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Socio-legal Systems and Implementation of the Nash Solution in Debreu-Hurwicz Equilibrium

Claus-Jochen Haake · **Walter Trockel**

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Abstract In this article we combine Debreu's (1952) social system with Hurwicz's (1994, 2008) ideas of embedding a "desired" game form into a "natural" game form that includes all feasible behavior, even if it is "illegal" according to the desired form. For the resulting socio-legal system we extend Debreu's concepts of a social system and its social equilibria to a socio-legal system with its Debreu-Hurwicz equilibria. We build on a more general version of social equilibrium due to Shafer and Sonnenschein (1975) that also generalizes the dc-mechanism of Koray and Yildiz (2018) which relates implementation via mechanisms with implementation via rights structures as introduced by Sertel (2001). In the second part we apply and illustrate these new concepts via an application in the narrow welfarist framework of two person cooperative bargaining. There we provide in a socio-legal system based on Nash's demand game an implementation of the Nash bargaining solution in Debreu-Hurwicz equilibrium.

Keywords socio-legal systems · implementation · social systems · generalized games · Nash demand game

JEL classification D02, C78, C72

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1 Introduction

In the inaugural issue of the [Review of] Economic Design in 1994, with Semih Koray among the associate editors, Murat Sertel as the editor-in-chief and William Thomson as the co-editor, the journal's Honorary Editor Leonid Hurwicz contributed the first article titled "Economic Design, adjustment processes, mechanisms and institutions".

In this seminal and visionary work he explained concepts like *mechanism design, institution, game form* and *equilibrium concepts* (by Nash, Walras, Lindahl) in their historical contexts. Hurwicz stressed the fundamental importance of enforcement and procedures making institutional arrangements effective, which he termed *genuine implementation*. In this context he wrote (p.12):

> "The problem of modeling enforcement poses some difficult problems. To begin with, there is a view that Nash equilibria are self-enforcing, by definition. What I believe is meant, is that if the genuine implementation apparatus is in place so that the rules of the game are obeyed and the outcome function is effective, then no player has an incentive unilaterally to defect. But, in general, there is nothing in a specific game form, prescribing particular strategy domains and outcome functions that would prevent players from resorting to 'illegal' strategies, nor is there automatic assurance that outcomes specified by the outcome function will occur unless the required apparatus is in place."

Nor is it enough to expand the game form so as to prescribe the procedures governing the enforcement process, since similar objections can be raised with respect to these expanded game form. This is the so called *infinite regress problem*. In section III, "Are Nash equilibria self-enforcing?", of his Nobel Lecture "But who will guard the guardians?", Hurwicz (2008, p. 579) writes:

> "Now we come to a very important and closely related issue. We are asking whether a given Nash equilibrium in a specified game is enforceable or not. And one occasionally hears the claim that there can be no enforcement problem with Nash equilibria because allegedly Nash equilibria are self-enforcing. I want to stress that I am denying this claim, but I want to give the other side an opportunity to give their arguments. Their argument is that, by definition, in a Nash equilibrium, no player can profit by a unilateral departure from his or her equilibrium strategy. Furthermore, collusions are infeasible in a non-cooperative game. Hence, the argument goes, there is no need (or possibility, really) for enforcement."

And in Section 5, "Successful Enforcement and Implementation", he states (p.581):

> "And I conclude this section by saying that implementation is successful if the equilibrium outcomes correspond to those of the desired game, i.e., those envisaged by the legislation. Expressed in this framework, a Nash equilibrium is not self-enforcing because, while it is unprofitable to move to alternative legal strategies, it may be profitable, in the absence of enforcement, to move to illegal strategies. Similarly, Nash equilibria are not self-implementing because implementing actions are required to ensure that the true outcome is the same as the legal outcome."

Clearly, in case of multiple equilibria no single one of them could be expected to be self-enforcing. But, Hurwicz is contrasting deliberate participation in a game with enforced participation in a legally prescribed subgame. In a previous note (Trockel and Haake, 2019) we had suggested a connection between Hurwicz's dichotomy of *legal* vs. *illegal* and Debreu's (1952) distinction between socially agreed *choices* and socially unaccepted *actions*.

Inspired by those ideas of enforcement and a dichotomy of legal and illegal games within true games and based on Debreu's (1952) concept of a social system and extensions due to Shafer and Sonnenschein (1975), Prakash and Sertel (1996) and of Koray and Yildiz (2018), we shall introduce socio-legal systems and their *Debreu-Hurwicz* equilibria in Section 2.

We will then factorize such a system into a system form, representing the rules and the legal outcomes of the game, and the individual characteristics of the agents, namely their preferences and their mutual constraints on strategies by all agents resulting in (further) restrictions on feasible outcomes. In Section 3 we will comment on the relation between non-cooperative strategic foundation of axiomatic solutions of coalitional games and implementation of social choice rules as elaborated in Trockel (2002, 2003) [see also Howard (1992), Serrano (1997), Dagan and Serrano (1998), Bergin and Duggan (1999), Haake and Trockel (2010)]. In section 4 we will illustrate our concept of a socio-legal system in a very special application to two-person bargaining games, departing from Nash's simple demand game and arriving at an implementation of the Nash bargaining solution in Debreu-Hurwicz equilibrium.

