DISSERTATION

On some Two-Dimensional Singular Stochastic Control Problems and their Free-Boundary Analysis

zur Erlangung des akademischen Grades Doktor der Mathematik (Dr. math.)

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Summary

In this thesis, we investigate two-dimensional singular stochastic control problems motivated by different applications in economics and finance. The main interest is to characterize the optimal control in the problems, and in particular to characterize the corresponding free-boundaries. We investigate three different settings, in which the two-dimensional nature is driven by various aspects. In Section 2, we propose and solve a dividend problem with capital injections in a finite time horizon setting. The surplus process of a firm is assumed to follow a stochastic dynamic, and due to the finite time horizon, the time itself becomes a state variable. In Section 3, we study a control problem regarding the inventory of a firm. We assume that the demand of a good follows some stochastic dynamics. In addition, we assume that drift and volatility parameters are Markov modulated, representing different scenarios of the economy. Finally, in Section 4, we study a control problem with interconnected dynamics. This problem is motivated by different applications as, for example, the inflation control. We consider X to be a process with some stochastic dynamics (e.g. the inflation rate), in which the drift can be controlled. In this model, the process X and the drift are state variables, which are interconnected.

In all these applications, we characterize the free-boundaries by combining and extending different techniques. In particular, in Section 2, we extend a result by El Karoui and Karatzas [40], which connects a singular stochastic control problem with a problem of optimal stopping. Hence, we can study the time-dependent free-boundary of the optimal stopping problem. Moreover, the optimal dividend strategy can be expressed as a solution to a Skorokhod reflection problem at the free-boundary. In Section 3, an application of the dynamic programming principle is used to derive a system of non-linear equations characterizing the constant free-boundaries. This system is solved numerically to provide a comparative static analysis. Finally, in Section 4, we derive the structure of the value function by employing the connection of the singular stochastic control problem to a Dynkin game of stopping. Moreover, by characterizing the value function as a viscosity solution to the corresponding dynamic programming equation, we can derive a second-order smooth-fit property as well as a necessary system of non-linear functional equations for the free-boundaries. Furthermore, in a particular modification of the model, these functional equations can be used to derive a system of first-order ordinary differential equations, which is explicitly computable.

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Contents

1 Introduction

In many real-world situations, an agent is confronted with an optimization problem of a certain performance criterion by adjusting the dynamics of the so-called *state process* via the so-called *control variable*. The state processes can be either modelled as a deterministic or as a stochastic process. Such control problems arise in many different areas as Finance, Economics, Physics, Engineering and Biology among others. For example, the state process may describe the evolution of a stock price (Finance), changes in demand of a good (Economics), or the movement of a certain particle (Physics).

There are two classical approaches to solve optimal control problems, namely the *maximum principle* and the *dynamic programming approach*. The maximum principle was developed by L.S. Pontryagin, see Pontryagin et al. [80] in order to solve deterministic problems. Later, it was also extended to the stochastic case, see for example Chapter 3 in Yong and Zhou [97]. It states necessary conditions for optimality, more precisely that an optimal control has to solve a two-point boundary value problem, the *Hamiltonian system*, along the optimal state trajectory together with a maximum condition on the *Hamiltonian*. In the stochastic case, the Hamiltonian system becomes a forward-backward stochastic differential equation. Moreover, the necessary conditions of the maximum principle are also sufficient for the control problem under additional convexity assumptions, see [97].

The second approach to solve optimal control problems is the dynamic programming principle, which was developed by R. Bellman in the 1950's, see Bellman [10]. He shows that in order to solve a particular control problem, one can examine a family of optimal control problems which differ in the initial values of their state process. The aim is to find a connection between those control problems which, under sufficient regularity conditions, leads to a (non-linear) partial differential equation (PDE) for the value function of the control problem in the case of a multi-dimensional state process. In case of a one-dimensional state space, it reads as an ordinary differential equation (ODE). This is the so-called *Hamilton-Jacobi-Bellman* (HJB) equation (or just Bellman equation in the deterministic case). More precisely, the HJB equation is a first-order PDE (ODE) in the deterministic case and a second-order PDE (ODE) in the stochastic case. In case a classical solution to the HJB equation exists, it can be used to obtain an optimal (feedback) control for the control problem. This method is known as the *verification technique* or *guess-and-verify-approach*, see Chapter 4 in [97]. However, it must be mentioned that the necessary existence of a classical solution to the HJB equation is often all but trivial to establish.

In 1983, Crandall and Lions, see [30], used the notion of *viscosity solution*, introduced by Evans in 1980, see [42], to study HJB equations arising in control problems. For a more recent introduction to viscosity solutions, see Fleming and Soner [49]. Roughly speaking, the viscosity solution solves the HJB only in a generalized sense. Therefore, it does not need to be smooth. Nevertheless, this is flexible enough to allow for existence, stability under uniform limits, and uniqueness. The limitation of the viscosity approach is that it usually only provides information about optimal controls if one is able to upgrade the regularity of the value function to a degree allowing for a verification theorem. In Section 4 we use this approach to solve a two-dimensional control problem with interconnected dynamics.

In this thesis, two classes of continuous-time stochastic control problems are examined: *Singular stochastic control* (SSC) problems and problems with *controls of bounded-velocity*. In SSC problems, the control affects the state process instantaneously. Moreover, the control variable denotes the cumulative amount of actions up to a certain point in time. Mathematically, the control process belongs to the set of processes with bounded variation. Furthermore, in a Markovian setting, the corresponding HJB equation becomes a second-order PDE (ODE) with a local gradient constraint, and as such it is related to a *free-boundary problem*. The optimal strategy is usually described by two regions, which split the state space: the *waiting* and the *action* region. As long as the state process lies inside the waiting region, no control is exerted. In the action region, it is optimal to exert as much control as needed in order to bring the controlled state process back to the waiting region. In mathematical terms, the optimal control in this particular setting is related to the solution of *Skorokhod reflection problem*, see Skorokhod [90]. To construct a solution of the Skorokhod reflection problem, it is essential to understand the geometry of the state space and, in particular, to specify properties like monotonicity, continuity or locally Lipschitz continuity of the free-boundary between the action and the waiting region.

One way to obtain these important regularity properties consists of exploiting a characteristic feature of SSC problems, namely, the link to *optimal stopping* (OS) problems, see El Karoui and Karatzas [40], [41], Karatzas and Shreve [58], among others. Typically, the gradient of the value function of a SSC in the direction of the controlled variable can be expressed as the value function of an OS problem. In particular, the waiting region in a SSC problem coincides with the waiting region of the associated OS problem, where the underlying process is uncontrolled. Therefore, to study the geometry of the waiting region in a SCC problem, one can use the literature and methodology available for OS problems. As a matter of fact, such a connection SSC-OS serves as (one of) the main tools to tackle the SSC problems in Section 2 and Section 4. In Section 2, the link SSC-OS enables us to solve an optimal dividend problem with capital injections by constructing the optimal control as a solution to the corresponding Skorokhod reflection problem. While in Section 4, the link SSC-OS is one of the key elements that is needed to determine the geometry of the state space and the properties of the value function in a two-dimensional SSC problem with interconnected dynamics. These properties allow us later to upgrade the regularity of the value function in such a way that we can derive necessary equations for the free-boundaries.

In Section 3, beside to singular controls, we consider controls with bounded-velocity. This class of models dates back to V.E. Beneš in 1973, see [11]. In such a setting, the controls are assumed to have trajectories that are absolutely continuous (w.r.t. the Lebesgue measure). In contrast to SSC problems, the control acts only with a rate on the state variable, but not instantaneously. Therefore, the HJB equation in bounded-velocity control problems has no longer a local gradient constraint and, furthermore, one can not establish an exploitable connection to a corresponding OS problem. In Section 3, we solve the considered optimal control problem for both controls of singular and of bounded-velocity type and we provide a comparison across the resulting optimal policies.

The focus of the thesis lies on two-dimensional SSC problems motivated by various economic applications. Despite that there is a comprehensive literature on one-dimensional SSC, the literature on multi-dimensional settings is still limited. One explanation might be that, as mentioned above, in case of multi-dimensional control problems the HJB equation takes the form of a second-order PDE with local gradient constraint, for which explicit solutions are typically not available. As a consequence, the guess-and-verify approach can not be employed in order to solve the control problem, unless specific degenerate settings are investigated (see [2], [33], [34], [70], [71], and [74] as well as the references in the next subsections). In the mentioned papers, the authors can guess the geometry of the state space and, by imposing suitable smoothness on a candidate value function, they perform a verification theorem to provide the optimal solution.

In this thesis, we propose three different two-dimensional SSC problems for which we are able to provide a detailed description of the value functions and the free-boundaries splitting the state spaces. The two-dimensional nature of the problem is due to the presence of a finite time horizon in the considered optimization problem in Section 2, due to the problem's coefficients that are Markov-modulated in Section 3 and due to a two-dimensional degenerate state variable in Section 4. Basing on the different two-dimensional nature of the problems, we use different approaches to tackle the problems.

We now continue with a more detailed view on the problems treated in this dissertation. In particular, we introduce the studied model, describe the solution approach and discuss the contribution to the literature for each Section.

An Optimal Dividend Problem with Capital Injections over a Finite Horizon[1] [Section 2]

The literature on optimal dividend problems started in 1957 with the work of de Finetti [36]. He proposes, for the first time in the literature, to measure an insurance portfolio by the discounted value of its future dividends' payments. Since then, the literature in Mathematics and Actuarial Mathematics experienced many scientific contributions on the optimal dividend problem, which has been typically modeled as a stochastic control problem subject to different specifications of the control processes and the surplus dynamics (see, among many others, the early works by Jeanblanc-Piqué and Shiryaev [51], Shreve et al. [89], the more recent works by Akyildirim et al. [1], De Angelis and Ekström [32] and Jiang and Pistorius [53], the review by Avanzi [5], and the book by Schmidli [87]).

Starting from the observation that ruin occurs almost surely when the fund's manager pays dividends by following the optimal strategy of de Finetti's problem, in Dickson and Waters [37], the authors proposed several modifications to the original

[1]Parts of this Introduction and of Section 2 are already published in a joint work with Giorgio Ferrari, see [47].

formulation of the optimal dividend problem. In particular, Dickson and Waters [37] suggest a model in which the shareholders are obliged to inject capital in order to avoid bankruptcy. This is the so-called *optimal dividend problem with capital injections.*

The literature on the optimal dividend problem with capital injections is not as rich as that on the classical de Finetti's problem. Kulenko and Schmidli [65] study an optimal dividend problem with capital injections in which the surplus process is reflected at the origin and evolves according to a classical Cramér-Lundberg risk model on $(0, \infty)$. Schmidli [86] solves an optimal dividend problem with capital injections and taxes in a diffusive setting. In Lokka and Zervos [69], the shareholders can choose the capital injections' policy and, in absence of any interventions, the surplus process follows a Brownian motion with drift. Other works in which the surplus process evolves as a general one-dimensional diffusion are the ones by Ferrari [46], Zhu and Yang [96], and Shreve et al. [89]. Optimal dividends and capital injections in a jump-diffusion setting are determined by Avanzi et al. [6]. In all these papers, the optimal dividend problem with capital injections is formulated as a SSC problem for a reflected process (i.e. a so-called *reflected follower problem*) over an infinite time horizon. Given the stationarity of the setting, in these works it is shown that (apart from a possible initial lump sum payment) it is optimal to pay just enough dividends in order to keep the surplus process in the interval $[0, b]$, for some constant $b > 0$, endogenously determined.

In Section 2, we propose and solve, for the first time in the literature, an optimal dividend problem with capital injections over a finite time horizon $T \in (0, \infty)$. This horizon might be seen as a pre-specified future date at which the fund is liquidated.

As it is common in the literature, in absence of any interventions, the surplus process evolves as a Brownian motion with drift μ and volatility σ (see [1], [32] and [69], among many others). This dynamics for the fund's value can be obtained as a suitable (weak) limit of a classical dynamics à la Cramér-Lundberg (see Appendix D.3 in Schmidli [87] for details). We assume that, after time-dependent transaction costs/taxes have been paid, shareholders receive a time-dependent instantaneous net proportion of leakages f from the surplus. Moreover, shareholders are *forced* to inject capital whenever the surplus attempts to become negative. By injecting capital, they incur a time-dependent marginal administration cost m. Finally, a surplus-dependent liquidation reward g is obtained at liquidation time T. Notice that, under suitable requirements on f, m and g (see Remark 2.4), injecting capital at the origin turns out to be optimal within the class of dividends/capital injections that keeps the surplus non-negative at any time with probability one (see also Kulenko and Schmidli [65], Scheer and Schmidli [85] and Schmidli [86]).

Within this setting, the fund's manager takes the point of view of the shareholders and thus aims at solving

$$V(t,x) := \sup_D \mathbb{E}\bigg[\int_0^{T-t} f(t+s)\ dD_s - \int_0^{T-t} m(t+s)\ dI^D_s + g(T, X^D_{T-t}(x)) \bigg], \quad (1.1)$$

for any initial time $t \in [0, T]$ and any initial value of the fund $x \in \mathbb{R}_+$. In (1.1) the fund's value evolves as

$$X_s^D(x) = x + \mu s + \sigma W_s - D_s + I_s^D, \quad s \geq 0,$$

and the optimization is performed over a suitable class of non-decreasing processes D. The quantity D_s represents the cumulative amount of dividends paid to shareholders up to time s, whereas I_s^D is the cumulative amount of capital injected by the shareholders up to time s. We take I^D as the minimal non-decreasing process which ensures that X^D stays non-negative, and it is flat off $\{t \geq 0 : X_t^D = 0\}$.
If we attempt to tackle problem (1.1) via a dynamic programming approach, the HJB equation for V takes the form of a parabolic PDE with gradient constraint (i.e. a variational inequality), and with a Neumann boundary condition at $x = 0$ (the latter is due to the fact that the state process X is reflected at the origin through the capital injections process). Proving that a solution to this PDE problem has enough regularity to characterize an optimal control is far from trivial.
Starting from the observation that the optimal dividend problem with capital injections (1.1) is actually a reflected follower problem (see, e.g., Baldursson [7], El Karoui and Karatzas [39] and Karatzas and Shreve [59] as early contributions) with costly reflection at the origin, and inspired by the results of El Karoui and Karatzas [40], we solve (1.1) without relying on PDE methods. Instead, we relate (1.1) to a (still complex but) more tractable optimization problem; i.e., to an optimal stopping problem with absorption at the origin and with value function u (cf. (2.6)).
If the optimal stopping time for this problem is given in terms of a continuous and strictly positive time-dependent boundary $b(\cdot)$ (cf. the structural Assumption 2.5), it follows that $V_x = u$, and the optimal dividends' payments strategy $D^\star$ is triggered by b (see Theorem 2.6). In fact, if the optimization starts at time $t \in [0, T]$, the couple $(D^\star, I^{D^\star})$ keeps the optimally controlled fund's value $X_s^{D^\star}$ non-negative and below the time-dependent critical level $b(s + t)$ at any instant in time $s \in [0, T - t]$. This result is obtained via an almost exclusively probabilistic study in which we suitably integrate in the space variable two different representations of the value function u of the auxiliary optimal stopping problem. It is worth noticing that although we borrow arguments from the study in El Karoui and Karatzas [40] on the connection between reflected follower problems and questions of optimal stopping (see also Karatzas and Shreve [59]), differently to El Karoui and Karatzas [40], in our performance criterion (1.1) we have a cost of reflection which requires a careful and not immediate adaptation of the ideas and results of El Karoui and Karatzas [40].
We then show that the structural Assumption 2.5, which is needed to establish the relation between (1.1) and the optimal stopping problem, does indeed hold in a canonical formulation of the optimal dividend problem with capital injections. Indeed, the marginal benefits and costs are constants discounted at a constant rate, and the liquidation value at time T is proportional to the terminal value of the fund. In particular, we show that the optimal dividend strategy is given in terms of an optimal boundary b that is decreasing, continuous, bounded, and null at terminal time.

Optimal Production Under Regime Switching[2] [Section 3]

The literature dealing with optimal production problems in continuous-time is very comprehensive. For example, Bensoussan et al. [14], Sethi and Thompson [88] and Khmelnitsky et al. [62] consider the demand of a product to be constant. Fleming et al. [48] consider the demand as a continuous-time Markov chain with a finite state space and Cadenillas et al. [19] investigate a cumulative demand that evolves as a drifted Brownian motion or as a geometric Brownian motion. Moreover, in both cases the drift and the variance parameters are modulated by a continuous-time Markov chain representing the regime of the economy. Among these, only [14] imposes the realistic assumption that the production rate is non-negative.

This motivated us to study a production problem with a non-negative production rate and in which the demand process follows a stochastic dynamic. We assume that a manager of a firm is faced with the problem of controlling the inventory of a certain product under regime switching. An exogenous given finite-state Markov chain $\epsilon = \{\epsilon_t, t \geq 0\}$ describes the current regime of the economy. The demand[3] of the product evolves as

$$dD_t = \mu_{\epsilon_t}\, dt + \sigma_{\epsilon_t}\, dW_t.$$

The objective of the manager is to maintain the inventory level as close as possible to a fixed target value depending on the state of the economy. In order to increase the inventory of the good, and thus to control it, the manager chooses a non-negative production strategy P to increase the inventory of the good, whose dynamic is given by

$$dX_t = dP_t - dD_t.$$

In this setting, the manager aims at solving

$$V(x,i) = \inf_P \mathbb{E}\left[\int_0^\infty e^{-\delta s}\alpha_{\epsilon_s}\left(X_s - \mathcal{I}_{\epsilon_s}\right)^2\, ds + \int_0^\infty e^{-\delta s} k_{\epsilon_s}\, dP_s\right] \qquad (1.2)$$

for an initial inventory level x and an initial regime i. We develop two versions of the model. In the first one, there is no upper bound for the production rate and the model is expressed as a singular stochastic control problem with regime switching. In the second one, an upper bound for the rate of production is introduced, and we consider a classical bounded-velocity stochastic control problem with regime switching in the spirit of V.E. Beneš [11]. To obtain explicit solutions, we concentrate on the case with two regimes. One regime may stand for a recessionary period, in which the demand is very low, and a second one that represents an expansionary period with high demand.

Applying dynamic programming to Problem (1.2), we show that the HJB equation solved by V takes the form of a system of second-order ODEs with gradient constraints in the case of singular controls, and we perform a guess-and-verify approach. Moreover, we derive an analytical representation for V and show that in

[2]This project started during a research visit at the University of Edmonton under the supervision of Abel Cadenillas.

[3]A possible negative demand can be seen as an oversupply of the good.

both versions that the optimal control is triggered by constant boundaries b_i for each state. In fact, in the case of singular controls the optimal production is such that the inventory level X stays always above the boundaries b_i and in the case of bounded-velocity controls the rate of production is maximal as long as the inventory is below the level b_i.

Furthermore, we are able to derive a system of non-linear equations for the boundaries and the coefficients in the analytical representation. Unfortunately, this system can only be solved numerically. Based on the numerical solutions, we provide a comparative statics analysis of the free-boundaries with respect to some model parameters. In particular, we are able to show that unexpectedly the boundaries in the two versions show different behavior with respect to demand uncertainty σ_i, which depends on the state. Indeed, the boundaries are decreasing with respect to σ_i in the SSC but not in the bounded-velocity control case. Moreover, we show that the boundaries in the bounded-velocity case are larger than in the singular case. Finally, if the upper bound for the bounded-velocity control diverges to infinity, we show that the value function and the free-boundaries of the first case converges to the value function and the boundaries of the singular control case.

A singular stochastic control problem with interconnected dynamics[4] [Section4]

In this Section, we study a singular stochastic control problem with interconnected dynamics. A purely controlled process Y, given by

$$Y_t = y + \xi_t^+ - \xi_t^-, \quad y \in \mathbb{R},$$

affects the drift component of a diffusive process X. The process X follows the dynamics

$$dX_t = \alpha Y_t - \theta X_t \; dt + \eta \; dW_t, \quad X_0 = x,$$

hence it evolves as an Ornstein-Uhlenbeck process for $\theta > 0$ and as a drifted Brownian Motion for $\theta = 0$.

The objective of a decision maker is to minimize a total expected cost functional, which consists of a time-integral over running costs plus proportional costs arising from adjusting the drift component Y. The problem is modeled as a Markovian two-dimensional degenerate singular stochastic control problem with controls of bounded variation. Our model can be seen as a generalization of the bounded-velocity control of a scalar Brownian motion, introduced by V.E. Beneš in 1974 [11], which has stimulated a subsequent large literature allowing for different specifications of the performance criterion and incorporating also other features like discretionary stopping and partial observation (see [4], [12], [54], [55], [56], [73], among many others). In whose formulation, the drift is chosen from a bounded set, e.g. $[-1, 1]$, and the resulting optimal control results to be of the so-called *bang-bang type* (see [11], [54],

[4] Parts of this introduction and of Section 4 are already published in two joint works with Giorgio Ferrari and Salvatore Federico, see [44] and [45].

[55], [56], among others, or Section 3 of this thesis). In opposite to them, our control and hence the drift is unlimited, and the resulting optimal control is of singular type.

The fact that the two state processes are coupled makes this problem quite involved and a guess-and-verify approach seems not to be applicable. The closest papers from the literature to our problem are Federico and Pham [43] and Chiarolla and Haussmann[26]. In fact, from a mathematical point of view, our model can be seen in between that of [26] (see also Chiarolla and Haussmann [25] for a finite time horizon version) and that of [43] (see also Merhi and Zervos [74]). On the one hand, we propose a degenerate version of the fully non-degenerate two-dimensional bounded-variation stochastic control of [26]; on the other hand, the problem of [43] can be obtained from our when the dynamics of the two components of the state process decouple. It is exactly the degeneracy of our state process that makes the determination of the structure of the value function possible in our problem, and it is the coupling between X and Y that makes our analysis much more involved than that in [43].

To the best of our knowledge, the only other paper dealing with a degenerate two-dimensional singular stochastic control problem where the dynamics of the two components of the state process are coupled are Koch and Vargiolu [64] and Pierre, Villeneuve and Warin [81]. In [81] the authors consider a dividend and investment problem for a cash constrained firm, and both a viscosity solution approach and a verification technique are employed to get qualitative properties of the value function. It is important to notice that, in contrast to our model, the problem in [81] is not convex, thus making it hard to prove any regularity of the value function further than its continuity. In [64] the authors study a two-dimensional singular stochastic control problem with interconnected dynamics and a finite fuel constraint, in which the control is assumed to be monotone. This problem, motivated by irreversible installation of solar panels, is solved explicitly and it turns out that the free-boundary can be characterized by a first-order ODE completed by a boundary condition, which is implied by the finite fuel constraint. Their problem is similar to our model in the case $\theta > 0$, as we also consider a mean-reverting state process (in [64] the price solar electricity) with drift affected by the purely controlled process (in [64] the amount of installed solar panels). However, in contrast to them, our control is not assumed to be monotone, and no finite fuel condition is imposed, which leads to several additional difficulties.

Therefore, in order to tackle our problem, we choose a direct approach instead of a guess-and-verify approach. First of all, we show for both formulations that the value function is differentiable with (locally) Lipschitz derivatives. Moreover, by exploiting a suitable approximation procedure, we can use a result of [26] to show that the derivative V_y is the value function of a related Dynkin game of optimal stopping. This fact, combined with the convexity of the value function V, provides first information about the state space of the problem. In particular, we show the existence of two monotone curves (free-boundaries) dividing the state space into three connected regions (continuation and action regions). Next, we show that V is a viscosity solution to the corresponding HJB equation (an ODE with gradient constraints). Moreover, from this result we show that V is also a classical solution in the continuation region (the region between the two monotone curves). Furthermore, we improve the regularity of V by proving a second-order smooth fit property for the mixed derivative. Together with the structure of V, this allows us to derive a necessary system of non-linear functional equations for the free-boundaries, which coincides, in the case of decoupled dynamics, with that of Proposition 5.5 in [43]. In [43], due to the decoupled dynamics, this system can be derived by an analytical approach, while, in our setting, we employ the local-time-space calculus of Peskir [76] and properties of one-dimensional regular diffusions (see Borodin and Salminen [17]). Since our system is highly complex, a statement about uniqueness of the solution is far from trivial. In addition, we show that $V_{yxx} \neq 0$ at the free-boundaries, so that, by an application of the implicit function theorem, we can show that the free-boundaries are locally Lipschitz. In the case where X evolves as a drifted Brownian Motion, this property allows us to differentiate the system of necessary equations for the free-boundaries and to derive a system of (explicitly computable) first-order ODEs. Moreover, this implies that the free-boundaries are actually continuously differentiable with locally Lipschitz derivatives (see Theorem 4.30). To the best of our knowledge, in the context of a fully degenerate two-dimensional singular stochastic control problem with interconnected dynamics, this result appears here for the first time. Unfortunately, the question of uniqueness still remains open, since no initial condition for the system of ODEs is derived, differently from [64], where it is instead implied by the finite fuel condition. Finally, we also discuss the structure of the optimal control rule.

2 An Optimal Dividend Problem with Capital Injections over a Finite Horizon [5]

2.1 Problem Formulation

In this section we introduce the optimal dividend problem that is the object of our study. Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space rich enough to accommodate an $\mathbb{F}$-Brownian motion $W := (W_t)_{t\geq 0}$. We assume that the filtration $\mathbb{F}$ satisfies the usual conditions. We assume that the fund's value is described by the one-dimensional process

$$X^D_s(x) = x + \mu s + \sigma W_s - D_s + I^D_s, \quad s \geq 0,$$

where $x \geq 0$ is the initial value of the fund, $\mu \in \mathbb{R}$, $\sigma > 0$, and W is an $\mathbb{F}$-standard Brownian motion. For any $s \geq 0$, D_s represents the cumulative amount of dividends paid to shareholders up to time s, whereas I^D_s is the cumulative amount of capital injected by the shareholders up to time s in order to avoid bankruptcy of the fund.

Define the (nonempty) set

$$\mathcal{A} = \Big\{\nu : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ : \mathbb{F} - \text{adapted s.t. } s \mapsto \nu_s(\omega) \text{ is a.s.} \\ \text{non-decreasing and left-continuous, and } \nu_0 = 0 \text{ a.s.}\Big\}.$$

For fixed $x \geq 0$, we assume that the fund's manager can pick a dividends' distribution strategy among the processes $D \in \mathcal{A}$ and such that a.s.

$$D_{s+} - D_s \leq X^D_s(x) \quad \text{for all } s \geq 0; \tag{2.1}$$

that is, bankruptcy can not be obtained with a single lump sum dividend's payment. For any such dividend policy D, the capital injections process I^D is given as the minimal cumulative amount of capital needed to ensure that $X^D(x)$ stays non-negative, and which is flat off $\{t \geq 0 : X^D_t(x) = 0\}$. In particular, for $x \geq 0$, we take the couple $(X^D(x), I^D)$ as the unique solution to the (discontinuous) Skorokhod reflection problem (see, e.g., Chaleyat-Maurel et al. [23] and Ma [72]):

$$\text{Find}(X^D(x), I^D) \text{ s.t. } \begin{cases} I^D \in \mathcal{A}, \quad X^D_s(x) = x + \mu s + \sigma W_s - D_s + I^D_s, \quad s \geq 0, \\ X^D_s(x) \geq 0 \quad \text{a.s. for any } s \geq 0, \\ \int_0^\infty X^D_s(x) d(I^D_s)^c = 0 \quad \text{a.s.}, \\ \Delta I^D_s := I^D_{s+} - I^D_s = 2X^D_{s+}(x) \quad \forall s \in \{s \geq 0 : \Delta I^D_s > 0\}. \end{cases} \tag{2.2}$$

Here, $(I^D)^c$ denotes the continuous part of I^D. Notice that, given (2.1), the process

$$I^D_t := 0 \vee \sup_{0\leq s\leq t} \left(D_s - (x + \mu s + \sigma W_s)\right), \quad t \geq 0, \quad I^D_0 = 0,$$

[5] This Section is already published in a joint work with Giorgio Ferrari, see [47].

uniquely solves (2.2) and $t \mapsto I_t^D$ is continuous (see, e.g., Propositions 2 and 3 in [23], or Theorem 3.1 and Corollary 3.2 in [59]). As a consequence, the last condition in (2.2) is not binding, since $\Delta I_t^D = 0$ a.s. for all $t \geq 0$.

Given a time horizon $T \in (0, \infty)$ representing, e.g., a finite liquidation time, the fund's manager takes the point of view of the shareholders, and is faced with the problem of choosing a dividends' distribution strategy D maximizing the performance criterion

$$J(t, x; D) = \mathbb{E}\left[\int_0^{T-t} f(t+s)\ dD_s - \int_0^{T-t} m(t+s)\ dI_s^D + g(T, X_{T-t}^D(x))\right], \quad (2.3)$$

for $(t, x) \in [0, T] \times \mathbb{R}_+$ given and fixed. That is, the fund's manager aims at solving

$$V(t, x) := \sup_{D \in \mathcal{D}(t,x)} J(t, x; D), \quad (t, x) \in [0, T] \times \mathbb{R}_+. \quad (2.4)$$

Here, for any $(t, x) \in [0, T] \times \mathbb{R}_+$, $\mathcal{D}(t, x)$ denotes the class of dividend payments belonging to $\mathcal{A}$ and satisfying (2.1), when the surplus process X^D starts from level x and the optimization runs up to time $T - t$. In the following, any $D \in \mathcal{D}(t, x)$ will be called *admissible* for $(t, x) \in [0, T] \times \mathbb{R}_+$.

In the reward functional (2.3) the term $\mathbb{E}[\int_0^{T-t} f(t+s)\ dD_s]$ is the total expected cash-flow from dividends. The function f might be seen as a time-dependent instantaneous net proportion of leakages from the surplus received by the shareholders after time-dependent transaction costs/taxes have been paid. The term $\mathbb{E}[\int_0^{T-t} m(t+s)\ dI_s^D]$ gives the total expected costs of capital injections, and m is a time-dependent marginal administration cost for capital injections. Finally, $\mathbb{E}\left[g(T, X_{T-t}^D(x))\right]$ is a liquidation value.

The functions f, m, and g satisfy the following conditions.

Assumption 2.1. $f : [0, T] \to \mathbb{R}_+, m : [0, T] \to \mathbb{R}_+, g : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ *are continuous, f and m are continuously differentiable with respect to t, and g is continuously differentiable with respect to x. Moreover,*

(i) $g_x(T, x) \geq f(T)$ *for any* $x \in (0, \infty)$,

(ii) $m(t) > f(t)$ *for any* $t \in [0, T]$.

Remark 2.2. Requirement (i) ensures that the marginal liquidation value is at least as high as the marginal profits from dividends. This will ensure that the value function of the optimal stopping problem considered below is not discontinuous at terminal time. From an economic point of view, it means that the additional profit of an unit is at least as high as paying this unit as dividends, which is a plausible assumption.

Condition (ii) means that the marginal costs for capital injections are bigger than the marginal profits from dividends. Notice that in the case in which $m < f$ the value function might be infinite, as it shown in the next example. Take $f(s) = \eta$, $m(s) = \kappa$ for all $s \in [0, T]$, and $\eta > \kappa$. For arbitrary $\beta > 0$ consider the admissible strategy $\widehat{D}_s := \beta s$, and notice that $\widehat{I}_s^D = \sup_{0 \leq u \leq s}(-x - \mu u - \sigma B_u + \beta u) \vee 0$. Then $\widehat{I}_s^D \leq \beta s + Y_s$, with $Y_s := \sup_{0\le u\le s}(-x-\mu u-\sigma B_u)\vee 0$, and using that $g\ge 0$ we obtain for the sub-optimal strategy $\widehat{D}$

$$\begin{aligned} V(t,x) &\ge \beta\eta(T-t)-\beta\kappa(T-t)-\kappa\mathbb{E}[Y_{T-t}] \\ &= \beta(T-t)(\eta-\kappa)-\kappa\mathbb{E}[Y_{T-t}]. \end{aligned}$$

However, the latter expression can be made arbitrarily large by increasing β if $\eta>\kappa$.

On the other hand, by taking $m(t)=f(t)=e^{-rt}$, is has been recently shown in Ferrari [46] for a problem with $T=+\infty$ (see Theorem 3.8 therein) that an optimal control may not exist, but only an ε-optimal control does exist.

In order to avoid pathological situations as the ones described above, here we assume Assumption 2.1-(ii).

Remark 2.3. Notice that our formulation is general enough to accommodate also a problem in which profits and costs are discounted at a deterministic time-dependent discount rate $(r_s)_{s\ge0}$. Indeed, if we consider the optimal dividend problem with capital injections

$$\begin{aligned} \widehat{V}(t,x) := \sup_{D\in\mathcal{D}(t,x)} \mathbb{E}\bigg[\int_0^{T-t} e^{-\int_t^{t+s} r_\alpha d\alpha}\,\widehat{f}(t+s)\,dD_s - \int_0^{T-t} e^{-\int_t^{t+s} r_\alpha d\alpha}\,\widehat{m}(t+s)\,dI^D_s \\ + e^{-\int_t^T r_\alpha d\alpha}\widehat{g}(T,X^D_{T-t}(x))\bigg], \end{aligned}$$

then, for any $(t,x)\in[0,T]\times\mathbb{R}_+$ we can set

$$f(t) := e^{-\int_0^t r_\alpha d\alpha}\,\widehat{f}(t), \quad m(t) := e^{-\int_0^t r_\alpha d\alpha}\,\widehat{m}(t), \quad g(t,x) := e^{-\int_0^t r_\alpha d\alpha}\widehat{g}(t,x),$$

and $V(t,x) := e^{-\int_0^t r_\alpha d\alpha}\,\widehat{V}(t,x)$ is of the form (2.4).

In Section 2.4 we will consider a problem with constant marginal profits and costs discounted at a constant rate $r>0$ (see (2.77), (2.78) and (2.79) in Section 2.4).

Remark 2.4. Notice that in our model shareholders are *forced* to inject capital whenever the surplus process attempts to become negative; that is, the capital injection process is not a control variable of their, and shareholders do not choose when and how to invest in the company.

Injecting capital at the origin, under the condition that bankruptcy is not allowed, can be shown to be optimal in the canonical formulation of the optimal dividend problem of Section 2.4 in which marginal costs and profits are constants discounted at a constant interest rate. Indeed, in such a case, due to discounting, shareholders will inject capital as late as possible in order to minimize the total costs of capital injections. See also Kulenko and Schmidli [65] and Schmidli [86] for a similar result in stationary problems. More in general, the policy "inject capital at the origin" is optimal when m is decreasing and $\min_{t\in[0,T]} m(t) > g_x(T,x)$ for all $x\in\mathbb{R}_+$. Under these conditions, shareholders postpone injection of capital, and inject only as much capital as necessary since any additional capital injection can not be compensated by the reward at terminal time.

The dynamic programming equation for V takes the form of a parabolic partial differential equation (PDE) with gradient constraint, and with a Neumann boundary condition at $x = 0$ (the latter is due to the fact that the state process X is reflected at the origin through the capital injections process). Indeed, it reads

$$\max\Big\{\partial_t V + \frac{1}{2}\sigma^2 \partial_{xx} V + \mu \partial_x V, f - \partial_x V\Big\} = 0, \quad \text{on} \quad [0,T) \times (0,\infty),$$

with boundary conditions $\partial_x V(0,t) = m(t)$ for all $t \in [0,T]$, and $V(T,x) = g(T,x)$ for any $x \in (0,\infty)$. Proving that such a PDE problem admits a solution that has enough regularity to characterize an optimal control is far from being trivial. Hence, a direct guess and verify approach will not work at this point.

In order to solve the optimal dividend problem (2.4) we then follow a different approach, and we relate (2.4) to an optimal stopping problem with absorbing condition at $x = 0$. This is obtained by borrowing arguments from the study of El Karoui and Karatzas in [40] on the connection between reflected follower problems and questions of optimal stopping (see also Baldursson [7] and Karatzas and Shreve [59]). However, differently to [40], in our performance criterion (2.3) we also have a cost of reflection which requires a careful and not immediate adaptation of the ideas and results of [40].

In particular, introducing a problem of optimal stopping with absorption at the origin, we show that a proper integration of the value function of the latter leads to the value function of the optimal control problem (2.4). This result is stated in the next section, and then proved in Section 2.3.

2.2 The Main Result

Let $S(x) := \inf\{s \geq 0 : x + \mu s + \sigma W_s = 0\}$, $x \geq 0$, and for any $s \geq 0$, introduce the absorbed drifted Brownian motion

$$A_s(x) := \begin{cases} x + \mu s + \sigma W_s, & s < S(x), \\ \Delta, & s \geq S(x), \end{cases} \tag{2.5}$$

where Δ is a cemetery state isolated from $\mathbb{R}_+$ (i.e. $\Delta < 0$).

Introducing the convention $g_x(T, \Delta) := 0$, for $(t,x) \in [0,T] \times \mathbb{R}_+$, consider the optimal stopping problem

$$\begin{aligned} u(t,x) := & \sup_{\tau \in [0,T-t]} \mathbb{E}\Big[f(t+\tau)\mathbb{1}_{\{\tau < (T-t)\wedge S(x)\}} + m(t+S(x))\mathbb{1}_{\{\tau \geq S(x)\}} \\ & \qquad + g_x\big(T, x + \mu(T-t) + \sigma W_{T-t}\big)\mathbb{1}_{\{\tau = T-t < S(x)\}}\Big] \\ = & \sup_{\tau \in \mathcal{T}(T-t)} \mathbb{E}\Big[f(t+\tau)\mathbb{1}_{\{A_\tau(x) > 0\}}\mathbb{1}_{\{\tau < T-t\}} + m(t+S(x))\mathbb{1}_{\{A_\tau(x) \leq 0\}} \\ & \qquad + g_x\big(T, A_{T-t}(x)\big)\mathbb{1}_{\{\tau = T-t\}}\Big], \end{aligned} \tag{2.6}$$

where $\mathcal{T}(T-t)$ denotes the set of all $\mathbb{F}$-stopping times with values in $[0, T-t]$ a.s. Problem (2.6) is an optimal stopping problem for the absorbed process A.

To establish the relation between (2.4) and (2.6) we need the following *structural assumption*, which will be standing in this section and in Section 2.3. Its validity has to be verified on a case by case basis. In particular, it holds in the optimal dividend problem considered in Section 2.4.

Assumption 2.5. *Assume that the continuation region of the stopping problem* (2.6) *is given by*

$$\begin{aligned}\mathcal{C} &:= \{(t,x) \in [0,T) \times (0,\infty) : u(t,x) > f(t)\} \\ &= \{(t,x) \in [0,T) \times (0,\infty) : x < b(t)\}, \end{aligned} \tag{2.7}$$

and that its stopping region by

$$\begin{aligned}\mathcal{S} &:= \{(t,x) \in [0,T) \times (0,\infty) : u(t,x) \leq f(t)\} \cup \big(\{T\} \times (0,\infty)\big) \\ &= \{(t,x) \in [0,T) \times (0,\infty) : x \geq b(t)\} \cup \big(\{T\} \times (0,\infty)\big),\end{aligned}$$

for a continuous function $b : [0,T) \to (0,\infty)$*. We refer to the function* b *as to the optimal stopping boundary of problem* (2.6)*. Further, assume that the stopping time*

$$\tau^\star(t,x) := \inf\{s \in [0,T-t) : A_s(x) \geq b(t+s)\} \wedge (T-t) \tag{2.8}$$

(with the usual convention $\inf \emptyset = +\infty$*) is optimal; that is,*

$$\begin{aligned}u(t,x) = \mathbb{E}\Big[& f(t+\tau^\star(t,x))\mathbb{1}_{\{\tau^\star(t,x)<(T-t)\wedge S(x)\}} + m(t+S(x))\mathbb{1}_{\{\tau^\star(t,x)\geq S(x)\}} \\ & + g_x(T, x+\mu(T-t)+\sigma W_{T-t})\mathbb{1}_{\{\tau^\star(t,x)=T-t<S(x)\}}\Big].\end{aligned} \tag{2.9}$$

For any $(t,x) \in [0,T]\times\mathbb{R}_+$, and with b the optimal stopping boundary of problem (2.6) (cf. Assumption 2.5), we define the processes $I^\star(t,x)$ and $D^\star(t,x)$ through the system

$$\begin{cases} D^\star_s(t,x) := \max\left\{0, \max\limits_{0\leq\theta\leq s}\big(x+\mu\theta+\sigma W_\theta + I^\star_\theta(t,x) - b(t+\theta)\big)\right\}, \\ I^\star_s(t,x) := \max\left\{0, \max\limits_{0\leq\theta\leq s}\big(-x-\mu\theta-\sigma W_\theta + D^\star_\theta(t,x)\big)\right\}, \end{cases} \tag{2.10}$$

for any $s \in [0,T-t]$, and with initial values $D^\star_0(t,x) = I^\star_0(t,x) = 0$ a.s. The existence and uniqueness of the solution to system (2.10) can be proved by an application of Tarski's fixed point theorem following arguments as those employed by Karatzas in the proof of Proposition 7 in Section 8 of [57]. It can be easily shown from (2.10) and the positivity of b that $D^\star$ satisfies (2.1), and, consequently, that $I^\star$ has continuous paths. The latter property of $I^\star$ implies that $t \mapsto D^\star_t$ is continuous apart for a possible initial jump at time zero of amplitude $(x-b(t))^+$. We can now state the following result.

Theorem 2.6. *Let Assumption 2.5 hold. Then, the process* $D^\star$ *defined through* (2.10) *provides the optimal dividends' distribution policy, and the value function* V *of* (2.4) *is such that*

$$V(t,x) = V(t,b(t)) - \int_x^{b(t)} u(t,y)\, dy, \quad (t,x) \in [0,T]\times\mathbb{R}_+. \tag{2.11}$$

Assume further that $\lim_{t\uparrow T} b(t) =: b(T) < \infty$. *Then*

$$\begin{aligned} V(t,b(t)) = &-\mu \int_0^{T-t} f'(t+s)s\ ds + \int_0^{T-t} f'(t+s)b(t+s)\ ds \\ &+ g(T,b(T)) + f(T)\mu(T-t) + f(t)b(t) - f(T)b(T). \end{aligned} \tag{2.12}$$

Consistently with the result of El Karoui and Karatzas in [40] (see also Karatzas and Shreve [59]), we find that also in our problem with costly reflection at the origin the value of an optimal stopping problem (namely, problem (2.6)) gives the marginal value of the value function (2.4). The optimal stopping boundary b thus triggers the timing at which it is optimal to pay an additional unit of dividends. Moreover, once the optimal stopping value function u and its corresponding free-boundary b are known, (2.11) and (2.12) provide a complete characterization of the optimal dividend problem's value function V. Notice that the condition $b(T) < \infty$ is satisfied in the case study of Section 2.4, where we actually prove that $b(T) = 0$. The proof of Theorem 2.6 is quite lengthy and technical, and it is relegated to Section 2.3.

2.3 On the Proof of Theorem 2.6

This section is entirely devoted to the proof of Theorem 2.6. This is done through a series of intermediate results which are proved by employing mostly probabilistic arguments. Assumption 2.5 will be standing throughout this section.

2.3.1 On a Representation of the Optimal Stopping Value Function

Here we derive an alternative representation for the value function of the optimal stopping problem (2.6), by borrowing ideas from El Karoui and Karatzas [40], Section 3. In the following we set $g_x(T,\Delta) = 0$.

The idea that we adopt here is to rewrite the optimal stopping problem (2.6) in terms of the function b of Assumption 2.5. To accomplish that, for given $(t,x) \in [0,T] \times \mathbb{R}_+$, define the payoff associated to the admissible stopping rule "never stop" as

$$G(t,x) := \mathbb{E}\left[m(t+S(x))\mathbb{1}_{\{S(x)\le T-t\}} + g_x(T, A_{T-t}(x))\right], \tag{2.13}$$

where we have used that $g_x(T,A_{T-t}(x))\mathbb{1}_{\{T-t<S(x)\}} = g_x(T,A_{T-t}(x))$ because of (2.5) and the fact that $g_x(T,\Delta) = 0$.

Also, introduce the function $\tilde{g} : [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ (depending parametrically on t) as

$$\tilde{g}(\alpha,q,y;t) := \begin{cases} g_x(T,y), & \alpha < q, \\ m(t+q), & \alpha \ge q, \end{cases} \tag{2.14}$$

and notice that $v := u - G$ admits the representation

$$v(t,x) = \sup_{\tau\in\mathcal{T}(T-t)} \mathbb{E}\left[\left(f(t+\tau) - \tilde{g}(T-t,S(x),A_{T-t}(x);t)\right)\mathbb{1}_{\{\tau<S(x)\wedge T-t\}\}}\right]. \tag{2.15}$$

Clearly, the stopping time $\tau^\star$ defined by (2.8) is also optimal for v since G is independent of $\tau \in \mathcal{T}(T-t)$. Therefore, we can expect that v can be expressed in terms of the optimal stopping boundary b. Following [40], we obtain such a representation for v by means of the theory of dual previsible projections ("balayée prévisible"), as it is shown in the following. From now on, $(t,x) \in [0,T] \times \mathbb{R}_+$ will be given and fixed.

We define the process $(C_\alpha)_{\alpha\in[0,T]}$ such that for any $\alpha \in [0, T-t]$

$$\begin{aligned}C_\alpha(t,x) :=& -\int_0^{\alpha\wedge S(x)\wedge T-t} f'(t+\theta)d\theta \\ &+ \big[f(T\wedge(t+S(x))) - \tilde{g}(T-t, S(x), A_{T-t}(x);t)\big]\mathbb{1}_{\{0<T-t\wedge S(x)\leq\alpha\}},\end{aligned} \tag{2.16}$$

as well as the stopping time

$$\sigma_\alpha(t,x) := \inf\{\theta \in [\alpha, T-t) : A_\theta(x) \geq b(t+\theta)\} \wedge (T-t), \tag{2.17}$$

with the convention $\inf\emptyset = +\infty$. The process $C_\cdot(t,x)$ is absolutely continuous on $[0,T-t)\wedge S(x)$ with a possible jump at $(T-t)\wedge S(x)$, and $\alpha \mapsto \sigma_\alpha(t,x)$ is a.s. non-decreasing and right-continuous.

Since the stopping time $\sigma_0(t,x)$ is optimal for $u(t,x)$ by Assumption 2.5, and therefore also for $v(t,x) = (u-G)(t,x)$, by using (2.16) we can write from (2.15)

$$v(t,x) = \mathbb{E}\left[C_{T-t}(t,x) - C_{\sigma_0(t,x)}(t,x)\right] = \mathbb{E}\Big[\widetilde{C}_{T-t}(t,x)\Big], \tag{2.18}$$

where we have introduced

$$\widetilde{C}_\alpha(t,x) := C_{\sigma_\alpha(t,x)}(t,x) - C_{\sigma_0(t,x)}(t,x), \quad \alpha \in [0,T-t]. \tag{2.19}$$

The process $\widetilde{C}_\cdot(t,x)$ is of bounded variation, since it is the composition of the process of bounded variation $C_\cdot(t,x)$ and of the non-decreasing process $\sigma_\cdot(t,x)$, but it is not $\mathbb{F}$-adapted. However, being v an excessive function, it is also the potential of an adapted, non-decreasing process $\Theta_\cdot(t,x)$ (cf. Section IV.4 in the book of Blumenthal and Getoor [16]), which is the dual predictable (or previsible) projection of $\widetilde{C}_\cdot(t,x)$ (see, e.g., Theorem 21.1 in Chapter VI of the book by Rogers and Williams [83] for further details on the dual predictable projection). In the following we provide the explicit representation of $\Theta_\cdot(t,x)$. This is obtained by employing the methodology of El Karoui and Karatzas in [41], Section 7.

Theorem 2.7. *The dual predictable projection $\Theta(t,x)$ of $\widetilde{C}(t,x)$ exists, is non-decreasing and it is given by*

$$\begin{aligned}\Theta_\alpha(t,x) =& \int_0^\alpha -f'(t+\theta)\mathbb{1}_{\{A_\theta(x)>b(t+\theta)\}}\,d\theta \\ &+ \Big[f(T\wedge(t+S(x))) - \tilde{g}(T-t,S(x),A_{T-t}(x);t)\Big]\mathbb{1}_{\{A_{T-t}(x)>b(T)\}}\mathbb{1}_{\{0<T-t\wedge S(x)\leq\alpha\}} \\ =& \int_0^{\alpha\wedge S(x)} -f'(t+\theta)\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta>b(t+\theta)\}}\,d\theta \\ &+ \Big[f(T\wedge(t+S(x))) - \tilde{g}(T-t,S(x),A_{T-t}(x);t)\Big]\mathbb{1}_{\{A_{T-t}(x)>b(T)\}}\mathbb{1}_{\{0<T-t\wedge S(x)\leq\alpha\}}\end{aligned} \tag{2.20}$$

for any $\alpha \in [0,T-t]$.

Theorem 2.7 can be proved by carefully adapting to our case the techniques presented in Section 7 of [41] (see also, Section 3 of [40]). In particular, differently to Section 7 of [41], here we deal with an absorbed drifted Brownian motion as a state variable of the optimal stopping problem (2.6) (instead of a Brownian motion). However, all the arguments and proofs of Section 7 of [41] carry over also to our setting with random time horizon $(T-t) \wedge S(x)$ (up to which the process A is in fact a drifted Brownian motion) upon using representation (2.15) of v (in which the function $\tilde{g}$ takes care of the random time horizon $(T-t) \wedge S(x)$) together with (2.17) and (2.19).

A consequence of Theorem 2.7 is the next result.

Corollary 2.8. *It holds that*

(i) $\big[f(T \wedge (t+S(x))) - \tilde{g}(T-t, S(x), A_{T-t}(x); t)\big]\mathbb{1}_{\{A_{T-t}(x) > b(T)\}} = 0$ *a.s.*

(ii) $\{t \in [0,T) : f'(t) \leq 0\} \supseteq \mathcal{S}$*;*

Proof. (i) On the set $\{A_{T-t}(x) > b(T)\}$ we obtain by the definition of $\tilde{g}$ (see (2.14)) that

$$f(T \wedge (t+S(x))) - \tilde{g}(T-t, S(x), A_{T-t}(x); t) = f(T) - g_x(T, A_{T-t}(x)). \quad (2.21)$$

Since $\Theta_{\cdot}(t,x)$ is non-decreasing, the last term in (2.21) has to be positive, thus implying $f(T) - g_x(T, A_{T-t}(x)) \geq 0$ on $\{A_{T-t}(x) > b(T)\}$. However, by Assumption 2.1-(i) one has $f(T) \leq g_x(T,x)$ for all $x \in (0,\infty)$. Hence the claim follows.

(ii) Since $\alpha \mapsto \Theta_\alpha(t,x)$ is a.s. non-decreasing, it follows from (i) above and (2.20) that $f'(t+\theta)\mathbb{1}_{\{A_\theta(x) > b(t+\theta)\}} \leq 0$ a.s. for a.e. $\theta \in [0, T-t]$. But $f'(\cdot)$, $A_{\cdot}(x)$ and $b(t+\cdot)$ are continuous up to $(T-t) \wedge S(x)$, and therefore the latter actually holds a.s. for all $\theta \in [0, T-t]$. Hence, $\{t \in [0,T) : f'(t) \leq 0\} \supseteq \mathcal{S}$. ☐

Remark 2.9. As a byproduct of Corollary 2.8-(i) (see in particular (2.21)), Assumption 2.1-(i), and of the fact that $A_{T-t}(x)$ has a transition probability that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$ (cf. (A.4)), one has $\big(f(T) - g_x(T,y)\big)\mathbb{1}_{\{y > b(T)\}} = 0$ for $y \geq 0$.

We can now obtain an alternative representation of the value function u of problem (2.6).

