DOCTORAL THESIS

Characteristic polynomials of random matrices and their role in an effective theory of strong interactions

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Abstract

The aim of this thesis is to investigate the impact of characteristic polynomials on the spectral eigenvalue statistics of random matrix models, with applications in effective field theory models of Quantum chromodynamics (QCD). The symmetries of the field theory lead to random matrix ensembles named chiral Gaussian Unitary Ensemble (chGUE(N)) and extensions thereof. The random matrix ensembles are comparable to the effective theory of QCD in a low-energy regime, where chiral symmetry breaking is predominant and it suffices to consider only the smallest eigenvalues of the QCD Dirac operator. We consider four members of the chGUE(N) symmetry class: the classical chGUE(N) consisting of Hermitian, chiral block matrices with complex entries and its extensions by N_f massive flavors describing dynamical quarks. Furthermore, we consider the chGUE(N) extended by external parameters describing effects of external sources like temperature and its combination with N_f massiv flavors. The correlations of the chGUE(N), and its extensions with external parameters, als well as its deformations with massive flavors, belong the class of determinantal point processes. This implies that correlation functions can be expressed as determinants of a correlation kernel. The random matrix ensembles we consider feature special biorthogonal structures leading to a sub-class of determinantal point processes called invertible polynomial ensembles. Such ensembles are characterised by a joint probability density function (JPDF) containing two determinants, which can be linked to orthogonal polynomials, if the considered model is independent of temperature. If temperature is present as an external source, the JPDF has biorthogonal structure and the usage of orthogonal polynomials becomes more involved. in this case, the correlation kernel can be expressed in terms of expectation values of ratios of characteristic polynomials. We will derive a multi-contour-integral representation of the expectation value of an arbitrary ratio of characteristic polynomials for invertible polynomial ensembles at finite matrix size N. Additionally, we perform a saddle point analysis and derive the large N asymptotic form of the correlation kernel for the chGUE(N) matrix models including temperature as an external source. The limiting kernels show determinantal structures comparable to existing results partially derived with supersymmetry and orthogonal polynomial methods. We show that the limiting kernel for non-zero temperature models is indeed equivalent to existing results for temperature independent models. Furthermore, we show that the resulting correlation functions for both zero and non-zero temperature models agree with existing formulae of the correlation functions derived via supersymmetry. This answers the question wether the correlations of the underlying physical field model are indeed universal in the low-energy regime, where random matrices can be used to model QCD effective field theories.

Declaration of Authorship

I, Tim Robert WÜRFEL, declare that this thesis titled, "Characteristic polynomials of random matrices and their role in an effective theory of strong interactions" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at the University of Bielefeld.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this university or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is my own work.
- I have acknowledged all main sources of help and indicated all work done in cooperation with others.

Principal Publications

The following publications should be considered part of this thesis:

[1] G. Akemann, E. Strahov, T. R. Würfel: *Averages of Products and Ratios of Characteristic Polynomials in Polynomial Ensembles*. Ann. Henri Poincaré **21**, 3973-4002 (2020).

Signed: 

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Chapter 1

Introduction

In this thesis we discuss random matrix models, which can be used to describe aspects of strong interactions in the standard model of elementary particles. The quantum field theory associated with strong interactions is called *Quantum chromodynamics* (QCD) and forms a complex system, which we approximate by an effective theory for its low-energy eigenvalues. The matrix models describe the temperature dependence of correlation functions of the *Dirac operator*, which contains the information on quark interactions in the QCD Lagrangian. This allows us to answer questions of universality for zero and non-zero temperature phases in the phase diagram of the effective random matrix model. To evaluate the eigenvalue correlation functions we compute expectation values of products and ratios of characteristic polynomials with respect to the eigenvalues of the Dirac operator. The used formulae are based on results from the principal publication [1].
In section 1.1 we discuss important features and applications of Random Matrix Theory (RMT). In particular, we introduce Gaussian random matrix models, which serve as effective low-energy matrix models for QCD. In section 1.2 we give a brief introduction to Quantum chromodynamics and its symmetries relevant for the RMT approach. We show how to connect QCD and RMT and give a brief overview of the developed matrix models to study QCD with and without temperature. An outline of the thesis is given in section 1.3.

1.1 Random Matrix Theory

Random Matrix Theory (RMT) is a theory, which combines the concepts of matrices and random variables. Starting with a set of random variables, a *random matrix* can be formed by arranging the random variables in matrix form. Thus, the matrix itself becomes a random variable and consequently methods from both probability theory and matrix theory are combined into one theory.
RMT is often used as a mathematical tool to analyze correlation functions of complex physical systems in form of an effective theory. This means the following: The theory models a certain effect or aspect of the complex system, while reducing its complexity in the process. This raises the question if and when results from the effective model can be used to describe the complex system. Alternatively speaking: When does the reduced complexity lead to different results in the effective model and the complex system? If a result of an effective model turns out to be true in any model of a given complex system and in particular also in the complex system itself, then this result is called *universal* in physical sense. Thus, the main objective in building an effective random matrix theory is to produce as many universal results as possible, while balancing the complexity of the matrix model and the corresponding complex system.
Random matrices depend only on the underlying probability distribution and symmetries

of the matrix. Therefore, random matrix models can be used to describe spectral properties of complex systems dominated by symmetries. To study spectral properties the Hamiltonian is replaced by a suitable random matrix. This is a sensible procedure in many applications, in both physics and mathematics.
RMT started in 1928, when Wishart introduced the notion of rectangular matrices with random entries. Those entries were chosen as independently, identically distributed random variables with respect to a Gaussian probability density [2]. The first connection between RMT and physics was suggested in the 1950s by Wigner, who used eigenvalue statistics of random matrices to understand the energy levels of heavy nuclei [3]. In particular, Wigner proposed that the spacings of energy levels could be modeled by eigenvalue spacings of large random matrices [4]. This idea had profound implications, because it was a vast generalization of concepts already known and used in Statistical Mechanics. Since the Hamiltonian of a many-body system, studied in Statistical Mechanics, has many degrees of freedom, we might as well view the entries of the Hamiltonian, in its finite-dimensional matrix representation, as random numbers. This immediately leads to a random matrix model describing the Hamiltonian.
The choice of distribution between entries of a random matrix impacts the spectral statistics and has to be chosen appropriately for the given physical context. In general, the random matrix model is a good approximation of the complex physical system, as long as the global symmetries of the random matrix match the symmetries of the Hamiltonian, which is a Hermitian operator. In other words: The random matrix model and the complex physical system belong to the same symmetry class. Therefore, the objective is to find a suitable random matrix model whose global symmetries match those of the physical system. Consequently, random matrix models are classified in terms of their symmetries. The first symmetry that is used in the classification is self-adjointness, or Hermiticity, because physical systems display real eigenvalues originating from Hermitian Hamilton operators. The random matrix ensemble, which shares this symmetry and has no other symmetry, is called *Gaussian Unitary Ensemble*. It is one of three classical Gaussian ensembles introduced by Dyson [5–7]. It is abbreviated as $\mathrm{GUE}(N)$ and contains all Hermitian $N \times N$ matrices H with entries drawn from a Gaussian probability distribution. The underlying matrix space is $\mathbb{C}^{N\times N}$ meaning each entry of H is a complex number $z = a + ib$. This leads to a conotation via the *Dyson index* $\beta = 2$, counting the degrees of freedom a, b. Overall we find

$$\frac{1}{\mathcal{N}} \exp\left[-\frac{1}{2}\mathrm{Tr}\, H^2\right] dH, \quad \text{where} \quad dH = \prod_{1\le i<j\le N} d\mathrm{Re}\,(H_{ij}) d\mathrm{Im}\,(H_{ij}) \prod_{k=1}^{N} dH_{kk}\,,$$

with normalization constant $\mathcal{N}$. The measure dH is the flat Lebesgue measure of all independent entries of the underlying space $\mathbb{C}^{N\times N}$. Changing the field $\mathbb{C}$ to $\mathbb{R}$ leads to the $\mathrm{GOE}(N)$,[1] meaning that the matrices H are not Hermitian, but real symmetric, which implies that the number of independent parameters of the underlying field is $\beta = 1$. In the language of symmetries, the change from complex to real space hints at the existence of an additional symmetry. This symmetry is anti-unitary and coincides with the symmetry of complex conjugation or charge conjugation. In physical settings this translates to the combination of time-reversal and rotational invariance. Breaking the rotational invariance, but at the same time keeping the time-reversal invariance is also possible. Thus, the Hamilton operator becomes self-dual Hermitian with entries taken from the real quaternion space $\mathbb{H}^{N\times N}$. Therefore, we find the last of the three classical ensembles associated with $\beta = 4$

[1] GOE is the shorthand notation for Gaussian Orthogonal Ensemble.

free parameters. The corresponding random matrix ensemble is denoted as GSE(N).[2]
The random matrix models we consider in this thesis are based on one of the chiral analogs of the three Dyson ensembles, namely the chiral analog of the GUE(N). The notion of *chirality* implies the existence of an additional symmetry. It turns out that chiral ensembles feature a special block form

$$H = \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} \tag{1.1}$$

with blocks W of dimension $N \times (N + \nu)$. Note: The fact that the matrix H is Hermitian, does not imply the Hermiticity of the block matrix W. Nevertheless, the chiral ensembles are classified as Hermitian random matrix ensembles on the level of H. The three chiral ensembles are denoted by chGUE(N),[3] chGOE(N) and chGSE(N) in analogy to the classical Dyson ensembles.
We consider the blocks of the random matrix H in Eq. (1.1) to be composed of independent random entries distributed with respect to a Gaussian probability measure. To study the eigenvalue spectrum of H we may instead study the matrix product $WW^\dagger$, which is called a *Wishart matrix*. To study the spectrum of $WW^\dagger$ instead of H is possible, because the spectral statistics of H are obtained via the exponential term $\mathrm{Tr}\, H^2$, which is equal to $\mathrm{Tr}\, WW^\dagger$ in case of H being chiral. Furthermore, ν counts the number of zero eigenvalues of $W^\dagger W$, which shares its non-zero spectrum with the Wishart matrix $WW^\dagger$. Consequently, the eigenvalues w_j of $WW^\dagger$ are connected to the eigenvalues of H, which we denote by h_j ($j = 1, \ldots, N$). The connection reads $h_j = \pm\sqrt{w_j}$ and shows a two-fold degeneracy. Following this, the probability measure describing the chGUE(N) reads

$$\frac{1}{\mathcal{N}_0} \exp\left[-\frac{1}{2}\mathrm{Tr}\, H^2\right] dH = \frac{1}{\mathcal{N}_0} \exp\left[-\mathrm{Tr}\, WW^\dagger\right] dW \ . \tag{1.2}$$

All three chiral ensembles form *Hermitian* random matrix models and were introduced as models for QCD by Verbaarschot [8, 9]. In this thesis we also consider the chGUE(N) model extended by addition of a matrix T to the block matrix W, i.e.

$$H = \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} \to \begin{pmatrix} 0 & W + T \\ W^\dagger + T^\dagger & 0 \end{pmatrix}. \tag{1.3}$$

This type of matrix model is used to study the effect of temperature in QCD. Note that the matrix T can be chosen diagonal with real entries [10–13]. Another type of extension arises from a deformation with $i\mu$ instead of T, which corresponds to the effect of non-zero chemical potential in QCD. This deformation breaks the Hermiticity of H, leading to a non-Hermitian random matrix model, in contrast to the deformation with T in Eq. (1.3).
In general, we distinguish between two classes of random matrix models - Hermitian and non-Hermitian models. Both classes appear in physics, mathematics and many other related areas of science. Possible applications encompass areas such as: physics [14], medicine [15], sociology [16, 17], engineering [18], finance [19] and pure mathematics [20, 21]. These references are overviews and provide important further references to respective applications. To give a more detailed overview on all the different areas, where RMT models appear, is beyond the scope of this thesis. Hence, we also refer to collections such as [22] for more informations. Because of the variety of applications, the study of random matrix ensembles

[2] GSE is the shorthand notation for Gaussian Symplectic Ensemble.
[3] The chGUE(N) ensemble is sometimes referred to as Wishart ensemble or Laguerre Unitary ensemble.

has become a research topic of its own. In particular, the study of eigenvalue and eigenvector correlations of random matrices [23–26], but also the already mentioned symmetry classifications [5, 27, 28] of random matrix models have drawn a lot of attention over the years. In particular, the symmetry classification of physical operators in strong interacting systems was a popular research topic [29, 30]. In recent years the study of sums and products [31] of random matrices has also attracted a lot of attention.
As mentioned before, we focus on the application of random matrix models to the theory of strong interactions. The connection between these two fields is well established, as we will see in the next subsection in more detail. See also [32] for a review and references therein for more information. The results and techniques we discuss in this thesis may be useful in adjacent applications of random matrix models and therefore are of its own interest.

1.2 Application to QCD

Quantum chromodynamics (QCD) is a field theory designed to describe the effects of the strong force in the Standard Model of Elementary Particles. The Standard Model is considered as one of the biggest achievements in modern particle physics, because of its predictions and their confirmation by experiments. The model combines three of the four fundamental forces of nature, namely the strong force, the electromagnetic force and the weak force. In this thesis we focus solely on the strong force and its effects described by the quantum field theory called QCD. The strong interaction has an effect on elementary particles called *quarks*, of which six *flavors* are established by experimental data. Those quarks are: up, down, strange, charm, top and bottom. In theoretical models one often keeps the number of quarks as a free parameter, which is then denoted by N_f - the number of flavors. We will adopt this notation throughout this thesis to cover models of QCD with different numbers of quarks, for example $N_f = 2$ and $N_f = 3$.
Quarks are fermions and described in QCD by quantum fields ψ_f.[4] The interaction between quarks is carried out by exchanging gauge bosons, which are called *gluons*. The gauge group containing the gluons is an $\mathrm{SU}(3)$ Lie group, leading to 8 gluons and three colors: *red*, *blue* and *green*. To incorporate models for different number of colors, we keep the number of colors as a free parameter, denoted by N_c. The gluons are described by bosonic fields A_μ, which are vector fields of spin 1, similar to quantum electrodynamics. Furthermore, the quark field ψ_f is a vector N_c components representing the effect of each color. Henceforth, we will not consider the structural impact of the color group in more detail.
One of the key features of QCD is *asymptotic freedom*, which describes the behaviour of quarks at high energies, or temperatures. Above a critical energy quarks and gluons become free particles and form a quark-gluon plasma (QGP). Both lattice data and experimental data of collider experiments show that at high temperatures a QGP phase exists [33, 34]. This hints at a more structured phase diagram. At vanishing chemical potential the phase transition to the QGP phase is in fact an analytic crossover [35, 36], which is conjectured to be extendable to small non-zero chemical potential.
Beyond the existence of the QGP phase, little is known about the shape and structure of the phase diagram of QCD, particularly at the physical point - the point in phase space, where the masses of the quarks in nature are realized. Therefore, the understanding of the phase diagram and in particular the phase transition from low to high temperatures is still of interest. In Fig. 1.1, a schematical depiction of the phase diagram of QCD is given as a function of temperature T and baryon chemical potential μ. The order parameter for the

[4] The quark field index f denotes the flavor.

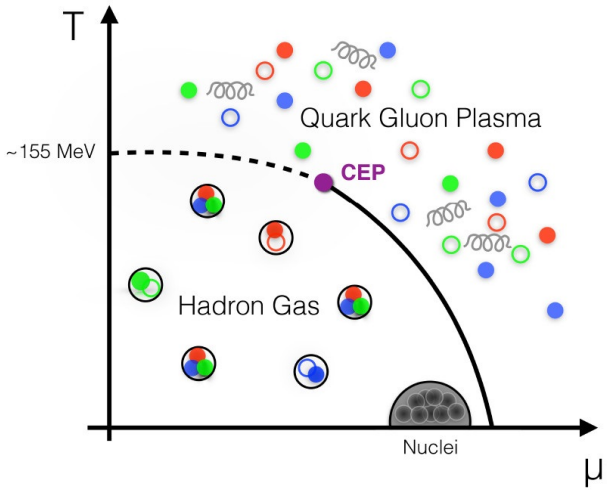

Figure 1.1: Schematic diagram of the $T - \mu$ phase space of QCD taken from [37]. Three phases are shown: The Quark Gluon Plasma, where quarks and gluons are asymptotically free, the nuclear phase, where nuclear matter is formed and the Hadron Gas phase, where quarks and gluons are confined in hadrons. The solid line represents a phase transition, being of first order up to the critical end-point (CEP). The dashed line represents an analytic crossover at a critical temperature T_C up to the CEP.

transition from low to high temperature is the chiral condensate Σ, which is zero in the QGP phase and non-zero in the hadron phase. At $\Sigma = 0$ asymptotic freedom allows for perturbative expansion of the Lagrangian of QCD in powers of the coupling constant g_{QCD}. In the hadronic phase, quarks and gluons appear together in colorless particles called either baryons, like proton and neutron, or mesons, like the pions $\pi^{\pm}$, π^0. This feature is called *confinement* and - crucially - perturbation theory in terms of quarks and gluons is no longer available. In QCD at low baryon density μ symmetry analysis of quarks and gluons leads to the concept of chiral symmetry breaking. At high temperatures the chiral symmetry is restored, while at low temperatures the chiral quark condensate, in the limit of massless quarks, breaks the chiral symmetry of the Dirac operator spontaneously. The Goldstone Theorem [38] leads to the existence of Goldstone Bosons. At $N_f = 2$ the quark mass breaks the chiral symmetry explicitly, but even for small quark masses one can still see the effects of the spontaneous symmetry breaking. In particular, we can identify the lightest mesons (pions) with the Goldstone Bosons and therefore use these particles as building blocks for effective field models. For $\Sigma \neq 0$ one possibility to study QCD is to discretize space-time in a finite volume $V = L^4$. This leads to what is called *lattice QCD*, which depends on lattice spacing a, lattice shape and lattice boundary conditions. The advantage of lattice QCD is that numerical simulations are possible to gain access to the QGP and phase transition regions of the phase diagram of QCD. However, the necessary parametrization of space-time requires a Euclidean metric space, which is achieved via Wick rotation of the time direction circumventing the *sign problem* in the original Minkowski space. For more details on lattice QCD we refer to the literature [39–44]. To compare lattice simulations to experimental data we have to send the lattice spacing $a \to 0$. This means we take the *continuum limit* first and afterwards the *thermodynamical limit* by sending $V \to \infty$.
In this thesis we do not consider lattice QCD to model the effects described by QCD in the low energy hadronic phase. As the title of the thesis suggests we use effective theories to model QCD in the low-energy regime. We consider models with and without temperature

quark flavor	quark mass (in MeV)	quark flavor	quark mass (in GeV)
u	4	c	1,3
d	8	b	4,4
s	160	t	175

Table 1.1: The bare masses of the 6 known quarks are shown, divided in groups relative to the QCD scale $\Lambda_{\mathrm{QCD}} \sim 1$ GeV [45].

to better understand the effects of temperature in the phase diagram. In general, an effective field theory uses the lightest particles on a given energy scale as building blocks to perform perturbation theory. One example of an effective theory for QCD is *Chiral Perturbation Theory* (chPT). This theory uses the Goldstone Bosons of chiral symmetry breaking as building blocks. These bosons can be identified with the lightest mesons of the theory - the pions. Consequently, the perturbation is done in terms of the pion masses and poin momenta.
A simple way to categorize the three methods - perturbative QCD, lattice QCD and low-energy effective field theory - can be done with respect to the physical quark masses, which are given in Tab. 1.1. The quarks can be grouped into light quarks (u, d, s) and heavy quarks (c, b, t) according to their mass. First principle calculations of QCD lead to perturbative QCD, which is only applicable above several GeV, courtesy of the asymptotic freedom. In contrast, a low-energy effective field theory can only be applied well below the QCD scale $\Lambda_{\mathrm{QCD}} \sim 1$ GeV. Thus, only theories containing 2 or 3 flavors make sense. Especially the $N_f = 2$ flavor theories are interesting, as the two corresponding quarks build up the pions $\pi^{\pm}$, π^0 and lead to chiral symmetry and their explicit and spontaneous breaking by the pion mass and the chiral condensate respectively. Lattice QCD serves as a tool to understand the region between the two perturbative schemes. In particular, the crossover along the temperature axis in the phase diagram at non-zero chemical potential has been tested with lattice simulations - see [46] for more information.
Starting from chPT one can take another step to arrive at random matrix models which can be used to describe chPT and thus QCD indirectly. It is clear, that such an approach can only serve as a description of certain aspects of QCD and never describe the whole field theory. The RMT models we introduce in chapter 2 of this thesis describe certain spectral properties of the QCD Dirac operator D, in particular they predict the spectral behaviour of the smallest eigenvalues of D.
In summary, QCD is a non-Abelian Yang-Mills gauge theory [47] of fermionic quark fields and gauge bosons (gluons) who all carry a color charge. A Quantum Field Theory (QFT), such as QCD, is characterised by the action, which is an integral of the Lagrangian over the 4-dimensional space-time light cone. We focus solely on the Euclidean formulation of QCD, as this is needed for both the comparison to lattice QCD data and in order to match with the random matrix models we introduce in chapter 2. The action of QCD reads

$$S_{\mathrm{QCD}} = \int \mathrm{d}^4x \, \mathcal{L}_{\mathrm{QCD}}(x), \quad \text{with} \quad \mathcal{L}_{\mathrm{QCD}} = \mathcal{L}_{\mathrm{gluon}} + \mathcal{L}_{\mathrm{quark}} \, .$$

The Lagrangian $\mathcal{L}_{\mathrm{QCD}}$ describing the interaction of quarks and gluons reads

$$\mathcal{L}_{\mathrm{QCD}} = \frac{1}{2}\mathrm{Tr}\,\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right] + \sum_{f=1}^{N_f} \bar{\psi}_f(x)(D + m_f)\psi_f(x) = \mathcal{L}_{\mathrm{gluon}} + \mathcal{L}_{\mathrm{quark}} \, . \tag{1.4}$$

$F_{\mu\nu}$ is the field strength tensor on SU (N_c) and $\bar{\psi}_f = \psi_f^\dagger \gamma_4$ is the Dirac conjugate field. The dynamics of the fermionic fields ψ_f are described by the Dirac operator D, while the gluons are described by fields A_μ, which also appear as part of the Dirac operator representation. Thus, the Dirac operator can be written as

$$D = \sum_{\mu=1}^{4} \gamma_\mu D_\mu, \quad \text{where } D_\mu = \partial_\mu + i g_{\text{QCD}} A_\mu \,, \tag{1.5}$$

with g_{QCD} the coupling constant of QCD. The fact that QCD is a field theory allows us to apply Feynman's path integral formalism [48] and make the connection to Statistical Mechanics. In particular, we describe the theory via partition functions, who contain all necessary informations of the considered field theory. The Euclidean partition function of QCD reads

$$Z_{\text{QCD}} = \text{Tr}\, \left[e^{-\beta H}\right] = \int \mathrm{d}A_\mu \prod_{f=1}^{N_f} \det(D + m_f)\, \exp\left(-S_{\text{YM}}\right) \,, \tag{1.6}$$

where β is the inverse temperature and S_{YM} is the Euclidean Yang-Mills action, which is basically the gluonic part of the Lagrangian of QCD in Eq. (1.4). The N_f-fold determinant over the Dirac operator originates from the Berezin integral over Grassmann variables [49] using the Matthews-Salam formula [50, 51]

$$\prod_{f=1}^{N_f} \det(D + m_f \mathbb{1}) = \int \mathrm{d}\bar{\psi}\mathrm{d}\psi\, \exp\left(-\int \mathrm{d}^4x\, \mathcal{L}_{\text{quark}}(x)\right) \,.$$

This explains the connection of Eq. (1.6) and the QCD Lagrangian in Eq. (1.4). The QCD partition function can also be interpreted as an average (or expectation value) over the Yang-Mills action of the fermion determinant. Depending on the notation we can write[5]

$$Z_{\text{QCD}} = \left\langle \prod_{f=1}^{N_f} \det(D + m_f \mathbb{1}) \right\rangle = \mathbb{E}\left[\prod_{f=1}^{N_f} \det(D + m_f \mathbb{1})\right] \,.$$

Note that the fermion determinant $\det(D + m_f \mathbb{1})$ is mathematically nothing more than the characteristic polynomial with respect to the Dirac operator D. This hints at the connection of characteristic polynomials and their expecation values to the spectral statistics of QCD. We will encounter more characteristic polynomials of random matrices in the models for QCD in chapters 3, 4 and 5.
In a fashion similar to the partition function one can also define the spectral density of the Dirac operator as an expectation value with respect to the Yang-Mills action, namely

$$\rho(\lambda) = \left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle = \mathbb{E}\left[\sum_n \delta(\lambda - \lambda_n)\right] \,.$$

[5]The bracket notation $\langle \mathcal{O} \rangle$ is often used in physics literature, while the notation $\mathbb{E}(\mathcal{O})$ is used more often in mathematical literature.

The spectral density is an important quantity in terms of universality and connected to chiral symmetry breaking, which can be seen in the Banks-Casher relation [52]

$$\rho_D(\lambda \approx 0) = \frac{1}{\pi} V\Sigma \,, \tag{1.7}$$

where V is the space-time volume. We can define the chiral condensate Σ as the expectation value of the quark vacuum ground state, i.e.

$$\Sigma = |\langle 0|\bar{\psi}\psi|0\rangle| \neq 0 \,.$$

Thus, the quark vaccum ground state breaks chiral symmetry spontaneously. More precisely, we can write [30]

$$\Sigma = |\langle 0|\bar{\psi}\psi|0\rangle| = \lim_{a\to 0}\lim_{m\to 0}\lim_{V\to\infty} \frac{\pi}{V}\int \frac{2m\rho(\lambda)\mathrm{d}\lambda}{m^2+\lambda^2} \,. \tag{1.8}$$

The order of the limits is crucial to provide a correct ultra-violett cut-off on the lattice or in the effective theory. The smallest eigenvalues of the Dirac operator make up a large portion of the chiral condensate, which we can deduce from the Banks-Casher relation, as the average distance of eigenvalues is directly related to the volume V, see Eq. (1.7). Thus, we can say the following: The global symmetries are manifest in the lowest eigenvalues of the Dirac operator in a finite volume $V = L^4$, low energy effective theories apply and the spectral gap is closed. ChPT applies, if $1/\Lambda_{\mathrm{QCD}} \ll L$, where Λ_{QCD} is the typical QCD scale and L is the size of the box making up the finite volume V. In chPT the expansion in pion fields with masses m_π and quantised momenta $p \propto 1/L$ leads to

$$\frac{1}{L} \sim \mathcal{O}(p), \quad m_\pi \sim \mathcal{O}(p), \quad m_f \sim \mathcal{O}(p^2) \,,$$

where the quark mass m_f follows from the Gell-Mann-Oakes-Renner relation $m_\pi^2 \sim m_f$. The product $m_\pi L$, pion mass and size of the box, is an indicator for the behaviour of zero and non-zero modes. Looking at the pion propagator

$$\Delta(p^2) = \frac{1}{V}\frac{1}{p^2+m_\pi^2}$$

we see that for $m_\pi L \sim \mathcal{O}(1)$ zero and non-zero models share a $1/L^2$ vanishing pion propagator. This regime is called p-regime, because the expansion is basically done by counting the number of derivatives (or momenta) in the chiral perturbation $\partial \sim p$ [53]. The modes in the p-regime share a Compton wave-length that is smaller than the size of the box.
There is another kinematic regime due to Gasser and Leutwyler [54] called ϵ-regime, where the partition function describing the system splits into a zero-momentum and a non-zero-momentum part. For $m_\pi L \ll \mathcal{O}(1)$ only the non-zero mode propagator vanishes - the zero modes produce a divergence in the chiral limit. Introducing a new parameter $\varepsilon \sim 1/L$ and expanding around the zero momentum part of the Lagrangian, we find [32, 55, 56]

$$m_\pi \sim \mathcal{O}(\varepsilon^2), \quad m_f \sim \mathcal{O}(\varepsilon^4), \quad V \sim \mathcal{O}(\varepsilon^{-4}) \,.$$

The term $m_\pi^2 V$ is of order one in this counting regime and the zero modes are no longer suppressed [56]. However, the wavelength of the pions must be much larger than the box length size of the finite volume enclosing the physical system.

The zero-momentum part of the theory is fully described by its global symmetries and can be modeled by RMT with the same symmetries [8]. To reach the ε-regime a rescaling scheme of eigenvalues and masses and also the spectral density is needed [9]:

$$\hat{\lambda}_n = \Sigma V \lambda_n, \quad \hat{m}_f = \Sigma V m_f, \quad \rho_{\text{micro}}(\hat{\lambda}) = \lim_{V\to\infty} \frac{1}{\Sigma V}\, \rho\left(\frac{\hat{\lambda}}{\Sigma V}\right) . \tag{1.9}$$

This is called the *microscopic limit* of QCD. The results we derive from our RMT models have to be compared to physical quantities taken in this limit. Consequently, we can only use results from RMT models directly as predictions for QCD in the ϵ-regime of chPT after taking the microscopic limit. In particular, the microscopic spectral density can be computed analytically using random matrix models as well as finite volume partition functions [57–60], when temperature is zero. To compare results of RMT with lattice data it is necessary to introduce the *quenched* approximation, which is achieved by taking the limit $N_f \to 0$. At $N_f = 0$ the result for the microscopic spectral density derived from RMT models reads [32]

$$\rho_{\text{micro}}(\hat{\lambda}) = \frac{\hat{\lambda}}{2}\left(J_\nu^2(\hat{\lambda}) - J_{\nu-1}(\hat{\lambda}) J_{\nu+1}(\hat{\lambda})\right), \tag{1.10}$$

where $J(\lambda)$ denotes the standard Bessel function of the first kind. The microscopic spectral density is plotted in Fig. 1.2 in comparison to lattice data of staggered fermions in the fundamental representation of SU(3) in the quenched approximation. In order to identify

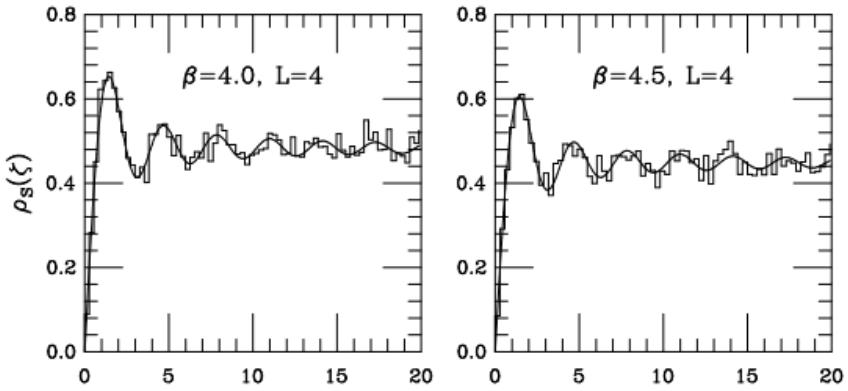

Figure 1.2: The microscopic spectral density ρ_S from Eq. (1.10) is compared to data of staggered fermions in the fundamental representation of SU(3). This was done for different lattice sizes in the quenched approximation in [61]. Similar comparisons for the smallest eigenvalues can be found in [61, 62].

a suitable RMT model we have to study the symmetries of the Dirac operator in a given gauge representation. Starting from Eq. (1.5) we see that the Euclidean 4×4 dimensional γ-matrices fullfill the Clifford algebra relation

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu} .$$

Additionally, the γ-matrices can be represented in their chiral representation, meaning they feature off-block structure, i.e. for $k = 1, 2, 3$ and Pauli matrices σ_k we have

$$\gamma_k = \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix}, \; \gamma_4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} .$$

Analyzing the global symmetries of the Dirac operator leads to its chiral representation, since the Dirac operator is anti-Hermitian $D^\dagger = -D$ and anticommutes with the matrix $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, i.e. $\{D, \gamma_5\} = 0$. For γ-matrices in their chiral representation this leads to

$$D = i \begin{pmatrix} 0 & d \\ d^\dagger & 0 \end{pmatrix} . \tag{1.11}$$

The off-block entry d is a differential operator and depends on the gauge fields A_μ and space-time. The commutation relation of γ_5 with the Dirac operator implies that all non-zero eigenvalues of D come in pairs and are purely imaginary.
Now we come back to the chiral symmetry breaking. From the construction of the γ_5-matrix we find the projectors

$$P_\pm = \frac{1}{2}(\mathbb{1}_4 \pm \gamma_5), \quad P_\pm^2 = P_\pm, \quad P_+P_- = P_-P_+ = 0, \quad P_+ + P_- = \mathbb{1}_4 .$$

We can use these to decompose the quark fields into left-handed (L) and right-handed (R) components, $\psi_{f,\mathrm{L/R}} = P_\pm \psi_f$, with respect to their chirality. The quark part of the QCD Lagrangian can be decomposed term by term and we obtain

$$\sum_{f=1}^{N_f} \bar\psi_f (D + m_f \mathbb{1}_4)\psi_f = \sum_{f=1}^{N_f} \bar\psi_{f,\mathrm{L}} D \psi_{f,\mathrm{L}} + \sum_{f=1}^{N_f} \bar\psi_{f,\mathrm{R}} D \psi_{f,\mathrm{R}} + \sum_{f=1}^{N_f} m_f (\bar\psi_{f,\mathrm{L}}\psi_{f,\mathrm{R}} + \bar\psi_{f,\mathrm{R}}\psi_{f,\mathrm{L}}) . \tag{1.12}$$

The terms above are invariant under different symmetry transformations. We see that the massless term is invariant under global rotations $\psi_{f,\mathrm{L}} = U_L \psi_{f,\mathrm{L}}$ and $\psi_{f,\mathrm{R}} = U_R \psi_{f,\mathrm{R}}$. The $U_{L/R} \in \mathrm{U}_{L/R}(N_f)$ are unitary matrices from groups with indices corresponding to different chirality. Hence, this term is invariant under

$$\mathrm{U}_L(N_f) \times \mathrm{U}_R(N_f) = \mathrm{SU}_L(N_f) \times \mathrm{SU}_R(N_f) \times \mathrm{U}_V(1) \times \mathrm{U}_A(1) .$$

The two phases $\mathrm{U}_V(1)$ and $\mathrm{U}_A(1)$ stand for additional conserved vector (V) and axial-vector (A) currents. The first of the two symmetries can be identified with the conservation of the baryon number. The $\mathrm{U}_A(1)$ symmetry is always broken by the anomaly.
The mass term violates the separation in left-handed and right-handed fields, as we can see in Eq. (1.12). In the case that all quark masses are taken to be equal, i.e. $m_f = m$ for all $f = 1, \ldots, N_f$, the mass term is invariant under $U_L = U_R$, which means, that one of the $\mathrm{SU}(N_f)$ groups remains. Hence, the breaking pattern with respect to the mass term reads:

$$\mathrm{SU}_L(N_f) \times \mathrm{SU}_R(N_f) \to \mathrm{SU}(N_f) .$$

This breaking pattern also applies, if the explicit breaking via the mass term is replaced by spontaneous breaking via the condensate, since

$$\Sigma = |\langle 0|\bar\psi\psi|0\rangle| = |\langle 0|\bar\psi_\mathrm{L}\psi_\mathrm{R} + \bar\psi_\mathrm{R}\psi_\mathrm{L}|0\rangle| \neq 0$$

is again invariant under $\mathrm{SU}(N_f)$, if $U_L = U_R$. The explicit breaking of chiral symmetry can be avoided in the already mentioned *quenched* approximation, which is realized when we send the number of flavors to zero, i.e. $N_f \to 0$. Physically, this means, we assume, that the quarks are heavy enough such that the mass term dominates the dynamics described by the Dirac operator. In that setting only the effects of the pure gauge fields on the Dirac spectrum are affecting the spectral statistics, which is of its own interest.

Now we are ready to move on to the random matrix models, which were introduced to describe the spectrum of the Dirac operator. Looking at Eq. (1.11) and comparing to Eq. (1.1) leads immediately to the suggestion of Shuryak and Verbaarschot from 1993 [8]. They proposed to replace the differential operators d, which build the off-blocks in the Dirac operator represenation in Eq. (1.11), with random matrices W. The number of eigenvalues of the RMT version of the Dirac operator is then denoted by $2N + \nu$ and should be proportional to the space-time volume V. The replacement leads to a block W of size $N \times (N + \nu)$, where ν denotes the zero-modes of the Dirac operator. All in all we have

$$D + m_f \mathbb{1} = \begin{pmatrix} 0 & id \\ id^\dagger & 0 \end{pmatrix} + m_f \mathbb{1} \to \begin{pmatrix} 0_N & iW \\ iW^\dagger & 0_{N+\nu} \end{pmatrix} + m_f \mathbb{1}_{2N+\nu} = D + m_f \mathbb{1}_{2N+\nu} \,, \tag{1.13}$$

where we have included the mass m_f corresponding to the quark flavor f. The matrix-valued RMT Dirac operator has $2N$ non-zero eigenvalues lying on $i\mathbb{R}$, and ν zero eigenvalues. The Yang-Mills action is replaced by a Gaussian probability distribution $P(W)$. In particular, Verbaarschot introduced the term $\mathrm{Tr}\, WW^\dagger$ inside an exponential in $P(W)$. Crucially, the symmetries of the Dirac operator blocks d are conserved in the symmetries of W. This includes possible anti-unitary symmetries the Dirac operator might possess depending on the gauge representation. The classification of possible anti-unitary symmetries can be found in the literature [27, 28]. In short, the matrix elements of W can become real, complex or real quaternion.[6] The three random matrix ensembles are exactly the three chiral Gaussian ensembles chGOE(N) (real), chGUE(N) (complex) and chGSE(N) (quaternion). These three cases correspond to special combinations of representations of the color gauge group and the number of colors N_c. The case of $N_c = 3$ colors and gauge group elements in the fundamental representation corresponds to the matrix model with complex entries, i.e. the chGUE(N). Therefore, the partition function coming from the chGUE(N) density, but in the presence of N_f massive, dynamical quarks, reads (compare Eq. (1.2))

$$Z_N = Z_N^{(N_f,\nu)}(m_1, \ldots, m_{N_f}) = \frac{1}{\mathcal{N}} \int dW \prod_{f=1}^{N_f} \det(D + m_f \mathbb{1}_{2N+\nu}) \exp\left[-\mathrm{Tr}\, WW^\dagger\right] \,, \tag{1.14}$$

where D is the matrix-valued Dirac operator from Eq. (1.13) and $\mathcal{N}$ is a constant we use to simplify the joint probability density function of the eigenvalues in later chapters.
As we have seen in the previous section the eigenvalues of the Wishart matrix $WW^\dagger$ are connected to the eigenvalues of the Dirac operator D via $\pm\sqrt{w_j} = y_j$,[7] where y_j are the eigenvalues of D and w_j the eigenvalues of $WW^\dagger$ respectively. To obtain results in the RMT model we have to access the spectrum of the random matrix model at the correct scaling. We will go into more detail on different scaling limits in chapter 2. For now it suffices to note that after the eigenvalues are appropriately rescaled with N, we can take the large N-limit and obtain results comparable to QCD in the microscopic limit.
To include temperature in the random matrix model a simple shift by a constant, deterministic matrix T was considered by several groups, for example [12, 13]. The change of the

[6] In arbitrary space-time dimensions one could expect all ten Hermitian matrix models [27, 28]. Since we are focused on 4 space-time dimensions, only the three chiral ensembles are possible [5].

[7] The eigenvalues of H are also called the squared singular values of W, when H has the chiral block-form as in Eq. (1.1).

model, as it was indicated already in Eq. (1.3), leads to a new Dirac operator

$$D = i \begin{pmatrix} 0 & W+T \\ W^\dagger + T^\dagger & 0 \end{pmatrix},$$

which replaces the Dirac operator in Eq. (1.14).
Results using the temperature dependent RMT models date back to the 90s. In particular, the works [12] and [13] studied the model using supersymmetric techniques and derived correlation functions in the large N limit, which can be compared to the microscopic limit of QCD in the quenched approximation [12].[8] Both works considered only quadratic matrices W, meaning $\nu = 0$. At the same time results for the correlation functions of zero-temperature models were derived [63, 64], which opens to the question of universality of the correlation functions with respect to temperature. The equivalence of results of both temperature phases, $T = 0$ and $T < T_C$, could not be answered with the techniques and knowledge available at the time. Since then the understanding of the underlying structures, even at finite N, has grown thanks to works such as [25, 65, 66]. Hence, the models can now be reviewed, also for $\nu \neq 0$, and the connection of the results from $T \neq 0$ [12, 13] to $T = 0$ [63, 64] can be seen in a new light, answering the question of universality. This is covered in chapters 4 and 5 of this thesis.

1.3 Outline

The thesis is organized as follows:

In chapter 2 the mathematical framework to study random matrix models of the chGUE(N) symmetry class is introduced. In particular, the focus is set on random matrix ensembles with biorthogonal structures leading to determinantal point processes and their correlation kernels. Expectation values of characteristic polynomials will be linked to the computation of correlation kernels. Scaling limits as well as questions universality will also be discussed.
In chapter 3 we derive results valid for polynomial ensembles at finite matrix size N. Formulae for expectation values of ratios of characteristic polynomials are derived leading to determinantal structures, which do not depend on N. The results of chapter 3 are based on the principal publication [1].
In chapter 4 the obtained results from chapter 3 will be used as a starting point of a large N analysis. Via saddle point approximation the large N limit of the correlation kernel is derived for the chGUE(N) with an external source describing temperature for both $N_f = 0$ and $N_f \neq 0$ massive flavors. In chapter 5 the question of universality of the asymptotic results of chapter 4 is discussed. In particular, it is shown that the results for non-zero temperature from chapter 4 are equivalent to previously derived results for zero temperature from [63]. Additionally, the equivalence to results obtained for the non-zero temperature model via supersymmetry [13] is shown. The results of both chapters are still unpublished. We then conclude the thesis in chapter 6.
The appendices contain: Properties of Vandermonde determinants in Appendix A, material concerning determinantal formulae for expectation values of characteristic polynomials in polynomial ensembles in Appendix B, derivation of JPDFs in Appendix C and additional results for Giambelli compatible point processes in Appendix D.

[8] In [13] the extension of [12] to N_f massive flavors was made.

Chapter 2

Mathematical and Physical Framework

In this chapter we introduce concepts and techniques from Random Matrix Theory, which are needed in chapters 3, 4 and 5. In particular, we discuss Hermitian random matrix models and their applications in Quantum chromodynamics and similar structured strong interacting field theories. As a result, we introduce the random matrix models from the chGUE(N) symmetry class via the Altland-Zirnbauer classification of Hermitian random matrix models and highlight the matrix-valued probability distributions we need for the discussions in chapters 4 and 5. Next, we illustrate how eigenvalue and singular value distribution functions can be obtained for the chGUE(N)-type models. In the process, we briefly discuss integraliblity of group integrals and obtain the joint probability density functions for the models with and without external sources in the chGUE(N) symmetry class. Additionally, we introduce properties of orthogonal polynomials and show their connection to expectation values of characteristic polynomials as well as partition functions. Then, we discuss the existence and usage of scaling limits with respect to universality in spectral correlation functions. Lastly, we introduce the notion of determinantal point processes and give an overview of important properties of the associated correlation kernels and correlation functions. Additionally, a discussion of biorthogonal ensembles and their subclasses is given, which leads to the notion of polynomial ensembles - the main subject of chapter 3.

2.1 Random matrix models for QCD

Quantum chromodynamics (QCD), as a highly involved quantum theory of non-Abelian fermionic fields and gauge fields, requires tools from group and representation theory, as well as field theory to tackle its spectral properties.
In this thesis we focus on the fundamental representation with $N_c \geq 3$ colors, so the corresponding random matrix model is the *chiral Gaussian Unitary Ensemble*.[1] The main idea to model QCD with random matrices, which are drawn from the chGUE(N), is to replace the Dirac operator D with a random matrix, whose off-diagonal blocks are rectangular $N \times (N + \nu)$ dimensional matrices W with independent Gaussian entries. The spectrum of the Dirac operator can be obtained from the non-zero eigenvalues of the Wishart matrix $WW^\dagger$, or equivalently from the squared singular values of W. In order to derive the spectral statistics of eigenvalues we first have to formulate the partition function describing the model on random matrix level and identify the joint probability density function of the eigenvalues (or singular values).

[1] We use the short-hand notation chGUE(N) to denote the chiral Gaussian Unitary Ensemble, which is sometimes referred to as complex Wishart Laguerre ensemble.

Random matrix models from the chGUE symmetry class

The classical chGUE(N) is defined on the space of $N \times (N+\nu)$ complex non-Hermitian matrices W with independent, identically distributed, Gaussian entries. A chiral, anti-Hermitian matrix D is obtained from W as a $2N+\nu$ dimensional matrix of the form

$$D = i \begin{pmatrix} 0_N & W \\ W^\dagger & 0_{N+\nu} \end{pmatrix}. \tag{2.1}$$

The partition function of the chGUE(N) is given as the integral over the measure $P_{\text{chGUE}}(W)dW$ and denoted by Z_N. The measure is the product of the flat Lebesgue measure dW, on the space $\mathbb{C}^{N\times(N+\nu)}$, and the density function $P_{\text{chGUE}}(W)$. Z_N serves as an analogon to the physical partition function, which is modeled by the random matrix H. In the context of QCD the partition function Z_N is related to finite-volume partition functions of the physical Dirac operator in the microscopic limit (see section 2.4). We can write

$$Z_N^{\text{chGUE}} = \int P_{\text{chGUE}}(W)dW\,, \quad \text{where} \quad P_{\text{chGUE}}(W) = \frac{1}{\mathcal{N}_0} \exp\left(-\text{Tr}\, WW^\dagger\right). \tag{2.2}$$

The constant $\mathcal{N}_0$ serves as a normalization constant to be determined later.
The derivation of the joint probability density function of the real, positive eigenvalues of the Wishart matrix product $WW^\dagger$ is done using a singular value decomposition of the matrix W. We demonstrate how to use this concept in the next subsection. At this point it is necessary to mention important extensions and generalizations of the chGUE(N). In particular, we have to make a connection to the Dirac operator with N_f massive quark flavors and include the effect of temperature on the spectral statistics in the model. The latter can be done by substracting a rectangular, deterministic matrix T from the matrix W. In this case the ensemble is called chGUE(N) with an external source. Comparing with Eq. (1.14) the density function reads, when $m_f \to 0$ for all $f = 1, \ldots, N_f$:

$$P_{\text{chGUE}}^{\text{ext}}(W) = \frac{1}{\mathcal{N}_{\text{ext}}} \exp\left(-\text{Tr}\,(W-T)(W^\dagger - T^\dagger)\right). \tag{2.3}$$

The study of random matrix ensembles with external sources goes back to the works of Brezin and Hikami [67, 68] and works by Guhr [69–71]. We are interested in the k-point correlation functions of the eigenvalues of $WW^\dagger$. Crucially, these functions can be studied and obtained analytically in the external source case. This follows from the existence of a special type of group integral, as we will see in more detail in the next subsection.
Another important type of model in the chGUE(N) symmetry class is given by the classical chGUE(N) enhanced by a deformation with a fixed number of determinants. In our case we conveniently choose the number of deforming determinants to be equal to the number of physical flavors N_f. Recalling Eq. (1.6) this identification connects to the inclusion of dynamical massive quarks via fermion determinants of the Dirac operator. The additional product of N_f determinants leads to a density of the form

$$\begin{aligned} P_{\text{chGUE}}^{\text{deformed}}(W) &= \frac{1}{\mathcal{N}_{\text{def}}} \prod_{f=1}^{N_f} \det(D+m_f)\, \exp\left(-\text{Tr}\, WW^\dagger\right) \\ &= \frac{1}{\mathcal{N}_{\text{def}}} \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f \mathbb{1}_N & iW \\ iW^\dagger & m_f \mathbb{1}_{N+\nu} \end{pmatrix} \exp\left(-\text{Tr}\, WW^\dagger\right). \end{aligned} \tag{2.4}$$

The combination of the deformation of the chGUE(N) with an external source leads to the main model we want to study in the following chapters of this thesis. We denote this model with the upper index *temp* to indicate that it includes temperature, as an external source, but at the same time includes N_f massive flavors. We obtain the density

$$\begin{aligned} P^{\text{temp}}_{\text{chGUE}}(W) &= \frac{1}{\mathcal{N}} \prod_{f=1}^{N_f} \det(D_{\text{temp}} + m_f) \ \exp\left(-\text{Tr}\, WW^\dagger\right) \\ &= \frac{1}{\mathcal{N}} \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f \mathbb{1}_N & i(W+T) \\ i(W^\dagger + T^\dagger) & m_f \mathbb{1}_{N+\nu} \end{pmatrix} \ \exp\left(-\text{Tr}\, WW^\dagger\right), \end{aligned} \tag{2.5}$$

which allows us to write the random matrix parition function as

$$Z^{\text{temp}}_N = \int P^{\text{temp}}_{\text{chGUE}}(W) dW = \frac{1}{\mathcal{N}} \int dW \prod_{f=1}^{N_f} \det(D_{\text{temp}} + m_f \mathbb{1}_{2N+\nu}) \exp\left[-\text{Tr}\, WW^\dagger\right] ,$$

where we have introduced the temperature dependent Dirac operator D_{temp} as

$$D_{\text{temp}} = i \begin{pmatrix} 0 & W+T \\ W^\dagger + T^\dagger & 0 \end{pmatrix} .$$

The expansion of the fermion fields into a sum over Matsubara frequencies is a common technique on the lattice to study the phase transitions in QCD phase space. The model introduced above shares the chiral and flavor structure of the QCD Dirac operator and includes the temperature dependence schematically in analogy to the addition of the lowest Matsubara frequency [10, 72, 73]. The effect of temperature is thus restricted to the lowest eigenvalues of the Dirac operator and conveniently leads to mean-field universal critical exponents. As we have seen before, the smallest eigenvalues dominate in the ε-regime, where RMT models can be applied to QCD. Thus, the correlation functions of these eigenvalues can be used to study the chiral phase transition along the temperature axis in phase space, where the spectrum of the Dirac operator is purely imaginary [10, 72, 73]. The phase transition along the temperature axis, is analytical and of second order with mean field critical exponents [10, 72, 73]. The critical temperature T_C divides the phase space into two domains: Below T_C chiral symmetry is spontaneously broken, and above T_C it is restored. The macroscopic spectral density below T_C is given by a semicircle, which splits into two disjoint semicircles above T_C, which is in agreement with with lattice simulations.
Starting from the matrix density function, the next objective is to derive and discuss the joint probability density function (JPDF) of the eigenvalues of the Dirac operator D_{temp} under the influence of temperature. This objective can be achieved by performing singular value decompositions of the matrices W and T simultaneously and is done in the next section.

2.2 Joint Probability Density Function

The matrix-valued density function of the chGUE(N) encodes the spectral information in terms of all independent entries of the given matrix product $WW^\dagger$, as we can see from Eq. (2.2). From a representation theory point of view the expression of the Wishart matrix $WW^\dagger$ in terms of all independent matrix entries is only one possible choice. In analogy with classic linear algebra, we can perform a change of basis to go to the eigenvalue basis of $WW^\dagger$.

This has the advantage of reducing the number of variables from N^2 independent entries to N eigenvalues. A change of basis leads to a Jacobian, which we have to take into account. In this case the Jacobian is non-trivial and leads to non-trivial involvement among the eigenvalues of $WW^\dagger$. From a physical standpoint the invariance of the spectrum under basis change is required, because Hamiltonians and all other possible observables should not depend on a particular choice of basis.

The expression $\det(y\mathbb{1}_{2N+\nu} - D)$ can be used to determine the eigenvalues of the Dirac operator exploiting its chiral block form, see Eq. (2.1). We need to find the zeros of this polynomial equation, which becomes possible via the block determinant formula

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det(D - CA^{-1}B), \quad \text{if } A \text{ is invertible,}$$
$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det D \det(A - BD^{-1}C), \quad \text{if } D \text{ is invertible.}$$

The matrix $y\mathbb{1}_{2N+\nu} - D$ can be treated via both cases above. In particular, we obtain for $y \neq 0$

$$\begin{aligned}\det(y\mathbb{1}_{2N+\nu} - D) &= \det\begin{pmatrix} y\mathbb{1}_N & iW \\ iW^\dagger & y\mathbb{1}_{2N+\nu} \end{pmatrix} = \det(y\mathbb{1}_N)\det\left(y\mathbb{1}_{N+\nu} - iW^\dagger\frac{1}{y}\mathbb{1}_N iW\right) \\ &= y^N \det\left(\frac{1}{y}(y^2\mathbb{1}_{N+\nu} + W^\dagger W)\right) = y^{-\nu}\det(y^2\mathbb{1}_{N+\nu} + W^\dagger W)\,,\end{aligned}$$

and similarly

$$\begin{aligned}\det(y\mathbb{1}_{2N+\nu} - D) &= \det\begin{pmatrix} y\mathbb{1}_N & iW \\ iW^\dagger & y\mathbb{1}_{2N+\nu} \end{pmatrix} = \det(y\mathbb{1}_{N+\nu})\det\left(y\mathbb{1}_N - iW\frac{1}{y}\mathbb{1}_{N+\nu} iW^\dagger\right) \\ &= y^{N+\nu}\det\left(\frac{1}{y}(y^2\mathbb{1}_N + WW^\dagger)\right) = y^\nu \det(y^2\mathbb{1}_N + WW^\dagger)\,.\end{aligned}$$

Combining both equations and substituting $x = y^2$ we arrive at a special case of Sylvester's determinant identity [74]

$$\det(x\mathbb{1}_{N+\nu} + W^\dagger W) = x^\nu \det(x\mathbb{1}_N + WW^\dagger)\,.$$

From the equations above we gain some insight into the derivation of the spectrum of the Dirac operator. In particular, we verify that the eigenvalues of D can be obtained by the eigenvalues of the product $WW^\dagger$, or $W^\dagger W$. The non-zero eigenvalues come in pairs, due to the substitution $x = y^2$, which originates from the chirality property of D and its anti-Hermiticity. The non-zero eigenvalues λ_n, $n = 1, \ldots, N$, of $WW^\dagger$ can be derived as the zeros of the characteristic polynomial

$$0 = \det(\widetilde{x}\mathbb{1}_N - WW^\dagger) = \prod_{n=1}^{N}(\widetilde{x} - \lambda_n)\,,$$

with $\widetilde{x} = -x$. Since the Dirac operator is anti-Hermitian, the product of its blocks - the Wishart matrix $WW^\dagger$ - is Hermitian, as we can see from Eq. (2.1). Hence, the eigenvalues of $WW^\dagger$ are real and positive, as D has a purely imaginary eigenvalue spectrum.

Singular Value Decomposition

By standard linear algebra we know that the transformation to an eigenvalue basis can be done via unitary matrices if the original matrix is Hermitian. This is the case for $WW^\dagger$ as we have seen above. Consequently, we consider a tranformation

$$\Lambda = UWW^\dagger U^\dagger$$

with a diagonal eigenvalue matrix Λ containing the eigenvalues of $WW^\dagger$. The matrix $U \in \mathrm{U}(N)$ contains the eigenvectors with respect to $WW^\dagger$.
To look at the decomposition with U alone does not suffice in our situation as we can see looking back at the partition function Z_N of the chGUE(N). The integration in Eq. (2.2) is done over the independent, real variables of the entries of W, not $WW^\dagger$. In principle, it is possible to make a change of variables to the eigenvalues of the positive definite matrix $WW^\dagger$. However, this yields a non-trivial Jacobian and can be circumvented by consideration of the singular values of W instead. Hence, we go over from the eigenvalues of $WW^\dagger$ to the singular values of W in order to unlock spectral information in our model and to keep the calculation simple.[2] The decomposition of W associated with its squared singular values $\lambda_n = w_n^2$, where $n = 1, \ldots, N$, reads

$$W = U_W W_D V_W^\dagger, \text{ where } W_D = \begin{pmatrix} \Lambda_W & 0_{N\times\nu} \end{pmatrix} \text{ and } \Lambda_W = \begin{pmatrix} w_1 & 0 & 0 & \ldots & 0 \\ 0 & w_2 & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & & 0 \\ 0 & 0 & \ldots & 0 & w_N \end{pmatrix}.$$

We have unitary matrices U_W and $V_W^\dagger$ with dimensions depending on the remaining degrees of freedom after decomposing W to W_D. Their direct product is an element of the coset space $[\mathrm{U}(N) \times \mathrm{U}(N+\nu)]/(\mathrm{U}(1))^N$. The matrices are comprised of vectors, which form the sets of left and right eigenvectors of W respectively.