2 Socio-legal systems

2.1 Social systems

The concept of an n-person non-cooperative game in normal or strategic form and the existence of an equilibrium for such a game, termed Nash equilibrium, have been extended by Debreu (1952) to a *social system* with a *social equilibrium*. The concept of a social system and its social equilibrium had been used as the basis for the general competitive equilibrium existence theorem for a competitive economy by Arrow and Debreu (1954). They denoted the specific social system used therein as *abstract economy*, also known as *generalized game*.

Consider a set $N := \{1,\dots,n\}$ of agents (or players) of the underlying society. Then a *social system* for N is according to Debreu (1952) defined by

$$\Gamma^S := (N, X_1,\dots,X_n,\mathscr{A}_1,\dots,\mathscr{A}_n,\pi_1,\dots,\pi_n),$$

where $X_i \neq \emptyset$, $i \in N$, are agents' *action sets* (or *strategy sets*). Denote by $X := \prod_{j\in N} X_j$ the set of *action profiles* and by $X_{-i} := \prod_{j\in N:\, j\neq i} X_j$, $i \in N$, the set of action profiles for players except player i. $\pi_i : X := \prod_{j\in N} X_j \to \Re$ are agents' *payoff functions*, and $\mathscr{A}_i : X_{-i} \Longrightarrow X_i$, $i \in N$, are agents' *constraints correspondences*.

For $x_{-i} \in X_{-i}$ the set $\mathscr{A}_i(x_{-i})$ is interpreted as the set of actions of player i that are declared permissible by the players in $N \setminus \{i\}$ via their choice of $x_{-i} \in X_{-i}$. In the special case, in which for all $i \in N$ and all $x_{-i} \in X_{-i}$ we have $\mathscr{A}_(x_{-i}) = X_i$ we are back to the class of non-cooperative games in strategic form. Here, the correspondences $\mathscr{A}_i, i \in N$ do not impose restrictions on actions and may therefore be skipped from the social system Γ^S.

It is clear, that (in fact under less restrictive assumptions) Debreu's social equilibrium existence proof implies Nash's (1950b) proof of the existence of Nash equilibria in non-cooperative games.

Koray and Yildiz (2018, fn. 19) in their section 7 "Deviation constrained mechanisms" refer to the fact that the equilibrium notion they are using for their *dc-mechanisms* is similar to Debreu's (1952) social equilibrium. They remark that the latter, however, is more restrictive than their concept, as it excludes a player's action x_i as arguments of his constraint correspondence $\mathscr{A}_i$.

The dc-mechanisms play a crucial role in their paper that is employing *right structures* (as introduced by Sertel, 2001) instead of mechanisms for implementation. Proposition 5 in Koray and Yildiz (2018) links their approach to mechanism theoretic implementation by using dc-mechanisms. The generalization of a social system by extending the domain X_{-i} of constraint correspondences to X for all $i \in N$ has already been used by Shafer and Sonnenschein (1975). They had replaced the payoff functions π_i, $i \in N$, defined as utilities of commodity bundles as *outcomes* (see Sections 2.2 and 3 therein) in a *generalized game form* by preference relations not being assumed to be transitive or asymmetric. So, their model of a social system comprises not only that of Debreu (1952) but also that of Koray and Yildiz (2018).

The range of applications of game theory can enormously and fruitfully be widened by using the idea of splitting a strategic game into two parts, namely players' individual characteristics, usually their preferences or utility functions, and the rules of the game given to the players by a designer or arbiter. Mathematically that means a factorization of π_i into an (player independent) *outcome function* $h : X \to A$ with some nonempty *set of outcomes* A and a utility function $u_i : A \to \Re$ for each player i, $i \in N$, such that $\pi_i := u_i \circ h$.

In applications, one usually takes the outcome set A as given, which clearly restricts the options for factorizing the π_i. But, in principle there may be a large set of candidates that can be used as an outcome set. The two extreme cases are $A = X$ and $A = \Re^n$ with $\pi_i = u_i \circ id_X$ and $\pi_i = \mathrm{proj}_i \circ h$, respectively.

2.2 Legal systems

While in a social system constraints are formalized for actions via actions with consequent restrictions on payoffs, in a *legal system* constraints on actions are determined by the rules. But they also imply constraints for the outcomes with consequences for the payoffs. Thus a legal restriction must be independent of a specific population of agents or players and expressible independent of any knowledge of players' individual characteristics.

This fact forces us to proceed partially on the level of game forms rather than of games. Hurwicz (2008) defined a *legal game* based on a *true* (or *natural*) *game* as follows. Given some non-cooperative game, termed the *true game*,

$$\Gamma := (X_1, \ldots, X_n, \pi_1, \ldots, \pi_n),$$

then a *legal game* is a game embedded in Γ defined as

$$\Gamma^L := \left(X_1^L, \ldots, X_n^L, \pi_1^L, \ldots, \pi_n^L\right),$$

with $\emptyset \neq X_i^L \subseteq X_i$ and $\pi_i^L := \pi_i|_{X_i^L}$. Consequently, actions in X_i^L are *legal actions* for player i, while those in $X_i \setminus X_i^L$ are termed *illegal actions*. As a result, the set of action profiles, in which every player takes a legal action, exhibits a product structure.

