative noise. The last-mentioned approach was used by [78] and [77], where generalized porous media equations with discontinuous nonlinearities are analyzed as explained above.

A different approach to the long-time behaviour of solutions to SPDEs is to analyze the existence and the structure of random attractors of random dynamical systems, as e. g. in [34, 33, 66, 71, 21, 72]. A property which has turned out to be very useful in this context is order preservation of trajectories which are driven by the same noise, see, e. g., [67, 3, 60, 27]. A close connection between random attractors and ergodic and mixing properties of random dynamical systems can be obtained in the case of synchronization (see [32]), which is on hand if the random attractor is a singleton. This case has been investigated in, e. g., [30, 59, 60, 115, 27].

Last but not least, we mention [8, 15, 68], where similar equations are considered under multiplicative noise, leading to finite-time absorption of the process into a subcritical region.

4.1.2 Notation

On a bounded open set $\mathcal{O} \subset \mathbb{R}$, we use the classical notation $L^p := L^p(\mathcal{O})$ for the Lebesgue space with exponent $p \in [1, \infty]$ with norm $\|\cdot\|_p$. We write $H_0^1 := H_0^1(\mathcal{O})$ for the Sobolev space of weakly differentiable functions with exponent 2 and zero trace, and its topological dual will be denoted by H^{-1}. A bounded linear operator $T : U \to H$, where U and H are separable Hilbert spaces, is called Hilbert-Schmidt if

$$\|T\|_{L_2(U,H)} := \sum_{k \in \mathbb{N}} \|Te_k\|_H^2 < \infty,$$

where $(e_k)_{k \in \mathbb{N}}$ is an orthonormal basis of U. For a Hilbert space H, $C_b(H)$ denotes the space of bounded continuous functions on H, $\mathcal{B}(H)$ denotes the Borel σ-algebra, and $\mathcal{B}_b(H)$ the set of bounded functions $H \to \mathbb{R}$ which are $\mathcal{B}(H)$-$\mathcal{B}(\mathbb{R})$-measurable. Multivalued operators on H, which arise in this work as subdifferentials of proper, convex and lower-semicontinuous functionals, are mappings $A : H \to 2^H$. We define the domain of A by

$$D(A) := \{x \in H : A(x) \neq \emptyset\}$$

and its range by

$$R(A) := \bigcup_{x \in H} A(x).$$

For a metric space V and $r > 0$, we denote by B_r^V the open ball with radius r with respect to the corresponding metric. If $V = L^\infty$, we use B_r^∞ for $B_r^{L^\infty}$. Within term manipulations, the constant C may vary from line to line.

4.2 Setting and main result

We consider a one-dimensional open bounded interval $\mathcal{O} \subset \mathbb{R}$ as the underlying domain. For simplicity, set $\mathcal{O} := (-1, 1)$.

Define by $\phi : \mathbb{R} \to 2^{\mathbb{R}}$ the multi-valued maximal monotone extension of

$$\mathbb{R} \ni x \mapsto x\mathbf{1}_{\{|x|>1\}},$$

and let $\psi : \mathbb{R} \to \mathbb{R}$ be its anti-derivative with $\psi(0) = 0$, i. e.

$$\psi(x) = \frac{1}{2}(|x|^2 - 1)\mathbf{1}_{\{|x|>1\}}.$$

Let furthermore $\varphi : H^{-1} \to [0, \infty]$ be defined as

$$\varphi(u) = \begin{cases} \int_{\mathcal{O}} \psi(u)\,\mathrm{d}x & \text{if } u \in L^2 \\ +\infty & \text{else,} \end{cases} \tag{4.2.1}$$

and consider the SPDE

$$\begin{aligned} \mathrm{d}X_t^x &\in -\partial\varphi(X_t^x)\,\mathrm{d}t + B\,\mathrm{d}W_t, \\ X_0^x &= x, \end{aligned} \tag{4.2.2}$$

where $x \in H^{-1}$, W is an Id-cylindrical Wiener process in some separable Hilbert space U, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $(\mathcal{F}_t)_{t \in [0,T]}$, and $B \in L_2(U, L^2)$ is a Hilbert-Schmidt operator. This leads to BW_t being a trace-class Wiener process in L^2, such that there are mutually orthogonal L^2 functions $(\xi_k)_{k \in \mathbb{N}}$ with

$$\sum_{k \in \mathbb{N}} \|\xi_k\|_2^2 < \infty, \tag{4.2.3}$$

for which

$$BW_t = \sum_{i=1}^{\infty} \beta_k(t)\xi_k, \tag{4.2.4}$$

where $(\beta_k)_{k\in\mathbb{N}}$ are independent one-dimensional standard Brownian motions. Additionally, we impose that there are $m \in \mathbb{N}$, $c_1, \dots, c_m \in \mathbb{R}$ such that

$$g \in L^2, \quad g(x) := \sum_{k=1}^{m} c_k \xi_k(x) > 1 \quad \text{for almost all } x \in \mathcal{O}. \tag{4.2.5}$$

Note that the well-posedness of the SPDE (4.2.2) has been shown in Chapter 3 in the sense of SVI-solutions, identifying x with an almost surely constant random variable $x \in L^2(\Omega, H^{-1})$. The process constructed there gives rise to a semigroup $(P_t)_{t\geq 0}$ of Markov transition kernels by

$$P_t(x, A) = \mathbb{E}\mathbf{1}_A(X_t^x) \quad \text{for } x \in H^{-1} \text{ and } A \in \mathcal{B}(H^{-1}), \tag{4.2.6}$$

which will be shown below in Lemma 4.3.7. By a slight abuse of notation, we will denote the induced semigroup on $\mathcal{B}_b(H^{-1})$ also by P_t, i. e.

$$P_t f(x) = \int_{H^{-1}} f(y) P_t(x, \mathrm{d}y) \quad \text{for } f \in \mathcal{B}_b(H^{-1}),\ x \in H^{-1}. \tag{4.2.7}$$

The main result of this article is the following:

Theorem 4.2.1. *In the setting described above, the semigroup $(P_t)_{t\geq 0}$ admits a unique invariant probability Borel measure μ on H^{-1}, i. e. for all $f \in C_b(H^{-1})$ we have*

$$\int_{H^{-1}} P_t f \mathrm{d}\mu = \int_{H^{-1}} f \mathrm{d}\mu.$$

We briefly mention the steps of the proof. After we introduce the main approximating object $X^{x,\varepsilon}$ to solutions X^x of (4.2.2), we prove a contraction principle, i. e.

$$\mathbb{P}\left(\|X_T^x - X_T^y\|_{H^{-1}} \leq \|x - y\|_{H^{-1}}\right) = 1 \quad \text{for all } T > 0,$$

which will be needed throughout the remaining proof. The lower bound technique of [91] is then applied in three steps: We first prove that solutions to (4.2.2) are likely to stay on average close to a ball in L^∞, i. e. for $\rho, \delta > 0$ there exists an $R > 0$ such that for sufficiently large $T > 0$

$$\frac{1}{T}\int_0^T \mathbb{P}(X_r^x \in C_\delta(R))\,\mathrm{d}r \geq 1 - \rho, \tag{4.2.8}$$

where $C_\delta(R)$ is the δ-neighbourhood of $B_R^\infty(0)$ in H^{-1}. We then analyze the deterministic equation

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} u^{\pm R} = -\partial\varphi(u^{\pm R}) + g, \\ &u^{\pm R}(0) \equiv \pm R, \end{aligned}$$

which will serve as the control process mentioned above and which converges for large times to a limit $u_\infty \in H^{-1}$. Finally, we show that with positive probability, X^x behaves “similar” to $u^{\pm R}$ if $x \in C_\delta(R)$, so that together with (4.2.8) we conclude that for all $x \in H^{-1}$, $\delta > 0$

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T P_r\left(x, B_{2\delta}^{H^{-1}}(u_\infty)\right)\mathrm{d}r > 0,$$

which implies the existence and uniqueness of an invariant measure by [91, Theorem 1].

4.3 Lemmas and proof

We recall the following notion from [91]:

Definition 4.3.1. We say that a transition semigroup $(P_t)_{t\geq 0}$ on some Hilbert space H has the e-property if the family of functions $(P_t f)_{t\geq 0}$ is equicontinuous at every point $x \in H$ for any bounded and Lipschitz continuous function $f : H \to \mathbb{R}$.

As mentioned before, the proof of the main theorem relies on the following sufficient condition of [91]:

Proposition 4.3.2 (Komorowski-Peszat-Szarek 2010). *Let $(P_t)_{t\geq 0}$ be the transition semigroup of a stochastically continuous Markov process taking values on a separable Hilbert space H. Assume that $(P_t)_{t\geq 0}$ satisfies the Feller- and the e-property. Furthermore, assume that there exists $z \in H$ such that for every $\delta > 0$ and $x \in H$*

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T P_r(x, B_\delta^H(z))\mathrm{d}r > 0. \tag{4.3.1}$$

Then the semigroup $(P_t)_{t\geq 0}$ admits a unique invariant probability Borel measure.

Most of the following arguments involve an approximating process, which will be introduced in the following lemmas. We first summarize some of the auxiliary statements from Chapter 3. For the quantitative estimates, see especially (3.4.6).

