Center for Mathematical Economics
Working Papers

656

October 2021

Language games under Knightian uncertainty about types

Zhaojun Xing

Center for Mathematical Economics (IMW)
Bielefeld University
Universitätsstraße 25
D-33615 Bielefeld · Germany

e-mail: imw@uni-bielefeld.de
uni-bielefeld.de/zwe/imw/research/working-papers
ISSN: 0931-6558

Language games under Knightian uncertainty about types

Zhaojun Xing*

Bielefeld University and Paris 1 University

October 28, 2021

Abstract

We concern a sender-receiver game of common interests having infinite types, e.g the set $[0, 1]^2$, but with finite signals. In our paper, we extend the game by introducing multiple priors over the type space and use incomplete preferences in Bewley's way. We characterize the equilibria under incomplete preferences by E-admissibility. Besides, it has the equivalence between the equilibria and Voronoi languages. Further, we demonstrates the existence of the indeterminacy of the game. At last, we present that vague words, e.g. cheap, big, red, etc., exist in the Knightian worlds but not in the Bayesian worlds, which means that vagueness comes from the way we view the world in Knightian method.

*I would like to thank my supervisors Frank Riedel and Jean-Marc Tallon and also Adam Dominiak, Frédéric Koessler, Max Nendel, Teddy Seidenfeld, Joel Sobel and Wei Xiong for their insightful comments and suggestions. Of course, all errors are my own. Financial support by the German Research Foundation (Deutsche Forschungsgemeinschaft/German Research Foundation) via grant Ri 1128-9-1 (Open Research Area in the Social Sciences, Ambiguity in Dynamic Environments) and by German-French University via the TiQE program are gratefully acknowledged.
E-mail: zhaojun.xing@hotmail.com

1 Introduction

Vague words are widely used in our daily life, e.g. long, child, red, chair, etc. However, using vague words is inefficient in Crawford and Sobel (1982)'s cheap-talk game if the sender and the receiver have common interests. Based on a cheap-talk game Jäger et al. (2011), that is a generalization (in sense of type space with multiple dimensions) of Crawford and Sobel's game, we show that using vague words is rational.

In classic Bayesian games, As Lipman (2009) says, it is not that people have a precise view of the world but communicate it vaguely; instead, they have a vague view of the world. Then we assume that the receiver has multiple priors rather than a single prior on the sender's private type set. We adopt Aumann (1962)'s attitude towards players' preferences, i.e. of all the axioms of utility theory, the completeness axiom is perhaps the most questionable [1]. And both players employ incomplete preferences in Bewley (2002)'s way. We find that there is a set of strict equilibria given multiple priors of the players on type set, where each strict equilibria is characterized by a Voronoi language Jäger et al. (2011). The set of strict equilibria forms a so called vague Voronoi language under some conditions, which consists of vague words and each each vague word features thick borderline cases.

As Crawford and Sobel (1982) demonstrates, the sender has motivation to hide her private information and uses noise or vague signals to confuse the receiver when she has conflict interests with the receiver. Based on Crawford and Sobel's model, most recent researches, e.g. Blume et al. (2007) and Kellner and Le Quement (2018), obtain vagueness relying on the assumption of conflict interests between the sender and the receiver. However, many human activities require coordination, e.g. meeting with one people in some

[1] In detail, he says, like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from the normative viewpoint. Does "rationality" demand that an individual make definite preference comparisons between all possible lotteries (even on a limited set of basic alternatives)? For example, certain decisions that our individual is asked to make might involve highly hypothetical situations, which he will never face in real life; he might feel that he cannot reach an "honest" decision in such cases. Other decision problems might be extremely complex, too complex for intuitive "insight," and our individual might prefer to make no decision at all in these problems. Or he might be willing to make rough preference statements such as, "I prefer a cup of cocoa to a 75-25 lottery of coffee and tea, but reverse my preference if the ratio is 25-75"; but he might be unwilling to fix the break-even point between coffee-tea lotteries and cocoa any more precisely. Is it "rational" to force decisions in such cases?

place without full information (see P5-P8 of Lewis (1969) for more examples). But Lipman (2009) shows that using vague words (represented by mixed strategies) is inefficient under aligned interests due to the concavity of players' utility functions. It seems irrational to use vague words under common interests. Then he asks a question "why is language vague?"

We extend the games from Jäger et al. (2011) with Knightian uncertainty to answer this question. The cheap-talk game progresses as follows. There are two players including a sender and a receiver in the game. The sender's private type set is a convex and compact one-dimensional or multiple-dimensional metric space, e.g. $[0, 1]^2$. The sender sends its type to the receiver via some word from a finite word list, e.g. $\{Left, Right\}$. Once the receiver obtains the word, she interprets it as some meaning which belongs to the type set. Due to aligned interests between the sender and the receiver, they want to cooperate to reduce the loss during the information transmission. However, due to the limited words, it is impossible to transmit infinite types to the receiver exactly without communication before gaming. So, there is a loss for almost all types (except the prototypical types) during the information transmission. Naturally, a question occurs that is how do the sender and the receiver act to reduce the loss. Jäger et al. (2011) finds that the sender should partition the type set into mutually exclusive cells, and each cell corresponds to each signal or word. All of these cells form a Voronoi tessellation. Here, "Voronoi" means a Voronoi tessellation or a Voronoi diagram, which is generated by a set of prototypical points. For each point there is one corresponding cell such that all points in this cell are closer to the prototypical point of this cell than any other prototypical point. As in Figure 1, the Voronoi tessellation is generated by the points $\{a, b\}$. All points in the left cell is closer to the

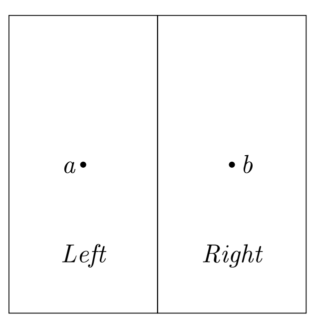


Figure 1: a Voronoi language with two words: *Left* and *Right*.

point a than the point b, and all points in the right cell is closer to the point b than the point a. The points a and b correspond to the generated *Left* and *Right* cell, respectively. Further, it requires that the receiver should explain each word as the Bayesian estimator of the cell that the word means. The sender's strategy and the receiver's strategy form a Voronoi language [2]. For example, in Figure 1, a Voronoi language that is consisting of the sender's strategy (sending the left cell to the word *Left* and sending the right cell to the word *Right*) and the receiver's strategy (interpreting the word *Left* to the point a and interpreting the word *Right* to the point b), where the points a and b are the centers of the mass of the left and right cell, respectively. Besides, they have demonstrated that the each strict Nash equilibrium [3] is a Voronoi language, and vice versa.

Now, we introduce multiple priors of the players on types. A decision rule called maximality Bewley (2002) is used to obtain players' optimal strategies. Finding optimal acts in decision problems or equilibria in games with incomplete preferences is not trivial. A natural idea is scalarizing incomplete preferences as Shapley (1959), Aumann (1962), Schervish et al. (2003), Rigotti and Shannon (2005),Bade (2005), Evren (2014) and Danan et al. (2016). Among them, Schervish et al. (2003), Rigotti and Shannon (2005) and Danan et al. (2016) consider the incompleteness coming from multiple priors, where the state spaces in their models are finite. In Appendix A, we extend the scalarization to an infinite-dimensional space in the level of decision theory. We scalarize an act by multiplying each prior from the prior set and then admit the acts which achieves maximum value under some prior. This decision rule is called E-admissibility (short for Expectation-admissibility) Levi (1980). Notice that an act or an option is maximal if there is no other act dominating this act under all distributions, and an act is E-admissible if there exists some distribution such that the act has the maximum expected value. Schervish et al. (2003) have proved the equivalence of maximality and E-admissibility within finite states. We extend their outcome in Appendix A under infinite states to solve our game with infinite types.

Naturally, the concept of (strict) Nash equilibrium should be updated for introducing Knightian uncertainty. We have defined (strictly) maximal equi-

[2]Geometrically, a Voronoi language is a centroidal Voronoi tessellation Du et al. (1999), which is generated by the centers of the mass of corresponding cells if the loss function is quadratic.

[3]Due to the plethora of Nash equilibria in all signaling games, like polling equilibrium, they focus on strict Nash equilibria.

librium and (strictly) E-admissible equilibrium, which correspond to maximality and E-admissibility, respectively. As we have an equivalence between maximality and E-admissibility, we are able to apply it in the game. By the feature of E-admissibility, we are able to focus on players' behavior with single prior, i.e. without Knightian uncertainty. In case of single prior, each strict Nash equilibrium is a Voronoi language. Via the equivalence, we are able to characterize all strict Nash equilibria under Knightian uncertainty. Further, we show the equivalences among the following three concepts in our games: strictly E-admissible Nash equilibrium, strictly maximal Nash equilibrium and Voronoi language.

Further, we study the equilibria under Knightian uncertainty. For each strictly maximal Nash equilibrium, there exists a continuum of strictly maximal Nash equilibrium surround it. This kind of indeterminacy also appears in financial markets Rigotti and Shannon (2005). In natural languages, this phenomenon is called vagueness, like the vague words 'cheap', 'small', 'red', 'game' etc. Besides, we define the concept named vague Voronoi languages inspired by the definition of vagueness in a conceptual spaces approach Douven et al. (2013) and Decock and Douven (2014), where vagueness comes from the multiplicity of prototypes (corresponding to Bayesian estimators in our paper) of each word. Each vague Voronoi language collects a set of strictly maximal Nash equilibria or Bayesian estimators. And we prove that vague Voronoi languages do not exist in the Bayesian worlds but exist in the Knightian worlds. At last, we study the efficiency of vague Voronoi languages and obtain an efficient vague Voronoi language.

This paper contains the following sections. Section 2 defines the game. Section 3 analyzes equilibria of the game. Section 4 presents the indeterminacy of the equilibria. Section 5 studies vague Voronoi languages. The efficiency of vague Voronoi languages is discussed in Section 6. Section 7 concludes the paper.

2 The model

In this section, we extend the cheap-talk games with high-dimensional types and few signals Jäger et al. (2011) to allow for Knightian uncertainty about types. The revised game progresses as follows. The sender s observes a private signal, her type $t \in T$, where T is a convex and compact subset of $\mathbb{R}^n$ with non-empty interior and $n \geq 1$, e.g. $T = [0,1]^2 \subseteq \mathbb{R}^2$. The sender chooses a word or signal w from a finite word list W with N words and sends it to the receiver. The receiver r receives the word w_j and interprets it as a point $i_j \in T$. Both players want the type t and the interpretation i_j to be as close as possible. A only deviation from the game of Jäger et al. (2011) is that the sender and the receiver share a set of priors over the type set T rather than a single prior, where the beliefs are measured by a nonempty set of Borel regular probability (also non-atomic) measures $\mathscr{P}$, where $\mathscr{P}$ is convex and closed [4] defined on T assigned with a Borel σ-algebra.

A pure strategy for the sender is a measurable function $w \in W^T : T \to W$. A behavioral strategy of the sender is a measurable mapping $\omega \in \Omega : T \to \Delta W$, where Ω is the set of the sender's all behavioral strategies and ΔW is the set of all probabilities over W. A pure strategy for the receiver is a vector $i = (i_1, \cdots, i_N) \in T^N$, where i_j denotes the receiver's interpretation of the word w_j. A behavioral strategy of the receiver is a mapping $\mu \in M : W \to \Delta T$, where M is the set of the receiver's all behavioral strategies and ΔT is the set of all probability measures over T. If the sender's type t is interpreted by the receiver as i when they communicate with the word w, then both players' losses are measured by $l(\|t - i_{w(t)}\|)$, where $\|\cdot\|$ is a Euclidean norm on T and $l : \mathbb{R}_+ \to \mathbb{R}$ is a continuous, convex and strictly increasing function, e.g. $l(d) = d^2$.