Hurwicz suggested then a strong version of successful enforcement of all players playing legally by requesting for each player $i \in N$ that for any $x \in X$ with $x_i \in X_i \setminus X_i^L$ there is an $x_i' \in X_i^L$ such that $\pi_i(x_i', x_{-i}) > \pi_i(x)$. That means, for each player taking any illegal action is dominated by some legal action, independent of the other players' choices. In this model the behavior of players' choices, either legal or illegal, is independent of the other players' choices.

2.3 Socio-legal systems

We will combine now both structures, namely social systems and legal systems in a new *socio-legal system*. In a first step we ignore players' characteristics, namely preferences or utility functions and constraints correspondences. Our focus is only on those parts of the system that are controlled by the designer. These are (a) the rules, formalized by an outcome function, and (b) the exact description of legal vs. illegal action profiles. Therefore, we have to express the concept of a game form $G = (N,X,h)$ in such a way that legality (of action profiles) is taken care of without the designer knowing players' characteristics.

Anticipating our notation for a bargaining game in Section 4, we denote now the concrete players' characteristics of a game, unknown to the designer by S and the set of all admissible S by $\mathscr{B}$.[1]

An essential task for the planner is to design a device by which players can distinguish legal from illegal action profiles. This is accomplished by a *legality correspondence* $\mathscr{L} : X \times \mathscr{B} \Longrightarrow X$, which is provided to the players. Knowing neither players' characteristics S nor the action profile x the players choose, the designer may "construct" $\mathscr{L}$ according to,e.g., legal principles.[2]

Formally, an action profile $x \in X$ is *legal (illegal)* for the population characterized by S, if $x \in \mathscr{L}(x,S)$ ($x \notin \mathscr{L}(x,S)$) holds.

Any possible population characterized by S is enabled to verify legality of an action profile, by evaluating $\mathscr{L}$ with S, which formally defines a correspondence $\mathscr{L}_S : X \Longrightarrow X$, $\mathscr{L}_S(x) = \mathscr{L}(x,S) \subseteq X$. Legality (for S) of an action profile x can now be rephrased as $x \in \mathscr{L}_S(x)$, so that any population S for which this x is a fixed point of $\mathscr{L}_S$ can approve x as a legal action profile. This shows that the S, despite being unknown to the designer, can be correctly and effectively used by him as an argument for the legality correspondence designed by him.

Also the outcome function h that maps action profiles to outcomes needs to allow a dependence on the characteristics S, though unknown to the designer, in the above sense. To this end, an outcome $h(x)$ from action profile $x \in X$ is described by a function $h(x) : \mathscr{B} \to X$. Consequently, we define as outcome set $A := \{g : \mathscr{B} \to X\}$.

Similar to the evaluation of legality, any specific population with characteristics S can evaluate an outcome $h_x = h(x,\cdot) : \mathscr{B} \to X$ via $\mathrm{ev}_S(h_x) = h_x(S) = h(x,S)$.

In order to enable the designer to make outcomes and the legality of action profiles dependent on the concrete S unknown to the designer, we split the outcome function h into two parts $h^L : X \to A$ and $h^{IL} : X \to A$ that become effective, depending on whether the action profile is legal or illegal, respectively. Again, write $h^L_x = h^L(x) : \mathscr{B} \to X \in A$ and $h^{IL}_x = h^{IL}(x) : \mathscr{B} \to X \in A$. The outcome function h is defined for all $x \in X$ and all $S \in \mathscr{B}$ via $h_x(S) := h^L_x(S)\,\mathbf{1}_{\mathscr{L}(x,S)}(x) + h^{IL}_x(S)\,\mathbf{1}_{X\setminus\mathscr{L}(x,S)}(x)$.

Recall that the players with characteristics S themselves can verify legality of x and are therefore able to execute the correct "part" of the outcome function, namely h^L or h^{IL}.

A *legal system form* is a tuple $G^L = (N,X,\mathscr{L},h)$. It is important to keep in mind that in contrast to players' constraints correspondences $\mathscr{A}_i$ the legality correspondence $\mathscr{L}$ is part of the designer's mechanism rather than of players' choices.

The verification of S has to be provided to the referee by respective populations of players via a jointly guaranteed and signed declaration. That is important for the enforcement of

[1] In Section 4 a population's aggregate characteristics will be described by (the shape of) a normalized bargaining game with utility possibility set denoted by S.

[2] In Section 4 we define a legality correspondence based on an equity rights principle.

the solution caused by the fact, that legality is made dependent on the specific S building the game although this is not known by the designer.

In the next step, we combine the legal system form with players' constraints correspondences $\mathscr{A}_i^S : X \Longrightarrow X_i$ and utility functions $u_i^S : X \to \Re$. A *socio-legal system* is a tuple[3,4]

$$\Gamma_S^{SL} := \left(N, X, \mathscr{L}, h, (\mathscr{A}_i^S)_{i\in N}, (u_i^S)_{i\in N}\right).$$

We define $\pi_i^S(x) := u_i^S(h(x)(S)) = u_i^S(h(x,S))$, $i \in N$ as players' payoffs.

While Debreu (1952) distinguishes between actions $x_i \in X_i$ and choices $x_i \in \mathscr{A}_i(x)$, Hurwicz (1994) distinguishes between legal and illegal actions. In this paper, we admit all $x \in X$ as possible arguments of the payoff functions $\pi_i = u_i^S \circ h$, even if they are no choice profiles or illegal. We think that the notion of a socio-legal system combining extensions of Debreu's social system and Hurwicz's legal games deserves for its equilibria a tribute to both.