Lemma 4.3.3. *Let ϕ^ε be the Yosida approximation of ϕ, as introduced in Appendix 4.B. Let $T > 0$ and $x \in L^2$, and consider the SPDE*

$$\begin{aligned} \mathrm{d}X_t^{x,\varepsilon} &= \varepsilon\Delta X_t^{x,\varepsilon}\mathrm{d}t + \Delta\phi^\varepsilon(X_t^{x,\varepsilon})\mathrm{d}t + B\,\mathrm{d}W_t, \\ X_0 &= x. \end{aligned} \tag{4.3.2}$$

Then, identifying x with a random variable $x \in L^2(\Omega, L^2)$ being almost surely constant, (4.3.2) allows for a unique variational solution $(X_t^{x,\varepsilon})_{t\in[0,T]}$ in the sense of [112, Definition 4.2.1] with respect to the Gelfand triple $L^2 \hookrightarrow H^{-1} \hookrightarrow (L^2)'$. Furthermore, $X^{x,\varepsilon}$ satisfies the regularity estimate

$$\mathbb{E}\sup_{t\in[0,T]} \|X_t^{x,\varepsilon}\|_2^2 + \varepsilon\mathbb{E}\int_0^T \|X_r^{x,\varepsilon}\|_{H_0^1}^2\,\mathrm{d}r \leq C(T)(\mathbb{E}\,\|x\|_2^2 + 1) \tag{4.3.3}$$

with a constant $C(T) > 0$ independent of ε. For $(x_n)_{n\in\mathbb{N}} \subset L^2$, $x_n \to x$ in H^{-1} for $n \to \infty$, we have

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} X^{x_n,\varepsilon} = X^x, \tag{4.3.4}$$

where the limits are taken in $L^2\left(\Omega, C([0,T], H^{-1})\right)$ and X^x is the SVI solution to (4.2.2). More precisely, the ε-limit is uniform on bounded sets of L^2 by the estimate

$$\mathbb{E}\sup_{t\in[0,T]} \|X^{y,\varepsilon} - X^y\|_{H^{-1}}^2 \leq \varepsilon\, C(T)(\|y\|_2^2 + 1) \tag{4.3.5}$$

for $y \in L^2$, and for the n-limit we have

$$\mathbb{E}\sup_{t\in[0,T]} \|X^{x_n} - X^x\|_{H^{-1}}^2 \leq C(T)\,\|x - x_n\|_{H^{-1}}^2\,. \tag{4.3.6}$$

Finally, for $x, y \in H^{-1}$ we have

$$\sup_{t\in[0,T]} \mathbb{E}\,\|X_t^x - X_t^y\|_{H^{-1}}^2 \leq C(T)\,\|x - y\|_{H^{-1}}^2\,. \tag{4.3.7}$$

Remark 4.3.4. We note that if $0 < T_1 < T_2 < \infty$, $x \in L^2$, $X^{x,\varepsilon}$ is a solution to (4.3.2) constructed on $[0, T_1]$ and $Y^{x,\varepsilon}$ is a solution to (4.3.2) constructed on $[0, T_2]$, then $(Y_t^{x,\varepsilon})_{t\in[0,T_1]}$ is also a solution to (4.3.2). By the uniqueness part of [112, Theorem 4.2.4], we have

$$X_t^{x,\varepsilon} = Y_t^{x,\varepsilon} \quad \text{for all } t \in [0, T_1].$$

Consequently, $X_t^{x,\varepsilon}$ is consistently defined for all $t \geq 0$, $x \in H^{-1}$, and the same is true for X_t^x by (4.3.4).

From [99, section 4.3], we recall the following disintegration result.

Lemma 4.3.5. *The solution to* (4.3.2) *is a time-homogeneous Markov process, such that we have*

$$\mathbb{E}f(X_{t+s}^{x,\varepsilon}) = \mathbb{E}_{\omega_1}\mathbb{E}_{\omega_2} f(X_t^{X_s^{x,\varepsilon}(\omega_1),\varepsilon}(\omega_2))$$

for any $f \in \mathcal{B}_b(L^2)$ *and* $t, s > 0$.

We need that solutions to (4.2.2) are almost surely contractive, which will be important in the subsequent analysis.

Lemma 4.3.6. *Let* $x, y \in H^{-1}$ *and let* $(X_t^x)_{t\geq 0}$ *and* $(X_t^y)_{t\geq 0}$ *be the SVI solutions to* (4.2.2) *with initial value* x *and* y*, respectively. Then for all* $T > 0$ *we have*

$$\mathbb{P}\left(\|X_T^x - X_T^y\|_{H^{-1}} \leq \|x-y\|_{H^{-1}}\right) = 1. \tag{4.3.8}$$

Proof. We first fix $T > 0$ for which we want to show the statement.

Step 1: First we prove contractivity on the level of approximate solutions and $x, y \in L^2$. For this, let $(X_t^{x,\varepsilon})_{t\in[0,T]}$ and $(X_t^{y,\varepsilon})_{t\in[0,T]}$ solve (4.3.2) with the respective initial value. Let furthermore $Z_t := X_t^{x,\varepsilon} - X_t^{y,\varepsilon}$, which solves

$$\begin{aligned}
\mathrm{d}Z_t &= \varepsilon\Delta(X_t^{x,\varepsilon} - X_t^{y,\varepsilon})\,\mathrm{d}t + (\Delta\phi^\varepsilon(X_t^{x,\varepsilon}) - \Delta\phi^\varepsilon(X_t^{y,\varepsilon}))\,\mathrm{d}t,\\
Z_0 &= x - y.
\end{aligned}$$

Then, by Ito's formula (see e. g. [112, Theorem 4.2.5]), and noting that $Z \in H_0^1$ $\mathbb{P}\otimes \mathrm{d}t$-almost surely by (4.3.3), we obtain $\mathbb{P}$-almost surely

$$\begin{aligned}
\|Z_t\|_{H^{-1}}^2 &= \|x-y\|_{H^{-1}}^2 + 2\varepsilon\int_0^t \langle \Delta Z_r, Z_r\rangle_{H^{-1}}\,\mathrm{d}r\\
&\quad + \int_0^t \langle \Delta\phi^\varepsilon(X_r^{x,\varepsilon}) - \Delta\phi^\varepsilon(X_r^{y,\varepsilon}), Z_r\rangle_{H^{-1}}\,\mathrm{d}r\\
&= \|x-y\|_{H^{-1}}^2 - 2\varepsilon\int_0^t \|Z_r\|_2^2\,\mathrm{d}r\\
&\quad - \int_0^t \langle \phi^\varepsilon(X_r^{x,\varepsilon}) - \phi^\varepsilon(X_r^{y,\varepsilon}), X_r^{x,\varepsilon} - X_r^{y,\varepsilon}\rangle_{L^2}\,\mathrm{d}r.
\end{aligned}$$

The last two terms (the latter because of the monotonicity of ϕ^ε) are negative, which yields

$$\mathbb{P}\left(\|X_T^{x,\varepsilon} - X_T^{y,\varepsilon}\|_{H^{-1}} - \|x-y\|_{H^{-1}} > 0\right) = 0. \tag{4.3.9}$$

Step 2: We now turn to SVI solutions to (4.2.2) with $x, y \in L^2$. Note that it is enough to show for arbitrary $n \in \mathbb{N}, \gamma > 0$ that

$$\mathbb{P}\left(\|X_T^x - X_T^y\|_{H^{-1}} - \|x-y\|_{H^{-1}} > \frac{1}{n}\right) \leq \gamma. \tag{4.3.10}$$

To obtain this, choose ε sufficiently small such that by (4.3.5)

$$\max\left\{\mathbb{E}\,\|X_T^{x,\varepsilon} - X_T^x\|_{H^{-1}}, \mathbb{E}\,\|X_T^{y,\varepsilon} - X_T^y\|_{H^{-1}}\right\} < \frac{\gamma}{4n},$$

which yields by Markov's inequality that

$$\mathbb{P}\left(\|X_T^{x,\varepsilon} - X_T^x\|_{H^{-1}} \geq \frac{1}{2n}\right) \leq \frac{\gamma}{2}$$

and the corresponding statement for X_T^y. Thus together with (4.3.9) we have

$$\begin{aligned}
&\mathbb{P}\left(\|X_T^x - X_T^y\|_{H^{-1}} - \|x-y\|_{H^{-1}} > \frac{1}{n}\right)\\
&\leq \mathbb{P}\left(\|X_T^x - X_T^{x,\varepsilon}\|_{H^{-1}} \geq \frac{1}{2n}\right) + \mathbb{P}\left(\|X_T^y - X_T^{y,\varepsilon}\|_{H^{-1}} \geq \frac{1}{2n}\right)\\
&\quad + \mathbb{P}\left(\|X_T^{x,\varepsilon} - X_T^{y,\varepsilon}\|_{H^{-1}} - \|x-y\|_{H^{-1}} > 0\right)\\
&\leq \gamma,
\end{aligned}$$

which yields (4.3.8) in the case $x, y \in L^2$.

