Dynamic Complexity in a Keynesian Macroeconomic Model

by

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1. Introduction

Neo-Keynesian macroeconomic models, as originally developed by BARRO/GROSSMAN [1976], BENASSY [1982], DRÈZE [1975], and MALINVAUD [1977], have been studied intensively in the late 1970s and early 1980s in an attempt to reconsider the Keynesian paradigm and to provide sound microeconomic foundations for phenomena such as underemployment and inflation. These models emphasize the possible emergence of different disequilibrium regimes according to the rationing of supply or demand in the different markets which depends on given wages and prices prevailing in the period under consideration. Such an analysis of allocations at non-Walrasian, *i.e.,* non market clearing prices, provides important insights into the possible nature of disequilibrium allocations.

Most contributions, however, consider static or steady state aspects of these models only, neither revealing sufficiently the temporary, sequential structure of the model nor providing an explicit dynamic analysis. The concentration of the analysis on static equilibrium aspects widely supports the supposition that this attempts to explain why prices are fixed *and* non-Walrasian. It is clear that such a static theory of equilibrium cannot explain the emergence of steady state non-Walrasian prices. Thus, the characterization of allocations given arbitrary prices provides insight only into the nature of possible configurations where all agents consider these as given in the short run, quite independently of how prices were determined in the first place. It follows therefore that this theory of allocations should be considered as the temporary, short run component of a general dynamic model for which an appropriate process to determine prices has to be specified. Hence, the apparent failure of the so called fix-price-approach to explain at the same time why prices are fixed *and* non-Walrasian is a premature judgment which ignores the necessity of embedding the temporary fixed price situation into a proper dynamic flex-price setting.

The usual discussion of Keynesian issues in the macroeconomic literature within purely static models is equally less convincing. Although traditional textbooks seem to suggest a dominating influence of fixed prices in Keynesian macroeconomic models, most Keynesians would agree that Keynes himself argued that prices and wages respond sluggishly to market signals and not that they are fixed or rigid. On the contrary, varying prices are, for example, a prerequisite for Keynes' own discussion of the effects of varying relative prices.[1]

The important consequence of the above considerations seems to be that the central issue to be discussed in a theory of unemployment, or more generally of disequilibrium, is not whether an assumption of price rigidity is justified or not, but rather whether the adjustment process of prices generates stable steady states with non-Walrasian prices. Thus rigidity can only be an outcome of a well specified adjustment process.

For the dynamic analysis it is necessary to specify appropriate adjustment rules. One price adjustment scenario could be that a non-participating outside agent (like a planner or auctioneer) chooses prices and wages depending on some adjustment criterion. If his information is primarily given by observing activities on markets the adjustment rule will be

[1] An elaborate discussion of Keynes' views on the flexibility of prices can be found in TOBIN [1993]: *"All Keynesian macroeconomics really requires is that product prices and money wages are not perfectly flexible, whatever may be the rationale for their behavior."* (TOBIN [1993], p. 56.

defined on observed market disequilibrium signals, *i.e.,* price changes would depend primarily on the temporary state of the economy. A second scenario is that of administered or regulated prices based on a particular government objective. Since the government also interacts with private economic agents on markets, it is clear that a government policy in general would be a rule which chooses prices and wages as well as taxes and government demand. A third scenario is that of private economic agents choosing prices and wages in each period. This clearly corresponds to the asymmetric situation where some agents have price setting power while others do not. The resulting rules must be the dynamic counterparts of strategic oligopoly theory where some agents choose some prices and some quantities simultaneously. Such choices will be influenced by agents' expectations, their objective functions, and their information about other agents' behavior. In such a context steady state non-Walrasian prices would describe a rigidity with a strategic best-response character. Under full information these may be non-voluntary.[2] WEINRICH [1994] stresses the information aspect and the role of risk aversion as a cause for temporary price inflexibility.[3] Whatever the price and wage setting rule or scenario is, the temporary allocation is determined at given prices and wages and all other relevant stock variables. Thus, given the price and wage setting rules and the definitional adjustments of all stock variables, the evolution of the economy is then described by the behavior of a dynamical system determined by the adjustment rules. Depending on the specification of the economic structure and the functional relationships, the system may exhibit convergence towards a fixed point or regular periodic orbits. In some cases even complex, chaotic motion may emerge. Thus the fixed non-Walrasian prices as known from static (dis-)equilibrium models will appear in the dynamic context exclusively as fixed points of the dynamical system.

Dynamic adjustment processes within the framework of temporary equilibrium models have been analyzed in the literature (cf. *e.g.,* BENASSY [1984], BÖHM [1978], HÉNIN/MICHEL [1982], PICARD [1983], ECKALBAR [1979, 1980], or HONKAPOHJA/ITO [1981, 1983]) and others. However, general analytical investigations of even the simplest dynamic models with rationing have encountered a seemingly insurmountable problem of tractability. Most macroeconomic models with rationing based on microeconomic principles of the form suggested by the theory of temporary equilibrium analysis cannot be written in the standard form of closed dynamical systems. One of the principle reasons for this impossibility is that the results of a complex system of interacting decisions and markets in each period has to be obtained before the change in the state variables like prices and wages can be computed. The dynamic process of price and wage determination requires as input a consistent list of rationing levels, demands, and trades, which is a solution to an extremely involved fixed-point problem. Such a fixed point may not be unique.

In the following, a simple macroeconomic model with rationing will be discussed which can be interpreted as a re-formulation of the Keynesian textbook model. The presence of rationing and the regime-dependent price-adjustment behavior imply a highly non-

[2] Cf. SCHMACHTENBERG [1987] for a discussion of an inconsistency in the Drèze disequilibrium concept when monopolistic competition prevails.

[3] For a dynamic embedding see BÖHM/NAEVE [1994].

linear dynamic structure of the model. Standard analytical tools alone cannot provide a sufficiently complete picture of the complexity of the dynamic model. A systematic numerical inquiry of the model shows that the dynamic behavior is characterized by complicated bifurcation sequences, co-existing period-n cycles, complex (chaotic)[4] attractors in phase space, and complicated basins of attraction.

2. The Model

The following presentation provides a brief description of the basic ingredients of the model. The behavioral assumptions and definitional concepts are described in greater detail in Böhm [1989, 1993] and Böhm/Lohmann/Lorenz [1994]. The present outline emphasizes the macroeconomic aspects of the model. A more detailed analysis of the microeconomic foundations of the consumption behavior within the framework of a stationary structure of overlapping generations of consumers is contained in Böhm [1989].

2.1. Behavioral Assumptions

The set of economic agents in each period consists of the government, young and old households, and firms. For simplicity, young and old households and firms are assumed to consist of a single agent in each sector. In each period $t = 1, 2, \ldots$, the firm produces a homogeneous output good with the help of labor as the single input factor. Profits of the firm are paid to the young household who offers labor when young and demands the produced good in both periods of his life. With fiat money as the only store of value, the young household transfers the difference between income and expenditures for consumption as savings to the next period. In the next period, this household acts as the old household and spends all his money balances for consumption purposes. At the end of this second period, the old household dies. The government purchases goods in the goods market and levies a proportional tax on the young household's income; budget deficits or surpluses of the government are financed via money creation or destruction.

At the beginning of each period, the commodity price, p_t, and the nominal wage rate, w_t, are given and remain unchanged for the length of the period. The nominal money stock, M_t, at the beginning of the period t is the result of the savings behavior of the household in the previous period. Possible disequilibria in period t affect the price and wage at the beginning of period $t + 1$. The savings behavior of the young household in period t determines the final nominal money stock which is transferred to period $t + 1$.

4 The term "chaos" is used here in an informal way as an additional descriptive term for irregualr, aperiodic behavior.

The Government

The government purchases goods in the market and levies an income tax on young consumers to finance these purchases. Let $g \geq 0$ denote the constant quantity of goods purchased by the government in each period. If Y_t denotes total household income in period t and $0 \leq tax \leq 1$ is the proportional tax rate on the young household's income, the budget deficit/surplus of the government is

$$\text{Deficit/Surplus} = p_t g - tax\, Y_t \tag{2.1}$$

The government finances deficits via money creation; in the case of a surplus the appropriate amount of money is taken out of circulation, *i.e.,*

$$\Delta M_{t+1} \equiv M_{t+1} - M_t = p_t g - tax\, Y_t \tag{2.2}$$

This money creation policy implies that the net asset position of the household sector changes if and only if the government budget is unbalanced. Other direct transfers or lump sum taxes are excluded in the model.

The Households

In each period, the household sector consists of old households and young households. The old households have an initial money balances, M_t, which is spent for the consumption of goods in period t. Each young household supplies a fixed amount of labor $L_{\max} > 0$ in each period. The net income, Y_t^{net}, in period t after payment of the income tax is

$$Y_t^{\text{net}} = (1 - tax)(W_t + \Pi_t), \tag{2.3}$$

with W_t as his labor income and Π_t as profits. The household can either spend his income for the consumption of goods in period t or save it in the form of money balances transferred to period $t+1$. Let $\theta^e_{t,t+1} \equiv p^e_{t,t+1}/p_t$ denote the expected rate of inflation of young households in period t and $c(\theta^e_{t,t+1})$ their propensity to consume out of current real net income.[5] Then the desired consumption, x_t^*, of both households in period t is

$$x_t^* = \frac{M_t}{p_t} + c(\theta^e_{t,t+1})\frac{Y_t^{\text{net}}}{p_t}, \quad 0 < c(\theta^e_{t,t+1}) < 1, \tag{2.4}$$

It follows that aggregate desired net savings in period t is described by

$$S_t^* = -M_t + \big(1 - c(\theta^e_{t,t+1})\big)Y_t^{\text{net}}. \tag{2.5}$$

[5] These features are easily derived from an overlapping generations structure of consumers with homothetic preferences. Cf. BÖHM [1989] for details.