Definition 1 A *Debreu-Hurwicz equilibrium* of a a socio-legal system Γ_S^{SL} is a profile of actions $x^* = (x_1^*, \ldots x_n^*) \in X$ satisfying:

(i) $x^* \in \prod_{i\in N} \mathscr{A}_i^S(x^*)$
(ii) $x^* \in \mathscr{L}(x^*, S)$
(iii) $x_i^* \in \underset{x_i : (x_i, x_{-i}^*) \in \mathscr{L}((x_i, x_{-i}^*), S) \cap \prod_{i\in N} \mathscr{A}_i^S(x_i, x_{-i}^*)}{\operatorname{argmax}} \pi_i^S(x_i, x_{-i}^*) \quad (i \in N).$

Conditions (i) and (ii) require the equilibrium action profile be feasible and legal, respectively. Condition (iii) ensures that for each player there is no profitable unilateral deviation to some other action, given that the resulting action profile remains feasible and legal. As we shall see in the next section, social systems, socio-legal systems and games underlying them usually may have different equilibria.

3 Implementation of solution-based social choice rules

In order to implement a solution (concept) for cooperative coalitional games in some equilibrium (concept) for non-cooperative strategic games one needs to overcome the following impediment: The literature in the previous century does not provide much insight into the exact relation between the Nash program and mechanism theory. Serrano (1997) observed: "The Nash program and the abstract theory of implementation are often regarded as unrelated research agendas". And Bergin and Duggan (1999) write: "Nevertheless, because the implementation-theoretic and traditional approaches both involve the construction of games or game forms whose equilibria have specific features, considerable confusion surrounds the relationship between them".

Solutions for cooperative games are different from social choice rules. While the former ones map cooperative games to sets of feasible payoff vectors for those games the latter ones map profiles of utility functions or preference relations defined on outcome sets to subsets of those outcome sets. Neither the domains nor the image spaces of these mappings coincide, except in very special cases.

[3] The superscript "SL" in Γ_S^{SL} reflects that we define a Socio-Legal system, while the subscript "S" refers to the players' characteristics.

[4] In our general framework we employ the extended constraints correspondences $\mathscr{A}_i : X \Longrightarrow X_i$ of Shafer and Sonnenschein (1976). Also our application in Section 4 is based on our general concept, although Debreu's $\mathscr{A}_i : X_{-i} \Longrightarrow X$ would be sufficient as the x_i are dummy variables there.

In fact there are instances in the literature where the term implementation is used in a framework of non-cooperative games where the mechanism theoretic aspect is not addressed at all. Sometimes games forms are simply confused with games. Starting from this situation Serrano (1997) attempted "to clarify the role of the mechanisms used in the Nash program for cooperative games". Dagan and Serrano (1998) extended Serrano's model, where characteristic forms are supplemented by physical allocations resulting from some production economy. In contrast to traditional terminology, they distinguished games in characteristic function form from games in coalitional form. The latter are defined by supplementing the former ones via adding outcome functions admitting it to define solutions as mappings to outcomes rather than to payoff vectors. These general outcomes extend Serrano's model, where characteristic forms are supplemented by physical allocations resulting from some production economy.

A very general model of non-cooperative foundation based on the explicit modeling of physical environments had been suggested by Bergin and Duggan (1999). There, cooperative solutions are alternatively defined as mappings resulting in outcomes rather than in payoff vectors. Having an outcome set available half of the before mention impediment has been removed.

An alternative approach not relying on a specific physical environment had been used by Trockel (2000) to implement the Nash bargaining solution in Nash equilibrium. Here the outcome set, needed for implementation, is derived endogenously from the data of the classes of games considered in the traditional non-cooperative foundation of an axiomatic cooperative solution. A general Embedding Principle for integrating the Nash Program of non-cooperative foundation of axiomatic cooperative solutions had been proven by Trockel (2002). The pretended impossibility of implementation of the Nash solution in Nash equilibrium [cf. Howard (1992), Serrano (1997), Dagan and Serrano (1998)] had been proven by Howard (1992) via demonstrating that the Nash social choice rule representing the Nash solution failed to be Maskin monotonic. However, his proof is based on a specific choice of the outcome set used for his Nash social choice rule. But there exist different ways for factorizing payoff functions of games into outcome functions and utility functions, with different options for outcome sets. Haake and Trockel (2010) contrast in a most simple framework Howards model with its non-Maskin monotonic Nash social choice rule with an equally simple alternative model in which the Nash social choice rule is in fact Maskin monotonic. That is consistent with the Nash implementation of the Nash solution via a Walrasian modification of Nash's (1953) demand game in Trockel (2000).

In our last section we will first represent Nash's demand game as a social system. It will turn out that the social equilibria of this social system coincide with the efficient Nash equilibria of the Nash demand game. We shall extend then this social system to a socio-legal system with a Walrasian legal restriction. For every relative weights $\alpha, 1-\alpha \in (0,1)$ of the two players we get a unique Debreu-Hurwicz equilibrium that coincides with the payoff vector of the $(\alpha, 1-\alpha)$-symmetric Nash solution of the Nash demand game. We will derive implementation in Debreu-Hurwicz equilibrium from this cooperative foundation.