For a prior $P \in \mathscr{P}$ of any player, the expected loss of both players using pure strategies and behavioral strategies are

$$L(w, i, P) = \int_T l\left(\left\|t - i_{w(t)}\right\|\right) P(dt),$$

and

$$L(\omega, \mu, P) = \int_T \sum_{k=1}^{N} \int_T l\left(\|t - i\|\right) \mu_k(di)\omega_k(t)P(dt),$$

respectively, where $\omega_k(t)$ denotes the probability that the sender uses the word w_k under the type t, and $\mu_k(di)$ is the probability of the receiver choosing i from T to interpret w_k. Note that we allow a set of priors of the players, and the expected losses of both players is the set $\{L(w, i, P) : P \in \mathscr{P}\}$.

As discussed, we assume that players use incomplete preferences in Bewley's way. Without completeness, an strategy for a player dominates another

[4] A set of probabilities $\mathscr{P}$ is closed with respect to the weak* topology $\sigma(ca_r(T), C(T))$, where $C(T)$ is the set of all continuous functionals defined on T and $ca_r(T)$ is the set of all regular signed Borel measures of bounded variation over T.

strategy if and only if it induces less loss under all priors in $\mathscr{P}$ if we fix the other player's strategy. Formally, fixing the receiver's strategy μ, the sender's strategy

$$\omega(\succeq) \succ \omega' \text{ if and only if } L(\omega, \mu, P)(\leq) < L(\omega', \mu, P) \text{ for all } P \in \mathscr{P}.$$

Similarly, fixing the sender's strategy ω, the sender's strategy

$$\mu(\succeq) \succ \mu' \text{ if and only if } L(\omega, \mu, P)(\leq) < L(\omega, \mu', P) \text{ for all } P \in \mathscr{P}.$$

3 Equilibria and Voronoi languages

As multiple priors introduced, the definition of equilibria should be updated. A strategy profile in a Bayesian game is a (Bayesian) Nash equilibrium under a prior if no one can obtain higher payoff unilaterally in the equilibrium. In the spirit of Nash equilibrium, the revised Nash equilibrium with Knightian uncertainty should be the profile such that nobody can obtain higher payoff unilaterally under all priors. Like the definition of Nash equilibrium, we define our equilibrium via the concept of best responses as follows.

Definition 1. *A sender's strategy ω is a (strictly) maximal best response to a receiver's strategy μ if there is no $\omega' \in \Omega \backslash \{\omega\}$ such that*

$$L(\omega', \mu, P)(\leq) < L(\omega, \mu, P)$$

for all $P \in \mathscr{P}$. Similarly, a receiver's strategy μ is a (strictly) maximal best response to a sender's strategy ω if there is no $\mu' \in M \backslash \{\mu\}$ such that

$$L(\omega, \mu', P)(\leq) < L(\omega, \mu, P)$$

for all $P \in \mathscr{P}$.

Definition 2. *A strategy profile (ω, μ) is a (strictly) maximal Nash equilibrium if ω is a (strictly) maximal best response to μ and μ is a (strictly) maximal best response to ω.*

Although there is no single expected utility function to fully characterize an incomplete preference, we are able to describe decision makers' behavior by focusing on each prior from the set of multiple priors. By collecting maximal acts under each prior, also called E-admissible acts Levi (1980), we are able to find all maximal acts under incomplete preferences. In the following, we define the notion of E-admissible Nash equilibrium to find solutions under incomplete preferences with multiple priors.

Definition 3. *A sender's strategy ω is an (a strictly) E-admissible best response to a receiver's strategy μ if there exists some $P \in \mathscr{P}$ such that*

$$L(\omega, \mu, P)(<) \leq L(\omega', \mu, P)$$

for all $\omega' \in \Omega \backslash \{\omega\}$. Similarly, a receiver's strategy μ is an (a strictly) E-admissible best response to a sender's strategy ω if there exists some $P \in \mathscr{P}$ such that

$$L(\omega, \mu, P)(<) \leq L(\omega, \mu', P)$$

for all $\mu' \in M \backslash \{\mu\}$.

Then, we provide the definition of E-admissible Nash equilibria by the E-admissible best responses.

Definition 4. *A strategy profile (ω, μ) is an or a (strictly) E-admissible Nash equilibrium if ω is an or a (strictly) E-admissible best response to μ and μ is an or a (strictly) E-admissible best response to ω.*

Remark 1. *By Definition 3 and 4, if we fix a prior P in $\mathscr{P}$ shared by both players, an E-admissible Nash equilibrium is a Nash equilibrium.*

Jäger et al. (2011) has demonstrated that each strict Nash equilibrium of the cheap-talk game under single prior is a Voronoi language. In the following, we demonstrate that this outcome can be generalized under multiple priors.

Definition 5. *A Voronoi language (see Jäger et al. (2011)) (w, i) consists of a Voronoi tessellation for the sender and a Bayesian estimator interpretation for the receiver under $P \in \mathscr{P}$, i.e.*

$$w(t) := w_{\underset{j=1,\cdots,N}{\arg\min} \|t - i_j\|}, \tag{1}$$

$$\textit{and } i_j = b(C_j) \ (\textit{for } C_j = \{t \in T : w(t) = w_j\}), \tag{2}$$

where the Bayesian estimator conditional on $C_j \subseteq T$ is

$$b(C_j) := \underset{i \in T}{\arg\min} \int_{C_j} l(\|t - i_j\|) P(dt).$$

Remark 2. *For the sake of simplicity, given a Voronoi language* (w, i)*, each Voronoi cell* C_j *mapped to the word* w_j *by the sender can be rewritten as*

$$C_j = \{t \in T : \|t - i_j\| \leq \|t - i_k\| \text{ for all } k \in \{1, \ldots, N\} \backslash \{j\}, = \text{holds if } j < k\} .$$

The sender partitions her own type set T in the way of being a Voronoi tessellation. With slight abuse of notation, we sometimes let i_j denote $i(w_j)$, i.e. the interpretation of the word w_j. Given the receiver's strategy $i = (i_1, \cdots, i_N)$, the sender maps any type t to the word w_j, where the interpretation i_j of the word w_j is the closest interpretation among $\{i_i, \cdots, i_N\}$ to t. If there are multiple interpretations closest to t, without loss of generality, we choose the interpretation whose index is the smallest one. Like a horizontal Voronoi language shown as the left one of Figure 2, the sender maps any type t in the cell C_1 to the word w_1 and the interpretation $i(w_1)$ of the word w_1 is the closest interpretation of $\{i(w_1), i(w_2)\}$ to the type t. Given the sender's strategy $w = (w_1, \cdots, w_n)$, once the receiver receives the word w_j, she updates her belief and chooses the interpretation i_j that minimizes the expected loss of the types mapping to w_j. For example, in the left Voronoi language of Figure 2, the receiver interprets the word w_1 and w_2 as the word $i(w_1)$ and $i(w_2)$, respectively, provided that the sender partitions the type set T into C_1 and C_2.

Example 1. Jäger et al. (2011) *If the loss function* $l(d) = d^2$*, the receiver's strategy is interpreting each word* w_j *as the center of the mass of the Voronoi cell* C_j*, i.e.*

$$i_j = \frac{\int_{C_j} tP(dt)}{\int_{C_j} P(dt)}.$$

Let $T = [0, 1]^2$*,* $w = \{w_1, w_2\}$*,* P *is the uniform distribution over* T *and* $l(d) = d^2$*. There are only two Voronoi languages shown as Figure 2. The sender partitions the type set into two cells* C_1 *and* C_2*, and maps* C_1 *and* C_2 *to* w_1 *and* w_2*, respectively. The receiver interprets* C_1 *and* C_2 *as* $i(w)_1$ *and* $i(w)_2$*, respectively.*

Theorem 1. *The following three statements are equivalent.*

1. *A language* (w, i) *is a Voronoi language with full vocabulary under a prior* $P \in \mathscr{P}$*,*

2. *A language* (w, i) *is a strictly maximal Nash equilibrium under* $\mathscr{P}$*,*

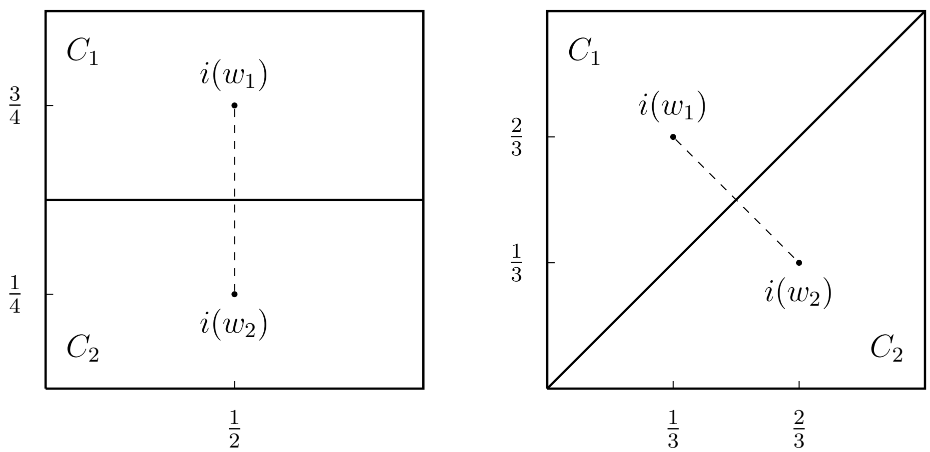


Figure 2: horizontal and diagonal Voronoi languages.

3. A language (w, i) is a strictly E-admissible Nash equilibrium under a prior $P \in \mathscr{P}$,

where the full vocabulary means the range of w is W, i.e. all words in the word list are fully used by the sender. In the following, we always assume that the word list are fully used in Voronoi languages if not particularly indicated.

Proof. 3 ⇒ 1. By Corollary 3, Lemma 5, and Lemma 9, we obtain the proof.

1 ⇒ 2. By the definition of the Voronoi language (w, i), the sender's strategy is $w(t) = \arg\min_{j=1,\cdots,N} \|t - i_j\|$. By the additivity of integral, for all $P \in \mathscr{P}$, the strategy $w(\cdot)$ minimize the expected loss

$$\int_T l\left(\left\|t - i_{w(t)}\right\|\right) P(dt).$$

By the definition of the Voronoi language, the receiver's strategy is

$$i_k = b(C_k) \text{ (for } C_k = \{t \in T : w(t) = w_k\}), \tag{3}$$

and

$$b(C) = \arg\min_{i \in T} \int_C l(\|t - i\|) P d(t),$$

where the probability distribution $P \in \mathscr{P}$ is one of the receiver's priors over T. Definitely, the sender's strategy minimize the expected loss under $P \in \mathscr{P}$.

Hence, by the Definition 2, the language (w, i) is a strictly maximal Nash equilibrium.

