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Bingguang Chen

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Preface

This thesis studies the large deviations of the two-dimensional stochastic Navier-Stokes equations (SNSE) with anisotropic viscosity. Consider the following SNSE with anisotropic viscosity on the two dimensional torus $\mathbb{T}^2$ for $\varepsilon > 0$:

$$\begin{aligned} &du^\varepsilon(t) = \partial_1^2 u^\varepsilon(t)dt - u^\varepsilon \cdot \nabla u^\varepsilon(t)dt + \sqrt{\varepsilon}\sigma(t, u^\varepsilon(t))dW(t), \\ &u^\varepsilon(0) = u_0, \end{aligned}$$

where W is an l^2-cylindrical Wiener process and σ is the random external force. As $\varepsilon \to 0$, u^ε will converge to the solution to the following deterministic equation:

$$\begin{aligned} &du^0(t) = \partial_1^2 u^0(t)dt - u^0 \cdot \nabla u^0(t)dt, \\ &u^0(0) = u_0. \end{aligned}$$

We will investigate the asymptotic behaviour of the trajectory

$$\frac{1}{\sqrt{\varepsilon}\lambda(\varepsilon)}(u^\varepsilon - u^0)$$

as $\varepsilon \to 0$, where $\lambda(\varepsilon)$ is some deviation scale which strongly influences the behaviour.

(1) The case $\lambda(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}$ provides small noise large deviation principle(LDP). We use the weak convergence method to prove that u^ε satisfies the large deviation principle.

(2) For $\lambda(\varepsilon) = 1$, we are in the domain of the central limit theorem(CLT). We show that $\frac{u^\varepsilon - u^0}{\sqrt{\varepsilon}}$ converges to the solution to a stochastic differential equation as $\varepsilon \to 0$.

(3) To fill in the gap between the CLT and LDP, we will study the so-called moderate deviation principle. In this part we may assume

$$\lambda(\varepsilon) \to \infty, \quad \sqrt{\varepsilon}\lambda(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

We prove that $\frac{1}{\sqrt{\varepsilon}\lambda(\varepsilon)}(u^\varepsilon - u^0)$ satisfies the large deviation principle.

Moreover, we study small time large deviation principle for the two-dimensional SNSE with anisotropic viscosity. Let u be the solution to original SNSE with anisotropic viscosity. For $\varepsilon > 0$, the law of $u(\varepsilon t)$ coincides with the law of

$$\begin{aligned} &du_\varepsilon = \varepsilon\partial_1^2 u_\varepsilon dt - \varepsilon u_\varepsilon \cdot \nabla u_\varepsilon dt + \sqrt{\varepsilon}\sigma(\varepsilon t, u_\varepsilon)dW(t), \\ &u_\varepsilon(0) = u_0. \end{aligned}$$

We prove that u_ε satisfies the large deviation principle. The proof is based on exponentially equivalence.

We also study the small time asymptotics of the dynamical Φ^4_1 model. The dynamical Φ^4_1 model is given by

$$d\phi(t) = \Delta\phi(t)dt - \phi(t)^3 dt + dW(t),\ \text{for } (t,x) \in [0,T] \times \mathbb{T},$$
$$\phi(0) = \phi_0,$$

The law of $\phi(\varepsilon t)$ coincides with the law of

$$d\phi_\varepsilon = \varepsilon\Delta\phi_\varepsilon dt - \varepsilon\phi_\varepsilon^3 dt + \sqrt{\varepsilon} dW(t),$$
$$\phi_\varepsilon(0) = \phi_0.$$

We prove that ϕ_ε satisfies the large deviation principle.

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Bielefeld, March 15, 2021, Bingguang Chen

Contents

Chapter 1

Introduction

This thesis is concerned on the large and moderate deviation principle for the two-dimensional stochastic Navier-Stokes (NS) equations with anisotropic viscosity and the small time asymptotics of the dynamical Φ_1^4 model.

1.1 Stochastic NS equations with anisotropic viscosity

Consider the following stochastic NS equation with anisotropic viscosity on the two dimensional (2D) torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$:

$$\begin{aligned} &du = \partial_1^2 u dt - u \cdot \nabla u dt + \sigma(t,u)dW(t) - \nabla p dt, \\ &\text{div } u = 0, \\ &u(0) = u_0, \end{aligned} \tag{1.1}$$

where $u(t,x)$ denotes the velocity field at time $t \in [0,T]$ and position $x \in \mathbb{T}^2$, p denotes the pressure field, σ is the random external force and W is an l^2-cylindrical Wiener process.

Let's first recall the classical NS equation which is given by

$$\begin{aligned} &du = \nu \Delta u dt - u \cdot \nabla u dt - \nabla p dt, \\ &\text{div } u = 0, \\ &u(0) = u_0, \end{aligned} \tag{1.2}$$

where $\nu > 0$ is the viscosity of the fluid. (1.2) describes the time evolution of an incompressible fluid. In 1934, J. Leray proved global existence of finite energy weak solutions for the deterministic case in the whole space $\mathbb{R}^d$ for $d = 2, 3$ in the seminar paper [Ler33]. For more results on deterministic NS equation, we refer to [CKN82], [Tem79], [Tem95], [KT01] and references therein. For the stochastic case, there exists a great amount of literature too. The existence and uniqueness of solutions and ergodicity property to the stochatic 2D NS equation have been obtained (see e.g. [FG95], [MR05], [HM06]). Large deviation principles for the two-dimensional stochastic NS equations have been established in [CM10] and [SS06]. Moderate deviation principles for the two-dimensional stochastic NS equations have been established in [WZZ15].

Compared to (1.2), (1.1) only has partial dissipation, which can be viewed as an intermediate equation between NS equation and Euler equation. Systems of this type appear in geophysical fluids (see for instance [CDGG06] and [Ped79]). Instead of putting the classical viscosity $-\nu\Delta$ in (1.2), meteorologists often modelize turbulent diffusion by putting a viscosity of the form: $-\nu_h\Delta_h - \nu_3\partial_{x_3}^2$, where ν_h and ν_3 are empiric constants, and ν_3 is usually much smaller than ν_h. We refer to the book of J. Pedlovsky [Ped79, Chapter 4] for a more complete discussion. However, for the 3 dimensional case there is no result concerning global existence of weak solutions.

In the 2D case, [LZZ18] investigates both the deterministic system and the stochastic system (1.1) for $H^{0,1}$ initial value (for the definition of space see Chapter 2). The main difference in obtaining the global well-posedness for (1.1) is that the L^2-norm estimate is not enough to establish $L^2([0,T],L^2)$ strong convergence due to lack of compactness in the second direction. In [LZZ18], the proof is based on an additional $H^{0,1}$-norm estimate.

1.1.1 Large and moderate deviations

For $\varepsilon > 0$, consider the equation:

$$\begin{aligned} &du^\varepsilon(t) = \partial_1^2 u^\varepsilon(t)dt - u^\varepsilon \cdot \nabla u^\varepsilon(t)dt + \sqrt{\varepsilon}\sigma(t,u^\varepsilon(t))dW(t),\\ &u^\varepsilon(0) = u_0. \end{aligned} \tag{1.3}$$

As $\varepsilon \to 0$, u^ε will converge to the solution to the following deterministic equation:

$$\begin{aligned} &du^0(t) = \partial_1^2 u^0(t)dt - u^0 \cdot \nabla u^0(t)dt,\\ &u^0(0) = u_0. \end{aligned} \tag{1.4}$$

We will investigate deviations of u^ε from the deterministic solution u^0. That is, the asymptotic behaviour of the trajectory

$$\frac{1}{\sqrt{\varepsilon}\lambda(\varepsilon)}(u^\varepsilon - u^0),$$

where $\lambda(\varepsilon)$ is some deviation scale which strongly influences the behaviour.

Small noise large deviation principle

The large deviation theory concerns the asymptotic behavior of a family of random variables X_ε and we refer to the monographs [DPZ09] and [Str84] for many historical remarks and extensive references. It asserts that for some tail or extreme event A, $P(X_\varepsilon \in A)$ converges to zero exponentially fast as $\varepsilon \to 0$ and the exact rate of convergence is given by the so-called rate function. The large deviation principle was first established by Varadhan in [Var66] and he also studied the small time asymptotics of finite dimensional diffusion processes in [Var67]. Since then, many important results concerning the large deviation principle have been established. For results on the large deviation principle for stochastic differential equations in finite dimensional case we refer to [FW84]. For the extensions to infinite dimensional diffusions or SPDE, we refer the readers to [BDM08], [CM10], [DM09], [Liu09], [LRZ13], [RZ08], [XZ09], [Zha00] and the references therein.

The case $\lambda(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}$ provides some large deviation estimates. In Chapter 3 we study the small noise large deviation for the stochastic NS equations with anisotropic viscosity by using the weak convergence approach. This approach is mainly based on a variational representation formula for certain functionals of infinite dimensional Brownian Motion, which is established by Budhiraja and Dupuis in [BD00]. The main advantage of the weak convergence approach is that one can avoid some exponential probability estimates, which might be very difficult to derive for many infinite dimensional models. To use the weak convergence approach, we need to prove two conditions in Hypothesis 2.5. In [Liu09] and [LRZ13], the authors use integration by parts and lead to some extra conditions on diffusion coefficient. In [CM10], the authors use time discretization and require time-regularity of diffusion coefficient. We use the argument in [WZZ15] (in which the authors prove a moderate deviation principle), i.e. first establishing the convergence in $L^2([0,T],L^2)$ and then by using this and Itô's formula to obtain $L^\infty([0,T],L^2)\bigcap L^2([0,T],H^{1,0})$ convergence. By this argument, we can drop the extra condition on diffusion coefficient.

Central limit theorem

If $\lambda(\varepsilon)=1$, we are in the domain of the central limit theorem (CLT). In Chapter 4 we will show that $\frac{u^\varepsilon-u^0}{\sqrt{\varepsilon}}$ converges to the solution of a stochastic differential equation as $\varepsilon\to 0$.

The central limit theorem is a traditional topic in the theory of probability and statistics. The classical CLT shows that the normalized sum of a series of independent and identically distributed random variables convergent in distribution to a standard normal random variable. For the study of the central limit theorem for stochastic (partial) differential equation, we refer the readers to [WZZ15], [CLWY18] and [WZ14].

Moderate deviation principle

To fill in the gap between the CLT and LDP, we will study the so-called moderate deviation principle (MDP). Here we may assume

$$\lambda(\varepsilon)\to\infty,\quad \sqrt{\varepsilon}\lambda(\varepsilon)\to 0 \text{ as } \varepsilon\to 0.$$

The moderate deviation principle refines the estimates obtained through the central limit theorem. It provides the asymptotic behaviour for $P(\|u^\varepsilon-u^0\|\geqslant\delta\sqrt{\varepsilon}\lambda(\varepsilon))$ while CLT gives bounds for $P(\|u^\varepsilon-u^0\|\geqslant\delta\sqrt{\varepsilon})$. MDP arises in the theory of statistical inference. It can provide us with the rate of convergence and a useful method for constructing asymptotic confidence intervals, see [Erm12], [GZ11], [KI03], [Kal83] and references therein. For the study of MDP for general Markov process see [Lim95]. Resulst of MDP for stochastic partial differential equations have been obtained in [WZ14], [BDG16], [DXZZ17] and references therein.

In Chapter 5 we study the moderate deviations by using the weak convergence approach. We need to prove two conditions in Hypothesis 2.5. We will use the argument in [WZZ15] too, i.e. we first establish the convergence in $L^2([0,T],L^2)$ and then by using this and Itô's formula, $L^\infty([0,T],L^2)\bigcap L^2([0,T],H^{1,0})$ convergence can be obtained. As mentioned above, due to the lack of compactness in the second direction, we need to do $H^{0,1}$ estimate for the skeleton equation (5.1), which requires $H^{0,2}$ estimates of solution to the deterministic equation (1.4). To obtain this, we use a commutator estimate (see Lemma 2.13) from [CDGG00]. This also leads to $H^{0,2}$ condition for the initial value.

1.1.2 Small time large deviation principle

In Chapter 6 we study the small time asymptotics (large deviations) of the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity. That is, the behaviour of the solution to

$$\begin{aligned} &du_\varepsilon = \varepsilon\partial_1^2 u_\varepsilon dt - \varepsilon u_\varepsilon \cdot \nabla u_\varepsilon dt + \sqrt{\varepsilon}\sigma(\varepsilon t, u_\varepsilon)dW(t), \\ &u_\varepsilon(0) = u_0. \end{aligned}$$

This describes the limiting behaviour of the solution $u(\varepsilon t)$ as ε goes to zero.

The study of the small time asymptotics of finite dimensional diffusion processes was initiated by Varadhan in the influential work [Var67]. The small time asymptotics (large deviation) of SPDEs were studied in [Zha00], [XZ09], [LRZ13] and references therein. Another motivation will be to get the following Varadhan identity through the small time asymptotics:

$$\lim_{t\to 0} 2t \log P(u(0) \in B, u(t) \in C) = -d^2(B, C),$$

where d is an appropriate Riemannian distance associated with the diffusion generated by the solutions of the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity. The small time asymptotics itself is also theoretically interesting, since the study involves the investigation of the small noise and the effect of the small, but highly nonlinear drift.

To prove the small time asymptotics, we follow the idea of [XZ09] to prove the solution to (1.1) is exponentially equivalent to the solution to the linear equation. The main difference compared to [XZ09] is that similar to [LZZ18] L^2-norm estimate is not enough due to less dissipation and we have to do $H^{0,1}$-norm estimate.

1.2 Small time asymptotics of Φ_1^4 model

In Chapter 7 we study small time behaviour of the dynamical Φ_1^4 model :

$$\begin{aligned} &d\phi(t) = \Delta\phi(t)dt - \phi(t)^3 dt + dW(t), \text{ for } (t,x) \in [0,T] \times \mathbb{T}, \\ &\phi(0) = \phi_0, \end{aligned} \tag{1.5}$$

where $\mathbb{T}$ is one dimensional torus and W is a cylindrical Wiener process on $L^2(\mathbb{T})$.

Equation (1.5) in d dimensional case describes the natural reversible dynamics for the Euclidean Φ_d^4 quantum field theory. It is formally given by the following probability measure

$$\nu(d\varphi) = N^{-1} \prod_{x\in\mathbb{T}^d} d\varphi(x) \exp[-\int_{\mathbb{T}^d} (\frac{1}{2}|\nabla\varphi(x)|^2 + \varphi^4(x))dx],$$

where N is a renormalization constant and φ is the real-valued field. This measure was investigated intensively in the 1970s and 1980s (see [GJ87] and the references therein). Parisi and Wu in [PW81] proposed a program named stochastic quantization of getting the measure as limiting distributions of stochastic processes, especially as solutions to nonlinear stochastic differential equations(see [JLM85]). The issue to study Φ_d^4 measure is to solve and study properties of (1.5) in d dimensional case.

The dynamical Φ_1^4 model with Dirichlet boundary condition (which also named as reaction-diffusion equations) was studied systematically in [DP04]. In [DP04] not only existence and uniqueness of solutions to this equation have been obtained, but also the strong Feller property and ergodicity. For more details and more properties we refer to [DP04, Section 4]. We can obtain the results on the torus case similarly.

In 2 and 3 dimensions, the equation (1.5) falls in the category of the singular SPDEs due to the irregular nature of the noise $dW(t)$. Solutions are expected to take value in distribution spaces of negative regularity, which means the cubic term in the equation is not well-defined in the classical sense and renormalization has to be done for the nonlinear term.

In two spatial dimensions, weak solutions to (1.5) have been first constructed in [AR91] by using Dirichlet form theory. In [DDP03] the authors decomposed (1.5) into the linear equation and a shifted equation (so called Da Prato-Debussche trick) and obtain a probalistical strong solution via a fixed-point argument and invariant measure $\nu(d\phi)$. Recently, global well-posedness to (1.5) via a PDE argument has been obtained in [MW17b]. See also [RZZ17] for a study of relation between weak solutions and strong solutions.

By Hairer's breakthrough work on regularity structures [Hai14], (1.5) in the three dimensional case is well-defined and local existance and uniqueness can be obtained. In [GIP15] Gubinelli, Imkeller and Perkowski introduced paracontrolled distributions method for singular SPDEs and by this method in [CC18] the authors also obtained local-in-time well-posedness result. Mourrat and Weber in [MW17a] gave existence and uniqueness of global-in-time solutions on $\mathbb{T}^3$ by energy estimate and mild formulation. Recently, Gubinelli and Hofmanová in [GH19] proved the global existence and uniqueness results for (1.5) on $\mathbb{R}^3$ based on maximum principle and localization technique.

The purpose is to study the small time asymptotics (large deviations) of the dynamical Φ_1^4 model. We try to estimate the limiting behavior of the solution in time interval $[0, t]$ as t goes to zero, which describes how fast the solution approximating its initial data in the sense of probability. The small time asymptotics in this case is also theoretically interesting, since the study involves the investigation of the small rough noise and the effect of the small nonlinear drift.

We also want to mention the following small time asymptotics result by Dirichlet form. By [AR91] and [ZZ18] we know that the dynamical Φ_d^4 model associated with a conservative and local Dirichlet form. Then the main result in [HR03] implies the following Varadhan-type small time asymptotics for the dynamical Φ_d^4 model:

$$\lim_{t\to 0} t\log P^{\nu}(\phi(0)\in A, \phi(t)\in B) = -\frac{d(A,B)^2}{2},$$

for all measurable sets A, B, where d is the intrinsic metric associated with the Dirichlet form of Φ_d^4 model (see [HR03] for the definition). However, these results is for the stationary case or holds for $\nu(d\phi)$-almost every starting point (see [HR03, Theorem 1.3] for a stronger version). The small time large deviation result in this thesis holds for every starting point and is of its own interest.

Let $\varepsilon > 0$, by the scaling property of the Brownian motion, it is easy to see that $\phi(\varepsilon t)$ coincides in law with the solution of the following equation:

$$\begin{aligned} d\phi_\varepsilon &= \varepsilon\Delta\phi_\varepsilon dt - \varepsilon\phi_\varepsilon^3 dt + \sqrt{\varepsilon} dW(t), \\ \phi_\varepsilon(0) &= \phi_0. \end{aligned} \tag{1.6}$$

To establish the small time large deviation, we follow the idea of [XZ09] to prove the solution to (1.6) is exponentially equivalent to the solution to the linear equation. In our case, due to the irregularity of the white noise, the Itô formula in [XZ09] cannot be uesd. Our calculations are based on the energy estimate for the shifted equation (see (7.5)) and the mild formulation.

In [HW15] the small noise large deviation principle for the dynamical Φ_d^4 model is established. The authors considered the solution as a continuous map F of the noise $\sqrt{\varepsilon}\xi$ and some renormalization terms which belong to the Wiener chaos with the help of the regularity structure, then the result follows from the large deviation for Wiener chaos and the contraction principle. However, this method seems not work for the small time asymptotics problem. By this method, we have to prove the large deviation principle for the solution to linear equations in a better space (compared to Theorem 7.2 in our paper), which seems not true since $e^{\varepsilon\Delta} \to I$ as $\varepsilon \to 0$ and the smoothing effect of heat flow will disappear.

1.3 Structure of the thesis

This thesis is organised in the following:

In Chapter 2 we collect some preliminaries. First we give the function spaces we are working on. Then we introduce the large deviation principle and the weak convergence method which is given by Budhiraja and Dupuis in [BD00]. We also list the existence and uniqueness results from [LZZ18]. In Section 2.5, we list some useful estimates.

In Chapter 3 we obtain the small noise large deviation principle for the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity. In Section 3.1 we introduce the skeleton equation which gives the rate function and measures the rate of the convergence. In Section 3.2 we prove that the rate function is good. In Section 3.3, we check the last hypothesis and hence prove the large deviation principle.

In Chapter 4 we study the central limit theorem for the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity. First we study the well-posedness for the limiting equation and then obtain the central limit theorem.

In Chapter 5 we obtain the moderate deviation principle for the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity. The structure of this chapter is similar to Chapter 3. In Section 5.1 we study the skeleton equation. In Section 5.2 we prove that the rate function is good. In Section 5.3 we establish the moderate deviation principle.

In Chapter 6 we obtain the small time large deviation principle for the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity. We start by establishing the large deviation principle for the linear equation in Section 6.1. Section 6.2 is devoted to the energy estimates. In Section 6.3 we approximate the initial data. In Section 6.4 we prove the exponential equivalence between the linear and nonlinear equation and hence the large deviation principle.

In Chapter 7 we obtain the small time large deviation principle for the dynamical Φ_1^4 model. Section 7.1 is devoted to the large deviation principle for the linear equation. In Section 7.2 we prove the exponential equivalence and finally establish the main result.

Chapter 2

Preliminary

2.1 Function spaces on torus

We first recall some definitions of function spaces for the two dimensional torus $\mathbb{T}^2$.

Let $\mathbb{T}^2 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} = (\mathbb{T}_h, \mathbb{T}_v)$ where h stands for the horizonal variable x_1 and v stands for the vertical variable x_2. For exponents $p, q \in [1, \infty)$, we denote the space $L^p(\mathbb{T}_h, L^q(\mathbb{T}_v))$ by $L^p_h(L^q_v)$, which is endowed with the norm

$$\|u\|_{L^p_h(L^q_v)(\mathbb{T}^2)} := \{\int_{\mathbb{T}_h} (\int_{\mathbb{T}_v} |u(x_1, x_2)|^q dx_2)^{\frac{p}{q}} dx_1\}^{\frac{1}{p}}.$$

Similar notation for $L^p_v(L^q_h)$. In the case $p, q = \infty$, we denote L^∞ the essential supremum norm. Throughout the paper, we denote various positive constants by the same letter C.

For $u \in L^2(\mathbb{T}^2)$, we consider the Fourier expansion of u:

$$u(x) = \sum_{k \in \mathbb{Z}^2} \hat{u}_k e^{ik\cdot x} \text{ with } \hat{u}_k = \overline{\hat{u}_{-k}},$$

where $\hat{u}_k := \frac{1}{(2\pi)^2} \int_{[0,2\pi]\times[0,2\pi]} u(x) e^{-ik\cdot x} dx$ denotes the Fourier coefficient of u on $\mathbb{T}^2$.

Define the Sobolev norm:

$$\|u\|^2_{H^s} := \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^s |\hat{u}_k|^2,$$

and the anisotropic Sobolev norm:

$$\|u\|^2_{H^{s,s'}} = \sum_{k \in \mathbb{Z}^2} (1 + |k_1|^2)^s (1 + |k_2|^2)^{s'} |\hat{u}_k|^2,$$

where $k = (k_1, k_2)$. We define the Sobolev spaces $H^s(\mathbb{T}^2)$, $H^{s,s'}(\mathbb{T}^2)$ as the completion of $C^\infty(\mathbb{T}^2)$ with the norms $\|\cdot\|_{H^s}$, $\|\cdot\|_{H^{s,s'}}$ respectively. The notation $L^p_v(H^s_h)$ is given by

$$\|u\|_{L^p_v(H^s_h)} := \left(\int_{\mathbb{T}_v} \|u(\cdot, x_2)\|^p_{H^s(\mathbb{T}_h)} dx_2\right)^{\frac{1}{p}}$$

Let us recall the definition of anisotropic dyadic decomposition of the Fourier space, which will lead to another represnetation of $H^{s,s'}$ in the sense of Besov space. For a general introduction to the theory of Besov space we refer to [BCD11], [Tri78], [Tri06].

Let $\chi^{(1)}, \theta^{(1)} \in \mathcal{D}$ be nonnegative radial functions on $\mathbb{R}$, such that

i. the support of $\chi^{(1)}$ is contained in a ball and the support of $\theta^{(1)}$ is contained in an annulus;

ii. $\chi^{(1)}(z) + \sum_{j\geq 0} \theta^{(1)}(2^{-j}z) = 1$ for all $z \in \mathbb{R}$.

iii. $\mathrm{supp}(\chi^{(1)}) \cap \mathrm{supp}(\theta^{(1)}(2^{-j}\cdot)) = \emptyset$ for $j \geq 1$ and $\mathrm{supp}\theta^{(1)}(2^{-i}\cdot) \cap \mathrm{supp}\theta^{(1)}(2^{-j}\cdot) = \emptyset$ for $|i-j| > 1$.

We call such $(\chi^{(1)}, \theta^{(1)})$ dyadic partition of unity. The Littlewood-Paley blocks in the vertical variable are now defined as $u = \sum_{j\geqslant -1} \Delta_j^v u$, where

$$\Delta_{-1}^v u = \mathcal{F}^{-1}(\chi^{(1)}(|k_2|)\hat{u}) \quad \Delta_j^v u = \mathcal{F}^{-1}(\theta^{(1)}(2^{-j}|k_2|)\hat{u}),\ k_2 \in \mathbb{Z},$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. The anisotropic Sobolev norm can also be defined as follows:

$$\|u\|_{H^{s,s'}} = \left(\sum_{j\geq -1} 2^{2js'} \|\Delta_j^v u\|^2_{L^2_v(H^s(\mathbb{T}_h))} \right)^{\frac{1}{2}}.$$

To formulate the stochastic Navier-Stokes equations with anisotropic viscosity, we need the following spaces:

$$H := \{u \in L^2(\mathbb{T}^2; \mathbb{R}^2); \mathrm{div}\ u = 0\},$$

$$V := \{u \in H^1(\mathbb{T}^2; \mathbb{R}^2); \mathrm{div}\ u = 0\},$$

$$\tilde{H}^{s,s'} := \{u \in H^{s,s'}(\mathbb{T}^2; \mathbb{R}^2); \mathrm{div}\ u = 0\}.$$

Moreover, we use $\langle \cdot, \cdot \rangle$ to denote the scalar product (which is also the inner product of L^2 and H)

$$\langle u, v \rangle = \sum_{j=1}^{2} \int_{\mathbb{T}^2} u^j(x) v^j(x) dx$$

and $\langle \cdot, \cdot \rangle_X$ to denote the inner product of Hilbert space X where $X = l^2$, V or $\tilde{H}^{s,s'}$.

Besov spaces

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions on $\mathbb{R}^d$, such that

i. the support of χ is contained in a ball and the support of θ is contained in an annulus;

ii. $\chi(z) + \sum_{j\geqslant 0} \theta(2^{-j}z) = 1$ for all $z \in \mathbb{R}^d$;

iii. $\mathrm{supp}(\chi) \cap \mathrm{supp}(\theta(2^{-j}\cdot)) = \emptyset$ for $j \geqslant 1$ and $\mathrm{supp}\theta(2^{-i}\cdot) \cap \mathrm{supp}\theta(2^{-j}\cdot) = \emptyset$ for $|i-j| > 1$.

We call such (χ, θ) dyadic partition of unity, and for the existence of dyadic partitions of unity we refer to [BCD11, Proposition 2.10]. The Littlewood-Paley blocks are now defined as

$$\Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u) \quad \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j}\cdot)\mathcal{F} u).$$

For $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, $u \in \mathcal{D}$ we define

$$\|u\|_{B^\alpha_{p,q}} := \Big(\sum_{j\geqslant -1} (2^{j\alpha} \|\Delta_j u\|_{L^p})^q\Big)^{1/q},$$

with the usual interpretation as l^∞ norm in case $q = \infty$. The Besov space $B^\alpha_{p,q}$ consists of the completion of $\mathcal{D}$ with respect to this norm and the Hölder-Besov space $\mathcal{C}^\alpha$ is given by $\mathcal{C}^\alpha(\mathbb{R}^d) = B^\alpha_{\infty,\infty}(\mathbb{R}^d)$. For $p, q \in [1, \infty)$,

$$B^\alpha_{p,q}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{B^\alpha_{p,q}} < \infty\}.$$

$$\mathcal{C}^\alpha(\mathbb{R}^d) \subsetneq \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{\mathcal{C}^\alpha(\mathbb{R}^d)} < \infty\}.$$

We point out that everything above and everything that follows can be applied to distributions on the torus (see [Sic85], [SW71]). More precisely, let $\mathcal{S}'(\mathbb{T}^d)$ be the space of distributions on $\mathbb{T}^d$. Besov spaces on the torus with general indices $p, q \in [1, \infty]$ are defined as the completion of $C^\infty(\mathbb{T}^d)$ with respect to the norm

$$\|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)} := (\sum_{j\geqslant -1} (2^{j\alpha}\|\Delta_j u\|_{L^p(\mathbb{T}^d)})^q)^{1/q},$$

and the Hölder-Besov space $\mathcal{C}^\alpha$ is given by $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}(\mathbb{T}^d)$. We write $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{B^\alpha_{\infty,\infty}(\mathbb{T}^d)}$ in the following for simplicity. For $p, q \in [1, \infty)$

$$B^\alpha_{p,q}(\mathbb{T}^d) = \{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)} < \infty\}.$$

$$\mathcal{C}^\alpha \subsetneq \{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_\alpha < \infty\}.$$

Here we choose Besov spaces as completions of smooth functions, which ensures that the Besov spaces are separable which has a lot of advantages for our analysis below.

In this thesis, we use the following notations:

$$C\mathcal{C}^\beta := C([0, T], \mathcal{C}^\beta),\ CL^\infty := C([0, T], L^\infty(\mathbb{T}^d)).$$

2.2 Large deviation principle

We recall the definition of the large deviation principle. For a general introduction to the theory we refer to [DPZ09], [DZ10].

Definition 2.1 (Large deviation principle). *Given a family of probability measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ on a metric space (E, ρ) and a lower semicontinuous function $I : E \to [0, \infty]$ not identically equal to $+\infty$. The family $\{\mu_\varepsilon\}$ is said to satisfy the large deviation principle(LDP) with respect to the rate function I if*
(U) for all closed sets $F \subset E$ we have

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(F) \leqslant -\inf_{x\in F} I(x),$$

(L) for all open sets $G \subset E$ we have

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G) \geqslant -\inf_{x\in G} I(x).$$

A family of random variable is said to satisfy large deviation principle if the law of these random variables satisfy large deviation princple.

Moreover, I is a good rate function if its level sets $I_r := \{x \in E : I(x) \leqslant r\}$ are compact for arbitrary $r \in (0, +\infty)$.

Definition 2.2 (Laplace principle). *A sequence of random variables $\{X^\varepsilon\}$ is said to satisfy the Laplace principle with rate function I if for each bounded continuous real-valued function h defined on E*

$$\lim_{\varepsilon\to 0} \varepsilon \log E\left[e^{-\frac{1}{\varepsilon}h(X^\varepsilon)}\right] = -\inf_{x\in E}\{h(x) + I(x)\}.$$

Given a probabilty space $(\Omega, \mathcal{F}, P)$, the random variables $\{Z_\varepsilon\}$ and $\{\overline{Z}_\varepsilon\}$ which take values in (E, ρ) are called exponentially equivalent if for each $\delta > 0$,

$$\lim_{\varepsilon\to 0} \varepsilon \log P(\rho(Z_\varepsilon, \overline{Z}_\varepsilon) > \delta) = -\infty.$$

Lemma 2.3 ([DZ10, Theorem 4.2.13])**.** *If an LDP with a rate function $I(\cdot)$ holds for the random variables $\{Z_\varepsilon\}$, which are exponentially equivalent to $\{\overline{Z}_\varepsilon\}$, then the same LDP holds for $\{\overline{Z}_\varepsilon\}$.*

2.3 Weak convergence approach

The weak convergence approach introduced by Budhiraja and Dupuis in [BD00] will play an important role in this thesis. The starting point is the equivalence between the large deviation principle and the Laplace principle. This result was first formulated in [Puk94] and it is essentially a consequence of Varadhan's lemma [Var66] and Bryc's converse theorem [Bry90].

Remark 2.4. *By [DZ10] we have the the equivalence between the large deviation principle and the Laplace principle in completely regular topological spaces. In [BD00] the authors give the weak convergence approach on a Polish space. Since the proof does not depend on the separability and the completeness, the result also holds in metric spaces.*

Let $\{W(t)\}_{t\geqslant 0}$ be a cylindrical Wiener process on l^2 w.r.t. a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ (i.e. the path of W take values in $C([0,T];U)$, where U is another Hilbert space such that the embedding $l^2 \subset U$ is Hilbert-Schmidt). Let E be a metric space and suppose $g^\varepsilon \colon C([0,T],U) \to E$ is a measurable map for $\varepsilon > 0$. Let

$$\mathcal{A} := \left\{ v : v \text{ is } l^2\text{-valued } \mathcal{F}_t\text{-predictable process and } \int_0^T \|v(s)(\omega)\|_{l^2}^2 ds < \infty \text{ a.s.} \right\},$$

$$S_N := \left\{ \phi \in L^2([0,T], l^2) : \int_0^T \|\phi(s)\|_{l^2}^2 ds \leqslant N \right\},$$
$$\mathcal{A}_N := \{ v \in \mathcal{A} : \ v(\omega) \in S_N \text{ P-a.s.} \}.$$

Here we will always refer to the weak topology on S_N in the following if we do not state it explicitly.

Now we formulate the following sufficient conditions for the Laplace principle of g^ε as $\varepsilon \to 0$.

Hypothesis 2.5. *There exists a measurable map $g^0 : C([0,T],U) \to E$ such that the following two conditions hold:*
1. Let $\{v^\varepsilon : \varepsilon > 0\} \subset \mathcal{A}_N$ for some $N < \infty$. If v^ε converges to v in distribution as S_N-valued random elements, then

$$g^\varepsilon \left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot v^\varepsilon(s) ds \right) \to g^0 \left(\int_0^\cdot v(s) ds \right)$$

in distribution as $\varepsilon \to 0$.
2. For each $N < \infty$, the set

$$K_N = \left\{ g^0 \left(\int_0^\cdot \phi(s) ds \right) : \phi \in S_N \right\}$$

is a compact subset of E.

Lemma 2.6 ([BD00, Theorem 4.4])**.** *If g^ε satisfies Hypothesis 2.5, then the family $\{g^\varepsilon(W(\cdot))\}$ satisfies the Laplace principle (hence large deviation principle) on E with the good rate function I given by*

$$I(f) = \inf_{\{\phi \in L^2([0,T], l^2): f = g^0(\int_0^\cdot \phi(s)ds)\}} \left\{ \frac{1}{2} \int_0^T \|\phi(s)\|_{l^2}^2 ds \right\}. \tag{2.1}$$

2.4 Existence and uniqueness of solutions

Due to the divergence free condition, we introduce the following Larey projection operator $P_H : L^2(\mathbb{T}^2) \to H$:

$$P_H : u \mapsto u - \nabla \Delta^{-1}(\text{div } u).$$

By applying the operator P_H to (1.1) we can rewrite the equation in the following form:

$$\begin{aligned} du(t) &= \partial_1^2 u(t)dt - B(u(t))dt + \sigma(t, u(t))dW(t), \\ u(0) &= u_0, \end{aligned} \tag{2.2}$$

where the nonlinear operator $B(u, v) = P_H(u \cdot \nabla v)$ with the notation $B(u) = B(u, u)$. Here we use the same symbol σ after projection for simplicity.

For $u, v, w \in V$, define

$$b(u, v, w) := \langle B(u, v), w \rangle.$$

We have $b(u, v, w) = -b(u, w, v)$ and $b(u, v, v) = 0$.

We introduce the precise assumptions on the diffusion coefficient σ. Given a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$. Let $L_2(l^2, U)$ denotes the Hilbert-Schmidt norms from l^2 to U for a Hilbert space U. We recall the following conditions for σ from [LZZ18]:

(i) **Growth condition**

There exists nonnegative constants K_i', K_i, $\tilde{K}_i$ $(i = 0, 1, 2)$ such that for every $t \in [0, T]$:

(A0) $\|\sigma(t, u)\|_{L_2(l^2, H^{-1})}^2 \leqslant K_0' + K_1' \|u\|_H^2$;
(A1) $\|\sigma(t, u)\|_{L_2(l^2, H)}^2 \leqslant K_0 + K_1 \|u\|_H^2 + K_2 \|\partial_1 u\|_H^2$;
(A2) $\|\sigma(t, u)\|_{L_2(l^2, H^{0,1})}^2 \leqslant \tilde{K}_0 + \tilde{K}_1 \|u\|_{H^{0,1}}^2 + \tilde{K}_2 (\|\partial_1 u\|_H^2 + \|\partial_1 \partial_2 u\|_H^2)$;

(ii)**Lipschitz condition**

There exists nonnegative constants L_1, L_2 such that:

(A3) $\|\sigma(t, u) - \sigma(t, v)\|_{L_2(l^2, H)}^2 \leqslant L_1 \|u - v\|_H^2 + L_2 \|\partial_1 (u - v)\|_H^2$.

The following theorem from [LZZ18] shows the well-posedness of equation (2.2):

Theorem 2.7 ([LZZ18, Theorem 4.1, Theorem 4.2])**.** *Under the assumptions (A0), (A1), (A2) and (A3) with $K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{5}, L_2 < \frac{1}{5}$, equation (2.2) has a unique probabilistically strong solution $u \in L^\infty([0, T], \tilde{H}^{0,1}) \cap L^2([0, T], \tilde{H}^{1,1}) \cap C([0, T], H^{-1})$ for $u_0 \in \tilde{H}^{0,1}$.*

2.5 Some useful estimates

We first present several lemmas from [LZZ18]. It follows from Minkowski inequality that

Lemma 2.8. *For $1 \leqslant q \leqslant p \leqslant \infty$, we have*

$$\|u\|_{L_h^p(L_v^q)} \leqslant \|u\|_{L_v^q(L_h^p)},$$

$$\|u\|_{L_v^p(L_h^q)} \leqslant \|u\|_{L_h^q(L_v^p)}.$$

Lemma 2.9 ([LZZ18, Lemma 3.4])**.** *Let u be a smooth function from $\mathbb{T}^2$ to $\mathbb{R}$, we have*

$$\|u\|^2_{L_v^2(L_h^\infty)} \leqslant C(\|u\|_{L^2}\|\partial_1 u\|_{L^2} + \|u\|^2_{L^2}),$$

$$\|u\|^2_{L_h^2(L_v^\infty)} \leqslant C(\|u\|_{L^2}\|\partial_2 u\|_{L^2} + \|u\|^2_{L^2}).$$

The following anisotropic estimate is from the proof of [LZZ18, Theorem 3.1]:

Lemma 2.10. *For smooth functions u, v from $\mathbb{T}^2$ to $\mathbb{R}$ with u satisfies the divergence free condition, we have*

$$\begin{aligned}|b(u,v,u)| \leqslant a\|\partial_1 u\|^2_{L^2} + C\|u\|^2_{L^2}\Big(&\|\partial_1 v\|^{\frac{2}{3}}_{L^2}\|\partial_1\partial_2 v\|^{\frac{2}{3}}_{L^2} + \|\partial_2 v\|^{\frac{2}{3}}_{L^2}\|\partial_1\partial_2 v\|^{\frac{2}{3}}_{L^2}\\ &+ \|\partial_1 v\|^2_{L^2} + \|\partial_1 v\|_{L^2} + \|\partial_2 v\|^2_{L^2} + \|\partial_2 v\|_{L^2}\\ &+ \|\partial_1 v\|^{\frac{1}{2}}_{L^2}\|\partial_1\partial_2 v\|^{\frac{1}{2}}_{L^2} + \|\partial_2 v\|^{\frac{1}{2}}_{L^2}\|\partial_1\partial_2 v\|^{\frac{1}{2}}_{L^2}\Big),\end{aligned}$$

where $a > 0$ is a constant small enough.

In particular, we have

$$|b(u,v,u)| \leqslant a\|\partial_1 u\|^2_{L^2} + C\|u\|^2_{L^2}(1 + \|v\|^2_{H^{1,1}}).$$

Proof We have

$$\begin{aligned}|b(u,v,u)| &= |\langle u^1\partial_1 v + u^2\partial_2 v, u\rangle|\\ &\leqslant (\|u^1\|_{L_h^\infty(L_v^2)}\|\partial_1 v\|_{L_h^2(L_v^\infty)} + \|u^2\|_{L_h^2(L_v^\infty)}\|\partial_2 v\|_{L_h^\infty(L_v^2)})\|u\|_{L^2},\end{aligned}$$

where $u = (u^1, u^2)$. Now we show the calculation of two terms in the right hand side separately.

For the first term, by Lemmas 2.8 and 2.9, we have

$$\begin{aligned}&\|u^1\|_{L_h^\infty(L_v^2)}\|\partial_1 v\|_{L_h^2(L_v^\infty)}\|u\|_{L^2}\\ \leqslant& C\|u\|_{L^2}\left(\|u^1\|_{L^2}\|\partial_1 u^1\|_{L^2} + \|u^1\|^2_{L^2}\right)^{\frac{1}{2}}\left(\|\partial_1 v\|_{L^2}\|\partial_1\partial_2 v\|_{L^2} + \|\partial_1 v\|^2_{L^2}\right)^{\frac{1}{2}}\\ \leqslant& C\|u\|_{L^2}\left(\|u^1\|_{L^2}\|\partial_1 u^1\|_{L^2}\|\partial_1 v\|_{L^2}\|\partial_1\partial_2 v\|_{L^2}\right)^{\frac{1}{2}} + C\|u\|_{L^2}\|u^1\|_{L^2}\|\partial_1 v\|_{L^2}\\ &+ C\|u\|_{L^2}(\|u^1\|_{L^2} + \|\partial_1 u^1\|_{L^2})\|\partial_1 v\|_{L^2} + C\|u\|_{L^2}\|u^1\|_{L^2}\|\partial_1 v\|^{\frac{1}{2}}_{L^2}\|\partial_1\partial_2 v\|^{\frac{1}{2}}_{L^2}.\end{aligned}$$

Then Young's inequality implies that

$$\begin{aligned}&C\|u\|_{L^2}\left(\|u^1\|_{L^2}\|\partial_1 u^1\|_{L^2}\|\partial_1 v\|_{L^2}\|\partial_1\partial_2 v\|_{L^2}\right)^{\frac{1}{2}}\\ \leqslant&\frac{a}{4}\|\partial_1 u\|^2_{L^2} + C\|\partial_1 v\|^{\frac{2}{3}}_{L^2}\|\partial_1\partial_2 v\|^{\frac{2}{3}}_{L^2}\|u\|^2_{L^2},\end{aligned}$$

and

$$C\|u\|_{L^2}\|\partial_1 u^1\|_{L^2}\|\partial_1 v\|_{L^2} \leqslant \frac{a}{4}\|\partial_1 u\|_{L^2}^2 + C\|\partial_1 v\|_{L^2}^2\|u\|_{L^2}^2.$$

Thus we have

$$\begin{aligned}&\|u^1\|_{L_h^\infty(L_v^2)}\|\partial_1 v\|_{L_h^2(L_v^\infty)}\|u\|_{L^2}\\ \leqslant&\frac{a}{2}\|\partial_1 u\|_{L^2}^2 + C\|u\|_{L^2}^2\Big(\|\partial_1 v\|_{L^2}^{\frac{2}{3}}\|\partial_1\partial_2 v\|_{L^2}^{\frac{2}{3}} + \|\partial_1 v\|_{L^2}^2 + \|\partial_1 v\|_{L^2} + \|\partial_1 v\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 v\|_{L^2}^{\frac{1}{2}}\Big).\end{aligned}$$

Do the same calculation for the second term and combine the divergence free condition $\partial_2 u^2 = -\partial_1 u^1$, we have

$$\begin{aligned}&\|u^2\|_{L_h^2(L_v^\infty)}\|\partial_2 v\|_{L_h^\infty(L_v^2)}\|u\|_{L^2}\\ \leqslant&\frac{a}{2}\|\partial_1 u\|_{L^2}^2 + C\|u\|_{L^2}^2\Big(\|\partial_2 v\|_{L^2}^{\frac{2}{3}}\|\partial_1\partial_2 v\|_{L^2}^{\frac{2}{3}} + \|\partial_2 v\|_{L^2}^2 + \|\partial_2 v\|_{L^2} + \|\partial_2 v\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 v\|_{L^2}^{\frac{1}{2}}\Big),\end{aligned}$$

which implies the first inequality.