Step 3: Finally consider $x, y \in H^{-1}$. By (4.3.7) we know that for $x, y \in H^{-1}$

$$\mathbb{E}\left\|X_T^x - X_T^y\right\|_{H^{-1}} \leq C\left\|x-y\right\|_{H^{-1}}.$$

In order to confirm (4.3.10), we choose $\tilde{x}, \tilde{y} \in L^2$ in a way that ($\|\cdot\| = \|\cdot\|_{H^{-1}}$)

$$\max\left\{\|x-\tilde{x}\|, \|y-\tilde{y}\|\right\} \leq \frac{1}{4n} \quad \text{and} \quad \max\left\{C\|x-\tilde{x}\|, C\|y-\tilde{y}\|\right\} \leq \frac{\gamma}{8n}.$$

Using

$$\|x-y\| = \|x-\tilde{x}+\tilde{x}-\tilde{y}+\tilde{y}-y\| \geq \|\tilde{x}-\tilde{y}\| - \|x-\tilde{x}\| - \|y-\tilde{y}\|$$

and, again by Markov's inequality,

$$\max\left\{\mathbb{P}\left(\left\|X_T^x - X_T^{\tilde{x}}\right\| \geq \frac{1}{4n}\right), \mathbb{P}\left(\left\|X_T^y - X_T^{\tilde{y}}\right\| \geq \frac{1}{4n}\right)\right\} \leq \frac{\gamma}{2},$$

we compute

$$\begin{aligned}
&\mathbb{P}\left(\left\|X_T^x - X_T^y\right\|_{H^{-1}} - \|x-y\|_{H^{-1}} > \frac{1}{n}\right) \\
&\leq \mathbb{P}\left(\left\|X_T^x - X_T^{\tilde{x}}\right\| \geq \frac{1}{4n}\right) + \mathbb{P}\left(\left\|X_T^{\tilde{x}} - X_T^{\tilde{y}}\right\| - \|\tilde{x}-\tilde{y}\| > 0\right) \\
&\quad + \mathbb{P}\left(\left\|X_T^{\tilde{y}} - X_T^y\right\| \geq \frac{1}{4n}\right) + \mathbb{P}\left(\|x-\tilde{x}\| \geq \frac{1}{4n}\right) + \mathbb{P}\left(\|y-\tilde{y}\| \geq \frac{1}{4n}\right) \leq \gamma,
\end{aligned}$$

which finishes the proof. □

Lemma 4.3.7. *The solution to* (4.2.2) *gives rise to a semigroup of Markov transition kernels by*

$$P_t(x, A) = \mathbb{E}\boldsymbol{1}_A(X_t^x) \quad \text{for } x \in H^{-1} \text{ and } A \in \mathcal{B}(H^{-1}).$$

The induced semigroup $(P_t)_{t\geq 0}$ on $\mathcal{B}_b(H^{-1})$, given by

$$P_t f(x) = \int_{H^{-1}} f(y) P_t(x, \mathrm{d}y),$$

has the Feller- and the e-property. For all $x \in H^{-1}$ and $f \in C_b(H^{-1})$,

$$[0,\infty) \ni t \mapsto P_t f(x) \tag{4.3.11}$$

is continuous at $t = 0$.

Remark 4.3.8. The semigroup $(P_t)_{t\geq 0}$ consisting of Markov transition kernels together with the obvious fact

$$P_0(x, A) = \mathbf{1}_A(x)$$

implies that there is a "canonical" Markov process with transition probabilities $(P_t)_{t\geq 0}$ (see e. g. [37, Section 2.2]).

Remark 4.3.9. Note that the last statement in Lemma 4.3.7 implies the stochastic continuity of $(P_t)_{t\geq 0}$ by [37, Proposition 2.1.1]. By [37, Theorem 2.2.2], the corresponding canonical process is then also stochastically continuous.

Proof of Lemma 4.3.7: The continuity of (4.3.11) follows from the construction as an almost surely continuous process, and the Feller property from the contractivity in Lemma 4.3.6. In both arguments, the dominated convergence theorem applies due to the continuity and boundedness of the test functions.

To prove the e-property for $(P_t)_{t\geq 0}$, it is sufficient to show that for $f : H^{-1} \to \mathbb{R}$ bounded and Lipschitz continuous, $P_t f$ $(t \geq 0)$ is Lipschitz continuous with Lipschitz constant independent of t and equal to the Lipschitz constant $[f]_{\mathrm{Lip}}$ of f. Using Lemma 4.3.6, we compute for $x, y \in H^{-1}$

$$\begin{aligned}
|P_t f(x) - P_t f(y)| &= \left|\mathbb{E}\left[f(X_t^x) - f(X_t^y)\right]\right| \\
&\leq \mathbb{E}\left|f(X_t^x) - f(X_t^y)\right| \\
&\leq \mathbb{E}\left[[f]_{\mathrm{Lip}}\left\|X_t^x - X_t^y\right\|_{H^{-1}}\right] \\
&\leq [f]_{\mathrm{Lip}}\|x-y\|_{H^{-1}},
\end{aligned}$$

as required.

We turn to the kernel properties of P_t: For $x \in H^{-1}$, $t \geq 0$, $P_t(x, \cdot)$ is the pushforward measure of X_t^x and thereby a probability measure. Moreover, let $A \in \mathcal{B}(H^{-1})$. Note that the class of all functions $f \in \mathcal{B}_b(H^{-1})$, for which

$$H^{-1} \ni x \mapsto P_t f(x) \tag{4.3.12}$$

is measurable, is monotone in the sense of [114, Theorem 0.2.2, i) and ii)]. As the family of bounded Lipschitz functions generates the Borel σ-algebra and is stable under pointwise multiplication,

$$H^{-1} \ni x \mapsto P_t \mathbf{1}_A(x)$$

is proven to be measurable by the monotone class theorem (see e. g. [114, Theorem 0.2.2]), as soon as we show measurability of (4.3.12) for bounded and Lipschitz continuous f. The latter, however, becomes clear by taking into account that $P_t f$ is Lipschitz continuous if f is Lipschitz continuous (see the proof of the e-property above).

To establish the semigroup property, we first note that the class of functions $f \in \mathcal{B}_b(H^{-1})$, for which the semigroup property

$$P_{t+s} f(x) = P_s(P_t f)(x) \quad \text{for all } t, s \geq 0,\, x \in H^{-1} \tag{4.3.13}$$

is satisfied, is also monotone, so that it is enough to prove the semigroup property for $f : H^{-1} \to \mathbb{R}$ being bounded and Lipschitz continuous.

For $x \in L^2$, recall the approximation of $(X_t^x)_{t\geq 0}$ by $(X_t^{x,\varepsilon})_{t\geq 0}$ as in (4.3.5). We compute for $f : H^{-1} \to \mathbb{R}$ bounded and Lipschitz continuous with constant $[f]_{\text{Lip}}$, $t, s > 0$

$$\begin{aligned} P_{t+s} f(x) = \mathbb{E} f(X_{t+s}^x) &= \lim_{\varepsilon\to 0} \mathbb{E} f(X_{t+s}^{x,\varepsilon}) \\ &= \lim_{\varepsilon\to 0} \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} f(X_t^{X_s^{x,\varepsilon}(\omega_1),\varepsilon}(\omega_2)) \\ &= \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} f(X_t^{X_s^x(\omega_1)}(\omega_2)) = P_s(P_t f)(x), \end{aligned} \tag{4.3.14}$$

where the steps are justified as follows. The semigroup property on the level of ε-approximations is known from Lemma 4.3.5. The first limit is clear by construction and the assumption of f being Lipschitz continuous together with (4.3.5). For the second limit, we compute

$$\begin{aligned} &\left| \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} f(X_t^{X_s^{x,\varepsilon}(\omega_1),\varepsilon}(\omega_2)) - \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} f(X_t^{X_s^x(\omega_1)}(\omega_2)) \right| \\ \leq\, & \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \left| f(X_t^{X_s^{x,\varepsilon}(\omega_1),\varepsilon}(\omega_2)) - f(X_t^{X_s^x(\omega_1)}(\omega_2)) \right| \\ \leq\, & \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \left| f(X_t^{X_s^{x,\varepsilon}(\omega_1),\varepsilon}(\omega_2)) - f(X_t^{X_s^{x,\varepsilon}(\omega_1)}(\omega_2)) \right| \\ & + \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \left| f(X_t^{X_s^{x,\varepsilon}(\omega_1)}(\omega_2)) - f(X_t^{X_s^x(\omega_1)}(\omega_2)) \right| \end{aligned} \tag{4.3.15}$$

For the first term, we use (4.3.5) and (4.3.3) (which in particular implies that $X_s^{x,\varepsilon} \in L^2$ almost surely) to compute

$$\begin{aligned} &\left(\mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \left| f(X_t^{X_s^{x,\varepsilon}(\omega_1),\varepsilon}(\omega_2)) - f(X_t^{X_s^{x,\varepsilon}(\omega_1)}(\omega_2)) \right| \right)^2 \\ \leq\, & \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \left[[f]_{\text{Lip}}^2 \left\| X_t^{X_s^{x,\varepsilon}(\omega_1),\varepsilon}(\omega_2) - X_t^{X_s^{x,\varepsilon}(\omega_1)}(\omega_2) \right\|_{H^{-1}}^2 \right] \\ \leq\, & [f]_{\text{Lip}}^2\, \mathbb{E}_{\omega_1} \left[C(t)\, \varepsilon\, (\|X_s^{x,\varepsilon}(\omega_1)\|_{L^2}^2 + 1) \right] \\ \leq\, & C(t)\, \varepsilon\, [f]_{\text{Lip}}^2\, C(s)(\|x\|_{L^2}^2 + 1) \to 0 \quad \text{for } \varepsilon \to 0. \end{aligned}$$

For the second term, we use Lemma 4.3.6 and again (4.3.5) to obtain

$$\begin{aligned} &\left(\mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \left| f(X_t^{X_s^{x,\varepsilon}(\omega_1)}(\omega_2)) - f(X_t^{X_s^x(\omega_1)}(\omega_2)) \right| \right)^2 \\ \leq\, & \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \left[[f]_{\text{Lip}}^2 \left\| X_t^{X_s^{x,\varepsilon}(\omega_1)}(\omega_2) - X_t^{X_s^x(\omega_1)}(\omega_2) \right\|_{H^{-1}}^2 \right] \\ \leq\, & [f]_{\text{Lip}}^2\, \mathbb{E}\, \|X_s^{x,\varepsilon} - X_s^x\|_{H^{-1}}^2 \to 0 \quad \text{for } \varepsilon \to 0. \end{aligned} \tag{4.3.16}$$

It remains to show (4.3.13) for general initial conditions $x \in H^{-1}$, f still being bounded and Lipschitz. In analogy to (4.3.14), we compute for a sequence $(x_n)_{n\in\mathbb{N}} \subset L^2$, $x_n \to x$ in H^{-1},

$$\begin{aligned} P_{t+s}f(x) = \mathbb{E}f(X^x_{t+s}) &= \lim_{n\to\infty} \mathbb{E}f(X^{x_n}_{t+s}) \\ &= \lim_{n\to\infty} \mathbb{E}_{\omega_1}\mathbb{E}_{\omega_2} f(X_t^{X_s^{x_n}(\omega_1)}(\omega_2)) \\ &= \mathbb{E}_{\omega_1}\mathbb{E}_{\omega_2} f(X_t^{X_s^{x}(\omega_1)}(\omega_2)) = P_s(P_t f)(x). \end{aligned} \tag{4.3.17}$$

The intermediate step on the level of the approximating sequence has been proved above. The first limit is clear by (4.3.6) and the Lipschitz continuity of f. The second limit is treated by the same steps as in (4.3.16), using (4.3.6) instead of (4.3.5). This concludes the proof. □

The following lemma is an energy estimate for the L^∞ norm.