Theorem 2.10. *For any* $(t,x) \in [0,T] \times \mathbb{R}_+$ *one has*

$$\begin{aligned} u(t,x) = \mathbb{E}\bigg[\int_0^{(T-t)\wedge S(x)} & -f'(t+\theta)\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta \geq b(t+\theta)\}}\, d\theta \\ & + m(t+S(x))\mathbb{1}_{\{S(x) \leq T-t\}} + g_x(T, A_{T-t}(x))\bigg]. \end{aligned} \quad (2.22)$$

Proof. Since by Theorem 2.7 $\Theta(t,x)$ is the dual predictable projection of $\widetilde{C}(t,x)$, from (2.18) we can write for any $(t,x) \in [0,T] \times \mathbb{R}_+$

$$v(t,x) = \mathbb{E}\left[\widetilde{C}_{T-t}(t,x)\right] = \mathbb{E}\left[\Theta_{T-t}(t,x)\right]. \quad (2.23)$$

Due to (2.20) and Corollary 2.8-(i), (2.23) gives

$$v(t,x) = \mathbb{E}\left[\int_0^{(T-t)\wedge S(x)} -f'(t+\theta)\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta \geq b(t+\theta)\}}\, d\theta\right]. \tag{2.24}$$

Here we have also used that the joint law of $S(x)$ and of the drifted Brownian motion is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^2$ (cf. (A.2)) to replace $\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta > b(t+\theta)\}}$ with $\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta \geq b(t+\theta)\}}$ inside the expectation in (2.20).

However, since by definition $v = u - G$, we obtain from (2.24) and (2.13) the alternative representation

$$u(t,x) = v(t,x) + G(t,x) = \mathbb{E}\bigg[\int_0^{(T-t)\wedge S(x)} -f'(t+\theta)\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta \geq b(t+\theta)\}}\, d\theta$$
$$+ m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}} + g_x(T, A_{T-t}(x))\bigg].$$

□

Remark 2.11. Notice that representation (2.22) coincides with that one might obtain by an application of Itô's formula if u were $C^{1,2}([0,T)\times(0,\infty))\cap C([0,T]\times\mathbb{R}_+)$, and satisfies (as it is customary in optimal stopping problems) the free-boundary problem

$$\begin{cases} \partial_t u + \frac{1}{2}\sigma^2\partial^2_{xx}u + \mu\partial_x u = 0, & 0 < x < b(t),\ t\in[0,T) \\ u = f, & x \geq b(t),\ t\in[0,T) \\ u(T,x) = g_x(T,x), & x > 0 \\ u(t,0) = m(t), & t\in[0,T]. \end{cases} \tag{2.25}$$

Indeed, in such a case an application of Dynkin's formula gives

$$\mathbb{E}\left[u(t+(T-t)\wedge S(x), Z_{(T-t)\wedge S(x)}(x))\right]$$
$$= u(t,x) + \mathbb{E}\left[\int_0^{(T-t)\wedge S(x)} f'(t+\theta)\mathbb{1}_{\{Z_\theta(x)\geq b(t+\theta)\}}\, d\theta\right],$$

where we have set $Z_s(x) := x + \mu s + \sigma W_s$, $s \geq 0$, to simplify exposition. Hence, using (2.25) we have from the latter

$$u(t,x) = \mathbb{E}\bigg[m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}} + g_x(T, x+\mu(T-t)+\sigma W_{T-t})\mathbb{1}_{\{S(x)>T-t\}}$$
$$- \int_0^{(T-t)\wedge S(x)} f'(t+\theta)\mathbb{1}_{\{Z_\theta(x)\geq b(t+\theta)\}}\, d\theta\bigg] = \mathbb{E}\bigg[m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}}$$
$$+ g_x(T, A_{T-t}(x))\mathbb{1}_{\{S(x)>T-t\}} - \int_0^{(T-t)\wedge S(x)} f'(t+\theta)\mathbb{1}_{\{Z_\theta(x)\geq b(t+\theta)\}}\, d\theta\bigg]$$
$$= \mathbb{E}\bigg[m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}} + g_x(T, A_{T-t}(x))$$
$$- \int_0^{(T-t)\wedge S(x)} f'(t+\theta)\mathbb{1}_{\{Z_\theta(x)\geq b(t+\theta)\}}\, d\theta\bigg],$$

where in the last step we have used that $g_x(T, A_{T-t}(x))\mathbb{1}_{\{S(x)>T-t\}} = g_x(T, A_{T-t}(x))$ because of (2.5) and the fact that $g_x(T, \Delta) = 0$.

Remark 2.12. Notice that the representation (2.22) immediately gives an integral equation for the optimal stopping boundary b. Indeed, since (2.22) holds for any $(t, x) \in [0, T] \times \mathbb{R}_+$, by taking $x = b(t)$, $t \leq T$, on both sides of (2.22), and by recalling that $u(t, b(t)) = f(t)$, we find that b solves

$$f(t) = \mathbb{E}\bigg[\int_0^{(T-t)\wedge S(b(t))} -f'(t+\theta)\mathbb{1}_{\{b(t)+\mu\theta+\sigma W_\theta \geq b(t+\theta)\}}\, d\theta \\ + m(t + S(b(t)))\mathbb{1}_{\{S(b(t))\leq T-t\}} + g_x(T, A_{T-t}(b(t)))\bigg]. \tag{2.26}$$

By adapting arguments as those in Section 25 of Peskir and Shiryaev [77], based on the superharmonic characterization of u, one might then prove that b is the unique solution to (2.26) among a suitable class of continuous and positive functions.

The next result follows from (2.22) by expressing the expected value as an integral with respect to the probability densities of the involved processes and random variables. Its proof can be found in the Appendix for the sake of completeness.

Corollary 2.13. *The function $u(t, \cdot)$ is continuously differentiable on $(0, \infty)$ for all $t \in [0, T)$.*

In the next section we will suitably integrate the two alternative representations of u (2.9) and (2.22) with respect to the space variable, and we will show that such integrations give the value function (2.4) of the optimal dividend problem. As a byproduct, we will also obtain the optimal dividend strategy $D^\star$.

2.3.2 Integrating the Optimal Stopping Value Function

In the next two propositions we integrate with respect to the space variable the two representations of u given by (2.9) and (2.22). The proofs will employ pathwise arguments. However, in order to simplify exposition, we will not stress the ω-dependence of the involved random variables and processes.

Proposition 2.14. *Let b the optimal stopping boundary of problem (2.6), recall*

$$I^0_s(x) = \max_{0\leq\theta\leq s}\{-x - \mu\theta - \sigma W_\theta\} \vee 0, \quad s \geq 0,$$

and define

$$R_s(x) := x + \mu s + \sigma W_s + I^0_s(x), \quad s \geq 0. \tag{2.27}$$

Then for any $(t, x) \in [0, T] \times \mathbb{R}_+$ one has

$$\int_x^{b(t)} u(t, y)\, dy = N(t, b(t)) - N(t, x), \tag{2.28}$$

where

$$N(t,x) := \mathbb{E}\Big[-\int_0^{T-t} \big(R_s(x) - b(t+s)\big)^+ f'(t+s)\ ds - \int_0^{T-t} m(t+s)\ dI^0_s(x) \\ + g(T, R_{T-t}(x))\Big]. \tag{2.29}$$

Proof. To prove (2.28) we use representation (2.22) of the value function of the optimal stopping problem (2.6). Using Fubini-Tonelli's Theorem we obtain

$$\int_x^{b(t)} u(t,y)\ dy = \int_x^{b(t)} \mathbb{E}\Big[\int_0^{(T-t)\wedge S(y)} -f'(t+s)\mathbb{1}_{\{y+\mu s+\sigma W_s \geq b(t+s)\}}\ ds \\ + m(t+S(y))\mathbb{1}_{\{S(y)\leq T-t\}} + g_x(T, A_{T-t}(y))\Big]\ dy \\ = \mathbb{E}\Big[-\int_0^{(T-t)} f'(t+s)\Big(\int_x^{b(t)} \mathbb{1}_{\{y+\mu s+\sigma W_s\geq b(t+s)\}}\mathbb{1}_{\{s\leq S(y)\}}\ dy\Big)\ ds \tag{2.30} \\ + \int_x^{b(t)} m(t+S(y))\mathbb{1}_{\{S(y)\leq T-t\}}\ dy + \int_x^{b(t)} g_x(T, A_{T-t}(y))\ dy\Big].$$

In the following we investigate separately the three summands of the last term on the right-hand side of (2.30).

Recalling $S(x) = \inf\{u \geq 0 : x + \mu u + \sigma W_u = 0\}$ it is clear that

$$S(y) \geq s \Leftrightarrow M_s \leq y \tag{2.31}$$

for any $(s, y) \in \mathbb{R}_+ \times (0, \infty)$, where we have defined

$$M_s := \max_{0\leq\theta\leq s}(-\mu\theta - \sigma W_\theta), \quad s \geq 0. \tag{2.32}$$

We can then rewrite (2.27) in terms of (2.32) and obtain

$$R_s(x) = (x \vee M_s) + \mu s + \sigma W_s, \quad s \geq 0. \tag{2.33}$$

By using (2.31) we find

$$\int_x^{b(t)} \mathbb{1}_{\{y+\mu s+\sigma W_s\geq b(t+s)\}}\mathbb{1}_{\{S(y)\geq s\}}\ dy = \int_{x\vee\big[b(t+s)-\mu s-\sigma W_s\big]}^{b(t)\vee\big[b(t+s)-\mu s-\sigma W_s\big]} \mathbb{1}_{\{S(y)\geq s\}}\ dy \\ = \int_{x\vee\big[b(t+s)-\mu s-\sigma W_s\big]}^{b(t)\vee\big[b(t+s)-\mu s-\sigma W_s\big]} \mathbb{1}_{\{M_s\leq y\}}\ dy \\ = \big[(b(t) \vee (b(t+s) - \mu s - \sigma W_s) \vee M_s) - (x \vee (b(t+s) - \mu s - \sigma W_s) \vee M_s)\big] \\ = \big[(b(t) \vee M_s) \vee (b(t+s) - \mu s - \sigma W_s) - (x \vee M_s) \vee (b(t+s) - \mu s - \sigma W_s)\big] \\ = \big[\big([(b(t) \vee M_s) + \mu s + \sigma W_s] \vee b(t+s)\big) - \big([(x \vee M_s) + \mu s + \sigma W_s] \vee b(t+s)\big)\big] \\ = \big[\big(R_s(b(t)) \vee b(t+s)\big) - \big(R_s(x) \vee b(t+s)\big)\big] \\ = \big[\big(R_s(b(t)) - b(t+s)\big)^+ - \big(R_s(x) - b(t+s)\big)^+\big]. \tag{2.34}$$

For the third summand of the last term of the right-hand side of (2.30) we have, due to the fact that $g_x(T,\Delta)=0$,

$$\begin{aligned}\int_x^{b(t)} g_x(T, A_{T-t}(y))dy &= \int_x^{b(t)} g_x(T, y+\mu(T-t)+\sigma W_{T-t})\mathbb{1}_{\{S(y)>T-t\}}dy \\ &= \int_x^{b(t)} g_x(T, y+\mu(T-t)+\sigma W_{T-t})\mathbb{1}_{\{M_{T-t}<y\}}dy \qquad (2.35)\\ &= \int_{x\vee M_{T-t}}^{b(t)\vee M_{T-t}} g_x(T, y+\mu(T-t)+\sigma W_{T-t})dy \\ &= g(T, R_{T-t}(b(t))) - g(T, R_{T-t}(x)),\end{aligned}$$

where in the last step we use (2.33). To prove that

$$\begin{aligned}&\int_x^{b(t)} m(t+S(y))\mathbb{1}_{\{S(y)\le T-t\}}dy \\ &= \int_0^{T-t} m(t+s)dI^0_s(x) - \int_0^{T-t} m(t+s)dI^0_s(b(t))\end{aligned} \qquad (2.36)$$

we have to distinguish two cases. In the following we let $(t,x)\in[0,T]\times\mathbb{R}_+$ be given and fixed, and we prove (2.36) by taking $x<b(t)$. The arguments are exactly the same if $b(t)<x$ by reversing the roles of x and $b(t)$.

Case 1. Here we take $x\in\{y\in\mathbb{R}_+ : S(y)\ge T-t\}$; that is, the initial point $x>0$ is such that the drifted Brownian motion is not reaching 0 before the time horizon. This implies that $R_s(x)$ in (2.27) equals $x+\mu s+\sigma W_s$ and so $I^0_s(x)=0$ for all $s\in[0,T-t]$. Hence, we can write

$$\begin{aligned}\int_x^{b(t)} m(t+S(y))\mathbb{1}_{\{S(y)\le T-t\}}dy = 0 = &\int_0^{T-t} m(t+s)dI^0_s(x) \\ &- \int_0^{T-t} m(t+s)dI^0_s(b(t)),\end{aligned} \qquad (2.37)$$

where we have used that $S(y)>S(x)\ge T-t$ for any $y>x$ and $\{x\}$ has zero Lebesgue measure to obtain the first equality, and the fact that $0=I^0_s(x)\ge I^0_s(b(t))\ge 0$ since $x<b(t)$.

Case 2. Here we take $x\in\{y\in\mathbb{R}_+ : S(y)<T-t\}$; i.e., the drifted Brownian motion reaches 0 before the time horizon. Define

$$z := \inf\{y\in\mathbb{R}_+ : S(y)\ge T-t\}, \qquad (2.38)$$

with the usual convention $\inf\emptyset=+\infty$. In the sequel we assume that $z<+\infty$, since otherwise there is no need for the following analysis to be performed. Note that, by continuity in time and in the initial datum of the paths of the drifted Brownian motion, we have $S(z)\le T-t$. Furthermore, it holds for all $y\in[x,z]$ that (cf. (2.32))

$$y+I^0_s(y)=M_s, \quad \forall s\ge S(y), \qquad (2.39)$$

$$I^0_s(y) = 0, \quad \forall s < S(y). \tag{2.40}$$

Using (2.39), (2.40), (2.31), and the change of variable formula in Section 4 of Chapter 0 of the book by Revuz and Yor [82] (see also equation (4.7) in Baldursson and Karatzas [8]) we obtain

$$\begin{aligned}
&\int_x^{z\wedge b(t)} m(t+S(y))\mathbb{1}_{\{S(y)\le T-t\}}dy = \int_x^{z\wedge b(t)} m(t+S(y))dy \\
&= \int_{S(x)}^{S(z\wedge b(t))} m(t+s)dM_s = \int_{S(x)}^{S(z\wedge b(t))} m(t+s)\left(dI^0_s(x) - dI^0_s(z\wedge b(t))\right) \qquad (2.41)\\
&= \int_0^{T-t} m(t+s)\left(dI^0_s(x) - dI^0_s(z\wedge b(t))\right) \\
&= \int_0^{T-t} m(t+s)dI^0_s(x) - \int_0^{T-t} m(t+s)dI^0_s(z\wedge b(t)).
\end{aligned}$$

For the integral $\int_{z\wedge b(t)}^{b(t)} m(t+S(y))\mathbb{1}_{\{S(y)\le T-t\}}\, dy$ we can use the result of Case 1 due to the definition of z (2.38). Then, combining (2.37) and (2.41) leads to (2.36).
Ù
By (2.34), (2.35) and (2.36), and recalling (2.29) and (2.30) we obtain (2.28). □

Proposition 2.15. *Let $(D^\star, I^\star)$ be the solution to system (2.10). Then, for any $(t,x) \in [0,T]\times\mathbb{R}_+$ one has*

$$\int_x^{b(t)} u(t,y)\, dy = M(t,b(t)) - M(t,x), \tag{2.42}$$

where b is the optimal stopping boundary of problem (2.6) and

$$\begin{aligned}
M(t,x) := \mathbb{E}\bigg[\int_0^{T-t} f(t+s)\, dD^\star_s(t,x) - \int_0^{T-t} m(t+s)\, dI^\star_s(t,x) \\
+ g(T, X^{D^\star}_{T-t}(x))\bigg]. \qquad (2.43)
\end{aligned}$$

Proof. For this proof we use instead the representation of u (cf. (2.9))

$$\begin{aligned}
u(t,x) = \mathbb{E}\Big[&f(t+\tau^\star(t,x))\mathbb{1}_{\{\tau^\star(t,x)<T-t\wedge S(x)\}} + m(t+S(x))\mathbb{1}_{\{\tau^\star(t,x)\ge S(x)\}} \\
&+ g_x(T, A_{T-t}(x))\mathbb{1}_{\{\tau^\star(t,x)=T-t<S(x)\}}\Big].
\end{aligned}$$

The proof is quite long and technical and it is organized in four steps. Moreover, in order to simplify exposition from now we set $t = 0$. Indeed, all the following arguments remain valid if $t \in (0,T]$ by obvious modifications.

If $x \ge b(0)$, then (2.42) clearly holds. Indeed, $\int_x^{b(0)} u(0,y)\, dy = -(x-b(0))f(0)$ since $\tau^\star(0,y) = 0$ for any $y \ge b(0)$. Also, from (2.43) $M(0,b(0)) - M(0,x) = M(0,b(0)) - \left[(x-b(0))f(0) + M(0,b(0))\right]$, since $D^\star(0,x)$ has an initial jump of size

$(x-b(0))$ which is such that $X_{0+}^{D^\star}(x)=b(0)$. Hence, in the following we prove (2.42) assuming that $x<b(0)$.

Step 1. Here we take $x\in\{y\in\mathbb{R}_+ : \tau^\star(0,y)<S(y)\}$; that is, the initial point $x>0$ is such that either the drifted Brownian motion reaches the boundary before hitting the origin, or the time horizon arises before hitting the origin. Define the process $(L_s)_{s\geq 0}$ such that

$$L_s := \max_{0\leq\theta\leq s}\{\mu\theta+\sigma W_\theta - b(\theta)\}, \qquad 0\leq s\leq T.$$

Then we have that for all $y\in[x,b(0)]$

$$\{\tau^\star(0,y)\leq s\}=\{L_s\geq -y\}, \tag{2.44}$$

$$\{\tau^\star(0,y)=T\}=\{L_T\leq -y\}, \tag{2.45}$$

$$D_s^\star(0,y)=\begin{cases}0, & 0\leq s\leq \tau^\star(0,y),\\ y+L_s, & \tau^\star(0,y)\leq s\leq S(y),\end{cases} \tag{2.46}$$

and

$$X_s^{D^\star}(y)=\begin{cases}y+\mu s+\sigma W_s, & 0\leq s\leq\tau^\star(0,y),\\ \mu s+\sigma W_s - L_s, & \tau^\star(0,y)\leq s\leq S(y),\end{cases}$$

and in particular (cf. (2.10)) $I_s^\star(0,y)=I_s^\star(0,b(0))=0$ for any $s\in[0,\tau^\star(0,y)]$.

Moreover, it follows by definition of $\tau^\star(0,x)$, $S(x)$ and $X^{D^\star}(x)$ that for all $y\in[x,b(0)]$ we have

$$0=\tau^\star(0,b(0))\leq\tau^\star(0,y)\leq\tau^\star(0,x),$$

$$\tau^\star(0,y)<\tau^\star(0,x)<S(x)\leq S(y), \tag{2.47}$$

and

$$\text{on } \{\tau^\star(0,x)<T\}\text{:}\quad X_s^{D^\star}(y)=X_s^{D^\star}(x), \quad \forall s>\tau^\star(0,x). \tag{2.48}$$

With these results at hand, we now show that for all $x\in[0,b(0)]$ such that $\tau^\star(0,x)<S(x)$ it holds that

$$\int_x^{b(0)} f(\tau^\star(0,y))1_{\{\tau^\star(0,y)<S(y)\}}dy=\int_0^T f(s)\ dD_s^\star(0,b(0))-\int_0^T f(s)\ dD_s^\star(0,x), \tag{2.49}$$

$$\int_x^{b(0)} g_x(T,y+\mu T+\sigma W_T)\mathbb{1}_{\{\tau^\star(0,y)=T<S(y)\}}\ dy=g(T,X_T^{D^\star}(b(0)))-g(T,X_T^{D^\star}(x)) \tag{2.50}$$

and

$$\int_x^{b(0)} m(S(y))1_{\{\tau^\star(0,y)\geq S(y)\}}dy=\int_0^T m(s)\ dI_s^\star(0,x)-\int_0^T m(s)\ dI_s^\star(0,b(0)). \tag{2.51}$$

We start with (2.49). By (2.48) we have that $dD_s^\star(0,x)=dD_s^\star(0,b(0))$ for all $\tau^\star(0,x)<s\leq T$. By (2.46), and since $\tau^\star(0,b(0))=0$ one also has

$$D_s^\star(0,b(0))=b(0)+L_s,\ \forall s\in[0,S(b(0))]. \tag{2.52}$$

Hence the right-hand side of (2.49) rewrites as

$$\begin{aligned}
&\int_0^T f(s)\ dD^\star_s(0,b(0)) - \int_0^T f(s)\ dD^\star_s(0,x) = \int_0^{\tau^\star(0,x)} f(s)\ dD^\star_s(0,b(0)) \\
&- \int_0^{\tau^\star(0,x)} f(s)\ dD^\star_s(0,x) = \int_0^{\tau^\star(0,x)} f(s)\ dD^\star_s(0,b(0)) = \int_0^{\tau^\star(0,x)} f(s)\ dL_s,
\end{aligned} \tag{2.53}$$

where we have used that $dD^\star_s(0,x) = 0$ for all $s \in [0,\tau^\star(0,x)]$ by (2.46). However, by using a change of variable formula as in Baldursson and Karatzas [8], equation (4.7), we obtain

$$\int_x^{b(0)} f(\tau^\star(0,y))1_{\{\tau^\star(0,y)<S(y)\}}dy = \int_x^{b(0)} f(\tau^\star(0,y))dy = \int_0^{\tau^\star(0,x)} f(s)\ dL_s, \tag{2.54}$$

where we have used (2.47) in the first step, and the fact that $L_.$ is the left-continuous inverse of $\tau^\star(0,y)$ (cf. (2.44)) in the last equality. Combining (2.53) and (2.54) equation (2.49) holds.

Next we show (2.50). Using (2.52) and again (2.48) we obtain for the right-hand side of (2.50) that

$$\begin{aligned}
&g(T, X^{D^\star}_T(b(0))) - g(T, X^{D^\star}_T(x)) \\
&= [g(T,\mu T + \sigma W_T - L_T) - g(T, x + \mu T + \sigma W_T)]\, \mathbb{1}_{\{\tau^\star(0,x)=T\}}.
\end{aligned}$$

Also, (2.45) and (2.47) yields

$$\begin{aligned}
&\int_x^{b(0)} g_x(T, y + \mu T + \sigma W_T)\mathbb{1}_{\{\tau^\star(0,y)=T\}}\ dy = \int_x^{b(0)} g_x(T, y + \mu T + \sigma W_T)\mathbb{1}_{\{y \le -L_T\}}\ dy \\
&= [g(T,\mu T + \sigma W_T - L_T) - g(T, x + \mu T + \sigma W_T)]\, \mathbb{1}_{\{\tau^\star(0,x)=T\}}.
\end{aligned}$$

Hence, we obtain (2.50).

Finally, for (2.51) there is nothing to show. In fact, the left-hand side is equal 0 by (2.47), while the right-hand side is zero since the processes $I^\star(0,x) = I^\star(0,b(0))$ coincide (cf. (2.48)).

Step 2. Take $x \in \{y \in \mathbb{R}_+ : \tau^\star(0,y) > S(y),\ \tau^\star(0,q) < S(q)\ \forall q \in (y, b(0))\}$. For a realization like that, such an x is such that the drifted Brownian motion touches the origin before hitting the boundary, but it does not cross the origin. This in particular implies that $I^\star_s(0,x) = 0$ for all $s \le \tau^\star(0,x)$. Hence the same arguments employed in Step 1 hold true, and (2.49) – (2.51) follow.

Step 3. Here we take $x \in \{y \in \mathbb{R}_+ : \tau^\star(0,y) > S(y)\}$; that is, the drifted Brownian motion hits the origin before reaching the boundary.

Define

$$z := \inf\{y \in [0,b(0)] : \tau^\star(0,y) < S(y)\} \tag{2.55}$$

which exists finite since $y \mapsto \tau^\star(0,y) - S(y)$ is decreasing and $\tau^\star(0,b(0)) = 0$ and $S(0) = 0$ a.s. We want to prove that

$$\int_x^z m(S(y))\mathbb{1}_{\{\tau^\star(0,y)\geq S(y)\}}\, dy = \int_0^T m(s)\, dI_s^\star(0,x) - \int_0^T m(s)\, dI_s^\star(0,z), \quad (2.56)$$

$$\int_x^z f(\tau^\star(0,y))1_{\{\tau^\star(0,y)<S(y)\}}dy = \int_0^T f(s)\, dD_s^\star(0,z) - \int_0^T f(s)\, dD_s^\star(0,x), \quad (2.57)$$

and

$$\int_x^z g_x(T, y+\mu T+\sigma W_T)\mathbb{1}_{\{\tau^\star(0,y)=T<S(y)\}}\, dy$$
$$= \left[g(T, X_T^{D^\star}(z)) - g(T, X_T^{D^\star}(x))\right]. \quad (2.58)$$

Recall the process $(M_s)_{s\geq 0}$ of (2.32) such that

$$M_s = \max_{0\leq\theta\leq s}(-\mu\theta - \sigma W_\theta), \quad s \geq 0,$$

and (cf. (2.31))

$$\{M_s \geq x\} = \{S(x) \leq s\} \quad \forall s \geq 0.$$

For all $y \in [x,z)$ and $s \in [0, \tau^\star(0,y)]$ we have

$$I_s^\star(0,y) = (M_s - y)^+ = \begin{cases} 0, & 0 \leq t \leq S(y) \\ M_s - y, & S(y) \leq s \leq \tau^\star(0,y), \end{cases} \quad (2.59)$$

and

$$X_s^{D^\star}(y) = \begin{cases} y + \mu s + \sigma W_s, & 0 \leq s \leq S(y) \\ \mu s + \sigma W_s + M_s, & S(y) \leq s \leq \tau^\star(0,y), \end{cases} = (y \vee M_s) + \mu s + \sigma W_s. \quad (2.60)$$

Also, it follows by (2.60) and (2.59) that for all $y \in [x,z)$

$$X_s^{D^\star}(y) = X_s^{D^\star}(z)\ \forall s \geq S(z). \quad (2.61)$$

Moreover, recall that

$$S(x) \leq S(y) \leq S(z), \quad (2.62)$$

$$\tau^\star(0,y) > S(y), \quad (2.63)$$

With these observations at hand we can now show (2.56)-(2.58).

By (2.61) we have that $dI_s^\star(0,x) = dI_s^\star(0,z)$ for all $s \geq S(z)$. Further, we have that $I_s^\star(0,z) = 0$ for all $s \leq S(z)$. Therefore, by (2.62) $I_s^\star(0,z) = I_s^\star(0,x) = 0$ for $s \leq S(x)$, and the right-hand side of (2.56) rewrites as

$$\int_0^T m(s)\, dI_s^\star(0,x) - \int_0^T m(s)\, dI_s^\star(0,z) = \int_{S(x)}^{S(z)} m(s)\, [dI_s^\star(0,x) - dI_s^\star(0,z)]$$
$$= \int_{S(x)}^{S(z)} m(s)\, dI_s^\star(0,x) = \int_{S(x)}^{S(z)} m(s)\, dM_s. \quad (2.64)$$

Here we have used (2.59) with $y = x$.

On the other hand, for the left-hand side of (2.56), we use the change of variable formula of Section 4 in Chapter 0 of Revuz and Yor [82]. This leads to

$$\int_x^z m(S(y))\mathbb{1}_{\{\tau^\star(0,y)\geq S(y)\}}\,dy = \int_x^z m(S(y))\,dy = \int_{S(x)}^{S(z)} m(s)\,dM_s, \qquad (2.65)$$

where we use (2.63), the fact that $\{z\}$ is a Lebesgue zero set, and that M is the right-continuous inverse of S (see (2.31)). Combining (2.64) and (2.65) proves (2.56).

Equation (2.57) follows by observing that (2.61)–(2.62) imply that the processes $D^\star(0,z)$ and $D^\star(0,x)$ coincide, and the left-hand side equals 0 by definition. Notice that for such an argument particular care has to be put when considering z of (2.55) as a starting point for the drifted Brownian motion. In particular, if the realization of the Brownian motion is such that $\tau^\star(0,z) < S(z)$, then by definition of z, the drifted Brownian motion only touches the boundary at time $\tau^\star(0,z)$, but does not cross it. Hence, we still have $D^\star_s(0,z) = 0$ for all $s \leq S(z)$, which implies (2.61) and therefore still $D^\star_s(0,z) = D^\star_s(0,x)$. In turn, this gives again that (2.57) holds also for such a particular realization of the Brownian motion.

Finally, to prove equation (2.58) remember that $x \in \{y \in \mathbb{R}_+ : \tau^\star(0,y) > S(y)\}$. By definition of z we obtain $\tau^\star(0,y) \geq S(y)$ for all $y \in [x,z)$ and the left-hand side of (2.58) equals zero. By (2.61) the processes $X^{D^\star}_s(z) = X^{D^\star}_s(x)$ coincides for all $s \geq S(z)$, and $S(z) \leq T$ a.s. by Lemma A.1 in the Appendix. Therefore, the right-hand side of (2.58) equals zero as well.

Step 4. For $x \in \{y \in \mathbb{R}_+ : \tau^\star(0,y) < S(y)\}$, (2.42) follows by the results of Step 1. If, instead, $x \in \{y \in \mathbb{R}_+ : \tau^\star(0,y) > S(y)\}$, then we can integrate u separately in the intervals $[x,z]$ and $[z,b(0)]$. When integrating u in the interval $[x,z]$ we use the results of Step 3. On the other hand, integrating u over $[z,b(0)]$ we have to distinguish two cases. Now, if z belongs to $\{y \in \mathbb{R}_+ : \tau^\star(0,y) < S(y)\}$, then we can still apply the results of Step 1 to conclude. If z belongs to $\{y \in \mathbb{R}_+ : \tau^\star(0,y) > S(y),\ \tau^\star(0,q) < S(q)\ \forall q \in (y,b(0))\}$, we can employ the results of Step 2 to obtain the claim. Thus, in any case, (2.42) holds. □

We now prove that the two functions N and M of (2.29) and (2.43), respectively, are such that $N = M$. To accomplish that we preliminary notice that by their definitions and strong Markov property, one has that the processes

$$N(t+s\wedge\tau^\star(t,x), R_{s\wedge\tau^\star(t,x)}(x)) - \int_0^{s\wedge\tau^\star(t,x)} m(t+\theta)\,dI^0_\theta(x), \quad 0 \leq s \leq T-t, \quad (2.66)$$

and

$$M(t+s\wedge\tau^\star(t,x), R_{s\wedge\tau^\star(t,x)}(x)) - \int_0^{s\wedge\tau^\star(t,x)} m(t+\theta)\,dI^\star_\theta(t,x), \quad 0 \leq s \leq T-t, \quad (2.67)$$

are $\mathbb{F}$-martingales for any $(t,x) \in [0,T]\times\mathbb{R}_+$. Moreover, by (2.28) one has $N(t,x) = N(t,b(t)) - \int_x^{b(t)} u(t,y)\,dy$ and, due to (2.42), $M(t,x) = M(t,b(t)) - \int_x^{b(t)} u(t,y)\,dy$. Hence,

$$\Psi(t) := M(t,x) - N(t,x), \quad t \in [0,T],$$

is independent of the x variable. We now prove that one actually has $\Psi = 0$ and therefore $N = M$.

Theorem 2.16. *It holds $\Psi(t) = 0$ for all $t \in [0,T]$. Therefore, $N = M$ on $[0,T] \times \mathbb{R}_+$.*

Proof. Since $(N-M)$ is independent of x, it suffices to show that $(N-M)(t,x) = 0$ at some x for any $t \leq T$. To accomplish that we show $\Psi'(t) = 0$ for any $t < T$, since by (2.28) and (2.42) we already know that

$$\Psi(T) = N(T,x) - M(T,x) = g(T,x) - g(T,x) = 0.$$

Then take $0 < x_1 < x_2, t_0 \in [0,T)$ and $\varepsilon > 0$ such that $t_0 + \varepsilon < T$ given and fixed, consider the rectangular domain $\mathcal{R} := (t_0 - \varepsilon, t_0 + \varepsilon) \times (x_1, x_2)$ such that $cl(\mathcal{R}) \subset \mathcal{C}$ (where $\mathcal{C}$ has been defined in (2.7)). Also, denote by $\partial_0\mathcal{R} := \partial\mathcal{R}\setminus(\{t_0 - \varepsilon\} \times (x_1, x_2))$. Then consider the problem

$$(P) \begin{cases} h_t(t,x) = \mathcal{L}h(t,x), & (t,x) \in \mathcal{R}, \\ h(t,x) = (N-M)(t,x), & (t,x) \in \partial_0\mathcal{R}, \end{cases}$$

where $\mathcal{L}$ is the second-order differential operator that acting on $\varphi \in C^{1,2}([0,T] \times \mathbb{R})$ gives

$$(\mathcal{L}\varphi)(t,x) = \mu\frac{\partial\varphi}{\partial x}(t,x) + \frac{1}{2}\sigma^2\frac{\partial^2\varphi}{\partial x^2}(t,x), \; (t,x) \in [0,T] \times \mathbb{R}.$$

By reversing time, $t \mapsto T - t$, Problem (P) corresponds to a classical initial value problem with uniformly elliptic operator (notice that $\sigma^2 > 0$) and parabolic boundary $\partial_0\mathcal{R}$. Since $N - M$ is continuous, and all the coefficients in the first equation of (P) are smooth (actually constant), by classical theory of partial differential equations of parabolic type (see, e.g., Chapter V in the book by Lieberman [67]) problem (P) admits an unique solution h that is continuous, with continuous derivatives h_t, h_x, h_{xx}. Moreover, by the Feynman-Kac's formula, such a solution admits the representation

$$h(t,x) = \mathbb{E}[(N-M)(t + \widehat{\tau}(t,x), Z_{\widehat{\tau}(t,x)}(x))],$$

where

$$\widehat{\tau}(t,x) := \inf\{s \in [0, T-t) : (t+s, Z_s(x)) \in \partial_0\mathcal{R}\} \wedge (T-t),$$

and $Z_s(x) = x + \mu s + \sigma W_s, s \geq 0$. Notice that we have $\widehat{\tau}(t,x) \leq \tau^\star(t,x)$ a.s., since $cl(\mathcal{R}) \subset \mathcal{C}$. Also, the integral terms in (2.66) and (2.67) are equal since $dI^0_\theta(x) = dI^\star_\theta(t,x) = 0$ for any $\theta \leq \widehat{\tau}(t,x) \leq \tau^\star(t,x)$. Hence by the martingale property of (2.66) and (2.67) we have

$$h(t,x) = (N-M)(t,x) \text{ in } \mathcal{R},$$

and, by arbitrariness of $\mathcal{R}$,

$$\Psi(t) = (N-M)(t,x) = h(t,x) \text{ in } \mathcal{C}.$$

Therefore, since $\Psi = N - M$ is independent of x, continuous in t and solves the first equation of (P) in $\mathcal{C}$, we obtain $\Psi'(t) = 0$ for any $t < T$. Hence $\Psi(t) = 0$ for any $t \leq T$ since $\Psi(T) = 0$, and thus $N(t,x) - M(t,x) = 0$ for any $t \leq T$ and for any $x \in (0, \infty)$.

□

In the following we show that the function N is an upper bound for the value function V of (2.4). We first prove the following result.

Theorem 2.17. *For any* $(t,x) \in \mathbb{R}_+ \times [0,T]$ *the process*

$$\widetilde{N}_s := N(t+s, R_s(x)) - \int_0^s m(t+u)\ dI_u^0(x), \quad 0 \leq s \leq T-t, \tag{2.68}$$

is an $\mathbb{F}$*-supermartingale.*

Proof. It is enough to show that $\mathbb{E}\big[\widetilde{N}_\theta\big] \leq \mathbb{E}[\widetilde{N}_\tau]$ for all bounded $\mathbb{F}$-stopping times θ, τ such that $\theta \geq \tau$ (see Karatzas and Shreve [60], Chapter 1, Problem 3.26).

By the strong Markov property and the definition of N (2.29), we get that for any bounded $\mathbb{F}$-stopping time ρ one has

$$\begin{aligned}
\mathbb{E}[\widetilde{N}_\rho] &= \mathbb{E}\left[N(t+\rho, R_\rho(x)) - \int_0^\rho m(t+s)\ dI_s^0(x)\right] \\
&= \mathbb{E}\left[-\int_\rho^{T-t} f'(t+s)[R_s(x) - b(t+s)]^+ ds \right.\\
&\quad \left. - \int_0^{T-t} m(t+s)\ dI_s^0(x) + g(R_{T-t}(x))\right] \\
&= N(t,x) + \mathbb{E}\left[\int_0^\rho f'(t+s)\big(R_s(x) - b(t+s)\big)^+ ds\right] =: N(t,x) + \Delta_\rho,
\end{aligned}$$

for any $(t,x) \in [0,T] \times \mathbb{R}_+$. Hence, taking θ, τ such that $T - t \geq \theta \geq \tau$ we get from the latter that $\mathbb{E}[\widetilde{N}_\theta] = N(t,x) + \Delta_\theta \leq N(t,x) + \Delta_\tau = \mathbb{E}[\widetilde{N}_\tau]$, where the inequality is due to the fact that $f' \leq 0$ on $\mathcal{S}$ (cf. Corollary 2.8-(ii)). This proves the claimed supermartingale property. □

To proceed further, we need the following properties of the function N of (2.29). Its proof is relegated to the Appendix.

Lemma 2.18. *The function* $N \in C^{1,2}([0,T) \times (0,\infty)) \cap C^0([0,T] \times \mathbb{R}_+)$.

Thanks to Lemma 2.18, an application of Itô's formula allows us to obtain the following (unique) Doob-Meyer decomposition of the $\mathbb{F}$-supermartingale $\widetilde{N}$ (cf. (2.68)).

Corollary 2.19. *The* $\mathbb{F}$*-supermartingale* $\widetilde{N}$ *of* (2.68) *is such that for all* $(t,x) \in [0,T] \times \mathbb{R}_+$ *and* $s \in [0, T-t]$

$$\begin{aligned}
&N(t+s, R_s(x)) - \int_0^s m(t+\theta)\ dI_\theta^0(x) \\
&= N(t,x) + \sigma \int_0^s u(t+\theta, R_\theta(x))\ dW_\theta + \Pi_s(t,x),
\end{aligned} \tag{2.69}$$

where $\Pi_\cdot(t,x)$ *is a continuous, non-increasing and* $\mathbb{F}$*-adapted process.*

Proof. By the Doob-Meyer decomposition, the $\mathbb{F}$-supermartingale in (2.68) can be (uniquely) written as the sum of an $\mathbb{F}$-martingale and a continuous, $\mathbb{F}$-adapted non-increasing process $(\Pi_s)_{s\geq 0}$. Applying the martingale representation theorem to the martingale part of $\widetilde{N}$, yields the decomposition

$$\widetilde{N}_s = N(t,x) + \int_0^s \phi_\theta \, dW_\theta + \Pi_s(t,x),$$

for some $\phi \in L^2(\Omega \times [0,T], \mathbb{P} \otimes dt)$. Finally, an application of Itô's lemma shows that $\phi_\theta = \sigma u(t+\theta, R_\theta(x))$ a.s. □

Theorem 2.20. *For any process $D \in \mathcal{D}(t,x)$ and any $(t,x) \in [0,T] \times \mathbb{R}_+$, the process*

$$Q_s(D;t,x) := \int_{[0,s]} f(t+\theta) \, dD_\theta - \int_0^s m(t+\theta) \, dI^D_\theta + N(t+s, X^D_s(x)), \quad (2.70)$$

$s \in [0, T-t]$, is such that

$$\mathbb{E}\left[Q_s(D;t,x)\right] \leq N(t,x), \quad \textit{for any } s \in [0, T-t]. \quad (2.71)$$

Proof. The proof is organized in 3 steps.

Step 1. For $D \equiv 0$, the proof is given by Theorem 2.17.

Step 2. Let $D_s := \int_0^s z_u \, du$, $s \geq 0$, where z is a bounded, non-negative, $\mathbb{F}$-progressively measurable process. To show (2.71) we use Girsanov's Theorem and we rewrite the state process $X^D_s(x) = x + \mu s + \sigma W_s + D_s - I^D_s$ as a new drifted Brownian motion reflected at the origin. We therefore introduce the exponential martingale

$$Z_s = \exp\left(\int_0^s \frac{z_u}{\sigma} \, dW_u - \frac{1}{2\sigma^2} \int_0^s z_u^2 \, du\right), \quad s \geq 0,$$

and we obtain that under the measure $\widehat{\mathbb{P}} = Z_T \mathbb{P}$, the process

$$\widehat{W}_s := W_s - \frac{1}{\sigma} \int_0^s z_u du, \quad s \geq 0,$$

is an $\mathbb{F}$- Brownian motion.

We can now rewrite the process Q of (2.70) under $\widehat{\mathbb{P}}$ as

$$Q_s(D;t,x) = \int_{[0,s]} f(t+\theta) \, dD_\theta - \int_0^s m(t+\theta) \, d\widehat{I}^D_\theta + N(t+s, \widehat{R}_s(x)), \quad (2.72)$$

for any $s \in [0, T-t]$, where under $\widehat{\mathbb{P}}$

$$\widehat{X}^D_s(x) = x + \mu s + \sigma \widehat{W}_s + \widehat{I}^D_s =: \widehat{R}_s(x).$$

Here $\widehat{I}^D_\cdot$ is flat off $\{s \geq 0 : \widehat{R}_s(x) = 0\}$ and reflects the drifted Brownian motion at the origin. By employing (2.69), equation (2.72) reads as

$$Q_s(D;t,x) = N(t,x) + \sigma \int_0^s u(t+u, \widehat{R}_u(x)) d\widehat{W}_u + \widehat{\Pi}_s(t,x), \quad (2.73)$$

for $s \in [0, T-t]$, where we have set

$$\widehat{\Pi}_s(t,x) := \Pi_s(t,x) + \int_0^s \Big(f(t+\theta) - u(t+\theta, R_\theta(x)) \Big) z_\theta d\theta, \quad s \in [0, T-t].$$

Since $\widehat{\Pi}$ is non-increasing due to the fact that $u \geq f$ and $\Pi_\cdot(t,x)$ is non-increasing, we can take expectations in (2.73) so to obtain

$$\mathbb{E}\left[Q_s(D;t,x)\right] \leq N(t,x), \quad \forall s \in [0, T-t].$$

Step 3. Since any arbitrary $D \in \mathcal{D}(t,x)$ can be approximated by an increasing sequence $(D^n)_{n\in\mathbb{N}}$ of absolutely continuous processes as the ones considered in Step 2 (see El Karoui and Karatzas [39], Lemmata 5.4, 5.5 and Proposition 5.6), we have for all $n \in \mathbb{N}$

$$\mathbb{E}\left[Q_s(D^n;t,x)\right] \leq N(t,x).$$

Applying monotone and dominated convergence theorem, this property holds for $Q(D;t,x)$ as well, for any $D \in \mathcal{D}(t,x)$. □

By Theorem 2.20 and the definition of Q as in (2.70) we immediately obtain

$$V(t,x) = \sup_{D\in\mathcal{D}(t,x)} J(t,x;D) = \sup_{D\in\mathcal{D}(t,x)} \mathbb{E}\left[Q_{T-t}(D;t,x)\right] \leq N(t,x). \tag{2.74}$$

Moreover, by definition (2.43) one has

$$M(t,x) = J(t,x;D^\star(t,x)) \leq V(t,x). \tag{2.75}$$

With all these results at hand, we can now finally prove Theorem 2.6.

Proof of Theorem 2.6. By combining (2.74), (2.75), and Theorem 2.16 we obtain the series of inequalities

$$N(t,x) \geq V(t,x) \geq M(t,x) = N(t,x)$$

which proves the claim that $V = M$, and the optimality of $D^\star$. It just remains to prove (2.12). To accomplish that we adapt and expand arguments as those used by El Karoui and Karatzas in the proof of Corollary 4.2 in [40].

Observe that optimality of $D^\star$ implies that for all $x > b(t)$

$$V(t,b(t)) + f(t)(x - b(t)) = V(t,x). \tag{2.76}$$

Using (2.29) and the fact that $V = N$ as proved above, we then find from (2.76)

$$\begin{aligned}
V(t,b(t)) &= V(t,x) - f(t)(x-b(t)) \\
&= \mathbb{E}\Big[- \int_0^{T-t} f'(t+s)(R_s(x) - b(t+s))^+ \, ds - \int_0^{T-t} m(t+s)\, dI_s^0(x) \\
&+ g(T, R_{T-t}(x)) - f(t)(x-b(t))\Big] \\
&= \mathbb{E}\Big[- \int_0^{T-t} f'(t+s)\Big[(R_s(x) - b(t+s))^+ - (x-b(t))\Big] \, ds \\
&- \int_0^{T-t} m(t+s)\, dI_s^0(x) + g(T, R_{T-t}(x)) - f(T)(x-b(t))\Big].
\end{aligned}$$

Recall (2.27), and observe that under the condition $b(T) < \infty$ we can write

$$\begin{aligned}
&\mathbb{E}\Big[g(T, R_{T-t}(x))\Big] = g(T,b(T)) + \mathbb{E}\Big[\Big(\int_{b(T)}^{R_{T-t}(x)} g_x(T,y)dy\Big)\mathbb{1}_{\{R_{T-t}(x) > b(T)\}} \\
&- \Big(\int_{R_{T-t}(x)}^{b(T)} g_x(T,y)dy\Big)\mathbb{1}_{\{R_{T-t}(x) \le b(T)\}}\Big] = g(T,b(T)) \\
&+ \mathbb{E}\Big[f(T)\big(R_{T-t}(x) - b(T)\big)\mathbb{1}_{\{R_{T-t}(x) > b(T)\}} - \Big(\int_{R_{T-t}(x)}^{b(T)} g_x(T,y)dy\Big)\mathbb{1}_{\{R_{T-t}(x) \le b(T)\}}\Big],
\end{aligned}$$

where the last equality follows from Remark 2.9. Therefore, we obtain that

$$\begin{aligned}
V(t,b(t)) &= \mathbb{E}\Big[- \int_0^{T-t} f'(t+s)\Big[(R_s(x) - b(t+s))^+ - (x-b(t))\Big] \, ds \\
&- \int_0^{T-t} m(t+s)\, dI_s^0(x) + g(T,b(T)) + f(T)\big(R_{T-t}(x) - b(T)\big)\mathbb{1}_{\{R_{T-t}(x) > b(T)\}} \\
&- f(T)\big(x - b(t)\big) - \Big(\int_{R_{T-t}(x)}^{b(T)} g_x(T,y)dy\Big)\mathbb{1}_{\{R_{T-t}(x) \le b(T)\}}\Big].
\end{aligned}$$

Notice now that $I_s^0(x) \to 0$, $R_s(x) \to \infty$, and $(R_s(x) - b(t+s))^+ - (x - b(t)) \to \mu s + \sigma W_s - b(t+s) + b(t)$ a.s. for any $s \ge 0$ when $x \uparrow \infty$ (cf. (2.27)). Then, letting $x \to \infty$ in the last expression for $V(t,b(t))$, and invoking the monotone and dominated convergence theorems, we find (after evaluating the expectations and rearranging terms)

$$\begin{aligned}
V(t,b(t)) &= \mathbb{E}\Big[- \int_0^{T-t} f'(t+s)\Big(\mu s + \sigma W_s - b(t+s) + b(t)\Big) \, ds \\
&+ g(T,b(T)) + f(T)\left(\mu(T-t) + \sigma W_{T-t} - b(T) + b(t)\right)\Big] \\
&= -\mu \int_0^{T-t} f'(t+s)s \, ds + \int_0^{T-t} f'(t+s)b(t+s) \, ds \\
&+ g(T,b(T)) + f(T)\mu(T-t) + f(t)b(t) - f(T)b(T).
\end{aligned}$$

□

Remark 2.21. As a byproduct of the fact that $V = N$ and of Lemma 2.18, we have that $V \in C^{1,2}([0,T) \times (0,\infty)) \cap C^0([0,T] \times \mathbb{R}_+)$. Moreover, from (2.11) and (2.6) we have that V satisfies the Neumann boundary condition $V_x(t,0) = m(t)$ for all $t \in [0,T]$.

Remark 2.22. The pathwise approach followed in this section seems to suggest that some of the intermediate results needed to prove Theorem 2.6 remain valid also in a more general setting in which profits and costs in (2.4) are discounted at a stochastic rate. We leave the analysis of this interesting problem for future work.

2.4 A Case Study with Discounted Constant Marginal Profits and Costs

In this section we consider the optimal dividend problem with capital injections

$$\begin{aligned}\widehat{V}(t,x) &:= \sup_{D \in \mathcal{D}(t,x)} \mathbb{E}\left[\int_0^{T-t} \eta e^{-rs}\, dD_s - \int_0^{T-t} \kappa e^{-rs}\, dI^D_s + \eta e^{-r(T-t)} X^D_{T-t}(x)\right] \\ &= e^{rt} V(t,x), \end{aligned} \tag{2.77}$$

where we have defined

$$\begin{aligned}V(t,x) := \sup_{D \in \mathcal{D}(t,x)} \mathbb{E}\Bigg[\int_0^{T-t} \eta e^{-r(t+s)}\, dD_s &- \int_0^{T-t} \kappa e^{-r(t+s)}\, dI^D_s \\ &+ \eta e^{-rT} X^D_{T-t}(x)\Bigg]. \end{aligned} \tag{2.78}$$

It is clear from (2.78) and (2.3) that such a problem can be accommodated in our general setting (2.4) by taking (cf. Assumption 2.1)

$$f(t) = \eta e^{-rt}, \quad m(t) = \kappa e^{-rt}, \quad g(t,x) = \eta e^{-rt} x, \tag{2.79}$$

for some $\kappa > \eta$ (see also Remark 2.3).

In $\widehat{V}$ of (2.77) the coefficient κ can be seen as a constant proportional administration cost for capital injections. On the other hand, if we immagine that transaction costs or taxes have to be paid on dividends, the coefficient η measures a constant net proportion of leakages from the surplus received by the shareholders.

Remark 2.23. Problem (2.77) is perhaps the most common formulation of the optimal dividend problem with capital injections (see, e.g., Kulenko and Schmidli [65], Lokka and Zervos [69], Zhu and Yang [96] and references therein). However, to the best of our knowledge, no previous work has considered such a problem in the case of a finite time horizon, whereas problem (2.77) has been extensively studied when $T = +\infty$ (see, e.g., Ferrari [46] and references therein). In particular, it has been shown, e.g., in [46] that in the case $T = +\infty$ the optimal dividend strategy is triggered by a boundary $b_\infty > 0$ that can be characterized as the solution to a non-linear algebraic equation (see Proposition 3.2 in [46]). In Proposition 3.6 of [46] such a trigger value is also shown to be the optimal stopping boundary of problem (2.80) below (when the optimization is performed over all the $\mathbb{F}$-stopping times).

Thanks to Theorem 2.6 we know that, whenever Assumption 2.5 is satisfied, the optimal control $D^\star$ for problem (2.78) is triggered by the optimal stopping boundary b of the optimal stopping problem

$$\begin{aligned} u(t,x) &= \sup_{\tau \in \mathcal{T}(T-t)} \mathbb{E}\Big[e^{-r\tau}\eta \mathbb{1}_{\{\tau < S(x)\}} + e^{-rS(x)}\kappa \mathbb{1}_{\{\tau \geq S(x)\}}\Big] \\ &= \sup_{\tau \in \mathcal{T}(T-t)} \mathbb{E}\Big[e^{-r\tau}\eta \mathbb{1}_{\{A_\tau(x) > 0\}} + e^{-rS(x)}\kappa \mathbb{1}_{\{A_\tau(x) \leq 0\}}\Big]. \end{aligned} \tag{2.80}$$

In the following we study the optimal stopping problem (2.80) and verify the requirements of Assumption 2.5.

Moreover, by taking the sub-optimal stopping time $\tau = 0$ in (2.80) clearly gives $u(t,x) \geq \eta$ for $(t,x) \in [0,T] \times (0,\infty)$. Therefore, we can define the continuation and the stopping region of problem (2.80) as

$$\mathcal{C} := \{(t,x) \in [0,T) \times (0,\infty) : u(t,x) > \eta\}, \; \mathcal{S} := \{(t,x) \in [0,T] \times (0,\infty) : u(t,x) = \eta\}.$$

Also, notice that we have $u(t,x) \leq \kappa$ for $(t,x) \in [0,T] \times \mathbb{R}_+$ since $\eta < \kappa$.