The second inequality holds from the first one and Young's Inequality.

□

Similar to the proof of Lemma 2.10, by Lemmas 2.8 and 2.9, we also have

Lemma 2.11. *For smooth functions u, v, w form $\mathbb{T}^2$ to $\mathbb{R}^2$ with divergence free condition, we have*

$$|b(u,v,w)| \leqslant C\|u\|_{H^{1,0}}\|v\|_{H^{1,1}}\|w\|_{L^2}.$$

Proof

$$\begin{aligned}&|b(u,v,w)|\\ \leqslant&(\|u^1\|_{L_h^\infty(L_v^2)}\|\partial_1 v\|_{L_h^2(L_v^\infty)} + \|u^2\|_{L_h^2(L_v^\infty)}\|\partial_2 v\|_{L_h^\infty(L_v^2)})\|w\|_{L^2}\\ \leqslant&C\Big((\|u^1\|_{L^2}\|\partial_1 u^1\|_{L^2} + \|u^1\|_{L^2}^2)^{\frac{1}{2}}(\|\partial_1 v\|_{L^2}\|\partial_1\partial_2 v\|_{L^2} + \|\partial_1 v\|_{L^2}^2)^{\frac{1}{2}}\\ &+ (\|u^2\|_{L^2}\|\partial_2 u^2\|_{L^2} + \|u^2\|_{L^2}^2)^{\frac{1}{2}}(\|\partial_2 v\|_{L^2}\|\partial_1\partial_2 v\|_{L^2} + \|\partial_2 v\|_{L^2}^2)^{\frac{1}{2}}\Big)\|w\|_{L^2}\\ \leqslant&C\|u\|_{H^{1,0}}\|v\|_{H^{1,1}}\|w\|_{L^2},\end{aligned}$$

where we used the divergence free condition to deal with the term $\partial_2 u^2$ in the last inequality. □

The next lemma is from the proof of [LZZ18, Lemma 3.5], which plays an important role in $H^{0,1}$-estimate.

Lemma 2.12. *For smooth function u form $\mathbb{T}^2$ to $\mathbb{R}^2$ with divergence free condition, we have*

$$|\langle\partial_2 u, \partial_2(u\cdot\nabla u)\rangle| \leqslant a\|\partial_1\partial_2 u\|_{L^2}^2 + C(1+\|\partial_1 u\|_{L^2}^2)\|\partial_2 u\|_{L^2}^2,$$

where $a > 0$ is a constant small enough.

Proof We have

$$\langle\partial_2 u, \partial_2(u\cdot\nabla u)\rangle = \langle\partial_2 u^1, \partial_2(u\cdot\nabla u^1)\rangle + \langle\partial_2 u^2, \partial_2(u\cdot\nabla u^2)\rangle,$$

where $u = (u^1, u^2)$.

For the first term on the right hand side, we have

$$\begin{aligned}\langle\partial_2u^1,\partial_2(u\cdot\nabla u^1)\rangle=&\langle\partial_2u^1,\partial_2(u^1\partial_1u^1+u^2\partial_2u^1)\rangle\\=&\langle\partial_2u^1,\partial_2u^1\partial_1u^1\rangle+\langle\partial_2u^1,u^1\partial_2\partial_1u^1\rangle\\&+\langle\partial_2u^1,\partial_2u^2\partial_2u^1\rangle+\langle\partial_2u^1,u^2\partial_2^2u^1\rangle\\=&\langle\partial_2u^1,u^1\partial_2\partial_1u^1\rangle+\langle\partial_2u^1,u^2\partial_2^2u^1\rangle\\=&\langle\partial_2u^1,u\cdot\nabla\partial_2u^1\rangle\\=&-\frac{1}{2}\int\text{div }u|\partial_2u^1|^2dx\\=&0,\end{aligned}$$

where we use the fact div $u=0$ in the third and sixth equality.

Similarly, for the second term, we have

$$\begin{aligned}\langle\partial_2u^2,\partial_2(u\cdot\nabla u^2)\rangle=&\langle\partial_2u^2,\partial_2u^1\partial_1u^2\rangle+\langle\partial_2u^2,u^1\partial_2\partial_1u^2\rangle\\&+\langle\partial_2u^2,\partial_2u^2\partial_2u^2\rangle+\langle\partial_2u^2,u^2\partial_2^2u^2\rangle\\=&\langle\partial_2u^2,\partial_2u^1\partial_1u^2\rangle+\frac{1}{2}\int u^1\partial_1(\partial_2u^2)^2dx\\&+\langle\partial_2u^2,\partial_2u^2\partial_2u^2\rangle+\frac{1}{2}\int u^2\partial_2(\partial_2u^2)^2dx\\=&\langle\partial_2u^2,\partial_2u^1\partial_1u^2\rangle+\langle\partial_2u^2,\partial_2u^2\partial_2u^2\rangle\\&-\frac{1}{2}\langle\partial_2u^2,\partial_1u^1\partial_2u^2\rangle-\frac{1}{2}\langle\partial_2u^2,\partial_2u^2\partial_2u^2\rangle\\=&\langle\partial_2u^2,\partial_2u^1\partial_1u^2\rangle+\langle\partial_2u^2,\partial_2u^2\partial_2u^2\rangle,\end{aligned}$$

where we use div $u=0$ in the last equality.

Then by Lemma 2.9 we have

$$\begin{aligned}&|\langle\partial_2u,\partial_2(u\cdot\nabla u)\rangle|\\=&|\langle\partial_2u^2,\partial_2u^1\partial_1u^2\rangle+\langle\partial_2u^2,\partial_2u^2\partial_2u^2\rangle|\\\leqslant&\left(\|\partial_2u^1\|_{L_h^\infty(L_v^2)}\|\partial_1u^2\|_{L_h^2(L_v^\infty)}+\|\partial_1u^1\|_{L_h^2(L_v^\infty)}\|\partial_2u^2\|_{L_h^\infty(L_v^2)}\right)\|\partial_2u^2\|_{L^2}\\\leqslant&C\left(\|\partial_2u\|_{L^2}+\|\partial_2u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2u\|_{L^2}^{\frac{1}{2}}\right)\left(\|\partial_1u\|_{L^2}+\|\partial_1u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2u\|_{L^2}^{\frac{1}{2}}\right)\|\partial_2u^2\|_{L^2}\\\leqslant&C\|\partial_1u\|_{L^2}\|\partial_2u\|_{L^2}^2+C\|\partial_1\partial_2u\|_{L^2}\|\partial_1u\|_{L^2}\|\partial_2u\|_{L^2}\\&+C\|\partial_1\partial_2u\|_{L^2}^{\frac{1}{2}}\left(\|\partial_1u\|_{L^2}\|\partial_2u\|_{L^2}^{\frac{1}{2}}+\|\partial_2u\|_{L^2}\|\partial_1u\|_{L^2}^{\frac{1}{2}}\right)\|\partial_2u^2\|_{L^2},\end{aligned}$$

where we use the following inequality in the last inequality:

$$\begin{aligned}&\|\partial_2u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2u\|_{L^2}\|\partial_1u\|_{L^2}^{\frac{1}{2}}\|\partial_2u^2\|_{L^2}\\=&\|\partial_2u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2u\|_{L^2}\|\partial_1u\|_{L^2}^{\frac{1}{2}}\|\partial_1u^1\|_{L^2}^{\frac{1}{2}}\|\partial_2u^2\|_{L^2}^{\frac{1}{2}}\\\leqslant&\|\partial_1\partial_2u\|_{L^2}\|\partial_1u\|_{L^2}\|\partial_2u\|_{L^2},\end{aligned}$$

where we use div $u=0$ in the first equality.

By Young's inequality, we have

$$C\|\partial_1\partial_2u\|_{L^2}\|\partial_1u\|_{L^2}\|\partial_2u\|_{L^2}\leqslant\frac{a}{2}\|\partial_1\partial_2u\|_{L^2}^2+C\|\partial_1u\|_{L^2}^2\|\partial_2u\|_{L^2}^2,$$

and

$$\begin{aligned}
&C\|\partial_1\partial_2 u\|_{L^2}^{\frac12}\left(\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2}^{\frac12}+\|\partial_2 u\|_{L^2}\|\partial_1 u\|_{L^2}^{\frac12}\right)\|\partial_2 u^2\|_{L^2}\\
\leqslant&\frac{a}{2}\|\partial_1\partial_2 u\|_{L^2}^2+C\left(\|\partial_1 u\|_{L^2}^{\frac43}\|\partial_2 u\|_{L^2}^{\frac23}+\|\partial_2 u\|_{L^2}^{\frac43}\|\partial_1 u\|_{L^2}^{\frac23}\right)\|\partial_2 u^2\|_{L^2}^{\frac43}\\
\leqslant&\frac{a}{2}\|\partial_1\partial_2 u\|_{L^2}^2+C\|\partial_1 u\|_{L^2}^{\frac43}\|\partial_2 u\|_{L^2}^2+C\|\partial_2 u\|_{L^2}^{\frac43}\|\partial_1 u\|_{L^2}^{\frac23}\|\partial_1 u^1\|_{L^2}^{\frac23}\|\partial_2 u^2\|_{L^2}^{\frac23}\\
\leqslant&\frac{a}{2}\|\partial_1\partial_2 u\|_{L^2}^2+C(1+\|\partial_1 u\|_{L^2}^2)\|\partial_2 u\|_{L^2}^2,
\end{aligned}$$

where we use div $u=0$ in the second inequality.

Thus we deduce that

$$|\langle\partial_2 u,\partial_2(u\cdot\nabla u)\rangle|\leqslant a\|\partial_1\partial_2 u\|_{L^2}^2+C(1+\|\partial_1 u\|_{L^2}^2)\|\partial_2 u\|_{L^2}^2.$$

□

The following estimates are obtained by [CDGG00] in dimension 3, we now present its 2-dimension version.

Lemma 2.13 ([CDGG00, Lemma 3])**.** *For any real number $s_0>\frac12$ and $s\geqslant s_0$, for any vector fields u and w, with divergence free condition, there exists constants C and $d_k(u,w)$ such that*

$$\begin{aligned}
|\langle\Delta_k^v(u\cdot\nabla w),\Delta_k^v w\rangle|\leqslant Cd_k2^{-2ks}\|w\|_{H^{\frac14,s}}(&\|u\|_{H^{\frac14,s_0}}\|\partial_1 w\|_{H^{0,s}}+\|u\|_{H^{\frac14,s}}\|\partial_1 w\|_{H^{0,s_0}}\\
&+\|\partial_1 u\|_{H^{0,s_0}}\|w\|_{H^{\frac14,s}}+\|\partial_1 u\|_{H^{0,s}}\|w\|_{H^{\frac14,s_0}}),
\end{aligned}$$

where $\sum_k d_k=1$.

Proof Define

$$F_k^h=\Delta_k^v(u^1\partial_1 w)\text{ and }F_k^v=\Delta_k^v(u^2\partial_2 w).$$

Let us start by proving the result for F_k^h. Recall the Bony decomposition (see [BCD11]) in vertical variables for tempered distributions a,b:

$$ab=T_a^v b+T_b^v a+R^v(a,b),$$

with

$$T_a^v b=\sum_j S_{j-1}^v a\Delta_j^v b\quad\text{ and }\quad R^v(a,b)=\sum_{|k-j|\leqslant1}\Delta_k^v a\Delta_j^v b,$$

where $S_{j-1}^v a=\sum_{j'\leqslant j-2}\Delta_{j'}^v a$.

Then we have by Hölder's inequality and Sobolev embedding $H^{\frac14}(\mathbb{T})\hookrightarrow L^4(\mathbb{T})$

$$\begin{aligned}
\langle\Delta_k^v(u^1\partial_1 w),\Delta_k^v w\rangle\leqslant&\|\Delta_k^v(u^1\partial_1 w)\|_{L_v^2(L_h^{\frac43})}\|\Delta_k^v w\|_{L_v^2(L_h^4)}\\
\leqslant&C\|\Delta_k^v(T_{u^1}^v\partial_1 w+T_{\partial_1 w}^v u^1+R^v(u^1,\partial_1 w))\|_{L_v^2(L_h^{\frac43})}\|\Delta_k^v w\|_{L_v^2(H_h^{\frac14})}\\
\leqslant&C\|\Delta_k^v(T_{u^1}^k\partial_1 w+T_{\partial_1 w}^v u^1+R^v(u^1,\partial_1 w))\|_{L_v^2(L_h^{\frac43})}2^{-ks}c_k\|w\|_{H^{\frac14,s}},
\end{aligned}\tag{2.3}$$

where $c_k=\frac{2^{ks}\|\Delta_v^k w\|_{L_v^2(H_h^{\frac14})}}{\|w\|_{H^{\frac14,s}}}\in l^2$. For the first term of the third line, we have

$$\|\Delta_k^v(T_{u^1}^k\partial_1 w)\|_{L_v^2(L_h^{\frac43})}$$

$$
\begin{aligned}
&\leqslant \sum_{|k-k'|\leqslant N_0} \|S^v_{k'-1}u^1\Delta^v_{k'}\partial_1 w\|_{L^2_v(L^{\frac{4}{3}}_h)} \leqslant \sum_{|k-k'|\leqslant N_0} \|S^v_{k'-1}u^1\|_{L^\infty_v(L^4_h)}\|\Delta^v_{k'}\partial_1 w\|_{L^2_v(L^2_h)}\\
&\leqslant C\sum_{|k-k'|\leqslant N_0} \|u^1\|_{H^{\frac{1}{4},s_0}} 2^{-k's}b_{k'}\|\partial_1 w\|_{H^{0,s}} \leqslant Cb^{(1)}_k 2^{-ks}\|u^1\|_{H^{\frac{1}{4},s_0}}\|\partial_1 w\|_{H^{0,s}},
\end{aligned}
$$

where $b_k = \frac{2^{ks}\|\Delta^v_k\partial_1 w\|_{L^2_v(L^2_h)}}{\|\partial_1 w\|_{H^{0,s}}} \in l^2$ and $b^{(1)}_k = 2^{ks}\sum_{|k-k'|\leqslant N_0} 2^{-k's}b_{k'} \in l^2$. Note here N_0 depends on the choice of Dyadic partition. For the second term, similarly we have

$$
\begin{aligned}
\|\Delta^v_k(T^k_{\partial_1 w}u^1)\|_{L^2_v(L^{\frac{4}{3}}_h)} \leqslant& \sum_{|k-k'|\leqslant N_0} \|S^v_{k'-1}\partial_1 w\|_{L^\infty_v(L^2_h)}\|\Delta^v_{k'}u^1\|_{L^2_v(L^4_h)}\\
\leqslant& C\sum_{|k-k'|\leqslant N_0} \|\partial_1 w\|_{H^{0,s_0}} 2^{-k's}a_{k'}\|u\|_{H^{\frac{1}{4},s}} \leqslant Ca^{(1)}_k 2^{-ks}\|\partial_1 w\|_{H^{0,s_0}}\|u\|_{H^{\frac{1}{4},s}},
\end{aligned}
$$

where $a_k = \frac{2^{ks}\|\Delta^v_k u\|_{L^2_v(H^{\frac{1}{4}}_h)}}{\|u\|_{H^{\frac{1}{4},s}}} \in l^2$ and $a^{(1)}_k = 2^{ks}\sum_{|k-k'|\leqslant N_0} 2^{-k's}\tilde{c}_k \in l^2$.

$$
\begin{aligned}
\|\Delta^v_k R^v(u^1,\partial_1 w)\|_{L^2_v(L^{\frac{4}{3}}_h)} \leqslant& \sum_{|k'-j|\leqslant 1, k'\geqslant k-N_0} \|\Delta^v_{k'}u^1\|_{L^2_v(L^4_h)}\|\Delta^v_j\partial_1 w\|_{L^\infty_v(L^2_h)}\\
\leqslant& C\sum_{k'\geqslant k-N_0} 2^{-k's}a_{k'}\|u\|_{H^{\frac{1}{4},s}}\|\partial_1 w\|_{H^{0,s_0}}\\
\leqslant& Ca^{(2)}_k 2^{-ks}\|u\|_{H^{\frac{1}{4},s}}\|\partial_1 w\|_{H^{0,s_0}},
\end{aligned}
$$

where $a^{(2)}_k = 2^{ks}\sum_{k'\geqslant k-N_0} 2^{-k's}a_{k'} = \sum_{k'\in\mathbb{Z}} I_{\{k'\leqslant N_0\}}2^{k's}a_{k-k'}$ and by Young's convolution inequality

$$
\|a^{(2)}\|_{l^2} \leqslant \|I_{\{k'\leqslant N_0\}}2^{k's}\|_{l^1}\|a\|_{l^2} < \infty.
$$

This implies that

$$
|\langle F^h_k, \Delta^v_k w\rangle| \leqslant Cc_k(b^{(1)}_k + a^{(1)}_k + a^{(2)}_k)2^{-2ks}\|w\|_{H^{\frac{1}{4},s}}(\|u\|_{H^{\frac{1}{4},s_0}}\|\partial_1 w\|_{H^{0,s}} + \|u\|_{H^{\frac{1}{4},s}}\|\partial_1 w\|_{H^{0,s_0}}),
$$

where $c_k(b^{(1)}_k + a^{(1)}_k + a^{(2)}_k) \in l^1$.

To estimate the term $\langle F^v_k, \Delta^v_k w\rangle$, write $\Delta^v_k(u^2\partial_2 w) = F^{v,1}_k + F^{v,2}_k$ with

$$
F^{v,1}_k = \Delta^v_k \sum_{k'\geqslant k-N_0} S^v_{k'+2}\partial_2 w\Delta^v_{k'}u^2 \quad \text{and} \quad F^{v,2}_k = \Delta^v_k \sum_{|k-k'|\leqslant N_0} S^v_{k'-1}u^2\Delta^v_{k'}\partial_2 w.
$$

For $F^{v,1}_k$, again we have by Hölder's inequality and Sobolev embedding,

$$
\begin{aligned}
\|F^{v,1}_k\|_{L^2_v(L^{\frac{4}{3}}_h)} \leqslant& \sum_{k'\geqslant k-N_0} \|S^v_{k'+2}\partial_2 w\|_{L^\infty_v(L^4_h)}\|\Delta^v_{k'}u^2\|_{L^2_v(L^2_h)}\\
\leqslant& C\sum_{k'\geqslant k-N_0} 2^{k'}\|S^v_{k'+2}w\|_{L^\infty_v(L^4_h)}2^{-k'}\|\Delta^v_{k'}\partial_2 u^2\|_{L^2_v(L^2_h)}\\
\leqslant& C\sum_{k'\geqslant k-N_0} \|w\|_{H^{\frac{1}{4},s_0}}2^{-k's}\tilde{c}_{k'}\|\partial_1 u\|_{H^{0,s}}\\
\leqslant& C2^{-ks}\tilde{c}^{(2)}_k\|w\|_{H^{\frac{1}{4},s_0}}\|\partial_1 u\|_{H^{0,s}},
\end{aligned}
$$

where we use Bernstein's inequality twice in the second inequality and divergence free condition in the third inequality. Note here $\tilde{c}_k = \frac{2^{ks}\|\Delta_k^v \partial_1 u\|_{L_v^2(L_h^2)}}{\|\partial_1 u\|_{H^{0,s}}} \in l^2$ and $\tilde{c}_k^{(2)} = 2^{ks}\sum_{k'\geqslant k-N_0} 2^{-k's}\tilde{c}_{k'} \in l^2$.

Then similar as (2.3) we have

$$|\langle F_k^{v,1}, \Delta_k^v w\rangle| \leqslant Cc_k\tilde{c}_k^{(2)}2^{-2ks}\|w\|_{H^{\frac14,s}}\|w\|_{H^{\frac14,s_0}}\|\partial_1 u\|_{H^{0,s}}.$$

The last term $F_k^{v,2}$ requires commutator estimates. Following a computation in [CL92], we have

$$\begin{aligned}\langle F_k^{v,2}, \Delta_k^v w\rangle =&\langle S_{k-1}^v u^2\Delta_k^v\partial_2 w, \Delta_k^v w\rangle + R_k(u,w) \quad \text{with}\\ R_k(u,v) =& \sum_{|k-k'|\leqslant N_0}\langle[\Delta_k^v, S_{k'-1}^v u^2]\Delta_{k'}^v\partial_2 w, \Delta_k^v w\rangle\\ &-\sum_{|k'-k|\leqslant N_0}\langle(S_{k-1}^v - S_{k'-1}^v)u^2\Delta_k^v\Delta_{k'}^v\partial_2 w, \Delta_k^v w\rangle.\end{aligned}$$

Using an integration by parts and divergence free condition, we have

$$\begin{aligned}|\langle S_{k-1}^v u^2\Delta_k^v\partial_2 w, \Delta_k^v w\rangle| =&\frac12|\langle S_k^v\partial_2 u^2\Delta_k^v w, \Delta_k^v w\rangle| = \frac12|\langle S_k^v\partial_1 u^1\Delta_k^v w, \Delta_k^v w\rangle|\\ \leqslant& C\|S_k^v\partial_1 u^1\|_{L_v^\infty(L_h^2)}\|\Delta_k^v w\|_{L_v^2(L_h^4)}^2\\ \leqslant& Cc_k^2 2^{-2ks}\|\partial_1 u\|_{H^{0,s_0}}\|w\|_{H^{\frac14,s}}^2.\end{aligned}\tag{2.4}$$

Note that the Fourier transform of $(S_{k-1}^v - S_{k'-1}^v)u^2$ is supported in $2^k\mathcal{A}$ since $|k-k'|\leqslant N_0$ where $\mathcal{A}$ is an annulus. We have by Bernstein's inequality

$$\begin{aligned}&\|\sum_{|k'-k|\leqslant N_0}(S_{k-1}^v - S_{k'-1}^v)u^2\Delta_k^v\Delta_{k'}^v\partial_2 w\|_{L_v^2(L_h^{\frac43})}\\ \leqslant&\sum_{|k'-k|\leqslant N_0}\|(S_{k-1}^v - S_{k'-1}^v)u^2\|_{L_v^\infty(L_h^2)}\|\Delta_k^v\Delta_{k'}^v\partial_2 w\|_{L_v^2(L_h^4)}\\ \leqslant& C\sum_{|k'-k|\leqslant N_0}2^k\|(S_{k-1}^v - S_{k'-1}^v)\partial_2 u^2\|_{L_v^\infty(L_h^2)}2^{-k}\|\Delta_k^v w\|_{L_v^2(L_h^4)}\\ \leqslant& C\sum_{|k'-k|\leqslant N_0}\|\partial_1 u^1\|_{H^{0,s_0}}2^{-ks}c_k\|w\|_{H^{\frac14,s}}.\end{aligned}$$

This similar as (2.3) implies that

$$|\langle\sum_{|k'-k|\leqslant N_0}(S_{k-1}^v - S_{k'-1}^v)u^2\Delta_{k'}^v\partial_2 w, \Delta_k^v w\rangle| \leqslant Cc_k^2 2^{-2ks}\|\partial_1 u\|_{H^{0,s_0}}\|w\|_{H^{\frac14,s}}^2.$$

To estimate the term $\langle[\Delta_k^v, S_{k'-1}^v u^2]\Delta_{k'}^v\partial_2 w, \Delta_k^v w\rangle$, we have for any function f,

$$\begin{aligned}&[\Delta_k^v, S_{k'-1}^v u^2]f(x_1,x_2)\\ =&2^k\int_{\mathbb{T}_v} h(2^k y_2)(S_{k'-1}^v u^2(x_1,x_2) - S_{k'-1}^v u^2(x_1,x_2-y_2))f(x_1,x_2-y_2)dy_2\\ =&\int_{\mathbb{T}_v\times[0,1]} h_1(2^k y_2)(S_{k'-1}^v\partial_2 u^2)(x_1,x_2+(t-1)y_2)f(x_1,x_2-y_2)dy_2dt\end{aligned}$$

$$= -\int_{\mathbb{T}_v\times[0,1]} h_1(2^k y_2)(S^v_{k'-1}\partial_1 u^1)(x_1, x_2+(t-1)y_2)f(x_1,x_2-y_2)dy_2dt,$$

where $h = \mathcal{F}^{-1}\chi^{(1)}, (k=-1)$ or $h=\mathcal{F}^{-1}\theta^{(1)}, (k\geqslant 0)$, $h_1(z)=zh(z)$ and we use divergence free condition in the last line. This implies

$$\|[\Delta^v_k, S^v_{k'-1}u^2]f(\cdot,x_2)\|_{L_h^{\frac{4}{3}}} \leqslant C\int |h_1(2^k y_2)| \|S^v_{k'-1}\partial_1 u^1\|_{L^\infty_v(L^2_h)}\|f(\cdot,x_2-y_2)\|_{L^4_h}dy_2$$

Then we get

$$\|[\Delta^v_k, S^v_{k'-1}u^2]f\|_{L^2_v(L_h^{\frac{4}{3}})} \leqslant C2^{-k}\|S^v_{k'-1}\partial_1 u^1\|_{L^\infty_v(L^2_h)}\|f\|_{L^2_v(H_h^{\frac{1}{4}})}.$$

Hence

$$\begin{aligned}
&|\sum_{|k-k'|\leqslant N_0}\langle[\Delta^v_k, S^v_{k'-1}u^2]\Delta^v_{k'}\partial_2 w, \Delta^v_k w\rangle| \\
\leqslant& C2^{-k}\sum_{|k-k'|\leqslant N_0}\|S^v_{k'-1}\partial_1 u^1\|_{L^\infty_v(L^2_h)}2^{k'}\|\Delta^v_{k'}w\|_{L^2_v(H_h^{\frac{1}{4}})}\|\Delta^v_k w\|_{L^2_v(H_h^{\frac{1}{4}})} \\
\leqslant& C\sum_{|k-k'|\leqslant N_0}\|\partial_1 u\|_{H^{0,s_0}}2^{-k's}c_{k'}\|w\|_{H^{\frac{1}{4},s}}2^{-ks}c_k\|w\|_{H^{\frac{1}{4},s}} \\
\leqslant& Cc_kc_k^{(1)}2^{-2ks}\|\partial_1 u\|_{H^{0,s_0}}\|w\|_{H^{\frac{1}{4},s}}\|w\|_{H^{\frac{1}{4}.s}},
\end{aligned}$$

where $c_k^{(1)} = 2^{ks}\sum_{|k-k'|\leqslant N_0}2^{-k's}c_{k'}\in l^2$

Combining all the term together, let

$$d'_k = c_k(b_k^{(1)}+a_k^{(1)}+a_k^{(2)}+\tilde{c}_k^{(2)}+c_k+c_k^{(1)})\in l^1 \quad \text{and } d_k=\frac{d'_k}{\|d'_k\|_{l^1}}$$

we finish the proof. □

The following remarkable result is from [BY82] and [Dav76]:

Lemma 2.14. *There exists a universal constant c such that, for any $p\geqslant 2$ and for all continuous martingale (M_t) with $M_0=0$ and stopping times τ,*

$$\|M^*_\tau\|_p \leqslant cp^{\frac{1}{2}}\|\langle M\rangle_\tau^{\frac{1}{2}}\|_p,$$

where $M^*_t=\sup_{0\leqslant s\leqslant t}|M_s|$ and $\|\cdot\|_p$ stands for the L^p norm with respect to the probability space.

We will need several important properties of Besov spaces on the torus and we recall the following Besov embedding theorems on the torus (c.f. [Tri78, Theorem 4.6.1], [GIP15, Lemma A.2]):

Lemma 2.15. *Let $1\leq p_1\leq p_2\leq\infty$ and $1\leq q_1\leq q_2\leq\infty$, and let $\alpha\in\mathbb{R}$. Then $B^\alpha_{p_1,q_1}(\mathbb{T}^d)$ is continuously embedded in $B^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}(\mathbb{T}^d)$.*

We recall the following Schauder estimates, i.e. the smoothing effect of the heat flow, for later use.

Lemma 2.16 ([GIP15, Lemma A.7])**.** *Let $u\in\mathcal{C}^\alpha$ for some $\alpha\in\mathbb{R}$. Then for every $\delta\geqslant 0$, there exists a constant C independent of u such that*

$$\|e^{t\Delta}u\|_{\alpha+\delta}\leqslant Ct^{-\delta/2}\|u\|_\alpha.$$

Chapter 3

Small noise large deviation principle

In this chapter, we consider the small noise large deviation principle for the stochastic Navier-Stokes equations with anisotropic viscosity.

Consider the following equation:

$$\begin{aligned} &du^\varepsilon(t) = \partial_1^2 u^\varepsilon(t)dt - B(u^\varepsilon(t))dt + \sqrt{\varepsilon}\sigma(t, u^\varepsilon(t))dW(t), \\ &u^\varepsilon(0) = u_0. \end{aligned} \tag{3.1}$$

By Lemma 2.7, under the assumptions (A0)-(A3) with $K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{5}, L_2 < \frac{1}{5}$, (3.1) has a unique strong solution $u^\varepsilon \in L^\infty([0,T], \tilde{H}^{0,1}) \bigcap L^2([0,T], \tilde{H}^{1,1}) \bigcap C([0,T], H^{-1})$ for $u_0 \in \tilde{H}^{0,1}$. It follows from Yamada-Watanabe theorem (See [LR15, Appendix E]) that there exists a Borel-measurable function

$$g^\varepsilon : C([0,T], U) \to L^\infty([0,T], H) \bigcap L^2([0,T], \tilde{H}^{1,0}) \bigcap C([0,T], H^{-1})$$

such that $u^\varepsilon = g^\varepsilon(W)$ a.s..

Let us introduce the skeleton equation associated to (3.1), for $\phi \in L^2([0,T], l^2)$:

$$\begin{aligned} &dz^\phi(t) = \partial_1^2 z^\phi(t)dt - B(z^\phi(t))dt + \sigma(t, z^\phi(t))\phi(t)dt, \\ &\text{div } z^\phi = 0, \\ &z^\phi(0) = u_0. \end{aligned} \tag{3.2}$$

Define $g^0 : C([0,T], U) \to L^\infty([0,T], H) \bigcap L^2([0,T], \tilde{H}^{1,0}) \bigcap C([0,T], H^{-1})$ by

$$g^0(h) := \begin{cases} z^\phi, & \text{if } h = \int_0^\cdot \phi(s)ds \text{ for some } \phi \in L^2([0,T], l^2); \\ 0, & \text{otherwise.} \end{cases}$$

Then the rate function can be written as

$$I(z) = \inf\left\{ \frac{1}{2}\int_0^T \|\phi(s)\|_{l^2}^2 ds : \ z = z^\phi, \ \phi \in L^2([0,T], l^2) \right\}, \tag{3.3}$$

where $z \in L^\infty([0,T], H) \bigcap L^2([0,T], \tilde{H}^{1,0}) \bigcap C([0,T], H^{-1})$.

The main result of this chapter is the following one:

Theorem 3.1. *Assume (A0)-(A3) hold with $K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{5}, L_2 = 0$ and $u_0 \in \tilde{H}^{0,1}$, then u^ε satisfies a large deviation principle on $L^\infty([0,T], H) \bigcap L^2([0,T], \tilde{H}^{1,0}) \bigcap C([0,T], H^{-1})$ with the good rate function I given by (3.3).*

3.1 Two equations

In this section we give existence and uniqueness of solutions to two equations which will be used in the proof of the main result. The first one we consider is the skeleton equation (3.2).

An element $z^\phi \in L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$ is called a (weak) solution to (3.2) if for any $\varphi\in (C_0^\infty([0,T]\times\mathbb{T}^2))^2$ with $\mathrm{div}\varphi=0$, and $t>0$,

$$\langle z^\phi(t),\varphi(t)\rangle=\langle u_0,\varphi(0)\rangle+\int_0^t\langle z^\phi,\partial_t\varphi\rangle-\langle\partial_1 z^\phi,\partial_1\varphi\rangle+\langle -B(z^\phi)+\sigma(s,z^\phi)\phi,\varphi\rangle ds.$$

The following Lemma gives existence and uniqueness of the weak solution to (3.2) which can be obtained by the same method as in [LZZ18].

Lemma 3.2. *Assume (A0)-(A3) hold with $L_2=0$. For all $u_0\in\tilde{H}^{0,1}$ and $\phi\in L^2([0,T],l^2)$ there exists a unique solution*

$$z^\phi\in L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})\bigcap C([0,T],H^{-1})$$

to (3.2).

Proof First we give some a priori estimates for z^ϕ. By taking H inner product of (3.2) with z^ϕ and using div $z^\phi=0$, we have

$$\begin{aligned}
&\|z^\phi(t)\|_H^2+2\int_0^t\|\partial_1 z^\phi(s)\|_H^2ds\\
=&\|u_0\|_H^2+2\int_0^t\langle z^\phi(s),\sigma(s,z^\phi(s))\phi(s)\rangle ds\\
\leqslant&\|u_0\|_H^2+2\int_0^t\|z^\phi(s)\|_H\|\sigma(s,z^\phi(s))\|_{L_2(l^2,H)}\|\phi(s)\|_{l^2}ds\\
\leqslant&\|u_0\|_H^2+2\int_0^t\left(\|z^\phi(s)\|_H^2\|\phi(s)\|_{l^2}^2+K_0+K_1\|z^\phi(s)\|_H^2+K_2\|\partial_1 z^\phi(s)\|_H^2\right)ds,
\end{aligned}$$

where we used (A1) in the last inequality.

Hence by Gronwall's inequality, we have

$$\|z^\phi(t)\|_H^2+\int_0^t\|\partial_1 z^\phi(s)\|_H^2ds\leqslant(\|u_0\|_H^2+C)e^{C\int_0^t(\|\phi(s)\|_{l^2}^2+1)ds}. \tag{3.4}$$

Similarly, we have

$$\begin{aligned}
&\|z^\phi(t)\|_{\tilde{H}^{0,1}}^2+2\int_0^t(\|\partial_1 z^\phi(s)\|_H^2+\|\partial_1\partial_2 z^\phi(s)\|_H^2)ds\\
=&\|u_0\|_{\tilde{H}^{0,1}}^2-2\int_0^t\langle\partial_2 z^\phi(s),\partial_2(z^\phi\cdot\nabla z^\phi)(s)\rangle ds+2\int_0^t\langle z^\phi(s),\sigma(s,z^\phi(s))\phi(s)\rangle_{\tilde{H}^{0,1}}ds\\
\leqslant&\|u_0\|_{\tilde{H}^{0,1}}^2+\int_0^t(\frac{1}{5}\|\partial_1\partial_2 z^\phi(s)\|_H^2+C(1+\|\partial_1 z^\phi(s)\|_H^2)\|\partial_2 z^\phi(s)\|_H^2)ds\\
&+2\int_0^t(\|z^\phi(s)\|_{\tilde{H}^{0,1}}^2\|\phi(s)\|_{l^2}^2+\|\sigma(s,z^\phi(s))\|_{L_2(l^2,\tilde{H}^{0,1})}^2)ds,
\end{aligned}$$

where we used Lemma 2.12 in the last inequality.

Hence by (A2) we deduce that

$$\begin{aligned}&\|z^{\phi}(t)\|^2_{\tilde{H}^{0,1}}+\int_0^t\|z^{\phi}(s)\|^2_{\tilde{H}^{1,1}}ds\\ \leqslant&\|u_0\|^2_{\tilde{H}^{0,1}}+C+C\int_0^t(1+\|\partial_1 z^{\phi}(s)\|^2_H+\|\phi(s)\|^2_{l^2})\|z^{\phi}(s)\|^2_{\tilde{H}^{0,1}}ds.\end{aligned}$$

Then by Gronwall's inequality and (3.4) we have

$$\|z^{\phi}(t)\|^2_{\tilde{H}^{0,1}}+\int_0^t\|z^{\phi}(s)\|^2_{\tilde{H}^{1,1}}ds\leqslant(\|u_0\|^2_{\tilde{H}^{0,1}}+C)e^{C(t,\phi,u_0)},\tag{3.5}$$

where

$$C(t,\phi,u_0)=C\left(\int_0^t(1+\|\phi(s)\|^2_{l^2})ds+(\|u_0\|^2_H+1)e^{C\int_0^t(1+\|\phi(s)\|^2_{l^2})ds}\right).$$

Now consider the following approximate equation:

$$\begin{cases}dz^{\phi}_{\epsilon}(t)=\partial_1^2 z^{\phi}_{\epsilon}(t)dt+\epsilon^2\partial_2^2 z^{\phi}_{\epsilon}(t)dt-B(z^{\phi}_{\epsilon}(t))dt+\sigma(t,z^{\phi}_{\epsilon}(t))\phi(t)dt,\\ \mathrm{div} z^{\phi}_{\epsilon}=0,\\ z^{\phi}_{\epsilon}(0)=u_0*j_{\epsilon},\end{cases}\tag{3.6}$$

where j is a smooth function on $\mathbb{R}^2$ with

$$j(x)=1,\ |x|\leqslant 1;\ j(x)=0,\ |x|\geqslant 2,$$

and

$$j_{\epsilon}(x)=\frac{1}{\epsilon^2}j(\frac{x}{\epsilon}).$$

It follows from classical theory on Navier-Stokes system that (3.6) has a unique global smooth solution z^{ϕ}_{ϵ} for any fixed ϵ. Furthermore, along the same line to (3.4) and (3.5) we have

$$\begin{aligned}&\|z^{\phi}_{\epsilon}(t)\|^2_H+\int_0^t\|\partial_1 z^{\phi}_{\epsilon}(s)\|^2_H ds+\epsilon^2\int_0^t\|\partial_2 z^{\phi}_{\epsilon}(s)\|^2_H ds\leqslant(\|u_0\|^2_H+C)e^{C\int_0^t(\|\phi(s)\|^2_{l^2}+1)ds},\\ &\|\partial_2 z^{\phi}_{\epsilon}(t)\|^2_H+\int_0^t\|\partial_1\partial_2 z^{\phi}_{\varepsilon}(s)\|^2_H ds+\epsilon^2\int_0^t\|\partial_2^2 z^{\phi}_{\epsilon}(s)\|^2_H ds\leqslant(\|u_0\|^2_{\tilde{H}^{0,1}}+C)e^{C(t,\phi,u_0)},\end{aligned}\tag{3.7}$$

The following follows a similar argument as in the proof of [LZZ18, Theorem 3.1]. By (3.7), we have $\{z^{\phi}_{\epsilon}\}_{\epsilon>0}$ is uniformly bounded in $L^{\infty}([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})$, hence bounded in $L^4([0,T],H^{\frac{1}{2}})$ (by interpolation) and $L^4([0,T],L^4(\mathbb{T}^2))$ (by Sobolev embedding). Thus $B(z^{\phi}_{\epsilon})$ is uniformly bounded in $L^2([0,T],H^{-1})$. Let $p\in(1,\frac{4}{3})$, we have

$$\begin{aligned}\int_0^T\|\sigma(s,z^{\phi}_{\epsilon}(s))\phi(s)\|^p_{H^{-1}}ds\leqslant&\int_0^T\|\sigma(s,z^{\phi}_{\epsilon}(s))\|^p_{L_2(l^2,H^{-1})}\|\phi(s)\|^p_{l^2}ds\\ \leqslant&C\int_0^T(1+\|\sigma(s,z^{\phi}_{\epsilon}(s))\|^4_{L_2(l^2,H^{-1})}+\|\phi(s)\|^2_{l^2})ds\end{aligned}$$

$$\leqslant C\int_0^T (1+\|z_\epsilon^\phi(s))\|_H^4+\|\phi(s)\|_{l^2}^2)ds<\infty,$$

where we used Young's inequality in the second line and (A0) in the third line. It comes out that

$$\{\partial_t z_\epsilon^\phi\}_{\epsilon>0} \text{ is uniformly bounded in } L^p([0,T],H^{-1}). \tag{3.8}$$

Thus by Aubin-Lions lemma (see [LZZ18, Lemma 3.6]), there exists a $z^\phi\in L^2([0,T],H)$ such that

$$z_\epsilon^\phi\to z^\phi \text{ strongly in } L^2([0,T],H) \text{ as } \epsilon\to 0 \text{ (in the sense of subsequence).}$$

Since $\{z_\epsilon^\phi\}_{\epsilon>0}$ is uniformly bounded in $L^\infty([0,T],\tilde H^{0,1})\bigcap L^2([0,T],\tilde H^{1,1})$, there exists a $\tilde z\in L^\infty([0,T],\tilde H^{0,1})\bigcap L^2([0,T],\tilde H^{1,1})$ such that

$$z_\epsilon^\phi\to\tilde z \text{ weakly in } L^2([0,T],\tilde H^{1,1}) \text{ as } \epsilon\to 0 \text{ (in the sense of subsequence).}$$

$$z_\epsilon^\phi\to\tilde z \text{ weakly star in } L^\infty([0,T],\tilde H^{0,1}) \text{ as } \epsilon\to 0 \text{ (in the sense of subsequence).}$$

By the uniqueness of weak convergence limit, we deduce that $z^\phi=\tilde z$. By (3.8) and [FG95, Theorem 2.2], we also have for any $\delta>0$

$$z_\epsilon^\phi\to z^\phi \text{ strongly in } C([0,T],H^{-1-\delta}) \text{ as } \epsilon\to 0 \text{ (in the sense of subsequence).}$$

Now we use the above convergence to prove that z^ϕ is a solution to (3.2). Note that for any $\varphi\in C^\infty([0,T]\times\mathbb{T}^2)$ with $\mathrm{div}\varphi=0$, for any $t\in[0,T]$, z_ϵ^ϕ satisfies

$$\begin{aligned}\langle z_\epsilon^\phi(t),\varphi(t)\rangle=&\langle u_0,\varphi(0)\rangle\\&+\int_0^t\langle z_\epsilon^\phi,\partial_t\varphi\rangle-\langle\partial_1 z_\epsilon^\phi,\partial_1\varphi\rangle-\epsilon^2\langle\partial_2 z_\epsilon^\phi,\partial_2\varphi\rangle+\langle -B(z_\epsilon^\phi)+\sigma(s,z_\epsilon^\phi)\phi,\varphi\rangle ds.\end{aligned} \tag{3.9}$$

By [Tem79, Chapter 3, Lemma 3.2] we have

$$\int_0^t\langle -B(z_\epsilon^\phi),\varphi\rangle ds\to\int_0^t\langle -B(z^\phi),\varphi\rangle ds \text{ as } \epsilon\to 0.$$

For the last term in the right hand side of (3.9), we have

$$\begin{aligned}&\int_0^t\langle\sigma(s,z_\epsilon^\phi)\phi-\sigma(s,z^\phi)\phi,\varphi\rangle ds\\ \leqslant&\int_0^t\|(\sigma(s,z_\epsilon^\phi)-\sigma(s,z^\phi))\phi\|_H\|\varphi\|_H ds\\ \leqslant& C\int_0^t\|\sigma(s,z_\epsilon^\phi)-\sigma(s,z^\phi)\|_{L_2(l^2,H)}\|\phi\|_{l^2}ds\\ \leqslant& C\left(\int_0^t\|z_\epsilon^\phi-z^\phi\|_H^2ds\right)^{\frac12}\left(\int_0^t\|\phi(s)\|_{l^2}^2ds\right)^{\frac12},\end{aligned}$$

where we used Hölder's inequality and (A3) with $L_2=0$ in the last inequality.