The expected inflation rate for the next period is assumed to depend on the last τ actual inflation rates in the past.

$$\theta^e_{t,t+1} = \Psi(\theta^{(1)}_t, \theta^{(2)}_t, \ldots, \theta^{(\tau)}_t), \qquad \theta^{(k)}_t := \frac{p_{t-\tau+k}}{p_{t-\tau+k-1}}, \quad k = 1, \ldots, \tau. \tag{2.6}$$

The function $\Psi(\cdot)$: $\mathbb{R}^\tau_+ \to \mathbb{R}_+$ is assumed to be continuous with the properties

$$\Psi(\theta, \theta, \ldots, \theta) = \theta \ \forall \ \theta > 0 \quad \text{and} \quad \Psi \equiv 1 \ \text{if} \ \tau = 0.$$

The class of such expectations functions is clearly very large and contains a wide variety of point estimates. A particularly simple specification of $\Psi(\cdot)$ in the form of an unweighted average will be introduced in Section 3.

The Firm

The firm produces a single good, y, with the help of labor, z, as the single input factor. The production function

$$F\colon \mathbb{R}_+ \to \mathbb{R}_+; \quad y_t = F(z_t), \tag{2.7}$$

is assumed to be a C^2, strictly monotonically increasing, and strictly concave function which fulfills the Inada conditions. The optimal desired production plan of the firm is determined by maximizing the current period's profit $p_t F(z_t) - w_t z_t$. This yields the *notional* demand for labor in period t,

$$z^*_t := h\big(w_t/p_t\big) := \ \arg\max \{p_t F(z) - w_t z\}, \tag{2.8}$$

and the notional supply of goods

$$y^*_t := F\big(h\,(w_t/p_t)\big). \tag{2.9}$$

2.2. Temporary Feasible States

For the given constant values of $L_{\max}$, g and *tax* the economic situation in period t is fully described by the vector $(w_t,\ p_t,\ M_t,\ p^e_{t,t+1})$ (or $(w_t/p_t,\ M_t/p_t,\ p^e_{t,t+1}/p_t)$). This implies that all feasible activities in a period t and its consequences for the dynamic adjustment are a result of these variables. Therefore, such a vector will be called a *temporary state vector*, or a *temporary state*.[6] For an arbitrary temporary state the three quantities $x^*_t + g$,

[6] Although the vector $(w_t,\ p_t,\ M_t,\ p^e_{t,t+1})$ (or its homogeneous counterpart $(w_t/p_t, M_t/p_t,$ $p^e_{t,t+1}/p_t)$) defines completely the temporary situation, *i.e.,* the state of the economy in period t, these variables are *not* the state variables from the point of view of dynamical systems theory (see Section 2.5 below).

$F(L_{\max})$, and y_t^* are not equal in general. A description of aggregate temporary activity in the economy is called an allocation of the economy in period t which is defined by a pair (y_t, L_t) of output y_t and employment L_t. According to the so called minimum rule only the minimum of supply, demand, and capacity can be traded in the commodity market. Therefore, an allocation which can be traded and which is producible is called a *temporary feasible allocation*, or simply a *feasible allocation*. A state together with a feasible allocation will be called a *feasible state*.

Definition: A pair (y_t, L_t) is called a *temporary feasible allocation* for a given vector of state variables $(M_t, p_t, w_t, \theta^e_{t,t+1}) \gg 0$ and constant government parameters g and *tax* iff

$$y_t = \min\left\{x_t^* + g, F(L_{\max}), y_t^*\right\} \qquad \text{and} \qquad L_t = F^{-1}(y_t). \tag{2.10}$$

Keynes' so called "effective demand" y_t^D is given by the amount of real income or quantity of output which solves the equation

$$y_t^D = x_t^* + g = \frac{M_t}{p_t} + c(\theta^e_{t,t+1})(1 - tax)y_t^D + g. \tag{2.11}$$

It is straightforward to show that (y_t, L_t) is feasible if and only if

$$y_t = \min\{y_t^D, F(L_{\max}), y_t^*\} \quad \text{and} \quad L_t = F^{-1}(y_t).$$

Therefore, let

$$\mathcal{D}(M_t/p_t, \theta^e_{t,t+1}, g, tax) := \frac{M_t/p_t + g}{1 - c(\theta^e_{t,t+1})(1 - tax)} \tag{2.12}$$

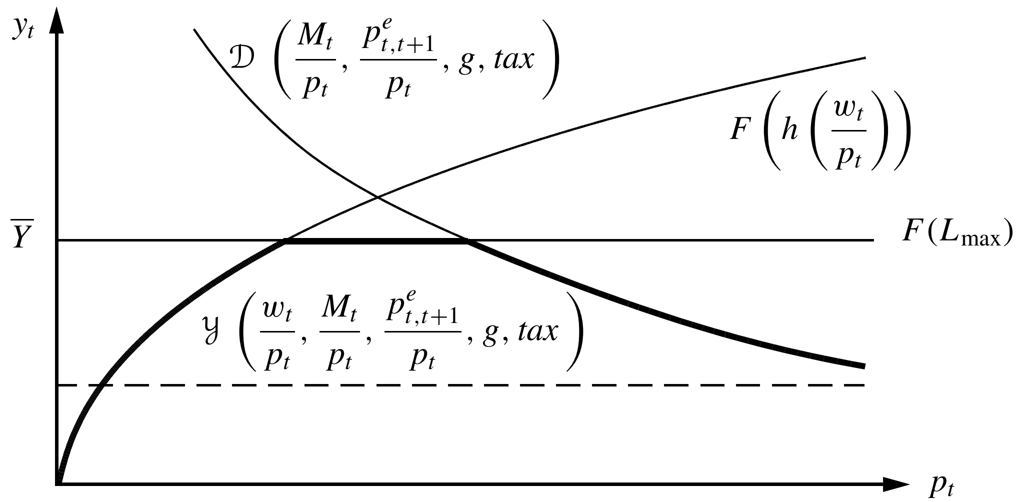
denote the effective demand function and define real money balances and real wages as $m_t := M_t/p_t$ and $\alpha_t := w_t/p_t$, respectively. Then, the unique feasible allocation (y_t, L_t) in period t is given by the two functions $\mathcal{Y}$ and $\mathcal{L}$ defined by (2.13) and (2.14) below.

$$y_t = \mathcal{Y}(\alpha_t, m_t, \theta^e_{t,t+1}, g, tax) := \min\left\{\mathcal{D}(m_t, \theta^e_{t,t+1}, g, tax), F(L_{\max}), y_t^*\right\} \tag{2.13}$$

$$L_t = \mathcal{L}(\alpha_t, m_t, \theta^e_{t,t+1}, g, tax) := F^{-1}\left(\mathcal{Y}(\alpha_t, m_t, \theta^e_{t,t+1}, g, tax)\right) \tag{2.14}$$

The function $y_t = \mathcal{Y}(\alpha_t, m_t, \theta^e_{t,t+1}, g, tax)$ is illustrated in Figure 1, holding the values for w_t and M_t fixed. The negatively sloped curve is the analog of the aggregate demand curve known from the textbook literature.[7] The horizontal curve describes the capacity

[7] The horizontal dashed line indicates Keynes' "effective demand" in the case of a zero money stock. The aggregate demand curve converges toward this line for $p_t \to \infty$.



The Set of Temporary Feasible Allocations (Heavy Solid Line)
For Given Values of M, g, tax , c, and w
Figure 1

output for a given labor supply. The positively sloped curve describes the notional (neo-classical) supply curve (or marginal cost curve) of the firm. The y values belonging to the set of temporary feasible states are denoted by the heavy solid line in the figure.

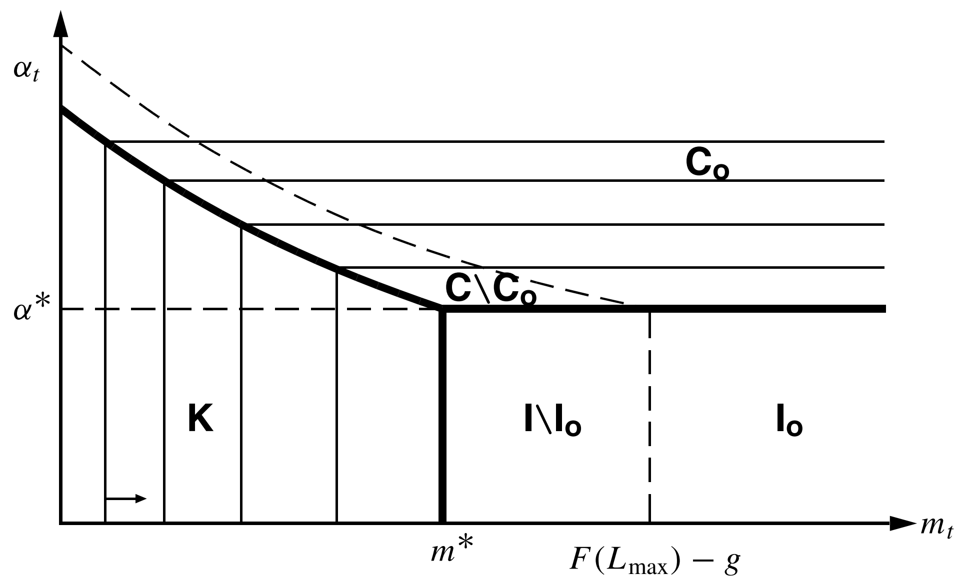
It has become useful to distinguish temporary feasible states of the economy according to the rationing nature of their associated feasible allocations. The model describes the economic interaction of agents on two different markets, namely the commodity market and the labor market. In each of the two markets either demand is rationed, or supply is rationed, or no rationing takes place at all. If these three rationing situations are mutually exclusive, it follows that nine different rationing combinations or regimes are possible.

- *Walrasian Equilibrium.* Neither the firm nor the household or the government is rationed in one of the two markets. Agents realize their notional supplies and demands in both markets (**W**).
- *Keynesian Unemployment.* The household is rationed in the labor market; the firm is rationed in the goods market (**K**).
- *Classical Unemployment.* The household is rationed in both the labor market and the goods market (**C**).
- *Repressed Inflation.* The household is rationed in the goods market; the firm is rationed in the labor market (**I**).
- *Underconsumption.* The firm is rationed in both the labor and the goods market (**U**).
- *Boundary Cases.* The four boundaries between the regimes **K**, **C**, **I**, and **U**.