The next question is how the measure dW behaves under the singular value decomposition defined above. Recall that the measure in our model has to be drawn from a complex rectangular matrix, that is, the underlying matrix space is $\mathbb{C}^{N\times(N+\nu)}$. The measure itself is equal to the flat Lebesgue measure, which means

$$dW = \prod_{n=1}^{N} \prod_{m=1}^{N+\nu} d\mathrm{Re}W_{nm}\, d\mathrm{Im}W_{nm}\,, \tag{2.6}$$

where the matrix entry W_{nm} has to be decomposed into its real and imaginary part. Taking a conjugation like $W \to UWV$ with unitary matrices of appropriate dimensions leaves the measure invariant, because the differential length element $(ds)^2 = \mathrm{Tr}\, dW dW^\dagger$ is invariant on $\mathbb{C}^{N\times(N+\nu)}$ under this transformation.[3] However, this only implies that the decomposition into squared singular values leads to a factorisation into a singular value dependent part and an eigenvector dependent part with respect to the measure. This particular factorisation is

[2] The squared singular values of W are exactly the eigenvalues of the product $WW^\dagger$.
[3] More details and information can be found in the literature [24, 75, 76].

known in the literature. We find, looking for example in [23]

$$dW = c \left(\prod_{n=1}^{N} \lambda_n^{\nu}\right) |\Delta_N(\lambda_1, \ldots, \lambda_N)|^2 \left(\prod_{n=1}^{N} d\lambda_n\right) d\mu(U_W) d\mu(V_W) \,. \tag{2.7}$$

The prefactor c is independent of the squared singular values and can be found in the literature for the chGUE(N) but also many other matrix ensembles for the same or similar matrix decompositions. The term $\Delta_N(\lambda_1, \ldots, \lambda_N)$ denotes the Vandermonde determinant of N elements, which can be written as

$$\begin{aligned} \Delta_N(\lambda_1, \ldots, \lambda_N) = \det\left[\lambda_j^{i-1}\right]_{i,j=1}^{N} = \prod_{1 \le i < j \le N} (\lambda_j - \lambda_i) = \begin{vmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_N \\ \vdots & \vdots & \vdots \\ \lambda_1^{N-1} & \ldots & \lambda_N^{N-1} \end{vmatrix} \\ = (-1)^{N(N-1)/2} \det\left[\lambda_j^{N-i}\right]_{i,j=1}^{N} \,. \end{aligned} \tag{2.8}$$

This determinant becomes zero, when two arguments λ_i and λ_j become equal. Therefore, it serves as a repulsory factor, keeping the eigenvalues apart from each other. More details and properties of the Vandermonde determinant can be found in Appendix A.
The measure $d\mu(X)$ denotes the *Haar measure* with respect to its argument [77]. This measure is left and right invariant under conjugation with corresponding group elements [78]. The integration over the Haar measures can be performed for all four models introduced in section 2.1. The two temperature independent models, which are the classical chGUE(N) given via Eq. (2.2) and the deformed chGUE(N) via Eq. (2.4), are unitary bi-invariant under the singular value decomposition. Thus, the integration over the Haar measures containing the eigenvector dependence is trivial. Consequently, we find for the classical chGUE(N)[4]

$$\mathcal{P}_{\text{chGUE}}(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N^{(0,0)}} \Delta_N^2(\lambda_1, \ldots \lambda_N) \left(\prod_{n=1}^{N} \lambda_n^{\nu} e^{-\lambda_n}\right), \tag{2.9}$$

with normalisation constant given as

$$Z_N^{(0,0)} = \left(\prod_{n=1}^{N} \int d\lambda_n\right) \Delta_N^2(\lambda_1, \ldots, \lambda_N) \left(\prod_{n=1}^{N} \lambda_n^{\nu} e^{-\lambda_n}\right).$$

Similarly, we find for the deformed chGUE(N)

$$\mathcal{P}_{\text{chGUE}}^{\text{deformed}}(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N^{(N_f,0)}} \Delta_N^2(\lambda_1, \ldots \lambda_N) \left(\prod_{n=1}^{N} \lambda_n^{\nu} e^{-\lambda_n} \prod_{f=1}^{N_f} (m_f^2 + \lambda_n)\right), \tag{2.10}$$

where the constant reads

$$Z_N^{(N_f,0)} = \left(\prod_{n=1}^{N} \int d\lambda_n\right) \Delta_N^2(\lambda_1, \ldots, \lambda_N) \left(\prod_{n=1}^{N} \lambda_n^{\nu} e^{-\lambda_n}\right) \left(\prod_{f=1}^{N_f} \prod_{n=1}^{N} (m_f^2 + \lambda_n)\right).$$

[4] More details regarding the derivation of joint probability density functions with and without external parameters can be found in Appendix C.

The temperature dependent models given in the Eqs. (2.3) and (2.5) are not unitary bi-invariant. In both ensembles we need to do a second singular value decomposition, with respect to the external source matrix $T = \frac{1}{2} U_T T_D V_T^\dagger$. The integration over the Haar measures is now non-trivial and we have to utilize the Berezin-Karpelevich integral [79][5]

$$\begin{aligned} I(U,V) &= \int_{\mathrm{U}(N)} d\mu(U) \int_{\mathrm{U}(N+\nu)} d\mu(V) \exp\left(\mathrm{ReTr}\left(U W_D V^\dagger T_D^\dagger\right)\right) \\ &= C \frac{\prod_{n=1}^N (a_n \lambda_n)^{-\frac{\nu}{2}}}{\Delta_N(a_1,\ldots,a_N)} \frac{\det[I_\nu(2\sqrt{a_i\lambda_j})]_{i,j=1}^N}{\Delta_N(\lambda_1,\ldots,\lambda_N)}, \end{aligned}$$

where we have used the substitution $4a_n = t_n^2$ for all $n = 1,\ldots,N$ to denote the squared singular values of T. Additionally, $I_\nu(z)$ denotes the modified Bessel function of the first kind with argument z. The JPDF for the chGUE(N) with an external source then reads

$$\begin{aligned} \mathcal{P}^{\mathrm{ext}}_{\mathrm{chGUE}}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N^{(0,T)}} &\left[\prod_{n=1}^N \left(\frac{\lambda_n}{a_n}\right)^{\nu/2} e^{-(\lambda_n+a_n)}\right] \\ &\times \det\left[I_\nu\left(2\sqrt{a_i\lambda_j}\right)\right]_{i,j=1}^N \Delta_N(\lambda_1,\ldots\lambda_N), \end{aligned} \tag{2.11}$$

with a constant

$$Z_N^{(0,T)} = \left(\prod_{n=1}^{N_f} \int d\lambda_n\right) \left[\prod_{n=1}^N \left(\frac{\lambda_n}{a_n}\right)^{\nu/2} e^{-(\lambda_n+a_n)}\right] \det\left[I_\nu\left(2\sqrt{a_i\lambda_j}\right)\right]_{i,j=1}^N \Delta_N(\lambda_1,\ldots\lambda_N).$$

For the ensemble from the chGUE(N) symmetry class, which contains both N_f massive flavors and additionally depends on temperature as an external source, the JPDF becomes[6]

$$\begin{aligned} \mathcal{P}^{\mathrm{temp}}_{\mathrm{chGUE}}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N^{(N_f,T)}} &\left[\prod_{n=1}^N \left(\frac{\lambda_n}{a_n}\right)^{\nu/2} e^{-(\lambda_n+a_n)}\right] \left(\prod_{f=1}^{N_f}\prod_{n=1}^N (m_f^2+\lambda_n)\right) \\ &\times \det\left[I_\nu\left(2\sqrt{a_i\lambda_j}\right)\right]_{i,j=1}^N \Delta_N(\lambda_1,\ldots\lambda_N)\,. \end{aligned} \tag{2.12}$$

The partition function is given by

$$\begin{aligned} Z_N^{(N_f,T)} = &\left(\prod_{n=1}^N \int d\lambda_n\right) \left[\prod_{n=1}^N \left(\frac{\lambda_n}{a_n}\right)^{\nu/2} e^{-(\lambda_n+a_n)}\right] \left(\prod_{f=1}^{N_f}\prod_{n=1}^N (m_f^2+\lambda_n)\right) \\ &\times \det\left[I_\nu\left(2\sqrt{a_i\lambda_j}\right)\right]_{i,j=1}^N \Delta_N(\lambda_1,\ldots\lambda_N)\,. \end{aligned}$$

Given the JPDF of eigenvalues (or singular values) the next challenge in unlocking the spectral statistics of the Dirac operator is to derive correlation functions with respect to these new variables (eigenvalues or singular values). By construction, the eigenvalues obtained for D are random variables, but are no longer independent. This becomes particularly clear, when we look at the JPDF, which contains the Vandermonde determinant, which serves as a repulsion between the eigenvalues. In order to obtain correlation functions between the

[5] There exist some variations and extensions of the Berezin-Karpelevich integral, which can be found in [65, 69–71, 80–83].

[6] For more details see Appendix C.

random variables $\lambda_1, \ldots, \lambda_N$ it is necessary to introduce the notion of the k-point correlation function $\rho_{k,N}$. This function describes the correlation between the first k variables $\lambda_1, \ldots, \lambda_k$ taken out of the set of variables $\{\lambda_1, \ldots, \lambda_N\}$. It is clear, that $k \leq N$ has to be assumed. The k-point correlation function can be obtained from the JPDF by integrating out the last $N - k$ variables, i.e.[7]

$$\rho_{k,N}(\lambda_1, \ldots, \lambda_k) \equiv \frac{N!}{(N-k)!} \left(\prod_{n=k+1}^{N} \int d\lambda_n \right) \mathcal{P}(\lambda_1, \ldots, \lambda_N) \,. \tag{2.13}$$

There are two values of k, which are particularly interesting. The first is $k = 1$, which is called the averaged *spectral density*[8] and is given as

$$\rho_{1,N}(\lambda) = N \left(\prod_{n=2}^{N} \int d\lambda_n \right) \mathcal{P}(\lambda_1, \ldots, \lambda_N) \,.$$

It describes the density of all variables at the position λ. The second value is $k = N$, which gives back the JPDF itself weighted with $N!$. The question that remains is, how to perform the integrals for any value of k in Eq. (2.13). We introduce the methods of orthogonal polynomials to tackle this question in the next section.

2.3 Orthogonal Polynomials

We introduce orthogonal polynomials (OP) as a tool to perform the integrals in the definition of the k-point correlation function for the chGUE(N) and extensions thereof given in Eq. (2.13). We denote polynomials that are orthogonal with respect to a *weight function* $w(\lambda)$ by $p_n(\lambda)$, $n = 1, \ldots, N$. Orthogonality is given in the following sense:

$$\langle p_j(\lambda), p_k(\lambda) \rangle = \int_a^b d\lambda p_j(\lambda) p_k(\lambda) w(\lambda) = h_j \delta_{jk} \,. \tag{2.14}$$

Here $\langle \cdot, \cdot \rangle$ denotes the scalar product with support $[a, b]$. The h_j are the norms of the OP $p_j(\lambda)$ and δ_{jk} is the Kronecker-δ-function.
For the classical chGUE(N) with JPDF given in Eq. (2.9) the weight function for an eigenvalue variable λ reads

$$w^{(0)}(\lambda) = \lambda^\nu e^{-\lambda} \,.$$

For the deformed chGUE(N) we have a slightly more complicated weight function:

$$w^{(N_f)}(\lambda) = \prod_{f=1}^{N_f} (\lambda + m_f^2) w^{(0)}(\lambda) \,.$$

We have to distinguish between these two models and the other two models, depending on temperature (compare Eqs. (2.11) and (2.12)), because the former two share JPDFs proportional to Δ_N^2 and the latter two do not. They have JPDFs that only contain one Vandermonde determinant and cannot be treated with orthogonal polynomials directly. We will see how to treat the two temperature depending models in section 2.5.

[7] The definition of the k-point correlation function holds for all four random matrix models we have introduced in this chapter.

[8] The adjective "averaged" is omitted most of the time, when considering $\rho_{1,N}(\lambda)$.

Correlation kernel and correlation functions

We start by considering a random matrix model featuring a JPDF of the form

$$\mathcal{P}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N}\Delta_N^2(\lambda_1,\ldots,\lambda_N)\prod_{n=1}^{N} w(\lambda_n)\,,$$

where the weight function factorises with respect to the λ_n. Then, we can write

$$\mathcal{P}(\lambda_1,\ldots,\lambda_N) = \frac{\prod_{n=0}^{N-1} h_n}{Z_N}\left(\prod_{n=1}^{N} w(\lambda_n)\right)\,\det\left[K_N(\lambda_n,\lambda_m)\right]_{n,m=1}^{N}\,, \tag{2.15}$$

where

$$K_N(\lambda,\lambda') = \sum_{n=0}^{N-1}\frac{p_n(\lambda)p_n(\lambda')}{h_n} \tag{2.16}$$

is the so-called *correlation kernel*.[9] This can be achieved by rewriting the Vandermonde determinant in terms of orthogonal polynomials. More details regarding this issue can be found in Appendix A. The kernel satisfies a reproducing property[10]

$$K_N(\lambda,\lambda') = \int dy K_N(\lambda,y)K_N(y,\lambda')w(y)\,.$$

The reproducing property of the kernel carries over to the determinant of the kernel, which is comprised and referred to as the *Dyson-Gaudin integration lemma* [84, 85] given in a simplified, but suitable, form in [76].

Lemma 2.1. *Let $J_N = (J_{ij})_{i,j=1}^{N} = J_N(\mathbf{x})$ be an $N\times N$ matrix whose entries depend on a real vector $\mathbf{x} = (x_1,\ldots,x_N)$ and have the form $J_{ij} = f(x_i,x_j)$, where f is a complex-valued function satisfying - for some measure $d\mu(x)$ - the reproducing kernel property:*

$$\int d\mu(y) f(x,y)f(y,z) = f(x,z)\,.$$

Then we have

$$\int d\mu(x_N)\det J_N(\mathbf{x}) = [q-(N-1)]\det J_{N-1}\,,$$

where $q = \int d\mu(x)f(x,x)$. The matrix $J_{N-1} = (J_{ij})_{i,j=1}^{N-1}$ has the same functional form as J_N with $\mathbf{x}$ replaced by a vector with dimension $N-1$, namely $(x_1,\ldots,x_{N-1})$.

Proof. See [76]. □

[9] There exists another convention for the definition of the kernel. Therein, the weight function $w(\lambda)$ from Eq. (2.15) is included in the definition of the kernel in Eq. (2.16).

[10] We have to include the weight function in the determinant in Eq. (2.15) to show the reproducing property. This is possible, since $\prod_l a_l \det(f(i,j)) = \det(\sqrt{a_i a_j} f(i,j))$.

This lemma allows us to perform the integrals in the k-point correlation function iteratively. We can write

$$\begin{aligned}\rho_{k,N}(\lambda_1,\ldots,\lambda_N) &= \frac{N!}{(N-k)!}\frac{\prod_{n=1}^{N} h_{n-1}}{Z_N}\left(\prod_{n=k+1}^{N}\int d\lambda_n\right)\det\left[K_N(\lambda_i,\lambda_j)\right]_{i,j=1}^{N}\prod_{n=1}^{N} w(\lambda_n)\\ &= \frac{1}{(N-k)!}(N-k)!\prod_{n=1}^{k} w(\lambda_n)\det\left[K_N(\lambda_i,\lambda_j)\right]_{i,j=1}^{k}\\ &= \prod_{n=1}^{k} w(\lambda_n)\det\left[K_N(\lambda_i,\lambda_j)\right]_{i,j=1}^{k}\ .\end{aligned}$$

Setting $k = 1$ yields the spectral density

$$\rho(\lambda) = \rho_{1,N}(\lambda) = w(\lambda)K_N(\lambda,\lambda)\ .$$

Similarly, setting $k = N$ recovers the JPDF

$$\rho_{N,N}(\lambda_1,\ldots,\lambda_N) = \prod_{n=1}^{N} w(\lambda_n)\det\left[K_N(\lambda_i,\lambda_j)\right]_{i,j=1}^{N}\ ,$$

where we have applied the normalization constant given as $Z_N = N!\prod_{n=1}^{N} h_{n-1}$. What have we learned from this? The kernel can be expressed via orthogonal polynomials and the k-point correlation function can be written as a determinant of a $k \times k$ matrix containing the kernel as entries. Therefore, we have to study and obtain the set of orthogonal polynomials for a given weight function in order to express the kernel. Thus, we now present some results for orthogonal polynomials with general weigths.

Properties of Orthogonal Polynomials with general weights

We consider orthogonal polynomials on the positive real half line with a general weight function $w(\lambda)$. We follow [32], with further information drawn from [23, 24]. We assume that $w(\lambda)$ is measurable and all moments $\int d\lambda w(\lambda)\lambda^k \leq \infty$ exist, for $k \in \mathbb{N}$. This allows to construct OPs via the Gram-Schmidt procedure.
The orthogonal polynomials $p_k(\lambda)$ are also considered to be monic, i.e. $p_k(\lambda) = \lambda^k + \mathcal{O}(\lambda^{k-1})$, and satisfy a recurrence relation

$$\lambda p_k(\lambda) = p_{k+1}(\lambda) + \alpha_k^k p_k(\lambda) + \alpha_k^{k-1} p_{k-1}(\lambda)\ ,$$

where

$$\alpha_k^l = h_l^{-1}\int d\lambda w(\lambda)\lambda p_k(\lambda)p_l(\lambda)\ .$$

This relation becomes even more symmetric by going from orthogonal to *orthonormal* polynomials $P_k(\lambda) = p_k(\lambda)/\sqrt{h_k}$, namely

$$\lambda P_k(\lambda) = c_k P_{k+1}(\lambda) + \alpha_k^k P_k(\lambda) + c_{k-1}P_{k-1}(\lambda),\quad c_{k-1} = \sqrt{\frac{h_k}{k_{k-1}}}\ .$$

The kernel becomes

$$K_N(\lambda,\lambda') = \sum_{n=0}^{N-1} P_n(\lambda)P_n(\lambda') \,.$$

Using the recurrence relation of the OPs we can derive the Christoffel Darboux formula of the kernel. This special formula reads [23]

$$K_N(\lambda,\lambda') = \sum_{n=0}^{N-1} P_n(\lambda)P_n(\lambda') = c_{N-1}\frac{P_N(\lambda)P_{N-1}(\lambda') - P_{N-1}(\lambda)P_N(\lambda')}{\lambda-\lambda'} \,. \tag{2.17}$$

The derivation of the OPs via the Gram Schmidt procedure is possible, but not always practical. Luckily, for the classical chGUE(N) the weight function is $w(\lambda) = x^\nu e^{-\lambda}$ and the monic polynomials orthogonal with respect to this weight are known already. They can be expressed via generalized Laguerre polynomials, hence the reason to call the chGUE(N) by the name Wishart-Laguerre ensemble. We can write

$$p_n(\lambda) = (-1)^n n! L_n^\nu(\lambda) \,, \quad h_n = n!\Gamma(n+\nu+1) \,. \tag{2.18}$$

The next model we considered was the deformation with N_f flavors. In that case the JPDF features a weight function of the form

$$w(\lambda) = \lambda^\nu e^{-\lambda}\prod_{f=1}^{N_f}(\lambda+m_f^2) \,.$$

The problem arising in this case is that the orthogonal polynomials are rather cumbersome and not elementary as for the classical chGUE(N), where generalized Laguerre polynomials appear. It is unclear, how the two sets of polynomials can be related, in particular, how we can trace back to the Laguerre polynomials in the case $N_f = 0$ from the more general $N_f \neq 0$ case. Therefore, we show another way of deriving the orthogonal polynomials and even the kernel for the deformed chGUE(N). We need to compute expectation values of products of characteristic polynomials.

Expectation values of characteristic polynomials

We start with the presentation of the Heine formula [86, 87][11]

$$p_N(\lambda) = \mathbb{E}_{\mathcal{P}}\left[\prod_{n=1}^{N}(\lambda-\lambda_n)\right] = \mathbb{E}_{\mathcal{P}}\left[D_N(\lambda)\right] , \tag{2.19}$$

where we have introduced the notation $D_N(\lambda) = \prod_{n=1}^{N}(\lambda-\lambda_n)$ for the characteristic polynomial and denoted the JPDF by $\mathcal{P}$. The Heine formula allows us to express an orthogonal polynomial as an expectation value of *one* characteristic polynomial. To show this, we need to recall the definition of the expectation value of an observable $\mathcal{O}$ with respect to the JPDF $\mathcal{P}$. We can write

$$\mathbb{E}_{\mathcal{P}}\left[\mathcal{O}(\lambda_1,\ldots,\lambda_N)\right] = \prod_{n=1}^{N}\int d\lambda_n \mathcal{P}(\lambda_1,\ldots,\lambda_N)\mathcal{O}(\lambda_1,\ldots,\lambda_N) \,. \tag{2.20}$$

[11] It is possible to obtain the orthogonal polynomial $p_N(\lambda)$ as a ratio of two determinants of size N by the Gram-Schmidt procedure, see [32, Eq. 14].

Setting $\mathcal{O} = D_N(\lambda)$ and choosing the JPDF as the JPDF of the classical chGUE(N) given in Eq. (2.9) we find

$$\begin{aligned}\mathbb{E}_{\mathcal{P}}\left[D_N(\lambda)\right] &= \frac{1}{Z_N}\left(\prod_{n=1}^{N}\int d\lambda_n\right)\Delta_N^2(\lambda_1,\ldots,\lambda_N)\prod_{n=1}^{N}w(\lambda_n)\prod_{n=1}^{N}(\lambda-\lambda_n)\\ &= \frac{1}{Z_N}\left(\prod_{n=1}^{N}\int d\lambda_n\right)\Delta_N(\lambda_1,\ldots,\lambda_N)\prod_{n=1}^{N}w(\lambda_n)\Delta_{N+1}(\lambda_1,\ldots,\lambda_N,\lambda)\,,\end{aligned}$$

where we have used an extension property of the Vandermonde determinant that is shown in Appendix A, see Lemma A.2.
Now we can write both Vandermonde determinants in terms of orthogonal polynomials with respect to the weight function and expand the determinants in sums over permutations from S_N and S_{N+1} respectively

$$\begin{aligned}\mathbb{E}_{\mathcal{P}}\left[D_N(\lambda)\right] &= \frac{1}{Z_N}\sum_{\sigma\in S_N}\sum_{\sigma'\in S_{N+1}}(-1)^{\sigma+\sigma'}\left(\prod_{n=1}^{N}\int d\lambda_n w(\lambda_n)p_{\sigma(n)-1}(\lambda_n)p_{\sigma'(n)-1}(\lambda_n)\right)p_{\sigma'(N+1)-1}(\lambda)\\ &= \frac{1}{Z_N}\sum_{\sigma\in S_N}\sum_{\sigma'\in S_{N+1}}(-1)^{\sigma+\sigma'}\prod_{n=1}^{N}h_{\sigma(n)-1}\delta_{\sigma(n)-1,\sigma'(n)-1}p_{\sigma'(N+1)-1}(\lambda)\,.\end{aligned}$$

The orthogonality of the polynomials ensures that $\sigma'(N+1)$ is fixed to $N+1$. Thus, we obtain

$$\mathbb{E}_{\mathcal{P}}\left[D_N(\lambda)\right] = \frac{1}{Z_N}\prod_{n=1}^{N}h_{n-1}\sum_{\sigma\in S_N}p_N(\lambda) = p_N(\lambda)$$

as desired.

The correlation kernel $K_N(\lambda,\lambda')$ can also be expressed as an expectation value of characteristic polynomials, namely [32, 88]

$$K_{N+1}(\lambda,\lambda') = \frac{1}{h_N}\mathbb{E}_{\mathcal{P}}\left[D_N(\lambda)D_N(\lambda')\right]. \tag{2.21}$$

Given an JPDF with two Vandermonde determinants, like the classical chGUE(N), the extension to arbitrary products, ratios and products of inverse characteristic polynomials is immediate and has been studied extensively in the past. Ensembles of this kind are called *orthogonal polynomial ensembles* to emphasize their close relation to orthogonal polynomials. Further information and useful formulae may be found in [89–91]. We summarize this class in terms of weight function and partition function

$$Z_N^{(N_f,0)} = \left(\prod_{n=1}^{N}\int_0^\infty d\lambda_n\, w^{(N_f)}(\lambda_n)\right)\Delta_N^2(\lambda_1,\ldots,\lambda_N)\,. \tag{2.22}$$

The weight function $w^{(N_f)}(\lambda)$ depends on the number flavors N_f in the model. Thus, it is either the classical chGUE(N) weight function $w^{(0)}(\lambda) = \lambda^\nu e^{-\lambda}$ or the deformed chGUE(N)

weight function

$$w^{(N_f)}(\lambda) = \lambda^{\nu} e^{-\lambda} \prod_{f=1}^{N_f} (\lambda + m_f^2) = w^{(0)}(\lambda) \prod_{f=1}^{N_f} (\lambda + m_f^2) \,. \tag{2.23}$$

For both ensembles above the kernel is accessible by the expectation value of a product of two characteristic polynomials with respect to the weight function of the corresponding JPDF, see Eq. (2.21). Consequently, we have for the classical chGUE(N)

$$K_N^{[0]}(\lambda, \lambda') = \frac{1}{h_{N-1}^{[0]}} \mathbb{E}_{\mathcal{P}_{\text{chGUE}}} \left[D_{N-1}(\lambda) D_{N-1}(\lambda') \right] , \tag{2.24}$$

and similarly for deformed chGUE(N)

$$K_N^{[N_f]}(\lambda, \lambda') = \frac{1}{h_{N-1}^{[N_f]}} \mathbb{E}_{\mathcal{P}^{\text{deformed}}_{\text{chGUE}}} \left[D_{N-1}(\lambda) D_{N-1}(\lambda') \right] \tag{2.25}$$

with norms $h_N^{[N_f]}$ of the OPs orthogonal with respect to the weight function Eq. (2.23).[12] The partition function for the deformed chGUE(N) can be expressed as an expectation value of characteristic polynomials with respect to the JPDF of the classical chGUE(N), namely

$$Z_N^{(N_f,0)} = (-1)^{NN_f} Z_N^{(0,0)} \mathbb{E}_{\mathcal{P}_{\text{chGUE}}} \left[\prod_{f=1}^{N_f} D_N(-m_f^2) \right] . \tag{2.26}$$

This means, we can circumvent the OPs with respect to the more involved weight function $w^{(N_f)}(\lambda)$ by calculating expectation values of products of characteristic polynomials with respect to the weight $w^{(0)}(\lambda)$, which is simpler and more importantly has OPs that are elementary. We also can obtain the kernel in Eq. (2.25) in a similar fashion. We can rewrite the expectation value and find [32]

$$\frac{\mathbb{E}_{\mathcal{P}^{\text{deformed}}_{\text{chGUE}}} \left[D_{N-1}(\lambda) D_{N-1}(\lambda') \right]}{h_{N-1}^{[N_f]}} = (-1)^{N_f} h_{N-1}^{[0]} \frac{\mathbb{E}_{\mathcal{P}_{\text{chGUE}}} \left[\prod_{f=1}^{N_f} D_{N-1}(-m_f^2) D_{N-1}(\lambda) D_{N-1}(\lambda') \right]}{\mathbb{E}_{\mathcal{P}_{\text{chGUE}}} \left[\prod_{f=1}^{N_f} D_N(-m_f^2) \right]} . \tag{2.27}$$

With this knowledge we can highlight the following result by Akemann and Vernizzi [90], which helps calculating the expectation values of characteristic polynomials we have encountered in the Eqs. (2.26) and (2.27).

Theorem 2.2. *Assume we start with a random matrix model given by a partition function Z_N of Δ_N^2-type, like in Eq.* (2.22)*, with factorizing weight function $w^{(K)}(x)$. Let $\{v_i \mid i = 1, \ldots, M\}$ and $\{u_i \mid i = 1, \ldots, L\}$ be two sets of numbers which are pairwise distinct among each set. Without loss of generality we assume $M \geq L$, where the empty set with $L = 0$ is permitted as well. Taking orthonormal polynomials $P_k(x)$ with respect to the weight $w^{(0)}(x)$ and together with norms h_k, the*

[12] We are a priori unable to compute the OPs directly, but this is not necessary to state the formula for the kernel. We only need to know that the OPs exist.

following statement can be proven:

$$\mathbb{E}_{\mathcal{P}}\left[\prod_{i=1}^{M} D_N(v_i)\prod_{j=1}^{L} D_N(u_j)\right] = \frac{\prod_{i=N}^{N+M-1}\sqrt{h_i}\prod_{j=N}^{N+L-1}\sqrt{h_j}}{\Delta_M(\{v\})\Delta_L(\{u\})}\det_{1\leq l,m\leq M}\left[\mathcal{B}(v_l,u_m)\right]$$

with the definition

$$\mathcal{B}(v_l,u_m) \equiv \begin{cases} \sum_{i=0}^{N+L-1} P_i(v_l)P_i(u_m) & \text{for } m = 1,\ldots,L, \\ P_{N+m-1}(v_l) & \text{for } m = L+1,\ldots,M. \end{cases}$$

Proof. This was proven in [90] for an even wider class of random matrix models, but crucially it applies also to orthogonal polynomial ensembles with eigenvalues on the positive real half-line. □

The result of the Theorem implies that for a given expectation value of a product of $M+L$ characteristic polynomials there exist many different determinantal representations depending on the numbers M and L and the polynomials $P_n(\lambda)$ which are orthonormal with respect to the weight $w^{(0)}(\lambda)$. It is also worth noting that we can apply the Christoffel-Darboux identity to the first L rows of the determinant of $\mathcal{B}(v_l,u_m)$ in the above Theorem. Thus, these rows contain kernels, while the last $M-L$ rows contain orthonormal polynomials. This representation will become very useful in chapter 5, when we discuss questions of universality.

2.4 Scaling Limits and universality

In previous sections we considered matrix models and their correlations at finite matrix size N. Scaling limits and concepts of universality arise when the matrix size N tends to infinity: $N \to \infty$. We call the limit $N \to \infty$ a *scaling limit* of an observable depending on variables $\lambda_1,\ldots,\lambda_N$, if the variables are rescaled with respect to N. The observables we consider in this thesis are the correlation kernel and the k-point correlation function. The $N \to \infty$ limit is used to apply the underlying random matrix model of the chGUE(N) symmetry class to the QCD Dirac operator spectrum.
We distinguish between the *global* scaling limit and different *local* scaling limits. The global scaling limit describes the limit as N tends to $+\infty$, where the spectral correlations are considered with respect to the full support of the underlying probability measure and corresponding spectrum. This implies that distances between eigenvalues, called spacings, can be as large as the spectrum and its support allows. As N tends to $+\infty$ the fluctuations between individual eigenvalues with small spacings are averaged out, and expectation values factorise. In contrast to the global scale we can also look specifically at local scales. Those scales are obtained, when we zoom into a specific point or area of the spectrum, for example the edges. The zoom-in is achieved by magnifying fluctuations between eigenvalues in the respective area of interest, which becomes possible via rescaling the eigenvalues with suitable powers $1/N^{\delta}$. The value of δ depends on the point of interest in the spectrum. Hence, the corresponding local scaling limit is obtained by taking the finite N result of the considered observable, then rescaling the underlying variables with $1/N^{\delta}$ and then taking $N \to \infty$. Consequently, the limiting result of the observable will differ depending on the region of the spectrum characterised by δ.
A scaling limit is called *universal*, when the limiting expression of the observable does not

depend on the specifics of the underlying random matrix model. This implies that the limiting expression of the observable appears in exactly the same scaling regime, but with a different random matrix model as a finite N starting point. This phenomenon is the mathematical version of universality. In the physical context another concept of universality is commonly used. Therein the fact that two different models or theories, for example RMT and chiral perturbation theory of QCD, yield the same operator spectral statistics in a local scaling regime is also denoted as universality. The only requirement here is that the two starting models or theories belong to the same symmetry class, which is why we described the symmetries of the Dirac operator in the introduction and used matrix models with chiral symmetry. Note also that the second type of universality is the reason that why we apply RMT to QCD in the first place. The RMT models for QCD with temperature dependence are oftentimes the simplest members of the underlying symmetry (or universality) class. Thus, we do not need to study QCD directly but can rather use the much simpler RMT models from the chGUE(N) symmetry class instead. Crucially, because of universality we still obtain useful results in the scaling limit, where this universality concept is met. Note that in the following chapters of this thesis we encounter both concepts of universality.
A well-known result representing a global scaling limit is the *macroscopic* spectral density of all eigenvalues for the chGUE(N), and their real and quaternion counterparts. The global spectral density is restricted to the positive part of the real line. As N tends to infinity, the density becomes the *Marchenko Pastur density* [92][13]

$$\rho_{\mathrm{MP}}(\lambda) = \frac{1}{2\pi}\frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{q\lambda}\mathbb{I}_{[\lambda_-,\lambda_+]}\,, \quad \text{with} \quad \lambda_\pm = (1 \pm \sqrt{q})^2\,.$$

In the above equation $\mathbb{I}_\Sigma$ is the indicator function on the interval Σ that is 1 on the interval and zero outside. The density $\rho_{\mathrm{MP}}(\lambda)$ forms a probability distribution together with the Lebesgue measure $d\lambda$ on the compact interval $[\lambda_-, \lambda_+]$. The number of zero-modes ν of the chGUE(N) is encoded in the parameter q as $\lim_{N\to\infty}\frac{N}{N+\nu} = q \in (0,1]$.[14] The Marchenko Pastur density is illustrated in Fig. 2.1 for $q = 1$ and also checked numerically for $q = 0.1$ with 1000 eigenvalues of the Wishart matrix $WW^\dagger$. The application to effective theories of QCD is achieved exactly by choosing $q = 1$. Another result for the global spectral density is

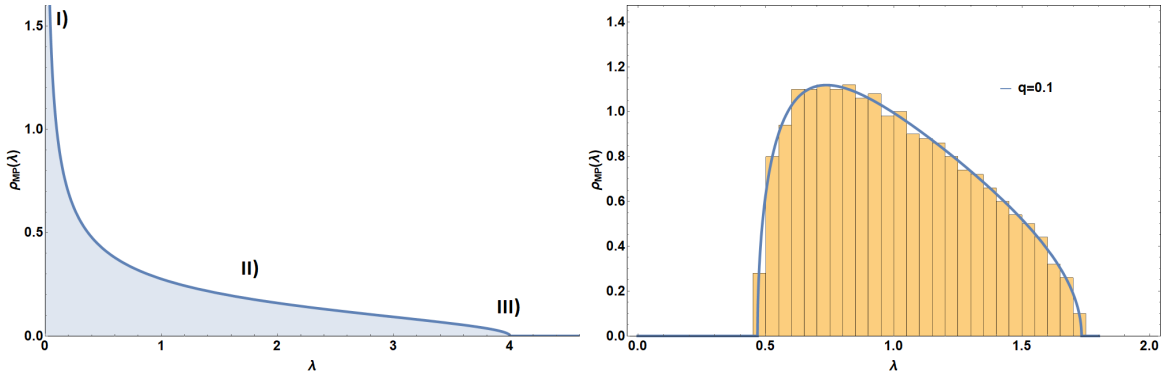

Figure 2.1: *Left:* The Marchenko-Pastur density for $q = 1$ is relevant for QCD applications and shows different local statistics: I) Bessel kernel statistics at the hard edge, II) sine kernel statistics in the bulk and III) Airy kernel statistics at the soft edge. *Right:* Normalized histogram of eigenvalues of Wishart matrix $WW^\dagger$ with $N = 1000$ and $q = 0.1$.

[13] $\rho_{\mathrm{MP}}(\lambda)$ was also derived by Dyson in [93] and was tested against numerics in [94].
[14] The zero modes have to be rescaled appropriately as $N \to N\nu$ for q to remain finite as $N \to \infty$.

found for the GUE(N) in the $N \to \infty$ limit. The limiting density originates from the rescaled 1-point correlation function $\rho_{1,N}(\lambda) = K_N(\lambda, \lambda) \to \rho(\lambda)$ and forms a semicircle on the real line as N tends to infinity. This is the famous Wigner semicircle [95]. The same limiting behaviour of the 1-point function in the large N-limit on the global scale is also observed for the GOE(N) and GSE(N), which makes the semicircle a universal result.
We have seen that the eigenvalues of the QCD Dirac operator y_n come in pairs and are connected to the eigenvalues of the Wishart matrix $WW^\dagger$ λ_n via $y_n = \pm\sqrt{\lambda_n}$. It is convenient to go from the eigenvalues of $WW^\dagger$ (which are the squared singular values of W) to the singular values of W. We have seen this in particular in the derivation of the JPDF in section 2.2. For $q = 1$ the Marchenko Pastur distribution then becomes a quarter circular law, i.e.

$$\rho_{\mathrm{MP}}(\lambda = x^2)d\lambda = \frac{1}{2\pi}\sqrt{\frac{4-\lambda}{\lambda}}\mathbb{I}_{[0,4]}d\lambda = \frac{1}{\pi}\sqrt{4-x^2}\mathbb{I}_{[0,2]}dx = \rho_{\mathrm{QC}}(x)dx \tag{2.28}$$

This can be checked numerically by computing the singular values of W, see Fig. 2.2.

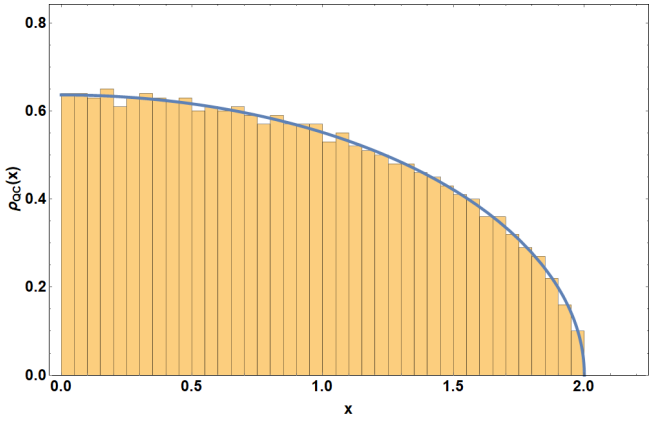

Figure 2.2: Quarter Circular Law: Computed are the singular values x of the block matrix W of the chGUE(N) with positive Dirac eigenvalues $\lambda = x^2$. Taking a matrix W of size $N = 2000$ and comparing the histogram with the quarter circular law shows good agreement between the numerical simulation and the analytical prediction.

Since the singular values of W only describe the $+\sqrt{\lambda_n}$ part of the Dirac spectrum, we obtain the full semicircle for the Dirac eigenvalues by mapping the quartercircle to the full real line. The *global* density of D at the origin $y = 0$ is then constant as expected. To study the behaviour of the Dirac spectrum in this limit with other techniques, like loop-equations, is also possible, see [96].
The concept of local scaling limits, sometimes referred to as *microscopic* limits, can be illustrated by the Marchenko Pastur density for $q = 1$, see Fig 2.1 and Eq. (2.28).
We can see that the density is defined on the interval $[\lambda_-, \lambda_+] = [0, 4]$. By rescaling the eigenvalues we can zoom in on different regions I), II) and III) of the support. Region II) is called *bulk* of the spectrum and contains all eigenvalues on the interior of the intervall (λ_-, λ_+). Looking at the correlation kernel associated with the Marchenko Pastur distribution it has been shown that in the bulk the *sine kernel* is found [23, 97, 98]

$$K_{\mathrm{bulk}}(x, y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}\,.$$

The sine kernel is highly universal in the bulk [85, 99–103].

For $q = 1$ the region III) is called *soft edge* of the spectrum. At the edge point $\lambda = 4$ the density vanishes like $\sqrt{N}$. For finite, but large N one can always find eigenvalues in the samples that are located to the right of the support $[0, 4]$ with non-zero probability. This means the edge of the spectrum lets eigenvalues through the barrier to the outside of the support of the density. Such a behaviour justifies the name *soft* edge. For all unitary ensembles, even those with regular external potentials, the kernel at the soft edge is universal and given by the *Airy kernel*

$$K_{\text{soft}}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y} ,$$

where $\text{Ai}(x)$ denotes the Airy function of argument x.[15]
The region I) of the Marchenko Pastur density is another type of edge called *hard edge* of the spectrum. The density blows up as $\lambda \to 0$ and builds an impregable wall for the eigenvalues that keeps them close to the origin. The rescaling to reach this region can be done via multiplication with a factor $N^{-1/2}$. For the chGUE(N) one finds the so-called *Bessel kernel* at the hard edge. The Bessel kernel for general ν takes the form

$$K_{\text{hard}}(x, y) = \frac{1}{2} \frac{x J_{\nu+1}(x) J_\nu(y) - y J_{\nu+1}(y) J_\nu(x)}{x^2 - y^2} ,$$

where $J_\nu(x)$ denotes the Bessel function of the first kind with argument x.
One of the important questions we want to answer in this thesis is wether the Bessel kernel is still universal when chGUE(N) models with external source describing temperature are considered at the hard edge in comparison to chGUE(N) models without the external source.

We have seen in the introduction that the application of RMT to effective theories of QCD, like chiral perturbation theory, is possible in the ε-regime leading to the microscopic limit of QCD, see Eq. (1.9). This implies that we have to be especially interested in the low-lying eigenvalues of the Dirac operator, since those eigenvalues determine a large portion of the chiral condensate and thus are primarily important in the chiral-non-chiral phase transition of QCD. This becomes apparent via the Banks-Casher relation in Eq. (1.7). Thus, on the RMT level we have to consider the region close to the origin, which features a hard edge for the eigenvalues of $WW^\dagger$ as described above. This local scaling limit is then comparable to the microscopic limit of QCD, after the appropriate rescaling of the eigenvalues λ_n by $N^{-1/2}$. We introduce the following scaling scheme for the masses m_f and eigenvalues λ_n:

$$m_f^2 = \frac{\mu_f^2}{4N\Xi} \quad \text{and} \quad \lambda_n = \frac{\zeta_n^2}{4N\Xi} ,$$

where μ_f and ζ_n are microscopic variables. As we zoom in at the origin and take $N \to \infty$ these variables remain finite and thus are comparable to their physical counterparts.[16] The value $\Xi = \Xi(a_1, \ldots, a_N)$ is the *temperature dependent* chiral condensate, which we can write as

$$\Xi = \begin{cases} \Sigma , & \text{for zero temperature models,} \\ \Xi(a_1, \ldots, a_N) , & \text{for non-zero temperature models.} \end{cases}$$

With this scaling scheme we are able to reach the hard edge limit, which appears for the

[15] The Airy kernel appeared first in [104, 105].
[16] The factor 4 ensures that comparisons with known results [13, 63] are possible in chapter 5.

squared singular values λ_n as we have seen from the Marchenko Pastur distribution near the origin. Furthermore, we are now able to compare results obtained in this limit and answer the question of universality for objects like the correlation kernel, the k-point function and the partition function.
We denote the k-point function, at the hard edge, with microscopic variables $\zeta_1, \ldots, \zeta_k$, describing the Dirac eigenvalues, by

$$\rho_S^{(N_f)}(\zeta_1, \ldots, \zeta_k) = \det_{1 \leq a,b \leq k} \left[K_S^{(N_f)}(\zeta_a, \zeta_b) \right]$$

with the corresponding correlation kernel also taken at the hard edge. The k-point function can also be written in terms of the k-point function of the squared singular values of W. Following the notation in [106] we have

$$\rho_S^{(N_f)}(\zeta_1, \ldots, \zeta_k) = |\zeta_1| \cdot \ldots \cdot |\zeta_k| \, \sigma_S^{(N_f)}(\zeta_1^2, \ldots, \zeta_k^2) \, ,$$

where σ_S denotes the k-point function for the squared singular values of W.
The kernel $K_S^{(N_f)}$ can be obtained from the finite N kernel $K_N^{(N_f)}$ via

$$K_S^{(N_f)}(\zeta_a, \zeta_b) = \sqrt{|\zeta_a \zeta_b|} \lim_{N \to \infty} \frac{1}{2N\Xi} K_N^{(N_f)} \left(x_1 = \frac{\zeta_a^2}{4N\Xi}, x_2 = \frac{\zeta_b^2}{4N\Xi} \right) \, .$$

For temperature independent random matrix models like the classical chGUE(N) it was also realized that finite-volume QCD partition functions $\mathcal{Z}_\nu$ can be expressed as large N limits of finite N partition functions Z_N, namely[17]

$$\mathcal{Z}_\nu^{(N_f+2k)}(\mu_1, \ldots, \mu_{N_f}, \{i\zeta_m\}_{m=1}^{2k}) = \lim_{N \to \infty} \frac{Z_N^{(N_f+2k,0)}(m_1^2, \ldots, m_{N_f}^2, \{z_m^2\}_{m=1}^{2k})}{C_N^{[N_f+2k]}} \tag{2.29}$$

with $m_f = \mu_f^2/4N$ and $z_m^2 = (i\zeta_m)^2/4N$.[18] This representation will be particularly helpful in showing universality in chapter 5.

2.5 Determinantal Point Processes

The random matrix models we introduced in section 2.1 form stochastic point processes with respect to the finite set of squared singular values (or eigenvalues) $\{\lambda_n\}_{n=1}^N$. Moreover, the joint probability density functions of the classical chGUE(N) given in Eq. (2.9) and the deformed chGUE(N) given in Eq. (2.10) are both of Δ^2-type and thus treatable via orthogonal polynomial methods as we have seen in previous sections of this chapter. Additionally, they form *determinantal point processes*,[19] which means all k-point correlation functions of the set $\{\lambda_1, \ldots, \lambda_N\}$ can be expressed as a determinant of a $k \times k$ matrix with kernel entries

$$\rho_{k,N}(\lambda_1, \ldots, \lambda_k) = \det \left[K_N(\lambda_m, \lambda_n) \right]_{m,n=1}^k \, .$$

[17] The form of the normalisation constant $C_N^{[N_f+2k]}$ is given in Eq. (5.8) in chapter 5.

[18] We have omitted the rescaling with the chiral condensate for the partition function, because it is usually included later in the expressions for the k-point functions and kernels as overall prefactors.

[19] More information on determinantal point processes can be found in [107, 108].

Note that there exists an invariance of the k-point correlation function under the following transformation of the kernel

$$K_N(x,y) \to K_N(x,y)\frac{f(x)}{f(y)} \tag{2.30}$$

for any non-zero function f.
The other two ensembles, the chGUE(N) with external source in Eq. (2.11) and the temperature depending deformed chGUE(N)-type ensemble in Eq. (2.12), are drawn from a more general class of ensembles. This class is called *biorthogonal ensembles* introduced by Borodin [109]. For biorthogonal ensembles classical methods like orthogonal polynomials are no longer easily applicable. Crucially, these ensembles still belong to the class of determinantal point processes, which was also shown by Borodin [109]. We aim to give a short overview of the important properties of biorthogonal ensembles and discuss sub-classes in this section.

Biorthogonal Ensembles

Biorthogonal ensembles, as described by Borodin [109], are given in terms of a JPDF of a set of real random variables $\{\lambda_1,\dots,\lambda_N\}$ drawn from an interval $I \subset \mathbb{R}$ with a determinantal structure, i.e.

$$\mathcal{P}(\lambda_1,\dots,\lambda_N) = \frac{1}{Z_N}\det\left[\psi_j(\lambda_i)\right]_{i,j=1}^N \det\left[\varphi_j(\lambda_i)\right]_{i,j=1}^N \tag{2.31}$$

with normalisation constant Z_N. The k-point functions are given via

$$\rho_{k,N}(\lambda_1,\dots,\lambda_k) = \det\left[K_N(\lambda_m,\lambda_n)\right]_{m,n=1}^k \ ,$$

where the kernel is given by

$$K_N(x,y) = \sum_{i,j=1}^N \psi_i(x)c_{i,j}\varphi_j(y) \quad \text{with} \quad \sum_{k=1}^N g_{i,k}c_{j,k} = \delta_{ij} \ .$$

The elements $g_{i,j}$ and $c_{i,j}$ can be collected in matrices $G = (g_{i,j})_{i,j=1}^N$ and $C = (c_{i,j})_{i,j=1}^N$ which are then connected via $C^T = G^{-1}$. This, in principle, reduces the solution of the correlation kernel to the linear algebra problem of computing the *Gram matrix* G and then its inverse [88, 110]. The Gram matrix elements are computed via the integral

$$g_{i,j} = \int_I d\lambda \psi_i(\lambda)\varphi_j(\lambda) \ .$$

Consequently, only the inversion of G remains as a challenge. Furthermore, it is possible to construct new biorthogonal functions $\eta_i \in \{\psi_1,\dots,\psi_N\}$ and $\zeta_j \in \{\varphi_1,\dots,\varphi_N\}$ via

$$\int_I d\lambda \eta_i(\lambda)\zeta_j(\lambda) = \delta_{ij} \ ,$$

which justifies the name *biorthogonal* and allows us to write the kernel as a single sum

$$K_N(x,y) = \sum_{k=1}^N \eta_k(x)\zeta_k(y) \ .$$

The normalization constant Z_N of any given biorthogonal ensemble can be computed using the extended Andreief formula, which is given in the following Proposition.

Proposition 2.3. *We consider the N-fold integration over two determinants of different sizes $(N+k)\times(N+k)$ and $(N+l)\times(N+l)$ with block structure, such that*

$$\prod_{n=1}^{N}\int_I dx_n \det\begin{bmatrix} R_{a,b} & |_{1\le a\le k}^{1\le b\le N+k} \\ \eta_b(x_a) & |_{1\le a\le N}^{1\le b\le N+k}\end{bmatrix} \det\begin{bmatrix} S_{b,a} & |_{1\le b\le N+l}^{1\le a\le l} & \varphi_b(x_a) & |_{1\le b\le N+l}^{1\le a\le N}\end{bmatrix}$$
$$= (-1)^{kl}N!\det\begin{bmatrix} 0_{k\times l} & & R_{a,b} & |_{1\le a\le k}^{1\le b\le N+k} \\ S_{b,a} & |_{1\le b\le N+l}^{1\le a\le l} & \int_I dx\varphi_b(x)\eta_a(x) & |_{1\le b\le N+l}^{1\le a\le N+k}\end{bmatrix}.$$

The functions η and φ are given such that the integrals on both sides are convergent. Apart from this they can be chosen arbitrarily.

Proof. See [111]. □

The special case $k=l=0$ leads back to the classical Andreief formula, i.e.

$$\left(\prod_{n=1}^{N}\int_I dx_n\right)\det[\psi_l(x_k)]_{k,l=1}^N\det[\phi_l(x_k)]_{k,l=1}^N = N!\det\left[\int_I dx\psi_k(x)\phi_l(x)\right]_{k,l=1}^N.$$

This allows us to compute the normalisation constant Z_N:

$$Z_N = \left(\prod_{n=1}^{N}\int_I d\lambda_n\right)\det\left[\psi_j(\lambda_i)\right]_{i,j=1}^N\det[\varphi_m(\lambda_n)]_{n,m=1}^N$$
$$= N!\det\left[\int_I d\lambda\psi_k(\lambda)\varphi_l(\lambda)\right]_{k,l=1}^N = N!\det G$$

with entries $g_{i,j}=\int_I d\lambda\psi_i(\lambda)\varphi_j(\lambda)$.

Subclasses of Biorthogonal Ensembles

There are many possible applications of biorthogonal ensembles in physics and mathematics depending on the choice of functions $\psi(\lambda)$ and $\varphi(\lambda)$. For example, choosing $\psi_j(\lambda)=\lambda^{j-1}$ and $\varphi_i(\lambda)=w(\lambda)\lambda^{i-1}$ leads to classical Δ^2-type ensembles with weight function $w(\lambda)$. In this sub-class of biorthogonal ensembles we may use orthogonal polynomial methods with respect to the weight function $w(\lambda)$. Therefore, this sub-class is denoted as the class of *orthogonal polynomial ensembles*. A simple example describing an orthogonal polynomial ensemble is given by the eigenvalues of $N\times N$ complex Hermitian random matrices H from the GUE(N), defined by the probability measure

$$P_{\mathrm{GUE}}(H)dH = c_N\exp[-\mathrm{Tr}\,[H^2]]dH\,,\quad c_N = 2^{\frac{N(N-1)}{2}}\pi^{-\frac{N^2}{2}}\,.$$

The probability density function of the real eigenvalues $x_1,\ldots,x_N$ of H reads [24]

$$\mathcal{P}_{\mathrm{GUE}}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N^{\mathrm{GUE}}}\Delta_N(\lambda_1,\ldots,\lambda_N)^2\exp\left[-\sum_{j=1}^{N}\lambda_j^2\right].$$

This is a biorthogonal ensemble, where the resulting functions are given as $\varphi_k(\lambda)=\lambda^{k-1}e^{-\lambda^2}$ and $\psi_k(\lambda)=\lambda^{k-1}$.

This sub-class also includes the classical chGUE(N) with Eqs. (2.2) and (2.9)

$$P_{\text{chGUE}}(W) = \frac{1}{\mathcal{N}_0} e^{-\text{Tr}\, WW^\dagger} ,$$
$$\mathcal{P}_{\text{chGUE}}(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N^{(0,0)}} \Delta_N^2(\lambda_1, \ldots \lambda_N) \left(\prod_{n=1}^{N} \lambda_n^\nu e^{-\lambda_n} \right) ,$$

and the deformed chGUE(N) with Eqs. (2.4) and (2.10)

$$P_{\text{chGUE}}^{\text{deformed}}(W) = \frac{1}{\mathcal{N}_{\text{def}}} \prod_{f=1}^{N_f} \det(D + m_f) \, \exp\left(-\text{Tr}\, WW^\dagger \right) ,$$
$$\mathcal{P}_{\text{chGUE}}^{\text{deformed}}(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N^{(N_f,0)}} \Delta_N^2(\lambda_1, \ldots \lambda_N) \left(\prod_{n=1}^{N} \lambda_n^\nu e^{-\lambda_n} \prod_{f=1}^{N_f} (m_f^2 + \lambda_n) \right) .$$

The intermediate step between general biorthogonal ensembles and orthogonal polynomial ensembles is achieved by ensembles whose JPDF features one Vandermonde determinant and one determinant each:

$$\mathcal{P}(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N} \Delta(\lambda_1, \ldots, \lambda_N) \det[\varphi_l(\lambda_k)]_{k,l=1}^N .$$

Looking at the definition of general biorthogonal ensembles in Eq. (2.31) this sub-class can be reached by setting $\psi_j(\lambda) = \lambda^{j-1}$. The existence of the one Vandermonde determinant in the JPDF hints at a closer connection to orthogonal polynomial methods in this sub-class. Following the terminology of Kuijlaars and Stivigny [112] this sub-class is called the class of *polynomial ensembles*. The biorthogonality relation and the computation of the Gram matrix entries is not necessarily simpler in this class, but in some cases it is possible to compute the $g_{i,j}$ explicitly. In this thesis we will use expectation values of characteristic polynomials and their connection to the correlation kernels to simplify the models, where the computation of the G matrix entries proves to be difficult.