Thus let $\epsilon\to 0$ in (3.9), we have $z^\phi\in L^\infty([0,T],\tilde H^{0,1})\bigcap L^2([0,T],\tilde H^{1,1})$ and

$$\partial_t z^\phi=\partial_1^2 z^\phi-B(z^\phi)+\sigma(t,z^\phi(t))\phi.$$

Since the right hand side belongs to $L^p([0,T],H^{-1})$, we deduce that

$$z^\phi \in L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})\bigcap C([0,T],H^{-1}).$$

For uniqueness, let $z_1^\phi, z_2^\phi \in L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})\bigcap C([0,T],H^{-1})$ be two solutions to (3.2) and $w^\phi = z_1^\phi - z_2^\phi$. Then we have

$$\begin{aligned}
&\|w^\phi(t)\|_H^2 + 2\int_0^t \|\partial_1 w^\phi(s)\|_H^2 ds\\
=&\|w^\phi(0)\|_H^2 - 2\int_0^t \langle w^\phi(s), B(z_1^\phi)(s) - B(z_2^\phi)(s)\rangle ds\\
&+2\int_0^t \langle w^\phi(s), \sigma(s,z_1^\phi(s))\phi(s) - \sigma(s,z_2^\phi(s))\phi(s)\rangle ds\\
\leqslant&\|w^\phi(0)\|_H^2 - 2\int_0^t b(w^\phi(s), z_2^\phi(s), w^\phi(s))ds\\
&+2\int_0^t \|w^\phi(s)\|_H\|\sigma(s,z_1^\phi(s)) - \sigma(s,z_2^\phi(s))\|_{L_2(l^2,H)}\|\phi(s)\|_{l^2}ds\\
\leqslant&\|w^\phi(0)\|_H^2 + \int_0^t \frac{1}{5}\|\partial_1 w^\phi(s)\|_H^2 ds + C\int_0^t (1+\|z_2^\phi(s)\|_{\tilde{H}^{1,1}}^2)\|w^\phi(s)\|_H^2 ds\\
&+\int_0^t (\|w^\phi(s)\|_H^2\|\phi(s)\|_{l^2}^2 + L_1\|w^\phi(s)\|_H^2)ds,
\end{aligned}$$

where we used Lemma 2.10 in the sixth line and (A3) with $L_2 = 0$ in the last line.

Then by Gronwall's inequality we have

$$\|w^\phi(t)\|_H^2 \leqslant \|w^\phi(0)\|_H^2 e^{C\int_0^t (1+\|z_2^\phi(s)\|_{\tilde{H}^{1,1}}^2 + \|\phi(s)\|_{l^2}^2)ds},$$

which along with the fact that $z_2^\phi \in L^2([0,T],\tilde{H}^{1,1})$ and $\phi \in L^2([0,T],l^2)$ implies that $w^\phi(t) = 0$. That is: $z_1^\phi = z_2^\phi$.

□

For next step, consider the following equation:

$$\begin{cases}
dZ_v^\varepsilon(t) = \partial_1^2 Z_v^\varepsilon(t)dt - B(Z_v^\varepsilon(t))dt + \sigma(t,Z_v^\varepsilon(t))v^\varepsilon(t)dt + \sqrt{\varepsilon}\sigma(t,Z_v^\varepsilon(t))dW(t),\\
\mathrm{div} Z_v^\varepsilon = 0,\\
Z_v^\varepsilon(0) = u_0,
\end{cases} \tag{3.10}$$

where $v^\varepsilon \in \mathcal{A}_N$ for some $N < \infty$. Here Z_v^ε should have been denoted $Z_{v^\varepsilon}^\varepsilon$ and the slight abuse of notation is for simplicity.

Lemma 3.3. *Assume (A0)-(A3) hold with $L_2 = 0$ and $v^\varepsilon \in \mathcal{A}_N$ for some $N < \infty$. Then $Z_v^\varepsilon = g^\varepsilon\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot v^\varepsilon(s)ds\right)$ is the unique strong solution to (3.10).*

Proof Since $v^\varepsilon \in \mathcal{A}_N$, by the Girsanov theorem (see [LR15, Appendix I]), $\tilde{W}(\cdot) := W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot v^\varepsilon(s)ds$ is an l^2-cylindrical Wiener-process under the probability measure

$$d\tilde{P} := \exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T v^\varepsilon(s)dW(s) - \frac{1}{2\varepsilon}\int_0^T \|v^\varepsilon(s)\|_{l^2}^2 ds\right\} dP.$$

Then $(Z_v^\varepsilon, \tilde{W})$ is the solution to (3.1) on the stochastic basis $(\Omega, \mathcal{F}, \tilde{P})$. By (A0) we have

$$\int_0^T \|\sigma(s, Z_v^\varepsilon(s))\|_{H^{-1}} ds < \infty.$$

Then (Z_v^ε, W) satisfies the condition of the definition of weak solution (see [LZZ18, Definition 4.1]) and hence is a weak solution to (3.10) on the stochastic basis $(\Omega, \mathcal{F}, P)$ and $Z_v^\varepsilon = g^\varepsilon\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot v^\varepsilon(s)ds\right)$.

If $\tilde{Z}_v^\varepsilon$ and Z_v^ε are two weak solutions to (3.10) on the same stochastic basis $(\Omega, \mathcal{F}, P)$. Let $W^\varepsilon = Z_v^\varepsilon - \tilde{Z}_v^\varepsilon$ and $q(t) = k\int_0^t (\|Z_v^\varepsilon(s)\|^2_{\tilde{H}^{1,1}} + \|v^\varepsilon(s)\|^2_{l^2})ds$ for some constant k. Applying Itô's formula to $e^{-q(t)}\|W^\varepsilon(t)\|_H^2$, we have

$$\begin{aligned}
&e^{-q(t)}\|W^\varepsilon(t)\|_H^2 + 2\int_0^t e^{-q(s)}\|\partial_1 W^\varepsilon(s)\|_H^2 ds \\
=&-k\int_0^t e^{-q(s)}\|W^\varepsilon(s)\|_H^2(\|Z_v^\varepsilon(s)\|^2_{\tilde{H}^{1,1}} + \|v^\varepsilon(s)\|^2_{l^2})ds - 2\int_0^t e^{-q(s)} b(W^\varepsilon, Z_v^\varepsilon, W^\varepsilon)ds \\
&+2\int_0^t e^{-q(s)}\langle \sigma(s, Z_v^\varepsilon)v^\varepsilon - \sigma(s, \tilde{Z}_v^\varepsilon)v^\varepsilon, W^\varepsilon(s)\rangle ds \\
&+2\sqrt{\varepsilon}\int_0^t e^{-q(s)}\langle W^\varepsilon(s), (\sigma(s, Z_v^\varepsilon) - \sigma(s, \tilde{Z}_v^\varepsilon))dW(s)\rangle \\
&+\varepsilon\int_0^t e^{-q(s)}\|\sigma(s, Z_v^\varepsilon) - \sigma(s, \tilde{Z}_v^\varepsilon)\|^2_{L_2(l^2, H)} ds.
\end{aligned}$$

By Lemma 2.10, there exists constants $\tilde{\alpha} \in (0,1)$ and $\tilde{C}$ such that

$$|b(W^\varepsilon, Z_v^\varepsilon, W^\varepsilon)| \leqslant \tilde{\alpha}\|\partial_1 W^\varepsilon\|_H^2 + \tilde{C}(1 + \|Z_v^\varepsilon\|^2_{\tilde{H}^{1,1}})\|W^\varepsilon\|_H^2.$$

We also have

$$\begin{aligned}
2|\langle \sigma(s, Z_v^\varepsilon)v^\varepsilon - \sigma(s, \tilde{Z}_v^\varepsilon)v^\varepsilon, W^\varepsilon\rangle| &\leqslant 2\|(\sigma(s, Z_v^\varepsilon) - \sigma(s, \tilde{Z}_v^\varepsilon))v^\varepsilon\|_H\|W^\varepsilon\|_H \\
&\leqslant \|\sigma(s, Z_v^\varepsilon) - \sigma(s, \tilde{Z}_v^\varepsilon)\|^2_{L_2(l^2, H)} + \|v^\varepsilon\|^2_{l^2}\|W^\varepsilon\|_H^2.
\end{aligned}$$

Let $k > 2\tilde{C}$ and we may assume $\varepsilon < \frac{16}{25}$, by (A3) with $L_2 = 0$ we have

$$\begin{aligned}
&e^{-q(t)}\|W^\varepsilon(t)\|_H^2 + (2 - 2\tilde{\alpha})\int_0^t e^{-q(s)}\|\partial_1 W^\varepsilon(s)\|_H^2 ds \\
\leqslant& C\int_0^t e^{-q(s)}\|W^\varepsilon(s)\|_H^2 ds + 2\sqrt{\varepsilon}\int_0^t e^{-q(s)}\langle W^\varepsilon(s), (\sigma(s, Z_v^\varepsilon) - \sigma(s, \tilde{Z}_v^\varepsilon))dW(s)\rangle.
\end{aligned}$$

By the Burkhölder-Davis-Gundy's inequality (see [LR15, Appendix D]), we have

$$\begin{aligned}
&2\sqrt{\varepsilon}|E[\sup_{r\in[0,t]}\int_0^r e^{-q(s)}\langle W^\varepsilon(s), (\sigma(s, Z_v^\varepsilon) - \sigma(s, \tilde{Z}_v^\varepsilon))dW(s)\rangle]| \\
\leqslant& 6\sqrt{\varepsilon}E\left(\int_0^t e^{-2q(s)}\|\sigma(s, Z_v^\varepsilon) - \sigma(s, \tilde{Z}_v^\varepsilon)\|^2_{L_2(l^2, H)}\|W^\varepsilon(s)\|_H^2 ds\right)^{\frac{1}{2}} \\
\leqslant& \sqrt{\varepsilon}E(\sup_{s\in[0,t]}(e^{-q(s)}\|W^\varepsilon(s)\|_H^2)) + 9\sqrt{\varepsilon}E\int_0^t e^{-q(s)}L_1\|W^\varepsilon(s)\|_H^2 ds,
\end{aligned}$$

where we used (A3) with $L_2 = 0$ and assume that $\tilde{\alpha} < 1$.

Thus we have

$$E(\sup_{s\in[0,t]}(e^{-q(s)}\|W^\varepsilon(s)\|_H^2)) \leqslant CE\int_0^t e^{-q(s)}\|W^\varepsilon(s)\|_H^2 ds.$$

By Gronwall's inequality we obtain $W^\varepsilon = 0$ P-a.s., i.e. $\tilde{Z}_v^\varepsilon = Z_v^\varepsilon$ P-a.s..

Then by the Yamada-Watanabe theorem, we have Z_v^ε is the unique strong solution to (3.10). □

3.2 Proof of Hypothesis 2

In this section we will show that I is a good rate function by checking the second part of Hypothesis 2.5. The proof follows essentially the same argument as in [WZZ15, Proposition 4.5].

Lemma 3.4. *Assume (A0)-(A3) hold with $L_2 = 0$. For all $N < \infty$, the set*

$$K_N = \left\{ g^0\left(\int_0^\cdot \phi(s)ds\right) : \phi \in S_N \right\}$$

is a compact subset in $L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$.

Proof By definition, we have

$$K_N = \left\{ z^\phi : \phi \in L^2([0,T],l^2),\ \int_0^T \|\phi(s)\|_{l^2}^2 ds \leqslant N \right\}.$$

Let $\{z^{\phi_n}\}$ be a sequence in K_N where $\{\phi_n\} \subset S_N$. Note that (3.5) implies that z^{ϕ_n} is uniformly bounded in $L^\infty([0,T],\tilde{H}^{1,0}) \cap L^2([0,T],\tilde{H}^{1,1})$. Thus by weak compactness of S_N, a similar argument as in the proof of Lemma 3.2 shows that there exists $\phi \in \mathcal{S}_N$ and $z' \in L^2([0,T],H)$ such that the following convergence hold as $n \to \infty$ (in the sense of subsequence):

$\phi_n \to \phi$ in $\mathcal{S}_N$ weakly,
$z^{\phi_n} \to z'$ in $L^2([0,T],\tilde{H}^{1,0})$ weakly,
$z^{\phi_n} \to z'$ in $L^\infty([0,T],H)$ weak-star,
$z^{\phi_n} \to z'$ in $L^2([0,T],H)$ strongly.
$z^{\phi_n} \to z'$ in $C([0,T],H^{-1-\delta})$ strongly for any $\delta > 0$.

Then for any $\varphi \in C^\infty([0,T]\times\mathbb{T}^2)$ with $\text{div}\varphi = 0$ and for any $t \in [0,T]$, z^{ϕ_n} satisfies

$$\langle z^{\phi_n}(t),\varphi(t)\rangle = \langle u_0,\varphi(0)\rangle + \int_0^t \langle z^{\phi_n},\partial_t\varphi\rangle - \langle \partial_1 z^{\phi_n},\partial_1\varphi\rangle + \langle -B(z^{\phi_n}) + \sigma(s,z^{\phi_n})\phi_n,\varphi\rangle ds. \tag{3.11}$$

Let $n \to \infty$, we have

$$\begin{aligned}&\int_0^t \langle \sigma(s,z^{\phi_n})\phi_n - \sigma(s,z')\phi,\varphi\rangle ds \\ =&\int_0^t \langle [\sigma(s,z^{\phi_n}) - \sigma(s,z')]\phi_n + \sigma(s,z')(\phi_n - \phi),\varphi\rangle ds\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_0^t \|(\sigma(s,z^{\phi_n})-\sigma(s,z'))\phi_n\|_H\|\varphi\|_H ds + \int_0^t \langle \sigma(s,z')(\phi_n-\phi),\varphi\rangle ds\\
&\leqslant C\int_0^t \|\sigma(s,z^{\phi_n})-\sigma(s,z')\|_{L_2(l^2,H)}\|\phi_n\|_{l^2} ds + \int_0^t \langle \sigma(s,z')(\phi_n-\phi),\varphi\rangle ds\\
&\leqslant C\left(\int_0^t \|z^{\phi_n}-z'\|_H^2 ds\right)^{\frac12}\left(\int_0^t \|\phi_n(s)\|_{l^2}^2 ds\right)^{\frac12} + \int_0^t \langle \sigma(s,z')(\phi_n-\phi),\varphi\rangle ds\\
&\to 0,
\end{aligned}
$$

where we used Hölder's inequality and (A3) with $L_2=0$ in the last inequality. By [Tem79, Chapter 3, Lemma 3.2] we also have

$$
\int_0^t \langle -B(z^{\phi_n}),\varphi\rangle ds \to \int_0^t \langle -B(z'),\varphi\rangle ds.
$$

Then we deduce that

$$
\langle z'(t),\varphi(t)\rangle = \langle u_0,\varphi(0)\rangle + \int_0^t \langle z',\partial_t\varphi\rangle - \langle \partial_1 z',\partial_1\varphi\rangle + \langle -B(z')+\sigma(s,z')\phi,\varphi\rangle ds,
$$

which implies that z' is a solution to (3.2). By the uniqueness of solution, we deduce that $z'=z^{\phi}$.

Our goal is to prove $z^{\phi_n}\to z^{\phi}$ in $L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$.

Let $w^n=z^{\phi_n}-z^{\phi}$, by a direct calculation, we have

$$
\begin{aligned}
&\|w^n(t)\|_H^2 + 2\int_0^t \|\partial_1 w^n(s)\|_H^2 ds\\
=&-2\int_0^t \langle w^n(s), B(z^{\phi_n})(s)-B(z^{\phi})(s)\rangle ds\\
&+2\int_0^t \langle w^n(s), \sigma(s,z^{\phi_n}(s))\phi_n(s)-\sigma(s,z^{\phi}(s))\phi(s)\rangle ds\\
=&-2\int_0^t b(w^n,z^{\phi},w^n)(s)ds + 2\int_0^t \langle w^n(s),(\sigma(s,z^{\phi_n}(s))-\sigma(s,z^{\phi}(s)))\phi_n(s)\rangle ds\\
&+2\int_0^t \langle w^n(s),\sigma(s,z^{\phi}(s))(\phi_n(s)-\phi(s))\rangle ds\\
\leqslant&\int_0^t \frac15\|\partial_1 w^n(s)\|_H^2 ds + C\int_0^t (1+\|z^{\phi}(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2 ds\\
&+C\int_0^t \|w^n(s)\|_H^2\|\phi_n(s)\|_{l^2} ds\\
&+\int_0^t \|w^n(s)\|_H\|\phi_n(s)-\phi(s)\|_{l^2}(K_0+K_1\|z^{\phi}(s)\|_H^2+K_2\|\partial_1 z^{\phi}(s)\|_H^2)^{\frac12} ds,
\end{aligned}
$$

where we used Lemma 2.10 in the sixth line, (A3) with $L_2=0$ in the seventh line and (A1) in the last line. Then we have

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|w^n(t)\|_H^2 + \int_0^T \|\partial_1 w^n(s)\|_H^2 ds\\
\leqslant& C\int_0^T (1+\|z^{\phi}(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&+C(\sup_{t\in[0,T]}\|z^{\phi_n}(t)\|_H+\sup_{t\in[0,T]}\|z^{\phi}(t)\|_H)\left(\int_0^T\|\phi_n(s)\|_{l^2}^2ds\right)^{\frac{1}{2}}\left(\int_0^T\|w^n(s)\|_H^2ds\right)^{\frac{1}{2}}\\
&+C\left(\int_0^T\|\phi_n(s)-\phi(s)\|_{l^2}^2ds\right)^{\frac{1}{2}}\left(\int_0^T(1+\|z^{\phi}(s)\|_H^2+\|\partial_1 z^{\phi}(s)\|_H^2)\|w^n(s)\|_H^2ds\right)^{\frac{1}{2}}\\
&\leqslant C\int_0^T(1+\|z^{\phi}(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2ds+C(N)\left(\int_0^T\|w^n(s)\|_H^2ds\right)^{\frac{1}{2}}\\
&+CN^{\frac{1}{2}}\left(\int_0^T(1+\|z^{\phi}(s)\|_H^2+\|\partial_1 z^{\phi}(s)\|_H^2)\|w^n(s)\|_H^2ds\right)^{\frac{1}{2}},
\end{aligned}
$$

where we used (3.4) and the fact that ϕ_n, ϕ are in $\mathcal{S}_N$.

For any $\epsilon>0$, let

$$A_\epsilon:=\{s\in[0,T];\|z^{\phi_n}(s)-z^{\phi}(s)\|_H>\epsilon\}.$$

Since $z^{\phi_n}\to z^{\phi}$ in $L^2([0,T],H)$ strongly, we have

$$\int_0^T\|w^n(s)\|_H^2ds\to 0,\ \text{as } n\to\infty$$

and $\lim_{n\to\infty}Leb(A_\epsilon)=0$, where $Leb(B)$ means the Lebesgue measure of $B\in\mathcal{B}(\mathbb{R})$. Thus we have

$$
\begin{aligned}
&\int_0^T(1+\|z^{\phi}(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2ds\\
\leqslant&\left(\int_{A_\epsilon}+\int_{[0,T]\setminus A_\epsilon}\right)(1+\|z^{\phi}(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2ds\\
\leqslant&C\epsilon+2\int_{A_\epsilon}(1+\|z^{\phi}(s)\|_{\tilde{H}^{1,1}}^2)(\|z^{\phi_n}(s)\|_H^2+\|z^{\phi}(s)\|_H^2)ds\\
\leqslant&C\epsilon+C\int_{A_\epsilon}(1+\|z^{\phi}(s)\|_{\tilde{H}^{1,1}}^2)ds\\
\to&\ C\epsilon \text{ as } n\to\infty,
\end{aligned}
$$

where we used (3.4) in the forth line and (3.5) in the last line. A similar argument also implies that

$$\int_0^T(1+\|z^{\phi}(s)\|_H^2+\|\partial_1 z^{\phi}(s)\|_H^2)\|w^n(s)\|_H^2ds\leqslant C\epsilon.$$

Hence we have

$$\sup_{t\in[0,T]}\|w^n(t)\|_H^2+\int_0^T\|\partial_1 w^n(s)\|_H^2ds\leqslant C\epsilon+C\sqrt{\epsilon}\text{ as } n\to\infty.$$

Since ϵ is arbitrary, we obtain that

$$z^{\phi^n}\to z^{\phi}\text{ strongly in } L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1}).$$

$\square$

3.3 Proof of Hypothesis 1

In this section we will prove the main result by checking the rest of Hypothesis 2.5.

Lemma 3.5. *Assume Z_v^ε is a solution to (3.10) with $v^\varepsilon \in \mathcal{A}_N$ and $\varepsilon < 1$ small enough. Then we have*

$$E(\sup_{t\in[0,T]} \|Z_v^\varepsilon(t)\|_H^4) + E\int_0^T \|Z_v^\varepsilon(s)\|_H^2 \|Z_v^\varepsilon(s)\|_{\tilde{H}^{1,0}}^2 ds + E\int_0^T \|\partial_1 Z_v^\varepsilon(s)\|_H^2 ds \leqslant C(N,u_0). \tag{3.12}$$

Moreover, there exists $k>0$ such that

$$E(\sup_{t\in[0,T]} e^{-kg(t)}\|Z_v^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2) + E\int_0^T e^{-kg(s)}\|Z_v^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds \leqslant C(N,u_0), \tag{3.13}$$

where $g(t)=\int_0^t \|Z_v^\varepsilon(s)\|_H^2 ds$ and $C(N,u_0)$ is a constant depend on N, u_0 but independent of ε.

Proof We prove (3.12) by two parts of estimates. For first step, applying Itô's formula to $\|Z_v^\varepsilon(t)\|_H^2$, we have

$$\begin{aligned}
&\|Z_v^\varepsilon(t)\|_H^2 + 2\int_0^t \|\partial_1 Z_v^\varepsilon(s)\|_H^2 ds\\
=&\|u_0\|_H^2 + 2\int_0^t \langle Z_v^\varepsilon(s), \sigma(s, Z_v^\varepsilon(s))v^\varepsilon(s)\rangle ds\\
&+2\sqrt{\varepsilon}\int_0^t \langle Z_v^\varepsilon(s), \sigma(s,Z_v^\varepsilon(s))dW(s)\rangle + \varepsilon\int_0^t \|\sigma(s,Z_v^\varepsilon(s))\|_{L_2(l^2,H)}^2 ds\\
\leqslant&\|u_0\|_H^2 + \int_0^t (\|Z^\varepsilon(s)\|_H^2\|v^\varepsilon(s)\|_{l^2}^2 + \|\sigma(s,Z_v^\varepsilon(s))\|_{L_2(l^2,H)}^2) ds\\
&+2\sqrt{\varepsilon}\int_0^t \langle Z_v^\varepsilon(s), \sigma(s,Z_v^\varepsilon(s))dW(s)\rangle + \varepsilon\int_0^t \|\sigma(s,Z_v^\varepsilon(s))\|_{L_2(l^2,H)}^2 ds\\
\leqslant&\|u_0\|_H^2 + \int_0^t \|Z_v^\varepsilon(s)\|_H^2\|v^\varepsilon(s)\|_{l^2}^2 ds + (1+\varepsilon)\int_0^t (K_0 + K_1\|Z_v^\varepsilon\|_H^2 + K_2\|\partial_1 Z_v^\varepsilon\|_H^2) ds\\
&+2\sqrt{\varepsilon}\int_0^t \langle Z_v^\varepsilon(s), \sigma(s,Z_v^\varepsilon(s))dW(s)\rangle,
\end{aligned}$$

where we used (A1) in the last inequality.

By Gronwall's inequality and $v^\varepsilon \in \mathcal{A}_N$,

$$\begin{aligned}
&\|Z_v^\varepsilon(t)\|_H^2 + (2-(1+\varepsilon)K_2)\int_0^t \|\partial_1 Z_v^\varepsilon(s)\|_H^2 ds\\
\leqslant&(\|u_0\|_H^2 + C + 2\sqrt{\varepsilon}\int_0^t \langle Z_v^\varepsilon(s), \sigma(s,Z_v^\varepsilon(s))dW(s)\rangle) e^{N+2K_1T}.
\end{aligned}$$

For the term in the right hand side, by the Burkhölder-Davis-Gundy inequality we have

$$2\sqrt{\varepsilon}e^{N+K_1T} E\left(\sup_{0\leqslant s\leqslant t} |\int_0^s \langle Z_v^\varepsilon(r), \sigma(r, Z_v^\varepsilon(r))dW(r)\rangle|\right)$$

$$
\begin{aligned}
&\leqslant 6\sqrt{\varepsilon}e^{N+K_1T}E\left(\int_0^t \|Z_v^\varepsilon(r)\|_H^2\|\sigma(r,Z_v^\varepsilon(r))\|_{L_2(l^2,H)}^2ds\right)^{\frac{1}{2}}\\
&\leqslant\sqrt{\varepsilon}E[\sup_{0\leqslant s\leqslant t}(\|Z_v^\varepsilon(s)\|_H^2)]+9\sqrt{\varepsilon}e^{2N+2K_1T}E\int_0^t[K_0+K_1\|Z_v^\varepsilon(s)\|_H^2+K_2\|\partial_1 Z_v^\varepsilon(s)\|_H^2]ds,
\end{aligned}
$$

where $(9\sqrt{\varepsilon}e^{2N+2K_1T}+1+\varepsilon)K_2-2<0$ (this can be done when $\varepsilon<(\frac{10}{9e^{2N+2K_1T}+1})^2$) and we used (A1) in the last inequality. Thus we have

$$
\begin{aligned}
&E[\sup_{s\in[0,t]}(\|Z_v^\varepsilon(t)\|_H^2)]+E\int_0^t\|\partial_1 Z_v^\varepsilon(s)\|_H^2ds\\
\leqslant &C(\|u_0\|_H^2+1)+C\int_0^t E[\sup_{r\in[0,s]}(\|Z_v^\varepsilon(r)\|_H^2)]ds.
\end{aligned}
$$

Then by Gronwall's inequality we have

$$
E(\sup_{0\leqslant t\leqslant T}\|Z_v^\varepsilon(t)\|_H^2)+E\int_0^T\|\partial_1 Z_v^\varepsilon(s)\|_H^2ds\leqslant C(1+\|u_0\|_H^2). \tag{3.14}
$$

The second step is similar to [LZZ18, Lemma 4.2]. By Itô's formula we have

$$
\begin{aligned}
\|Z_v^\varepsilon(t)\|_H^4=&\|u_0\|_H^4-4\int_0^t\|Z_v^\varepsilon\|_H^2\|\partial_1 Z_v^\varepsilon(s)\|_H^2ds\\
&+4\int_0^t\|Z_v^\varepsilon(s)\|_H^2\langle\sigma(s,Z_v^\varepsilon(s))v^\varepsilon(s),Z_v^\varepsilon(s)\rangle ds\\
&+2\varepsilon\int_0^t\|Z_v^\varepsilon(s)\|_H^2\|\sigma(s,Z_v^\varepsilon(s))\|_{L_2(l^2,H)}^2ds\\
&+4\varepsilon\int_0^t\|\sigma(s,Z_v^\varepsilon(s))^*(Z_v^\varepsilon)\|_{l^2}^2ds\\
&+4\sqrt{\varepsilon}\int_0^t\|Z_v^\varepsilon(s)\|_H^2\langle Z_v^\varepsilon(s),\sigma(s,Z_v^\varepsilon(s))dW(s)\rangle_H\\
=:&\|u_0\|_H^4-4\int_0^t\|Z_v^\varepsilon\|_H^2\|\partial_1 Z_v^\varepsilon(s)\|_H^2ds+I_1+I_2+I_3+I_4.
\end{aligned} \tag{3.15}
$$

By (A1) we have

$$
\begin{aligned}
I_1(t)\leqslant&4\int_0^t\|Z_v^\varepsilon(s)\|_H^2\|\sigma(s,Z_v^\varepsilon(s))\|_{L_2(l^2,H)}\|v^\varepsilon(s)\|_{l^2}\|Z_v^\varepsilon(s)\|_H ds\\
\leqslant&2\int_0^t\|Z_v^\varepsilon(s)\|_H^2(K_0+K_1\|Z_v^\varepsilon(s)\|_H^2+K_2\|\partial_1 Z_v^\varepsilon(s)\|_H^2+\|v^\varepsilon(s)\|_{l^2}^2\|Z_v^\varepsilon(s)\|_H^2)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2+I_3\leqslant&6\varepsilon\int_0^t\|\sigma(s,Z_v^\varepsilon(s))\|_{L_2(l^2,H)}^2\|Z_v^\varepsilon(s)\|_H^2ds\\
\leqslant&6\varepsilon\int_0^t(K_0+K_1\|Z_v^\varepsilon(s)\|_H^2+K_2\|\partial_1 Z_v^\varepsilon(s)\|_H^2)\|Z_v^\varepsilon(s)\|_H^2ds.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\|Z_v^\varepsilon(t)\|_H^4+(4-2K_2-6\varepsilon K_2)\int_0^t\|Z_v^\varepsilon(s)\|_H^2\|\partial_1 Z_v^\varepsilon(s)\|_H^2ds\\
\leqslant&\|u_0\|_H^4+I_4+(2+6\varepsilon)K_0\int_0^t\|Z_v^\varepsilon(s)\|_H^2ds+\int_0^t(2K_1+6\varepsilon K_1+2\|v^\varepsilon(s)\|_{l^2}^2)\|Z_v^\varepsilon(s)\|_H^4)ds.
\end{aligned}
$$

Since $v^\varepsilon\in\mathcal{A}_N$, by Gronwall's inequality we have

$$
\begin{aligned}
&\|Z_v^\varepsilon(t)\|_H^4+(4-2K_2-6\varepsilon K_2)\int_0^t\|Z_v^\varepsilon(s)\|_H^2\|\partial_1 Z_v^\varepsilon(s)\|_H^2ds\\
\leqslant&\left(\|u_0\|_H^4+I_4+(2+6\varepsilon)K_0\int_0^t\|Z_v^\varepsilon(s)\|_H^2ds\right)e^{8K_1T+N}.
\end{aligned}
$$

The Burkhölder-Davis-Gundy inequality, the Young's inequality and (A1) imply that

$$
\begin{aligned}
E(\sup_{s\in[0,t]}I_4(s))\leqslant&12\sqrt{\varepsilon}E\left(\int_0^t\|\sigma(s,Z_v^\varepsilon(s))\|_{L_2(l^2,H)}^2\|Z_v^\varepsilon(s)\|_H^6ds\right)^{\frac12}\\
\leqslant&\sqrt{\varepsilon}E(\sup_{s\in[0,t]}\|Z_v^\varepsilon(s)\|_H^4)\\
&+36\sqrt{\varepsilon}E\int_0^t(K_0+K_1\|Z_v^\varepsilon(s)\|_H^2+K_2\|\partial_1 Z_v^\varepsilon(s)\|_H^2)\|Z_v^\varepsilon(s)\|_H^2ds.
\end{aligned}
$$

Let ε small enough such that $2K_2+6\varepsilon K_2+36\sqrt{\varepsilon}K_2e^{8K_1T+N}<4$ and $\sqrt{\varepsilon}e^{8K_1T+N}<1$ (for instance $\varepsilon<(\frac{10}{3+18e^{8K_1T+N}})^2$). Then the above estimates and (3.12) imply that

$$
\begin{aligned}
&E(\sup_{s\in[0,t]}\|Z_v^\varepsilon(s)\|_H^4)+\int_0^t\|Z_v^\varepsilon(s)\|_H^2\|Z_v^\varepsilon(s)\|_{\tilde H^{1,0}}^2ds\\
\leqslant&C(N,u_0)+CE\int_0^t\|Z_v^\varepsilon(s)\|_H^4ds,
\end{aligned}
$$

which by Gronwall's inequality yields that

$$
E(\sup_{s\in[0,t]}\|Z_v^\varepsilon(s)\|_H^4)+\int_0^t\|Z_v^\varepsilon(s)\|_H^2\|Z_v^\varepsilon(s)\|_{\tilde H^{1,0}}^2ds\leqslant C(N,u_0).
$$

For (3.13), let $h(t)=kg(t)+\int_0^t\|v^\varepsilon(s)\|_{l^2}^2ds$ for some universal constant k. Applying Itô's formula to $e^{-h(t)}\|Z_v^\varepsilon(t)\|_{\tilde H^{0,1}}^2$ (by applying Itô's formula to its finite- dimension projection first and then passing to the limit), we have

$$
\begin{aligned}
&e^{-h(t)}\|Z_v^\varepsilon(t)\|_{\tilde H^{0,1}}^2+2\int_0^t e^{-h(s)}(\|\partial_1 Z_v^\varepsilon(s)\|_H^2+\|\partial_1\partial_2 Z_v^\varepsilon(s)\|_H^2)ds\\
=&\|u_0\|_{\tilde H^{0,1}}^2-\int_0^t e^{-h(s)}(k\|\partial_1 Z_v^\varepsilon(s)\|_H^2+\|v^\varepsilon(s)\|_{l^2}^2)\|Z_v^\varepsilon(s)\|_{\tilde H^{0,1}}^2ds\\
&+2\int_0^t e^{-h(s)}\langle\partial_2 Z_v^\varepsilon(s),\partial_2(Z_v^\varepsilon\cdot\nabla Z_v^\varepsilon)(s)\rangle ds+2\int_0^t e^{-h(s)}\langle Z_v^\varepsilon(s),\sigma(s,Z_v^\varepsilon(s))v^\varepsilon(s)\rangle_{\tilde H^{0,1}}ds\\
&+2\sqrt{\varepsilon}\int_0^t e^{-h(s)}\langle Z_v^\varepsilon(s),\sigma(s,Z_v^\varepsilon(s))dW(s)\rangle_{\tilde H^{0,1}}+\varepsilon\int_0^t e^{-h(s)}\|\sigma(s,Z_v^\varepsilon(s))\|_{L_2(l^2,\tilde H^{0,1})}^2ds.
\end{aligned}
$$

By Lemma 2.12, there exists a constant C_1 such that

$$|\langle \partial_2 Z_v^\varepsilon, \partial_2(Z_v^\varepsilon \cdot \nabla Z_v^\varepsilon)\rangle| \leqslant \frac{1}{2}\|\partial_1\partial_2 Z_v^\varepsilon\|_H^2 + C_1(1+\|\partial_1 Z_v^\varepsilon\|_H^2)\|\partial_2 Z_v^\varepsilon\|_H^2.$$

By Young's inequality,

$$2|\langle Z_v^\varepsilon(s), \sigma(s, Z_v^\varepsilon(s))v^\varepsilon(s)\rangle_{\tilde{H}^{0,1}}| \leqslant \|Z_v^\varepsilon\|_{\tilde{H}^{0,1}}^2\|v^\varepsilon\|_{l^2}^2 + \|\sigma(s, Z_v^\varepsilon)\|_{L_2(l^2,\tilde{H}^{0,1})}^2.$$

Choosing $k > 2C_1$, we have

$$\begin{aligned}
&e^{-h(t)}\|Z_v^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2 + \int_0^t e^{-h(s)}(\|\partial_1 Z_v^\varepsilon(s)\|_H^2 + \|\partial_1\partial_2 Z_v^\varepsilon(s)\|_H^2)ds\\
\leqslant&\|u_0\|_{\tilde{H}^{0,1}}^2 + C\int_0^t e^{-h(s)}\|\partial_2 Z_v^\varepsilon(s)\|_H^2 ds + (1+\varepsilon)\int_0^t e^{-h(s)}\|\sigma(s, Z_v^\varepsilon(s))\|_{L_2(l^2,\tilde{H}^{0,1})}^2 ds\\
&+ 2\sqrt{\varepsilon}\int_0^t e^{-h(s)}\langle Z_v^\varepsilon(s), \sigma(s, Z_v^\varepsilon(s))dW(s)\rangle_{\tilde{H}^{0,1}}.
\end{aligned}$$

By the Burkhölder-Davis-Gundy inequality we have

$$\begin{aligned}
&2\sqrt{\varepsilon}E\left(\sup_{s\in[0,t]}|\int_0^s e^{-h(r)}\langle Z_v^\varepsilon(r), \sigma(r, Z_v^\varepsilon(r))dW(r)\rangle_{\tilde{H}^{0,1}}|\right)\\
\leqslant&6\sqrt{\varepsilon}E\left(\int_0^t e^{-2h(s)}\|Z_v^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2\|\sigma(s, Z_v^\varepsilon(s))\|_{L_2(l^2,\tilde{H}^{0,1})}^2 ds\right)^{\frac{1}{2}}\\
\leqslant&\sqrt{\varepsilon}E[\sup_{s\in[0,t]}(e^{-h(s)}\|Z_v^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2)]\\
&+ 9\sqrt{\varepsilon}E\int_0^t e^{-h(s)}[\tilde{K_0} + \tilde{K_1}\|Z_v^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 + \tilde{K_2}(\|\partial_1 Z_v^\varepsilon(s)\|_H^2 + \|\partial_1\partial_2 Z_v^\varepsilon(s)\|_H^2)]ds,
\end{aligned}$$

where $(9\sqrt{\varepsilon}+1+\varepsilon)\tilde{K_2} - 1 < 0$ (this can be done if $\varepsilon < \frac{9}{400}$) and we used (A2) in the last inequality.

Combine the above estimates, we have

$$\begin{aligned}
&E(\sup_{s\in[0,t]} e^{-h(s)}\|Z_v^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2) + E\int_0^t e^{-h(s)}\|Z_v^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds\\
\leqslant&C(\|u_0\|_{\tilde{H}^{0,1}}^2 + 1 + E\int_0^t e^{-h(s)}\|Z_v^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 ds)
\end{aligned}$$

Then Gronwall's inequality implies that

$$E(\sup_{0\leqslant t\leqslant T} e^{-h(t)}\|Z_v^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2) + E\int_0^T e^{-h(s)}\|Z_v^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds \leqslant C(1+\|u_0\|_{\tilde{H}^{0,1}}^2).$$

Since $v^\varepsilon \in \mathcal{S}_N$, we deduce that

$$E(\sup_{t\in[0,T]} e^{-kg(t)}\|Z_v^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2) + E\int_0^T e^{-kg(s)}\|Z_v^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds \leqslant C(1+\|u_0\|_{\tilde{H}^{0,1}}^2)e^N. \tag{3.16}$$

□

Similar as [LZZ18, lemma 4.3], we have the following tightness lemma:

Lemma 3.6. *Assume Z_v^ε is a solution to (3.10) with $v^\varepsilon \in \mathcal{A}_N$ and $\varepsilon < 1$ small enough. There exists $\varepsilon_0 > 0$, such that $\{Z_v^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ is tight in the space*

$$\chi = C([0,T],H^{-1})\bigcap L^2([0,T],H)\bigcap L^2_w([0,T],\tilde{H}^{1,1})\bigcap L^\infty_{w^*}([0,T],\tilde{H}^{0,1}),$$

where L^2_w denotes the weak topology, $L^\infty_{w^*}$ denotes the weak star topology and χ equipped with the topology τ_χ generated by the four subspace topology of the four intersecting spaces.