Lemma 4.3.10. *Let $x \in H^{-1}$, $\delta, \rho > 0$ and for $R > 0$*

$$C_\delta(R) := \left\{u \in H^{-1} : \exists v \in B_R^\infty(0) \text{ such that } \|u-v\|_{H^{-1}} < \delta\right\},$$

where $B_R^\infty(0) := \{v \in L^\infty : \|v\|_\infty < R\}$. Then there exists $R = R(\rho, x) > 3$ such that for all $T > 1$ we have

$$\frac{1}{T}\int_0^T \mathbb{P}(X_r^x \in C_\delta(R))\,\mathrm{d}r \ge 1-\rho. \tag{4.3.18}$$

for solutions X^x to (4.2.2).

Proof. We first consider the approximating solutions from (4.3.2) with initial value $\tilde{x} \in L^2$, for which we know by (4.3.3) that they are in H_0^1, $\mathbb{P}\otimes\mathrm{d}t$-almost surely. We choose $\tilde{x}$ in a way that

$$\|x-\tilde{x}\|_{H^{-1}} \le \frac{\delta}{2}. \tag{4.3.19}$$

Note also that ϕ^ε is weakly differentiable for $\varepsilon > 0$ and

$$(\phi^\varepsilon)' \ge \frac{1}{2}\mathbf{1}_{\mathbb{R}\setminus[-1,1]} \tag{4.3.20}$$

for $0 < \varepsilon < 1$ by (4.B.2). Ito's formula (see e. g. [112, Theorem 4.2.5]) on the Gelfand triple $H_0^1 \hookrightarrow L^2 \hookrightarrow H^{-1}$ then yields

$$\begin{aligned} \left\|X_t^{\tilde{x},\varepsilon}\right\|_2^2 = \|\tilde{x}\|_2^2 &+ \int_0^t 2\,{}_{H_0^1}\!\left\langle X_r^{\tilde{x},\varepsilon}, \Delta(\varepsilon X_r^{\tilde{x},\varepsilon} + \phi^\varepsilon(X_r^{\tilde{x},\varepsilon}))\right\rangle_{H^{-1}}\mathrm{d}r \\ &+ \int_0^t 2\left\langle X_r^{\tilde{x},\varepsilon}, B\,\mathrm{d}W_r\right\rangle_{L^2} + \int_0^t 2\,\|B\|^2_{L_2(U,L^2)}\,\mathrm{d}r. \end{aligned}$$

Abbreviating the last two summands by K and using the chain rule for Sobolev functions (see e. g. [126, Theorem 2.1.11]) and (4.3.20), we obtain

$$\begin{aligned} \left\|X_t^{\tilde{x},\varepsilon}\right\|_2^2 &= \|\tilde{x}\|_2^2 - 2\varepsilon\int_0^t \left\|\nabla X_r^{\tilde{x},\varepsilon}\right\|_2^2\mathrm{d}r \\ &\quad - \int_0^t\int_{\mathcal{O}} 2\left\langle \nabla X_r^{\tilde{x},\varepsilon}, \nabla\phi^\varepsilon(X_r^{\tilde{x},\varepsilon})\right\rangle\,\mathrm{d}x\,\mathrm{d}r + K \\ &\le \|\tilde{x}\|_2^2 - 2\int_0^t\int_{\mathcal{O}} (\phi^\varepsilon)'(X_r^{\tilde{x},\varepsilon})(\nabla X_r^{\tilde{x},\varepsilon})^2\,\mathrm{d}x\,\mathrm{d}r + K \\ &\le \|\tilde{x}\|_2^2 - \int_0^t\int_{\mathcal{O}} \mathbf{1}_{\{|X_r^{\tilde{x},\varepsilon}|>1\}}(\nabla X_r^{\tilde{x},\varepsilon})^2\,\mathrm{d}x\,\mathrm{d}r + K \\ &= \|\tilde{x}\|_2^2 - \int_0^t\int_{\mathcal{O}} \left(\mathbf{1}_{\{|X_r^{\tilde{x},\varepsilon}|>1\}}\nabla X_r^{\tilde{x},\varepsilon}\right)^2\,\mathrm{d}x\,\mathrm{d}r + K. \end{aligned} \tag{4.3.21}$$

Defining $A \in \mathrm{Lip}(\mathbb{R})$ by

$$x \mapsto A(x) = \mathrm{sgn}(x)\,(|x|-1)\,\mathbf{1}_{\{|x|>1\}},$$

we see that almost everywhere

$$A'(X_r^{\tilde{x},\varepsilon}) = \mathbf{1}_{\left\{|X_r^{\tilde{x},\varepsilon}|>1\right\}}.$$

Thus, using the chain rule for Sobolev functions and the continuous embedding $H_0^1 \hookrightarrow L^\infty$, we continue (4.3.21) by

$$\begin{aligned}\left\|X_t^{\tilde{x},\varepsilon}\right\|_2^2 &\leq \|\tilde{x}\|_2^2 - \int_0^t \int_{\mathcal{O}} \left(\nabla A(X_r^{\tilde{x},\varepsilon})\right)^2 \,\mathrm{d}x\,\mathrm{d}r + K \\ &\leq \|\tilde{x}\|_2^2 - C\int_0^t \left\|A(X_r^{\tilde{x},\varepsilon})\right\|_\infty^2 \,\mathrm{d}r + K \\ &= \|\tilde{x}\|_2^2 - C\int_0^t \left(\left\|X_r^{\tilde{x},\varepsilon}\right\|_\infty - 1\right)_+^2 \,\mathrm{d}r + K.\end{aligned} \tag{4.3.22}$$

For the remaining part

$$K = \int_0^t 2\left\langle X_r^{\tilde{x},\varepsilon}, B\,\mathrm{d}W_r\right\rangle_{L^2} + \int_0^t 2\,\|B\|_{L_2(U,L^2)}^2 \,\mathrm{d}r$$

we notice that the first summand vanishes in expectation and that the second one can be estimated from above by Ct by the assumptions on B. Thus, taking expectations in (4.3.22) provides

$$\mathbb{E}\int_0^t (\left\|X_r^{\tilde{x},\varepsilon}\right\|_\infty - 1)_+^2 \,\mathrm{d}r \leq C\left(\|\tilde{x}\|_2^2 + t\right), \tag{4.3.23}$$

where we emphasize that C does not depend on ε. By the Markov inequality, we then use (4.3.23) to compute

$$\begin{aligned}\frac{1}{T}\int_0^T \mathbb{P}\left(\left(\left\|X_r^{\tilde{x},\varepsilon}\right\|_\infty - 1\right)_+^2 > R\right)\mathrm{d}r &\leq \frac{1}{T}\int_0^T \frac{\mathbb{E}\left(\left\|X_r^{\tilde{x},\varepsilon}\right\|_\infty - 1\right)_+^2}{R}\,\mathrm{d}r \\ &\leq \frac{C}{TR}\left(\|\tilde{x}\|_2^2 + T\right),\end{aligned}$$

which for $T > 1$ becomes smaller than $\frac{\rho}{2}$ by choosing R large enough, uniformly in ε. For technical reasons, we impose $R > 3$ without loss of generality. For $T > 1$ fixed, we now choose ε small enough such that

$$\mathbb{E}\sup_{t\in[0,T]}\left\|X_t^{\tilde{x}} - X_t^{\tilde{x},\varepsilon}\right\|_{H^{-1}} \leq \frac{\rho\delta}{4}. \tag{4.3.24}$$

By Markov's inequality, (4.3.24) yields

$$\mathbb{P}\left(\sup_{t\in[0,T]}\left\|X_t^{\tilde{x}} - X_t^{\tilde{x},\varepsilon}\right\|_{H^{-1}} \geq \frac{\delta}{2}\right) \leq \frac{\rho}{2}.$$

By Lemma 4.3.6 and (4.3.19) we have for $t > 0$

$$\left\|X_t^x - X_t^{\tilde{x}}\right\|_{H^{-1}} \leq \frac{\delta}{2} \quad \text{almost surely,}$$

which we use to conclude for R as chosen above

$$\begin{aligned}&\frac{1}{T}\int_0^T \mathbb{P}(X_r^x \in C_\delta(R))\,\mathrm{d}r \\ &\geq \frac{1}{T}\int_0^T \mathbb{P}(X_r^{\tilde{x}} \in C_{\frac{\delta}{2}}(R)\,\mathrm{d}r \\ &= 1 - \frac{1}{T}\int_0^T \mathbb{P}(X_r^{\tilde{x}} \notin C_{\frac{\delta}{2}}(R))\,\mathrm{d}r \\ &\geq 1 - \frac{1}{T}\int_0^T \mathbb{P}\left(\left\|X_r^{\tilde{x}} - X_r^{\tilde{x},\varepsilon}\right\|_{H^{-1}} \geq \frac{\delta}{2} \text{ or } \left\|X_r^{\tilde{x},\varepsilon}\right\|_\infty \geq R\right)\mathrm{d}r \\ &\geq 1 - \frac{1}{T}\int_0^T \mathbb{P}\left(\left\|X_r^{\tilde{x}} - X_r^{\tilde{x},\varepsilon}\right\|_{H^{-1}} \geq \frac{\delta}{2}\right) + \mathbb{P}\left(\left\|X_r^{\tilde{x},\varepsilon}\right\|_\infty \geq \sqrt{R}+1\right)\mathrm{d}r \\ &\geq 1 - \frac{\rho}{2} - \frac{1}{T}\int_0^T \mathbb{P}\left((\left\|X_r^{\tilde{x},\varepsilon}\right\|_\infty - 1)_+^2 \geq R\right)\mathrm{d}r \\ &\geq 1 - \rho,\end{aligned} \tag{4.3.25}$$

as required. $\square$

We continue with the analysis of the deterministic control process, for which we cite a translated version of [26, Théorème 3.11]. For the definition of weak and strong solutions, see Definition 4.A.1.