The basic underlying principle of this classification is that rationing occurs on one side of each market only, *i.e.,* either on the demand or on the supply side. Due to the specific definition of feasibility which imposes $y_t = F(L_t)$ rationing of the producer on both markets simultaneously is a degenerate case. Thus, all underconsumption states are essentially degenerate and **U**$=$ **K** $\cap$ **U** $=$ **U** $\cap$ **I**. This degeneracy has been known and recognized in the literature for a long time.[8] It is also well known that it depends on the specific definition of feasibility and that it disappears if output and sales differ, *e.g.,* if inefficient production plans or inventory holding of the firm are introduced.[9]

The fact that firms cannot be rationed in both markets at the same time implies that the underconsumption regime cannot be observed in the present model. States $(w_t, p_t, M_t, p^e_{t,t+1})$ which would define an underconsumption regime in an isolated consideration of the demand and supply sides of both markets, actually belong to either the Keynesian or the Inflationary regime when the activities of the firm on both markets are considered simultaneously. Geometrically speaking, the regime of underconsumption shrinks to the boundary between the Inflationary and Keynesian regimes. The three possible regimes $\mathbf{K} \cap \mathbf{U}$, $\mathbf{U}$, and $\mathbf{U} \cap \mathbf{I}$ shrink to the set $\mathbf{K} \cap \mathbf{I}$. Summarizing, for a given vector $(p_t, w_t, M_t, p^e_{t,t+1}) \gg 0$ and $g \geq 0$ and $0 \leq tax \leq 1$, there exists a unique positive feasible allocation (y_t, L_t) of output and employment levels; the temporary state is classified as Classical, Keynesian, Inflationary, or one of the four boundary cases (including the Walrasian equilibrium).

As four temporary state variables define the different regimes they can be represented in different ways in a planar diagram. One of these representations is contained in Figure 2 with $\alpha_t = w_t/p_t$ and $m_t = M_t/p_t$ as coordinates for given $\theta^e_{t,t+1}$. The coordinates α^* and m^* denote the Walrasian equilibrium. The Classical regime, **C**, is characterized by high real wages such that firms realize their notional labor demand; the inflationary regime, **I**, is characterized by a high real money stock and low real wages such that the notional goods demand and the notional labor supply is high; and the Keynesian regime, **K**, is characterized by low real money balances.



Temporary Feasible States in (α_t, m_t) – Space
Figure 2

The boundary between the Classical and the Inflationary regime is a horizontal straight line such that the real wage rate is constant along the boundary. This follows from the

[8] Cf. Böhm [1980], Malinvaud [1977], or Muellbauer/Portes [1978].

[9] Cf., for example, Böhm [1978, 1989], Muellbauer/Portes [1978], and Neary/Stiglitz [1983].

fact that on the boundary there is excess demand for goods but that the labor market is in equilibrium, *i.e.,* $L_{\max} = h(\alpha_t)$. The monotonicity of h implies that the equality holds for a unique α, *i.e.,* $\alpha^* = h^{-1}(L_{max})$. The boundary between the Inflationary and Keynesian regime is a vertical straight line at $m = m^*$. This follows from the fact that on the boundary the maximal possible production, $F(L_{\max})$, equals real aggregate demand, y^D. The condition $F(L_{\max}) = y^D = (M_t/p_t + g)/\bigl(1 - c(1 - tax)\bigr)$ yields a unique value of m for each $\theta^e_{t,t+1}$ which again is identical with m^*. The non-constancy of α_t on the boundary stems from the fact that the $L_{\max} = F^{-1}(y^D)$ is independent of α. The boundary between the Keynesian and Classical regime is determined by the condition that the notional commodity supply of the firm equals aggregate demand, y^d, *i.e.,* $F\bigl(h(\alpha_t)\bigr) = (m_t + g)/\bigl(1 - c(1 - tax)\bigr)$. Differentiation yields $d\alpha/dm < 0$ and $d\alpha^2/dm^2 > 0$.

In Figure 2, the Classical and Inflationary regimes are both partitioned into the sets $\mathbf{I_o}$ and $\mathbf{I}\backslash\mathbf{I_o}$, and $\mathbf{C_o}$ and $\mathbf{C}\backslash\mathbf{C_o}$, respectively. The sets $\mathbf{I_o}$ and $\mathbf{C_o}$ represent those states in the Inflationary and Classical regimes, respectively, where output is less than autonomous demand $m + g$. Therefore, the boundary between the set $\mathbf{I_o}$ and the remaining part of $\mathbf{I}$ is defined by the condition $F(L_{\max}) = m_t + g$, whereas $F\bigl(f(\alpha)\bigr) = m_t + g$ defines the boundary of $\mathbf{C_o}$.

Figure 2 also contains a few level sets of the function L, the so called iso-employment curves in the Classical and Keynesian regime. The horizontal lines in the Classical regime and the vertical lines in the Keynesian regime indicate lower employment levels the further the lines are located away from the Walrasian equilibrium. The employment level in the Inflationary regime is equal to the constant labor supply level $L_{\max}$.

The fact that the type of disequilibrium is uniquely associated with the temporary state in any period t implies that the dynamic analysis of the temporary state variables provides the appropriate insight into possible regime switching over time.

2.3. The Dynamics of Money Balances (Government Deficit)

The young household transfers that part of net income which is not consumed in period t to the next period $t+1$ in the form of money. Let x_t denote actual real consumption of the young household in t. Thus, gross savings, *i.e.,* the money stock of the young household at the beginning of the next period, is

$$M_{t+1} = (1 - tax)p_t y_t - p_t x_t, \tag{2.15}$$

with $y_t = \mathcal{Y}(\alpha_t, m_t, \theta^e_{t,t+1})$ as defined above.

The actual consumption of the young and old households and of the government depends on the specific rationing scheme. If demand rationing occurs, it is assumed that the young household is rationed first. When the young household is rationed to zero and demand is still larger than the feasible y_t, then the old household is rationed. The government is rationed only when the old household has already been rationed to zero. The assumed rationing scheme implies that the actual consumption of the young household is

$$x_t = \max\,\{0, y_t - g - m_t\}\,. \tag{2.16}$$

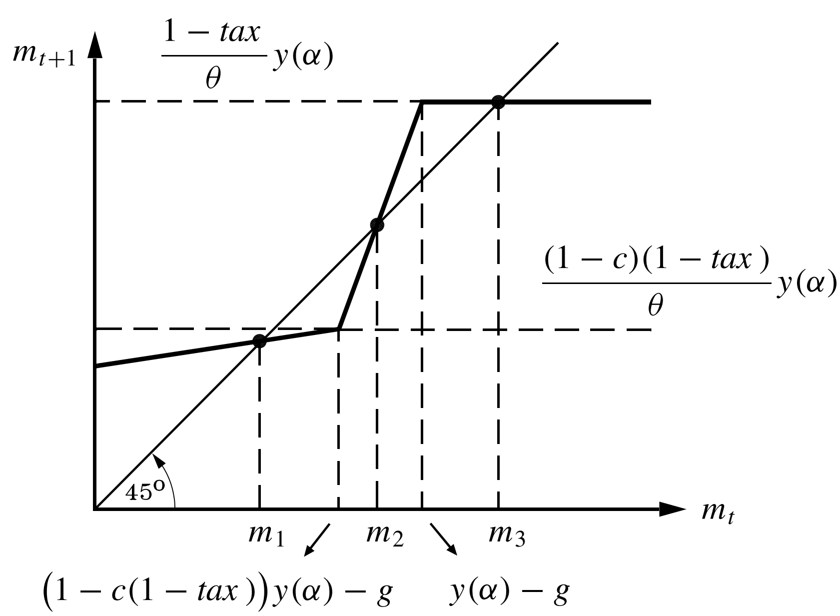
Substitution in (2.15) and considering real balances yields

$$m_{t+1} = \frac{1}{\theta_t}\left[\min\,\{y_t, g + m_t\} - tax\; y_t\right], \tag{2.19}$$

with $\theta_t := p_{t,t+1}/p_t$. (2.19) reveals a piecewise linear structure which plays an important role in the dynamics of the model. Let $y(\alpha) = \min\left\{F\big(h(\alpha)\big), F(L_{\max})\right\}$. Then it is straightforward to exhibit that (2.19) possesses a distinct partition into three linear sections for a given vector (α, θ).

$$m_{t+1} = \frac{1}{\theta_t}\begin{cases} \dfrac{(1-c)(1-tax\,)}{1-c(1-tax\,)}(m_t + g) & \text{iff} \quad g + m_t \leq \\ & \qquad \big(1 - c(1-tax\,)\big)y(\alpha), \\ m_t + g - tax\; y(\alpha) & \text{iff} \quad y(\alpha) \geq g + m_t > \\ & \qquad \big(1 - c(1-tax\,)\big)y(\alpha), \\ (1-tax\,)y(\alpha) & \text{iff} \quad y(\alpha) < g + m_t. \end{cases} \tag{2.20}$$

Figure 3 illustrates equation (2.20) for $\alpha < \alpha^*$ and $\theta_t < 1$. The piecewise-linear, one-dimensional system (2.19) has three fixed points m_1, m_2, and m_3. Obviously, in this case m_2 is unstable while m_1 and m_3 are stable. Thus, if no other state variable changed, initial real money balances would converge towards m_1 if $m_{t_0} < m_2$ or to m_3 if $m_{t_0} > m_2$.



The Dynamics of the Money Balances for $\alpha < \alpha*$ and $\theta < 1$

Figure 3

2.4. Price and Wage Dynamics

A given vector $(w_t, p_t, M_t, p^e_{t,t+1})$ of the temporary state variables in t together with the government parameters g and tax defines the location of the economy in one of the different regimes. In all but the Walrasian state, rationing takes place in at least one of the two markets. The fact that supply differs from demand on the labor market and/or the goods market implies that wages and/or prices tend to change in such disequilibrium situations. According to the renowned 'law of supply and demand' it will be assumed in the following that

$$w_{t+1} \begin{cases} > w_t & \text{iff labor demand is rationed in } t, \\ < w_t & \text{iff labor supply is rationed in } t, \end{cases} \tag{2.21}$$

and

$$p_{t+1} \begin{cases} > p_t & \text{iff goods demand is rationed in } t, \\ < p_t & \text{iff goods supply is rationed in } t. \end{cases} \tag{2.22}$$

In order to allow for several actual implementations of this principle a general formulation of the law is assumed in the following.