Since the reward process $\phi_t := e^{-rt}\eta \mathbb{1}_{\{t < S(x)\}} + e^{-rS(x)}\kappa \mathbb{1}_{\{t \geq S(x)\}}$ is upper semi-continuous in expectation along stopping times (thanks to the fact that $\eta < \kappa$), Theorem 2.9 in Kobylanski and Quenez [63] ensures that the first time the value process (i.e. the Snell envelope of the reward process) equals the reward process is optimal. In our Markovian setting we thus have that the stopping time

$$\tau^\star(t,x) := \inf\{s \in [0, T-t) : \, (t+s, A_s(x)) \in \mathcal{S}\} \wedge (T-t), \tag{2.81}$$

for $(t,x) \in [0,T] \times \mathbb{R}_+$, is optimal. Further, defining $Z_s(x) := x + \mu s + \sigma W_s$, $s \geq 0$, the process

$$e^{-r(s \wedge \tau^\star(t,x) \wedge S(x))} u(t + (s \wedge \tau^\star(t,x) \wedge S(x)), Z_{(s \wedge \tau^\star(t,x) \wedge S(x))}(x)), \quad s \in [0, T-t], \tag{2.82}$$

is an $\mathbb{F}$-martingale (cf. Proposition 1.6 and Remark 1.7 in Kobylanski and Quenez [63]).

The next proposition proves some preliminary properties of u.

Proposition 2.24. *The value function u of* (2.80) *satisfies the following:*

(i) $u(T,x) = \eta$ *for any* $x > 0$ *and* $u(t,0) = \kappa$ *for any* $t \in [0,T]$*;*

(ii) $t \mapsto u(t,x)$ *is non-increasing for any* $x > 0$*;*

(iii) $x \mapsto u(t,x)$ *is non-increasing for any* $t \in [0,T]$*.*

Proof. We prove each item separately.

(i) The first property easily follows from definition (2.80).

(ii) The second property is due to the fact that $\mathcal{T}(T - \cdot)$ shrinks and the expected value on the right-hand side of (2.80) is independent of $t \in [0,T]$.

(iii) Fix $t \in [0,T]$, $x_2 > x_1 \geq 0$ and notice that $S(x_2) > S(x_1)$. Then, from (2.80) we can write

$$\begin{aligned}
&u(t,x_2) - u(t,x_1) \\
&\leq \sup_{\tau \in \mathcal{T}(T-t)} \mathbb{E}\Big[e^{-r\tau}\eta \mathbb{1}_{\{\tau < S(x_2)\}} - e^{-r\tau}\eta \mathbb{1}_{\{\tau < S(x_1)\}} + e^{-rS(x_2)}\kappa \mathbb{1}_{\{\tau \geq S(x_2)\}} - e^{-rS(x_1)}\kappa \mathbb{1}_{\{\tau \geq S(x_1)\}}\Big] \\
&= \sup_{\tau \in \mathcal{T}(T-t)} \mathbb{E}\Big[\mathbb{1}_{\{S(x_1) \leq \tau < S(x_2)\}}\Big(e^{-r\tau}\eta - e^{-rS(x_1)}\kappa\Big) + \Big(e^{-rS(x_2)} - e^{-rS(x_1)}\Big)\kappa \mathbb{1}_{\{\tau \geq S(x_2)\}}\Big] \\
&\leq \sup_{\tau \in \mathcal{T}(T-t)} \mathbb{E}\Big[e^{-rS(x_1)}(\eta - \kappa)\mathbb{1}_{\{S(x_1) \leq \tau < S(x_2)\}} + \Big(e^{-rS(x_2)} - e^{-rS(x_1)}\Big)\kappa \mathbb{1}_{\{\tau \geq S(x_2)\}}\Big] \leq 0,
\end{aligned}$$

where we have used that $\eta < \kappa$ in the last step. □

Since $x \mapsto u(t,x)$ is non-increasing for each $t \in [0,T]$, setting

$$b(t) := \inf\{x > 0 : u(t,x) \leq \eta\}, \quad t \in [0,T], \tag{2.83}$$

it is clear that

$$\mathcal{C} = \{(t,x) \in [0,T) \times (0,\infty) : x < b(t)\}, \ \mathcal{S} = \{(t,x) \in [0,T] \times (0,\infty) : x \geq b(t)\}.$$

Moreover, the optimal stopping time of (2.81) reads

$$\tau^\star(t,x) := \inf\{s \in [0, T-t) : \ A_s(x) \geq b(t+s)\} \wedge (T-t). \tag{2.84}$$

In the following we will refer to b as to the *free-boundary*. The next theorem proves preliminary properties of b.

Proposition 2.25. *The free-boundary b is such that*

(i) $t \mapsto b(t)$ is non-increasing;

(ii) One has $b(t) > 0$ for all $t \in [0,T)$. Moreover, there exists $b_\infty > 0$ such that $b(t) \leq b_\infty$ for any $t \in [0,T]$.

Proof. We prove each item separately.

(i) The claimed monotonicity of b immediately follows from (ii) of Proposition 2.24.

(ii) To show that $b(t) > 0$ for any $t \in [0,T)$ it is enough to observe that $u(t,0) = \kappa > \eta$ for all $t \in [0,T)$.

To prove $b(t) < \infty$ notice that $u(t,x) \leq u_\infty(x)$ for all $(t,x) \in [0,T] \times \mathbb{R}_+$, where

$$u_\infty(x) := \sup_{\tau \geq 0} \mathbb{E}\left[\eta e^{-r\tau}\mathbb{1}_{\{\tau < S(x)\}} + \kappa e^{-rS(x)}\mathbb{1}_{\{\tau \geq S(x)\}}\right].$$

Hence, setting $b_\infty := \inf\{x > 0 : u_\infty(x) = \eta\}$ (which exists finite, e.g., by Proposition 3.2 in Ferrari [46]; see also Remark 2.23 above), we have $b(t) \leq b_\infty$ for all $t \in [0,T]$. □

The proof of the next proposition is quite lenghty, and it is therefore postponed in the Appendix in order to simplify the exposition.

Proposition 2.26. *The function $(t,x) \mapsto u(t,x)$ is lower semicontinuous on $[0,T) \times (0,\infty)$.*

The lower semicontinuity of u implies that the martingale of (2.82) has right-continuous sample paths, and that the stopping region is closed. The latter fact in turn plays an important role when proving continuity of the free-boundary, as it is shown in the next proposition.

Proposition 2.27. *The free-boundary b is such that $t \mapsto b(t)$ is continuous on $[0,T)$. Moreover, $b(T) := \lim_{t\uparrow T} b(t) = 0$.*

Proof. We prove the two properties separately.

Here we show that b is continuous, and this proof is divided in two parts. We start with the right-continuity. Note that, by lower semicontinuity of u (cf. Proposition 2.26), the stopping region $\mathcal{S}$ is closed. Then fix an arbitrary point $t \in [0,T)$, take any sequence $(t_n)_{n\geq 1}$ such that $t_n \downarrow t$, and notice that $(t_n, b(t_n)) \in \mathcal{S}$, by definition. Setting $b(t+) := \lim_{t_n \downarrow t} b(t_n)$ (which exists due to Proposition 2.25-(i)), we have $(t_n, b(t_n)) \to (t, b(t+))$, and since $\mathcal{S}$ is closed $(t, b(t+)) \in \mathcal{S}$. Therefore, it holds $b(t+) \geq b(t)$ by definition (2.83) of b. However, $b(\cdot)$ is non-increasing, and therefore $b(t) = b(t+)$.

Next we show left-continuity for all $t \in (0,T)$ and for this we adapt to our setting ideas as those in the proof of Proposition 4.2 in De Angelis and Ekström [32]. Suppose that b makes a jump at some $t \in (0,T)$. By Proposition 2.25-(i) we have $\lim_{t_n \uparrow t} b(t_n) := b(t-) \geq b(t)$. We employ a contradiction scheme to show $b(t-) = b(t)$, and we assume $b(t-) > b(t)$. Let $x := \frac{b(t-)+b(t)}{2}$, recall $Z_s(x) = x + \mu s + \sigma W_s$, $s \geq 0$, and define

$$\tau_\varepsilon := \inf\{s \geq 0 : Z_s(x) \notin (b(t-), b(t))\} \wedge \varepsilon$$

for $\varepsilon \in (0,t)$. Then noticing that $\tau_\varepsilon < \tau^\star(t-\varepsilon, x) \wedge S(x)$, by the martingale property of (2.82) we can write

$$\begin{aligned}
u(t-\varepsilon, x) &= \mathbb{E}\left[e^{-r\tau_\varepsilon} u(t-\varepsilon+\tau_\varepsilon, Z_{\tau_\varepsilon}(x))\right] \\
&= \mathbb{E}\left[e^{-r\varepsilon} u(t, Z_\varepsilon(x)) \mathbb{1}_{\{\tau_\varepsilon = \varepsilon\}} + e^{-r\tau_\varepsilon} u(t-\varepsilon+\tau_\varepsilon, Z_{\tau_\varepsilon}(x)) \mathbb{1}_{\{\tau_\varepsilon < \varepsilon\}}\right] \\
&\leq \mathbb{E}\left[e^{-r\varepsilon} \eta \mathbb{1}_{\{\tau_\varepsilon = \varepsilon\}} + e^{-r\tau_\varepsilon} \kappa \mathbb{1}_{\{\tau_\varepsilon < \varepsilon\}}\right] \\
&\leq e^{-r\varepsilon}\eta + \kappa \mathbb{P}\left(\tau_\varepsilon < \varepsilon\right),
\end{aligned}$$

where the last step follows from the fact that $u \leq \kappa$, and that $Z_{\tau_\varepsilon}(x) \geq b(t)$ on the set $\{\tau_\varepsilon = \varepsilon\}$. Since $e^{-r\varepsilon}\eta + \kappa\mathbb{P}(\tau_\varepsilon < \varepsilon) = \eta(1 - r\varepsilon) + \kappa o(\varepsilon)$ as $\varepsilon \downarrow 0$, we have found a contradiction to $u(t,x) \geq \eta$. Therefore, $b(t-) = b(t)$ and b is continuous on $[0,T)$.

To prove the claimed limit, notice that if $b(T) := \lim_{t\uparrow T} b(t) > 0$, then any point (T,x) with $x \in (0, b(T))$ belongs to $\mathcal{C}$. However, we know that $(T,x) \in \mathcal{S}$ for all $x > 0$, and we thus reach a contradiction. □

Thanks to the previous results all the requirements of Assumption 2.5 are satisfied for problem (2.80). Hence Theorem 2.6 holds, and one has that V of (2.78) and u of (2.80) are such that $V_x = u$ on $[0,T] \times \mathbb{R}_+$. In particular, by (2.77) and Theorem 2.6 we can write

$$\widehat{V}(t,x) = \widehat{V}(t,b(t)) - e^{rt} \int_x^{b(t)} u(t,y)\, dy,$$

where by (2.12), (2.79), and the fact that $b(T) = 0$ we have

$$\widehat{V}(t,b(t)) = \eta b(t) + \frac{\mu\eta}{r}\big(1 - e^{-r(T-t)}\big) - r\eta \int_t^T e^{-r(u-t)} b(u) du.$$

Moreover, the optimal dividend distributions' policy $D^\star$ given through (2.10) is triggered by the free-boundary b whose properties have been derived in Theorem 2.27.

2.4.1 A Comparative Statics Analysis.

We conclude by providing the monotonicity of the free-boundary with respect to some of the problem's parameters. In the following, for any given and fixed $t \in [0,T]$, we write $b(t;\cdot)$ in order to stress the dependence of the free-boundary point $b(t)$ with respect to a given parameter. Similarly, we write $u(t,x;\cdot)$ when we need to consider the dependence of $u(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}_+$, with respect to a given problem's parameter.

Proposition 2.28. *Let $t \in [0,T]$ be given and fixed. It holds that*

(i) $\kappa \mapsto b(t;\kappa)$ *is non-decreasing;*

(ii) $\eta \mapsto b(t;\eta)$ *is non-increasing;*

(iii) $r \mapsto b(t;r)$ *is non-increasing;*

(iv) $\mu \mapsto b(t;\mu)$ *is non-increasing.*

Proof. Recalling that

$$u(t,x) = \sup_{\tau \in \mathcal{T}(T-t)} \mathbb{E}\Big[e^{-r\tau}\eta \mathbb{1}_{\{\tau < S(x)\}} + e^{-rS(x)}\kappa \mathbb{1}_{\{\tau \geq S(x)\}}\Big], \quad (t,x) \in [0,T] \times \mathbb{R}_+,$$

one can easily show that

(1) $\kappa \mapsto u(t,x;\kappa)$ is non-decreasing,

(2) $\eta \mapsto u(t,x;\eta) - \eta = \sup_{\tau \in \mathcal{T}(T-t)} \mathbb{E}\Big[\eta\big(e^{-r\tau}\mathbb{1}_{\{\tau < S(x)\}} - 1\big) + e^{-rS(x)}\kappa \mathbb{1}_{\{\tau \geq S(x)\}}\Big]$ is non-increasing,

(3) $r \mapsto u(t,x;r)$ is non-increasing.

Moreover, let $\mu_2 > \mu_1$ and denote by $S(x;\mu_2)$ (resp. $S(x;\mu_1)$) the hitting time of the origin of the drifted Brownian Motion with drift μ_2 (resp. μ_1). Since $S(x;\mu_2) \geq S(x;\mu_1)$ a.s. we obtain

$$\begin{aligned}
u(t,x;\mu_2) - u(t,x;\mu_1) &\leq \sup_{\tau \in \mathcal{T}(T-t)} \mathbb{E}\Big[e^{-r\tau}\eta \left(\mathbb{1}_{\{\tau < S(x;\mu_2)\}} - \mathbb{1}_{\{\tau < S(x;\mu_1)\}} \right) \\
&\quad + \kappa \left(e^{-rS(x;\mu_2)} \mathbb{1}_{\{\tau \geq S(x,\mu_2)\}} - e^{-rS(x;\mu_1)} \mathbb{1}_{\{\tau \geq S(x;\mu_1)\}} \right) \Big] \\
&\leq \sup_{\tau \in \mathcal{T}(T-t)} \mathbb{E}\Big[e^{-r\tau}\eta \mathbb{1}_{\{S(x,\mu_1) \leq \tau < S(x;\mu_2)\}} - \kappa e^{-rS(x;\mu_1)} \mathbb{1}_{\{S(x,\mu_2) > \tau \geq S(x,\mu_1)\}} \\
&\quad + \kappa \mathbb{1}_{\{\tau \geq S(x;\mu_2)\}} \left(e^{-rS(x;\mu_2)} - e^{-rS(x;\mu_1)} \right) \Big] \\
&= \sup_{\tau \in \mathcal{T}(T-t)} \mathbb{E}\Big[\mathbb{1}_{\{S(x,\mu_1) \leq \tau < S(x;\mu_2)\}} \left(e^{-r\tau}\eta - e^{-rS(x;\mu_1)}\kappa \right) \\
&\quad + \mathbb{1}_{\{\tau \geq S(x;\mu_2)\}} \left(e^{-rS(x;\mu_2)} - e^{-rS(x;\mu_1)} \right) \Big] \\
&\leq 0.
\end{aligned}$$

Given the previous monotonicity properties of u, we can now prove items (i)-(iv).

(i) Taking $\kappa_2 > \kappa_1$ and using (1) and (2.83) we have

$$b(t;\kappa_2) := \inf\{x > 0 : u(t,x;\kappa_2) \leq \eta\} \geq \inf\{x > 0 : u(t,x;\kappa_1) \leq \eta\} = b(t;\kappa_1).$$

(ii) Taking $\eta_2 > \eta_1$ and using (2) and (2.83) we have

$$b(t;\eta_2) := \inf\{x > 0 : u(t,x;\eta_2) - \eta_2 \leq 0\} \leq \inf\{x > 0 : u(t,x;\eta_1) - \eta_1 \leq 0\} = b(t;\eta_1).$$

(iii) Taking $r_2 > r_1$ and using (3) and (2.83) we have

$$b(t;r_2) := \inf\{x > 0 : u(t,x;r_2) \leq \eta\} \leq \inf\{x > 0 : u(t,x;r_1) \leq \eta\} = b(t;r_1).$$

(iv) Taking $\mu_2 > \mu_1$ and that $u(t,x;\mu_2) - u(t,x;\mu_1) \leq 0$ and (2.83) we have

$$b(t;\mu_2) := \inf\{x > 0 : u(t,x;\mu_2) \leq \eta\} \leq \inf\{x > 0 : u(t,x;\mu_1) \leq \eta\} = b(t;\mu_1).$$

□

The last proposition allows us to draw some economic implications. Increasing the parameters η, r, and μ, leads, at each time t, to an earlier dividends' distribution. This result is quite intuitive since an higher interest rate r lowers future profits due to discounting, an higher η increases the marginal value of dividends, and an higher μ increases the surplus' trend and lowers the probability of bankruptcy, hence of capital injections. On the other hand, an increase of κ postpones the dividends' distribution since capital injections become more expensive, and the fund's manager thus acts in a more cautious way.

Proving the monotonicity of the free-boundary with respect to the surplus' volatility σ seems not to be feasible by following the arguments of the proof of

Proposition 2.28. One should then rely on a careful numerical analysis of the dynamic programming equation associated to the optimal dividend problem, and we believe that such a study falls outside the scopes of this work. However, we conjecture that an increase of σ should postpone the dividends' distribution. Indeed, the larger σ is, the higher becomes the risk of the need of costly capital injections. As a consequence, the fund's manager wants to wait longer before distributing an additional unit of dividends. Such a monotonicity of the free-boundary with respect to σ has been recently proved by Ferrari in Proposition 4.1 of [46] in the case of a stationary optimal dividend problem with capital injections.

2.5 Conclusion

In this part of the thesis, we proposed and solved a dividend problem with capital injections over a finite time horizon. Mathematically, it is formulated as a two-dimensional singular stochastic control problem. Because the problem reads as a reflected follower problem with costly reflection at the origin, we were able to extend the model of El Karoui and Karatzas [40] by including costs for reflection. More precisely, we related the singular stochastic control problem to a more tractable optimal stopping problem with absorption at the origin. The main result is the following: If one is able to prove that the time-dependent free-boundary of the optimal stopping problem is continuous and non-negative, it turns out that the optimal dividend strategy is triggered by the free-boundary. In particular, it is optimal to pay dividends if the surplus process X is above the free-boundary. This result is derived by almost exclusively probabilistic arguments. Afterwards, we investigated a common formulation of the optimal dividend problem with capital injections. In this formulation, we were able to show that the free-boundary of the related optimal stopping problem is continuous, non-increasing and non-negative. Hence, our main result was applicable and characterized the optimal dividend strategy of the control problem by the free-boundary of the optimal stopping problem. Moreover, we studied the dependence of the free-boundary on some model parameters.

3 Optimal Production under Regime Switching[6]

3.1 Problem Formulation

We assume a firm that faces an uncertainty concerning the demand of its product, which is modeled by a Brownian motion W and a continuous-time Markov chain ϵ with finite-state space. The Brownian motion W describes random fluctuations in the demand, while the Markov chain ϵ describes uncertain long-term conditions. For instance, one regime may represent a recessionary period with a low demand rate for the product and the other regime may represent an expansionary period with a high demand rate. For example, one regime may represent a high demand rate for a medical item when there is an epidemic and the other regime may represent a low demand rate for the same medical item in times without an epidemic. At every point in time, the management of the firm has full information about these two sources of uncertainty.

Formally, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, rich enough to accommodate a standard Brownian motion $W = \{W_t,\, t \geq 0\}$ and a continuous-time Markov chain $\epsilon = \{\epsilon_t, t \geq 0\}$ with state space $\mathcal{S} = \{1, \cdots, N\}$, $N \geq 2$. We assume that ϵ and W are independent and that the Markov chain ϵ has a strongly irreducible generator $Q = [q_{ij}]_{N\times N}$, where $q_{ii} = -\lambda_i < 0$ and $\sum_{j\in\mathcal{S}} q_{ij} = 0$ for every $i \in \mathcal{S}$. We denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the $\mathbb{P}$-augmentation of the filtration $\{\mathcal{F}_t^{(W,\epsilon)}, t \geq 0\}$ generated by the Brownian motion and the Markov chain.

We assume that the cumulative demand process $D = \{D_t, t \geq 0\}$ satisfies the stochastic differential equation

$$dD_t = \mu_{\epsilon_t}\, dt + \sigma_{\epsilon_t}\, dW_t,$$

where the parameter μ_i represents the current drift of the demand and $\sigma_i > 0$ the current volatility of the demand for each state $i \in \mathcal{S}$.

Given a production strategy P, the inventory of a good $X = \{X_t, t \geq 0\}$ is given by

$$dX_t = -dD_t + dP_t = -\mu_{\epsilon_t}\, dt - \sigma_{\epsilon_t}\, dW_t + dP_t, \quad X_0 = x.$$

The management of the firm faces the problem of choosing a production strategy P such that the inventory is close to an exogenously given target value $\mathcal{I}_i$, depending on the state of the economy, with proportional costs for production. Hence, the manager wants to minimize the cost functional

$$J(x,i;P) := \mathbb{E}\left[\int_0^\infty e^{-\delta t}\alpha_{\epsilon_t}(X_t - \mathcal{I}_{\epsilon_t})^2 dt + \int_0^\infty e^{-\delta t} k_{\epsilon_t} dP_t\right],$$

for $(x,i) \in \mathbb{R}\times\mathcal{S}$. That is, the manger aims at solving

$$V(x,i) := \inf_{P\in\mathcal{A}} J(x,i;P), \quad (x,i) \in \mathbb{R}\times\mathcal{S}. \tag{3.1}$$

[6]This project started during a research visit at the University of Edmonton under the supervision of Abel Cadenillas.

The set $\mathcal{A}$ denotes the set of all admissible production strategies of the firm.

In this section, we consider two different admissible sets. For both, we assume that the production rate is non-negative. Hence, the production is irreversible meaning that the inventory decreases only because of the demand. The differences between both cases lies in the upper bound for the production rate. At first, we consider a model in which the production rate is unbounded from above. Then, (3.1) becomes a singular stochastic control problem with regime switching. In the second formulation, we consider an upper bound for the production rate and (3.1) becomes a bounded-velocity control problem with regime switching. In both cases, we apply the dynamic programming method to obtain an analytical solution for the optimal production control and the value function.

From an economic point of view, the first case implies the possibility of an immediate production of any amount of a good. This is a critical assumption, which does not hold in many applications. Hence, we compare the solution with the case of a bounded production rate at the end of this section.

3.2 The Singular Stochastic Control Case

In this section, we study the case of an unbounded production rate. The set of admissible strategies is defined by the (non-empty) set

$$\mathcal{A}_U := \Big\{ P : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ : \mathbb{F}\text{-adapted s.t.} t \mapsto P_t \text{ is a.s. non-decreasing,} \\ \text{left-continuous and } P_0 = 0 \text{ and s.t. } J(x,i;P) < \infty. \Big\}$$

Under these conditions, Problem (3.1) reads as follows.

Problem 3.1. The management aims at choosing the optimal production policy $\hat{P} \in \mathcal{A}_U$, which solves the problem

$$\begin{aligned} V(x,i) &:= \inf_{P \in \mathcal{A}_U} J(x,i;P) \\ &:= \inf_{P \in \mathcal{A}_U} \mathbb{E}\left[\int_0^\infty e^{-\delta s} \alpha_{\epsilon_s} (X_s - \mathcal{I}_{\epsilon_s})^2 ds + \int_0^\infty e^{-\delta s} k_{\epsilon_s} dP_s \right]. \end{aligned}$$

We now provide a preliminary growth property of the value function.

Lemma 3.2. *The set of admissible strategies $\mathcal{A}_U$ is non-empty and for all $(x,i) \in \mathbb{R} \times \mathcal{S}$ we have $V(x,i) \leq C(1+x^2)$ for some constant $C > 0$.*

Proof. Taking the no-production strategy $P^0 \equiv 0$. First, we observe that by standard estimates, one obtains that $J(x,i;P^0) \leq C(1+x^2)$ for some constant $C > 0$. Hence, P^0 is admissible and we get

$$0 \leq V(x,i) \leq J(x,i;P^0) \leq C(1+x^2).$$

□

Remark 3.3. For each admissible strategy $P \in \mathcal{A}_U$, we observe that

$$\int_0^\infty e^{-\delta s}\alpha_{\epsilon_s}(X_s - \mathcal{I}_{\epsilon_s})^2 ds \leq J(x,i;P) < \infty.$$

Since $\alpha_i > 0$ for all $i \in \mathcal{S}$, we also obtain that

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-\delta T} X_T^2\right] = 0. \tag{3.2}$$

The last equation becomes important in the proof of the verification theorem 3.5.

We observe that Problem 3.1 is a stochastic singular control problem with regime switching. We define by $\Lambda^P := \{t \geq 0 : P_{t+} \neq P_t\}$ the set of times where $P \in \mathcal{A}_U$ has a discontinuity. The set Λ^P is countable because P jumps only a countable number of times during the interval $[0,\infty)$. Moreover, we denote by P^d the discontinuous part of P, that is, $P_t^d := \sum_{0\leq s\leq t,\, s\in\Lambda}(P_{s^+} - P_s)$ and by P^c the continuous part of P, that is, $P_t^c := P_t - P_t^d$.

3.2.1 Verification Theorem

By the dynamic programming principle, we expect that V identifies with a suitable solution to the Hamilton-Jacobi-Bellman equation

$$\min\left\{(\mathcal{L}-\delta)v(x,i) + \alpha_i(x-\mathcal{I}_i)^2,\ \ v'(x,i) + k_i\right\} \ = \ 0, \tag{3.3}$$

for any $(x,i) \in \mathbb{R}\times\mathcal{S}$. $\mathcal{L}$ denotes the infinitesimal generator of (X,ϵ) acting on functions $g(\cdot,i) \in C^2(\mathbb{R})$ and, for a given $i \in \mathcal{S}$, it yields

$$\mathcal{L}g(x,i) := \frac{1}{2}\sigma_i^2 g_{xx}(x,i) - \mu_i g_x(x,i) - \lambda_i g(x,i) + \sum_{i\neq j\in\mathcal{S}} q_{ij}g(x,j). \tag{3.4}$$

Equation (3.3) can be derived, assuming that an optimal control exists, by investigating the two possible actions at each time: (i) do not produce for a small amount of time, and continue optimally; (ii) adjust the inventory by a lump sum increase, and continue optimally. In Theorem 3.5, we proof that a solution to (3.3), under some conditions, identifies with the value function of Problem 3.1

We note that, due to (3.4), hte HJB equation (3.3) leads to a system of N variational inequalities with state-dependent gradient constraints, which are coupled through the transition rates q_{ij}.

From (3.3), we can define, for each solution v to the HJB equation and regime $i \in \mathcal{S}$, the continuation region

$$\mathcal{C}^v(i) := \left\{\, x \in \mathbb{R} :\ (\mathcal{L}-\delta)v(x,i) + \alpha_i(x-\mathcal{I}_i)^2 = 0,\ \ k_i + v'(x,i) > 0\right\}$$

and the intervention region

$$\Sigma^v(i) := \left\{x \in \mathbb{R} : (\mathcal{L}-\delta)v(x,i) + \alpha_i(x-\mathcal{I}_i)^2 \geq 0,\ \ k_i + v'(x,i) = 0\,\right\}.$$

The regions $\mathcal{C}^v(i)$ and $\Sigma^v(i)$ provides a partion of $\mathbb{R}$.

For a solution v to (3.3), one can construct a control P^v, which holds the inventory inside the continuation region.

Definition 3.4. An $\mathbb{F}$-adapted, non-negative, and non-decreasing control process P^v is associated with the function v above if

$$\begin{aligned} &(i) \quad X_t^{P^v} := x - \int_0^t \mu_{\epsilon_s}\, ds - \int_0^t \sigma_{\epsilon_s}\, dW_s + P_t^v, \\ &(ii) \quad X_t^{P^v} \in \overline{\mathcal{C}(\epsilon_t)}, \quad \text{for every} \quad t \in (0, \infty), \quad \mathbb{P}-a.s., \\ &(iii) \quad \int_0^\infty \mathbb{1}_{\{X_s^{P^v} \in \mathcal{C}(\epsilon_s)\}}\, dP_s^v \;=\; 0, \quad \mathbb{P}-a.s. \end{aligned} \tag{3.5}$$

It turns out, see Theorem 3.5, that the associated control will identify the optimal control for Problem 3.1, if one can construct a suitable solution to the HJB equation. Moreover, $\mathcal{C}^v(i)$ identifies the region in which the firm is not producing. The region $\Sigma^v(i)$ instead identifies the region, in which production is the optimal.

Next, we present a verification theorem that provides sufficient conditions under which a solution to the HJB equation (3.3) identifies with the value function of Problem 3.1.

Theorem 3.5. *Let $v(\cdot, i) \in C^2(\mathbb{R} \setminus N_i)$, $i \in \mathcal{S}$, where N_i are finite subsets of $\mathbb{R}$, be a convex function on $\mathbb{R}$ with quadratic growth. Suppose that v satisfies the HJB equation* (3.3) *for all $(x, i) \in \mathbb{R} \times \mathcal{S}$. Then, for all admissible strategies $P \in \mathcal{A}_U$ it holds that*

$$v(x, i) \leq J(x, i; P).$$

Moreover, if the associated stochastic control P^v fulfills

$$\lim_{T \to \infty} \mathbb{E}\left[e^{-\delta T} v(X_{T+}^{P^v}, \epsilon_T)\right] = 0,$$

then

$$v(x, i) = J(x, i; P^v).$$

Therefore, v coincides with the value function V of Problem 3.1 and the control process P^v is the optimal production policy.

Proof. Consider an admissible control P and the corresponding semimartingale

$$X_t \;=\; x - \int_0^t \mu_{\epsilon_s} ds - \int_0^t \sigma_{\epsilon_s} dW_s + P_t^c + P_t^d.$$

Let $v(\cdot, i)$, $i \in \mathcal{S}$, be a solution to the HJB equation (3.3) and define $f(\cdot, \cdot, i)$, $i \in \mathcal{S}$, by $f(t, x, i) = e^{-\delta t} v(x, i)$. Following a procedure similar to Sotomayor and Cadenillas [94], we get, by applying Itô's formula for Markov modulated processes (see also Björk [15]),

$$
\begin{aligned}
df(t, X_t, \epsilon_t) &= \left(\frac{1}{2}\sigma_{\epsilon_t}^2 f_{xx}(t, X_t, \epsilon_t) - \mu_{\epsilon_t} f_x(t, X_t, \epsilon_t) + f_t(t, X_t, \epsilon_t)\right) dt \\
&+ f_x(t, X_t, \epsilon_t)\sigma_{\epsilon_t}\, dW_t + f_x(t, X_t, \epsilon_t)\, dP_t^c \\
&+ \Big(f(t, X_{t+}, \epsilon_t) - f(t, X_t, \epsilon_t)\Big) I_{\{t \in \Lambda\}} \\
&+ \left(-\lambda_{\epsilon_t} f(t, X_t, \epsilon_t) + \sum_{j \neq \epsilon_t} q_{\epsilon_t j} f(t, X_t, j)\right) dt + dM_t^f \\
&= e^{-\delta t}\left(\mathcal{L} - \rho\right) v(X_t, \epsilon_t) dt - \sigma_{\epsilon_t}\, e^{-\delta t} v_x(X_t, \epsilon_t)\, dW_t \\
&+ e^{-\delta t} v_x(X_t, \epsilon_t)\, dP_t^c + e^{-\delta t}\left(v(X_{t+}, \epsilon_t) - v(X_t, \epsilon_t)\right) I_{\{t \in \Lambda\}} + dM_t^f,
\end{aligned}
$$

where the process $M^f = \{M_t^f, t \geq 0\}$ is a square integrable martingale when $f(\cdot, \cdot, i)$, $i \in \mathcal{S}$, is bounded (see equation (5) in [15]). We observe that $v(\cdot, i)$, $i \in \mathcal{S}$, and $v_x(\cdot, i)$, $i \in \mathcal{S}$, are not necessarily bounded. However, we assume that $v(\cdot, i)$, $i \in \mathcal{S}$, is convex. Let a and b be real numbers satisfying $-\infty < a < X_0 = x < b < +\infty$ and define $\tau_a := \inf\{t \geq 0 : X_t = a\}$, $\tau_b := \inf\{t \geq 0 : X_t = b\}$ and $\tau := \tau_a \wedge \tau_b$. Then, for every time $t \in [0, \infty)$, we have

$$
\begin{aligned}
e^{-\delta(t\wedge\tau)} v(X_{(t\wedge\tau)^+} \epsilon_{t\wedge\tau}) &= v(X_0, \epsilon_0) + \int_0^{t\wedge\tau} e^{-\delta s}\left(\mathcal{L} - \delta\right) v(X_s, \epsilon_s) ds \\
- \int_0^{t\wedge\tau} \sigma_{\epsilon_s} e^{-\delta s} v_x(X_s, \epsilon_s) dW_s &+ \int_0^{t\wedge\tau} e^{-\delta s} v_x(X_s, \epsilon_s) dP_s^c \\
+ \sum_{0 \leq s \leq t\wedge\tau,\, s \in \Lambda} & e^{-\delta s}\left(v(X_{s^+}, \epsilon_s) - v(X_s, \epsilon_s)\right) + M_{t\wedge\tau}^f - M_0^f.
\end{aligned}
$$

Taking conditional expectations, we have

$$
\begin{aligned}
\mathbb{E}\left[e^{-\delta(t\wedge\tau)} v(X_{(t\wedge\tau)^+}, \epsilon_{t\wedge\tau})\right] &= v(x, i) + \mathbb{E}\left[\int_0^{t\wedge\tau} e^{-\delta s}\left(\mathcal{L} - \delta\right) v(X_s, \epsilon_s) ds\right] \\
- \mathbb{E}\left[\int_0^{t\wedge\tau} \sigma_{\epsilon_s} e^{-\delta s} v_x(X_s, \epsilon_s) dW_s\right] &+ \mathbb{E}\left[\int_0^{t\wedge\tau} e^{-\delta s} v_x(X_s, \epsilon_s) dP_s^c\right] \\
+ \mathbb{E}\left[\sum_{0 \leq s \leq t\wedge\tau,\, s \in \Lambda} e^{-\delta s}\left(v(X_{s^+}, \epsilon_s) - v(X_s, \epsilon_s)\right)\right] &+ \mathbb{E}\left[M_{t\wedge\tau}^f - M_0^f\right].
\end{aligned} \tag{3.6}
$$

The HJB equation (3.3) guarantees that $\left(\mathcal{L} - \delta\right) v(X_s, \epsilon_s) \geq -\alpha_{\epsilon_s}(X_s - \mathcal{I}_{\epsilon_s})^2$. Moreover, $v'(x, i) \geq -k_i$ for $x \in \mathbb{R}$ (recall equation (3.3)) and the mean value theorem implies that $v(y_1, i) - v(y_2, i) \geq -k_i(y_1 - y_2)$ for every $y_1, y_2 \in \mathbb{R}$, $y_1 > y_2$, and for every $i \in \mathcal{S}$. Hence, replacing $i = \epsilon_t$, $y_1 = X_t$ and $y_2 = X_{t+}$, we obtain $v(X_{t+}, \epsilon_t) - v(X_t, \epsilon_t) \geq -k_{\epsilon_t}(X_{t+} - X_t)$. By observing that $X_{t^+} - X_t = P_{t^+} - P_t$ we get from equation (3.6), that

$$\mathbb{E}\left[e^{-\delta(t\wedge\tau)}v(X_{(t\wedge\tau)^+},\epsilon_{t\wedge\tau})\right] \geq v(x,i) - \mathbb{E}\left[\int_0^{t\wedge\tau}\sigma_{\epsilon_s}e^{-\delta s}v_x(X_s,\epsilon_s)dW_s\right]$$
$$-\mathbb{E}\left[\int_0^{t\wedge\tau}e^{-\delta s}k_{\epsilon_s}dP^c_s\right] - \mathbb{E}\left[\sum_{0\leq s\leq t\wedge\tau, s\in\Lambda}e^{-\delta s}k_{\epsilon_s}\left(P_{s^+}-P_s\right)\right]$$
$$+\mathbb{E}\left[M^f_{t\wedge\tau}-M^f_0\right] - \mathbb{E}\left[\int_0^{t\wedge\tau}e^{-\delta s}\alpha_{\epsilon_s}(X_s-\mathcal{I}_{\epsilon_s})^2ds\right]$$
$$= v(x,i) - \mathbb{E}\left[\int_0^{t\wedge\tau}\sigma_{\epsilon_s}e^{-\delta s}v_x(X_s,\epsilon_s)dW_s\right] - \mathbb{E}\left[\int_0^{t\wedge\tau}e^{-\delta s}k_{\epsilon_s}dP_s\right]$$
$$+\mathbb{E}\left[M^f_{t\wedge\tau}- M^f_0\right] - \mathbb{E}\left[\int_0^{t\wedge\tau}e^{-\delta s}\alpha_{\epsilon_s}(X_s-\mathcal{I}_{\epsilon_s})^2ds\right]. \tag{3.7}$$

We note that $v(X_s,\epsilon_s)$ and $v_x(X_s,\epsilon_s)$ are bounded when $s\in[0,t\wedge\tau]$. Then, $\{M^f_{t\wedge\tau}, t\geq 0\}$ is a square integrable martingale and, hence, $\mathbb{E}\left[M^f_{t\wedge\tau}-M^f_0\right]=0$, for any $t\geq 0$. Furthermore, $\sigma^2_{\epsilon_s}\,e^{-2\delta s}(v_x(X_s,\epsilon_s))^2$ is bounded when $s\in[0,t\wedge\tau]$, and

$$\mathbb{E}\left[\int_0^{t\wedge\tau}\sigma_{\epsilon_s}\,e^{-\delta s}\,v_x(X_s,\epsilon_s)\,dW_s\right] \;=\; 0.$$

Letting $a\downarrow-\infty$ and $b\uparrow+\infty$, we get $\tau_a\to+\infty$ and $\tau_b\to+\infty$. Then, $\tau\to\infty$ and (3.7) leads to

$$v(x,i)\leq \mathbb{E}\left[e^{-\delta t}v(X_t+,\epsilon_t)\right]+\mathbb{E}\left[\int_0^{t\wedge\tau}e^{-\delta s}k_{\epsilon_s}dP_s\right]$$
$$+\mathbb{E}\left[\int_0^{t\wedge\tau}e^{-\delta s}\alpha_{\epsilon_s}(X_s-\mathcal{I}_{\epsilon_s})^2ds\right]. \tag{3.8}$$

Since P is assumed to be admissible, we can combine (3.2) and the quadratic growth of v to obtain

$$\lim_{t\uparrow\infty}\mathbb{E}\left[e^{-\delta t}v(X_{t^+},\epsilon_t)\right]\leq\mathbb{E}\left[e^{-\delta t}(1+X^2_{t^+})\right]=0.$$

Taking $t\to\infty$, we get from (3.8)

$$v(x,i)\leq\mathbb{E}\left[\int_0^{\infty}e^{-\delta s}k_{\epsilon_s}dP_s\right]+\mathbb{E}\left[\int_0^{\infty}e^{-\delta s}\alpha_{\epsilon_s}(X_s-\mathcal{I}_{\epsilon_s})^2ds\right]$$
$$=J(x,i;P).$$

Taking now the associated control P^v, see Definition 3.4, the inequality in (3.7) and therefore also in (3.8) becomes an equality. Hence

$$v(x,i)=\mathbb{E}\left[e^{-\delta t}v(X^{P^v}_t+,\epsilon_t)\right]+\mathbb{E}\left[\int_0^{t\wedge\tau}e^{-\delta s}k_{\epsilon_s}dP^v_s\right]$$
$$+\mathbb{E}\left[\int_0^{t\wedge\tau}e^{-\delta s}\alpha_{\epsilon_s}(X_s-\mathcal{I}_{\epsilon_s})^2ds\right]. \tag{3.9}$$

By assumption, P^v fulfills

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-\delta T} v(X_{T+}^{P^v}, \epsilon_T)\right] = 0.$$

Taking $t \to \infty$, we obtain from (3.9) that

$$\begin{aligned} v(x,i) &= \mathbb{E}\left[\int_0^\infty e^{-\delta s}\alpha_{\epsilon_s}(X_s^{P^v} - \mathcal{I}_{\epsilon_x})^2 ds + \int_0^\infty e^{-\delta s} k_{\epsilon_s} dP_s^v\right] \\ &= J(x,i;P^v). \end{aligned}$$

□

3.2.2 Construction of the Solution

In this section, we construct a candidate value function v that solves the HJB equation (3.3).

To construct such a solution, we conjecture, as stated in Theorem 3.5, that v is a convex function with quadratic growth. We define $b_i := \sup\{x \in \mathbb{R} : v_x(x,i) \leq -k_i\}$ for each $i \in \mathcal{S}$. Inspired by the HJB equation (3.3), we expect v to solve

$$\frac{1}{2}\sigma_i^2\, v_{xx}(x,i) - \mu_i v_x(x,i) - \delta v(x,i) + \alpha_i(x - \mathcal{I}_i)^2 = \lambda_i\, v(x,i) - \sum_{j\neq i} q_{ij}\, v(x,j)$$

for all $i \in \mathcal{S}$ and $x \in [b_i, \infty)$. For all $i \in \mathcal{S}$ and $x \in (-\infty, b_i)$, we expect v (due to the conjectured convexity) to satisfy $v_x(x,i) = -k_i$.

For simplicity, we assume in the remainder of this section that the economy shifts only between two regimes, i.e., $\mathcal{S} = \{1,2\}$. Under this assumption, the generator of ϵ is given by

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}.$$

The relation between b_1 and b_2 depends on the relations among the different parameters. We only consider the case $b_1 < b_2$. The case $b_1 > b_2$ can be treated similar. Due to our conjectures, we consider three possibilities for the initial level x of the inventory: $x \in (-\infty, b_1)$, $x \in [\,b_1, b_2)$, and $x \in [\,b_2, \infty)$.

For the construction, we need the following lemma, which we adapt from Remark 2.1 in Guo [50] (see also Lemma 3.1 in [94]).

Lemma 3.6. *For $i \in \mathcal{S}$, consider the real function $\phi_i(z) = -\sigma_i^2 z^2/2 - \tilde{\mu}_i z + (\lambda_i + \delta)$ where $\tilde{\mu}_i$ is a function of μ_i. Since σ_1, σ_2, λ_1 and λ_2 are positive, the equation $\phi_1(z)\,\phi_2(z) = \lambda_1\lambda_2$ has four real roots such that $z_1 < z_2 < 0 < z_3 < z_4$.*

We start with the case $x \in (b_2, \infty)$. By (3.3) we expect v to solve the system of differential equations

$$\begin{aligned} 0 &= -(\lambda_1 + \delta)v(x,1) - \mu_1 v_x(x,1) + \frac{1}{2}\sigma_1^2 v_{xx}(x,1) + \lambda_1 v(x,2) + \alpha_1(x - \mathcal{I}_1)^2, \\ 0 &= -(\lambda_2 + \delta)v(x,2) - \mu_2 v_x(x,2) + \frac{1}{2}\sigma_2^2 v_{xx}(x,2) + \lambda_2 v(x,1) + \alpha_2(x - \mathcal{I}_2)^2. \end{aligned} \tag{3.10}$$

This system is a system of two second-order ODEs and, according to Lemma 3.6, its solution is given by

$$\begin{aligned} v(x,1) &= A_1\, e^{\gamma_1(x-b_2)} \;+\; A_2\, e^{\gamma_2(x-b_2)} \;+\; A_3\, e^{\gamma_3(x-b_2)} \;+\; A_4\, e^{\gamma_4(x-b_2)} \\ &\quad + R_1(x-b_2)^2 + S_1(x-b_2) + T_1, \\ v(x,2) &= B_1\, e^{\gamma_1(x-b_2)} \;+\; B_2\, e^{\gamma_2(x-b_2)} \;+\; B_3\, e^{\gamma_3(x-b_2)} \;+\; B_4\, e^{\gamma_4(x-b_2)} \\ &\quad + R_2(x-b_2)^2 + S_2(x-b_2) + T_2, \end{aligned}$$

where A_j are constants for each $j = 1,2,3,4$ and

$$B_j = \frac{\phi_1^1(\gamma_j)}{\lambda_1}\, A_j = \frac{\lambda_2}{\phi_2^1(\gamma_j)}\, A_j. \tag{3.11}$$

The real values $\gamma_1 < \gamma_2 < 0 < \gamma_3 < \gamma_4$ above are the real roots of the characteristic equation $\phi_1^1(\gamma)\phi_2^1(\gamma) = \lambda_1\lambda_2$, where

$$\phi_i^1(\gamma) := -\frac{1}{2}\,\sigma_i^2\gamma^2 + \mu_i\,\gamma + (\lambda_i + \delta), \qquad i = 1,2.$$

Furthermore, R_i, S_i and T_i are the solution of the system

$$\begin{aligned} 0 &= -(\lambda_1+\delta)R_1 + \lambda_1 R_2 + \alpha_1, \\ 0 &= -(\lambda_1+\delta)S_1 - 2\mu_1 R_1 + \lambda_1 S_2 + 2\alpha_1(b_2 - \mathcal{I}_1), \\ 0 &= -(\lambda_1+\delta)T_1 - \mu_1 S_1 + \sigma_1^2 R_1 + \lambda_1 T_2 + \alpha_1(b_2 - \mathcal{I}_1)^2, \\ 0 &= -(\lambda_2+\delta)R_2 + \lambda_2 R_1 + \alpha_2, \\ 0 &= -(\lambda_2+\delta)S_2 - 2\mu_2 R_2 + \lambda_2 S_1 + 2\alpha_2(b_2 - \mathcal{I}_2), \\ 0 &= -(\lambda_2+\delta)T_2 - \mu_2 S_2 + \sigma_2^2 R_2 + \lambda_2 T_1 + \alpha_2(b_2 - \mathcal{I}_2)^2. \end{aligned} \tag{3.12}$$

By Theorem 3.5, we conjecture that $v(\cdot,i)$ admits quadratic growth. Thus, we set $B_3 = B_4 = A_3 = A_4 = 0$. Hence, the solution of the system (3.10) simplifies to

$$v(x,1) = A_1\, e^{\gamma_1(x-b_2)} \;+\; A_2\, e^{\gamma_2(x-b_2)} \;+ R_1(x-b_2)^2 + S_1(x-b_2) + T_1, \tag{3.13}$$

$$v(x,2) = B_1\, e^{\gamma_1(x-b_2)} \;+\; B_2\, e^{\gamma_2(x-b_2)} \;+ R_2(x-b_2)^2 + S_2(x-b_2) + T_2. \tag{3.14}$$

Next, we consider $x \in (b_1, b_2]$. We expect $v(x,2)$ to satisfy $v_x(x,2) = -k_2$. Therefore, we define $v(x,2) := -k_2(x-b_2) + D_2$ for a constant $D_2 \in \mathbb{R}$. Moreover, $v(x,1)$ should satisfy

$$-(\lambda_1+\delta)v(x,1) - \mu_1 v_x(x,1) + \frac{1}{2}\sigma_1^2 v_{xx}(x,1) + \lambda_1 v(x,2) + \alpha_1(x-\mathcal{I}_1)^2 = 0.$$

Solving this ODE, we obtain

$$v(x,1) = \widetilde{A}_1\, e^{\tilde\gamma_1(x-b_2)} + \widetilde{A}_2\, e^{\tilde\gamma_2(x-b_2)} + \tilde R_1(x-b_2)^2 + \tilde S_1(x-b_2) + \tilde T_1, \tag{3.15}$$

$$v(x,2) = -k_2(x-b_2) + D_2, \tag{3.16}$$

where $\tilde{\gamma}_1 < 0 < \tilde{\gamma}_2$ are the real roots of the equation

$$\phi_1^2(\tilde{\gamma}) := -\frac{1}{2}\sigma_1^2\tilde{\gamma}^2 + \mu_1\tilde{\gamma} + (\lambda_1 + \delta) = 0.$$

Furthermore, $\tilde{R}_1$, $\tilde{S}_1$ and $\tilde{T}_1$ are the solution of the system

$$\begin{aligned}
0 &= -(\lambda_1 + \delta)\tilde{R}_1 + \alpha_1, \\
0 &= -(\lambda_1 + \delta)\tilde{S}_1 - 2\mu_1\tilde{R}_1 - \lambda_1 k_2 + 2\alpha_1(b_2 - \mathcal{I}_1), \\
0 &= -(\lambda_1 + \delta)\tilde{T}_1 - \mu_1\tilde{S}_1 + \sigma_1^2\tilde{R}_1 + \lambda_1 D_2 + \alpha_1(b_2 - \mathcal{I}_1)^2.
\end{aligned}$$

Finally, for $x \in (-\infty, b_1]$, we expect v to satisfy $v_x(x,i) = -k_i$. Therefore, we set

$$v(x,1) := -k_1(x - b_1) + D_1, \tag{3.17}$$

$$v(x,2) := -k_2(x - b_2) + D_2 \tag{3.18}$$

for some constant $D_1 \in \mathbb{R}$.

It remains to find the free-boundaries b_1 and b_2, and the coefficients and constants in the equations (3.13)-(3.14),(3.15)-(3.16) and (3.17)-(3.18). Therefore, we impose the *smooth-fit* conditions. Thus, we expect v to solve the system of equations

$$\begin{array}{rcll}
v(b_2-,i) &=& v(b_2+,i) & \text{for both } i = 1,2, \\
v(b_1-,1) &=& v(b_1+,1), & \\
v_x(b_1+,1) &=& -k_1, & \\
v_x(b_2+,2) &=& -k_2, & \\
v_x(b_2-,1) &=& v_x(b_2+,1), & \\
v_{xx}(b_i+,i) &=& 0 & \text{for both } i = 1,2.
\end{array} \tag{3.19}$$

If our candidate value function v satisfy system (3.19), it follows that $v(\cdot,i) \in C^2(\mathbb{R} \setminus b_{3-i})$, which is also required in Theorem 3.5.

Remark 3.7. Note that the system (3.19) is a non-linear system of eight equations and eight unknowns. In the general case, we can neither provide existence nor uniqueness of the solution. We investigate later a concrete example, in which we solve the system numerically to provide the values of the free-boundaries and the coefficients.

3.2.3 Verification of the Solution

In this subsection, we prove that our candidate value function v coincides with the true value function of Problem 3.1.

To prove this, we need to verify our conjectures as well as the conditions of Theorem 3.5. Therefore, we first sho that we can upgrade the regularity of our candidate value function. Denote in the following the n-th derivative with respect to x, $n \geq 3$, by $v^{(n)}(x,i)$.