4 Debreu-Hurwicz Implementation of the Nash Solution for two-person bargaining games

Two-person cooperative bargaining games have been introduced and intensively analyzed in Nash (1950a) and Nash (1953), respectively. A *two person bargaining game* is a pair (S,D) where $D \in S$ and S is a convex, compact subset of $\Re^2$ and there is $z \in S$ with $z >> D$.

Here S and D are interpreted, respectively, as the set of feasible payoff vectors for the two players and a status quo or threat point determining the payoffs in case of non-agreement. For simplicity, we assume that the Pareto boundary of S, denoted by ∂S, is differentiable and contains no line segments parallel to the two axes in $\Re^2$.

In the fifth section of Nash (1953) "The Formal Negotiation Model" consists of four stages in the first of which players determine a point D by choosing their threats. With the argument "The Stages two and four do not involve any decisions by players [...] the game consisting of the second move alone may be considered separately" Nash focused on this game of stage two together with the already determined D and called it *demand game*. This Nash demand game was supposed to provide by its Nash equilibria a non-cooperative foundation or support of the Nash bargaining solution. We do not restate Nash's original axioms characterizing this solution here, but just follow Nash in characterizing the unique Nash solution point of an arbitrary game (S,D) as the point $N(S,D) \in S$ maximizing the Nash product $g^N(z,D) := (z_1 - D_1)(z_2 - D_2)$ over points $z \in S$ with $z \geq D$.

In the chapter "Nash Program", Serrano (2008) writes: "In the first paper on the Nash program, Nash (1953) provides a non-cooperative approach to his axiomatically derived [Nash] solution. This is done by a simple demand game." For a skeptical comment concerning the success of this attempt by Nash see Trockel (2003, pp.156, 157) [cf. also Duman and Trockel (2016)].

The problematic multiplicity of Nash equilibria of Nash's demand game, preventing a non-cooperative support, and hence in particular an implementation in Nash equilibrium, can be avoided by a *Walrasian* modification of Nash's demand game. Based on this, Proposition 1 of Trockel (2000) provides a non-cooperative support, which is then, by use of the embedding principle in Trockel (2002, Section 4), extended to a Nash implementation of a Nash solution based Nash social choice rule.

We proceed in two steps extending Nash's demand game first to a social system and then to a socio-legal system. This will stepwise shrink the sets of equilibria and eliminate the inefficient equilibrium with the first modification, while the second modification only admits of one equilibrium, which payoff coincides with the payoff in the (symmetric) Nash solution.

Having established the non-cooperative support for the Nash solution, we shall then derive an implementation of the Nash solution in Debreu-Hurwicz equilibrium. It is of fundamental importance to see that an analogous implementation is possible for any asymmetric version of the Nash solution, which again supports the insight from the literature that the only property that distinguishes the symmetric from the asymmetric Nash solution is its symmetry or equitability. A very convincing way to this insight is Shapley's (1969) introduction of the λ-transfer value that coincides with the Nash solution on two-person bargaining games. Essentially without loss of generality we focus on bargaining games (S,D) with $D = 0$, denoted just by S, and $\max\{z_i | z \in S, z \geq D\} = 1$, $i = 1,2$. We denote the class of considered bargaining problems by $\mathscr{B}$. Recall that there is no conflict with the notation used in Section 2, as the set S aggregates players' characteristics that lead to a bargaining problem described by S.

The Nash demand game $\Gamma_S = (N, X, \pi_1^S, \pi_2^S)$ is defined by $X := [0,1]^2$, $\pi_i^S : X \to \Re$, $x \mapsto x_i \mathbf{1}_S(x)$, $i = 1,2$. As one readily verifies, its set of Nash equilibria is $\partial S \cup \{(1,1)\}$ with the set of equilibrium payoff vectors $\partial S \cup \{(0,0)\}$.

Describe the Pareto frontier ∂S as the graph of either function f_1^S or f_2^S with $f_1^S : [0,1] \to [0,1]$, $i = 1,2$ and $f_2^S = (f_1^S)^{-1}$.[5]

We may transform the game Γ_S into a social system $\Gamma_S^S = (N, X, \mathscr{A}_1^S, \mathscr{A}_2^S, \text{proj}_1, \text{proj}_2)$ with $\mathscr{A}_i^S(x) := [0, f_{3-i}^S(x_{3-i})]$, $i = 1,2$. Thus, the constraints correspondences reflect feasibility of an action profile x in S, while the projections as payoff functions reflect that a player's action is to choose an envisaged payoff in $[0,1]$.

The set of social equilibria coincides numerically with the set of social equilibrium payoff vectors and equals ∂S. The point $(1,1)$ is not a social equilibrium as it fails to be a member of $\mathscr{A}^S((1,1)) := \mathscr{A}_1^S((1,1)) \times \mathscr{A}_2^S((1,1))$. In contrast to the original demand game Γ_S, the social system Γ_S^S supports (and as we will see later) also implements the Pareto social choice rule. Like the game Γ_S, the social system Γ_S^S does not support or even implement the Nash solution given as a Nash social choice rule.