Consider a disequilibrium signal for each of the two markets given by a real number $s^c \in [-1, 1]$ or $s^\ell \in [-1, 1]$ which indicates the sign and size of actual rationing in the commodity and the labor market, respectively. The dependence of s^c and s^ℓ on the state variables w_t, p_t, M_t, and $p^e_{t,t+1}$, or, due to the homogeneity properties, on α_t, m_t, and $\theta^e_{t,t+1}$ is described by two functions σ^c and σ^ℓ:

$$\begin{aligned} \sigma^c &: \mathbb{R}^3_{++} \to [-1, +1]: \quad s^c_t = \sigma^c(\alpha_t, m_t, \theta^e_{t,t+1}) \\ \sigma^\ell &: \mathbb{R}^3_{++} \to [-1, +1]: \quad s^\ell_t = \sigma^\ell(\alpha_t, m_t, \theta^e_{t,t+1}). \end{aligned} \tag{2.23}$$

The signs of σ^c and σ^ℓ in the different regimes are described in Böhm [1989]. For the boundary case between the Keynesian and Inflationary regime (except the Walrasian equilibrium), *i.e.,* for $(\alpha_t, m_t, \theta^e_{t,t+1}) \in (\mathbf{K} \cap \mathbf{I}) \setminus \mathbf{W}$, it is assumed that $\sigma^\ell(\alpha_t, m_t, \theta^e_{t,t+1}) > \sigma^c(\alpha_t, m_t, \theta^e_{t,t+1})$. The functions σ^c and σ^ℓ are assumed to be continuous except on the boundary $\mathbf{K} \cap \mathbf{I}$.

On the basis of the disequilibrium signals s^c and s^ℓ a price adjustment function $\mathcal{P}$ and a wage adjustment function $\mathcal{W}$ determine the actual price and wage changes. The functions $\mathcal{P}$ and $\mathcal{W}$

$$\mathcal{P}: [-1, 1] \to (-1, +\infty), \qquad \mathcal{P}(s^c_t) \stackrel{!}{=} \frac{p_{t+1}}{p_t} - 1 = \hat{p} \tag{2.26}$$

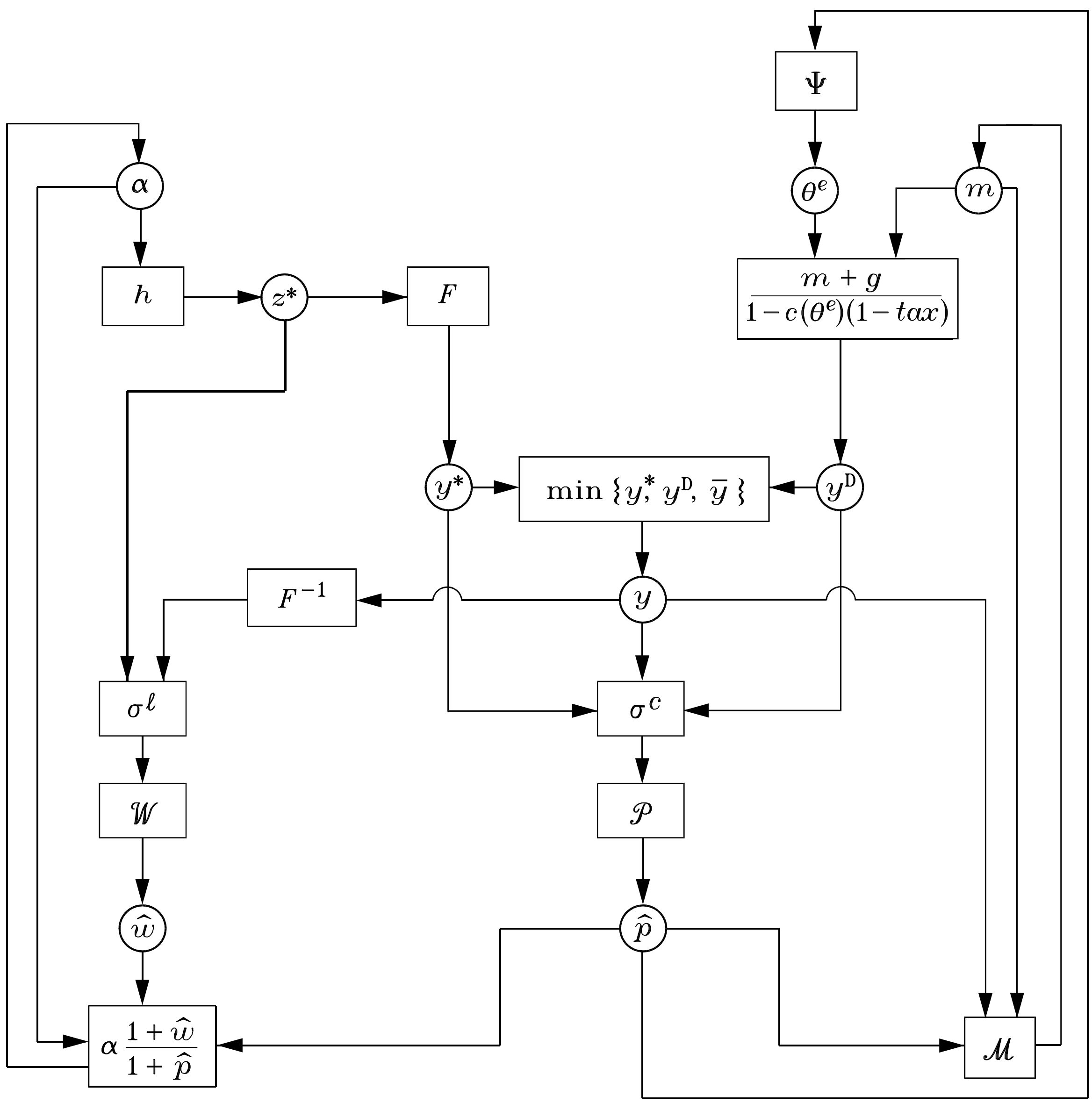
and

$$\mathcal{W}: [-1, 1] \to (-1, +\infty), \qquad \mathcal{W}(s^\ell_t) \stackrel{!}{=} \frac{w_{t+1}}{w_t} - 1 = \hat{w} \tag{2.27}$$

determine the actual growth rates. The functions $\mathcal{P}$ and $\mathcal{W}$ are assumed to be continuous and monotonically increasing with $\mathcal{P}(0) = 0$ and $\mathcal{W}(0) = 0$.

2.5. The Complete Dynamical System

The model described in the previous sections is a dynamic feedback system, *i.e.,* a system whose output in any period t is used as its input for period $t+1$. The feedback structure of the model is illustrated in Figure 7. Boxes represent functions (with possibly multiple inflows and a single outflow); circles indicate the definition of variables which serve as arguments in functions.



The Feedback Structure of the Complete Model
Figure 7

The dynamic behavior of the economy with this feedback structure is defined by the system of equations (2.28):

$$\alpha_{t+1} = \mathcal{A}(\alpha_t, s_t^\ell, s_t^c) := \alpha_t \frac{1 + \mathcal{W}(s_t^\ell)}{1 + \mathcal{P}(s_t^c)}, \tag{2.28.i}$$

$$m_{t+1} = \mathcal{M}(y_t, m_t, s_t^c, g, tax) := \frac{\min\{y_t, m_t + g\} - tax\, y_t}{1 + \mathcal{P}(s_t^c)}, \tag{2.28.ii}$$

$$\theta^e_{t,t+1} = \Psi(\theta_t^{(1)}, \theta_t^{(2)}, \dots, \theta_t^{(\tau)}), \tag{2.28.iii}$$

subject to the conditions in (2.29):

$$y_t = \mathcal{Y}(\alpha_t, m_t, \theta^e_{t,t+1}, g, tax), \tag{2.29.i}$$

$$s_t^j = \sigma^j(\alpha_t, m_t, \theta^e_{t,t+1}), \quad j = c, \ell. \tag{2.29.ii}$$

They generate a unique sequence (or path) $\{(\alpha_t, m_t, \theta^e_{t,t+1})\}_{t_0}^T$ of temporary states.

Equations (2.28) and (2.29) reveal the two-step structure of the dynamic process. The evolution of the system via the functions (2.28.i-iii) can be defined only after the feasible allocation (y_t, L_t) and their disequilibrium signals (s_t^c, s_t^ℓ) have been determined, which in itself depend on price expectations $p^e_{t,t+1}$. On the other hand, given a temporary feasible state $(\alpha_t, m_t, \theta^e_{t,t+1})$, the expectation formation process Ψ requires that the past τ inflation rates are used in order to determine the expected inflation rate in next period's temporary state. Therefore, the proper mathematical description of the state of the economy in period t is the list $(\alpha_t, m_t, \theta_t^{(1)}, \dots \theta_t^{(\tau)})$, a vector in $\mathbb{R}^2_+ \times \mathbb{R}^\tau_+$. Hence, the state space of the dynamical system is $\mathbb{R}^{2+\tau}_+$.

Let $\boldsymbol{\theta}_t = (\theta_t^{(1)}, \theta_t^{(2)}, \dots, \theta_t^{(\tau)})$ denote the vector of the τ past inflation rates relevant in period t. In order to describe the time map generating $\boldsymbol{\theta}_{t+1}$ using equations (2.28) and (2.29) define the projection of $\boldsymbol{\theta} \in \mathbb{R}^\tau$ onto its last $(\tau - 1)$ coordinates as

$$\Pi_{-1}(\boldsymbol{\theta}_t) = \Pi_{-1}(\theta_t^{(1)}, \theta_t^{(2)}, \dots, \theta_t^{(\tau)}) := (\theta_t^{(2)}, \theta_t^{(3)}, \dots, \theta_t^{(\tau)}) \tag{2.30}$$

Then, the price adjustment process $\mathcal{P}$ together with the shift defined by Π_{-1} yields the new vector of inflation rates $(\theta_{t+1}^{(1)}, \dots, \theta_{t+1}^{(\tau)}) = \left(\theta_{t+1}^{(1)}, \Pi_{-1}(\boldsymbol{\theta}_t)\right)$. Thus, the function Θ: $\mathbb{R}^{2+\tau}_+ \to \mathbb{R}^\tau_+$ given by the two component functions $\Theta := (1 + \mathcal{P}, \Pi_{-1})$ defines the dynamics of the vector of inflation rates. Therefore, the list $\mathcal{F} := (\mathcal{A}, \mathcal{M}, \Theta)$ defines the proper dynamical system. Figure 8 provides an illustration of $\mathcal{F}$ indicating the commutative relationships of all functions involved.