Proof Note that the law of Z_v^ε is defined on the path space $C([0,T],H^{-1})$. First we should point out that it can be restricted to χ. We denote the space $C([0,T],H^{-1})$ by X with Borel σ-algebra $\mathcal{B}(X)$.

For $N\in\mathbb{N}$, let

$$Y_N := \{w \in L^2([0,T],\tilde{H}^{1,1}) : \|w\|_{L^2([0,T],\tilde{H}^{1,1})} \leqslant N\},$$

equipped with the weak topology on $L^2([0,T],\tilde{H}^{1,1})$. Then Y_N is compact and metrizable, hence separable and complete.

Similarly, let

$$Z_N := \{w \in L^\infty([0,T],\tilde{H}^{0,1}) : \|w\|_{L^\infty([0,T],\tilde{H}^{0,1})} \leqslant N\},$$

equipped with the weak star topology on $L^\infty([0,T],\tilde{H}^{0,1})$. Then Z_N is compact and metrizable, hence separable and complete.

Define

$$\chi_N = C([0,T],H^{-1})\bigcap L^2([0,T],H)\bigcap Y_N\bigcap Z_N := X_1\cap X_2\cap X_3\cap X_4,$$

where X_i are complete separable metric spaces with metric d_i, $i=1,2,3,4$. Let χ_N be equipped with the metric $d=\max\{d_1,d_2,d_3,d_4\}$. Then χ_N is separable. To show that χ_N is complete, it is enough to show that if $w_k\in\chi_N, k\in\mathbb{N}$ and $w_k\to w^{(i)}\in X_i$ in d_i for every $1\leqslant i\leqslant 4$, then $w^{(1)}=w^{(2)}=w^{(3)}=w^{(4)}$. This is true since obviously we have the continuous embedding

$$X_i\subset \mathcal{M}([0,T],H^{-2}),\quad 1\leqslant i\leqslant 4,$$

where $\mathcal{M}$ denotes the space of Radon measures. Hence (χ_N,d) is a complete separable metric space. Furthermore, the following embeddings are continuous and hence measurable:

$$(\chi_N,d)\subset X.$$

Therefore by Kuratowski's theorem we have for the Borel σ-algebra $\mathcal{B}(\chi_N)$ of (χ_N,d),

$$\chi_N\in\mathcal{B}(X),\quad \mathcal{B}(\chi_N)=\mathcal{B}(X)\cap\chi_N.$$

Consequently, $\chi=\cup\chi_N\in\mathcal{B}(X)$.

Note that χ_N is a τ_χ-closed subset of χ. Let $A\subset\chi$ be τ_χ-closed. Then $A\cap\chi_N$ is τ_χ-closed too, hence

$$\begin{aligned}A\cap\chi_N &\in\mathcal{B}(\chi_N)\\ &=\mathcal{B}(X)\cap\chi_N=\{B\in\mathcal{B}(X):B\subset\chi_N\}\\ &\subset\{B\in\mathcal{B}(X):B\subset\chi\}\\ &\subset\mathcal{B}(X)\cap\chi.\end{aligned}$$

Hence

$$A=\bigcup_{N=1}^{\infty} A\cap\chi_N\in\mathcal{B}(X)\cap\chi$$

and

$$\mathcal{B}(\tau_\chi)\subset\mathcal{B}(X)\cap\chi.$$

Since $\chi\subset X$ continuously, hence measurably, we have $\mathcal{B}(X)\cap\chi\subset\mathcal{B}(\tau_\chi)$. Then

$$\mathcal{B}(\tau_\chi)=\mathcal{B}(X)\cap\chi.$$

Thus any probability measure on X can be restricted on χ.

Let k be the same constant as in the proof of (3.13) and let

$$\begin{aligned}K_R:=\Big\{u\in C([0,T],H^{-1}):\sup_{t\in[0,T]}\|u(t)\|_H^2+\int_0^T\|u(t)\|_{\tilde{H}^{1,0}}^2dt+\|u\|_{C^{\frac{1}{16}}([0,T],H^{-1})}\\ +\sup_{t\in[0,T]}e^{-k\int_0^t\|\partial_1u(s)\|_H^2ds}\|u(t)\|_{\tilde{H}^{0,1}}^2+\int_0^Te^{-k\int_0^t\|\partial_1u(s)\|_H^2ds}\|u(t)\|_{\tilde{H}^{1,1}}^2dt\leqslant R\Big\},\end{aligned}$$

where $C^{\frac{1}{16}}([0,T],H^{-1})$ is the Hölder space with the norm:

$$\|f\|_{C^{\frac{1}{16}}([0,T],H^{-1})}=\sup_{0\leqslant s<t\leqslant T}\frac{\|f(t)-f(s)\|_{H^{-1}}}{|t-s|^{\frac{1}{16}}}.$$

Then from the proof of [LZZ18, Lemma 4.3], we know that for any $R>0$, K_R is relatively compact in χ.

Now we only need to show that for any $\delta>0$, there exists $R>0$, such that $P(Z_v^\varepsilon\in K_R)>1-\delta$ for any $\varepsilon\in(0,\varepsilon_0)$, where ε_0 is the constant such that Lemma 3.5 hold.

By Lemma 3.5 and Chebyshev inequality, we can choose R_0 large enough such that

$$P\left(\sup_{t\in[0,T]}\|Z_v^\varepsilon(t)\|_H^2+\int_0^T\|Z_v^\varepsilon(t)\|_{\tilde{H}^{1,0}}^2dt>\frac{R_0}{3}\right)<\frac{\delta}{4},$$

and

$$P\left(\sup_{t\in[0,T]}e^{-k\int_0^t\|\partial_1u(s)\|_H^2ds}\|u(t)\|_{\tilde{H}^{0,1}}^2+\int_0^Te^{-k\int_0^t\|\partial_1u(s)\|_H^2ds}\|u(t)\|_{\tilde{H}^{1,1}}^2dt>\frac{R_0}{3}\right)<\frac{\delta}{4},$$

where k is the same constant as in (3.13).

Fix R_0 and let

$$\begin{aligned}\hat{K}_{R_0}=\Big\{u\in C([0,T],H^{-1}):\sup_{t\in[0,T]}\|u(t)\|_H^2+\int_0^T\|u(t)\|_{\tilde{H}^{1,0}}^2dt\leqslant\frac{R_0}{3}\text{ and}\\ \sup_{t\in[0,T]}e^{-k\int_0^t\|\partial_1u(s)\|_H^2ds}\|u(t)\|_{\tilde{H}^{0,1}}^2+\int_0^Te^{-k\int_0^t\|\partial_1u(s)\|_H^2ds}\|u(t)\|_{\tilde{H}^{1,1}}^2dt\leqslant\frac{R_0}{3}\Big\}.\end{aligned}$$

Then $P(Z_v^\varepsilon\in C([0,T],H^{-1})\setminus\hat{K}_{R_0})<\frac{\delta}{2}$.

Now for $Z_v^\varepsilon\in\hat{K}_{R_0}$, we have $\partial_1^2Z_v^\varepsilon$ is uniformly bounded in $L^2([0,T],H^{-1})$. Similar as in Lemma 3.2, Z_v^ε is uniformly bounded in $L^4([0,T],H^{\frac{1}{2}})$ and $L^4([0,T],L^4(\mathbb{T}^2))$, thus $B(Z_v^\varepsilon)$ is uniformly bounded in $L^2([0,T],H^{-1})$. By Hölder's inequality, we have

$$\sup_{s,t\in[0,T],s\neq t}\frac{\|\int_s^t\partial_1^2Z_v^\varepsilon(r)+B(Z_v^\varepsilon(r))dr\|_{H^{-1}}^2}{|t-s|}\leqslant\int_0^T\|\partial_1^2Z_v^\varepsilon(r)+B(Z_v^\varepsilon(r))\|_{H^{-1}}^2dr\leqslant C(R_0),$$

where $C(R_0)$ is a constant depend on R_0. For any $p\in(1,\frac{4}{3})$, by Hölder's inequality, we have

$$\begin{aligned}\sup_{s,t\in[0,T],s\neq t}\frac{\|\int_s^t\sigma(r,Z_v^\varepsilon(r))v^\varepsilon(r)dr\|_{H^{-1}}^p}{|t-s|^{p-1}}\leqslant&\int_0^T\|\sigma(r,Z_v^\varepsilon(r))v^\varepsilon(r)\|_{H^{-1}}^pdr\\ \leqslant&\int_0^T\|\sigma(r,Z_v^\varepsilon(r))\|_{L_2(l^2,H^{-1})}^p\|v^\varepsilon(r)\|_{l^2}^pdr\\ \leqslant&C\int_0^T(1+\|Z_v^\varepsilon(r)\|_H^4+\|v^\varepsilon(r)\|_{l^2}^4)dr\\ \leqslant&C(R_0),\end{aligned}$$

where we used Young's inequality and (A0) in the third inequality.

Moreover, for any $0\leqslant s\leqslant t\leqslant T$, by Hölder's inequality we have

$$\begin{aligned}E\|\int_s^t\sigma(r,Z_v^\varepsilon(r))dW(r)\|_{H^{-1}}^4\leqslant&CE\left(\int_s^t\|\sigma(r,Z_v^\varepsilon(r))\|_{L_2(l^2,H^{-1})}^2dr\right)^2\\ \leqslant&C|t-s|E\int_s^t\|\sigma(r,Z_v^\varepsilon(r))\|_{L_2(l^2,H^{-1})}^4dr\\ \leqslant&C|t-s|^2(1+E(\sup_{t\in[0,T]}\|Z_v^\varepsilon(t)\|_H^4))\\ \leqslant&C|t-s|^2,\end{aligned}$$

where we used (A0) in the third inequality and (3.12) in the last inequality. Then by Kolmogorov's continuity criterion, for any $\alpha\in(0,\frac{1}{4})$, we have

$$E\left(\sup_{s,t\in[0,T],s\neq t}\frac{\|\int_s^t\sigma(r,Z_v^\varepsilon(r))dW(r)\|_{H^{-1}}^4}{|t-s|^{2\alpha}}\right)\leqslant C.$$

Choose $p=\frac{8}{7},\alpha=\frac{1}{8}$ in the above estimates, we deduce that there exists $R>R_0$ such that

$$\begin{aligned}&P\left(\|Z_v^\varepsilon\|_{C^{\frac{1}{16}}([0,T],H^{-1})}>\frac{R}{3},Z_v^\varepsilon\in\hat{K}_{R_0}\right)\\ \leqslant&\frac{E\left(\sup_{s,t\in[0,T],s\neq t}\frac{\|Z_v^\varepsilon(t)-Z_v^\varepsilon(s)\|_{H^{-1}}}{|t-s|^{\frac{1}{16}}}1_{\{Z_v^\varepsilon\in\hat{K}_{R_0}\}}\right)}{\frac{R}{3}}<\frac{\delta}{2}.\end{aligned}$$

Combining the fact that $P(Z_v^\varepsilon\in C([0,T],H^{-1})\setminus\hat{K}_{R_0})<\frac{\delta}{2}$, we finish the proof. □

Lemma 3.7. *Assume (A0)-(A3) hold with $K_2<\frac{2}{21},\tilde{K}_2<\frac{1}{5},L_2=0$. Let $\{v^\varepsilon\}_{\varepsilon>0}\subset\mathcal{A}_N$ for some $N<\infty$. Assume v^ε converge to v in distribution as S_N-valued random elements, then*

$$g^\varepsilon\left(W(\cdot)+\frac{1}{\sqrt{\varepsilon}}\int_0^\cdot v^\varepsilon(s)ds\right)\to g^0\left(\int_0^\cdot v(s)ds\right)$$

in distribution as $\varepsilon\to0$.

Proof The proof follows essentially the same argument as in [WZZ15, Proposition 4.7].

By Lemma 3.3, we have $Z_v^\varepsilon = g^\varepsilon\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot v^\varepsilon(s)ds\right)$. By a similar but simple argument as in the proof of Lemmas 3.2 and 3.5, there exists a unique strong solution $Y^\varepsilon \in L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})\bigcap C([0,T],H^{-1})$ satisfying

$$\begin{aligned} dY^\varepsilon(t) =& \partial_1^2 Y^\varepsilon(t)dt + \sqrt{\varepsilon}\sigma(t,Z_v^\varepsilon(t))dW(t),\\ \text{div } Y^\varepsilon =& 0,\\ Y^\varepsilon(0) =& 0, \end{aligned}$$

and

$$\lim_{\varepsilon\to 0}\left[E\sup_{t\in[0,T]}\|Y^\varepsilon(t)\|_H^2 + E\int_0^T\|Y^\varepsilon(t)\|_{\tilde{H}^{1,0}}^2 dt\right] = 0,$$

$$\lim_{\varepsilon\to 0}\left[E\sup_{t\in[0,T]}(e^{-kg(t)}\|Y^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2) + E\int_0^T e^{-kg(t)}\|Y^\varepsilon(t)\|_{\tilde{H}^{1,1}}^2 dt\right] = 0,$$

where $g(t) = \int_0^t \|Z_v^\varepsilon(s)\|_H^2 ds$ and k are the same as in (3.13).

Set

$$\Xi := \left(\chi, \mathcal{S}_N, L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T].H^{-1})\right).$$

The above limit implies that $Y^\varepsilon \to 0$ in $L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T].H^{-1})$ almost surely as $\varepsilon\to 0$ (in the sense of subsequence). By Lemma 3.6 the family $\{(Z_v^\varepsilon, v^\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0)}$ is tight in $(\chi,\mathcal{S}_N)$. Let $(Z_v, v, 0)$ be any limit point of $\{(Z_v^\varepsilon, v^\varepsilon, Y^\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0)}$. Our goal is to show that Z_v has the same law as $g^0\left(\int_0^\cdot v(s)ds\right)$ and Z_v^ε convergence in distribution to Z_v in the space $L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$.

By the Skorokhod Theorem, there exists a stochastic basis $(\tilde{\Omega},\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]},\tilde{P})$ and, on this basis, Ξ-valued random variables $(\tilde{Z}_v,\tilde{v},0)$, $(\tilde{Z}_v^\varepsilon,\tilde{v}^\varepsilon,\tilde{Y}^\varepsilon)$, such that $(\tilde{Z}_v^\varepsilon,\tilde{v}^\varepsilon,\tilde{Y}^\varepsilon)$ (respectively $(\tilde{Z}_v,\tilde{v},0)$) has the same law as $(Z_v^\varepsilon, v^\varepsilon, Y^\varepsilon)$ (respectively $(Z_v, v, 0)$), and $(\tilde{Z}_v^\varepsilon,\tilde{v}^\varepsilon,\tilde{Y}^\varepsilon)\to(\tilde{Z}_v,\tilde{v},0)$, $\tilde{P}$-a.s.

We have

$$\begin{aligned} d(\tilde{Z}_v^\varepsilon(t)-\tilde{Y}^\varepsilon(t)) =& \partial_1^2(\tilde{Z}_v^\varepsilon(t)-\tilde{Y}^\varepsilon(t))dt - B(\tilde{Z}_v^\varepsilon(t))dt + \sigma(t,\tilde{Z}_v^\varepsilon(t))\tilde{v}^\varepsilon(t)dt,\\ \tilde{Z}_v^\varepsilon(0)-\tilde{Y}^\varepsilon(0) =& u_0, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} & P(\tilde{Z}_v^\varepsilon - \tilde{Y}^\varepsilon \in L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1}))\\ =& P(Z_v^\varepsilon - Y^\varepsilon \in L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1}))\\ =& 1. \end{aligned}$$

Let $\tilde{\Omega}_0$ be the subset of $\tilde{\Omega}$ such that for $\omega\in\tilde{\Omega}_0$,

$$(\tilde{Z}_v^\varepsilon,\tilde{v}^\varepsilon,\tilde{Y}^\varepsilon)(\omega)\to(\tilde{Z}_v,\tilde{v},0)(\omega) \text{ in } \Xi,$$

and

$$e^{-k\int_0^\cdot \|\tilde{Z}_v^\varepsilon(\omega,s)\|_H^2 ds}\tilde{Y}^\varepsilon(\omega)\to 0 \text{ in } L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})\bigcap C([0,T],H^{-1}),$$

then $P(\tilde{\Omega}_0) = 1$. For any $\omega \in \tilde{\Omega}_0$, fix ω, we have $\sup_\varepsilon \int_0^T \|\tilde{Z}_v^\varepsilon(\omega,s)\|_H^2 ds < \infty$, then we deduce that

$$\lim_{\varepsilon\to 0}\left(\sup_{t\in[0,T]}\|\tilde{Y}^\varepsilon(\omega,t)\|_{\tilde{H}^{0,1}} + \int_0^T \|\tilde{Y}^\varepsilon(\omega,t)\|_{\tilde{H}^{1,1}}^2 dt\right) = 0. \tag{3.18}$$

Now we show that

$$\sup_{t\in[0,T]}\|\tilde{Z}_v^\varepsilon(\omega,t) - \tilde{Z}_v(\omega,t)\|_H^2 + \int_0^T \|\tilde{Z}_v^\varepsilon(\omega,t) - \tilde{Z}_v(\omega,t)\|_{\tilde{H}^{1,0}}^2 dt \to 0 \text{ as } \varepsilon \to 0. \tag{3.19}$$

Let $Z^\varepsilon = \tilde{Z}_v^\varepsilon(\omega) - \tilde{Y}^\varepsilon(\omega)$, then by (3.17) we have

$$dZ^\varepsilon(t) = \partial_1^2 Z^\varepsilon(t)dt - B(Z^\varepsilon(t) + \tilde{Y}^\varepsilon(t))dt + \sigma(t, Z^\varepsilon(t) + \tilde{Y}^\varepsilon(t))\tilde{v}^\varepsilon(t)dt. \tag{3.20}$$

Since $Z^\varepsilon(\omega) \to \tilde{Z}_v(\omega)$ in χ, by a very similar argument as in Lemma 3.4 we deduce that $\tilde{Z}_v = z^{\tilde{v}} = g^0\left(\int_0^\cdot \tilde{v}(s)ds\right)$. Moreover, note that $\tilde{Z}_v^\varepsilon(\omega) \to z^{\tilde{v}}(\omega)$ weak star in $L^\infty([0,T],\tilde{H}^{0,1})$, then the uniform boundedness principle implies that

$$\sup_\varepsilon \sup_{t\in[0,T]}\|\tilde{Z}_v^\varepsilon(\omega)\|_{\tilde{H}^{0,1}} < \infty. \tag{3.21}$$

Let $w^\varepsilon = Z^\varepsilon - z^{\tilde{v}}$, then we have

$$\begin{aligned}\|w^\varepsilon(t)\|_H^2 + 2\int_0^t \|\partial_1 w^\varepsilon(s)\|_H^2 ds =& -2\int_0^t \langle w^\varepsilon(s), B(Z^\varepsilon + \tilde{Y}^\varepsilon) - B(z^{\tilde{v}})\rangle ds \\ &+ 2\int_0^t \langle w^\varepsilon(s), \sigma(s, Z^\varepsilon + \tilde{Y}^\varepsilon)\tilde{v}^\varepsilon(s) - \sigma(s, z^{\tilde{v}})\tilde{v}(s)\rangle ds.\end{aligned}$$

By Lemmas 2.10 and 2.11, we have

$$\begin{aligned}&\int_0^t \langle w^\varepsilon(s), B(Z^\varepsilon + \tilde{Y}^\varepsilon) - B(z^{\tilde{v}})\rangle ds \\ =& \int_0^t b(\tilde{Y}^\varepsilon, z^{\tilde{v}}, w^\varepsilon) + b(\tilde{Y}^\varepsilon, \tilde{Y}^\varepsilon, w^\varepsilon) + b(w^\varepsilon, \tilde{Y}^\varepsilon + z^{\tilde{v}}, w^\varepsilon) + b(z^{\tilde{v}}, \tilde{Y}^\varepsilon, w^\varepsilon)ds \\ \leqslant& \int_0^t [\frac{1}{2}\|\partial_1 w^\varepsilon(s)\|_H^2 + \frac{1}{2}\|\tilde{Y}^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 + C(1 + \|z^{\tilde{v}}(s)\|_{\tilde{H}^{1,1}}^2 + \|\tilde{Y}^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2)\|w^\varepsilon(s)\|_H^2]ds \\ &+ C\int_0^t \|\tilde{Y}^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 \|w^\varepsilon(s)\|_H ds \\ \leqslant& \int_0^t \frac{1}{2}\|\partial_1 w^\varepsilon(s)\|_H^2 ds + C\int_0^t \|\tilde{Y}^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds + C\int_0^t (1 + \|z^{\tilde{v}}(s)\|_{\tilde{H}^{1,1}}^2)\|w^\varepsilon(s)\|_H^2 ds,\end{aligned}$$

where we used the fact that by (3.18) and (3.21) w^ε are uniformly bounded in $L^\infty([0,T],H)$ in the last inequality. By (A1) and (A3) with $L_2 = 0$ we have

$$\begin{aligned}&\int_0^t \langle w^\varepsilon(s), \sigma(s, Z^\varepsilon + \tilde{Y}^\varepsilon)v^\varepsilon(s) - \sigma(s, z^{\tilde{v}})\tilde{v}(s)\rangle ds \\ =& \int_0^t \langle w^\varepsilon(s), (\sigma(s, Z^\varepsilon + \tilde{Y}^\varepsilon) - \sigma(s, z^{\tilde{v}}))\tilde{v}^\varepsilon(s)\rangle ds + \int_0^t \langle w^\varepsilon(s), \sigma(s, z^{\tilde{v}})(\tilde{v}^\varepsilon(s) - \tilde{v}(s))\rangle ds\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\int_0^t (\|w^\varepsilon(s)\|_H\|\tilde v^\varepsilon(s)\|_{l^2}(\|w^\varepsilon(s)\|_H^2+\|\tilde Y^\varepsilon(s)\|_H^2)^{\frac12}ds\\
&\quad+\int_0^t \|w^\varepsilon(s)\|_H\|\tilde v^\varepsilon(s)-\tilde v(s)\|_{l^2}(K_0+K_1\|z^{\tilde v}(s)\|_H^2+K_2\|\partial_1 z^{\tilde v}(s)\|_H^2)^{\frac12}ds\\
&\leqslant CN^{\frac12}\left(\int_0^t (\|w^\varepsilon(s)\|_H^2+\|\tilde Y^\varepsilon(s)\|_H^2 ds\right)^{\frac12}\\
&\quad+CN^{\frac12}\left(\int_0^t \|w^\varepsilon(s)\|_H^2(K_0+K_1\|z^{\tilde v}(s)\|_H^2+K_2\|\partial_1 z^{\tilde v}(s)\|_H^2)ds\right)^{\frac12},
\end{aligned}
$$

where we used the fact that w^ε are uniformly bounded in $L^\infty([0,T],H)$ and that $\tilde v^\varepsilon$, $\tilde v$ are in $\mathcal{A}_N$. Thus we have

$$
\begin{aligned}
&\|w^\varepsilon(t)\|_H^2+\int_0^t \|\partial_1 w^\varepsilon(s)\|_H^2 ds\\
\leqslant& C\int_0^t (1+\|z^{\tilde v}(s)\|_{\tilde H^{1,1}}^2)\|w^\varepsilon(s)\|_H^2 ds + C\int_0^t \|\tilde Y^\varepsilon(s)\|_{\tilde H^{1,1}}^2 ds\\
&+CN^{\frac12}\left(\int_0^t (\|w^\varepsilon(s)\|_H^2+\|\tilde Y^\varepsilon(s)\|_H^2)ds\right)^{\frac12}+CN^{\frac12}\left(\int_0^t (1+\|z^{\tilde v}(s)\|_{\tilde H^{1,1}}^2)\|w^\varepsilon(s)\|_H^2 ds\right)^{\frac12}.
\end{aligned}
$$

Since $Z^\varepsilon(\omega)\to z^{\tilde v}(\omega)$ strongly in $L^2([0,T],H)$ and $\tilde Y^\varepsilon\to 0$ in $L^2([0,T],\tilde H^{1,1})$, the same argument used in Lemma 3.4 implies

$$
\sup_{t\in[0,T]}\|\tilde Z_v^\varepsilon(\omega,t)-z^{\tilde v}(\omega,t)\|_H^2+\int_0^T \|\tilde Z_v^\varepsilon(\omega,t)-z^{\tilde v}(\omega,t)\|_{\tilde H^{1,0}}^2 dt\to 0 \text{ as } \varepsilon\to 0. \tag{3.22}
$$

The proof is thus complete.

□

Proof of Theorem 3.1. *The result holds from Lemmas 2.6, 3.4 and 3.7.* □

Chapter 4

Central limit theorem

In this chapter, we will establish the central limit theorem. Let u^ε be the solution to (1.3) and u^0 the solution to (1.4). Then we have the following estimates from Lemma 3.5, Lemma 4.1, Lemma 4.2 and Lemma 4.4 in [LZZ18]:

Lemma 4.1. *Assume (A0)-(A3) hold with $K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{5}, L_2 < \frac{1}{5}$, there exists $\varepsilon_0 > 0$ such that*

$$\sup_{\varepsilon\in(0,\varepsilon_0)} E\left(\sup_{t\in[0,T]} \|u^\varepsilon(t)\|_H^2 + \int_0^T \|u^\varepsilon(s)\|_{\tilde{H}^{1,0}}^2 ds\right) \leqslant C.$$

Particularly,

$$\sup_{t\in[0,T]} \|u^0(t)\|_{\tilde{H}^{0,1}}^2 + \int_0^T \|u^0(s)\|_{\tilde{H}^{1,1}}^2 ds \leqslant C.$$

We have the following $\tilde{H}^{0,2}$ estimate for u^0:

Lemma 4.2. *Given $u_0 \in \tilde{H}^{0,2}$, the unique solution u^0 to (1.4) satisfies the following estimate:*

$$\sup_{t\in[0,T]} \|u^0(t)\|_{\tilde{H}^{0,2}}^2 + \int_0^T \|u^0(t)\|_{\tilde{H}^{1,2}}^2 dt \leqslant C. \tag{4.1}$$

Proof Let's start by proving a priori estimates for u^0. Applying the operator Δ_k^v and using an L^2 energy estimate, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_k^0(t)\|_H^2 + \|\partial_1 u_k^0(t)\|_H^2 \leqslant \langle \Delta_k^v(u^0\cdot\nabla u^0), u_k^0\rangle,$$

where we denote by u_k^0 the term $\Delta_k^v u^0$. By Lemma 2.13 with $s=2, s_0=1$ and $u=v=u^0$, there exists $d_k \in l^1$ such that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_k^0(t)\|_H^2 + \|\partial_1 u_k^0(t)\|_H^2\\ \leqslant& C d_k 2^{-4k}\left(\|u^0\|_{\tilde{H}^{\frac{1}{4},2}}\|u^0\|_{\tilde{H}^{\frac{1}{4},1}}\|\partial_1 u^0\|_{\tilde{H}^{0,2}} + \|u^0\|_{\tilde{H}^{\frac{1}{4},2}}^2\|\partial_1 u^0\|_{\tilde{H}^{0,1}}\right).\end{aligned}$$

Now multiplying by 2^{4k} and taking sum over k gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^0(t)\|_{\tilde{H}^{0,2}}^2 + \|\partial_1 u^0(t)\|_{\tilde{H}^{0,2}}^2 \leqslant C\left(\|u^0\|_{\tilde{H}^{\frac{1}{4},2}}\|u^0\|_{\tilde{H}^{\frac{1}{4},1}}\|\partial_1 u^0\|_{\tilde{H}^{0,2}} + \|u^0\|_{\tilde{H}^{\frac{1}{4},2}}^2\|\partial_1 u^0\|_{\tilde{H}^{0,1}}\right).$$

By interpolation inequalities (see [BCD11, Theorem 2.80]) we have

$$\|u^0\|_{\tilde{H}^{\frac{1}{4},s}} \leqslant \|u^0\|^{\frac{3}{4}}_{\tilde{H}^{0,s}} \|u^0\|^{\frac{1}{4}}_{\tilde{H}^{1,s}},$$

where $s = 1, 2$. Thus we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^0(t)\|^2_{\tilde{H}^{0,2}} + \|\partial_1 u^0(t)\|^2_{\tilde{H}^{0,2}} \\
\leqslant & C\Big(\|u^0\|^{\frac{3}{4}}_{\tilde{H}^{0,2}}\|u^0\|_{\tilde{H}^{\frac{1}{4},1}}\|\partial_1 u^0\|^{\frac{5}{4}}_{\tilde{H}^{0,2}} + \|u^0\|_{\tilde{H}^{0,2}}\|u^0\|_{\tilde{H}^{\frac{1}{4},1}}\|\partial_1 u^0\|_{\tilde{H}^{0,2}} \\
&+ \|u^0\|^{\frac{3}{2}}_{\tilde{H}^{0,2}}\|\partial_1 u^0\|^{\frac{1}{2}}_{\tilde{H}^{0,2}}\|\partial_1 u^0\|_{\tilde{H}^{0,1}} + \|u^0\|^2_{\tilde{H}^{0,2}}\|\partial_1 u^0\|_{\tilde{H}^{0,1}}\Big) \\
\leqslant & \alpha\|\partial_1 u^0\|^2_{\tilde{H}^{0,2}} + C\|u^0\|^{\frac{8}{3}}_{\tilde{H}^{\frac{1}{4},1}}\|u^0\|^2_{\tilde{H}^{0,2}} + C\|u^0\|^2_{\tilde{H}^{\frac{1}{4},1}}\|u^0\|^2_{\tilde{H}^{0,2}} \\
&+ C\|\partial_1 u^0\|^{\frac{4}{3}}_{\tilde{H}^{0,1}}\|u^0\|^2_{\tilde{H}^{0,2}} + \|\partial_1 u^0\|_{\tilde{H}^{0,1}}\|u^0\|^2_{\tilde{H}^{0,2}} \\
\leqslant & \alpha\|\partial_1 u^0\|^2_{\tilde{H}^{0,2}} + C\|u^0\|^2_{\tilde{H}^{0,1}}\|u^0\|^{\frac{2}{3}}_{\tilde{H}^{1,1}}\|u^0\|^2_{\tilde{H}^{0,2}} + C\|u^0\|^{\frac{3}{2}}_{\tilde{H}^{0,1}}\|u^0\|^{\frac{1}{2}}_{\tilde{H}^{1,1}}\|u^0\|^2_{\tilde{H}^{0,2}} \\
&+ C\|\partial_1 u^0\|^{\frac{4}{3}}_{\tilde{H}^{0,1}}\|u^0\|^2_{\tilde{H}^{0,2}} + \|\partial_1 u^0\|_{\tilde{H}^{0,1}}\|u^0\|^2_{\tilde{H}^{0,2}} \\
\leqslant & \alpha\|\partial_1 u^0\|^2_{\tilde{H}^{0,2}} + C(1 + \|u^0\|^2_{\tilde{H}^{0,1}})(1 + \|u^0\|^2_{\tilde{H}^{1,1}})\|u^0\|^2_{\tilde{H}^{0,2}},
\end{aligned}$$

where we used Young's inequality in the third inequality and $\alpha < \frac{1}{2}$. Then Gronwall's inequality implies that

$$\begin{aligned}
&\sup_{t\in[0,T]}\|u^0(t)\|^2_{\tilde{H}^{0,2}} + \int_0^T \|\partial_1 u^0(t)\|^2_{\tilde{H}^{0,2}} dt \\
\leqslant & \|u_0\|^2_{\tilde{H}^{0,2}} \exp\left(C \sup_{t\in[0,T]}(1 + \|u^0(t)\|^2_{\tilde{H}^{0,1}})\int_0^T (1 + \|u^0(t)\|^2_{\tilde{H}^{1,1}})dt\right).
\end{aligned}$$

Then by Lemma 4.1, we get the result.

□

The next proposition is about the convergence of u^ε.

Proposition 4.3. *Assume (A0)-(A3) hold with $K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{5}, L_2 < \frac{1}{5}$, then there exists a constant $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, we have*

$$E\left(\sup_{t\in[0,T]}\|u^\varepsilon(t) - u^0(t)\|^2_H + \int_0^T \|u^\varepsilon(s) - u^0(s)\|^2_{\tilde{H}^{1,0}} ds\right) \leqslant C\varepsilon. \tag{4.2}$$

Proof Applying Itô's formula to $\|u^\varepsilon(t) - u^0(t)\|^2_H$, we have

$$\begin{aligned}
&\|u^\varepsilon(t) - u^0(t)\|^2_H \\
= & -2\int_0^t \|\partial_1(u^\varepsilon - u^0)(s)\|^2_H ds - 2\int_0^t \langle u^\varepsilon(s) - u^0(s), B(u^\varepsilon(s)) - B(u^0(s))\rangle ds \\
&+ 2\sqrt{\varepsilon}\int_0^t \langle u^\varepsilon(s) - u^0(s), \sigma(s, u^\varepsilon(s))dW(s)\rangle + \varepsilon\int_0^t \|\sigma(s, u^\varepsilon(s))\|^2_{L_2(l^2,H)} ds.
\end{aligned}$$

By Lemma 2.11 we have

$$|\langle u^\varepsilon(s) - u^0(s), B(u^\varepsilon(s)) - B(u^0(s))\rangle|$$

$$
\begin{aligned}
=&|b(u^\varepsilon, u^\varepsilon, u^\varepsilon - u^0) - b(u^0, u^0, u^\varepsilon - u^0)| \\
=&|b(u^\varepsilon - u^0, u^0, u^\varepsilon - u^0)| \\
\leqslant&\frac{1}{4}\|\partial_1(u^\varepsilon - u^0)\|_H^2 + C(1+\|u^0\|_{\tilde{H}^{1,1}}^2)\|u^\varepsilon - u^0\|_H^2.
\end{aligned}
$$

By the Burkhölder-Davis-Gundy's inequality (see [LR15, Appendix D]), we have

$$
\begin{aligned}
&2\sqrt{\varepsilon}E\left(\sup_{s\in[0,t]}\left|\int_0^t \langle u^\varepsilon(s) - u^0(s), \sigma(s, u^\varepsilon(s))dW(s)\rangle\right|\right) \\
\leqslant&6\sqrt{\varepsilon}E\left(\int_0^t \|u^\varepsilon(s) - u^0(s)\|_H^2 \|\sigma(s, u^\varepsilon(s)\|_{L_2(l^2,H)}^2 ds\right)^{\frac{1}{2}} \\
\leqslant&6\sqrt{\varepsilon}E\left(\sup_{s\in[0,t]}\|u^\varepsilon(s) - u^0(s)\|_H^2 \int_0^t (K_0 + K_1\|u^\varepsilon(s)\|_H^2 + K_2\|\partial_1 u^\varepsilon(s)\|_H^2)ds\right)^{\frac{1}{2}} \\
\leqslant&\frac{1}{2}E\left(\sup_{s\in[0,t]}\|u^\varepsilon(s) - u^0(s)\|_H^2\right) + C\varepsilon E\left(\int_0^t (1+\|u^\varepsilon(s)\|_H^2 + \|\partial_1 u^\varepsilon(s)\|_H^2)ds\right),
\end{aligned}
$$

where we used (A1) in the last second line. Thus by above estimates and (A1) we deduce that

$$
\begin{aligned}
&E\left(\sup_{s\in[0,t]}\|u^\varepsilon(s) - u^0(s)\|_H^2 + \int_0^t \|u^\varepsilon(s) - u^0(s)\|_{\tilde{H}^{1,0}}^2 ds\right) \\
\leqslant&C\int_0^t (1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2)E(\sup_{l\in[0,s]}\|u^\varepsilon(l) - u^0(l)\|_H^2)ds \\
&+ C\varepsilon E\left(\int_0^t (1+\|u^\varepsilon(s)\|_H^2 + \|\partial_1 u^\varepsilon(s)\|_H^2)ds\right).
\end{aligned}
$$

Then Gronwall's inequality and Lemma 4.1 imply that

$$
\begin{aligned}
&E\left(\sup_{s\in[0,T]}\|u^\varepsilon(s) - u^0(s)\|_H^2 + \int_0^T \|u^\varepsilon(s) - u^0(s)\|_{\tilde{H}^{1,0}}^2 ds\right) \\
\leqslant&C\varepsilon E\left(\int_0^T (1+\|u^\varepsilon(s)\|_H^2 + \|\partial_1 u^\varepsilon(s)\|_H^2)ds\right) e^{C\int_0^T (1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2)ds} \\
\leqslant&C\varepsilon.
\end{aligned}
$$

□

Let V^0 be the solution to the following SPDE:

$$
\begin{aligned}
&dV^0(t) = \partial_1^2 V^0(t)dt - B(V^0(t), u^0(t))dt - B(u^0(t), V^0(t))dt + \sigma(t, u^0(t))dW(t), \\
&V^0(0) = 0.
\end{aligned}
\tag{4.3}
$$

4.1 Well-posedness of the limiting equation

In this section we give existence and uniqueness of the solution to the limiting equation.

Lemma 4.4. *Assume that u^0 satisfies (4.1). Then under the assumptions (A0), (A1), (A2), equation (4.3) has a unique probabilistically strong solution*

$$
V^0 \in L^\infty([0,T], \tilde{H}^{0,1}) \cap L^2([0,T], \tilde{H}^{1,1}) \cap C([0,T], H^{-1}).
$$

Proof The proof follows a very similar Galerkin approximation argument as in [LZZ18, Section 4], we show some key steps here.

Let $\{e_k, k \geqslant 1\}$ be an orthonormal basis of H whose elements belong to H^2 and orthogonal in $\tilde{H}^{0,1}$ and $\tilde{H}^{1,0}$. Let $\mathcal{H}_n = \text{span}\{e_1, \dots, e_n\}$ and let P_n denote the orthogonal projection from H to $\mathcal{H}_n$. For l^2-cylindrical Wiener process $W(t)$, let $W_n(t) = \Pi_n W(t) := \sum_{j=1}^n \psi_j \beta_j(t)$, where β_j is a sequence of independent Brownian motions and ψ_j is an orthonormal basis of l^2. Set $F : H^1 \to H^{-1}$ with $F(u) = -B(u, u^0) - B(u^0, u) + \partial_1^2 u$.

Fix $n \geqslant 1$ and for $v \in \mathcal{H}_n$ consider the following equation on $\mathcal{H}_n$:

$$\begin{aligned} d\langle V_n(t), v\rangle =&\langle P_n F(V_n), v\rangle dt + \langle P_n \sigma(t, u^0(t)) dW_n(t), v\rangle \\ V_n(0) =& P_n u_0. \end{aligned} \tag{4.4}$$

Then by [LR15, Theorem 3.1.1] there exists unique global strong solution V_n to (4.4). Moreover, $V_n \in C([0,T], \mathcal{H}_n)$.

We first prove a priori estimates. Applying Itô's formula to $\|V_n\|^2_{\tilde{H}^{0,1}}$, we have

$$\begin{aligned} \|V_n(t)\|^2_{\tilde{H}^{0,1}} + 2\int_0^t \|\partial_1 V_n(s)\|^2_{\tilde{H}^{0,1}} ds =& \|P_n u_0\|^2_{\tilde{H}^{0,1}} - 2\int_0^t \langle B(V_n, u^0) + B(u^0, V_n), V_n\rangle_{\tilde{H}^{0,1}} ds \\ &+ 2\int_0^t \langle \sigma(s, u^0(s)) dW_n(s), V_n(s)\rangle_{\tilde{H}^{0,1}} \\ &+ \int_0^t \|P_n \sigma(s, u^0(s))\Pi_n\|^2_{L_2(l^2, \tilde{H}^{0,1})} ds. \end{aligned}$$

By Lemma 2.11 and Young's inequality, we have

$$\begin{aligned} &|\langle B(V_n, u^0) + B(u^0, V_n), V_n\rangle_{\tilde{H}^{0,1}}| \\ \leqslant& |b(V_n, u^0, V_n)| + |b(\partial_2 V_n, u^0, \partial_2 V_n)| + |b(V_n, \partial_2 u^0, \partial_2 V_n)| + |b(\partial_2 u^0, V_n, \partial_2 V_n)| \\ \leqslant& C\Big(\|V_n\|_{\tilde{H}^{1,0}} \|u^0\|_{\tilde{H}^{1,1}} \|V_n\|_H + \|\partial_2 V_n\|_{\tilde{H}^{1,0}} \|u^0\|_{\tilde{H}^{1,1}} \|\partial_2 V_n\|_H \\ &+ \|V_n\|_{\tilde{H}^{1,0}} \|\partial_2 u^0\|_{\tilde{H}^{1,1}} \|\partial_2 V_n\|_H + \|\partial_2 u^0\|_{\tilde{H}^{1,0}} \|V_n\|_{\tilde{H}^{1,1}} \|\partial_2 V_n\|_H\Big) \\ \leqslant& \alpha \|V_n\|^2_{\tilde{H}^{1,1}} + C\|u^0\|^2_{\tilde{H}^{1,2}} \|V_n\|^2_{\tilde{H}^{0,1}}, \end{aligned}$$

where $\alpha < \frac{1}{2}$.

The growth condition and Lemma 4.1 imply that

$$\int_0^t \|P_n \sigma(s, u^0(s))\Pi_n\|^2_{L_2(l^2, \tilde{H}^{0,1})} ds \leqslant C\int_0^t (1 + \|u^0\|^2_{\tilde{H}^{1,1}}) ds \leqslant C.$$

Similarly, by the Burkhölder-Davis-Gundy's inequality, we have

$$\begin{aligned} &2E\left(\sup_{s\in[0,t]} \left|\int_0^t \langle \sigma(s, u^0(s)) dW_n(s), V_n(s)\rangle_{\tilde{H}^{0,1}}\right|\right) \\ \leqslant& 6E\left(\int_0^t \|P_n \sigma(s, u^0(s))\Pi_n\|^2_{L_2(l^2, \tilde{H}^{0,1})} \|V_n(s)\|^2_{\tilde{H}^{0,1}} ds\right)^{\frac{1}{2}} \\ \leqslant& \beta E\left(\sup_{s\in[0,t]} \|V^0(s)\|^2_{\tilde{H}^{0,1}}\right) + C\int_0^t (1 + \|u^0\|^2_{\tilde{H}^{1,1}}) ds \end{aligned}$$

$$\leqslant \beta E\left(\sup_{s\in[0,t]}\|V^0(s)\|^2_{\tilde{H}^{0,1}}\right)+C,$$

where $\beta<\frac{1}{2}$.