Starting from an orthogonal polynomial ensemble a member of this more general sub-class is obtained by breaking the unitary bi-invariance via an external source [67, 68]. The GUE(N) with an external source [67, 70] containing an additional constant, deterministic Hermitian matrix A of size $N \times N$ in its probability measure is a simple example for a polynomial ensemble. We have[20]

$$P_{\text{ext1}}(H)dH = c_N \exp[-\text{Tr}\,[(H - A)^2]]dH ,$$

and a JPDF given as

$$\mathcal{P}_{\text{ext1}}(x_1, \ldots, x_N) = \frac{1}{\mathcal{Z}_N^{\text{ext1}}} \Delta_N(x_1, \ldots, x_N) \det\left[\exp[-(x_j - a_k)^2] \right]_{j,k=1}^N , \tag{2.32}$$

where we have to identify $\varphi_k(x) = e^{-(x-a_k)^2}$. This follows from a group integral, which is a version of the Harish-Chandra–Itzykson–Zuber integral [113, 114], and from multiplying the Gaussian term inside the determinant. We refer to [67] for the derivation. Notice also

[20] We choose A to be diagonal, $A = \text{diag}(a_1, \ldots, a_N)$ with $a_j \in \mathbb{R}$ for $j = 1, \ldots, N$, without loss of generality.

that the second determinant in Eq. (2.32) cannot be reduced to a Vandermonde determinant in general. The class of polynomial ensembles naturally includes the chGUE(N) with an external source matrix T leading to additional parameters $\{a_n\}_{n=1}^N$ (compare Eqs. (2.3) and (2.11)) [65]

$$\begin{aligned} P^{\text{ext}}_{\text{chGUE}}(W) &= \frac{1}{\mathcal{N}_{\text{ext}}} \exp\left(-\text{Tr}\,(W-T)(W^\dagger - T^\dagger)\right) \\ \mathcal{P}^{\text{ext}}_{\text{chGUE}}(\lambda_1,\ldots,\lambda_N) &= \frac{1}{Z_N^{(0,T)}} \left[\prod_{n=1}^N \left(\frac{\lambda_n}{a_n}\right)^{\nu/2} e^{-(\lambda_n+a_n)}\right] \\ &\quad \times \det\left[I_\nu\left(2\sqrt{a_i\lambda_j}\right)\right]_{i,j=1}^N \Delta_N(\lambda_1,\ldots\lambda_N) \\ &= \frac{1}{Z_N^{(0,T)}} \det\left[\varphi_i(\lambda_j)\right]_{i,j=1}^N \Delta_N(\lambda_1,\ldots\lambda_N)\,. \end{aligned}$$

The modified Bessel function of second kind I_ν, inside the determinant with function

$$\varphi_k(\lambda) = \left(\frac{\lambda}{a_k}\right)^{\nu/2} e^{-(\lambda+a_k)} I_\nu\left(2\sqrt{a_k\lambda}\right),$$

follows from the Berezin-Karpelevich integral, cf. [82]. In principle we may also allow the parameter $\nu > -1$ to take real values.
The more general chGUE(N) ensemble with N_f massive flavors and temperature dependence defined in Eq. (2.5) and JPDF given in Eq. (2.12) also belongs to the class of polynomial ensembles by setting

$$\varphi_k(\lambda) = \left(\frac{\lambda}{a_k}\right)^{\nu/2} e^{-(\lambda+a_k)} I_\nu\left(2\sqrt{a_k\lambda}\right) \prod_{f=1}^{N_f}(m_f^2+\lambda),$$

which leads to

$$\begin{aligned} P^{\text{temp}}_{\text{chGUE}}(W) &= \frac{1}{\mathcal{N}} \prod_{f=1}^{N_f} \det(D_{\text{temp}} + m_f)\ \exp\left(-\text{Tr}\, WW^\dagger\right), \\ \mathcal{P}^{\text{temp}}_{\text{chGUE}}(\lambda_1,\ldots,\lambda_N) &= \frac{1}{Z_N^{(N_f,T)}} \det\left[\varphi_i(\lambda_j)\right]_{i,j=1}^N \Delta_N(\lambda_1,\ldots\lambda_N). \end{aligned}$$

More examples from the class of polynomial ensembles are the so-called Muttalib-Borodin ensembles [109, 115] and the Polya ensembles [116, 117]. In addition to the application to temperature dependent QCD, polynomial ensembles appear as distributions of eigenvalues (or singular values) of random matrices [65–68, 70], in sums and products of random matrices [31, 118–120] and in the context of counting intersection numbers of moduli spaces on Riemann surfaces [121]. Polymomial ensembles still have some invariance properties in terms of their correlation kernels and k-point functions, as well as their associated biorthogonal functions [117, 122].
In the next chapter we will review some important results for polynomial ensembles with emphasis on the impact of expectation values of characteristic polynomials in this sub-class of determinantal point processes.

Chapter 3

Expectation Values of Characteristic Polynomials in Polynomial Ensembles

In this chapter we look at the impact of expectation values of characteristic polynomials in a sub-class of determinantal point processes. This sub-class contains special biorthogonal ensembles [109] which are called polynomial ensembles [123]. We start the discussion in this chapter with the definition of a polynomial ensemble, as we will use it in the rest of the chapter and then continue with presenting results derived in part in the principal publication [1].

Definition 3.1. *Consider a set of real random variables $x_1, \ldots, x_N \in I$ defined by the joint probability density function (JPDF) $\mathcal{P}(x_1, \ldots, x_N)$, where $I \subset \mathbb{R}$ is an interval. The variables $x_1, \ldots, x_N$ form a* polynomial ensemble *[123], if their JPDF takes the form*

$$\mathcal{P}(x_1, \ldots, x_N) = \frac{1}{Z_N} \Delta_N(x_1, \ldots, x_N) \det[\varphi_l(x_k)]_{k,l=1}^N ,$$

where $\Delta_N(x_1, \ldots, x_N) = \prod_{j>i}(x_j - x_i)$ is the Vandermonde determinant of N variables. The $\varphi_1, \ldots \varphi_N$ are certain integrable, real-valued functions on I, such that the normalisation constant Z_N, which reads

$$Z_N = \left(\prod_{n=1}^N \int_I dx_n \right) \Delta_N(x_1, \ldots, x_N) \det[\varphi_l(x_k)]_{k,l=1}^N = N! \det G , \tag{3.1}$$

exists and is non-zero. The matrix entries $g_{k,l}$ are supposed to be finite and can be computed via

$$g_{k,l} = \int_I dx x^{k-1} \varphi_l(x) , \tag{3.2}$$

such that the Gram matrix $G = (g_{i,j})_{i,j=1}^N$ is non-singular.

The motivation to study polynomial ensembles in this thesis originates from effective models for the theory of strong interactions under the influence of external parameters like temperature. We have introduced the models with and without temperature in section 2.1. For more information regarding the zero-temperature models, we refer to [32, 63, 64], and similarly to [12, 13] for results in the non-zero temperature case.

We will collect important results for expectation values of characteristic polynomials in polynomial ensembles in section 3.1. We will denote a characteristic polynomial of the variables $x_1, \ldots x_N$ as $D_N(x) = \prod_{n=1}^N (x - x_n)$ with respect to the value x. The expectation value was

already introduced in Eq. (2.20) for a given JPDF of N variables:

$$\mathbb{E}_{\mathcal{P}}\left[\mathcal{O}(x_1,\ldots,x_N)\right] = \left(\prod_{n=1}^{N}\int_I dx_n\right)\mathcal{P}(x_1,\ldots,x_N)\mathcal{O}(x_1,\ldots,x_N)\,. \tag{3.3}$$

With this preparation we are ready to state results for expectation values of products and ratios of characteristic polynomials in polynomial ensembles.
We start from the analog of the Heine formula, going to products and then arbitrary ratios of characteristic polynomials. We then proceed to introduce the definition of *invertible* polynomial ensembles in section 3.2. We discuss examples of this class and present the first main result of this thesis, which is a multi-integral representation of an arbitrary ratio of characteristic polynomials. Additionally, we provide a procedure called *reweighting* in section 3.3, which allows us to trace more general polynomial ensembles back to invertible polynomial ensembles. This procedure will become important in chapters 4 and 5, when we consider the most general model in this thesis of temperature dependent Dirac eigenvalues in the presence of N_f massive flavors.
This chapter is based on the work in [1].

3.1 Determinantal structures in polynomial ensembles

We focus on results derived for expectation values of characteristic polynomials and their impact in polynomial ensemble settings in this section in order to keep this thesis compact. To mention all the authors and results in this field is beyond the scope of this thesis, so we refer to the following list of references [1, 25, 65, 66, 91, 109, 112, 120, 122, 124–126], where the results of this section are drawn from, and where more informations and references may be found.
The analog of the Heine formula, presented in chapter 2, see Eq. (2.19), for polynomial ensembles reads [65, 124]

$$\mathbb{E}_{\mathcal{P}}\left[D_N(z)\right] = \frac{1}{\det G}\begin{vmatrix} g_{1,1} & \cdots & g_{1,N} & z^0 \\ g_{2,1} & \cdots & g_{2,N} & z^1 \\ \vdots & \vdots & \vdots & \vdots \\ g_{N+1,1} & \cdots & g_{N+1,N} & z^N \end{vmatrix}. \tag{3.4}$$

This formula can be extended to an arbitrary number of characteristic polynomials. The resulting formula and a proof can be found in Appendix B.
Similarly, the expectation value of an inverse characteristic polynomial for any given polynomial ensemble was computed in [65] and later also stated in [124]. The result reads

$$\mathbb{E}_{\mathcal{P}}\left[\frac{1}{D_N(y)}\right] = \frac{1}{\det G}\begin{vmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N-1,1} & g_{N-1,2} & \cdots & g_{N-1,N} \\ \int_0^\infty du\,\frac{\varphi_1(u)}{y-u} & \int_0^\infty du\,\frac{\varphi_2(u)}{y-u} & \ldots & \int_0^\infty du\,\frac{\varphi_N(u)}{y-u} \end{vmatrix}. \tag{3.5}$$

The next step is then to consider expectation values of *ratios* of characteristic polynomials. For polynomial ensembles with *real* eigenvalues[1] it has been shown [14, 127] that the k-point

[1] All ensembles we have introduced in chapter 2 feature real eigenvalues - or squared singular values, which are real.

function can be obtained via a ratio of characteristic polynomials:

$$\rho_{k,N}(x_1,\ldots,x_k) = \operatorname*{Res}_{z_1=x_1}\ldots\operatorname*{Res}_{z_k=x_k}\left(\frac{\partial_k}{\partial y_1\ldots\partial y_k}\mathbb{E}_{\mathcal{P}}\left[\prod_{i=1}^{k}\frac{D_N(y_i)}{D_N(z_i)}\right]\right)_{y_i=z_i}.$$

The residue in the equation above is defined through

$$f(x) \equiv \operatorname*{Res}_{z=x}\int_I dt\frac{f(t)}{z-t}\,. \tag{3.6}$$

This immediately leads to the question how to compute ratios of characteristic polynomials. For a simple ratio of two characteristic polynomials we find [25, 128]

$$\mathbb{E}_{\mathcal{P}}\left[\frac{D_N(z)}{D_N(y)}\right] = \frac{(-1)}{\det G}\begin{vmatrix} g_{1,1} & \cdots & g_{1,N} & z^0 \\ g_{2,1} & \cdots & g_{2,N} & z^1 \\ \vdots & \vdots & \vdots & \vdots \\ g_{N,1} & \cdots & g_{N,N} & z^{N-1} \\ \int_0^\infty du\frac{z-u}{y-u}\varphi_1(u) & \ldots & \int_0^\infty du\frac{z-u}{y-u}\varphi_N(u) & 0 \end{vmatrix}.$$

In [1] we showed that the following Theorem of Borodin, Olshanski, Strahov [129], derived originally for orthogonal polynomial ensembles, also holds for polynomial ensembles:

Theorem 3.2. *Assume that $x_1,\ldots,x_N$ form a polynomial ensemble in the sense of Definition 3.1. Let $u_1,\ldots,u_M \in \mathbb{C}\backslash\mathbb{R}$ and $z_1,\ldots,z_M \in \mathbb{C}$ for any $M \in \mathbb{N}$ be pairwise distinct variables. Then*

$$\mathbb{E}_{\mathcal{P}}\left[\prod_{m=1}^{M}\frac{D_N(z_m)}{D_N(u_m)}\right] = \left[\det\left(\frac{1}{u_i-z_j}\right)_{i,j=1}^{M}\right]^{-1}\det\left[\frac{1}{u_i-z_j}\mathbb{E}_{\mathcal{P}}\left(\frac{D_N(z_j)}{D_N(u_i)}\right)\right]_{i,j=1}^{M},$$

where $D_N(z) = \prod_{n=1}^{N}(z-x_n)$ denotes the characteristic polynomial associated with the random variables $x_1, \ldots, x_N$.

The proof can be found in Appendix D using the notion of Giambelli compatible point processes, which is an interesting sub-class of determinantal point processes.
There is another application for the ratio of characteristic polynomials shown in [65]. We can use the ratio to derive the correlation kernel $K_N(x_1,x_2)$ via the following formula:

$$K_N(x_1,x_2) = \frac{1}{x_1-x_2}\operatorname*{Res}_{y=x_2}\left(\mathbb{E}_{\mathcal{P}}\left[\frac{D_N(x_1)}{D_N(y)}\right]\right). \tag{3.7}$$

We will capitalize on this equation in chapter 4 to derive the correlation kernel for our temperature dependent models for QCD. In order to do so for both models, with and without N_f massive flavors, we will need not only the ratio of two characteristic polynomials but rather a general ratio of arbitrary numbers of characteristic polynomials in both numerator and denominator of the ratio. We will discuss this in the next section. More details on determinantal structures and more formulae can be found in Appendix B.

3.2 Results for Invertible Polynomial Ensembles

In this section we present one of the main results from [1], which features arbitrary ratios of characteristic polynomials,

$$\mathbb{E}\left[\frac{\prod_{m=1}^{M} D_N(z_m)}{\prod_{l=1}^{L} D_N(y_l)}\right],$$

allowing the number of characteristic polynomials in the numerator M and denominator, $L \leq N$, to differ. We impose the following conditions on the parameters y_n and z_n: The parameters $y_1, \ldots, y_L$ must be drawn from $\mathbb{C}\backslash\mathbb{R}$, while the parameters $z_1, \ldots, z_M$ can be taken from $\mathbb{C}$ - in both cases the parameters have to be pairwise distinct. We will not consider the most general polynomial ensembles in the sense of Definition 3.1. Instead we impose certain conditions on the functions $\varphi_j(x)$ which are specified below. We obtain a sub-class of polynomial ensembles, which we call *invertible* polynomial ensembles.

Definition 3.3. *Consider a polynomial ensemble from Definition 3.1 and assume that the function $\varphi_l(x) = \varphi(a_l, x)$ is analytic in both arguments, for $l = 1, \ldots, N$, where $a_1, \ldots, a_N$ are real parameters. Additionally, assume that there exists a family $\{\pi_k\}_{k=0}^{\infty}$ of monic polynomials such that each polynomial π_k of degree k can be represented as*

$$\pi_k(a) = \int_I dx x^k \varphi(a, x), \quad k = 0, 1, \ldots. \tag{3.8}$$

In addition, assume that Eq. (3.8) *is invertible, i.e. there exists a function $F : I' \times \mathbb{C} \to \mathbb{C}$ such that*

$$z^k = \int_{I'} ds F(s, z)\pi_k(s), \quad k = 0, 1, \ldots, \tag{3.9}$$

where I' is a certain contour in the complex plane. Then, we will refer to such a polynomial ensemble as an invertible polynomial ensemble.

Remark 3.4. The condition in Eq. (3.8) together with Eq. (3.1) immediately implies that for any invertible polynomial ensemble the normalising partition function simplifies as follows:

$$Z_N = N!\Delta_N(a_1, \ldots, a_N). \tag{3.10}$$

Here, we use the Andreief formula in Proposition 2.3 and that the Vandermonde determinant can be rewritten as a determinant of arbitrary monic polynomials.

We are now ready to discuss the first simple example of a polynomial ensemble that is invertible according to our Definition 3.3.

Example 3.5. We have already introduced the GUE(N) with an external source in Eq. (2.32). There, the eigenvalues take real values, $I = \mathbb{R}$, and the functions $\varphi_l(x)$ can be chosen as

$$\varphi_l(x) = \varphi(a_l, x) = \frac{e^{-(x-a_l)^2}}{\sqrt{\pi}}, \tag{3.11}$$

which are analytic. From [130, Eq. 8.951] we know the following representation of the standard Hermite polynomials $H_n(t)$ of degree n:

$$H_n(t) = \frac{(2i)^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx e^{-(x+it)^2} x^n,$$

which can be made monic as follows, $2^{-n}H_n(x) = x^n + O(x^{n-2})$. This leads directly to the first condition for invertibility via the integral

$$(2i)^{-n}H_n(ia) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} ds s^n e^{-(s-a)^2} ,$$

from which we can read off

$$\pi_k(a) = \int_{-\infty}^{\infty} dx x^k \frac{e^{-(x-a)^2}}{\sqrt{\pi}} ,$$

with $\pi_k(a) = (2i)^{-k}H_k(ia)$, for $k = 0, 1, \ldots$, which is again monic. Thus, the first condition in Eq. (3.8) is satisfied. For the second condition in Eq. (3.9) we use the integral [130, Eq. 7.374.6]

$$y^n = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, 2^{-n}H_n(x)e^{-(x-y)^2} .$$

Renaming $y = iz$ and $x = is$ we obtain

$$z^k = \int_{I'} ds F(s,z)\pi_k(s) , \quad \text{for } k = 0, 1, \ldots, \quad \text{with} \quad F(s,z) = \frac{i}{\sqrt{\pi}}e^{(s-z)^2}$$

and $I' = i\mathbb{R}$. This implies that also the second condition is satisfied and the GUE(N) with an external source A is indeed an invertible polynomial ensemble.

Remark 3.6. Example 3.5 is the simplest case of a much wider class of polynomial ensembles of Polya type convolved with fixed matrices, as introduced in [131, Theorem II.3]. Such polynomials ensembles are generalising the form in Eq. (3.11) to

$$\varphi(a_l, x) = f(x - a_l) ,$$

such that the function f is $(N-1)$-times differentiable on $\mathbb{R}$, analytic on $\mathbb{C}$, and the moments of its derivatives exist,[2]

$$\left| \int_{-\infty}^{\infty} dx x^k \frac{\partial^j f(x)}{\partial x^j} \right| < \infty , \quad \forall k, j = 0, 1, \ldots, N-1 .$$

It immediately follows that its generalised moment matrix leads to polynomials, upon shifting the integration variable, and thus Eq. (3.8) is satisfied. It is not too difficult to show using Fourier transformation of f that also the condition in Eq. (3.9) of Definition 3.3 is satisfied and thus these ensembles are indeed invertible.

Example 3.7. Our second example is the chGUE(N) with an external source, see Eq. (2.11). We have $I = \mathbb{R}_+$, since the variables x_n now originate from a singular value decomposition and thus are real and positive. The functions $\varphi_l(x)$ can be chosen as

$$\varphi_l(x) = \varphi(a_l, x) = \left(\frac{x}{a_l}\right)^{\nu/2} e^{-(x+a_l)} I_\nu(2\sqrt{a_l x}) ,$$

which are analytic, with positive, real numbers a_l describing the temperature dependence in our model. The following integral is known, see e.g. [130, Eq. 6.631.10] after analytic

[2] The unconvoluted polynomial ensemble has $\varphi_j(x) = \partial^j f(x)/\partial x^j$.

continuation,

$$\int_0^\infty x^{n+\frac{\nu}{2}} e^{-x} I_\nu(2\sqrt{ax})dx = n! a^{\nu/2} e^a L_n^\nu(-a) \ .$$

Here, $L_n^\nu(y)$ is the standard generalised Laguerre polynomial of degree n, which is made monic as follows, $n!L_n^\nu(-x) = x^n + O(x^{n-1})$. Then, the first condition (3.8) is satisfied,

$$\pi_k(a) = \int_0^\infty dx x^k \left(\frac{x}{a}\right)^{\frac{\nu}{2}} e^{-(x+a)} I_\nu(2\sqrt{ax}) \ ,$$

with $\pi_k(a) = k!L_k^\nu(-a)$ for $k = 0, 1, \ldots$. For the second condition given in Eq. (3.9) we consider the following integral, see [130, Eq. 7.421.6], which is also called Hankel transform,

$$\int_0^\infty dt t^{\nu/2} e^{-t} n! L_n^\nu(t) J_\nu\left(2\sqrt{zt}\right) = z^n z^{\nu/2} e^{-z} \ .$$

Bringing factors on the other side and making the substitution $t = -s$ to make the same monic polynomials $n!L_n^\nu(-s)$ as above appear in the integrand, we obtain after using the identity of Bessel functions $I_\nu(x) = i^{-\nu} J_\nu(ix)$

$$z^k = \int_{I'} ds F(s,z)\pi_k(s) \ , \ \text{ for } k = 0, 1, \ldots, \text{ and } \ F(s,z) = (-1)^\nu \left(\frac{s}{z}\right)^{\nu/2} e^{s+z} I_\nu\left(2\sqrt{zs}\right) \quad (3.12)$$

with $I' = \mathbb{R}_- = (-\infty, 0]$. Consequently, also the chGUE(N) with an external source is an invertible polynomial ensemble.

Remark 3.8. The chGUE(N) with an external source is sometimes defined without the factors $a^{-\nu/2}e^{-a}$ in the literature, in contrast to our definition in Eqs. (2.11), (2.12). This turns out to be a disadvantage, since the conditions of Definition 3.3 for invertible polynomial ensembles are not satisfied. Therefore, we consider the models after including these factors in order to meet the conditions of Definition 3.3. Consequently, we derive the following formulae for expectation values below.[3]

We can now state the main results of this chapter. First, we have a formula for expectation values of products and ratios of characteristic polynomials in the case of invertible polynomial ensembles.

Theorem 3.9. *Consider a polynomial ensemble from Definition 3.1 formed by $x_1, \ldots, x_N$, and assume that this ensemble is invertible in the sense of Definition 3.3. Then we have for $L \leq N$*

$$\begin{aligned}
\mathbb{E}_{\mathcal{P}}\left[\frac{\prod_{m=1}^M D_N(z_m)}{\prod_{l=1}^L D_N(y_l)}\right] &= \frac{(-1)^{\frac{L(L-1)}{2}}}{L!\Delta_M(z_1,\ldots,z_M)} \left[\prod_{j=1}^M \int_{I'} ds_j F(s_j, z_j) \prod_{n=1}^N (s_j - a_n)\right] \\
&\times \left[\prod_{l=1}^L \int_I dv_l \left(\frac{v_l}{y_l}\right)^{N-L} \frac{\prod_{m=1}^M (z_m - v_l)}{\prod_{j=1}^L (y_j - v_l)}\right] \\
&\times \left[\prod_{l=1}^L \oint_{C_l} \frac{du_l}{2\pi i} \frac{1}{\prod_{n=1}^N (u_l - a_n)} \frac{\varphi(u_l, v_l)}{\prod_{j=1}^M (s_j - u_l)}\right] \\
&\times \Delta_M(s_1,\ldots,s_M)\Delta_L(v_1,\ldots,v_L)\Delta_L(u_1,\ldots,u_L) \ ,
\end{aligned} \quad (3.13)$$

[3] Looking more closely at the derivation of the JPDF for the chGUE(N) with external source - see Appendix B - we find that the relevant factor can also be absorbed into $\mathcal{N}_{\text{ext}}$ and is recovered by changing $\mathcal{N}_{\text{ext}}$.

where $D_N(z) = \prod_{n=1}^{N}(z - x_n)$ denotes the characteristic polynomial associated with the random variables $x_1, \ldots, x_N$, the parameters $y_1, \ldots, y_L \in \mathbb{C}\backslash\mathbb{R}$ and $z_1, \ldots, z_M \in \mathbb{C}$ are pairwise distinct, and all contours C_l with $l = 1, \ldots, N$ encircle the points $a_1, \ldots, a_N$ counter-clockwise.

We would like to emphasize that Theorem 3.9 generalises Theorem 5.1 in [25] for the ratio of two characteristic polynomials derived for the polynomial ensemble with $\varphi(a, x) = x^{\mathcal{L}} e^{-x} I_0(2\sqrt{ax})$ to general ratios in invertible polynomial ensembles. Our result is well suited for the asymptotic analysis when $N \to \infty$ as the number of integrations does not depend on N. Theorem 3.9 features $M + L$ real integrations and L complex contour integrations and both M and L are fixed natural numbers, independent of N. We proceed with the proof of Theorem 3.9.

Proof. Denoting by S_N the symmetric group of a set of N variables with its elements being the permutations of the variables in the set, we will utilise the following Lemma that was proven in [132].

Lemma 3.10. *Let L be an integer with $1 \leq L \leq N$, and let $x_1, \ldots, x_N$ and $y_1, \ldots, y_L$ denote two sets of parameters that are pairwise distinct. Then the following identity holds*

$$\prod_{l=1}^{L} \frac{y_l^{N-L}}{\prod_{n=1}^{N}(y_l - x_n)} = \sum_{\sigma \in S_N/(S_{N-L} \times S_L)} \frac{\Delta_L(x_{\sigma(1)}, \ldots, x_{\sigma(L)})\Delta_{N-L}(x_{\sigma(L+1)}, \ldots, x_{\sigma(N)}) \prod_{n=1}^{L} x_{\sigma(n)}^{N-L}}{\Delta_N(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) \prod_{n,l=1}^{L}(y_l - x_{\sigma(n)})(-1)^{L(L-N)}}$$

on the coset of the permutation group.

As shown in [132] this follows from the Cauchy-Littlewood formula and the determinantal formula for the Schur polynomials, which are defined in Appendix D, see Eq. (D.1). We can use this identity to reduce the number of variables in the inverse characteristic polynomials from N to L. Applied to the averages of products and ratios of characteristic polynomials, we obtain the formula

$$\begin{aligned} \mathbb{E}_{\mathcal{P}}\left[\frac{\prod_{m=1}^{M} D_N(z_m)}{\prod_{l=1}^{L} D_N(y_l)}\right] &= \frac{N!(-1)^{L(L-N)}}{(N-L)!L!Z_N}\left[\prod_{n=1}^{N}\int_I dx_n \prod_{m=1}^{M}(z_m - x_n)\right] \det[\varphi_l(x_k)]_{k,l=1}^{N} \\ &\quad \times \frac{\prod_{k=1}^{L}\left(\frac{x_k}{y_k}\right)^{N-L}}{\prod_{n,l=1}^{L}(y_l - x_n)} \Delta_L(x_1, \ldots, x_L)\Delta_{N-L}(x_{L+1}, \ldots, x_N)\,, \end{aligned}$$

where we used the fact that each term in the sum over permutations gives the same contribution to the expectation. Hence, we can undo the permutations under the sum by a change of variables, and replace the sum over $S_N/(S_{N-L} \times S_L)$ by the cardinality of the coset space $N!/(N-L)!L!$. Expanding the determinant over the $\det[\varphi_l(x_k)]_{k,l=1}^{N}$ is the next logical step. Then we separate the integration over the first L variables $x_{l=1,\ldots,L}$ and the following $N - L$

variables $x_{n=L+1,\dots,N}$, by also splitting the characteristic polynomials accordingly. This gives

$$\begin{aligned}\mathbb{E}_{\mathcal{P}}\left[\frac{\prod_{m=1}^{M} D_N(z_m)}{\prod_{l=1}^{L} D_N(y_l)}\right] &= \frac{(-1)^{L(L-N)}N!}{(N-L)!L!Z_N}\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\\ &\times\left[\prod_{l=1}^{L}\int_I dx_l\ \varphi_{\sigma(l)}(x_l)\frac{x_l^{N-L}}{y_l^{N-L}}\frac{\prod_{m=1}^{M}(z_m-x_l)}{\prod_{j=1}^{L}(y_j-x_l)}\right]\Delta_L(x_1,\dots,x_L)\\ &\times\left[\prod_{k=L+1}^{N}\int_I dx_k\ \varphi_{\sigma(k)}(x_k)\prod_{m=1}^{M}(z_m-x_k)\right]\Delta_{N-L}(x_{L+1},\dots,x_N)\,.\end{aligned} \tag{3.14}$$

Because we are aiming at an expression that will be amenable to taking the large-N limit, we now focus on the integrals over $N-L$ variables in the second line, which we denote by J. Here, we make use of one of the properties of the Vandermonde determinant, namely the absorption of the M characteristic polynomials in J into a larger Vandermonde determinant, see Eq. (A.2), to write

$$J = \left[\prod_{k=L+1}^{N}\int_I dx_k\ \varphi_{\sigma(k)}(x_k)\right]\frac{\Delta_{N-L+M}(x_{L+1},\dots,x_N,z_1,\dots,z_M)}{\Delta_M(z_1,\dots,z_M)}\,.$$

We use the representation of the Vandermonde determinant given in Eq. (A.1) to pull the integrations $\int_I dx_k\ \varphi_{\sigma(k)}(x_k)$ into the corresponding columns and use the definition of the Gram matrix and its entries, see Eq. (3.2), to obtain

$$J = \frac{1}{\Delta_M(z_1,\dots,z_M)}\begin{vmatrix} g_{1,\sigma(L+1)} & \cdots & g_{1,\sigma(N)} & 1 & \dots & 1\\ g_{2,\sigma(L+1)} & \cdots & g_{2,\sigma(N)} & z_1 & \dots & z_M\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ g_{N+M-L,\sigma(L+1)} & \cdots & g_{N+M-L,\sigma(N)} & z_1^{N+M-L-1} & \dots & z_M^{N+M-L-1}\end{vmatrix}\,.$$

The first property of an invertible polynomial ensemble, given in Eq. (3.8), enables us to rewrite J as

$$\begin{aligned}J = &\frac{1}{\Delta_M(z_1,\dots,z_M)}\\ &\times\begin{vmatrix} \pi_0(a_{\sigma(L+1)}) & \dots & \pi_0(a_{\sigma(N)}) & 1 & \dots & 1\\ \pi_1(a_{\sigma(L+1)}) & \dots & \pi_1(a_{\sigma(N)}) & z_1 & \dots & z_M\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \pi_{N+M-L-1}(a_{\sigma(L+1)}) & \dots & \pi_{N+M-L-1}(a_{\sigma(N)}) & z_1^{N+M-L-1} & \dots & z_M^{N+M-L-1}\end{vmatrix}\,.\end{aligned}$$

The second property of invertible polynomial ensembles given in Eq. (3.9) allows us to replace again the determinant of monic polynomials by a Vandermonde determinant of size $N-L+M$ to obtain

$$J = \frac{\Delta_{N-L}(a_{\sigma(L+1)},\dots,a_{\sigma(N)})}{\Delta_M(z_1,\dots,z_M)}\left[\prod_{j=1}^{M}\int_{I'} dt_j F(t_j,z_j)\prod_{n=L+1}^{N}(t_j-a_{\sigma(n)})\right]\Delta_M(t_1,\dots,t_M)\,.$$

Let us come back to the expectation value of characteristic polynomials in the form of Eq. (3.14) and insert what we have derived for J above. This gives

$$\begin{aligned}\mathbb{E}_{\mathcal{P}}\left[\frac{\prod_{m=1}^{M} D_N(z_m)}{\prod_{l=1}^{L} D_N(y_l)}\right] &= \frac{(-1)^{L(L-N)}N!}{(N-L)!L!Z_N\Delta_M(z_1,\ldots,z_M)} \\ &\times \left[\prod_{j=1}^{M}\int_{I'} dt_j F(t_j,z_j)\prod_{n=1}^{N}(t_j-a_n)\right]\Delta_M(t_1,\ldots,t_M) \\ &\times \left[\prod_{l=1}^{L}\int_{I} dx_l \left(\frac{x_l}{y_l}\right)^{N-L}\frac{\prod_{m=1}^{M}(z_m-x_l)}{\prod_{j=1}^{L}(y_j-x_l)}\right]\Delta_L(x_1,\ldots,x_L) \\ &\times \sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\Delta_{N-L}(a_{\sigma(L+1)},\ldots,a_{\sigma(N)})\prod_{l=1}^{L}\frac{\varphi(a_{\sigma(l)},x_l)}{\prod_{j=1}^{M}(t_j-a_{\sigma(l)})}\,.\end{aligned} \tag{3.15}$$

The integrals are now put into a form to apply the following Lemma, that will allow us to simplify (and eventually get rid of) the sum over permutations.

Lemma 3.11. *Let S_N denote the permutation group of $\{1,\ldots,N\}$, and let S_L be the subgroup of S_N realized as the permutation group of the first L elements $\{1,\ldots,L\}$. Also, let S_{N-L} be the subgroup of S_N realised as the permutation group of the remaining $N-L$ elements $\{L+1,\ldots,N\}$. Assume that F is a complex valued function on S_N which satisfies the condition $F(\sigma h) = F(\sigma)$ for each $\sigma \in S_N$, and each $h \in S_L \times S_{N-L}$. Then we have*

$$\sum_{\sigma\in S_N} F(\sigma) = (N-L)!L! \sum_{1\le l_1<\ldots<l_L\le N} F\left(\left(l_1,\ldots,l_L,1,\ldots,\check{l}_1,\ldots,\check{l}_L\ldots,N\right)\right), \tag{3.16}$$

where $(i_1,\ldots,i_N)$ is one-line notation for the permutation $\begin{pmatrix}1 & 2 & \ldots & N\\ i_1 & i_2 & \ldots & i_N\end{pmatrix}$, and notation $\check{l}_p$ means that l_p is removed from the list.

Proof. Recall that if G is a finite group, and H is its subgroup, then there are transversal elements $t_1,\ldots,t_k \in G$ for the left cosets of H such that $G = t_1H \uplus \ldots \uplus t_kH$, where $\uplus$ denotes disjoint union. It follows that if F is a function on G with the property $F(gh) = F(g)$ for any $g \in G$, and any $h \in H$, then

$$\sum_{g\in G} F(g) = |H| \sum_{i=1}^{k} F(t_i)\,, \tag{3.17}$$

where $|H|$ denotes the number of elements in H. In our situation $G = S_N$, $H = S_L \times S_{N-L}$, and each transversal element can be represented as a permutation

$$\left(l_1,\ldots,l_L,1,\ldots,\check{l}_1,\ldots,\check{l}_L\ldots,N\right),$$

written in one-line notation, where $1 \le l_1 < \ldots < l_L \le N$. Moreover, each collection of numbers $l_1,\ldots,l_L$ satisfying the condition $1 \le l_1 < \ldots < l_L \le N$ gives a transversal element for the left cosets of $H = S_L \times S_{N-L}$ in $G = S_N$. We conclude that Eq. (3.17) is reduced to Eq. (3.16). □

Assume that $\Phi(x_1,\ldots,x_L)$ is antisymmetric under permutations σ of its L variable, i.e.

$$\Phi(x_{\sigma(1)},\ldots,x_{\sigma(L)}) = \mathrm{sgn}\,(\sigma)\Phi(x_1,\ldots,x_L),$$

and that $L \leq N$. Let F be the function on S_N defined by

$$F(\sigma) = \mathrm{sgn}\,(\sigma)\Delta_{N-L}(a_{\sigma(L+1)},\ldots,a_{\sigma(N)}) \left[\prod_{k=1}^{L}\int_I dx_k f(a_{\sigma(k)},x_k)\right]\Phi(x_1,\ldots,x_L)\,,$$

where f is a function of two variables. Clearly, F satisfies the condition $F(\sigma h) = F(\sigma)$ for each $\sigma \in S_N$, and each $h \in S_L \times S_{N-L}$. Application of Lemma 3.11 to this function gives

$$\begin{aligned}\sum_{\sigma\in S_N} F(\sigma) = (N-L)!L! \sum_{1\leq l_1<\ldots<l_L\leq N} &\mathrm{sgn}\left(\left(l_1,\ldots,l_L,1,\ldots,\check{l}_1,\ldots,\check{l}_L\ldots,N\right)\right)\\ &\times \Delta_{N-L}^{(l_1,\ldots,l_L)}(a_1,\ldots,a_N)\left[\prod_{k=1}^{L}\int_I dx_k f(a_{l_k},x_k)\right]\Phi(x_1,\ldots,x_L)\,,\end{aligned}$$

where the reduced Vandermonde determinant is defined in Eq. (A.6). Taking into account that

$$\mathrm{sgn}\left(\left(l_1,\ldots,l_L,1,\ldots,\check{l}_1,\ldots,\check{l}_L\ldots,N\right)\right) = (-1)^{l_1+\ldots+l_L-\frac{L(L+1)}{2}}\,,$$

we obtain the formula

$$\begin{aligned}&\sum_{\sigma\in S_N}\mathrm{sgn}\,(\sigma)\Delta_{N-L}(a_{\sigma(L+1)},\ldots,a_{\sigma(N)})\left[\prod_{k=1}^{L}\int_I dx_k f(a_{\sigma(k)},x_k)\right]\Phi(x_1,\ldots,x_L)\\ &= (N-L)!L!\sum_{1\leq l_1<\ldots<l_L\leq N}(-1)^{l_1+\ldots+l_L-\frac{L(L+1)}{2}}\Delta_{N-L}^{(l_1,\ldots,l_L)}(a_1,\ldots,a_N)\\ &\quad\times\left[\prod_{k=1}^{L}\int_I dx_k f(a_{l_k},x_k)\right]\Phi(x_1,\ldots,x_L)\,,\end{aligned}\tag{3.18}$$

valid for any antisymmetric function $\Phi(x_{\sigma(1)},\ldots,x_{\sigma(L)})$, and for any function $f(x,y)$ such that the integrals in the equation above exist.
The formula in Eq. (3.18) enables us to rewrite Eq. (3.15) as

$$\begin{aligned}&\mathbb{E}_{\mathcal{P}}\left[\frac{\prod_{m=1}^{M}D_N(z_m)}{\prod_{l=1}^{L}D_N(y_l)}\right] = \frac{(-1)^{L(L-N)}N!}{Z_N\Delta_M(z_1,\ldots,z_M)}\\ &\times\left[\prod_{j=1}^{M}\int_{I'}dt_j F(t_j,z_j)\prod_{n=1}^{N}(t_j-a_n)\right]\Delta_M(t_1,\ldots,t_M)\\ &\times\left[\prod_{l=1}^{L}\int_I dx_l\left(\frac{x_l}{y_l}\right)^{N-L}\frac{\prod_{m=1}^{M}(z_m-x_l)}{\prod_{j=1}^{L}(y_j-x_l)}\right]\Delta_L(x_1,\ldots,x_L)\\ &\times\sum_{1\leq l_1<\ldots<l_L\leq N}(-1)^{l_1+\ldots+l_L-\frac{L(L+1)}{2}}\Delta_{N-L}^{(l_1,\ldots,l_L)}(a_1,\ldots,a_N)\prod_{i=1}^{L}\frac{\varphi(a_{l_i},x_i)}{\prod_{j=1}^{M}(t_j-a_{l_j})}.\end{aligned}\tag{3.19}$$

We note that due to Eq. (A.7) it holds

$$\frac{\Delta_{N-L}^{(l_1,\ldots,l_L)}(a_1,\ldots,a_N)}{\Delta_N(a_1,\ldots,a_N)} = \frac{(-1)^{l_1+\ldots+l_L+NL}\Delta_L(a_{l_1},\ldots,a_{l_L})}{\prod\limits_{\substack{n=1\\n\neq l_1}}^{N}(a_{l_1}-a_n)\ldots\prod\limits_{\substack{n=1\\n\neq l_L}}^{N}(a_{l_L}-a_n)}.$$

In addition, we apply the result presented in Eq. (3.10) to eliminate Z_N, cancel signs, and see that the strict ordering of the indices $l_1 < l_2 < \ldots < l_L$ can be relaxed,

$$L!\sum_{1\leq l_1<\ldots<l_L\leq N} \to \sum_{l_1=1}^{N}\cdots\sum_{l_L=1}^{N}.$$

Finally, we see that the sum in Eq. (3.19) can be written as contour integrals, because of the formula

$$\frac{1}{2\pi i}\oint_C du\frac{f(u)}{\prod_{n=1}^{N}(u-a_n)} = \sum_{l=1}^{N}\frac{f(a_l)}{\prod_{\substack{n=1\\n\neq l}}^{N}(a_l-a_n)},$$

where the contour C encircles the points $a_1,\ldots,a_N$ counter-clockwise. The leads to the formula in the statement of Theorem 3.9. □

It is well known that every polynomial ensemble is a determinantal point process, as we have discussed in chapter 2. For invertible polynomial ensembles (see Definition 3.3) Theorem 3.9 enables us to deduce a double contour integration formula for the correlation kernel.

Proposition 3.12. *Consider an invertible polynomial ensemble, i.e. a polynomial ensemble from Definition 3.1, where the functions $\varphi_l(x) = \varphi(a_l, x)$ satisfy the conditions specified in Definition 3.3. The correlation kernel $K_N(x,y)$ of this ensemble can be written as*

$$K_N(x,y) = \frac{1}{2\pi i}\int_{I'} dsF(s,x)\prod_{n=1}^{N}(s-a_n)\oint_C du\frac{\varphi(u,y)}{(s-u)\prod_{n=1}^{N}(u-a_n)},$$

where C encircles the points $a_1,\ldots,a_N$ counter-clockwise, and where $\varphi(u,y)$ and $F(s,x)$ are defined by Eq. (3.8) and Eq. (3.9) correspondingly.

Proof. We use the following fact valid for any polynomial ensemble formed by $x_1,\ldots,x_N$ on $I \subseteq \mathbb{R}$, see Ref. [65][4]. Assume that

$$\mathbb{E}_{\mathcal{P}}\left(\frac{D_N(x)}{D_N(z)}\right) = \int_I dv\frac{x-v}{z-v}\Phi_N(x,v),$$

where the function $v \to \Phi_N(x,v)$ is analytic at y, $y \in I$. Then the correlation kernel of the determinantal process formed by $x_1,\ldots,x_N$ is given by

$$K_N(x,y) = \Phi_N(x,y).$$

[4]Because we take the residue of the right hand side at $z = y$, any ratio $f(v)/f(z)$ can be multiplied under the integral for regular functions f, without changing the values of the kernel, see also Eq. (2.30).

In our case Theorem 3.9 gives

$$\mathbb{E}_{\mathcal{P}}\left(\frac{D_N(x)}{D_N(z)}\right) = \frac{1}{2\pi i}\int_I dv \left(\frac{v}{z}\right)^{N-1}\frac{x-v}{z-v}$$
$$\times\left[\int_{I'} dsF(s,x)\prod_{n=1}^{N}(s-a_n)\oint_C du\frac{\varphi(u,v)}{(s-u)\prod_{n=1}^{N}(u-a_n)}\right],$$

which leads to the formula for the correlation kernel in the statement of the Proposition. □

Special cases

In Proposition 3.12 we have used Eq. (3.13) in the case $M = L = 1$. Another case of interest is that corresponding to products of characteristic polynomials. In this case $L = 0$, and we obtain that only the first set of integrals remains in Eq. (3.13), i.e.

$$\mathbb{E}_{\mathcal{P}}\left[\prod_{m=1}^{M} D_N(z_m)\right] = \frac{\det[\mathcal{B}_i(z_j)]_{i,j=1}^{M}}{\Delta_M(z_1,\ldots,z_M)},$$

where

$$\mathcal{B}_i(z) = \int_{I'} dsF(s,z)\, s^{i-1}\prod_{n=1}^{N}(s-a_n), \tag{3.20}$$

after pulling the M integrations over the s_j's into the Vandermonde determinant of size M. This result also could have been derived directly using Lemma A.2.
As a final special case of interest we look at the ratio of $M+1$ characteristic polynomials over a single one at $L = 1$. This object is needed in the next chapter to tackle the main model containing N_f massive flavors and temperature dependence as given in Eq. (2.12) [13]. Theorem 3.9 gives

$$\begin{aligned}&\mathbb{E}_{\mathcal{P}}\left[\frac{\prod_{m=1}^{M+1} D_N(z_m)}{D_N(y)}\right]\\ =\;& \frac{1}{\Delta_{M+1}(z_1,\ldots,z_{M+1})}\left(\prod_{j=1}^{M+1}\int_{I'} ds_j F(s_j,z_j)\prod_{n=1}^{N}(s_j-a_n)\right)\Delta_{M+1}(s_1,\ldots,s_{M+1})\\ &\times\int_I dv\left(\frac{v}{y}\right)^{N-1}\frac{\prod_{m=1}^{M+1}(z_m-v)}{(y-v)}\oint_C\frac{du}{2\pi i}\frac{1}{\prod_{n=1}^{N}(u-a_n)}\frac{\varphi(u,v)}{\prod_{j=1}^{M+1}(s_j-u)}.\end{aligned} \tag{3.21}$$

Following [65], we may use the Lagrange extrapolation formula

$$\prod_{j=1}^{M+1}\frac{1}{u-s_j} = \sum_{m=1}^{M+1}\frac{1}{u-s_m}\prod_{\substack{j=1\\ j\neq m}}^{M+1}\frac{1}{s_m-s_j},$$

to introduce the reduced Vandermonde, see Eq. (A.6) and Lemma A.3, and rewrite

$$\frac{\Delta_{M+1}(s_1,\ldots,s_{M+1})}{\prod_{j=1}^{M+1}(s_j-u)} = (-1)^{M+1}\sum_{m=1}^{M+1}\frac{(-1)^{M+1-m}}{u-s_m}\Delta_M^{(m)}(s_1,\ldots,s_{M+1}).$$

This leads to the following rewriting of Eq. (3.21)

$$\begin{aligned}
&\mathbb{E}_{\mathcal{P}}\left[\frac{\prod_{m=1}^{M+1} D_N(z_m)}{D_N(y)}\right] \\
&= \frac{1}{\Delta_{M+1}(z_1,\ldots,z_{M+1})}\int_I dv \left(\frac{v}{y}\right)^{N-1}\frac{\prod_{m=1}^{M+1}(z_m-v)}{(y-v)}\oint_C \frac{du}{2\pi i}\frac{\varphi(u,v)}{\prod_{n=1}^{N}(u-a_n)} \\
&\quad\times\sum_{m=1}^{M+1}(-1)^m\left(\prod_{j=1}^{M+1}\int_{I'} ds_j F(s_j,z_j)\prod_{n=1}^{N}(s_j-a_n)\right)\frac{1}{u-s_m}\Delta_M^{(m)}(s_1,\ldots,s_{M+1}) \\
&= \frac{1}{\Delta_{M+1}(z_1,\ldots,z_{M+1})}\int_I dv \left(\frac{v}{y}\right)^{N-1}\frac{\prod_{m=1}^{M+1}(z_m-v)}{(y-v)}\oint_C \frac{du}{2\pi i}\frac{\varphi(u,v)}{\prod_{n=1}^{N}(u-a_n)} \\
&\quad\times\det\begin{bmatrix}\mathcal{A}(z_1,u) & \ldots & \mathcal{A}(z_{M+1},u)\\ \mathcal{B}_1(z_1) & \ldots & \mathcal{B}_1(z_{M+1})\\ \vdots & \ldots & \vdots \\ \mathcal{B}_M(z_1) & \ldots & \mathcal{B}_M(z_{M+1})\end{bmatrix},
\end{aligned} \tag{3.22}$$

where we have defined

$$\mathcal{A}(z,u) = \int_{I'} dsF(s,z)\frac{-1}{u-s}\prod_{n=1}^{N}(s-a_n)\,. \tag{3.23}$$

In the second step in Eq. (3.22) we have first pulled all the s-integrals except the one over s_m into the Vandermonde determinant $\Delta_M^{(m)}(s_1,\ldots,s_{M+1})$, leading to a determinant of size M with matrix elements $\mathcal{B}_i(z_j)$ from Eq. (3.20). We then recognise that the sum is a Laplace expansion of a determinant of size $M+1$ with respect to the first row, containing the matrix elements $\mathcal{A}(z_j,u)$ from Eq. (3.23). This reveals the determinantal form of the corresponding kernel.

3.3 Reweighting of expectation values

We have seen in the previous section that the computation of expectation values depends heavily on the matrix elements $g_{k,l}$. Polynomial ensembles, formed by variables $x_1,\ldots,x_N$ as in Definition 3.1, are characterised by φ-functions such that the constant Z_N can be computed, which reduces the problem to the computation of the Gram matrix entries $g_{k,l}$ - see Eq. (3.2). For certain polynomial ensembles it is possible to derive results for expectation values of characteristic polynomials by *reweighting* the expectation value. The idea is to express the expectation value in terms of other expectation values with respect to simpler φ-type functions. This procedure can be seen as an analogy to the computation of the kernel for the deformed chGUE(N) via the classical chGUE(N) as demonstrated in chapter 2, see Eqs. (2.25) and (2.27). Given a polynomial ensemble with $\varphi_l(x)=\varphi(a_l,x)$ and

$$\varphi_l(x) = \psi_l(x)\left[\prod_{m=1}^{M}(z_m-x)\right]\left[\prod_{l=1}^{L}(y_l-x)\right]^{-1} \tag{3.24}$$

we require that $\psi_l(x) = \psi(a_l, x)$ is a non-zero, analytical function with an integral

$$\int_I dx x^{k-1} \psi(a_l, x) \,,$$

which has to be finite for all $k = 1, 2, \ldots$. Then we can write the JPDF via Eq. (3.24) in terms of a new polynomial ensemble:

$$\det\left[\varphi_l(x_k)\right]_{k,l=1}^N = \left(\prod_{m=1}^M \prod_{n=1}^N (z_m - x_n)\right) \left(\prod_{l=1}^L \prod_{n=1}^N (y_l - x_n)\right)^{-1} \det\left[\psi_l(x_k)\right]_{k,l=1}^N \,.$$

The JPDF with respect to the function $\varphi(a_l, x)$ is written via Eq. (3.24) as

$$\mathcal{P}_\varphi(x_1, \ldots, x_N) = \frac{1}{Z_N} \Delta(x_1, \ldots, x_N) \det[\psi_l(x_k)]_{k,l=1}^N \frac{\prod_{m=1}^M D_N(z_m)}{\prod_{l=1}^L D_N(y_l)} \,.$$

The constant Z_N corresponding to the $\varphi(a_l, x)$ functions can be rewritten as follows:

$$\begin{aligned} Z_N &= Z_N \frac{Z'_N}{Z'_N} = Z'_N \left(\prod_{n=1}^N \int_I dx_n\right) \frac{1}{Z'_N} \Delta(x_1, \ldots, x_N) \det\left[\varphi_l(x_k)\right]_{k,l=1}^N \\ &= Z'_N \left(\prod_{n=1}^N \int_I dx_n\right) \frac{1}{Z'_N} \Delta(x_1, \ldots, x_N) \det\left[\psi_l(x_k)\right]_{k,l=1}^N \frac{\prod_{m=1}^M D_N(z_m)}{\prod_{l=1}^L D_N(y_l)} \\ &= Z'_N \, \mathbb{E}_\psi \left[\frac{\prod_{m=1}^M D_N(z_m)}{\prod_{l=1}^L D_N(y_l)}\right] , \end{aligned} \tag{3.25}$$

where in the last line the expectation value $\mathbb{E}_\psi$ is taken with respect to the functions ψ, which form another polynomial ensemble with the JPDF

$$\mathcal{P}_\psi(x_1, \ldots, x_N) = \frac{1}{Z'_N} \Delta(x_1, \ldots, x_N) \det[\psi_l(x_k)]_{k,l=1}^N$$

and constant $Z'_N = N! \det G'$, where the entries $g'_{k,l}$ of G' are given by

$$g'_{k,l} = \int_I dx x^{k-1} \psi(a_l, x) \,.$$

Denoting the expectation value with respect to the polynomial ensemble with φ-function by $\mathbb{E}_\varphi$ we can prove the following Lemma:

Lemma 3.13. *Given a polynomial ensemble via Definition 3.1 with φ-functions of the form*

$$\varphi_l(x) = \psi_l(x) \left[\prod_{m=1}^M (z_m - x)\right] \left[\prod_{l=1}^L (y_l - x)\right]^{-1} ,$$

where $L \leq N$. Assume that $\psi_l(x) = \psi(a_l, x)$ such that another polynomial ensemble with

$$\mathcal{P}_\psi(x_1, \ldots, x_N) = \frac{1}{Z'_N} \Delta(x_1, \ldots, x_N) \det[\psi_l(x_k)]_{k,l=1}^N$$

is formed with constant Z'_N given by $Z'_N = N! \det G'$, where the Gram matrix G' has entries $g'_{k,l}$ given by the finite integrals

$$g'_{k,l} = \int_I dx x^{k-1} \psi_l(x) \, .$$

The expectation value of an arbitary ratio of characteristic polynomials with respect to φ can be rewritten as a product of two expectation values with respect to ψ:

$$\mathbb{E}_\varphi \left[\frac{\prod_{k=1}^K D_N(a_k)}{\prod_{h=1}^H D_N(b_l)} \right] = \mathbb{E}_\psi \left[\frac{\prod_{k=1}^K D_N(a_k)}{\prod_{h=1}^H D_N(b_l)} \frac{\prod_{m=1}^M D_N(z_m)}{\prod_{l=1}^L D_N(y_l)} \right] \times \left(\mathbb{E}_\psi \left[\frac{\prod_{m=1}^M D_N(z_m)}{\prod_{l=1}^L D_N(y_l)} \right] \right)^{-1} .$$

Proof. The left side of the above equation can be written as

$$\mathbb{E}_\varphi \left[\frac{\prod_{k=1}^K D_N(a_k)}{\prod_{h=1}^H D_N(b_l)} \right] = \left(\prod_{n=1}^N \int_I dx_N \right) \left[\frac{\prod_{k=1}^K D_N(a_k)}{\prod_{h=1}^H D_N(b_l)} \right] \mathcal{P}_\varphi(x_1, \ldots, x_N) \, .$$

The JPDF becomes

$$\begin{aligned} \mathcal{P}_\varphi(x_1, \ldots, x_N) &= \frac{1}{Z_N} \Delta(x_1, \ldots, x_N) \det \left[\varphi_l(x_k) \right]_{k,l=1}^N \\ &\overset{\text{Eq. (3.25)}}{=} \frac{1}{Z'_N} \Delta(x_1, \ldots, x_N) \det \left[\psi_l(x_k) \right]_{k,l=1}^N \\ &\quad \times \left[\frac{\prod_{m=1}^M D_N(z_m)}{\prod_{l=1}^L D_N(y_l)} \right] \left(\mathbb{E}_\psi \left[\frac{\prod_{m=1}^M D_N(z_m)}{\prod_{l=1}^L D_N(y_l)} \right] \right)^{-1} . \end{aligned}$$

This leads to

$$\begin{aligned} \mathbb{E}_\varphi \left[\frac{\prod_{k=1}^K D_N(a_k)}{\prod_{h=1}^H D_N(b_l)} \right] &= \left(\mathbb{E}_\psi \left[\frac{\prod_{m=1}^M D_N(z_m)}{\prod_{l=1}^L D_N(y_l)} \right] \right)^{-1} \left(\prod_{n=1}^N \int_I dx_N \right) \\ &\quad \times \frac{1}{Z'_N} \Delta(x_1, \ldots, x_N) \det \left[\psi_l(x_k) \right]_{k,l=1}^N \left[\frac{\prod_{k=1}^K D_N(a_k)}{\prod_{h=1}^H D_N(b_l)} \right] \left[\frac{\prod_{m=1}^M D_N(z_m)}{\prod_{l=1}^L D_N(y_l)} \right] \\ &= \mathbb{E}_\psi \left[\frac{\prod_{k=1}^K D_N(a_k)}{\prod_{h=1}^H D_N(b_l)} \frac{\prod_{m=1}^M D_N(z_m)}{\prod_{l=1}^L D_N(y_l)} \right] \left(\mathbb{E}_\psi \left[\frac{\prod_{m=1}^M D_N(z_m)}{\prod_{l=1}^L D_N(y_l)} \right] \right)^{-1} , \end{aligned}$$

which finishes the proof. □

The important requirement that needs to be met in Lemma 3.13 is that the original polynomial ensemble features a φ-function that consist of characteristic polynomials, or products, or ratios of characteristic polynomials and another function ψ, which forms a polynomial ensemble itself. In the context of this thesis this leads to the following Remark.

Remark 3.14. We know that the chGUE(N) with an external source forms an invertible polynomial ensemble, see Example 3.7. Furthermore, the extension with N_f massive flavors of this model, given in Eq. (2.12), forms also a polynomial ensemble, which fullfills the

requirements to apply Lemma 3.13 via the identification

$$\varphi(a_l, x) = (-1)^{N_f} \left(\frac{x}{a_l}\right)^{\nu/2} e^{-(x+a_l)} I_\nu\left(2\sqrt{a_l x}\right) \prod_{f=1}^{N_f} (-m_f^2 - x)$$

and by comparing Eq. (2.11) we have

$$\psi(a_l, x) = \left(\frac{x}{a_l}\right)^{\nu/2} e^{-(x+a_l)} I_\nu\left(2\sqrt{a_l x}\right) \ .$$

We can express expectation values of the N_f massive flavor model in terms of expectation values of the simpler chGUE(N) with an external source. This becomes important in the derivation of both correlation kernels in the next chapter.