The dimension of the system depends on the number τ, *i.e.,* the length of the memory in the expectations formations process. For $\tau = 0$, *i.e.,* the case of no adaptation of $\theta^e_{t,t+1}$ to past inflation rates, the system is two-dimensional because there is no intertemporal link between expected and actual inflation rates.

Although the emphasis of the numerical experiments in Sections 3-5 is put on the detection of cyclic behavior of the system $\mathcal{F}$, it is useful to introduce the following notations and concepts.

$$
\begin{tikzcd}
(\alpha_t, m_t, \boldsymbol{\theta}_t) \arrow[r, "\mathcal{F}"] \arrow[d, "\Psi"] & (\alpha_{t+1}, m_{t+1}, \boldsymbol{\theta}_{t+1}) \arrow[d, "\Psi"] \\
\big((\alpha_t, m_t, \theta^e_{t,t+1}), \boldsymbol{\theta}_t\big) \arrow[d, "\mathcal{Y}, \mathcal{L}, \sigma"] & \big((\alpha_{t+1}, m_{t+1}, \theta^e_{t+1,t+2}), \boldsymbol{\theta}_{t+1}\big) \\
\big(\alpha_t, m_t, (y_t, L_t, s^c_t, s^\ell_t), \boldsymbol{\theta}_t\big) \arrow[uur, "\mathcal{A}, \mathcal{M}, \Theta"] &
\end{tikzcd}
$$

The Dynamical System $\mathcal{F}$
Figure 8

> **Definition**: A sequence $\{(\alpha_t, m_t, \theta^e_{t,t+1})\}_{t=t_0}^{\infty}$ is called a *quasi-stationary path* if for all pairs t, t' one has: $\alpha_t = \alpha_{t'}$, $m_t = m_{t'}$, and $\theta^e_{t,t+1} = \theta^e_{t',t'+1}$.

Therefore a quasi-stationary path is characterized by constant allocations.

> **Definition**: A sequence $\{(\alpha_t, m_t, \theta^e_{t,t+1})\}_{t=t_0}^{\infty}$ is called a *stationary path* if it is quasi-stationary and $\theta^e_{t,t+1} = 1 \; \forall \, t$.

Let $\hat{M}_t \equiv M_{t+1}/M_t$. Then the quasi-stationarity implies that $\hat{w}_t + 1 = \hat{w} + 1 = \hat{p}_t + 1 = \theta = \hat{M}_t$. Furthermore, for $\theta \gtrless 1$, a constant m_t implies that nominal balances M_t shrink or expand with the same rate. As money is created or destroyed only when the government deficit is positive or negative, respectively, it follows that $\theta \gtreqless 1$ only if $g \gtreqless tax\, y_t$. A quasi-stationary path can never be located in the Classical region because this region is characterized by $\hat{p} > 0 > \hat{w}$. Hence the real wage rate cannot be stationary. Finally, quasi-stationary states are characterized by perfect foresight, *i.e.*, $\theta^e_{t,t+1} = \theta_{t+1}$. Since a constant expected inflation rate θ^e implies a constant actual inflation rate θ, both must coincide due to the property of the expectation function.

Since every state $(\alpha_t, m_t, \theta^e_{t,t+1})$ is generated by the dynamic system $\mathcal{F}$, the following lemma is immediately obvious.

> **Lemma**: A vector (α, m, θ^e) is a quasi-stationary path if and only if it defines a fixed point of the dynamical system $\mathcal{F}$.

A few more or less immediate observations yield a further characterization of quasi-stationary states. The magnitude of θ determines the location of the quasi-stationary state in the regimes **I**, **W**, or **K**:

$$\begin{array}{lll} \theta > 1 & \text{iff} & (\alpha, m, \theta) \in \mathbf{I} \\ \theta = 1 & \text{iff} & (\alpha, m, \theta) \in \mathbf{W} \\ \theta < 1 & \text{iff} & (\alpha, m, \theta) \in \mathbf{K} \end{array}$$

As $\theta \gtreqless 1$ implies $\hat{M} \gtreqless 1$ in a quasi-stationary state, the following theorem is straightforward.

> **Theorem**: Assume that the assumptions about households and the firm hold true. Then, quasi-stationary states (α, m, θ) satisfy
>
> $$\begin{array}{lll} (\alpha, m, \theta) \in \mathbf{K} & \text{iff} & g < tax\, F(L_{\max}), \\ (\alpha, m, \theta) \in \mathbf{W} & \text{iff} & g = tax\, F(L_{\max}), \\ (\alpha, m, \theta) \in \mathbf{I} & \text{iff} & g > tax\, F(L_{\max}). \end{array}$$

This shows that the disequilibrium type of the stationary state is uniquely determined by the government parameters and technology. In particular, none of the behavioral assumptions for the consumption sector or the adjustment features of the markets have an influence.

3. The Structural Instability of the Dynamical System

Due to its piece-wise definition and the necessity of calculating feasible allocations and the disequilibrium signals, the dynamical system $\mathcal{F}$ is a functionally complicated, nonlinear dynamical system which cannot be studied in a complete fashion with the help of standard analytical tools anymore. In order to get an insight into its dynamic behavior, the system is studied in the following section with the help of numerical experiments. For most of the numerical simulations studied below a standard set of parameters for given specifications of the behavioral functions is used. An outline of the anticipated future extensions and systematic investigation of parameter dependencies and the influence of different functional forms is contained in the final section.

3.1. A Specification of the Functional Forms and the Standard Parameter Set

In order to perform the following numerical experiments it is necessary to specify functional forms for the behavioral assumptions in a more detailed manner.

The young household is assumed to have an intertemporal CES utility function

$$u(x_t, x_{t+1}) = \begin{cases} \dfrac{1}{\rho}\left(x_t^{\rho} + \delta x_{t+1}^{\rho}\right) & \text{if } \rho \neq 0 \\ \ln x_t + \delta \ln x_{t+1} & \text{if } \rho = 0, \end{cases} \tag{3.1}$$

with $\delta > 0$ as the time discount factor and $-\infty < \rho < 1$ as the parameter of substitution. It follows that the notional demand of the young household is

$$x_t^* = \begin{cases} \dfrac{1}{1 + \delta^{1/(1-\rho)} (\theta_{t,t+1}^e)^{\rho/(\rho-1)}} \, y_t^{\text{net}} & \text{if } \rho \neq 0, \\ \dfrac{1}{1+\delta} \, y_t^{\text{net}} & \text{if } \rho = 0. \end{cases} \tag{3.2}$$

The factor premultiplying real net income y_t^{net} in formula (3.2) is the marginal propensity to consume, $c(\theta_{t,t+1}^e)$. Note that in case of $\rho = 0$, *i.e.,* the standard Cobb-Douglas case, the marginal propensity to consume is independent of the expected inflation rate.

Planned savings of the young household follows as

$$S_t^* = \big(1 - c(\theta_{t,t+1}^e)\big) Y_t^{\text{net}}. \tag{3.3}$$

The planned consumption of the old household in period t is, according to the assumptions made in Section 2, equal to the initial real money balances at the beginning of t.

According to (2.6) the expected inflation rate, $\theta_{t,t+1}^e$, depends on the last τ actual inflation rates. For simplicity and as a first approach, an unweighted average of past inflation rates is used here to determine the expected inflation rate:

$$\theta_{t,t+1}^e = \frac{p_{t,t+1}^e}{p_t} = \frac{1}{\tau} \sum_{k=1}^{\tau} \theta_t^{(k)}, \quad k = 1, \ldots, \tau. \tag{3.4}$$

Several other forecasting rules are analyzed in LORENZ/LOHMANN [1996].

The firm is completely described by its production function. The function

$$y_t = \frac{A}{B} z_t^B, \tag{3.5}$$

with $A > 0$ and $0 < B < 1$, is a simple Cobb-Douglas-type production function with B as the elasticity of production and A as a scaling parameter.

The disequilibrium signal in the goods market, s_t^c, is defined as the relative excess demand in the market when $y_t^D > y_t$ and as the ratio of the difference between feasible and notional output and the notional output in all other cases. The adjustment coefficients are assumed to be constant positive numbers $0 < \gamma < 1$ and $0 < \kappa < 1$, one for each side of the market. Thus, the function $\mathcal{P}(s_t^c)$ is given by

$$\mathcal{P}(s_t^c) = \begin{cases} \gamma \dfrac{y_t^D - y_t}{y_t^D} & \text{if } y_t^D > y_t, \\ \kappa \dfrac{y_t - y_t^*}{y_t^*} & \text{otherwise.} \end{cases} \tag{3.6}$$

The disequilibrium signal in the labor market, s_t^ℓ, is the relative difference between actual employment and the constant labor supply if supply is larger than employment and the relative difference between notional demand and employment in all other cases. Thus, the function $\mathcal{W}(s_t^\ell)$ is given by

$$\mathcal{W}(s_t^\ell) = \begin{cases} \lambda \dfrac{L_t - L_{\max}}{L_{\max}} & \text{if } L_{\max} > L_t, \\ \mu \dfrac{z_t^* - L_t}{z_t^*} & \text{otherwise,} \end{cases} \tag{3.7}$$

with $0 < \lambda < 1$ and $0 < \mu < 1$ as adjustment coefficients. Given these functional specifications and the admissible values for the parameters, the system possesses a unique quasi-stationary state (α, m, θ) (cf. KAAS [1995]).

The set of parameters for the functional specifications consists of 12 numbers plus initial conditions. Some of them are pure scale parameters (like $L_{\max}$ and A). These are set equal to one. The remaining define a wide range of possible numerical specifications in a ten-dimensional parameter space. The complexities and dependencies which may be generated have not been studied fully. The results which were obtained for one of the simplest configurations (the so-called standard parameter set) show that the underlying dynamical system displays a high degree of complexity.