Then we get

$$E\left(\sup_{s\in[0,t]}\|V_n(s)\|^2_{\tilde{H}^{0,1}}\right)+E\int_0^t\|V_n(s)\|^2_{\tilde{H}^{1,1}}ds$$
$$\leqslant C+C\int_0^t\left(\|u^0\|^2_{\tilde{H}^{1,2}}+1\right)E\left(\sup_{r\in[0,s]}\|V_n(r)\|^2_{\tilde{H}^{0,1}}\right)ds.$$

Then by Gronwall's inequality and (4.1), we have

$$E\left(\sup_{s\in[0,t]}\|V_n(s)\|^2_{\tilde{H}^{0,1}}\right)+E\int_0^t\|V_n(s)\|^2_{\tilde{H}^{1,1}}ds\leqslant C\exp\left(C\int_0^t\left(\|u^0\|^2_{\tilde{H}^{0,2}}+1\right)ds\right)\leqslant C. \tag{4.5}$$

The rest part of the existence proof is very similar as in the proof of [LZZ18, Theorem 4.1], we only need to point out that the convergence of $F(V_n)$ holds as $n\to\infty$: From the proof we could obtain that there exists another stochastic basis $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ and random variables $\tilde{V}_n$ with same law of V_n such that $\tilde{V}_n\to\tilde{V}$ in $C([0,T],H^{-1})\cap L^2([0,T],H)$, $\tilde{P}$-a.s. (in the sense of subsequence). Fix $l\in C^\infty(\mathbb{T}^2)$ with $\mathrm{div} l=0$. Since $F(V_n)$ is actually linear term, the convergence of $\tilde{V}_n$ in $L^2([0,T],H)$ implies that

$$\int_0^t\langle F(\tilde{V}_n),P_nl\rangle ds\to\int_0^t\langle F(\tilde{V}),l\rangle ds,\tilde{P}\text{-a.s.}$$

For uniqueness, assume V_1^0,V_2^0 are two solutions in $L^\infty([0,T],\tilde{H}^{0,1})\cap L^2([0,T],\tilde{H}^{1,1})\cap C([0,T],H^{-1})$ with the same initial condition, let $w=V_1-V_2$, then $w(0)=0$ and w satisfies

$$dw(t)=\partial_1^2w(t)dt-B(w(t),u^0(t))dt-B(u^0(t),w(t))dt.$$

Then similarly as the proof of the uniqueness for the deterministic Navier-Stokes equation with anisotropic viscosity, we know that $w=0$.

□

Remark 4.5. *Note here we do not need assumption (A3) and $L^4(\Omega)$ estimate of V_n since the drift term $\sigma(t,u^0)$ does not depend on V_n.*

4.2 Central limit theorem

In this section we give the main theorem of this chapter.

Theorem 4.6. *Assume (A0)-(A3) hold with $K_2<\frac{2}{21},\tilde{K}_2<\frac{1}{5},L_2<\frac{1}{5}$, then for $u_0\in\tilde{H}^{0,2}$ we have*

$$\lim_{\varepsilon\to0}E\left(\sup_{t\in[0,T]}\|\frac{u^\varepsilon(t)-u^0(t)}{\sqrt{\varepsilon}}-V^0(t)\|^2_H+\int_0^T\|\frac{u^\varepsilon(t)-u^0(t)}{\sqrt{\varepsilon}}-V^0(t)\|^2_{\tilde{H}^{1,0}}dt\right)=0$$

Proof Let $V^\varepsilon = \frac{u^\varepsilon(t)-u^0(t)}{\sqrt{\varepsilon}}$. Then we have

$$\begin{aligned} dV^\varepsilon(t) &= \partial_1^2 V^\varepsilon(t)dt - B(V^\varepsilon(t), u^\varepsilon(t))dt - B(u^0(t), V^\varepsilon(t))dt + \sigma(t, u^\varepsilon(t))dW(t), \\ V^\varepsilon(0) &= 0, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} d(V^\varepsilon - V^0) =&\partial_1^2(V^\varepsilon - V^0)dt - (B(V^\varepsilon, u^\varepsilon) - B(V^0, u^0))dt \\ &- B(u^0, V^\varepsilon - V^0)dt + (\sigma(t, u^\varepsilon) - \sigma(t, u^0))dW(t). \end{aligned}$$

By Itô's formula, we have

$$\begin{aligned} &\|V^\varepsilon(t) - V^0(t)\|_H^2 + 2\int_0^t \|\partial_1(V^\varepsilon(s) - V^0(s))\|_H^2 ds \\ =&-2\int_0^t \langle B(V^\varepsilon, u^\varepsilon) - B(V^0, u^0), V^\varepsilon - V^0\rangle ds \\ &+ 2\int_0^t \langle(\sigma(s, u^\varepsilon) - \sigma(s, u^0))dW(s), V^\varepsilon(s) - V^0(s)\rangle \\ &+ \int_0^t \|\sigma(s, u^\varepsilon) - \sigma(s, u^0)\|_{L_2(l^2,H)}^2 ds \\ \leqslant& 2\int_0^t |b(V^\varepsilon - V^0, u^0, V^\varepsilon - V^0)|ds \\ &+ 2\int_0^t |b(V^\varepsilon, u^\varepsilon - u^0, V^\varepsilon - V^0)|ds \\ &+ 2\left|\int_0^t \langle(\sigma(s, u^\varepsilon) - \sigma(s, u^0))dW(s), V^\varepsilon(s) - V^0(s)\rangle\right| \\ &+ \int_0^t \|\sigma(s, u^\varepsilon) - \sigma(s, u^0)\|_{L_2(l^2,H)}^2 ds \\ =:& I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Taking the supremum and the expectation, we obtain that

$$\begin{aligned} &E\left(\sup_{s\in[0,t]} \|V^\varepsilon(s) - V^0(s)\|_H^2 + 2\int_0^t \|\partial_1(V^\varepsilon(s) - V^0(s))\|_H^2 ds\right) \\ \leqslant& E(I_1(t) + I_2(t) + \sup_{s\in[0,t]} I_3(s) + I_4(t)). \end{aligned}$$

By Lemma 2.11, we have

$$EI_1(t) \leqslant 2E\int_0^t \left(\frac{1}{4}\|V^\varepsilon - V^0\|_{\tilde{H}^{1,0}}^2 + C\|u^0\|_{\tilde{H}^{1,1}}^2 \|V^\varepsilon - V^0\|_H^2\right) ds.$$

By Lemma 2.11, we have

$$\begin{aligned} EI_2(t) =& 2\sqrt{\varepsilon}E\int_0^t |b(V^\varepsilon, V^\varepsilon, V^\varepsilon - V^0)|ds \\ =& 2\sqrt{\varepsilon}E\int_0^t |b(V^\varepsilon, V^\varepsilon, V^0)|ds = 2\sqrt{\varepsilon}E\int_0^t |b(V^\varepsilon, V^0, V^\varepsilon)|ds \end{aligned}$$

$$\leqslant \sqrt{\varepsilon} CE \int_0^t (\|V^\varepsilon\|^2_{\tilde{H}^{1,0}} \|V^\varepsilon\|^2_H + \|V^0\|^2_{\tilde{H}^{1,1}}) ds.$$

By the Burkhölder-Davis-Gundy inequality and (A3), we have

$$\begin{aligned} E\left(\sup_{s\in[0,t]} I_3(s)\right) &\leqslant 6E\left(\int_0^t \|\sigma(s,u^\varepsilon)-\sigma(s,u^0)\|^2_{L_2(l^2,H)} \|V^\varepsilon - V^0\|^2_H ds\right)^{\frac{1}{2}} \\ &\leqslant 6E\left(\sup_{s\in[0,t]} \|V^\varepsilon - V^0\|^2_H \int_0^t \|\sigma(s,u^\varepsilon)-\sigma(s,u^0)\|^2_{L_2(l^2,H)} ds\right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{2} E\left(\sup_{s\in[0,t]} \|V^\varepsilon - V^0\|^2_H\right) + CE\left(\int_0^t \|u^\varepsilon - u^0)\|^2_H + \|\partial_1(u^\varepsilon - u^0)\|^2_H ds\right). \end{aligned}$$

By (A1), we have

$$EI_4(t) \leqslant CE\left(\int_0^t \|u^\varepsilon - u^0\|^2_H + \|\partial_1(u^\varepsilon - u^0)\|^2_H ds\right).$$

The above estimates together with Lemma 4.3 and Lemma 4.7 below induce that

$$\begin{aligned} &E\left(\sup_{s\in[0,t]} \|V^\varepsilon(s) - V^0(s)\|^2_H + \int_0^t \|V^\varepsilon(s) - V^0(s)\|^2_{\tilde{H}^{1,0}} ds\right) \\ \leqslant & CE\int_0^t \left(\|u^0(s)\|^2_{\tilde{H}^{1,1}} \sup_{l\in[0,s]} \|V^\varepsilon(l) - V^0(l)\|^2_H\right) ds \\ &+ \sqrt{\varepsilon} CE \int_0^t (\|V^\varepsilon\|^2_{\tilde{H}^{1,0}} \|V^\varepsilon\|^2_H + \|V^0\|^2_{\tilde{H}^{1,1}}) ds \\ &+ CE\left(\int_0^t \|u^\varepsilon - u^0\|^2_H + \|\partial_1(u^\varepsilon - u^0)\|^2_H ds\right) \\ \leqslant & CE\int_0^t \left((1 + \|u^0(s)\|^2_{\tilde{H}^{1,1}}) \sup_{l\in[0,s]} \|V^\varepsilon(l) - V^0(l)\|^2_H\right) ds + C(\sqrt{\varepsilon} + \varepsilon). \end{aligned}$$

Then by Gronwall's inequality and Lemma 4.1 we have

$$\begin{aligned} &E\left(\sup_{s\in[0,t]} \|V^\varepsilon(s) - V^0(s)\|^2_H + \int_0^t \|V^\varepsilon(s) - V^0(s)\|^2_{\tilde{H}^{1,0}} ds\right) \\ \leqslant & C(\sqrt{\varepsilon} + \varepsilon) \exp\left(C\int_0^t (1 + \|u^0(s)\|^2_{\tilde{H}^{1,1}}) ds\right) \leqslant C(\sqrt{\varepsilon} + \varepsilon). \end{aligned}$$

Let $\varepsilon \to 0$, we complete the proof.

□

It remains to establish the following lemma.

Lemma 4.7. *Assume (A0)-(A3) hold with $K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{5}, L_2 < \frac{1}{5}$. Let V^ε be the solution to* (4.6)*, then there exists a constant $\varepsilon_0 > 0$ such that*

$$\sup_{\varepsilon\in(0,\varepsilon_0)} E\int_0^T \|V^\varepsilon(s)\|^2_H \|V^\varepsilon(s)\|^2_{\tilde{H}^{1,0}} ds < \infty.$$

Proof Applying Itô's formula to $\|V^\varepsilon\|_H^4$, we have

$$
\begin{aligned}
d\|V^\varepsilon\|_H^4 \leqslant & 2\|V^\varepsilon\|_H^2\Big(-2\|\partial_1 V^\varepsilon\|_H^2 dt - 2b(V^\varepsilon, u^\varepsilon, V^\varepsilon)dt \\
& + 2\langle \sigma(t,u^\varepsilon)dW(t), V^\varepsilon\rangle + \|\sigma(t,u^\varepsilon)\|_{L_2(l^2,H)}^2 dt\Big) + 4\|\left(\sigma(t,u^\varepsilon(t))\right)^* V^\varepsilon\|_{l^2}^2 dt.
\end{aligned}
$$

Taking the supremum and the expectation, we have

$$
\begin{aligned}
& E\left(\sup_{s\in[0,t]}\|V^\varepsilon(s)\|_H^4 + 4\int_0^t \|V^\varepsilon(s)\|_H^2\|\partial_1 V^\varepsilon(s)\|_H^2 ds\right) \\
\leqslant & 4E\left(\int_0^t \|V^\varepsilon(s)\|_H^2 |b(V^\varepsilon(s), u^\varepsilon(s), V^\varepsilon(s))| ds\right) \\
& + 6E\left(\int_0^t \|V^\varepsilon(s)\|_H^2 \|\sigma(s,u^\varepsilon(s))\|_{L_2(l^2,H)}^2 ds\right) \\
& + 4E\left(\sup_{s\in[0,t]}\left|\int_0^t \|V^\varepsilon(s)\|_H^2 \langle\sigma(s,u^\varepsilon(s))dW(s), V^\varepsilon(s)\rangle\right|\right) \\
=: & I_1 + I_2 + I_3.
\end{aligned}
$$

Recall that $V^\varepsilon = \frac{u^\varepsilon - u^0}{\sqrt{\varepsilon}}$. By Lemma 2.11, we have

$$
\begin{aligned}
I_1(t) = & 4E\left(\int_0^t \|V^\varepsilon(s)\|_H^2 |b(V^\varepsilon(s), u^0(s) + \sqrt{\varepsilon}V^\varepsilon(s), V^\varepsilon(s))| ds\right) \\
= & 4E\left(\int_0^t \|V^\varepsilon(s)\|_H^2 |b(V^\varepsilon(s), u^0(s), V^\varepsilon(s))| ds\right) \\
\leqslant & E\left(\int_0^t \|V^\varepsilon(s)\|_H^2 (\|\partial_1 V^\varepsilon(s)\|_H^2 + C(1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2)\|V^\varepsilon(s)\|_H^2) ds\right) \\
\leqslant & E\int_0^t \|V^\varepsilon(s)\|_H^2 \|\partial_1 V^\varepsilon(s)\|_H^2 ds + CE\left(\int_0^t (1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2)\sup_{l\in[0,s]}\|V^\varepsilon(l)\|_H^4 ds\right).
\end{aligned}
$$

Note that Proposition 4.3 implies the boundedness of u^0 in $L^2([0,T],\tilde{H}^{1,1})$. By (A1) we have

$$
\begin{aligned}
I_2(t) \leqslant & CE\left(\int_0^t \|V^\varepsilon(s)\|_H^2 (1 + \|u^\varepsilon(s)\|_H^2 + \|\partial_1 u^\varepsilon(s)\|_H^2) ds\right) \\
\leqslant & CE\left(\int_0^t \|V^\varepsilon(s)\|_H^2 (1 + \|u^0(s)\|_H^2 + \varepsilon\|V^\varepsilon(s)\|_H^2 + \|\partial_1 u^0(s)\|_H^2 + \varepsilon\|\partial_1 V^\varepsilon(s)\|_H^2) ds\right) \\
\leqslant & C + \varepsilon CE\left(\sup_{s\in[0,t]}\|V^\varepsilon(s)\|_H^4\right) + \varepsilon CE\left(\int_0^t \|V^\varepsilon(s)\|_H^2 \|\partial_1 V^\varepsilon(s)\|_H^2 ds\right).
\end{aligned}
$$

By the Burkholder-Davis-Gundy inequality, (A1) and Proposition 4.3, we have

$$
\begin{aligned}
& I_3(t) \\
\leqslant & CE\left(\int_0^t \|V^\varepsilon(s)\|_H^6 \|\sigma(s,u^\varepsilon(s))\|_{L_2(l^2,H)}^2 ds\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant CE\left(\sup_{s\in[0,t]}\|V^\varepsilon(s)\|_H^2\left(\int_0^t\|V^\varepsilon(s)\|_H^2(1+\|u^\varepsilon(s)\|_H^2+\|\partial_1 u^\varepsilon(s)\|_H^2)ds\right)^{\frac{1}{2}}\right)\\
&\leqslant\frac{1}{2}E\left(\sup_{s\in[0,t]}\|V^\varepsilon(s)\|_H^4\right)\\
&\quad+CE\left(\int_0^t\|V^\varepsilon(s)\|_H^2(1+\|u^0(s)\|_H^2+\varepsilon\|V^\varepsilon(s)\|_H^2+\|\partial_1 u^0(s)\|_H^2+\varepsilon\|\partial_1 V^\varepsilon(s)\|_H^2)ds\right)\\
&\leqslant(\frac{1}{2}+\varepsilon C)E\left(\sup_{s\in[0,t]}\|V^\varepsilon(s)\|_H^4\right)+C+\varepsilon CE\left(\int_0^t\|V^\varepsilon(s)\|_H^2\|\partial_1 V^\varepsilon(s)\|_H^2ds\right).
\end{aligned}
$$

Combining the above estimates, there exists constants C_0 and C_1,

$$
\begin{aligned}
&E\left((\frac{1}{2}-C_0\varepsilon)\sup_{s\in[0,t]}\|V^\varepsilon(s)\|_H^2+(3-C_1\varepsilon)\int_0^t\|V^\varepsilon(s)\|_H^2\|\partial_1 V^\varepsilon(s)\|_H^2ds\right)\\
\leqslant& C+CE\left(\int_0^t(1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2)\sup_{l\in[0,s]}\|V^\varepsilon(l)\|_H^4ds\right).
\end{aligned}
$$

When $\varepsilon<\varepsilon_0:=\min\{\frac{1}{4C_0},\frac{3}{2C_1}\}$, by Gronwall's inequality, we have

$$
E\left(\sup_{s\in[0,t]}\|V^\varepsilon(s)\|_H^4+\int_0^t\|V^\varepsilon(s)\|_H^2\|\partial_1 V^\varepsilon(s)\|_H^2ds\right)\leqslant C\exp\left(\int_0^t(1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2)ds\right).
$$

Again by Lemma 4.1 we complete the proof.

□

Chapter 5

Moderate deviation principle

In this chapter, we will prove that $Z^\varepsilon := \frac{1}{\sqrt{\varepsilon}\lambda(\varepsilon)}(u^\varepsilon - u^0)$ satisfies LDP on

$$L^\infty([0,T],H)\cap L^2([0,T],\tilde{H}^{1,0})\cap C([0,T],H^{-1})$$

if $\lambda(\varepsilon)$ satisfies:

$$\lambda(\varepsilon)\to\infty,\quad \sqrt{\varepsilon}\lambda(\varepsilon)\to 0 \text{ as } \varepsilon\to 0.$$

Let us introduce the following skeleton equation associated to $Z^\varepsilon = \frac{1}{\sqrt{\varepsilon}\lambda(\varepsilon)}(u^\varepsilon - u^0)$, for $\phi\in L^2([0,T],l^2)$:

$$\begin{aligned}&dX^\phi(t) = \partial_1^2 X^\phi(t)dt - B(X^\phi(t),u^0(t))dt - B(u^0(t),X^\phi(t))dt + \sigma(t,u^0(t))\phi(t)dt,\\ &X^\phi(0)=0.\end{aligned} \tag{5.1}$$

Define $g^0: C([0,T],U)\to L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$ by

$$g^0(h) := \begin{cases} X^\phi, & \text{if } h=\int_0^\cdot \phi(s)ds \text{ for some } \phi\in L^2([0,T],l^2);\\ 0, & \text{otherwise.}\end{cases}$$

Then the rate function can be written as

$$I(g) = \inf\left\{\frac{1}{2}\int_0^T \|\phi(s)\|_{l^2}^2 ds:\ g = X^\phi,\ \phi\in L^2([0,T],l^2)\right\}, \tag{5.2}$$

where $g\in L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$.

The main result of this section is the following:

Theorem 5.1. *Assume (A0)-(A3) hold with $K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{5}, L_2 < \frac{1}{5}$ and $u_0\in\tilde{H}^{0,2}$, then Z^ε satisfies a large deviation principle on*

$$L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$$

with speed $\lambda^2(\varepsilon)$ and with the good rate function I given by (5.2), more precisely, it holds that
(U) for all closed sets $F\subset L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$ we have

$$\limsup_{\varepsilon\to0}\frac{1}{\lambda^2(\varepsilon)}\log P\left(\frac{u^\varepsilon-u^0}{\sqrt{\varepsilon}\lambda(\varepsilon)}\in F\right)\leqslant -\inf_{g\in F} I(g),$$

(L) for all open sets $G\subset L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$ we have

$$\limsup_{\varepsilon\to0}\frac{1}{\lambda^2(\varepsilon)}\log P\left(\frac{u^\varepsilon-u^0}{\sqrt{\varepsilon}\lambda(\varepsilon)}\in G\right)\geqslant -\inf_{g\in G} I(g).$$

By Lemma 2.6, we should check that Hypothesis 2.5 holds with ε replaced by λ^{-2}.

5.1 Two equations

In this section we give existence and uniqueness of solutions to two equations which will be used in the proof of the main result. The first one we consider is the skeleton equation (5.1).

Proposition 5.2. *Assume (A0)-(A2) hold. For all* $u_0 \in \tilde{H}^{0,2}$ *and* $\phi \in L^2([0,T], l^2)$ *there exists a unique solution*

$$X^\phi \in L^\infty([0,T], \tilde{H}^{0,1}) \bigcap L^2([0,T], \tilde{H}^{1,1}) \bigcap C([0,T], H^{-1})$$

to (5.1).

Proof We start by giving a priori estimates. Using an $\tilde{H}^{0,1}$ energy estimate, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|X^\phi\|^2_{\tilde{H}^{0,1}} + \|\partial_1 X^\phi\|^2_{\tilde{H}^{0,1}} \\ =& -\langle B(X^\phi, u^0) + B(u^0, X^\phi), X^\phi\rangle_{\tilde{H}^{0,1}} + \langle \sigma(t, u^0(t))\phi(t), X^\phi\rangle_{\tilde{H}^{0,1}}. \end{aligned}$$

The first two terms on the roght hand side can be dealt by the same calculation as in the proof of Lemma 4.4. For the third term we have

$$\begin{aligned} |\langle \sigma(t, u^0(t))\phi(t), X^\phi\rangle_{\tilde{H}^{0,1}}| \leqslant& \|\sigma(t,u^0)\|_{L_2(l^2, \tilde{H}^{0,1})}\|\phi(t)\|_{l^2}\|X^\phi(t)\|_{\tilde{H}^{0,1}} \\ \leqslant& \tilde{K}_0 + \tilde{K}_1\|u\|^2_{\tilde{H}^{0,1}} + \tilde{K}_2(\|\partial_1 u\|^2_H + \|\partial_1\partial_2 u\|^2_H) + C\|\phi\|^2_{l^2}\|X^\phi\|^2_{\tilde{H}^{0,1}} \\ \leqslant& C + C\|\phi\|^2_{l^2}\|X^\phi\|^2_{\tilde{H}^{0,1}}, \end{aligned}$$

where we used (A2) in the second line. Thus we deduce that

$$\begin{aligned} &\|X^\phi(t)\|^2_{\tilde{H}^{0,1}} + \int_0^t \|X^\phi(s)\|^2_{\tilde{H}^{1,1}} ds \\ \leqslant& C + C\int_0^t \left(1 + \|u^0\|^2_{\tilde{H}^{1,2}} + \|\phi\|^2_{l^2}\right)\|X^\phi\|^2_{\tilde{H}^{0,1}} ds. \end{aligned}$$

By Gronwall's inequality we have

$$\begin{aligned} &\|X^\phi(t)\|^2_{\tilde{H}^{0,1}} + \int_0^t \|X^\phi(s)\|^2_{\tilde{H}^{1,1}} ds \\ \leqslant& C\exp\left(\int_0^t \left(1 + \|u^0\|^2_{\tilde{H}^{1,2}} + \|\phi\|^2_{l^2}\right) ds\right) \leqslant C, \end{aligned}$$

where we used Lemma 4.2.

The existence results will be given by compactness arguments (see [LZZ18, Theorem 3.1]). We put them in the following for the use in the proof of next lemma.

Consider the approximate equation:

$$\begin{cases} dX^\phi_\epsilon(t) = \partial_1^2 X^\phi_\epsilon(t)dt + \epsilon^2\partial_2^2 X^\phi_\epsilon(t)dt - B(X^\phi_\epsilon, u^0)dt - B(u^0, X^\phi_\epsilon)dt + \sigma(t, u^0(t))\phi(t)dt, \\ X^\phi_\epsilon(0) = 0. \end{cases} \tag{5.3}$$

It follows from classical theory on Navier-Stokes system that (5.3) has a unique global smooth solution z_ϵ^ϕ for any fixed ϵ. Furthermore, we have

$$\|X_\epsilon^\phi(t)\|^2_{\tilde{H}^{0,1}} + \int_0^t \|X_\epsilon^\phi(s)\|^2_{\tilde{H}^{1,1}} ds \leqslant C.$$

Then we have that $\{X_\epsilon^\phi\}_{\epsilon>0}$ is uniformly bounded in $L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})$, hence bounded in $L^4([0,T],H^{\frac{1}{2}})$ (by interpolation) and $L^4([0,T],L^4(\mathbb{T}^2))$ (by Sobolev embedding). Thus $B(X_\epsilon^\phi,u^0)$ and $B(u^0,X_\epsilon^\phi)$ are uniformly bounded in $L^2([0,T],H^{-1})$. Let $p\in(1,\frac{4}{3})$, we have

$$\begin{aligned}\int_0^T \|\sigma(s,u^0(s))\phi(s)\|^p_{H^{-1}} ds \leqslant& \int_0^T \|\sigma(s,u^0(s))\|^p_{L_2(l^2,H^{-1})}\|\phi(s)\|^p_{l^2} ds\\ \leqslant& C\int_0^T (1+\|\sigma(s,u^0(s))\|^4_{L_2(l^2,H^{-1})}+\|\phi(s)\|^2_{l^2})ds\\ \leqslant& C\int_0^T (1+\|u^0(s))\|^4_{H}+\|\phi(s)\|^2_{l^2})ds<\infty,\end{aligned}$$

where we used Young's inequality in the second line and (A0) in the third line. It comes out that

$$\{\partial_t X_\epsilon^\phi\}_{\epsilon>0} \text{ is uniformly bounded in } L^p([0,T],H^{-1}). \tag{5.4}$$

Thus by Aubin-Lions lemma (see [LZZ18, Lemma 3.6]), there exists a $X^\phi\in L^2([0,T],H)$ such that

$$X_\epsilon^\phi \to X^\phi \text{ strongly in } L^2([0,T],H) \text{ as } \epsilon\to 0 \text{ (in the sense of subsequence).}$$

Since $\{X_\epsilon^\phi\}_{\epsilon>0}$ is uniformly bounded in $L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})$, there exists a $\tilde{X}\in L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})$ such that

$$X_\epsilon^\phi \to \tilde{X} \text{ weakly in } L^2([0,T],\tilde{H}^{1,1}) \text{ as } \epsilon\to 0 \text{ (in the sense of subsequence).}$$

$$X_\epsilon^\phi \to \tilde{X} \text{ weakly star in } L^\infty([0,T],\tilde{H}^{0,1}) \text{ as } \epsilon\to 0 \text{ (in the sense of subsequence).}$$

By the uniqueness of weak convergence limit, we deduce that $X^\phi=\tilde{X}$. By (5.4) and [FG95, Theorem 2.2], we also have for any $\delta>0$

$$X_\epsilon^\phi \to X^\phi \text{ strongly in } C([0,T],H^{-1-\delta}) \text{ as } \epsilon\to 0 \text{ (in the sense of subsequence).}$$

Now we use the above convergence to prove that X^ϕ is a solution to (5.1). Note that for any $\varphi\in C^\infty([0,T]\times\mathbb{T}^2)$ with $\mathrm{div}\varphi=0$, for any $t\in[0,T]$, z_ϵ^ϕ satisfies

$$\begin{aligned}\langle X_\epsilon^\phi(t),\varphi(t)\rangle =& \int_0^t \langle X_\epsilon^\phi,\partial_t\varphi\rangle - \langle\partial_1 X_\epsilon^\phi,\partial_1\varphi\rangle - \epsilon^2\langle\partial_2 X_\epsilon^\phi,\partial_2\varphi\rangle\\ &+\langle -B(X_\epsilon^\phi,u^0)-B(u^0,X_\epsilon^\phi)+\sigma(s,u^0)\phi,\varphi\rangle ds.\end{aligned} \tag{5.5}$$

Let $\epsilon\to 0$ in (5.5), we have $X^\phi\in L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})$ and

$$\partial_t X^\phi = \partial_1^2 X^\phi - B(X^\phi,u^0) - B(u^0,X^\phi) + \sigma(t,u^0(t))\phi.$$

Since the right hand side belongs to $L^p([0,T],H^{-1})$, we deduce that

$$X^\phi\in L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})\bigcap C([0,T],H^{-1}).$$

The uniqueness part is exactly the same as in Lemma 4.4. □

Recall $Z^\varepsilon = \frac{u^\varepsilon - u^0}{\sqrt{\varepsilon}\lambda(\varepsilon)}$, then

$$\begin{aligned} dZ^\varepsilon(t) =& \partial_1^2 Z^\varepsilon(t)dt - B(Z^\varepsilon(t), u^0(t) + \sqrt{\varepsilon}\lambda(\varepsilon)Z^\varepsilon(t))dt - B(u^0(t), Z^\varepsilon(t))dt \\ &+ \lambda^{-1}(\varepsilon)\sigma(t, u^0(t) + \sqrt{\varepsilon}\lambda(\varepsilon)Z^\varepsilon(t))dW(t), \end{aligned} \tag{5.6}$$

with initial value $Z^\varepsilon(0) = 0$. The uniqueness of solution to (5.6) is very similar to that of (2.2). Then it follows from Yamada-Watanabe theorem (See [LR15, Appendix E]) that there exists a Borel-measurable function

$$g^\varepsilon : C([0,T], U) \to L^\infty([0,T], H) \bigcap L^2([0,T], \tilde{H}^{1,0}) \bigcap C([0,T], H^{-1})$$

such that $Z^\varepsilon = g^\varepsilon(W)$ a.s..

Now consider the following equation:

$$\begin{aligned} dX^\varepsilon(t) =& \partial_1^2 X^\varepsilon(t)dt - B(X^\varepsilon(t), u^0(t) + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(t))dt - B(u^0(t), X^\varepsilon(t))dt \\ &+ \sigma(t, u^0(t) + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(t))v^\varepsilon(t)dt + \lambda^{-1}(\varepsilon)\sigma(t, u^0(t) + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(t))dW(t), \\ X^\varepsilon(0) =& 0, \end{aligned} \tag{5.7}$$

where $v^\varepsilon \in \mathcal{A}_N$ for some $N < \infty$. Here X^ε should have been denoted $X^\varepsilon_{v^\varepsilon}$ and the slight abuse of notation is for simplicity.

Lemma 5.3. *Assume (A0)-(A3) hold with $K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{5}, L_2 < \frac{1}{5}$ and $v^\varepsilon \in \mathcal{A}_N$ for some $N < \infty$. Then $X^\varepsilon = g^\varepsilon\left(W(\cdot) + \lambda(\varepsilon)\int_0^\cdot v^\varepsilon(s)ds\right)$ is the unique strong solution to (5.7).*

Proof Since $v^\varepsilon \in \mathcal{A}_N$, by the Girsanov theorem (see [LR15, Appendix I]), $\tilde{W}(\cdot) := W(\cdot) + \lambda(\varepsilon)\int_0^\cdot v^\varepsilon(s)ds$ is an l^2-cylindrical Wiener-process under the probability measure

$$d\tilde{P} := \exp\left\{-\lambda(\varepsilon)\int_0^T v^\varepsilon(s)dW(s) - \frac{1}{2}\lambda^2(\varepsilon)\int_0^T \|v^\varepsilon(s)\|_{l^2}^2 ds\right\} dP.$$

Then $(X^\varepsilon, \tilde{W})$ is the solution to (5.6) on the stochastic basis $(\Omega, \mathcal{F}, \tilde{P})$. Thus (X^ε, W) satisfies the condition of the definition of weak solution (see [LZZ18, Definition 4.1]) and hence is a weak solution to (5.7) on the stochastic basis $(\Omega, \mathcal{F}, P)$ and $X^\varepsilon = g^\varepsilon\left(W(\cdot) + \lambda(\varepsilon)\int_0^\cdot v^\varepsilon(s)ds\right)$.

If $\tilde{X}^\varepsilon$ and X^ε are two weak solutions to (5.7) on the same stochastic basis $(\Omega, \mathcal{F}, P)$. Let $W^\varepsilon = X^\varepsilon - \tilde{X}^\varepsilon$ and $q(t) = k\int_0^t(\|u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 + \|v^\varepsilon(s)\|_{l^2}^2)ds$ for some constant k. Applying Itô's formula to $e^{-q(t)}\|W^\varepsilon(t)\|_H^2$, we have

$$\begin{aligned} &e^{-q(t)}\|W^\varepsilon(t)\|_H^2 + 2\int_0^t e^{-q(s)}\|\partial_1 W^\varepsilon(s)\|_H^2 ds \\ =& -k\int_0^t e^{-q(s)}\|W^\varepsilon(s)\|_H^2(\|u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 + \|v^\varepsilon(s)\|_{l^2}^2)ds \\ &- 2\int_0^t e^{-q(s)} b(W^\varepsilon, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon, W^\varepsilon)ds \\ &+ 2\int_0^t e^{-q(s)}\langle \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)v^\varepsilon - \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)v^\varepsilon, W^\varepsilon(s)\rangle ds \\ &+ 2\lambda^{-1}(\varepsilon)\int_0^t e^{-q(s)}\langle W^\varepsilon(s), (\sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon) - \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon))dW(s)\rangle \end{aligned}$$

$$+\lambda^{-2}(\varepsilon)\int_0^t e^{-q(s)}\|\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)-\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)\|^2_{L_2(l^2,H)}ds.$$

By Lemma 2.11, there exists constants $\tilde{\alpha}\in(0,1)$ and $\tilde{C}$ such that

$$|b(W^\varepsilon,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon,W^\varepsilon)|\leqslant\tilde{\alpha}\|\partial_1W^\varepsilon\|^2_H+\tilde{C}(1+\|u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon\|^2_{\tilde{H}^{1,1}})\|W^\varepsilon\|^2_H.$$

We also have

$$\begin{aligned}&2|\langle\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)v^\varepsilon-\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)v^\varepsilon,W^\varepsilon\rangle|\\ \leqslant&2\|(\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)-\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon))v^\varepsilon\|_H\|W^\varepsilon\|_H\\ \leqslant&\|\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)-\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)\|^2_{L_2(l^2,H)}+\|v^\varepsilon\|^2_{l^2}\|W^\varepsilon\|^2_H.\end{aligned}$$

By (A3), we have

$$\begin{aligned}&\|\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)-\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)\|^2_{L_2(l^2,H)}\\ \leqslant&\sqrt{\varepsilon}\lambda(\varepsilon)(L_1\|W^\varepsilon\|^2_H+L_2\|\partial_1W^\varepsilon\|^2_H).\end{aligned}$$

By the Burkhölder-Davis-Gundy's inequality (see [LR15, Appendix D]), we have

$$\begin{aligned}&2\lambda^{-1}(\varepsilon)|E[\sup_{r\in[0,t]}\int_0^r e^{-q(s)}\langle W^\varepsilon(s),(\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)-\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon))dW(s)\rangle]|\\ \leqslant&6\lambda^{-1}(\varepsilon)E\left(\int_0^t e^{-2q(s)}\|\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)-\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)\|^2_{L_2(l^2,H)}\|W^\varepsilon(s)\|^2_Hds\right)^{\frac{1}{2}}\\ \leqslant&\sqrt{\varepsilon}E(\sup_{s\in[0,t]}(e^{-q(s)}\|W^\varepsilon(s)\|^2_H))+9\sqrt{\varepsilon}E\int_0^t e^{-q(s)}(L_1\|W^\varepsilon(s)\|^2_H+L_2\|\partial_1W^\varepsilon(s)\|^2_H)ds,\end{aligned}$$

where we used (A3).

Let $k>2\tilde{C}$ and we may assume $\sqrt{\varepsilon}\lambda(\varepsilon)<1$, by (A3) we have

$$\begin{aligned}&e^{-q(t)}\|W^\varepsilon(t)\|^2_H+(2-2\tilde{\alpha}-L_2\varepsilon\lambda^2(\varepsilon))\int_0^t e^{-q(s)}\|\partial_1W^\varepsilon(s)\|^2_Hds\\ \leqslant&C\int_0^t e^{-q(s)}\|W^\varepsilon(s)\|^2_Hds\\ &+2\lambda^{-1}(\varepsilon)\int_0^t e^{-q(s)}\langle W^\varepsilon(s),(\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)-\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon))dW(s)\rangle.\end{aligned}$$

Let ε be small enough such that $1-\sqrt{\varepsilon}-L_2\varepsilon\lambda^2(\varepsilon)-9\sqrt{\varepsilon}L_2>0$. Then we have

$$E(\sup_{s\in[0,t]}(e^{-q(s)}\|W^\varepsilon(s)\|^2_H))\leqslant CE\int_0^t e^{-q(s)}\|W^\varepsilon(s)\|^2_Hds.$$

By Gronwall's inequality we obtain $W^\varepsilon=0$ P-a.s., i.e. $\tilde{X}^\varepsilon=X^\varepsilon$ P-a.s..

Then by the Yamada-Watanabe theorem, we have X^ε is the unique strong solution to (5.7). □

5.2 Proof of Hypothesis 2

In this section we will show that I is a good rate function by checking the second part of Hypothesis 2.5.

Lemma 5.4. *Assume (A0)-(A2) hold. For all $N < \infty$, the set*

$$K_N = \left\{ g^0 \left(\int_0^{\cdot} \phi(s) ds \right) : \phi \in S_N \right\}$$

is a compact subset in $L^\infty([0,T], H) \bigcap L^2([0,T], \tilde{H}^{1,0}) \bigcap C([0,T], H^{-1})$.

Proof By definition, we have

$$K_N = \left\{ X^\phi : \phi \in L^2([0,T], l^2), \int_0^T \|\phi(s)\|_{l^2}^2 ds \leqslant N \right\}.$$

Let $\{X^{\phi_n}\}$ be a sequence in K_N where $\{\phi_n\} \subset S_N$. Note that X^{ϕ_n} is uniformly bounded in $L^\infty([0,T], \tilde{H}^{1,0}) \cap L^2([0,T], \tilde{H}^{1,1})$. Thus by weak compactness of S_N, a similar argument as in the proof of Lemma 5.2 shows that there exists $\phi \in \mathcal{S}_N$ and $X' \in L^2([0,T], H)$ such that the following convergence hold as $n \to \infty$ (in the sense of subsequence):

$\phi_n \to \phi$ in $\mathcal{S}_N$ weakly,
$X^{\phi_n} \to X'$ in $L^2([0,T], \tilde{H}^{1,0})$ weakly,
$X^{\phi_n} \to X'$ in $L^\infty([0,T], H)$ weak-star,
$X^{\phi_n} \to X'$ in $L^2([0,T], H)$ strongly.
$X^{\phi_n} \to X'$ in $C([0,T], H^{-1-\delta})$ strongly for any $\delta > 0$.
Then for any $\varphi \in C^\infty([0,T] \times \mathbb{T}^2)$ with $\mathrm{div}\varphi = 0$ and for any $t \in [0,T]$, X^{ϕ_n} satisfies

$$\begin{aligned} &\langle X^{\phi_n}(t), \varphi(t) \rangle = \langle u_0, \varphi(0) \rangle \\ &+ \int_0^t \langle X^{\phi_n}, \partial_t \varphi \rangle - \langle \partial_1 X^{\phi_n}, \partial_1 \varphi \rangle + \langle -B(X^{\phi_n}, u^0) - B(u^0, X^{\phi_n}) + \sigma(s, u^0)\phi_n, \varphi \rangle ds. \end{aligned} \tag{5.8}$$

Let $n \to \infty$, we deduce that X' is a solution to (5.1). By the uniqueness of solution, we deduce that $X' = X^\phi$.

Our goal is to prove $X^{\phi_n} \to X^\phi$ in $L^\infty([0,T], H) \bigcap L^2([0,T], \tilde{H}^{1,0}) \bigcap C([0,T], H^{-1})$.

Let $w^n = X^{\phi_n} - X^\phi$, by a direct calculation, we have

$$\begin{aligned} &\|w^n(t)\|_H^2 + 2\int_0^t \|\partial_1 w^n(s)\|_H^2 ds \\ =& -2\int_0^t \langle w^n(s), B(X^{\phi_n}(s) - X^\phi(s), u^0(s)) \rangle ds \\ &-2\int_0^t \langle w^n(s), B(u^0(s), X^{\phi_n}(s) - X^\phi(s)) \rangle ds \\ &+2\int_0^t \langle w^n(s), \sigma(s, u^0(s))(\phi_n(s) - \phi(s)) \rangle ds \\ \leqslant& 2\int_0^t |b(w^n, u^0, w^n)(s)| ds + 2\int_0^t |\langle w^n(s), \sigma(s, u^0(s))(\phi_n(s) - \phi(s)) \rangle| ds \end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_0^t \|\partial_1 w^n(s)\|_H^2 + C(1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2 ds \\
&\quad + C\int_0^t \|w^n(s)\|_H \|\phi_n(s)-\phi(s)\|_{l^2}(1+\|u^0(s)\|_H^2+\|\partial_1 u^0(s)\|_H^2)^{\frac{1}{2}} ds,
\end{aligned}
$$

where we used Lemma 2.11 and (A1) in the last inequality.