$2 \Rightarrow 3$. First, fix the sender's strategy w. i is a strictly maximal response to w. So the function $t \mapsto l\left(\left\|t - i_{w(t)}\right\|\right)$ is a strictly maximal act among the set

$$\left\{t \mapsto l\left\|t - i_{w(t)}\right\| : i \in T^N\right\}.$$

where by the continuity of l and the norm $\|\cdot\|$, the above set is compact and convex due to the compactness and convexity of T^N. By Theorem 5, we know that $t \mapsto l\left(\left\|t - i_{w(t)}\right\|\right)$ is an strictly E-admissible maximal act among the set. i is a strictly E-admissible best response to w. Similarly, we can prove w is a strictly E-admissible best response to i. At last, we obtain the proof. □

The rest of this section paves the way for the proof of Theorem 1.

Lemma 1. *Any strictly E-admissible Nash equilibrium is a strictly maximal Nash equilibrium.*

Proof. By the definitions of both equilibria, each strictly E-admissible Nash equilibrium is a strict Nash equilibrium under the game with some single prior $P \in \mathscr{P}$. Then it is impossible for each player to deviate unilaterally to obtain lower loss under all priors $\mathscr{P}$. So, we have the proof. □

Lemma 2. *There exists a strictly E-admissible Nash equilibrium.*

Proof. Fix any $P \in \mathscr{P}$, the game degenerates into the game without Knightian uncertainty or the Bayesian game. By Lemma 1, Theorem 1 and Theorem 2 of Jäger et al. (2011), there exists a Voronoi language and it is a strict Nash equilibrium. By Remark 1, the strict Nash equilibrium is a strictly E-admissible Nash equilibrium. □

Corollary 1. *There exists a strictly maximal Nash equilibrium.*

Proof. By Lemma 1, any strictly E-admissible Nash equilibrium is a strictly maximal Nash equilibrium. Then by Lemma 2, we get the proof. □

Lemma 3. *Any strictly maximal Nash equilibrium consists of pure strategies.*

Proof. Suppose that there exists a strictly maximal Nash equilibrium (ω^*, μ^*) consisting of non-degenerate behavioral strategies. Fix any t in T. Given a non-degenerate behavioral strategy (ω^*, μ^*), the loss of any type t during information transmission is

$$\sum_{k=1}^{N} \int_T l\left(\|t-i\|\right) \mu_k^*(di)\omega_k^*(t).$$

Firstly, fix the sender's strategy ω^*. For each word w_k^*, due to the non-degenerate strategy used by the receiver, the interpretation μ_k^* maps w_k^* to a lottery on T and the support of the probability measure μ_k^* is not a singleton. Meanwhile, because of the convexity of l and the Euclidean norm, there exists some interpretation μ_k and $i_k \in T$ satisfying $\mu_k(i_k) = 1$, such that

$$\int_T l\left(\|t-i\|\right) \mu_k^*(di) \geq l\left(\|t-i_k\|\right).$$

Abusing notation slightly, we rewrite μ_k as i_k. For each word w_k^*, we find i_k to interpret w_k^*. Repeating the above procedure, we obtain an interpretation vector or the receiver's strategy $i = (i_1, \cdots, i_N)$ for the sender's strategy $w^* = (w_1^*, \cdots, w_N^*)$. By summing up all words $\{w_1^*, \cdots, w_N^*\}$, it has

$$\sum_{k=1}^{N} \int_T l\left(\|t-i\|\right) \mu_k^*(di)\omega_k^*(t) \geq \sum_{k=1}^{N} l\left(\|t-i_k\|\right) \omega_k^*(t),$$

which induces

$$\begin{aligned} &\int_T \sum_{k=1}^{N} \int_T l\left(\|t-i\|\right) \mu_k^*(di)\omega_k^*(t)P(dt) \geq \int_T \sum_{k=1}^{N} l\left(\|t-i\|\right) \omega_k^*(t)P(dt) \\ &\Leftrightarrow L(\omega^*, \mu^*, P) \geq L(\omega^*, i, P). \end{aligned}$$

for all $P \in \mathscr{P}$. Then μ^* is dominated by i and is not a strictly maximal Nash equilibrium. Hence, the receiver uses pure strategies. Similarly, by the convexity of l and the Euclidean norm, we are able to prove that the sender also uses pure strategies in any strictly maximal Nash equilibrium. □

Corollary 2. *Any strictly E-admissible Nash equilibrium consists of pure strategies.*

Proof. Any strictly E-admissible Nash equilibrium is a strictly maximal Nash equilibrium. □

From now on, in this paper we only consider pure strategies.

Lemma 4. *Given a strictly maximal Nash equilibrium* (w^*, i^*)*, the sender's strategy is a Voronoi tessellation corresponding to* i^**, i.e.* $w^*(t) = w_{\underset{j=1,\cdots,N}{\arg\min}\|t-i_j^*\|}$.

Proof. Given the receiver's strategy i^* and any t in T, by the monotonicity of l, the sender uses the word $w^*(t) := w_{\underset{j=1,\cdots,N}{\arg\min}\|t-i_j^*\|}$ to minimize the loss $l(\|t-i^*_{w^*(t)}\|)$. By the additivity of integral, for all $P \in \mathscr{P}$, the strategy $w^*(\cdot)$ minimizes the expected loss

$$\int_T l\left(\left\|t - i^*_{w^*(t)}\right\|\right) P(dt).$$

□

Remark 3. *Since* $w^*(t) = \underset{j=1,\cdots,N}{\arg\min}\|t - i_j^*\|$ *for all* $t \in T$*, even at the ex ante stage the sender's optimal strategy is independent of her priors on* T.

Corollary 3. *Given a strictly E-admissible Nash equilibrium* (w^*, i^*)*, the sender's strategy is a Voronoi tessellation corresponding to* i^**, i.e.* P*-almost everywhere* $w^*(t) = w_{\underset{j=1,\cdots,N}{\arg\min}\|t-i_j^*\|}$.

Proof. By Lemma 1, we get the proof. □

Lemma 5. *Given a strictly E-admissible Nash equilibrium* (w^*, i^*) *under* $P \in \mathscr{P}$*, the receiver's strategy* $i^* = (i_k^*)_{k\in\{1,\ldots,N\}}$ *is a Bayesian estimator interpretation vector. For each cell* C_k^*,

$$i_k^* = \arg\min_{i\in T} \int_{C_k^*} l(\|t - i_{w_k}\|)P(dt) \text{ and } C_k^* = \{t \in T : w^*(t) = w_k\}.$$

Proof. By Remark 1, a strictly E-admissible Nash equilibrium is a Nash equilibrium. Then by Jäger et al. (2011), we know that the receiver's strategy is a Bayesian estimator. □

Lemma 6. *The function* $t \mapsto l(\|t - i_{w^*(t)}\|)$ *is continuous if* w^* *is a Voronoi tessellation corresponding to* i*, i.e.* $w^*(t) = w_{\underset{j=1,\cdots,N}{\arg\min}\|t-i_j\|}$.

Proof. Given any type t, the value of the function moves continuously in the interior of the cell $w^{-1}(w^*(t))$. If a type jumps from one cell to its adjacent cell, it also moves continuously since the point on the boundary is equal distance to the closest interpretations. □

Lemma 7. *The mapping $\phi : T^N \to C_c(T)$ defined by $i \mapsto l(\|t - i_{w^*(t)}\|)$ is continuous on T^N and the set* $\text{cl}(\text{conv}(\{\phi(i) : i \in T^N\}))$ [5] *is convex and compact if the topology $C_c(T)$ equipped with the sup norm and $w^*(t) = w_{\underset{j=1,\cdots,N}{\arg\min}\|t-i_j\|}$.*

Proof. To prove the continuity of ϕ, due to the continuity of l it is sufficient to prove the continuity of $i \mapsto \|t - i_{w^*(t)}\|$. For any $i, i' \in T^N$ and $\epsilon > 0$, if $\|i' - i\| < \epsilon$, then

$$\begin{aligned}
& \sup_{t\in T} \left| \|t - i'_{w^*(t)}\| - \|t - i_{w^*(t)}\| \right| \\
\leq\ & \sup_{t\in T} \|t - i'_{w^*(t)} - t + i_{w^*(t)}\| \\
=\ & \|i'_{w^*(t)} - i_{w^*(t)}\| \\
\leq\ & \|i' - i\| \\
<\ & \epsilon.
\end{aligned}$$

Hence, we obtain the continuity of ϕ.

Although $i \in T^N$ is a convex set, obviously, $\{\phi(i) : i \in T^N\}$ may be not. $\{\phi(i) : i \in T^N\}$ is compact due to the continuity of ϕ respect to i and the compactness of T^N. By Theorem 5.35 of Aliprantis and Border (2006), i.e. the closed convex hull of a compact set is compact in a completely metrizable locally convex space, $\text{cl}(\text{conv}(\{\phi(i) : i \in T^N\}))$ is compact with respect to $C_c(T)$. And the closed set of a convex set is also convex in the topological vector space $C_c(T)$. So we obtain the proof. □

Lemma 8. *Given a strictly maximal Nash equilibrium (w^*, i^*), the receiver's strategy i^* is a Bayesian estimator interpretation vector under some prior $P \in \mathscr{P}$.*

Proof. By Lemma 4, w^* is a Voronoi tessellation corresponding to the interpretation i^*. By Lemma 6, the function $t \mapsto l(\|t - i_{w^*(t)}\|)$ (short for $l(t, w^*, i)$) is continuous. So, the function is an act as we have defined in Appendix A. Given the equilibrium (w^*, i^*), by the definition of strictly maximal Nash equilibrium, $l(t, w^*, i^*)$ is a strictly maximal act among $\{l(t, w^*, i) : i \in T^N\}$. Next, we will show that $l(t, w^*, i^*)$ is also a strictly maximal act in $\text{cl}(\text{conv}(\{l(t, w^*, i) : i \in T^N\}))$, where $\text{cl}(\text{conv}(\{l(t, w^*, i) : i \in T^N\}))$ is a convex and compact set in $C_c(T)$ proved in Lemma 7. Suppose that

[5] $\text{cl}(\text{conv}(\{\phi(i) : i \in T^N\}))$ is the closed and convex hull of $\{\phi(i) : i \in T^N\}$.

the act is not, it has some mixed act $\lambda l(t,w^*,i^1)+(1-\lambda)l(t,w^*,i^2)\in \mathrm{cl}(\mathrm{conv}(\{l(t,w^*,i):i\in T^N\}))\backslash\{l(t,w^*,i):i\in T^N\}$ such that

$$\int_T l(t,w^*,i^*)P(dt)\geq \int_T \lambda l(t,w^*,i^1)+(1-\lambda)l(t,w^*,i^2)P(dt)$$

for all $P\in\mathscr{P}$, where $i^1\neq i^2\in T^N$. Due to the convexity of l and the Euclidean norm, it has

$$\int_T \lambda l(t,w^*,i^1)+(1-\lambda)l(t,w^*,i^2)P(dt)\geq \int_T l(t,w^*,\lambda i^1+(1-\lambda)i^2)P(dt).$$

Since $\lambda i^1+(1-\lambda)i^2\in T^N$, $l(t,w^*,i^*)$ is not strictly maximal in $\{l(t,w^*,i): i\in T^N\}$. So, we obtain a contradiction.