3.4 Summary

This chapter has been concerned with the impact of characteristic polynomials in polynomial ensembles. We reviewed the definition of polynomial ensembles as a sub-class of determinantal point processes and biorthogonal ensembles introduced by Borodin [109]. We collected known results for polynomial ensembles and realized how the correlation kernel $K_N(x, y)$ of a polynomial ensemble and additionally the k-point correlation function may be computed using expectation values of ratios of characteristic polynomials. From chapter 2 we knew that the random matrix models we considered to study the temperature dependence of effective theories of QCD are all part of the class of polynomial ensembles - the classical chGUE(N) and the deformed chGUE(N) are actually *orthogonal* polynomial ensembles, while the chGUE(N) with an external source and the temperature dependent chGUE(N) with N_f massive flavors are both polynomial ensembles, following the Definition of polynomial ensembles given in Definition 3.1.
Next, we introduced the new notion of *invertible* polynomial ensembles in Definition 3.3. For this new sub-class of polynomial ensembles we stated and have proven a multi-contour-integral formula for the expectation value of an arbitrary ratio of characteristic polynomials, see Theorem 3.9. The number of integrations in our formula depends on the number of characteristic polynomials in the ratio: M in the numerator and L in the denominator. Crucially, neither the number of integrations nor the appearing determinants in the integrands depend on the ensemble size N. This simplifies the large N analysis in chapter 4.
We also showed that the chGUE(N) with an external source is indeed an invertible polynomial ensemble, along with other examples like the Polya ensembles. An open question at this point in time is how large the class of invertible polynomial ensembles acutally is. In general, it remains unclear, if it is possible to show that any polynomial ensemble is indeed invertible.
We proceeded to discuss the *reweighting* of expectation values allowing us to trace expectation values of a given polynomial ensemble containing ratios characteristic polynomials in its JPDF back to expectation values of a simpler polynomial ensemble. This allows us to express the correlation kernel of our main model, the chGUE(N) with temperature dependence and N_f massive flavors, in terms of expectation values with respect to the chGUE(N) with an external source, which is an invertible polynomial ensemble. Thus, our Theorem 3.9 becomes applicable and is used to derive $K_N(x, y)$ in the next chapter.

Chapter 4

Asymptotic Analysis of Correlation Kernels

In this chapter we take a closer look at the spectral statistics of the random matrix models introduced for effective theories of strong interacting theories like Quantum chromodynamics. We have derived the joint probability density functions of four chGUE(N)-type models in chapter 2: We introduced the classical chiral Gaussian Unitary Ensemble, chGUE(N), and its extension by N_f massive flavors. These two ensembles are classical orthogonal polynomial ensembles with JPDF of Δ^2-type. The models have been studied extensively in particular as models for zero-temperature QCD [8, 9, 58–60, 63]. The correlation kernels and k-point functions for both ensembles have been known and can be used as comparisons for the main results for non-zero temperature QCD random matrix models we derive in this chapter.
The two ensembles for non-zero temperature QCD we introduced are the chGUE(N) with an external source, see Eq. (2.3) and (2.11), and its extension with N_f massive flavors in Eq. (2.5) and (2.12). Both ensembles have been studied in the past using supersymmetry methods to derive correlation kernels and k-point functions [12, 13] in the large N limit. The JPDFs are given as

$$\begin{aligned}\mathcal{P}^{\text{ext}}_{\text{chGUE}}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N^{(0,T)}}&\left[\prod_{n=1}^{N}\left(\frac{\lambda_n}{a_n}\right)^{\nu/2} e^{-(\lambda_n+a_n)}\right]\\ &\times \det\left[I_\nu\left(2\sqrt{a_i\lambda_j}\right)\right]_{i,j=1}^{N}\Delta_N(\lambda_1,\ldots\lambda_N)\end{aligned} \tag{4.1}$$

for the chGUE(N) with an external source and

$$\begin{aligned}\mathcal{P}^{\text{temp}}_{\text{chGUE}}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N^{(N_f,T)}}&\left[\prod_{n=1}^{N}\left(\frac{\lambda_n}{a_n}\right)^{\nu/2} e^{-(\lambda_n+a_n)}\right]\left(\prod_{f=1}^{N_f}\prod_{n=1}^{N}(m_f^2+\lambda_n)\right)\\ &\times \det\left[I_\nu\left(2\sqrt{a_i\lambda_j}\right)\right]_{i,j=1}^{N}\Delta_N(\lambda_1,\ldots\lambda_N)\end{aligned} \tag{4.2}$$

for its extension with N_f massive flavors. We will use the notion of polynomial ensembles introduced in chapter 3 to derive a useful finite N representation of the correlation kernels for both ensembles in section 4.1.
The large N limit of the random matrix models with and without temperature have to be compared with the microscopic limit of low-energy effective theories of QCD. The application of the random matrix models to the theory of strong interactions is only possible in this limit. Random matrix models without external parameters are directly comparable to

effective theories, like chPT, of QCD at zero temperature in the $N \to \infty$ limit. This comparison is no longer clear at non-zero finite temperature. As far as we know, there exists no direct comparable counterpart on the effect theory level. Nevertheless, we find a structure of the correlation kernel in the large N-limit that can be identified with the zero-temperature results of QCD in this limit. The only difference is a rescaling of the order parameter - the chiral condensate - with respect to temperature.
We discuss the large N asymptotics of the kernels derived in section 4.1 in the two sections 4.2 and 4.3, for $N_f = 0$ and $N_f \neq 0$ respectively.

4.1 Correlation kernels at finite N

The two ensembles describing temperature dependence drawn from the chGUE(N) symmetry class introduced in Eq. (4.1) and (4.2) are both polynomial ensembles in the sense of Definition 3.1. Ensembles of this type are characterised by a function φ. For the chGUE(N) with an external source - Eq. (4.1) - this function can be written as

$$\varphi_l(x) = \varphi(a_l, x) = \left(\frac{x}{a_l}\right)^{\nu/2} e^{-(x+a_l)}\, I_\nu\left(2\sqrt{a_l x}\right) \quad \forall\, l = 1, \ldots, N. \tag{4.3}$$

For the model extended with N_f massive flavors - Eq. (4.2) - the function φ becomes

$$\varphi_l(x) = \varphi(a_l, x) = \left(\frac{x}{a_l}\right)^{\nu/2} e^{-(x+a_l)} \prod_{f=1}^{N_f}(x + m_f^2)\, I_\nu\left(2\sqrt{a_l x}\right) \quad \forall\, l = 1, \ldots, N.$$

The goal in this section is to derive the correlation kernels $K_N(x, y)$ for both models above. We make use of the results presented and derived in chapter 3. In particular we know that for polynomial ensembles the following formula for the correlation kernel holds [65], [26]:

$$K_N(x_1, x_2) = \frac{1}{x_1 - x_2} \operatorname*{Res}_{y=x_2} \left(\mathbb{E}_{\mathcal{P}} \left[\frac{D_N(x_1)}{D_N(y)} \right] \right), \tag{4.4}$$

where $D_N(z) = \prod_{n=1}^{N}(z - x_n)$ is the characteristic polynomial and the expectation $\mathbb{E}$ is taken with respect to the JPDF denoted by $\mathcal{P}$. It is convenient to replace the index $\mathcal{P}$ in the expectation value $\mathbb{E}_{\mathcal{P}}$ by either 0 or N_f. When we consider the chGUE(N) with an external source we use the index 0 and we use the index N_f for the extension with N_f flavors. Eq. (4.4) indicates the necessity to compute the expectation value of a ratio of characteristic polynomials. Moreover, the resulting form of the correlation kernel should be suitable for a following large N limit analysis. The results of chapter 3, like Theorem 3.9 and Proposition 3.12, can be applied, if the ensemble considered is an invertible polynomial ensembles following Definition 3.3. Because of Example 3.7 we know that the chGUE(N) with an external source is indeed an invertible polynomial ensemble with φ-function Eq. (4.3). For the chGUE(N) with an external source we can directly apply Proposition 3.12, which gives us the kernel:[1]

$$K_N^{(0)}(x_1, x_2) = \frac{1}{2\pi i} \int_{I'} ds F(s, x_1) \prod_{n=1}^{N}(s - a_n) \oint_C du \frac{\varphi(u, x_2)}{(s-u)\prod_{n=1}^{N}(u - a_n)}\,.$$

[1] The upper index on $K_N^{(0)}(x, y)$ indicates that we consider the chGUE(N) with an external source without N_f massive flavors.

We use Eqs. (4.3), (3.12) and $I' = \mathbb{R}_- = (-\infty, 0]$, namely

$$\varphi(u, x_2) = \left(\frac{x_2}{u}\right)^{\nu/2} e^{-(x_2+u)} I_\nu\left(2\sqrt{ux_2}\right) \text{ and } F(s, x_1) = (-1)^\nu \left(\frac{s}{x_1}\right)^{\nu/2} e^{s+x_1} I_\nu\left(2\sqrt{x_1 s}\right). \tag{4.5}$$

Via the substitution $s \to -s$ we obtain

$$\begin{aligned} K_N^{(0)}(x_1, x_2) = \frac{(-1)}{2\pi i} \left(\frac{x_2}{x_1}\right)^{\frac{\nu}{2}} e^{x_1 - x_2} \int_0^\infty ds s^{\nu/2} e^{-s} J_\nu\left(2\sqrt{x_1 s}\right) \prod_{n=1}^N (s + a_n) \\ \times \oint_C du \frac{u^{-\nu/2} e^{-u} I_\nu(2\sqrt{ux_2})}{(u+s)\prod_{n=1}^N (a_n - u)}. \end{aligned}$$

Alternatively, we could apply Theorem 3.9 for one ratio of characteristic polynomials, setting $M = L = 1$, and then applying the residue from Eq. (3.6). This leads to the same result and the calculation can be found in Appendix B. Furthermore, we can remove the prefactors $(x_2/x_1)^{\nu/2} e^{x_1 - x_2}$ due to the invariance of the kernel discussed in chapter 2, see Eq. (2.30). We conclude that the kernel for $N_f = 0$ corresponding to the chGUE(N) with an external source and JPDF given in Eq. (4.1) can be expressed as

$$\begin{aligned} K_N^{(0)}(x_1, x_2) = \frac{-1}{2\pi i} \int_0^\infty ds s^{\nu/2} e^{-s} J_\nu(2\sqrt{sx_1}) \prod_{n=1}^N (a_n + s) \\ \times \oint_C du \frac{u^{-\nu/2} e^{-u} I_\nu(2\sqrt{ux_2})}{\prod_{n=1}^N (a_n - u)} \frac{1}{u+s} \end{aligned} \tag{4.6}$$

with integration contour C, which encircles the points $a_1, \ldots, a_N$ counter-clockwise and leaves the real number $-s$ outside. This generalises results for quadratic matrices derived by Fyodorov, Strahov and Grela [25] to rectangular matrices by introducing $\nu \neq 0$.

Adding massive flavors and reweighting of expectation values

For the chGUE(N) with an external source and N_f massive flavors - compare Eq. (4.2) - we cannot directly apply the results for invertible polynomial ensembles from chapter 3, since we are not able to show that the requirements of Definition 3.3 can be fullfilled for this ensemble. Instead we can use that this model is an extension with N_f characteristic polynomials of an invertible polynomial ensemble, namely the chGUE(N) with an external source. This allows us to utilize Remark 3.14 and also Lemma 3.13. In short, we are able to express the expectation value of a single ratio with respect to $\mathcal{P}_{N_f}$ in terms of two expectation values with respect to the invertible polynomial ensemble that we denote as chGUE(N) with an external source. This is necessary to compute the correlation kernel via Eq. (4.4). With Remark 3.14 and Lemma 3.13 we can write

$$\mathbb{E}_{N_f}\left[\frac{D_N(x_1)}{D_N(y)}\right] = \mathbb{E}_0\left[\frac{D_N(x_1)}{D_N(y)}\left(\prod_{f=1}^{N_f} D_N(-m_f^2)\right)\right]\left(\mathbb{E}_0\left[\prod_{f=1}^{N_f} D_N(-m_f^2)\right]\right)^{-1}. \tag{4.7}$$

Note that the second expectation value in the line above is independent of the variables x_1 and y, which are the original kernel variables on the left-hand side of the equation. Thus, its resulting form will be a prefactor, which can be computed independently of the first expectation value and the kernel variables.

The reweighting allows us to use the results from the previous chapter for the two expectation values on the right-hand side in Eq. (4.7). In particular, we can use Theorem 3.9 and its special cases from chapter 3. We had for a product of M characteristic polynomials and its expectation value a formula given in chapter 3, namely

$$\mathbb{E}_0\left[\prod_{m=1}^{M} D_N(z_m)\right] = \frac{\det[\mathcal{B}_i(z_j)]_{i,j=1}^{M}}{\Delta_M(z_1,\ldots,z_M)}\,,$$

where

$$\mathcal{B}_i(z) = \int_{I'} ds F(s,z)\, s^{i-1} \prod_{n=1}^{N}(s-a_n)\,, \tag{4.8}$$

which yields for $M = N_f$ using Eq. (4.5) the following formula:

$$\begin{aligned} &B_i(m_f^2) \equiv \int_0^\infty ds s^{\nu/2} e^{-s} I_\nu(2\sqrt{m_f^2 s}) s^{i-1} \prod_{n=1}^{N}(s+a_n), \\ &\text{with} \quad \mathcal{B}_i(-m_f^2) = (-1)^N (-1)^\nu m_f^{-\nu} e^{-m_f^2} (-1)^{i-1} B_i(m_f^2) \end{aligned} \tag{4.9}$$

and overall[2]

$$\mathbb{E}_0\left[\prod_{f=1}^{N_f} D_N(-m_f^2)\right] = (-1)^{NN_f}(-1)^{-\nu N_f}\left(\prod_{f=1}^{N_f} m_f^{-\nu} e^{-m_f^2}\right)\frac{\det[B_i(m_f^2)]_{i,f=1}^{N_f}}{\Delta_{N_f}(m_1^2,\ldots,m_{N_f}^2)}\,. \tag{4.10}$$

The second expectation value

$$\mathbb{E}_0\left[\frac{D_N(x_1)}{D_N(y)}\left(\prod_{f=1}^{N_f} D_N(-m_f^2)\right)\right]$$

can be treated directly via Theorem 3.9 with $L = 1$ and $M = N_f + 1$. We have derived a formula for this special case of a product of $N_f + 1$ over one characteristic polynomial back in chapter 3, see Eq. (3.22). For $z_{M+1} = x_1$ and $M = N_f$ we obtain

$$\begin{aligned} \mathbb{E}_0\left[\frac{\prod_{m=1}^{M+1} D_N(z_m)}{D_N(y)}\right] &= \frac{1}{\Delta_{M+1}(z_1,\ldots,z_{M+1})}\int_I dx \left(\frac{x}{y}\right)^{N-1}\frac{\prod_{m=1}^{M+1}(z_m-x)}{(y-x)} \\ &= \oint_C \frac{du}{2\pi i}\frac{1}{\prod_{n=1}^{N}(u-a_n)}\frac{\varphi(u,x)}{w(u)}\det\begin{bmatrix} \mathcal{A}(z_1,u) & \ldots & \mathcal{A}(z_{M+1},u) \\ \mathcal{B}_1(z_1) & \ldots & \mathcal{B}_1(z_{M+1}) \\ \vdots & \ldots & \vdots \\ \mathcal{B}_M(z_1) & \ldots & \mathcal{B}_M(z_{M+1}) \end{bmatrix}, \end{aligned}$$

where we can apply Eq. (4.5), namely

$$\varphi(u,x_2) = \left(\frac{x_2}{u}\right)^{\nu/2} e^{-(x_2+u)}\, I_\nu\left(2\sqrt{ux_2}\right) \quad \text{and} \quad F(s,x_1) = (-1)^\nu\left(\frac{s}{x_1}\right)^{\nu/2} e^{s+x_1} I_\nu\left(2\sqrt{x_1 s}\right)\,.$$

[2]Note that for the large N analysis in the next section it is convenient to rewrite the determinant. We replace the monomials s^{i-1} with monic Hermite polynomials $h_{i-1}(s)$.

With Eq. (3.23), $I' = \mathbb{R}_- = (-\infty, 0]$ and $s \to -s$ we find

$$\begin{aligned}\mathcal{A}(z,u) &= \int_{I'} ds F(s,z) \frac{-1}{u-s} \prod_{n=1}^{N}(s-a_n) \\ &= (-1)^N \begin{cases} (-1)^\nu m_f^{-\nu} e^{-m_f^2} A(m_f^2,u), & \text{for } z = -m_f^2 \quad \text{with } f = 1,\dots,N_f, \\ x_1^{-\nu/2} e^{x_1} \widehat{A}(x_1,u), & \text{for } z = x_1, \end{cases}\end{aligned}$$

where we use the definitions

$$\begin{aligned} A(m_f^2,u) &\equiv \int_0^\infty ds s^{\nu/2} e^{-s} I_\nu\left(2\sqrt{m_f^2 s}\right) \frac{-1}{u+s} \prod_{n=1}^{N}(s+a_n)\,, \\ \widehat{A}(x_1,u) &\equiv \int_0^\infty ds (-1)^\nu (-1)^{\nu/2} s^{\nu/2} e^{-s} \underbrace{I_\nu(2\sqrt{x_1 s} i)}_{i^\nu J_\nu(2\sqrt{x_1 s})} \frac{-1}{u+s} \prod_{n=1}^{N}(s+a_n)\,. \end{aligned}$$

The $\mathcal{B}_i(z_m)$ are given in Eq. (4.8) leading to Eq. (4.9) and a special case, namely

$$\begin{aligned} &\widehat{B}_i(x_1) \equiv \int_0^\infty ds s^{\nu/2} e^{-s} J_\nu\left(2\sqrt{x_1 s}\right) s^{i-1} \prod_{n=1}^{N}(s+a_n)\,, \\ &\text{with} \quad \mathcal{B}_i(x_1) = (-1)^N x_1^{-\nu/2} e^{x_1} (-1)^{i-1} \widehat{B}_i(x_1)\,. \end{aligned}$$

This leads to[3]

$$\begin{aligned} &\mathbb{E}_0\left[\frac{D_N(x_1)}{D_N(y)}\left(\prod_{f=1}^{N_f} D_N(-m_f^2)\right)\right] = (-1)^{NN_f}(-1)^{\nu N_f}(-1)^N \left(\prod_{f=1}^{N_f} m_f^{-\nu} e^{-m_f^2}\right) x_1^{-\nu/2} e^{x_1} \\ &\times \frac{(-1)^{N_f}(-1)^N \left(\prod_{f=1}^{N_f}(x_1+m_f^2)\right)^{-1}}{\Delta_{N_f}(m_1^2,\dots,m_{N_f}^2)} \int_0^\infty dv \left(\frac{v}{y}\right)^{N-1} \frac{x_1-v}{(y-v)} \left(\prod_{m=1}^{N_f}(m_f^2+v)\right) v^{\nu/2} e^{-v} \\ &\times \oint_C \frac{du}{2\pi i} \frac{u^{-\nu/2} e^{-u} I_\nu(2\sqrt{uv})}{\prod_{n=1}^{N}(a_n-u)} \det \begin{bmatrix} A(m_1^2,u) & \dots & A(m_{N_f}^2,u) & \widehat{A}(x_1,u) \\ B_1(m_1^2) & \dots & B_1(m_{N_f}^2) & \widehat{B}_1(x_1) \\ \vdots & \dots & \vdots & \vdots \\ B_{N_f}(m_1^2) & \dots & B_{N_f}(m_{N_f}^2) & \widehat{B}_{N_f}(x_1) \end{bmatrix} \\ &= (-1)^{NN_f}(-1)^{\nu N_f} \left(\prod_{f=1}^{N_f} m_f^{-\nu} e^{-m_f^2}\right) x_1^{-\nu/2} e^{x_1} \frac{\left(\prod_{f=1}^{N_f}(x_1+m_f^2)\right)^{-1}}{\Delta_{N_f}(m_1^2,\dots,m_{N_f}^2)} \\ &\times \oint_C \frac{du}{2\pi i} \frac{u^{-\nu/2} e^{-u}}{\prod_{n=1}^{N}(a_n-u)} \det \begin{bmatrix} \widehat{A}(x_1,u) & A(m_1^2,u) & \dots & A(m_{N_f}^2,u) \\ \widehat{B}_1(x_1) & B_1(m_1^2) & \dots & B_1(m_{N_f}^2) \\ \vdots & \vdots & \dots & \vdots \\ \widehat{B}_{N_f}(x_1) & B_{N_f}(m_1^2) & \dots & B_{N_f}(m_{N_f}^2) \end{bmatrix} \\ &\times \int_0^\infty dv \left(\frac{v}{y}\right)^{N-1} \frac{x_1-v}{(y-v)} \left(\prod_{m=1}^{N_f}(m_f^2+v)\right) v^{\nu/2} e^{-v} I_\nu(2\sqrt{uv})\,. \end{aligned}$$

[3] It is always possible to replace s^{i-1} with Hermite polynomials $h_{i-1}(s)$ due to invariance of the determinant.

Derivation of the kernel with N_f flavors and temperature dependence

With this preparation we can write the correlation kernel as follows:

$$\begin{aligned} K_N^{(N_f)}(x_1,x_2) &= \frac{1}{x_1-x_2}\operatorname*{Res}_{y=x_2}\left(\mathbb{E}_{N_f}\left[\frac{D_N(x_1)}{D_N(y)}\right]\right) \\ &= \frac{1}{x_1-x_2}\operatorname*{Res}_{y=x_2}\left(\mathbb{E}_0\left[\frac{D_N(x_1)}{D_N(y)}\left(\prod_{f=1}^{N_f}D_N(-m_f^2)\right)\right]\left(\mathbb{E}_0\left[\prod_{f=1}^{N_f}D_N(-m_f^2)\right]\right)^{-1}\right) \\ &= \frac{1}{x_1-x_2}\left(\mathbb{E}_0\left[\prod_{f=1}^{N_f}D_N(-m_f^2)\right]\right)^{-1}\operatorname*{Res}_{y=x_2}\left(\mathbb{E}_0\left[\frac{D_N(x_1)}{D_N(y)}\left(\prod_{f=1}^{N_f}D_N(-m_f^2)\right)\right]\right). \end{aligned}$$

Using Eq. (4.10) and (4.11) we obtain

$$\begin{aligned} K_N^{(N_f)}(x_1,x_2) = \frac{1}{x_1-x_2}&\left(\det[B_i(m_f^2)]_{i,f=1}^{N_f}\right)^{-1}\frac{x_1^{-\nu/2}e^{x_1}}{2\pi i}\oint_C du\frac{u^{-\nu/2}e^{-u}}{\prod_{n=1}^N(a_n-u)} \\ &\times\det\begin{bmatrix}\hat{A}(x_1,u) & A(m_1^2,u) & \dots & A(m_{N_f}^2,u)\\ \hat{B}_1(x_1) & B_1(m_1^2) & \dots & B_1(m_{N_f}^2)\\ \vdots & \vdots & \dots & \vdots\\ \hat{B}_{N_f}(x_1) & B_{N_f}(m_1^2) & \dots & B_{N_f}(m_{N_f}^2)\end{bmatrix} \\ &\times\operatorname*{Res}_{y=x_2}\left(\int_0^\infty dv\left(\frac{v}{y}\right)^{N-1}\frac{x_1-v}{y-v}\,v^{\nu/2}e^{-v}I_\nu(2\sqrt{uv})\prod_{f=1}^{N_f}\frac{m_f^2+v}{m_f^2+x_1}\right). \end{aligned}$$

The next step is to evaluate the residue, which is defined via Eq. (3.6). Moreover, we can apply the following identity

$$\left(\frac{v}{y}\right)^{N-1}\frac{1}{y-v} = \frac{1}{y-v} - \sum_{k=0}^{N-2}\frac{v^k}{y^{k+1}} \tag{4.11}$$

to simplify the evaluation of the residue. This leads to two terms, where the first reads

$$\begin{aligned} &\operatorname*{Res}_{y=x_2}\left(\int_0^\infty dv\frac{v^{\nu/2}e^{-v}}{y-v}(x_1-v)I_\nu(2\sqrt{uv})\prod_{f=1}^{N_f}\frac{m_f^2+v}{m_f^2+x_1}\right) \\ &= (x_1-x_2)x_2^{\nu/2}e^{-x_2}I_\nu(2\sqrt{ux_2})\prod_{f=1}^{N_f}\frac{m_f^2+x_2}{m_f^2+x_1}, \end{aligned} \tag{4.12}$$

which follows from Eq. (3.6). The second term reads

$$\begin{aligned} &\operatorname*{Res}_{y=x_2}\left(\int_0^\infty dv\frac{v^{k+\nu/2}e^{-v}}{y^{k+1}}(x_1-v)I_\nu(2\sqrt{uv})\prod_{f=1}^{N_f}\frac{m_f^2+v}{m_f^2+x_1}\right) \\ &= \int_0^\infty dv v^{k+\nu/2}e^{-v}(x_1-v)I_\nu(2\sqrt{uv})\prod_{f=1}^{N_f}\frac{m_f^2+v}{m_f^2+x_1}\operatorname*{Res}_{y=x_2}\left(\frac{1}{y^{k+1}}\right), \end{aligned} \tag{4.13}$$

where we used the linearity of the residue. We also use that for all $y \neq 0$ the above expression vanishes directly, because the residue is zero by Eq. (3.6). For $y = 0$ we have a pol of order $k+1$, which leads to

$$\operatorname*{Res}_{y=x_2=0}\left(\frac{1}{y^{k+1}}\right) = \lim_{y\to 0}\frac{1}{k!}\frac{\partial^k}{\partial y^k} y^{k+1}\frac{1}{y^{k+1}} = 0\,, \tag{4.14}$$

and we realize that the contribution of this term is actually always equal to zero.
Combining Eqs. (4.11), (4.12), (4.13) and (4.14), we can write

$$\begin{aligned} K_N^{(N_f)}(x_1,x_2) &= \frac{x_1^{-\nu/2}x_2^{\nu/2}}{x_1-x_2}e^{x_1-x_2}\left(\det[B_i(m_f^2)]_{i,f=1}^{N_f}\right)^{-1}\oint_C\frac{du}{2\pi i}\frac{u^{-\nu/2}e^{-u}I_\nu(2\sqrt{ux_2})}{\prod_{n=1}^N(a_n-u)} \\ &\times\det\begin{bmatrix}\hat{A}(x_1,u) & A(m_1^2,u) & \dots & A(m_{N_f}^2,u)\\ \hat{B}_1(x_1) & B_1(m_1^2) & \dots & B_1(m_{N_f}^2)\\ \vdots & \vdots & \dots & \vdots\\ \hat{B}_{N_f}(x_1) & B_{N_f}(m_1^2) & \dots & B_{N_f}(m_{N_f}^2)\end{bmatrix}(x_1-x_2)\prod_{f=1}^{N_f}\frac{m_f^2+x_2}{m_f^2+x_1} \\ &= \left(\frac{x_2}{x_1}\right)^{\nu/2}e^{x_1-x_2}\prod_{f=1}^{N_f}\frac{x_2+m_f^2}{x_1+m_f^2}\frac{\det\begin{bmatrix}K_N^{(0)}(x_1,x_2) & \hat{K}_N^{(0)}(m_1^2,x_2) & \dots & \hat{K}_N^{(0)}(m_{N_f}^2,x_2)\\ \hat{B}_1(x_1) & B_1(m_1^2) & \dots & B_1(m_{N_f}^2)\\ \vdots & \vdots & \dots & \vdots\\ \hat{B}_{N_f}(x_1) & B_{N_f}(m_1^2) & \dots & B_{N_f}(m_{N_f}^2)\end{bmatrix}}{\det[B_i(m_f^2)]_{i,f=1}^{N_f}}\,. \end{aligned}$$

We can again remove the prefactors, as in the $N_f = 0$ case, and identify the kernel $K_N^{(0)}(x_1,x_2)$ as the first entry of the N_f+1 dimensional determinant.
A similar expression is given by $\hat{K}_N^{(0)}(m_f^2,x_2)$, namely

$$\hat{K}_N^{(0)}(m_f^2,x_2) \equiv \int_0^\infty ds s^{\nu/2}e^{-s}I_\nu\left(2\sqrt{sm_f^2}\right)\prod_{n=1}^N(a_n+s)\oint_C\frac{du}{2\pi i}\frac{u^{-\nu/2}e^{-u}I_\nu(2\sqrt{ux_2})}{\prod_{n=1}^N(a_n-u)}\frac{-1}{u+s}\,.$$

Overall we find a determinantal structure, with dimension N_f+1, of the kernel:[4]

$$K_N^{(N_f)}(x_1,x_2) = \sqrt{\frac{\prod_{f=1}^{N_f}(x_2+m_f^2)}{\prod_{f=1}^{N_f}(x_1+m_f^2)}}\frac{\det\begin{bmatrix}K_N^{(0)}(x_1,x_2) & \hat{K}_N^{(0)}(m_1^2,x_2) & \dots & \hat{K}_N^{(0)}(m_{N_f}^2,x_2)\\ \hat{B}_1(x_1) & B_1(m_1^2) & \dots & B_1(m_{N_f}^2)\\ \vdots & \vdots & \dots & \vdots\\ \hat{B}_{N_f}(x_1) & B_{N_f}(m_1^2) & \dots & B_{N_f}(m_{N_f}^2)\end{bmatrix}}{\det[B_i(m_f^2)]_{i,f=1}^{N_f}}\,. \tag{4.15}$$

[4] We keep the factor of $\sqrt{\frac{\prod_{f=1}^{N_f}(x_2+m_f^2)}{\prod_{f=1}^{N_f}(x_1+m_f^2)}}$ to compare with results in chapter 5.

4.2 Asymptotic analysis for $N_f = 0$

In this section we study the large N limit of the correlation kernel for the chGUE(N) with an external source, derived in the previous section, see Eq. (4.6). We can state and prove the following Proposition:

Proposition 4.1. *For the chGUE(N) with an external source and JPDF given by Eq.* (4.1) *the correlation kernel at finite matrix size N reads (Eq.* (4.6)*):*

$$K_N^{(0)}(x_1,x_2) = \frac{-1}{2\pi i}\int_0^\infty ds s^{\nu/2}e^{-s}J_\nu(2\sqrt{sx_1})\left(\prod_{n=1}^N(a_n+s)\right)\oint_C du\frac{u^{-\nu/2}e^{-u}I_\nu(2\sqrt{ux_2})}{\prod_{n=1}^N(a_n-u)}\frac{1}{u+s}\,,$$

where the complex contour integration is taken counter-clockwise around the real, positive values $a_1,\ldots,a_N$ and leaving the real value $-s$ outside.
Rescaling the variables x_1, x_2 as

$$x_1 = \frac{\zeta_a^2}{4N\Xi}\,, \quad \textit{and} \quad x_2 = \frac{\zeta_a^2}{4N\Xi}\,,$$

where Ξ can be identified with the temperature dependent chiral condensate, leads to the following large N limit of the kernel:

$$\lim_{N\to\infty}\frac{1}{2N\Xi}K_N^{(0)}\left(x_1=\frac{\zeta_a^2}{4N\Xi},x_2=\frac{\zeta_b^2}{4N\Xi}\right) = \frac{1}{2}\int_0^1 d\tau J_\nu(\zeta_a\sqrt{\tau})J_\nu(\zeta_b\sqrt{\tau})\,. \tag{4.16}$$

Remark 4.2. The resulting integral in Proposition 4.1 can be evaluated as

$$\frac{1}{2}\int_0^1 d\tau J_\nu(\zeta_a\sqrt{\tau})J_\nu(\zeta_b\sqrt{\tau}) = \frac{\zeta_a J_{\nu+1}(\zeta_a)J_\nu(\zeta_b)-\zeta_b J_{\nu+1}(\zeta_b)J_\nu(\zeta_a)}{\zeta_a^2-\zeta_b^2} \equiv \mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_b)$$

for $x_1 \neq x_2$.[5] Similarly, we find for $x_1 = x_2$

$$\frac{1}{2}\int_0^1 d\tau J_\nu(\zeta_a\sqrt{\tau})J_\nu(\zeta_b\sqrt{\tau}) = \frac{1}{2}\left(J_\nu^2(\zeta_a) - \frac{2\nu}{\zeta_a}J_\nu(\zeta_a)J_{\nu+1}(\zeta_a) + J_{\nu+1}^2(\zeta_a)\right)\,.$$

We will see this in the following proof of Proposition 4.1 explicitly.

Proof. We rescale the variables $s \to Ns$, $u \to Nu$, $a_n \to Na_n$ $\forall n = 1,\ldots,N$ and introduce a rescaled kernel

$$k(\rho,\eta) = \frac{1}{N}K_N^{(0)}\left(x_1=\frac{\rho}{N},x_2=\frac{\eta}{N}\right)\,.$$

We introduce $\mathcal{L}_1(u) = u + \frac{1}{N}\sum_{n=1}^N \log(a_n-u)$ to write

$$e^{-Nu}\left(\prod_{n=1}^N(a_n-u)\right)^{-1} = e^{-Nu}e^{-N\frac{1}{N}\sum_{n=1}^N\log(a_n-u)} = e^{-N\mathcal{L}_1(u)}\,.$$

[5] The right-hand side of Eq. (4.16) is the well-known Bessel kernel [133], which will be denoted by $\mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_b)$.

Similarly, we have

$$e^{-Ns}\left(\prod_{n=1}^{N}(s+a_n)\right) = e^{-Ns}e^{N\frac{1}{N}\sum_{n=1}^{N}\log(a_n+s)} = e^{-N\mathcal{L}_2(s)}\,,$$

where $\mathcal{L}_2(s) = s - \frac{1}{N}\sum_{n=1}^{N}\log(a_n+s)$. The rescaled kernel takes the form

$$k(\rho,\eta) = (-1)\int_0^\infty ds\, s^{\nu/2}e^{-N\mathcal{L}_2(s)}J_\nu\left(2\sqrt{\rho s}\right)\frac{1}{2\pi i}\oint_C du\,\frac{u^{-\nu/2}I_\nu\left(2\sqrt{\eta u}\right)}{s+u}e^{-N\mathcal{L}_1(u)}\,.$$

To evalutate this kernel as $N\to\infty$, we perform a saddle point approximation of the two expressions $e^{-N\mathcal{L}_1(u)}$ and $e^{-N\mathcal{L}_2(s)}$. Therefore, we do a saddle point analysis and try to find appropriate saddle points of $\mathcal{L}_1(u)$ and $\mathcal{L}_2(s)$. Calculating the derivatives with respect to u and s respectively yields[6]

$$\begin{aligned}
\mathcal{L}_1(u) &= u + \frac{1}{N}\sum_{n=1}^{N}\log(a_n-u)\,, \quad \text{and} \quad \mathcal{L}_2(s) = s - \frac{1}{N}\sum_{n=1}^{N}\log(a_n+s)\,,\\
\mathcal{L}_1'(u) &= 1 - \frac{1}{N}\sum_{n=1}^{N}\frac{1}{a_n-u}\,, \quad \text{and} \quad \mathcal{L}_2'(s) = 1 - \frac{1}{N}\sum_{n=1}^{N}\frac{1}{a_n+s}\,,\\
\mathcal{L}_1''(u) &= -\frac{1}{N}\sum_{n=1}^{N}\frac{1}{(a_n-u)^2}\,, \quad \text{and} \quad \mathcal{L}_2''(s) = +\frac{1}{N}\sum_{n=1}^{N}\frac{1}{(a_n+s)^2}\,.
\end{aligned}$$

We immediately see the connections[7]

$$\mathcal{L}_1(-x) = -\mathcal{L}_2(x), \quad \text{and} \quad \mathcal{L}_1'(-x) = \mathcal{L}_2'(x), \quad \text{and} \quad \mathcal{L}_1''(-x) = -\mathcal{L}_2''(x)\,,$$

which implies the following: If $\bar{s}$ is a saddle point of $\mathcal{L}_2(s)$, then the value $\bar{u}$ is a saddle point of $\mathcal{L}_1(u)$.[8]

We define the critical value

$$t_c \equiv \frac{1}{N}\sum_{n=1}^{N}\frac{1}{a_n}\,.$$

The temperature encoding values a_n are positive and real. Thus, we can distinguish between three cases

$$\text{I)}\ t_c = 1, \quad \text{II)}\ t_c < 1, \quad \text{and} \quad \text{III)}\ t_c > 1\,.$$

Because of the rescaling $a_n \to Na_n$, the rescaled a_n are drawn from the interval $[0,1)$ as $N\to\infty$. Thus, we can estimate in the large N-limit

$$t_c = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{a_n} > \frac{1}{N}\sum_{n=1}^{N}1 = 1\,.$$

Therefore, $t_c > 1$ (case III)) will always be true and we can neglect the other two cases.

[6] Note that $\mathcal{L}_1(u)$ is an analytic function with complex derivative, if we restrict the logarithm $\log(a_n-u)$ such that the branch cut of the logarithm is on the negative real line and the derivative of $\log z$ is $1/z$.

[7] The connection is only valid for u being a real-valued variable. This is in general not the case.

[8] The saddle point connection $\bar{u} = -\bar{s}$ is valid even if u is a complex variable and s is a real variable.

The next step is to analyze the saddle points of $\mathcal{L}_2(s)$. We follow [12, 13] and consider the function

$$h(s) = 1 - \frac{1}{N}\sum_{n=1}^{N}\frac{1}{a_n + s}\,.$$

Using t_c as defined above, we have

$$h(0) = 1 - t_c < 0, \quad \text{for case } \mathbb{III}) \quad \Leftrightarrow \quad t_c > 1\,.$$

The derivative of $h(s)$ is given as

$$h'(s) = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{(a_n + s)^2} > 0, \quad \forall\, s \in \mathbb{R}_+\,.$$

As $s \to \pm\infty$, we obtain $h(s) \to 1$. Thus, we have N real, negative poles of $h(s)$ at the points $-a_1, \ldots, -a_N$. For $s \in [0, \infty]$ the function $h(s)$ is smooth, continuous and monotonically increasing, with limit 1 as $s \to \infty$. Therefore, in case $\mathbb{III}$), there is only one real, positive zero of $h(s)$ in $[0, \infty]$.[9]
We denote the real, positive zero of $h(s)$ as $\bar{s}$ and investigate the derivative of $h(s)$ at that point:

$$h'(\bar{s}) = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{(a_n + \bar{s})^2} > 0 \quad \forall\, s \in \mathbb{R}\backslash\Omega\,,$$

where $\Omega = \{-a_1, \ldots, -a_N\}$. Hence, the second derivative of $\mathcal{L}_2(s)$ is greater than zero at the point $\bar{s}$ and we have found a suitable saddle point for the s-integration. Since all a_n are living in the intervall $[0, 1)$, we can also give an upper bound of the derivative of $h(s)$. We find

$$h'(\bar{s}) = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{(a_n + \bar{s})^2} < \frac{1}{N}\sum_{n=1}^{N}\frac{1}{\bar{s}^2} = \bar{s}^{-2} = \text{const.} \tag{4.17}$$

This implies that the second derivative of $\mathcal{L}_2(s)$ is bounded from above by $\bar{s}$.

The analysis of $\mathcal{L}_1(u)$ is similar. We introduce

$$g(u) = 1 - \frac{1}{N}\sum_{n=1}^{N}\frac{1}{a_n - u}\,.$$

Since the contour C encircles the points in the set $\widetilde{\Omega} = \{a_1, \ldots, a_N\}$ and leaves $-s$ outside, $g(u)$ is an analytic function on the contour C. Thus, we can deform the contour such that no singularities from $\widetilde{\Omega}$ are crossed and still obtain the same value for the integral. Additionally, we have

$$g(0) = 1 - t_c < 0, \quad \text{for case } \mathbb{III}) \quad \Leftrightarrow \quad t_c > 1\,.$$

This leads us to

$$g'(u) = -\frac{1}{N}\sum_{n=1}^{N}\frac{1}{(a_n - u)^2} < 0 \quad \forall\, u \in \mathbb{R}\backslash\widetilde{\Omega}\,.$$

If the variable u takes real values, $g(u)$ is continuous and decreases monotonically with $\lim_{u\to\pm\infty} g(u) = 1$. Since $g(u)$ is analytic and continuous in u with N real poles, we can

[9] $h(s)$ has N zeros in total. Because $h(s)$ has N real, negative poles, $N - 1$ zeros must be real and negative by constitution of $h(s)$. The remaining zero is then determined by the value of t_c.

assume that, without loss of generality, the contour C can be deformed such that the zeros of $g(u)$ are also real.[10] Because of the continuity of $g(u)$ and the positions of the poles on the real line, there are exactly $N-1$ real, positive zeros at all times. Depending on the case $\mathbb{I}$), $\mathbb{II}$) or $\mathbb{III}$), the last zero is either positive, negative or exactly positioned at zero. In case $\mathbb{III}$) the zero is real and *negative*, leading to a maximum of $\mathcal{L}_1(u)$ instead of a minimum, which would be required for a sensible saddle point approximation. This problem can be circumvented and will be addressed later in the proof.

Recall the connection between $\mathcal{L}_1(u)$ and $\mathcal{L}_2(s)$ in terms of saddle points: We have seen that $\bar{u} = -\bar{s}$. We have identified a suitable saddle point $\bar{s}$ for $\mathcal{L}_2(s)$ and expand via

$$s = \bar{s} + \frac{1}{\sqrt{N}}x \quad \text{and} \quad ds = \frac{1}{\sqrt{N}}dx\,.$$

As we have discussed previousely the connected saddle point $\bar{u} = -\bar{s}$ does not imply the required maximum, if we view u as a real variable. Nevertheless, we can expand for complex u with a suitable branch cut of the logarithm in $\mathcal{L}_1(u)$ via

$$u = \bar{u} + i\frac{1}{\sqrt{N}}y \quad \text{and} \quad du = i\frac{1}{\sqrt{N}}dy\,.$$

This implies that the integration over u gets rotated onto the real line as $N \to \infty$ and the maximum we obtained at $\mathcal{L}_1(\bar{u})$ becomes a minimum as required for the saddle point expansion. Additionally, these expansions lead to

$$\frac{1}{u+s} = \frac{\sqrt{N}}{x+iy}\,.$$

After the identification of the saddle points and the introduction of the necessary expansions of the integration variables, we can compute the integrals. We perfom steps similar to [25]. Recall the kernel

$$k(\rho,\eta) = (-1)\int_0^\infty ds\, f_2(s) e^{-N\mathcal{L}_2(s)} I(s)\,,$$

where we have introduced $f_2(s) = s^{\nu/2}J_\nu(2\sqrt{\rho s})$ and

$$I(s) = \frac{1}{2\pi i}\oint_C du\, \frac{f_1(u)}{u+s} e^{-N\mathcal{L}_1(u)}$$

with $f_1(u) = u^{-\nu/2}I_\nu(2\sqrt{\eta u})$.
Looking at the contour integration, a deformation through the saddle point $\bar{u}$ adds a contribution from the pole at $-s$, if and only if $\bar{u} < -s$. We include this contribution by writing

$$\oint_C du\, \frac{f_1(u)}{u+s} e^{-N\mathcal{L}_1(u)} = \oint_{\widetilde{C}} du\, \frac{f_1(u)}{u+s} e^{-N\mathcal{L}_1(u)} - \oint_{C_s} du\, \Theta(-s-\bar{u}) \frac{f_1(u)}{u+s} e^{-N\mathcal{L}_1(u)}, \tag{4.18}$$

where $\widetilde{C}$ is a contour, which enlarges C by also encircling the pole at $-s$. In contrast, C_s is a simple contour encircling only the pole at $-s$. The function $\Theta(x)$ is the Heaviside step function with argument x, being 1, when $x \geq 0$ and zero otherwise. Applying the Basic

[10] The reader might note: It is feasable to consider to show that $g(u)$ has only real zeros using the fundemental theorem of algebra.

Residue Theorem to the second term in the above expression yields

$$I(s) = \frac{1}{2\pi i}\oint_{\widetilde{C}} du\, \frac{f_1(u)}{u+s} e^{-N\mathcal{L}_1(u)} - \frac{\Theta(-s-\bar{u})}{2\pi i} 2\pi i f_1(-s) e^{-N\mathcal{L}_1(-s)} .$$

Furthermore, we have $\mathcal{L}_1(-s) = -\mathcal{L}_2(s)$ and

$$f_1(-s) = (-1)^{\nu/2} s^{-\nu/2} I_\nu(2\sqrt{s\eta}i) = \underbrace{(-1)^{\nu/2} i^{-\nu}}_{=1} s^{-\nu/2} J_\nu(2\sqrt{\eta s}) .$$

The expansion via $u = \bar{u} + i\frac{1}{\sqrt{N}}y$ yields for the exponential

$$\begin{aligned} -N\mathcal{L}_1(u) &= -N\mathcal{L}_1\left(\bar{u} + i\frac{1}{\sqrt{N}}y\right) = -N\left[\mathcal{L}_1(\bar{u}) + \mathcal{L}_1'(\bar{u})\frac{iy}{\sqrt{N}} + \frac{1}{2}\mathcal{L}_1''(\bar{u})\left(\frac{iy}{\sqrt{N}}\right)^2 + \mathcal{O}(N^{-3/2})\right] \\ &= -N\mathcal{L}_1(\bar{u}) + \frac{1}{2}\mathcal{L}_1''(\bar{u})y^2 + \mathcal{O}(N^{-1/2}) = N\mathcal{L}_2(\bar{s}) - \frac{1}{2}\mathcal{L}_2''(\bar{s})y^2 + \mathcal{O}(N^{-1/2}) . \end{aligned}$$

Similarly, we find for the function $f_1(u)$:

$$\begin{aligned} f_1\left(\bar{u} + i\frac{1}{\sqrt{N}}y\right) &= f_1(\bar{u}) + f_1'(\bar{u})\left(\frac{iy}{\sqrt{N}}\right) + \mathcal{O}(N^{-1}) \\ &= f_1(\bar{u})\left(1 + \frac{f_1'(\bar{u})}{f_1(\bar{u})}\left(\frac{iy}{\sqrt{N}}\right)\right) + \mathcal{O}(N^{-1}) . \end{aligned}$$

The consequence of the line above is that for $f_1(\bar{u}) \neq 0$ the second term and also all higher orders vanish in the large N-limit. Therefore, we obtain

$$\begin{aligned} I(s) &= \frac{1}{2\pi i}\oint_{\widetilde{C}} du\, \frac{f_1(u)}{u+s} e^{-N\mathcal{L}_1(u)} - \Theta(-s+\bar{s}) f_1(-s) e^{N\mathcal{L}_2(s)} \\ &= \frac{1}{2\pi i}\frac{i}{\sqrt{N}}\int_{-\infty}^{\infty} dy\, \frac{f_1(-\bar{s})}{\bar{u} + i\frac{1}{\sqrt{N}y+s}} e^{N\mathcal{L}_2(\bar{s})} e^{-\frac{1}{2}\mathcal{L}_2''(\bar{s})y^2} - \Theta(-s+\bar{s}) f_1(-s) e^{N\mathcal{L}_2(s)} . \end{aligned}$$

This leads to the kernel $k(\rho,\eta)$ taking the form

$$\begin{aligned} k(\rho,\eta) = \Bigg[&\int_0^\infty ds\, f_2(s) e^{-N\mathcal{L}_2(s)} f_1(-s) e^{N\mathcal{L}_2(s)} \Theta(\bar{s}-s) \\ &- \frac{1}{2\pi\sqrt{N}}\int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}\mathcal{L}_2''(s)y^2} \int_0^\infty ds\, f_2(s) e^{-N\mathcal{L}_2(s)} e^{N\mathcal{L}_2(\bar{s})} f_1(\bar{s}) \frac{1}{\bar{u} + iN^{-1/2}y + s}\Bigg] . \end{aligned}$$

The next step is expand the integration over s around $\bar{s}$. We find

$$\begin{aligned} -N\mathcal{L}_2(s) &= -N\mathcal{L}_2\left(\bar{s} + \frac{1}{\sqrt{N}}x\right) = -N\left[\mathcal{L}_2(\bar{s}) + \mathcal{L}_2'(\bar{s})\frac{x}{\sqrt{N}} + \frac{1}{2}\mathcal{L}_2''(\bar{s})\left(\frac{x}{\sqrt{N}}\right)^2 + \mathcal{O}(N^{-3/2})\right] \\ &= -N\mathcal{L}_2(\bar{s}) - \frac{1}{2}\mathcal{L}_2''(\bar{s})x^2 + \mathcal{O}(N^{-1/2}) \end{aligned}$$

and

$$f_2\left(\bar{s} + \frac{1}{\sqrt{N}}x\right) = f_2(\bar{s}) + f_2'(\bar{s})\left(\frac{x}{\sqrt{N}}\right) + \mathcal{O}(N^{-1})$$
$$= f_2(\bar{s})\left(1 + \frac{f_2'(\bar{s})}{f_2(\bar{s})}\left(\frac{x}{\sqrt{N}}\right)\right) + \mathcal{O}(N^{-1})\,.$$

Again all higher order terms vanish for large N and the kernel becomes

$$k(\rho,\eta) = \left[\int_0^{\bar{s}} ds\; f_2(s)f_1(-s) - \frac{1}{2\pi\sqrt{N}} e^{N\mathcal{L}_2(\bar{s})} f_1(\bar{s}) \int_{-\infty}^{\infty} dy\; e^{-\frac{1}{2}\mathcal{L}_2''(s)y^2}\right.$$
$$\left.\times \int_{-\infty}^{\infty} dx\; \frac{1}{\sqrt{N}} f_2(\bar{s}) e^{-N\mathcal{L}_2(\bar{s})} e^{-\frac{1}{2}\mathcal{L}_2''(\bar{s})x^2} \frac{\sqrt{N}}{x+iy}\right]$$
$$= \left[\int_0^{\bar{s}} ds\; f_2(s)f_1(-s) - \frac{1}{\sqrt{N}} \underbrace{\frac{1}{2\pi} f_2(\bar{s}) f_1(\bar{s}) \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; e^{-\frac{1}{2}\mathcal{L}_2''(\bar{s})y^2} e^{-\frac{1}{2}\mathcal{L}_2''(\bar{s})x^2} \frac{1}{x+iy}}_{\text{independent of } N}\right]$$

In the last line above we have used the upper bound for $\mathcal{L}_2''(\bar{s})$ from Eq. (4.17), which implies that the integration with respect to x and y is always a finite Gaussian integral, independent of N. Thus, the second term vanishes for large N and we only need to consider the first term further. For $\bar{s} > 0$ we find

$$\int_0^{\bar{s}} ds\; f_1(-s)f_2(s) = \int_0^{\bar{s}} ds\; s^{-\nu/2} J_\nu(2\sqrt{\eta s}) s^{\nu/2} J_\nu(2\sqrt{\rho s}) = \int_0^{\bar{s}} ds\; J_\nu(2\sqrt{\eta s}) J_\nu(2\sqrt{\rho s})\,.$$

Doing the substitution $s = \bar{s}\tau$ we obtain

$$\int_0^{\bar{s}} ds\; J_\nu(2\sqrt{\eta s}) J_\nu(2\sqrt{\rho s}) = \bar{s} \int_0^1 d\tau\; J_\nu(\sqrt{4\eta\bar{s}}\sqrt{\tau}) J_\nu(\sqrt{4\rho\bar{s}}\sqrt{\tau})\,.$$

Using another substitution, namely, $\zeta_a^2 = 4\bar{s}\rho$ and $\zeta_b^2 = 4\bar{s}\eta$ and setting $\bar{s} = \Xi$ we can write

$$k\left(\rho = \frac{\zeta_a^2}{4\Xi}, \eta = \frac{\zeta_b^2}{4\Xi}\right) = \Xi \int_0^1 d\tau\; J_\nu(\zeta_a\sqrt{\tau}) J_\nu(\zeta_b\sqrt{\tau})\,.$$

The above integral can be found in [130]. For $a \neq b$ we find

$$\int_0^1 d\tau J_\nu(a\sqrt{\tau}) J_\nu(b\sqrt{\tau}) = 2\frac{bJ_{\nu-1}(b)J_\nu(a) - aJ_{\nu-1}(a)J_\nu(b)}{a^2 - b^2}$$

and for $a = b$

$$\int_0^1 d\tau J_\nu^2(a\sqrt{\tau}) = J_\nu^2(a) - \frac{2\nu}{a} J_\nu(a) J_{\nu+1}(a) + J_{\nu+1}^2(a)\,.$$

Together with the known formula for Bessel functions $J_\nu(x)$,

$$J_{\nu-1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu+1}(x)\,,$$

the rescaled kernel becomes, for $\zeta_a^2 \neq \zeta_b^2$,

$$k\left(\rho=\frac{\zeta_a^2}{4\Xi},\eta=\frac{\zeta_b^2}{4\Xi}\right)=2\Xi\frac{\zeta_a J_{\nu+1}(\zeta_a)J_\nu(\zeta_b)-\zeta_b J_{\nu+1}(\zeta_b)J_\nu(\zeta_a)}{\zeta_a^2-\zeta_b^2}\,. \tag{4.19}$$

For $\zeta_a^2=\zeta_b^2$ we find

$$k\left(\rho=\frac{\zeta_a^2}{4\Xi},\eta=\frac{\zeta_b^2}{4\Xi}\right)=\Xi\left(J_\nu^2(\zeta_a)-\frac{2\nu}{\zeta_a}J_\nu(\zeta_a)J_{\nu+1}(\zeta_a)+J_{\nu+1}^2(\zeta_a)\right)\,. \tag{4.20}$$

Finally, we obtain for $\zeta_a^2 \neq \zeta_b^2$ using Eq. (4.19)

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{2N\Xi}K_N^{(0)}\left(x_1=\frac{\zeta_a^2}{4N\Xi},x_2=\frac{\zeta_b^2}{4N\Xi}\right)&=\lim_{N\to\infty}\frac{1}{2\Xi}k\left(\rho=\frac{\zeta_a^2}{4\Xi},\eta=\frac{\zeta_b^2}{4\Xi}\right)\\&=\frac{\zeta_a J_{\nu+1}(\zeta_a)J_\nu(\zeta_b)-\zeta_b J_{\nu+1}(\zeta_b)J_\nu(\zeta_a)}{\zeta_a^2-\zeta_b^2}\end{aligned}$$

and for $\zeta_a^2=\zeta_b^2$ using Eq. (4.20)

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{2N\Xi}K_N^{(0)}\left(x_1=\frac{\zeta_a^2}{4N\Xi},x_2=\frac{\zeta_b^2}{4N\Xi}\right)&=\lim_{N\to\infty}\frac{1}{2\Xi}\,k\left(\rho=\frac{\zeta_a^2}{4\Xi},\eta=\frac{\zeta_b^2}{4\Xi}\right)\\&=\frac{1}{2}\left(J_\nu^2(\zeta_a)-\frac{2\nu}{\zeta_a}J_\nu(\zeta_a)J_{\nu+1}(\zeta_a)+J_{\nu+1}^2(\zeta_a)\right),\end{aligned}$$

which finishes the proof. □

4.3 Asymptotic analysis for $N_f \neq 0$

In this section we study the large N limit of the correlation kernel of the chGUE(N) with an external source extended by N_f massive flavors. The kernel at finite N was derived in the previous section, see Eq. (4.15). We can state and prove the following Proposition:

Proposition 4.3. *For the chGUE(N) with an external source and N_f massive flavors the JPDF was given by Eq.* (4.2)*. The correlation kernel at finite matrix size N reads (see Eq.* (4.15)*):*

$$\begin{aligned}K_N^{(N_f)}(x_1,x_2)=&\left(\det[B_i(m_f^2)]_{i,f=1}^{N_f}\right)^{-1}\sqrt{\frac{\prod_{f=1}^{N_f}(x_2+m_f^2)}{\prod_{f=1}^{N_f}(x_1+m_f^2)}}\\&\times\det\begin{bmatrix}K_N^{(0)}(x_1,x_2) & \hat{K}_N^{(0)}(m_1^2,x_2) & \dots & \hat{K}_N^{(0)}(m_{N_f}^2,x_2)\\ \hat{B}_1(x_1) & B_1(m_1^2) & \dots & B_1(m_{N_f}^2)\\ \vdots & \vdots & \dots & \vdots\\ \hat{B}_{N_f}(x_1) & B_{N_f}(m_1^2) & \dots & B_{N_f}(m_{N_f}^2)\end{bmatrix}.\end{aligned} \tag{4.21}$$