Note that ϕ_n, ϕ are in $\mathcal{S}_N$, we have

$$
\begin{aligned}
&\|w^n(t)\|_H^2 + \int_0^t \|\partial_1 w^n(s)\|_H^2 ds \\
\leqslant & C\int_0^t (1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2 ds \\
&+ C\left(\int_0^t \|w^n(s)\|_H^2(1+\|u^0(s)\|_H^2+\|\partial_1 u^0(s)\|_H^2) ds\right)^{\frac{1}{2}}\left(\int_0^t \|\phi_n(s)-\phi(s)\|_{l^2}^2\right)^{\frac{1}{2}} \\
\leqslant & C\int_0^t (1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2 ds \\
&+ C\sqrt{N}\left(\int_0^t \|w^n(s)\|_H^2(1+\|u^0(s)\|_H^2+\|\partial_1 u^0(s)\|_H^2) ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

For any $\epsilon > 0$, let

$$
A_\epsilon := \{s \in [0,T]; \|w^n(s)\|_H > \epsilon\}.
$$

Since $X^{\phi_n} \to X^\phi$ in $L^2([0,T],H)$ strongly, we have

$$
\int_0^T \|w^n(s)\|_H^2 ds \to 0, \text{ as } n \to \infty
$$

and $\lim_{n\to\infty} Leb(A_\epsilon) = 0$, where $Leb(B)$ means the Lebesgue measure of $B \in \mathcal{B}(\mathbb{R})$. Thus we have

$$
\begin{aligned}
&\int_0^T (1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2 ds \\
\leqslant & \left(\int_{A_\epsilon} + \int_{[0,T]\setminus A_\epsilon}\right)(1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2 ds \\
\leqslant & C\epsilon + 2\int_{A_\epsilon}(1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2)(\|X^{\phi_n}(s)\|_H^2 + \|X^\phi(s)\|_H^2) ds \\
\leqslant & C\epsilon + C\int_{A_\epsilon}(1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2) ds \\
\to & C\epsilon \text{ as } n \to \infty,
\end{aligned}
$$

where we used Lemma 4.1 in the last line. A similar argument also implies that

$$
\int_0^T (1+\|u^0(s)\|_H^2+\|\partial_1 u^0(s)\|_H^2)\|w^n(s)\|_H^2 ds \leqslant C\epsilon.
$$

Hence we have

$$
\sup_{t\in[0,T]} \|w^n(t)\|_H^2 + \int_0^T \|\partial_1 w^n(s)\|_H^2 ds \leqslant C\epsilon + C\sqrt{\epsilon} \text{ as } n \to \infty.
$$

Since ϵ is arbitrary, we obtain that

$$
X^{\phi^n} \to X^\phi \text{ strongly in } L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1}).
$$

□

5.3 Proof of Hypothesis 1

In this section we will prove the main result by checking the rest of Hypothesis 2.5.

Lemma 5.5. *Assume X^ε is a solution to (5.7) with $v^\varepsilon \in \mathcal{A}_N$ and $\varepsilon < 1$ small enough. Then we have*

$$E(\sup_{t\in[0,T]} \|X^\varepsilon(t)\|_H^4) + E\int_0^T (\|X^\varepsilon(s)\|_H^2 + 1)\|X^\varepsilon(s)\|_{\tilde{H}^{1,0}}^2 ds \leqslant C(N). \tag{5.9}$$

Moreover, there exists $k > 0$ such that

$$E(\sup_{t\in[0,T]} e^{-kg(t)}\|X^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2) + E\int_0^T e^{-kg(s)}\|X^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds \leqslant C(N), \tag{5.10}$$

where $g(t) = \int_0^t \|\partial_1 X^\varepsilon(s)\|_H^2 ds$ and $C(N)$ is a constant depend on N but independent of ε.

Proof We prove (5.9) by two steps of estimates. For the first step, applying Itô's formula to $\|X^\varepsilon(t)\|_H^2$, we have

$$\begin{aligned}
&\|X^\varepsilon(t)\|_H^2 + 2\int_0^t \|\partial_1 X^\varepsilon(s)\|_H^2 ds \\
=& -2\int_0^t b(X^\varepsilon, u^0, X^\varepsilon)ds + 2\int_0^t \langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))v^\varepsilon(s)\rangle ds \\
&+ 2\lambda^{-1}(\varepsilon)\int_0^t \langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))dW(s)\rangle \\
&+ \lambda^{-2}(\varepsilon)\int_0^t \|\sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))\|_{L_2(l^2,H)}^2 ds \\
\leqslant& \int_0^t (\frac{1}{2}\|\partial_1 X^\varepsilon(s)\|_H^2 + C(1 + \|u^0\|_{\tilde{H}^{1,1}}^2)\|X^\varepsilon\|_H^2)ds \\
&+ \int_0^t (\|X^\varepsilon(s)\|_H^2\|v^\varepsilon(s)\|_{l^2}^2 + \|\sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))\|_{L_2(l^2,H)}^2)ds \\
&+ 2\lambda^{-1}(\varepsilon)\int_0^t \langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))dW(s)\rangle \\
&+ \lambda^{-2}(\varepsilon)\int_0^t \|\sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))\|_{L_2(l^2,H)}^2 ds \\
\leqslant& \int_0^t (\frac{1}{2}\|\partial_1 X^\varepsilon(s)\|_H^2 + C(1 + \|u^0\|_{\tilde{H}^{1,1}}^2)\|X^\varepsilon\|_H^2)ds + \int_0^t \|X^\varepsilon(s)\|_H^2\|v^\varepsilon(s)\|_{l^2}^2 ds \\
&+ (1 + \lambda^{-2}(\varepsilon))\int_0^t (K_0 + K_1\|u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon\|_H^2 + K_2\|\partial_1(u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)\|_H^2)ds \\
&+ 2\lambda^{-1}(\varepsilon)\int_0^t \langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))dW(s)\rangle,
\end{aligned}$$

where we used (A1) in the last inequality.

Note that $v^\varepsilon \in \mathcal{A}_N$, by Lemma 4.1 and Gronwall's inequality,

$$\|X^\varepsilon(t)\|_H^2 + (\frac{3}{2} - \varepsilon K_2 - \lambda^2(\varepsilon)\varepsilon K_2)\int_0^t \|\partial_1 X^\varepsilon(s)\|_H^2 ds$$

$$\leqslant (C+2\lambda^{-1}(\varepsilon)\int_0^t \langle X^\varepsilon(s),\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))dW(s)\rangle)e^{C_1(N)}.$$

For the term on the right hand side, by the Burkhölder-Davis-Gundy inequality we have

$$\begin{aligned}
&2\lambda^{-1}(\varepsilon)e^{C_1(N)}E\left(\sup_{0\leqslant s\leqslant t}|\int_0^s\langle X^\varepsilon(r),\sigma(r,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(r))dW(r)\rangle|\right)\\
\leqslant &6\lambda^{-1}(\varepsilon)e^{C_1(N)}E\left(\int_0^t\|X^\varepsilon(r)\|_H^2\|\sigma(r,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(r))\|^2_{L_2(l^2,H)}ds\right)^{\frac{1}{2}}\\
\leqslant &\lambda^{-1}(\varepsilon)E[\sup_{0\leqslant s\leqslant t}(\|X^\varepsilon(s)\|_H^2)]\\
&+9\lambda^{-2}(\varepsilon)e^{C_1(N)}E\int_0^t[K_0+K_1\|u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s)\|_H^2+K_2\|\partial_1(u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))\|_H^2]ds,
\end{aligned}$$

where $(9\varepsilon e^{C_1(N)}+\varepsilon\lambda^2(\varepsilon)+\varepsilon)K_2-\frac{3}{4}<0$ (this can be done since $\sqrt{\varepsilon}\lambda(\varepsilon)\to 0$) and we used (A1) in the last inequality. Thus we have

$$\begin{aligned}
&E[\sup_{s\in[0,t]}(\|X^\varepsilon(t)\|_H^2)]+E\int_0^t\|\partial_1X^\varepsilon(s)\|_H^2ds\\
\leqslant &C(N)+C(N)\int_0^t E[\sup_{r\in[0,s]}(\|X^\varepsilon(r)\|_H^2)]ds.
\end{aligned}$$

Then by Gronwall's inequality we have

$$E(\sup_{0\leqslant t\leqslant T}\|X^\varepsilon(t)\|_H^2)+E\int_0^T\|\partial_1X^\varepsilon(s)\|_H^2ds\leqslant C(N). \tag{5.11}$$

Now by Itô's formula we have

$$\begin{aligned}
\|X^\varepsilon(t)\|_H^4=&-4\int_0^t\|X^\varepsilon(s)\|_H^2\|\partial_1X^\varepsilon(s)\|_H^2ds-4\int_0^t\|X^\varepsilon(s)\|_H^2b(X^\varepsilon,u^0,X^\varepsilon)ds\\
&+4\int_0^t\|X^\varepsilon(s)\|_H^2\langle\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))v^\varepsilon(s),X^\varepsilon(s)\rangle ds\\
&+2\lambda^{-2}(\varepsilon)\int_0^t\|X^\varepsilon(s)\|_H^2\|\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))\|^2_{L_2(l^2,H)}ds\\
&+4\lambda^{-2}(\varepsilon)\int_0^t\|\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))^*(X^\varepsilon)\|_{l^2}^2ds\\
&+4\lambda^{-1}(\varepsilon)\int_0^t\|X^\varepsilon(s)\|_H^2\langle X^\varepsilon(s),\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))dW(s)\rangle_H\\
=:&-4\int_0^t\|X^\varepsilon\|_H^2\|\partial_1X^\varepsilon(s)\|_H^2ds+I_0+I_1+I_2+I_3+I_4.
\end{aligned} \tag{5.12}$$

By Lemma 2.11,

$$|I_0(t)|\leqslant 4\int_0^t\|X^\varepsilon\|_H^2(\frac{1}{4}\|\partial_1X^\varepsilon\|_H^2+C(1+\|u^0\|^2_{\tilde{H}^{1,1}})\|X^\varepsilon\|_H^2))ds.$$

By (A1) we have

$$
\begin{aligned}
I_1(t) \leqslant & 4\int_0^t \|X^\varepsilon(s)\|_H^2 \|\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))\|_{L_2(l^2,H)}\|v^\varepsilon(s)\|_{l^2}\|X^\varepsilon(s)\|_H ds\\
\leqslant & 2\int_0^t \|X^\varepsilon(s)\|_H^2 (K_0+K_1\|u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s)\|_H^2\\
& + K_2\|\partial_1(u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))\|_H^2 + \|v^\varepsilon(s)\|_{l^2}^2\|X^\varepsilon(s)\|_H^2)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2+I_3 \leqslant & 6\lambda^{-2}(\varepsilon)\int_0^t \|\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))\|_{L_2(l^2,H)}^2\|X^\varepsilon(s)\|_H^2 ds\\
\leqslant & 6\lambda^{-2}(\varepsilon)\int_0^t (K_0+K_1\|u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s)\|_H^2\\
& + K_2\|\partial_1(u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))\|_H^2)\|X^\varepsilon(s)\|_H^2 ds.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& \|X^\varepsilon(t)\|_H^4 + (3-2\varepsilon\lambda^2(\varepsilon)K_2 - 6\varepsilon K_2)\int_0^t \|X^\varepsilon(s)\|_H^2\|\partial_1 X^\varepsilon(s)\|_H^2 ds\\
\leqslant & I_4 + C + C\int_0^t (1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2 + \|v^\varepsilon(s)\|_{l^2}^2)\|X^\varepsilon(s)\|_H^4)ds.
\end{aligned}
$$

Since $v^\varepsilon \in \mathcal{A}_N$, by Gronwall's inequality we have

$$
\begin{aligned}
& \|X^\varepsilon(t)\|_H^4 + (3-2\varepsilon\lambda^2(\varepsilon)K_2 - 6\varepsilon K_2)\int_0^t \|X^\varepsilon(s)\|_H^2\|\partial_1 X^\varepsilon(s)\|_H^2 ds\\
\leqslant & (I_4 + C)\, e^{C_2(N)}.
\end{aligned}
$$

Then the Burkhölder-Davis-Gundy inequality, the Young's inequality and (A1) imply that

$$
\begin{aligned}
E(\sup_{s\in[0,t]} I_4(s)) \leqslant & 12\lambda^{-1}(\varepsilon)E\left(\int_0^t \|\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))\|_{L_2(l^2,H)}^2\|X^\varepsilon(s)\|_H^6 ds\right)^{\frac{1}{2}}\\
\leqslant & \lambda^{-1}(\varepsilon)E(\sup_{s\in[0,t]}\|X^\varepsilon(s)\|_H^4) + 36\lambda^{-1}(\varepsilon)E\int_0^t (K_0+K_1\|u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s)\|_H^2\\
& + K_2\|\partial_1(u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))\|_H^2)\|X^\varepsilon(s)\|_H^2 ds.
\end{aligned}
$$

Let ε small enough such that $3-2\varepsilon\lambda^2(\varepsilon)K_2-6\varepsilon K_2-36\varepsilon K_2 e^{C_2(N)}>0$ and $\lambda^{-1}(\varepsilon)e^{C_2(N)}<1$. Then the above estimates and (5.9) imply that

$$
\begin{aligned}
& E(\sup_{s\in[0,t]}\|X^\varepsilon(s)\|_H^4) + \int_0^t \|X^\varepsilon(s)\|_H^2\|X^\varepsilon(s)\|_{\tilde{H}^{1,0}}^2 ds\\
\leqslant & C(N) + C(N)E\int_0^t \|X^\varepsilon(s)\|_H^4 ds,
\end{aligned}
$$

which by Gronwall's inequality yields that

$$
E(\sup_{s\in[0,t]}\|X^\varepsilon(s)\|_H^4) + \int_0^t \|X^\varepsilon(s)\|_H^2\|X^\varepsilon(s)\|_{\tilde{H}^{1,0}}^2 ds \leqslant C(N).
$$

For (5.10), let $h(t) = kg(t) + \int_0^t \|v^\varepsilon(s)\|_{l^2}^2 ds$ for some universal constant k. Applying Itô's formula to $e^{-h(t)}\|X^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2$ (by applying Itô's formula to its finite- dimension projection first and then passing to the limit), we have

$$\begin{aligned}
&e^{-h(t)}\|X^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2 + 2\int_0^t e^{-h(s)}(\|\partial_1 X^\varepsilon(s)\|_H^2 + \|\partial_1\partial_2 X^\varepsilon(s)\|_H^2)ds \\
=& -\int_0^t e^{-h(s)}(k\|\partial_1 X^\varepsilon(s)\|_H^2 + \|v^\varepsilon(s)\|_{l^2}^2)\|X^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 ds \\
&-2\int_0^t e^{-h(s)} b(X^\varepsilon, u^0, X^\varepsilon)ds - 2\int_0^t e^{-h(s)}\langle \partial_2 X^\varepsilon(s), \partial_2(X^\varepsilon\cdot\nabla(u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon))(s)\rangle ds \\
&-2\int_0^t e^{-h(s)}\langle \partial_2 X^\varepsilon(s), \partial_2(u^0\cdot\nabla X^\varepsilon)(s)\rangle ds \\
&+2\int_0^t e^{-h(s)}\langle X^\varepsilon(s), \sigma(s, u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))v^\varepsilon(s)\rangle_{\tilde{H}^{0,1}} ds \\
&+2\lambda^{-1}(\varepsilon)\int_0^t e^{-h(s)}\langle X^\varepsilon(s), \sigma(s, u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))dW(s)\rangle_{\tilde{H}^{0,1}} \\
&+\lambda^{-2}(\varepsilon)\int_0^t e^{-h(s)}\|\sigma(s, u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(s))\|_{L_2(l^2,\tilde{H}^{0,1})}^2 ds.
\end{aligned}$$

By Lemma 2.11, we have

$$2|b(X^\varepsilon, u^0, X^\varepsilon)| \leqslant \alpha\|\partial_1 X^\varepsilon\|_H^2 + C(1+\|u^0\|_{\tilde{H}^{1,1}}^2)\|X^\varepsilon\|_H^2,$$

where $\alpha < \frac{1}{3}$. By Lemma 2.12, there exists C_1,

$$2\sqrt{\varepsilon}\lambda(\varepsilon)|\langle \partial_2 X^\varepsilon, \partial_2(X^\varepsilon\cdot\nabla X^\varepsilon)\rangle| \leqslant \alpha\|\partial_1\partial_2 X^\varepsilon\|_H^2 + C_1(1+\|\partial_1 X^\varepsilon\|_H^2)\|\partial_2 X^\varepsilon\|_H^2.$$

By Lemma 2.11, we have

$$\begin{aligned}
2|\langle \partial_2 X^\varepsilon, \partial_2(X^\varepsilon\cdot\nabla u^0))\rangle| \leqslant& 2|b(\partial_2 X^\varepsilon, u^0, \partial_2 X^\varepsilon)| + 2|b(X^\varepsilon, \partial_2 u^0, \partial_2 X^\varepsilon)| \\
\leqslant& \alpha(\|X^\varepsilon\|_{\tilde{H}^{1,0}}^2 + \|\partial_2 X^\varepsilon\|_{\tilde{H}^{1,0}}^2) + C\|u^0\|_{\tilde{H}^{1,2}}^2\|\partial_2 X^\varepsilon\|_H^2.
\end{aligned}$$

Similarly,

$$|\langle \partial_2 X^\varepsilon(s), \partial_2(u^0\cdot\nabla X^\varepsilon)(s)\rangle| = |b(\partial_2 u^0, X^\varepsilon, \partial_2 X^\varepsilon)| \leqslant \alpha\|X^\varepsilon\|_{\tilde{H}^{1,1}}^2 + C\|u^0\|_{\tilde{H}^{1,1}}^2\|\partial_2 X^\varepsilon\|_H^2.$$

By Young's inequality,

$$2|\langle X^\varepsilon(s), \sigma(s, u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)v^\varepsilon(s)\rangle_{\tilde{H}^{0,1}}| \leqslant \|X^\varepsilon\|_{\tilde{H}^{0,1}}^2\|v^\varepsilon\|_{l^2}^2 + \|\sigma(s, u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)\|_{L_2(l^2,\tilde{H}^{0,1})}^2.$$

Choosing $k > 2C_1\sqrt{\varepsilon}\lambda(\varepsilon)$, we have

$$\begin{aligned}
&e^{-h(t)}\|X^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2 + (2-3\alpha)\int_0^t e^{-h(s)}\|X^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds \\
\leqslant& C\int_0^t e^{-h(s)}(1+\|u^0\|_{\tilde{H}^{1,2}}^2)\|X^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 ds \\
&+ (1+\lambda^{-2}(\varepsilon))\int_0^t e^{-h(s)}\|\sigma(s, u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)\|_{L_2(l^2,\tilde{H}^{0,1})}^2 ds
\end{aligned}$$

$$+ 2\lambda^{-1}(\varepsilon)\int_0^t e^{-h(s)}\langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)dW(s)\rangle_{\tilde{H}^{0,1}}.$$

By (A2) we have

$$\begin{aligned}&(1+\lambda^{-2}(\varepsilon))\|\sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)\|^2_{L_2(l^2,\tilde{H}^{0,1})} \leqslant C(1+\|u^0\|^2_{\tilde{H}^{1,1}})\\&+(1+\lambda^{-2}(\varepsilon))\left(\tilde{K}_0 + \tilde{K}_1\varepsilon\lambda^2(\varepsilon)\|X^\varepsilon\|^2_{\tilde{H}^{0,1}} + \tilde{K}_2\varepsilon\lambda^2(\varepsilon)(\|\partial_1 X^\varepsilon\|^2_H + \|\partial_1\partial_2 X^\varepsilon\|^2_H)\right).\end{aligned}$$

By the Burkhölder-Davis-Gundy inequality we have

$$\begin{aligned}&2\lambda^{-1}(\varepsilon)E\left(\sup_{s\in[0,t]}|\int_0^s e^{-h(r)}\langle X^\varepsilon(r), \sigma(r, u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)dW(r)\rangle_{\tilde{H}^{0,1}}|\right)\\&\leqslant 6\lambda^{-1}(\varepsilon)E\left(\int_0^t e^{-2h(s)}\|X^\varepsilon(s)\|^2_{\tilde{H}^{0,1}}\|\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)\|^2_{L_2(l^2,\tilde{H}^{0,1})}ds\right)^{\frac{1}{2}}\\&\leqslant \lambda^{-1}(\varepsilon)E[\sup_{s\in[0,t]}(e^{-h(s)}\|X^\varepsilon(s)\|^2_{\tilde{H}^{0,1}})] + \lambda^{-1}(\varepsilon)C\int_0^t e^{-h(s)}(1+\|u^0\|^2_{\tilde{H}^{1,1}})ds\\&\quad + 9\varepsilon\lambda(\varepsilon)E\int_0^t e^{-h(s)}[\tilde{K}_1\|X^\varepsilon(s)\|^2_{\tilde{H}^{0,1}} + \tilde{K}_2(\|\partial_1 X^\varepsilon(s)\|^2_H + \|\partial_1\partial_2 X^\varepsilon(s)\|^2_H)]ds,\end{aligned}$$

where we choose ε small enough such that $(9\varepsilon\lambda(\varepsilon)+\varepsilon\lambda^2(\varepsilon)+\varepsilon)\tilde{K}_2 < 1-3\alpha$ and we used (A2) in the last inequality.

Combine the above estimates, we have

$$\begin{aligned}&E(\sup_{s\in[0,t]} e^{-h(s)}\|X^\varepsilon(s)\|^2_{\tilde{H}^{0,1}}) + E\int_0^t e^{-h(s)}\|X^\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds\\&\leqslant C + CE\left(\int_0^t e^{-h(s)}(1+\|u^0(s)\|^2_{\tilde{H}^{1,2}})\|X^\varepsilon(s)\|^2_{\tilde{H}^{0,1}}ds\right)\end{aligned}$$

Then Gronwall's inequality and (4.1) imply that

$$E(\sup_{0\leqslant t\leqslant T} e^{-h(t)}\|X^\varepsilon(t)\|^2_{\tilde{H}^{0,1}}) + E\int_0^T e^{-h(s)}\|X^\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds \leqslant C.$$

Since $v^\varepsilon \in \mathcal{S}_N$, we deduce that

$$E(\sup_{t\in[0,T]} e^{-kg(t)}\|X^\varepsilon(t)\|^2_{\tilde{H}^{0,1}}) + E\int_0^T e^{-kg(s)}\|X^\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds \leqslant C. \tag{5.13}$$

□

Similar as [LZZ18, lemma 4.3], we have the following tightness lemma:

Lemma 5.6. *Assume X^ε is a solution to (5.7) with $v^\varepsilon \in \mathcal{A}_N$ and ε small enough. There exists $\varepsilon_0 > 0$, such that $\{X^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ is tight in the space*

$$\chi = C([0,T],H^{-1})\bigcap L^2([0,T],H)\bigcap L^2_w([0,T],\tilde{H}^{1,1})\bigcap L^\infty_{w^*}([0,T],\tilde{H}^{0,1}),$$

where L^2_w denotes the weak topology and $L^\infty_{w^*}$ denotes the weak star topology.

Proof Similar as in the proof of Lemma 3.6, the law of Z_v^ε on $C([0,T],H^{-1})$ can be restricted on χ.

Let k be the same constant as in the proof of (5.10) and let

$$K_R := \Big\{ u \in C([0,T],H^{-1}) : \sup_{t\in[0,T]} \|u(t)\|_H^2 + \int_0^T \|u(t)\|^2_{\tilde{H}^{1,0}} dt + \|u\|_{C^{\frac{1}{16}}([0,T],H^{-1})} \\ + \sup_{t\in[0,T]} e^{-k\int_0^t \|\partial_1 u(s)\|_H^2 ds} \|u(t)\|^2_{\tilde{H}^{0,1}} + \int_0^T e^{-k\int_0^t \|\partial_1 u(s)\|_H^2 ds} \|u(t)\|^2_{\tilde{H}^{1,1}} dt \leqslant R \Big\},$$

where $C^{\frac{1}{16}}([0,T],H^{-1})$ is the Hölder space with the norm:

$$\|f\|_{C^{\frac{1}{16}}([0,T],H^{-1})} = \sup_{0\leqslant s<t\leqslant T} \frac{\|f(t)-f(s)\|_{H^{-1}}}{|t-s|^{\frac{1}{16}}}.$$

Then from the proof of [LZZ18, Lemma 4.3], we know that for any $R>0$, K_R is relatively compact in χ.

Now we only need to show that for any $\delta>0$, there exists $R>0$, such that $P(X^\varepsilon \in K_R) > 1-\delta$ for any $\varepsilon \in (0,\varepsilon_0)$, where ε_0 is the constant such that Lemma 5.5 hold.

By Lemma 5.5 and Chebyshev inequality, we can choose R_0 large enough such that

$$P\left(\sup_{t\in[0,T]} \|X^\varepsilon(t)\|_H^2 + \int_0^T \|X^\varepsilon(t)\|^2_{\tilde{H}^{1,0}} dt > \frac{R_0}{3} \right) < \frac{\delta}{4},$$

and

$$P\left(\sup_{t\in[0,T]} e^{-k\int_0^t \|\partial_1 X^\varepsilon(s)\|_H^2 ds} \|X^\varepsilon(t)\|^2_{\tilde{H}^{0,1}} + \int_0^T e^{-k\int_0^t \|\partial_1 X^\varepsilon(s)\|_H^2 ds} \|X^\varepsilon(t)\|^2_{\tilde{H}^{1,1}} dt > \frac{R_0}{3} \right) < \frac{\delta}{4},$$

where k is the same constant as in (5.10).

Fix R_0 and let

$$\hat{K}_{R_0} = \Big\{ u \in C([0,T],H^{-1}) : \sup_{t\in[0,T]} \|u(t)\|_H^2 + \int_0^T \|u(t)\|^2_{\tilde{H}^{1,0}} dt \leqslant \frac{R_0}{3} \text{ and} \\ \sup_{t\in[0,T]} e^{-k\int_0^t \|\partial_1 u(s)\|_H^2 ds} \|u(t)\|^2_{\tilde{H}^{0,1}} + \int_0^T e^{-k\int_0^t \|\partial_1 u(s)\|_H^2 ds} \|u(t)\|^2_{\tilde{H}^{1,1}} dt \leqslant \frac{R_0}{3} \Big\}.$$

Then $P(X^\varepsilon \in C([0,T],H^{-1}) \setminus \hat{K}_{R_0}) < \frac{\delta}{2}$.

Now for $X^\varepsilon \in \hat{K}_{R_0}$, we have $\partial_1^2 X^\varepsilon$ is uniformly bounded in $L^2([0,T],H^{-1})$. Similar as in Lemma 5.2, X^ε is uniformly bounded in $L^4([0,T],H^{\frac{1}{2}})$ and $L^4([0,T],L^4(\mathbb{T}^2))$, thus $B(X^\varepsilon, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon)$ and $B(u^0, X^\varepsilon)$ are uniformly bounded in $L^2([0,T],H^{-1})$. By Hölder's inequality, we have

$$\sup_{s,t\in[0,T],s\neq t} \frac{\|\int_s^t \partial_1^2 X^\varepsilon(r) + B(X^\varepsilon, u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon) + B(u^0,X^\varepsilon) dr\|^2_{H^{-1}}}{|t-s|} \\ \leqslant \int_0^T \|\partial_1^2 X^\varepsilon(r) + B(X^\varepsilon, u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon) + B(u^0,X^\varepsilon)\|^2_{H^{-1}} dr \leqslant C(R_0),$$

where $C(R_0)$ is a constant depend on R_0. For any $p \in (1, \frac{4}{3})$, by Hölder's inequality, we have

$$\begin{aligned}
&\sup_{s,t\in[0,T],s\neq t} \frac{\| \int_s^t \sigma(r, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(r))v^\varepsilon(r)dr\|_{H^{-1}}^p}{|t-s|^{p-1}} \\
\leqslant& \int_0^T \|\sigma(r, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(r))v^\varepsilon(r)\|_{H^{-1}}^p dr \\
\leqslant& \int_0^T \|\sigma(r, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(r))\|_{L_2(l^2,H^{-1})}^p \|v^\varepsilon(r)\|_{l^2}^p dr \\
\leqslant& C\int_0^T (1 + \|u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(r)\|_H^4 + \|v^\varepsilon(r)\|_{l^2}^4) dr \\
\leqslant& C(R_0),
\end{aligned}$$

where we used Young's inequality and (A0) in the third inequality.

Moreover, for any $0 \leqslant s \leqslant t \leqslant T$, by Hölder's inequality we have

$$\begin{aligned}
&E\| \int_s^t \sigma(r, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(r))dW(r)\|_{H^{-1}}^4 \\
\leqslant& CE\left(\int_s^t \|\sigma(r, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(r))\|_{L_2(l^2,H^{-1})}^2 dr\right)^2 \\
\leqslant& C|t-s|E\int_s^t \|\sigma(r, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(r))\|_{L_2(l^2,H^{-1})}^4 dr \\
\leqslant& C|t-s|^2(1 + E(\sup_{t\in[0,T]} \|u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(t)\|_H^4)) \\
\leqslant& C|t-s|^2,
\end{aligned}$$

where we used (A0) in the third inequality and (5.9) in the last inequality. Then by Kolmogorov's continuity criterion, for any $\alpha \in (0, \frac{1}{4})$, we have

$$E\left(\sup_{s,t\in[0,T],s\neq t} \frac{\| \int_s^t \sigma(r, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(r))dW(r)\|_{H^{-1}}^4}{|t-s|^{2\alpha}}\right) \leqslant C.$$

Choose $p = \frac{8}{7}, \alpha = \frac{1}{8}$ in the above estimates, we deduce that there exists $R > R_0$ such that

$$\begin{aligned}
&P\left(\|X^\varepsilon\|_{C^{\frac{1}{16}}([0,T],H^{-1})} > \frac{R}{3}, X^\varepsilon \in \hat{K}_{R_0}\right) \\
\leqslant& \frac{E\left(\sup_{s,t\in[0,T],s\neq t} \frac{\|X^\varepsilon(t)-X^\varepsilon(s)\|_{H^{-1}}}{|t-s|^{\frac{1}{16}}} 1_{\{X^\varepsilon\in\hat{K}_{R_0}\}}\right)}{\frac{R}{3}} < \frac{\delta}{2}.
\end{aligned}$$

Combining the fact that $P(X^\varepsilon \in C([0,T], H^{-1}) \setminus \hat{K}_{R_0}) < \frac{\delta}{2}$, we finish the proof. ☐

Lemma 5.7. *Let $\{v^\varepsilon\}_{\varepsilon>0} \subset \mathcal{A}_N$ for some $N < \infty$. Assume v^ε converge to v in distribution as S_N-valued random elements, then*

$$g^\varepsilon\left(W(\cdot) + \lambda(\varepsilon)\int_0^\cdot v^\varepsilon(s)ds\right) \to g^0\left(\int_0^\cdot v(s)ds\right)$$

in distribution as $\varepsilon \to 0$.

Proof The proof follows essentially the same argument as in [WZZ15, Proposition 4.7].

By Lemma 5.3, we have $X^\varepsilon = g^\varepsilon\left(W(\cdot)+\lambda(\varepsilon)\int_0^\cdot v^\varepsilon(s)ds\right)$. By a similar argument as in the proof of Lemmas 5.2 and 5.5, there exists a unique strong solution

$$Y^\varepsilon \in L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})\bigcap C([0,T],H^{-1})$$

satisfying

$$\begin{aligned} dY^\varepsilon(t) =&\partial_1^2 Y^\varepsilon(t)dt+\lambda^{-1}(\varepsilon)\sigma(t,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon(t))dW(t),\\ Y^\varepsilon(0)=&0, \end{aligned}$$

and

$$\lim_{\varepsilon\to 0}\left[E\sup_{t\in[0,T]}\|Y^\varepsilon(t)\|_H^2+E\int_0^T\|Y^\varepsilon(t)\|^2_{\tilde{H}^{1,0}}dt\right]=0,$$

$$\lim_{\varepsilon\to 0}\left[E\sup_{t\in[0,T]}(e^{-kg(t)}\|Y^\varepsilon(t)\|^2_{\tilde{H}^{0,1}})+E\int_0^T e^{-kg(t)}\|Y^\varepsilon(t)\|^2_{\tilde{H}^{1,1}}dt\right]=0,$$

where $g(t)=\int_0^t\|\partial_1 X^\varepsilon(s)\|_H^2ds$ and k are the same as in (5.10).

Set

$$\Xi := \left(\chi,\mathcal{S}_N, L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T].H^{-1})\right).$$

The above limit implies that $Y^\varepsilon\to 0$ in $L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T].H^{-1})$ almost surely as $\varepsilon\to 0$ (in the sense of subsequence). By Lemma 5.6 the family $\{(X^\varepsilon,v^\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0)}$ is tight in $(\chi,\mathcal{S}_N)$. Let $(X_v,v,0)$ be any limit point of $\{(X^\varepsilon,v^\varepsilon,Y^\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0)}$. Our goal is to show that X_v has the same law as $g^0\left(\int_0^\cdot v(s)ds\right)$ and X^ε convergence in distribution to X_v in the space $L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$.

By Jakubowski-Skorokhod's representation theorem (see [Jak98] or [LZZ18, Theorem 4.3]), there exists a stochastic basis $(\tilde{\Omega},\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]},\tilde{P})$ and, on this basis, Ξ-valued random variables $(\tilde{X}_v,\tilde{v},0)$, $(\tilde{X}^\varepsilon,\tilde{v}^\varepsilon,\tilde{Y}^\varepsilon)$, such that $(\tilde{X}^\varepsilon,\tilde{v}^\varepsilon,\tilde{Y}^\varepsilon)$ (respectively $(\tilde{X}_v,\tilde{v},0)$) has the same law as $(X^\varepsilon,v^\varepsilon,Y^\varepsilon)$ (respectively $(X_v,v,0)$), and $(\tilde{X}^\varepsilon,\tilde{v}^\varepsilon,\tilde{Y}^\varepsilon)\to(\tilde{X}_v,\tilde{v},0)$, $\tilde{P}$-a.s.

We have

$$\begin{aligned} d(\tilde{X}^\varepsilon(t)-\tilde{Y}^\varepsilon(t)) =&\partial_1^2(\tilde{X}^\varepsilon(t)-\tilde{Y}^\varepsilon(t))dt-B(\tilde{X}^\varepsilon,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)dt\\ &-B(u^0,\tilde{X}^\varepsilon)dt+\sigma(t,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon(t))\tilde{v}^\varepsilon(t)dt,\\ \tilde{X}^\varepsilon(0)-\tilde{Y}^\varepsilon(0)=&0, \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} &P(\tilde{X}^\varepsilon-\tilde{Y}^\varepsilon\in L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1}))\\ =&P(X^\varepsilon-Y^\varepsilon\in L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1}))\\ =&1. \end{aligned}$$

Let $\tilde{\Omega}_0$ be the subset of $\tilde{\Omega}$ such that for $\omega\in\tilde{\Omega}_0$,

$$(\tilde{X}^\varepsilon,\tilde{v}^\varepsilon,\tilde{Y}^\varepsilon)(\omega)\to(\tilde{X}_v,\tilde{v},0)(\omega)\text{ in }\Xi,$$

and

$$e^{-k\int_0^\cdot \|\tilde{X}^\varepsilon(\omega,s)\|_H^2 ds}\tilde{Y}^\varepsilon(\omega) \to 0 \text{ in } L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})\bigcap C([0,T],H^{-1}),$$

then $P(\tilde{\Omega}_0)=1$. For any $\omega\in\tilde{\Omega}_0$, fix ω, we have $\sup_\varepsilon \int_0^T \|\tilde{X}^\varepsilon(\omega,s)\|_H^2 ds<\infty$, then we deduce that

$$\lim_{\varepsilon\to 0}\left(\sup_{t\in[0,T]}\|\tilde{Y}^\varepsilon(\omega,t)\|_{\tilde{H}^{0,1}}+\int_0^T \|\tilde{Y}^\varepsilon(\omega,t)\|_{\tilde{H}^{1,1}}^2 dt\right)=0. \tag{5.15}$$

Now we show that

$$\sup_{t\in[0,T]}\|\tilde{X}^\varepsilon(\omega,t)-\tilde{X}_v(\omega,t)\|_H^2+\int_0^T \|\tilde{X}^\varepsilon(\omega,t)-\tilde{X}_v(\omega,t)\|_{\tilde{H}^{1,0}}^2 dt\to 0 \text{ as } \varepsilon\to 0. \tag{5.16}$$

Let $U^\varepsilon=\tilde{X}^\varepsilon(\omega)-\tilde{Y}^\varepsilon(\omega)$, then by (5.14) we have

$$\begin{aligned} dU^\varepsilon(t) =&\partial_1^2 U^\varepsilon(t)dt - B(U^\varepsilon+\tilde{Y}^\varepsilon, u^0+\sqrt{\varepsilon}\lambda(\varepsilon)(U^\varepsilon+\tilde{Y}^\varepsilon))dt \\ &- B(u^0, U^\varepsilon+\tilde{Y}^\varepsilon)+\sigma(t,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)(U^\varepsilon(t)+\tilde{Y}^\varepsilon(t)))\tilde{v}^\varepsilon(t)dt. \end{aligned} \tag{5.17}$$

Since $U^\varepsilon(\omega)\to\tilde{X}_v(\omega)$ in χ, by a very similar argument as in Lemma 5.4 we deduce that $\tilde{X}_v=X^{\tilde{v}}=g^0\left(\int_0^\cdot \tilde{v}(s)ds\right)$. Moreover, note that $\tilde{X}^\varepsilon(\omega)\to X^{\tilde{v}}(\omega)$ weak star in $L^\infty([0,T],\tilde{H}^{0,1})$, then the uniform boundedness principle implies that

$$\sup_\varepsilon\sup_{t\in[0,T]}\|\tilde{X}^\varepsilon(\omega)\|_{\tilde{H}^{0,1}}<\infty. \tag{5.18}$$

Let $w^\varepsilon=U^\varepsilon-X^{\tilde{v}}$, then we have

$$\begin{aligned} &\|w^\varepsilon(t)\|_H^2+2\int_0^t \|\partial_1 w^\varepsilon(s)\|_H^2 ds \\ =&-2\int_0^t \langle w^\varepsilon(s), B(U^\varepsilon+\tilde{Y}^\varepsilon, u^0+\sqrt{\varepsilon}\lambda(\varepsilon)(U^\varepsilon+\tilde{Y}^\varepsilon))-B(X^{\tilde{v}},u^0)\rangle ds \\ &-2\int_0^t \langle w^\varepsilon(s), B(u^0, w^\varepsilon+\tilde{Y}^\varepsilon)\rangle ds \\ &+2\int_0^t \langle w^\varepsilon(s), \sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)(U^\varepsilon+\tilde{Y}^\varepsilon))\tilde{v}^\varepsilon(s)-\sigma(s,u^0)\tilde{v}(s)\rangle ds \\ =:&I_1+I_2+I_3. \end{aligned}$$

By Lemma 2.11, we have

$$\begin{aligned} &|I_1+I_2| \\ =&|\int_0^t b(w^\varepsilon, u^0+\sqrt{\varepsilon}\lambda(\varepsilon)(X^{\tilde{v}}+\tilde{Y}^\varepsilon), w^\varepsilon)+b(\tilde{Y}^\varepsilon,u^0,w^\varepsilon) \\ &+\sqrt{\varepsilon}\lambda(\varepsilon)b(X^{\tilde{v}}+\tilde{Y}^\varepsilon, X^{\tilde{v}}+\tilde{Y}^\varepsilon, w^\varepsilon)+b(u^0,\tilde{Y}^\varepsilon,w^\varepsilon)ds| \\ \leqslant&\int_0^t [\frac{1}{2}\|\partial_1 w^\varepsilon(s)\|_H^2+C(1+\|u^0(s)\|_{\tilde{H}^{1,1}}^2+\|X^{\tilde{v}}(s)\|_{\tilde{H}^{1,1}}^2+\|\tilde{Y}^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2)\|w^\varepsilon(s)\|_H^2]ds \\ &+\int_0^t [\|\tilde{Y}^\varepsilon(s)\|_{\tilde{H}^{1,0}}^2+C\|u^0(s)\|_{\tilde{H}^{1,1}}^2\|w^\varepsilon(s)\|_H^2]ds \end{aligned}$$

$$
\begin{aligned}
&+\sqrt{\varepsilon}\lambda(\varepsilon)\int_0^t [\|X^{\tilde{v}}(s)\|^2_{\tilde{H}^{1,0}}+\|\tilde{Y}^\varepsilon(s)\|^2_{\tilde{H}^{1,0}}+(\|X^{\tilde{v}}(s)\|^2_{\tilde{H}^{1,1}}+\|\tilde{Y}^\varepsilon(s)\|^2_{\tilde{H}^{1,1}})\|w^\varepsilon(s)\|^2_H]ds\\
&+\int_0^t [\|\tilde{Y}^\varepsilon(s)\|^2_{\tilde{H}^{1,1}}+C\|u^0(s)\|^2_{\tilde{H}^{1,0}}\|w^\varepsilon(s)\|^2_H]ds\\
\leqslant&\int_0^t [\frac{1}{2}\|\partial_1 w^\varepsilon(s)\|^2_H+C(1+\|u^0(s)\|^2_{\tilde{H}^{1,1}}+\|X^{\tilde{v}}(s)\|^2_{\tilde{H}^{1,1}})\|w^\varepsilon(s)\|^2_H]ds\\
&+C\int_0^t \|\tilde{Y}^\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds+\sqrt{\varepsilon}\lambda(\varepsilon)\int_0^t \|X^{\tilde{v}}(s)\|^2_{\tilde{H}^{1,0}}ds.
\end{aligned}
$$

where we used the fact that by (5.15) and (5.18) w^ε are uniformly bounded in $L^\infty([0,T],H)$ in the last inequality. By (A1) and (A3) we have

$$
\begin{aligned}
|I_3(t)|=&\int_0^t \langle w^\varepsilon(s),(\sigma(s,u^0+\sqrt{\varepsilon}\lambda(\varepsilon)[U^\varepsilon+\tilde{Y}^\varepsilon])-\sigma(s,u^0))\tilde{v}^\varepsilon(s)\rangle ds\\
&+\int_0^t \langle w^\varepsilon(s),\sigma(s,u^0)(\tilde{v}^\varepsilon(s)-\tilde{v}(s))\rangle ds\\
\leqslant& C(\sqrt{\varepsilon}\lambda(\varepsilon))^{\frac{1}{2}}\int_0^t (\|w^\varepsilon(s)\|_H\|\tilde{v}^\varepsilon(s)\|_{l^2}(\|w^\varepsilon(s)\|^2_{\tilde{H}^{1,0}}+\|X^{\tilde{v}}(s)\|^2_{\tilde{H}^{1,0}}+\|\tilde{Y}^\varepsilon(s)\|^2_{\tilde{H}^{1,0}})^{\frac{1}{2}}ds\\
&+\int_0^t \|w^\varepsilon(s)\|_H\|\tilde{v}^\varepsilon(s)-\tilde{v}(s)\|_{l^2}(K_0+K_1\|u^0(s)\|^2_H+K_2\|\partial_1 u^0(s)\|^2_H)^{\frac{1}{2}}ds\\
\leqslant&(\sqrt{\varepsilon}\lambda(\varepsilon))^{\frac{1}{2}}\left(CN+C_1\int_0^t (\|w^\varepsilon(s)\|^2_{\tilde{H}^{1,0}}+\|X^{\tilde{v}}(s)\|^2_{\tilde{H}^{1,0}}+\|\tilde{Y}^\varepsilon(s)\|^2_{\tilde{H}^{1,0}}ds\right)\\
&+CN^{\frac{1}{2}}\left(\int_0^t \|w^\varepsilon(s)\|^2_H(K_0+K_1\|u^0(s)\|^2_H+K_2\|\partial_1 u^0(s)\|^2_H)ds\right)^{\frac{1}{2}},
\end{aligned}
$$

where we used the fact that w^ε are uniformly bounded in $L^\infty([0,T],H)$ and that $\tilde{v}^\varepsilon$, $\tilde{v}$ are in $\mathcal{A}_N$. Note here C_1 is a positive constant. Thus choose ε small enough such that $\frac{1}{2}+(\sqrt{\varepsilon}\lambda(\varepsilon))^{\frac{1}{2}}C_1<1$, we have

$$
\begin{aligned}
&\|w^\varepsilon(t)\|^2_H+\int_0^t \|\partial_1 w^\varepsilon(s)\|^2_H ds\\
\leqslant& C\int_0^t (1+\|u^0(s)\|^2_{\tilde{H}^{1,1}}+\|X^{\tilde{v}}(s)\|^2_{\tilde{H}^{1,1}})\|w^\varepsilon(s)\|^2_H ds\\
&+C\int_0^t \|\tilde{Y}^\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds+\sqrt{\varepsilon}\lambda(\varepsilon)\int_0^t \|X^{\tilde{v}}(s)\|^2_{\tilde{H}^{1,0}}ds\\
&+C(\sqrt{\varepsilon}\lambda(\varepsilon))^{\frac{1}{2}}\left(N+\int_0^t (\|w^\varepsilon(s)\|^2_H+\|X^{\tilde{v}}(s)\|^2_{\tilde{H}^{1,0}}+\|\tilde{Y}^\varepsilon(s)\|^2_{\tilde{H}^{1,0}}ds\right)\\
&+CN^{\frac{1}{2}}\left(\int_0^t (1+\|u^0(s)\|^2_{\tilde{H}^{1,1}})\|w^\varepsilon(s)\|^2_H ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

Since $U^\varepsilon(\omega)\to X^{\tilde{v}}(\omega)$ strongly in $L^2([0,T],H)$ and $\tilde{Y}^\varepsilon\to 0$ in $L^2([0,T],\tilde{H}^{1,1})$, the same argument used in Lemma 5.4 implies

$$
\sup_{t\in[0,T]}\|\tilde{X}^\varepsilon(\omega,t)-X^{\tilde{v}}(\omega,t)\|^2_H+\int_0^T \|\tilde{X}^\varepsilon(\omega,t)-X^{\tilde{v}}(\omega,t)\|^2_{\tilde{H}^{1,0}}dt\to 0 \text{ as } \varepsilon\to 0. \quad (5.19)
$$

The proof is thus complete.