Proposition 4.3.11. *Let H be a Hilbert space and $A : H \supseteq D(A) \to H$ a maximal monotone operator of the form $A = \partial\varphi$ for some $\varphi : H \to [0,\infty]$ convex, proper and lower-semicontinuous. Suppose that for all $\alpha \in \mathbb{R}$ the set*

$$M_\alpha := \{x \in H : \varphi(x) + \|x\|^2 \leq \alpha\} \tag{4.3.26}$$

is strongly compact. Let $f \in L^1_{\mathrm{loc}}([0,\infty); H)$ such that $\lim_{t\to\infty} f(t) =: f_\infty$ exists, $f - f_\infty \in L^1([0,\infty); H)$ and $f_\infty \in R(\partial\varphi)$. For $x \in \overline{D(\partial\varphi)}$, let u^x be a weak solution to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} u^x &\in -\partial\varphi(u^x) + f, \\ u(0) &= x. \end{aligned}$$

Then $\lim_{t\to\infty} u^x(t) =: u_\infty$ exists and

$$f_\infty \in \partial\varphi(u_\infty). \tag{4.3.27}$$

Remark 4.3.12. Note that existence even of strong solutions to (4.3.28) is guaranteed by [26, Théorèmes 3.4 and 3.6] for $t \in [0,T]$, $T > 0$. By uniqueness, we can extend the solution to $[0,\infty)$, analogous to Remark 4.3.4. In particular, for $t > 0$ and $x \in \overline{D(\partial\varphi)}$ we have $u^x(t) \in D(\partial\varphi)$.

From the definition of g in (4.2.5), recall especially that $g \in L^2$ and $g > 1$ almost everywhere in $\mathcal{O}$. For $x \in \overline{D(\partial\varphi)}$, consider the deterministic evolution equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} u^x &\in -\partial\varphi(u^x) + g, \\ u^x(0) &= x \end{aligned} \tag{4.3.28}$$

on H^{-1}, where φ is defined as in (4.2.1).

Lemma 4.3.13. *Let $R > 1$. For the initial states $x \equiv \pm R$, Proposition 4.3.11 can be applied to problem* (4.3.28) *by replacing both $f(t)$ and f_∞ by g. In this case,*

$$u_\infty = ((-\Delta)^{-1} g) \vee 1. \tag{4.3.29}$$

Proof. The functional φ as defined in (4.2.1) is obviously not constantly ∞. Furthermore, it is convex and lower-semicontinuous by [7, Proposition 2.10].

In order to verify the compactness of the sets M_α, $\alpha \in \mathbb{R}$, as defined in (4.3.26), we first show that M_α is a bounded subset of L^2. This is obvious for $\alpha \leq 0$ such that we can restrict to $\alpha > 0$ in the following. Indeed, if for $u \in H^{-1}$ $\varphi(u) \leq \alpha < \infty$, then $u \in L^2$ by (4.2.1). Then, we compute

$$\begin{aligned} \int_{\mathcal{O}} u^2 \mathrm{d}x &\leq |\mathcal{O}| + \int_{\{|u|\geq 1\}} (|u| - 1 + 1)^2 \mathrm{d}x \\ &\leq |\mathcal{O}| + \int_{\{|u|\geq 1\}} (|u| - 1)^2 + 2(|u| - 1) + 1 \, \mathrm{d}x \\ &\leq |\mathcal{O}| + 2\varphi(u) + 2\,|\mathcal{O}|^{\frac{1}{2}} \left(\int_{\{|u|\geq 1\}} (|u| - 1)^2 \mathrm{d}x \right)^{\frac{1}{2}} + |\mathcal{O}| \\ &\leq 2\,|\mathcal{O}| + 2\varphi(u) + 2\sqrt{2}\,|\mathcal{O}|^{\frac{1}{2}} \varphi^{\frac{1}{2}}(u) \leq C\,(1 + \alpha) < \infty. \end{aligned}$$

Since the canonical embedding $L^2 \hookrightarrow H^{-1}$ is compact, it follows that $\overline{M_\alpha}$ is compact. As φ is lower-semicontinuous, so is $\varphi + \|\cdot\|^2_{H^{-1}}$, and thus M_α is also closed. Hence, M_α is compact, as required.

We recall from [7, Proposition 2.10] that $\partial\varphi$ can be characterized by

$$\partial\varphi = \left\{ \begin{matrix} [u,w] \in (H^{-1} \cap L^1) \times H^{-1} : \\ w = -\Delta v, v \in H^1_0, v(x) \in \phi(u(x)) \text{ for a. e. } x \in \mathcal{O} \end{matrix} \right\},$$

with
$$D(\partial\varphi) = \left\{u \in H^{-1} \cap L^1 : \exists\, v \in H_0^1 \text{ such that } v \in \phi(u) \text{ almost everywhere}\right\}.$$

To show that the constant functions $\pm R$ are elements of $\overline{D(\partial\varphi)}$, we define for $n \in \mathbb{N}$

$$v_n := n(1-x) \wedge n(x+1) \wedge R \in H_0^1,$$

and $u_n := v_n \vee 1$. We then have $u_n \in H^{-1} \cap L^1$ and $v_n \in \phi(u_n)$, and thus $u_n \in D(\partial\varphi)$. Since $u_n \to R$ in H^{-1}, we have that the constant function $R \in \overline{D(\partial\varphi)}$. For the constant function with value $-R$, analogous considerations apply.

Finally, to show (4.3.29), we first prove that

$$u_\infty = ((-\Delta)^{-1} g) \vee 1 \tag{4.3.30}$$

satisfies (4.3.27) with f_∞ replaced by g. Setting $v := (-\Delta)^{-1} g$, we have $v \in H_0^1$, as g was assumed to be in $L^2 \subset H^{-1}$, and consequently $v \vee 1 \in H^{-1} \cap L^1$. Furthermore, $v > 0$ almost everywhere by the strong maximum principle (see [79, Theorem 8.19]) and thus $v \in \phi(v \vee 1)$ a. e., such that $v \vee 1 \in D(\partial\varphi)$. Since additionally $g = -\Delta v$, we have $g \in R(\partial\varphi)$ and $g \in \partial\varphi(v \vee 1)$.

We conclude by noticing that (4.3.30) is the only choice for u_∞ such that (4.3.27) is satisfied. This becomes clear by the strict monotonicity of $\phi|_{\mathbb{R}\setminus(-1,1)}$ and the strict positivity of $(-\Delta)^{-1} g$ by the strong maximum principle. □

Similarly to Lemma 4.3.3, we define approximations $u^{x,\varepsilon}$ for equation (4.3.28) by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} u_t^{x,\varepsilon} &= \varepsilon \Delta u_t^{x,\varepsilon} + \Delta \phi^\varepsilon(u_t^{x,\varepsilon}) + g \quad \text{for } t \in (0, S], \\ u_0^{x,\varepsilon} &= x, \end{aligned} \tag{4.3.31}$$

where $S > 0$ and g still satisfies assumption (4.2.5). Analogous to the approximation of X^x, there is a unique variational solution to (4.3.31), and if $x \in \overline{D(\partial\varphi)} \cap L^2$, so that (4.3.28) has a strong solution, we obtain

$$\sup_{t \in [0,S]} \|u_t^{x,\varepsilon} - u_t^x\|_{H^{-1}}^2 \leq \varepsilon\, C(S)(\|x\|_2^2 + 1) \tag{4.3.32}$$

analogous to (4.3.5).