The 1×1 entry of the matrix in the numerator of the above ratio of determinants is given by an expression which is equal to the kernel for $N_f = 0$, i.e.

$$K_N^{(0)}(x_1, x_2) = \frac{-1}{2\pi i} \int_0^\infty ds s^{\nu/2} e^{-s} J_\nu(2\sqrt{s x_1}) \prod_{n=1}^N (a_n + s) \oint_C du \frac{u^{-\nu/2} e^{-u} I_\nu(2\sqrt{u x_2})}{\prod_{n=1}^N (a_n - u)} \frac{1}{u+s} .$$

Similarly, the other entries of the first row, $\hat{K}_N^{(0)}(m_f^2, x_2)$, are given as

$$\hat{K}_N^{(0)}(m_f^2, x_2) \equiv \frac{-1}{2\pi i} \int_0^\infty ds s^{\nu/2} e^{-s} I_\nu\left(2\sqrt{s m_f^2}\right) \prod_{n=1}^N (a_n + s) \oint_C du \frac{u^{-\nu/2} e^{-u} I_\nu(2\sqrt{u x_2})}{\prod_{n=1}^N (a_n - u)} \frac{1}{u+s} .$$

Furthermore, we have for the other entries of the matrix in the numerator:

$$\hat{B}_i(x_1) = \int_0^\infty dt\, t^{\nu/2} e^{-t} J_\nu(2\sqrt{x_1 t}) \prod_{n=1}^N (a_n + t)\, t^{i-1} ,$$
$$\text{and} \quad B_i(m_f^2) = \int_0^\infty dt\, t^{\nu/2} e^{-t} I_\nu\left(2\sqrt{m_f^2 t}\right) \prod_{n=1}^N (a_n + t)\, t^{i-1} .$$

Rescaling the variables x_1, x_2 as

$$x_1 = \frac{\zeta_a^2}{4N\Xi} , \quad x_2 = \frac{\zeta_a^2}{4N\Xi} , \quad \text{and} \quad m_f^2 = \frac{\mu_f^2}{4N\Xi} ,$$

where Ξ can be identified with the temperature dependent chiral condensate, leads to the following large N limit of the kernel $K_N^{(N_f)}(x_1, x_2)$:

$$\lim_{N\to\infty} \frac{1}{2N\Xi} K_N^{(N_f)}\left(x_1 = \frac{\zeta_a^2}{4N\Xi}, x_2 = \frac{\zeta_b^2}{4N\Xi}\right) = \sqrt{\frac{\prod_{f=1}^{N_f}(\zeta_b^2 + \mu_f^2)}{\prod_{f=1}^{N_f}(\zeta_a^2 + \mu_f^2)}}$$
$$\times \frac{\det \begin{bmatrix} \mathcal{B}_{JJ}(\zeta_a, \zeta_b) & \mathcal{B}_{IJ}(\mu_1, \zeta_b) & \dots & \mathcal{B}_{IJ}(\mu_{N_f}, \zeta_b) \\ J_\nu(\zeta_a) & I_\nu(\mu_1) & \dots & I_\nu(\mu_{N_f}) \\ \vdots & \vdots & \dots & \vdots \\ \zeta_a^{N_f-1} J_{\nu+N_f-1}(\zeta_a) & (-\mu_1)^{N_f-1} I_{\nu+N_f-1}(\mu_1) & \dots & (-\mu_{N_f})^{N_f-1} I_{\nu+N_f-1}(\mu_{N_f}) \end{bmatrix}}{\det\left[(-\mu_f)^{j-1} I_{\nu+j-1}(\mu_f)\right]_{j,f=1}^{N_f}} , \tag{4.22}$$

where we have introduced the shorthand notations for the Bessel kernel:

$$\mathcal{B}_{JJ}(\zeta_a, \zeta_b) \equiv \frac{\zeta_a J_{\nu+1}(\zeta_a) J_\nu(\zeta_b) - \zeta_b J_{\nu+1}(\zeta_b) J_\nu(\zeta_a)}{\zeta_a^2 - \zeta_b^2} ,$$
$$\mathcal{B}_{IJ}(\mu_f, \zeta_b) \equiv \frac{\mu_f I_{\nu+1}(\mu_f) J_\nu(\zeta_b) + \zeta_b J_{\nu+1}(\zeta_b) I_\nu(\mu_f)}{\mu_f^2 + \zeta_b^2} .$$

Proof. The given determinantal form of the kernel in Eq. (4.21) indicates that the large N-limit affects the entries of the matrices directly. The determinantal structure is preserved in the large N limit. The root prefactor takes the desired form by putting in the rescaled variables and will be neglected in the analysis until the end of the proof.
Before we start with the entries directly, we use the determinantal structure of the kernel to replace the monomials t^{i-1} with monic Herminte polynomials $h_{i-1}(t)$ in the entries $\hat{B}(x_1)$ and $B(m_f^2)$ respectively. This is possible, because we can add and substract rows in the determinants of the kernel without changing its value. Then, we can consider the large N-limit of all four different entry-types directly. We have to consider

$$
\begin{aligned}
K_N^{(0)}(x_1,x_2) &= \frac{-1}{2\pi i}\int_0^\infty ds s^{\nu/2}e^{-s}J_\nu(2\sqrt{sx_1})\prod_{n=1}^N(a_n+s)\oint_C du\frac{u^{-\nu/2}e^{-u}I_\nu(2\sqrt{ux_2})}{\prod_{n=1}^N(a_n-u)}\frac{1}{u+s},\\
\hat{K}_N^{(0)}(m_f^2,x_2) &\equiv \frac{-1}{2\pi i}\int_0^\infty ds s^{\nu/2}e^{-s}I_\nu\left(2\sqrt{sm_f^2}\right)\prod_{n=1}^N(a_n+s)\oint_C du\frac{u^{-\nu/2}e^{-u}I_\nu(2\sqrt{ux_2})}{\prod_{n=1}^N(a_n-u)}\frac{1}{u+s},\\
\hat{B}_i(x_1) &= \int_0^\infty dt\, t^{\nu/2}e^{-t}J_\nu(2\sqrt{x_1t})\prod_{n=1}^N(a_n+t)\,h_{i-1}(t),\\
B_i(m_f^2) &= \int_0^\infty dt\, t^{\nu/2}e^{-t}I_\nu\left(2\sqrt{m_f^2t}\right)\prod_{n=1}^N(a_n+t)\,h_{i-1}(t).
\end{aligned}
$$

The first entry-type is exactly the kernel for $N_f = 0$. Therefore, we can directly apply the large-N-limit derived in Proposition 4.1, namely

$$
\lim_{N\to\infty}\frac{1}{2N\Xi}K_N^{(0)}\left(x_1=\frac{\zeta_a^2}{4N\Xi},x_2=\frac{\zeta_b^2}{4N\Xi}\right)=\frac{\zeta_aJ_{\nu+1}(\zeta_a)J_\nu(\zeta_b)-\zeta_bJ_{\nu+1}(\zeta_b)J_\nu(\zeta_a)}{\zeta_a^2-\zeta_b^2}
$$

for $x_1 \neq x_2$, which is equal to $\mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_b)$. Similarly, we have for $x_1 = x_2$:

$$
\lim_{N\to\infty}\frac{1}{2N\Xi}K_N^{(0)}\left(x_1=\frac{\zeta_a^2}{4N\Xi},x_2=\frac{\zeta_a^2}{4N\Xi}\right)=\frac{1}{2}\left(J_\nu^2(\zeta_a)-\frac{2\nu}{\zeta_a}J_\nu(\zeta_a)J_{\nu+1}(\zeta_a)+J_{\nu+1}^2(\zeta_a)\right).
$$

The second entry-type $\hat{K}_N^{(0)}(m_f^2,x_2)$ differs from $K_N^{(0)}(x_1,x_2)$ by the change

$$
J_\nu(2\sqrt{sx_1}) \to I_\nu\left(2\sqrt{sm_f^2}\right).
$$

Since $I_\nu(x) = i^{-\nu}J_\nu(ix)$, we can replace m_f^2 with $-x_1$ and multiply everything with a factor of $i^{-\nu}$ to get from $\hat{K}_N^{(0)}(m_f^2,x_2)$ to $K_N^{(0)}(x_1,x_2)$. Thus, the large N-analysis is analogous, if we rescale the masses by $m_f^2 = \frac{\mu_f^2}{4N\Xi}$. From Proposition 4.1 we obtain with $\zeta_a = i\mu_f$

$$
\begin{aligned}
i^{-\nu}\lim_{N\to\infty}\frac{1}{2N\Xi}K_N^{(0)}\left(x_1=-m_f^2,x_2\right) &= i^{-\nu}\frac{\zeta_aJ_{\nu+1}(\zeta_a)J_\nu(\zeta_b)-\zeta_bJ_{\nu+1}(\zeta_b)J_\nu(\zeta_a)}{\zeta_a^2-\zeta_b^2}\\
&= \frac{\mu_fI_{\nu+1}(\mu_f)J_\nu(\zeta_b)+\zeta_bJ_{\nu+1}(\zeta_b)I_\nu(\mu_f)}{\mu_f^2+\zeta_b^2},
\end{aligned}
$$

which can be used to write

$$\lim_{N\to\infty} \frac{1}{2N\Xi} K_N^{(0)} \left(m_f^2 = \frac{\mu_f^2}{4N\Xi}, x_2 = \frac{\zeta_b^2}{4N\Xi} \right) = \frac{\mu_f I_{\nu+1}(\mu_f) J_\nu(\zeta_b) + \zeta_b J_{\nu+1}(\zeta_b) I_\nu(\mu_f)}{\mu_f^2 + \zeta_b^2} \,.$$

The right-hand side in the equation above is equal to $\mathcal{B}_{IJ}(\mu_f, \zeta_B)$, containing Bessel functions $J_\nu(z)$ and $I_\nu(z)$, as desired.
The remaining two entry-types are also closely related in their large N asymptotics. To obtain the desired result of the whole correlation kernel, we note that the determinantal structure of the kernel $K_N^{(N_f)}(x_1, x_2)$ with its ratio of two determinants allows us to cancel factors, which appear in both numerator and denominator. This becomes important in the large N asymptotics of the entries of $\hat{B}_j$ and B_j.
First we look at

$$\hat{B}_j(x_1) = \int_0^\infty dt\; t^{\nu/2} e^{-t} J_\nu(2\sqrt{x_1 t}) \prod_{n=1}^{N} (a_n + t)\; h_{j-1}(t) \,.$$

The first step is to rescale with[11]

$$t \to Nt, \quad a_n \to N a_n \quad \text{and} \quad x_1 = \frac{\zeta_a^2}{4N\Xi} \,.$$

Additionally, we express the argument of the Hermite polynomial as

$$t \to \sqrt{\frac{N\mathcal{L}_2''(\Xi)}{2}} (t - \Xi) \,.$$

The next step is to expand the integration around the saddle point Ξ by

$$t = \Xi + \sqrt{\frac{2}{N\mathcal{L}_2''}}\; p \,,$$

which leads to

$$\begin{aligned}
\hat{B}_j(x_1) &= \int_0^\infty dt\; t^{\nu/2} e^{-t} J_\nu(2\sqrt{x_1 t}) \prod_{n=1}^{N} (a_n + t)\; h_{j-1}(t) \\
&= N^\gamma \sqrt{\frac{2}{\mathcal{L}_2''(\Xi)}} \int_{-\infty}^{\infty} dp \left(\Xi + \sqrt{\frac{2}{N\mathcal{L}_2''(\Xi)}} p \right)^{\nu/2} J_\nu \left(\zeta_a \sqrt{1 + \frac{1}{\Xi} \sqrt{\frac{2}{N\mathcal{L}_2''(\Xi)}} p} \right) \\
&\times e^{-N\mathcal{L}_2(\Xi)} e^{-p^2} e^{-\mathcal{O}(N^{1/2})} h_{j-1}(p) \,.
\end{aligned}$$

The expansion around Ξ in the large N limit leads to subleading contributions in the exponential of the order $\mathcal{O}(N^{1/2})$, which can be omitted as $N \to \infty$ as $e^{-\mathcal{O}(N^{1/2})} \to 1$. Furthermore, we use

$$h_{j-1}(p) = 2^{1-j} (-1)^{j-1} e^{p^2} \frac{d^{j-1}}{dp^{j-1}} e^{-p^2} \,.$$

[11] For convenience we rescale the Hermite polynomial argument with N by first rephrasing it as a monomial, then rescaling, and then rewriting it back as a Hermite polynomial.

Now we do integration by parts ($j-1$-times), where we can use that the arising boundary terms vanish in each step. This leads to

$$\begin{aligned}\hat{B}_j(x_1) &= N^\gamma \sqrt{\frac{2}{\mathcal{L}_2''(\Xi)}} e^{-N\mathcal{L}_2(\Xi)} 2^{1-j} \int_{-\infty}^{\infty} dp e^{-p^2} \\ &\times \frac{d^{j-1}}{dp^{j-1}} \left(\Xi + \sqrt{\frac{2}{N\mathcal{L}_2''(\Xi)}} p\right)^{\nu/2} J_\nu \left(\zeta_a \sqrt{1 + \frac{1}{\Xi}\sqrt{\frac{2}{N\mathcal{L}_2''(\Xi)}} p}\right) .\end{aligned}$$

Inserting $\left(\frac{\Xi}{\zeta_a^2}\right)^{\nu/2} \left(\frac{\zeta_a^2}{\Xi}\right)^{\nu/2}$ and substituting $z = \sqrt{\zeta_a^2 + \frac{\zeta_a^2}{\Xi}\sqrt{\frac{2}{N\mathcal{L}_2''(\Xi)}} p}$ gives us

$$dp = \frac{\Xi}{\zeta_a^2}\sqrt{2N\mathcal{L}_2''(\Xi)} z dz, \quad \frac{d^{j-1}}{dp^{j-1}} = \left(\frac{\Xi}{\zeta_a^2}\right)^{-j+1} \left[2N\mathcal{L}_2''(\Xi)\right]^{\frac{-j+1}{2}} \left(\frac{d}{zdz}\right)^{j-1} .$$

Thus, we obtain

$$\begin{aligned}\hat{B}_j(x_1) &= N^\gamma \sqrt{\frac{2}{\mathcal{L}_2''(\Xi)}} e^{-N\mathcal{L}_2(\Xi)} 2^{1-j} \left(\frac{\Xi}{\zeta_a^2}\right)^{\nu/2-j+1} \left[2N\mathcal{L}_2''(\Xi)\right]^{\frac{-j+1}{2}} \\ &\times \int_{-\infty}^{\infty} dp e^{-p^2} \left(\frac{d}{zdz}\right)^{j-1} z^\nu J_\nu(z) \\ &= \widetilde{C}_j(N) \zeta_a^{-\nu+2(j-1)} \int_{-\infty}^{\infty} dp e^{-p^2} \Bigg[z^{2\nu} \left(\frac{d}{zdz}\right)^{j-1} z^{-\nu} J_\nu(z) \\ &+ z^{-\nu} J_\nu(z) \left(\frac{d}{zdz}\right)^{j-1} z^{2\nu} \Bigg] .\end{aligned}$$

The prefactor $\widetilde{C}_j(N)$ can be extracted from the determinants in the kernel from each row in both numerator and denominator, because it appears also in the expansion of the entry-type $B_j(m_f^2)$, namely

$$\begin{aligned}B_j(m_f^2) &= N^\gamma \sqrt{\frac{2}{\mathcal{L}_2''(\Xi)}} e^{-N\mathcal{L}_2(\Xi)} 2^{1-j} \left(\frac{\Xi}{\mu_f^2}\right)^{\nu/2} \left(\frac{\Xi}{\mu_f^2}\right)^{-j+1} \left[2N\mathcal{L}_2''(\Xi)\right]^{\frac{-j+1}{2}} \\ &\times \int_{-\infty}^{\infty} dp e^{-p^2} \left(\frac{d}{z_f dz_f}\right)^{j-1} z_f^\nu I_\nu(z_f) \\ &= \widetilde{C}_j(N) \mu_f^{-\nu+2(j-1)} \int_{-\infty}^{\infty} dp e^{-p^2} \Bigg[z_f^{2\nu} \left(\frac{d}{z_f dz_f}\right)^{j-1} z_f^{-\nu} I_\nu(z_f) \\ &+ z_f^{-\nu} I_\nu(z_f) \left(\frac{d}{z_f dz_f}\right)^{j-1} z_f^{2\nu} \Bigg] .\end{aligned}$$

In the lines above we have used

$$m_f = \frac{\mu_f^2}{4N\Xi}, \quad \text{and} \quad z_f = \sqrt{\mu_f^2 + \frac{\mu_f^2}{\Xi}\sqrt{\frac{2}{N\mathcal{L}_2''(\Xi)}} p} .$$

We can now neglect the $\widetilde{C}_j(N)$ factor and apply

$$\left(\frac{d}{zdz}\right)^m z^{-\nu} J_\nu(z) = (-1)^m z^{-\nu-m} J_{\nu+m}(z),$$
$$\left(\frac{d}{zdz}\right)^m z^{-\nu} I_\nu(z) = z^{-\nu-m} I_{\nu+m}(z),$$
$$\left(\frac{d}{zdz}\right)^{j-1} z^{2\nu} = \begin{cases} z^{2\nu}, & \text{for } j = 1, \\ \prod_{n=0}^{j-1}(2\nu - 2n) z^{2\nu-2(j-1)}, & \text{for } j > 1. \end{cases}$$

For $j > 1$ we can write

$$\hat{B}_j(x_1) \to \zeta_a^{-\nu+2(j-1)} \int_{-\infty}^{\infty} dp e^{-p^2} \left[(-1)^{j-1} z^{\nu-(j-1)} J_{\nu+j-1}(z) + \prod_{n=0}^{j-1}(2\nu - 2n) z^{\nu-2(j-1)} J_\nu(z) \right],$$
$$B_j(m_f^2) \to \mu_f^{-\nu+2(j-1)} \int_{-\infty}^{\infty} dp e^{-p^2} \left[z_f^{\nu-(j-1)} I_{\nu+j-1}(z_f) + \prod_{n=0}^{j-1}(2\nu - 2n) z_f^{\nu-2(j-1)} I_\nu(z_f) \right],$$

and for $j = 1$ we obtain

$$\hat{B}_1(x_1) \to \zeta_a^{-\nu} \int_{-\infty}^{\infty} dp e^{-p^2} \left[z^\nu J_\nu(z) + z^\nu J_\nu(z) \right] = 2\zeta_a^{-\nu} \int_{-\infty}^{\infty} dp e^{-p^2} z^\nu J_\nu(z),$$
$$B_1(m_f^2) \to \mu_f^{-\nu} \int_{-\infty}^{\infty} dp e^{-p^2} \left[z_f^\nu I_\nu(z_f) + z_f^{\nu)} I_\nu(z_f) \right] = 2\mu_f^{-\nu} \int_{-\infty}^{\infty} dp e^{-p^2} z_f^\nu I_\nu(z_f).$$

The factor of 2 can be extracted from the determinants and cancels. Additionally, we see that for large N z and z_f approach α and μ_f respectively. Thus, both become independent of p. Therefore, we write

$$\hat{B}_1(x_1) \overset{N\to\infty}{\to} \zeta_a^{-\nu} \zeta_a^\nu J_\nu(\zeta_a) = J_\nu(\zeta_a),$$
$$B_1(m_f^2) \overset{N\to\infty}{\to} \mu_f^{-\nu} \mu_f^\nu I_\nu(\mu_f) = I_\nu(\mu_f),$$

and for $j > 1$

$$\hat{B}_j(x_1) \overset{N\to\infty}{\to} \zeta_a^{-\nu+2(j-1)} \left[(-1)^{j-1} \zeta_a^{\nu-(j-1)} J_{\nu+j-1}(\zeta_a) + \prod_{n=0}^{j-1}(2\nu - 2n) \zeta_a^{\nu-2(j-1)} J_\nu(\zeta_a) \right]$$
$$= (-1)^{j-1} \left[\zeta_a^{j-1} J_{\nu+j-1}(\zeta_a) + (-1)^{j-1} \prod_{n=0}^{j-1}(2\nu - 2n) J_\nu(\zeta_a) \right],$$
$$B_j(m_f^2) \overset{N\to\infty}{\to} \mu_f^{-\nu+2(j-1)} \left[\mu_f^{\nu-(j-1)} I_{\nu+j-1}(\mu_f) + \prod_{n=0}^{j-1}(2\nu - 2n) \mu_f^{\nu-2(j-1)} I_\nu(\mu_f) \right]$$
$$= (-1)^{j-1} \left[(-\mu_f)^{j-1} I_{\nu+j-1}(\mu_f) + (-1)^{j-1} \prod_{n=0}^{j-1}(2\nu - 2n) I_\nu(\mu_f) \right].$$

The factors $(-1)^{j-1}$ can be taken out of the determinants in numerator and denominator of the kernel and can thus be neglected. The second of the two terms in the above expressions of $B_j(m_f^2)$ and $\hat{B}_j(x_1)$ are multiples of the $j = 1$ row. By determinantal transformation

(substracting of rows) we then obtain the final large N result:

$$\lim_{N\to\infty}\frac{1}{2N\Xi}K_N^{(N_f)}\left(x_1=\frac{\zeta_a^2}{4N\Xi},x_2=\frac{\zeta_b^2}{4N\Xi}\right)=\sqrt{\frac{\prod_{f=1}^{N_f}(\zeta_b^2+\mu_f^2)}{\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)}}$$
$$\det\begin{bmatrix}\mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_b) & \mathcal{B}_{\mathrm{IJ}}(\mu_1,\zeta_b) & \dots & \mathcal{B}_{\mathrm{IJ}}(\mu_{N_f},\zeta_b)\\ J_\nu(\zeta_a) & I_\nu(\mu_1) & \dots & I_\nu(\mu_{N_f})\\ \vdots & \vdots & \dots & \vdots\\ \zeta_a^{N_f-1}J_{\nu+N_f-1}(\zeta_a) & (-\mu_1)^{N_f-1}I_{\nu+N_f-1}(\mu_1) & \dots & (-\mu_{N_f})^{N_f-1}I_{\nu+N_f-1}(\mu_{N_f})\end{bmatrix}$$
$$\times\frac{}{\det\left[(-\mu_f)^{j-1}I_{\nu+j-1}(\mu_f)\right]_{j,f=1}^{N_f}}.$$

This finishes the proof.

□

4.4 Summary

This chapter is concerned with the asymptotic behavior of correlation kernels for the chGUE(N) with an external source and its extension with N_f massive flavors. In section 4.1 we derived a finite N representation of the corresponding correlation kernels. We found a determinantal structure depending on the number of flavors N_f, see Eq. (4.15). This representation of the kernel is particularly suitable for large N analysis, which is needed for comparison with the microscopic limit of the theory of strong interactions with temperature.
In section 4.2 we performed a saddle point analysis and the large N analysis of the kernel for the chGUE(N) with an external source, which is equivalent to the quenched approximation of the effective model for QCD without N_f massive flavors. We found that the correlation kernel of the chGUE(N) with an external source becomes equal to the Bessel kernel in the large N limit. This is in agreement with the predicted hard edge behaviour of the Marchenko Pastur distribution close to the origin. Furthermore, this implies that the correlation kernels at the hard edge for the chGUE(N) with and also without external source belong to the same universality class. We will discuss this in more detail in chapter 5.
In section 4.3 we extended the large N analysis to the chGUE(N) with external source and N_f massive flavors. The determinantal structure at finite N from section 4.1 carries over to the large N limit. We find that the matrix entries in the ratio of determinants are either equal to Bessel kernels or Bessel functions of the first kind, see. Eq. (4.22).
The limiting kernels need to be compared to existing results for both zero and non-zero temperature to assess questions of universality. This will be done in the next chapter.

Chapter 5

Universality

In this chapter we discuss how the large N limit results derived in chapter 4 for the correlation kernel can be compared to existing results for zero temperature models [63, 64] and non-zero temperature models [12, 13]. The large N asymptotic behavior of the correlation kernel of the deformed chGUE(N) was studied in [63, 64]. The model discussed in both references does not include temperature in contrast to the chGUE(N) with external source discussed in chapter 4. The JPDF of the model discussed in [63, 64] is expressed via the partition function

$$Z_N^{(N_f,0)} = \left(\prod_{n=1}^{N} \int_0^\infty d\lambda_n \, w^{(N_f)}(\lambda_n) \right) \Delta_N^2(\lambda_1, \ldots, \lambda_N), \tag{5.1}$$

where the weight function $w^{(N_f)}(\lambda_n)$ is given as

$$w^{(N_f)}(\lambda) = e^{-\lambda} \prod_{f=1}^{N_f} (\lambda + m_f^2) = w^{(0)}(\lambda) \prod_{f=1}^{N_f} (\lambda + m_f^2). \tag{5.2}$$

This is a special case of the JPDF of the deformed chGUE(N) given in chapter 2. Setting $\nu = 0$ in Eq. (2.10) we obtain Eqs. (5.1) and (5.2).
We present a method to compare both results in section 5.1 using the Theorem 2.2 of Akemann and Vernizzi [90] presented in chapter 2, which reads:

Theorem 5.1. *Assume we start with a random matrix model given by a partition function Z_N of Δ_N^2-type, like in Eq.* (5.1)*, with factorizing weight function $w^{(K)}(x)$. Let $\{v_i \mid i = 1, \ldots, M\}$ and $\{u_i \mid i = 1, \ldots, L\}$ be two sets of numbers which are pairwise distinct among each set. Without loss of generality we assume $M \geq L$, where the empty set with $L = 0$ is permitted as well. Taking orthonormal polynomials $P_k(x)$ with respect to the weight $w^{(0)}(x)$ and together with norms h_k, the following statement can be proven:*

$$\mathbb{E}_{\mathcal{P}} \left[\prod_{i=1}^{M} D_N(v_i) \prod_{j=1}^{L} D_N(u_j) \right] = \frac{\prod_{i=N}^{N+M-1} \sqrt{h_i} \prod_{j=N}^{N+L-1} \sqrt{h_j}}{\Delta_M(\{v\}) \Delta_L(\{u\})} \det_{1 \leq l,m \leq M} [\mathcal{B}(v_l, u_m)]$$

with the definition

$$\mathcal{B}(v_l, u_m) \equiv \begin{cases} \sum_{i=0}^{N+L-1} P_i(v_l) P_i(u_m) & \text{for } m = 1, \ldots, L, \\ P_{N+m-1}(v_l) & \text{for } m = L+1, \ldots, M. \end{cases}$$

The comparison with our result for the kernel leads to an equivalence, which displays the universality of the correlation kernel we derived in chapter 4.

In section 5.2 we connect the results of [63, 64], and consequently our results, with existing results for the k-point function derived via supersymmetry. The JPDF of the model in [12] coincides with the chGUE(N) with external source for $\nu = 0$, compare Eq. (2.11)

$$\mathcal{P}^{\text{ext}}_{\text{chGUE}}(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N^{(0,T)}} \left(\prod_{n=1}^{N} e^{-\lambda_n} \right) \det \left[I_0 \left(2\sqrt{a_i \lambda_j} \right) \right]_{i,j=1}^{N} \Delta_N(\lambda_1, \ldots \lambda_N). \tag{5.3}$$

In [13] this model is extended, adding N_f massive flavors. The JPDF is comparable to Eq. (2.12) for $\nu = 0$. Thus, the authors considered

$$\begin{aligned} \mathcal{P}^{\text{temp}}_{\text{chGUE}}(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N^{(N_f,T)}} \left(\prod_{n=1}^{N} e^{-\lambda_n} \right) \left(\prod_{f=1}^{N_f} \prod_{n=1}^{N} (m_f^2 + \lambda_n) \right) \\ \times \det \left[I_0 \left(2\sqrt{a_i \lambda_j} \right) \right]_{i,j=1}^{N} \Delta_N(\lambda_1, \ldots \lambda_N). \end{aligned} \tag{5.4}$$

We show that the representation of the k-point function, in its large N limit, derived in [12] for $N_f = 0$ and in [13] for $N_f \neq 0$ is equivalent to the k-point function obtained via the kernel of [63, 64] for the zero-temperature model. This also links our large N result of the correlation kernel to the results of [12, 13].

5.1 Equivalence with results for zero temperature models

In the references [63, 64] the authors, Damgaard and Nishigaki in [63], as well as Wilke, Guhr and Wettig in [64], looked at the chGUE(N) with N_f massive flavors, without temperature and topology in the large N limit. Thus, our result from chapter 4 for the correlation kernel with N_f flavors - see Eq. (4.22) - has to be compared to their result, after setting $\nu = 0$ and $a_n = 0$ for all $n = 1, \ldots, N$. This comparison would help to answer the question wether there exists a universal expression for the correlation kernel of this symmetry class in the large N limit at the hard edge, which is independent of the influence of external parameters like temperature in our application.
In the large N limit, as well as for finite N, the k-point correlation funcions of the chGUE(N) symmetry class form determinantal point processes. Thus, we can write

$$\begin{aligned} \rho_S^{(N_f)}(\zeta_1, \ldots, \zeta_k) &= \lim_{N \to \infty} \frac{1}{(N\Sigma)^k} \rho_N^{(N_f)} \left(\frac{\zeta_1}{2N\Sigma}, \ldots, \frac{\zeta_k}{2N\Sigma} \right) \\ &= \frac{1}{\Sigma^k} \det_{1 \leq a,b \leq k} \left[K_S^{(N_f)} \left(\frac{\zeta_a}{2\Sigma}, \frac{\zeta_b}{2\Sigma} \right) \right] . \end{aligned} \tag{5.5}$$

Σ denotes the chiral condensate, the order parameter of the chiral phase transition of QCD. It is used as a rescaling parameter to properly describe the physical ingredients of this model of QCD in the low-energy, small temperature, regime.
The large N result for the correlation kernel, stated in [63, 64], has a determinantal structure, as does our result - compare Eq. (4.22). Interestingly, the dimension of the matrices inside the determinants of [63, 64] are different from our results. In [63, 64] the N_f massive flavors and the 2 variables of the kernel appear as entries of a matrix and its determinant in terms of Bessel functions. Crucially, the variables do not combine directly to $N_f = 0$ kernels as entries of the given matrix as was the case in Eq. (4.22) (recall the first row of Eq. (4.22)). This leads to a determinantal structure of size $N_f + 2$ in contrast to our $N_f + 1$. More precisely, the

authors of [63, 64] found the following form of the correlation kernel for $\nu = 0$[1]

$$K_S^{(N_f)}\left(\frac{\zeta_a}{2\Sigma},\frac{\zeta_b}{2\Sigma};\left\{\frac{\mu_f}{2\Sigma}\right\}_{f=1}^{N_f}\right) = \frac{-\Sigma\sqrt{|\zeta_a\zeta_b|}}{\zeta_a^2-\zeta_b^2} \times \frac{\det\begin{pmatrix} J_0(\zeta_a) & \zeta_a J_1(\zeta_a) & \dots & \zeta_a^{N_f+1}J_{N_f+1}(\zeta_a) \\ J_0(\zeta_b) & \zeta_b J_1(\zeta_b) & \dots & \zeta_b^{N_f+1}J_{N_f+1}(\zeta_b) \\ I_0(\mu_1) & -\mu_1 I_1(\mu_1) & \dots & (-\mu_1)^{N_f+1}I_{N_f+1}(\mu_1) \\ \vdots & \vdots & \dots & \vdots \\ I_0(\mu_{N_f}) & -\mu_{N_f} I_1(\mu_{N_f}) & \dots & (-\mu_{N_f})^{N_f+1}I_{N_f+1}(\mu_{N_f}) \end{pmatrix}}{\prod_{f=1}^{N_f}\sqrt{(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}\det\limits_{1\le f,g\le N_f}\left[(-\mu_f)^{g-1}I_{g-1}(\mu_f)\right]}. \tag{5.6}$$

Recall that the RMT ensemble underlying the analysis in [63, 64] is of the orthogonaly polynomial ensemble class, with two Vandermonde determinants in the JPDF. Thus, orthogonal polynomials methods are applicable leading to the correlation kernel via the Christoffel-Darboux formula, which was done in chapter 2. The route via expectation values of characteristic polynomials is also possible and Theorem 5.1 can be applied to obtain many different expressions of the correlation kernel. This will be the main idea in this section to show the equivalence of our result - see Proposition 4.3, Eq. (4.22) - with the result obtained in Eq. (5.6). To make the application of Theorem 5.1 simpler, we use another result for the large N correlation kernel. It was shown in [57, 58] by Akemann and Damgaard that the correlation kernel in its large N limit, as well as the k-point correlation functions can be expressed in terms of finite-volume partition functions

$$K_S^{(N_f)}\left(\zeta_a,\zeta_b;\{\mu_f\}_{f=1}^{N_f}\right) = (-1)^\nu\sqrt{|\zeta_a\zeta_b|}\prod_{f=1}^{N_f}\sqrt{(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)} \times \frac{\mathcal{Z}_\nu^{(N_f+2)}(\mu_1,\dots,\mu_{N_f},i\zeta_a,i\zeta_b)}{\mathcal{Z}_\nu^{(N_f)}(\mu_1,\dots,m_{N_f})} \tag{5.7}$$

with k-point correlation functions

$$\rho_S^{(N_f)}(\zeta_1,\dots,\zeta_k) = \det_{1\le a,b\le k}\left[K_S^{(N_f)}\left(\zeta_a,\zeta_b;\{\mu_f\}_{f=1}^{N_f}\right)\right].$$

Finite-volume partition functions as limits of RMT partition functions

It is known that the finite-volume partition function of N_f massive flavors, given in Eq. (5.7), can be written as (see for example [80])

$$\mathcal{Z}_\nu^{(N_f)}(\mu_1,\dots,\mu_{N_f}) = \frac{\det\limits_{1\le a,b\le N_f}\left[\mu_a^{b-1}I_{\nu+b-1}(\mu_b)\right]}{\Delta_{N_f}(\mu_1^2,\dots,\mu_{N_f}^2)}.$$

[1]The kernel in [63] is given without the rescaling factor of 2Σ in the arguments of the determinants. We can include these factors, as they are canceled in the ratio of determinants in Eq. (5.6), when also including the prefactor Σ^k in the correlation functions.

We will demonstrate how the kernel in Eq. (5.6), can be brought into the same form as our kernel in Eq. (4.22), using the representation of the finite-volume partition function in terms of the large N-limit of RMT partition functions for zero temperature, i.e.

$$\mathcal{Z}_\nu^{(N_f+2k)}(\mu_1,\ldots,\mu_{N_f},\{i\zeta_m\}_{m=1}^{2k}) = \lim_{N\to\infty}\frac{Z_N^{(N_f+2k,0)}(m_1^2,\ldots,m_{N_f}^2,\{z_m^2\}_{m=1}^{2k})}{C_N^{[N_f+2k]}},$$

which we introduced back in chapter 2 - see Eq. (2.29). The constant $C_N^{[N_f+2k]}$ reads

$$\begin{aligned} C_N^{[N_f+2k]} &= 2^{(N_f+2k)(N_f+2k-1)/2}2^{\nu(N_f+2k)}\left(\prod_{j=1}^{N_f+2k}\Gamma(j)\right)\left(\prod_{j=1}^{N}\Gamma(j+\nu)\Gamma(N_f+2k+j)\right) \\ &\quad\times N!N^{\nu(N_f+2k)}\left(\prod_{f=1}^{N_f}\mu_f^{-\nu}\right)(-1)^{\nu k}\left(\prod_{m=1}^{2k}\zeta_m^{-\nu}\right). \end{aligned} \tag{5.8}$$

Because of Eq. (5.7) and (5.8) we only need to compute Eq. (5.8) for $k = 1$ to show the equivalence between the representations Eq. (5.6) and Eq. (4.22).
It is also convenient to recall that, for zero flavors $N_f = 0$, the deformed chGUE(N) considered in [63, 64], becomes the classical chGUE(N) with weight function $w(x) = w^{(0)}(x)$, which leads to orthogonal polynomials proportional to the generalized Laguerre polynomials, recall Eq. (2.18). Thus, we obtain the partition function $Z_N^{(0,0)}$ of the chGUE(N) in terms of norms h_n of these orthogonal polynomials. Those norms are simple products of Γ-functions and we find

$$Z_N^{(0,0)} = N!\prod_{j=1}^{N}h_{j-1} = N!\prod_{j=1}^{N}\Gamma(j+\nu)\Gamma(j)\,. \tag{5.9}$$

By Eqs. (5.7), (5.8) and (5.8) it remains to show that Eq. (5.8) leads to two different determinantal representations for the finite-volume partition function by applying Theorem 5.1. Taking Eq. (5.8) for $k = 1$ leads to

$$\mathcal{Z}_\nu^{(N_f+2)}(\mu_1,\ldots,\mu_{N_f},i\zeta_a,i\zeta_b) = \lim_{N\to\infty}\frac{Z_N^{(N_f+2,0)}(m_1^2,\ldots,m_{N_f}^2,z_a^2,z_b^2)}{C_N^{[N_f+2]}} \tag{5.10}$$

with

$$m_f^2 = \frac{\mu_f^2}{4N},\ \text{for}\ f=1,\ldots,N_f,\quad z_a^2 = -\frac{\zeta_a^2}{4N},\quad\text{and}\quad z_b^2 = -\frac{\zeta_b^2}{4N}\,. \tag{5.11}$$

The constant $C_N^{[N_f+2]}$ reads

$$\begin{aligned} C_N^{[N_f+2]} &= 2^{(N_f+2)(N_f+1)/2}2^{\nu(N_f+2)}\left(\prod_{j=1}^{N_f+2}\Gamma(j)\right)\left(\prod_{j=1}^{N}\Gamma(j+\nu)\Gamma(N_f+2+j)\right) \\ &\quad\times N!N^{\nu(N_f+2)}\left(\prod_{f=1}^{N_f}\mu_f^{-\nu}\right)(-1)^{\nu}\zeta_a^{-\nu}\zeta_b^{-\nu}\,. \end{aligned} \tag{5.12}$$

Writing partition functions as expectation values

The expression of the expectation value can be expressed in terms of other expectation values, which we already discussed in chapter 2. In particular, we can apply the following fact: The partition function $Z_N^{(M,0)}$ of the deformed chGUE(N) can for any number of flavors M be expressed as an expectation value of M characteristic polynomials, with respect to the classical chGUE(N) weight function - recall Eq. (2.26):

$$Z_N^{(M,0)} = (-1)^{NM} Z_N^{(0,0)} \mathbb{E}\left[\prod_{f=1}^{M} D_N(-m_f^2)\right]. \tag{5.13}$$

This helps us to express the finite N partition function in Eq. (5.10) as an expectation value of $N_f + 2$ characteristic polynomials with respect to the classical chGUE(N), namely[2]

$$Z_N^{(N_f+2,0)}(m_1^2,\ldots,m_{N_f}^2,z_a^2,z_b^2) = \frac{Z_N^{(0,0)}}{(-1)^{N(N_f+2)}} \mathbb{E}\left[\prod_{f=1}^{N_f} D_N(-m_f^2) D_N(-z_a^2) D_N(-z_b^2)\right]. \tag{5.14}$$

The expectation value can be expressed via Proposition 5.1 in many different ways, depending on how we divide the characteristic polynomials into sets of M and L characteristic polynomials in the expression above. We need two particular cases of Theorem 5.1 as representations for the expectation value. The first is obtained by taking $M = N_f + 2$ and $L = 0$ and the second by taking $M = N_f + 1$ and $L = 1$. The first case gives the kernel representation in Eq. (5.6) and the second case leads to our result in Proposition 4.3, Eq. (4.22).

Kernel representation equivalent to [63, 64]

In the case $M = N_f + 1$ and $L = 0$ the expectation value in Eq. (5.14) becomes

$$\mathbb{E}\left[\prod_{f=1}^{N_f} D_N(-m_f^2) D_N(-z_a^2) D_N(-z_b^2)\right] = \left(\prod_{i=N}^{N+N_f+1} \sqrt{h_i}\right) \frac{\det\limits_{1\le k,j\le N_f+2}\left[\mathcal{C}_{k,j}\right]}{\Delta_{N_f+2}(\{-m^2\},-z_a^2,-z_b^2)}$$

via Theorem 5.1, where

$$\mathcal{C}_{k,j} = \frac{(-1)^{N+j-1}\,(N+j-1)!}{\sqrt{h_{N+j-1}}} \begin{cases} L_{N+j-1}^{\nu}(-m_k^2), & \text{for } k = 1,\ldots,N_f,\ j = 1,\ldots,N_f+2, \\ L_{N+j-1}^{\nu}(-z_a^2), & \text{for } k = N_f+1,\ j = 1,\ldots,N_f+2, \\ L_{N+j-1}^{\nu}(-z_b^2), & \text{for } k = N_f+2,\ j = 1,\ldots,N_f+2. \end{cases} \tag{5.15}$$

We can extract the prefactors and obtain a determinant of Laguerre polynomials. For these polynomials we know [130, Eq. 8.971.5]

$$L_N^{\nu-1}(x) = L_N^{\nu}(x) - L_{N-1}^{\nu}(x), \tag{5.16}$$

[2] Note that we omit the index at the expectation value, which indicates that the underlying JPDF is the simplest model we have introduced, namely the classical chGUE(N) without flavors, so $N_f = 0$.

which we can use to transform the determinant as

$$\det\begin{bmatrix} L_N^\nu(-m_1^2) & L_N^\nu(-m_2^2) & \dots & L_N^\nu(-m_{N_f}^2) & L_N^\nu(-z_a^2) & L_N^\nu(-z_b^2) \\ L_{N+1}^\nu(-m_1^2) & L_{N+1}^\nu(-m_2^2) & \dots & L_{N+1}^\nu(-m_{N_f}^2) & L_{N+1}^\nu(-z_a^2) & L_{N+1}^\nu(-z_b^2) \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ L_{N+N_f+1}^\nu(-m_1^2) & L_{N+N_f+1}^\nu(-m_2^2) & \dots & L_{N+N_f+1}^\nu(-m_{N_f}^2) & L_{N+N_f+1}^\nu(-z_a^2) & L_{N+N_f+1}^\nu(-z_b^2) \end{bmatrix}$$
$$= \det\begin{bmatrix} L_N^\nu(-m_1^2) & \dots & L_N^\nu(-m_{N_f}^2) & L_N^\nu(-z_a^2) & L_N^\nu(-z_b^2) \\ L_{N+1}^{\nu-1}(-m_1^2) & \dots & L_{N+1}^{\nu-1}(-m_{N_f}^2) & L_{N+1}^{\nu-1}(-z_a^2) & L_{N+1}^{\nu-1}(-z_b^2) \\ \vdots & \dots & \vdots & \vdots & \vdots \\ L_{N+N_f+1}^{\nu-N_f-1}(-m_1^2) & \dots & L_{N+N_f+1}^{\nu-N_f-1}(-m_{N_f}^2) & L_{N+N_f+1}^{\nu-N_f-1}(-z_a^2) & L_{N+N_f+1}^{\nu-N_f-1}(-z_b^2) \end{bmatrix} \equiv \det\mathcal{F}\,.$$

The expectation value is then equal to

$$\underbrace{\frac{\prod_{i=0}^{N_f+1}\sqrt{h_{N+i}}}{\prod_{j=1}^{N_f+2}\sqrt{h_{N+j-1}}}}_{=1}\frac{(-1)^{N(N_f+2)+1}}{(-1)^{N_f(N_f-1)/2}}\frac{(4N)^{(N_f+2)(N_f+1)/2}\prod_{j=1}^{N_f+2}(-1)^{j-1}\Gamma(N+j)\det\mathcal{F}}{\Delta_{N_f}(\{\mu^2\})(\zeta_a^2-\zeta_b^2)\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}$$
$$= (-1)^{N(N_f+2)}\prod_{j=N+1}^{N+N_f+2}\Gamma(j)\frac{2^{(N_f+2)(N_f+1)}N^{N_f(N_f-1)/2}N^{2N_f+1}\det\mathcal{F}}{\Delta_{N_f}(\{\mu^2\})(\zeta_a^2-\zeta_b^2)\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}\,.$$

To compute the whole partition function in Eq (5.14) we have to include the prefactor, namely $Z_N^{(0,0)}(-1)^{N(N_f+2)}$. The partition function $Z_N^{(0,0)}$ can be computed via norms, i.e.

$$Z_N^{(0)} = N!\prod_{j=1}^{N}h_{j-1} = N!\prod_{j=1}^{N}\Gamma(j+\nu)\Gamma(j)\,.$$

We obtain for the partition function $Z_N^{(N_f+2,0)}$ the expression

$$Z_N^{(0,0)}\prod_{j=N+1}^{N+N_f+2}\Gamma(j)\frac{2^{(N_f+2)(N_f+1)}N^{N_f(N_f-1)/2}N^{2N_f+1}\det\mathcal{F}}{\Delta_{N_f}(\{\mu^2\})(\zeta_a^2-\zeta_b^2)\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}$$
$$= N!\prod_{j=1}^{N}\Gamma(j+\nu)\prod_{j=1}^{N+N_f+2}\Gamma(j)\frac{2^{(N_f+2)(N_f+1)}N^{N_f(N_f-1)/2}N^{2N_f+1}\det\mathcal{F}}{\Delta_{N_f}(\{\mu^2\})(\zeta_a^2-\zeta_b^2)\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}\,.$$

The constant $C_N^{[N_f+2]}$, as defined in Eq. (5.12), leads us to

$$Z_N^{(N_f+2)}(m_1^2,\dots,m_{N_f}^2,z_a^2,z_b^2) = \frac{C_N^{[N_f+2]}}{2^{\nu(N_f+2)}}\frac{\prod_{j=1}^{N_f+2}N^{-\nu+j-1}\det\mathcal{F}}{\Delta_{N_f}(\{\mu^2\})(\zeta_a^2-\zeta_b^2)\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}$$
$$\times\frac{(-1)^\nu\zeta_a^\nu\zeta_b^\nu}{2^{-(N_f+2)(N_f+1)/2}}\left(\prod_{f=1}^{N_f}\mu_f^\nu\right).$$

Hence, we have

$$\mathcal{Z}_\nu^{(N_f+2)}(\mu_1,\ldots,\mu_{N_f},i\zeta_a,i\zeta_b) = \frac{(-1)^\nu \zeta_a^\nu \zeta_b^\nu}{2^{\nu(N_f+2)}} \frac{2^{(N_f+2)(N_f+1)/2}\left(\prod_{f=1}^{N_f}\mu_f^\nu\right)}{\Delta_{N_f}(\{\mu^2\})(\zeta_a^2-\zeta_b^2)\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)} \times \lim_{N\to\infty}\prod_{j=1}^{N_f+2} N^{-\nu+j-1}\det\mathcal{F}\,.$$

Thus, the large N limit is reduced to the large N limit of the determinant of the matrix $\mathcal{F}$. The next step is to apply the large N asymptotics of Laguerre polynomials, namely [130, Eq. 8.978.2]

$$\lim_{N\to\infty} N^{-p} L_{N+j}^p\left(\frac{z}{N}\right) = z^{-p/2} J_p(2\sqrt{z}), \tag{5.17}$$

which is true for any fixed parameter j. Via Eq. (5.11) the determinant of $\mathcal{F}$ becomes in the large N limit:

$$\lim_{N\to\infty}\prod_{j=1}^{N_f+2} N^{-\nu+j-1}\det\mathcal{F} = \left(\prod_{f=1}^{N_f}\mu_f^{-\nu}\right)\zeta_a^{-\nu}\zeta_b^{-\nu}2^{\nu(N_f+2)}2^{-(N_f+2)(N_f+1)/2}$$
$$\det\begin{bmatrix} I_\nu(\mu_1) & \ldots & I_\nu(\mu_{N_f}) & J_\nu(\zeta_a) & J_\nu(\zeta_b) \\ \mu_1 I_{\nu-1}(\mu_1) & \ldots & \mu_{N_f} I_{\nu-1}(\mu_{N_f}) & \zeta_a J_{\nu-1}(\zeta_a) & \zeta_b J_{\nu-1}(\zeta_b) \\ \vdots & \ldots & \vdots & \vdots & \vdots \\ \mu_1^{N_f+1} I_{\nu-N_f-1}(\mu_1) & \ldots & \mu_{N_f}^{N_f+1} I_{\nu-N_f-1}(\mu_{N_f}) & \zeta_a^{N_f+1} J_{\nu-N_f-1}(\zeta_a) & \zeta_b^{N_f+1} J_{\nu-N_f-1}(\zeta_b) \end{bmatrix}.$$

This means we obtain overall

$$\mathcal{Z}_\nu^{(N_f+2)}(\mu_1,\ldots,\mu_{N_f},i\zeta_a,i\zeta_b) = \frac{(-1)^\nu}{\Delta_{N_f}(\{\mu^2\})(\zeta_a^2-\zeta_b^2)\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}$$
$$\det\begin{bmatrix} I_\nu(\mu_1) & \ldots & I_\nu(\mu_{N_f}) & J_\nu(\zeta_a) & J_\nu(\zeta_b) \\ \mu_1 I_{\nu-1}(\mu_1) & \ldots & \mu_{N_f} I_{\nu-1}(\mu_{N_f}) & \zeta_a J_{\nu-1}(\zeta_a) & \zeta_b J_{\nu-1}(\zeta_b) \\ \vdots & \ldots & \vdots & \vdots & \vdots \\ \mu_1^{N_f+1} I_{\nu-N_f-1}(\mu_1) & \ldots & \mu_{N_f}^{N_f+1} I_{\nu-N_f-1}(\mu_{N_f}) & \zeta_a^{N_f+1} J_{\nu-N_f-1}(\zeta_a) & \zeta_b^{N_f+1} J_{\nu-N_f-1}(\zeta_b) \end{bmatrix}.$$

The determinant in the second line in the above equation can be transformed by applying rules for Bessel functions (see [130, Eq. 8.486.1] and [130, Eq. 8.471.1])

$$zI_{\nu-1}(z) = 2\nu I_\nu(z) + zI_{\nu+1}(z), \quad \text{and} \quad zJ_{\nu-1}(z) = 2\nu J_\nu(z) - zJ_{\nu+1}(z). \tag{5.18}$$

By iteration of row subtractions in the determinant and using the equation above, the indices of the Bessel I- and J-functions can be raised from $\nu - m$ to $\nu + m$ for $m = 0,\ldots,N_f+1$. Additionally, we obtain powers of -1 for the Bessel J-function entries. These powers can be

extracted from the determinant and we obtain

$$\mathcal{Z}_\nu^{(N_f+2)}(\mu_1,\ldots,\mu_{N_f},i\zeta_a,i\zeta_b) = \frac{(-1)^\nu}{\Delta_{N_f}(\{\mu^2\})(\zeta_a^2-\zeta_b^2)\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}$$
$$\det\begin{bmatrix} I_\nu(\mu_1) & \ldots & I_\nu(\mu_{N_f}) & J_\nu(\zeta_a) & J_\nu(\zeta_b) \\ \mu_1 I_{\nu+1}(\mu_1) & \ldots & \mu_{N_f} I_{\nu+1}(\mu_{N_f}) & -\zeta_a J_{\nu+1}(\zeta_a) & -\zeta_b J_{\nu+1}(\zeta_b) \\ \vdots & \ldots & \vdots & \vdots & \vdots \\ \mu_1^{N_f+1} I_{\nu+N_f+1}(\mu_1) & \ldots & \mu_{N_f}^{N_f+1} I_{\nu+N_f+1}(\mu_{N_f}) & (-\zeta_a)^{N_f+1} J_{\nu+N_f+1}(\zeta_a) & (-\zeta_b)^{N_f+1} J_{\nu+N_f+1}(\zeta_b) \end{bmatrix}$$
$$= \frac{(-1)^\nu(-1)^{(N_f+2)(N_f+1)/2}}{\Delta_{N_f}(\{\mu^2\})(\zeta_a^2-\zeta_b^2)\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}$$
$$\det\begin{bmatrix} I_\nu(\mu_1) & \ldots & I_\nu(\mu_{N_f}) & J_\nu(\zeta_a) & J_\nu(\zeta_b) \\ -\mu_1 I_{\nu+1}(\mu_1) & \ldots & -\mu_{N_f} I_{\nu+1}(\mu_{N_f}) & \zeta_a J_{\nu+1}(\zeta_a) & \zeta_b J_{\nu+1}(\zeta_b) \\ \vdots & \ldots & \vdots & \vdots & \vdots \\ (-\mu_1)^{N_f+1} I_{\nu+N_f+1}(\mu_1) & \ldots & (-\mu_{N_f})^{N_f+1} I_{\nu+N_f+1}(\mu_{N_f}) & \zeta_a^{N_f+1} J_{\nu+N_f+1}(\zeta_a) & \zeta_b^{N_f+1} J_{\nu+N_f+1}(\zeta_b) \end{bmatrix}.$$

Putting this representation of the finite-volume partition function into the kernel formula Eq. (5.7) we obtain with Eq. (5.8)

$$K_S^{(N_f)}\left(\zeta_a,\zeta_b;\{\mu_f\}_{f=1}^{N_f}\right) = (-1)^\nu\sqrt{|\zeta_a\zeta_b|}\prod_{f=1}^{N_f}\sqrt{(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}\frac{\mathcal{Z}_\nu^{(N_f+2)}(\mu_1,\ldots,\mu_{N_f},i\zeta_a,i\zeta_b)}{\mathcal{Z}_\nu^{(N_f)}(\mu_1,\ldots,m_{N_f})}$$
$$= \sqrt{|\zeta_a\zeta_b|}\frac{\Delta_{N_f}(\mu_1^2,\ldots,\mu_{N_f}^2)}{\det\limits_{1\le a,b\le N_f}\left[\mathcal{B}_{a,b}(\{\mu_f\}_{f=1}^{N_f})\right]}\frac{(-1)(-1)^{N_f(N_f-1)/2}}{\Delta_{N_f}(\{\mu^2\})(\zeta_a^2-\zeta_b^2)\prod_{f=1}^{N_f}\sqrt{(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}}$$
$$\det\begin{bmatrix} I_\nu(\mu_1) & \ldots & I_\nu(\mu_{N_f}) & J_\nu(\zeta_a) & J_\nu(\zeta_b) \\ -\mu_1 I_{\nu+1}(\mu_1) & \ldots & -\mu_{N_f} I_{\nu+1}(\mu_{N_f}) & \zeta_a J_{\nu+1}(\zeta_a) & \zeta_b J_{\nu+1}(\zeta_b) \\ \vdots & \ldots & \vdots & \vdots & \vdots \\ (-\mu_1)^{N_f+1} I_{\nu+N_f+1}(\mu_1) & \ldots & (-\mu_{N_f})^{N_f+1} I_{\nu+N_f+1}(\mu_{N_f}) & \zeta_a^{N_f+1} J_{\nu+N_f+1}(\zeta_a) & \zeta_b^{N_f+1} J_{\nu+N_f+1}(\zeta_b) \end{bmatrix}$$
$$= \frac{-\sqrt{|\zeta_a\zeta_b|}}{(\zeta_a^2-\zeta_b^2)\prod_{f=1}^{N_f}\sqrt{(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}}\left[\det_{1\le f,g\le N_f}\left((-\mu_f)^{g-1}I_{\nu+g-1}(\mu_f)\right)\right]^{-1}$$
$$\det\begin{bmatrix} I_\nu(\mu_1) & \ldots & I_\nu(\mu_{N_f}) & J_\nu(\zeta_a) & J_\nu(\zeta_b) \\ -\mu_1 I_{\nu+1}(\mu_1) & \ldots & -\mu_{N_f} I_{\nu+1}(\mu_{N_f}) & \zeta_a J_{\nu+1}(\zeta_a) & \zeta_b J_{\nu+1}(\zeta_b) \\ \vdots & \ldots & \vdots & \vdots & \vdots \\ (-\mu_1)^{N_f+1} I_{\nu+N_f+1}(\mu_1) & \ldots & (-\mu_{N_f})^{N_f+1} I_{\nu+N_f+1}(\mu_{N_f}) & \zeta_a^{N_f+1} J_{\nu+N_f+1}(\zeta_a) & \zeta_b^{N_f+1} J_{\nu+N_f+1}(\zeta_b) \end{bmatrix}.$$

This result is equal to the kernel obtained in [63, 64], if we set $\nu = 0$. The next step is to show that for the second case, $M = N_f + 1$ and $L = 1$, we find the expression of our correlation kernel from Eq. (4.22).