□

Proof of Theorem 5.1. *The result holds from Lemmas 2.6, 5.4 and 5.7.* □

Chapter 6

Small time asymptotics

In this chapter, we consider the small time behaviour. We need the following additional assumption (A3') and (A4). Note that (A3') is stronger than (A3).

(A3') $\|\sigma(t,u)-\sigma(s,v)\|^2_{L_2(l^2,H)} \leqslant L_0|t-s|^\alpha + L_1\|u-v\|^2_H$.

(A4) $\|\sigma(t,u)\|^2_{L_2(l^2,V)} \leqslant \overline{K}_0 + \overline{K}_1\|u\|^2_V$.

Remark 6.1. *A typical example of σ is similar as in [LZZ18, Remark 4.2]. For $u = (u^1,u^2) \in H^{1,1}$ and $y \in l^2$, let*

$$\sigma(t,u)y = \sum_{k=1}^{\infty} b_k g(u)\langle y, \psi_k\rangle_{l^2},$$

where $\{\psi_k\}_{k\geqslant 0}$ is the orthonormal basis of l^2, $\{b_k\}_{k\geqslant 0}$ are functions from $\mathbb{T}^2$ to $\mathbb{R}$ and g is a differentiable function from $\mathbb{R}^2$ to $\mathbb{R}$. Assume that $|g(x)-g(y)| \leqslant C|x-y|$ for all $x,y\in\mathbb{R}^2$ and some constant C depends on g. Also suppose that $\mathrm{div}(b_k g(u)) = 0$ and $b_k, \partial_1 b_k, \partial_2 b_k \in L^\infty$, $\sum_{k=1}^\infty \|b_k\|^2_{L^\infty} \leqslant M$, $\sum_{k=1}^\infty \|\partial_1 b_k\|^2_{L^\infty} \leqslant M$ and $\sum_{k=1}^\infty \|\partial_2 b_k\|^2_{L^\infty} \leqslant M$. From the conditions of g, it is easy to obtain $|g(u)| \leqslant C|u| + C$, $|\partial_1 g(u)| \leqslant C$ and $|\partial_2 g(u)| \leqslant C$. In this case, σ satisfies (A0)-(A4) and (A3'):

$$\begin{aligned}
\|\sigma(t,u)\|^2_{L_2(l^2,H)} \leqslant& \sum_{k=1}^\infty \|b_k g(u)\|^2_H \leqslant CM(\|u\|^2_H + 1);\\
\|\sigma(t,u)\|^2_{L_2(l^2,H^{0,1})} \leqslant& \sum_{k=1}^\infty \|b_k g(u)\|^2_H + \sum_{k=1}^\infty \|\partial_2(b_k g(u))\|^2_H\\
\leqslant& CM(\|u\|^2_H+1)\\
&+\sum_{k=1}^\infty \|\partial_2 b_k g(u) + b_k(\partial_1 g(u)\partial_2 u^1 + \partial_2 g(u)\partial_2 u^2)\|^2_H\\
\leqslant& CM(1+\|u\|^2_H + \|\partial_2 u\|^2_H);\\
\|\sigma(t,u)\|^2_{L_2(l^2,V)} \leqslant& CM(\|u\|^2_H+1) + \sum_{k=1}^\infty \|\partial_1(b_k g(u))\|^2_H + \sum_{k=1}^\infty \|\partial_2(b_k g(u))\|^2_H\\
\leqslant& CM(1+\|u\|^2_H + \|\partial_1 u\|^2_H + \|\partial_2 u\|^2_H);\\
\|\sigma(t,u)-\sigma(s,v)\|^2_{L_2(l^2,H)} \leqslant& MC\|u-v\|^2_H.
\end{aligned}$$

Let $\varepsilon > 0$ and u be the solution to (2.2), by the scaling property of the Brownian motion, $u(\varepsilon t)$ coincides in law with the solution to the following equation:

$$\begin{aligned} du_\varepsilon &= \varepsilon \partial_1^2 u_\varepsilon dt - \varepsilon B(u_\varepsilon)dt + \sqrt{\varepsilon}\sigma(\varepsilon t, u_\varepsilon)dW(t), \\ u_\varepsilon(0) &= u_0. \end{aligned} \tag{6.1}$$

Define a functional I^{u_0} on $L^\infty([0,T],H)\bigcap C([0,T],H^{-1})$ by

$$I^{u_0}(g) = \inf_{h\in\Gamma_g}\{\frac{1}{2}\int_0^T \|h(t)\|_{l^2}^2 dt\},$$

where

$$\Gamma_g = \{h \in L^2([0,T],l^2) : g(t) = u_0 + \int_0^t \sigma(0,g(s))h(s)ds,\ t\in[0,T]\}.$$

The main theorem of this chapter is the following one:

Theorem 6.2. *Assume (A0), (A1), (A2), (A3'), (A4) hold with $K_2 = \tilde{K}_2 = 0$ and $u_0 \in \tilde{H}^{0,1}$, then u_ε satisfies a large deviation principle on $L^\infty([0,T],H)\bigcap C([0,T],H^{-1})$ with the good rate function I^{u_0}.*

We aim to prove that u_ε is exponentially equivalent to the solution to the following equation:

$$v_\varepsilon(t) = u_0 + \sqrt{\varepsilon}\int_0^t \sigma(\varepsilon s, v_\varepsilon(s))dW(s). \tag{6.2}$$

Because of the non-linear form $b(\cdot,\cdot,\cdot)$ and the anisotropic viscosity, we split the proof into several lemmas.

6.1 LDP for linear equation

In this section we prove that v_ε satisfies a large deviation principle.

Lemma 6.3. *Assume $u_0 \in \tilde{H}^{0,1}$, then v_ε satisfies a large deviation principle on the space $L^\infty([0,T],H)\bigcap C([0,T],H^{-1})$ with the good rate function I^{u_0}.*

Proof Let z_ε be the solution to the stochastic equation:

$$z_\varepsilon(t) = u_0 + \sqrt{\varepsilon}\int_0^t \sigma(0, z_\varepsilon(s))dW(s).$$

By [DPZ09, Theorem 12.11], we know that z_ε satisfies a large deviation principle with the good rate function I^{u_0}. Applying Itô's formula to $\|v_\varepsilon - z_\varepsilon\|_H^2$, we obtain

$$\begin{aligned} \|v_\varepsilon(t) - z_\varepsilon(t)\|_H^2 =& 2\sqrt{\varepsilon}\int_0^t \langle v_\varepsilon(s) - z_\varepsilon(s), [\sigma(\varepsilon s, v_\varepsilon(s)) - \sigma(0, z_\varepsilon(s))]dW(s)\rangle \\ &+ \varepsilon\int_0^t \|\sigma(\varepsilon s, v_\varepsilon(s)) - \sigma(0, z_\varepsilon(s))\|_{L_2(l^2,H)}^2 ds. \end{aligned}$$

Then by (A3') and Lemma 2.14, we get for $p \geqslant 2$,

$$
\begin{aligned}
&\left(E[\sup_{0\leqslant t\leqslant T}\|v_\varepsilon(t)-z_\varepsilon(t)\|_H^{2p}]\right)^{\frac{2}{p}}\\
\leqslant& C\varepsilon\left(E[\sup_{0\leqslant t\leqslant T}\int_0^t\langle v_\varepsilon(s)-z_\varepsilon(s),(\sigma(\varepsilon s,v_\varepsilon(s))-\sigma(0,z_\varepsilon(s)))dW(s)\rangle]^p\right)^{\frac{2}{p}}\\
&+C\varepsilon^2\left(E[\int_0^T\|\sigma(\varepsilon s,v_\varepsilon(s))-\sigma(0,z_\varepsilon(s))\|^2_{L_2(l^2,H)}ds]^p\right)^{\frac{2}{p}}\\
\leqslant& C\varepsilon p\left(E\left[\int_0^T\|v_\varepsilon(s)-z_\varepsilon(s)\|_H^2\|\sigma(\varepsilon s,v_\varepsilon(s))-\sigma(0,z_\varepsilon(s))\|^2_{L_2(l^2,H)}ds\right]^{\frac{p}{2}}\right)^{\frac{2}{p}}\\
&+C\varepsilon^2\left(\varepsilon^{2\alpha}T^{2+2\alpha}+T\int_0^T\left(E[\sup_{0\leqslant l\leqslant s}\|v_\varepsilon(l)-z_\varepsilon(l)\|_H^{2p}]\right)^{\frac{2}{p}}ds\right)\\
\leqslant& C\varepsilon p\left(\varepsilon^{2\alpha}+\int_0^T\left(E[\sup_{0\leqslant l\leqslant s}\|v_\varepsilon(l)-z_\varepsilon(l)\|_H^{2p}]\right)^{\frac{2}{p}}ds\right)\\
&+C\varepsilon^2\left(\varepsilon^{2\alpha}+\int_0^T\left(E[\sup_{0\leqslant l\leqslant s}\|v_\varepsilon(l)-z_\varepsilon(l)\|_H^{2p}]\right)^{\frac{2}{p}}ds\right).
\end{aligned}
$$

By Gronwall's inequality, we have

$$
\left(E[\sup_{0\leqslant t\leqslant T}\|v_\varepsilon(t)-z_\varepsilon(t)\|_H^{2p}]\right)^{\frac{2}{p}}\leqslant C(\varepsilon^{1+2\alpha}p+\varepsilon^{2+2\alpha})e^{C(\varepsilon p+\varepsilon^2)}.
$$

Then Chebyshev's inequality implies that

$$
\begin{aligned}
\varepsilon\log P(\sup_{0\leqslant t\leqslant T}\|v_\varepsilon(t)-z_\varepsilon(t)\|_H^2>\delta)\leqslant&\varepsilon\log E[\sup_{0\leqslant t\leqslant T}\|v_\varepsilon(t)-z_\varepsilon(t)\|_H^{2p}]-\varepsilon p\log\delta\\
\leqslant&\frac{\varepsilon p}{2}(C+C\varepsilon p+C\varepsilon^2+\log(\varepsilon^{1+2\alpha}p+\varepsilon^{2+2\alpha})-2\log\delta).
\end{aligned}
$$

Let $p=\frac{1}{\varepsilon}$ and $\varepsilon\to 0$, we get that v_ε and z_ε are exponentially equivalent, which by Lemma 2.3 implies the result. □

6.2 Energy estimates

In this section, we give some energy estimates.

Lemma 6.4. *Let* $F_{u_\varepsilon}(t)=\sup_{0\leqslant s\leqslant t}\|u_\varepsilon(s)\|_H^2+\varepsilon\int_0^t\|\partial_1 u_\varepsilon(s)\|_H^2ds$*, then*

$$
\lim_{M\to\infty}\sup_{0<\varepsilon\leqslant 1}\varepsilon\log P(F_{u_\varepsilon}(T)>M)=-\infty.
$$

Proof

Since $b(u_\varepsilon,u_\varepsilon,u_\varepsilon)=0$, applying Itô's formula to $\|u_\varepsilon(t)\|_H^2$, we have

$$
\|u_\varepsilon(t)\|_H^2+2\varepsilon\int_0^t\|\partial_1 u_\varepsilon(s)\|_H^2ds
$$

$$=\|u_0\|_H^2+2\sqrt{\varepsilon}\int_0^t\langle u_\varepsilon(s),\sigma(\varepsilon s,u_\varepsilon(s))dW(s)\rangle+\varepsilon\int_0^t\|\sigma(\varepsilon s,u_\varepsilon(s))\|^2_{L_2(l^2,H)}ds.$$

Then it follows from (A1) with $K_2=0$ that

$$\|u_\varepsilon(t)\|_H^2+\varepsilon\int_0^t\|\partial_1 u_\varepsilon(s)\|_H^2ds\leqslant\|u_0\|^2_{\tilde H^{0,1}}+C\varepsilon t+C\varepsilon\int_0^t\|u_\varepsilon(s)\|_H^2ds$$
$$+2\sqrt{\varepsilon}\int_0^t\langle u_\varepsilon,\sigma(\varepsilon s,u_\varepsilon(s))dW(s)\rangle.$$

Take supremum over t, for $p\geqslant 2$, we have

$$(E[F_{u_\varepsilon}(T)]^p)^{\frac1p}\leqslant\|u_0\|^2_{\tilde H^{0,1}}+C\varepsilon T+C\varepsilon\int_0^T(E[F_{u_\varepsilon}(t)]^p)^{\frac1p}dt$$
$$+2\sqrt{\varepsilon}(E[\sup_{0\leqslant t\leqslant T}|\int_0^t\langle u_\varepsilon,\sigma(\varepsilon s,u_\varepsilon(s))dW(s)\rangle|]^p)^{\frac1p}.$$

For the term in the last line, by Lemma 2.14 and [XZ09, (3.12)], we have

$$2\sqrt{\varepsilon}(E[\sup_{0\leqslant t\leqslant T}|\int_0^t\langle u_\varepsilon,\sigma(\varepsilon s,u_\varepsilon(s))dW(s)\rangle|]^p)^{\frac1p}$$
$$\leqslant C\sqrt{\varepsilon p}\left[\int_0^T 1+(E\|u_\varepsilon(s)\|_H^{2p})^{\frac2p}ds\right]^{\frac12}.$$

Combining the above estimate, we arrive at

$$(E[F_{u_\varepsilon}(T)]^p)^{\frac2p}\leqslant C\left(\|u_0\|^2_{\tilde H^{0,1}}+\varepsilon T\right)^2+C\varepsilon^2\int_0^T(E[F_{u_\varepsilon}(t)]^p)^{\frac2p}ds$$
$$+C\varepsilon pT+C\varepsilon p\int_0^T(E[F_{u_\varepsilon}(t)]^p)^{\frac2p}dt.$$

Then Gronwall's inequality implies

$$(E[F_{u_\varepsilon}(T)]^p)^{\frac2p}\leqslant C\left[\|u_0\|^4_{\tilde H^{0,1}}+\varepsilon^2+\varepsilon p\right]e^{C\varepsilon^2+C\varepsilon p}.$$

Let $p=\frac1\varepsilon$, by Chebyshev's inequality, we have

$$\begin{aligned}&\varepsilon\log P(F_{u_\varepsilon}(T)>M)\\ \leqslant&-\log M+\log(E[F_{u_\varepsilon}(T)]^p)^{\frac1p}\\ \leqslant&-\log M+\log\sqrt{\|u_0\|^4_{\tilde H^{0,1}}+\varepsilon^2+1}+C(\varepsilon^2+1).\end{aligned}$$

Take supremum over ε and let $M\to\infty$, we finish the proof. □

Lemma 6.5. *For $M>0$, define a random time*

$$\tau_{M,\varepsilon}=T\wedge\inf\{t:\|u_\varepsilon(t)\|_H^2>M,\ \text{or}\ \varepsilon\int_0^t\|\partial_1u_\varepsilon(s)\|_H^2ds>M\}.$$

Then $\tau_{M,\varepsilon}$ is a stopping time with respect to $\mathcal{F}_{t+}=\cap_{s>t}\mathcal{F}_s$.
Similarly, Let

$$\tau'_{M,\varepsilon}=T\wedge\inf\{t:\|u_\varepsilon(t)\|^2_{\tilde H^{0,1}}>M,\ \text{or}\ \varepsilon\int_0^t\|u_\varepsilon(s)\|^2_{\tilde H^{1,1}}ds>M\},$$

then $\tau'_{M,\varepsilon}$ is a stopping time with respect to $\mathcal{F}_{t+}$.

Proof The problem comes with the continuity of $u_\varepsilon(t)$. Since $\int_0^t \|\partial_1 u_\varepsilon(s)\|_H^2 ds$ is a continuous adapted process, we only need to prove that $\hat{\tau} = \inf\{t > 0 : \|u_\varepsilon(t)\|_H^2 > M\}$ is a stopping time.

Since $u_\varepsilon \in L^\infty([0,T], H) \bigcap C([0,T], H^{-1})$, $u_\varepsilon(t)$ is weakly continuous on H, which implies the lower semi-continuity of u_ε on H.

By definition of $\hat{\tau}$, for $t > 0$

$$\bigcap_{s\in(0,t]} \{\|u_\varepsilon(s)\|_H^2 \leqslant M\} \subset \{\hat{\tau} \geqslant t\} \subset \bigcap_{s\in(0,t)} \{\|u_\varepsilon(s)\|_H^2 \leqslant M\}.$$

On the contrary, if $\omega \in \{\hat{\tau} \geqslant t\}$, for any $s < t$, $\|u_\varepsilon(s)(\omega)\|_H^2 \leqslant M$. Then lower semi-continuity implies

$$\|u_\varepsilon(t)(\omega)\|_H^2 \leqslant \liminf_{s<t, s\to t} \|u_\varepsilon(s)\|_H^2 \leqslant M.$$

Hence we have

$$\{\hat{\tau} \geqslant t\} = \bigcap_{s\in(0,t]} \{\|u_\varepsilon(s)\|_H^2 \leqslant M\}.$$

Note that for $\omega \in \bigcap_{s\in(0,t]\cap\mathbb{Q}} \{\|u_\varepsilon(s)\|_H^2 \leqslant M\}$, we have for any $s \in (0,t]$, by the lower semi-continuity,

$$\|u_\varepsilon(s)(\omega)\|_H^2 \leqslant \liminf_{s'\to s} \|u_\varepsilon(s')\|_H^2 \leqslant \liminf_{s'\to s, s'\in\mathbb{Q}} \|u_\varepsilon(s')\|_H^2 \leqslant M,$$

which means

$$\bigcap_{s\in(0,t]} \{\|u_\varepsilon(s)\|_H^2 \leqslant M\} = \bigcap_{s\in(0,t]\cap\mathbb{Q}} \{\|u_\varepsilon(s)\|_H^2 \leqslant M\}.$$

Then we have for $t > 0$

$$\{\hat{\tau} \geqslant t\} = \bigcap_{s\in(0,t]} \{\|u_\varepsilon(s)\|_H^2 \leqslant M\} = \bigcap_{s\in(0,t]\cap\mathbb{Q}} \{\|u_\varepsilon(s)\|_H^2 \leqslant M\} \in \mathcal{F}_t,$$

which implies the result.

For $\tau'_{M,\varepsilon}$, the result follows from the fact that u_ε is weakly continuous in $\tilde{H}^{0,1}$ since $u_\varepsilon \in L^\infty([0,T], \tilde{H}^{0,1}) \bigcap C(0,T], H^{-1})$. □

Lemma 6.6. *Let* $G_{u_\varepsilon}(t) = \sup_{0\leqslant s\leqslant t} \|u_\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 + \varepsilon \int_0^t \|u_\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds$. *For fixed* M_1, *we have*

$$\lim_{M\to\infty} \sup_{0<\varepsilon\leqslant 1} \varepsilon \log P(G_{u_\varepsilon}(\tau_{M_1,\varepsilon}) > M) = -\infty.$$

Proof Let k be a positive constant and $f_\varepsilon(t) = 1 + \|\partial_1 u_\varepsilon(t)\|_H^2$. Applying Itô's formula to $e^{-k\varepsilon\int_0^t f_\varepsilon(s)ds}\|u_\varepsilon(t)\|_{\tilde{H}^{0,1}}^2$ (by applying Itô's formula to its finite- dimension projection first and then passing to the limit), we obtain

$$\begin{aligned}
&e^{-k\varepsilon\int_0^t f_\varepsilon(s)ds}\|u_\varepsilon(t)\|_{\tilde{H}^{0,1}}^2 + 2\varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}(\|\partial_1 u_\varepsilon(s)\|_H^2 + \|\partial_1\partial_2 u_\varepsilon(s)\|_H^2)ds \\
=&\|u_0\|_{\tilde{H}^{0,1}}^2 - k\varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr} f_\varepsilon(s)\|u_\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 ds \\
&- 2\varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\langle\partial_2 u_\varepsilon(s), \partial_2(u_\varepsilon\cdot\nabla u_\varepsilon)(s)\rangle ds
\end{aligned}$$

$$+ 2\sqrt{\varepsilon}\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\langle u_\varepsilon(s),\sigma(\varepsilon s,u_\varepsilon(s))dW(s)\rangle_{\tilde{H}^{0,1}}$$
$$+ \varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|\sigma(\varepsilon s,u_\varepsilon(s))\|^2_{L_2(l^2,\tilde{H}^{0,1})}ds.$$

The fourth and the fifth line can be dealt in the same way as in the proof of Lemma 6.4. For the third line, by Lemma 2.12, we have

$$|\langle\partial_2 u_\varepsilon,\partial_2(u_\varepsilon\cdot\nabla u_\varepsilon)\rangle|\leqslant\frac{1}{2}\|\partial_1\partial_2 u_\varepsilon\|^2_H+C_1 f_\varepsilon\|\partial_2 u_\varepsilon\|^2_H,$$

where C_1 is a constant. Therefore by (A2) with $\tilde{K}_2=0$ we get

$$\begin{aligned}
&e^{-k\varepsilon\int_0^t f_\varepsilon(s)ds}\|u_\varepsilon(t)\|^2_{\tilde{H}^{0,1}}+\varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|u_\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds\\
\leqslant&\|u_0\|^2_{\tilde{H}^{0,1}}-k\varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}f_\varepsilon(s)\|u_\varepsilon(s)\|^2_{\tilde{H}^{0,1}}ds\\
&+2C_1\varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}f_\varepsilon(s)\|u_\varepsilon(s)\|^2_{\tilde{H}^{0,1}}ds\\
&+2\sqrt{\varepsilon}\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\langle u_\varepsilon(s),\sigma(\varepsilon s,u_\varepsilon(s))dW(s)\rangle_{\tilde{H}^{0,1}}\\
&+\varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}[\tilde{K_0}+(\tilde{K_1}+1)\|u_\varepsilon(s)\|^2_{\tilde{H}^{0,1}}]ds.
\end{aligned}$$

For the last second line, similar to [XZ09, (3.12)], we have

$$\begin{aligned}
&2\sqrt{\varepsilon}(E[\sup_{0\leqslant s\leqslant t}|\int_0^s e^{-k\varepsilon\int_0^r f_\varepsilon(l)dl}\langle u_\varepsilon(r),\sigma(\varepsilon r,u_\varepsilon(r))dW(r)\rangle_{\tilde{H}^{0,1}}|]^p)^{\frac{1}{p}}\\
\leqslant&C\sqrt{\varepsilon p}(E[\int_0^t e^{-2k\varepsilon\int_0^r f_\varepsilon(l)dl}\|u_\varepsilon(r)\|^2_{\tilde{H}^{0,1}}\|\sigma(\varepsilon r,u_\varepsilon(r))\|^2_{L_2(l^2,\tilde{H}^{0,1})}dr]^{\frac{p}{2}})^{\frac{1}{p}}\\
\leqslant&C\sqrt{\varepsilon p}(E[\int_0^t e^{-2k\varepsilon\int_0^r f_\varepsilon(l)dl}\|u_\varepsilon(r)\|^2_{\tilde{H}^{0,1}}(1+\|u_\varepsilon(r)\|^2_{\tilde{H}^{0,1}})dr]^{\frac{p}{2}})^{\frac{1}{p}}\\
\leqslant&C\sqrt{\varepsilon p}(E[\int_0^t e^{-2k\varepsilon\int_0^r f_\varepsilon(l)dl}(1+\|u_\varepsilon(r)\|^4_{\tilde{H}^{0,1}})dr]^{\frac{p}{2}})^{\frac{1}{p}}\\
\leqslant&C\sqrt{\varepsilon p}\left[\int_0^t 1+(E[e^{-pk\varepsilon\int_0^r f_\varepsilon(l)dl}\|u_\varepsilon(s)\|^{2p}_{\tilde{H}^{0,1}}])^{\frac{2}{p}}ds\right]^{\frac{1}{2}},
\end{aligned}$$

where we used (A2) with $K_2=0$ in the third line.

Let $k>2C_1$ and using Lemma 2.14, we have for $p\geqslant 2$

$$\begin{aligned}
&\left(E\left[\sup_{0\leqslant s\leqslant t\wedge\tau_{M_1,\varepsilon}}e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|u_\varepsilon(s)\|^2_{\tilde{H}^{0,1}}+\varepsilon\int_0^{t\wedge\tau_{M_1,\varepsilon}}e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|u_\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds\right]^p\right)^{\frac{2}{p}}\\
\leqslant&C(\|u_0\|^2_{\tilde{H}^{0,1}}+\varepsilon)^2+C\varepsilon^2\int_0^t\left(E\left[\sup_{0\leqslant r\leqslant s\wedge\tau_{M_1,\varepsilon}}e^{-k\varepsilon\int_0^r f_\varepsilon(l)dl}\|u_\varepsilon(r)\|^2_{\tilde{H}^{0,1}}\right]^p\right)^{\frac{2}{p}}ds\\
&+C\varepsilon p+C\varepsilon p\int_0^t\left(E\left[\sup_{0\leqslant r\leqslant s\wedge\tau_{M_1,\varepsilon}}e^{-k\varepsilon\int_0^r f_\varepsilon(l)dl}\|u_\varepsilon(r)\|^2_{\tilde{H}^{0,1}}\right]^p\right)^{\frac{2}{p}}ds.
\end{aligned}$$

Applying Gronwall's inequality, we obtain

$$\begin{aligned}&\left(E\left[\sup_{0\leqslant t\leqslant\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^t f_\varepsilon(s)ds}\|u_\varepsilon(t)\|^2_{\tilde{H}^{0,1}}+\varepsilon\int_0^{\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|u_\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds\right]^p\right)^{\frac{2}{p}}\\ \leqslant& C\left[\|u_0\|^4_{\tilde{H}^{0,1}}+\varepsilon^2+\varepsilon p\right]e^{C(\varepsilon^2+\varepsilon p)}.\end{aligned}$$

Hence by the definition of $\tau_{M_1,\varepsilon}$, we have

$$\begin{aligned}&(E\left[G_{u_\varepsilon}(\tau_{M_1,\varepsilon})\right]^p)^{\frac{2}{p}}\\ \leqslant&\left(E\left[\left(\sup_{0\leqslant t\leqslant\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^t f_\varepsilon(s)ds}\|u_\varepsilon(t)\|^2_{\tilde{H}^{0,1}}+\varepsilon\int_0^{\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|u_\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds\right)^p e^{pk\varepsilon\int_0^t f_\varepsilon(s)ds}\right]\right)^{\frac{2}{p}}\\ \leqslant& e^{C(M_1+\varepsilon)}\left(E\left[\sup_{0\leqslant t\leqslant\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^t f_\varepsilon(s)ds}\|u_\varepsilon(t)\|^2_{\tilde{H}^{0,1}}+\varepsilon\int_0^{\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|u_\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds\right]^p\right)^{\frac{2}{p}}\\ \leqslant& Ce^{C(M_1+\varepsilon)}\left[\|u_0\|^4_{\tilde{H}^{0,1}}+\varepsilon^2+\varepsilon p\right]e^{C(\varepsilon^2+\varepsilon p)}.\end{aligned}$$

Let $p=\frac{2}{\varepsilon}$, by Chebyshev's inequality, we have

$$\begin{aligned}&\varepsilon\log P(G_{u_\varepsilon}(\tau_{M_1,\varepsilon})>M)\\ \leqslant&\varepsilon\log\frac{E\left[G_{u_\varepsilon}(\tau_{M_1,\varepsilon})\right]^p}{M^p}\\ \leqslant&-2\log M+C+C(M_1+\varepsilon)+C(\varepsilon^2+\varepsilon p)+\log[\|u_0\|^4_{\tilde{H}^{0,1}}+\varepsilon^2+\varepsilon p].\end{aligned}$$

Take supremum over ε and let $M\to\infty$, we finish the proof. □

6.3 Approximating the initial value

Since V is dense in $\tilde{H}^{0,1}$, there exists a sequence $\{u_0^n\}\subset V$ such that

$$\lim_{n\to+\infty}\|u_0^n-u_0\|_{\tilde{H}^{0,1}}=0.$$

Let $u_{n,\varepsilon}$ be the solution to (6.1) with the initial data u_0^n. Similarly, let $v_{n,\varepsilon}$ be the solution to (6.2) with the initial data u_0^n.

For $M>0$, define a random time (which is also a stopping time with respect to $\mathcal{F}_{t+}$ by Lemma 6.5)

$$\tau^n_{M,\varepsilon}:=T\wedge\inf\{t:\|u_{n,\varepsilon}(t)\|^2_H>M,\ \text{or}\ \varepsilon\int_0^t\|\partial_1u_{n,\varepsilon}(s)\|^2_Hds>M\}.$$

From the proof of Lemma 6.4 and Lemma 6.6, it follows that

Lemma 6.7.

$$\lim_{M\to\infty}\sup_n\sup_{0<\varepsilon\leqslant1}\varepsilon\log P(F_{u_{n,\varepsilon}}(T)>M)=-\infty.$$

For fixed M_1, we have

$$\lim_{M\to\infty}\sup_n\sup_{0<\varepsilon\leqslant1}\varepsilon\log P(G_{u_{n,\varepsilon}}(\tau^n_{M_1,\varepsilon})>M)=-\infty.$$

The following lemma for $v_{n,\varepsilon}$ is from [XZ09]:

Lemma 6.8 ([XZ09, Lemma 3.2])**.**

$$\lim_{M\to\infty}\sup_{0<\varepsilon\leqslant 1}\varepsilon\log P\left(\sup_{0\leqslant t\leqslant T}\|v_{n,\varepsilon}(t)\|_V^2>M\right)=-\infty.$$

Lemma 6.9. *For any* $\delta>0$*,*

$$\lim_{n\to\infty}\sup_{0<\varepsilon\leqslant 1}\varepsilon\log P\left(\sup_{0\leqslant t\leqslant T}\|u_{n,\varepsilon}(t)-u_\varepsilon(t)\|_H^2>\delta\right)=-\infty.$$

Proof

Clearly, for $M_1,M_2>0$

$$\begin{aligned}&P\left(\sup_{0\leqslant t\leqslant T}\|u_{n,\varepsilon}(t)-u_\varepsilon(t)\|_H^2>\delta\right)\\ \leqslant&P\left(\sup_{0\leqslant t\leqslant T}\|u_{n,\varepsilon}(t)-u_\varepsilon(t)\|_H^2>\delta,F_{u_\varepsilon}(T)\leqslant M_1,G_{u_\varepsilon}(T)\leqslant M_2\right)\\ &+P\left(F_{u_\varepsilon}(T)>M_1\right)+P\left(F_{u_\varepsilon}(T)\leqslant M_1,G_{u_\varepsilon}(T)>M_2\right)\\ \leqslant&P\left(\sup_{0\leqslant t\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}}\|u_{n,\varepsilon}(t)-u_\varepsilon(t)\|_H^2>\delta\right)\\ &+P\left(F_{u_\varepsilon}(T)>M_1\right)+P\left(G_{u_\varepsilon}(\tau_{M_1,\varepsilon})>M_2\right),\end{aligned}\tag{6.3}$$

where $\tau_{M_1,\varepsilon}$ and $\tau'_{M_2,\varepsilon}$ are introduced in Lemma 6.5.

For the first term on the right hand of (6.3), let k be a positive constant and

$$U_\varepsilon=1+\|u_\varepsilon\|_{\tilde H^{1,1}}^2.$$

Applying Itô's formula to $e^{-\varepsilon k\int_0^t U_\varepsilon(s)ds}\|u_\varepsilon(t)-u_{n,\varepsilon}(t)\|_H^2$, we get

$$\begin{aligned}&e^{-\varepsilon k\int_0^t U_\varepsilon(s)ds}\|u_\varepsilon(t)-u_{n,\varepsilon}(t)\|_H^2+2\varepsilon\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\|\partial_1(u_\varepsilon(s)-u_{n,\varepsilon}(s))\|_H^2ds\\ =&\|u_0-u_{n,0}\|_H^2-k\varepsilon\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}U_\varepsilon(s)\|u_\varepsilon(s)-u_{n,\varepsilon}(s)\|_H^2ds\\ &-2\varepsilon\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\left(b(u_\varepsilon,u_\varepsilon,u_\varepsilon-u_{n,\varepsilon})(s)-b(u_{n,\varepsilon},u_{n,\varepsilon},u_\varepsilon-u_{n,\varepsilon})(s)\right)ds\\ &+\varepsilon\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\|\sigma(\varepsilon s,u_\varepsilon(s))-\sigma(\varepsilon s,u_{n,\varepsilon}(s))\|_{L_2(l^2,H)}^2ds\\ &+2\sqrt{\varepsilon}\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\langle u_\varepsilon(s)-u_{n,\varepsilon}(s),(\sigma(\varepsilon s,u_\varepsilon(s))-\sigma(\varepsilon s,u_{n,\varepsilon}(s)))dW(s)\rangle.\end{aligned}$$

Notice that by the property of the trilinear form b and Lemma 2.10, we have

$$\begin{aligned}&|b(u_\varepsilon,u_\varepsilon,u_\varepsilon-u_{n,\varepsilon})-b(u_{n,\varepsilon},u_{n,\varepsilon},u_\varepsilon-u_{n,\varepsilon})|\\ =&|b(u_\varepsilon,u_\varepsilon,u_\varepsilon-u_{n,\varepsilon})-b(u_{n,\varepsilon},u_\varepsilon,u_\varepsilon-u_{n,\varepsilon})|\\ =&|b(u_\varepsilon-u_{n,\varepsilon},u_\varepsilon,u_\varepsilon-u_{n,\varepsilon})|\end{aligned}$$

$$\leqslant \frac{1}{2}\|\partial_1(u_\varepsilon - u_{n,\varepsilon})\|_H^2 + C_1 U_\varepsilon \|u_\varepsilon - u_{n,\varepsilon}\|_H^2,$$

where C_1 is a constant.

Therefore,

$$\begin{aligned}
&e^{-\varepsilon k \int_0^t U_\varepsilon(s)ds}\|u_\varepsilon(t) - u_{n,\varepsilon}(t)\|_H^2 \\
\leqslant &\|u_0 - u_{n,0}\|_{\tilde{H}^{0,1}}^2 - k\varepsilon \int_0^t e^{-\varepsilon k \int_0^s U_\varepsilon(r)dr} U_\varepsilon(s)\|u_\varepsilon(s) - u_{n,\varepsilon}(s)\|_H^2 ds \\
&+ 2\varepsilon C_1 \int_0^t e^{-\varepsilon k \int_0^s U_\varepsilon(r)dr} U_\varepsilon(s)\|u_\varepsilon(s) - u_{n,\varepsilon}(s)\|_H^2 ds \\
&+ L\varepsilon \int_0^t e^{-\varepsilon k \int_0^s U_\varepsilon(r)dr} \|u_\varepsilon(s) - u_{n,\varepsilon}(s)\|_H^2 ds \\
&+ 2\sqrt{\varepsilon} \int_0^t e^{-\varepsilon k \int_0^s U_\varepsilon(r)dr} \langle u_\varepsilon(s) - u_{n,\varepsilon}(s), (\sigma(\varepsilon s, u_\varepsilon(s)) - \sigma(\varepsilon s, u_{n,\varepsilon}(s)))dW(s)\rangle,
\end{aligned}$$

where we used (A3') in the forth line.

Choosing $k > 2C_1$ and using Lemma 2.14 and (A3'), by the similar calculation as in the proof of Lemma 6.6 we have for $p \geqslant 2$

$$\begin{aligned}
&\left(E\left[\sup_{0\leqslant s\leqslant t\wedge\tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} e^{-\varepsilon k \int_0^s U_\varepsilon(r)dr}\|u_\varepsilon(s) - u_{n,\varepsilon}(s)\|_H^2\right]^p\right)^{\frac{2}{p}} \\
\leqslant &2\|u_0 - u_{n,0}\|_{\tilde{H}^{0,1}}^4 + C\varepsilon^2 \int_0^t \left(E\left[\sup_{0\leqslant r\leqslant s\wedge\tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} e^{-\varepsilon k \int_0^r U_\varepsilon(l)dl}\|u_\varepsilon(r) - u_{n,\varepsilon}(r)\|_H^2\right]^p\right)^{\frac{2}{p}} ds \\
&+ C\varepsilon p \int_0^t \left(E\left[\sup_{0\leqslant r\leqslant s\wedge\tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} e^{-\varepsilon k \int_0^r U_\varepsilon(l)dl}\|u_\varepsilon(r) - u_{n,\varepsilon}(r)\|_H^2\right]^p\right)^{\frac{2}{p}} ds.
\end{aligned}$$

Applying Gronwall's inequality, we obtain

$$\left(E\left[\sup_{0\leqslant s\leqslant t\wedge\tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} e^{-\varepsilon k \int_0^s U_\varepsilon(r)dr}\|u_\varepsilon(s) - u_{n,\varepsilon}(s)\|_H^2\right]^p\right)^{\frac{2}{p}} \leqslant C\|u_0 - u_{n,0}\|_{\tilde{H}^{0,1}}^4 e^{C(\varepsilon^2+\varepsilon p)}.$$

Hence, by the definition of the stopping times,

$$\begin{aligned}
&\left(E\left[\sup_{0\leqslant s\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} \|u_\varepsilon(s) - u_{n,\varepsilon}(s)\|_H^2\right]^p\right)^{\frac{2}{p}} \\
\leqslant &\left(E\left[\Big(\sup_{0\leqslant s\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} e^{-\varepsilon k \int_0^s U_\varepsilon(r)dr}\|u_\varepsilon(s) - u_{n,\varepsilon}(s)\|_H^2\Big)^p e^{kp\varepsilon \int_0^{\tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} U_\varepsilon(s)ds}\right]\right)^{\frac{2}{p}} \\
\leqslant &e^{C(\varepsilon+M_2)k}\left(E\left[\sup_{0\leqslant s\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} e^{-\varepsilon k \int_0^s U_\varepsilon(r)dr}\|u_\varepsilon(s) - u_{n,\varepsilon}(s)\|_H^2\right]^p\right)^{\frac{2}{p}} \\
\leqslant &Ce^{C(\varepsilon+M_2)k}\|u_0 - u_{n,0}\|_{\tilde{H}^{0,1}}^4 e^{C(\varepsilon^2+\varepsilon p)}.
\end{aligned}$$

Fix M_1, M_2, let $p = \frac{2}{\varepsilon}$, then Chebyshev's inequality implies that

$$\begin{aligned}
&\sup_{0<\varepsilon\leqslant 1} \varepsilon \log P\left(\sup_{0\leqslant t\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} \|u_{n,\varepsilon}(t)-u_\varepsilon(t)\|_H^2 > \delta\right)\\
\leqslant &\sup_{0<\varepsilon\leqslant 1} \varepsilon \log \frac{E\left[\sup_{0\leqslant t\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} \|u_{n,\varepsilon}(t)-u_\varepsilon(t)\|_H^{2p}\right]}{\delta^p}\\
\leqslant &C(\varepsilon+M_2) - 2\log\delta + \log\|u_0-u_{n,0}\|^4_{\tilde{H}^{0,1}} + C(\varepsilon^2+\varepsilon p) + C\\
\to &-\infty, \text{ as } n\to\infty.
\end{aligned}$$

By Lemma 6.4, for any $R > 0$, there exists a constant M_1 such that for any $\varepsilon \in (0,1]$,

$$P\left(F_{u_\varepsilon}(T) > M_1\right) \leqslant e^{-\frac{R}{\varepsilon}}.$$

For such a M_1, by Lemma 6.6, there exists a constant M_2 such that for any $\varepsilon \in (0,1]$,

$$P\left(G_{u_\varepsilon}(\tau_{M_1,\varepsilon}) > M_2\right) \leqslant e^{-\frac{R}{\varepsilon}}.$$

For such M_1, M_2, there exists a positive integer N, such that for any $n \geqslant N$ and $\varepsilon \in (0,1]$,

$$P\left(\sup_{0\leqslant t\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} \|u_{n,\varepsilon}(t)-u_\varepsilon(t)\|_H^2 > \delta\right) \leqslant e^{-\frac{R}{\varepsilon}}.$$

Then by (6.3), we see that there exists a positive integer N, such that for any $n \geqslant N$, $\varepsilon \in (0,1]$,

$$P\left(\sup_{0\leqslant t\leqslant T} \|u_{n,\varepsilon}(t)-u_\varepsilon(t)\|_H^2 > \delta\right) \leqslant 3e^{-\frac{R}{\varepsilon}}.$$

Since R is arbitrary, the lemma follows. □

The following lemma for v_ε is from [XZ09]:

Lemma 6.10 ([XZ09, Lemma 3.4])**.** *For any $\delta > 0$,*

$$\lim_{n\to\infty} \sup_{0<\varepsilon\leqslant 1} \varepsilon \log P\left(\sup_{0\leqslant t\leqslant T} \|v_{n,\varepsilon}(t)-v_\varepsilon(t)\|_H^2 > \delta\right) = -\infty.$$

6.4 Exponential equivalence

In this section we prove the main results by showing the exponential equivalence.