Then we are able to apply Theorem 5. It has that $l(t,w^*,i^*)$ is an E-admissible act in $\mathrm{cl}(\mathrm{conv}(\{l(t,w^*,i):i\in T^N\}))$ and also in its subset $\{l(t,w^*,i):i\in T^N\}$. So there exists some $P\in\mathscr{P}$ such that

$$\int_T l(t,w^*,i^*)P(dt)<\int_T l(t,w^*,i)P(dt)$$

for all $i\in T^N$. At last, by Lemma 5, we obtain the proof. □

Remark 4. *Although the sender have some priors, both strictly maximal Nash equilibria and strictly E-admissible Nash equilibria only depend on the receiver's belief. Hence, we can relax the assumption that both the sender and the receiver know the receiver's prior is $\mathscr{P}$ without the assumption that they share a common prior set.*

Lemma 9. *Any strictly E-admissible Nash equilibrium (w,i) is a language with full vocabulary, where full vocabulary means that the range of w is W.*

Proof. Prove by contradiction. Given a strictly E-admissible Nash equilibrium (w,i), suppose that without loss of generality, $w^{-1}(w_1)=\emptyset$, i.e. the first word has not been used, and the second word is used. Then we split $C_2=w^{-1}(w_2)$ into C_2' and C_2''. C_2' and C_2'' map to w_1 and w_2, respectively, and nothing else changes. We denote the new strategy profile (w',i). The difference between the expected loss of (w, i) and of (w', i) is

$$\int_{C_2} l\left(\|t-i_2\|\right)P(dt) - \left(\int_{C_2'} l\left(\|t-i_1\|\right)P(dt) + \int_{C_2''} l\left(\|t-i_2\|\right)P(dt)\right)$$
$$= \int_{C_2} l\left(\|t-i_2\|\right)P(dt) - \left(\int_{C_2'} l\left(\|t-i_1\|\right)P(dt) + \int_{C_2-C_2'} l\left(\|t-i_2\|\right)P(dt)\right)$$
$$= \int_{C_2'} l\left(\|t-i_2\|\right)P(dt) - \int_{C_2'} l\left(\|t-i_1\|\right)P(dt)$$

By Lemma 5, if (w, i) is a strictly E-admissible Nash equilibrium, it should have

$$i_k = \arg\min_{i_k\in T} \int_{C_k} l\left(\|t-i_k\|\right)P(dt).$$

Since $C_1 = w^{-1}(w_1) = \emptyset$, i_1 can be any point in T. Then let i_1 be

$$\arg\min_{i\in T} \int_{C_2'} l\left(\|t-i\|\right)P(dt).$$

We obtain

$$\int_{C_2'} l\left(\|t-i_2\|\right)P(dt) - \int_{C_2'} l\left(\|t-i_1\|\right)P(dt) \geq 0.$$

So (w', i) dominate (w, i), and it generates a contradiction. □

4 Indeterminacy

In this section, we focus on the role of Knightian uncertainty in equilibria.

Given a strictly maximal Nash equilibrium (w, i) under $\mathscr{P}$, w is a Voronoi tessellation generated by the interpretation vector i. The equilibrium (w, i) can be fully characterized by the interpretation vector i and we use i to denote equilibrium (w, i). The next proposition says that the strictly maximal Nash equilibria moves continuously on the probability set $\mathscr{P}$.

Proposition 1. *The correspondence $\nu : \mathscr{P} \twoheadrightarrow T^N$, where $\nu(P)$ is the set of all strict equilibria under the probability P, is upper hemicontinuous, i.e. $\forall P \in \mathscr{P}$ if for every open neighborhood U_i of $\nu(P)$, there is a neighborhood U_P of P such that $P' \in U_P$ implies $\nu(P') \subset U_i$. Further, given some $P \in \mathscr{P}$, if there are any two different strictly maximal equilibria $i^1, i^2 \in \nu(P)$, then there exist two open neighborhoods U_{i^1} of i^1 and U_{i^2} of i^2 such that $U_{i^1} \cap U_{i^2} = \emptyset$.*

Proof. First, let us consider the signaling game with two words under the type space $T = [0, 1]$. Besides, let us assume the loss function $l(d) = d^2$. If the equilibrium $i^* = (i_1^*, i_2^*)$, then there exists some $P^* \in \mathscr{P}$ such that

$$\begin{cases} f_1(P^*, i^*) := \dfrac{\int_0^{\frac{i_1^*+i_2^*}{2}} tP^*(dt)}{\int_0^{\frac{i_1^*+i_2^*}{2}} P^*(dt)} = i_1^* \\ f_2(P^*, i^*) := \dfrac{\int_{\frac{i_1^*+i_2^*}{2}}^1 tP^*(dt)}{\int_{\frac{i_1^*+i_2^*}{2}}^1 P^*(dt)} = i_2^*, \end{cases}$$

where the mapping $f(P, i) = (f_1(P, i), f_2(P, i)) : \mathscr{P} \times T^2 \to T^2$ satisfies $f(P^*, i^*) = i^*$. Notice that i_1^* and i_2^* are the centers of the mass of $[0, \frac{i_1^*+i_2^*}{2})$ and $[\frac{i_1^*+i_2^*}{2}, 1]$, respectively. Next, let

$$\nu(P) := \{i \in T^2 : f(P, i) = i\}.$$

The graph of ν is

$$\operatorname{Gr} \nu = \{(P, i) \in \mathscr{P} \times T^2 : f(P, i) = i\}$$

Due to the continuity [6] of f on $\mathscr{P} \times T^2$, the set $\operatorname{Gr} \nu$ is closed. By the closed graph theorem (Theorem 17.11 of Aliprantis and Border (2006)), ν is upper hemicontinuous given the compactness of T^2.

In the similar way, we extend this outcome to higher-dimensional space T under more general loss function l with N words. Given the equilibrium $i^* = (i_1^*, \cdots, i_N^*)$, for each $j \in \{1, \cdots, N\}$ it has

$$f_j(P^*, i^*) := \arg\min_{i_j \in T} \int_T l(\|t - i_j\|) 1_{C_j^*} P^*(dt) = i_j^*$$

where

$$C_j^* = \left\{ t \in T : \|t - i_j^*\| \leq \|t - i_k^*\| \text{ for all } k \in \{1, \ldots, N\} \right\}.$$

Due to the convexity of l and nonlinearity of l with respect to i, by Theorem 7.15 of Lehmann and Casella (1998), there is a unique i_j^* minimizing f_j,

[6] $\mathscr{P}$ is is equipped with the weak* topology metricized by the Prokhorov metric. And $\mathscr{P} \times T^2$ is equipped with the product topology.

i.e. the Bayesian estimator i_j^* is unique conditional on C_j^*. Then, f_j is a well-defined function.

Similarly, we construct the mapping $f(P,i) = (f_1(P,i), \cdots, f_N(P,i)) : \mathscr{P} \times T^N \to T^N$ such that $f(P^*, i^*) = i^*$. Next, we show that $f(P,i)$ is continuous on $\mathscr{P} \times T^N$. It is sufficient to demonstrate the continuity of

$$\int_T l(\|t - i_j\|) 1_{C_j} P(dt),$$

on $\mathscr{P} \times T^N$. $t \mapsto l(\|t - i_j\|) 1_{C_j}$ is continuous on $C_j \subseteq T$, given that w^* is a Voronoi tessellation. By the definition of weak* convergence, if $P^\lambda \overset{weak^*}{\longrightarrow} P^*$, then

$$\int_T l(\|t - i_j\|) 1_{C_j} P^\lambda(dt) \to \int_T l(\|t - i_j\|) 1_{C_j} P^*(dt).$$

Further, given a sequence of Voronoi languages $\{i^\lambda\}_{\lambda \in \Lambda}$ with $i^\lambda \to i^*$, it has a sequence of P^*-measurable functions $(l(\|t - i_j^\lambda\|) 1_{C_j^\lambda})_{\lambda \in \Lambda}$ convergence in measure to $l(\|t - i_j^*\|) 1_{C_j^*}$. Then by Lebesgue's dominated convergence theorem, it has

$$\int_T l(\|t - i_j^\lambda\|) 1_{C_j^\lambda} P^*(dt) \to \int_T l(\|t - i_j\|) 1_{C_j} P^*(dt).$$

So, $f(P,i)$ is continuous on its domain.

As before, let

$$\nu(P) := \{i \in T^2 : f(P,i) = i\}.$$

The graph of ν is

$$\mathrm{Gr}\, \nu = \{(P,i) \in \mathscr{P} \times T^2 : f(P,i) = i\}.$$

$\mathrm{Gr}\, \nu$ is closed due to the continuity of f on $\mathscr{P} \times T^N$. By the closed graph theorem, ν is upper hemicontinuous given the compactness of T^N.

Second, it is sufficient to prove any strictly maximal equilibrium is a local minimum point of the total loss $L(P,i)$. Since L is continuous on interpretations I, given a strict (strictly maximal) equilibrium under P, it is a local minimum point due to common interests between the sender and the receiver. □

The above proposition confirms equilibria moving continuously on a set of probabilities. It seems that there are infinite equilibria around any equilibrium. However, it is possible that the correspondence ν is constant on its domain. Of course, the constant correspondence is continuous. Henceforth, we seem any Voronoi language with essentially same partition of type space as one Voronoi language. The following example shows that there is a unique Voronoi language under multiple priors.

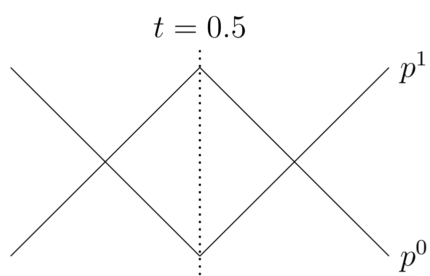
Example 2. *Given $T = [0,1]$ and two word list $\{w_1, w_2\}$, $\mathscr{P}$ is a set of probabilities* $\operatorname{conv}(\{P^0, P^1\})$*, where the densities of P^0 and P^1 are*

$$p^0(t) = \begin{cases} \frac{3}{4} + t & 0 \le t < \frac{1}{2} \\ \frac{7}{4} - t & \frac{1}{2} \le t \le 1 \end{cases},$$

and

$$p^1(x) = \begin{cases} \frac{5}{4} - t & 0 \le t < \frac{1}{2} \\ \frac{1}{4} + t & \frac{1}{2} \le t \le 1 \end{cases},$$

respectively. Although the players have a set of convex and closed priors $\mathscr{P}$, due to the symmetric of priors, the sender always partitions T into $[0, 0.5)$ and $[0.5, 1]$ if she has two words. And the receiver's interpretations for w_1, w_2 are also symmetric under all priors. It means that there is no indeterminacy.



Although the prior set in the above example contain multiple priors, close probabilities (in weak* topology) are not in, like a probability P^0_ϵ, $\epsilon \to 0^+$, with the density

$$p^0_\epsilon(t) = \begin{cases} \frac{3}{4} + t - \epsilon & 0 \le t < \frac{1}{2} \\ \frac{7}{4} - t + \epsilon & \frac{1}{2} \le t \le 1 \end{cases}.$$

If both players have sufficient uncertainty, there exists a continuum of equilibria around any strictly maximal Nash equilibrium.