For these approximating deterministic equations, we need order-preservation in the initial value. A partial order on H^{-1} can be defined as follows:

Definition 4.3.14. We write $u \leq v$ in H^{-1}, if for all $\eta \in H_0^1, \eta \geq 0$ almost everywhere, one has

$$u(\eta) \leq v(\eta).$$

Lemma 4.3.15. *Let $u, v, w \in H^{-1}$. Then $u \leq v \leq w$ in H^{-1} implies*

$$\|v\|_{H^{-1}} \leq \|u\|_{H^{-1}} + \|w\|_{H^{-1}}.$$

Proof. For arbitrary $\eta \in H_0^1, \|\eta\|_{H_0^1} \leq 1$, we compute

$$\begin{aligned} v(\eta) &= v(\eta \wedge 0) + v(\eta \vee 0) \\ &= -v(-(\eta \wedge 0)) + v(\eta \vee 0) \\ &\leq -u(-(\eta \wedge 0)) + w(\eta \vee 0) \\ &= u(\eta \wedge 0) + w(\eta \wedge 0) \\ &\leq \|u\|_{H^{-1}} + \|w\|_{H^{-1}}, \end{aligned}$$

where for the last step we note that both $\eta \wedge 0$ and $\eta \vee 0$ are H_0^1 functions with norm less than η (see e. g. [126, Corollary 2.1.8]). □

For the approximate deterministic dynamics governed by (4.3.31), we then have the following comparison principle:

Lemma 4.3.16. *Let $x, y \in L^\infty \subseteq L^2$ and $x \leq y$ almost everywhere, and let $u^{x,\varepsilon}$ and $u^{y,\varepsilon}$ be the solutions to* (4.3.31) *with the corresponding initial values. Then*

$$u_t^{x,\varepsilon} \leq u_t^{y,\varepsilon} \quad \textit{in } H^{-1}, \textit{ for all } t > 0.$$

Proof. Note that $u^{x,\varepsilon}$ for $x \in L^\infty$ is also a weak solution in the sense of [119, Chapter 5] with $\Phi = \varepsilon \mathrm{Id} + \phi^\varepsilon$. By [119, Theorem 5.7], the claimed comparison principle is satisfied. □

Corollary 4.3.17. *Let $R > 0$. As a consequence of Lemmas 4.3.15 and 4.3.16, we have for $x \in L^\infty$, $\|x\|_\infty \leq R$ and arbitrary $u \in H^{-1}$*

$$\|u_t^{x,\varepsilon} - u\|_{H^{-1}} \leq \left\|u_t^{R,\varepsilon} - u\right\|_{H^{-1}} + \left\|u_t^{-R,\varepsilon} - u\right\|_{H^{-1}} \quad \textit{for } t \geq 0.$$

Proof. It is enough to read off Definition 4.3.14 that $-R \leq x \leq R$ almost everywhere implies $-R \leq x \leq R$ in H^{-1}, and that the order is invariant under translation by a fixed element of H^{-1}. □

We now compare the approximations $u^{x,\varepsilon}$ to the solution of the stochastic equation (4.3.2), with a noise conditioned on suitable events.

Lemma 4.3.18. *Let $R, S > 0, 0 < \beta \leq 1, x \in L^\infty, \|x\|_\infty \leq R$ and let $u^{x,\varepsilon}$ be the solution to* (4.3.31). *Furthermore, let $X^{x,\varepsilon}$ be the solution to* (4.3.2) *up to time S with the same initial condition x. Assume that*

$$\sup_{t\in[0,S]} \left\|W_t^B - tg\right\|_2 \leq \beta, \tag{4.3.33}$$

where for simplicity we write $W_t^B = BW_t$. Then for $0 < \varepsilon \leq 1$ we have

$$\|X_S^{x,\varepsilon} - u_S^{x,\varepsilon}\|_{H^{-1}} \leq C(R, S)\beta.$$

Proof. We consider the transformed processes

$$\begin{aligned} Y_t^{x,\varepsilon} &= X_t^{x,\varepsilon} - W_t^B \quad \text{and} \\ v_t^{x,\varepsilon} &= u_t^{x,\varepsilon} - tg, \end{aligned}$$

so that by

$$\|X_S^{x,\varepsilon} - u_S^{x,\varepsilon}\|_{H^{-1}} \leq \|Y_S^{x,\varepsilon} - v_S^{x,\varepsilon}\|_{H^{-1}} + \left\|W_S^B - Sg\right\|_{H^{-1}},$$

we can focus on $\|Y_S^{x,\varepsilon} - v_S^{x,\varepsilon}\|_{H^{-1}}^2$ using (4.3.33) and the continuity of the embedding $L^2 \hookrightarrow H^{-1}$. For the following equalities, recall that $X^{x,\varepsilon} \in H_0^1$ $\mathbb{P}\otimes \mathrm{d}t$–almost everywhere due to (4.3.3) and $u_r^{x,\varepsilon} \in H_0^1$ for almost every $r \in [0, S]$ by [119, Theorem 5.7]. Thus,

$$\begin{aligned} \varepsilon X_r^{x,\varepsilon} + \phi^\varepsilon(X_r^{x,\varepsilon}) &\in H_0^1 \quad \mathbb{P}\otimes \mathrm{d}t\text{–a. e.} \\ \text{and} \quad \varepsilon u_r^{x,\varepsilon} + \phi^\varepsilon(u_r^{x,\varepsilon}) &\in H_0^1 \quad \text{for a. e. } r \in [0, S], \end{aligned} \tag{4.3.34}$$

by the Lipschitz continuity of ϕ^ε and the chain rule for Sobolev functions (e. g. [126, Theorem 2.1.11]), which allows to write

$$\begin{aligned}
&\frac{1}{2}\|Y_S^{x,\varepsilon} - v_S^{x,\varepsilon}\|_{H^{-1}}^2 \\
&= \int_0^S \langle Y_r^{x,\varepsilon} - v_r^{x,\varepsilon}, \Delta\left(\varepsilon X_r^{x,\varepsilon} + \phi^\varepsilon(X_r^{x,\varepsilon})\right) - \Delta\left(\varepsilon u_r^{x,\varepsilon} + \phi^\varepsilon(u_r^{x,\varepsilon})\right)\rangle_{H^{-1}}\,\mathrm{d}r \\
&= -\int_0^S \langle Y_r^{x,\varepsilon} - v_r^{x,\varepsilon}, \varepsilon X_r^{x,\varepsilon} + \phi^\varepsilon(X_r^{x,\varepsilon}) - (\varepsilon u_r^{x,\varepsilon} + \phi^\varepsilon(u_r^{x,\varepsilon}))\rangle_{L^2}\,\mathrm{d}r \\
&= -\int_0^S \left\langle Y_r^{x,\varepsilon} + W_r^B - (v_r^{x,\varepsilon} + rg), \varepsilon(Y_r^{x,\varepsilon} + W_r^B - (v_r^{x,\varepsilon} + rg))\right\rangle_{L^2}\mathrm{d}r \\
&\quad - \int_0^S \left\langle Y_r^{x,\varepsilon} + W_r^B - (v_r^{x,\varepsilon} + rg), \phi^\varepsilon(Y_r^{x,\varepsilon} + W_r^B) - \phi^\varepsilon(v_r^{x,\varepsilon} + rg)\right\rangle_{L^2}\mathrm{d}r \\
&\quad + \int_0^S \left\langle W_r^B - rg, \varepsilon(Y_r^{x,\varepsilon} + W_r^B - (v_r^{x,\varepsilon} + rg)) + \phi^\varepsilon(Y_r^{x,\varepsilon} + W_r^B) - \phi^\varepsilon(v_r^{x,\varepsilon} + rg)\right\rangle_{L^2}\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^S \left\|W_r^B - rg\right\|_2 \left\|\varepsilon(Y_r^{x,\varepsilon} + W_r^B) + \phi^\varepsilon(Y_r^{x,\varepsilon} + W_r^B) - \varepsilon(v_r^{x,\varepsilon} + rg) - \phi^\varepsilon(v_r^{x,\varepsilon} + rg)\right\|_2 \mathrm{d}r \\
&\leq \left(\int_0^S \left\|W_r^B - rg\right\|_2^2 \mathrm{d}r\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^S \Big(\varepsilon \left\|Y_r^{x,\varepsilon} + W_r^B\right\|_2 + \left\|\phi^\varepsilon(Y_r^{x,\varepsilon} + W_r^B)\right\|_2 \right. \\
&\qquad\qquad \left. + \varepsilon \left\|v_r^{x,\varepsilon} + rg\right\|_2 + \left\|\phi^\varepsilon(v_r^{x,\varepsilon} + rg)\right\|_2 \Big)^2 \mathrm{d}r\right)^{\frac{1}{2}} \\
&\leq S^{\frac{1}{2}} \beta \left(4 \int_0^S \varepsilon^2 \left\|Y_r^{x,\varepsilon} + W_r^B\right\|_2^2 + \left\|\phi^\varepsilon(Y_r^{x,\varepsilon} + W_r^B)\right\|_2^2 \right. \\
&\qquad\qquad \left. + \varepsilon^2 \left\|v_r^{x,\varepsilon} + rg\right\|_2^2 + \left\|\phi^\varepsilon(v_r^{x,\varepsilon} + rg)\right\|_2^2 \mathrm{d}r\right)^{\frac{1}{2}}.
\end{aligned}
$$

Note that the monotonicity of ϕ^ε has been used for the first inequality. It remains to show that the last factor can be bounded in terms of R and S uniformly in $\beta \leq 1$.

To see this boundedness, first notice by (4.B.3) in Appendix 4.B that $|\phi^\varepsilon(x)| \leq |x|$ for all $x \in \mathbb{R}$, $\varepsilon > 0$, so that it is enough to prove suitable bounds on

$$
\int_0^S \left\|Y_r^{x,\varepsilon} + W_r^B\right\|_2^2 \mathrm{d}r \quad \text{and} \quad \int_0^S \left\|v_r^{x,\varepsilon} + rg\right\|_2^2 \mathrm{d}r.
$$

To this end, we compute

$$
\frac{1}{2} \left\|Y_S^{x,\varepsilon}\right\|_{H^{-1}}^2 = \|x\|_{H^{-1}}^2 + \int_0^S \langle \varepsilon\Delta(X_r^{x,\varepsilon}) + \Delta\phi^\varepsilon(X_r^{x,\varepsilon}), Y_r^{x,\varepsilon}\rangle_{H^{-1}} \mathrm{d}r \tag{4.3.35}
$$

by (4.3.3), and further, noting $Y_r^{x,\varepsilon} \in L^2$ by (4.3.3) and (4.2.4),

$$
\begin{aligned}
(4.3.35) &= \|x\|_{H^{-1}}^2 - \int_0^S \langle \varepsilon X_r^{x,\varepsilon} + \phi^\varepsilon(X_r^{x,\varepsilon}), Y_r^{x,\varepsilon}\rangle_{L^2} \mathrm{d}r \\
&= \|x\|_{H^{-1}}^2 - \int_0^S \left\langle \varepsilon(Y_r^{x,\varepsilon} + W_r^B) + \phi^\varepsilon(Y_r^{x,\varepsilon} + W_r^B), Y_r^{x,\varepsilon} + W_r^B \right\rangle_{L^2} \mathrm{d}r \\
&\quad + \int_0^S \left\langle \varepsilon(Y_r^{x,\varepsilon} + W_r^B) + \phi^\varepsilon(Y_r^{x,\varepsilon} + W_r^B), W_r^B \right\rangle_{L^2} \mathrm{d}r.
\end{aligned} \tag{4.3.36}
$$