Lemma 6.11. *For any $\delta > 0$, and every positive integer n,*

$$\lim_{\varepsilon\to 0} \varepsilon \log P\left(\sup_{0\leqslant t\leqslant T} \|u_{n,\varepsilon}(t)-v_{n,\varepsilon}(t)\|_H^2 > \delta\right) = -\infty.$$

Proof For $M > 0$, recall the definition of $\tau^n_{M,\varepsilon}$ and define the following random time:

$$\tau^{2,n}_{M,\varepsilon} := T \wedge \inf\{t : \|u_{n,\varepsilon}(t)\|^2_{\tilde{H}^{0,1}} > M, \text{ or } \varepsilon\int_0^t \|u_{n,\varepsilon}(s)\|^2_{\tilde{H}^{1,1}} ds > M\},$$

which is a stopping time with respect to $\mathcal{F}_{t+}$ by Lemma 6.5.

Moreover, define

$$\tau^{3,n}_{M,\varepsilon} := T \wedge \inf\{t : \|v_{n,\varepsilon}(t)\|^2_V > M\},$$
$$\tau^{1,n}_{M,\varepsilon} := \tau^{n}_{M,\varepsilon} \wedge \tau^{3,n}_{M,\varepsilon}.$$

We should point out that $\tau^{3,n}_{M,\varepsilon}$ is a stopping time with respect to $\mathcal{F}_t$ under the condition $v_{n,\varepsilon} \in C([0,T],V)$. Now we prove that $v_{n,\varepsilon} \in C([0,T],V)$.

By Itô's formula and Gronwall's inequality there exists a constant $C(\varepsilon)$ such that

$$E(\sup_{s\in[0,t]} \|v_{n,\varepsilon}(s)\|^2_V) \leqslant C(\varepsilon).$$

For $0 \leqslant s < t \leqslant T$, by (A4) we have

$$\begin{aligned} E\|v_{n,\varepsilon}(t) - v_{n,\varepsilon}(s)\|^2_V \leqslant & \varepsilon E \int_s^t \|\sigma(\varepsilon r, v_{n,\varepsilon}(r))\|^2_{L_2(l^2,V)} dr \\ \leqslant & \varepsilon \int_s^t (\overline{K}_0 + \overline{K}_1 E(\sup_{l\in[0,r]} \|v_{n,\varepsilon}(l)\|^2_V)) dr \\ \leqslant & \varepsilon(\overline{K}_0 + \overline{K}_1 C(\varepsilon))|t-s|. \end{aligned}$$

Then Kolmogorov's continuity criterion implies that $v_{n,\varepsilon} \in C([0,T],V)$.

Now for $M_1, M_2 > 0$, similarly to (6.3), we have

$$\begin{aligned} & P\left(\sup_{0\leqslant t\leqslant T} \|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|^2_H > \delta\right) \\ \leqslant & P\left(\sup_{0\leqslant t\leqslant \tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}} \|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|^2_H > \delta\right) \\ & + P(F_{u_{n,\varepsilon}}(T) > M_1) + P(G_{u_{n,\varepsilon}}(\tau^n_{M_1,\varepsilon}) > M_2) + P\left(\sup_{0\leqslant t\leqslant T} \|v_{n,\varepsilon}(t)\|^2_V > M_1\right) \end{aligned} \tag{6.4}$$

Let $U_{n,\varepsilon} = 1 + \|u_{n,\varepsilon}\|^2_{\tilde{H}^{1,1}}$, applying Itô's formula to $e^{-k\varepsilon\int_0^t U_{n,\varepsilon}(s)ds}\|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|^2_H$ for some constant $k > 0$, we get

$$\begin{aligned} & e^{-k\varepsilon\int_0^t U_{n,\varepsilon}(s)ds}\|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|^2_H + 2\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|\partial_1(u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s))\|^2_H ds \\ = & -k\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr} U_{n,\varepsilon}(s)\|u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s)\|^2_H ds \\ & + 2\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\langle u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s), \partial_1^2 v_{n,\varepsilon}(s)\rangle ds \\ & - 2\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr} b(u_{n,\varepsilon}(s), u_{n,\varepsilon}(s), u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s)) ds \\ & + \varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|\sigma(\varepsilon s, u_{n,\varepsilon}(s)) - \sigma(\varepsilon s, v_{n,\varepsilon}(s))\|^2_{L_2(l^2,H)} ds \\ & + 2\sqrt{\varepsilon}\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\langle u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s), (\sigma(\varepsilon s, u_{n,\varepsilon}(s)) - \sigma(\varepsilon s, v_{n,\varepsilon}(s)))dW(s)\rangle. \end{aligned} \tag{6.5}$$

For the second term on the right hand side of (6.5), we have

$$
\begin{aligned}
&\Big|\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\langle u_{n,\varepsilon}(s)-v_{n,\varepsilon}(s),\partial_1^2 v_{n,\varepsilon}(s)\rangle ds\Big|\\
\leqslant&\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|\partial_1(u_{n,\varepsilon}(s)-v_{n,\varepsilon}(s))\|_H\|\partial_1 v_{n,\varepsilon}(s)\|_H ds\\
\leqslant&\frac{1}{4}\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|\partial_1(u_{n,\varepsilon}(s)-v_{n,\varepsilon}(s))\|_H^2 ds+C\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|v_{n,\varepsilon}(s)\|_V^2 ds,
\end{aligned}
$$

where we use Young's inequality in the last inequality.

For the third term on the right hand side of (6.5), by Lemmas 2.10 and 2.11 we have

$$
\begin{aligned}
&|b(u_{n,\varepsilon},u_{n,\varepsilon},u_{n,\varepsilon}-v_{n,\varepsilon})|\\
=&|b(u_{n,\varepsilon}-v_{n,\varepsilon},u_{n,\varepsilon},u_{n,\varepsilon}-v_{n,\varepsilon})+b(v_{n,\varepsilon},u_{n,\varepsilon},u_{n,\varepsilon}-v_{n,\varepsilon}|\\
\leqslant&\frac{1}{4}\|\partial_1(u_{n,\varepsilon}-v_{n,\varepsilon})\|_H^2+CU_{n,\varepsilon}\|u_{n,\varepsilon}-v_{n,\varepsilon}\|_H^2+C\|v_{n,\varepsilon}\|_V\|u_{n,\varepsilon}\|_{\tilde{H}^{1,1}}\|u_{n,\varepsilon}-v_{n.\varepsilon}\|_H \qquad (6.6)\\
\leqslant&\frac{1}{4}\|\partial_1(u_{n,\varepsilon}-v_{n,\varepsilon})\|_H^2+C\|v_{n,\varepsilon}\|_V^2+C_1U_{n,\varepsilon}\|u_{n,\varepsilon}-v_{n,\varepsilon}\|_H^2,
\end{aligned}
$$

where C_1 is a constant.

Thus we obtain

$$
\begin{aligned}
&e^{-k\varepsilon\int_0^t U_{n,\varepsilon}(s)ds}\|u_{n,\varepsilon}(t)-v_{n,\varepsilon}(t)\|_H^2+\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|\partial_1(u_{n,\varepsilon}(s)-v_{n,\varepsilon}(s))\|_H^2 ds\\
\leqslant&-k\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}U_{n,\varepsilon}(s)\|u_{n,\varepsilon}(s)-v_{n,\varepsilon}(s)\|_H^2 ds+C\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|v_{n,\varepsilon}(s)\|_V^2 ds\\
&+C_1\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}U_{n,\varepsilon}(s)\|u_{n,\varepsilon}(s)-v_{n,\varepsilon}(s)\|_H^2 ds\\
&+L_1\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|u_{n,\varepsilon}(s)-v_{n,\varepsilon}(s)\|_H^2 ds\\
&+2\sqrt{\varepsilon}\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\langle u_{n,\varepsilon}(s)-v_{n,\varepsilon}(s),(\sigma(\varepsilon s,u_{n,\varepsilon}(s))-\sigma(\varepsilon s,v_{n,\varepsilon}(s)))dW(s)\rangle,
\end{aligned}
$$

where we used (A3') in the fourth line.

Hence, choosing $k>C_1+C_2$, by Lemma 2.14 and the similar techniques in the previous lemma and the definition of stopping times, we deduce that for $p\geqslant 2$

$$
\begin{aligned}
&\left(E\left[\sup_{0\leqslant s\leqslant t\wedge\tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}} e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|u_{n,\varepsilon}(s)-v_{n,\varepsilon}(s)\|_H^2\right]^p\right)^{\frac{2}{p}}\\
\leqslant&CM_1^2\varepsilon^2+C(\varepsilon^2+\varepsilon p)\int_0^t\left(E\left[\sup_{0\leqslant r\leqslant s\wedge\tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}} e^{-k\varepsilon\int_0^r U_{n,\varepsilon}(l)dl}\|u_{n,\varepsilon}(r)-v_{n,\varepsilon}(r)\|_H^2\right]^p\right)^{\frac{2}{p}}ds.
\end{aligned}
$$

Then Gronwall's inequality implies that

$$
\begin{aligned}
&\left(E\left[\sup_{0\leqslant t\leqslant \tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}}\|u_{n,\varepsilon}(t)-v_{n,\varepsilon}(t)\|_H^2\right]^p\right)^{\frac{2}{p}}\\
\leqslant&\left(E\left[\sup_{0\leqslant t\leqslant \tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}}(e^{-k\varepsilon\int_0^t U_{n,\varepsilon}(s)ds}\|u_{n,\varepsilon}(t)-v_{n,\varepsilon}(t)\|_H^2)^p e^{kp\varepsilon\int_0^{\tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}}U_{n,\varepsilon}(s)ds}\right]\right)^{\frac{2}{p}}\\
\leqslant& e^{C(\varepsilon+M_2)}CM_1^2\varepsilon^2e^{C(\varepsilon^2+\varepsilon p)}.
\end{aligned}
\tag{6.7}
$$

By Lemmas 6.7 and 6.8, we know that for any $R>0$, there exists M_1 such that

$$
\sup_{0<\varepsilon\leqslant 1}\varepsilon\log P\left(F_{u_{n,\varepsilon}}(T)>M_1\right)\leqslant -R,
$$

$$
\sup_{0<\varepsilon\leqslant 1}\varepsilon\log P\left(\sup_{0\leqslant t\leqslant T}\|v_{n,\varepsilon}(t)\|_V^2>M_1\right)\leqslant -R.
$$

For such a constant M_1, by Lemma 6.7, there exists M_2 such that

$$
\sup_{0<\varepsilon\leqslant 1}\varepsilon\log P\left(G_{u_{n,\varepsilon}}(\tau^n_{M_1,\varepsilon})>M_2\right)\leqslant -R.
$$

Then for such M_1, M_2, let $p=\frac{2}{\varepsilon}$ in (6.7), we obtain

$$
\begin{aligned}
&\varepsilon\log P\left(\sup_{0\leqslant t\leqslant \tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}}\|u_{n,\varepsilon}(t)-v_{n,\varepsilon}(t)\|_H^2>\delta\right)\\
\leqslant&\log\left(E\left[\sup_{0\leqslant t\leqslant \tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}}\|u_{n,\varepsilon}(t)-v_{n,\varepsilon}(t)\|_H^2\right]^p\right)^{\frac{2}{p}}-\log\delta^2\\
\leqslant& C(\varepsilon+M_2)+\log[CM_1^2\varepsilon^2]+C(\varepsilon^2+1)-\log\delta^2\\
\to&-\infty\quad\text{as }\varepsilon\to 0,
\end{aligned}
$$

where we used Chebyshev's inequality in the first inequality. Thus there exists a $\varepsilon_0\in(0,1)$ such that for any $\varepsilon\in(0,\varepsilon_0)$,

$$
P\left(\sup_{0\leqslant t\leqslant \tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}}\|u_{n,\varepsilon}(t)-v_{n,\varepsilon}(t)\|_H^2>\delta\right)\leqslant e^{-\frac{R}{\varepsilon}}.
$$

Putting the above estimate together, by (6.4) we see that for $\varepsilon\in(0,\varepsilon_0)$

$$
P\left(\sup_{0\leqslant t\leqslant T}\|u_{n,\varepsilon}(t)-v_{n,\varepsilon}(t)\|_H^2>\delta\right)\leqslant 4e^{-\frac{R}{\varepsilon}}.
$$

Since R is arbitrary, we finish the proof. □

Proof of Theorem 6.2. *By Lemma 6.3, v_ε satisfies a large deviation principle with the rate function I^{u_0}. Our task remain is to show that u_ε and v_ε are exponentially equivalent, then the result follows from Lemma 2.3.*

By Lemmas 6.9 and 6.10, for any $R > 0$, there exists a N_0 such that for any $\varepsilon \in (0,1]$,

$$P\left(\sup_{0\leqslant t\leqslant T}\|u_\varepsilon(t)-u_{N_0,\varepsilon}(t)\|_H^2>\frac{\delta}{3}\right)\leqslant e^{-\frac{R}{\varepsilon}},$$

and

$$P\left(\sup_{0\leqslant t\leqslant T}\|v_\varepsilon(t)-v_{N_0,\varepsilon}(t)\|_H^2>\frac{\delta}{3}\right)\leqslant e^{-\frac{R}{\varepsilon}}.$$

Then by Lemma 6.11, for such N_0, there exists a ε_0 such that for any $\varepsilon \in (0,\varepsilon_0)$,

$$P\left(\sup_{0\leqslant t\leqslant T}\|u_{N_0,\varepsilon}(t)-v_{N_0,\varepsilon}(t)\|_H^2>\frac{\delta}{3}\right)\leqslant e^{-\frac{R}{\varepsilon}}.$$

Therefore we deduce that for $\varepsilon \in (0,\varepsilon_0)$

$$P\left(\sup_{0\leqslant t\leqslant T}\|u_\varepsilon(t)-v_\varepsilon(t)\|_H^2>\delta\right)\leqslant 3e^{-\frac{R}{\varepsilon}}.$$

Since R is arbitrary, we finish the proof. □

Chapter 7

Small time asymptotics for Φ_1^4 model

In this chapter we consider the equation

$$\begin{aligned} d\phi(t) &= \Delta\phi(t)dt - \phi^3(t)dt + dW(t), \\ \phi(0) &= \phi_0, \end{aligned}$$

where $\phi_0 \in \mathcal{C}^{-\beta}$ for $0 < \beta < \frac{1}{4}$ and W is a cylindrical Wiener process on $L^2(\mathbb{T})$. By a similar argument as [DP04, Theorem 4.8], we obtain that the equation has a unique solution $\phi \in C\mathcal{C}^{-\beta}$.

Let $\varepsilon > 0$, by the scaling property of the Brownian motion, it is easy to see that $\phi(\varepsilon t)$ coincides in law with the solution to the following equation:

$$\begin{aligned} d\phi_\varepsilon &= \varepsilon\Delta\phi_\varepsilon dt - \varepsilon\phi_\varepsilon^3 dt + \sqrt{\varepsilon}dW, \\ \phi_\varepsilon(0) &= \phi_0. \end{aligned}$$

Our purpose is to establish a large deviation principle for ϕ_ε. The main result is the following Theorem:

Theorem 7.1. *Assume $\phi_0 \in \mathcal{C}^{-\beta}$ for $0 < \beta < \frac{1}{4}$ and $\alpha > 0$ small enough, then ϕ_ε satisfies LDP on $C\mathcal{C}^{-\frac{1}{2}-\alpha}$ with the good rate function I^{ϕ_0}, where I^{ϕ_0} is given in Theorem 7.2.*

7.1 The linear case

In this section we concentrate on the following linear equations on the torus $\mathbb{T}$:

$$\begin{aligned} dZ_\varepsilon(t) &= \varepsilon\Delta Z_\varepsilon(t)dt + \sqrt{\varepsilon}dW(t), \\ Z_\varepsilon(0) &= \phi_0. \end{aligned} \tag{7.1}$$

where $W(t)$ is an $L^2(\mathbb{T})$ cylindrical Wiener process and $\phi_0 \in \mathcal{C}^{-\beta}$ for $0 < \beta < \frac{1}{4}$. We will prove that the solutions to (7.1) satisfy a large deviation principle.

The mild solutions to (7.1) are given by

$$Z_\varepsilon(t) = e^{\varepsilon t\Delta}\phi_0 + \sqrt{\varepsilon}\int_0^t e^{\varepsilon(t-s)\Delta}dW(s).$$

Theorem 7.2. *Assume $\phi_0 \in \mathcal{C}^{-\beta}$ for $0 < \beta < \frac{1}{4}$. Let $\mu_{\varepsilon,\phi_0} = \mathcal{L}(Z_\varepsilon(\cdot))$ and $\alpha > 0$ small enough. Define a functional I on $C\mathcal{C}^{-\frac{1}{2}-\alpha}$ by*

$$I^{\phi_0}(g) = \inf_{h\in\Gamma_g}\{\frac{1}{2}\int_0^T \|h'(t)\|^2_{L^2(\mathbb{T})}dt\},$$

where

$$\Gamma_g = \{h \in C\mathcal{C}^{-\frac{1}{2}-\alpha} : h(\cdot) \text{ is absolutely continuous}, \ g(t) = \phi_0 + \int_0^t h'(s)ds\}.$$

Then μ_{ε,ϕ_0} satisfies a large deviation principle with the rate function $I^{\phi_0}(\cdot)$. Moreover, I^{ϕ_0} is a good rate function.

Proof

Let x_ε be the solution to the stochastic equation

$$x_\varepsilon(t) = \phi_0 + \sqrt{\varepsilon}\int_0^t dW(s).$$

Since x_ε is Gaussian on $C\mathcal{C}^{-\frac{1}{2}-\alpha}$, by [DPZ09, Theorem 12.9], we know that $x_\varepsilon - \phi_0$ satisfy a large deviation principle with the rate function I^0. Combing the deterministic initial data, we deduce that x_ε satisfy a large deviation principle with the rate function I^{ϕ_0}.

Now we prove that I^{ϕ_0} is a good rate function. Consider the level set for $r \in (0,\infty)$

$$I^{\phi_0}_r = \{g \in C\mathcal{C}^{-\frac{1}{2}-\alpha} : I^{\phi_0}(g) \leqslant r\}.$$

For any $g \in I^{\phi_0}_r$, we have for $s,t \in [0,T]$

$$\|g(t)-g(s)\|_{-\frac{1}{2}-\alpha} \leqslant C\|g(t)-g(s)\|_{L^2(\mathbb{T})} \leqslant C\int_s^t \|g'(l)\|_{L^2(\mathbb{T})}dl \leqslant C(2r)^{\frac{1}{2}}|t-s|^{\frac{1}{2}},$$

where we use Lemma 6.4 in the first inequality and Hölder's inequality in the last inequality. Since the constant C does not depend on g, $I^{\phi_0}_r$ is equicontinuous. For each $t \in [0,T]$, let $I^{\phi_0}_{r,t} := \{g(t), g \in I^{\phi_0}_r\}$. For any $a \in I^{\phi_0}_{r,t}$, there exists $g \in I^{\phi_0}_r$ such that $a = g(t)$. Then Hölder's inequality implies

$$\|a-\phi_0\|_{L^2(\mathbb{T})} = \|g(t)-g(0)\|_{L^2(\mathbb{T})} \leqslant Cr^{\frac{1}{2}}.$$

Thus $I^{\phi_0}_{r,t}$ is contained in a ball $B_{L^2}(\phi_0, Cr^{\frac{1}{2}})$. By [Tri06, Proposition 4.6], the embedding $L^2(\mathbb{T}) \hookrightarrow \mathcal{C}^{-\frac{1}{2}-\alpha}$ is compact, which implies that $I^{\phi_0}_{r,t}$ is relatively compact in $\mathcal{C}^{-\frac{1}{2}-\alpha}$ for any t. Then the generalized Aerelà-Ascoli theorem implies that $I^{\phi_0}_r$ is compact, i.e., I^{ϕ_0} is a good rate function.

By Lemma 2.3, the task remain is to show that Z_ε and x_ε are exponentially equivalent, that is, for any $\delta > 0$,

$$\lim_{\varepsilon\to 0}\varepsilon \log P(\sup_{0\leqslant t\leqslant T}\|Z_\varepsilon(t) - x_\varepsilon(t)\|_{-\frac{1}{2}-\alpha} > \delta) = -\infty.$$

Let $w_\varepsilon = Z_\varepsilon - x_\varepsilon$, we have

$$\frac{d}{dt}w_\varepsilon(t) = \varepsilon\Delta w_\varepsilon(t) + \varepsilon\Delta x_\varepsilon(t), \quad w_\varepsilon(0) = 0.$$

The mild formulation of w_ε is given by

$$\begin{aligned} w_\varepsilon(t) &= \varepsilon \int_0^t e^{\varepsilon(t-s)\Delta} \Delta x_\varepsilon(s) ds \\ &= \varepsilon \int_0^t e^{\varepsilon(t-s)\Delta} \Delta \phi_0 ds + \varepsilon\sqrt{\varepsilon} \int_0^t e^{\varepsilon(t-s)\Delta} \Delta W(s) ds. \end{aligned}$$

Now we estimate every term in the second line. By Lemma 6.6, we have

$$\begin{aligned} \sup_{0\leqslant t\leqslant T} \|\varepsilon \int_0^t e^{\varepsilon(t-s)\Delta} \Delta\phi_0 ds\|_{-\frac{1}{2}-\alpha} &\leqslant \sup_{0\leqslant t\leqslant T} C\varepsilon \int_0^t \frac{1}{[\varepsilon(t-s)]^{\frac{3}{4}-\frac{\alpha-\beta}{2}}} \|\Delta\phi_0\|_{-2-\beta} ds \\ &\leqslant C\varepsilon^{\frac{1}{4}+\frac{\alpha-\beta}{2}} \|\phi_0\|_{-\beta}. \end{aligned}$$

Similarly, we have for $0 < \kappa_1 < \frac{\alpha}{2}$,

$$\begin{aligned} &\sup_{0\leqslant t\leqslant T} \|\varepsilon\sqrt{\varepsilon} \int_0^t e^{\varepsilon(t-s)\Delta} \Delta W(s) ds\|_{-\frac{1}{2}-\alpha} \\ \leqslant &\sup_{0\leqslant t\leqslant T} C\varepsilon\sqrt{\varepsilon} \int_0^t \frac{1}{[\varepsilon(t-s)]^{1-\kappa_1}} \|\Delta W(s)\|_{-\frac{5}{2}-\alpha+2\kappa_1} ds \\ \leqslant &C\sqrt{\varepsilon}\varepsilon^{\kappa_1} \sup_{0\leqslant t\leqslant T} \|W(t)\|_{-\frac{1}{2}-\alpha+2\kappa_1}. \end{aligned}$$

We should point out that the constant C above is independent of ε and may change from line to line.

For the cylindrical Wiener process W, we have for $s, t \in [0, T]$, $0 < \kappa_1 < \frac{\alpha}{3}$

$$\begin{aligned} E|\triangle_j(W(t) - W(s))|^2 &= E|\sum_{k\in\mathbb{Z}} \theta_j(k) e_k \langle W(t) - W(s), e_k\rangle|^2 \\ &\leqslant C|t-s|(1 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{2^{j(1+2\alpha-6\kappa_1)}}{|k|^{1+2\alpha-6\kappa_1}}) \leqslant C|t-s| 2^{j(1+2\alpha-6\kappa_1)}, \end{aligned}$$

where $e_k = 2^{-\frac{1}{2}} e^{i\pi kx}$ and we use $k \in \text{supp}\theta_j \subset 2^j\mathcal{A}$($\mathcal{A}$ is an annulus).

By Nelson's hypercontractive estimate in [Nel73], for $p > 2$, there exists a constant C independent of p such that

$$\begin{aligned} E\|\triangle_j(W(t) - W(s))\|^p_{L^p(\mathbb{T})} &= \int E|\triangle_j(W(t) - W(s))|^p(x) dx \\ &\leqslant C^p p^{\frac{p}{2}} \int (E|\triangle_j(W(t) - W(s))|^2(x))^{\frac{p}{2}} dx. \end{aligned}$$

Then we obtain for $\frac{1}{p} < \kappa_1$

$$E\|W(t) - W(s)\|^p_{B_{p,p}^{-\frac{1}{2}-\alpha+2\kappa_1+\frac{1}{p}}(\mathbb{T})} \leqslant C^p |t-s|^{\frac{p}{2}} p^{\frac{p}{2}} \sum_{j\geqslant -1} 2^{j(-\kappa_1+\frac{1}{p})p}.$$

Thus Lemma 6.4 and Kolmogorov's continuity criterion imply that for $p > \frac{1}{\kappa_1}$

$$(E[\sup_{0\leqslant t\leqslant T} \|W\|^p_{-\frac{1}{2}-\alpha+2\kappa_1}])^{\frac{1}{p}} \leqslant C(E[\sup_{0\leqslant t\leqslant T} \|W\|^p_{B_{p,p}^{-\frac{1}{2}-\alpha+2\kappa_1+\frac{1}{p}}(\mathbb{T})}])^{\frac{1}{p}} \leqslant C p^{\frac{1}{2}}.$$

Hence, with the above estimates in hand, we have

$$\begin{aligned}(E\sup_{0\leqslant t\leqslant T}\|w_\varepsilon(t)\|^p_{-\frac{1}{2}-\alpha})^{\frac{1}{p}} &\leqslant C\varepsilon^{\frac{1}{4}+\frac{\alpha-\beta}{2}}\|\phi_0\|_{-\beta}+C\sqrt{\varepsilon}\varepsilon^{\kappa_1}(E[\sup_{0\leqslant t\leqslant T}\|W\|_{-\frac{1}{2}-\alpha+2\kappa_1}]^p)^{\frac{1}{p}}\\ &\leqslant C\varepsilon^{\kappa_1}(1+\sqrt{\varepsilon}p^{\frac{1}{2}}),\end{aligned}$$

where C is the constant independent of ε, p and may change from line to line.

Therefore Chebyshev's inequality implies that

$$\begin{aligned}\varepsilon\log P(\sup_{0\leqslant t\leqslant T}\|w_\varepsilon(t)\|_{-\frac{1}{2}-\alpha}>\delta) &\leqslant \varepsilon\log\frac{E\sup_{0\leqslant t\leqslant T}\|w_\varepsilon(t)\|^p_{-\frac{1}{2}-\alpha}}{\delta^p}\\ &\leqslant \varepsilon p(\log C\varepsilon^{\kappa_1}(1+\sqrt{\varepsilon}p^{\frac{1}{2}})-\log\delta).\end{aligned}$$

Let $p=\frac{1}{\varepsilon}$ and $\varepsilon\to 0$, the proof is complete.

□

Now we follow the notations from [GP16, Section 9] and give some estimates of Z_ε: We represent the white noise in terms of its spatial Fourier transform. Let $E=\mathbb{Z}\setminus\{0\}$ and let $W(s,k)=\langle W(s),e_k\rangle$, where $\{e_k:=2^{-\frac{1}{2}}e^{i\pi kx}\}_{k\in\mathbb{Z}}$ is the Fourier basis of $L^2(\mathbb{T})$. Here for simplicity we assume that $\langle W(s),e_0\rangle=0$ and restrict ourselves to the flow of $\int_{\mathbb{T}}u(x)dx=0$. In the following we view $W(s,k)$ as a Gaussian process on $\mathbb{R}\times E$ with covariance given by

$$E[\int_{\mathbb{R}\times E}f(\eta)w(d\eta)\int_{\mathbb{R}\times E}g(\eta')w(d\eta')]=\int_{\mathbb{R}\times E}f(\eta_1)g(\eta_{-1})d\eta,$$

where $\eta_a=(s_a,k_a)$ and the measure $d\eta_a=ds_a dk_a$ is the product measure of the Lebesgue measure ds on $\mathbb{R}$ and the counting measure dk on E .

Let $\overline{Z}_\varepsilon=Z_\varepsilon-e^{\varepsilon t\Delta}\phi_0$, then

$$\overline{Z}_\varepsilon(t,x)=\int_{\mathbb{R}\times E}\sqrt{\varepsilon}e_k(x)e^{-\varepsilon(t-s)\pi|k|^2}1_{\{0<s<t\}}W(d\eta).$$

Now we have the following calculations: for $s,t\in[0,T]$,

$$\begin{aligned}&E[|\triangle_j(\overline{Z}_\varepsilon(t)-\overline{Z}_\varepsilon(s))|^2]\\ =&E[|\int\theta_j(k_1)(\sqrt{\varepsilon}e_{k_1}e^{-\varepsilon(t-s_1)\pi|k_1|^2}1_{\{0<s_1<t\}}-\sqrt{\varepsilon}e_{k_1}e^{-\varepsilon(s-s_1)\pi|k_1|^2}1_{\{0<s_1<s\}})W(d\eta_1)|^2]\\ \leqslant&\varepsilon C\int\theta_j^2(k_1)(e^{-2\varepsilon(t-s_1)\pi|k_1|^2}1_{\{s<s_1<t\}}+|e^{-\varepsilon(t-s)\pi|k_1|^2}-1|^2e^{-2\varepsilon(s-s_1)\pi|k_1|^2}1_{\{0<s_1<s\}})d\eta_1\\ \leqslant&C\int\theta_j(k_1)^2\frac{(\varepsilon|t-s||k_1|^2)^{2\kappa}}{|k_1|^2}dk_1\\ \leqslant&C\varepsilon^{2\kappa}|t-s|^{2\kappa}2^j\frac{1}{(2^j)^{2-4\kappa}}=C\varepsilon^{2\kappa}|t-s|^{2\kappa}2^{j(-1+4\kappa)},\end{aligned}\tag{7.2}$$

where we use $1-e^x \leqslant |x|^\kappa$ for $\kappa \in (0,1), x<0$ in the fourth inequality and $k \in \mathrm{supp}\theta_j \subset 2^j\mathcal{A}(\mathcal{A}$ is an annulus) in the last inequality. Here the constant C is independent of ε and may change from line to line.

By Nelson's hypercontractive estimate in [Nel73], we have for $p>2$, there exists a constant C independent of p, ε such that

$$\begin{aligned} E\|\triangle_j(\overline{Z}_\varepsilon(t)-\overline{Z}_\varepsilon(s))\|^p_{L^p(\mathbb{T})} &= \int E|\triangle_j(\overline{Z}_\varepsilon(t)-\overline{Z}_\varepsilon(s))|^p(x)dx \\ &\leqslant C^p p^{\frac{p}{2}} \int (E|\triangle_j(\overline{Z}_\varepsilon(t)-\overline{Z}_\varepsilon(s))|^2(x))^{\frac{p}{2}}dx. \end{aligned}$$

Let $\kappa=\frac14-\kappa'$ for $\kappa'>0$ small enough, we obtain

$$E\|\overline{Z}_\varepsilon(t)-\overline{Z}_\varepsilon(s)\|^p_{B^{\kappa'}_{p,p}(\mathbb{T})} \leqslant C^p p^{\frac{p}{2}}(\varepsilon|t-s|)^{(\frac14-\kappa')p}.$$

Then Lemma 6.4 and Kolmogorov's continuity criterion implies that for $p>\frac{1}{\kappa'}$, we have

$$E\|\overline{Z}_\varepsilon\|^p_{CL^\infty} \leqslant E\|\overline{Z}_\varepsilon\|^p_{C\mathcal{C}^{\kappa'-\frac1p}} \leqslant E\|\overline{Z}_\varepsilon\|^p_{C([0,T];B^{\kappa'}_{p,p}(\mathbb{T}))} \leqslant C^p\varepsilon^{(\frac14-\kappa')p}p^{\frac{p}{2}}. \tag{7.3}$$

Remark 7.3. *We want to emphasize that (7.3) only holds for $\overline{Z}_\varepsilon$ due to $\overline{Z}_\varepsilon(0)=0$. For the stationary one this does not hold since the expectation of the stationary one does not depend on ε.*

7.2 Exponential equivalence

Theorem 7.2 implies that Z_ε satisfies a large deviation principle on the space $C\mathcal{C}^{-\frac12-\alpha}$ with the rate function I^{ϕ_0}. By Lemma 2.3, our task is to show that ϕ_ε and Z_ε are exponentially equivalent in $C\mathcal{C}^{-\frac12-\alpha}$. That is:

Theorem 7.4. *For any $\delta>0$,*

$$\lim_{\varepsilon\to0}\varepsilon\log P(\sup_{0\leqslant t\leqslant T}\|\phi_\varepsilon(t)-Z_\varepsilon(t)\|_{-\frac12-\alpha}>\delta)=-\infty. \tag{7.4}$$

Proof At the beginning of the proof, we should point out that the constant C in the following is independent of ε, p and may change from line to line.

Let $Y_\varepsilon(t):=\phi_\varepsilon(t)-Z_\varepsilon(t)$, then Y_ε is the solution to the following shifted equation:

$$\begin{aligned} &dY_\varepsilon(t)=\varepsilon\Delta Y_\varepsilon(t)dt-\varepsilon(Y_\varepsilon(t)+Z_\varepsilon(t))^3dt, \\ &Y_\varepsilon(0)=0. \end{aligned} \tag{7.5}$$

For $p\geqslant1$, we have

$$\begin{aligned} &\frac{1}{2p}\frac{d}{dt}\|Y_\varepsilon\|^{2p}_{L^{2p}(\mathbb{T})} \\ =&\varepsilon\langle\Delta Y_\varepsilon, Y_\varepsilon^{2p-1}\rangle-\varepsilon\langle Y_\varepsilon^3, Y_\varepsilon^{2p-1}\rangle-3\varepsilon\langle Y_\varepsilon^2Z_\varepsilon, Y_\varepsilon^{2p-1}\rangle-3\varepsilon\langle Y_\varepsilon Z_\varepsilon^2, Y_\varepsilon^{2p-1}\rangle-\varepsilon\langle Z_\varepsilon^3, Y_\varepsilon^{2p-1}\rangle. \end{aligned}$$

Then

$$\frac{1}{2p}\|Y_\varepsilon(t)\|^{2p}_{L^{2p}(\mathbb{T})}+\varepsilon\int_0^t[(2p-1)\langle\nabla Y_\varepsilon(s), Y_\varepsilon^{2p-2}(s)\nabla Y_\varepsilon(s)\rangle+\|Y_\varepsilon^{2p+2}(s)\|_{L^1(\mathbb{T})}]ds$$

$$
\begin{aligned}
&= - \varepsilon \int_0^t [3\langle Y_\varepsilon^{2p+1}(s), Z_\varepsilon(s)\rangle + 3\langle Y_\varepsilon^{2p}(s), Z_\varepsilon^2(s)\rangle + \langle Y_\varepsilon^{2p-1}(s), Z_\varepsilon^3(s)\rangle]ds \\
&\leqslant \varepsilon \int_0^t (a\|Y_\varepsilon(s)^{2p+2}\|_{L^1(\mathbb{T})} + C\|Z_\varepsilon(s)\|_{L^\infty(\mathbb{T})}^{2p+2})ds,
\end{aligned}
$$

where we use Hölder's inequality and Young's inequality in the last inequality and $a \in (0,1)$. Take $p=3$, for $t \in [0,T]$, we have

$$
\begin{aligned}
\|Y_\varepsilon(t)\|_{L^6(\mathbb{T})}^6 &\leqslant \varepsilon C \int_0^t \|Z_\varepsilon(s)\|_{L^\infty(\mathbb{T})}^8 ds \\
&\leqslant \varepsilon C \int_0^t (\|e^{\varepsilon s \Delta}\phi_0\|_{\beta'}^8 + \|\overline{Z}_\varepsilon(s)\|_{L^\infty(\mathbb{T})}^8)ds \\
&\leqslant \varepsilon C \int_0^t (\frac{1}{(\varepsilon s)^{\frac{8(\beta'+\beta)}{2}}}\|\phi_0\|_{-\beta}^8 + \|\overline{Z}_\varepsilon(s)\|_{L^\infty(\mathbb{T})}^8)ds \\
&\leqslant C(\varepsilon^{1-4(\beta'+\beta)}\|\phi_0\|_{-\beta}^8 + \varepsilon\|\overline{Z}_\varepsilon\|_{CL^\infty}^8),
\end{aligned}
\tag{7.6}
$$

where $0 < \beta' < \frac{1}{4} - \beta$ and we use Lemma 6.6 in the third inequality.

Thus Young's inequality and the mild formulation of Y_ε given by

$$
Y_\varepsilon(t) = \varepsilon \int_0^t e^{\varepsilon(t-s)\Delta}[-Y_\varepsilon^3 - 3Y_\varepsilon^2 Z_\varepsilon - 3Y_\varepsilon Z_\varepsilon^2 - Z_\varepsilon^3]ds
$$

imply that

$$
\begin{aligned}
\sup_{t\in[0,T]} \|Y_\varepsilon(t)\|_{L^2(\mathbb{T})} &\leqslant \varepsilon C \int_0^T (\|Y_\varepsilon(s)\|_{L^6(\mathbb{T})}^3 + \|Z_\varepsilon(s)\|_{L^\infty(\mathbb{T})}^3)ds \\
&\leqslant \varepsilon C \int_0^T (\|Y_\varepsilon(s)\|_{L^6(\mathbb{T})}^3 + \frac{1}{(\varepsilon s)^{\frac{3(\beta'+\beta)}{2}}}\|\phi_0\|_{-\beta}^3 + \|\overline{Z}_\varepsilon(s)\|_{L^\infty(\mathbb{T})}^3)ds \\
&\leqslant C(\varepsilon^{\frac{3}{2}-2(\beta'+\beta)} + \varepsilon^{\frac{3}{2}}\|\overline{Z}_\varepsilon\|_{CL^\infty}^4 + \varepsilon^{1-\frac{3(\beta'+\beta)}{2}} + \varepsilon\|\overline{Z}_\varepsilon\|_{CL^\infty}^3),
\end{aligned}
$$

where we use Lemma 6.6 in the second inequality and (7.6) in the last inequality.

Thus by (7.3) we have for $3q > \frac{1}{\kappa'}$

$$
\begin{aligned}
(E \sup_{t\in[0,T]} \|Y_\varepsilon(t)\|_{L^2(\mathbb{T})}^q)^{\frac{1}{q}} &\leqslant C(\varepsilon^{\frac{3}{2}-2(\beta'+\beta)} + \varepsilon^{\frac{3}{2}}(E[\|\overline{Z}_\varepsilon\|_{CL^\infty}^{4q}])^{\frac{1}{q}} + \varepsilon^{1-\frac{3(\beta'+\beta)}{2}} + \varepsilon(E[\|\overline{Z}_\varepsilon\|_{CL^\infty}^{3q}])^{\frac{1}{q}}) \\
&\leqslant C(\varepsilon^{\frac{3}{2}-2(\beta'+\beta)} + \varepsilon^{\frac{5}{2}-4\kappa'}q^2 + \varepsilon^{1-\frac{3(\beta'+\beta)}{2}} + \varepsilon^{\frac{7}{4}-3\kappa'}q^{\frac{3}{2}}).
\end{aligned}
$$

Therefore, by Chebyshev's inequality and Lemma 6.4 we have

$$
\begin{aligned}
&\varepsilon \log P(\sup_{0\leqslant t\leqslant T} \|Y_\varepsilon(t)\|_{-\frac{1}{2}-\alpha} > \delta) \\
\leqslant&\varepsilon \log \frac{E\sup_{t\in[0,T]}\|Y_\varepsilon(t)\|_{L^2(\mathbb{T})}^q}{\delta^q} \\
\leqslant&\varepsilon q[\log[C(\varepsilon^{\frac{3}{2}-2(\beta'+\beta)} + \varepsilon^{\frac{5}{2}-4\kappa'}q^2 + \varepsilon^{1-\frac{3(\beta'+\beta)}{2}} + \varepsilon^{\frac{7}{4}-3\kappa'}q^{\frac{3}{2}})] - \log\delta].
\end{aligned}
$$

Let $q = \frac{1}{\varepsilon}$, we deduce that

$$
\lim_{\varepsilon\to 0} \varepsilon \log P(\sup_{0\leqslant t\leqslant T} \|Y_\varepsilon(t)\|_{-\frac{1}{2}-\alpha} > \delta) = -\infty.
$$

□

Then Theorem 7.1 follows from Lemma 2.3 and Theorem 7.4 .

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