Theorem 2. *If $\mathscr{P}$ contains an open ball, for each strictly maximal Nash equilibrium i^* and any open neighborhood U_{i^*} of i^*, there exists a continuum of strictly maximal Nash equilibria in U_{i^*}.*

Proof. Since i^* a maximal equilibrium, by Theorem 1, it is an E-admissible equilibrium under some $P^* \in \mathscr{P}$. By Proposition 1, it has some open neighborhood U_{i^*} of i^* and a neighborhood or open ball U_{P^*} of P^* such that $\nu(P') \subseteq U_{i^*}$ if $P' \in U_{P^*}$. By Proposition 1, ν maps to a unique value in some local region. Then, there exists some open ball $B_{P^*} \subseteq U_{P^*}$ such that every $\nu(P')$ is singleton if $P' \in B_{P^*}$. Then $\nu(P)$ is a continuous function on B_{P^*}. As B_{P^*} is connected in the compact and metrizable space of all probability measures, $\nu(B_P*)$ is connected in T^N due to the continuity of ν. If $\nu(B_{P^*})$ is not a singleton, by the continuity of $\nu(B_{P^*})$, we obtain the continuum.

In the following, we prove that $\nu(B_{P^*})$ is not a singleton. The proof idea is to destroy the symmetry happened in Example 2.

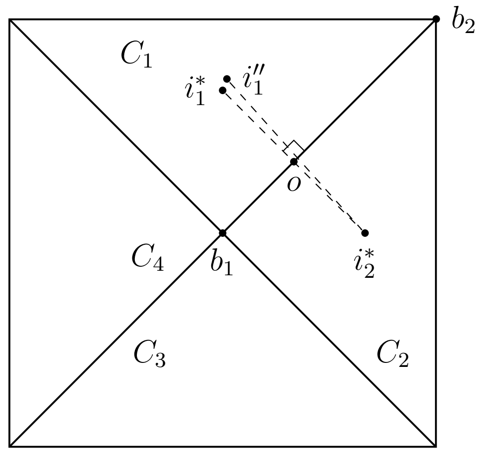


Figure 3: A Voronoi language with four words

For any Voronoi tessellation and any two adjacent intervals or cells, the two generating points of the two cells forms a segment, like the segment $i_1^* i_2^*$ in Figure 3. The segment intersects with the boundary point [7] or plane of the two cells at some point, such as the intersection point o in Figure 3. Then the distance of each generating point to the intersection point is same, e.g. $\|i_1^* - o\| = \|i_2^* - o\|$ in Figure 3.

Suppose that $\nu(B_{P^*})$ is a singleton. Then the Voronoi languages share the same Voronoi tessellation but possible not share the generators. So the set of the Voronoi languages can be rewritten as $\{(w, i^\kappa) : \kappa \in K\}$, where

[7] In one-dimensional type space, the two generating points pass through the boundary point.

K is an index set. Without loss of generality, we focus on any two cells [8] denoted as C_1 and C_2. The equilibrium $i^* = (i_1^*, i_2^*, \cdots, i_N^*)$ is a strictly E-admissible Nash equilibrium under P^*. Since B_{P^*} is an open ball, we are able to find a probability $P'' \in B_{P^*}$, where P'' shares the same probability value with the probability P^* in cells $C_2, \cdots, C_N$, and deviates a little bit from P^* in the cell C_1 such that the new Bayesian estimator for C_1 is i_1'' satisfying $\|i_1'' - o\| \neq \|i_2^* - o\|$ shown as Figure 3. It is possible because the value of P'' on C_2 dose not change and the new Bayesian estimator under P'' should be i_2^* as before. As we have assumed, the Voronoi language under P'' shares the same the Voronoi tessellation under P^*. But under P'', $\|i_1'' - o\| \neq \|i_2^* - o\|$ and it generates a contradiction. So we obtain the proof.

□

5 Vague Voronoi languages

5.1 Vagueness

In this paper, we follow a conservative definition of vagueness by Perirce that is a word is vague if it admits borderline cases (pp.14-15 of Keefe and Smith (1996)). For example, if the values in [0,0.5] and (0.5, 1] are called small and big, respectively, then the boundary between the two words is sharp and there are no thick borderline cases. The both words have thick borderline cases if values in [0,0.4] and (0.6,1] are called small and big, respectively, (0.4,0.6) are called small or big. Notice that, values in (0.4,0.6) are hard to be called definitely small (or big) but values in [0,0.4] is definitely small. Now, we go back our games. The existence of vagueness is the existence of many equilibria such that the borderline cases formed by the sender have positive measure and the interpretations or Bayesian estimators of the receiver of the equilibria are connected. Here, the interpretations are usually called prototypical or typical points of a concept in prototype theory. The reason why we require the interpretations are connected in the following definition is that Berlin and Kay 1969 (see section 1.5) find that their test subjects point at more than one chip as being a typical instance (seem as the receivers interpretation in this paper) of a color, these chips are always adjacent. Besides, we can image that if 0.14 and 0.16 are typically small but any point in (0.14,0.16) is not typically small, it seems strange. Then,

[8]It is possible because the number of words is greater than 2.

Gärdenfors (2000) (see p.138) assumes that Voronoi concepts could be generated by prototype areas, where each area is a circle and then is definitely connected. And Douven et al. (2013) and Decock and Douven (2014) relax Gärdenfors's assumption of any prototypical area to be connected.

5.2 Definitions

By Figure 4, what a color denoting on the figure is a connected region, like 'red' and 'orange'. The meaning of the word 'red' should be some point of the most left region and never be some point in the most right area. A feature of the meaning of natural words usually is connected. In our model, it says that all the types sent to a word should be a connected set. To achieve this, we follow the idea of Douven et al. (2013) and Decock and Douven (2014) and assume the set of interpretations should be connected. Then we provide the definition of vague Voronoi languages as follows.

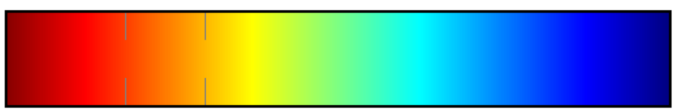


Figure 4: a color bar

Definition 6. *Given a convex and compact set* $T \subseteq \mathbb{R}^n$, *and a finite word list* $\{w_1, \cdots, w_N\}$, *a set of languages*

$$\mathscr{V} := \{(w^\lambda, i^\lambda) : \lambda \in \Lambda\}$$

indexed by a set Λ *is called a vague Voronoi language if*

- *each language in* $\mathscr{V}$ *is a Voronoi language,*
- *(thickness) its boundary set*

$$\mathscr{B}(\mathscr{V}) := T - \bigcup_{k \leq N} \bigcap_{\lambda \in \Lambda} C_k^\lambda$$

  is positive measured under Lebesgue measure and $C_k^\lambda = \{t \in T : w^\lambda(t) = w_k\}$,

- *(connectedness) for each word w_k in the word list $\{w_1, \cdots, w_N\}$ the interpretation set of the word w_k, i.e. $I_k(\mathscr{V}) = \{i_k^\lambda : \lambda \in \Lambda\}$ is connected,*

- *(maximality) and $\mathscr{V}$ is maximal, i.e. there is no set of Voronoi languages $\mathscr{V}'$ satisfying the above two conditions such that $\mathscr{V}$ is a proper subset of $\mathscr{V}'$.*

Remark 5. *For each λ, since the sender's strategy w^λ divides T into N cells, it has $T - \bigcup_{k \leq N} C_k^\lambda \equiv \emptyset$. Then the thickness of boundary requires that there exists at least two different Voronoi languages $(w^{\lambda_1}, i^{\lambda_1})$ and $(w^{\lambda_2}, i^{\lambda_2})$ such that for one word w_k the difference of the two corresponding cells $C_k^{\lambda_1} \triangle C_k^{\lambda_2} := (C_k^{\lambda_1} \setminus C_k^{\lambda_2}) \bigcup (C_k^{\lambda_2} \setminus C_k^{\lambda_1})$ is positive measurable. In another word, the word w_k denotes two different meanings or types in different Voronoi languages. Then the word w_k is vague.*

The connectedness means that we only collect connected Voronoi languages. In Example 1, given $T = [0,1]^2$, $w = \{w_1, w_2\}$ and the uniform distribution P, as Figure 2 there are only two Voronoi languages named by horizontal Voronoi language and diagonal Voronoi language. We do not collect them to form a vague Voronoi language. Although in Example 1 the sender use w_1 to denote C_1 of both horizontal Voronoi language and diagonal Voronoi language, in our natural language we would like to call C_1 of horizontal Voronoi language as "Up" and C_1 of diagonal Voronoi language as "Upper-left". To avoid aggregating essentially different words, we require the connectedness.

The maximality ensures that a vague Voronoi language is unique in some local region.

Remark 6. *For each λ, the set C_k^λ is a convex set, which is an intersection set of half-spaces. And the set $\bigcap_{\lambda \in \Lambda} C_k^\lambda$ definitely expressed by the word w_k is also convex since it is an intersection set of convex sets. This verifies the thought from Gärdenfors (2000) that the meaning of a word is convex.*

Intuitively, the higher uncertainty of belief the higher vagueness of a language is. This following proposition verifies the intuition.

Proposition 2. *Given two convex and closed set of probabilities $\mathscr{P}_1 \subseteq \mathscr{P}_2$, then if there is a vague Voronoi language $\mathscr{V}^{\mathscr{P}_1}$ under $\mathscr{P}_1$, then there is a vague Voronoi language $\mathscr{V}^{\mathscr{P}_2}$ under $\mathscr{P}_2$ such that $\mathscr{B}(\mathscr{V}^{\mathscr{P}_1}) \subseteq \mathscr{B}(\mathscr{V}^{\mathscr{P}_2})$.*

Proof. Due to $\mathscr{P}_1 \subseteq \mathscr{P}_2$, by Definition 3 and 2, the set of all strictly E-admissible equilibria under $\mathscr{P}_1$ is a subset of the set of all strictly E-admissible equilibria under $\mathscr{P}_2$. By Theorem 1, under some convex and closed set $\mathscr{P}$, the set of all Voronoi languages coincides with the set of all strictly E-admissible equilibria. So, the set of all Voronoi languages under $\mathscr{P}_1$ is a subset of the set of all Voronoi languages under $\mathscr{P}_2$. The vague Voronoi language $\mathscr{V}^{\mathscr{P}_1}$ can be extended to $\mathscr{V}^{\mathscr{P}_2}$. By the monotonicity of the operator $\mathscr{B}$ from Definition 6, we can obtain $\mathscr{B}(\mathscr{V}^{\mathscr{P}_1}) \subseteq \mathscr{B}(\mathscr{V}^{\mathscr{P}_2})$. □

5.3 Examples

Example 3. *Let T be $[0,1]$, $l(d) = d^2$, $W = \{w_1, w_2\}$ and $\mathscr{P} = \{P^\lambda : (1-\lambda)P^0 + \lambda P^1, \lambda \in [0,1]\}$, where the density of P^0 is*

$$p^0(t) = \begin{cases} \frac{3}{2} & 0 \le t \le \frac{2}{5} \\ \frac{2}{3} & \frac{2}{5} < t \le 1 \end{cases},$$

and the density of P^1 is

$$p^1(x) = \begin{cases} \frac{1}{2} & 0 \le t \le \frac{2}{5} \\ \frac{4}{3} & \frac{2}{5} < t \le 1 \end{cases},$$