A	B	$L_{\max}$	δ	ρ	g	tax	γ	κ	λ	μ	τ	α_0	m_0
1.0	0.9	1.0	1.0	0	var.	0.25	0.6	0.6	0.6	0.6	10	0.6	0.6

The Standard Parameter Set

Table 3

Table 3 lists the standard parameter set. For some values of the substitution parameter ρ multiple stationary states exist. This points towards an interesting bifurcation behavior which will not be studied here. For $\rho = 0$ (*i.e.,* intertemporal preferences of the Cobb-Douglas type) stationary states are unique. Only this situation is studied below.[10] It provides already a vast range of interesting dynamic features. Since the expectation lag, τ, and the form of the expectations hypothesis play no role in this case, the state space of the dynamic model is $\mathbb{R}_+^2$. Hence all state space diagrams below portray the true behavior and not projections of higher dimensional dynamics.

[10] The influence of different expectations hypotheses and varying lags is studied in LORENZ/LOHMANN [1996].

3.2. The Bifurcation Behavior

While most parameters in the standard set influence the dynamic behavior of the system, the effects of varying the governments policy parameters, namely government demand and the tax rate, usually attract the highest attention in macroeconomics. This subsection concentrates on the effects of varying only the government demand.

Figure 9 contains a bifurcation diagram for varying values of g.[11] The real wage rate as one of the three state variables is plotted horizontally. For high values of g the real wage rate converges toward a stable fixed point. Lower values of g imply an unstable fixed point and the emergence of period-2 cycles. Even lower values of g lead to period-doubling and complicated behavior in the form of either quasi-periodic or chaotic behavior.[12]

In contrast to the renowned persistent period-doubling scenario in the logistic map, the bifurcation diagram in Figure 9 is characterized by a quick emergence of complicated behavior followed by windows with regular periodic behavior in which a variety of bifurcation patterns can be observed. For $g \approx [0.25, 0.35]$ and $g \approx [0.7, 0.8]$, windows with regular behavior and period-doubling behavior can be observed. Other windows, *e.g.,* the window in the vicinity of $g = 0.45$, are characterized by a more complicated behavior in the form of discontinuous jumps from one odd-period cycle to another odd-period cycle.

For *ceteris paribus* variations of the other parameters in the standard set similar bifurcation diagrams can be observed. The diagrams differ in the number of windows and the bifurcation behavior in these windows. In addition some also show period halfing as well as period doubling features.

Regular Periodic Behavior

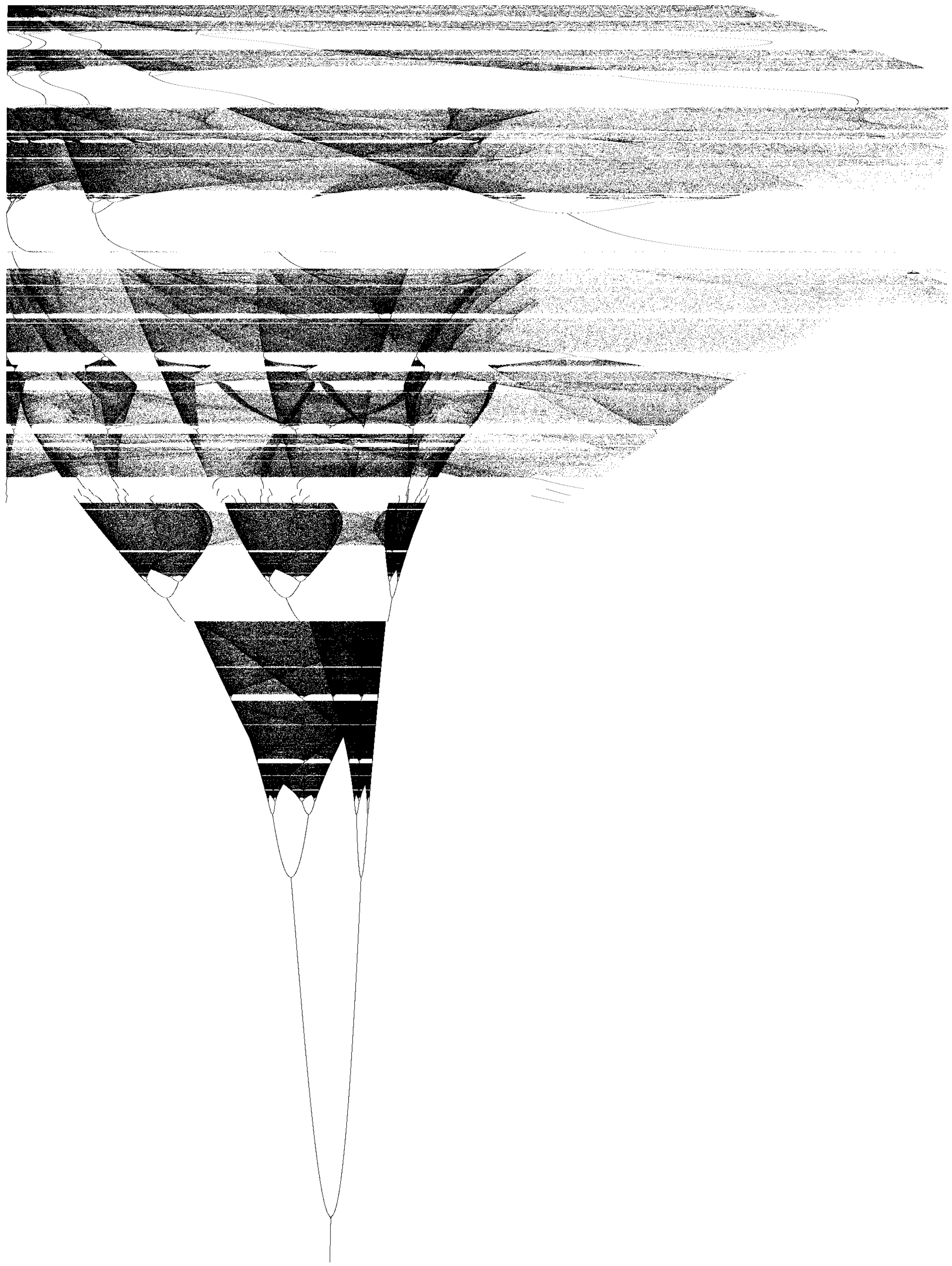
It is obvious from the bifurcation diagram in Figure 9 that regular, periodic behavior exists in the system for high values of g and in the windows. The following Figures 13 and 14 contain two examples of low-periodic regular behavior in (α_t, m_t) space. The thin vertical, horizontal, and upward-bending curves represent the boundaries between the three distint regimes outlined in Figures 3 and 6. The intersection of the three boundary lines defines the Walrasian equilibrium.

Figure 13 illustrates a period-5 cycle with one component of the cycle in the Classical regime and four components in the Keynesian regime. The five components of the cycle are marked by solid circles; the diamond-shaped mark indicates the location of the steady state for the assumed parameter constellation.[13] The circular organization of the components of the cycle in Figure 13 can also be observed for other low-periodic cycles. The period-9 cycle in Figure 14, however, displays a more complicated dynamic pattern. The components of the cycle are located in all three regimes; the steady state lies in the

[11] Cf. Lorenz [1993], pp. 128f., for details of computing bifurcation diagrams.

[12] Details on the notion of quasi-periodic and chaotic behavior can be found later in this section.

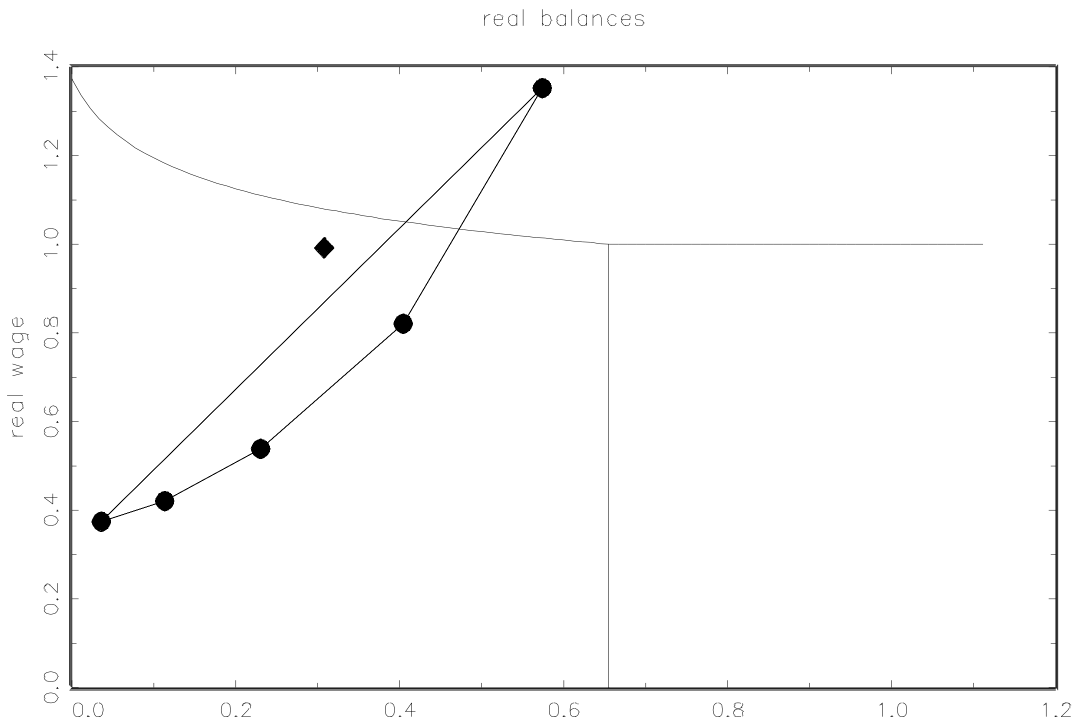
[13] In contrast to two-dimensional continuous-time dynamical systems, periodic orbits do not necessarily have to encircle the fixed point of the dynamical system.