Lemma 3.8. *Let b_i, $i = 1,2$, A_j, $j = 1,2$, $\widetilde{A}_j$, $j = 1,2$, and D_i, $i = 1,2$, be the solution of the system of equations* (3.19). *Let B_j, $j = 1,2$, be defined by* (3.11). *Then, the function v defined by*

$$v(x,1) := \begin{cases} -k_1(x-b_1)+D_1 & x \in (-\infty, b_1), \\ \widetilde{A}_1\, e^{\tilde{\gamma}_1(x-b_2)} + \widetilde{A}_2\, e^{\tilde{\gamma}_2(x-b_2)} + \tilde{R}_1(x-b_2)^2 + \tilde{S}_1(x-b_2) + \tilde{T}_1 & x \in [b_1, b_2), \\ A_1\, e^{\gamma_1(x-b_2)} + A_2\, e^{\gamma_2(x-b_2)} + R_1(x-b_2)^2 + S_1(x-b_2) + T_1 & x \in [b_2, \infty), \end{cases}$$

and

$$v(x,2) := \begin{cases} -k_2(x-b_2)+D_2 & x \in (-\infty, b_2), \\ B_1\, e^{\gamma_1(x-b_2)} + B_2\, e^{\gamma_2(x-b_2)} + R_2(x-b_2)^2 + S_2(x-b_2) + T_2 & x \in [b_2, \infty), \end{cases}$$

is such that $v(\cdot,1) \in C^\infty(\mathbb{R} \setminus \{b_1,b_2\}) \cap C^4(\mathbb{R} \setminus \{b_1\}) \cap C^2(\mathbb{R})$ and $v(\cdot,2) \in C^\infty(\mathbb{R} \setminus \{b_2\}) \cap C^2(\mathbb{R})$. Moreover, for fixed $i \in \{1,2\}$ and $x > b_i$, we have

$$-(\lambda_i+\delta)v_{xx}(x,i) - \mu_i v^{(3)}(x,i) + \frac{1}{2}\sigma_1^2 v^{(4)}(x,i) + \lambda_i v_{xx}(x,3-i) + 2\alpha_i = 0. \quad (3.20)$$

Proof. First, note that by construction and system (3.19), it follows that $v(\cdot,2) \in C^\infty(\mathbb{R} \setminus \{b_2\}) \cap C^2(\mathbb{R})$. The regularity of $v(\cdot,1)$ can be shown as follows. By construction and system (3.19), $v(\cdot,1) \in C^\infty(\mathbb{R} \setminus \{b_1,b_2\}) \cap C^2(\mathbb{R} \setminus \{b_2\}) \cap C^1(\mathbb{R})$ and solves, for $x > b_1$,

$$-(\lambda_1+\delta)\, v(x,1) - \mu_1\, v_x(x,1) + \frac{1}{2}\, \sigma_1^2\, v_{xx}(x,1) + \lambda_1\, v(x,2) + \alpha_1(x-\mathcal{I}_1)^2 = 0.$$

Hence, we can rewrite this equation and obtain

$$v_{xx}(x,1) = \frac{2}{\sigma_1^2}\left((\lambda_1+\delta)\, v(x,1) + \mu_1\, v_x(x,1) - \lambda_1\, v(x,2)\right) \quad x > b_1.$$

Since $v(\cdot,2) \in C^2(\mathbb{R})$ and $v(\cdot,1) \in C^1(\mathbb{R})$ by (3.19), we obtain that $v(\cdot,1) \in C^4((b_1,\infty))$ by differentiating the last equation. Hence, we obtain that $v(\cdot,1) \in C^\infty(\mathbb{R} \setminus \{b_1,b_2\}) \cap C^4(\mathbb{R} \setminus \{b_1\}) \cap C^2(\mathbb{R})$. Furthermore, fixing $i \in \{1,2\}$ and $x > b_i$, we know that

$$-(\lambda_i+\delta)v(x,i) - \mu_i v_x(x,i) + \frac{1}{2}\sigma_1^2 v_{xx}(x,i) + \lambda_i v(x,3-i) + \alpha_i(x-I_i)^2 = 0.$$

Therefore, the proven regularity allows us to differentiate this equation two times with respect to x, and we get

$$-(\lambda_i+\delta)v_{xx}(x,i) - \mu_i v^{(3)}(x,i) + \frac{1}{2}\sigma_1^2 v^{(4)}(x,i) + \lambda_i v_{xx}(x,3-i) + 2\alpha_i = 0.$$

□

The proven regularity allows us to apply Itô's formula for Markov-modulated processes, as in the proof of Theorem 3.5, to $v_{xx}(\cdot,i)$. This allows us to prove convexity of the candidate for value function, see Theorem 3.10.

Before verifying that our candidate solution coincides with the true value function of Problem 3.1, we check that the associate control P^v fulfills the assumption of the Verification Theorem 3.5

Lemma 3.9. *For the constructed candidate value function v, the associated control P^v given by Definition 3.5, is such that*

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-\delta T} v(X_{T+}^{P^v}, \epsilon_T)\right] = 0.$$

Proof. First, notice that, given the structure of our candidate value function v, P^v solves the Skorokhod reflection problem

- $X_t^{P^v} \geq b_{\epsilon_t}$ $\mathbb{P}-a.s.$ for each $t > 0$,
- $\int_0^t \mathbb{1}_{\{X^{P^v} > b_{\epsilon_t}\}} \, dP^v = 0$ $\mathbb{P}-a.s.$ for each $t > 0$.

In particular, P^v is the minimal effort needed to hold the inventory process X, given the current state $i \in \mathcal{S}$, above the free-boundary b_i (see for example Karatzas and Schreve [60] and Skorokhod [90] for classical references). Moreover, we note that the candidate value function v is bounded from below by some constant $\underline{C}$. Hence, it holds that

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-\delta T} v(X_{T+}^{P^v}, \epsilon_T)\right] \geq \lim_{T\to\infty} e^{-\delta T}\underline{C} = 0.$$

Therefore, we only have to show the opposite direction. Since v has quadratic growth by construction, it is enough to show that

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-\delta T} v(X_{T+}^{P^v}, \epsilon_T)\right] \leq \lim_{T\to\infty} C_1 e^{-\delta T}\mathbb{E}\left[1 + \left(X_T^{P^v}\right)^2\right] = 0,$$

where $C_1 > 0$ is a constant. Define $\bar{\mu} := \max_{i\in\mathcal{S}} |\mu_i|$, $\bar{\sigma} := \max_{i\in\mathcal{S}} \sigma_i$ and

$$\bar{P}_t := \sup_{s\leq t} (b_2 - x)^+ + \int_0^s \mu_{\epsilon_u} \, du + \int_0^s \sigma_{\epsilon_u} \, dW_u.$$

The process $\bar{P}$ is a non-decreasing process such that $X_t^{\bar{P}} \geq b_2$ for all $t > 0$. In particular, since $b_2 > b_1$ by assumption, and P^v is the solution to the Skorokhod reflection problem above (hence it is the minimal process to keep the inventory above $b_{\epsilon_t} \leq b_2$), it holds that $X_t^{P^v} \leq X_t^{\bar{P}}$ for all $t \geq 0$. Moreover,

$$\begin{aligned} X_t^{\bar{P}} &= x + (b_2 - x)^+ + \sup_{s\leq t}\left(\int_0^s \mu_{\epsilon_u} \, du - \int_0^t \mu_{\epsilon_u} \, du + \int_0^s \sigma_{\epsilon_u} \, dW_u - \int_0^t \sigma_{\epsilon_u} \, dW_u\right) \\ &\leq x + (b_2 - x)^+ + \bar{\mu} t + 2\bar{\sigma} \sup_{s\leq t} |W_s|. \end{aligned}$$

Hence, for some constant $C_2 > 0$, we have that

$$\begin{aligned} e^{-\delta T}\mathbb{E}\left[\left(X_T^{P^v}\right)^2\right] &\leq e^{-\delta T}\mathbb{E}\left[b_1^2 + \left(X_T^{\bar{P}}\right)^2\right] \\ &\leq e^{-\delta T}\left(b_1^2 + C_2\left(x + (b_2 - x)^+ + \bar{\mu} T\right)^2 + 4\bar{\sigma}^2 C_2 \mathbb{E}\left[\left(\sup_{s\leq T} |W_s|\right)^2\right]\right) \\ &\leq e^{-\delta T}\left(b_1^2 + C_2\left(x + (b_2 - x)^+ + \bar{\mu} T\right)^2 + 16\bar{\sigma}^2 C_2 \mathbb{E}\left[|W_T|^2\right]\right), \end{aligned}$$

where the last inequality follows by Doob's maximal inequality. Since $\mathbb{E}\left[|W_T|^2\right] = T^2$, it follows, by taking $T \to \infty$, that

$$\lim_{T\to\infty} e^{-\delta T}\mathbb{E}\left[\left(X_T^{P^v}\right)^2\right] = 0.$$

Hence,

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-\delta T} v(X_{T+}^{P^v}, \epsilon_T)\right] \le \lim_{T\to\infty} C_1 e^{-\delta T}\mathbb{E}\left[1 + \left(X_T^{P^v}\right)^2\right] = 0.$$

□

Theorem 3.10. *Let b_i, $i = 1, 2$, A_j, $j = 1, 2$, $\widetilde{A}_j$, $j = 1, 2$, and D_i, $i = 1, 2$, be the solution of the system of equations* (3.19). *Let B_j, $j = 1, 2$, be defined by* (3.11). *Then, the function v given by*

$$v(x,1) := \begin{cases} -k_1(x-b_1) + D_1 & x \in (-\infty, b_1), \\ \widetilde{A}_1\, e^{\tilde{\gamma}_1(x-b_2)} + \widetilde{A}_2\, e^{\tilde{\gamma}_2(x-b_2)} + \tilde{R}_1(x-b_2)^2 + \tilde{S}_1(x-b_2) + \tilde{T}_1 & x \in [b_1, b_2), \\ A_1\, e^{\gamma_1(x-b_2)} + A_2\, e^{\gamma_2(x-b_2)} + R_1(x-b_2)^2 + S_1(x-b_2) + T_1 & x \in [b_2, \infty), \end{cases}$$

and

$$v(x,2) := \begin{cases} -k_2(x-b_2) + D_2 & x \in (-\infty, b_2), \\ B_1\, e^{\gamma_1(x-b_2)} + B_2\, e^{\gamma_2(x-b_2)} + R_2(x-b_2)^2 + S_2(x-b_2) + T_2 & x \in [b_2, \infty), \end{cases}$$

is the value function V of Problem 3.1 and the optimal production policy is given by P^v, described by Definition 3.4

Proof. To prove that the function v defined above coincides with the value function of Problem 3.1, it is enough to show that it satisfies the conditions of Theorem 3.5. Note that $v(\cdot, i)$ is of quadratic growth by construction. We start showing that v is convex. We know already that $v_{xx}(x, i) = 0$ for all $x \le b_i$ and $i \in \{1, 2\}$.

By Lemma 3.8, we know that $v(\cdot, i) \in C^4(\mathbb{R} \setminus \{b_i\})$. Hence, for fixed $i \in \{1, 2\}$ and $x > b_i$, we can apply Itô's formula for Markov modulated processes and obtain, as in the proof of Theorem 3.5, for $v_{xx}(\cdot, i)$ that

$$\mathbb{E}\left[e^{-\delta\tau} v_{xx}(X_\tau, \epsilon_\tau)\right] = v_{xx}(x, i) + \mathbb{E}\left[\int_0^\tau e^{-\delta s}\left(\mathcal{L} - \delta\right) v_{xx}(X_s, \epsilon_s)\, ds\right],$$

where $\tau := \inf\{t \ge 0 : X_t \le b_{\epsilon_t}\}$. We obtain that

$$\left(\mathcal{L} - \delta\right) v_{xx}(X_s^{x,i}, \epsilon_s) = -2\alpha_{\epsilon_s}, \quad \forall s < \tau.$$

Hence,

$$v_{xx}(x, i) = \mathbb{E}\left[2\int_0^\tau e^{-\delta s}\alpha_{\epsilon_s}\, ds + e^{-\delta\tau} v_{xx}(X_\tau, \epsilon_\tau)\right]. \tag{3.21}$$

Since $v_{xx}(x, i) = 0$ for all $x \le b_i$ by construction, (3.21) implies, together with the fact that $\alpha_i > 0$, that $v_{xx}(x, i) > 0$ for all $i \in \{1, 2\}$ and $x > b_i$. Moreover, $v_{xx}(x, i) = 0$ for all $x \le b_i$ by construction. This proves the convexity of $v(\cdot, i)$ for $i = 1, 2$.

Next, we show that the candidate value function v satisfies the HJB equation (3.3). By construction and system (3.19), we have that $v_x(x, i) = -k_i$ for all $x \le b_i$ and $\left(\mathcal{L} - \delta\right) v(x, i) = 0$ for all $x \ge b_i$. Therefore, we need to show that for $i \in \{1, 2\}$

i) $v_x(x,i) \geq -k_i, \quad x \geq b_i,$

ii) $(\mathcal{L}-\delta)\, v(x,i) + \alpha_i(x-\mathcal{I}_i)^2 \geq 0, \quad x \leq b_i.$

Item i) follows from the convexity and equation three and four of system (3.19). Now we show item ii). Define

$$g(x,i) := (\mathcal{L}-\delta)\, v(x,i) + \alpha_i(x-\mathcal{I}_i)^2.$$

From Lemma 3.8, we obtain that

$$g(\cdot,1) \in \mathcal{C}^0(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}\setminus\{b_1\}), \quad g(\cdot,2) \in \mathcal{C}^0(\mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R}\setminus\{b_2\}).$$

Moreover, by construction we have that $g(x,i)=0$ for all $x \geq b_i$ and $i \in \mathcal{S}$. We want to show that $g(x,i) \geq 0$ for all $x \leq b_i$ and $i \in \{1,2\}$.

Let $x \in [b_1, b_2)$. We have $v(x,2) = -k_2(x-b_2) + D_2$. Hence,

$$\begin{aligned} g(x,2) &= \mu_2 k_2 - (\delta+\lambda_2)[-k_2(x-b_2)+D_2] + \lambda_2 v(x,1) + \alpha_2(x-\mathcal{I}_2)^2, \\ g_x(x,2) &= (\delta+\lambda_2)k_2 + \lambda_2 v_x(x,1) + 2\alpha_2(x-\mathcal{I}_2), \end{aligned}$$

and, due to the convexity,

$$g_{xx}(x,2) = \lambda_2 v_{xx}(x,1) + 2\alpha_2 > 0.$$

We want to show that $g(x,2) \geq 0$. Since $g(b_2-,2) = g(b_2+,2) = 0$, it is enough to show that $g(x,2)$ is decreasing, i.e., $g_x(x,2) \leq 0$. Since $g_{xx}(x,2) \geq 0$, we have that $g_x(x,2)$ is increasing. Hence, $g_x(x,2) \leq g_x(b_2-,2)$. Thus, it is enough to show that

$$0 \geq g_x(b_2-,2) = k_2(\delta+\lambda_2) + \lambda_2 v_x(b_2-,1) + 2\alpha_2(x-\mathcal{I}_2). \tag{3.22}$$

We investigate $\lambda_2 v_x(b_2-,1)$. By construction of v and the continuity of v_x, we obtain that

$$\begin{aligned} \lambda_2 v_x(b_2-,1) &= \lambda_2 v_x(b_2+,1) = \lambda_2[S_1 + \sum_{i=1}^4 \gamma_i A_i] = \lambda_2[S_1 + \sum_{i=1}^4 \gamma_i \frac{\phi_2^1(\gamma_i)}{\lambda_2} B_i] \\ &= \lambda_2 S_1 + \sum_{i=1}^4 \gamma_i B_i[-\frac{1}{2}\sigma_2^2\gamma_i^2 + \mu_2\gamma_i + (\lambda_2+\delta)] \\ &= \lambda_2 S_1 - \frac{1}{2}\sigma_2^2 \sum_{i=1}^4 \gamma_i^3 B_i + \mu_2 \sum_{i=1}^4 \gamma_i^2 B_i + (\lambda_2+\delta)\sum_{i=1}^4 \gamma_i B_i \\ &= \lambda_2 S_1 - \frac{1}{2}\sigma_2^2 v^{(3)}(b_2+,2) + \mu_2\left(v_{xx}(b_2+,2) - 2R_2\right) \\ &\quad + (\lambda_2+\delta)\left(v_x(b_2+,2) - S_2\right) \\ &\leq \lambda_2 S_1 - 2\mu_2 R_2 - k_2(\lambda_2+\delta) - (\lambda_2+\delta)S_2, \end{aligned} \tag{3.23}$$

where the last step follows from the fact that $v_{xx}(b_2+,2)=0$ by the smooth-fit conditions and $v^{(3)}(b_2+,2) \geq 0$. The fact that $v^{(3)}(b_2+,2) \geq 0$ follows from convexity.

Assume, in the opposite, $v^{(3)}(b_2+,2) < 0$. This implies, since $v_{xx}(b_2+,2) = 0$, that $v_{xx}(b_2+\varepsilon,2) < 0$ for ε small enough, which contradicts convexity. Hence, using (3.12), we combine (3.22) and (3.23) to obtain

$$g_x(b_2-,2) \le \lambda_2 S_1 - 2\mu_2 R_2 + 2\alpha_2(b_2 - \mathcal{I}_2) - (\lambda_2+\delta)S_2 = 0.$$

Therefore, we obtain, that

$$g_x(x,2) \le g_x(b_2-,2) \le 0, \quad x \ge b_1 \tag{3.24}$$

and

$$g(x,2) \ge g(b_2-,2) = 0, \quad x \ge b_1. \tag{3.25}$$

Finally, we consider $x < b_1$. From the construction of v, we obtain

$$\begin{aligned} g(x,i) &= \mu_i k_i - \delta v(x,i) - \lambda_i v(x,i) + \lambda_i v(x,3-i) + \alpha_i(x-\mathcal{I})^2, \\ g_x(x,i) &= \delta k_i + \lambda_i k_i - \lambda_i k_{3-i} + 2\alpha_i(x-\mathcal{I}), \\ g_{xx}(x,i) &= 2\alpha_i > 0. \end{aligned}$$

Hence, we have

$$g_x(x,i) \le g_x(b_1-,i).$$

For $i=2$, we obtain from (3.24) that $g_x(b_1-,2) = g_x(b_1+,2) \le 0$. Hence, by (3.25), it holds that $g(x,2) \ge g(b_1-,2) = g(b_1+,2) \ge 0$.

For $i=1$, we note that $g(x,1)=0$ for all $x \ge b_1$. Thus, we have $g_x(x,1) = 0$ for all $x \ge b_1$. Hence,

$$\begin{aligned} 0 = g_x(b_1+,1) &= \frac{1}{2}\sigma_1^2 v^{(3)}(b_1+,1) - \mu_1 v_{xx}(b_1+,1) - (\delta+\lambda_1)v_x(b_1+,1) \\ &\quad + \lambda_1 v_x(b_1+,2) + 2\alpha_1(b_1 - \mathcal{I}_1). \end{aligned} \tag{3.26}$$

Next, we argue by contradiction that $v^{(3)}(b_1+,1) \ge 0$. Therefore, assume that $v^{(3)}(b_1+,1) < 0$. By the fact that $v_{xx}(b_1+,1) = 0$, this implies that $v_{xx}(b_1+\varepsilon,1) < 0$ for ε small enough. This contradicts the convexity of v.

Using that $v_{xx}(b_1+,1) = 0$, $v^{(3)}(b_1+,1) \ge 0$, and that $v_x(b_1+,i) = -k_i$, we obtain from (3.26)

$$0 \ge (\delta+\lambda_1)k_1 - \lambda_1 k_2 + 2\alpha_1(b_1 - \mathcal{I}_1). \tag{3.27}$$

Let now $x \le b_1$. Since $g_{xx}(x,1) = 2\alpha_1$ we have that

$$g_x(x,1) \le g_x(b_1-,1) = (\delta+\lambda_1)k_1 - \lambda_1 k_2 + 2\alpha_1(b_1 - \mathcal{I}_1) \le 0,$$

where the last inequality follows from (3.27). Hence, $g(x,1)$ is decreasing and by continuity we have

$$g(x,1) \ge g(b_1-,x) = g(b_1+,x) = 0.$$

This implies that v is a solution to the HJB equation (3.3).

Finally, we show that the associated control P^v is admissible. From the construction of P^v, it is enough to show that $J(x,i;P^v)$ is finite. Arguing as in the proof of Theorem 3.5, we obtain

$$v(x,i) = \mathbb{E}\left[e^{-\delta T} v(X_{T+}, \epsilon_T)\right] + \mathbb{E}\left[\int_0^t e^{-\delta s} \alpha_{\epsilon_s} \left(X_s - \mathcal{I}_{\epsilon_s}\right)^2 \, ds\right] + \mathbb{E}\left[\int_0^t e^{-\delta s} k_{\epsilon_S} \, dP_s^v\right]. \tag{3.28}$$

From Lemma 3.9, we know that

$$\lim_{T \to \infty} \mathbb{E}\left[e^{-\delta T} v(X_{T+}^{P^v}, \epsilon_T)\right] = 0.$$

Therefore, taking $T \to \infty$ in (3.28), leads to

$$\infty > v(x,i) = J(x,i;P^v).$$

Hence, v and its associated control P^v fulfill all the conditions of Theorem 3.5, and v is indeed the true value function. □

Summarizing our results, we have proven that our candidate value function v coincides with the true value function of Problem 3.1. Moreover, the associated control P^v is the optimal production strategy.

Therefore, the optimal production policy, given the regime is i, works as follows: (a) do not produce as long as the inventory level is above the threshold b_i, and (b) adjust the inventory by a lump sum production to keep the inventory level above b_i whenever the inventory level falls below b_i. We observe that the firm also increase production just because the regime is changing. This happens whenever the level of the inventory lies in the interval (b_1, b_2) and the regime changes from $i = 1$ to $i = 2$.

Remark 3.11. We note that the verification theorem implies that the system (3.19) has at most one solution such that $b_1 < b_2$. If there would be two distinct solutions, then both leads to the same value function, which is a contradiction.

3.2.4 Comparative Statics and Numerical Examples

In this subsection, we study the influence of some model parameters on the free-boundaries $b_i, i \in \mathcal{S}$, which characterize the optimal production strategy. First, we present the results that can be obtained analytically and afterwards those which were derived numerically. To derive the analytical results, we use the link between SSC and OS for monotone follower problems. Following the technique of *switching paths at appropiate random times*, see Karatzas and Shreve [58], we obtain $V_x(x,i) = u(x,i)$, where

$$u(x,i) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\int_0^\tau e^{-\delta s} 2\alpha_{\epsilon_s}(X_s^{x,0} - \mathcal{I}_{\epsilon_s}) \, ds - e^{-\delta \tau} k_{\epsilon_\tau}\right].$$

Here, $\mathcal{T}$ denotes the set of stopping times and $X^{x,0}$ the inventory process in absence of any production. As a consequence from $V_x(x,i) = u(x,i)$ it follows that

$$b_i = \sup\{x \in \mathbb{R} : u(x,i) \leq -k_i\}. \tag{3.29}$$

Lemma 3.12. *For each $i \in \mathcal{S}$, the function $x \mapsto u(x,i)$ is non-decreasing.*

Proof. The claim follows immediately from the fact that $x \mapsto X_s^{x,0}$ is non-decreasing. □

In the following, we denote by $b_i(a)$ the free-boundary b_i, given the parameter a. Moreover, we denote by $u(x,i;a)$ the function $u(\cdot,i)$ with respect to the parameter a.

Lemma 3.13. *For given $i \in \mathcal{S} = \{1,2\}$, the free-boundaries b_i have the following properties:*

i) $I_j \mapsto b_i(I_j)$ is non-decreasing for every $j \in \mathcal{S}$,

ii) $\mu_j \mapsto b_i(\mu_j)$ is non-decreasing for every $j \in \mathcal{S}$,

iii) $k_i \mapsto b_i(k_i)$ is non-increasing,

iv) $k_{3-i} \mapsto b_i(k_{3-i})$ is non-decreasing.

Proof. We define

$$\hat{u}(x,i) := u(x,i) + k_i = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\int_0^\tau e^{-\delta s} 2\alpha_{\epsilon_s}(X_s^{x,0} - \mathcal{I}_{\epsilon_s})\, ds \right.$$
$$\left. + \left(1 - e^{-\delta\tau}\right) k_i \mathbb{1}_{\{\epsilon_\tau = i\}} + \left(k_i - e^{-\delta\tau} k_{3-i}\right) \mathbb{1}_{\{\epsilon_\tau = 3-i\}}\right].$$

One can easily show that

1) $I_j \mapsto u(x,i;I_j)$ is non-increasing for every $j \in \mathcal{S}$,

2) $\mu_j \mapsto u(x,i;\mu_j)$ is non-increasing for every $j \in \mathcal{S}$ since $\mu_j \mapsto X_t^{x,0} = x - \int_0^t \mu_{\epsilon_s}\, dt - \int_0^t \sigma_{\epsilon_s}\, dt$ is non-increasing,

3) $k_i \mapsto \hat{u}(x,i;k_i)$ is non-decreasing,

4) $k_{3-i} \mapsto \hat{u}(x,i;k_{3-i})$ is non-increasing.

Thus, we can prove each result separately.

i) Taking $j \in \mathcal{S}$, $I_j^1 > I_j^2$ and using (3.29), Lemma 3.12 and 1), we have

$$b_i(I_j^2) := \sup\{x : u(x,i;I_j^2) \le -k_i\} \le \sup\{x : u(x,i;I_j^1) \le -k_i\} =: b_i(I_j^1).$$

ii) Taking $j \in \mathcal{S}$, $\mu_j^1 > \mu_j^2$ and using (3.29), Lemma 3.12 and 2), we have

$$b_i(\mu_j^2) := \sup\{x : u(x,i;\mu_j^2) \le -k_i\} \le \sup\{x : u(x,i;\mu_j^1) \le -k_i\} =: b_i(\mu_j^1).$$

iii) Taking $k_i^1 > k_i^2$ and using (3.29), Lemma 3.12 and 3), we have

$$b_i(k_i^2) := \sup\{x : \hat{u}(x,i;k_i^2) \le 0\} \ge \sup\{x : u(x,i;k_i^1) \le 0\} =: b_i(k_i^1).$$

iv) Taking $k^1_{3-i} > k^2_{3-i}$ and using (3.29), Lemma 3.12 and 4), we have

$$b_i(k^2_{3-i}) := \sup\{x : \hat{u}(x, i; k^2_{3-i}) \leq 0\} \leq \sup\{x : u(x, i; k^1_{3-i}) \leq 0\} =: b_i(k^1_{3-i}).$$

□

The results so far are obtained analytically. But to study the dependence of the free-boundaries b_i with respect to σ_i and λ_i, we have to apply numerical methods in order to solve system (3.19). This is done by using Mathematica. Therefore, we introduce a benchmark case with the following parameter values:

$$\text{Regime 1: } \mu_1 = 0.2,\ \sigma_1 = 0.2,\ \alpha_1 = 0.2,\ I_1 = 2,\ k_1 = 2,\ \lambda_1 = 0.2$$
$$\text{Regime 2: } \mu_2 = 1,\ \sigma_2 = 0.2,\ \alpha_2 = 0.2,\ I_2 = 2,\ k_2 = 2,\ \lambda_2 = 0.2$$

and a discount rate of $\delta = 0.2$.

In this setting, the two regimes are clearly distinguishable by the drift. In particular, regime 1 is connected to a regime with lower demand. Hence we call it the *low regime*. Regime 2 is connected to a regime with higher demand, called the *high regime*.

Considering the above parameters, the solution to system (3.19) can be obtained numerically and is given by

$$A_1 = -0.206,\ A_2 = -15.429,\ \tilde{A}_1 = -0.308,\ \tilde{A}_2 = 0.0,\ D_1 = 5.844,\ D_2 = 8.382.$$

For the free-boundaries, we obtain

$$b_1 = 0,914, \quad b_2 = 0.977.$$

Figure 1 illustrates the value function for both regimes.

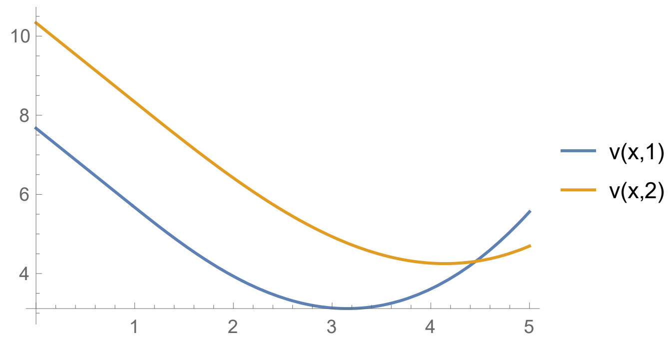


Figure 1: Graphical illustration of the value function for the benchmark case.

Starting from the benchmark case, we study the dependence of b_1 and b_2 on the uncertainty parameters σ_i. The results are presented in Figure 2.

One can see that both free-boundaries are decreasing in the volatility parameters σ_i. Hence, a higher uncertainty in the demand lowers in both regimes the value up to which the firm would produce. Therefore, higher uncertainty leads to less production. Moreover, if the uncertainty in the high regime is too high in comparison to the one in the low regime, the role of the two regimes switches (see the empty area in Figure 2).

Next, we investigate the dependence of the free-boundaries on λ_i.

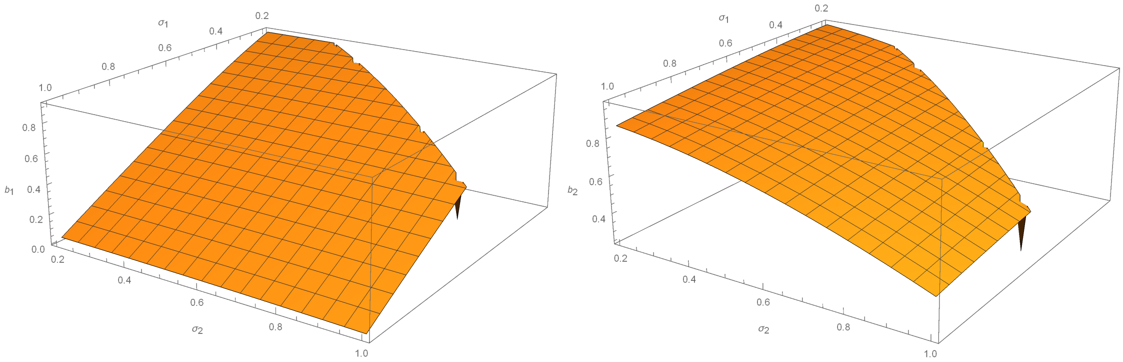


Figure 2: Graphical illustration of the dependence of the free-boundaries on σ_i. The other parameters are assumed to be the same as in the benchmark case. In the empty space in the right corner, the roles of state 1 and state 2 switches and b_1 becomes bigger than b_2. We let this part be empty.

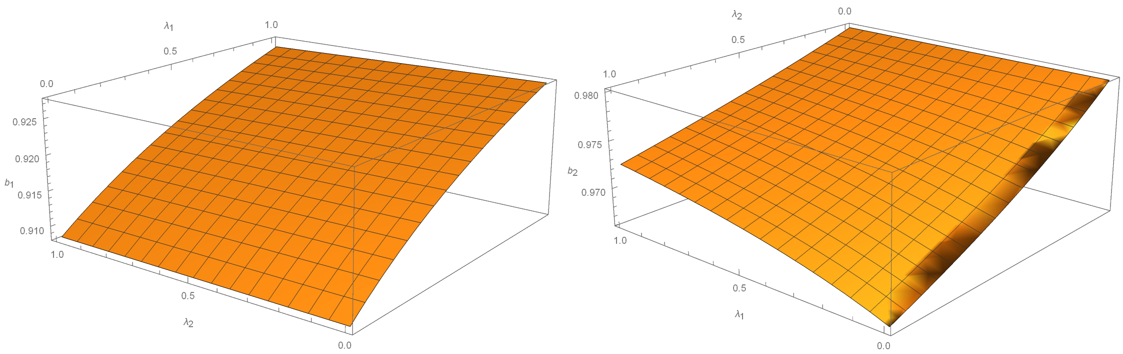


Figure 3: Graphical illustration of the dependence of the free-boundaries on λ_i. The other parameters are assumed to be the same as in the benchmark case.

In Figure 3 we see that both free-boundaries are increasing in λ_1, but decreasing in λ_2. Hence, a higher probability of leaving the low regime increases both free-boundaries, and leads to more production, because it is more likely to be in the regime with higher demand. Analogously, if the probability of leaving the high regime increases, it leads to lower production in both states.

For completeness, we introduce Figure 4, which shows the behavior of the free-boundaries with respect to μ_i, see Lemma 3.13. More precisely,one see that both free-boundaries converge to each other if the drift components become equal.

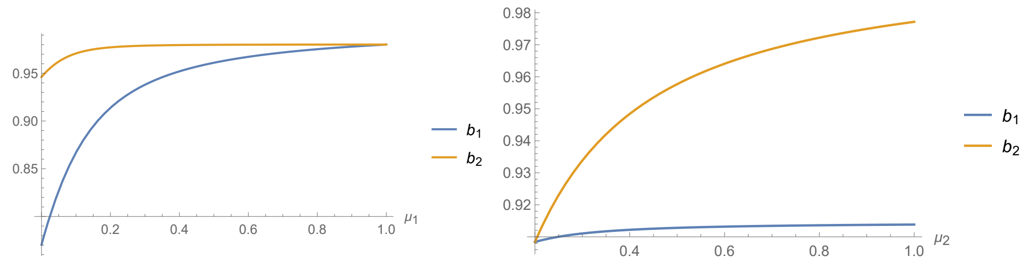


Figure 4: Graphical illustration of the dependence of the free-boundaries on μ_i. The other parameters are assumed to be the same as in the benchmark case.

3.3 The Bounded-Velocity Control Case

In this section we consider the model with bounded production rates. In this case, the production process $P = \{P_t, t \geq 0\}$ is represented as $dP_t = p_t\, dt$, where p is the production rate. Let K_i, $i \in \mathcal{S}$, be positive real numbers. The set of admissible strategies is defined by the (non-empty) set

$$\mathcal{A}_B := \left\{ p : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ : \mathbb{F}\text{−adapted s.t. } p(\omega, t) \leq K_{\epsilon_t} \right\}.$$

We note that for an admissible control p, the inventory is given by

$$X_t \;=\; x + \int_0^t (p_s - \mu_{\epsilon_s})\; ds - \int_0^t \sigma_{\epsilon_s}\, dW_s$$

for every $t \in [0, \infty)$.

Under these assumptions, Problem 3.1 reads as follows.

Problem 3.14. The management aims at choosing the optimal production rate $\hat{p} \in \mathcal{A}_B$, which solves the problem

$$\begin{aligned} V(x,i) &:= \inf_{p \in \mathcal{A}_B} J(x,i;p) \\ &:= \inf_{p \in \mathcal{A}_B} \mathbb{E}\left[\int_0^\infty e^{-\delta s} \alpha_{\epsilon_s} (X_s - \mathcal{I}_{\epsilon_s})^2 ds + \int_0^\infty e^{-\delta s} k_{\epsilon_s} p_s ds \right]. \end{aligned}$$

Remark 3.15. We note that

$$J(x,i;p) < \infty$$

for all control processes $p \in \mathcal{A}_B$. Moreover, as in the SSC case, see Lemma 3.2, it holds that

$$0 \leq V(x,i) \leq C\left(1 + x^2\right) \tag{3.30}$$

for some constant $C > 0$.

3.3.1 Verification Theorem

By the dynamic programming principle, we expect V to identify with a suitable solution to the Hamilton-Jacobi-Bellman equation

$$\inf_{p\in[0,K_i]}\{(\mathcal{L}^p-\delta)\,v(x,i)+k_ip\}+\alpha_i(x-\mathcal{I}_i)^2=0 \tag{3.31}$$

for any $(x,i)\in\mathbb{R}\times\mathcal{S}$. $\mathcal{L}^p$ denotes, given a production strategy p, the infinitesimal generator of (X,ϵ), acting on functions $g(\cdot,i)\in C^2(\mathbb{R})$, and, for given $i\in\mathcal{S}$, it yields

$$\mathcal{L}^p g(x,i):=\frac{1}{2}\sigma_i^2 g_{xx}+(p-\mu_i)g_x-\lambda_i g(x,i)+\sum_{i\neq j\in\mathcal{S}} q_{ij}g(x,j).$$

Since we consider here, in contrast to the SSC case, only controls of bounded-velocity, the corresponding HJB equation (3.31) results in a system of N ODEs, coupled through the transition rates q_{ij}, *without* gradient constrains.

Next, we present a verification theorem that provides sufficient conditions under which a solution to the HJB equation (3.31) identifies with the value function of Problem 3.14.

Theorem 3.16. *Let $v(\cdot,i)\in C^2(\mathbb{R}\setminus N_i)$, $i\in\mathcal{S}$, where N_i are finite subsets of $\mathbb{R}$ be a solution to the HJB equation (3.31) for all $(x,i)\in\mathbb{R}\times\mathcal{S}$. Moreover, assume that v is convex with quadratic growth, so*

$$|v(x,i)|\le C\left(1+x^2\right).$$

Then, the control $\hat{p}$ defined by

$$\hat{p}_t=\arg\inf_{p\in[0,K_{\epsilon_t}]}\{(\mathcal{L}^p-\delta)\,v(x,i)+k_ip\},$$

$t\in[0,\infty)$, is the optimal control for Problem 3.14. Moreover, it holds that $v(x,i)=J(x,i;\hat{p})$ and v is equal to the value function V of Problem 3.14.

Proof. Consider an admissible control $p=(p_t)_{\{t\in[0,\infty)\}}$. Let $v(\cdot,i)$, $i\in\mathcal{S}$ be a solution of the HJB equation (3.31) and define the function $f(\cdot,\cdot,i)$, $i\in\mathcal{S}$ by $f(t,x,i)=e^{-\delta t}v(x,i)$. Following a procedure similar to Sotmayor and Cadenillas [94], an application of Itô's formula for Markov modulated processes (see also [15]), leads to

$$\begin{aligned}df(t,X_t,\epsilon_t)&=\left(\frac{1}{2}\sigma_{\epsilon_t}^2 f_{xx}(t,X_t,\epsilon_t)+(p_t-\mu_{\epsilon_t})f_x(t,X_t,\epsilon_t)+f_t(t,X_t,\epsilon_t)\right)dt\\&-f_x(t,X_t,\epsilon_t)\sigma_{\epsilon_t}dW_t+\left(\lambda_{\epsilon_t}f(t,X_t,\epsilon_t)+\sum_{\epsilon_t\neq j\in\mathcal{S}}q_{\epsilon_t j}f(t,X_t,j)\right)dt+dM_t^f.\\&=e^{-\delta t}\left(\mathcal{L}^p-\delta\right)v(X_t,\epsilon_t)dt-\sigma_{\epsilon_t}e^{-\delta t}v_x(X_t,\epsilon_t)dW_t+\,dM_t^f.\end{aligned}$$

Here, the process $M^f=\{M_t^f,t\ge 0\}$ is a square integrable martingale when $f(\cdot,\cdot,i)$, $i\in\mathcal{S}$, is bounded (see equation (5) in [15]).

We observe that $v(\cdot, i)$ and $v_x(\cdot, i)$ are not necessarily bounded. Let a and b be real numbers such that $-\infty < a < X_0 = x < b < +\infty$ and define $\tau_a := \inf\{t \geq 0 : X_t = a\}$, $\tau_b := \inf\{t \geq 0 : X_t = b\}$ and $\tau := \tau_a \wedge \tau_b$. Then, for every $t \in [0, \infty)$, we get

$$e^{-\delta(t\wedge\tau)} v(X_{t\wedge\tau}, \epsilon_{t\wedge\tau}) = v(X_0, \epsilon_0) + \int_0^{t\wedge\tau} e^{-\delta s} \left(\mathcal{L}^p - \delta\right) v(X_s, \epsilon_s) ds \\ - \int_0^{t\wedge\tau} \sigma_{\epsilon_s} e^{-\delta s} v_x(X_s, \epsilon_s) dW_s + M^f_{t\wedge\tau} - M^f_0.$$

Taking expectations, we have

$$\mathbb{E}\left[e^{-\delta(t\wedge\tau)} v(X_{t\wedge\tau}, \epsilon_{t\wedge\tau})\right] = v(x, i) + \mathbb{E}\left[\int_0^{t\wedge\tau} e^{-\delta s} \left(\mathcal{L}^p - \delta\right) v(X_s, \epsilon_s) ds\right] \\ - \mathbb{E}\left[\int_0^{t\wedge\tau} \sigma_{\epsilon_s} e^{-\delta s} v_x(X_s, \epsilon_s) dW_s\right] + \mathbb{E}\left[M^f_t - M^f_0\right]. \quad (3.32)$$

We note that $v(X_s, \epsilon_s)$, $v_x(X_s, \epsilon_s)$ and $\sigma_{\epsilon_s} e^{-\delta s}\, v_x(X_s, \epsilon_s)$ are bounded for every $s \in [0, t \wedge \tau]$. Therefore, $\{M^f_{t\wedge\tau}, t \geq 0\}$ is a square integrable martingale and

$$\mathbb{E}\left[\int_0^{t\wedge\tau} \sigma_{\epsilon_s} e^{-\delta s} v'(X_s, \epsilon_s) dW_s\right] = 0.$$

Then, using equation (3.31) and taking $a \downarrow -\infty$ and $b \uparrow \infty$, which implies $\tau \to \infty$, we get from (3.32) that

$$v(x, i) \leq \mathbb{E}\left[e^{-\delta t} v(X_t, \epsilon_t)\right] + \mathbb{E}\left[\int_0^t e^{-\delta s} \alpha_{\epsilon_s} (X_s - \mathcal{I}_{\epsilon_s})^2 ds\right] \\ + \mathbb{E}\left[\int_0^t e^{-\delta s} k_{\epsilon_s} p_s ds\right]. \quad (3.33)$$

Let now $\bar{K} := \max_{i\in\mathcal{S}} K_i$, $\bar{\sigma} := \max_{i\in\mathcal{S}} \sigma_i$ and $\underline{\mu} := \min i \in \mathcal{S}\mu_i$. Then, it holds that

$$X_t = x + \int_0^t \left(p_s - \mu_{\epsilon_s}\right) ds - \int_0^t \sigma_{\epsilon_s}\, dW_s \\ \leq x + (\bar{K} - \underline{\mu})t + \bar{\sigma} \sup_{s\leq t} |W_s|.$$

By the quadratic growth condition of v, we obtain that

$$\mathbb{E}\left[e^{-\delta t} |v(X_t, \epsilon_t)|\right] \leq \mathbb{E}\left[e^{-\delta t} C \left(1 + X_t^2\right)\right] \\ \leq e^{-\delta t} C \left(1 + C_1 \left((x + (\bar{K} - \underline{\mu})t)^2 + \bar{\sigma}^2 \mathbb{E}\left[\left(\sup_{s\leq t} |W_s|\right)^2\right]\right)\right) \\ \leq e^{-\delta t} C \left(1 + C_1 \left((x + (\bar{K} - \underline{\mu})t)^2 + 4\bar{\sigma}^2 \mathbb{E}\left[W_t^2\right]\right)\right) \\ = e^{-\delta t} C \left(1 + C_1 \left((x + (\bar{K} - \underline{\mu})t)^2 + 4\bar{\sigma}^2 t\right)\right) \to_{t\to\infty} 0.$$

Taking $t \to \infty$ in (3.33), we observe from the last inequality that

$$v(x,i) \le \mathbb{E}\left[\int_0^\infty e^{-\delta s}\alpha_{\epsilon_s}(X_s - \mathcal{I}_{\epsilon_s})^2 ds + \int_0^\infty e^{-\delta s} k_{\epsilon_s} p_s ds\right] = J(x,i;p).$$

In particular, the inequality in (3.33) becomes an equality for $p = \hat{p}$. Hence $v(x,i) = J(x,i;\hat{p}) = V(x,i)$.

□

3.3.2 Construction of the Solution

In this section we construct a candidate value function that satisfies the HJB equation (3.31).

We note that for $t \in [0,\infty)$,

$$\hat{p}_t = \arg \inf_{p \in [0, K\_\epsilon_t]} \{p_t [k_{\epsilon_t} + v_x(X_t, \epsilon_t)]\} = \begin{cases} K_{\epsilon_t} & \text{if} \quad v_x(X_t,\epsilon_t) < -k_{\epsilon_t}, \\ 0 & \text{if} \quad v_x(X_t,\epsilon_t) \ge -k_{\epsilon_t}. \end{cases} \tag{3.34}$$

Hence, the candidate for optimal control $\hat{p}$ has the form $\hat{p}_t = \varphi(X_t, \epsilon_t)$ for $t \in [0,\infty)$, where $\varphi(\cdot, i)$, $i \in \mathcal{S}$, is a measurable function defined by

$$\varphi(x,i) := \begin{cases} K_i & \text{if} \quad v_x(x,i) < -k_i, \\ 0 & \text{if} \quad v_x(x,i) \ge -k_i, \end{cases}$$

for $x \in \mathbb{R}$. Hence, the candidate optimal control is of so-called *bang-bang-type* and it switches, given the regime $i \in \mathcal{S}$, between the extreme values 0 (no production) and K_i (maximal production).

For simplicity, we assume, as in the SSC case, that the economy shifts only between two regimes, that is, $\mathcal{S} = \{1,2\}$. Under this assumption, we have that

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}.$$

By (3.30), we conjecture that v is convex and admits quadratic growth. We define $b_i := \sup\{x \in \mathbb{R} : v_x(x,i) \le k_i\}$. The relation between b_1 and b_2 depends on the relations among the different parameters. We only consider the case $b_1 < b_2$. The case $b_1 > b_2$ can be treated similar. we expect the candidate value function v to solve, for $x \in (-\infty, b_i)$,

$$\frac{1}{2}\sigma_i^2 v_{xx}(x,i) + (K_i - \mu_i)v_x(x,i) - (\lambda_i + \delta)\, v(x,i) + \lambda_i v(x,3-i) + \alpha_i(x - \mathcal{I}_i)^2 + K_i k_i = 0$$

and, for $x \in [b_i, \infty)$,

$$\frac{1}{2}\sigma_i^2 v_{xx}(x,i) - \mu_i v_x(x,i) - (\lambda_i + \delta)\, v(x,i) + \lambda_i v(x,3-i) + \alpha_i(x - \mathcal{I}_i)^2 + K_i k_i = 0.$$

Due to our conjectures, we consider three possibilities for the initial level of the inventory: $x \in (-\infty, b_1)$, $x \in [b_1, b_2)$, and $x \in [b_2, \infty)$.

When $x \in [b_2, \infty)$, equation (3.31) leads to the following system of differential equations:

$$\begin{aligned} &-(\lambda_1+\delta)\, v(x,1) - \mu_1\, v_x(x,1) + \frac{1}{2}\,\sigma_1^2\, v_{xx}(x,1) + \lambda_1\, v(x,2) + \alpha_1(x-\mathcal{I}_1)^2 = 0,\\ &-(\lambda_2+\delta)\, v(x,2) - \mu_2\, v_x(x,2) + \frac{1}{2}\,\sigma_2^2\, v_{xx}(x,2) + \lambda_2\, v(x,1) + \alpha_2(x-\mathcal{I}_2)^2 = 0. \end{aligned} \tag{3.35}$$

Consider the characteristic equation for (3.35), $\phi_1^1(\gamma)\phi_2^1(\gamma) = \lambda_1\lambda_2$, where

$$\phi_i^1(\gamma) := -\frac{1}{2}\,\sigma_i^2\gamma^2 + \mu_i\,\gamma + (\lambda_i+\delta), \qquad i = 1,2.$$

Lemma 3.6 proves that $\phi_1^1(\gamma)\ \phi_2^1(\gamma) = \lambda_1\lambda_2$ has 4 real roots: $\gamma_1 < \gamma_2 < 0 < \gamma_3 < \gamma_4$. Then, the solution to the system of differential equations (3.35) is

$$\begin{aligned} v(x,1) &= A_1\, e^{\gamma_1(x-b_2)} \ + \ A_2\, e^{\gamma_2(x-b_2)} \ + \ A_3\, e^{\gamma_3(x-b_2)} \ + \ A_4\, e^{\gamma_4(x-b_2)}\\ &\quad + R_1(x-b_2)^2 + S_1(x-b_2) + T_1,\\ v(x,2) &= B_1\, e^{\gamma_1(x-b_2)} \ + \ B_2\, e^{\gamma_2(x-b_2)} \ + \ B_3\, e^{\gamma_3(x-b_2)} \ + \ B_4\, e^{\gamma_4(x-b_2)}\\ &\quad + R_2(x-b_2)^2 + S_2(x-b_2) + T_2, \end{aligned}$$

where, for each $j = 1,2,3,4$,

$$B_j = \frac{\phi_1^1(\gamma_j)}{\lambda_1}\, A_j = \frac{\lambda_2}{\phi_2^1(\gamma_j)}\, A_j. \tag{3.36}$$

Furthermore, R_i, S_i and T_i are the solution to the system

$$\begin{aligned} 0 &= -(\lambda_1+\delta)R_1 + \lambda_1 R_2 + \alpha_1,\\ 0 &= -(\lambda_1+\delta)S_1 - 2\mu_1 R_1 + \lambda_1 S_2 + 2\alpha_1(b_2-\mathcal{I}_1),\\ 0 &= -(\lambda_1+\delta)T_1 - \mu_1 S_1 + \sigma_1^2 R_1 + \lambda_1 T_2 + \alpha_1(b_2-\mathcal{I}_1)^2,\\ 0 &= -(\lambda_2+\delta)R_2 + \lambda_2 R_1 + \alpha_2,\\ 0 &= -(\lambda_2+\delta)S_2 - 2\mu_2 R_2 + \lambda_2 S_1 + 2\alpha_2(b_2-\mathcal{I}_2),\\ 0 &= -(\lambda_2+\delta)T_2 - \mu_2 S_2 + \sigma_2^2 R_2 + \lambda_2 T_1 + \alpha_2(b_2-\mathcal{I}_2)^2. \end{aligned}$$

Recall that we conjecture that $v(\cdot,i)$ admits quadratic growth. Thus, we set $B_3 = B_4 = A_3 = A_4 = 0$. Hence, the solution of the system (3.35) simplifies to

$$v(x,1) = A_1\, e^{\gamma_1(x-b_2)} \ + \ A_2\, e^{\gamma_2(x-b_2)} \ + R_1(x-b_2)^2 + S_1(x-b_2) + T_1, \tag{3.37}$$

$$v(x,2) = B_1\, e^{\gamma_1(x-b_2)} \ + \ B_2\, e^{\gamma_2(x-b_2)} \ + R_2(x-b_2)^2 + S_2(x-b_2) + T_2. \tag{3.38}$$

Nest, we consider $x \in [\,b_1, b_2)$, Now, the HJB equation (3.31) reads as

$$\begin{aligned} &0 = -(\lambda_1+\delta)v(x,1) - \mu_1\, v_x(x,1) + \frac{1}{2}\,\sigma_1^2\, v_{xx}(x,1) + \lambda_1\, v(x,2) + \alpha_1(x-\mathcal{I}_1)^2,\\ &0 = -(\lambda_2+\delta)v(x,2) + (K_2-\mu_2)v_x(x,2) + \frac{1}{2}\sigma_2^2 v_{xx}(x,2) + \lambda_2 v(x,1)\\ &\quad + K_2 k_2 + \alpha_2(x-\mathcal{I}_2)^2. \end{aligned} \tag{3.39}$$

Consider the characteristic equation for the system (3.39), $\phi_1^2(\tilde{\gamma})\,\phi_2^2(\tilde{\gamma}) = \lambda_1\lambda_2$, where

$$\phi_1^2(\tilde{\gamma}) := -\frac{1}{2}\,\sigma_1^2\tilde{\gamma}^2 + \mu_1\,\tilde{\gamma} + (\lambda_1 + \delta),$$
$$\phi_2^2(\tilde{\gamma}) := -\frac{1}{2}\,\sigma_2^2\tilde{\gamma}^2 + (\mu_2 - K_2)\,\tilde{\gamma} + (\lambda_2 + \delta).$$

From Lemma 3.6, $\phi_1^2(\tilde{\gamma})\,\phi_2^2(\tilde{\gamma}) = \lambda_1\lambda_2$ has 4 real roots: $\tilde{\gamma}_1 < \tilde{\gamma}_2 < 0 < \tilde{\gamma}_3 < \tilde{\gamma}_4$. Then, we find that the solution to the system of equations (3.39) is

$$\begin{aligned} v(x,1) &= \widetilde{A}_1\,e^{\tilde{\gamma}_1(x-b_1)} + \widetilde{A}_2\,e^{\tilde{\gamma}_2(x-b_1)} + \widetilde{A}_3\,e^{\tilde{\gamma}_3(x-b_1)} + \widetilde{A}_4\,e^{\tilde{\gamma}_4(x-b_1)} \\ &\quad + \tilde{R}_1(x-b_1)^2 + \tilde{S}_1(x-b_1) + \tilde{T}_1, \end{aligned} \tag{3.40}$$
$$\begin{aligned} v(x,2) &= \widetilde{B}_1\,e^{\tilde{\gamma}_1(x-b_1)} + \widetilde{B}_2\,e^{\tilde{\gamma}_2(x-b_1)} + \widetilde{B}_3\,e^{\tilde{\gamma}_3(x-b_1)} + \widetilde{B}_4\,e^{\tilde{\gamma}_4(x-b_1)} \\ &\quad + \tilde{R}_2(x-b_1)^2 + \tilde{S}_2(x-b_1) + \tilde{T}_2, \end{aligned} \tag{3.41}$$

where, for each $j = 1,2,3,4$,

$$\widetilde{B}_j = \frac{\phi_1^2(\tilde{\gamma}_j)}{\lambda_1}\,\widetilde{A}_j = \frac{\lambda_2}{\phi_2^2(\tilde{\gamma}_j)}\,\widetilde{A}_j. \tag{3.42}$$

Furthermore, $\tilde{R}_i$, $\tilde{S}_i$ and $\tilde{T}_i$ are the solution of the system

$$\begin{aligned} 0 &= -(\lambda_1+\delta)\tilde{R}_1 + \lambda_1\tilde{R}_2 + \alpha_1, \\ 0 &= -(\lambda_1+\delta)\tilde{S}_1 - 2\mu_1\tilde{R}_1 + \lambda_1\tilde{S}_2 + 2\alpha_1(b_1 - \mathcal{I}_1), \\ 0 &= -(\lambda_1+\delta)\tilde{T}_1 - \mu_1\tilde{S}_1 + \sigma_1^2\tilde{R}_1 + \lambda_1\tilde{T}_2 + \alpha_1(b_1 - \mathcal{I}_1)^2, \\ 0 &= -(\lambda_2+\delta)\tilde{R}_2 + \lambda_2\tilde{R}_1 + \alpha_2, \\ 0 &= -(\lambda_2+\delta)\tilde{S}_2 + 2(K_2 - \mu_2)\tilde{R}_2 + \lambda_2\tilde{S}_1 + 2\alpha_2(b_1 - \mathcal{I}_2), \\ 0 &= -(\lambda_2+\delta)\tilde{T}_2 + (K_2 - \mu_2)\tilde{S}_2 + \sigma_2^2\tilde{R}_2 + \lambda_2\tilde{T}_1 + K_2k_2 + \alpha_2(b_1 - \mathcal{I}_2)^2. \end{aligned}$$

Finally, for $x \in (-\infty, b_1)$, the function v is assumed to solve the system

$$\begin{aligned} 0 &= -(\lambda_1+\delta)\,v(x,1) + (K_1 - \mu_1)\,v_x(x,1) + \frac{1}{2}\,\sigma_1^2\,v_{xx}(x,1) + \lambda_1\,v(x,2) \\ &\quad + K_1k_1 + \alpha_1(x - \mathcal{I}_1)^2, \\ 0 &= -(\lambda_2+\delta)\,v(x,2) + (K_2 - \mu_2)\,v_x(x,2) + \frac{1}{2}\,\sigma_2^2\,v_{xx}(x,2) + \lambda_2\,v(x,1) \\ &\quad + K_2k_2 + \alpha_2(x - \mathcal{I}_2)^2. \end{aligned} \tag{3.43}$$

Consider the characteristic equation for (3.43), $\phi_1^3(\hat{\gamma})\,\phi_2^3(\hat{\gamma}) = \lambda_1\lambda_2$, where

$$\phi_i^3(\hat{\gamma}) := -\frac{1}{2}\,\sigma_i^2\hat{\gamma}^2 - (K_i - \mu_i)\,\hat{\gamma} + (\lambda_i + \delta).$$

From Lemma 3.6, $\phi_1^3(\hat{\gamma}\phi_2^3(\hat{\gamma}) = \lambda_1\lambda_2$ has 4 real roots: $\hat{\gamma}_1 < \hat{\gamma}_2 < 0 < \hat{\gamma}_3 < \hat{\gamma}_4$. Then, the solution to the system of differential equations (3.43) is given by:

$$\begin{aligned} v(x,1) &= \widehat{A}_1 e^{\hat{\gamma}_1(x-b_1)} + \widehat{A}_2 e^{\hat{\gamma}_2(x-b_1)} + \widehat{A}_3 e^{\hat{\gamma}_3(x-b_1)} + \widehat{A}_4 e^{\hat{\gamma}_4(x-b_1)} \\ &\quad + \widehat{R}_1(x-b_1)^2 + \widehat{S}_1(x-b_1) + \widehat{T}_1, \\ v(x,2) &= \widehat{B}_1 e^{\hat{\gamma}_1(x-b_1)} + \widehat{B}_2 e^{\hat{\gamma}_2(x-b_1)} + \widehat{B}_3 e^{\hat{\gamma}_3(x-b_1)} + \widehat{B}_4 e^{\hat{\gamma}_4(x-b_1)} \\ &\quad + \widehat{R}_2(x-b_1)^2 + \widehat{S}_2(x-b_1) + \widehat{T}_2, \end{aligned}$$

where, for each $j = 1, 2, 3, 4$,

$$\widehat{B}_j = \frac{\phi_1^3(\hat{\gamma}_j)}{\lambda_1}\,\widehat{A}_j = \frac{\lambda_2}{\phi_2^3(\hat{\gamma}_j)}\,\widehat{A}_j. \tag{3.44}$$

Furthermore, $\widehat{R}_i$, $\widehat{S}_i$ and $\widehat{T}_i$ are the solution of the system

$$\begin{aligned} 0 &= -(\lambda_1+\delta)\widehat{R}_1 + \lambda_1\widehat{R}_2 + \alpha_1, \\ 0 &= -(\lambda_1+\delta)\widehat{S}_1 + 2(K_1-\mu_1)\widehat{R}_1 + \lambda_1\widehat{S}_2 + 2\alpha_1(b_1-\mathcal{I}_1), \\ 0 &= -(\lambda_1+\delta)\widehat{T}_1 + (K_1-\mu_1)\widehat{S}_1 + \sigma_1^2\widehat{R}_1 + \lambda_1\widehat{T}_2 + K_1k_1 + \alpha_1(b_1-\mathcal{I}_1)^2, \\ 0 &= -(\lambda_2+\delta)\widehat{R}_2 + \lambda_2\widehat{R}_1 + \alpha_2, \\ 0 &= -(\lambda_2+\delta)\widehat{S}_2 + 2(K_2-\mu_2)\widehat{R}_2 + \lambda_2\widehat{S}_1 + 2\alpha_2(b_1-\mathcal{I}_2), \\ 0 &= -(\lambda_2+\delta)\widehat{T}_2 + (K_2-\mu_2)\widehat{S}_2 + \sigma_2^2\widehat{R}_2 + \lambda_2\widehat{T}_1 + K_2k_2 + \alpha_2(b_1-\mathcal{I}_2)^2. \end{aligned}$$

Recall that we are conjecturing that the function v admits quadratic growth. Hence, we set $\widehat{A}_1 = \widehat{A}_2 = \widehat{B}_1 = \widehat{B}_2 = 0$. Therefore,

$$v(x,1) = \widehat{A}_3 e^{\hat{\gamma}_3(x-b_1)} + \widehat{A}_4 e^{\hat{\gamma}_4(x-b_1)} + \widehat{R}_1(x-b_1)^2 + \widehat{S}_1(x-b_1) + \widehat{T}_1, \tag{3.45}$$

$$v(x,2) = \widehat{B}_3 e^{\hat{\gamma}_3(x-b_1)} + \widehat{B}_4 e^{\hat{\gamma}_4(x-b_1)} + \widehat{R}_2(x-b_1)^2 + \widehat{S}_2(x-b_1) + \widehat{T}_2 \tag{3.46}$$

is the solution for the system (3.43), where (3.44) is satisfied for $j = 1, 2$.