Obviously, the probability $P^{0.5}$, i.e. $0.5P^0 + 0.5P^1$, is the uniform distribution on $[0,1]$. For each probability distribution P^λ, if let the both players share the prior P^λ, there exist a Voronoi language by Lemma 2. For example, let $P^\lambda = P^0$, we can get a Voronoi language as follows. The sender divides her type space T into two cells $[0, b^0]$ and $(b^0, 1]$, where $b^0 \in (0,1)$, and she sends the word w_1 if her type belongs to $[0, b^0]$ otherwise w_2. As we know, if both players share the uniform prior $P^{0.5}$, $b^{0.5}$ should be 0.5. Compared to $P^{0.5}$, P^0 is small biased. The sender decreases the value of boundary, i.e. $b^0 < 0.5$, to reduce the expected loss. The receiver should interpret w_1 as $i_1^0 \in (0, b^0)$ and w_2 as $i_2^0 \in (b^0, 1)$. Since b^0 is the center point between i_1^0 and i_2^0, we have

$$b^0 = \frac{i_1^0 + i_2^0}{2}. \tag{4}$$

Besides, i_1^0 is the center of mass of the cell $[0, b^0]$. We obtain

$$\begin{aligned} i_1^0 &= \frac{\int_{[0,b^0]} tp^0(t)dt}{\int_{[0,b^0]} p^0(t)dt} \\ &= \frac{\int_0^{\frac{2}{5}} \frac{3}{2}tdt + \int_{\frac{2}{5}}^{b^0} \frac{2}{3}tdt}{\int_0^{\frac{2}{5}} \frac{3}{2}dt + \int_{\frac{2}{5}}^{b^0} \frac{2}{3}dt} \\ &= \frac{\frac{3}{25} + \frac{2}{3}(\frac{b^{0\,2}}{2} - \frac{2}{25})}{\frac{3}{5} + \frac{2}{3}(b^0 - \frac{2}{5})}. \end{aligned} \tag{5}$$

Meanwhile, i_2^0 is the center of mass of the cell $[b^0, 1]$. Then, it has

$$\begin{aligned} i_2^0 &= \frac{\int_{[b^0,1]} tp^0(t)dt}{\int_{[b^0,1]} p^0(t)dt} \\ &= \frac{\int_{b^0}^1 \frac{2}{3}tdt}{\int_{b^0}^1 \frac{2}{3}dt} \\ &= \frac{1 + b^0}{2}. \end{aligned} \tag{6}$$

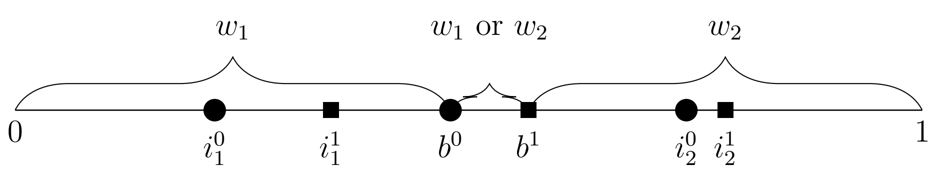
Combining Formula (4), (5), and (6), we have

$$\begin{cases} b^0 = \dfrac{1}{40}(-5 + 3\sqrt{65}) \approx 0.480 \\ i_1^0 = -\dfrac{11}{16} + \dfrac{9}{80} \approx 0.220 \\ i_2^0 = \dfrac{7}{16} + \dfrac{3}{80}\sqrt{65} \approx 0.740 \end{cases}$$

If both players share the uniform prior P^1, it also has a unique Voronoi language. Compared to $P^{0.5}$, P^1 is great biased. The sender increases the value of boundary, i.e. $b^1 > 0.5$, to reduce the expected loss. The receiver should interpret w_1 as $i_1^1 \in (0, b^1)$ and w_2 as $i_2^1 \in (b^1, 1)$. Similarly, we obtain *the following equation system.*

$$\begin{cases} b^1 = \dfrac{1}{2}(i_1^1 + i_2^1) \\ i_1^1 = \dfrac{\int_0^{\frac{2}{5}} \frac{1}{2} t dt + \int_{\frac{2}{5}}^{b^1} \frac{4}{3} t dt}{\int_0^{\frac{2}{5}} \frac{1}{2} dt + \int_{\frac{2}{5}}^{b^1} \frac{4}{3} dt} = \dfrac{\frac{1}{25} + \frac{2}{3}({b^1}^2 - \frac{4}{25})}{\frac{4}{3}b - \frac{1}{3}} \\ i_2^1 = \dfrac{\int_{b^1}^1 \frac{4}{3} t dt}{\int_{b^1}^1 \frac{4}{3} dt} = \dfrac{1 + b^1}{2} \end{cases}$$

We have $b^1 \approx 0.566$, $i_1^1 \approx 0.348$, and $i_2^1 \approx 0.783$. Similarly, for each prior P we obtain a unique language (w^P, i^P). Actually, the set of all Voronoi languages $\mathscr{V} := \{(w^P, i^P) : P \in \mathscr{P}\}$ forms a vague Voronoi language. $[0, b^0)$ is definitely named w_1 and $(b^1, 1]$ is definitely named w_2. But $[b^0, b^1]$ is sometimes called w_1 and sometimes called w_2. In the following, we will check whether $\mathscr{V}$ is a vague language.



First, the thickness is satisfied due to $b^1 - b^0 > 0$. Second, we need to check the connectedness. As in one-dimensional type space, essentially, there is one Voronoi language for each prior P. Then the correspondence $\nu(P)$ degenerates into the mapping $\nu(P) = i^P = (i_1^P, i_2^P)$. By the continuity of ν and the connectedness of $\mathscr{P}$, we are able to obtain that both sets $I_1(\mathscr{V})$ and $I_2(\mathscr{V})$ are connected, where $I_1(\mathscr{V}) := \{i_1^P : P \in \mathscr{P}\}$ and $I_2(\mathscr{V}) := \{i_2^P : P \in \mathscr{P}\}$. The maximality is trivial.

In the following, we provide some example under bivariate normal distributions and with more words. Usually, finding Voronoi languages with general distributions in a higher-dimensional space is not easy as Example 3. Fortunately, we are able to approximate Voronoi languages by the algorithm of Lloyd (1982). In detail, first step, randomly select the receiver's interpretations from the type set, i.e. randomly start from some generating points from the type set. Second step, generate the receiver's partition strategy

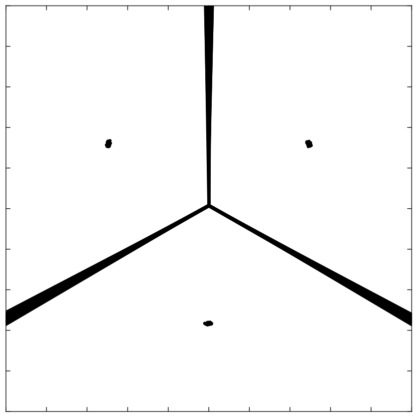


Figure 5: a vague Voronoi languages with three words under probabilities of the convex hull of $\mathcal{N}((0,0), \mathrm{diag}(0.81, 0.81))$ and $\mathcal{N}((0,0), \mathrm{diag}(1,1))$.

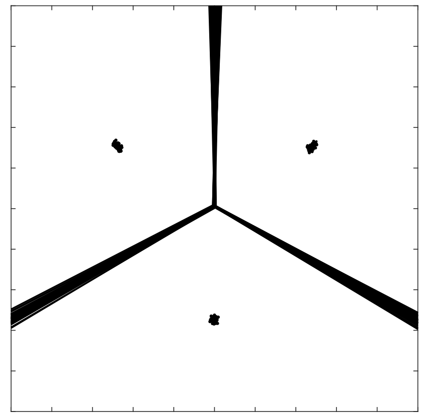


Figure 6: a vague Voronoi languages with three words under probabilities of the convex hull of $\mathcal{N}((0,0), \mathrm{diag}(0.49, 0.49))$ and $\mathcal{N}((0,0), \mathrm{diag}(1,1))$.

according to the receiver's interpretations, i.e. Voronoi tessellation is generated. Third step, compute the receiver's expected estimator of the sender' strategy, i.e. compute the center of mass of each Voronoi cell. Forth step, move the generating points to the centers of mass of the cells calculated in the third step and then repeatedly executes from the second step. In this way we obtain Voronoi languages. And we are able to collect Voronoi language under each prior to form vague Voronoi languages.

Example 4. *Let T be $[-1,1]$, $l(d) = d^2$, $W = \{w_1, w_2, w_3\}$ and $\mathscr{P} = \{(1-\lambda)P^0 + \lambda P^1 : \lambda \in [0,1]\}$, where P^0 is a bivariate normal distribution with a mean vector $\mu^0 = (0,0)$ and a covariance matrix is a diagonal matrix* $\mathrm{diag}(0.81, 0.81)$*, i.e. $t \sim \mathcal{N}((0,0), \mathrm{diag}(0.81, 0.81))$. And P^1 is a bivariate normal distribution with a mean vector $\mu^1 = (0,0)$ and a covariance matrix being a diagonal matrix* $\mathrm{diag}(1,1)$*, i.e .$t \sim \mathcal{N}((0,0), \mathrm{diag}(1,1))$. A vague Voronoi language is shown as Figure 5.*

Example 5. *Let T be $[-1,1]$, $l(d) = d^2$, $W = \{w_1, w_2, w_3\}$ and $\mathscr{P} = \{(1-\lambda)P^0 + \lambda P^1 : \lambda \in [0,1]\}$, where P^0 is a bivariate normal distribution with a mean vector $\mu^0 = (0,0)$ and a covariance matrix is a diagonal matrix* $\mathrm{diag}(0.49, 0.49)$*, i.e. $t \sim \mathcal{N}((0,0), \mathrm{diag}(0.49, 0.49))$. And P^1 is a bivariate* *normal distribution with a mean vector* $\mu^1 = (0, 0)$ *and a covariance matrix being a diagonal matrix* $\mathrm{diag}(1, 1)$*, i.e .t* $\sim \mathcal{N}((0, 0), \mathrm{diag}(1, 1))$*. A vague Voronoi language is shown as Figure 6.*

Comparing Example 4 and Example 5, the priors of the latter are more uncertain than the former's. Naturally, by Figure 5 and 6, we see that the boundary region of the latter is greater than the former's due to higher uncertainty.

Example 6. *Let* T *be* $[-1, 1]$*,* $l(d) = d^2$*,* $W = \{w_1, \cdots, w_{27}\}$ *and* $\mathscr{P} = \{(1-\lambda)P^0 + \lambda P^1 : \lambda \in [0, 1]\}$*, where* P^0 *is a bivariate normal distribution with a mean vector* $\mu^0 = (0, 0)$ *and a covariance matrix is a diagonal matrix* $\mathrm{diag}(1, 1)$*, and* P^0 *is a bivariate normal distribution with a mean vector* $\mu^1 = (0, 0)$ *and a covariance matrix is a diagonal matrix* $\mathrm{diag}(1.44, 1.44)$*. A vague Voronoi language is shown as Figure 7.*

Example 7. *Let* T *be* $[-1, 1]$*,* $l(d) = d^2$*,* $W = \{w_1, \cdots, w_{27}\}$ *and* $\mathscr{P} = \{(1-\lambda)P^0 + \lambda P^1 : \lambda \in [0, 1]\}$*, where* P^0 *is a bivariate normal distribution with a mean vector* $\mu^0 = (0, 0)$ *and a covariance matrix is a diagonal matrix* $\mathrm{diag}(0.035, 0.035)$*, and* P^0 *is a bivariate normal distribution with a mean vector* $\mu^1 = (0, 0)$ *and a covariance matrix is a diagonal matrix* $\mathrm{diag}(0.04, 0.04)$*. A vague Voronoi language is shown as Figure 8.*

Comparing Example 6 and Example 7, the densities in Example 6 are far more evenly than then densities in Example 7. Points are heavily concentrated around $(0, 0)$ in Example 7. So as shown in Figure 8, the sender pay more attention on the types around $(0, 0)$ and use more words around $(0, 0)$ to reduce loss.