Kernel representation equivalent to Proposition 4.3

Coming back to Eq. (5.14)

$$Z_N^{(N_f+2,0)}(m_1^2,\dots,m_{N_f}^2,z_a^2,z_b^2) = Z_N^{(0,0)}(-1)^{N(N_f+2)}\mathbb{E}\left[\prod_{f=1}^{N_f} D_N(-m_f^2)D_N(-z_a^2)D_N(-z_b^2)\right],$$

we apply Theorem 5.1 for the second case we mentioned, namely taking $M = N_f + 1$ and $L = 1$. This leads to

$$\mathbb{E}\left[\prod_{f=1}^{N_f} D_N(-m_f^2)D_N(-z_a^2)D_N(-z_b^2)\right] = \left(\prod_{i=N}^{N+N_f}\sqrt{h_i}\right)\sqrt{h_N}\frac{\det\limits_{1\le k,j\le N_f+1}\left[\mathcal{C}_{k,j}\right]}{\Delta_{N_f+1}(\{-m^2\},-z_a^2)}, \tag{5.19}$$

where

$$\mathcal{C}_{k,j} = \begin{cases} \hat{L}(-m_j^2,-z_b^2), & \text{for } k=1,\ j=1,\dots,N_f,\\ \hat{L}(-z_a^2,-z_b^2), & \text{for } k=1,\ j=N_f+1,\\ \frac{(-1)^{N+k-1}\,(N+k-1)!}{\sqrt{h_{N+k-1}}}L_{N+k-1}^{\nu}(-m_j^2), & \text{for } k=2,\dots,N_f+1,\ j=1,\dots,N_f,\\ \frac{(-1)^{N+k-1}\,(N+k-1)!}{\sqrt{h_{N+k-1}}}L_{N+k-1}^{\nu}(-z_a^2), & \text{for } k=2,\dots,N_f+1,\ j=N_f+1. \end{cases} \tag{5.20}$$

The first two cases contain a shorthand notation for the kernel with respect to the orthogonal polynomials $P(x)$ given via the Christoffel-Darboux formula as

$$\begin{aligned}\hat{L}(x,y) &= \sum_{i=0}^{N} P_i(x)P_i(y) = \sqrt{\frac{h_{N+1}}{h_N}}\frac{P_{N+1}(x)P_N(y)-P_N(x)P_{N+1}(y)}{x-y}\\ &= \sqrt{\frac{h_{N+1}}{h_N}}\frac{(-1)^{N+1}(N+1)!}{\sqrt{h_{N+1}}}\frac{(-1)^N N!}{\sqrt{h_N}}\frac{L_{N+1}^{\nu}(x)L_N^{\nu}(y)-L_N^{\nu}(x)L_{N+1}^{\nu}(y)}{x-y}\\ &= \frac{(-1)(N+1)!}{\Gamma(N+\nu+1)}\frac{L_{N+1}^{\nu}(x)L_N^{\nu}(y)-L_N^{\nu}(x)L_{N+1}^{\nu}(y)}{x-y}.\end{aligned} \tag{5.21}$$

Applying the rescaled variables allows us to write

$$\begin{aligned}&\hat{L}\left(-\frac{\mu_j^2}{4N},\frac{\zeta_b^2}{4N}\right) = \frac{4N\,(N+1)!}{\Gamma(N+\nu+1)}\frac{L_{N+1}^{\nu}\left(-\frac{\mu_j^2}{4N}\right)L_N^{\nu}\left(\frac{\zeta_b^2}{4N}\right)-L_N^{\nu}\left(-\frac{\mu_j^2}{4N}\right)L_{N+1}^{\nu}\left(\frac{\zeta_b^2}{4N}\right)}{\mu_j^2+\zeta_b^2}\\ &= \frac{4N^{2\nu}\,\Gamma(N+2)}{\Gamma(N+\nu+1)}\frac{N^{-\nu+1}L_{N+1}^{\nu-1}\left(-\frac{\mu_j^2}{4N}\right)N^{-\nu}L_{N+1}^{\nu}\left(\frac{\zeta_b^2}{4N}\right)-N^{-\nu}L_{N+1}^{\nu}\left(-\frac{\mu_j^2}{4N}\right)N^{-\nu+1}L_{N+1}^{\nu-1}\left(\frac{\zeta_b^2}{4N}\right)}{\mu_j^2+\zeta_b^2}\\ &\equiv \frac{\Gamma(N)N^{\nu+1}}{\Gamma(N+\nu+1)}\frac{\Gamma(N+2)}{\Gamma(N)N^2}\,N^{\nu+1}\,2^{2\nu+1}\mu_j^{-\nu}\zeta_b^{-\nu}\hat{\mathcal{L}}_N(\mu_j,\zeta_b).\end{aligned} \tag{5.22}$$

Using the known asymptotic relation for Γ-functions [130, Eq. 8.328.2]

$$\lim_{N\to\infty}\frac{\Gamma(N+\alpha)}{\Gamma(N)N^{\alpha}} = 1, \tag{5.23}$$

we can neglect the first two terms in the last line of Eq. (5.22). Considering the large N limit by applying Eq. (5.17), we only have to consider $\hat{\mathcal{L}}_N(\mu_j, \zeta_b)$. We find

$$\begin{aligned}\lim_{N\to\infty} \hat{\mathcal{L}}_N(\mu_j,\zeta_b) &= 2^{-2\nu+1}\frac{\mu_j^\nu \zeta_b^\nu}{\mu_j^2+\zeta_b^2}\Big(\mu_j^{-\nu}2^{\nu-1}\mu_j I_{\nu-1}(\mu_j)\zeta_b^{-\nu}2^\nu J_\nu(\zeta_b) \\ &\quad - \mu_j^{-\nu}2^\nu I_\nu(\mu_j)\zeta_b^{-\nu}2^{\nu-1}\zeta_b J_{\nu-1}(\zeta_b)\Big) \\ &= \frac{1}{\mu_j^2+\zeta_b^2}\Big[\mu_j I_{\nu+1}(\mu_j)J_\nu(\zeta_b) + I_\nu(\mu_j)\zeta_b J_{\nu+1}(\zeta_b)\Big] = \mathcal{B}_{\mathrm{IJ}}(\mu_j,\zeta_b).\end{aligned} \tag{5.24}$$

Similarly, we have

$$\begin{aligned}&\hat{L}\left(\frac{\zeta_a^2}{4N},\frac{\zeta_b^2}{4N}\right) = \frac{(-1)\,4N\,(N+1)!}{\Gamma(N+\nu+1)}\frac{L^\nu_{N+1}\left(\frac{\zeta_a^2}{4N}\right)L^\nu_N\left(\frac{\zeta_b^2}{4N}\right) - L^\nu_N\left(\frac{\zeta_a^2}{4N}\right)L^\nu_{N+1}\left(\frac{\zeta_b^2}{4N}\right)}{\zeta_a^2-\zeta_b^2} \\ &= \frac{(-1)\,4N^{2\nu}\,\Gamma(N+2)}{\Gamma(N+\nu+1)}\frac{N^{-\nu+1}L^{\nu-1}_{N+1}\left(\frac{\zeta_a^2}{4N}\right)N^{-\nu}L^\nu_{N+1}\left(\frac{\zeta_b^2}{4N}\right) - N^{-\nu}L^\nu_{N+1}\left(\frac{\zeta_a^2}{4N}\right)N^{-\nu+1}L^{\nu-1}_{N+1}\left(\frac{\zeta_b^2}{4N}\right)}{\zeta_a^2-\zeta_b^2} \\ &\equiv \frac{\Gamma(N)N^{\nu+1}}{\Gamma(N+\nu+1)}\frac{\Gamma(N+2)}{\Gamma(N)N^2}\,N^{\nu+1}\,2^{2\nu+1}\zeta_a^{-\nu}\zeta_b^{-\nu}\hat{\mathcal{L}}_N(\zeta_a,\zeta_b)\,,\end{aligned} \tag{5.25}$$

which leads to

$$\begin{aligned}\lim_{N\to\infty} \hat{\mathcal{L}}_N(\zeta_a,\zeta_b) &= 2^{-2\nu+1}\frac{(-1)\zeta_a^\nu\zeta_b^\nu}{\zeta_a^2-\zeta_b^2}\Big(\zeta_a^{-\nu}2^{\nu-1}\zeta_a J_{\nu-1}(\zeta_a)\zeta_b^{-\nu}2^\nu J_\nu(\zeta_b) \\ &\quad - \zeta_a^{-\nu}2^\nu J_\nu(\zeta_a)\zeta_b^{-\nu}2^{\nu-1}\zeta_b J_{\nu-1}(\zeta_b)\Big) \\ &= \frac{1}{\zeta_a^2-\zeta_b^2}\Big[\zeta_a J_{\nu+1}(\zeta_a)J_\nu(\zeta_b) - J_\nu(\zeta_a)\zeta_b J_{\nu+1}(\zeta_b)\Big] = \mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_b)\,.\end{aligned} \tag{5.26}$$

Note that in the case of equal arguments $\zeta_a = \zeta_b$ we have to apply l'Hospital's rule to $\mathcal{B}_{\mathrm{JJ}}$ and obtain[3]

$$\begin{aligned}\mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_a) &= \frac{1}{2\zeta_a}\left[\left(\frac{\partial}{\partial\zeta_a}\Big[\zeta_a J_{\nu+1}(\zeta_a)\Big]\right)J_\nu(\zeta_a) - \left(\frac{\partial}{\partial\zeta_a}J_\nu(\zeta_a)\right)\zeta_a J_{\nu+1}(\zeta_a)\right] \\ &= \frac{1}{2\zeta_a}\left[\zeta_a J_\nu^2(\zeta_a) - \nu J_{\nu+1}(\zeta_a)J_\nu(\zeta_a) - \left(\frac{\partial}{\partial\zeta_a}J_\nu(\zeta_a)\right)\zeta_a J_{\nu+1}(\zeta_a)\right] \\ &= \frac{1}{2}\left[J_\nu^2(\zeta_a) - \frac{\nu}{\zeta_a}J_{\nu+1}(\zeta_a)J_\nu(\zeta_a) - \frac{1}{2}J_{\nu+1}(\zeta_a)\left(J_{\nu-1}(\zeta_a) - J_{\nu+1}(\zeta_a)\right)\right] \\ &= \frac{1}{2}\left(J_\nu^2(\zeta_a) - \frac{2\nu}{\zeta_a}J_\nu(\zeta_a)J_{\nu+1}(\zeta_a) + J_{\nu+1}^2(\zeta_a)\right)\,,\end{aligned} \tag{5.27}$$

where we have used [130, Eq. 8.471.2] and [130, Eq. 8.472.1]. Looking back at Eqs. (5.19), (5.20) we collect an overall factor of $N^{\nu+1}2^{2\nu+1}\zeta_b^{-\nu}$ from the first row of the matrix $\mathcal{C}_{k,j}$. The rest of the calculation is similar to computations we have done previously, so we can be

[3]The case of $\mathcal{B}_{\mathrm{IJ}}$ for equal arguments is similar, with $\mu_j = i\zeta_b$.

brief. The rows number 2 to N_f+1 of the matrix $\mathcal{C}_{k,j}$ contain Laguerre polynomials, which become Bessel functions in the large N limit, as we have demonstrated previously, see Eqs. (5.15) and (5.17). Hence, we can write

$$\begin{aligned}
\mathbb{E}\left[\prod_{f=1}^{N_f} D_N(-m_f^2)D_N(-z_a^2)D_N(-z_b^2)\right] &= \left(\prod_{i=N}^{N+N_f}\sqrt{h_i}\right)\sqrt{h_N}\frac{\det\limits_{1\le k,j\le N_f+1}\left[\mathcal{C}_{k,j}\right]}{\Delta_{N_f+1}(\{-m^2\},-z_a^2)} \\
&= \left(\prod_{i=0}^{N_f}\sqrt{h_{i+N}}\right)\sqrt{h_N}\frac{\det\limits_{1\le k,j\le N_f+1}\left[\mathcal{C}_{k,j}\right]}{\Delta_{N_f}(\{\mu^2\})\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)}(4N)^{N_f(N_f+1)/2}(-1)^{N_f(N_f-1)/2} \\
&= \left(\prod_{i=1}^{N_f}\sqrt{h_{i+N}}\right)h_N\frac{(4N)^{N_f(N_f+1)/2}(-1)^{N_f(N_f-1)/2}}{\Delta_{N_f}(\{\mu^2\})\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)}\left(\prod_{j=1}^{N_f}\frac{(-1)^{N+j}(N+j)!}{\sqrt{h_{N+j}}}\right) \\
&\times N^{\nu+1}2^{2\nu+1}\zeta_b^{-\nu}\left(\prod_{k=1}^{N_f}N^{\nu-k+1}\right)\det_{1\le k,j\le N_f+1}\left[\hat{\mathcal{C}}_{k,j}\right],
\end{aligned} \tag{5.28}$$

where we have

$$\hat{\mathcal{C}}_{k,j}=\begin{cases}\mu_j^{-\nu}\hat{\mathcal{L}}_N(\mu_j,\zeta_b), & \text{for } k=1\,,\ j=1,\dots,N_f,\\ \zeta_a^{-\nu}\hat{\mathcal{L}}_N(\zeta_a,\zeta_b), & \text{for } k=1\,,\ j=N_f+1,\\ N^{-\nu+k-2}L_{N+k-1}^{\nu-k+2}(-m_j^2), & \text{for } k=2,\dots,N_f+1,\ j=1,\dots,N_f,\\ N^{-\nu+k-2}L_{N+k-1}^{\nu-k+2}(-z_a^2), & \text{for } k=2,\dots,N_f+1,\ j=N_f+1.\end{cases}$$

Note that we have transformed the rows containing Laguerre polynomials according to Eq. (5.16), which gives us the correct dependence of the upper index on the corresponding row k. Furthermore, we have included the necessary powers of N in order to perform the large N asymptotics of Laguerre polynomials via Eq. (5.17). This leads to

$$\begin{aligned}
\mathbb{E}\left[\prod_{f=1}^{N_f} D_N(-m_f^2)D_N(-z_a^2)D_N(-z_b^2)\right] &= 2^{N_f(N_f+1)}2^{2\nu+1}\zeta_b^{-\nu}\frac{\det\limits_{1\le k,j\le N_f+1}\left[\hat{\mathcal{C}}_{k,j}\right]}{\Delta_{N_f}(\{\mu^2\})\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)} \\
&\times h_N N^{\nu+1}N^{(\nu+1)N_f}N^{-N_f(N_f+1)/2}N^{N_f(N_f+1)/2}(-1)^{N_f(N_f-1)/2}\left(\prod_{j=1}^{N_f}(-1)^{N+j}(N+j)!\right) \\
&= N^{(\nu+1)(N_f+1)}(-1)^{(N+1)N_f}h_N\left(\prod_{j=1}^{N_f}\Gamma(N+j+1)\right)\frac{2^{N_f(N_f+1)}2^{2\nu+1}\zeta_b^{-\nu}\det\limits_{1\le k,j\le N_f+1}\left[\hat{\mathcal{C}}_{k,j}\right]}{\Delta_{N_f}(\{\mu^2\})\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)}\,.
\end{aligned}$$

At this point we include the additional factors of $Z_N^{(0,0)}$ and $(-1)^{NN_f}$ needed for the computation of $Z_N^{(N_f+2,0)}$. This is analogous to Eq. (5.14) and we can organize all prefactors such that the constant $C_N^{[N_f+2]}$ (see Eq. (5.12)) appears, which is then canceled. As $N\to\infty$ we obtain the final form of the correlation kernel.

We have

$$
\begin{aligned}
Z_N^{(N_f+2,0)}(m_1^2,\dots,m_{N_f}^2,z_a^2,z_b^2) &= Z_N^{(0,0)}(-1)^{N(N_f+2)}\mathbb{E}\left[\prod_{f=1}^{N_f} D_N(-m_f^2)D_N(-z_a^2)D_N(-z_b^2)\right] \\
&= (-1)^{N_f}N!\left(\prod_{j=1}^{N}\Gamma(j+\nu)\right)2^{2\nu}\zeta_b^{-\nu}\frac{\Gamma(N+\nu+1)}{\Gamma(N)N^{\nu+1}}\,\Gamma(N)\,N^{\nu+1}\,2^{N_f(N_f+1)+1} \\
&\times N^{(\nu+1)(N_f+1)}\left(\prod_{j=1}^{N}\Gamma(j)\right)\Gamma(N+1)\left(\prod_{j=1}^{N_f}\Gamma(N+j+1)\right)\frac{\det\limits_{1\le k,j\le N_f+1}\left[\hat{\mathcal{C}}_{k,j}\right]}{\Delta_{N_f}(\{\mu^2\})\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)} \\
&= C_N^{[N_f+2]}(-1)^{\nu}\zeta_a^{\nu}\left(\prod_{f=1}^{N_f}\mu_f^{\nu}\right)2^{-(N_f+2)(N_f+1)/2}2^{-\nu N_f}(-1)^{N_f}2^{N_f(N_f+1)+1} \\
&\times\frac{\Gamma(N)N^{N_f+2}}{\Gamma(N+N_f+2)}\frac{\Gamma(N+\nu+1)}{\Gamma(N)N^{\nu+1}}\frac{\det\limits_{1\le k,j\le N_f+1}\left[\hat{\mathcal{C}}_{k,j}\right]}{\Delta_{N_f}(\{\mu^2\})\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)}\,.
\end{aligned}
$$

In the last line we have two terms, which become unity as N goes to infinity. The finite-volume partition function becomes

$$
\begin{aligned}
\mathcal{Z}_\nu^{(N_f+2)}(\mu_1,\dots,\mu_{N_f},i\zeta_a,i\zeta_b) &= 2^{-(N_f+2)(N_f+1)/2}2^{-\nu N_f}2^{N_f(N_f+1)+1} \\
&\frac{(-1)^{\nu}\zeta_a^{\nu}\left(\prod_{f=1}^{N_f}\mu_f^{\nu}\right)}{\Delta_{N_f}(\{\mu^2\})\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)}(-1)^{N_f}\lim_{N\to\infty}\det_{1\le k,j\le N_f+1}\left[\hat{\mathcal{C}}_{k,j}\right]
\end{aligned}
$$

and the limit of the matrix $\hat{\mathcal{C}}_{k,j}$ is given as

$$
\lim_{N\to\infty}\hat{\mathcal{C}}_{k,j} = \begin{cases}
\mu_j^{-\nu}\mathcal{B}_{\mathrm{IJ}}(\mu_j,\zeta_b), & \text{for } k=1\,,\ j=1,\dots,N_f, \\
\zeta_a^{-\nu}\mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_b), & \text{for } k=1\,,\ j=N_f+1, \\
2^{\nu-k+2}\mu_j^{-\nu+k-2}I_{\nu-k+2}(\mu_j), & \text{for } k=2,\dots,N_f+1,\ j=1,\dots,N_f, \\
2^{\nu-k+2}\zeta_a^{-\nu+k-2}J_{\nu-k+2}(\zeta_a), & \text{for } k=2,\dots,N_f+1,\ j=N_f+1.
\end{cases}
$$

The next step is to transform the determinant to bring the indices of the Bessel functions from $\nu-k+2$ to $\nu+k-2$ by applying Eq. (5.18). This leads to a matrix $\widetilde{\mathcal{C}}$, which contains powers of (-1) in the ζ_a column. Pulling out prefactors we obtain

$$
\begin{aligned}
\mathcal{Z}_\nu^{(N_f+2)}(\mu_1,\dots,\mu_{N_f},i\zeta_a,i\zeta_b) &= 2^{-(N_f+2)(N_f+1)/2}2^{N_f(N_f+1)+1}\left(\prod_{k=1}^{N_f}2^{-k+1}\right) \\
&\frac{(-1)^{\nu}}{\Delta_{N_f}(\{\mu^2\})\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)}(-1)^{N_f}\det_{1\le k,j\le N_f+1}\left[\widetilde{\mathcal{C}}_{k,j}\right].
\end{aligned}
$$

The factors of 2 all cancel and we can bring the last column in the determinant to the front. Taking out the factors of (-1) we find

$$\mathcal{Z}_\nu^{(N_f+2)}(\mu_1,\ldots,\mu_{N_f},i\zeta_a,i\zeta_b) = \frac{(-1)^\nu}{\Delta_{N_f}(\{\mu^2\})\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)}(-1)^{N_f(N_f-1)/2}$$
$$\det\begin{bmatrix} \mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_b) & \mathcal{B}_{\mathrm{IJ}}(\mu_1,\zeta_b) & \ldots & \mathcal{B}_{\mathrm{IJ}}(\mu_{N_f},\zeta_b) \\ J_\nu(\zeta_a) & I_\nu(\mu_1) & \ldots & I_\nu(\mu_{N_f}) \\ \vdots & \ldots & \ldots & \vdots \\ \zeta_a^{N_f-1}J_{\nu+N_f-1}(\zeta_a) & (-\mu_1)^{N_f-1}I_{\nu+N_f-1}(\mu_1) & \ldots & (-\mu_{N_f})^{N_f-1}I_{\nu+N_f-1}(\mu_{N_f}) \end{bmatrix}.$$

The correlation kernel reads

$$K_S^{(N_f)}\left(\zeta_a,\zeta_b;\{\mu_f\}_{f=1}^{N_f}\right) = (-1)^\nu\sqrt{|\zeta_a\zeta_b|}\prod_{f=1}^{N_f}\sqrt{(\zeta_a^2+\mu_f^2)(\zeta_b^2+\mu_f^2)}\frac{\mathcal{Z}_\nu^{(N_f+2)}(\mu_1,\ldots,\mu_{N_f},i\zeta_a,i\zeta_b)}{\mathcal{Z}_\nu^{(N_f)}(\mu_1,\ldots,m_{N_f})}$$
$$= \sqrt{|\zeta_a\zeta_b|}\sqrt{\frac{\prod_{f=1}^{N_f}(\zeta_b^2+\mu_f^2)}{\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)}}\frac{\Delta_{N_f}(\mu_1^2,\ldots,\mu_{N_f}^2)}{\det\limits_{1\le a,b\le N_f}\left[\mathcal{B}_{a,b}(\{\mu_f\}_{f=1}^{N_f})\right]}\frac{(-1)^{N_f(N_f-1)/2}}{\Delta_{N_f}(\mu_1^2,\ldots,\mu_{N_f}^2)\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)}$$
$$\det\begin{bmatrix} \mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_b) & \mathcal{B}_{\mathrm{IJ}}(\mu_1,\zeta_b) & \ldots & \mathcal{B}_{\mathrm{IJ}}(\mu_{N_f},\zeta_b) \\ J_\nu(\zeta_a) & I_\nu(\mu_1) & \ldots & I_\nu(\mu_{N_f}) \\ \vdots & \ldots & \ldots & \vdots \\ \zeta_a^{N_f-1}J_{\nu+N_f-1}(\zeta_a) & (-\mu_1)^{N_f-1}I_{\nu+N_f-1}(\mu_1) & \ldots & (-\mu_{N_f})^{N_f-1}I_{\nu+N_f-1}(\mu_{N_f}) \end{bmatrix}$$
$$= \sqrt{|\zeta_a\zeta_b|}\sqrt{\frac{\prod_{f=1}^{N_f}(\zeta_b^2+\mu_f^2)}{\prod_{f=1}^{N_f}(\zeta_a^2+\mu_f^2)}}\left(\det_{1\le a,b\le N_f}\left[(-\mu_a)^{b-1}I_{\nu+b-1}(\mu_a)\right]\right)^{-1}$$
$$\det\begin{bmatrix} \mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_b) & \mathcal{B}_{\mathrm{IJ}}(\mu_1,\zeta_b) & \ldots & \mathcal{B}_{\mathrm{IJ}}(\mu_{N_f},\zeta_b) \\ J_\nu(\zeta_a) & I_\nu(\mu_1) & \ldots & I_\nu(\mu_{N_f}) \\ \vdots & \ldots & \ldots & \vdots \\ \zeta_a^{N_f-1}J_{\nu+N_f-1}(\zeta_a) & (-\mu_1)^{N_f-1}I_{\nu+N_f-1}(\mu_1) & \ldots & (-\mu_{N_f})^{N_f-1}I_{\nu+N_f-1}(\mu_{N_f}) \end{bmatrix}.$$

This is indeed the form of our correlation kernel obtained in chapter 4.

The consequence of the above computation is the following: The correlation kernel obtained in chapter 4 for the chGUE(N) with N_f massive flavors and temperature dependence becomes universal in the large N limit with respect to its temperature dependence. In particular, it is equivalent to one representation of the kernel of the chGUE(N) without temperature in the presence of N_f massive flavors considered in [63, 64]. What does this mean physically? The effect of temperature, in microscopic limit of QCD, is reduced to the replacement of the chiral condensate Σ by the temperature dependent chiral condensate Ξ, which contains the information of the temperature singular values a_n. Otherwise the form of the correlation kernel is universal.

5.2 Equivalence with results obtained via SUSY techniques

In [12] Guhr and Wettig considered a random matrix ensemble for QCD with temperature in the quenched approximation without topology. Thus, they considered a version of the chGUE(N) with an external source, with JPDF given by Eq. (5.3). They derived the k-point correlation functions in the large N limit via supersymmetry and found

$$\rho_S^{(0)}(\zeta_1,\ldots,\zeta_k) = \lim_{N\to\infty} \frac{1}{(2N\Xi)^k} \det_{1\le a,b\le k} \left[K_N^{(0)}(\zeta_a,\zeta_b)\right],$$

with kernel $K_N^{(0)}(\zeta_a,\zeta_b)$ given by

$$K_N^{(0)}(\zeta_a,\zeta_b) = 2N\Xi\zeta_a \frac{\zeta_a J_1(\zeta_a)J_0(\zeta_b) - \zeta_b J_1(\zeta_b)J_0(\zeta_a)}{\zeta_a^2-\zeta_b^2} = 2N\Xi\zeta_a \mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_b)\,.$$

This is the standard Bessel kernel [133] for $\nu = 0$[4] and leads to correlation functions

$$\rho_S^{(0)}(\zeta_1,\ldots,\zeta_k) = \left(\prod_{j=1}^k |\zeta_j|\right) \det_{1\le a,b\le k} \left[\mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_b)\right].$$

In comparison, we found in Proposition 4.1 and in particular Eq. (4.16), that the correlation kernel for the chGUE(N) with an external source and $\nu \neq 0$ in the large N limit is given as

$$\lim_{N\to\infty} \frac{1}{2N\Xi} K_N^{(0)}\left(x_1 = \frac{\zeta_a^2}{4N\Xi}, x_2 = \frac{\zeta_b^2}{4N\Xi}\right) = \frac{\zeta_a J_{\nu+1}(\zeta_a)J_\nu(\zeta_b) - \zeta_b J_{\nu+1}(\zeta_b)J_\nu(\zeta_a)}{\zeta_a^2-\zeta_b^2}\,.$$

Thus, we can see immediately, that our results is equal to the result of Guhr and Wettig in [12], when ν is set to zero.
In [13] Seif, Guhr and Wettig extended the ensemble of [12] to N_f massive flavors and used supersymmmetry to obtain the microscopic limit of the k-point correlation functions. Their result reads

$$\rho_S^{(N_f)}(\zeta_1,\ldots,\zeta_k) = \left(\prod_{j=1}^k |\zeta_j|\right) \frac{\det\limits_{1\le a,b\le N_f+k} [\mathcal{A}_{a,b}]}{\det\limits_{1\le a,b\le N_f} [\mathcal{B}_{a,b}]}, \tag{5.29}$$

with

$$\mathcal{A}_{a,b} = \begin{cases} \mathcal{B}_{\mathrm{JJ}}(\zeta_b,\zeta_a), & \text{for } 1\le a,b\le k,\\ \mathcal{B}_{\mathrm{IJ}}(\mu_{b-k},\zeta_a), & \text{for } 1\le a\le k, k+1\le b\le k+N_f,\\ \zeta_b^{a-k-1} J_{a-k-1}(\zeta_b), & \text{for } k+1\le a\le k+N_f, 1\le b\le k,\\ (-\mu_{b-k})^{a-k-1} I_{a-k-1}(\mu_{b-k}), & \text{for } k+1\le a,b\le k+N_f, \end{cases}$$

and $\mathcal{B}_{a,b} = (-\mu_b)^{a-1} I_{a-1}(\mu_b)$, for $1\le a,b\le N_f$.

The ensemble considered in [13] is comparable to the extension of the chGUE(N) with an external source by N_f massive flavors we introduced in chapter 2, if we assume again $\nu = 0$. The underlying JPDF of this special case is given in Eq. (5.4). Because the model we considered in chapter 4 in the large N limit is equal to the ensemble of Seif, Guhr and Wettig for

[4] We do not distinguish between $\nu = 0$ and $\nu \neq 0$ in the short-hand notation $\mathcal{B}_{\mathrm{JJ}}(\zeta_a,\zeta_b)$.

$\nu = 0$, the determinantal representation for the microscopic k-point correlation function derived in [13] should in principle be representable as a determinant of a matrix of dimension $k \times k$ of the correlation kernel. The kernel itself should be in a determinantal representation, similar to the result we derived in the large N limit in chapter 4. The question in this section is how we can relate those two results. The problem is that it is unclear how to rewrite the $k \times k$ determinant of our kernel, which is a ratio of a determinant of a $(N_f + 1) \times (N_f + 1)$ matrix over a determinant of a $N_f \times N_f$ matrix, to obtain the result of Seif, Guhr and Wettig [13], whose representation of the k-point correlation function is a ratio of determinant of a $(N_f + k) \times (N_f + k)$ matrix over a determinant of a $N_f \times N_f$ matrix. The fact that the underlying ensemble is basically the same hints at the existence of a kind of master formula, which is know for ensembles without temperature, such as [63, 64]. Unfortunately, we were not able to derive such a master formula. Instead we will demonstrate that the k-point correlations functions of the chGUE(N) ensemble, independent of temperature, studied in [63, 64] can be represented not only in terms of a determinantal point process of a correlation kernel, see Eq. (5.5) and (5.6), but also in the determinantal form of [13], see Eq. (5.29). Hence, we will show the equivalence of the results of section 5.1 [63, 64] and the results of Seif, Guhr and Wettig [13]. By the equivalence shown in the previous section, this shows the equivalence of our results with the results of Seif, Guhr and Wettig [13].

Expressing correlation functions as ratios of finite-volume partition functions

We recall a result derived in [59, 60] for the microscopic k-point correlation function, namely[5]

$$\begin{aligned}\rho_S^{(N_f)}(\zeta_1, \ldots, \zeta_k) = (-1)^{k\nu} \left(\prod_{j=1}^{k} |\zeta_j| \prod_{f=1}^{N_f} (\zeta_j^2 + \mu_f^2) \right) \Delta_k^2 \left(\zeta_1^2, \ldots, \zeta_k^2\right) \\ \times \frac{\mathcal{Z}_\nu^{(N_f+2k)}(\mu_1, \ldots, \mu_{N_f}, \{i\zeta_1\}, \ldots, \{i\zeta_k\})}{\mathcal{Z}_\nu^{(N_f)}(\mu_1, \ldots, m_{N_f})} .\end{aligned} \tag{5.30}$$

The finite-volume partition functions can be expressed in terms of limits of finite N RMT partition functions. This was used extensively in the previous section, in particular, via Eq. (5.8) and (5.8). The above representation of the microscopic k-point correlation function must be equal to the representation given in Eq. (5.5)

$$\rho_S^{(N_f)}(\zeta_1, \ldots, \zeta_k) = \lim_{N\to\infty} \frac{1}{(N\Sigma)^k} \rho_N^{(N_f)} \left(\frac{\zeta_1}{2N\Sigma}, \ldots, \frac{\zeta_k}{2N\Sigma} \right) = \frac{1}{\Sigma^k} \det_{1\le a,b\le k} \left[K_S^{(N_f)} \left(\frac{\zeta_a}{2\Sigma}, \frac{\zeta_b}{2\Sigma} \right) \right] ,$$

where the kernel can be written as (compare Eq. (5.6) or (5.7))

$$K_S^{(N_f)} \left(\zeta_a, \zeta_b; \{\mu_f\}_{f=1}^{N_f} \right) = \sqrt{|\zeta_a \zeta_b|} \prod_{f=1}^{N_f} \sqrt{(\zeta_a^2 + \mu_f^2)(\zeta_b^2 + \mu_f^2)} \frac{\mathcal{Z}_\nu^{(N_f+2)}(\mu_1, \ldots, \mu_{N_f}, i\zeta_a, i\zeta_b)}{(-1)^\nu \mathcal{Z}_\nu^{(N_f)}(\mu_1, \ldots, m_{N_f})} .$$

The equality of the two representations of microscopic k-point correlation functions was shown for the classical chGUE(N) and its deformation with N_f massive flavors - see [59, 60, 134]. This observation is crucial for the equivalence we want to demonstrate in this section, since the ensembles considered in [63, 64] are exactly these two chGUE(N)-type ensembles.

[5] The bracket notation $\{i\zeta_j\}$ implies that each ζ is 2-fold degenerate.

From the equivalence shown in the previous section we can conclude that the representations of the microscopic k-point correlation functions is then also true for the chGUE(N) with an external source and its extension with N_f flavors. Those two ensembles coincide with the ensembles of [12, 13], if $\nu = 0$. Thus, we can use Eq. (5.7) and Theorem 5.1 to obtain different representations for the finite-volume partition function $\mathcal{Z}_\nu^{(N_f+2k)}(\mu_1,\ldots,\mu_{N_f})$ also in the setting of [12, 13]. This allows us to show the equivalence of the results of [63, 64] with [12, 13] and consequently also our results from chapter 4.

We will use Eq. (5.8), but this time we will take k to be general, i.e.

$$\mathcal{Z}_\nu^{(N_f+2k)}(\mu_1,\ldots,\mu_{N_f},\{i\zeta_m\}_{m=1}^{2k}) = \lim_{N\to\infty}\frac{Z_N^{(N_f+2k,0)}(m_1^2,\ldots,m_{N_f}^2,\{z_m^2\}_{m=1}^{2k})}{C_N^{[N_f+2k]}}$$

with k variables ζ, which each are 2-fold degenerate.[6] With the rescaling scheme

$$\mu_f^2 = \frac{\mu_f^2}{4N},\quad z_m^2 = \frac{(i\zeta_m)^2}{4N},\ \text{for}\ m=1,\ldots,k,\quad z_m^2 = \frac{(i\zeta_{m-k})^2}{4N},\ \text{for}\ m=k+1,\ldots,2k,$$

we obtain determinantal representations of the finite-volume partition function $\mathcal{Z}_\nu^{(N_f+2k)}$. We will find a determinant of a N_f+k dimensional matrix with k rows containing Bessel kernels and N_f rows containing Bessel-J- and Bessel-I-functions.

Expectation values of characteristic polynomials for k variables

To derive the needed representation of the finite-volume partition function, we need the expression of the finite N-partition function $Z_N^{(N_f+2k,0)}$ as an expectation value of N_f+2k characteristic polynomials, i.e.

$$Z_N^{(N_f+2k,0)}(m_1^2,\ldots,m_{N_f}^2,\{z_m\}_{m=1}^{2k}) = Z_N^{(0,0)}(-1)^{NN_f}\mathbb{E}\left[\left(\prod_{f=1}^{N_f}D_N(-m_f^2)\right)\left(\prod_{m=1}^{2k}D_N(-z_m^2)\right)\right].$$

We divide the N_f+2k characteristic polynomials into to sets, the first containing $M=N_f+k$ and the second containing $L=k$ characteristic polynomials. This circumvents the degeneracy of the ζ_j variables and we obtain a N_f+k dimensional determinantal representation of the finite N partition function via Theorem 5.1

$$\mathbb{E}\left[\left(\prod_{f=1}^{N_f}D_N\left(-\frac{\mu_f^2}{4N}\right)\prod_{m=1}^{k}D_N\left(\frac{\zeta_m^2}{4N}\right)\right)\left(\prod_{m=1}^{k}D_N\left(\frac{\zeta_m^2}{4N}\right)\right)\right] = \prod_{i=N}^{N+N_f+k-1}\sqrt{h_i}\prod_{j=N}^{N+k-1}\sqrt{h_j}$$
$$\times\frac{(-1)^{N_f(N_f-1)/2}(4N)^{(N_f+k)(N_f+k-1)/2}(4N)^{k(k-1)/2}}{\Delta_{N_f}(\mu_1^2,\ldots,\mu_{N_f}^2)\left(\prod_{f=1}^{N_f}\prod_{m=1}^{k}(\zeta_m^2+\mu_f^2)\right)\Delta_k^2(\zeta_1^2,\ldots,\zeta_k^2)}\det_{1\le i,j\le N_f+k}\left[\mathcal{C}_{i,j}\right].$$

[6] The degeneracy is not problematic in our calculations due to the special determinantal representation of the finite-volume partition function we will derive by using Theorem 5.1.

The matrix $\mathcal{C}_{i,j}$ is given as

$$\mathcal{C}_{i,j} = \begin{cases} \hat{L}(-m_j^2, -z_i^2), & \text{for } i = 1,\ldots,k\ ,\ j = 1,\ldots,N_f, \\ \hat{L}(-z_{j-N_f}^2, -z_i^2), & \text{for } i = 1,\ldots,k\ ,\ j = N_f+1,\ldots,N_f+k, \\ \frac{(-1)^{N+i-1}\,(N+i-1)!}{\sqrt{h_{N+i-1}}} L_{N+i-1}^{\nu}(-m_j^2), & \text{for } i = k+1,\ldots,N_f+k\ ,\ j = 1,\ldots,N_f, \\ \frac{(-1)^{N+i-1}\,(N+i-1)!}{\sqrt{h_{N+i-1}}} L_{N+i-1}^{\nu}(-z_{j-N_f}^2), & \text{for } i = k+1,\ldots,N_f+k\ ,\ j = N_f+1,\ldots,N_f+k. \end{cases}$$

Here we have used the generalized Laguerre polynomials according to Eq. (2.18). The quantity $\hat{L}(x,y)$ can be written similar to Eq. (5.21), namely

$$\begin{aligned} \hat{L}(x,y) &= \sum_{i=0}^{N+k-1} P_i(x)P_i(y) = \sqrt{\frac{h_{N+k}}{h_{N+k-1}}}\frac{P_{N+k}(x)P_{N+k-1}(y) - P_{N+k-1}(x)P_{N+k}(y)}{x-y} \\ &= \frac{(-1)(N+k)!}{\Gamma(N+k+\nu)}\frac{L_{N+k}^{\nu}(x)L_{N+k-1}^{\nu}(y) - L_{N+k-1}^{\nu}(x)L_{N+k}^{\nu}(y)}{x-y}, \end{aligned}$$

which leads to results similar to Eq. (5.22) and (5.25) using the Laguerre polynomials and their properties given in Eq. (5.16). We find

$$\begin{aligned} &\hat{L}\left(-\frac{\mu_j^2}{4N}, \frac{\zeta_i^2}{4N}\right) = \frac{4N\ (N+k)!}{\Gamma(N+k+\nu)}\frac{L_{N+k}^{\nu}\left(-\frac{\mu_j^2}{4N}\right)L_{N+k-1}^{\nu}\left(\frac{\zeta_i^2}{4N}\right) - L_{N+k-1}^{\nu}\left(-\frac{\mu_j^2}{4N}\right)L_{N+k}^{\nu}\left(\frac{\zeta_i^2}{4N}\right)}{\mu_j^2+\zeta_i^2} \\ &= \frac{4N\ (N+k)!}{\Gamma(N+k+\nu)}\frac{1}{\mu_j^2+\zeta_i^2}\left[L_{N+k}^{\nu-1}\left(-\frac{\mu_j^2}{4N}\right)L_{N+k}^{\nu}\left(\frac{\zeta_i^2}{4N}\right) - L_{N+k}^{\nu}\left(-\frac{\mu_j^2}{4N}\right)L_{N+k}^{\nu-1}\left(\frac{\zeta_i^2}{4N}\right)\right] \\ &\equiv \frac{\Gamma(N)N^{k+\nu}}{\Gamma(N+k+\nu)}\frac{\Gamma(N+k+1)}{\Gamma(N)N^{k+1}}\ N^{\nu+1}\ 2^{2\nu+1}\mu_j^{-\nu}\zeta_i^{-\nu}\hat{\mathcal{L}}_N(\mu_j,\zeta_i) \end{aligned}$$

and

$$\begin{aligned} &\hat{L}\left(\frac{\zeta_j^2}{4N}, \frac{\zeta_i^2}{4N}\right) = \frac{(-1)\ 4N\ (N+k)!}{\Gamma(N+k+\nu)}\frac{L_{N+k}^{\nu}\left(\frac{\zeta_j^2}{4N}\right)L_{N+k-1}^{\nu}\left(\frac{\zeta_i^2}{4N}\right) - L_{N+k-1}^{\nu}\left(\frac{\zeta_j^2}{4N}\right)L_{N+k}^{\nu}\left(\frac{\zeta_i^2}{4N}\right)}{\zeta_j^2-\zeta_i^2} \\ &= \frac{(-1)\ 4N\ (N+k)!}{\Gamma(N+k+\nu)}\frac{1}{\zeta_j^2-\zeta_i^2}\left[L_{N+k}^{\nu-1}\left(\frac{\zeta_j^2}{4N}\right)L_{N+k}^{\nu}\left(\frac{\zeta_i^2}{4N}\right) - L_{N+k}^{\nu}\left(\frac{\zeta_j^2}{4N}\right)L_{N+k}^{\nu-1}\left(\frac{\zeta_i^2}{4N}\right)\right] \\ &\equiv \frac{\Gamma(N)N^{k+\nu}}{\Gamma(N+k+\nu)}\frac{\Gamma(N+k+1)}{\Gamma(N)N^{k+1}}\ N^{\nu+1}\ 2^{2\nu+1}\zeta_j^{-\nu}\zeta_i^{-\nu}\hat{\mathcal{L}}_N(\zeta_j,\zeta_i)\ . \end{aligned}$$

Doing the same asymptotics for $\hat{L}_N$ as we did in the previous section, we obtain again Bessel kernels given in Eqs. (5.24) and (5.26). The case $\zeta_j = \zeta_i$ leads to the kernel obtained in Eq. (5.27) as before. Hence, we arrive at the following expression for the expectation value,

which is comparable to Eq. (5.28)

$$\mathbb{E}\left[\left(\prod_{f=1}^{N_f} D_N\left(-\frac{\mu_f^2}{4N}\right)\prod_{m=1}^{k} D_N\left(\frac{\zeta_m^2}{4N}\right)\right)\left(\prod_{m=1}^{k} D_N\left(\frac{\zeta_m^2}{4N}\right)\right)\right] = \prod_{i=0}^{N_f+k-1}\sqrt{h_{i+N}}\prod_{j=0}^{k-1}\sqrt{h_{j+N}}$$
$$\frac{(-1)^{N_f(N_f-1)/2}(4N)^{(N_f+k)(N_f+k-1)/2}(4N)^{k(k-1)/2}}{\Delta_{N_f}(\mu_1^2,\ldots,\mu_{N_f}^2)\left(\prod_{f=1}^{N_f}\prod_{m=1}^{k}(\zeta_m^2+\mu_f^2)\right)\Delta_k^2(\zeta_1^2,\ldots,\zeta_k^2)}\det_{1\le i,j\le N_f+k}\left[\mathcal{C}_{i,j}\right]$$
$$= \frac{(-1)^{N_f(N+k)}2^{(N_f+2k)(N_f+2k-1)}2^{-2k(N_f+k)}2^{(2\nu+1)k}N^{(N_f+k)(N_f+k-1)/2}N^{k(k-1)/2}}{\Delta_{N_f}(\mu_1^2,\ldots,\mu_{N_f}^2)\left(\prod_{f=1}^{N_f}\prod_{m=1}^{k}(\zeta_m^2+\mu_f^2)\right)\Delta_k^2(\zeta_1^2,\ldots,\zeta_k^2)}$$
$$\times\left(\prod_{f=1}^{N_f}\Gamma(N+k+f)\right)\left(\prod_{j=0}^{k-1}h_{j+N}\right)N^{(\nu+1)k}\left(\prod_{i=1}^{k}\zeta_i^{-\nu}\right)\left(\prod_{j=1}^{N_f}N^{\nu-j+1}\right)\det_{1\le i,j\le N_f+k}\left[\hat{\mathcal{C}}_{i,j}\right], \tag{5.31}$$

where the matrix $\hat{\mathcal{C}}_{i,j}$ is given as

$$\hat{\mathcal{C}}_{i,j} = \begin{cases} \mu_j^{-\nu}\hat{\mathcal{L}}_N(\mu_j,\zeta_i), & \text{for } i=1,\ldots,k\,,\ j=1,\ldots,N_f,\\ \zeta_{j-N_f}^{-\nu}\hat{\mathcal{L}}_N(\zeta_{j-N_f},\zeta_i), & \text{for } i=1,\ldots,k\,,\ j=N_f+1,\ldots,N_f+k,\\ N^{-\nu+i-k-1}L_{N+i-1}^{\nu-i+k+1}(-m_j^2), & \text{for } i=k+1,\ldots,N_f+k\,,\ j=1,\ldots,N_f,\\ N^{-\nu+i-k-1}L_{N+i-1}^{\nu-i+k+1}(-z_{j-N_f}^2), & \text{for } i=k+1,\ldots,N_f+k\,,\ j=N_f+1,\ldots,N_f+k. \end{cases}$$

The non-determinantal prefactors in front of the expectation value in Eq. (5.31) can be rewritten using the norms of Laguerre polynomials from Eq. (5.9) and the asymptotics of Γ-function in Eq. (5.23). We find

$$(-1)^{N_f k}2^{(N_f+2k)(N_f+2k-1)}2^{-2k(N_f+k)}2^{(2\nu+1)k}\left(\prod_{f=1}^{N_f}\frac{\Gamma(N+k+f)}{\Gamma(N)N^{k+f}}\right)$$
$$\times\left(\prod_{i=1}^{k}\zeta_i^{-\nu}\right)N!\left(\prod_{j=1}^{N}\Gamma(j+\nu)\Gamma(j)\right)\left(\prod_{j=1}^{k}\frac{\Gamma(N+j)\Gamma(N+j+\nu)}{[\Gamma(N)]^2N^{2j+\nu}}\right)[\Gamma(N)]^{N_f+2k}$$
$$\times N^{(N_f+k)(N_f+k-1)/2}N^{k(k-1)/2}N^{\nu k}N^{k(k+1)}N^{(\nu+1)k}N^{\nu N_f}N^{-N_f(N_f-1)/2}N^{N_f k}N^{N_f(N_f+1)/2}$$
$$= C_N^{[N_f+2k]}\left(\prod_{i=1}^{k}\zeta_i^{\nu}\right)\left(\prod_{f=1}^{N_f}\mu_f^{\nu}\right)(-1)^{k\nu}(-1)^{N_f k}2^{-\nu N_f+k}2^{(N_f+2k)(N_f+2k-1)/2}2^{-2k(N_f+k)}\,.$$

In the last line we have neglected terms that become unity in the large N limit. Therefore, we can write[7]

$$Z_N^{(N_f+2k)}(m_1^2,\ldots,m_{N_f}^2,\{z_m\}_{m=1}^{2k}) = C_N^{[N_f+2k]}\left(\prod_{i=1}^{k}\zeta_i^{\nu}\right)\left(\prod_{f=1}^{N_f}\mu_f^{\nu}\right)(-1)^{k\nu}(-1)^{N_f k}$$
$$\times\frac{2^{-\nu N_f+k}2^{(N_f+2k)(N_f+2k-1)/2}2^{-2k(N_f+k)}}{\Delta_{N_f}(\mu_1^2,\ldots,\mu_{N_f}^2)\left(\prod_{f=1}^{N_f}\prod_{m=1}^{k}(\zeta_m^2+\mu_f^2)\right)\Delta_k^2(\zeta_1^2,\ldots,\zeta_k^2)}\det_{1\le i,j\le N_f+k}\left[\hat{\mathcal{C}}_{i,j}\right].$$

[7]Note that we omit the terms, which become equal to unity in the large N limit.

Correlation functions equivalent to [13]

The finite volume partition function reads

$$\mathcal{Z}_\nu^{(N_f+2k)}(\mu_1,\dots,\mu_{N_f},\{i\zeta_m\}_{m=1}^{2k}) = \lim_{N\to\infty}\frac{Z_N^{(N_f+2k)}(m_1^2,\dots,m_{N_f}^2,\{z_m^2\}_{m=1}^{2k})}{C_N^{[N_f+2k]}}$$
$$= \frac{2^{-\nu N_f+k}2^{(N_f+2k)(N_f+2k-1)/2}2^{-2k(N_f+k)}}{\Delta_{N_f}(\mu_1^2,\dots,\mu_{N_f}^2)\left(\prod_{f=1}^{N_f}\prod_{m=1}^{k}(\zeta_m^2+\mu_f^2)\right)\Delta_k^2(\zeta_1^2,\dots,\zeta_k^2)}$$
$$\times\left(\prod_{i=1}^{k}\zeta_i^\nu\right)\left(\prod_{f=1}^{N_f}\mu_f^\nu\right)(-1)^{k\nu}(-1)^{N_fk}\det_{1\le i,j\le N_f+k}\left[\lim_{N\to\infty}\hat{\mathcal{C}}_{i,j}\right]$$

and the limit of the matrix $\hat{\mathcal{C}}_{i,j}$ is given as

$$\lim_{N\to\infty}\hat{\mathcal{C}}_{i,j} = \begin{cases}\mu_j^{-\nu}\mathcal{B}_{\mathrm{IJ}}(\mu_j,\zeta_i), & \text{for } i=1,\dots,k\,,\ j=1,\dots,N_f,\\ \zeta_{j-N_f}^{-\nu}\mathcal{B}_{\mathrm{JJ}}(\zeta_{j-N_f},\zeta_i), & \text{for } i=1,\dots,k\,,\ j=N_f+1,\dots,N_f+k,\\ 2^{\nu-i+k+1}\mu_j^{-\nu+i-k-1}I_{\nu-i+k+1}(\mu_j), & \text{for } i=k+1,\dots,N_f+k\,,\ j=1,\dots,N_f,\\ 2^{\nu-i+k+1}\zeta_{j-N_f}^{-\nu+i-k-1}J_{\nu-i+k+1}(\zeta_{j-N_f}), & \text{for } i=k+1,\dots,N_f+k\,,\ j=N_f+1,\dots,N_f+k.\end{cases}$$

We can extract the factors of 2 from the determinant. This leads to

$$\mathcal{Z}_\nu^{(N_f+2k)}(\mu_1,\dots,\mu_{N_f},\{i\zeta_m\}_{m=1}^{2k}) = \frac{(-1)^{k\nu}(-1)^{N_fk}\det_{1\le i,j\le N_f+k}\left[\widetilde{\mathcal{C}}_{i,j}\right]}{\Delta_{N_f}(\mu_1^2,\dots,\mu_{N_f}^2)\left(\prod_{f=1}^{N_f}\prod_{m=1}^{k}(\zeta_m^2+\mu_f^2)\right)\Delta_k^2(\zeta_1^2,\dots,\zeta_k^2)}$$
$$\times\left(\prod_{j=1}^{N_f}2^{\nu-j+1}\right)2^{(N_f+2k)(N_f+2k-1)/2}2^{-2k(N_f+k)}2^{-\nu N_f+k}$$

and all factors of 2 cancel. Using the factor $(-1)^{N_fk}$ to permute the last k columns with the first N_f columns and applying Eq. (5.30) leads to the microscopic k-point correlation function

$$\rho_S^{(N_f)}(\zeta_1,\dots,\zeta_k) = \left(\prod_{j=1}^{k}|\zeta_j|\prod_{f=1}^{N_f}(\zeta_j^2+\mu_f^2)\right)\Delta_k^2(\{\zeta^2\})\frac{\mathcal{Z}_\nu^{(N_f+2k)}(\mu_1,\dots,\mu_{N_f},\{i\zeta_1\},\dots,\{i\zeta_k\})}{(-1)^{k\nu}\mathcal{Z}_\nu^{(N_f)}(\mu_1,\dots,m_{N_f})}$$
$$= \frac{\left(\prod_{j=1}^{k}|\zeta_j|\right)\left(\prod_{m=1}^{k}\prod_{f=1}^{N_f}(\zeta_m^2+\mu_f^2)\right)\Delta_k^2(\zeta_1^2,\dots,\zeta_k^2)}{\Delta_{N_f}(\mu_1^2,\dots,\mu_{N_f}^2)\left(\prod_{f=1}^{N_f}\prod_{m=1}^{k}(\zeta_m^2+\mu_f^2)\right)\Delta_k^2(\zeta_1^2,\dots,\zeta_k^2)}(-1)^{k\nu}$$
$$\times(-1)^{k\nu}\frac{\Delta_{N_f}(\mu_1^2,\dots,\mu_{N_f}^2)}{\det_{1\le a,b\le N_f}[\mathcal{B}_{a,b}]}\det_{1\le a,b\le N_f+k}[\mathcal{A}_{a,b}]$$
$$= \left(\prod_{j=1}^{k}|\zeta_j|\right)\frac{\det_{1\le a,b\le N_f+k}[\mathcal{A}_{a,b}]}{\det_{1\le a,b\le N_f}[\mathcal{B}_{a,b}]}\,.$$

This is exactly the representation of the microscopic k-point correlation function found by Seif, Guhr and Wettig [13] for $\nu = 0$. Thus, the equivalence of the correlation functions for the classical chGUE(N) and the deformed chGUE(N) on one hand and the chGUE(N) with an external source and its extension with N_f flavors on the other hand is shown. Our results from chapter 4 are equivalent to the results of [63, 64], as we have seen in the previous section. Now we have demonstrated that the results of [63, 64] are equivalent to the result of Seif, Guhr and Wettig [12, 13]. Thus, we conclude that our results also have to be equivalent to the results of Seif, Guhr and Wettig, provided that we set $\nu = 0$ in our results of chapter 4. Furthermore, we have shown that the equivalence also holds for $\nu \neq 0$, which itself is a new result.

5.3 Summary

This chapter is concerned with the assessment of universality of the correlation kernels for the chGUE(N) with an external source and its extension with N_f massive flavors. The external source describes temperature effects in the comparison of the random matrix model with an effective field theory model of the theory of strong interactions (QCD). In the large N limit the correlation kernel derived in chapter 4 is compared to zero-temperature results of [63, 64]. We find that the determinantal structures of our kernel and the kernel derived in [63, 64] are equivalent in section 5.1. Thus, we showed universality of the correlation kernel and correlation function formulae derived in chapter 4. The only change is the scale with respect to temperature obtained via the replacement of the chiral condensate $\Sigma \to \Xi$. Note that both models operate in the low-energy, small temperature, regime below the critical temperature. This hints at an analytical phase transition in the QCD phase diagram along the temperature axis.
In section 5.2 we discuss how results for the k-point function using supersymmetry methods for the non-zero temperature models derived in [12, 13] can be compared to the zero-temperature results of [63, 64]. We showed that the results of both sources are equivalent, which was conjectured in [13]. This also connects our result to the results of [12, 13] via the equivalence show in section 5.1. The equivalence we showed holds for arbitary number of zero modes ν, including $\nu = 0$, which represents a new result beyond the conjecture of [13].