Starting with Γ_S^S, we now come to our next step of establishing a socio-legal system, mainly by adding a legality correspondence and an outcome function. The set of action profiles X as well as the constraint correspondences $\mathscr{A}_i^S$, $i = 1,2$ remain as in the social system. Our goal is to construct for each $S \in \mathscr{B}$ a socio-legal system
$\Gamma_S^{SL} := \left(N, X, \mathscr{L}, h, \mathscr{A}_1^S, \mathscr{A}_2^S, u_1^S, u_2^S\right)$ with payoff functions $\pi_i^S(x) = u_i^S(h(x)(S)) = u_i^S(h(x,S))$, $i = 1,2$ such that the payoff vector in (the unique) Debreu-Hurwicz equilibrium coincides with the Nash bargaining solution $(N(S)$. However, it is crucial to distinguish the Nash solution as a mapping on $\mathscr{B}$ from a social choice rule that represents the Nash solution in a social choice interpretation of a bargaining game. Whether such a social choice representation is Nash implementable strongly depends on the choice of an outcome set. For details see Howard (1992) and Haake and Trockel (2010). By "removing" the bargaining problem (players' characteristics), i.e., by moving to the socio-legal system form, we are able to provide an implementation result for the Nash solution in Debreu-Hurwicz equilibrium.

In order to define the remaining ingredients of Γ_S^{SL} we need to introduce some further quantities, motivated by the interpretation of S as a production economy (cf. Trockel (1996, 2000) for details). Any $x_i \in X_i = [0,1]$ generates a unique efficient point $s_i(x_i) \in \partial S$ such that $(s_i(x_i))_i = x_i$.[6] For every point $y \in \partial S$ we determine its value under the efficiency price system $p(y)$ that is normalized by $p(y) \cdot y = 1$. In particular, we get $p^i(x_i) = (p_1^i(x_i), p_2^i(x_i)) := p(s_i(x_i))$ with $p(s_i(x_i)) \cdot s_i(x_i) = 1$, $i = 1,2$.[7] The legal constraints later expressed by $\mathscr{L}$ are based on an *equity of rights principle*[8] that suggests equal benefits of $1/2$ for both agents when dividing a value $p(s_i(x_i)) \cdot s_i(x_i)$ among them.

Now, as mentioned above, legality of an action profile aims at relocating an equity of rights principle in the following way. Suppose, player i chooses x_i defining the efficient point $s^i(x_i)$ and the efficient price system $p^i(x_i)$. He is "allowed" to spend half of the value $p(s_i(x_i)) \cdot s^1(x_i) = 1$ on "his payoff"at the price $p_i^i(x_i))$, $i = 1,2$. Therefore, his available "demand" is $d_i^S(x_i) = \frac{1}{2}\frac{1}{p_i(s_i(x_i))}$, $i = 1,2$. That means that each x_i defines a quantity $d_i^S(x_i)$ that should not be exceeded due to the equity rights principle. In contrast, when we interpret x_i as player i's claim for a payoff, the claim need not be over-satisfied. Consequently, we define the legality correspondence $\mathscr{L} : X \times \mathscr{B} \Longrightarrow X$ through $(x,S) \mapsto \mathscr{L}(x,S) := [0, \min\{x_1, d_1^S(x_1)\}] \times [0, \min\{x_2, d_2^S(x_2)\}]$. Figure 1 illustrates the construction of $\mathscr{L}(x,S)$.

[5] f_1^S and f_2^S may be viewed as canonic parameterizations of the Pareto frontier. Mor precisely, for each $x_1, x_2 \in [0,1]$ we have $(x_1, f_1^S(x_1)), (f_2^S(x_2), x_2) \in \partial S$.

[6] Phrased differently, $s_1(x_1) = (x_1, f_1^S(x_1))$ and $s_2(x_2) = (f_2^S(x_2), x_2)$.

[7] For the sake of better readability we omit the superscript S at the efficient point $s_i(x_i)$ and the price system p, although both are of course depending on the bargaining problem.

[8] We briefly discuss this in our last section "Concluding Remarks".