From (4.B.3) in Appendix 4.B, we obtain the lower bound $|\phi^\varepsilon(x)| \geq \frac{1}{2}|x|$ for $|x| \geq 1 + \varepsilon$ and $\varepsilon \leq 1$, so that for $u \in L^2$ we have the estimate

$$
\|u\|_2^2 \leq \int_{\{|u| \geq 1+\varepsilon\}} 2u\phi^\varepsilon(u)\,\mathrm{d}x + 4|\mathcal{O}| \leq 2\langle u, \phi^\varepsilon(u)\rangle_{L^2} + 4|\mathcal{O}|. \tag{4.3.37}
$$

Using (4.3.37) and Young's inequality for the last two summands, once weighted by $\frac{1}{2}$, we continue by

$$
\begin{aligned}
(4.3.36) &\leq \|x\|_{H^{-1}}^2 - \int_0^S \varepsilon \left\|Y_r^{x,\varepsilon} + W_r^B\right\|_2^2 + \frac{1}{2}\left\|Y_r^{x,\varepsilon} + W_r^B\right\|_2^2 - C\,\mathrm{d}r \\
&\quad + \int_0^S \frac{\varepsilon}{2}\left\|Y_r^{x,\varepsilon} + W_r^B\right\|_2^2 + \frac{\varepsilon}{2}\left\|W_r^B\right\|_2^2 + \frac{1}{4}\left\|\phi^\varepsilon(Y_r^{x,\varepsilon} + W_r^B)\right\|_2^2 + \left\|W_r^B\right\|_2^2 \mathrm{d}r \\
&\leq \|x\|_{H^{-1}}^2 - \frac{1}{4}\int_0^S \left\|Y_r^{x,\varepsilon} + W_r^B\right\|_2^2 \mathrm{d}r + \frac{3}{2}\int_0^S \left\|W_r^B\right\|_2^2 + C\,\mathrm{d}r.
\end{aligned} \tag{4.3.38}
$$

By (4.3.33), assumption (4.2.5) and $\beta \leq 1$, we have for $r \in [0, S]$

$$
\left\|W_r^B\right\|_2 \leq \left\|W_r^B - rg\right\|_2 + \|rg\|_2 \leq \beta + S\|g\|_2 \leq C(S),
$$

such that (4.3.38) yields, by dropping the left-hand side and relabelling the constants,

$$\int_0^S \left\| Y_r^{x,\varepsilon} + W_r^B \right\|_2^2 \,\mathrm{d}r \le 4C(S, \|x\|_{H^{-1}}^2). \tag{4.3.39}$$

To obtain a bound that only depends on S and R, note that $x \in L^\infty, \|x\|_\infty \le R$ by assumption, such that

$$\|x\|_{H^{-1}} \le C \|x\|_2 \le 2\, C\, |\mathcal{O}|^{\frac{1}{2}} R,$$

which, together with (4.3.39), yields the desired bound. A similar estimate for $\int_0^S \|v_r^{x,\varepsilon} + rg\|_2^2 \,\mathrm{d}r$ can be obtained by analogous computations. □

We need to ensure that (4.3.33) is realized for each $\beta > 0$ with non-zero probability.

Lemma 4.3.19. *As in* (4.2.4) *we denote*

$$W_t^B = BW_t = \sum_{i=1}^{\infty} \beta_k(t)\xi_k,$$

with $\sum_{k\in\mathbb{N}} \|\xi_k\|_2^2 < \infty$. *Let* g *and* m *be defined as in* (4.2.5)*, and let the degeneracy assumption on* $(\xi_k)_{k\in\mathbb{N}}$ *in* (4.2.5) *be satisfied. Then for all* $S \ge 0, \beta > 0$ *we have*

$$\mathbb{P}\left(\sup_{t\in[0,S]} \left\| W_t^B - tg \right\|_2^2 \le \beta \right) > 0.$$

Proof. We use the orthogonality of $(\xi_k)_{k\in\mathbb{N}}$ to write, for $m^* > m$,

$$\begin{aligned} &\left\| W_t^B - tg \right\|_2^2 = \\ &= \left\| \sum_{k=1}^{m} \xi_k(\beta_k(t) - tc_k) \right\|_2^2 + \left\| \sum_{k=m+1}^{m^*} \xi_k\beta_k(t) \right\|_2^2 + \left\| \sum_{k=m^*+1}^{\infty} \xi_k\beta_k(t) \right\|_2^2 \\ &= \sum_{k=1}^{m} \|\xi_k\|_2^2 \,|\beta_k(t) - tc_k|^2 + \sum_{k=m+1}^{m^*} \|\xi_k\|_2^2 \,|\beta_k(t)|^2 + \sum_{k=m^*+1}^{\infty} \|\xi_k\|_2^2 \,|\beta_k(t)|^2 . \end{aligned} \tag{4.3.40}$$

For the first term, we note that the event

$$\max_{k\in\{1,\dots,m\}} \sup_{t\in[0,S]} |\beta_k(t) - c_k t|^2 \le \frac{\beta}{3\sum_{k=1}^m \|\xi_k\|_2} \tag{4.3.41}$$

has positive probability by the following reasoning: As the $(\beta_k)_{k=1}^m$ are independent, it is enough to show for each $k \in \{1, \dots, m\}$ that

$$\mathbb{P}\left(\sup_{t\in[0,S]} |\beta_k(t) - c_k t| \le \varepsilon \right) > 0 \tag{4.3.42}$$

for any fixed $S > 0, \varepsilon > 0$. To see this, note that $\beta_k(t) - c_k t$ is again a standard Brownian motion with respect to some probability measure $\mathbb{P}_Q$, which is absolutely continuous with respect to $\mathbb{P}$ by Girsanov's theorem. From [68, Lemma B.1], we obtain for a standard Brownian motion β_1 that

$$\mathbb{P}\left(\sup_{t\in[0,S]} |\beta_1(t)| \le \epsilon \right) > 0, \tag{4.3.43}$$

which is equivalent to

$$\mathbb{P}_Q\left(\sup_{t\in[0,S]} |\beta_k(t) - c_k t| \le \varepsilon \right) > 0.$$

Absolute continuity then yields (4.3.42). For the third term in (4.3.40), we compute

$$\begin{aligned} \mathbb{E} \sup_{t\in[0,S]} \sum_{k>m^*} |\beta_k(t)|^2 \|\xi_k\|_2^2 &\le \sum_{k>m^*} \|\xi_k\|_2^2 \,\mathbb{E} \sup_{t\in[0,S]} |\beta_k(t)|^2 \\ &\le 4S \sum_{k>m^*} \|\xi_k\|_2^2 =: R(m^*) \searrow 0 \end{aligned}$$

for $m^* \to \infty$, where we used the squared version of the Burkholder-Davis-Gundy inequality. Choosing m^* so large that $R(m^*) \le \frac{\beta}{3}$ we obtain

$$\mathbb{P}\left(\sup_{t\in[0,S]} \sum_{k>m^*} \|\xi_k\|_2^2 \,|\beta_k(t)|^2 \le \frac{\beta}{3}\right) \ge 1 - \frac{R(m^*)}{\frac{\beta}{3}} > 0.$$

Having chosen m^* in this way, we can now conclude by (4.3.43) that also for the second term of (4.3.40) we have

$$\mathbb{P}\left(\sup_{t\in[0,S]} \sum_{k=m+1}^{m^*} \|\xi_k\|_2^2 \,|\beta_k(t)|^2 \le \frac{\beta}{3}\right) > 0,$$

which proves the claim by independence. $\square$

The following lemma combines all results up to now.

Lemma 4.3.20. *Let $\delta > 0, R > 1$ and let $g \in L^2$ satisfy assumption* (4.2.5)*. Recall u_∞ from Lemma 4.3.13 as the long-time limit of solutions u^R, u^{-R} to* (4.3.28)*. Then there exist $\gamma, S > 0$ such that for every initial value $x \in C_\delta(R)$, where $C_\delta(R)$ is the δ-neighbourhood of $B_R^\infty(0)$ in H^{-1}, we have*

$$\mathbb{P}(\|X_S^x - u_\infty\|_{H^{-1}} < 2\delta) \ge \gamma.$$

Proof. Recall that u^R, u^{-R} are well-defined by Remark 4.3.12 and Lemma 4.3.13. According to Lemma 4.3.13, we can choose $S > 0$ such that we have

$$\max\left\{\left\|u^R(t) - u_\infty\right\|_{H^{-1}}, \left\|u^{-R}(t) - u_\infty\right\|_{H^{-1}}\right\} \le \frac{\delta}{8} \quad \text{for all } t \ge S. \tag{4.3.44}$$