Chapter 6

Concluding Remarks and Outlook

Characteristic polynomials play an important role in Random Matrix Theory (RMT), in particular as observables linked to correlation functions of random matrices with real spectra. Random matrices form ensembles characterized and classified by global symmetries. Therefore, the ensembles can be used to describe physical and statistical systems in the same symmetry class, which is due to the concept of universality. From the many different applications of RMT we focused on the application to the theory of strong interactions, whose quantum field theory is called Quantum chromodynamics (QCD). In particular, we looked at the microscopic spectrum of the QCD Dirac operator in the low-energy regime, where chiral symmetry breaking is the predominant factor determining the spectral statistics. The random matrix ensembles we considered in this thesis share the symmetries of the Dirac operator, in particular its chiral symmetry and consist of random matrices with entries distributed according to a Gaussian distribution. This leads to the name chiral Gaussian Unitary Ensemble - chGUE(N). The random matrix models allow analytical computation of spectral statistics of eigenvalues and eigenvectors. Thus, in the low-energy regime of QCD we may use spectral results of small eigenvalues of the RMT Dirac operator, instead of its quantum field theory version, to understand the spectrum.
Our discussions were made on models drawn from the chGUE(N) symmetry class, in particular those with external sources, which are used to model temperature effects. The external sources in our models appear as finite-rank matrix perturbations of the eigenvalues in the joint probability density function (JPDF) of the respective random matrix model. In absence of external sources we considered the chGUE(N) and its deformation by N_f massive flavors. Those ensembles have known correlation functions both at finite matrix size N and in its large N limit both on the macroscopic and microscopic scales.
The large N limit at the hard edge is of interest, since this limit corresponds to the microscopic spectrum of the QCD Dirac operator close to the origin - the spectral region, where random matrices match the Dirac operator and its statistics. The equivalence of results of ensembles for zero [63, 64] and non-zero temperature [12, 13] was conjectured in [13], and one of the reasons to redo the analysis on a more refined level. Hence, we also allowed the number of zero-modes ν to be non-zero in our calculations, which only slightly changed the results. In any case, the JPDF of the chGUE(N) (and its deformation with N_f flavors) contains two Vandermonde determinants allowing the application of orthogonal polynomials to derive the correlation functions and leads to the notion of orthogonal polynomial ensembles. The addition of external sources breaks the underlying unitary bi-invariance and one Vandermonde determinant becomes a general determinant leading to the more general notion of polynomial ensembles.
We have discussed several results within the class of polynomial ensembles. We introduced the notion of invertible polynomial ensembles in chapter 3 and identified several ensembles that belong to this subclass - most notably the chGUE(N) with an external source. We

derived a multi-integral representation of the expectation value of an arbitrary ratio of characteristic polynomials for general invertible polynomial ensembles. Crucially, the number of integrations and sizes of determinants in our resulting formula - Theorem 3.9 - are both independent of N, making the result a suitable object to study in the large N limit. Before the statement of Theorem 3.9 we generalized existing results of [25, 65, 124, 132] in chapter 3. We also derived an integral representation of the correlation kernel for general invertible polynomial ensembles in Proposition 3.12. In preparation of the large N analysis of the chGUE(N) ensembles with external sources introduced in chapter 2 we also considered special cases of Theorem 3.9, leading to explizit formulae for the expectation value of $i)$ a product of M characteristic polymomials and $ii)$ a ratio of M characteristic polynomials over one single characteristic polynomial.
An open problem that we consider worthy of further investigation is wether the notion of invertibility introduced in Definition 3.3 can be extended to more involved random matrix ensembles, or even extended to ensembles on the complex plane. This may include Segal-Bargmann transforms and poses a mathematical challenge without physical application, as far as we know.

In the second part of this thesis we derived a finite N representation of the correlation kernel for two models of the chGUE(N) symmetry class, featuring external sources - the chGUEN(N) with an external source and its extension with N_f massive flavors. The external source models the effect of temperature on the spectrum of the random matrix.
Using the results of chapter 3, we were able to express the correlation kernel through expectation values of ratios of characteristic polynomials leading to multi-integral representations via Theorem 3.9. By allowing $\nu \neq 0$ zero modes in our models, the results obtained in chapter 4 generalize exisiting results of [25, 65] and [12, 13]. We also performed a saddle point analysis to obtain an expression of the correlation kernels in the large N limit, which gives us insight into the non-zero temperature regime of the random matrix ensemble and thus is also comparable to the Dirac spectrum along the temperature axis.
We found a determinantal representation of size $N_f + 1$ of the correlation kernel combining Bessel kernels and Bessel functions. This result is compared to existing results [12, 13, 63, 64] in chapter 5. It is known that for zero-temperature models like the classical chGUE(N) the order parameter of chiral symmetry breaking, the chiral condensate Σ, only enters the large N limit result of the correlation kernel as a rescaling parameter [63, 64]. For models containing the external sources we find that the order parameter Σ is replaced by a temperature depending version $\Sigma(T) = \Xi$, where Ξ is determined as the saddle point in the large N analysis. This Ξ contains the effects of the external sources in our models and crucially this is the only dependence of the limiting kernels on the external sources. Thus, we conclude that the limiting kernel we obtain is universal on the random matrix level.
In chapter 5 we compared the limiting kernels derived in chapter 4 with existing results of random matrix models without external sources [63, 64] and with external sources [12, 13]. We demonstrated how to obtain equivalent expressions for our correlation kernel that agree with the results of both [63, 64] using zero-temperature models, and [12, 13] using non-zero temperature models. Physically, the results derived in chapter 3 and 4 may also be applicable in other fields extending on the applications in the work of Fyodorov, Strahov and Grela [25]. In particular, the analysis of expectation values of ratios of characteristic polynomials may be useful in the analysis of eigenvector statistics for certain rectangular matrix ensembles, as indicated in [25].

Appendix A

Properties of Vandermonde Determinants

In this appendix we collect properties of the Vandermonde determinant, which we define in various equivalent ways. We give formulae for extending the number of variables in the Vandermonde determinant and also for reducing the number of variables. The properties presented here can also be found in [1].

Definition A.1. *The Vandermonde determinant of N pairwise distinct variables $x_1, \ldots, x_N$ is denoted by $\Delta_N(x_1, \ldots, x_N)$ and can be represented in the following equivalent ways:*

$$\begin{aligned}\Delta_N(x_1,\ldots,x_N) &= \det\left[x_j^{i-1}\right]_{i,j=1}^{N} = \prod_{1\le i<j\le N}(x_j - x_i) = \begin{vmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_N \\ \vdots & \vdots & \vdots \\ x_1^{N-1} & \ldots & x_N^{N-1} \end{vmatrix} \\ &= (-1)^{N(N-1)/2} \det\left[x_j^{N-i}\right]_{i,j=1}^{N} .\end{aligned} \tag{A.1}$$

For $N = 1$ it follows that $\Delta_1(x_1) = 1$, and we also set $\Delta_0 = 1$.

The Vandermonde determinant can be extended from N to $N + M$ variables, when we multiply by M characteristic polynomials. We call the larger Vandermonde the extended Vandermonde determinant and find the following statement:

Lemma A.2. *The following extension formula holds for a Vandermonde determinant of size N. Let the M parameters $\{z_1, \ldots, z_M\}$ be pairwise distinct. Then it holds that*

$$\prod_{m=1}^{M}\prod_{n=1}^{N}(z_m - x_n)\Delta_N(x_1,\ldots,x_N) = \frac{\Delta_{N+M}(x_1,\ldots,x_N,z_1,\ldots,z_M)}{\Delta_M(z_1,\ldots,z_M)} . \tag{A.2}$$

Proof. We perform an induction over M. Defining $z_1 \equiv x_{N+1}$, the $M = 1$ case can seen from inserting the definition of the Vandermonde, Eq. (A.1), in product form, namely

$$\begin{aligned}\prod_{n=1}^{N}(z_1 - x_n)\Delta(x_1,\ldots,x_N) &= \prod_{n=1}^{N}(x_{N+1} - x_n)\prod_{1\le i<j\le N}(x_j - x_i) = \prod_{1\le i<j\le N+1}(x_j - x_i) \\ &= \Delta_{N+1}(x_1,\ldots,x_N,z_1) .\end{aligned} \tag{A.3}$$

We now assume that Eq. (A.2) is valid for any M. The induction step $M \to M+1$ is straightforward:

$$\begin{aligned}\prod_{m=1}^{M+1}\prod_{n=1}^{N}(z_m - x_n)\Delta_N(x_1,\ldots,x_N) &= \prod_{n=1}^{N}(z_{M+1}-x_n)\frac{\Delta_{N+M}(x_1,\ldots,x_N,z_1,\ldots,z_M)}{\Delta_M(z_1,\ldots,z_M)} \\ &= \frac{\Delta_{N+M+1}(x_1,\ldots,x_N,z_1,\ldots,z_{M+1})}{\Delta_{M+1}(z_1,\ldots,z_{M+1})}\,.\end{aligned} \tag{A.4}$$

Using the induction assumption, multiplying by a factor of unity $\frac{\prod_{l=1}^{M}(z_{M+1}-z_l)}{\prod_{l=1}^{M}(z_{M+1}-z_l)}$ and using the definition from Eq. (A.1) in product form, the formula in Eq. (A.2) for $M+1$ follows. □

For extended Vandermonde determinants we can permute entries, i.e.

$$\Delta_{N+M}(x_1,\ldots,x_N,z_1,\ldots,z_M) = (-1)^{NM}\Delta_{N+M}(z_1,\ldots,z_M,x_1,\ldots,x_N)\,. \tag{A.5}$$

This is achieved by permuting rows in the determinant form in Eq. (A.1).
Next we introduce a notation for the Vandermonde determinant with a reduced number of indices. For $L \leq N$ ordered indices $l_1,\ldots,l_L$ we define the reduced Vandermonde determinant of size $N-L$ by

$$\Delta_{N-L}^{(l_1,\ldots,l_L)}(x_1,\ldots,x_N) \equiv \Delta_{N-L}(x_1,\ldots,x_{l_1-1},x_{l_1+1},\ldots,x_{l_L-1},x_{l_L+1},\ldots,x_N)\,, \tag{A.6}$$

where the parameters x_j with $j = l_1,\ldots,l_L$ are absent. From the Definition A.1 we obtain that for $L = N$ both sides are equal to unity. Furthermore the reduced Vandermonde can be obtained from the product of two Vandermonde determinants by the following Lemma.

Lemma A.3. *For $L \leq N$ the Vandermonde determinant of size $N-L$ obtained by removing the variables x_{l_j} with $1 \leq l_1 < \ldots < l_L \leq N$ from the variables $x_1,\ldots,x_N$ is given via*

$$\Delta_{N-L}^{(l_1,\ldots,l_L)}(x_1,\ldots,x_N) = \prod_{j=1}^{L}(-1)^{N-l_j}\frac{\Delta_N(x_1,\ldots,x_N)\Delta_L(x_{l_1},\ldots,x_{l_L})}{\prod_{j=1}^{L}\prod_{\substack{n=1\\n\neq l_j}}^{N}(x_{l_j}-x_n)}\,. \tag{A.7}$$

Proof. The proof is again done by induction. For $L = 1$ we have for the right hand side of Eq. (A.7)[1]

$$\begin{aligned}(-1)^{N-l_1}\frac{\Delta_N(x_1,\ldots,x_N)}{\prod_{\substack{n=1\\n\neq l_1}}^{N}(x_{l_1}-x_n)} &= \prod_{n=1}^{l_1-1}\frac{1}{(x_{l_1}-x_n)}\prod_{n=l_1+1}^{N}\frac{-1}{(x_{l_1}-x_n)}\prod_{1\leq i<j\leq N}(x_j-x_i) \\ &= \prod_{\substack{1\leq i<j\leq N\\ i,j\neq l_1}}(x_j-x_i) = \Delta_{N-1}^{(l_1)}(x_1,\ldots,x_N)\,.\end{aligned} \tag{A.8}$$

For the induction step we assume that Eq. (A.7) holds for any $N > L \geq 1$. From the definition of the reduced Vandermonde in Eq. (A.6), which can be expressed as a product,

[1] Note: Our conventions imply that empty products are equal to unity.

we see that

$$\begin{aligned}\Delta_{N-L}^{(l_1,\ldots,l_L)}(x_1,\ldots,x_N) &= (-1)^{N-l_{L+1}} \prod_{\substack{n=1\\ n\neq l_1\ldots,l_{L+1}}}^{N} (x_{l_{L+1}}-x_n)\, \Delta_{N-L-1}^{(l_1,\ldots,l_{L+1})}(x_1,\ldots,x_N) \\ &= \prod_{\substack{n=1\\ n\neq l_{L+1}}}^{N} (x_{l_{L+1}}-x_n)\, \frac{(-1)^{N-l_{L+1}}}{\prod_{j=1}^{L}(x_{l_{L+1}}-x_{l_j})}\, \Delta_{N-L-1}^{(l_1,\ldots,l_{L+1})}(x_1,\ldots,x_N)\,.\end{aligned} \tag{A.9}$$

Using the induction assumption for the left hand side and solving this equation for the reduced Vandermonde determinant of size $N-L-1$ on the right hand side, we obtain

$$\begin{aligned}\Delta_{N-(L+1)}^{(l_1,\ldots,l_{L+1})}(x_1,\ldots,x_N) &= \frac{(-1)^{N-l_{L+1}}\prod_{j=1}^{L}(x_{l_{L+1}}-x_{l_j})}{\prod_{1=n\neq l_{L+1}}^{N}(x_{l_{L+1}}-x_n)}\, \frac{\Delta_N(x_1,\ldots,x_N)\Delta_L(x_{l_1},\ldots,x_{l_L})}{\prod_{j=1}^{L}(-1)^{N-l_j}\prod_{\substack{n=1\\ n\neq l_j}}^{N}(x_{l_j}-x_n)} \\ &= \prod_{j=1}^{L+1}(-1)^{N-l_j}\frac{\Delta_N(x_1,\ldots,x_N)\Delta_{L+1}(x_{l_1},\ldots,x_{l_{L+1}})}{\prod_{j=1}^{L+1}\prod_{\substack{n=1\\ n\neq l_j}}^{N}(x_{l_j}-x_n)}\,,\end{aligned} \tag{A.10}$$

which finishes the proof. □

Rewriting Vandermonde determinants

We now demonstrate how to use orthogonal polynomials to rewrite Vandermonde determinants. This is particularly useful in chapter 2, where we introduced the random matrix models of the chGUE(N) symmetry class. The main objective is to obtain the correlation kernel for these models, who belong to the class of determinantal point processes, meaning that their correlation functions can be expressed through a determinant over a single object, which is called correlation kernel.

One of the important features of the chGUE(N) and the deformed chGUE(N) is that the JPDF contains two Vandermonde determinants. This key feature can be used to introduce orthogonal polynomials in a way presented in [135, chapter 10]. We also refer to [24, 32, 76, 84, 136, 137] for further information. The starting point is the Vandermonde determinant $\Delta_N(\lambda_1,\ldots,\lambda_N)$, which remains invariant under addition and substraction of multiples of rows and columns, which can be seen easily from Definition A.1. Hence, we can replace the monomials λ^k inside the Vandermonde determinant with monic polynomials $\pi_k(\lambda)$ - which can, but need not, be chosen orthogonal to a weight function - i.e.

$$\Delta_N(\lambda_1,\ldots,\lambda_N) = \det\left(\pi_{k-1}(\lambda_j)\right)_{k,j=1}^{N}. \tag{A.11}$$

Choosing the polynomials to satisfy orthogonality with respect to the scalar product in Eq. (2.14) allows us to compute the normalization constant Z_N with respect to the weight functions $w(\lambda)$.

Setting $\pi_n(\lambda) = p_n(\lambda)$ we find for the classical chGUE(N), or the deformed chGUE(N) [32]

$$\begin{aligned} Z_N &= \left(\prod_{n=1}^{N}\int d\lambda_n w(\lambda_n)\right)\Delta_N^2(\lambda_1,\ldots,\lambda_N) = \left(\prod_{n=1}^{N}\int d\lambda_n w(\lambda_n)\right)\left(\det\left[p_{n-1}(\lambda_k)\right]_{n,k=1}^{N}\right)^2 \\ &= \sum_{\sigma,\sigma'\in S_N}\prod_{n=1}^{N}\int d\lambda_n w(\lambda_n) p_{\sigma(n)-1}(\lambda_n) p_{\sigma'(n)-1}(\lambda_n) \\ &= \prod_{n=1}^{N} h_{n-1}\sum_{\sigma\in S_N} 1 = N!\prod_{n=1}^{N} h_{N-1}\,. \end{aligned} \tag{A.12}$$

In a first step we have applied the expansion of determinants into sums over permutations and then in a second step the orthogonality of the polynomials leading to a product over norms.

Starting again from Eq. (A.11) we can include the norms h_n by

$$\frac{\prod_{n=0}^{N-1}\sqrt{h_n}}{\prod_{n=0}^{N-1}\sqrt{h_n}}\det\left(p_{k-1}(\lambda_j)\right)_{k,j=1}^{N} = \prod_{n=0}^{N-1}\sqrt{h_n}\det\left(\frac{p_{k-1}(\lambda_j)}{\sqrt{h_{k-1}}}\right)_{k,j=1}^{N}. \tag{A.13}$$

This becomes useful, because of the following general relation for a matrix $A = (a_{nm})_{n,m=1}^{N}$:

$$(\det A)^2 = \det A\det A = \det(A^T A) = \det\left(\sum_{k=1}^{N} a_{kn}a_{km}\right). \tag{A.14}$$

Thus, we write[2]

$$\Delta_N^2(\lambda_1,\ldots,\lambda_N) = \prod_{n=0}^{N-1} h_n \det\left[\sum_{k=0}^{N-1}\frac{p_k(\lambda_n)p_k(\lambda_m)}{h_k}\right]_{m,n=1}^{N}. \tag{A.15}$$

The expression above can be used to compute the correlation kernel $K_N(\lambda,\lambda')$ via the orthogonal polynomials $p_n(\lambda)$. Additionally, the expression inside the determinant on the right-hand-side can be rewritten, such that one term of the sum remains. The resulting expression inside the determinant is given in Eq. (2.17) and is known as the Christoffel-Darboux formula.

[2] We have shifted the index: $k \to k-1$.

Appendix B

Determinantal Formulae for Characteristic Polynomials

In this appendix we give determinantal formulae for products of normal and inverse characteristic polynomials as well as special ratios of characteristic polynomials for polynomial ensembles. Some of these formulae were derived previously in the literature, see for example [25, 65, 124, 125]. We extended on some of the existing results in preparation for the principal publication [1]. New results and their proofs not included in [1] are therefore collected in this appendix. We start with the definition of polynomial ensembles from chapter 3 - see Definition 3.1.

Definition B.1. *Consider a set of real random variables $x_1, \ldots, x_N \in I$ defined by the joint probability density function (JPDF) $\mathcal{P}(x_1, \ldots, x_N)$, where $I \subset \mathbb{R}$ is an interval. The variables $x_1, \ldots, x_N$ form a polynomial ensemble [123], if their JPDF takes the form*

$$\mathcal{P}(x_1, \ldots, x_N) = \frac{1}{Z_N} \Delta_N(x_1, \ldots, x_N) \det[\varphi_l(x_k)]_{k,l=1}^N,$$

where $\Delta_N(x_1, \ldots, x_N) = \prod_{j>i}(x_j - x_i)$ is the Vandermonde determinant of N variables. The $\varphi_1, \ldots \varphi_N$ are certain integrable, real-valued functions on I, such that the normalisation constant

$$Z_N = \left(\prod_{n=1}^N \int_I dx_n \right) \Delta_N(x_1, \ldots, x_N) \det[\varphi_l(x_k)]_{k,l=1}^N = N! \det G \tag{B.1}$$

exists and is non-zero. The matrix entries $g_{k,l}$ are supposed to be finite and can be computed via

$$g_{k,l} = \int_I dx x^{k-1} \varphi_l(x), \tag{B.2}$$

such that the Gram matrix $G = (g_{i,j})_{i,j=1}^N$ is non-singular.

With this definition we can state and prove some determinantal formulae for expectation values of characteristic polynomials.

Recall the analog of the Heine formula for polynomial ensembles, see Eq. (3.4):

Proposition B.2. *Consider a general polynomial ensemble from Definition B.1, then the following formula holds:*

$$\mathbb{E}_{\mathcal{P}}\left[D_N(z)\right] = \frac{1}{\det G} \begin{vmatrix} g_{1,1} & \cdots & g_{1,N} & \eta_1(z) \\ g_{2,1} & \cdots & g_{2,N} & \eta_2(z) \\ \vdots & \vdots & \vdots & \vdots \\ g_{N+1,1} & \cdots & g_{N+1,N} & \eta_{N+1}(z) \end{vmatrix}.$$

Proof. This result was stated and proven in [65, Prop. 2] and [124, Prop. 2.5]. □

Remark. The extension to an arbitrary number of characteristic polynomials is immediate. We can look at the expectation value of M characteristic polynomials, i.e.

$$\mathbb{E}_{\mathcal{P}}\left[\prod_{m=1}^{M} D_N(z_m)\right].$$

The resulting determinantal formulae is stated (and proven) in the Corollary below.

Corollary B.3. *Consider a general polynomial ensemble from Definition B.1, then the following formula holds:*

$$\mathbb{E}_{\mathcal{P}}\left[\prod_{m=1}^{M} D_N(z_m)\right] = \frac{1}{\Delta_M(z_1,\ldots,z_M)}\frac{1}{\det G}\begin{vmatrix} g_{1,1} & \cdots & g_{1,N} & z_1^0 & \cdots & z_M^0 \\ g_{2,1} & \cdots & g_{2,N} & z_1^1 & \cdots & z_M^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{N+M,1} & \cdots & g_{N+M,N} & z_1^{N+M-1} & \cdots & z_M^{N+M-1} \end{vmatrix}$$

Proof. Starting with the definition of the expectation value we can write

$$\begin{aligned}\mathbb{E}_{\mathcal{P}}\left[\prod_{m=1}^{M} D_N(z_m)\right] &= \left(\prod_{n=1}^{N}\int_I dx_n\right)\mathcal{P}(x_1,\ldots,x_N)\prod_{m=1}^{M}\prod_{n=1}^{N}(z_m-x_n)\\ &= \frac{1}{Z_N}\left(\prod_{n=1}^{N}\int_I dx_n\right)\prod_{k=1}^{M}\prod_{n=1}^{N}(z_k-x_n)\Delta_N(x_1,\ldots,x_N)\det\left[\varphi_l(x_k)\right]_{k,l=1}^{N}.\end{aligned}$$

We extend the Vandermonde determinant via Lemma A.2 using the additional characteristic polynomials. We also apply Eq. (B.1) to rewrite Z_N. This gives us

$$\begin{aligned}\mathbb{E}_{\mathcal{P}}\left[\prod_{m=1}^{M} D_N(z_m)\right] &= \frac{1}{\Delta_M(z_1,\ldots,z_M)}\frac{1}{N!\det G}\\ &\quad\times\left(\prod_{n=1}^{N}\int_I dx_n\right)\Delta_M(x_1,\ldots,x_N,z_1,\ldots,z_M)\det\left[\varphi_l(x_k)\right]_{k,l=1}^{N}.\end{aligned}$$

In our next step we permute rows of the Vandermonde determinant such that the z_j become the first M variables, meaning

$$\begin{aligned}\Delta_{M+N}(x_1,\ldots,x_N,z_1,\ldots,z_M) &= (-1)^{NM}\Delta_{M+N}(z_1,\ldots,z_M,x_1,\ldots,x_N)\\ &= (-1)^{NM}\begin{vmatrix} 1 & z_1 & \ldots & z_1^{N+M-1} \\ \vdots & \vdots & \vdots & \\ 1 & z_M & \ldots & z_M^{N+M-1} \\ -- & -- & -- & ---- \\ 1 & x_1 & \ldots & x_1^{N+M-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & \ldots & x_N^{N+M-1} \end{vmatrix}.\end{aligned}$$

Now we need to apply the extended Andreief formula - recall Proposition 2.3.

Proposition B.4. *We consider the N-fold integration over two determinants of different sizes* $(N+k)\times(N+k)$ *and* $(N+l)\times(N+l)$ *with block structure, such that*

$$\prod_{n=1}^{N}\int_I dx_n \det\begin{bmatrix} R_{a,b} & |_{1\leq a\leq k}^{1\leq b\leq N+k} \\ \eta_b(x_a) & |_{1\leq a\leq N}^{1\leq b\leq N+k} \end{bmatrix} \det\begin{bmatrix} S_{b,a} & |_{1\leq b\leq N+l}^{1\leq a\leq l} & \varphi_b(x_a) & |_{1\leq b\leq N+l}^{1\leq a\leq N} \end{bmatrix}$$
$$= (-1)^{kl}N!\det\begin{bmatrix} 0_{k\times l} & & R_{a,b} & |_{1\leq a\leq k}^{1\leq b\leq N+k} \\ S_{b,a} & |_{1\leq b\leq N+l}^{1\leq a\leq l} & \int_I dx\varphi_b(x)\eta_a(x) & |_{1\leq b\leq N+l}^{1\leq a\leq N+k} \end{bmatrix}.$$

The functions η *and* φ *are given such that the integrals on both sides are convergent. Apart from this they can be chosen arbitrarily.*

Proof. See [111]. □

We apply the Proposition for determinants of sizes $(N+M)\times(N+M)$ and $N\times N$. Together with the definition of the G-matrix entries in Eq. (B.2) this leads to

$$\left(\prod_{n=1}^{N}\int_I dx_n\right)\Delta_{M+N}(x_1,\ldots,x_N,z_1,\ldots,z_M)\det\left[\varphi_l(x_k)\right]_{k,l=1}^{N}$$
$$= (-1)^{NM}N!\det\begin{bmatrix} 1 & z_1 & \ldots & z_1^{N+M-1} \\ \vdots & \vdots & \vdots & \\ 1 & z_M & \ldots & z_M^{N+M-1} \\ -- & -- & -- & ---- \\ g_{1,1} & g_{2,1} & \cdots & g_{N+M,1} \\ \vdots & \vdots & \vdots & \vdots \\ g_{1,N} & g_{2,N} & \cdots & g_{N+M,N} \end{bmatrix}$$
$$= N!\det\begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,N} & z_1^0 & z_2^0 & \cdots & z_M^0 \\ g_{2,1} & g_{2,2} & \cdots & g_{2,N} & z_1^1 & z_2^1 & \cdots & z_M^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{N+M,1} & g_{N+M,2} & \cdots & g_{N+M,N} & z_1^{N+M-1} & z_2^{N+M-1} & \cdots & z_M^{N+M-1} \end{bmatrix}.$$

This finishes the proof. □

We also recall the result derived for one inverse characteristic polynomial - see Eq. (3.5):

Proposition B.5. *Consider a general polynomial ensemble from Definition B.1, then the following formula holds:*

$$\mathbb{E}_{\mathcal{P}}\left[D_N^{-1}(y)\right] = \frac{1}{\det G}\begin{vmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N-1,1} & g_{N-1,2} & \cdots & g_{N-1,N} \\ \int_I du\frac{\varphi_1(u)}{y-u} & \int_I du\frac{\varphi_2(u)}{y-u} & \ldots & \int_I du\frac{\varphi_N(u)}{y-u} \end{vmatrix}. \tag{B.3}$$

Proof. This result was stated and proven in [65, Prop. 2] and [124, Prop. 2.5]. □

We can also show an equivalent determinantal formula for this particular case of one inverse characteristic polynomial. The following Proposition holds:

Proposition B.6. *Consider a general polynomial ensemble from Definition B.1, then the following formula holds:*

$$\begin{aligned}\mathbb{E}_{\mathcal{P}}\left[D_N^{-1}(y)\right] &= \frac{1}{\det G}\begin{vmatrix} g_{1,1} & \cdots & g_{1,N} \\ \vdots & \ddots & \dots \\ g_{N-1,1} & \cdots & g_{N-1,N} \\ \int_I \frac{du\varphi_1(u)}{y-u}\left(\frac{u}{y}\right)^{N-1} & \cdots & \int_I \frac{du\varphi_N(u)}{y-u}\left(\frac{u}{y}\right)^{N-1} \end{vmatrix} \\ &= \int_I \frac{du}{y-u}\left(\frac{u}{y}\right)^{N-1}\sum_{j=1}^{N} c_{N,j}\varphi_j(u)\,, \end{aligned} \tag{B.4}$$

where C is the inverse of the $N \times N$ moment matrix G, and $c_{i,j}$ are the matrix elements of C^T.

Proof. Eq. (B.4) was stated in [25] following [65, 124], without the factors of $(u/y)^{N-1}$ leading to Eq. (B.3) and Proposition B.5. The equivalence of Proposition B.6 and Proposition B.5 can be seen as follows. Expanding the geometric series inside the determinant without these factors we have

$$\begin{aligned}&\frac{1}{\det G}\begin{vmatrix} g_{1,1} & \cdots & g_{1,N} \\ \vdots & \ddots & \vdots \\ g_{N-1,1} & \cdots & g_{N-1,N} \\ \int_I du\varphi_1(u)\sum_{j=0}^{\infty}\frac{u^j}{y^{j+1}} & \cdots & \int_I du\varphi_N(u)\sum_{j=0}^{\infty}\frac{u^j}{y^{j+1}} \end{vmatrix} \\ &= \begin{vmatrix} g_{1,1} & \cdots & g_{1,N} \\ \vdots & \ddots & \vdots \\ g_{N-1,1} & \cdots & g_{N-1,N} \\ \sum_{j=N}^{\infty}\frac{g_{j,1}}{y^{j+1}} & \cdots & \sum_{j=N}^{\infty}\frac{g_{j,N}}{y^{j+1}} \end{vmatrix}\det[c_{i,j}]_{i,j=1}^{N}\,. \end{aligned}$$

If we perform the integrals in the last row we obtain an infinite series over generalised moment matrix entries $g_{k,l}$. The first $N-1$ of these can be removed by subtraction of the upper $N-1$ rows. Rewriting the last row as integrals and resumming the series we arrive at the first line of Eq. (B.4).
The second line in Eq. (B.4) is obtained as follows. Using that $\det[c_{i,j}]_{i,j=1}^{N} = 1/Z_N$ and then multiplying the matrix C with the matrix inside the determinant from the right, this leads to an identity matrix, except for the last row, as C is the inverse of the finite, $N \times N$ dimensional matrix. Laplace expanding with respect to the last column leads to the desired result. □

Remark. As for the analog of the Heine formula, the extension to an arbitrary number of inverse characteristic polynomials can be done. For L inverse characteristic polynomials the expectation value

$$\mathbb{E}_{\mathcal{P}}\left[\prod_{l=1}^{L} D_N^{-1}(y_l)\right]$$

can be evaluated. The result is stated in the following Proposition.

Proposition B.7. *Consider a general polynomial ensemble from Definition B.1, then the following formula holds:*

$$\mathbb{E}_{\mathcal{P}}\left[\prod_{l=1}^{L} D_N^{-1}(y_l)\right] = \frac{1}{\Delta_L(y_1,\dots,y_L)}\frac{1}{\det G}\begin{vmatrix} g_{1,1} & \cdots & g_{1,N} \\ g_{2,1} & \cdots & g_{2,N} \\ \vdots & \vdots & \vdots \\ g_{N-L,1} & \cdots & g_{N-L,N} \\ \psi_{1,1} & \cdots & \psi_{1,N} \\ \psi_{2,1} & \cdots & \psi_{2,N} \\ \vdots & \vdots & \vdots \\ \psi_{L,1} & \cdots & \psi_{L,N} \end{vmatrix},$$

where

$$\psi_{l,m} \equiv \int_I dx \frac{\varphi_m(x)}{y_l - x}\,.$$

Proof. We start the proof with a closer look at the JPDF of a polynomial ensemble, which we can write as follows:

$$\begin{aligned}\mathcal{P}(x_1,\dots,x_N) &= \frac{1}{Z_N}\Delta_N(x_1,\dots,x_N)\det\left[\varphi_l(x_k)\right]_{k,l=1}^{N} \\ &= \frac{1}{Z_N}\Delta_L(x_1,\dots,x_L)\Delta_{N-L}(x_{L+1},\dots,x_N)\left(\prod_{i=1}^{L}\prod_{j=L+1}^{N}(x_j-x_i)\right)\det\left[\varphi_l(x_k)\right]_{k,l=1}^{N}.\end{aligned} \tag{B.5}$$

From the above equation we can deduce two things: Firstly, the JPDF is symmetric under permutations, which will be very helpful later. Secondly, we can decompose the Vandermonde such that the first L integration variables x_l are separated from the last $N-L$ variables. This will help us to compute the L inverse characteristic polynomials. To do this, we utilise a Lemma introduced in chapter 3, namely Lemma 3.10 from [132]:

Lemma B.8. *Let L be an integer with $1 \leq L \leq N$, and let $x_1,\dots,x_N$ and $y_1,\dots,y_L$ denote two sets of parameters that are pairwise distinct. Then the following identity holds:*

$$\prod_{l=1}^{L}\frac{y_l^{N-L}}{\prod_{n=1}^{N}(y_l-x_n)} = \sum_{\sigma\in S_N/(S_{N-L}\times S_L)}\frac{\Delta_L(x_{\sigma(1)},\dots,x_{\sigma(L)})\Delta_{N-L}(x_{\sigma(L+1)},\dots,x_{\sigma(N)})\prod_{n=1}^{L}x_{\sigma(n)}^{N-L}}{\Delta_N(x_{\sigma(1)},\dots,x_{\sigma(N)})\prod_{n,l=1}^{L}(y_l-x_{\sigma(n)})(-1)^{L(L-N)}}$$

on the coset of the permutation group.

With this Lemma we can write the expectation value over L inverse characteristic polynomials as follows

$$\begin{aligned}\mathbb{E}_{\mathcal{P}}\left[\prod_{l=1}^{L}D_N^{-1}(y_l)\right] &= \left(\prod_{n=1}^{N}\int_I dx_n\right)\left[\frac{\mathcal{P}(x_1,\dots,x_N)}{\prod_{l=1}^{L}\prod_{n=1}^{N}(y_l-x_n)}\right] \\ &= \left(\prod_{n=1}^{N}\int_I dx_n\right)\mathcal{P}(x_1,\dots,x_N)\sum_{\sigma\in S_N/(S_{N-L}\times S_L)}(-1)^{L(L-N)}\left(\prod_{n=1}^{L}\frac{x_{\sigma(n)}^{N-L}}{y_n^{N-L}}\right) \\ &\quad\left(\frac{1}{\prod_{l=1}^{L}\prod_{n=1}^{L}(y_l-x_{\sigma(n)})}\right)\frac{\Delta_L(x_{\sigma(1)},\dots,x_{\sigma(L)})\Delta_{N-L}(x_{\sigma(L+1)},\dots,x_{\sigma(N)})}{\Delta_N(x_{\sigma(1)},\dots,x_{\sigma(N)})}\,.\end{aligned} \tag{B.6}$$

Since the JPDF is invariant under permutations, we can replace the sum over the permutation group $S_N/(S_{N-L} \times S_L)$ by its cardinality and drop the permutation dependence. Furthermore, we can write the Vandermonde determinants inside the sum as follows:

$$\frac{\Delta_L(x_1,\ldots,x_L)\Delta_{N-L}(x_{L+1},\ldots,x_N)}{\Delta_N(x_1,\ldots,x_N)} = \frac{1}{\prod_{i=1}^{L}\prod_{j=L+1}^{N}(x_j - x_i)} \,.$$

Using the above equation and Eq. (B.5) in Eq. (B.6) we get

$$\begin{aligned}
\mathbb{E}_{\mathcal{P}}\left[\prod_{l=1}^{L} D_N^{-1}(y_l)\right] &= \frac{N!}{(N-L)!L!}\left(\prod_{n=1}^{N}\int_I dx_n\right)(-1)^{L(L-N)} \\
&\left(\prod_{n=1}^{L}\frac{x_n^{N-L}}{y_n^{N-L}}\right)\left[\prod_{l=1}^{L}\prod_{n=1}^{L}(y_l - x_n)\right]^{-1}\left[\prod_{i=1}^{L}\prod_{j=L+1}^{N}(x_j - x_i)\right]^{-1}\det\left[\varphi_n(x_k)\right]_{k,n=1}^{N} \\
&\frac{1}{Z_N}\Delta_L(x_1,\ldots,x_L)\Delta_{N-L}(x_{L+1},\ldots,L_N)\left[\prod_{i=1}^{L}\prod_{j=L+1}^{N}(x_j - x_i)\right].
\end{aligned}$$

The above equation shows that the first L of the N variables decouple from the last $N-L$ variables. Recalling the extended Andreief integral formula given in Proposition B.4, we can rewrite the integration over the $N-L$ variables $x_{L+1},\ldots,x_N$ using the Vandermonde $\Delta(x_{L+1},\ldots,x_N)$ and the determinant over the φ-functions, $\det[\varphi_n(x_k)]$. This leads to

$$\begin{aligned}
&\left(\prod_{n=L+1}^{N}\int_I dx_n\right)\Delta_{N-L}(x_{L+1},\ldots,x_N)\det\left[\varphi_j(x_k)\right]_{k,j=1}^{N} \\
&= (N-L)!\det\begin{bmatrix}
\varphi_1(x_1) & \ldots & \varphi_N(x_1) \\
\vdots & \vdots & \vdots \\
\varphi_1(x_L) & \ldots & \varphi_N(x_L) \\
-- & -- & -- \\
g_{1,1} & \cdots & g_{1,N} \\
\vdots & \vdots & \vdots \\
g_{N-L,1} & \cdots & g_{N-L,N}
\end{bmatrix} \equiv (N-L)!\det\mathcal{Q}\,,
\end{aligned}$$

where we have used the definition of the G-matrix entries from Eq. (B.2). Together with $Z_N = N!\det G$ we obtain

$$\begin{aligned}
\mathbb{E}_{\mathcal{P}}\left[\prod_{l=1}^{L} D_N^{-1}(y_l)\right] &= \frac{(-1)^{L(L-N)}}{L!\det G}\left(\prod_{n=1}^{L}\int_I dx_n\right)\left(\prod_{n=1}^{L}\frac{x_n^{N-L}}{y_n^{N-L}}\right)\frac{\Delta_L(x_1,\ldots,x_L)}{\left[\prod_{l=1}^{L}\prod_{n=1}^{L}(y_l - x_n)\right]}\det\mathcal{Q} \\
&= \frac{(-1)^{L(L-N)}}{L!\det G}\left(\prod_{l=1}^{L} y_l^{L-N}\right)\left(\prod_{n=1}^{L}\int_I dx_n\right)\underbrace{\left[\prod_{l=1}^{L}\prod_{n=1}^{L}\frac{x_n^{N-L}}{y_l - x_n}\right]\Delta_L(x_1,\ldots,x_L)}_{=\det\left[\Phi_i(x_j)\right]_{i,j=1}^{L}}\det\mathcal{Q}\,,
\end{aligned}$$

where we have introduced the function

$$\Phi_i(x_j) \equiv \frac{x_j^{N-L} x_j^{i-1}}{\prod_{l=1}^{L}(y_l - x_j)} \,. \tag{B.7}$$

From [132] we take the identity

$$\det\left[\Phi_i(x_j)\right]_{i,j=1}^{L} = \frac{1}{\Delta_L(y_1,\ldots,x_L)} \det\left[\frac{x_i^{N-L}}{y_j - x_i}\right]_{i,j=1}^{L}, \tag{B.8}$$

which enables us to apply Andreiefs formula for the remaining L variables, after we permute the rows in the matrix $\mathcal{Q}$ to bring the L rows containing φ-functions to be the last L rows of $\mathcal{Q}$. We obtain

$$\begin{aligned}
\mathbb{E}_{\mathcal{P}}\left[\prod_{l=1}^{L} D_N^{-1}(y_l)\right] &= \frac{1}{L!\det G}\left(\prod_{l=1}^{L} y_l^{L-N}\right)\frac{1}{\Delta_L(y_1,\ldots,y_L)} \\
&\left(\prod_{n=1}^{L}\int_0^\infty dx_n\right)\det\left[\frac{x_i^{N-L}}{y_j - x_i}\right]_{i,j=1}^{L}\det\begin{bmatrix} g_{1,1} & \cdots & g_{1,N} \\ \vdots & \vdots & \vdots \\ g_{N-L,1} & \cdots & g_{N-L,N} \\ -- & -- & -- \\ \varphi_1(x_1) & \ldots & \varphi_N(x_1) \\ \vdots & \vdots & \vdots \\ \varphi_1(x_L) & \ldots & \varphi_N(x_L) \end{bmatrix} \\
&= \frac{1}{\Delta_L(y_1,\ldots,y_L)\det G}\left(\prod_{l=1}^{L} y_l^{L-N}\right)\det\begin{bmatrix} g_{1,1} & \cdots & g_{1,N} \\ \vdots & \vdots & \vdots \\ g_{N-L,1} & \cdots & g_{N-L,N} \\ -- & -- & -- \\ \widehat{\psi}_{1,1} & \ldots & \widehat{\psi}_{N,1} \\ \vdots & \vdots & \vdots \\ \widehat{\psi}_{1,L} & \ldots & \widehat{\psi}_{N,L} \end{bmatrix},
\end{aligned}$$

where we have introduced new functions

$$\widehat{\psi}_{l,m} \equiv \int_I dx \frac{x^{N-L}}{y_l - x}\varphi_m(x) \,.$$

We can rewrite the integrand of $\widehat{\psi}_{l,m}$ as follows:

$$\frac{x^{N-L}}{y_l - x}\varphi_m(x) = \left(\frac{y_l^{N-L}}{y_l - x} + \frac{x^{N-L} - y_l^{N-L}}{y_l - x}\right)\varphi_m(x) \,.$$

This allows us to write

$$\widehat{\psi}_{l,m} = y_l^{N-L}\underbrace{\int_I dx \frac{\varphi_m(x)}{y_l - x}}_{=\psi_{l,m}} + \int_I dx \frac{x^{N-L} - y_l^{N-L}}{y_l - x}\varphi_m(x) \,.$$

We can use the binomial theorem for the second term, i.e.

$$\int_I dx \frac{x^{N-L} - y_l^{N-L}}{y_l - x} \varphi_m(x) = - \sum_{k=1}^{N-L-1} y_l^{N-L-k-1} \underbrace{\int_I dx x^k \varphi_m(x)}_{=g_{k+1,m}}.$$

We can get rid of this term by subtracting the first $N-L$ rows from the determinant with suitable prefactors. The determinant is invariant under such transformations and we obtain

$$\det \begin{bmatrix} g_{1,1} & \cdots & g_{1,N} \\ \vdots & \vdots & \vdots \\ g_{N-L,1} & \cdots & g_{N-L,N} \\ -- & -- & -- \\ \widehat{\psi}_{1,1} & \cdots & \widehat{\psi}_{N,1} \\ \vdots & \vdots & \vdots \\ \widehat{\psi}_{1,L} & \cdots & \widehat{\psi}_{N,L} \end{bmatrix} = \det \begin{bmatrix} g_{1,1} & \cdots & g_{1,N} \\ \vdots & \vdots & \vdots \\ g_{N-L,1} & \cdots & g_{N-L,N} \\ -- & -- & -- \\ y_1^{N-L}\psi_{1,1} & \cdots & y_1^{N-L}\psi_{N,1} \\ \vdots & \vdots & \vdots \\ y_L^{N-L}\psi_{1,L} & \cdots & y_L^{N-L}\psi_{N,L} \end{bmatrix}.$$

Taking out the prefactors in the last L rows of the determinant yields an overall factor of

$$\left(\prod_{l=1}^{L} y_l^{N-L} \right), \tag{B.9}$$

which cancels the corresponding prefactor in the expectation value and we obtain the desired result:

$$\mathbb{E}_{\mathcal{P}} \left[\prod_{l=1}^{L} D_N^{-1}(y_l) \right] = \frac{1}{\Delta_L(y_1, \ldots, y_L)} \frac{1}{\det G} \begin{vmatrix} g_{1,1} & \cdots & g_{1,N} \\ g_{2,1} & \cdots & g_{2,N} \\ \vdots & \vdots & \vdots \\ g_{N-L,1} & \cdots & g_{N-L,N} \\ \psi_{1,1} & \cdots & \psi_{1,N} \\ \psi_{2,1} & \cdots & \psi_{2,N} \\ \vdots & \vdots & \vdots \\ \psi_{L,1} & \cdots & \psi_{L,N} \end{vmatrix}.$$

□

The next step is to consider ratios of characteristic polynomials. The simplest ratio contains one characteristic polynomials in both numerator and denominator. The Proposition below follows.

Proposition B.9. *Consider a general polynomial ensemble from Definition B.1, then the following formula holds:*

$$\mathbb{E}_{\mathcal{P}} \left[\frac{D_N(z)}{D_N(y)} \right] = \frac{(-1)}{\det G} \begin{vmatrix} g_{1,1} & \cdots & g_{1,N} & z^0 \\ g_{2,1} & \cdots & g_{2,N} & z^1 \\ \vdots & \vdots & \vdots & \vdots \\ g_{N,1} & \cdots & g_{N,N} & z^{N-1} \\ \int_I du \frac{z-u}{y-u} \varphi_1(u) & \ldots & \int_I du \frac{z-u}{y-u} \varphi_N(u) & 0 \end{vmatrix}.$$

Proof. In [25] it was stated that in principle the ratio of two characteristic polynomials is given by the following formula - see [25, Prop. 5.2]:

$$\mathbb{E}_{\mathcal{P}}\left[\frac{D_N(z)}{D_N(y)}\right] = \int_I \frac{du}{\det G}\frac{z-u}{y-u}\sum_{j=1}^{N}\varphi_j(u)\begin{vmatrix} g_{1,1} & \cdots & g_{1,i-1} & z^0 & g_{1,i+1} & \cdots & g_{1,N} \\ g_{2,1} & \cdots & g_{2,i-1} & z^1 & g_{2,i+1} & \cdots & g_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{N,1} & \cdots & g_{N,i-1} & z^{N-1} & g_{N,i+1} & \cdots & g_{N,N} \end{vmatrix}.$$

This equation can be derived by the procedure described in [65, Prop. 2]. To obtain the desired result we have to rewrite the right-hand side of the above equation by shifting the column containing the powers of z to the last column of the determinant. This leads to

$$\begin{aligned}&\frac{(-1)}{\det G}\sum_{j=1}^{N}(-1)^{N+1-j}\int_I du\frac{z-u}{y-u}\varphi_j(u)\begin{vmatrix} g_{1,1} & \cdots & g_{1,i-1} & g_{1,i+1} & \cdots & g_{1,N} & z^0 \\ g_{2,1} & \cdots & g_{2,i-1} & g_{2,i+1} & \cdots & g_{2,N} & z^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{N,1} & \cdots & g_{N,i-1} & g_{N,i+1} & \cdots & g_{N,N} & z^{N-1} \end{vmatrix} \\ &= \frac{(-1)}{\det G}\begin{vmatrix} g_{1,1} & \cdots & g_{1,N} & z^0 \\ g_{2,1} & \cdots & g_{2,N} & z^1 \\ \vdots & \vdots & \vdots & \vdots \\ g_{N,1} & \cdots & g_{N,N} & z^{N-1} \\ \int_I du\frac{z-u}{y-u}\varphi_1(u) & \cdots & \int_I du\frac{z-u}{y-u}\varphi_N(u) & 0 \end{vmatrix}.\end{aligned}$$

This concludes the proof. □

Remark. As an extension of a single ratio of characteristic polynomials, we can consider the product of M characteristic polynomials over one single characteristic polynomial, i.e.

$$\mathbb{E}_{\mathcal{P}}\left[\frac{\prod_{m=1}^{M} D_N(z_m)}{D_N(y)}\right].$$

The result is stated in the following Proposition.

Proposition B.10. *Consider a general polynomial ensemble from Definition B.1, then the following formula holds:*

$$\begin{aligned}\mathbb{E}_{\mathcal{P}}\left[\frac{\prod_{m=1}^{M} D_N(z_m)}{D_N(y)}\right] &= \frac{1}{\Delta_M(z_1,\ldots,z_M)}\frac{(-1)^M}{\det G} \\ &\times\begin{vmatrix} g_{1,1} & \cdots & g_{1,N} & z_1^0 & \cdots & z_M^0 \\ g_{2,1} & \cdots & g_{2,N} & z_1^1 & \cdots & z_M^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{N+M-1,1} & \cdots & g_{N+M-1,N} & z_1^{N+M-1} & \cdots & z_M^{N+M-1} \\ \psi_1(y) & \ldots & \psi_N(y) & 0 & \ldots & 0 \end{vmatrix},\end{aligned}$$

where we have used the notation

$$\psi_l(y) \equiv \int_I du\frac{\prod_{m=1}^{M}(z_m-u)}{y-u}\varphi_l(u)\,. \tag{B.10}$$

Proof. The following algebraic identity holds true [65]

$$\frac{1}{\prod_{n=1}^{N}(y-x_n)} = \sum_{n=1}^{N} \frac{1}{y-x_n} \prod_{\substack{p=1 \\ p\neq n}}^{N} \frac{1}{x_n - x_p} \,.$$

Recall the reduced Vandermonde determinant and its properties, in particular Lemma A.3 and Eqs. (A.6), (A.7). Then we can always write

$$\Delta_N(x_1,\ldots,x_N) = (-1)^{N-l}\Delta_{N-1}^{(l)}(x_1,\ldots,x_N) \prod_{\substack{p=1 \\ p\neq l}}^{N}(x_l - x_p) \,.$$

This leads to

$$\frac{\Delta_N(x_1,\ldots,x_N)}{\prod_{n=1}^{N}(y-x_n)} = \sum_{j=1}^{N} \frac{(-1)^{N-j}\Delta_{N-1}^{(j)}(x_1,\ldots,x_N)}{y-x_j} \,. \tag{B.11}$$

Looking at the expectation value we find

$$\begin{aligned}
\mathbb{E}_{\mathcal{P}}\left[\frac{\prod_{m=1}^{M} D_N(z_m)}{D_N(y)}\right] &= \left(\prod_{n=1}^{N}\int_I dx_n\right) \mathcal{P}(x_1,\ldots,x_N) \frac{\prod_{m=1}^{M}\prod_{n=1}^{N}(z_m - x_n)}{\prod_{j=1}^{N}(y-x_n)} \\
&= \frac{1}{Z_N}\left(\prod_{n=1}^{N}\int_I dx_n\right) \det\left[\varphi_l(x_k)\right]_{k,l=1}^{N} \frac{\Delta(x_1,\ldots,x_N)\prod_{k=1}^{M}\prod_{j=1}^{N}(z_k - x_j)}{\prod_{j=1}^{N}(y-x_j)} \,.
\end{aligned}$$

With Eq. (B.11) and using $Z_N = N!\det G$ we obtain

$$\begin{aligned}
\mathbb{E}_{\mathcal{P}}\left[\frac{\prod_{m=1}^{M} D_N(z_m)}{D_N(y)}\right] &= \frac{1}{N!\det G}\sum_{l=1}^{N}(-1)^{N-l}\left(\prod_{n=1}^{N}\int_I dx_n\right) \\
&\times \det\left[\varphi_m(x_k)\right]_{k,m=1}^{N} \frac{\Delta_{N-1}^{(l)}(x_1,\ldots,x_N)\prod_{k=1}^{M}\prod_{j=1}^{N}(z_k - x_j)}{y-x_l} \\
&= \frac{1}{\Delta_M(z_1,\ldots,z_M)N!\det G}\sum_{l=1}^{N}(-1)^{N-l}\left(\prod_{n=1}^{N}\int_I dx_n\right)\det\left[\varphi_m(x_k)\right]_{k,m=1}^{N} \\
&\times \frac{\prod_{k=1}^{M}(z_k - x_l)}{y-x_l}\Delta_{N-1}^{(l)}(x_1,\ldots,x_N,z_1,\ldots,z_M) \\
&= \frac{(-1)^{(N-1)M}}{\Delta_M(z_1,\ldots,z_M)N!\det G}\sum_{l=1}^{N}(-1)^{N-l}\left(\prod_{\substack{n=1 \\ n\neq l}}^{N}\int_I dx_n\right)\Delta_{N-1}^{(l)}(z_1,\ldots,z_M,x_1,\ldots,x_N) \\
&\times \int_I dx_l \,\frac{\prod_{k=1}^{M}(z_k - x_l)}{y-x_l}\det\left[\varphi_m(x_k)\right]_{k,m=1}^{N} \,.
\end{aligned}$$

The reduced Vandermonde determinant can be rewritten such that the z_k variables are the first M variables, leading to a prefactor $(-1)^{(N-1)M}$. Then we pull the integration over x_l as well as the factors $\prod_{k=1}^{M}(z_k - x_l)$ and $\frac{1}{(y-x_l)}$ into the l-th row of the φ-determinant, which contains the x_l variable. Additionally, we identify the entries in that row with the functions

$\psi_l(y)$ via Eq. (B.10). Consequently, we obtain a determinant of the form

$$\det\begin{pmatrix} \varphi_1(x_1) & \dots & \varphi_N(x_1) \\ \vdots & \vdots & \vdots \\ \varphi_1(x_{l-1}) & \dots & \varphi_N(x_{l-1}) \\ \psi_1(y) & \dots & \psi_N(y) \\ \varphi_1(x_{l+1}) & \dots & \varphi_N(x_{l+1}) \\ \vdots & \vdots & \vdots \\ \varphi_1(x_N) & \dots & \varphi_N(x_N) \end{pmatrix} = (-1)^{l-1}\det\begin{pmatrix} \psi_1(y) & \dots & \psi_N(y) \\ \varphi_1(x_1) & \dots & \varphi_N(x_1) \\ \vdots & \vdots & \vdots \\ \varphi_1(x_{l-1}) & \dots & \varphi_N(x_{l-1}) \\ \varphi_1(x_{l+1}) & \dots & \varphi_N(x_{l+1}) \\ \vdots & \vdots & \vdots \\ \varphi_1(x_N) & \dots & \varphi_N(x_N) \end{pmatrix}$$

$$= (-1)^{l-1}\det\begin{pmatrix} \psi_1(y) & \varphi_1(x_1) & \dots & \varphi_1(x_{l-1}) & \varphi_1(x_{l-1}) & \dots & \varphi_1(x_N) \\ \psi_2(y) & \varphi_2(x_1) & \dots & \varphi_2(x_{l-1}) & \varphi_2(x_{l-1}) & \dots & \varphi_2(x_N) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_N(y) & \varphi_N(x_1) & \dots & \varphi_N(x_{l-1}) & \varphi_N(x_{l-1}) & \dots & \varphi_N(x_N) \end{pmatrix}.$$

Applying Andreief's integration formula - Proposition B.4 - for the remaining $N-1$ integrations over the remaining x variables, we combine the two determinants and make use of the definition of the G-matrix entries. We find

$$\frac{(-1)^{(N-1)M}}{\Delta_M(z_1,\dots,z_M)N!\det G}\sum_{l=1}^{N}(-1)^{N-l}\left(\prod_{\substack{j=1\\ j\neq l}}^{N}\int_0^\infty dx_j\right)\Delta^{(l)}_{N+M-1}(z_1,\dots,z_M,x_1,\dots,x_N)$$

$$(-1)^{l-1}\det\begin{pmatrix} \psi_1(y) & \varphi_1(x_1) & \dots & \varphi_1(x_{l-1}) & \varphi_1(x_{l-1}) & \dots & \varphi_1(x_N) \\ \psi_2(y) & \varphi_2(x_1) & \dots & \varphi_2(x_{l-1}) & \varphi_2(x_{l-1}) & \dots & \varphi_2(x_N) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_N(y) & \varphi_N(x_1) & \dots & \varphi_N(x_{l-1}) & \varphi_N(x_{l-1}) & \dots & \varphi_N(x_N) \end{pmatrix}$$

$$= \frac{(-1)^{(N-1)M}(-1)^{N-1}}{\Delta_M(z_1,\dots,z_M)N!\det G}\sum_{l=1}^{N}(N-1)!(-1)^M\det\begin{pmatrix} 0 & z_1^0 & z_1^1 & \dots & z_1^{N+M-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & z_M^0 & z_M^1 & \dots & z_M^{N+M-1} \\ \psi_1(y) & g_{1,1} & g_{2,1} & \dots & g_{N+M-1,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_N(y) & g_{1,N} & g_{2,N} & \dots & g_{N+M-1,N} \end{pmatrix}$$

$$= \frac{(-1)^M(-1)^{(N-1)(M+1)}}{\Delta_M(z_1,\dots,z_M)\det G}\det\begin{pmatrix} z_1^0 & \dots & z_M^0 & g_{1,1} & \dots & g_{1,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1^{N+M-1} & \dots & z_M^{N+M-1} & g_{N+M-1,1} & \dots & g_{N+M-1,N} \\ 0 & \dots & 0 & \psi_1(y) & \dots & \psi_N(y) \end{pmatrix}.$$