5.4 No vagueness in Bayesian worlds

Proposition 3. *If both players share a single prior* P *over* T*, then there exists no vague Voronoi language.*

Proof. By Proposition 1, each strict Nash equilibrium is locally unique in some local region. Then it has finite strict Nash equilibria when the range of i is a convex and compact set T^N. So it never has a continuum of equilibria. □

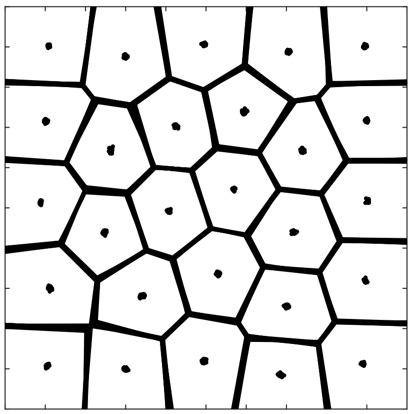


Figure 7: a vague Voronoi languages with 27 words under probabilities of the convex hull of $\mathcal{N}((0,0), \mathrm{diag}(1,1))$ and $\mathcal{N}((0,0), \mathrm{diag}(1.44, 1.44))$.

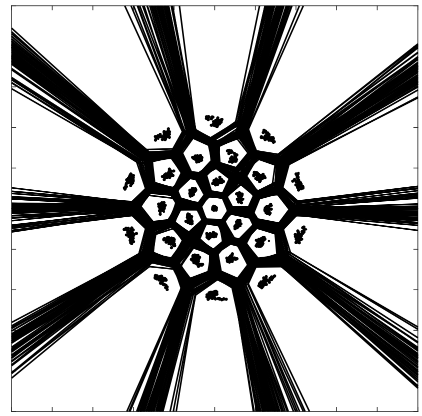


Figure 8: a vague Voronoi languages with 27 words under probabilities of the convex hull of $\mathcal{N}((0,0), \mathrm{diag}(0.035, 0.035))$ and $\mathcal{N}((0,0), \mathrm{diag}(0.04, 0.04))$.

5.5 Existence of vagueness in Knightian worlds

Even $\mathscr{P}$ is a set of multiple beliefs, it does not always generate a vague Voronoi languages. Show an example in one-dimensional type space as follows.

Example 8. *Given $T = [-1, 1]$ and two word list $\{w_1, w_2\}$, $\mathscr{P}$ is a set of normal distributions with probabilities $\{t \sim \mathcal{N}(0, \sigma^2) : \sigma \in [1, 3]\}$. If there are two words, the sender always partitions T into $[0, 0.5)$ and $[0.5, 1]$ for any distribution from $\mathscr{P}$. It means that there is no vague Voronoi languages even there are multiple priors.*

As before, if the prior set contains sufficient uncertainty, there exists a vague Voronoi language.

Theorem 3. *There exists a vague Voronoi language if the probabilities $\mathscr{P}$ contains a open ball.*

Proof. Now let us check the four conditions of the definition. The existence of Voronoi languages has been proved by Jäger et al. (2011) and the maximality is trivial. Each efficient language is a Voronoi language, where under a prior of the receiver P a language (w, i) is called efficient if it minimizes the loss $L(w, i, P)$ among all languages. We manage to find a set of some efficient languages that is connected and thick.

Step 1. Prove that there exists a set of efficient languages such that its interpretation set is connected.

Let $\psi : \mathscr{P} \twoheadrightarrow T^N$ be a constant correspondence. Here, we let $\psi(P) = T^N$ for all $P \in \mathscr{P}$, where N is the number of words. Since T^N is a compact set and ψ is constant, then ψ a compact-valued correspondence. Let a value function $-L : \text{Graph}\psi \to \mathbb{R}$, i.e.

$$-L(P, i) = -\sum_{j \in N} \int_{t \in w^{-1}(w_j)} l(\|t - i_j\|) P(dt),$$

and $-L$ is the opposite value of the total loss of the language (w, i). Here, we require that the sender's strategy w is a Voronoi tessellation uniquely generated by the interpretation vector i. By Lemma 6, L is continuous with respect to i. The correspondence of maximizers $\nu : \mathscr{P} \twoheadrightarrow T$ is

$$\nu(P) := \left\{ i \in \psi(P) : \max_i -L(P, i) \right\}.$$

By Berge's maximum theorem (see Theorem 17.31 of Aliprantis and Border (2006)), ν is upper hemicontinuous. Now let us focus on an efficient language (w^λ, i^λ) with a prior P^λ. Suppose that i^λ is a global maximization point of $-L(P^\lambda, i)$. Due to the convexity of l and the norm, i^λ is the unique point to maximize $-L(P^\lambda, i)$ in a neighborhood $U(i^\lambda, \delta)$ of i^λ. By upper hemicontinuity of ν, we are able to find a neighborhood $U(P, \zeta)$ of P such that for all $P \in U(P^\lambda, \zeta)$ it has $\nu(P) \in U(i^\lambda, \delta)$, where $\nu(P)$ is a singleton valued mapping in the range $U(i^\lambda, \delta)$. Then we can find a non-singleton, convex and connected set $U(P^\lambda, \eta) \subset U(P^\lambda, \zeta)$, and $U(P^\lambda, \eta)$ induces a connected set of interpretations contained in $U(i^\lambda, \delta)$ due to the continuity of $\nu(P)$.

Step 2. Prove the set of efficient languages with connected interpretations having thick boundary. That is to prove the set of efficient languages is not a singleton. The proof idea is breaking the symmetry that is similar to the proof in Theorem 2. □

Remark 7. *By Proposition 3 and Theorem 3, we know that Knightian uncertainty is the source of vagueness in our model. By Theorem 3, there are infinite strict equilibria around an strict equilibrium under Knightian* *uncertainty. This phenomenon called robust indeterminacy also appears in financial markets Rigotti and Shannon (2005), where there is a continuum of equilibrium allocations and prices under Knightian uncertainty.*

6 Efficiency

A language (w, i) [9] is efficient Jäger et al. (2011) if it minimizes the expected loss $L(w, i, P)$, where P measures the common belief of the players. Now, given multiple priors of the receiver, usually no strategy profile (w, i) minimizes $L(w, i, P)$ for all $P \in \mathscr{P}$ under incomplete preference, i.e. there may be a set of strategy profiles being Pareto optimal. In the following, we attempt to characterize these profiles and find the relation with the concept of vague Voronoi language.

Definition 7. *Given a convex and compact set $T \subseteq \mathbb{R}^n$, a closed and convex set of probabilities $\mathscr{P}$, and a finite word list $\{w_1, \cdots, w_N\}$, a set of languages*

$$\mathscr{V} = \{(w^\lambda, i^\lambda) : \lambda \in \Lambda\}$$

indexed by a set Λ is efficient if

- *(efficiency) for each language $(w, i) \in \mathscr{V}$, it is Pareto optimal, i.e. the language (w, i) is efficient if there is no $(w', i') \in W^T \times T^N \backslash \{(w, i)\}$ such that $L(w, i, P) \geq L(w', i', P)$ for all $P \in \mathscr{P}$,*

- *(connectedness) for each word k in the word list $\{w_1, \cdots, w_k\}$ the interpretation set of the word w_k, i.e. $I_k(\mathscr{V}) = \{i_k^\lambda : \lambda \in \Lambda\}$ is connected,*

- *(maximality) and $\mathscr{V}$ is maximal, i.e. there is no set of Voronoi languages $\mathscr{V}'$ satisfying the above condition such that $\mathscr{V}$ is a proper subset of $\mathscr{V}'$.*

Proposition 4. *If a non-singleton set of languages $\mathscr{V}$ is efficient, then it is a Vague Voronoi language.*

Proof. First, we show each language satisfying the efficiency is a Voronoi language. Obviously, by Definition 1 and 2, all languages in the set satisfying the efficiency should be strictly maximal Nash equilibria due to the

[9] Like before, non-degenerate behavioral strategies are always dominated by pure strategies under our games. Definitely, non-degenerate behavioral strategies are never efficient.

common interests between the players. Then by Theorem 1, we know that the language is a Voronoi language. Second, the set $\mathscr{V}$ is a non-singleton and the boundary set of $\mathscr{V}$ should be positive measurable. Other conditions are trivial. So we obtain the proof. □

Theorem 4. *There exists a set of languages $\mathscr{V}$ that is efficient.*

Proof. It has been proved in the proof of Theorem 3. The vague Voronoi language constructed in Theorem 3 is efficient. □

7 Conclusion

In this paper, we construct a sender-receiver game within Knightian uncertainty when the type space is finite-dimensional and the word list is finite. We use the incomplete rules maximality to deal with Knightian uncertainty. To obtain equilibria, we use E-admissibility to solve the game, where E-admissibility is equivalent to maximality under some weak conditions. Further, the strictly maximal equilibrium and the strictly E-admissible Nash equilibrium are defined. It has demonstrated that both equilibria are equivalent and are Voronoi languages. Under Knightian uncertainty, a strict equilibrium surround with a continuum of strict equilibria and players use vague languages. To characterize vagueness in our natural languages, we provide a definition of vague Voronoi languages, which is an aggregation of Voronoi languages or strict equilibria. We have demonstrated that the existence of vague Voronoi languages or vagueness in Knightian worlds. And there is no vagueness if the players' prior is characterized in Bayesian way. At last, we show that an set of efficient languages is a vague Voronoi language.

A Decisions under Knightian uncertainty

Now, let us remind formal definitions of the two incomplete decision rules and discuss the relation between them. A set of Borel regular probability measures $\mathscr{P}$ represents Knightian uncertainty on a subset B of all acts $A := C_c(T)$ [10], where $\mathscr{P}$ is defined on a compact Polish space (separable completely

[10] Here, since T is compact, then $C_c(T) = C(T)$ and its topological dual space $C_c(T)' = ca_r(T) = ca(T)$ is the set of all countably additive Borel signed measures on T (see Corollary 14.15 of Aliprantis and Border (2006)).

metrizable topological space) T assigned with a Borel $\sigma-$algebra, and A is a set of all continuous real-valued function on T with compact support and is paired with the sup norm. Due to the continuity of every act $\mathbf{a}$ over the compact set T, the expectation $\mathbb{E}_P(\mathbf{a}) = \int_T \mathbf{a} dP < \infty$ is well defined. Then the definitions of the both rules are shown as follows.

Definition 8. *Given a set $B \subseteq A$, an act $\mathbf{a}^*$ is (strictly) maximal in B if there is no act $\mathbf{a} \in B \backslash \{\mathbf{a}^*\}$ such that $\mathbb{E}_P(\mathbf{a})(\geq) > \mathbb{E}_P(\mathbf{a}^*)$ for all $P \in \mathscr{P}$.*

Definition 9. *An act $\mathbf{a}^*$ is (strictly) E-admissible in B if there exists some $P \in \mathscr{P}$ such that $\mathbb{E}_P(\mathbf{a}^*)(>) \geq \mathbb{E}_P(\mathbf{a})$ for all $\mathbf{a} \in B \backslash \{\mathbf{a}^*\}$.*

Often, maximality and E-admissibility are not equivalent. In Schervish et al. (2003), an equivalence of maximality and E-admissibility with finite set of states is discussed. But in our setting, the set of states T can be an infinite-dimensional space. We wonder whether there is the equivalence even when the set of states T is infinite. The following theorem provides an affirmative answer.