Let $u^{x,\varepsilon}$ be defined as in Lemma 4.3.18. As shown there, we can choose $0 < \beta \le 1$ such that

$$\sup_{t\in[0,S]} \left\|W_t^B - tg\right\|_2 \le \beta \quad \text{implies } \|X_S^{x,\varepsilon} - u_S^{x,\varepsilon}\|_{H^{-1}} < \frac{\delta}{4}, \tag{4.3.45}$$

uniformly for all $\varepsilon \in (0,1]$, $x \in B_R^\infty(0)$. We then define

$$\gamma := \frac{2}{3}\,\mathbb{P}\left(\sup_{t\in[0,S]} \left\|W_t^B - tg\right\|_2^2 \le \beta\right), \tag{4.3.46}$$

which is strictly positive by Lemma 4.3.19. We then choose $\varepsilon \in (0,1]$ small enough such that for $u^{R,\varepsilon}$ and $u^{-R,\varepsilon}$ as in (4.3.31) we have

$$\max\left\{\left\|u_S^{R,\varepsilon} - u_S^R\right\|_{H^{-1}}, \left\|u_S^{-R,\varepsilon} - u_S^{-R}\right\|_{H^{-1}}\right\} \le \frac{\delta}{8}, \tag{4.3.47}$$

which is possible by (4.3.32), and such that

$$\mathbb{E}\sup_{r\in[0,S]} \|X_r^{x,\varepsilon} - X_r^x\|_{H^{-1}} \le \frac{\gamma\delta}{8} \tag{4.3.48}$$

is satisfied uniformly for $x \in B_R^\infty(0)$ by (4.3.5) (note that the squared form in (4.3.5) is a stronger statement than needed for (4.3.48) by Jensen's inequality). For every $x \in B_R^\infty(0)$, (4.3.48) implies

$$\mathbb{P}\left(\|X_S^x - X_S^{x,\varepsilon}\|_{H^{-1}} \le \frac{\delta}{4}\right) \ge 1 - \frac{\gamma}{2}, \tag{4.3.49}$$

and Corollary 4.3.17, (4.3.44) and (4.3.47) yield

$$\begin{aligned}
\|u_S^{x,\varepsilon} - u_\infty\|_{H^{-1}} &\le \left\|u_S^{R,\varepsilon} - u_\infty\right\|_{H^{-1}} + \left\|u_S^{-R,\varepsilon} - u_\infty\right\|_{H^{-1}} \\
&\le \left\|u_S^{R,\varepsilon} - u_S^R\right\|_{H^{-1}} + \left\|u_S^R - u_\infty\right\|_{H^{-1}} \\
&\quad + \left\|u_S^{-R,\varepsilon} - u_S^{-R}\right\|_{H^{-1}} + \left\|u_S^{-R} - u_\infty\right\|_{H^{-1}} \\
&\le 4\,\frac{\delta}{8} = \frac{\delta}{2}.
\end{aligned}$$

Hence, still for $x \in B_R^\infty(0)$, we conclude,

$$\begin{aligned}
&\mathbb{P}\left(\|X_S^x - u_\infty\|_{H^{-1}} < \delta\right) \\
&\geq \mathbb{P}\left(\|X_S^x - X_S^{x,\varepsilon}\|_{H^{-1}} < \frac{\delta}{4} \text{ and } \|X_S^{x,\varepsilon} - u_S^{x,\varepsilon}\|_{H^{-1}} < \frac{\delta}{4}\right) \\
&= 1 - \mathbb{P}\left(\|X_S^x - X_S^{x,\varepsilon}\|_{H^{-1}} \geq \frac{\delta}{4} \text{ or } \|X_S^{x,\varepsilon} - u_S^{x,\varepsilon}\|_{H^{-1}} \geq \frac{\delta}{4}\right) \\
&\geq 1 - \mathbb{P}\left(\|X_S^x - X_S^{x,\varepsilon}\|_{H^{-1}} \geq \frac{\delta}{4}\right) - \mathbb{P}\left(\|X_S^{x,\varepsilon} - u_S^{x,\varepsilon}\|_{H^{-1}} \geq \frac{\delta}{4}\right) \\
&\geq \mathbb{P}\left(\|X_S^{x,\varepsilon} - u_S^{x,\varepsilon}\|_{H^{-1}} < \frac{\delta}{4}\right) - \frac{\gamma}{2} \\
&\geq \mathbb{P}\left(\sup_{t\in[0,S]} \left\|W_t^B - tg\right\|_2^2 \leq \beta\right) - \frac{\gamma}{2} = \gamma.
\end{aligned}$$

The claim for $x \in C_\delta(R)$ follows immediately by Lemma 4.3.6. □

Proof of Theorem 4.2.1. Lemma 4.3.7, Remark 4.3.8 and Remark 4.3.9 prove all requirements of Proposition 4.3.2 except (4.3.1). To see this remaining statement, we estimate for $0 < \rho < 1$ and $R(\rho, x)$ given in Lemma 4.3.10

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{T} \int_0^T P_r\left(x, B_{2\delta}^{H^{-1}}(u_\infty)\right) \mathrm{d}r \\
&= \liminf_{T\to\infty} \frac{1}{T} \int_0^T P_{r+S}\left(x, B_{2\delta}^{H^{-1}}(u_\infty)\right) \mathrm{d}r \\
&= \liminf_{T\to\infty} \frac{1}{T} \int_0^T \int_{H^{-1}} P_S\left(y, B_{2\delta}^{H^{-1}}(u_\infty)\right) P_r(x, \mathrm{d}y)\, \mathrm{d}r \\
&\geq \liminf_{T\to\infty} \frac{1}{T} \int_0^T \int_{C_\delta(R(\rho,x))} P_S\left(y, B_{2\delta}^{H^{-1}}(u_\infty)\right) P_r(x, \mathrm{d}y)\, \mathrm{d}r \\
&\geq \gamma \liminf_{T\to\infty} \frac{1}{T} \int_0^T P_r(x, C_\delta(R(\rho, x)))\, \mathrm{d}r > \gamma\,(1-\rho) > 0,
\end{aligned}$$

where we used the semigroup property of $(P_t)_{t\geq 0}$, Lemma 4.3.20 and Lemma 4.3.10. The result then follows by Proposition 4.3.2. □

4.A Solutions to monotone evolution equations

For the reader's convenience, we cite and translate [26, Definition 3.1]:

Definition 4.A.1. Let H be a Hilbert space, $f \in L^1([0,T]; H)$, $A : H \supseteq D(A) \to H$ a maximal monotone operator. A function $u \in C([0,T]; H^{-1})$ is called a **strong solution** to

$$\frac{\mathrm{d}}{\mathrm{d}t} u \in -Au + f, \tag{4.A.1}$$

if u is absolutely continuous on compact subsets of $(0,T)$ (which implies that u is differentiable almost everywhere in $(0,T)$) and for almost all $t \in (0,T)$

$$u(t) \in D(A)$$

and

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) \in -Au(t) + f(t).$$

We call $u \in C([0,T]; H^{-1})$ a **weak solution** to (4.A.1) if there are sequences $f_n \in L^1([0,T]; H)$ and $u_n \in C([0,T]; H)$ $(n \in \mathbb{N})$ such that u_n is a strong solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u_n \in -Au_n + f_n,$$

$f_n \to f$ in $L^1([0,T]; H)$ and $u_n \to u$ uniformly in $[0,T]$ for $n \to \infty$.

Remark 4.A.2. We observe that each strong solution is also a weak solution.

4.B Yosida approximation for the specific function ϕ

Recall from section 4.2 that the multivalued function $\phi : \mathbb{R} \to \mathbb{R}$ is defined as the maximal monotone extension of

$$\mathbb{R} \ni x \mapsto x\mathbf{1}_{\{|x|>1\}}.$$

We want to explicitly calculate its resolvent function $R^\varepsilon : \mathbb{R} \to \mathbb{R}$ and its Yosida approximation $\phi^\varepsilon : \mathbb{R} \to \mathbb{R}$. For theoretical details, see [74, Appendix C].

The resolvent $R^\varepsilon(x)$ is defined as the solution s to

$$s + \varepsilon\phi(s) \ni x. \tag{4.B.1}$$

Note that (4.B.1) has exactly one solution by the maximal monotonicity of ϕ. For $x \in [-1, 1]$ we have

$$0 \in \phi(x),$$

thus (4.B.1) is solved by $s = x$. Consequently $R^\varepsilon(x) = x$.

For $x \in (1, 1 + \varepsilon]$ we have

$$\frac{x-1}{\varepsilon} \in [0, 1] = \phi(1).$$

Thus, $s = 1$ solves the equation by

$$x = 1 + \varepsilon\frac{x-1}{\varepsilon} \in 1 + \varepsilon\phi(1),$$

which yields $R^\varepsilon(x) = 1$. If $x \in [-1 - \varepsilon, 1)$, the same argument yields $R^\varepsilon(x) = -1$.

For $|x| > 1 + \varepsilon$, we have $\left|\frac{x}{1+\varepsilon}\right| > 1$ such that

$$x = \frac{x}{1+\varepsilon} + \varepsilon\frac{x}{1+\varepsilon} \in \frac{x}{1+\varepsilon} + \varepsilon\phi\left(\frac{x}{1+\varepsilon}\right),$$

yielding $R^\varepsilon(x) = \frac{x}{1+\varepsilon}$. By definition of the Yosida approximation,

$$\phi^\varepsilon(x) = \frac{x - R^\varepsilon(x)}{\varepsilon},$$

it is now easy to conclude that

$$\phi^\varepsilon(x) = \begin{cases} 0, & |x| \le 1 \\ \frac{x-1}{\varepsilon} & x \in (1, 1+\varepsilon] \\ \frac{x+1}{\varepsilon} & x \in [-1-\varepsilon, 1) \\ \frac{x}{1+\varepsilon} & |x| > 1 + \varepsilon \end{cases}. \tag{4.B.2}$$

In particular, for $\varepsilon \le 1$ and $|x| \ge 1 + \varepsilon$, we observe that

$$|\phi^\varepsilon(x)| \ge \frac{|x|}{2}. \tag{4.B.3}$$

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Marius Neuss
Max–Planck–Institut für Mathematik in den Naturwissenschaften
04103 Leipzig, Germany
marius.neuss@mis.mpg.de

Gedruckt auf alterungsbeständigem Papier nach DIN ISO 9706.