In the last step we have transposed the matrix, then switched the first row to last and used the fact, that the summation over l gives an additional factor of N.
The next step is to rearrange the columns again, bringing the columns containing the G-matrix entries to the front, which yields a factor of $(-1)^{NM}$. Managing all the powers of (-1) yields

$$(-1)^{NM}(-1)^{(N-1)(M+1)} = (-1)^{NM}(-1)^{2M}(-1)^{-NM} = 1.$$

Overall we find

$$\mathbb{E}_{\mathcal{P}}\left[\frac{\prod_{m=1}^{M} D_N(z_m)}{D_N(y)}\right] = \frac{(-1)^M}{\Delta_M(z_1,\ldots,z_M)\det G} \times \det\begin{pmatrix} g_{1,1} & \cdots & g_{1,N} & z_1^0 & \cdots & z_M^0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{N+M-1,1} & \cdots & g_{N+M-1,N} & z_1^{N+M-1} & \cdots & z_M^{N+M-1} \\ \psi_1(y) & \cdots & \psi_N(y) & 0 & \cdots & 0 \end{pmatrix},$$

which concludes the proof. □

Alternative derivation of the correlation kernel for $N_f = 0$

In this section we want to derive the correlation kernel of the chGUE(N) with an external source for finite N using an alternative approach to chapter 4. We know from chapter 3, that the chGUE(N) with an external source forms an invertible polynomial ensemble, which allows us to use the Theorem 3.9 and also Proposition 3.12. In chapter 4 we used Proposition 3.12 directly via Eq. (4.3). Alternatively, we can apply Theorem 3.9, in the case $M = L = 1$, leading to:[1]

$$\mathbb{E}_{\mathcal{P}}\left[\frac{D_N(x_1)}{D_N(y)}\right] = \int_{I'} ds F(s,x_1) \prod_{n=1}^{N}(s-a_n) \int_I dv \left(\frac{v}{y}\right)^{N-1} \frac{x_1 - v}{y - v} \times \left[\oint_C \frac{du}{2\pi i} \frac{1}{\prod_{n=1}^{N}(u-a_n)} \frac{\varphi(u,v)}{s-u}\right].$$

This expression is useful to derive the kernel, because of Eq. (3.7), i.e.

$$K_N(x_1,x_2) = \frac{1}{x_1 - x_2} \operatorname*{Res}_{y=x_2}\left(\mathbb{E}_{\mathcal{P}}\left[\frac{D_N(x_1)}{D_N(y)}\right]\right), \tag{B.12}$$

where the residue is defined via - see Eq. (3.6):

$$f(x) \equiv \operatorname*{Res}_{z=x} \int_I dt \frac{f(t)}{z-t}. \tag{B.13}$$

Looking at the chGUE(N) with an external source, we find for general N_f the φ-function

$$\varphi(a_l,x) = \left(\frac{x}{a_l}\right)^{\frac{\nu}{2}} e^{-(x+a_l)} I_\nu\left(2\sqrt{a_l x}\right) \quad \forall l = 1,\ldots,N, \tag{B.14}$$

which was also given in Eq. (4.3). Additionally, we have

$$I = \mathbb{R}_+, \quad I' = \mathbb{R}_-, \quad \text{and} \quad F(s,x_1) = (-1)^\nu \left(\frac{s}{x_1}\right)^{\nu/2} e^{s+x_1} I_\nu\left(2\sqrt{x_1 s}\right), \tag{B.15}$$

[1] The equation has to be compared to Eq. (3.13) for $M = L = 1$

which allows us to write

$$\begin{aligned}\mathbb{E}_{\mathcal{P}}\left[\frac{D_N(x_1)}{D_N(y)}\right] = x_1^{-\nu/2}e^{x_1}\int_0^\infty ds(-1)^\nu(-1)^{\nu/2}s^{\nu/2}e^{-s}\underbrace{I_\nu(2i\sqrt{x_1 s})}_{=(-1)^{\nu/2}J_\nu(2\sqrt{x_1 s})}(-1)^{N+1}\prod_{n=1}^N(s+a_n)\\ \times\int_0^\infty dv\left(\frac{v}{y}\right)^{N-1}\frac{(x_1-v)}{(y-v)}\oint_C\frac{du}{2\pi i}\frac{u^{-\nu/2}e^{-u}}{\prod_{n=1}^N(u-a_n)}\frac{v^{\nu/2}e^{-v}I_\nu(2\sqrt{uv})}{s+u}\\ =(-1)x_1^{-\nu/2}e^{x_1}\int_0^\infty dtt^{\nu/2}e^{-t}J_\nu(2\sqrt{x_1 t})\prod_{n=1}^N(t+a_n)\\ \times\int_0^\infty dx\left(\frac{x}{y}\right)^{N-1}\frac{(x_1-x)}{(y-x)}\oint_C\frac{du}{2\pi i}\frac{u^{-\nu/2}e^{-u}}{\prod_{n=1}^N(a_n-u)}\frac{x^{\nu/2}e^{-x}I_\nu(2\sqrt{ux})}{t+u},\end{aligned}$$

with a contour C, which encircles the points $a_1,\ldots,a_N$ counter-clockwise and leaves the real number $-t$ outside.
Now we use the following identity:

$$\left(\frac{x}{y}\right)^{N-1}\frac{1}{y-x} = \frac{1}{y-x} - \sum_{k=0}^{N-2}\frac{x^k}{y^{k+1}}.$$

With this identity we can write

$$\begin{aligned}\mathbb{E}_{\mathcal{P}}\left[\frac{D_N(x_1)}{D_N(y)}\right] =&(-1)x_1^{-\nu/2}e^{x_1}\int_0^\infty dss^{\nu/2}e^{-s}J_\nu(2\sqrt{x_1 s})\prod_{n=1}^N(s+a_n)\\ &\times\int_0^\infty dv\left(\frac{v}{y}\right)^{N-1}\frac{(x_1-v)}{(y-v)}\oint_C\frac{du}{2\pi i}\frac{u^{-\nu/2}e^{-u}}{\prod_{n=1}^N(a_n-u)}\frac{v^{\nu/2}e^{-v}I_\nu(2\sqrt{uv})}{s+u}\\ =&x_1^{-\nu/2}e^{x_1}\int_0^\infty dss^{\nu/2}e^{-s}J_\nu(2\sqrt{x_1 s})\prod_{n=1}^N(s+a_n)\\ &\times\frac{(-1)}{2\pi i}\oint_C du\frac{u^{-\nu/2}e^{-u}}{(s+u)\prod_{n=1}^N(a_n-u)}\left[\int_0^\infty dv\frac{v^{\nu/2}e^{-v}}{y-v}(x_1-v)I_\nu(2\sqrt{uv})\right.\\ &\left.-\sum_{k=0}^{N-2}\int_0^\infty dv\frac{v^{k+\nu/2}e^{-v}}{y^{k+1}}(x_1-v)I_\nu(2\sqrt{uv})\right].\end{aligned}\tag{B.16}$$

The kernel is then obtained via Eq. (B.12) by applying the residue. The residue is taken with respect to the variable y, which is coupled only to the v-integration. Since the residue is per definition a linear operation, we can permute it with all the other integrations. Therefore, we need to evaluate only the following two expressions:

$$\begin{aligned}&\operatorname*{Res}_{y=x_2}\left(\int_0^\infty dv\frac{v^{\nu/2}e^{-v}}{y-v}(x_1-v)I_\nu(2\sqrt{uv})\right),\\ &\operatorname*{Res}_{y=x_2}\left(\int_0^\infty dv\frac{x^{k+\nu/2}e^{-v}}{y^{k+1}}(x_1-v)I_\nu(2\sqrt{uv})\right) = \int_0^\infty dvv^{k+\nu/2}e^{-v}(x_1-v)I_\nu(2\sqrt{uv})\operatorname*{Res}_{y=x_2}\left(\frac{1}{y^{k+1}}\right).\end{aligned}$$

The first expression can be evaluated with Eq. (B.13) to be

$$\operatorname*{Res}_{y=x_2}\left(\int_0^\infty dv \frac{v^{\nu/2}e^{-v}}{y-v}(x_1-v)I_\nu(2\sqrt{uv})\right) = (x_1-x_2)x_2^{\nu/2}e^{-x_2}I_\nu(2\sqrt{ux_2})\,.$$

The second expression vanishes for all $y \neq 0$, since the residue of y^{-k-1} is zero. For $y = 0$ we have a pol of order $k+1$, which leads to the formula

$$\operatorname*{Res}_{y=x_2=0}\left(\frac{1}{y^{k+1}}\right) = \lim_{y\to 0}\frac{1}{k!}\frac{\partial^k}{\partial y^k}y^{k+1}\frac{1}{y^{k+1}} = 0\,.$$

This implies that the second expression is always equal to zero. Hence, the terms in the sum in Eq. (B.16) all vanish when we apply the residue. Thus, we are left with

$$\begin{aligned}
K_N^{(0)}(x_1,x_2) &= \frac{1}{x_1-x_2}\frac{(-1)x_1^{-\nu/2}e^{x_1}}{2\pi i}\int_0^\infty ds s^{\nu/2}e^{-s}J_\nu(2\sqrt{sx_1})\prod_{n=1}^N(s+a_n)\\
&\times \oint_C du \frac{u^{-\nu/2}e^{-u}}{(s+u)\prod_{n=1}^N(a_n-u)}\operatorname*{Res}_{y=x_2}\left(\int_0^\infty dx\frac{x_1-x}{y-x}x^{\nu/2}e^{-x}I_\nu(2\sqrt{ux})\right)\\
&= \frac{1}{x_1-x_2}\frac{(-1)x_1^{-\nu/2}e^{x_1}}{2\pi i}\int_0^\infty ds s^{\nu/2}e^{-s}J_\nu(2\sqrt{sx_1})\prod_{n=1}^N(s+a_n)\\
&\times \oint_C du \frac{u^{-\nu/2}e^{-u}}{(s+u)\prod_{n=1}^N(a_n-u)}(x_1-x_2)x_2^{\nu/2}e^{-x_2}I_\nu(2\sqrt{ux_2})\\
&= \frac{-1}{2\pi i}\left(\frac{x_2}{x_1}\right)^{\nu/2}e^{x_1-x_2}\int_0^\infty ds s^{\nu/2}e^{-s}J_\nu\left(2\sqrt{sx_1}\right)\prod_{n=1}^N(s+a_n)\\
&\times \oint_C du \frac{u^{-\nu/2}e^{-u}}{(s+u)\prod_{n=1}^N(a_n-u)}I_\nu(2\sqrt{ux_2})\,.
\end{aligned}$$

Omitting the prefactors

$$\left(\frac{x_2}{x_1}\right)^{\nu/2}e^{x_1-x_2}\,,$$

we obtain the same result as in chapter 4. This is possible, because the kernel is invariant under such transformations.

Appendix C

Derivation of Joint Probability Density Functions

In this appendix we provide additional information on how to derive the joint probability density functions (JPDF) for the random matrix models introduced in chapter 2. In particular, we consider the classical chGUE(N) [2] and the deformed chGUE(N) with N_f massive flavors [9], see also [135, chapter 13] and [138] for an overview. We also consider the chGUE(N) with an external source [67–71] and its extension with N_f massive flavors [10–13], see also [139–141] for additional information on external source models. We have introduced all four matrix models via the matrix-valued density $P(W)$, the partition function Z_N and the measure dW. For the simplest model, the classical chGUE(N) we had in Eq. (2.2):

$$Z_N^{\text{chGUE}} = \int P_{\text{chGUE}}(W)dW, \quad \text{where} \quad P_{\text{chGUE}}(W) = \frac{1}{\mathcal{N}_0} \exp\left(-\operatorname{Tr} WW^\dagger\right). \tag{C.1}$$

For the deformed chGUE(N) the matrix-valued density $P(W)$ was given in Eq. (2.4):

$$\begin{aligned} P_{\text{chGUE}}^{\text{deformed}}(W) &= \frac{1}{\mathcal{N}_{\text{def}}} \prod_{f=1}^{N_f} \det(D + m_f) \, \exp\left(-\operatorname{Tr} WW^\dagger\right) \\ &= \frac{1}{\mathcal{N}_{\text{def}}} \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f \mathbb{1}_N & iW \\ iW^\dagger & m_f \mathbb{1}_{N+\nu} \end{pmatrix} \exp\left(-\operatorname{Tr} WW^\dagger\right). \end{aligned} \tag{C.2}$$

For the models with external sources T we also stated the densities $P(W)$ in chapter 2. For the chGUE(N) with an external source we had (compare Eq. (2.3)):

$$P_{\text{chGUE}}^{\text{ext}}(W) = \frac{1}{\mathcal{N}_{\text{ext}}} \exp\left(-\operatorname{Tr}\,(W - T)(W^\dagger - T^\dagger)\right). \tag{C.3}$$

The extension of the chGUE(N) with an external source by N_f massive flavors has a density of the form:

$$\begin{aligned} P_{\text{chGUE}}^{\text{temp}}(W) &= \frac{1}{\mathcal{N}} \prod_{f=1}^{N_f} \det(D_{\text{temp}} + m_f) \, \exp\left(-\operatorname{Tr} WW^\dagger\right) \\ &= \frac{1}{\mathcal{N}} \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f \mathbb{1}_N & i(W + T) \\ i(W^\dagger + T^\dagger) & m_f \mathbb{1}_{N+\nu} \end{pmatrix} \exp\left(-\operatorname{Tr} WW^\dagger\right), \end{aligned} \tag{C.4}$$

which was given in Eq. (2.5).

The matrix-valued function $P(W)$ for the chGUE(N) is supposed to be a density, since the matrices are random variables. This means that we have to meet the requirement of normalization to some constant with respect to the measure dW. This normalization is further specified by the fact that $P(W)$ is a probability density. Thus, the combination $P(W)dW$ has to form a probability measure, i.e. it has to be normalized to unity under integration

$$1 = \int dW P(W) \, . \tag{C.5}$$

The fact that both $WW^\dagger$ and W in our model are random variables carries over to their entries and eigenvalues. Hence, normalization with respect to the measure dW has to also carry over to the measure over the squared singular values of W. The question is: How can we ensure that this connection is well-defined, when the measure decomposes as we have seen in Eq. (2.6)? We get

$$dW = \prod_{n=1}^{N} \prod_{m=1}^{N+\nu} d\mathrm{Re}W_{nm} d\mathrm{Im}W_{nm} \, ,$$

which implies that the decomposition contains parts depending solely on the eigenvectors, and parts depending solely on the singular values. Any observable $\mathcal{O}(W)$ in our model that we impose on the matrix-variable-level would have to be checked wether and how it decomposes in terms of the singular value decomposition. We need this information in order to determine how $\mathcal{O}(W)$ affects the pure eigenvalue statistics and how it affects the eigenvector statistics. This is a general problem in Random Matrix Theory and can only be solved for explicitly given observables.
In this thesis we are focusing on the eigenvalue statistics of the Dirac operator. This means we consider only observables $\mathcal{O}(W)$ that are purely depending on the non-zero eigenvalues of $WW^\dagger$ and **not** on the eigenvectors. That implies that we impose an invariance with respect to unitary matrices U, V, i.e. $\mathcal{O}(W) = \mathcal{O}(UWV)$. Any observable or ensemble that fullfills this invariant is called *unitary bi-invariant*. Consequently, we assume that $\mathcal{O}(W) = \mathcal{O}(\lambda_1, \ldots, \lambda_N)$ can be ensured by the existence of a suitable JPDF of the squared sinuglar values of W. This assumption can be quantified by introducing the expectation value of the observable $\mathcal{O}(W)$:

$$\mathbb{E}_P \left[\mathcal{O}(W)\right] = \int dW P(W) \mathcal{O}(W) = \left(\prod_{n=1}^{N} \int d\lambda_n \right) \mathcal{P}(\lambda_1, \ldots, \lambda_N) \mathcal{O}(\lambda_1, \ldots, \lambda_N) \, , \tag{C.6}$$

where the quantity $\mathcal{P}(\lambda_1, \ldots, \lambda_N)$ is the required JPDF of the squared singular values. The integration over the squared singular values λ_n, $n = 1, \ldots, N$, is over the positive real halfline $\mathbb{R}_+$, since the eigenvalues of $WW^\dagger$ are supposed to be real and positive. The normalization in Eq. (C.5) then leads to the normalization property of the JPDF

$$1 = \left(\prod_{n=1}^{N} \int d\lambda_n \right) \mathcal{P}(\lambda_1, \ldots, \lambda_N) \, .$$

From Eq. (C.6) we see that unitary bi-invariance of $\mathcal{O}(W)$ is required to ensure that the JPDF indeed only depends on the squared singular values. If the invariance is broken by $P(W)$, which is the case when an external source T is introduced to the model, then in general the preferred eigenspaces of $\mathcal{O}(W)$ would impact the JPDF and also the pure eigenvalue statistics of the underlying physical system. There are some instances however, where we can still find a JPDF of squared singular values, despite the breaking of unitary bi-invariance. This

becomes possible by performing the integration with respect to the Haar-measures $d\mu(U)$ and $d\mu(V)$ explicitly and thus, integrating out the dependence on the eigenvectors. We will showcase this for the models considered in the Eqs. (C.1), (C.3), (C.2) and (C.4).

The Jacobian originating from the change of measure was already given in Eq. (2.7) and reads

$$dW = c \left(\prod_{n=1}^{N} \lambda_n^{\nu} \right) |\Delta_N(\lambda_1, \ldots, \lambda_N)|^2 \left(\prod_{n=1}^{N} d\lambda_n \right) d\mu(U_W) d\mu(V_W) \, .$$

The Jacobian has to be included in the JPDF $\mathcal{P}(\lambda_1, \ldots, \lambda_N)$. This also applies to the integration over the Haar measures $d\mu(U)$, $d\mu(V)$. For the simplest model, the classical chGUE(N), with matrix-valued density $P(W)$ given in Eq. (C.1), the unitary bi-invariance is unbroken and we obtain

$$\begin{aligned} P(W)dW &= P(U_W W_D V_W^{\dagger}) c \Delta_N^2(\lambda_1, \ldots \lambda_N) \left(\prod_{n=1}^{N} \lambda_n^{\nu} \right) d\mu(U_W) d\mu(V_W^{\dagger}) \left(\prod_{n=1}^{N} d\lambda_n \right) \\ &= \frac{c}{\mathcal{N}_0} \Delta_N^2(\lambda_1, \ldots \lambda_N) \left(\prod_{n=1}^{N} \lambda_n^{\nu} \right) d\mu(U_W) d\mu(V_W^{\dagger}) \\ &\times \exp\left(-\mathrm{Tr}\, (U_W W_D V_W^{\dagger})(U_W W_D V_W^{\dagger})^{\dagger} \right) \left(\prod_{n=1}^{N} d\lambda_n \right) \\ &= \mathcal{P}_{\mathrm{chGUE}}(\lambda_1, \ldots, \lambda_N) \left(\prod_{n=1}^{N} d\lambda_n \right) . \end{aligned}$$

Taking a closer look at the exponential term, and using the invariance of the trace under cyclic permutation, we can write

$$\begin{aligned} \exp\left(-\mathrm{Tr}\, (U_W W_D V_W^{\dagger})(U_W W_D V_W^{\dagger})^{\dagger} \right) &= \exp\left(-\mathrm{Tr}\, W_D W_D^{\dagger} \right) = \exp\left(-\mathrm{Tr}\, \Lambda_W^2 \right) \\ &= \exp\left(-\sum_{n=1}^{N} w_n^2 \right) = \prod_{n=1}^{N} e^{-\lambda_n} . \end{aligned}$$

Note that the dependence on U_W and $V_W^{\dagger}$ has dropped out completely. Hence, the integration over the Haar measure yields an integral $I(U, V) = c'$, where c' is a constant, which does not depend on the singular values. Therefore, we can write the JPDF for the classical chGUE(N) as

$$\mathcal{P}_{\mathrm{chGUE}}(\lambda_1, \ldots, \lambda_N) = \frac{c \cdot c'}{\mathcal{N}_0} \Delta_N^2(\lambda_1, \ldots \lambda_N) \left(\prod_{n=1}^{N} \lambda_n^{\nu} e^{-\lambda_n} \right) .$$

From a physical standpoint the normalization is realized in terms of a partition function $Z_N^{(N_f, T)}$, which depends on the deformation by dynamical, massive flavors N_f and the influence of temperature T. This can be reconciled with the probabilistic approach of Random Matrix Theory by choosing the constant $\mathcal{N}_0$ to be $\frac{Z_N^{(0,0)}}{cc'}$ for the classical chGUE(N), since this model does not contain temperature ($T = 0$) and has no flavors ($N_f = 0$). The partition

function $Z_N^{(0,0)}$ is written as the integral over the unnormalized JPDF we obtained above, i.e.

$$Z_N^{(0,0)} = \left(\prod_{n=1}^{N}\int d\lambda_n\right)\Delta_N^2(\lambda_1,\ldots,\lambda_N)\left(\prod_{n=1}^{N}\lambda_n^\nu e^{-\lambda_n}\right) .$$

The final form of the JPDF of the chGUE(N) then reads

$$\mathcal{P}_{\text{chGUE}}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N^{(0,0)}}\Delta_N^2(\lambda_1,\ldots\lambda_N)\left(\prod_{n=1}^{N}\lambda_n^\nu e^{-\lambda_n}\right) .$$

The deformed chGUE(N), with density $P(W)$ given in Eq. (C.2), is another example, where the decomposition into squared singular values does not break the unitary bi-invariance. By comparing Eqs. (C.1) and (C.2), we find that the deformation with N_f fermion determinants is invariant with respect to the singular value decomposition, due to

$$\begin{aligned}
\prod_{f=1}^{N_f}\det(D+m_f\mathbb{1}_{2N+\nu}) &= \prod_{f=1}^{N_f}\det(m_f\mathbb{1}_{N+\nu})\det(m_f\mathbb{1}_N + m_f^{-1}WW^\dagger)\\
&= \left(\prod_{f=1}^{N_f}m_f^\nu\right)\left(\prod_{f=1}^{N_f}\det(m_f^2\mathbb{1}_n + (U_W W_D V_W^\dagger)(U_W W_D V_W^\dagger)^\dagger)\right)\\
&= \left(\prod_{f=1}^{N_f}m_f^\nu\right)\left(\prod_{f=1}^{N_f}\det(m_f^2\mathbb{1}_n + W_D W_D^\dagger)\right)\\
&= \left(\prod_{f=1}^{N_f}m_f^\nu\right)\left(\prod_{f=1}^{N_f}\prod_{n=1}^{N}(m_f^2+\lambda_n)\right) .
\end{aligned} \tag{C.7}$$

We choose the constant $\mathcal{N}_{\text{def}}$ as follows:

$$\mathcal{N}_{\text{def}} = \frac{Z_N^{(N_f,0)}}{cc'}\left(\prod_{f=1}^{N_f}m_f^\nu\right) .$$

This leads us to

$$\begin{aligned}
\mathcal{P}_{\text{chGUE}}^{\text{deformed}}(\lambda_1,\ldots,\lambda_N) &= \frac{1}{Z_N^{(N_f,0)}}\Delta_N^2(\lambda_1,\ldots\lambda_N)\left(\prod_{n=1}^{N}\lambda_n^\nu e^{-\lambda_n}\prod_{f=1}^{N_f}(m_f^2+\lambda_n)\right)\\
&= \frac{Z_N^{(0)}}{Z_N^{(N_f)}}\left(\prod_{f=1}^{N_f}\prod_{n=1}^{N}(m_f^2+\lambda_n)\right)\mathcal{P}_{\text{chGUE}}(\lambda_1,\ldots,\lambda_N) .
\end{aligned}$$

The parition function $Z_N^{(N_f,0)}$ does not depend on temperature ($T=0$) but on the N_f massive flavors ($N_f \neq 0$). The partition function is of the form

$$Z_N^{(N_f,0)} = \left(\prod_{n=1}^{N}\int d\lambda_n\right)\Delta_N^2(\lambda_1,\ldots,\lambda_N)\left(\prod_{n=1}^{N}\lambda_n^\nu e^{-\lambda_n}\right)\left(\prod_{f=1}^{N_f}\prod_{n=1}^{N}(m_f^2+\lambda_n)\right) .$$

The complete factorisation of the eigenvectors and singular values, as we have seen in the

two ensembles above, is also known for other ensembles such as the GOE(N), GUE(N) and GSE(N). These three classical ensembles where studied in this sense by Hua [75]. Further information can also be found in classical literature, for example [23, 24, 75].
One important consequence of the factorisation is that methods of orthogonal polynomials become applicable to the eigenvalue (and singular value) statistics. We discussed some important features and results from this field in section 2.3.

Adding external sources, like T in the models of Eqs. (C.3) and (C.4), breaks the unitary bi-invariance, i.e. $P(W) \neq P(UWV^\dagger)$. We obtain group integrals with respect to the Haar measures that are a priori non-trivial and not necessarily constant with respect to the singular values. For example, the density given for the classical chGUE with an external source, Eq. (C.3), becomes

$$\begin{aligned}
P(W)dW &= P(U_W W_D V_W^\dagger)\, c\, \Delta_N^2(\lambda_1, \dots \lambda_N) \left(\prod_{n=1}^N \lambda_n^\nu\right) d\mu(U_W) d\mu(V_W^\dagger) \left(\prod_{n=1}^N d\lambda_n\right) \\
&= \frac{c}{\mathcal{N}_{\text{ext}}} \Delta_N^2(\lambda_1, \dots \lambda_N) \left(\prod_{n=1}^N \lambda_n^\nu\right) d\mu(U_W) d\mu(V_W^\dagger) \\
&\times \exp\Big(-\text{Tr}\,(W-T)(W^\dagger - T^\dagger)\Big) \left(\prod_{n=1}^N d\lambda_n\right) \\
&= \frac{c}{\mathcal{N}_{\text{ext}}} \Delta_N^2(\lambda_1, \dots \lambda_N) \left(\prod_{n=1}^N \lambda_n^\nu\right) \exp\Big(-\text{Tr}\, WW^\dagger\Big) \exp\Big(-\text{Tr}\, TT^\dagger\Big) \\
&\times \exp\Big(\text{Tr}\,(WT^\dagger + TW^\dagger)\Big)\, d\mu(U_W) d\mu(V_W^\dagger) \left(\prod_{n=1}^N d\lambda_n\right) .
\end{aligned}$$

The problem is how to proceed with the exponential term in the last line above, especially the mixed term in the exponential containing $WT^\dagger + TW^\dagger$. The solution is to do a singular value composition for W as before, and another for T at the same time, namely

$$T = \frac{1}{2} U_T T_D V_T^\dagger, \text{ where } T_D = \begin{pmatrix} \Lambda_T & 0_{N\times\nu} \end{pmatrix} \text{ and } \Lambda_T = \begin{pmatrix} t_1 & 0 & 0 & \dots & 0 \\ 0 & t_2 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & & 0 \\ 0 & 0 & \dots & 0 & t_N \end{pmatrix} .$$

This allows us to write the mixed term as

$$\begin{aligned}
\exp\Big(\text{Tr}\,(WT^\dagger + TW^\dagger)\Big) &= \exp\left(\frac{1}{2}\text{Tr}\,(U_W W_D V_W^\dagger V_T T_D^\dagger U_T^\dagger + U_T T_D V_T^\dagger V_W W_D^\dagger U_W^\dagger)\right) \\
&= \exp\left(\frac{1}{2}\text{Tr}\,(U_T^\dagger U_W W_D V^\dagger T_D^\dagger + U_W^\dagger U_T T_D V W_D^\dagger)\right) \\
&= \exp\left(\frac{1}{2}\text{Tr}\,(U W_D V^\dagger T_D^\dagger + U^\dagger T_D V W_D^\dagger)\right) \\
&= \exp\left(\frac{1}{2}\text{Tr}\,(U W_D V^\dagger T_D^\dagger + (U W_D V^\dagger T_D^\dagger)^\dagger)\right) \\
&= \exp\Big(\text{ReTr}\,\Big(U W_D V^\dagger T_D^\dagger\Big)\Big) .
\end{aligned}$$

In the last line we have used that $2\mathrm{Re}\, z = z + \bar{z}$ holds for any complex number z and that $\mathrm{Tr}\, A^T = \mathrm{Tr}\, A$ holds for any square matrix A. Furthermore, we can make use of the invariance of the Haar measure, allowing us to write $U = U_T^\dagger U_W$ - without changing the value of $d\mu(U_W) = d\mu(U)$ - and similarly for $V^\dagger$. We also have

$$\exp\left(-\mathrm{Tr}\, TT^\dagger\right) = \exp\left(-\frac{1}{4}\mathrm{Tr}\, T_D T_D^\dagger\right) = \exp\left(-\frac{1}{4}\sum_{n=1}^{N} t_n^2\right) = \prod_{n=1}^{N} e^{-a_n}\,,$$

where we have used the substitution $4a_n = t_n^2$ for all $n = 1, \ldots, N$ in the last step. Combining everything we have learned so far yields

$$\begin{aligned} P(W)dW = \frac{c}{\mathcal{N}_{\mathrm{ext}}}\left(\prod_{n=1} N e^{-a_n}\right)\Delta_N^2(\lambda_1, \ldots \lambda_N)\left(\prod_{n=1}^{N}\lambda_n^\nu e^{-\lambda_n}\right) \\ \times d\mu(U_W)d\mu(V_W^\dagger)\exp\left(\mathrm{ReTr}\,\left(UW_D V^\dagger T_D^\dagger\right)\right)\left(\prod_{n=1}^{N} d\lambda_n\right)\,. \end{aligned}$$

Thus, the resulting group integral $I(U, V)$ we have to compute is

$$I(U,V) = \int_{\mathrm{U}(N)} d\mu(U)\int_{\mathrm{U}(N+\nu)} d\mu(V)\exp\left(\mathrm{ReTr}\,\left(UW_D V^\dagger T_D^\dagger\right)\right)\,. \tag{C.8}$$

Integrals of this form are usually called Berezin-Karpelevich-type integrals [79]. The solution to the specific integral in Eq. (C.8) reads [65]

$$\begin{aligned} I(U,V) &= \int_{\mathrm{U}(N)} d\mu(U)\int_{\mathrm{U}(N+\nu)} d\mu(V)\exp\left(\mathrm{ReTr}\,\left(UW_D V^\dagger T_D^\dagger\right)\right) \\ &= C\frac{\prod_{n=1}^{N}(a_n\lambda_n)^{-\frac{\nu}{2}}}{\Delta_N(a_1,\ldots,a_N)}\frac{\det[I_\nu(2\sqrt{a_i\lambda_j})]_{i,j=1}^{N}}{\Delta_N(\lambda_1,\ldots,\lambda_N)}\,, \end{aligned}$$

where we have introduced the modified Bessel function of the first kind $I_\nu(x)$ with argument x, which has the series expansion [130]

$$x^{-\frac{\nu}{2}} I_\nu\left(2\sqrt{x}\right) = \sum_{n=0}^{\infty}\frac{x^n}{n!(n+\nu)!}\,.$$

The proportionality constant C of the integral $I(U, V)$ can be found in [80] and is of no further interest to us. Gathering all knowledge obtained so far allows us to write down the JPDF for the chGUE(N) with external source as

$$\begin{aligned} \mathcal{P}_{\mathrm{chGUE}}^{\mathrm{ext}}(\lambda_1,\ldots,\lambda_N) = \frac{c\cdot C}{\mathcal{N}_{\mathrm{ext}}}\frac{\prod_{n=1}^{N} a_n^{-\frac{\nu}{2}} e^{-a_n}}{\Delta_N(a_1,\ldots,a_N)}\left(\prod_{n=1}^{N}\lambda_n^{\nu/2} e^{-\lambda_n}\right) \\ \times\det\left[I_\nu\left(2\sqrt{a_i\lambda_j}\right)\right]_{i,j=1}^{N}\Delta_N(\lambda_1,\ldots\lambda_N)\,, \end{aligned}$$

which allows us to set

$$\mathcal{N}_{\mathrm{ext}} = \frac{Z_N^{(0,T)}}{c\cdot C}\frac{1}{\Delta_N(a_1,\ldots,a_N)}\,.$$

Finally, we arrive at

$$\begin{aligned}\mathcal{P}^{\text{ext}}_{\text{chGUE}}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N^{(0,T)}}\left[\prod_{n=1}^{N}\left(\frac{\lambda_n}{a_n}\right)^{\nu/2} e^{-(\lambda_n+a_n)}\right] \\ \times \det\left[I_\nu\left(2\sqrt{a_i\lambda_j}\right)\right]_{i,j=1}^{N}\Delta_N(\lambda_1,\ldots\lambda_N)\,,\end{aligned} \tag{C.9}$$

with a partition function now depending on temperature ($T \neq 0$), but not on dynamical flavors ($N_f = 0$), i.e.

$$\begin{aligned}Z_N^{(0,T)} = \left(\prod_{n=1}^{N_f}\int d\lambda_n\right)\left[\prod_{n=1}^{N}\left(\frac{\lambda_n}{a_n}\right)^{\nu/2} e^{-(\lambda_n+a_n)}\right] \\ \times \det\left[I_\nu\left(2\sqrt{a_i\lambda_j}\right)\right]_{i,j=1}^{N}\Delta_N(\lambda_1,\ldots\lambda_N)\,.\end{aligned}$$

The remaining model we have to consider is the combination of external source and deformation with N_f flavors, given in Eq. (C.4). Note that T is a deterministic matrix, which means we can shift $W \to W - T$ and keep the measure dW invariant at the same time. This allows us to write the density $P^{\text{temp}}_{\text{chGUE}}(W)$ as

$$\begin{aligned}P^{\text{temp}}_{\text{chGUE}}(W) &= \frac{1}{\mathcal{N}}\prod_{f=1}^{N_f}\det(D + m_f\mathbb{1}_{2N+\nu})\,\exp\left[-\text{Tr}\,(W-T)(W^\dagger - T^\dagger)\right] \\ &= \frac{\mathcal{N}_{\text{ext}}}{\mathcal{N}}\prod_{f=1}^{N_f}\det(D + m_f\mathbb{1}_{2N+\nu})P^{\text{ext}}_{\text{chGUE}}(W)\,,\end{aligned} \tag{C.10}$$

where we have the temperature independent Dirac operator D given by

$$D = i\begin{pmatrix}0_N & W\\ W^\dagger & 0_{N+\nu}\end{pmatrix}\,.$$

This operator was introduced in Eq. (2.1) back in chapter 2. From the second line in Eq. (C.10) we see that the difference to the simpler, external source model from before is the addition of N_f fermion determinants. After the shift $W \to W - T$ these are T independent. Additionally, we know from Eq. (C.7) that they are also unitary bi-invariant with respect to the singular value decomposition of W. This implies that we can take the result of Eq. (C.9) and upgrade it with Eq. (C.7). This leads us to the result for the JPDF, i.e.

$$\begin{aligned}\mathcal{P}^{\text{temp}}_{\text{chGUE}}(\lambda_1,\ldots,\lambda_N) = \frac{c\cdot C}{\mathcal{N}}\frac{\prod_{n=1}^{N}a_n^{-\frac{\nu}{2}}e^{-a_n}}{\Delta_N(a_1,\ldots,a_N)}\left(\prod_{n=1}^{N}\lambda_n^{\nu/2}e^{-\lambda_n}\right)\left(\prod_{f=1}^{N_f}m_f^\nu\right) \\ \times\left(\prod_{f=1}^{N_f}\prod_{n=1}^{N}(m_f^2\mathbb{1}_n + \lambda_n)\right)\det\left[I_\nu\left(2\sqrt{a_i\lambda_j}\right)\right]_{i,j=1}^{N}\Delta_N(\lambda_1,\ldots\lambda_N)\,.\end{aligned}$$

We can set the constant $\mathcal{N}$ to be

$$\mathcal{N} = \frac{Z_N^{(N_f,T)}}{c \cdot C} \frac{1}{\Delta_N(a_1,\ldots,a_N)} \left(\prod_{f=1}^{N_f} m_f^\nu \right). \tag{C.11}$$

Consequently, we obtain

$$\begin{aligned}
\mathcal{P}_{\text{chGUE}}^{\text{temp}}(\lambda_1,\ldots,\lambda_N) &= \frac{1}{Z_N^{(N_f,T)}} \left[\prod_{n=1}^{N} \left(\frac{\lambda_n}{a_n} \right)^{\nu/2} e^{-(\lambda_n + a_n)} \right] \left(\prod_{f=1}^{N_f} \prod_{n=1}^{N} (m_f^2 + \lambda_n) \right) \\
&\times \det \left[I_\nu \left(2\sqrt{a_i \lambda_j} \right) \right]_{i,j=1}^{N} \Delta_N(\lambda_1,\ldots\lambda_N) \\
&= \frac{Z_N^{(0,T)}}{Z_N^{(N_f,T)}} \left(\prod_{f=1}^{N_f} \prod_{n=1}^{N} (m_f^2 + \lambda_n) \right) \mathcal{P}_{\text{chGUE}}^{\text{ext}}(\lambda_1,\ldots,\lambda_N) \\
&= \frac{Z_N^{(N_f,0)}}{Z_N^{(N_f,T)}} \left(\prod_{n=1}^{N} (\lambda_n a_n)^{-\frac{\nu}{2}} e^{-a_n} \right) \frac{\det \left[I_\nu \left(2\sqrt{a_i \lambda_j} \right) \right]_{i,j=1}^{N}}{\Delta_N(\lambda_1,\ldots,\lambda_N)} \\
&\times \mathcal{P}_{\text{chGUE}}^{\text{deformed}}(\lambda_1,\ldots,\lambda_N) \\
&= \frac{Z_N^{(0,0)}}{Z_N^{(N_f,T)}} \left(\prod_{n=1}^{N} (\lambda_n a_n)^{-\frac{\nu}{2}} e^{-a_n} \right) \frac{\det \left[I_\nu \left(2\sqrt{a_i \lambda_j} \right) \right]_{i,j=1}^{N}}{\Delta_N(\lambda_1,\ldots,\lambda_N)} \\
&\times \left(\prod_{f=1}^{N_f} \prod_{n=1}^{N} (m_f^2 + \lambda_n) \right) \mathcal{P}_{\text{chGUE}}(\lambda_1,\ldots,\lambda_N) \,.
\end{aligned}$$

Setting $N_f = 0$, but keeping $T \neq 0$, yields the JPDF of the chGUE(N) with external source as is indicated in the third line above, as the fermion determinants are absent in this case. Setting $N_f \neq 0$ and $T = 0$ yields the JPDF of the deformed chGUE(N) with N_f fermion determinants, since the group integral $I(U,V)$ behaves as $I(U,V) \to c'$ in the limit $a_n \to 0$ for all $n = 1,\ldots,N$. This is indicated in the second to last line above and equivalent to saying that the ratio of determinants and partition functions is a constant equaling unity in this limit. The case $N_f = 0$ and $T = 0$ yields, in combination, the JPDF of the classical chGUE(N) in the last line above.

Appendix D

Polynomial Ensembles as Giambelli compatible Point Processes

In this appendix we collect result presented in [1] regarding a sub-class of determinantal point processes. The ensembles we consider share what is called the *Giambelli compatability property*, which is defined below. We show that any polynomial ensemble, as defined in chapter 3, has this property and becomes a *Giambelli compatible point process*. Following [129] this leads to a certain determinantal representation of the expectation value of a ratio of characteristic polynomials, where both numerator and denominator feature an equal number of characteristic polynomials. In contrast to the rest of this thesis we use a different convention for the Vandermonde determinant. We follow the notation in [1]. We also adopt notation and definitions from Macdonald [142].

Let Λ be the algebra of symmetric functions. The Schur functions s_λ indexed by Young diagrams λ form an orthonormal basis in Λ. Recall that Young diagrams can be written in the Frobenius notation, namely

$$\lambda = (p_1, \ldots, p_d | q_1, \ldots, q_d)\,,$$

where d equals the number of boxes on the diagonal of λ, p_j with $j = 1, \ldots, d$ denotes the number of boxes in the j-th row of λ to the right of the diagonal, and q_l with $l = 1, \ldots, d$ denotes the number of boxes in the l-th column of λ below the diagonal. The Schur functions satisfy the *Giambelli formula*:

$$s_{(p_1,\ldots,p_d|q_1,\ldots,q_d)} = \det\left[s_{(p_i|q_j)}\right]_{i,j=1}^{d}\,.$$

The Schur polynomial $s_\lambda(x_1, \ldots, x_N)$ is the specialization of s_λ to the variables x_1, …, x_N. The Schur polynomial $s_\lambda(x_1, \ldots, x_N)$ corresponding to the Young diagram λ with $l(\lambda) \leq N$ rows of lengths $\lambda_1 \geq \ldots \geq \lambda_{l(\lambda)} > 0$, can be defined by

$$s_\lambda(x_1, \ldots, x_N) = \frac{1}{\Delta_N(x_1, \ldots, x_N)} \det\left[x_i^{\lambda_j + N - j}\right]_{i,j=1}^{N}\,. \tag{D.1}$$

If $l(\lambda) > N$, then $s_\lambda(x_1, \ldots, x_N) \equiv 0$ (by definition).

The *Giambelli compatible* point processes form a class of point processes whose different probabilistic quantities of interest can be studied using the Schur symmetric functions. This class of point processes was introduced by Borodin, Olshanski, and Strahov in [129] to prove determinantal identities for averages of analogs of characteristic polynomials for ensembles

originating from Random Matrix Theory, the theory of random partitions, and from representation theory of the infinite symmetric group. In the context of random point processes formed by N-point random configurations on a subset of $\mathbb{R}$ the Giambelli compatible point processes can be defined as follows.

Definition D.1. *Assume that a point process is formed by an N-point configuration $(x_1, \ldots, x_N)$ on $I \subseteq \mathbb{R}$. If the Giambelli formula*

$$s_{(p_1,\ldots,p_d|q_1,\ldots,q_d)}(x_1, \ldots, x_N) = \det\left[s_{(p_i|q_j)}(x_1, \ldots, x_N)\right]_{i,j=1}^{d},$$

which is valid for the Schur polynomial $s_\lambda(x_1, \ldots, x_N)$ parameterized by an arbitrary Young diagram $\lambda\,(p_1, \ldots, p_d|q_1, \ldots, q_d)$, can be extended to averages, i.e.

$$\mathbb{E}\left[s_{(p_1,\ldots,p_d|q_1,\ldots,q_d)}(x_1, \ldots, x_N)\right] = \det\left[\mathbb{E}\left[s_{(p_i|q_j)}(x_1, \ldots, x_N)\right]\right]_{i,j=1}^{d},$$

then the random point process is called Giambelli compatible point process.

What we show next is that polynomial ensembles introduced in chapter 3 can be understood as Giambelli compatible point processes. Namely, the following Theorem holds true.

Theorem D.2. *Any polynomial ensemble in the sense of Definition 3.1 is a Giambelli compatible point process.*

As it is explained by Borodin, Olshanski, and Strahov in [129] the Giambelli compatibility of point processes implies determinantal formulas for averages of ratios of characteristic polynomials. Namely, we obtain

Theorem D.3. *Assume that $x_1, \ldots, x_N$ form a polynomial ensemble. Let $u_1, \ldots, u_M \in \mathbb{C}\backslash\mathbb{R}$ and $z_1, \ldots, z_M \in \mathbb{C}$ for any $M \in \mathbb{N}$ be pairwise distinct variables. Then*

$$\mathbb{E}\left[\prod_{m=1}^{M}\frac{D_N(z_m)}{D_N(u_m)}\right] = \left[\det\left(\frac{1}{u_i - z_j}\right)_{i,j=1}^{M}\right]^{-1}\det\left[\frac{1}{u_i - z_j}\mathbb{E}\left(\frac{D_N(z_j)}{D_N(u_i)}\right)\right]_{i,j=1}^{M}, \tag{D.2}$$

where $D_N(z) = \prod_{n=1}^{N}(z - x_n)$ denotes the characteristic polynomial associated with the random variables $x_1, \ldots, x_N$.

Proof of Theorem D.2

Let $x_1, \ldots, x_N$ form a polynomial ensemble on I^N, where $I \subseteq \mathbb{R}$. The probability density function of this ensemble is given in Definition 3.1. Denote by $\widetilde{s}_\lambda$ the expectation of the Schur polynomial $s_\lambda\,(x_1, \ldots, x_N)$ with respect to this ensemble,

$$\widetilde{s}_\lambda = \mathbb{E}\left(s_\lambda\,(x_1, \ldots, x_N)\right). \tag{D.3}$$

Our aim is to show that $\widetilde{s}_\lambda$ satisfies the Giambelli formula, i.e.

$$\widetilde{s}_\lambda = \det\left[\widetilde{s}_{(p_i|q_j)}\right]_{i,j=1}^{d}, \tag{D.4}$$

where λ is an arbitrary Young diagram, $\lambda = (p_1, \ldots, p_d|q_1, \ldots, q_d)$ in the Frobenius coordinates. According to Definition D.1 this will mean that the polynomial ensemble under

considerations is a Giambelli compatible point process.
The proof of Eq. (D.4) below is based on the following general fact due to Macdonald, see Macdonald [142], Example I.3.21.

Proposition D.4. *Let $\{h_{r,s}\}$ with integer $r \in \mathbb{Z}$ and non-negative integer $s \in \mathbb{N}$ be a collection of commuting indeterminates such that we have*

$$\forall s \in \mathbb{N} : h_{0,s} = 1 \text{ and } \forall r < 0 \; h_{r,s} = 0 \,, \tag{D.5}$$

and set

$$\widetilde{s}_\lambda = \det \left[h_{\lambda_i - i + j, j-1} \right]_{i,j=1}^{k} ,$$

where k is any number such that $k \geq l(\lambda)$. Then we have

$$\widetilde{s}_\lambda = \det \left[\widetilde{s}_{(p_i|q_j)} \right]_{i,j=1}^{d} ,$$

where λ is an arbitrary Young diagram, $\lambda = (p_1, \ldots, p_d | q_1, \ldots, q_d)$ in the Frobenius coordinates.

Clearly, in order to apply Proposition D.4 to $\widetilde{s}_\lambda$ defined by Eq. (D.3) we need to construct a collection of indeterminates $\{h_{r,s}\}$ such that

$$\mathbb{E}\left(s_\lambda \left(x_1, \ldots, x_N \right) \right) = \det \left[h_{\lambda_i - i + j, j-1} \right]_{i,j=1}^{k} \tag{D.6}$$

will hold true for an arbitrary Young diagram λ, for an arbitrary $k \geq l(\lambda)$, and such that the condition in Eq. (D.5) will be satisfied.
By Andréiéf's integration formula given in Proposition 2.3 and the expression for the normalisation constant Z_N in Eq. (3.1) we can write[1]

$$\mathbb{E}\left[s_\lambda (x_1, \ldots, x_N) \right] = \frac{\det \left[\int_I dx x^{\lambda_i + N - i} \varphi_j(x) \right]_{i,j=1}^{N}}{\det \left[\int_I dx x^{N-i} \varphi_j(x) \right]_{i,j=1}^{N}} \,, \tag{D.7}$$

where we used Eq. (A.1) and Eq. (D.1). Notice that at this point it matters that we consider polynomial ensembles and not more general bi-orthogonal ensembles. In the latter case the Vandermonde determinant in the denominator of the Schur function in Eq. (D.1) would not cancel, the Andréiéf formula would not apply and we would not know how to compute such expectation values. Set

$$A_{n,m} = \int_I dx x^n \varphi_m(x); \;\; n = 0, 1, \ldots; \;\; m = 1, \ldots, N,$$

and denote by $Q = \left(Q_{i,j} \right)_{i,j=1}^{N}$ the inverse[2] of $\tilde{G} = \left(\tilde{g}_{i,j} \right)_{i,j=1}^{N}$, where $\tilde{g}_{i,j} = \int_I dx x^{N-i} \varphi_j(x)$. With this notation we can rewrite Eq. (D.7) as

$$\mathbb{E}\left(s_\lambda (x_1, \ldots, x_N) \right) = \det \left[\sum_{\nu=1}^{N} A_{\lambda_i + N - i, \nu} Q_{\nu, j} \right]_{i,j=1}^{N} . \tag{D.8}$$

[1] Due to the difference in notation with respect to the Vandermonde in this Appendix and Appendix A we have a monomial of degree x^{N-i} and not x^{i-1} in both numerator and denominator.

[2] Notice that due to Eq. (3.2) we have $\det[G] = (-1)^{N(N-1)/2} \det[\tilde{G}]$. Again due to the different convention of the Vandermonde determinant.

Since Q is the inverse of $\tilde{G}$, we have

$$\sum_{j=1}^{N} \tilde{g}_{i,j} Q_{j,k} = \delta_{i,k}, \;\; 1 \leq i,k \leq N,$$

or

$$\sum_{j=1}^{N} A_{N-i,j} Q_{j,k} = \delta_{i,k}, \;\; 1 \leq i,k \leq N. \tag{D.9}$$

The following Proposition will imply Theorem D.2.

Proposition D.5. *Let $\{h_{r,s}\}$, with integer $r \in \mathbb{Z}$ and non-negative integer $s \in \mathbb{Z}_{\geq 0}$, be a collection of indeterminates defined by*

$$h_{r,s} \equiv \begin{cases} \sum_{\nu=1}^{N} A_{N+r-s-1,\nu} Q_{\nu,s+1}, & s \in \{0,1,\ldots,N-1\}, \quad r \geq 0, \\ \delta_{r,0}, & s \geq N, \quad r \geq 0, \\ 0, & s \geq 0, \quad r < 0. \end{cases} \tag{D.10}$$

The collection of indeterminates $\{h_{r,s}\}$ satisfies the condition in Eq. (D.5)*. Moreover, with this collection of indeterminates $\{h_{r,s}\}$ formula in Eq.* (D.6) *holds true for an arbitrary Young diagram λ, and for an arbitrary $k \geq l(\lambda)$.*

Proof. We divide the proof of into several steps. First, the collection of indeterminates $\{h_{r,s}\}$ defined by Eq. (D.10) is shown to satisfy the condition in Eq. (D.5). Next, we prove that Eq. (D.6) holds true for an arbitrary Young diagram λ, and for an arbitrary $k \geq l(\lambda)$.
Step 1. First, we want to show that

$$\det\left[h_{\lambda_i - i + j, j-1}\right]_{i,j=1}^{k} = \det\left[h_{\lambda_i - i + j, j-1}\right]_{i,j=1}^{l(\lambda)}, \tag{D.11}$$

for any $k \geq l(\lambda)$.
Let λ be an arbitrary Young diagram, and assume that $k > l(\lambda)$. Consider the diagonal entries of the $k \times k$ matrix

$$\left(h_{\lambda_i - i + j, j-1}\right)_{i,j=1}^{k}$$

for $i = j \in \{l(\lambda)+1, \ldots, k\}$. By definition of the $h_{r,s}$ these entries are all equal to 1, since $\lambda_i = 0$ for $i \in \{l(\lambda)+1, \ldots, k\}$ implying $h_{0,s} = 1$ by Eq. (D.5). For $r < 0$ we have $h_{r,s} = 0$ (see Eq. (D.10)) and the matrix $\left(h_{\lambda_i - i + j, j-1}\right)_{i,j=1}^{k}$ has the form

$$\left(\begin{array}{ccc|ccccc} \star & \ldots & \star & \star & \ldots & \ldots & \ldots & \star \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \star & \ldots & \star & \star & \ldots & \ldots & \ldots & \star \\ \hline 0 & \ldots & 0 & 1 & \star & \ldots & \ldots & \star \\ \vdots & & \vdots & 0 & 1 & \star & \ldots & \star \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \vdots & \vdots & & \ddots & \ddots & \star \\ 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 & 1 \end{array}\right),$$

where the first row from the top with zeros has the label $l(\lambda)+1$, and the first column from the left with ones has the label $l(\lambda)+1$. The determinant of such a block matrix reduces to the product of the determinants of the blocks, which gives the relation in Eq. (D.11).

Step 2. Assume now that $l(\lambda) > N$. Then it trivially holds that

$$\mathbb{E}\left(s_\lambda(x_1,\ldots,x_N)\right) = 0\,,$$

by the very definition of the Schur polynomials. Here, we would like to show that it equally holds that

$$\det\left[h_{\lambda_i-i+j,j-1}\right]_{i,j=1}^{l(\lambda)} = 0\,,$$

if $l(\lambda) > N$.
We have $h_{r,s} = \delta_{r,0}$ for $s \geq N$ and $r \geq 0$. This implies that the matrix $\left(h_{\lambda_i-i+j,j-1}\right)_{i,j=1}^{l(\lambda)}$, which we can write out as

$$\left(\begin{array}{cccc|ccccc}
h_{\lambda_1,0} & \star & \ldots & \star & | & h_{\lambda_1+N,N} & \ldots & \ldots & h_{\lambda_1-1+l(\lambda),l(\lambda)-1} \\
\star & h_{\lambda_2,1} & \ddots & \vdots & | & \vdots & & & \vdots \\
\vdots & \ddots & \ddots & \star & | & \vdots & & & \vdots \\
\star & \ldots & \star & h_{\lambda_N,N-1} & | & h_{\lambda_N+1,N} & \ldots & \ldots & h_{\lambda_N-N+l(\lambda),l(\lambda)-1} \\
\hline
\star & \ldots & \ldots & \star & | & h_{\lambda_{N+1},N} & \ldots & \ldots & h_{\lambda_{N+1}-N-1+l(\lambda),l(\lambda)-1} \\
\vdots & & & \vdots & | & \star & \ddots & & \vdots \\
\vdots & & & \vdots & | & \vdots & \ddots & \ddots & \vdots \\
\star & \ldots & \ldots & \star & | & \star & \ldots & \star & h_{\lambda_{l(\lambda)},l(\lambda)-1}
\end{array}\right)$$

has the form

$$\left(\begin{array}{cccc|cccc}
h_{\lambda_1,0} & \star & \ldots & \star & | & 0 & \ldots & 0 \\
\star & h_{\lambda_2,1} & \ddots & \vdots & | & \vdots & & \vdots \\
\vdots & \ddots & \ddots & \star & | & \vdots & & \vdots \\
\star & \ldots & \star & h_{\lambda_N,N-1} & | & 0 & \ldots & 0 \\
\hline
\star & \ldots & \ldots & \star & | & 0 & \ldots & 0 \\
\vdots & & & \vdots & | & \star & \ddots & \vdots \\
\vdots & & & \vdots & | & \vdots & \ddots & \vdots \\
\star & \ldots & \ldots & \star & | & \star & \ldots & 0
\end{array}\right).$$

Thus, we can again apply the formula for determinants of block matrices to obtain

$$\det\left[h_{\lambda_i-i+j,j-1}\right]_{i,j=1}^{l(\lambda)} = \det\left[h_{\lambda_i-i+j,j-1}\right]_{i,j=1}^{N} \cdot 0 = 0,$$

which is true for any $l(\lambda) > N$ and therefore the condition in Eq. (D.6) is satisfied in this case.

Step 3. Now we wish to prove that

$$\sum_{\nu=1}^{N} A_{N-i+\lambda_i,\nu} Q_{\nu,j} = h_{\lambda_i-i+j,j-1} \tag{D.12}$$

is valid for any Young diagram with $l(\lambda) \leq N$, and for $1 \leq i, j \leq N$. Assume that $\lambda_i - i + j \geq 0$. Then Eq. (D.12) turns into the first equation in Eq. (D.10) with $r = \lambda_i - i + j$, $s = j - 1$. Assume that $\lambda_i - i + j < 0$. Then $i - \lambda_i > j$. Clearly, $i - \lambda_i \in \{1, \ldots, N\}$ in this case, and we have

$$\sum_{\nu=1}^{N} A_{N-i+\lambda_i,\nu} Q_{\nu,j} = \delta_{i-\lambda_i,j} = 0,$$

where we have used Eq. (D.9). Also, if $\lambda_i - i + j < 0$, and $1 \leq i, j \leq N$, then $h_{\lambda_i-i+j,j-1} = 0$ as it follows from Eq. (D.10). We conclude that Eq. (D.12) holds true for $\lambda_i - i + j < 0$ as well.

Finally, the results obtained in Step 1-Step 3 together with the formula in Eq. (D.8) give the desired result of Eq. (D.6). □

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