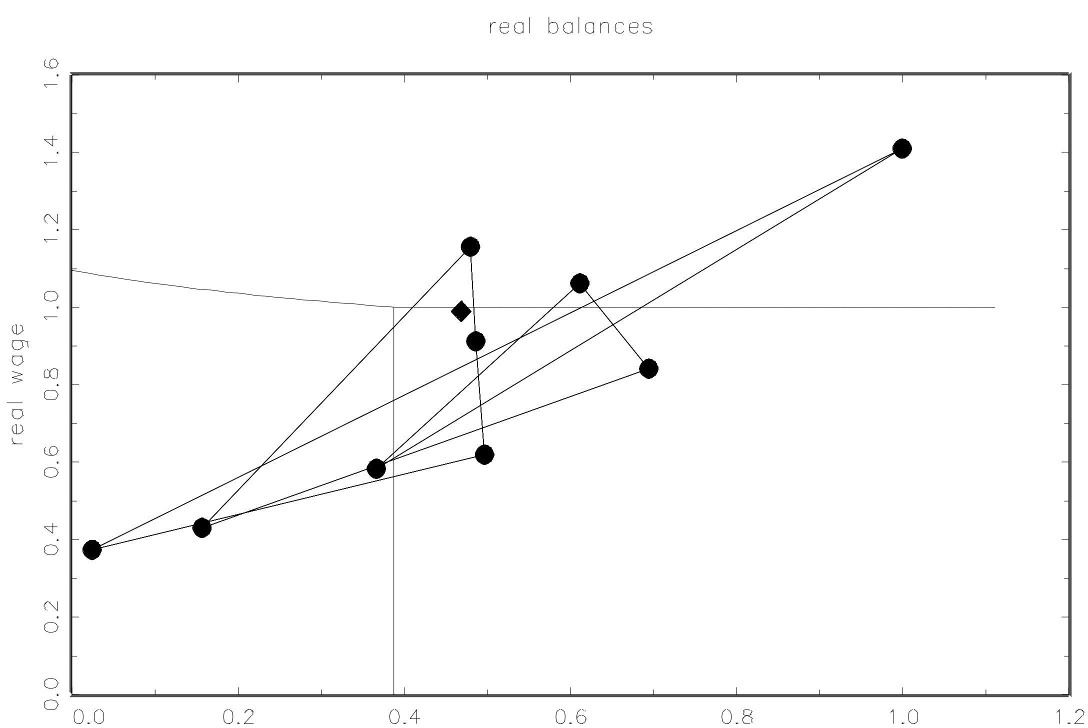
The Bifurcation Diagram for Varying Values of $g \in [0, 1.17]$ (From Top to Bottom);
The Real Wage Rate $\alpha_t \in [0.2, 2.2]$ is Plotted Horizontally
Figure 9

Inflationary regime close to the boundary with the Classical regime. The period-9 cycle consists of three rounds through the Classical, Keynesian, and Inflationary regimes.

The cycles in Figures 13 – 14 and other periodic orbits have the property that one or several of their components may be located in the Classical regime. However, two or



A period-5 cycle; standard parameter set, $g = 0.04$
Figure 13



A period-9 cycle; standard parameter set, $g = 0.307$
Figure 14

more components of the cycles are never sequentially met by the system in this regime. In contrast, a trajectory can stay in the Inflationary and Keynesian regime for more than one iteration.

For different values of g in the windows of the bifurcation diagram one obtains similar state-space diagrams. Cycles with higher periods may look quite complicated. Sometimes components are very close to each other. As a general rule, however, it can be stated that individual components always visit more than one regime, but not always all three. In general, cycles do not form geometric objects or are not located sequentially on geometric objects homeomorphic to a circle. All cycles of order larger than two tend to have a counterclockwise orientation in state space.

Co-Existing Period-n Cycles

The bifurcation diagram in Figure 9 and the period-n cycles in Figures 11-14 were plotted for constant values of the parameters in the standard set. These include the initial values of the state-space variables α and m. One of the surprising properties of the system consists in the fact that cycles of different order co-exist for the same set of parameters in the standard set, but for different initial conditions. Thus convergence and the ultimate cyclical behavior depends in a crucial way on initial conditions.

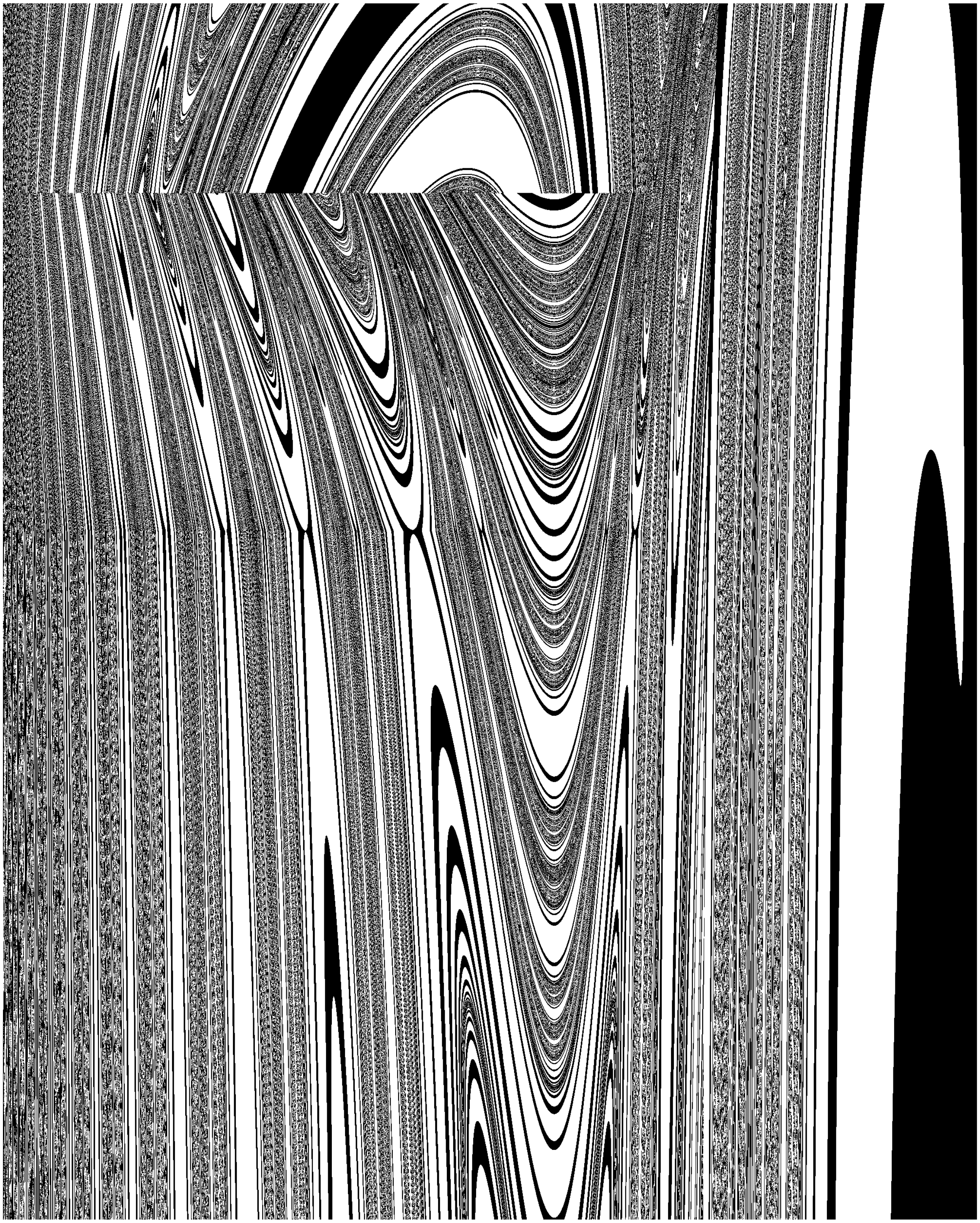
Figure 15 portrays the basins of attraction of a period-9 cycle and of a period-15 cycle. Both cycles were detected with the help of bifurcation analysis, choosing initial conditions as bifurcation parameters. The graphical procedure for the basins of attractions was carried out in the following way: An initial vector of (α_0, m_0) was iterated maximally 300 times. If during the iteration the orbit approached an ε-neighborhood of the period-9 cycle the initial point was marked as a black pixel. Given the time series $\{y_t\}$ this means for the period-9 cycle $x^9 = (x_1, \ldots, x_9)$ that there exists a t such that $|x_i - y_{t+i}| < \varepsilon$ for all $i = 1, \ldots, 9$. Similarly, if the orbit approached an ε-neighborhood of the period-15 cycle, the initial point was marked with a white pixel. If the orbit approached neither the period-9 cycle nor the period-15 cycle within the maximum number of iterations, the initial point was marked in color. No colored points which remained were found. Hence, Figure 15 does not include any colored pixel, indicating that the sets of initial points converging towards one or the other of the two cycles are mutually disjoint and exhaustive.[14]

The basin of attraction of the period-9 cycle (black areas) and the basin of attraction of the period-15 cycle (white areas) obviously are complicated geometric objects with seemingly fractal basin boundaries.[15] It follows that a minor variation in the initial state can imply the eventual settlement of the system on a completely different cycle. This situation describes the notion of a *sensitive dependence of the order of a cycle on initial conditions*.

---

[14] This statement assumes that a higher numerical resolution of the state space does not change the computed results in an essential manner.

[15] A thorough computer-assisted proof of the fractal character of the basin boundary necessitates the calculation of so-called *saddle-straddle trajectories* and their fractal dimensions. Cf. NUSSE/YORKE [1989] for details.