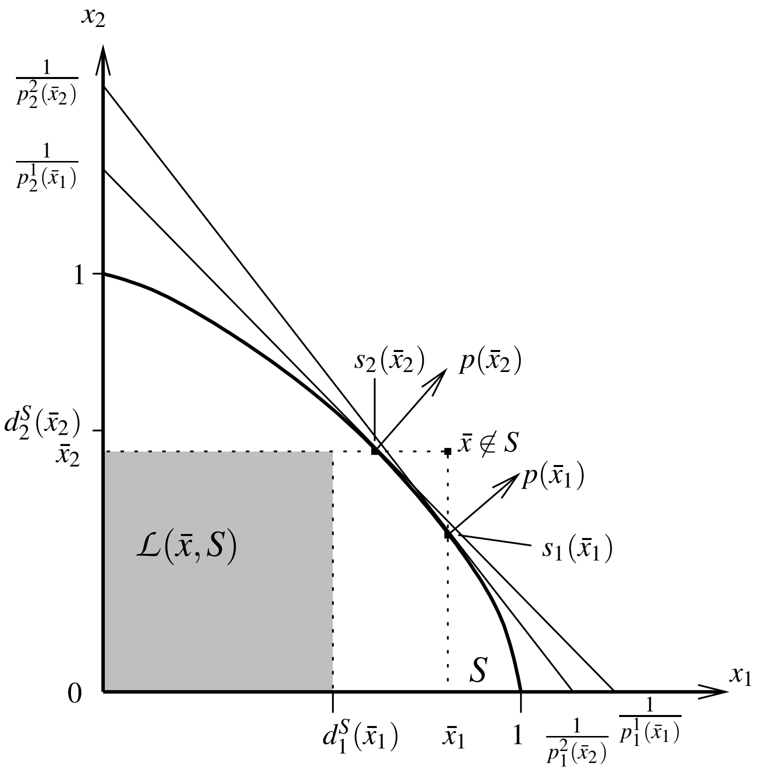

Fig. 1 Here $\min\{\bar{x}_1, d_1(\bar{x}_1)\} = d_1(\bar{x}_1) = \frac{1}{2p_1^1(\bar{x}_1)}$ and $\min\{\bar{x}_2, d_2(\bar{x}_2)\} = \bar{x}_2$. The action profile $\bar{x}$ is not legal, because $\bar{x} \notin \mathscr{L}(\bar{x}, S)$. The finally resulting $h(\bar{x})(S) = (d_1(\bar{x}_1), \bar{x}_2)$ is located in the interior of S. That shows that non-feasible (here in the sense of socially unaccepted) action profiles are also illegal.

We now define the outcome function $h : X \to A$ and proceed analogously to Section 2. Thus we choose the outcome set $A := X^{\mathscr{B}}$ and define the outcome function $h : X \to A$ by $h(x) : \mathscr{B} \to X$, $S \mapsto h(x)(S) := x\mathbf{1}_{\mathscr{L}(x,S)}(x)$. That means, the outcome $h(x)$ from an action profile x is the mapping that assigns x to each population of players represented by S, for which x is legal, and 0 (status quo) otherwise. For the utility functions u_i^S, $i = 1, 2$, we choose the projections $u_i^S = \text{proj}_i : X \to [0, 1]$. But, recall that according to h, only legal action profiles can lead to a nonzero payoff. Finally, the payoff functions π_i^S are thus given by $\pi_i^S(x) := x_i \mathbf{1}_{\mathscr{L}(x,S)}(x)$. This restriction of the payoff functions π_i^S results from the restriction imposed on the outcome function h by $\mathscr{L}$ as described in detail in Section 2.3.

Up to now, we have defined a socio-legal system Γ_S^{SL}. Th following two propositions establish the support of the Nash solution of S by the unique Debreu-Hurwicz equilibrium of Γ_S^{SL} (Proposition 1) and extend that result to an implementation of a Nash solution-based social choice rule in Debreu-Hurwicz equilibrium (Proposition 2).

The proofs make use of the following game used in Trockel (2000) to support the Nash solution in Nash equilibrium. Define the non-cooperative game $\Gamma_S^* := (X, \pi_1^{*S}, \pi_2^{*S})$ with payoff functions $\pi_i^{*S}(x) := \min\{x_i, d_i^S(x_i)\}$. The game $\Gamma_S^* := (X, \pi_1^{*S}, \pi_2^{*S})$ has a unique Nash equilibrium x^{*S} that coincides with the Nash solution point $N(S)$ of the bargaining game S (see Proposition 1 in Trockel (2000)). An extension of this support result had been extended to a Nash implementation in Trockel (2000, Section 6) using the "embedding principle" due to Trockel (2002, Section 4).

Proposition 1 *The Nash solution N on $\mathscr{B}$, defined by $\{N(S)\} = \mathrm{argmax}_{z \in S: z \geq 0} z_1 z_2$ can be supported by the unique Debreu-Hurwicz equilibrium x^{*S} of the socio-legal system Γ_S^{SL}, $S \in \mathscr{B}$, as defined above, i.e., $N(S) = \pi^S(x^{*S})(= x^{*S})$.*

Proof In order to form an equilibrium, x^{*S} has to satisfy (i) $x^{*S} \in \mathscr{L}(x^{*S}, S) \cap \mathscr{A}^S(x^{*S})$ with $\mathscr{A}^S(x^{*S}) = \mathscr{A}_1^S(x^{*S}) \times \mathscr{A}_2^S(x^{*S})$, (ii) x_i^{*S} maximizes $\pi_i^S = u_i^S(h(x,S))$ on S.

For $x^{*S} \notin S$ we have $x^{*S} \notin \mathscr{A}^S(x^{*S})$. For $x^{*S} \in S \setminus \partial S$ for either player there exists the possibility to improve by unilateral deviation that decreases the distance between x_i^{*S} and $d_i^S(x^{*S})$. The only remaining candidates for an equilibrium are the points in ∂S. Take an arbitrary point $x^{*S} \in \partial S \setminus N(S)$. Then w.l.o.g. $x_1^{*S} > N(S) > d_1^S(x_1^{*S})$. But then moving from x_1^{*S} to $N_1(S)$ would be an improvement for player 1. So, the only remaining candidate for a Debreu-Hurwicz equilibrium is $x^{*S} = N(S)$. As we have $x_i^{*S} = d_i^S(x_i^{*S}) = N_i(S)$, both players maximize at x^{*S} their payoffs over all feasible and legal action profiles.

Proposition 2 *The Nash solution on the set $\mathscr{B}$ of bargaining games can be (weakly) implemented in Debreu-Hurwicz equilibrium of socio-legal systems.*

The proof of Proposition 2 is based on Section 4.3 in Trockel (2003).