In order to find the thresholds b_1 and b_2, and the coefficients in the functions (3.37)-(3.38), (3.40)-(3.41) and (3.45)-(3.46), we impose the smooth-fit condition. Thus, we expect v to solve, for each $i = 1, 2$, the system

$$\begin{array}{rcl} v(b_i-,i) &=& v(b_i+,i), \\ v(b_{3-i}-,i) &=& v(b_{3-i}+,i), \\ v_x(b_i-,i) &=& -k_i, \\ v_x(b_i+,i) &=& -k_i, \\ v_x(b_{3-i}-,i) &=& v_x(b_{3-i}+,i). \end{array} \tag{3.47}$$

The solution of the system (3.47) provides the values for b_1, b_2 and the values for A_j, $j = 1, 2$, $\widetilde{A}_j$, $j = 1, 2, 3, 4$, and $\widehat{A}_j$, $j = 3, 4$. The values for the corresponding B_j, $\widetilde{B}_j$ and $\widehat{B}_j$ can be found from (3.36), (3.42), and (3.44).

Remark 3.17. Note that system (3.47) is a non-linear system of ten equations and ten unknowns. As in the singular stochastic control case, we can neither provide existence nor uniqueness of the solution. Therefore, we will solve this system numerically to provide the existence of a solution and to study comparative statics for the free-boundaries b_1 and b_2.

3.3.3 Verification of the Solution

In this subsection, we prove that our candidate value function v coincides with the true value function V of Problem 3.14. First, we investigate the regularity of the candidate for the value function.

Lemma 3.18. *Let A_j, $j = 1, 2$, $\widetilde{A}_j$, $j = 1, 2, 3, 4$, and $\widehat{A}_j$, $j = 3, 4$, be the solution of the system of equations* (3.47). *Let B_j, $j = 1, 2$, $\widetilde{B}_j$, $j = 1, 2, 3, 4$, and $\widehat{B}_j$, $j = 3, 4$, be defined by* (3.36), (3.42) *and* (3.44). *Suppose that $b_1 < b_2$ and that $v_{xx}(b_i, i) \geq 0$. Then, the function v given by*

$$v(x,1) = \begin{cases} \widehat{A}_3 e^{\hat{\gamma}_3(x-b_1)} + \widehat{A}_4 e^{\hat{\gamma}_4(x-b_1)} + \widehat{R}_1(x-b_1)^2 + \widehat{S}_1(x-b_1) + \widehat{T}_1 & x \in (-\infty, b_1), \\ \widetilde{A}_1 e^{\tilde{\gamma}_1(x-b_1)} + \widetilde{A}_2 e^{\tilde{\gamma}_2(x-b_1)} + \widetilde{A}_3 e^{\tilde{\gamma}_3(x-b_1)} + \widetilde{A}_4 e^{\tilde{\gamma}_4(x-b_1)} & \\ +\tilde{R}_1(x-b_1)^2 + \tilde{S}_1(x-b_1) + \tilde{T}_1 & x \in [b_1, b_2), \\ A_1 e^{\gamma_1(x-b_2)} + A_2 e^{\gamma_2(x-b_2)} + R_1(x-b_2)^2 + S_1(x-b_2) + T_1 & x \in [b_2, \infty), \end{cases}$$

and

$$v(x,2) = \begin{cases} \widehat{B}_3 e^{\hat{\gamma}_3(x-b_1)} + \widehat{B}_4 e^{\hat{\gamma}_4(x-b_1)} + \widehat{R}_2(x-b_1)^2 + \widehat{S}_2(x-b_1) + \widehat{T}_2 & x \in (-\infty, b_1), \\ \widetilde{B}_1 e^{\tilde{\gamma}_1(x-b_1)} + \widetilde{B}_2 e^{\tilde{\gamma}_2(x-b_1)} + \widetilde{B}_3 e^{\tilde{\gamma}_3(x-b_1)} + \widetilde{B}_4 e^{\tilde{\gamma}_4(x-b_1)} & \\ +\tilde{R}_2(x-b_1)^2 + \tilde{S}_2(x-b_1) + \tilde{T}_2 & x \in [b_1, b_2), \\ B_1 e^{\gamma_1(x-b_2)} + B_2 e^{\gamma_2(x-b_2)} + R_2(x-b_2)^2 + S_2(x-b_2) + T_2 & x \in [b_2, \infty), \end{cases}$$

is such that $v(\cdot, i) \in \mathcal{C}^\infty(\mathbb{R} \setminus \{b_1, b_2\}) \cap \mathcal{C}^4(\mathbb{R} \setminus \{b_i\}) \cap \mathcal{C}^2(\mathbb{R})$.

Proof. By construction and the smooth-fit conditions, see (3.47), the candidate the value function v is such that $v(\cdot, i) \in \mathcal{C}^\infty(\mathbb{R} \setminus \{b_1, b_2\}) \cap \mathcal{C}^1(\mathbb{R})$. Remember that the candidate for value function solves the HJB equations, e.g., for $x \in [b_2, \infty)$, v solves

$$\begin{aligned} 0 &= -(\lambda_1 + \delta)\, v(x,1) - \mu_1\, v'(x,1) + \frac{1}{2}\,\sigma_1^2\, v_{xx}(x,1) + \lambda_1\, v(x,2) + \alpha_1(x - \mathcal{I}_1)^2, \\ 0 &= -(\lambda_2 + \delta)\, v(x,2) - \mu_2\, v_x(x,2) + \frac{1}{2}\,\sigma_2^2\, v_{xx}(x,2) + \lambda_2\, v(x,1) + \alpha_2(x - \mathcal{I}_2)^2, \end{aligned} \tag{3.48}$$

for $x \in [b_1, b_2)$, v solves

$$\begin{aligned} 0 &= -(\lambda_1 + \delta)\, v(x,1) - \mu_1\, v_x(x,1) + \frac{1}{2}\,\sigma_1^2\, v_{xx}(x,1) + \lambda_1\, v(x,2) + \alpha_1(x - \mathcal{I}_1)^2, \\ 0 &= -(\lambda_2 + \delta)\, v(x,2) + (K_2 - \mu_2)\, v_x(x,2) + \frac{1}{2}\,\sigma_2^2\, v_{xx}(x,2) + \lambda_2\, v(x,1) \\ &\quad + K_2 k_2 + \alpha_2(x - \mathcal{I}_2)^2, \end{aligned} \tag{3.49}$$

and for $x \in (-\infty, b_1)$, v solves

$$\begin{aligned} 0 &= -(\lambda_1 + \delta)\, v(x,1) + (K_1 - \mu_1)\, v_x(x,1) + \frac{1}{2}\,\sigma_1^2\, v_{xx}(x,1) + \lambda_1\, v(x,2) \\ &\quad + K_1 k_1 + \alpha_1(x - \mathcal{I}_1)^2, \\ 0 &= -(\lambda_2 + \delta)\, v(x,2) + (K_2 - \mu_2)\, v_x(x,2) + \frac{1}{2}\,\sigma_2^2\, v_{xx}(x,2) + \lambda_2\, v(x,1) \\ &\quad + K_2 k_2 + \alpha_2(x - \mathcal{I}_2)^2. \end{aligned} \tag{3.50}$$

Using $v_x(b_2, 2) = -k_2$. we obtain from the second equation in (3.48) and (3.49) that $v(\cdot, 2) \in \mathcal{C}^2((b_1, \infty))$ and further, differentiating the first equation in (3.48) and (3.49), that $v(\cdot, 1) \in \mathcal{C}^4((b_1, \infty))$. A similar argument holds at the other free-boundary. Using that $v_x(b_1, 1) = -k_1$, combining the first equation in (3.49) and (3.50) gives that $v(\cdot, 1) \in \mathcal{C}^2((-\infty, b_2))$ and further, differentiating the second equation in (3.49) and (3.50), that $v(\cdot, 2) \in \mathcal{C}^4((-\infty, b_2))$. Hence, $v(\cdot, i) \in \mathcal{C}^\infty(\mathbb{R} \setminus \{b_1, b_2\}) \cap \mathcal{C}^4(\mathbb{R} \setminus \{b_i\}) \cap \mathcal{C}^2(\mathbb{R})$.

□

Theorem 3.19. *Let A_j, $j = 1, 2$, $\widetilde{A}_j$, $j = 1, 2, 3, 4$, and $\widehat{A}_j$, $j = 3, 4$, be the solution of the system of equations* (3.47). *Let B_j, $j = 1, 2$, $\widetilde{B}_j$, $j = 1, 2, 3, 4$, and $\widehat{B}_j$, $j = 3, 4$, be defined by* (3.36), (3.42) *and* (3.44). *Suppose that $b_1 < b_2$ and that $v_{xx}(b_i, i) \geq 0$. Then, the function v given by*

$$v(x,1) = \begin{cases} \widehat{A}_3 e^{\hat{\gamma}_3(x-b_1)} + \widehat{A}_4 e^{\hat{\gamma}_4(x-b_1)} + \widehat{R}_1(x-b_1)^2 + \widehat{S}_1(x-b_1) + \widehat{T}_1 & x \in (-\infty, b_1), \\ \widetilde{A}_1 e^{\tilde{\gamma}_1(x-b_1)} + \widetilde{A}_2 e^{\tilde{\gamma}_2(x-b_1)} + \widetilde{A}_3 e^{\tilde{\gamma}_3(x-b_1)} + \widetilde{A}_4 e^{\tilde{\gamma}_4(x-b_1)} & \\ +\tilde{R}_1(x-b_1)^2 + \tilde{S}_1(x-b_1) + \tilde{T}_1 & x \in [b_1, b_2), \\ A_1 e^{\gamma_1(x-b_2)} + A_2 e^{\gamma_2(x-b_2)} + R_1(x-b_2)^2 + S_1(x-b_2) + T_1 & x \in [b_2, \infty), \end{cases}$$

and

$$v(x,2) = \begin{cases} \widehat{B}_3 e^{\hat{\gamma}_3(x-b_1)} + \widehat{B}_4 e^{\hat{\gamma}_4(x-b_1)} + \widehat{R}_2(x-b_1)^2 + \widehat{S}_2(x-b_1) + \widehat{T}_2 & x \in (-\infty, b_1), \\ \widetilde{B}_1 e^{\tilde{\gamma}_1(x-b_1)} + \widetilde{B}_2 e^{\tilde{\gamma}_2(x-b_1)} + \widetilde{B}_3 e^{\tilde{\gamma}_3(x-b_1)} + \widetilde{B}_4 e^{\tilde{\gamma}_4(x-b_1)} & \\ +\tilde{R}_2(x-b_1)^2 + \tilde{S}_2(x-b_1) + \tilde{T}_2 & x \in [b_1, b_2), \\ B_1 e^{\gamma_1(x-b_2)} + B_2 e^{\gamma_2(x-b_2)} + R_2(x-b_2)^2 + S_2(x-b_2) + T_2 & x \in [b_2, \infty), \end{cases}$$

is the value function V of Problem 3.14. Furthermore, $\hat{p}$ defined by

$$\hat{p}_t = \begin{cases} K_i & \text{if} \quad \epsilon_t = i \quad \text{and} \quad X_t \in (-\infty, b_i), \\ 0 & \text{if} \quad \epsilon_t = i \quad \text{and} \quad X_t \in [b_i, \infty), \end{cases}$$

is the optimal production rate policy for Problem 3.14.

Proof. To prove that the candidate value function v defined above is the true value function of Problem 3.14, it is enough to show that it satisfies all the conditions of Theorem 3.16 and the conjectures we made in Section 3.3.2.

First, we show that v is convex. Given the regularity of v, see Lemma 3.18, we can apply Itô's formula, for given $i \in \mathcal{S}$ and $x \neq b_i$, as in the proof of Theorem 3.16, to $e^{-\delta s} v_{xx}(X_s^{x,i}, \epsilon_s)$. Then, we get that

$$\mathbb{E}\left[e^{-\delta\tau} v_{xx}(X_\tau, \epsilon_\tau)\right] = v_{xx}(x,i) + \mathbb{E}\left[\int_0^\tau e^{-\delta s} \left(\mathcal{L}^p - \delta\right) v_{xx}(X_s, \epsilon_s)\, ds\right],$$

where τ is a stopping time. In the following, we denote the n-th derivative with respect to x of $v^{(n)}(x,i)$. Since v solves the Hamilton-Jacobi-Bellmann equation, the regularity of v implies

$$\frac{1}{2}\sigma_i^2 v^{(4)}(x,i) + (K - \mu_i) v^{(3)}(x,i) - (\delta + \lambda_i) v_{xx}(x,i) + \lambda_i v(x, 3-i) = -2\alpha_i$$

when $x < b_i$ and

$$\frac{1}{2}\sigma_i^2\, v^{(4)}(x,i) - \mu_i\, v^{(3)}(x,i) - (\delta+\lambda_i)v_{xx}(x,i) + \lambda_i v(x,3-i) = -2\alpha_i$$

when $x \geq b_i$.
Taking $\tau := \inf\{t \geq 0 : (X_t, \epsilon_t) \in \{(b_1,1),(b_2,2)\}\}$, we obtain, for $i \in \{1,2\}$ and $x > b_i$,

$$v_{xx}(x,i) = \mathbb{E}\left[2\int_0^\tau e^{-\delta s}\alpha_{\epsilon_s}\, ds + e^{-\delta\tau}v_{xx}(b_1,1)\mathbb{1}_{\{(X_\tau^{x,i},\epsilon_\tau)=(b_1,1)\}} \right. \\ \left. + e^{-\delta\tau}v_{xx}(b_2,2)\mathbb{1}_{\{(X_\tau^{x,i},\epsilon_\tau)=(b_2,2)\}}\right].$$

We recall that we assume $v_{xx}(b_i,i) \geq 0$ and $\alpha_i > 0$, $i \in \mathcal{S}$. Hence, $v_{xx}(x,i) \geq 0$ (in particular $v_{xx}(x,i) > 0$) and $v(\cdot,i)$ is convex. Since v is convex, it follows immediately by construction that v solves (3.31), e.g., v solves

$$\inf_{p\in[0,K_i]}\{(\mathcal{L}^p - \delta)\, v(x,i) + k_i p\} + \alpha_i(x-\mathcal{I}_i)^2 = 0.$$

Moreover, the quadratic growth condition is also fulfilled by construction. Hence, by Theorem 3.16, v identifies with the true value function and $\hat{p}$ is the optimal control. □

Remark 3.20. The assumption that $v_{xx}(b_i,i) \geq 0$ is crucial for the proof. In the SSC case, according to the smooth-fit conditions, see (3.19), we have that $v_{xx}(b_i,i) = 0$. While in the bounded case, it is a priori not clear whether the condition holds. Therefore, at this point the assumption is not restrictive since a violation of it would imply that $v(\cdot,i)$ is not convex, thus our construction would not be applicable. We can show numerically, that the assumption holds for various parameter values. Unfortunately, because of the complex structure of (3.47), it seems not possible to prove a priori that a solution to (3.47) fulfilling the above assumption exists.

Summarizing our results, we have proven that our candidate value function v coincides with the true value function of Problem 3.14. Moreover, the optimal control is given by (3.34). Therefore, the optimal production policy, given the regime i, works as follows: (a) do not produce when the inventory level is above the threshold b_i, and (b) produce with the maximum rate K_i whenever the inventory level falls below b_i. We observe that the firm should also increase production because the regime is changing. This happens when the level of the inventory lies in the interval (b_1, b_2) and the regime changes from $i = 1$ to $i = 2$.

3.3.4 Comparative Statics and Numerical Examples

In this subsection, we study the influence of some parameters on the free-boundaries b_i. In contrast to the SSC case, our analysis is completely numerical, since there is no link between our classical bounded-velocity stochastic control problem and an OS problem. The numerical results are computed by using Mathematica.

Therefore, we basically consider the same benchmark case with the following parameter values:

$$\text{State 1: } \mu_1 = 0.2,\ \sigma_1 = 0.2,\ \alpha_1 = 0.2,\ I_1 = 2,\ k_1 = 2,\ \lambda_1 = 0.2,\ K_1 = 1,$$
$$\text{State 2: } \mu_2 = 1,\ \sigma_2 = 0.2,\ \alpha_2 = 0.2,\ I_2 = 2,\ k_2 = 2,\ \lambda_2 = 0.2,\ K_2 = 1,$$

and a discount rate of $\delta = 0.2$.

Again, we call state 1 the *low regime* and state 2 the *high regime*.

We obtain the following solution to the system (3.47):

$$\widehat{A}_3 = -14,267,\ \widehat{A}_4 = 0,001, \tilde{A}_1 = -0.002,\ \tilde{A}_2 = -1,085,\ \tilde{A}_3 = -0,006,$$

$$\tilde{A}_4 = 0,000,\ A_1 = -0,138,\ A_2 = -14,465,\ b_1 = 0,976,\ b_2 = 1,201.$$

To verify that this solution leads to the true value function, we have to show that $v_{xx}(b_i, i) \geq 0$ holds true. Computing these expressions gives

$$v_{xx}(b_1, 1) = 0.055, \quad v_{xx}(b_2, 2) = 0.100.$$

Therefore, the candidate value function v is indeed the true value function. Figure 5 visualizes the value function.

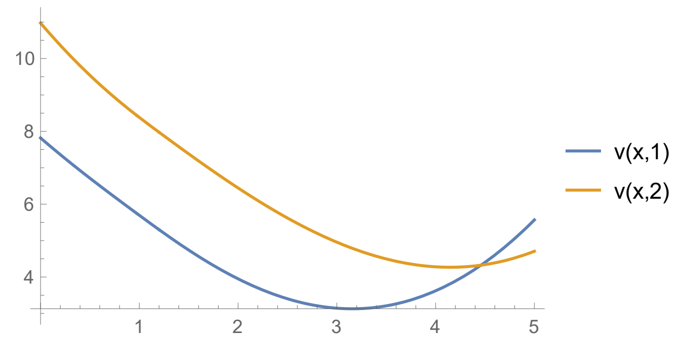


Figure 5: Graphical illustration of the value function for the benchmark case.

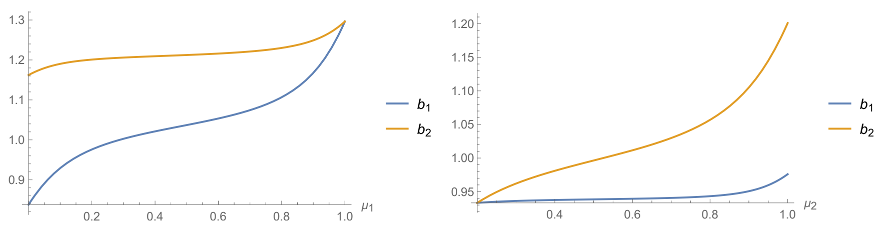


Figure 6: Graphical illustration of the dependence of the free-boundaries on μ_i. The other parameters are assumed to be the same as in the benchmark case..

In a first step, we study the dependence of the free-boundaries on μ_i. Figure 6 visualize that both free-boundaries increases if one of the drift components increases.

Moreover, as expected, for the same drift the free-boundaries are equal. This result is plausible, since a rise of μ_i increases the expected future demand causing the firm to compensate by expanding the production.

Next, we investigate the dependence of the free-boundaries on λ_i.

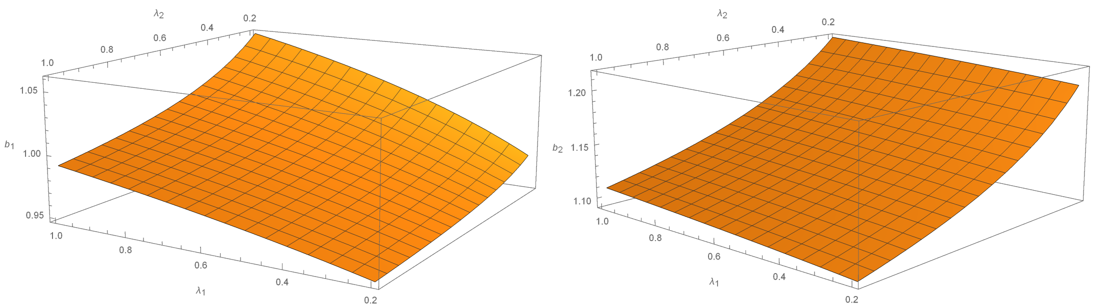


Figure 7: Graphical illustration of the dependence of the free-boundaries on λ_i. The other parameters are assumed to be the same as in the benchmark case.

Figure 7 illustrates that both free-boundaries are increasing in λ_1 and decreasing in λ_2. Hence, we obtain the same results as in the SSC case. The reason for this behavior is quite clear. If the probability of leaving the low regime is increasing, both free-boundaries increase because more time is spend in the high regime. If instead the probability of leaving the high regime is increasing, more time is spend in the low regime, thus production decreases.

Next, we study the dependence of the free-boundaries on σ_i.

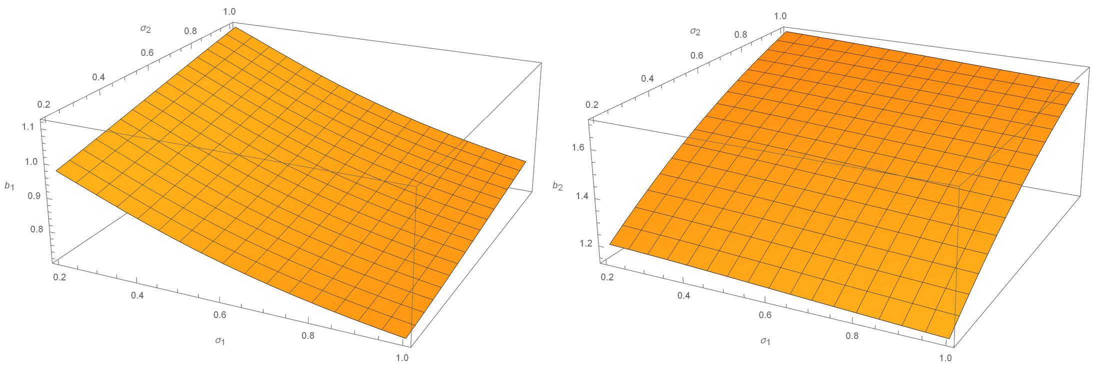


Figure 8: Graphical illustration of the dependence of the free-boundaries on σ_i. The other parameters are assumed to be the same as in the benchmark case.

Figure 8 presents our results showing an interesting effect. On the one hand, both free-boundaries are decreasing in σ_1 as in the SSC case, On the other hand, they are increasing in σ_2. This means that a higher uncertainty in the low regime causes the firm to reduce the production, but a higher uncertainty in the high regime leads to more production. This is different in comparison to the SSC case.

Finally, we investigate the dependence of the free-boundaries on k_i. In the SSC case, see Lemma 3.13, we see that $k_i \mapsto b_i(k_i)$ is non-increasing while $k_{3-i} \mapsto b_i(k_{3-i})$

is non-decreasing. Figure 9 shows the numerical results of the dependence of free-boundaries on k_i in the bounded-velocity control case. One can see that, at least in this parameter setting, the free-boundaries evolve in the same way as in the SSC case.

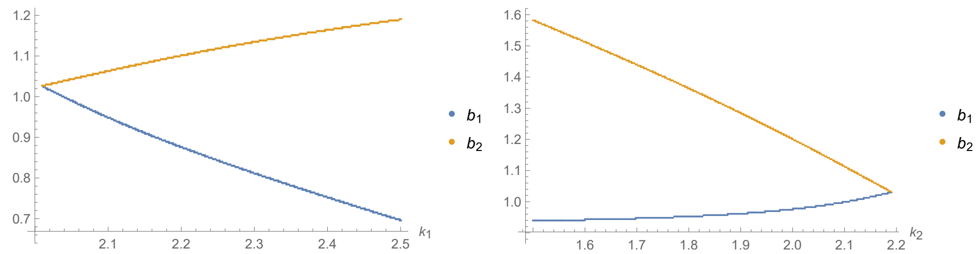


Figure 9: Graphical illustration of the dependence of the free-boundaries on k_i. The other parameters are assumed to be the same as in the benchmark case.

3.4 Comparison Between Different Models

3.4.1 Comparison Between the Singular and the Bounded-Velocity Control Cases

In this section we compare the two different cases investigated so far starting with the two benchmark cases. We note that, apart from the additional parameters $K_1 = K_2 = 1$, the benchmark cases coincide.

In the following, we denote by b_i^U the free-boundaries of the SSC case and by b_i^B the free-boundaries of the bounded-velocity case. For the benchmark cases, we obtain that

$$b_1^B = 0,976 > 0,914 = b_1^U \quad \text{and} \quad b_2^B = 1.201 > 0,977 = b_2^U.$$

Hence, the firm starts earlier to produce in the bounded-velocity control case compared to the SSC case. This is reasonable, because in the SSC case the firm can produce instantaneously any amount of the good, hence it can tolerate a lower inventory level. While in the bounded-velocity control case it can happen that the inventory level decreases even though that the production is maximal.

Next, we denote by V_B the value function of the bounded-velocity control case and by V_U the value function of the SSC case. If K_i is increasing, one expects that the value functions of the bounded-velocity control case converges from above to the value function of the SSC case. This is due to the fact that the bounded-velocity controls are also admissible in the SSC case. This is illustrated in Figure 10, where the relative distance between the value functions is shown with respect to several values for K_i. For simplicity, we assume that $K_1 = K_2 = K$.

A similar result can be obtained for the free-boundaries, see Figure 11.

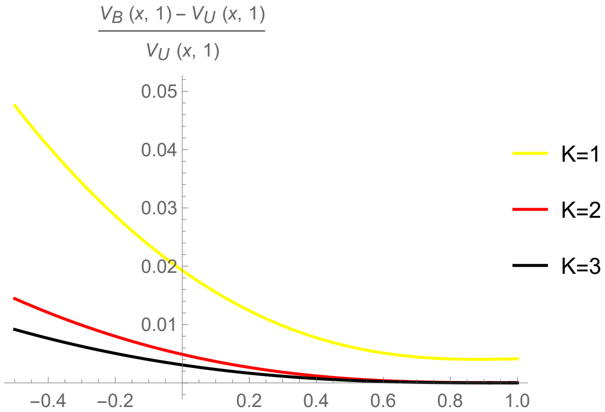


Figure 10: Graphical illustration of the relative change of the value functions in the different settings for certain values of K_i. The other parameters are assumed to be the same as in the benchmark case.

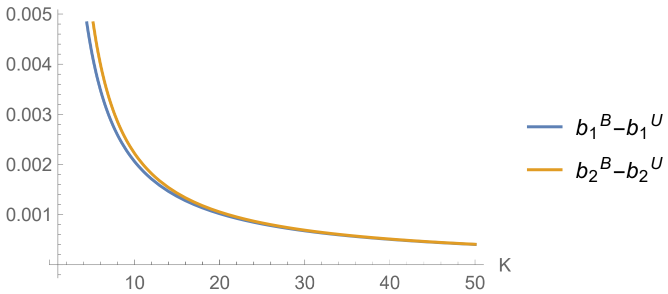


Figure 11: Graphical illustration of the absolute difference of the free-boundaries in the different settings for several values of K. The other parameters are assumed to be the same as in the benchmark case.

3.4.2 The Singular Stochastic Control Case: A Comparison with the Single Regime Case

The problem of a single regime can be solved rather easily and therefore the deatls are omitted. In particular, the free-boundary b in this problem can be calculated explicitly:

$$b = \frac{-\delta(k - \frac{2\alpha}{\delta\gamma_1}) + 2\mu\frac{\alpha}{\delta} + 2\alpha I}{2\alpha},$$

where γ_1 is the negative solution of $\frac{1}{2}\sigma^2\gamma^2 - \mu\gamma - \delta = 0$. Denoting by b_{ls} the free-boundary for the single regime case with the parameter values of the low regime ($\mu = 0.2$, $\sigma = 0.2$, $\delta = 0.2$, $\alpha = 0.2$, $I = 2$ and $k = 2$) and by b_{us} the free-boundary for the single regime case with the parameter values of the high regime ($\mu = 1.0$, $\sigma = 0.2$, $\delta = 0.2$, $\alpha = 0.2$, $I = 2$ and $k = 2$), Figure 12 shows the corresponding values of the free-boundaries of the single regime and the two regime case.

One can see that the free-boundaries in the case with regime switching lie between the ones of the corresponding single regime cases. To some extend, this can be understood as an *average effect*. For example, if we extend the single regime model

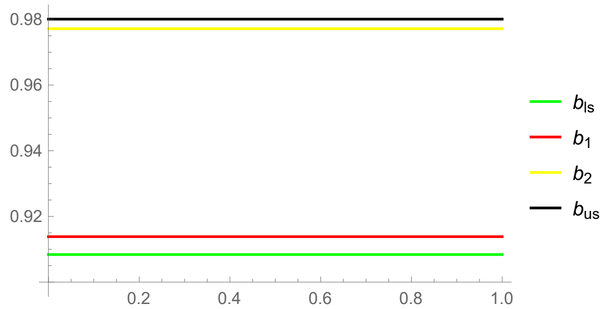


Figure 12: Graphical illustration of the different free-boundaries of the one regime and the two regime model. The parameters are assumed to be the same as in the benchmark case.

(in the low regime) with a second regime (the high regime), the firm reacts to the new possibility of a higher demand rate by increasing the level above which the inventory is supposed to be maintained.

3.4.3 The Bounded-Velocity Control Case: A Comparison with the Single Regime Case

If there is no regime switching, the free-boundary b can be calculated explicitly:

$$b = -\frac{2K\alpha\gamma_1\gamma_2 - 2\alpha\gamma_1\delta - 2\alpha\gamma_2\delta - 2I\alpha\gamma_1\gamma_2\delta + k\gamma_1\gamma_2\delta^2 - 4\alpha\gamma_1\gamma_2\mu}{2\alpha\gamma_1\gamma_2\delta},$$

where γ_1 is the positive solution of $\frac{1}{2}\sigma^2\gamma^2 + (K-\mu)\gamma - \delta = 0$ and γ_2 is the negative solution of $\frac{1}{2}\sigma^2\gamma^2 - \mu\gamma - \delta = 0$.

Again, we consider the low regime (with parameter values $\mu = 0.2$, $\sigma = 0.2$, $\delta = 0.2$, $\alpha = 0.2$, $I = 2$, $k = 2$ and $K = 1$) and the high regime (with parameter values $\mu = 1$, $\sigma = 0.2$, $\delta = 0.2$, $\alpha = 0.2$, $I = 2$, $k = 2$ and $K = 1$). Denoting by b_{ls} the free-boundary for the single regime case with the parameter values of the low regime and by b_{us} the free-boundary for the single regime case with the parameter values of the high regime, the same average effect as in the SSC case occurs, see Figure 13.

3.5 Conclusion

In this part of the thesis, we derived an analytical solution for the optimal production problem of a firm when it is confronted with an uncertain demand for its product. Mathematically, the demand uncertainty is modeled by two components; a Brownian motion capturing random short-term fluctuations in the economy and a continuous-time Markov chain determining the uncertain long-term conditions. We investigated two scenarios. One, in which a firm can immediately produce any amount of a good and one in which the production is expressed by a non-negative rate bounded from above. From a mathematical point of view, the first case is modeled as a SSC problem, while the second one is modeled as a classical stochastic control problem

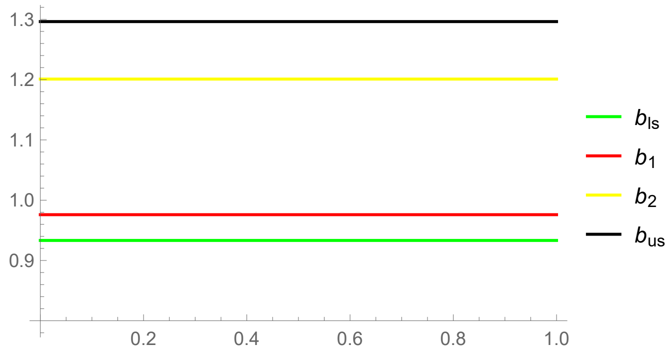


Figure 13: Graphical illustration of the different free-boundaries of the no regime and the two regime model. The parameters are assumed to be the same as in the benchmark case.

with bounded-velocity controls. For both scenarios, we find an analytical solution for the value function using a guess-and-verify approach. Moreover, the optimal production strategy is characterized by constant free-boundaries. In particular, the firm starts to produce when the inventory falls below these free-boundaries. The values of the free-boundaries are derived numerically. In the scope of a comparative static analysis we studied the dependence of the free-boundaries on some model parameters. The most remarkable result is that, in a particular benchmark case, the free-boundaries of the two scenarios show a different behavior with respect to the uncertainty parameter in the high regime. More precisely, the free-boundaries are decreasing in the SSC case, but increasing in the case of bounded-velocity controls.

4 A Singular Stochastic Control Problem with Interconnected Dynamics [7]

4.1 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space rich enough to accommodate an $\mathbb{F}$-Brownian motion $W := (W_t)_{t\geq 0}$. We assume that the filtration $\mathbb{F}$ satisfies the usual conditions.

Introducing the (nonempty) set

$$\begin{aligned}\mathcal{A} := \{\xi : \Omega \times \mathbb{R}_+ \to \mathbb{R} :\ &(\xi_t)_{t\geq 0} \text{ is } \mathbb{F}\text{-adapted and such that } t \mapsto \xi_t \text{ is a.s.}\\ &\text{càdlàg and (locally) of finite variation}\},\end{aligned} \tag{4.1}$$

for any $\xi \in \mathcal{A}$, we denote by ξ^+ and ξ^- the two non-decreasing $\mathbb{F}$-adapted càdlàg processes providing the minimal decomposition of ξ; i.e. $\xi = \xi^+ - \xi^-$ and the (random) Borel-measures induced on $[0, \infty)$ by ξ^+ and ξ^- have disjoint supports. In the following, for any $\xi \in \mathcal{A}$, we set $\xi^{\pm}_{0-} = 0$ a.s. and we denote by $|\xi|_t := \xi^+_t + \xi^-_t$, $t \geq 0$, its total variation.

For $\xi \in \mathcal{A}$, $(x, y) \in \mathbb{R}^2$, and $\alpha > 0$, we consider the purely controlled dynamics

$$Y^{y,\xi}_t = y + \xi^+_t - \xi^-_t, \quad t \geq 0, \qquad Y^{y,\xi}_{0-} = y, \tag{4.2}$$

as well as the diffusive

$$\begin{cases} dX^{x,y,\xi}_t = \left(\alpha Y^{y,\xi}_t - \theta X^{x,y,\xi}_t\right)\, dt + \eta\, dW_t, \quad t > 0, \\ X^{x,\xi}_0 = x, \end{cases} \tag{4.3}$$

where $\eta > 0$ is the volatility, $\alpha > 0$ measures the strength of interaction, and $\theta \geq 0$. If $\theta > 0$, the process $X^{x,y,\xi}$ evolves as an Ornstein Uhlenbeck process with mean reversion speed θ, in the case $\theta = 0$ it evolves as a drifted Brownian Motion; the unique strong solution to (4.3) is given by

$$X^{x,y,\xi}_t = \begin{cases} xe^{-\theta t} + e^{-\theta t}\alpha \int_0^t e^{\theta s} Y^{y,\xi}_s\, ds + \eta e^{-\theta t} \int_0^t e^{\theta s}\, dW_s, & \theta > 0, \\ x + \alpha \int_0^t Y^{y,\xi}_s\, ds + \eta W_t, & \theta = 0. \end{cases} \tag{4.4}$$

Remark 4.1. The assumption $\alpha > 0$ is not necessary for the following analysis; all results, up to small modifications, can be obtained with the same methods also for $\alpha < 0$. In order to simplify the exposition, we only consider $\alpha > 0$.

This model can capture different practical problems. For example, in the case of an Ornstein Uhlenbeck process, see [44], one can say that $Y^{y,\xi}$ describes the evolution of the key interest rate, purely controlled by a central bank, and $X^{x,y,\xi}$ gives the value of inflation. Another reasonable application for our model might be the problem of controlling the CO_2 emissions of a company. $X^{x,y,\xi}$ describes the

[7] This section is already published in two joint works with Giorgio Ferrari and Salvatore Federico, see [44] and [45].

total CO_2 emissions and $Y^{y,\xi}$ the number of production units that do not employ fossil fuel. This number can adjusted by the manager and affects negatively ($\alpha < 0$) the equilibrium value of CO_2 emissions of the firm. For $\theta = 0$, one can think about the problem of controlling the position of a satellite, which is disturbed by random fluctuations and can be adjusted by controlling its velocity. Alternatively, $X^{x,y,\xi}$ might describe a random demand of a product, whose instantaneous trend $Y^{y,\xi}$ can be affected via production (see, e.g., the review [95]).

The controller is faced with the problem of choosing, for given $(x, y) \in \mathbb{R}^2$ and $\rho > 0$, a process $\xi \in \mathcal{A}$ such that the cost functional

$$J(x, y; \xi) := \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} f(X_t^{x,y,\xi}, Y_t^{y,\xi})\, dt + \int_0^\infty e^{-\rho t} K \;\, d|\xi|_t\bigg] \tag{4.5}$$

is minimized; that is, it aims at solving

$$V(x, y) := \inf_{\xi \in \mathcal{A}} J(x, y; \xi), \qquad (x, y) \in \mathbb{R}^2. \tag{4.6}$$

In (4.5) and in the following, the integrals with respect to $d|\xi|$ and $d\xi^{\pm}$ are intended in the Lebesgue-Stieltjes' sense; in particular, for $\zeta \in \{|\xi|, \xi^+, \xi^-\}$, we set $\int_0^s (\,\cdot\,) d\zeta_t := \int_{[0,s]} (\,\cdot\,) d\zeta_t$ in order to take into account a possible mass at time zero of the Borel (random) measure $d\zeta$. The function $f : \mathbb{R}^2 \to \mathbb{R}^+$ satisfies the following standing assumption.

Assumption 4.2. *There exists constants $p > 1$, and $C_0, C_1, C_2 > 0$ such that the following hold true:*

(i) $0 \le f(z) \le C_0\big(1 + |z|\big)^p$, *for every* $z = (x, y) \in \mathbb{R}^2$*;*

(ii) for every $z = (x, y), z' = (x', y') \in \mathbb{R}^2$,

$$|f(z) - f(z')| \le C_1\big(1 + f(z) + f(z')\big)^{1-\frac{1}{p}} |z - z'|;$$

(iii) for every $z = (x, y), z' = (x', y') \in \mathbb{R}^2$ *and* $\lambda \in (0, 1)$,

$$0 \le \lambda f(z) + (1-\lambda) f(z') - f(\lambda z + (1-\lambda) z') \le C_2 \lambda (1-\lambda)(1 + f(z) + f(z'))^{\left(1-\frac{2}{p}\right)^+} |z - z'|^2;$$

(iv) $x \mapsto f_y(x, y)$ *is non-decreasing for any* $y \in \mathbb{R}$.

Remark 4.3. (i) By Assumption 4.2-(iii), f is convex and locally semiconcave; then, by Corollary 3.3.8 in [22],

$$f \in C^{1,Lip}_{\text{loc}}(\mathbb{R}^2; \mathbb{R}) = W^{2,\infty}_{\text{loc}}(\mathbb{R}^2; \mathbb{R}).$$

(ii) A function f satisfying Assumption 4.2 is, for example,

$$f(x, y) = |x - \hat{x}|^p + |y - \hat{y}|^p,$$

with $p \ge 2$ even for some $\hat{x}, \hat{y} \in \mathbb{R}$.

We now provide some preliminary properties of the value function, whose classical proof exploits the linear structure of the state equations, e.g. for all $(x,y),(\hat{x},\hat{y}) \in \mathbb{R}^2$, $\xi \in \mathcal{A}$ and $t \geq 0$, it holds, by using (4.4), that

$$X_t^{x,y,\xi} - X_t^{\hat{x},\hat{y},\xi} = \begin{cases} (x-\hat{x})e^{-\theta t} + \frac{\alpha}{\theta}(y-\hat{y})(1-e^{-\theta t}), & \theta > 0, \\ (x-\hat{x}) + \alpha(y-\hat{y})t, & \theta = 0. \end{cases}$$

Proposition 4.4. *Let Assumption 4.2 hold and let $p > 1$ be the constant appearing in the assumption. There exist constants $\widehat{C}_0, \widehat{C}_1, \widehat{C}_2 > 0$ such that the following hold:*

(i) $0 \leq V(z) \leq \widehat{C}_0\big(1+|z|^p\big)$, *for every* $z = (x,y) \in \mathbb{R}^2$;

(ii) for every $z = (x,y), z' = (x',y') \in \mathbb{R}^2$,

$$|V(z) - V(z')| \leq \widehat{C}_1\big(1+|z|+|z'|\big)^{p-1}|z-z'|;$$

(iii) for every $z = (x,y), z' = (x',y') \in \mathbb{R}^2$ *and* $\lambda \in (0,1)$,

$$0 \leq \lambda V(z) + (1-\lambda)V(z') - V(\lambda z + (1-\lambda)z') \leq \widehat{C}_2\lambda(1-\lambda)(1+|z|+|z'|)^{(p-2)^+}|z-z'|^2.$$

In particular, by (iii), V is convex and locally semiconcave, hence, by Corollary 3.3.8 in [22],

$$V \in C^{1,Lip}_{loc}(\mathbb{R}^2;\mathbb{R}) = W^{2,\infty}_{loc}(\mathbb{R}^2;\mathbb{R}).$$

Proof. Due to (4.2) and (4.3), the properties of f required in (i), (ii) and (iii) of Assumption 4.2 are straightly inherited by V (see, e.g., the proof of Theorem 2.1 of [28], that can easily adapted to our infinite time-horizon setting, or that of Theorem 2.1 in [24]). ☐

4.2 The Related Dynkin Game and Preliminary Properties of the Free-Boundaries

In this section, we derive a representation of V_y (the derivative in direction of the controlled variable). In particular, we identify V_y with the value of a zero-sum game of optimal stopping (*Dynkin ame*). This representation allows us to obtain further properties of V_y and to obtain a decomposition of the state space by two curves (the free-boundaries). Moreover, we derive preliminary properties of these curves. In order to simplify the notation, in the following we write $X^{x,y}$, instead of $X^{x,y,0}$, to identify the solution to (4.3) for $\xi \equiv 0$.

Theorem 4.5. *Let $(x,y) \in \mathbb{R}^2$. Denote by $\mathcal{T}$ the set of all $\mathbb{F}$-stopping times, and for $(\sigma,\tau) \in \mathcal{T} \times \mathcal{T}$ consider the stopping functional*

$$\begin{aligned}\Psi(\sigma,\tau;x,y) := \mathbb{E}\bigg[\int_0^{\tau\wedge\sigma} & e^{-\rho t}\Big(f_y(X_t^{x,y},y) + \alpha V_x(X_t^{x,y},y)\Big)\,dt \\ & - e^{-\rho\tau}K\mathbb{1}_{\{\tau\leq\sigma\}} + e^{-\rho\sigma}K\mathbb{1}_{\{\tau>\sigma\}}\bigg],\end{aligned} \tag{4.7}$$

where V_x is the partial derivative of V with respect to x (which exists continuous by Proposition 4.4). One has that the game has a value, i.e.

$$\inf_{\sigma\in\mathcal{T}}\sup_{\tau\in\mathcal{T}}\Psi(\sigma,\tau;x,y)=\sup_{\tau\in\mathcal{T}}\inf_{\sigma\in\mathcal{T}}\Psi(\sigma,\tau;x,y),$$

and such a value is given by

$$V_y(x,y)=\inf_{\sigma\in\mathcal{T}}\sup_{\tau\in\mathcal{T}}\Psi(\sigma,\tau;x,y)=\sup_{\tau\in\mathcal{T}}\inf_{\sigma\in\mathcal{T}}\Psi(\sigma,\tau;x,y). \tag{4.8}$$

Moreover, the couple of $\mathbb{F}$-stopping times $(\tau^\star(x,y),\sigma^\star(x,y)):=(\tau^\star,\sigma^\star)$ such that

$$\sigma^\star:=\inf\left\{t\geq 0:\ V_y(X_t^{x,y},y)\geq K\right\},\quad \tau^\star:=\inf\left\{t\geq 0:\ V_y(X_t^{x,y},y)\leq -K\right\} \tag{4.9}$$

(with the usual convention $\inf\emptyset=+\infty$*) forms a saddle-point; that is,*

$$\forall\tau\in\mathcal{T}\quad \Psi(\sigma^\star,\tau;x,y)\leq V_y(x,y)=\Psi(\sigma^\star,\tau^\star;x,y)\leq\Psi(\sigma,\tau^\star;x,y)\quad\forall\sigma\in\mathcal{T}.$$

The proof of Theorem 4.5 can be obtained by approximating our degenerate singular stochastic control problem by a fully diffusive setting and applying the results from Theorems 3.11 and 3.13 in [26]. The details are presented in Appendix B.1

Remark 4.6. This game reflects some interesting interpretations. The game is a two-player zero-sum game, where both players play against each other and have the possibility to stop the game. Player 1 can choose σ and Player 2 τ. If Player 1 stops the game, she pays $e^{-\rho\sigma}K$ to Player 2 and if Player 2 stops first, she pays $e^{-\rho\tau}K$ to Player 1. As long as the game is running, Player 1 is paying Player 2 a running cost at a rate $f_y(X_t^{x,y},y)+\alpha V_x(X_t^{x,y},y)$. The optimal strategy of the players can then be seen as a dynamic equilibrium between acting (and hence paying the linear cost K) and waiting (which results in the running cost), where τ is the optimal time to increase Y and σ the optimal time to decrease Y.

It is remarkable that the running cost consists of two parts. The first part describes an immediate change in the running cost function f of the control problem if one would act on y. The second part reflects that a changed level of y also modify the evolution of $X_t^{x,y}$ because of the interconnected dynamics. Hence, an indirect change in the cost arises, which is given through the term $\alpha V_x(X_t^{x,y},y)$. Moreover, the second part implies that the Dynkin game depends on the optimal strategies of the control problem.