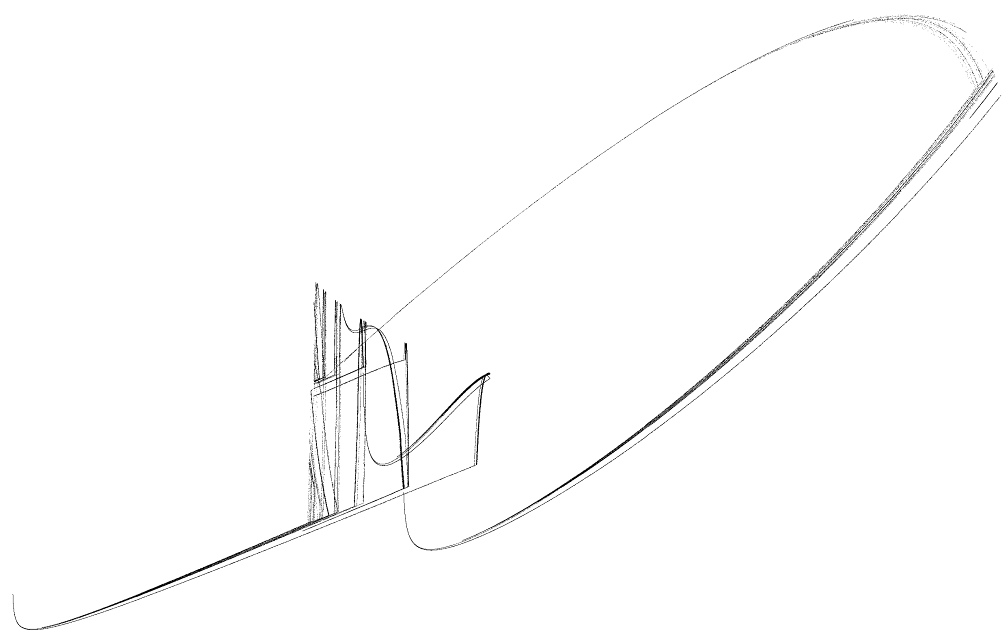
The Basins of Attraction of a Period-9 Cycle (Black Areas) and a Period-15 Cycle (White Areas). Real Money Balances are Plotted Vertically ($m_t \in [0.01, 1.51]$); Real Wages are Plotted Horizontally ($w_t \in [0.01, 1.51]$); $g = 0.45$
Figure 15

Similar scenarios could be observed for different parameter sets. For the standard parameter set (except the α_0 and m_0 values) co-existing cycles of order 4 and 26 could be detected for $g = 0.23$. For a different value of the tax rate ($tax = 0.466$), co-existing cycles of the order 9 and 24 exist. For other combinations of government expenditure, the tax rate and the adjustment parameters co-existing cycles of order 1 and 7, 2 and 7, 3 and 13, 4 and 8, and 9 and 15 could be detected. The basins of attraction of the detected co-existing cycles look very similar to the basins in Figure 15.

Complex Dynamic Behavior

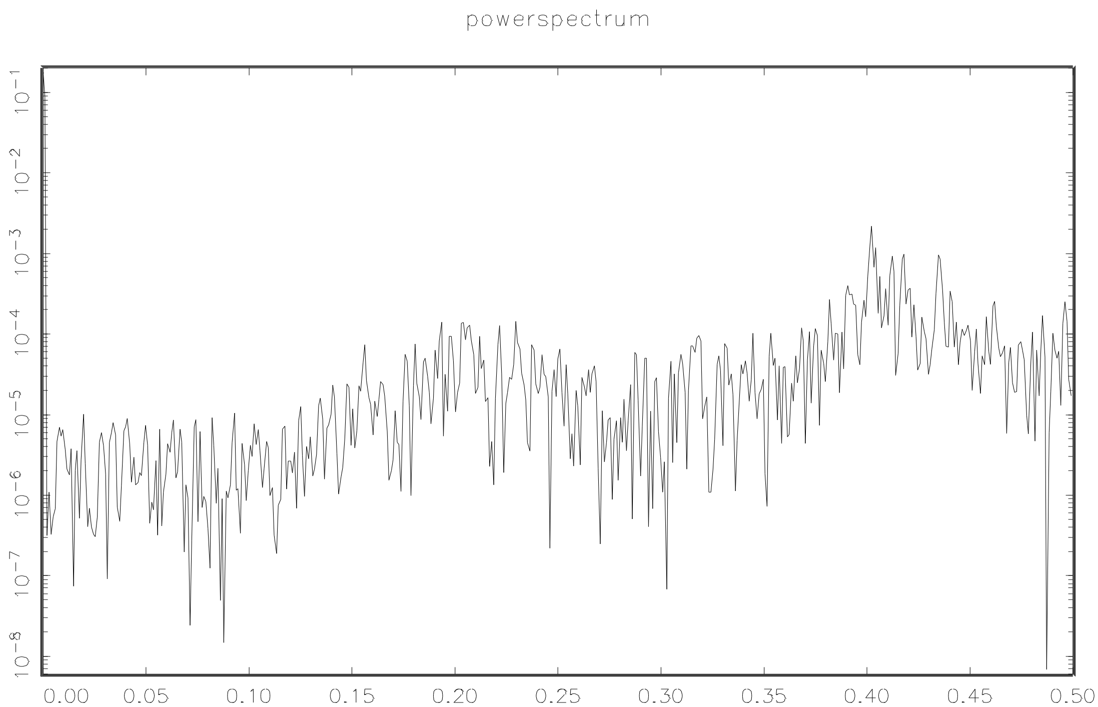
The bifurcation diagram in Figure 9 suggests that – in addition to the regular periodic behavior – the dynamical system is characterized by complicated dynamic patterns in the form of quasi-periodic or chaotic behavior.

The existence of complex dynamic patterns is exemplarily illustrated in Figure 16. Each point in the figure represents a projection of the state-space vector $(\alpha_t, m_t, \theta^e_{t,t+1})$ to the (α_t, m_t) plane. Since the parameter ρ is set to zero $\theta^e_{t,t+1}$ plays no role, so that this can be identified as the true state space. The sequence of (α_t, m_t) vectors moves quite arbitrarily in state space. For this reason the state-space points have not been connected by straight lines (as, for example, in the previous Figures 11 –14).



A Chaotic Attractor; Standard Parameter Set; $g = 0.5$, $tax = 0.45$
Figure 16

Although consecutive points are not located close to each other the sequence $\{\alpha_t, m_t\}_{t=t_1}^{T}$ of state-space points forms a geometric object, parts of which are reminiscent of the characteristic boomerang shape of the renowned Henon attractor. Since the transient phase t_0 until t_1 was excluded from plotting and since the geometric shapes of the objects do not



The Power Spectrum for the Attractor in Figure 16
Figure 18

change for higher values of T, we conclude that the object in Figure 16 indeed constitutes an attractor.[16]

The frequency-domain representation of the α_t time series is contained in Figure 18. The absence of distinguished peaks in the power spectrum indicates the aperiodicity of the motion. However, since it is impossible to distinguish between quasi-periodic and chaotic motion with the help of spectral analyses, the largest Lyapunov exponent was calculated for the parameter set in Figure 16. Since a direct computation of the Lyapunov spectrum from the (known) dynamical system requires the determination of the appropriate Jacobian matrices in the different regimes, only the largest Lyapunov exponent was estimated with the help of the algorithm described in WOLF/SWIFT/SWINNEY/VASTANO [1985]. For varying values of the evolution time, the scaling parameters and the embedding dimension, the largest exponent, λ^L, was found to be definitely bound away from zero and located in the interval $0.5 \leq \lambda^L \leq 0.6$. It can therefore be concluded that the object in Figure 16 is indeed a *chaotic* attractor according to the definition that an attractor is chaotic when the largest Lyapunov exponent is positive.

For different values of government demand and the tax rate similar chaotic attractors emerge. It has been observed that for higher values of government demand the associated attractors split into disconnected parts and/or shrink in size. While the geometric

[16] We will not elaborate upon the question whether the computed object is indeed an attractor or an example of an extremely long lasting transients emerging, for example, as a result of the motion near a complicated unstable manifold. The discussion of this topic in the case of the Henon map uncovers the theoretical difficulty of answering the question; from an applied point of view it is essentially irrelevant whether a (very) long time series represents a complicated transient or a complicated attractor.

complexity decreases in these cases, however, this does not necessarily imply that the univariate time series of the real wage rate or real money balances look more harmonic.

4. Summary and Outline of Future Work

The model presented in this paper represents one of the simplest dynamic Keynesian macroeconomic models with a microeconomic foundation of the behavioral assumptions and a consistent feasible-state scenario. The numerical experiments that have been performed with the model for the standard parameter set and varying magnitudes of government demand and the tax rate indicate that the highly nonlinear structure of the dynamical system gives rise to a variety of dynamic patterns including regular periodic behavior and complex irregular motion.

The numerical experiments described in Section 3 concentrate on variations in g and *tax*. These government parameters have mainly been chosen because the effects of changes in government expenditure and taxes have traditionally attracted most attention in macroeconomics. However, preliminary results obtained from varying different parameters in the standard parameter set (like the parameter B in the production function) suggest the importance of other parameters as well. More insight into the influence of different parameters on the dynamic behavior of the model can only be obtained from a systematic investigation of simultaneous parameter variations. The development of concise visualization techniques in the necessarily extensive computation process constitutes a major task in the project.

Aside from the assumptions on the production technology of the firm and the consumption and savings behavior of the households, the expectations hypothesis (2.6) constitutes the third key assumption of the model. The influence of various standard expectations hypotheses on the dynamic behavior of the previous model are investigated in LORENZ/LOHMANN [1996]. While different expectations hypotheses definitely influence the particular bifurcation behavior, the basic complexity in the form of co-existing periodic attractors and emerging chaotic attractor persists for all different hypotheses. Future work will concentrate on more elaborate versions with agents attempting to learn about periodic or aperiodic motion. Since the last τ actual inflation rates are included in the determination of the current expected inflation rate, $\theta^e_{t,t+1}$, it might be presumed that the length of the memory, *i.e.,* the value of τ, should affect the dynamic behavior of the model: when the economy exhibits a regular period-n orbit, agents should learn about this periodic behavior provided that $n \leq \tau$.

Two future modifications of the economic structure of the model are particularly important. First, the price and wage adjustment rules (3.6) and (3.7) reflect an implementation of the vague 'law of supply and demand' known from the analysis of competitive market mechanisms. However, in a rigorously designed macroeconomic model, prices and wages should be determined by the agents of the model. A future version of the model will therefore assume a monopolistic price-setting behavior of the firm.[17] Sec-

[17] Compare again TOBIN [1993] for the view that Keynesian economics actually relies on the concept of monopolistic competition.

ond, in the present version firms and households are treated asymmetrically in the sense that they solve an atemporal decision problem while the young household is concerned with the intertemporal allocation of his resources. This asymmetry is responsible for the shrinking of the underconsumption regime in Table 1. A future version of the model allows for inventory holdings of the firm which serve as a buffer between production and supply.

5. References

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