Proof ((Sketch)) As there is a bijective mapping from $\mathscr{B}$ to the set of socio-legal systems based on games in $\mathscr{B}$ as defined above, our Proposition 1 and Proposition 1 in Trockel (2000) imply the identity of the unique equilibria (Nash and Debreu-Hurwicz) in the games Γ_S and socio-legal system Γ_S^{SL} based on S as defined above. So, both equilibria support the same Nash solution of the respective $S \in \mathscr{B}$.

Using the *embedding principle* (Trockel, 2002, Section 4, Proposition) the support result can be extended to an implementation of the Nash bargaining solution on $\mathscr{B}$ in Nash equilibria of games Γ_S, $S \in \mathscr{B}$, hence in Debreu-Hurwicz equilibria of socio-legal systems Γ_S^{SL}, $S \in \mathscr{B}$.

Notice, however, that $N : \mathscr{B} \to X$, $S \mapsto N(S) \in S$ is a different mathematical object compared to a social choice rule $\mathscr{N}$ associating with each element of the set of utility (or preference) profiles on some outcome set A.

But, if we define A as the set of all singleton valued solutions on $\mathscr{B}$, i.e., $A := \{L : \mathscr{B} \to X, S \mapsto L(S) \in S\}$ we identify each element $S \in \mathscr{B}$ with a utility profile $u^S = (u_1^S, u_2^S)$ on A defined by $u_i^S(L) := L_i(S)$. In this case $N : \mathscr{B} \to X$, $u^S (\approx S) \mapsto \{N(u^S)\}$ is a social choice rule. But this rule cannot be Nash implemented. In contrast, the social choice rule $\mathscr{N} : \mathscr{B} \Longrightarrow A$, $S \mapsto [N_S] \subseteq A$ with $[N_S] := \{L \in A \,|\, L(S) = N(S)\}$ can be (weakly) implemented.

For all solutions $L \in [N_S] = \mathscr{N}(S)$ one has $u_i^S(L) = L_i(S) = N_i(S) = u_i^S(N)$, the generated payoff is always $N(S) = (N_1(S), N_2(S)) \in \mathscr{N}(S)$.

5 Concluding Remarks

Our main contribution in this article is the new concept of a socio-legal system that combines Debreu's (1952) social system with Hurwicz's (1994) distinction between natural games and imbedded legal games. We illustrate the effect of moving from a game to a social system and then further to a socio-legal system on their respective equilibria by the example of the Nash demand game. While the Nash demand game contains already implicitly the social constraints defining the shape of a game S (except their impact on the action profile $(1,1)$, the additional Walrasian legality correspondence is implicitly taken care of by Trockel's (2000) modification of Nash's demand game. Our wider concept of a socio-legal system follows Shafer and Sonnenschein (1975) by admitting in Debreu's constraint correspondences the determination of each player's socially accepted action set not only through all other players' actions, but through all players' actions, including the constrained player's one [see also Koray and Yildiz (2018)].

We generalize also Hurwicz's classification of legal and illegal actions in a given natural or true game by renunciation of Hurwicz's product structure. Instead, this brings us to distinguish between legal and illegal action profiles.

The crucial difference between social and socio-legal systems lies in the fact that constraints on action profiles generated by players' actions transforming games into social systems are supplemented by additional constraints on players' actions determined by the designer that extend these social systems to socio-legal systems.

In the printed version of his Nobel lecture Hurwicz (2008) considered a cascade of game forms on various layers where higher level agents determine by the outcomes of their games the rules for the games in lower level layers. There social constraints created by actions in higher level games create the legal constraints for the lower level games. Legal constraints on "guardians"' actions are generated through socially constrained actions of those "who guard the guardians".

In the second part of our paper we illustrated our new structure. Socio-legal systems are built in two steps from cooperative two-person bargaining games. By each of the two steps the set of respective equilibria shrinks; by the only inefficient equilibrium in the first step to a social systems and by all efficient equilibria but one in the second step to a socio-legal system. The remaining equilibrium supports the symmetric Nash bargaining solution. But, notice that any α and $1-\alpha$ in $(0,1)$ instead of $\frac{1}{2}$ and $\frac{1}{2}$ in the "budget distribution" of the Walrasian legality constraints would have uniquely supported and implemented a corresponding α-(a)symmetric Nash solution!

As the Nash demand game, disguised as social system, has all efficient points of S as social equilibria we needed an additional law of equity to get the unique support and implementation of the symmetric Nash equilibrium in Debreu-Hurwicz equilibrium of the Walrasian socio-legal system derived from S.

Our result confirms Shapley's (1969) famous derivation of the Shapley transfer value based on efficiency plus equity, which characterizes for two person games the symmetric Nash solution [cf. Trockel (1996)]. Future research may clarify via which legality correspondences used for bargaining games the various bargaining solutions may be supported and implemented in Debreu-Hurwicz equilibrium (cf. Moulin (1984), Haake (2009) regarding the Kalai-Smorodinsky solution).

As a final remark we want to stress the fact that our implementation as all others known to us are incomplete in the following sense. Despite their potential to provide agents with necessary information important for their action choices that the designer does not have they do not contain an explicit reliable and effective modeling of enforcement. That important feature would be part of the unfinished Hurwicz program of "genuine implementation" (cf. Hurwicz (1994, 2008). A potentially promising path within the context of this program based on blockchains as guardians is indicated in Chiu and Koeppl (2019).

Declarations

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