From (4.8), it follows that $-K\leq V_y(x,y)\leq K$ for any $(x,y)\in\mathbb{R}^2$. This suggest to define the following partition of $\mathbb{R}^2$;

$$\begin{cases}\mathcal{I}:=\left\{(x,y)\in\mathbb{R}^2:\ \ V_y(x,y)=-K\right\},\\ \mathcal{C}:=\left\{(x,y)\in\mathbb{R}^2:\ \ -K<V_y(x,y)<K\right\},\\ \mathcal{D}:=\left\{(x,y)\in\mathbb{R}^2:\ \ V_y(x,y)=K\right\}.\end{cases} \tag{4.10}$$

By continuity of V_y (see Proposition 4.4), one obtains that the $\mathcal{I}$ and $\mathcal{D}$ are closed sets and $\mathcal{C}$ is open. Moreover, defining $b_1 : \mathbb{R} \to \overline{\mathbb{R}}$ and $b_2 : \mathbb{R} \to \overline{\mathbb{R}}$ as

$$\begin{aligned} b_1(x) &:= \inf\{y \in \mathbb{R} \mid V_y(x,y) > -K\} = \sup\{y \in \mathbb{R} \mid V_y(x,y) = -K\}, \quad x \in \mathbb{R}, \\ b_2(x) &:= \sup\{y \in \mathbb{R} \mid V_y(x,y) < K\} = \inf\{y \in \mathbb{R} \mid V_y(x,y) = K\}, \quad x \in \mathbb{R}, \end{aligned} \tag{4.11}$$

(with the usual conventions $\inf \emptyset = \infty$, $\inf \mathbb{R} = -\infty$, $\sup \emptyset = -\infty$, $\sup \mathbb{R} = \infty$) one obtain the representation

$$\mathcal{C} = \{(x,y) \in \mathbb{R}^2 : \ b_1(x) < y < b_2(x)\},$$

$$\mathcal{I} = \{(x,y) \in \mathbb{R}^2 : \ y \le b_1(x)\}, \quad \mathcal{D} = \{(x,y) \in \mathbb{R}^2 : \ y \ge b_2(x)\}.$$

The Dynkin game representation allows us to proof easily the following result.

Lemma 4.7. *$V_y(\cdot, y)$ is non-decreasing for all $y \in \mathbb{R}$.*

Proof. From the convexity of V (see Proposition 4.4), we have that $x \mapsto V_x(x,y)$ is non-decreasing and by Assumption 4.2-(iv) it holds that $x \mapsto f_y(x,y)$ is non-decreasing. Hence, by (4.7), $\alpha > 0$ and the linear structure of $X_t^{x,y}$ in x, we have that $x \mapsto \Psi(\sigma,\tau;x,y)$ is non-decreasing for all $y \in \mathbb{R}$ and $\sigma, \tau \in \mathcal{T}$. The claim follows then by (4.8). □

Using the proved monotonicity of V_y as well as its continuity, we obtain the following preliminary properties of b_1 and b_2.

Proposition 4.8. *The following hold:*

(i) $b_1 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, $b_2 : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$;

(ii) b_1 *and* b_2 *are non-increasing;*

(iii) $b_1(x) < b_2(x)$ *for all* $x \in \mathbb{R}$;

(iv) b_1 *is left-continuous and* b_2 *is right-continuous.*

Proof. (i) We argue by contradiction. Assume that there exists a $x_0 \in \mathbb{R}$ such that $b_1(x_0) = \infty$. By (4.11), we have $V_y(x_0, y) = -K$ for all $y \in \mathbb{R}$ and hence

$$V(x_0, y + y') = V(x_0, y) - Ky'$$

for all $y, y' \in \mathbb{R}$. Since V is non-negative and $V(x_0, y) \le J(x_0, y; 0) < \infty$, we obtain

$$Ky' \le V(x_0, y) \le J(x_0, y; 0) < \infty$$

for all $y, y' \in \mathbb{R}$. Since the right-hand side is independent of y', a contradiction is obtained by choosing y' sufficiently large. A similar argument shows that b_2 takes values in $\mathbb{R} \cup \{\infty\}$.

(ii) The claimed monotonicty follows immediately from their definition and Lemma 4.7.

(iii) The fact that $b_1(x) < b_2(x)$ for all $x \in \mathbb{R}$ follows from the convexity of V with respect to y and the continuity of $V_y(x, \cdot)$ for all $x \in \mathbb{R}$.

(iv) We only prove the statement for b_1, since the argument for b_2 can be proved analogously. By (ii) above, we have that $b_1(x) \leq b_1(x - \epsilon)$ for all $\epsilon > 0$. Hence, $b_1(x) \leq \lim_{\epsilon \to 0} b_1(x - \epsilon) = b_1(x-)$. On the other hand, the sequence $(x - \epsilon, b_1(x - \epsilon))_{\epsilon > 0} \subset \mathcal{I}$ and, since $\mathcal{I}$ is closed, we get that $(x, b_x(x-)) \in \mathcal{I}$. Therefore, $b_1(x) \geq b_1(x-)$ and combining the two results give $b_1(x) = b_1(x-)$.

□

Let us now define

$$\bar{b}_1 := \sup_{x \in \mathbb{R}} b_1(x), \quad \underline{b}_1 := \inf_{x \in \mathbb{R}} b_1(x), \quad \bar{b}_2 := \sup_{x \in \mathbb{R}} b_2(x), \quad \underline{b}_2 := \inf_{x \in \mathbb{R}} b_2(x), \tag{4.12}$$

together with the pseudo-inverses of b_1 and b_2 by

$$g_1(y) := \sup\{x \in \mathbb{R} : b_1(x) \geq y\}, \quad g_2(y) := \inf\{x \in \mathbb{R} : b_2(x) \leq y\} \tag{4.13}$$

(again, with the usual conventions $\inf \emptyset = \infty$, $\inf \mathbb{R} = -\infty$, $\sup \emptyset = -\infty$, $\sup \mathbb{R} = \infty$).

The pseudo-inverses will play an important role later on, and we give some preliminary properties of them.

Proposition 4.9. *The following holds:*

(i) $g_1(y) = \inf\{x \in \mathbb{R} : V_y(x, y) > -K\}, \quad g_2(y) = \sup\{x \in \mathbb{R} : V_y(x, y) < K\}$;

(ii) the functions g_1, g_2 *are non-increasing;*

(iii) $g_1(y) < g_2(y)$ *for any* $y \in \mathbb{R}$*;*

(iv) If $\bar{b}_2 < \infty$*, then* $g_2(y) = -\infty$ *for all* $y \geq \bar{b}_2$ *and if* $\underline{b}_1 > -\infty$*, then* $g_1(y) = \infty$ *for all* $y \leq \underline{b}_1$*.*

Proof. Item (i) follows from the definition of g_1 and g_2 and (ii) by Proposition 4.8-(ii). Claim (iii) is due to (i), Lemma 4.7 and the continuity of V_y for all $y \in \mathbb{R}$. We show (iv) by contradiction. Assume that $\bar{b}_2 < \infty$ and suppose that $\lim_{y \to \infty} g_2(y) = \bar{g} > -\infty$. But then, it holds that

$$V_y(x, y) < K$$

for all $x < \bar{g}$ and $y \in \mathbb{R}$. Hence $b_2(x) = \infty$ for all $x < \bar{g}$ and a contradiction is reached. An analogous argument proves the statement for g_1. □

4.3 The Structure of the Value Function

Up to now, we derived a representation of the derivative V_y of the value function of the control problem and we decomposed the state space into three connected regions defined in terms of V_y. This regions are separated by non-increasing curves. In this section, we exploit these results to determine the structure of the value function V. In particular, we investigate the corresponding HJB equation for our problem.

For any given and fixed $y \in \mathbb{R}$, denote by $\mathcal{L}^y$ the infinitesimal generator associated to the uncontrolled process $X^{x,y,0}$. Acting on $g \in C^2(\mathbb{R};\mathbb{R})$ it yields, for $x \in \mathbb{R}$,

$$\big(\mathcal{L}^y g\big)(x) := \begin{cases} \frac{\eta^2}{2} g''(x) + (\alpha y - \theta x)\, g'(x), & \theta > 0, \\ \frac{\eta^2}{2} g''(x) + \alpha y g'(x), & \theta = 0. \end{cases}$$

Any solution $\beta(\cdot, y)$ to the second-order ordinary differential equation (ODE)

$$\big(\mathcal{L}^y \beta(\cdot, y)\big)(x) - \rho\beta(x,y) = 0, \quad x \in \mathbb{R},$$

can be written as

$$\beta(x,y) = A(y)\psi(x,y) + B(y)\varphi(x,y), \quad x \in \mathbb{R}.$$

The functions ψ and φ are the strictly increasing and decreasing fundamental solutions to the ODE above and are given by (see page 279 and 280 in [52], among others)

$$\psi(x,y) = \begin{cases} \exp\left(-\frac{\theta\left(x - \frac{\alpha}{\theta}y\right)^2}{2\eta^2}\right) D_{-\frac{\rho}{\theta}}\left(-\frac{x-\frac{\alpha}{\theta}y}{\eta}\sqrt{2\theta}\right), & \theta > 0, \\ \exp\left(\frac{x}{\eta^2}\left(-\alpha y + \sqrt{(\alpha y)^2 + 2\eta\rho}\right)\right), & \theta = 0, \end{cases} \tag{4.14}$$

and

$$\varphi(x,y) = \begin{cases} \exp\left(-\frac{\theta\left(x - \frac{\alpha}{\theta}y\right)^2}{2\eta^2}\right) D_{\frac{\rho}{\theta}}\left(-\frac{x-\frac{\alpha}{\theta}y}{\eta}\sqrt{2\theta}\right), & \theta > 0, \\ \exp\left(\frac{x}{\eta^2}\left(-\alpha y - \sqrt{(\alpha y)^2 + 2\eta\rho}\right)\right), & \theta = 0, \end{cases} \tag{4.15}$$

where

$$D_\beta(x) := \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\beta)} \int_0^\infty t^{-\beta-1} e^{-\frac{t^2}{2} - xt} dt, \quad \beta < 0,$$

is the Cylinder function of order β and $\Gamma(\,\cdot\,)$ is the Euler's Gamma function (see, e.g., Chapter VIII in [9]). Moreover, ψ and φ are strictly convex.

By the dynamic programming principle, we expect that V identifies with a suitable solution of the variational inequality

$$\max\Big\{ -v_y(x,y) - K,\ v_y(x,y) - K,\ [(\rho - \mathcal{L}^y)v(\cdot,y)](x) - f(x,y) \Big\} = 0, \tag{4.16}$$

for all $(x,y) \in \mathbb{R}^2$.

Assuming enough regularity for v for applying Itô's formula, (4.16) can be derived by investigating the three possible cases for the control: (i) immediately adjust Y by a lump sum increase with marginal cost K and continue optimally, (ii) immediately adjust Y by a lump sum decrease with marginal cost K and continue optimally, and (iii) wait for a small amount of time and continue optimally. A formal derivation of the dynamic programming can be found in [49]. In principle, since the HJB involves an ODE instead of a PDE, it might be possible to follow now a guess-and-verify approach by imposing certain smoothness assumptions. However, it would be hard to prove optimality of a candidate value function due to the interaction between X and Y. Therefore, we use a direct approach as in [43] and show that V is a viscosity solution to (4.16). This enables us to determine the structure of V (see Proposition 4.14 below) and to prove certain smoothness properties (cf. Theorem 4.16). This will allow us later to derive necessary equations for the free-boundaries (cf. Theorem 4.26).

Definition 4.10.

(i) A function $v \in C^0(\mathbb{R}^2;\mathbb{R})$ is called a viscosity subsolution *to* (4.16) *if, for every $(x,y) \in \mathbb{R}^2$ and every $\beta \in C^{2,1}(\mathbb{R}^2;\mathbb{R})$ such that $v-\beta$ attains a local maximum at (x,y), it holds*

$$\max\Big\{ -\beta_y(x,y)-K,\ \beta_y(x,y)-K,\ \rho\beta(x,y)-[\mathcal{L}^y\beta(\cdot,y)](x)-f(x,y)\Big\} \leq 0.$$

(ii) A function $v \in C^0(\mathbb{R}^2;\mathbb{R})$ is called a viscosity supersolution *to* (4.16) *if, for every $(x,y) \in \mathbb{R}^2$ and every $\beta \in C^{2,1}(\mathbb{R}^2;\mathbb{R})$ such that $v-\beta$ attains a local minimum at (x,y), it holds*

$$\max\Big\{ -\beta_y(x,y)-K,\ \beta_y(x,y)-K,\ \rho\beta(x,y)-[\mathcal{L}^y\beta(\cdot,y)](x)-f(x,y)\Big\} \geq 0.$$

(iii) A function $v \in C^0(\mathbb{R}^2;\mathbb{R})$ is called a viscosity solution *to* (4.16) *if it is both a viscosity subsolution and supersolution.*

Following the arguments developed in Theorem 5.1 in Section VIII.5 of [49], one can show the following result.

Proposition 4.11. *The value function V is a viscosity solution to* (4.16).

Remark 4.12. Recall that by Proposition 4.4-(iii) our value function V lies in the class $W^{2,\infty}_{\text{loc}}(\mathbb{R}^2;\mathbb{R})$. Hence, by Lemma 5.4 in Chapter 4 of [97] it is also a *strong solution* to (4.16) (in the sense, e.g., of [20]; see the same reference also for relations between these notions of solutions); that is, it solves (4.16) in the pointwise sense almost everywhere.

We have decided to employ the concept of viscosity solution since our analysis will later make use of the variational inequality (4.16) on sets of null Lebesgue measure (regular lines) (see Proposition 4.13 and Proposition 4.16 below). Because the viscosity property holds *for all* (and not merely for a.e.) points of the state space $\mathbb{R}^2$, the concept of viscosity solution is still able to provide information on V on regular lines.

For later use, notice that the function

$$\widehat{V}(x,y) := J(x,y,0) = \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} f(X_t^{x,y}, y)\,dt\bigg], \quad (x,y) \in \mathbb{R}^2, \tag{4.17}$$

is finite by Assumption 4.2-(i) and standard estimates, and continuously differentiable with respect to y and x, given the assumed regularity of f_x and f_y in Assumption 4.2-(iii). By Feynman-Kac's theorem, it follows that $\widehat{V}$ identifies with a classical particular solution to the inhomogeneous linear ODE

$$[(\mathcal{L}^y - \rho)\beta(\cdot,y)](x) + f(x,y) = 0, \quad x \in \mathbb{R}. \tag{4.18}$$

Recall now the regions $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{D}$ from (4.10), and that $V_y = -K$ on $\mathcal{I}$, while $V_y = K$ on $\mathcal{D}$. The next proposition provides the structure of V inside $\mathcal{C}$.

Proposition 4.13. *Recall* (4.12) *and let* $y_o \in (\underline{b}_1, \bar{b}_2)$.

(i) The function $V(\cdot, y_o)$ *is a viscosity solution to*

$$\rho\beta(x,y_o) - [\mathcal{L}^{y_o}\beta(\cdot,y_o)](x) - f(x,y_o) = 0, \quad x \in (g_1(y_o), g_2(y_o)). \tag{4.19}$$

(ii) $V(\cdot, y_o) \in C^{3,Lip}_{loc}((g_1(y_o), g_2(y_o)); \mathbb{R})$.

(iii) There exist constants $A(y_o)$ *and* $B(y_o)$ *such that for all* $x \in (g_1(y_o), g_2(y_o))$

$$V(x,y_o) = A(y_o)\psi(x,y_o) + B(y_o)\varphi(x,y_o) + \widehat{V}(x,y_o),$$

where the functions ψ *and* φ *have been defined in* (4.14),(4.15) *and* $\widehat{V}$ *is as in* (4.17).

Proof. We prove each item separately.

Proof of (i). We show the subsolution property; that is, we prove that for any $x_o \in (g_1(y_o), g_2(y_o))$ and $\beta \in C^2((g_1(y_o), g_2(y_o)); \mathbb{R})$ such that $V(\cdot,y_o) - \beta(\cdot,y_o)$ attains a local maximum at x_o it holds that

$$\rho\beta(x_o,y_o) - [\mathcal{L}^{y_o}\beta(\cdot,y_o)](x_o) - f(x_o,y_o) \le 0.$$

First of all, we claim that

$$(V_y(x_o,y_o), \beta'(x_o), \beta''(x_o)) \in D_x^{2,1,+}V(x_o,y_o),$$

where $D^{2,1,+}V(x_o,y_o)$ is the superdifferential of V at (x_o,y_o) of first order with respect to y and of second order with respect to x (see Section 5 in Chapter 4 of [97]). This means that we have to show that

$$\limsup_{(x,y)\to(x_o,y_o)} \frac{V(x,y) - V(x_o,y_o) - V_y(x_o,y_o)(y-y_o) - \beta'(x_o)(x-x_o) - \frac{1}{2}\beta''(x_o)(x-x_o)^2}{|y-y_o| + |x-x_o|^2} \le 0. \tag{4.20}$$

In order to prove (4.20), notice first that $V(x_o,\cdot)$ is continuously differentiable, and therefore

$$\lim_{y\to y_o}\frac{V(x,y)-V(x,y_o)-V_y(x_o,y_o)(y-y_o)}{|y-y_o|}=0 \quad \text{uniformly in } x\in(x_o-1,x_o+1). \tag{4.21}$$

Using now Lemma 5.4 in [97], we have that

$$(\beta'(x_o),\beta''(x_o))\in D_x^{2,+}V(x_o,y_o),$$

where $D_x^{2,+}V(x_o,y_o)$ denotes the superdifferential of $V(\cdot,y_o)$ at x_o of second order (with respect to x); i.e.

$$\limsup_{x\to x_o}\frac{V(x,y_o)-V(x_o,y_o)-\beta'(x_o)(x-x_o)-\frac{1}{2}\beta''(x_o)(x-x_o)^2}{|x-x_o|^2}\leq 0. \tag{4.22}$$

Adding and substracting $V(x,y_o)$ in the numerator of (4.20), and using (4.21) and (4.22), we obtain (4.20).

Using again Lemma 5.4 in [97], we can then construct a function $\widehat{\beta}\in C^{2,1}(\mathbb{R}^2;\mathbb{R})$ such that $V-\widehat{\beta}$ attains a local maximum in (x_o,y_o) and

$$\Big(\widehat{\beta}_y(x_o,y_o),\widehat{\beta}_x(x_o,y_o),\widehat{\beta}_{xx}(x_o,y_o)\Big)=(V_y(x_o,y_o),\beta'(x_o),\beta''(x_o)). \tag{4.23}$$

Since $(x_o,y_o)\in\mathcal{C}$ we know that $-K<V_y(x_o,y_o)<K$, and because V is a viscosity solution to (4.16), we obtain by (4.23) that

$$\rho\beta(x_o,y_o)-[\mathcal{L}^{y_o}\beta(\cdot,y_o)](x_o)-f(x_o,y_o)\leq 0,$$

thus completing the proof of the subsolution property. The supersolution property can be shown in an analogous way and the proof is therefore omitted.

Proof of (ii). Let $a,b\in\mathbb{R}$ be such that $(a,y_o),(b,y_o)\in\mathcal{C}$ and $a<b$. Introduce the Dirichlet boundary value problem

$$\begin{cases}(\mathcal{L}^{y_o}-\rho)q(x)+f(x,y_o)=0, & x\in(a,b),\\ q(a,y_o)=V(a,y_o), & q(b,y_o)=V(b,y_o).\end{cases} \tag{4.24}$$

Since $f(\cdot,y_o)\in C^{1,Lip}_{loc}((g_1(y_o),g_2(y_o));\mathbb{R})$, by assumption, and $V(\cdot,y_o)\in C([a,b];\mathbb{R})$, by classical results problem (4.24) admits a unique classical solution $\hat{q}\in C^0([a,b];\mathbb{R})\cap C^{3,Lip}_{loc}((a,b);\mathbb{R})$. The latter is also a viscosity solution, and by (i) above and standard uniqueness results for viscosity solutions of linear equations it must coincide with $V(\cdot,y_o)$. Hence, we have that $V(\cdot,y_o)\in C^{3,Lip}_{loc}((g_1(y_o),g_2(y_o));\mathbb{R})$ and $V(\cdot,y_o)$ is a classical solution to

$$[(\mathcal{L}^{y_o}-\rho)V(\cdot,y_o)](x)+f(x,y_o)=0, \quad x\in(g_1(y_o),g_2(y_o)),$$

given the arbitrariness of (a,b) and the fact that $\mathcal{C}$ is open.

Proof of (iii). Since any solution to the homogeneous linear ODE $(\mathcal{L}^{y_o}-\rho)q=0$ is given by a linear combination of its increasing fundamental solution ψ and decreasing fundamental solution φ, we conclude by (ii) and the superposition principle. □

We are now able to provide the structure of the value function V.

Proposition 4.14. *Define the sets*

$$\mathcal{O}_1 := \{x \in \mathbb{R} : \, b_1(x) > -\infty\} \quad \mathcal{O}_2 := \{x \in \mathbb{R} : \, b_2(x) < \infty\}.$$

There exist functions

$$A, B \in W^{2,\infty}_{loc}((\underline{b}_1, \bar{b}_2); \mathbb{R}) = C^{1,Lip}_{loc}((\underline{b}_1, \bar{b}_2); \mathbb{R}), \quad z_{1,2} : \mathcal{O}_{1,2} \to \mathbb{R}$$

such that the value function defined in (4.6) *can be written as*

$$V(x,y) = \begin{cases} A(y)\psi(x,y) + B(y)\varphi(x,y) + \widehat{V}(x,y) & \text{on } \bar{\mathcal{C}}, \\ z_1(x) - Ky & \text{on } \mathcal{I}, \\ z_2(x) + Ky & \text{on } \mathcal{D}, \end{cases} \tag{4.25}$$

where $\bar{\mathcal{C}}$ denotes the closure of $\mathcal{C}$,

$$z_1(x) := V(x, b_1(x)) + Kb_1(x), \quad x \in \mathcal{O}_1$$

and

$$z_2(x) := V(x, b_2(x)) - Kb_2(x), \quad x \in \mathcal{O}_2.$$

Proof. We start by deriving the structure of V within $\mathcal{C}$. Using Lemma 4.13, we already know the existence of functions $A, B : (\underline{b}_1, \bar{b}_2) \to \mathbb{R}$ such that

$$V(x,y) = A(y)\psi(x,y) + B(y)\varphi(x,y) + \widehat{V}(x,y), \quad (x,y) \in \mathcal{C}. \tag{4.26}$$

Take now $y_o \in (\underline{b}_1, \bar{b}_2)$. Since $g_1(y) < g_2(y)$ for any $y \in \mathbb{R}$ (cf. Proposition 4.9-(ii)), we can find x and $\tilde{x}$, $x \neq \tilde{x}$, such that $(x,y), (\tilde{x},y) \in \mathcal{C}$ for any given $y \in (y_o - \varepsilon, y_o + \varepsilon)$, for a suitably small $\varepsilon > 0$. Now, by evaluating (4.26) at the points (x,y) and $(\tilde{x},y)$, we obtain a linear algebraic system that we can solve with respect to $A(y)$ and $B(y)$ so to obtain

$$A(y) = \frac{(V(x,y) - \widehat{V}(x,y))\varphi(\tilde{x},y) - (V(\tilde{x},y) - \widehat{V}(\tilde{x},y)\varphi(x,y)}{\psi(x,y)\varphi(\tilde{x},y) - \psi(\tilde{x},y)\varphi(x,y)}, \tag{4.27}$$

$$B(y) = \frac{(V(\tilde{x},y) - \widehat{V}(\tilde{x},y)\psi(x,y) - (V(x,y) - \widehat{V}(x,y))\psi(\tilde{x},y)}{\psi(x,y)\varphi(\tilde{x},y) - \psi(\tilde{x},y)\varphi(x,y)}. \tag{4.28}$$

The denominators of the last two expressions do not vanish due to the strict monotonicity of ψ and φ, and to the fact that $x \neq \tilde{x}$. Since y_o was arbitrary and V, $\widehat{V}$, V_y, and $\widehat{V}_y$ are continuous with respect to y, we therefore obtain that A and B belong to $W^{2,\infty}_{\text{loc}}((\underline{b}_1, \bar{b}_2); \mathbb{R}) = C^{1,Lip}_{\text{loc}}((\underline{b}_1, \bar{b}_2); \mathbb{R})$. The structure of V in the closure of $\mathcal{C}$, denoted by $\overline{\mathcal{C}}$, is then obtained by Proposition 4.13 and by recalling that V is continuous on $\mathbb{R}^2$ and that A, B, and $\widehat{V}$ are also continuous.

Given the definition of z_1 and z_2, the structure of V inside the regions $\mathcal{I}$ and $\mathcal{D}$ follow by (4.10) and the continuity of V. □

Remark 4.15. Actually, by (4.27) and (4.28) one has that A and B belong to $W^{2,\infty}$ up to $\underline{b}_1$ (resp. $\bar{b}_2$) if $\underline{b}_1$ (resp. $\bar{b}_2$) is finite.

So far, we know that $V \in C^{1,Lip}_{\text{loc}}(\mathbb{R}^2;\mathbb{R})$ and we know it's structure. Therefore, we have already a first-order smooth-fit condition for the value function. Now, the provide also a second-order smooth-fit principle for the value function V. More precisely, we show that V_{yx} is jointly continuous. Notice that

$$V_{yx}(x,y) = 0 \qquad \forall (x,y) \in \mathbb{R}^2 \setminus \overline{\mathcal{C}}.$$

The next result shows that one actually has continuity of V_{yx} on the whole $\mathbb{R}^2$.

Theorem 4.16. *One has that*

$$\lim_{\substack{(x,y)\to\ (x_o,y_o)\\ (x,y)\in\mathcal{C}}} V_{yx}(x,y) = 0 \qquad \forall (x_o,y_o) \in \partial\mathcal{C}. \tag{4.29}$$

Hence, $V_{yx} \in C(\mathbb{R}^2;\mathbb{R})$.

Proof. We prove (4.29) only at $\partial^1\mathcal{C} := \{(x,y) \in \mathbb{R}^2 : V_y(x,y) = -K\}$, and we distinguish two different cases for $(x_o,y_o) \in \partial^1\mathcal{C}$.

Case (a). Assume that $y_o = b_1(x_o)$. Define the function

$$\bar{V}(x,y) := A(y)\psi(x,y) + B(y)\varphi(x,y) + \widehat{V}(x,y), \quad (x,y) \in \mathbb{R}^2, \tag{4.30}$$

where A, B are the functions of Proposition 4.14. Then, one clearly has that $\bar{V} \in C^{2,1}(\mathbb{R}^2;\mathbb{R})$. Moreover, the mixed derivative $\bar{V}_{yx}$ exists and is continuous. Since $\bar{V} = V$ in $\bar{\mathcal{C}}$, by Lemma 4.7 we conclude that $\bar{V}_{yx} \geq 0$ in $\mathcal{C}$. Then by continuity of $\bar{V}_{yx}$, in order to show (4.29) it is enough to show that

$$\bar{V}_{yx}(x_o,y_o) \leq 0.$$

Assume, by contradiction, $\bar{V}_{yx}(x_o,y_o) > 0$. Due to the continuity of $\bar{V}$, we can then find an $\varepsilon > 0$ such that

$$\bar{V}_{yx}(x,y) \geq \varepsilon \quad \forall (x,y) \in N_{x_o,y_o}, \tag{4.31}$$

where N_{x_o,y_o} is a suitable neighborhood of the point $(x_o,y_o) \in \partial^1\mathcal{C}$. Notice now that $\bar{V}_y(x_o,y_o) = V_y(x_o,y_o) = -K$, because $(x_o,y_o) \in \partial^1\mathcal{C}$, and $\bar{V} = V$ in $N_{x_o,y_o} \cap \bar{\mathcal{C}}$. Then, by assumption that $\bar{V}_{yx}(x_o,y_o) > 0$, we can apply the implicit function theorem to $\bar{V}_y(x,y) + K$, getting the existence of a continuous function $\bar{g}_1 : (y_o - \delta, y_o + \delta) \to \mathbb{R}$, for a suitable $\delta > 0$, such that $\bar{V}_y(\bar{g}_1(y), y) = -K$ in $(y_o - \delta, y_o + \delta)$. Moreover, taking into account the regularity of A, B, we have that $\bar{g}_1 \in W^{1,\infty}(y_o - \delta, y_o + \delta)$ as

$$\bar{g}_1'(y) = -\frac{\bar{V}_{yy}(g_1(y),y)}{\bar{V}_{yx}(g_1(y),y)} \quad \text{a.e. in } (y_o - \delta, y_o + \delta).$$

Hence, by (4.31) and the fact that $A, B \in W^{2,\infty}_{\text{loc}}((\underline{b}_1, \bar{b}_2);\mathbb{R})$ (see also Remark 4.15 for the case $y_o = \underline{b}_1$), there exists $M_\varepsilon > 0$ such that

$$|\bar{g}_1(y) - \bar{g}_1(\tilde{y})| \leq M_\varepsilon |y - \tilde{y}| \quad \forall y, \tilde{y} \in (y_o - \delta, y_o + \delta). \tag{4.32}$$

Furthermore, recalling the definition of g_1 in (4.13), $\bar{g}_1$ and g_1 coincide in $(y_o-\delta, y_o+\delta)$. Therefore, g_1 is continuous in $(y_o-\delta, y_o+\delta)$, and this fact immediately implies that b_1 - which is non-increasing by Proposition 4.8 - is actually strictly decreasing in a neighborhood $(x_o-\vartheta, x_o+\vartheta)$, for a suitable $\vartheta > 0$. Hence, $g_1 = b_1^{-1}$ over $b_1((x_o-\vartheta, x_o+\vartheta))$, and from (4.32) we find

$$M_\varepsilon|b_1(x)-b_1(\tilde{x})| \geq |\bar{g}_1(b_1(x))-\bar{g}_1(b_1(\tilde{x}))| = |x-\tilde{x}|, \quad \forall x,\tilde{x} \in (y_o-\delta, y_o+\delta). \quad (4.33)$$

Recalling again that b_1 is strictly decreasing in $(x_o-\vartheta, x_o+\vartheta)$, hence differentiable a.e. overthere, from (4.33), we obtain

$$\exists\ b_1'(x) \geq \frac{1}{M_\epsilon} \quad \forall x \in \mathcal{X}, \tag{4.34}$$

where $\mathcal{X}$ is a dense set (actually of full Lebesgue measure) in $(x_0-\vartheta, x_0]$.

Consider now the function $(x_0-\vartheta, x_0] \ni x \mapsto V(x, y_o) \in \mathbb{R}_+$. Since b_1 is strictly decreasing, we have that the set $K := \{(x, y_o) : x \in (x_0-\vartheta, x_0]\} \subset \mathcal{I}$, and therefore by Proposition 4.14 that

$$V(x, y_o) = -Ky_o + z_1(x) \quad \forall x \in (x_0-\vartheta, x_0]. \tag{4.35}$$

Furthermore, defining the function

$$(x_0-\vartheta, x_0] \to \mathbb{R}, \quad x \mapsto z_1(x) = V(x, b_1(x)) + Kb_1(x) = \bar{V}(x, b_1(x)) + Kb_1(x),$$

and applying the chain rule we get that

$$\exists\ z_1'(x) = \bar{V}_x(x, b_1(x)) + \bar{V}_y(x, b_1(x))b_1'(x) + Kb_1'(x), \quad \forall x \in \mathcal{X}. \tag{4.36}$$

Since by definition of b_1 we have that $\bar{V}_y(x, b_1(x)) = V_y(x, b_1(x)) = -K$, we obtain from (4.36)

$$z_1'(x) = \bar{V}_x(x, b_1(x)), \quad \forall x \in \mathcal{X}.$$

Using this result together with (4.35) we obtain existence of $V_x(x, y_o)$ for all $x \in \mathcal{X}$ and moreover

$$V_x(x, y_o) = z_1'(x) = \bar{V}_x(x, b_1(x)) \quad \forall x \in \mathcal{X}. \tag{4.37}$$

Using again the chain rule in (4.37) we obtain existence of $V_{xx}(x, y_o)$ for all $x \in \mathcal{X}$ and

$$V_{xx}(x, y_o) = z_1''(x) = \bar{V}_{xx}(x, b_1(x)) + \bar{V}_{xy}(x, b_1(x))b_1'(x) \quad \forall x \in \mathcal{X}. \tag{4.38}$$

Combining (4.38) with (4.34) and (4.31) one obtains

$$V_{xx}(x, y_o) \geq \bar{V}_{xx}(x, b_1(x)) + \frac{\varepsilon}{M_\varepsilon} \quad \forall x \in \mathcal{X}. \tag{4.39}$$

Using now that V is a viscosity solution to (4.16) (in particular a supersolution) by Proposition 4.11, that V_{xx} exists for all points $x \in \mathcal{X}$, and (4.37) and (4.39), we obtain that

$$\begin{aligned} f(x, y_o) &\leq \rho V(x, y_o) - [\mathcal{L}^{y_o} V(\cdot, y_o)](x) \\ &\leq \rho V(x, y_o) - [\mathcal{L}^{y_o} \bar{V}(\cdot, y_o)](x) - \frac{1}{2}\eta^2\frac{\varepsilon}{M_\varepsilon} \end{aligned} \tag{4.40}$$

for all $x \in \mathcal{X}$. Since $\mathcal{X}$ is dense in $(x_o - \vartheta, x_o]$, we can take a sequence $(x^n)_{n\in\mathbb{N}} \subset \mathcal{X}$ such that $x^n \uparrow x_o$. Evaluating (4.40) at $x = x^n$, taking limits as $n \uparrow \infty$, using the left-continuity of b_1, the fact that $y_o = b_1(x_o)$, and the fact that $\bar{V} \in C^{1,2}(\mathbb{R}^2;\mathbb{R})$, we obtain

$$f(x_o, y_o) \leq \rho\bar{V}(x_o, y_o) - [\mathcal{L}^{y_o}\bar{V}(\cdot, y_o)](x_o) - \frac{1}{2}\eta^2\frac{\varepsilon}{M_\varepsilon} \tag{4.41}$$

On the other hand, since $\rho\bar{V}(x,y) - [\mathcal{L}^y\bar{V}(\cdot,y)](x) = \rho V(x,y) - [\mathcal{L}^y V(\cdot,y)](x) = f(x,y)$ for all $(x,y) \in \mathcal{C}$, using that $\bar{V} \in C^{1,2}(\mathbb{R}^2;\mathbb{R})$ and $(x_o, y_o) \in \bar{\mathcal{C}}$, we obtain by continuity of $\bar{V}$ that

$$f(x_o, y_o) = \rho\bar{V}(x_o, y_o) - [\mathcal{L}^{y_o}\bar{V}(\cdot, y_o)](x_o). \tag{4.42}$$

Combining now (4.42) and (4.41) leads to $\frac{\varepsilon}{M_\varepsilon} \leq 0$. This gives the desired contradiction.

Case (b). Assume now that $x_o = g_1(y_o)$ and $y_o < b_1(x_o)$, with $b_1(x_o) < \infty$ due to Proposition 4.8-(i). Notice that such a case occurs if the function b_1 has a jump at x_o. Defining the segment $\Gamma := \{(x_o, y) : y \in [y_o, b_1(x_o)]\}$, it follows that $\Gamma \subset \partial^1\mathcal{C}$. Moreover, letting again $\bar{V}$ as in (4.30), we have that $V_y = \bar{V}_y = -K$ in Γ, so that

$$\bar{V}_y(x,y) + K = \bar{V}_y(x,y) - \bar{V}_y(x_o,y) = \int_{x_o}^{x} \bar{V}_{yx}(u,y)\ du, \quad \forall y \in [y_o, b_1(x_o)],\ \forall x \geq x_o. \tag{4.43}$$

Using now that A', B' are locally Lipschitz by Proposition 4.14, we can take the derivative with respect to y in (4.43) (in the Sobolev sense) and we obtain

$$\bar{V}_{yy}(x,y) = \int_{x_o}^{x} \bar{V}_{yxy}(u,y)\ du \quad \text{for a.e. } \ y \in [y_o, b_1(x_o)],\ x \geq x_o.$$

The convexity of V and the fact that $\bar{V} = V$ in $\bar{\mathcal{C}}$, yields $\bar{V}_{yy} \geq 0$ (again in the Sobolev sense) and therefore

$$0 \leq \int_{x_o}^{x} \bar{V}_{yxy}(u,y)\ du \quad \text{for a.e.} \quad \text{for a.e. } \ y \in [y_o, b_1(x_o)],\ x \geq x_o.$$

Dividing now both sides by $(x - x_o)$, letting $x \to x_o$, and invoking the mean value theorem one has

$$0 \leq \bar{V}_{yxy}(x_o, y) \quad \text{for a.e. } \ y \in [y_o, b_1(x_o)].$$

This implies that $\bar{V}_{yx}$ is non-decreasing with respect to $y \in [y_o, b_1(x_o)]$ and, since $\bar{V} = V$ in $\bar{\mathcal{C}}$ and Lemma 4.7, we get

$$0 \leq \bar{V}_{yx}(x_o, y_o) \leq \bar{V}_{yx}(x_o, b_1(x_o))$$

If we now assume, as in Case (a) above, that $\bar{V}_{yx}(x_o, y_o) > 0$, then we must also have $\bar{V}_{yx}(x_o, b_1(x_o)) > 0$. We are therefore left with the assumption employed in the contradiction scheme of Case (a), and we can thus apply again the rationale of that case to obtain a contradiction. This completes the proof. □

Lemma 4.17. *It holds $V_{yxx} \in L^\infty_{\text{loc}}(\mathbb{R} \times (\underline{b}_1, \bar{b}_2);\mathbb{R})$.*

Proof. Notice that $\psi_{yxx}(x,y) \in L^\infty_{\text{loc}}(\mathbb{R} \times (\underline{b}_1, \bar{b}_2);\mathbb{R})$, $\varphi_{yxx}(x,y) \in L^\infty_{\text{loc}}(\mathbb{R} \times (\underline{b}_1, \bar{b}_2);\mathbb{R})$ and $\widehat{V}_{yxx} \in L^\infty_{\text{loc}}(\mathbb{R}^2)$ by direct calculations, and $A_y, B_y \in W^{1,\infty}_{\text{loc}}((\underline{b}_1, \bar{b}_2);\mathbb{R})$ by Proposition 4.14. Hence, $V_{yxx} \in L^\infty_{\text{loc}}(\mathbb{R} \times (\underline{b}_1, \bar{b}_2);\mathbb{R})$ by (4.25). □

4.4 Further Properties of the Free-Boundaries

In this section, we show further properties of the free-boundaries and derive a system of functional equations characterizing them. These equations are basically derived by the smooth-fit conditions of the value function, see Theorem 4.16. For this, we need the following additional assumptions on f:

Assumption 4.18.

(i) $\lim_{x\to\pm\infty} f_x(x,y) = \pm\infty$.

(ii) f_{yx} *exists continuous.*

(iii) One of the following holds true:

  (a) $x \mapsto f_y(x,y)$ *is strictly increasing for any* $y \in \mathbb{R}$*;*

  (b) $f_{yx} \equiv 0$ *and* $f(\cdot, y)$ *is strictly convex for any* $y \in \mathbb{R}$*.*

Remark 4.19. The functions f discussed in Remark 4.3 satisfy the previous assumptions.

Proposition 4.20.

(i) Let Assumption 4.18-(i) hold. Then

$$\bar{b}_1 = \lim_{x\downarrow-\infty} b_1(x) = \infty, \quad \underline{b}_2 = \lim_{x\uparrow\infty} b_2(x) = -\infty;$$

hence, by Proposition 4.8-(iii), one also has $\underline{b}_1 = -\infty$ *and* $\bar{b}_2 = \infty$*.*

(ii) Define

$$\zeta_1(y) := \sup\{x \in \mathbb{R} : -\alpha V_x(x,y) - f_y(x,y) - \rho K \geq 0\}, \quad y \in \mathbb{R},$$

$$\zeta_2(y) := \inf\{x \in \mathbb{R} : -\alpha V_x(x,y) - f_y(x,y) + \rho K \leq 0\}, \quad y \in \mathbb{R}.$$

Then, for any $y \in \mathbb{R}$*, we have*

$$g_1(y) < \zeta_1(y) < \zeta_2(y) < g_2(y).$$

Proof. We prove the two claims separately.

Proof of (i). Here we show that $\lim_{x\downarrow-\infty} b_1(x) = \infty$. The fact that $\lim_{x\uparrow\infty} b_2(x) = -\infty$ can be proved by similar arguments. We argue by contradiction assuming $\bar{b}_1 := \lim_{x\downarrow-\infty} b_1(x) < \infty$. Take $y_o > \bar{b}_1$, so that $\tau^\star = \tau^\star(x, y_o) = \infty$ for all $x \in \mathbb{R}$, the latter being the stopping time defined in (4.9). Then, take $x_o < g_2(y_o)$ such that $(x_o, y_o) \in \mathcal{C}$. Clearly, every $x < x_o$ belongs to $\mathcal{C}$, and therefore, by the representation (4.25), we obtain that it must be $B(y_o) = 0$; indeed, otherwise, by taking limits as $x \to -\infty$ and using (4.15), we would contradict Proposition 4.4. Moreover, since for any $y \in \mathbb{R}$ one has $\psi_x(x,y) \to 0$ when $x \to -\infty$ (cf. (4.14)), we then have by dominated convergence for $\theta > 0$

$$\lim_{x\to-\infty} V_x(x,y_0) = \lim_{x\to-\infty} \widehat{V}_x(x,y_o) = \lim_{x\to-\infty} \mathbb{E}\left[\int_0^\infty e^{-(\rho+\theta)t} f_x(X_t^{x,y_o}, y_o)dt\right] = -\infty \quad (4.44)$$

and for $\theta = 0$

$$\lim_{x\to-\infty} V_x(x, y_0) = \lim_{x\to-\infty} \widehat{V}_x(x, y_o) = \lim_{x\to-\infty} \mathbb{E}\left[\int_0^\infty e^{-\rho t} f_x(X_t^{x,y_o}, y_o)dt\right] = -\infty. \quad (4.45)$$

Now, setting

$$\hat{\sigma}_x := \inf\{t \geq 0 : X_t^{x,y_o} \geq x_o\},$$

for $x < x_o$, we have by monotonicity of $f_y(\cdot, y)$ (cf. Assumption 4.2-(iv))

$$\begin{aligned} -K < V_y(x, y_o) &= \inf_{\sigma\in\mathcal{T}} \mathbb{E}\bigg[\int_0^\sigma e^{-\rho t}\Big(\alpha V_x(X_t^{x,y_o}, y_o) + f_y(X_t^{x,y_o}, y_o)\Big)\, dt + e^{-\rho\sigma}K\bigg] \\ &\leq \mathbb{E}\bigg[\int_0^{\hat{\sigma}_x} e^{-\rho t}\Big(\alpha V_x(X_t^{x,y_o}, y_o) + f_y(x_o, y_o)\Big)\, dt + K\bigg]. \end{aligned}$$

The latter implies

$$2K + \frac{|f_y(x_o, y_o)|}{\rho} \geq -\alpha\mathbb{E}\bigg[\int_0^{\hat{\sigma}_x} e^{-\rho t} V_x(X_t^{x,y_o}, y_o)\, dt\bigg]$$

Hence, letting $x \downarrow -\infty$, using (4.44) in the case $\theta > 0$ or (4.45) for the case $\theta = 0$, and invoking the dominated convergence theorem we get a contradiction.

Proof of (ii). Fix $y \in \mathbb{R}$. Recall that $V_y(\cdot, y) \in C(\mathbb{R}; \mathbb{R})$ by Proposition 4.4, $V_{yx}(\cdot, y) \in C(\mathbb{R}; \mathbb{R})$ by Theorem 4.16, and $V_{yxx}(\cdot, y) \in L^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R})$ by direct calculations on the representation of V given in Proposition 4.14. Also, it is readily verified from (4.8) that $-K \leq V_y(\cdot, y) \leq K$ on $\mathbb{R}^2$. Then, the semiharmonic characterization of [78] (see equations (2.27)–(2.29) therein, suitably adjusted to take care of the integral term appearing in (4.8)), together with the above regularity of $V_y(\cdot, y)$, allow to obtain by standard means that $(V_y(\cdot, y), g_1(y), g_2(y))$ solves

$$\begin{cases} \big(\mathcal{L}^y - \rho\big)V_y(x, y) = -\alpha V_x(x, y) - f_y(x, y) & \text{on } g_1(y) < x < g_2(y), \\ \big(\mathcal{L}^y - \rho\big)V_y(x, y) \leq -\alpha V_x(x, y) - f_y(x, y) & \text{on a.e. } x < g_1(y), \\ \big(\mathcal{L}^y - \rho\big)V_y(x, y) \geq -\alpha V_x(x, y) - f_y(x, y) & \text{on a.e. } x > g_2(y), \\ -K \leq V_y(x, y) \leq K & x \in \mathbb{R}, \\ V_y(g_1(y), y) = -K \quad \text{and} \quad V_y(g_2(y), y) = K, \\ V_{yx}(g_1(y), y) = 0 \quad \text{and} \quad V_{yx}(g_2(y), y) = 0. \end{cases} \quad (4.46)$$

In particular, we have that $V_y(x, y) = K$ for any $x > g_2(y)$, and therefore from the third equation in (4.46) we obtain

$$-\rho K \geq -\alpha V_x(x, y) - f_y(x, y) =: \Lambda(x, y), \quad \forall x > g_2(y).$$

Since the mapping $x \mapsto \Lambda(x, y)$ is non-increasing for any given $y \in \mathbb{R}$ by the convexity of V and the assumption on f_y (cf. Assumption 4.2), we obtain that

$$g_2(y) \geq \zeta_2(y) = \inf\{x \in \mathbb{R} : -\alpha V_x(x, y) - f_y(x, y) + \rho K \leq 0\}.$$

To show that $g_2(y) > \zeta_2(y)$, we suppose that there exists some y_o such that $g_2(y_o) = \zeta_2(y_o)$. Then $V_y(\zeta_2(y_o), y_o) = K$. Let now $\tau^\star := \tau^\star(\zeta_2(y_o), y_o)$ be the optimal stopping time for the sup player when the Dynkin game (4.8) starts at time zero from the point $(\zeta_2(y_o), y_o)$, and for $\varepsilon > 0$ define

$$q_\varepsilon := q_\varepsilon(\zeta_2(y_o), y_o) := \inf\{t \geq 0 : X_t^{\zeta_2(y_o), y_o} \geq \zeta_2(y_o) + \varepsilon\}.$$

Then by using that $f_y(\cdot, y_o) + \alpha V_x(\cdot, y_o)$ is non-decreasing and locally Lipschitz by Assumption 4.2-(iii) and Proposition 4.4(iii), we have from (4.8) for some constant $C(y_o) > 0$

$$\begin{aligned}
K = V_y(\zeta_2(y_o), y_o) &\leq \mathbb{E}\Big[\int_0^{\tau^\star \wedge q_\varepsilon} e^{-\rho t}\big(f_y + \alpha V_x\big)(X_s^{\zeta_2(y_o), y_o}, y_o) ds\Big] \\
&+ \mathbb{E}\Big[K e^{-\rho q_\varepsilon} \mathbb{1}_{\{\tau^\star > q_\varepsilon\}} - K e^{-\rho \tau^\star} \mathbb{1}_{\{\tau^\star < q_\varepsilon\}}\Big] \\
&\leq \big(f_y + \alpha V_x\big)(\zeta_2(y_o) + \varepsilon, y_o) \frac{1}{\rho} \mathbb{E}\Big[1 - e^{-\rho(\tau^\star \wedge q_\varepsilon)}\Big] \\
&+ \mathbb{E}\Big[K e^{-\rho q_\varepsilon} \mathbb{1}_{\{\tau^\star > q_\varepsilon\}} - K e^{-\rho \tau^\star} \mathbb{1}_{\{\tau^\star < q_\varepsilon\}}\Big] \\
&\leq \frac{1}{\rho}\Big[\big(f_y + \alpha V_x\big)(\zeta_2(y_o), y_o) + \varepsilon C(y_o)\Big] \mathbb{E}\Big[1 - e^{-\rho(\tau^\star \wedge q_\varepsilon)}\Big] \\
&+ \mathbb{E}\Big[K e^{-\rho q_\varepsilon} \mathbb{1}_{\{\tau^\star > q_\varepsilon\}} - K e^{-\rho \tau^\star} \mathbb{1}_{\{\tau^\star < q_\varepsilon\}}\Big].
\end{aligned}$$

Using now that, by definition of ζ_2, it must be $\big(f_y + \alpha V_x\big)(\zeta_2(y_o), y_o) = \rho K$, and rearranging terms, we get that

$$0 \leq \frac{\varepsilon C(y_o)}{\rho} \mathbb{E}\Big[1 - e^{-\rho(\tau^\star \wedge q_\varepsilon)}\Big] - 2K \mathbb{E}\Big[e^{-\rho \tau^\star} \mathbb{1}_{\{\tau^\star < q_\varepsilon\}}\Big]. \tag{4.47}$$

Notice now that (cf. eq. (4.3) in [31], among others)

$$\mathbb{E}\Big[e^{-\rho \tau^\star} \mathbb{1}_{\{\tau^\star < q_\varepsilon\}}\Big] = \frac{\psi(\zeta_2(y_o), y_o)\varphi(\zeta_2(y_o) + \varepsilon, y_o) - \psi(\zeta_2(y_o) + \varepsilon, y_o)\varphi(\zeta_2(y_o), y_o)}{\psi(g_1(y_o), y_o)\varphi(\zeta_2(y_o) + \varepsilon, y_o) - \psi(\zeta_2(y_o) + \varepsilon, y_o)\varphi(g_1(y_o), y_o)}$$

and

$$\mathbb{E}\Big[e^{-\rho q_\varepsilon} \mathbb{1}_{\{\tau^\star > q_\varepsilon\}}\Big] = \frac{\psi(g_1(y_o), y_o)\varphi(\zeta_2(y_o), y_o) - \psi(\zeta_2(y_o), y_o)\varphi(g_1(y_o), y_o)}{\psi(g_1(y_o), y_o)\varphi(\zeta_2(y_o) + \varepsilon, y_o) - \psi(\zeta_2(y_o) + \varepsilon, y_o)\varphi(g_1(y_o), y_o)}.$$

Then, because

$$1 - e^{-\rho(\tau^\star \wedge q_\varepsilon)} = 1 - e^{-\rho \tau^*} \mathbb{1}_{\{\tau^\star < q_\varepsilon\}} - e^{-\rho q_\varepsilon} \mathbb{1}_{\{\tau^\star > q_\varepsilon\}},$$

using the last two formulas in (4.47) and performing a first-order Taylor's expansion around $\varepsilon = 0$ of the terms on the right-hand side of (4.47), one finds that the first term on the right-hand side of (4.47) is positive and converges to zero as $\varepsilon \downarrow 0$ with order ε^2, while the second term is negative and converges to zero with order ε. We thus reach a contradiction in (4.47) for ε small enough, and therefore it can not exist y_o at which $g_2(y_o) = \zeta_2(y_o)$.

The statement $g_1(y) < \zeta_1(y)$ can be shown analogously.

Moreover, by monotonicity and continuity of $x \mapsto -\alpha V_x(x, y) - f_y(x, y)$ we have for any $y \in \mathbb{R}$ that

$$\begin{aligned}\zeta_1(y) &= \sup\{x \in \mathbb{R} : -\alpha V_x(x, y) - f_y(x, y) - 2\rho K + \rho K \geq 0\}\\ &< \sup\{x \in \mathbb{R} : -\alpha V_x(x, y) - f_y(x, y) + \rho K \geq 0\}\\ &= \inf\{x \in \mathbb{R} : -\alpha V_x(x, y) - f_y(x, y) + \rho K \leq 0\} = \zeta_2(y).\end{aligned}$$

□

From Proposition 4.20-(i), we immediately obtain the following corollary.

Corollary 4.21. *Let Assumption 4.18-(i) hold. Then the functions g_1, g_2 defined in* (4.13) *are finite.*

Proposition 4.22. *Let Assumption 4.18 hold. Then the functions b_1, b_2 are strictly decreasing.*

Proof. We prove the claim only for b_1, since analogous arguments apply to prove it for b_2.

Case (a). We assume here that item (a) of Assumption 4.18-(iii) holds, i.e. that $x \mapsto f_y(x, y)$ is strictly increasing for any $y \in \mathbb{R}$. By Proposition 4.14, we can differentiate the first line of (4.25) with respect to y and get by Proposition 4.13-(i) that V_y solves inside $\mathcal{C}$ the equation

$$[\mathcal{L}^y V_y(\cdot, y)](x) - \rho V_y(x, y) = -f_y(x, y) - \alpha V_x(x, y). \tag{4.48}$$

By continuity, (4.48) also holds on $\overline{\mathcal{C}}$, i.e.

$$[\mathcal{L}^y V_y(\cdot, y)](x) - \rho V_y(x, y) = -f_y(x, y) - \alpha V_x(x, y) \quad \forall (x, y) \in \overline{\mathcal{C}}.$$

In particular it holds on $\partial^1 \mathcal{C} := \overline{\mathcal{C}} \cap \mathcal{I}$. Assume now, by contradiction, that the boundary b_1 is constant on $(x_o, x_o + \varepsilon)$, for some $x_o \in \mathbb{R}$ and some $\varepsilon > 0$. Then, setting $y_o := b_1(x_o)$, we will have $V_{yxx}(\cdot, y_o) = V_{yx}(\cdot, y_o) = 0$ and $V_y(\cdot, y_o) = -K$ on $(x_o, x_o + \varepsilon)$. Hence, we obtain from (4.48) that

$$-\rho K = f_y(x, y_o) + \alpha V_x(x, y_o), \quad \forall x \in (x_o, x_o + \varepsilon),$$

and thus

$$-f_{yx}(x, y_o) = \alpha V_{xx}(x, y_o), \quad \forall x \in (x_o, x_o + \varepsilon).$$

But now $\alpha V_{xx}(x, y_o) \geq 0$ for any $x \in (x_o, x_o + \varepsilon)$ by convexity of $V(\cdot, y_o)$, while, by assumption, f_{yx} must be strictly positive on a subset of $(x_o, x_o + \varepsilon)$ with positive measure. Hence a contradiction is reached.

Case (b). We assume here that item (b) of Assumption 4.18-(iii) holds, i.e. that $f_{yx} \equiv 0$ and that $f(\cdot, y)$ is strictly convex for any $y \in \mathbb{R}$. Arguing exactly as in Case (a), we obtain that

$$-\rho K = f_y(x, y_o) + \alpha V_x(x, y_o), \quad \forall x \in (x_o, x_o + \varepsilon).$$

Hence, we obtain that

$$V_x(\cdot, y_o) \equiv -\frac{\rho K}{\alpha} - \frac{f_y(x, y_o)}{\alpha}, \;\; V_{xx}(\cdot, y_0) \equiv 0 \qquad \text{on } \;(x_o, x_o + \varepsilon).$$

On the other hand, by continuity of $V(\cdot, y_o)$, it solves (4.19) on $(x_o, x_o{+}\varepsilon)$. Therefore, for $\theta > 0$, we get

$$(\alpha y_o - \theta x)\Big[-\frac{\rho K}{\alpha} - \frac{f_y(x, y_o)}{\alpha}\Big] - \rho V(x, y_o) + f(x, y_o) = 0, \quad \forall x \in (x_o, x_o + \varepsilon)$$

and, for $\theta = 0$,

$$y_o\Big[-\rho K - f_y(x, y_o)\Big] - \rho V(x, y_o) + f(x, y_o) = 0, \quad \forall x \in (x_o, x_o + \varepsilon).$$

But this implies that $f_{xx}(\cdot, y_o) \equiv 0$ on $(x_o, x_o{+}\varepsilon)$, which contradicts that f is strictly convex. □

From the result above, it immediately follows the next corollary.

Corollary 4.23. *Let Assumption 4.18 hold. Then the functions g_1, g_2 defined in* (4.13) *are continuous.*

The next result will be of fundamental importance to show the locally Lipschitz property of g_i, $i = 1, 2$ and, in the next section, to determine a system of differential equations for those curves.

Proposition 4.24. *Let Assumption 4.18 hold. Then*

$$\exists \lim_{\substack{(x,y)\to (x_o,y_o) \\ (x,y)\in\mathcal{C}}} V_{yxx}(x, y) \neq 0 \quad \forall\, (x_o, y_o) \in \partial\mathcal{C}. \tag{4.49}$$

Proof. We provide the proof only for any $(x_o, y_o) \in \partial^2\mathcal{C} := \overline{\mathcal{C}} \cap \mathcal{D}$, as the other case can be treated similarly.

First of all, we notice that the limit in (4.49) exists since, by Proposition 4.14, the function $V : \mathcal{C} \to \mathbb{R}$ can be differentiated twice with respect to x and once with respect to y with continuity up to the boundary $\partial\mathcal{C}$.

Case (a). We assume here that item (a) of Assumption 4.18-(iii) holds, i.e. that $x \mapsto f_y(x, y)$ is strictly increasing for any $y \in \mathbb{R}$. Suppose, by contradiction, that for some $y_o \in \mathbb{R}$ one has

$$\lim_{\substack{(x,y)\to (g_2(y_o),y_o) \\ (x,y)\in\mathcal{C}}} V_{yxx}(x, y) = 0. \tag{4.50}$$

Then taking limits as $(x, y) \to (g_2(y_o), y_o)$ for $(x, y) \in \mathcal{C}$ in (4.19) we find, using that $V_{yx}(g_2(y_o), y_o) = 0$ by Proposition 4.16 and that $V_y(g_2(y_o), y_o) = K$,

$$-\rho K + f_y(g_2(y_o), y_o) = -\alpha V_x(g_2(y_o), y_o). \tag{4.51}$$

Since $g_2(y_o) > \zeta_2(y_o)$ by Proposition 4.20, and by definition of ζ_2, it must be

$$-\rho K + f_y(x, y_o) = -\alpha V_x(x, y_o) \quad \forall\, x \in (\zeta_2(y_o), g_2(y_o)),$$

which also implies that $-\alpha V_{xx}(x, y_o) = f_{yx}(x, y_o)$ for any $x \in (\zeta_2(y_o), g_2(y_o))$. We then conclude as in Step 1 of the proof of Proposition 4.22.

Case (b). We assume here that item (b) of Assumption 4.18-(iii) holds, which implies that there exists q such that $f_y(x, y) = q(y)$ for any $(x, y) \in \mathbb{R}^2$. Suppose again, with the aim of reaching a contradiction, that for some $y_o \in \mathbb{R}$ one has (4.50). Then taking limits as $(x, y) \to (g_2(y_o), y_o)$ for $(x, y) \in \mathcal{C}$ in (4.19) we find, using that $V_{yx}(g_2(y_o), y_o) = 0$ by Proposition 4.16 and that $V_y(g_2(y_o), y_o) = K$,

$$-\rho K + q(y_o) = -\alpha V_x(g_2(y_o), y_o).$$

As before, because $g_2(y_o) > \zeta_2(y_o)$ by Proposition 4.20, and by definition of ζ_2, it must be

$$-\rho K + q(y_o) = -\alpha V_x(x, y_o) \quad \forall\, x \in (\zeta_2(y_o), g_2(y_o));$$

that is, V is an affine function of x in that interval. However, using the latter and (4.19), we also have for $\theta > 0$

$$\frac{\rho K - q(y_o)}{\alpha}(\alpha y_o - \theta x) - \rho V(x, y_o) = -f(x, y_o) \quad \forall\, x \in (\zeta_2(y_o), g_2(y_o)),$$

and for $\theta = 0$

$$\rho K - q(y_o) y_o - \rho V(x, y_o) = -f(x, y_o) \quad \forall\, x \in (\zeta_2(y_o), g_2(y_o)).$$

Hence, we reach a contradiction since $f(\cdot, y_o)$ is strictly convex by assumption, while $V(\cdot, y_o)$ is affine. □

Proposition 4.25. *Let Assumption 4.18 hold. Then the functions g_1, g_2 are locally Lipschitz.*

Proof. Define the function

$$\bar{V}(x, y) := A(y)\psi(x, y) + B(y)\varphi(x, y) + \widehat{V}(x, y), \quad (x, y) \in \mathbb{R}^2, \tag{4.52}$$

where A, B are the functions of Proposition 4.14. Then, one clearly has that $\bar{V} \in C^{2,1}(\mathbb{R}^2; \mathbb{R})$, and $\bar{V} = V$ in $\mathbb{R}^2 \cap \bar{\mathcal{C}}$. Moreover, the mixed derivative $\bar{V}_{yx}$ exists and is continuous, and standard differentiation yield

$$\bar{V}_{yx}(x, y) = A_y(y)\psi_x(x, y) + B_y(y)\varphi_x(x, y) + \big(A(y)\psi_{yx}(x, y) + B(y)\varphi_{yx}(x, y)\big) + \widehat{V}_{yx}(x, y).$$

Since A_y and B_y are locally Lipschitz by Proposition 4.14, and ψ and φ are smooth (cf. (4.14) and (4.15)), we deduce that $\bar{V}_{yx}(x, \cdot)$ is locally Lipschitz.

Let now $y_o \in \mathbb{R}$. Then, for any given $x_o \in \mathbb{R}$ such that $(x_o, y_o) \in \partial\mathcal{C}$, we know by Proposition 4.24 that $\bar{V}_{yxx}(x_o, y_o) \neq 0$, while $\bar{V}_{yx}(x_o, y_o) = 0$ by Theorem 4.16. By the implicit function theorem (see, e.g., the Corollary at p. 256 in [29] or Theorem 3.1 in [75]) we therefore gain that for any $i = 1, 2$ there exists a unique continuous function $\bar{g}_i : (y_o - \delta, y_o + \delta) \to (x_o - \delta', x_o + \delta')$, for suitable $\delta, \delta' > 0$, such that $\bar{V}_{yx}(\bar{g}_i(y), y) = 0$ in $(y_o - \delta, y_o + \delta)$. Also, the aforementioned properties of $\bar{V}_{yxy}$ and $\bar{V}_{yxx}$ imply that there exists $C(y_o) > 0$ such that

$$|\bar{g}_i(y_2) - \bar{g}_i(y_1)| \leq C(y_o)|y_2 - y_1|, \quad \forall\, y_1, y_2 \in (y_o - \delta, y_o + \delta).$$

Recalling now that $\bar{V}_{yx}(g_i(y), y) = 0$, we can identify $\bar{g}_i = g_i$, $i = 1, 2$, in $(y_o - \delta, y_o + \delta)$ and therefore g_i is locally Lipschitz therein. Given the arbitrariness of the point (x_o, y_o) the proof is complete. □

4.5 A System of Equations for the Free-Boundaries

Before proving the main result of this section (i.e. Theorem 4.26 below), we need to introduce some of the characteristics of the process $X^{x,y}$. For an arbitrary $x_o \in \mathbb{R}$, and for any given and fixed $y \in \mathbb{R}$, the scale function density of the process $X^{x,y}$ is defined, for $x \in \mathbb{R}$, as

$$S_x(x, y) := \begin{cases} \exp\left\{-\int_{x_o}^{x} \frac{2(\alpha y - \theta z)}{\eta^2}\, dz\right\}, & \theta > 0 \\ \exp\left\{-\frac{2\alpha y x}{\eta^2}\right\}, & \theta = 0, \end{cases} \tag{4.53}$$

while the density of the speed measure is

$$m_x(x, y) := \frac{2}{\eta^2 S_x(x, y)}, \quad x \in \mathbb{R}. \tag{4.54}$$

Note that we set $x_0 = 0$ in the case $\theta = 0$. This simplifies our calculations in Section 4.6. For later use we also denote by p the transition density of $X^{x,y}$ with respect to the speed measure; then, letting $A \mapsto \mathbb{P}_t(x, A, y)$, $A \in \mathcal{B}(\mathbb{R})$, $t > 0$ and $y \in \mathbb{R}$, be the probability of starting at time 0 from level $x \in \mathbb{R}$ and reaching the set $A \in \mathcal{B}(\mathbb{R})$ in t units of time, we have (cf., e.g., p. 13 in [17])

$$\mathbb{P}_t(x, A, y) = \int_A p(t, x, z, y) m_x(z, y) dz.$$

The density p can be taken positive, jointly continuous in all variables and symmetric (i.e. $p(t, x, z, y) = p(t, z, x, y)$). Furthermore, our analysis will involve the Green function G that, for given and fixed $y \in \mathbb{R}$, is defined as (see again [17], p. 19)

$$G(x, z, y) := \int_0^\infty e^{-\rho t} p(t, x, z, y) dt = \begin{cases} w^{-1}\psi(x, y)\varphi(z, y) & \text{for } x \leq z, \\ w^{-1}\varphi(x, y)\psi(z, y) & \text{for } x \geq z, \end{cases} \tag{4.55}$$

where w denotes the positive constant (normalized) Wronskian between ψ and φ given by

$$w := \frac{\psi_x(x, z)\varphi(x, z) - \varphi_x(x, z)\psi(x, z)}{S_x(x, z)}.$$

Theorem 4.26. *Let Assumption 4.18 hold and for any* $(x,y) \in \mathbb{R}^2$ *define*

$$H(x,y) := \alpha V_x(x,y) + f_y(x,y) \tag{4.56}$$

The free-boundaries g_1 *and* g_2 *as in* (4.13), *and the coefficients* $A, B \in W^{2;\infty}_{loc}(\mathbb{R};\mathbb{R})$ *solve the following system of functional and ordinary differential equations*

$$0 = \int_{g_1(y)}^{g_2(y)} \psi(z,y)H(z,y)m_x(z,y)\ dz - K\frac{\psi_x(g_1(y),y)}{S_x(g_1(y),y)} - K\frac{\psi_x(g_2(y),y)}{S_x(g_2(y),y)}, \tag{4.57}$$

$$0 = \int_{g_1(y)}^{g_2(y)} \varphi(z,y)H(z,y)m_x(z,y)\ dz - K\frac{\varphi_x(g_1(y),y)}{S_x(g_1(y),y)} - K\frac{\varphi_x(g_2(y),y)}{S_x(g_2(y),y)}, \tag{4.58}$$

and

$$\begin{aligned} A'(y)\psi_x(g_1(y),y) + B'(y)\varphi_x(g_1(y),y) + \widehat{V}_{yx}(g_1(y),y) \\ + \big[A(y)\psi_{yx}(g_1(y),y) + B(y)\varphi_{yx}(g_1(y),y)\big] = 0 \end{aligned} \tag{4.59}$$

$$\begin{aligned} A'(y)\psi_x(g_2(y),y) + B'(y)\varphi_x(g_2(y),y) + \widehat{V}_{yx}(g_2(y),y) \\ + \big[A(y)\psi_{yx}(g_2(y),y) + B(y)\varphi_{yx}(g_2(y),y)\big] = 0. \end{aligned} \tag{4.60}$$

Proof. Fix $(x,y) \in \mathbb{R}^2$, and, for $n \in \mathbb{N}$, set $\tau_n := \inf\{t \geq 0 : |X^{x,y}_t| \geq n\}$, $n \in \mathbb{N}$. Propositions 4.4 and 4.16 guarantee that V_y and V_{yx} are continuous functions on $\mathbb{R}^2$. Moreover, by Lemma 4.17, we have that $V_{yxx} \in L^\infty_{\text{loc}}(\mathbb{R}^2)$. Such a regularity of V_y allows us to apply the local time-space calculus of [76] to the process $(e^{-\rho s}V_y(X^{x,y}_s, y))_{s\geq 0}$ on the time interval $[0,\tau_n]$. Taking expectations (so that the term involving the stochastic integral vanishes) and noticing that $\mathbb{P}(X^{x,y}_s = g_1(y)) = \mathbb{P}(X^{x,y}_s = g_2(y)) = 0$, $s > 0$, for any $(x,y) \in \mathbb{R}^2$, we obtain

$$\begin{aligned} &\mathbb{E}\Big[e^{-\rho\tau_n}V_y(X^{x,y}_{\tau_n}, y)\Big] \\ &= V_y(x,y) + \mathbb{E}\bigg[\int_0^{\tau_n} e^{-\rho s}\big[(\mathcal{L}^y - \rho)V_y(\cdot,y)\big](X^{x,y}_s)\ \mathbb{1}_{\{X^{x,y}_s \neq g_1(y)\}}\mathbb{1}_{\{X^{x,y}_s \neq g_2(y)\}}\ ds\bigg] \\ &= V_y(x,y) - \mathbb{E}\bigg[\int_0^{\tau_n} e^{-\rho s}\big(\alpha V_x(X^{x,y}_s, y) + f_y(X^{x,y}_s, y)\big)\mathbb{1}_{\{g_1(y) < X^{x,y}_s < g_2(y)\}}\ ds\bigg] \\ &+ \mathbb{E}\bigg[\int_0^{\tau_n} \rho K e^{-\rho s}\mathbb{1}_{\{X^{x,y}_s < g_1(y)\}}\ ds - \int_0^{\tau_n} \rho K e^{-\rho s}\mathbb{1}_{\{X^{x,y}_s > g_2(y)\}}\ ds\bigg]. \end{aligned} \tag{4.61}$$

Rearranging (4.61) we have

$$\begin{aligned} V_y(x,y) &= \mathbb{E}\Big[e^{-\rho\tau_n}V_y(X^{x,y}_{\tau_n}, y)\Big] \\ &+ \mathbb{E}\bigg[\int_0^{\tau_n} e^{-\rho s}\big(\alpha V_x(X^{x,y}_s, y) + f_y(X^{x,y}_s, y)\big)\mathbb{1}_{\{g_1(y) < X^{x,y}_s < g_2(y)\}}\ ds\bigg] \\ &- \mathbb{E}\bigg[\int_0^{\tau_n} \rho K e^{-\rho s}\mathbb{1}_{\{(X^{x,y}_s, y) \in \mathcal{I}\}}\ ds + \int_0^{\tau_n} \rho K e^{-\rho s}\mathbb{1}_{\{(X^{x,y}_s, y) \in \mathcal{D}\}}\ ds\bigg]. \end{aligned} \tag{4.62}$$

We now aim at taking limits as $n \uparrow \infty$ in the right-hand side of the latter. To this end notice that $\tau_n \uparrow \infty$ a.s. when $n \uparrow \infty$, and therefore $\lim_{n\uparrow\infty} \mathbb{E}[e^{-\rho\tau_n} V_y(X^{x,y}_{\tau_n}, y)] = 0$ since $V_y \in [-K, K]$. Also, recalling (4.4), Proposition 4.4-(ii), and using standard estimates based on Burkholder-Davis-Gundy's inequality, one has

$$\mathbb{E}\bigg[\int_0^\infty e^{-\rho s}\big(\alpha|V_x(X^{x,y}_s, y)| + |f_y(X^{x,y}_s, y)|\big)\, ds\bigg] < +\infty.$$

Hence, thanks to the previous observations we can take limits as $n \uparrow \infty$, invoke the dominated convergence theorem, and obtain from (4.62) that

$$\begin{aligned} V_y(x,y) &= \mathbb{E}\bigg[\int_0^\infty e^{-\rho s} H(X^{x,y}_s, y)\mathbb{1}_{\{(X^{x,y}_s, y)\in\mathcal{C}\}}\, ds\bigg] \\ &\quad - \mathbb{E}\bigg[\int_0^\infty \rho K e^{-\rho s}\mathbb{1}_{\{(X^{x,y}_s, y)\in\mathcal{I}\}}\, ds\bigg] + \mathbb{E}\bigg[\int_0^\infty \rho K e^{-\rho s}\mathbb{1}_{\{(X^{x,y}_s, y)\in\mathcal{D}\}}\, ds\bigg] \\ &=: I_1(x,y) - I_2(x,y) + I_3(x,y). \end{aligned} \tag{4.63}$$

With the help of the Green function (4.55) and Fubini's theorem, we can now rewrite each I_i, $i = 1, 2, 3$, so to find

$$\begin{aligned} I_1(x,y) &= \mathbb{E}\bigg[\int_0^\infty e^{-\rho s} H(X^{x,y}_s, y)\mathbb{1}_{\{g_1(y) < X^{x,y}_s < g_2(y)\}}\, ds\bigg] \\ &= \int_0^\infty e^{-\rho s}\Big(\int_{-\infty}^\infty H(z,y)\mathbb{1}_{\{g_1(y)<z<g_2(y)\}} p(s,x,z,y) m_x(z,y)\, dz\Big) ds \\ &= \int_{-\infty}^\infty G(x,z,y) H(z,y)\mathbb{1}_{\{g_1(y)<z<g_2(y)\}} m_x(z,y)\, dz \\ &= \frac{1}{w}\varphi(x,y)\int_{-\infty}^x \psi(z,y) H(z,y)\mathbb{1}_{\{g_1(y)<z<g_2(y)\}} m_x(z,y)\, dz \\ &\quad + \frac{1}{w}\psi(x,y)\int_x^\infty \varphi(z,y) H(z,y)\mathbb{1}_{\{g_1(y)<z<g_2(y)\}} m_x(z,y)\, dz, \end{aligned} \tag{4.64}$$

$$\begin{aligned} I_2(x,y) &= \mathbb{E}\bigg[\int_0^\infty \rho K e^{-\rho s}\mathbb{1}_{\{(X^{x,y}_s, y)\in\mathcal{I}\}}\, ds\bigg] \\ &= \rho K\int_0^\infty e^{-\rho s}\Big(\int_{-\infty}^\infty p(s,x,z,y)\mathbb{1}_{\{z\le g_1(y)\}} m_x(z,y)\, dz\Big) ds \\ &= \rho K\int_{-\infty}^\infty G(x,z,y)\mathbb{1}_{\{z\le g_1(y)\}} m'(z,y)\, dz \\ &= \frac{1}{w}\rho K\varphi(x,y)\int_{-\infty}^x \psi(z,y)\mathbb{1}_{\{z\le g_1(y)\}} m_x(z,y)\, dz \\ &\quad + \frac{1}{w}\rho K\psi(x,y)\int_x^\infty \varphi(z,y)\mathbb{1}_{\{z\le g_1(y)\}} m_x(z,y)\, dz, \end{aligned} \tag{4.65}$$

and, similarly,

$$\begin{aligned} I_3(x;y) &= \mathbb{E}\Big[\int_0^\infty \rho K e^{-\rho s}\mathbb{1}_{\{(X_s^{x,y},y)\in\mathcal{D}\}}\,ds\Big] \\ &= \frac{1}{w}\rho K\varphi(x,y)\int_{-\infty}^{x}\psi(z,y)\mathbb{1}_{\{z\geq g_2(y)\}}m_x(z;y)\,dz \\ &+ \frac{1}{w}\rho K\psi(x,y)\int_x^\infty \varphi(z,y)\mathbb{1}_{\{z\geq g_2(y)\}}m_x(z,y)\,dz. \end{aligned} \tag{4.66}$$

Now, by plugging (4.64), (4.65), and (4.66) into (4.63), and then by imposing that $V_y(g_1(y),y)=-K$ and $V_y(g_2(y),y)=K$, we obtain the two equations

$$-K = \frac{1}{w}\psi(g_1(y),y)\int_{g_1(y)}^{g_2(y)}\varphi(z,y)H(z,y)m_x(z)\,dz - I_2(g_1(y),y)+I_3(g_1(y),y)$$

and

$$K = \frac{1}{w}\varphi(g_2(y),y)\int_{g_1(y)}^{g_2(y)}\psi(z,y)H(z,y)m_x(z)\,dz - I_2(g_2(y),y)+I_3(g_2(y),y)$$

Finally, rearranging terms and using the fact that (cf. Chapter II in [17])

$$\frac{\psi_x(\cdot,y)}{S_x(\cdot,y)} = \rho\int_{-\infty}^{\cdot}\psi(z,y)m_x(z,y)\,dz$$

and

$$\frac{\varphi_x(\cdot,y)}{S_x(\cdot,y)} = -\rho\int_{\cdot}^{\infty}\varphi(z,y)m'(z,y)\,dz,$$

yield (4.57) and (4.58).

Equations (4.57) and (4.58) involve the coefficients $A(y)$ and $B(y)$ through the function H since $V_x(x,y)=A(y)\psi_x(x,y)+B(y)\varphi_x(x,y)+\widehat{V}_x(x,y)$, for any $g_1(y)<x<g_2(y)$, by (4.25). In order to obtain equations for A and B, we use (4.25) together with the second-order smooth-fit principle $V_{yx}(g_1(y),y)=V_{yx}(g_2(y),y)=0$, and we find that, given the boundary functions g_1 and g_2, A and B solve the system of ODEs (4.59) and (4.60). □

Remark 4.27. Notice that equations (4.57) and (4.58) are consistent with those obtained in Proposition 5.5 of [43]; in particular, one obtains, as a special case, those in Proposition 5.5 of [43] by taking $\alpha=0$ in ours (4.57) and (4.58). However, the nature of our equations is different. While the equations in [43] are algebraic, ours (4.57) and (4.58) are functional. Indeed, from (4.59) and (4.60) we see that A and B depend on the whole boundaries g_1 and g_2 (and not only on the points $g_1(y)$ and $g_2(y)$, for a fixed $y\in\mathbb{R}$), so that, once those coefficients are substituted into the expression for V_x, they give rise to a functional nature of (4.57) and (4.58).

In contrast to the lengthy analytic approach followed in [43], Equations (4.57) and (4.58) are derived via simple and handy probabilistic means using Itô's formula and properties of linear regular diffusions. We believe that this different approach has also a methodological value. Indeed, if we would have tried to derive equations for the free-boundaries imposing the continuity of V_y and V_{yx} at the points $(g_1(y), y)$ and $(g_2(y), y)$, $y \in \mathbb{R}$, we would have ended up with a system of complex and unhandy (algebraic and differential) equations from which it would have been difficult to observe their consistency with Proposition 5.5 of [43]. In the spirit of [3] (see also [84]), we also would like to mention that (4.57) and (4.58) can be seen as optimality conditions in terms of an integral representation based on the minimal y-harmonic mappings ψ and φ for the underlying diffusion $X^{x,y}$. As such, those equations could have been alternatively derived by applying the analytic representation of y-potentials obtained in Corollary 4.5 of [66].

In Theorem 4.26 we provide equations for the free-boundaries g_1 and g_2 and for the coefficients A, and B, but we don't prove uniqueness of the solution to (4.57), (4.58), (4.59) and (4.60). We admit that we do not know how to establish such a uniqueness claim. The results so far holding both for $\theta = 0$ and $\theta > 0$. In the next section, we will restrict our analyis only to the case $\theta = 0$. Due to the more simpler structure of ψ and ϕ (see (4.14), (4.15)), we can extend the results of Theorem 4.26.

4.6 A System of Differential Equations for the Free-Boundaries

In this section we derive, for the case $\theta = 0$, a first-order system of nonlinear differential equations for the free-boundaries g_1 and g_2, i.e. we will be able to write

$$\begin{cases} g_1'(y) = G_1(g_1(y), g_2(y), y), \\ g_2'(y) = G_2(g_1(y), g_2(y), y), \end{cases}$$

for some explicitly determined maps G_1, G_2, whose regularity will allow also to establish a $C^{1,\mathrm{Lip}}$ regularity for g_1, g_2. To the best of our knowledge, for a two-dimensional degenerate singular stochastic control problem with interconnected dynamics as ours, a similar result appears here for the first time.

For a comment on the complexity in the case $\theta > 0$, see Remark 4.29.

From (4.57), (4.58), (4.59) and (4.60), we obtain the following result.

Theorem 4.28. *Let Assumption 4.18 hold and let $\theta = 0$. Recall* (4.14), (4.15), (4.25). *Let Moreover, for $z \in \mathbb{R}$ let*

$$\lambda(z) := \sqrt{z^2 + 2\rho\eta^2}, \tag{4.67}$$

$$r_1(z) := \frac{-z + \sqrt{z^2 + 2\rho\eta^2}}{\eta^2} > 0, \tag{4.68}$$

$$r_2(z) := \frac{-z - \sqrt{z^2 + 2\rho\eta^2}}{\eta^2} < 0, \tag{4.69}$$

and for $y \in \mathbb{R}$, $i, j = 1, 2$, $j \neq i$,

$$\gamma_i(y) := -\frac{2}{\eta^2} \int_{g_1(y)}^{g_2(y)} e^{-r_j(\alpha y)u} \big(f_y(u, y) + \alpha \widehat{V}_x(u, y)\big) du + K r_i(\alpha y) \Big(e^{-r_j(\alpha y) g_1(y)} + e^{-r_j(\alpha y) g_2(y)}\Big).$$

Then, we obtain that

$$A(y) = -\frac{r_2(\alpha y)\eta^2\lambda(\alpha y)}{8\alpha\rho}\left[\frac{\gamma_1(y)\left(e^{-\frac{2}{\eta^2}\lambda(\alpha y)g_1(y)} - e^{-\frac{2}{\eta^2}\lambda(\alpha y)g_2(y)}\right) - \frac{2\lambda(\alpha y)\gamma_2(y)}{\eta^2}\big(g_2(y)-g_1(y)\big)}{\sinh^2\left(\frac{\lambda(\alpha y)}{\eta^2}\big(g_2(y)-g_1(y)\big)\right) - \left(\frac{\lambda(\alpha y)}{\eta^2}\big(g_2(y)-g_1(y)\big)\right)^2}\right] \tag{4.70}$$

$$B(y) = -\frac{r_1(\alpha y)\eta^2\lambda(\alpha y)}{8\alpha\rho}\left[\frac{\gamma_2(y)\left(e^{\frac{2}{\eta^2}\lambda(\alpha y)g_2(y)} - e^{\frac{2}{\eta^2}\lambda(\alpha y)g_1(y)}\right) - \frac{2\lambda(\alpha y)\gamma_1(y)}{\eta^2}\big(g_2(y)-g_1(y)\big)}{\sinh^2\left(\frac{\lambda(\alpha y)}{\eta^2}\big(g_2(y)-g_1(y)\big)\right) - \left(\frac{\lambda(\alpha y)}{\eta^2}\big(g_2(y)-g_1(y)\big)\right)^2}\right], \tag{4.71}$$

$$A'(y) = \frac{\eta^2}{2\rho}e^{\frac{2\alpha y}{\eta^2}g_1(y)}\left[\frac{M(g_1(y),g_2(y),y)}{e^{r_1(\alpha y)(g_2(y)-g_1(y))} - e^{r_2(\alpha y)(g_2(y)-g_1(y))}}\right] \tag{4.72}$$

$$B'(y) = \frac{\eta^2}{2\rho}e^{\frac{2\alpha y}{\eta^2}g_1(y)}\left[\frac{N(g_1(y),g_2(y),y)}{e^{r_1(\alpha y)(g_2(y)-g_1(y))} - e^{r_2(\alpha y)(g_2(y)-g_1(y))}}\right], \tag{4.73}$$

where

$$\begin{aligned}
&M(x_1,x_2,y) := r_2(\alpha y)\big(e^{r_2(\alpha y)x_1}\widehat{V}_{yx}(x_2,y) - e^{r_2(\alpha y)x_2}\widehat{V}_{yx}(x_1,y)\big)\\
&+\alpha A(y)r_2'(\alpha y)r_2(\alpha y)\Big(e^{r_1(\alpha y)x_2+r_2(\alpha y)x_1}\big(r_2(\alpha y)x_2+1\big) - e^{r_1(\alpha y)x_1+r_2(\alpha y)x_2}\big(r_1(\alpha y)x_1+1\big)\Big)\\
&+\alpha B(y)r_2^2(\alpha y)r_2'(\alpha y)e^{r_2(\alpha y)(x_1+x_2)}(x_2-x_1)
\end{aligned}$$

and

$$\begin{aligned}
&N(x_1,x_2,y) := r_1(\alpha y)\big(e^{r_1(\alpha y)x_2}\widehat{V}_{yx}(x_1,y) - e^{r_1(\alpha y)x_1}\widehat{V}_{yx}(x_2,y)\big)\\
&+\alpha B(y)r_2'(\alpha y)r_1(\alpha y)\Big(e^{r_1(\alpha y)x_2+r_2(\alpha y)x_1}\big(r_2(\alpha y)x_1+1\big) - e^{r_1(\alpha y)x_1+r_2(\alpha y)x_2}\big(r_2(\alpha y)x_2+1\big)\Big)\\
&-\alpha A(y)r_1^2(\alpha y)r_1'(\alpha y)e^{r_1(\alpha y)(x_1+x_2)}(x_2-x_1).
\end{aligned}$$

Proof. Exploiting, given g_1 and g_2, (4.25), one has from (4.57) and (4.58) that A and B solve the linear system

$$\begin{aligned}
&A(y)\left[\alpha\int_{g_1(y)}^{g_2(y)}\psi(z,y)\psi_x(z,y)m_x(z,y)\,dz\right] + B(y)\left[\alpha\int_{g_1(y)}^{g_2(y)}\psi(z,y)\varphi_x(z,y)m_x(z,y)\,dz\right]\\
&= K\left[\frac{\psi_x(g_1(y),y)}{S_x(g_1(y),y)} + \frac{\psi_x(g_2(y),y)}{S_x(g_2(y),y)}\right] - \int_{g_1(y)}^{g_2(y)}\psi(z,y)\Big(f_y(z,y)+\alpha\widehat{V}_x(z,y)\Big)m_x(z,y)\,dz,
\end{aligned} \tag{4.74}$$

$$\begin{aligned}
&A(y)\left[\alpha\int_{g_1(y)}^{g_2(y)}\varphi(z,y)\psi_x(z,y)m_x(z,y)\,dz\right] + B(y)\left[\alpha\int_{g_1(y)}^{g_2(y)}\varphi(z,y)\varphi_x(z,y)m_x(z,y)\,dz\right]\\
&= K\left[\frac{\varphi_x(g_1(y),y)}{S_x(g_1(y),y)} + \frac{\varphi_x(g_2(y),y)}{S_x(g_2(y),y)}\right] - \int_{g_1(y)}^{g_2(y)}\varphi(z,y)\Big(f_y(z,y)+\alpha\widehat{V}_x(z,y)\Big)m_x(z,y)\,dz.
\end{aligned} \tag{4.75}$$

By using expressions for ψ, φ, S_x and m_x (cf. (4.14), (4.15), (4.53) and (4.54)) one can explicitly evaluate the integrals appearing on the left-hand sides of (4.74) and (4.75). Then, solving the latter two equations with respect to A and B one finds after some simple but tedious algebra (4.70) and (4.71). Notice indeed that the denominator appearing in (4.70) and (4.71) is nonzero since $g_1 \neq g_2$ and one has $\sinh^2(z) - z^2 > 0$ for any $z \neq 0$.

In order to find (4.72) and (4.73) we solve (4.59) and (4.60) with respect to $A'(y)$ and $B'(y)$, and use (4.14), (4.15), (4.67), (4.68) and (4.69). □

Remark 4.29. In the case of $\theta > 0$, the uncontrolled process is of Ornstein-Uhlenbeck type and this made it not possible to determine explicit expressions for $A(y)$ and $B(y)$ as in (4.70) and (4.71) above. Indeed, the complex form of the functions ψ and φ associated to the Ornstein-Uhlenbeck process does not allow to conclude that the determinant of the coefficients' matrix arising when one tries to solve (4.74) and (4.75) with respect to $A(y)$ and $B(y)$ is nonzero.

We have now found explicit, given g_1 and g_2, formulas for A, B, A' and B'. Exploiting the connection between them, we obtain the following result for the function g_1 and g_2.

Theorem 4.30. *Let $D := \{(x_1, x_2, y) \in \mathbb{R}^3 : x_1 \neq x_2\} \times \mathbb{R}$.*
There exist explicitly computable[8] *functions $G_i \in C^{0,Lip}_{loc}(D; \mathbb{R})$, $i = 1, 2$ such that*

$$\begin{cases} g_1'(y) = G_1(g_1(y), g_2(y), y) \\ g_2'(y) = G_2(g_1(y), g_2(y), y). \end{cases} \tag{4.76}$$

In particular, $g_i \in C^{1,Lip}_{loc}(\mathbb{R}; \mathbb{R})$ for $i = 1, 2$.

Proof. Recall Proposition 4.14 and (4.56). In particular, for any (x, y) such that $g_1(y) \leq x \leq g_2(y)$ – i.e. for any $(x, y) \in \bar{\mathcal{C}}$ – we have by (4.25)

$$V_x(x, y) = A(y)\psi_x(x, y) + B(y)\varphi_x(x, y) + \widehat{V}_x(x, y),$$

with A, B belonging to $W^{2,\infty}_{\text{loc}}(\mathbb{R}; \mathbb{R})$. Defining then the function

$$\bar{H}(x, y) = f_y(x, y) + \alpha\big(A(y)\psi_x(x, y) + B(y)\varphi_x(x, y) + \widehat{V}_x(x, y)\big), \quad (x, y) \in \mathbb{R}^2,$$

one has $\bar{H} = H$ on $\bar{\mathcal{C}}$.

Introduce now $\Phi_i : D \to \mathbb{R}$ defined as

$$\Phi_1(x_1, x_2, y) := \int_{x_1}^{x_2} \psi(z, y)\bar{H}(z, y)m_x(z, y)\, dz$$
$$- K\rho \int_{-\infty}^{x_1} \psi(z, y)m_x(z, y)\, dz - K\rho \int_{-\infty}^{x_2} \psi(z, y)m_x(z, y)\, dz$$

[8] Cf. Remark 4.31.

$$\Phi_2(x_1,x_2,y) := \int_{x_1}^{x_2} \varphi(z,y)\bar{H}(z,y)m_x(z,y)\,dz$$
$$+ K\rho \int_{x_2}^{\infty} \varphi(z,y)m_x(z,y)\,dz + K\rho \int_{x_1}^{\infty} \varphi(z,y)m_x(z,y)\,dz.$$

Observing that (cf. Chapter II in [17])

$$\frac{\psi_x(\cdot,y)}{S_x(\cdot,y)} = \rho \int_{-\infty}^{\cdot} \psi(z,y)m_x(z,y)\,dz, \qquad \frac{\varphi_x(\cdot,y)}{S_x(\cdot,y)} = -\rho \int_{\cdot}^{\infty} \varphi(z,y)m_x(z,y)\,dz,$$

one can readily see that, by (4.57)-(4.58), for any $y \in \mathbb{R}$ one has

$$\Phi_1(g_1(y),g_2(y),y) = 0 \quad \text{and} \quad \Phi_2(g_1(y),g_2(y),y) = 0. \tag{4.77}$$

Thanks to Assumption 4.2 and Proposition 4.14, one has that $\bar{H} \in C^{1,\mathrm{Lip}}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R})$. Hence, for any $i = 1,2$, the map $(x_1,x_2) \mapsto \Phi_i(x_1,x_2,y)$ belongs to $C^2(D;\mathbb{R})$ for each $y \in \mathbb{R}$ and the map $y \mapsto \Phi_i(x_1,x_2,y)$ belongs to $C^{1,\mathrm{Lip}}_{\mathrm{loc}}(D;\mathbb{R})$ for each $(x_1,x_2) \in \mathbb{R}^2$. Recalling Proposition 4.25 we can take the total derivative on both terms appearing in (4.77) we obtain for a.e. $y \in \mathbb{R}$ that

$$\underbrace{\begin{pmatrix} \frac{\partial \Phi_1}{\partial x_1}(g_1(y),g_2(y),y) & \frac{\partial \Phi_1}{\partial x_2}(g_1(y),g_2(y),y) \\ \frac{\partial \Phi_2}{\partial x_1}(g_1(y),g_2(y),y) & \frac{\partial \Phi_2}{\partial x_2}(g_1(y),g_2(y),y) \end{pmatrix}}_{=:\Lambda(g_1(y),g_2(y),y)} \begin{pmatrix} g_1'(y) \\ g_2'(y) \end{pmatrix} = - \begin{pmatrix} \frac{\partial \Phi_1}{\partial y}(g_1(y),g_2(y),y) \\ \frac{\partial \Phi_2}{\partial y}(g_1(y),g_2(y),y) \end{pmatrix}. \tag{4.78}$$

The determinant of the matrix Λ, denoted by $|\Lambda|$, is given by

$$|\Lambda|(g_1(y),g_2(y),y) = \big(\bar{H}(g_1(y),y) + K\rho\big)\big(\bar{H}(g_2(y),y) - K\rho\big)m_x(g_1(y),y)m_x(g_2(y),y)\cdot$$
$$\cdot \Big(\psi(g_2(y),y)\varphi(g_1(y),y) - \psi(g_1(y),y)\varphi(g_2(y),y)\Big). \tag{4.79}$$

We now aim at showing that $|\Lambda|(g_1(y),g_2(y),y)$ does not vanish for any $y \in \mathbb{R}$ under Assumption 4.18-(iii). On the one hand, if item (a) of that assumption holds, i.e. $x \mapsto f_y(x,y)$ is strictly increasing, then we have that $x \mapsto H(x,y)$ is such as well. Since $\bar{H} = H$ on $\bar{\mathcal{C}}$ and $g_2(y) > \zeta_2(y) > \zeta_1(y) > g_1(y)$ by Proposition 4.20-(ii), we have

$$\bar{H}(g_1(y),y) + K\rho < 0, \quad \bar{H}(g_2(y),y) - K\rho > 0,$$

and

$$\psi(g_2(y),y)\varphi(g_1(y),y) - \psi(g_1(y),y)\varphi(g_2(y),y) > 0;$$

therefore, $|\Lambda|(g_1(y),g_2(y),y) < 0$. On the other hand, if item (b) of Assumption 4.18-(iii) holds, i.e. if $f_{yx} \equiv 0$ and $f(\cdot,y)$ is strictly convex for any $y \in \mathbb{R}$, we can argue by contradiction as in Case (b) of the proof of Corollary 4.23. To this end, suppose, for example, that $\bar{H}(g_1(y_o),y_o) + K\rho = H(g_1(y_o),y_o) + K\rho = 0$, for some $y_o \in \mathbb{R}$. Denoting $f_y(x,y) = q(y)$ it then follows that

$$-\rho K + q(y_o) = -\alpha V_x(x,y_o) \quad \forall\, x \in (g_1(y_o),\zeta_1(y_o)),$$

by definition of ζ_1 (cf. Proposition 4.20); that is, V is an affine function of x in that interval. However, using the latter and (4.19), we also have

$$\alpha y_o\Big(\frac{\rho K - q(y_o)}{\alpha}\Big) - \rho V(x, y_o) = -f(x, y_o) \quad \forall\, x \in (g_1(y_o), \zeta_1(y_o)),$$

and we reach a contradiction since f is strictly convex in x by assumption while V is affine. The same argument also implies that $\bar{H}(g_1(y_o), y_o) + K\rho \neq 0$. We have then proved that in any case one has $|\Lambda|(g_1(y), g_2(y), y) \neq 0$ under Assumption 4.18-(iii).

We can therefore invert the matrix Λ appearing in (4.78) and obtain that for a.e. $y \in \mathbb{R}$

$$\begin{cases} g_1'(y) = \dfrac{\Big[\frac{\partial \Phi_1}{\partial x_2}\frac{\partial \Phi_2}{\partial y} - \frac{\partial \Phi_2}{\partial x_2}\frac{\partial \Phi_1}{\partial y}\Big](g_1(y), g_2(y), y)}{|\Lambda|(g_1(y), g_2(y), y)} =: G_1(g_1(y), g_2(y), y) \\[2ex] g_2'(y) = \dfrac{\Big[\frac{\partial \Phi_2}{\partial x_1}\frac{\partial \Phi_1}{\partial y} - \frac{\partial \Phi_1}{\partial x_1}\frac{\partial \Phi_2}{\partial y}\Big](g_1(y), g_2(y), y)}{|\Lambda|(g_1(y), g_2(y), y)} =: G_2(g_1(y), g_2(y), y) \end{cases} \tag{4.80}$$

Observe now that, given the aforementioned regularity of $\frac{\partial \Phi_i}{\partial x_j}$, $i, j = 1, 2$, and of $\frac{\partial \Phi_i}{\partial y}$, $i = 1, 2$, we have $G_i \in C^{0,\mathrm{Lip}}_{\mathrm{loc}}(D; \mathbb{R})$; hence, $g_i \in C^{1,\mathrm{Lip}}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$. □

Remark 4.31. Notice that the right-hand sides of (4.80) are indeed functions only of $(g_1(y), g_2(y), y)$. To see that, it is enough to feed (4.70) and (4.71), and (4.72) and (4.73) in the right-hand sides of (4.80), upon noticing that for any $i, j = 1, 2$, $\frac{\partial \Phi_i}{\partial x_j}$ depend on $A(y), B(y)$, while, for any $i = 1, 2$, $\frac{\partial \Phi_i}{\partial y}$ depend on $A'(y), B'(y)$.

Remark 4.32.
i) In the proof of Proposition 5.6 of [43] (see page 2213 therein; see also Step 4 in the proof of Lemma 7 in [74] and the proof of Proposition 6 in [34]), a system of ODEs for the free-boundaries is determined with the aim of proving that the free-boundaries belong to C^1 and are strictly monotone. In our problem, proving strict monotonicity of g_1 and g_2 would require to establish a strict sign for G_1 and G_2 (cf. (4.80)). However, the interaction between our dynamics – and the consequent dependency of ψ, φ, and m_x on y – makes the partial derivatives $\frac{\partial \Phi_i}{\partial y}$ appearing in (4.80) much more complex than the analogous quantities in [43] or [74], and this in turn makes it unclear that $G_i < 0$, $i = 1.2$ (although expected).

ii) System (4.76) provides a system of ODEs for the free-boundaries g_1 and g_2. For a full characterization of the g_is, one need to provide boundary conditions. So far, it is not clear how to identify those. A possible idea might be to think about the parameter α (the strength of interaction) as a variable. If one can derive a system of ODEs in α, the case $\alpha = 0$ could be used as a boundary condition, since in this case the values for the boundaries are unique, see [43].

4.7 A Discussion on Theorem 4.30 and on the Optimal Control

4.7.1 On Theorem 4.30

Given the full degeneracy of our setting, the fact that the free-boundaries g_i, $i = 1, 2$, belong to the class $C^{1,\text{Lip}}_{\text{loc}}(\mathbb{R}; \mathbb{R})$ is, to the best of our knowledge, a remarkable result. Indeed, the lack of uniform ellipticity of the diffusion coefficient makes it already difficult to obtain a preliminary (locally) Lipschitz property of g_is by invoking results from PDE theory ([21] and [79], among others) or techniques as those in [91], [92], and [93]. Also the probabilistic approach developed in [35] is not directly applicable since our free-boundaries are associated to a Dynkin game rather than to an optimal stopping problem.

It is also worth stressing that Theorem 4.30 not only provides regularity of the free-boundaries, but also a system of ODEs. To the best of our knowledge, a similar result appears here for the first time. Clearly, in order to provide a complete characterization of g_is, (4.76) should be complemented by boundary conditions. The determination of those is a non trivial task. As a matter of fact, we have not been able to identify a relevant value of y for which the values of the free-boundaries can be determined. The only information available is that the free-boundaries diverge for large (in absolute value) levels of y; but this is clearly not enough. Even enforcing a finite-fuel constraint like $\underline{y} \leq Y^{y,\xi}_t \leq \overline{y}$ a.s. for any $t \geq 0$ would not help in order to obtain boundary conditions. Indeed, differently to the case with monotone controls (see [64]), here the drift process Y can be pushed back into $(\underline{y}, \overline{y})$ once any of the boundary points of that interval is reached. Also, it is not clear to us how to obtain some kind of asymptotic growth of the free-boundaries in order to restrict the functional class where to look for uniqueness of (4.76).

4.7.2 On The Optimal Control

So far, we investigated the structure of the value function and the properties of the free-boundaries g_is characterizing the state space. A key question now is the existence and uniqueness of an optimal control. Existence of an optimal control for problem (4.6) can be shown relying on (a suitable version of) Komlós' theorem, by following arguments similar to those employed in the proof of Proposition 3.4 in [43] (see also Theorem 3.3 in [61]). In fact, one also has uniqueness of the optimal control if the running cost function is strictly convex. In this section, we discuss the structure of the optimal control by relating it to the solution of a Skorokhod reflection problem at $\partial\mathcal{C}$, where $\mathcal{C}$ is given by (4.10). Moreover, we discuss conditions under which a solution to the reflection problem exists. First, we introduce the corresponding Skorokhod reflection problem.

Problem 4.33. Let $(x, y) \in \overline{\mathcal{C}}$ be given and fixed. Find a process $\widehat{\xi} \in \mathcal{A}$ such that $\widehat{\xi}_{0^-} = 0$ a.s. and, letting $(\widehat{X}^{x,y}_t, \widehat{Y}^{y}_t)_{t\geq 0} := (X^{x,y,\widehat{\xi}}_t, Y^{y,\widehat{\xi}}_t)_{t\geq 0}$ and denoting by $(\widehat{\xi}^+_t, \widehat{\xi}^-_t)_{t\geq 0}$ its minimal decomposition, we have

$$(\widehat{X}^{x,y}_t, \widehat{Y}^{y}_t) \in \overline{\mathcal{C}} \quad \text{for all } t \geq 0, \quad \mathbb{P} - \text{a.s.}$$

and

$$\widehat{\xi}_t^+ = \int_{(0,t]} \mathbb{1}_{\{(\widehat{X}_s^{x,y}, \widehat{Y}_s^y) \in \mathcal{I}\}} d\widehat{\xi}_s^+, \qquad \widehat{\xi}_t^- = \int_{(0,t]} \mathbb{1}_{\{(\widehat{X}_s^{x,y}, \widehat{Y}_s^y) \in \mathcal{D}\}} d\widehat{\xi}_s^-.$$

Before discussing the existence of a solution to problem (4.33), we provide a verification theorem which gives its optimality.

Theorem 4.34. *Let $(x, y) \in \mathbb{R}^2$ and suppose that a solution $\widehat{\xi} = \widehat{\xi}^+ - \widehat{\xi}^-$ to Problem 4.33 exists. Define the process $\xi^\star := \xi_t^{\star,+} - \xi_t^{\star,-}$, $t \geq 0$, where*

$$\xi_t^{\star,+} := \widehat{\xi}_t^+ + (g_1(y) - x)^+, \qquad \xi_t^{\star,-} := \widehat{\xi}_t^- + (x - g_2(y))^+, \quad \textit{for all } t \geq 0,$$

and with $\xi_{0-}^\star = 0$ a.s. Then $\xi^\star$ is optimal for problem (4.6). *Moreover, if f is strictly convex, it is the unique optimal control.*

Proof. Being the process $\xi^\star$ clearly admissible, it is enough to show that

$$V(x,y) \geq \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} f(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star}) dt + \int_0^\infty e^{-\rho t} K d\xi_t^{\star,+} + \int_0^\infty e^{-\rho t} K d\xi_t^{\star,-}\bigg]. \quad (4.81)$$

To accomplish that, let $(K_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact subsets such that $\bigcup_{n\in\mathbb{N}} K_n = \mathbb{R}^2$, and for any given $n \geq 1$, define the bounded stopping time $\tau_n := \inf\{t \geq 0 : (X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star}) \notin K_n\} \wedge n$. We already know by Proposition 4.14 that $V \in C^{2,1}(\bar{\mathcal{C}}; \mathbb{R})$; moreover, by construction, the process $\xi^\star$ is that $(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star}) \in \bar{\mathcal{C}}$ for all $t \geq 0$ a.s. Hence, we can apply Itô's formula on the (stochastic) time interval $[0, \tau_n]$ to the process $(e^{-\rho t} V(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star}))_{t\geq 0}$, take expectations, and obtain (upon noticing that the expectation of the resulting stochastic integral vanishes due to the continuity of V_x)

$$\begin{aligned} V(x,y) &= \mathbb{E}\Big[e^{-\rho\tau_n} V(X_{\tau_n}^{x,y,\xi^\star}, Y_{\tau_n}^{y,\xi^\star})\Big] - \mathbb{E}\bigg[\int_0^{\tau_n} e^{-\rho t} [(\mathcal{L}^y - \rho)V(\cdot, Y_t^{y,\xi^\star})](X_t^{x,y,\xi^\star})\, dt\bigg] \\ &\quad - \mathbb{E}\bigg[\int_0^{\tau_n} e^{-\rho t} V_y(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star})\, d\xi_t^{\star,c}\bigg] \\ &\quad - \mathbb{E}\bigg[\sum_{0\leq t\leq\tau_n} e^{-\rho t} \Big(V(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star}) - V(X_t^{x,y,\xi^\star}, Y_{t-}^{y,\xi^\star})\Big)\bigg]. \end{aligned} \quad (4.82)$$

Here $\xi^{\star,c}$ denotes the continuous part of $\xi^\star$. Notice now that

$$[(\mathcal{L}^y - \rho)V(\cdot, Y_t^{y,\xi^\star})](X_t^{x,y,\xi^\star}) = -f(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star})$$

due to Proposition 4.13-(i) and the fact that $V \in C^{2,1}(\bar{\mathcal{C}}; \mathbb{R})$ by Proposition 4.14. Therefore,

$$\mathbb{E}\bigg[\int_0^{\tau_n} e^{-\rho t} [(\mathcal{L}^y - \rho)V(\cdot, Y_t^{y,\xi^\star})](X_t^{x,y,\xi^\star})\, dt\bigg] = -\mathbb{E}\bigg[\int_0^{\tau_n} e^{-\rho t} f(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star})\, dt\bigg]. \quad (4.83)$$

Letting $\Delta\xi_t^{\star,\pm} := \xi_t^{\star,\pm} - \xi_{t-}^{\star,\pm}$, $t \geq 0$, notice now that

$$V(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star}) - V(X_t^{x,y,\xi^\star}, Y_{t-}^{y,\xi^\star}) = \mathbb{1}_{\{\delta\xi_t^{\star,+}>0\}} \int_0^{\delta\xi_t^{\star,+}} V_y(X_t^{x,y,\xi^\star}, Y_{t-}^{y,\xi^\star} + u)du$$
$$- \mathbb{1}_{\{\delta\xi_t^{\star,-}>0\}} \int_0^{\delta\xi_t^{\star,-}} V_y(X_t^{x,y,\xi^\star}, Y_{t-}^{y,\xi^\star} - u)du. \tag{4.84}$$

Since the support of the (random) measure induced on $\mathbb{R}_+$ by $\xi^{\star,+}$ is $\mathcal{I}$, and that of (random) the measure induced on $\mathbb{R}_+$ by $\xi^{\star,-}$ is $\mathcal{D}$, and $V_y = -K$ on $\mathcal{I}$ and $V_y = K$ on $\mathcal{D}$, we therefore conclude by using (4.84) that

$$\mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t} V_y(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star})\, d\xi_t^{\star,\mathrm{c}} + \sum_{0\leq t\leq\tau_n} e^{-\rho t}\Big(V(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star}) - V(X_t^{x,y,\xi^\star}, Y_{t-}^{y,\xi^\star})\Big)\Big]$$
$$= -\mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t}\Big(K\, d\xi_t^{\star,+} + K\, d\xi_t^{\star,-}\Big)\Big]. \tag{4.85}$$

$$\mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t} V_y(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star})\, d\xi_t^{\star,\mathrm{c}} + \sum_{0\leq t\leq\tau_n} e^{-\rho t}\Big(V(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star}) - V(X_t^{x,y,\xi^\star}, Y_{t-}^{y,\xi^\star})\Big)\Big]$$
$$= -\mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t}\Big(K\, d\xi_t^{\star,+} + K\, d\xi_t^{\star,-}\Big)\Big]. \tag{4.86}$$

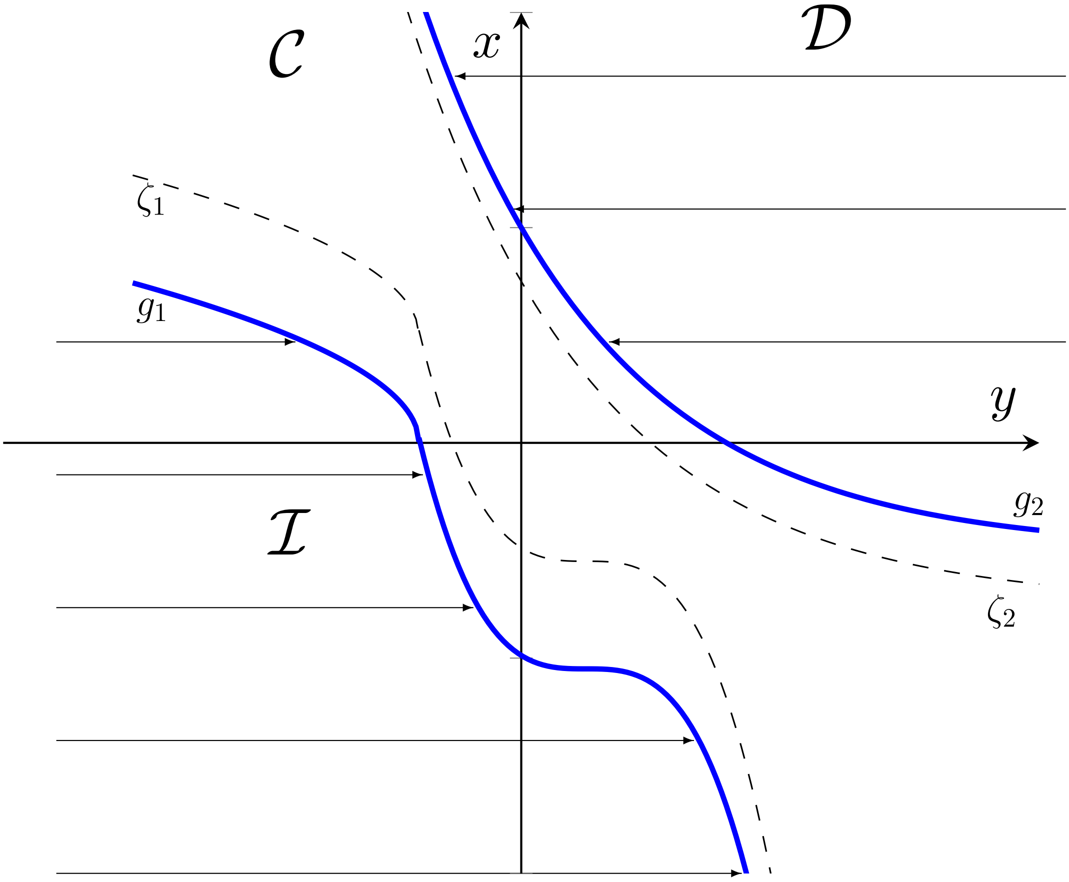
Then using (4.83) and (4.85) in (4.82), we obtain

$$V(x,y) \geq \mathbb{E}\Big[\int_0^{\tau_n} e^{-\rho t} f(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star})\, dt + \int_0^{\tau_n} e^{-\rho t} K\, d\xi_t^{\star,+} + \int_0^{\tau_n} e^{-\rho t} K\, d\xi_t^{\star,-}\Big],$$

where the non-negativity of V has also been employed. Taking now limits as $n \uparrow \infty$ in the right-hand side of the latter, and invoking the monotone convergence theorem (due to non-negativity of f and of K) we obtain (4.81).

Finally, uniqueness of the optimal control can be shown thanks to the strict convexity of f by arguing as in the proof of Proposition 3.4 in the Appendix A of [43]. □

The following picture provides an illustrative description of the expected behavior of the optimal control rule $\xi^\star$. This should be such that the jumps of the two-dimensional process $(X_t^{x,y,\xi^\star}, Y_t^{y,\xi^\star})_{t\geq 0}$ are induced by the optimal control only at initial time, if the initial data (x,y) lie in the interior of $\mathcal{I}$ or $\mathcal{D}$, or at those times at which the process meets jumps of the free-boundaries. The size of those interventions should be such that the process is immediately brought to the closest point on $\partial\mathcal{C}$, from where it evolves according to (4.2) and (4.3) and in such a way that it is kept inside the closure of $\mathcal{C}$ in a minimal way.



Question: does a solution to Problem 4.33 exist?

The latter is per se an interesting and not trivial problem, whose solution in multi-dimensional settings strongly hinges on the smoothness of the reflection boundary itself; sufficient conditions can be found in the seminal papers [38] and [68]. Unfortunately, our information on $\partial\mathcal{C}$ do not suffice to apply the results of the aforementioned works since we are not able to exclude horizontal segments of the free-boundaries g_1 and g_2 (cf. Case (1) and Case (2) in [38]). Indeed, although we can provide explicit formulas for the maps G_1 and G_2 appearing in (4.76), their complex expressions makes it hard to show that they are strictly negative (see also Remark 4.32).

An alternative and more constructive way of obtaining a solution to Problem 4.33 is the one followed in [26], where the needed reflected diffusion is constructed (weakly) by means of a Girsanov's transformation of probability measures (see Section 5 in [26]). The next proposition shows that this is possible also in our problem when f satisfies suitable additional requirements.

Proposition 4.35. *Suppose that there exists $C > 0$ such that $|f_x| \leq C$, and that $f_y(x,y) = \beta(y)$, for some strictly increasing function $\beta : \mathbb{R} \to \mathbb{R}$ such that $\lim_{y\to\pm\infty} \beta(y) = \pm\infty$. Then there exists a weak solution (in the sense of weak solutions to SDEs) to Problem 4.33.*

Proof. The proof is organized in two steps. For simplicity, we just perform the proof in the case $\theta > 0$, we case $\theta = 0$ can be treated similary.

Step 1. We here show that $\underline{b}_2 > -\infty$ and $\bar{b}_1 < +\infty$. Using the convexity of $f(\cdot, y)$, (4.4), and the assumed requirement on f_x, one easily finds for all $(x, y) \in \mathbb{R}^2$ that

$$\frac{V(x+\varepsilon, y) - V(x,y)}{\varepsilon} \leq \sup_{\xi \in \mathcal{A}} \mathbb{E}\bigg[\int_0^\infty e^{-(\rho+\theta)t} f_x(X_t^{x+\varepsilon,y,\xi}, y)dt\bigg] \leq \frac{C}{\rho+\theta} =: C'.$$

Analogously, for any $(x, y) \in \mathbb{R}^2$,

$$\frac{V(x,y) - V(x-\varepsilon, y)}{\varepsilon} \geq \inf_{\xi \in \mathcal{A}} \mathbb{E}\bigg[\int_0^\infty e^{-(\rho+\theta)t} f_x(X_t^{x-\varepsilon,y,\xi}, y)dt\bigg] \geq -C'.$$

Hence, by the existence of $V_x(\cdot, y)$, we have that $|V_x| \leq C'$.

Since, by assumption, $f_y(x, y) = \beta(y)$, for some strictly increasing function $\beta : \mathbb{R} \to \mathbb{R}$ such that $\lim_{y \to \pm\infty} \beta(y) = \pm\infty$, it follows from arguments similar to those employed to prove (ii) of Proposition 4.20 that

$$\{(x, y) \in \mathbb{R}^2 : y \geq b_2(x)\} \subseteq \{(x, y) \in \mathbb{R}^2 : y \geq \beta^{-1}(\rho K - \alpha C')\}.$$

Hence, $\underline{b}_2 > -\infty$.

Analogously, one has that

$$\{(x, y) \in \mathbb{R}^2 : y \leq b_1(x)\} \subseteq \{(x, y) \in \mathbb{R}^2 : y \leq \beta^{-1}(\alpha C' - \rho K)\};$$

therefore, $\bar{b}_1 < +\infty$.

Step 2. We here follow the approach developed in Section 5 of [26] in order to construct a weak solution (in the sense of weak solutions to SDEs) to Problem 4.33. Let $B := (B_t)_{t \geq 0}$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{G} := (\mathcal{G}_t)_{t \geq 0}, \mathbb{Q})$, where $\mathbb{G}$ satisfies the usual hypotheses. The smallest such filtration is the augmented filtration generated by B, that we denote by $\mathbb{F}$.

Following, e.g., the arguments of Section 4.3 in [43] one can construct a couple of $\mathbb{F}$-progressively measurable (since $\mathbb{F}$-adapted and right-continuous) processes $\xi^\star := (\xi_t^{\star,+}, \xi_t^{\star,-})_{t \geq 0}$ such that

$$\begin{cases} dX_t = -\theta X_t dt + \eta dB_t, \quad t > 0, \qquad X_0 = x \in \mathbb{R}, \\ Y_t^\star = y + \xi_t^{\star,+} - \xi_t^{\star,-}, \quad t \geq 0, \qquad Y_{0^-}^\star = y \in \mathbb{R}, \end{cases}$$

$$(X_t, Y_t^\star) \in \overline{\mathcal{C}} \quad \text{for all } t \geq 0, \quad \mathbb{Q} - \text{a.s.}, \tag{4.87}$$

and

$$\xi_t^{\star,+} = \int_{(0,t]} \mathbb{1}_{\{(X_s, Y_s^\star) \in \mathcal{I}\}} d\xi_s^{\star,+}, \qquad \xi_t^{\star,-} = \int_{(0,t]} \mathbb{1}_{\{(X_s, Y_s^\star) \in \mathcal{D}\}} d\xi_s^{\star,-}. \tag{4.88}$$

Since $\underline{b}_2 > -\infty$ and $\bar{b}_1 < +\infty$, one has that $\bar{b}_1 \vee y \geq Y_t^\star \geq \underline{b}_2 \wedge y$ for all $t \geq 0$, $\mathbb{Q}$-a.s.

It thus follows by Girsanov's theorem (Corollary 5.2 in Chapter 3.5 of [60]) that the process

$$W_t := B_t + \int_0^t \frac{-\alpha}{\eta} Y_s^\star ds, \quad t \geq 0,$$

is a standard Brownian motion on $(\Omega, \mathcal{F}^B, \mathbb{F}^B := (\mathcal{F}_t^B)_{t\geq 0}, \mathbb{P})$, where $\mathbb{F}^B$ is the (uncompleted) filtration generated by B, $\mathcal{F}^B := \mathcal{F}_\infty^B$, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F}^B)$ such that

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_T^B} = \exp\Big(- \int_0^T \frac{-\alpha}{\eta} Y_s^\star dB_s - \frac{1}{2}\int_0^T \frac{\alpha^2}{\eta^2}\big(Y_s^\star\big)^2 ds\Big), \quad T < \infty.$$

Hence, $\mathbb{P}$-a.s., $(X_t, Y_t^\star, \xi_t^\star)_{t\geq 0}$ solves (4.2) and (4.3), and satisfies (4.87) and (4.88); that is, it is a (weak) solution to Problem 4.33. □

Remark 4.36. Notice that the result of Proposition 4.35 is particularly relevant in the problem of optimal inflation management discussed in the introduction. Indeed, as a byproduct of Proposition 4.35 we have that the key interest rate stays bounded under the optimal monetary policy of the central bank.

In general, the constructive approach of [26] also gives a strong solution to Problem 4.33 if one can show show that the free-boundaries b_1 and b_2 are globally Lipschitz-continuous, a property that is assumed in [26]. In fact, in such a case, after constructing pathwise the solution to Problem 4.33 when $\alpha = 0$ in the dynamics of X (see, e.g., Section 5 in [26] or Section 4.3 in [43] for such a construction), one can still introduce back the linear term $\alpha Y^\star$ via a Girsanov's transformation. The Lipschitz property of the free-boundaries does indeed guarantee that the exponential process needed for the change of measure is an exponential martingale. Hence, a weak solution to Problem 4.33 exists and a strong solution could then be obtained via a pathwise uniqueness claim whose proof uses, once more, the global Lipschitz-continuity of the free boundaries (see Remark 5.2 in [26]).

It is worth noticing that in certain obstacle problems in $\mathbb{R}^d$, $d \geq 1$, the Lipschitz property is the preliminary regularity needed to upgrade - via a bootstrapping procedure and suitable technical conditions - the regularity of the free boundary to $C^{1,\delta}$-regularity, for some $\delta \in (0,1)$, and eventually to C^∞-regularity (see [21] and [79], among others, for details; see also [35] for Lipschitz-regularity results related to optimal stopping boundaries). In multi-dimensional singular stochastic control problems, Lipschitz regularity of the free boundary has been obtained, e.g., in a series of early papers by Soner and Shreve ([91], [92], and [93]), via fine PDE techniques, and in the more recent [18], via more probabilistic arguments. In all those works the control process is monotone and the state process is a linearly controlled Brownian motion. Obtaining global Lipschitz-continuity of the free-boundaries for the two-dimensional degenerate bounded-variation control problem (4.6) is a non trivial task that we leave for future research.

4.8 Conclusion

In this part of the thesis, we studied a two-dimensional singular stochastic control problem with interconnected dynamics. We considered both, a drifted Brownian motion and an Ornstein-Uhlenbeck process, where the drift components can be adjusted. We solved the problem using a direct approach. First, we proved some preliminary properties of the value function V such as differentiablility and convexity. Moreover, we related V_y (the derivative of the value function with respect to the controlled variable) to the value of a Dynkin game of optimal stopping (see Section 4.2). This allowed us to derive preliminary properties of the free-boundaries. Next, we showed that V is a viscosity solution to the corresponding HJB equation, which evolves as an ODE with gradient constraint. This fact enabled us to show that V is also a classical solution to the HJB in the continuation region. Furthermore, we upgraded the regularity of the value function V by proving a second-order smooth-fit condition for the mixed derivative V_{yx} and that $V_{yxx} \neq 0$ at the free-boundaries. These properties combined with the structure of V made it possible to derive a necessary system of non-linear functional equations for the free-boundaries. Moreover, the fact that $V_{yxx} \neq 0$ at the free-boundaries allowed us to apply the implicit function theorem and to prove a locally Lipschitz property of the free-boundaries (see Proposition 4.25). If X evolves as a drifted Brownian motion, the Lipschitz property makes it possible to differentiate the system of necessary functional equations and, thus, to derive an explicitly computable system of first-order ODEs. This result is the most remarkable one of this section. Unfortunately, so far we can not provide any initial conditions for the system of ODEs. Hence, the question of uniqueness still remains open. We let this problem left for future research. Finally, we discussed the construction of the optimal control, which turns out to be the solution of a Skorokhod reflection problem at the free-boundaries. Under additional assumptions, we proved that a weak solution to the Skorokhod reflection problem exists, allowing to construct the optimal control.

Appendices

A Appendix Section 2

A.1 Proof of Corollary 2.13

Notice that from (2.22) we can write for any $x > 0$ and $t \in [0, T]$

$$\begin{aligned} u(t,x) &= \mathbb{E}\bigg[\int_0^{T-t} -f'(t+\theta)\mathbb{1}_{\{x+\mu\theta+\sigma W_\theta \geq b(t+\theta)\}}\mathbb{1}_{\{\theta<S(x)\}}\, d\theta \\ &\quad + m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}} + g_x(T, A_{T-t}(x))\bigg] \\ &= \int_0^{T-t} -f'(t+\theta)\mathbb{P}\big(x+\mu\theta+\sigma W_\theta \geq b(t+\theta), S(x) > \theta\big)\, d\theta \\ &\quad + \mathbb{E}\big[m(t+S(x))\mathbb{1}_{\{S(x)\leq T-t\}}\big] + \mathbb{E}\big[g_x(T, A_{T-t}(x))\big], \end{aligned} \tag{A.1}$$

where Fubini's theorem and the fact that f' is deterministic has been used for the integral term above.

We now investigate the three summands separately. By using Proposition 3.2.1.1 in Jeanblanc et al. [52], and recalling that the stopping boundary b is strictly positive by Assumption 2.5, we have

$$\begin{aligned} &\mathbb{P}\Big(x+\mu\theta+\sigma W_\theta \geq b(t+\theta), S(x) > \theta\Big) \\ &= \mathbb{P}\Big(x+\mu\theta+\sigma W_\theta \geq b(t+\theta), \inf_{s\leq\theta}(x+\mu s+\sigma W_s) > 0\Big) \\ &= \mathbb{P}\Big(\frac{\mu}{\sigma}\theta + W_\theta \geq \frac{b(t+\theta)-x}{\sigma}, \inf_{s\leq\theta}\Big(\frac{\mu}{\sigma}s + W_s\Big) > -\frac{x}{\sigma}\Big) \\ &= \mathcal{N}\Bigg(\frac{\frac{x-b(t+\theta)}{\sigma}+\frac{\mu}{\sigma}\theta}{\sqrt{\theta}}\Bigg) - e^{-2\frac{\mu x}{\sigma^2}}\mathcal{N}\Bigg(\frac{-\frac{b(t+\theta)+x}{\sigma}+\frac{\mu}{\sigma}\theta}{\sqrt{\theta}}\Bigg). \end{aligned} \tag{A.2}$$

Here $\mathcal{N}(\,\cdot\,)$ denotes the cumulative distribution function of a standard Gaussian random variable. Note that the last term in (A.2) is continuously differentiable with respect to x for any $\theta > 0$.

For the second summand in the last expression on the right-hand side of (A.1) we first rewrite $S(x)$, for $x \geq 0$, as

$$\begin{aligned} S(x) &= \inf\{s \geq 0 : x+\mu s+\sigma W_s = 0\} = \inf\{s \geq 0 : \frac{\mu}{\sigma}s + W_s = -\frac{x}{\sigma}\} \\ &\overset{\mathcal{L}}{=} \inf\{s \geq 0 : -\frac{\mu}{\sigma}s + \widehat{W}_s = \frac{x}{\sigma}\} \end{aligned}$$

where $\widehat{W}$ is a standard Brownian motion. Hence equation (3.2.3) in Jeanblanc et al. [52] applies and allows us to write the probability density of $S(x)$ as

$$\rho_{S(x)}(u) := \frac{d\mathbb{P}(S(x) \in du)}{du} = \frac{x}{\sigma\sqrt{2\pi u^3}}e^{-\frac{(\frac{x}{\sigma}+\frac{\mu}{\sigma}u)^2}{2u}}, \quad u \geq 0. \tag{A.3}$$

For the third summand we notice that the absorbed process $A_{T-t}(x)$ of (2.5) is the drifted Brownian motion started in x and killed at the origin. Denote by $\rho_A(t,x,y)$ its transition density of moving from x to y in t units of time. Then, by employing the result of Borodin and Salminen [17], Section 15 in Appendix 1 (suitably adjusted to our case with $\sigma \neq 1$), we obtain

$$\begin{aligned}\rho_A\left(T-t,x,y\right) &:= \frac{d\mathbb{P}(A_{T-t}(x)\in dy)}{dy}\\ &= \frac{1}{\sqrt{2\pi(T-t)\sigma^2}}\exp\left(-\left(\frac{\mu(x-y)}{\sigma^2}\right)-\frac{\mu^2}{2\sigma^2}(T-t)\right)\\ &\times\left(\exp\left(-\frac{(x-y)^2}{2\sigma^2(T-t)}\right)-\exp\left(-\frac{(x+y)^2}{2\sigma^2(T-t)}\right)\right). \end{aligned} \tag{A.4}$$

Feeding (A.2), (A.3) and (A.4) back into (A.1) we obtain

$$\begin{aligned}u(t,x) = &\int_0^{T-t} -f'(t+\theta)\left[\mathcal{N}\left(\frac{\frac{x-b(t+\theta)}{\sigma}+\frac{\mu}{\sigma}\theta}{\sqrt{\theta}}\right)-e^{-2\frac{\mu x}{\sigma^2}}\mathcal{N}\left(\frac{-\frac{b(t+\theta)+x}{\sigma}+\frac{\mu}{\sigma}\theta}{\sqrt{\theta}}\right)\right]d\theta\\ &+\int_0^{T-t} m(t+u)\rho_{S(x)}(u)\,du+\int_0^\infty g_x(T,y)\rho_A\left(T-t,x,y\right)\,dy,\end{aligned}$$

and it is easy to see by the dominated convergence theorem that $x\mapsto u(t,x)$ is continuously differentiable on $(0,\infty)$ for any $t<T$.

A.2 Proof of Lemma 2.18

By (2.28) and Corollary 2.13 the function N of (2.29) is twice-continuously differentiable with respect to x on $(0,\infty)$. To show that N is also continuously differentiable with respect to t on $[0,T)$ we express the expected value on the right-hand side of (2.29) as an integral with respect to the probability densities of the involved processes. We thus start computing the transition density of the reflected Brownian motion R of (2.33), which we call ρ_R. By Appendix 1, Chapter 14, in Borodin and Salminen [17] (easily adapted to our case with $\sigma\neq 1$) we have

$$\begin{aligned}\rho_R(u,x,y) := \frac{d\mathbb{P}(R_u(x)\in dy)}{dy} = \frac{1}{\sqrt{2\pi u\sigma^2}}\exp\left(-\frac{\mu}{\sigma}\left(\frac{x-y}{\sigma}\right)-\frac{\mu^2}{2\sigma^2}u\right)\times\\ \left(\exp\left(-\frac{(x-y)^2}{2\sigma^2u}\right)-\exp\left(-\frac{(x+y)^2}{2\sigma^2u}\right)\right)-\frac{\mu}{2\sigma}\mathrm{Erfc}\left(\frac{x+y+\mu u}{\sqrt{2\sigma^2u}}\right),\end{aligned}$$

where $\mathrm{Erfc}(x) := \int_{-\infty}^x \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy$ for $x\in\mathbb{R}$. Hence, by using Fubini's Theorem, (2.29) reads as

$$\begin{aligned}
N(t,x) &= \mathbb{E}\Big[-\int_0^{T-t} (R_s(x)-b(t+s))^+ f'(t+s)\,ds - \int_0^{T-t} m(t+s)\,dI^0_s(x) \\
&\quad + g(T,R_{T-t}(x))\Big] = -\int_t^T \mathbb{E}\Big[(R_{u-t}(x)-b(u))^+\Big] f'(u)\,du \\
&\quad - \mathbb{E}\Big[\int_0^{T-t} m(t+s)\,dI^0_s(x)\Big] + \mathbb{E}\Big[g(T,R_{T-t}(x))\Big] \\
&= -\int_t^T \Big(\int_0^\infty (y-b(u))^+ \rho_R(u-t,x,y)\,dy\Big) f'(u)\,du \\
&\quad - \mathbb{E}\Big[\int_t^T m(u)\,dI^0_{u-t}(x)\Big] + \int_0^\infty g(T,y)\rho_R(T-t,x,y)\,dy. \qquad \text{(A.5)}
\end{aligned}$$

Recalling that m is continuously differentiable by Assumption 2.1 and using an integration by parts, we can write

$$\begin{aligned}
\mathbb{E}\Big[\int_t^T m(u)\,dI^0_{u-t}(x)\Big] &= \mathbb{E}\Big[m(T)I^0_{T-t}(x) - \int_t^T I^0_{u-t}(x)m'(u)\,du\Big] \\
&= m(T)\mathbb{E}\big[I^0_{T-t}(x)\big] - \int_t^T \mathbb{E}\big[I^0_{u-t}(x)\big]m'(u)\,du \\
&= m(T)\mathbb{E}\big[0\vee(\sigma\xi_{T-t}-x)\big] - \int_t^T \mathbb{E}\big[0\vee(\sigma\xi_{u-t}-x)\big]m'(u)\,du,
\end{aligned}$$

where we have used that $I^0_s(x) = 0\vee(\sigma\xi_s - x)$ with $\xi_s := \sup_{\theta\le s}(-\frac{\mu}{\sigma}\theta - W_\theta)$. Since (cf. Chapter 3.2.2 in Jeanblanc et al. [52])

$$\mathbb{P}\left(\xi_s \le z\right) = \mathcal{N}\left(\frac{z-\frac{\mu}{\sigma}s}{\sqrt{s}}\right) - \exp\left(2\frac{\mu}{\sigma}z\right)\mathcal{N}\left(\frac{-z-\frac{\mu}{\sigma}s}{\sqrt{s}}\right),$$

we get

$$\mathbb{E}\Big[0\vee(\sigma\xi_{u-t}-x)\Big] = \int_{\frac{x}{\sigma}}^\infty (\sigma z - x)\rho_\xi(u-t,z)\,dz,$$

where we have defined $\rho_\xi(s,z) := \frac{d\mathbb{P}(\xi_s\le z)}{dz}$. Because $\rho_\xi(\cdot,z)$ and $\rho_R(\cdot,x,y)$ are continuously differentiable on $(0,T]$, it follows that $N(t,x)$ as in (A.5) is continuously differentiable with respect to t, for any $t<T$. The continuity of N on $[0,T]\times\mathbb{R}_+$ also follows from the previous equations.

A.3 Proof of Proposition 2.26

Let $(t,x)\in[0,T)\times(0,\infty)$ be given and fixed, and take any sequence $(t_n,x_n)\subset[0,T)\times(0,\infty)$ such that $(t_n,x_n)\to(t,x)$. Then, let $\tau^\star := \tau^\star(t,x)$ be the optimal stopping time for $u(t,x)$ of (2.84). From (2.80) and the fact that $\tau^\star \le T-t$ a.s. we then find

$$u(t,x) - u(t_n,x_n) \le \mathbb{E}\left[\eta e^{-r\tau^\star}\mathbb{1}_{\{\tau^\star<S(x)\}} + \kappa e^{-rS(x)}\mathbb{1}_{\{\tau^\star\ge S(x)\}}\right.$$

$$
\begin{aligned}
&-\eta e^{-r(\tau^\star\wedge(T-t_n))}\mathbb{1}_{\{\tau^\star\wedge(T-t_n)<S(x_n)\}} - \kappa e^{-rS(x_n)}\mathbb{1}_{\{\tau^\star\wedge(T-t_n)\geq S(x_n)\}}\Big] \\
&= \mathbb{E}\Bigg[\mathbb{1}_{\{\tau^\star\leq T-t_n\}}\Bigg\{\eta e^{-r\tau^\star}\left(\mathbb{1}_{\{\tau^\star\geq S(x_n)\}} - \mathbb{1}_{\{\tau^\star\geq S(x)\}}\right) \\
&+ \kappa\left(e^{-rS(x)}\mathbb{1}_{\{\tau^\star\geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{\tau^\star\geq S(x_n)\}}\right)\Bigg\}\Bigg] \\
&+ \mathbb{E}\Bigg[\mathbb{1}_{\{\tau^\star> T-t_n\}}\Bigg\{\eta e^{-r\tau^\star}\mathbb{1}_{\{\tau^\star<S(x)\}} - \eta e^{-r(T-t_n)}\mathbb{1}_{\{T-t_n<S(x_n)\}} \\
&+ \kappa\left(e^{-rS(x)}\mathbb{1}_{\{\tau^\star\geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{T-t_n\geq S(x_n)\}}\right)\Bigg\}\Bigg] \\
&\leq \mathbb{E}\Bigg[\mathbb{1}_{\{\tau^\star\leq T-t_n\}}\Bigg\{\eta e^{-r\tau^\star}\mathbb{1}_{\{S(x_n)\leq\tau^\star<S(x)\}} \\
&+ \kappa\left(\left|e^{-rS(x)} - e^{-rS(x_n)}\right|\mathbb{1}_{\{\tau^\star\geq S(x_n)\vee S(x_n)\}} + e^{-rS(x)}\mathbb{1}_{\{S(x_n)>\tau^\star\geq S(x)\}}\right)\Bigg\}\Bigg] \\
&+ \mathbb{E}\Bigg[\mathbb{1}_{\{\tau^\star> T-t_n\}}\Bigg\{\eta e^{-r(T-t_n)}\left(\mathbb{1}_{\{T-t_n<S(x)\}} - \mathbb{1}_{\{T-t_n<S(x_n)\}}\right) \\
&+ \kappa\mathbb{1}_{\{T-t>S(x)\}}\left(e^{-rS(x)}\mathbb{1}_{\{\tau^\star\geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{T-t_n\geq S(x_n)\}}\right) \\
&+ \kappa\mathbb{1}_{\{T-t=S(x)\}}\left(e^{-rS(x)}\mathbb{1}_{\{\tau^\star\geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{T-t_n\geq S(x_n)\}}\right) \\
&+\kappa\mathbb{1}_{\{T-t<S(x)\}}\left(e^{-rS(x)}\mathbb{1}_{\{\tau^\star\geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{T-t_n\geq S(x_n)\}}\right)\Bigg\}\Bigg] \\
&\leq \mathbb{E}\left[\eta e^{-r\tau^\star}\mathbb{1}_{\{S(x_n)\leq\tau^\star<S(x)\}} + \kappa\left(\left|e^{-rS(x)} - e^{-rS(x_n)}\right| + \mathbb{1}_{\{S(x_n)>\tau^\star\geq S(x)\}}\right)\right] \\
&+ \mathbb{E}\Bigg[\mathbb{1}_{\{\tau^\star> T-t_n\}}\Bigg\{\eta e^{-r(T-t_n)}\mathbb{1}_{\{S(x_n)\leq T-t_n<S(x)\}} \\
&+ \kappa\mathbb{1}_{\{T-t>S(x)\}}\left(e^{-rS(x)}\mathbb{1}_{\{T-t\geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{T-t_n\geq S(x_n)\}}\right) \\
&+ \kappa\mathbb{1}_{\{T-t=S(x)\}} +\kappa\mathbb{1}_{\{T-t<S(x)\}}\mathbb{1}_{\{\tau^\star\geq S(x)\}}\Bigg\}\Bigg].
\end{aligned}
$$

Rearranging terms and taking limit inferior as $n\uparrow\infty$ on both sides one obtains

$$
\begin{aligned}
&\underline{\lim}_{n\to\infty}u(t_n,x_n) \geq u(t,x) - \overline{\lim}_{n\to\infty}\mathbb{E}\Bigg[\eta e^{-r\tau^\star}\mathbb{1}_{\{S(x_n)\leq\tau^\star<S(x)\}} \\
&+ \kappa\left(\left|e^{-rS(x)} - e^{-rS(x_n)}\right| + \mathbb{1}_{\{S(x_n)>\tau^\star\geq S(x)\}}\right)\Bigg] \\
&- \overline{\lim}_{n\to\infty}\mathbb{E}\Bigg[\mathbb{1}_{\{\tau^\star>T-t_n\}}\Bigg\{\eta e^{-r(T-t_n)}\mathbb{1}_{\{S(x_n)\leq T-t_n<S(x)\}} \\
&+ \kappa\mathbb{1}_{\{T-t>S(x)\}}\left(e^{-rS(x)}\mathbb{1}_{\{T-t\geq S(x)\}} - e^{-rS(x_n)}\mathbb{1}_{\{T-t_n\geq S(x_n)\}}\right) \\
&+ \kappa\mathbb{1}_{\{T-t=S(x)\}} + \kappa\mathbb{1}_{\{S(x)\leq\tau^\star\leq T-t<S(x)\}}\Bigg\}\Bigg] \\
&\geq u(t,x) - \mathbb{E}\left[\kappa\mathbb{1}_{\{S(x)=\tau^\star\}}\right] - \mathbb{E}\Bigg[\eta e^{-r(T-t)}\mathbb{1}_{\{T-t=S(x)\}} + \kappa\mathbb{1}_{\{T-t=S(x)\}}\Bigg] \\
&= u(t,x) - \kappa\mathbb{P}\left(\tau^\star = S(x)\right) - \left(\eta e^{-r(T-t)} + \kappa\right)\mathbb{P}\left(T-t=S(x)\right).
\end{aligned}
$$

The last inequality follows by interchanging expectations and limits by the dominated convergence theorem, using that $S(x_n) \to S(x)$, carefully investigating the involved limits superior, and observing that $\{\tau^\star \geq T-t\} = \{\tau^\star = T-t\}$ since $\tau^\star \in \Lambda(T-t)$.

Using now that $\{T-t = S(x)\}$ is a $\mathbb{P}$-null set by (A.3), and the fact that $\mathbb{P}(\tau^\star = S(x)) = 0$ since the free-boundary is strictly positive on $[0,T)$, we then obtain

$$\underline{\lim}_{n\to\infty} u(t_n, x_n) \geq u(t,x),$$

which proves the claimed lower semicontinuity of u on $[0,T)\times(0,\infty)$.

A.4 Lemma A.1

Lemma A.1. *Recall that (cf.* (2.55)*)*

$$z = \inf\{y \in [0, b(0)] : \tau^\star(0,y) < S(y)\}.$$

Then it holds that

$$S(z) \leq T \quad a.s.$$

Proof. In order to simplify exposition, in the following we shall stress the dependence on ω only when strictly necessary. Suppose that there exists a set $\Omega_0 \subset \Omega$ s.t. $\mathbb{P}(\Omega_0) > 0$, and that for any $\omega \in \Omega_0$ we have $S(z) > T$. Then take $\omega_0 \in \Omega_0$, recall that $Z_s(x) = x + \mu s + \sigma W_s$ for any $x > 0$ and $s \geq 0$, and notice that $\min_{0\leq s\leq T} Z_s(z;\omega_0) = \ell := \ell(\omega_0) > 0$. Then, defining $\widehat{z}(\omega_o) := \widehat{z} = z - \frac{\ell}{2}$, one has

$$\min_{0\leq s\leq T} Z_s(\widehat{z};\omega_0) = \min_{0\leq s\leq T}\left(z + \mu s + \sigma W_s(\omega_0) - \frac{\ell}{2}\right) = \ell - \frac{\ell}{2} = \frac{\ell}{2} > 0.$$

Hence, $S(\widehat{z}) > T \geq \tau^\star(0,\widehat{z})$, but this contradicts the definition of z since $\widehat{z} < z$. Therefore we conclude that $S(z) \leq T$ a.s. □

B Appendix Section 4

B.1 Proof of Theorem 4.5

We want to suitably employ the results of Theorems 3.11 and 3.13 of [26]. However, in contrast to the fully diffusive setting of [26], in our model the process Y is purely controlled so that the two-dimensional process (X, Y) is degenerate. The idea of the proof is then to perturb the dynamics of Y (cf. (4.2)) by adding a Brownian motion $B := (B_t)_{t\geq 0}$ with volatility coefficient $\delta > 0$, so to be able to apply Theorems 3.11 and 3.13 of [26] for any given and fixed δ. The claims of Theorem 4.5 (in particular (4.8)) will then follow by an opportune limit procedure as $\delta \downarrow 0$. We perform the proof only for the case $\theta > 0$, as the case $\theta = 0$ follows by similar arguments. Suppose that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is rich enough to accommodate a second Brownian motion $B := (B_t)_{t\geq 0}$, independent of W. Then, given $(x, y) \in \mathbb{R}^2$, $\delta > 0$, and $\xi \in \mathcal{A}$ (cf. (4.1)), we denote by $(X^{\xi;\delta}, Y^{\xi;\delta}) := (X_t^{\xi;\delta}, Y_t^{\xi;\delta})_{t\geq 0}$ the unique strong solution to

$$\begin{pmatrix} dY_t \\ dX_t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha & -\theta \end{pmatrix} \begin{pmatrix} Y_t \\ X_t \end{pmatrix} dt + \begin{pmatrix} \delta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} dB_t \\ dW_t \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\xi_t. \tag{B.1}$$

with initial data $X_{0^-} = x$ and $Y_{0^-} = y$. In order to simplify the notation, in the the rest of this proof we will not stress the dependency on (x, y) of the subsequent involved processes. In the case $\xi \equiv 0$, we simply write $(X^\delta, Y^\delta) := (X_t^{0;\delta}, Y_t^{0;\delta})_{t\geq 0}$.

Notice that (B.1) can be easily obtained from equation (2.2) of [26] by taking $c = 1$, by suitably defining the matrices b and σ therein, and by setting $x_1 = y$ and $x_2 = x$. Then we define the perturbed optimal control problem

$$V^\delta(x, y) := \inf_{\xi\in\mathcal{A}} \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} f(X_t^{\xi;\delta}, Y_t^{\xi;\delta})\, dt + K \int_0^\infty e^{-\rho t} \quad d|\xi|_t\bigg].$$

By estimates as those leading to Proposition 4.4 it can be shown that there exist constants $\tilde{C}_0, \tilde{C}_1, \tilde{C}_2$ (which are independent of δ, for all δ sufficiently small) such that for any $\lambda \in (0, 1)$, any $z := (x, y) \in \mathbb{R}^2$ and $z' := (x', y') \in \mathbb{R}^2$, we have

(i) $0 \leq V^\delta(z) \leq \tilde{C}_0\big(1 + |z|\big)^p$,

(ii) $|V^\delta(z) - V^\delta(z')| \leq \tilde{C}_1\big(1 + |z| + |z'|\big)^{p-1}|z - z'|$,

(iii) $0 \leq \lambda V^\delta(z) + (1-\lambda)V^\delta(z') - V^\delta(\lambda z + (1-\lambda)z') \leq \tilde{C}_2\lambda(1-\lambda)\big(1 + |z| + |z'|\big)^{(p-2)^+}|z - z'|^2$,

where $p > 1$ is the same of Assumption 4.2. Hence V^δ is convex and locally semi-concave, and therefore $V^\delta \in W^{2,\infty}_{\text{loc}}(\mathbb{R}^2; \mathbb{R})$. In particular, there exists a version of $V^\delta \in C^{1,\text{Lip}}_{\text{loc}}(\mathbb{R}^2; \mathbb{R})$.

Let $(X_t^\xi, Y_t^\xi)_{t\geq 0} := (X_t^{\xi;0}, Y_t^{\xi;0})_{t\geq 0}$. By (4.2), (4.4), and (B.1) one easily finds for $p \in [1, \infty)$

$$\mathbb{E}[|(X_t^{\xi;\delta}, Y_t^{\xi;\delta}) - (X_t^\xi, Y_t^\xi)|^p] \leq C_t\delta^p, \quad \forall \xi \in \mathcal{A} \text{ and } t \geq 0,$$

for some C_t that is at most of polynomial growth with respect to t. Using now the latter and Assumption 4.2-(ii), it can be shown that $V^\delta(x,y) \to V(x,y)$ as $\delta \downarrow 0$ for each $(x,y) \in \mathbb{R}^2$. Let $\mathcal{B}_N := \{z \in \mathbb{R}^2 : |z| < N\}$, for some $N > 0$. Since items (i)-(iii) above imply that $V^\delta \in W^{2,p}(\mathcal{B}_N)$ for any $p > 2$ and $W^{2,p}(\mathcal{B}_N)$ is reflexive, there exists a sequence $\delta_n \downarrow 0$ as $n \uparrow \infty$ such that V^{δ_n} converges weakly in $W^{2,p}(\mathcal{B}_N)$. Because $V^{\delta_n} \to V$ pointwise and weak limits are unique, we have that $V^{\delta_n} \rightharpoonup V$ weakly in $W^{2,p}(\mathcal{B}_N)$. Since the embedding $W^{2,p}(\mathcal{B}_N) \hookrightarrow C^1(\mathcal{B}_N)$ is compact for $p > 2$ (2 being the dimension of our space), it follows that

$$V^{\delta_n} \to V \text{ locally uniformly in } \mathbb{R}^2, \tag{B.2}$$

$$V_x^{\delta_n} \to V_x \text{ locally uniformly in } \mathbb{R}^2, \tag{B.3}$$

and

$$V_y^{\delta_n} \to V_y \text{ locally uniformly in } \mathbb{R}^2. \tag{B.4}$$

Moreover, by Theorem 3.11 in [26] (easily adjusted to take care of our general convex function f satisfying Assumption 4.2, and upon noticing that $b_{11} = 0$ in our setting, cf. (B.1)) we have that V_y^δ is the unique (given V_x^δ) solution to the pointwise variational inequality:

$$\begin{cases} V_y^\delta \in W^{2,q}_{\text{loc}}(\mathbb{R}^2),\ \forall q \geq 2, \qquad -K \leq V_y^\delta \leq K \quad \text{a.e. in } \mathbb{R}^2, \\ (\mathcal{L}^y - \rho)V_y^\delta \leq -\alpha V_x^\delta - f_y(x,y) \quad \text{a.e. in } \mathcal{I}^\delta, \\ (\mathcal{L}^y - \rho)V_y^\delta \geq -\alpha V_x^\delta - f_y(x,y) \quad \text{a.e. in } \mathcal{D}^\delta, \\ (\mathcal{L}^y - \rho)V_y^\delta = -\alpha V_x^\delta - f_y(x,y) \quad \text{a.e. in } \mathcal{C}^\delta, \end{cases} \tag{B.5}$$

where we have set

$$\mathcal{I}^\delta := \left\{(x,y) \in \mathbb{R}^2 : \quad V_y^\delta(x,y) = -K\right\}, \quad \mathcal{D}^\delta := \left\{(x,y) \in \mathbb{R}^2 : \quad V_y^\delta(x,y) = K\right\},$$

and

$$\mathcal{C}^\delta := \left\{(x,y) \in \mathbb{R}^2 : \quad -K < V_y^\delta(x,y) < K\right\}.$$

Define

$$\tau^{\star;\delta} := \inf\{t \geq 0 : V_y^\delta(X_t^\delta, Y_t^\delta) \leq -K\},$$

$$\sigma^{\star,\delta} := \inf\{t \geq 0 : V_y^\delta(X_t^\delta, Y_t^\delta) \geq K\},$$

$$\tau^\star := \inf\{t \geq 0 : V_y(X_t, y) \leq -K\},$$

$$\sigma^\star := \inf\{t \geq 0 : V_y(X_t, y) \geq K\},$$

as well as, for a given $M > 0$,

$$\tau_M^\delta := \inf\{t \geq 0 : |X_t^\delta| + |Y_t^\delta| \geq M\},$$

$$\tau_M := \inf\{t \geq 0 : |X_t| + |y| \geq M\}.$$

Now, by (B.5) we know that for each $\delta > 0$ given and fixed, V_y^δ is regular enough to apply a weak version of Itô's lemma (see, e.g., Theorem 8.5 at p. 185 of [13]) so that for any stopping time ζ and some fixed $T > 0$ one obtains

$$\begin{aligned} V_y^\delta(x,y) =& \mathbb{E}\bigg[- \int_0^{\tau_M^\delta \wedge \tau_M \wedge \zeta \wedge T} e^{-\rho s}(\mathcal{L}^y - \rho) V_y^\delta(X_s^\delta, Y_s^\delta) \ ds \\ &+ e^{-\rho(\tau_M^\delta \wedge \tau_M \wedge \zeta \wedge T)} V_y^\delta\Big(X^\delta_{\tau_M^\delta \wedge \tau_M \wedge \zeta \wedge T}, Y^\delta_{\tau_M^\delta \wedge \tau_M \wedge \zeta \wedge T}\Big)\bigg]. \end{aligned} \tag{B.6}$$

Given an $\mathbb{F}$-stopping time τ, set $\zeta := \sigma^{\star,\delta} \wedge \sigma^\star \wedge \tau$ in (B.6), and use that V^δ solves a.e. the variational inequality (B.5) to find

$$\begin{aligned} V_y^\delta(x,y) \geq& \ \mathbb{E}\bigg[\int_0^{\tau_M^\delta \wedge \tau_M \wedge \sigma^{\star,\delta} \wedge \sigma^\star \wedge \tau \wedge T} e^{-\rho s}\big(\alpha V_x^\delta(X_s^\delta, Y_s^\delta) + f_y(X_s^\delta, Y_s^\delta)\big) \ ds \\ &+ e^{-\rho(\tau_M^\delta \wedge \tau_M \wedge \sigma^{\star,\delta} \wedge \sigma^\star \wedge \tau \wedge T)} V_y^\delta\big(X^\delta_{\tau_M^\delta \wedge \tau_M \wedge \sigma^{\star,\delta} \wedge \sigma^\star \wedge \tau \wedge T}, Y^\delta_{\tau_M^\delta \wedge \tau_M \wedge \sigma^{\star,\delta} \wedge \sigma^\star \wedge \tau \wedge T}\big)\bigg] \\ \geq& \ \mathbb{E}\bigg[\int_0^{\tau_M^\delta \wedge \tau_M \wedge \sigma^{\star,\delta} \wedge \sigma^\star \wedge \tau \wedge T} e^{-\rho s}(\alpha V_x^\delta(X_s^\delta, Y_s^\delta) + f_y(X_s^\delta, Y_s^\delta)) \ ds \\ &+ \mathbb{1}_{\{\sigma^{\star,\delta} < \tau_M^\delta \wedge \tau_M \wedge \sigma^\star \wedge \tau \wedge T\}} e^{-\rho \sigma^{\star,\delta}} K - \mathbb{1}_{\{\tau \leq \tau_M^\delta \wedge \tau_M \wedge \sigma^{\star,\delta} \wedge \sigma^\star \wedge T\}} e^{-\rho\tau} K \\ &+ \mathbb{1}_{\{\tau_M^\delta \wedge \tau_M \wedge \sigma^\star \wedge T < \sigma^{\star,\delta} \wedge \tau\}} e^{-\rho(\tau_M^\delta \wedge \tau_M \wedge \sigma^\star \wedge T)} V_y^\delta\big(X^\delta_{\tau_M^\delta \wedge \tau_M \wedge \sigma^\star \wedge T}, Y^\delta_{\tau_M^\delta \wedge \tau_M \wedge \sigma^\star \wedge T}\big)\bigg]. \end{aligned} \tag{B.7}$$

Recalling (B.1), thanks to the estimates (i)-(iii) above, the uniform convergence of $V_y^{\delta_n}$ to V_y (cf. (B.4)), and the fact that there exists $C_T > 0$ such that

$$\mathbb{E}[\sup_{0 \leq s \leq T} |(X_t^{\delta_n}, Y_t^{\delta_n}) - (X_t, y)|^q] \leq C_T \delta_n^q,$$

with $X_t := X_t^{0;0}$ and $1 \leq q < \infty$, it can be shown that (see Theorem 3.7 in Section 3 of Chapter 3 of Chapter [13] – in particular p. 322 – and especially Lemma 4.17 in [27] for a detailed proof in a related but different setting) $\tau_M^{\delta_n} \wedge \tau_M \wedge \sigma^{\star,\delta_n} \wedge \sigma^\star \wedge \tau \wedge T \to \tau_M \wedge \sigma^\star \wedge \tau \wedge T$ as $n \uparrow \infty$, $\mathbb{P}$-a.s. Therefore, taking limits in (B.7) with $\delta = \delta_n$ as $n \uparrow \infty$, using the latter convergence of stopping times and (B.2)-(B.3), one finds

$$\begin{aligned} V_y(x,y) \geq& \ \mathbb{E}\bigg[\int_0^{\sigma^\star \wedge \tau_M \wedge \tau \wedge T} e^{-\rho s}\big(\alpha V_x(X_s, y) + f_y(X_s, y)\big) \ ds + e^{-\rho\sigma^\star} K \mathbb{1}_{\{\sigma^\star < \tau_M \wedge \tau \wedge T\}} \\ &- e^{-\rho\tau} K \mathbb{1}_{\{\tau \leq \sigma^\star \wedge \tau_M \wedge T\}} + e^{-\rho(\tau_M \wedge T)} V_y(X_{\tau_M \wedge T}, y) \mathbb{1}_{\{\tau_M \wedge \sigma^\star \wedge T < \sigma^\star \wedge \tau\}}\bigg]. \end{aligned}$$

Letting now $M \uparrow \infty$ and $T \uparrow \infty$ and invoking the dominated convergence theorem we obtain

$$\begin{aligned} V_y(x,y) \geq& \ \mathbb{E}\bigg[\int_0^{\sigma^\star \wedge \tau} e^{-\rho s}\big(\alpha V_x(X_s, y) + f_y(X_s, y)\big) \ ds \\ &+ e^{-\rho\sigma^\star} K \mathbb{1}_{\{\sigma^\star < \tau\}} - e^{-\rho\tau} K \mathbb{1}_{\{\tau \leq \sigma^\star\}}\bigg], \end{aligned} \tag{B.8}$$

for any $\mathbb{F}$-stopping time τ.

Analogously, picking $\zeta = \tau^{\star,\delta_n} \wedge \tau^\star \wedge \sigma$, for any $\mathbb{F}$-stopping time σ, in (B.6), and taking limits as $n \uparrow \infty$, and then as $M \uparrow \infty$ and $T \uparrow \infty$, yield

$$V_y(x,y) \leq \mathbb{E}\bigg[\int_0^{\sigma\wedge\tau^\star} e^{-\rho s}\big(\alpha V_x(X_s,y) + f_y(X_s,y)\big)\ ds + e^{-\rho\sigma} K \mathbb{1}_{\{\sigma<\tau^\star\}} - e^{-\rho\tau^\star} K \mathbb{1}_{\{\tau^\star \leq \sigma\}}\bigg]. \tag{B.9}$$

Finally, the choice $\zeta = \tau^{\star,\delta_n} \wedge \tau^\star \wedge \sigma^{\star,\delta_n} \wedge \sigma^\star$ leads (after taking limits) to

$$V_y(x,y) = \mathbb{E}\bigg[\int_0^{\sigma^\star\wedge\tau^\star} e^{-\rho s}\big(\alpha V_x(X_s,y) + f_y(X_s,y)\big)\ ds + e^{-\rho\sigma^\star} K \mathbb{1}_{\{\sigma^\star<\tau^\star\}} - e^{-\rho\tau^\star} K \mathbb{1}_{\{\tau^\star \leq \sigma^\star\}}\bigg]. \tag{B.10}$$

Combining (B.8), (B.9), and (B.10) completes the proof.

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