Theorem 5. *Given a nonempty convex and compact set [11] of acts $B \subseteq A$, and a nonempty convex and closed [12] set of priors $\mathscr{P}$, an act $\mathbf{a}^*$ is a strictly maximal act in B if and only if it is a strictly E-admissible act in B.*

Proof. Obviously, an E-admissible act is a maximal act. In the following, we prove that the maximal act $\mathbf{a}^*$ is an E-admissible act in B.

Step 1. We manage to find a positive linear functional to separate $\mathbf{a}^*$ from B such that $\mathbf{a}^*$ achieves the maximum value under the linear functional.

Let $C_A = \{\mathbf{a} \in A : \mathbb{E}_P(\mathbf{a}) \geq 0 \text{ for all } P \in \mathscr{P}\}$, and C_A is a convex cone for the linearity of $\mathbb{E}_P$. By the linearity of $\mathbb{E}$ and the closed graph theorem, C_A is a closed set. Then for any positive value ϵ and constant act ϵ^T, it has that

$$\left\{\mathbf{a}^* + \frac{1}{2}\epsilon^T\right\} + C_A := \left\{\mathbf{a}^* + \frac{1}{2}\epsilon^T + \mathbf{c}_A : \mathbf{c}_A \in C_A\right\}$$

is convex and closed. Here, in order to make sure the separated two sets are disjoint, we add the infinitesimal act $\frac{1}{2}\epsilon^T$ to a^*. Besides, B is convex, and

$$\left\{\mathbf{a}^* + \frac{1}{2}\epsilon^T\right\} + C_A \bigcap B = \emptyset,$$

[11] It is compact with respect to the topology with the sup norm.

[12] Here, a set of probabilities is closed with respect to the weak* topology $\sigma(ca_r(T), C_c(T))$, where $ca_r(T)$ is the set of all regular signed Borel measures of bounded variation over T.

where $\left\{\mathbf{a}^* + \frac{1}{2}\epsilon^T\right\} + C_A$ is definitely nonempty, closed, and convex. By the strong separating hyperplane theorem (Theorem 5.79 of Aliprantis and Border (2006)), there exists some nonzero continuous linear functional $\mathbb{L} \in A'$ such that

$$\begin{aligned} &\mathbb{L}(\mathbf{a}^* + \mathbf{c}_A + \frac{1}{2}\epsilon^T) \geq \alpha + \epsilon > \alpha \geq \mathbb{L}(\mathbf{a}) \\ \Rightarrow\ &\mathbb{L}(\mathbf{a}^* + \mathbf{c}_A) \geq \alpha + \frac{1}{2}\epsilon > \alpha \geq \mathbb{L}(\mathbf{a}) \\ \Rightarrow\ &\mathbb{L}(\mathbf{a}^* + \mathbf{c}_A) > \mathbb{L}(\mathbf{a}), \end{aligned}$$

for all $\mathbf{a} \in B$ and all $\mathbf{c}_A \in C_A$, where A' is the topological dual of A. By $\mathbb{L}(\mathbf{a}^* + \mathbf{c}_A) \geq \mathbb{L}(\mathbf{a})$ for all $\mathbf{a} \in B$ and all $\mathbf{c}_A \in C_A$, we obtain

$$\begin{aligned} \mathbb{L}(\mathbf{c}_A) &> \mathbb{L}(\mathbf{a}) - \mathbb{L}(\mathbf{a}^*) \\ &= \mathbb{L}(\mathbf{a}^*) - \mathbb{L}(\mathbf{a}^*) \\ &= 0 \end{aligned}$$

if let $\mathbf{a} = \mathbf{a}^*$. In another word, it has $\mathbb{L} \in C_{A'}$, where

$$C_{A'} = \{\mathbb{L} \in A' : \mathbb{L}(\mathbf{c}_A) \geq 0 \text{ for any } \mathbf{c}_A \in C_A\}.$$

More, we have $\mathbb{L}(\mathbf{a}^*) > \mathbb{L}(\mathbf{a})$ for all $\mathbf{a} \in B$ if $\mathbf{c}_A$ is 0^T, where 0^T is a constant act that assigns 0 to all elements of T.

Step 2. Based on the functional $\mathbb{L}$, by a Riesz representation theorem, it induces a probability measure.

Given $c = \mathbb{L}(1^T)$, we obtain a normalized functional $\frac{1}{c}\mathbb{L}$. By the definition of C_A, we know that $\mathbf{a} \in C_A$ if $\mathbf{a} \geq 0^T$. Since $\mathbb{L}(\mathbf{c}_A) \geq 0$ for all $\mathbf{c}_A \in C_A$, it has $\mathbb{L}(\mathbf{a}) \geq 0$ for all $\mathbf{a} \geq 0^T$. So $\frac{1}{c}\mathbb{L}$ is a positive linear functional. Since T is a compact set, by Riesz-Markov Theorem (Theorem 14.12 of Aliprantis and Border (2006)), there exists a unique positive regular Borel measure satisfying $\frac{1}{c}\mathbb{L}(\mathbf{a}) = \int_T \mathbf{a} dQ$. Because $Q(T) = \frac{1}{c}\mathbb{L}(1^T) = 1$, Q is a probability measure. Of course, it has $\frac{1}{c}\mathbb{L} = \mathbb{E}_Q$.

Step 3. We prove the probability $Q \in \mathscr{P}$. By Theorem 14.14, the norm dual of $A := C_c(T)$ is $ca_r(T)$. Then $\langle ca_r(T), C_c(T)\rangle$ is a dual pair with a bilinear map defined as

$$\langle \mu, \mathbf{a}\rangle = \int \mathbf{a} d\mu,$$

where $\mu \in ca_r(T)$ and $\mathbf{a} \in A$.

Suppose that $Q \notin \mathscr{P}$. By Theorem 5.93 of Aliprantis and Border (2006), dual pairs are weakly dual, i.e. the topological dual of $(ca_r(T), \sigma(ca_r(T), C_c(T)))$ is $C_c(T)$, where the set $ca_r(T)$ is equipped with a weak* topology $\sigma(ca_r(T), C_c(T))$. Due to the compactness of the $\{Q\}$ (see Lemma1 15.21 of Aliprantis and Border (2006)) and closeness of $\mathscr{P}$, by the strong separating hyperplane theorem (Theorem 5.79 of Aliprantis and Border (2006)), we can strongly separate $\{Q\}$ and $\mathscr{P}$ by some $\mathbf{a} \in C_c(T)$, such that $\mathbb{E}_Q(\mathbf{a}) = \langle Q, \mathbf{a} \rangle \leq \beta$ and $\mathbb{E}_P(\mathbf{a}) = \langle P, \mathbf{a} \rangle \geq \beta + \epsilon'$ for all $P \in \mathscr{P}$, where ϵ' is some positive real number. If $\beta < 0$, then there exist two positive number n and d such that

$$-\frac{1}{n}(\beta + \epsilon') \leq d < -\frac{1}{n}\beta.$$

Then we obtain

$$\mathbb{E}_Q(\frac{1}{n}\mathbf{a} + d^T) = \frac{1}{n}\mathbb{E}_Q(\mathbf{a}) + d \leq \frac{\beta}{n} + d < 0,$$

and

$$\mathbb{E}_P(\frac{1}{n}\mathbf{a} + d^T) = \frac{1}{n}\mathbb{E}_P(\mathbf{a}) + d \geq \frac{\beta + \epsilon'}{n} + d \geq 0$$

for all $P \in \mathscr{P}$. Hence it should have

$$\frac{1}{n}\mathbf{a} + d^T \in C_A.$$

At last, provided that $\frac{1}{c}\mathbb{L} \in C_{A'}$, it produces a contradiction for $\mathbb{E}_Q(\frac{1}{n}\mathbf{a} + d^T) = \frac{1}{c}\mathbb{L}(\frac{1}{n}\mathbf{a} + d^T) \geq 0$. Similarly, we also get a contradiction whenever $\beta \geq 0$. At last, we have $Q \in \mathscr{P}$.

□

References

[1] Charalambos D. Aliprantis and Kim Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer Science & Business Media, 2006.

[2] Robert J. Aumann. Utility Theory without the Completeness Axiom. *Econometrica*, 30(3):445–462, 1962.

[3] Sophie Bade. Nash equilibrium in games with incomplete preferences. *Economic Theory*, 26(2):309–332, 2005.

[4] Truman F. Bewley. Knightian decision theory. Part I. *Decisions in Economics and Finance*, 25(2):79–110, 2002.

[5] Andreas Blume, Oliver J. Board, and Kohei Kawamura. Noisy talk. *Theoretical Economics*, 2(4):395–440, 2007.

[6] Vincent P. Crawford and Joel Sobel. Strategic Information Transmission. *Econometrica*, 50(6):1431–1451, 1982.

[7] Eric Danan, Thibault Gajdos, Brian Hill, and Jean-Marc Tallon. Robust Social Decisions. *American Economic Review*, 106(9):2407–2425, 2016.

[8] Lieven Decock and Igor Douven. What Is Graded Membership? *Noûs*, 48(4):653–682, 2014.

[9] Igor Douven, Lieven Decock, Richard Dietz, and Paul Égré. Vagueness: A Conceptual Spaces Approach. *Journal of Philosophical Logic*, 42(1): 137–160, 2013.

[10] Q. Du, V. Faber, and M. Gunzburger. Centroidal Voronoi Tessellations: Applications and Algorithms. *SIAM Review*, 41(4):637–676, 1999.

[11] Özgür Evren. Scalarization methods and expected multi-utility representations. *Journal of Economic Theory*, 151:30–63, 2014.

[12] Peter Gärdenfors. *Conceptual Spaces: The Geometry of Thought*. MIT Press, 2000.

[13] Gerhard Jäger, Lars P. Metzger, and Frank Riedel. Voronoi languages: Equilibria in cheap-talk games with high-dimensional types and few signals. *Games and Economic Behavior*, 73(2):517–537, 2011.

[14] Rosanna Keefe and Peter Smith. *Vagueness: A Reader*. MIT Press, 1996.

[15] Christian Kellner and Mark T. Le Quement. Endogenous ambiguity in cheap talk. *Journal of Economic Theory*, 173:1–17, 2018.

[16] Erich L. Lehmann and George Casella. *Theory of Point Estimation*. Springer, New York, 2nd edition, 1998.

[17] Isaac Levi. *The Enterprise of Knowledge*. MIT Press, London, 1980.

[18] David K. Lewis. *Convention: A Philosophical Study*. Wiley-Blackwell, 1969.

[19] Barton L. Lipman. Why is language vague? *mimeo*, 2009.

[20] Stuart Lloyd. Least squares quantization in PCM. *IEEE Transactions on Information Theory*, 28(2):129–137, 1982.

[21] Luca Rigotti and Chris Shannon. Uncertainty and risk in financial markets. *Econometrica*, 73(1):203–243, 2005.

[22] Mark Schervish, Teddy Seidenfeld, Joseph Kadane, and Isaac Levi. Extensions of Expected Utility Theory and Some Limitations of Pairwise Comparisons. *Proceedings of the Third International Symposium on Imprecise Probabilities and Their Applications*, pages 496–510, 2003.

[23] L. S. Shapley. Equilibrium points in games with vector payoffs. *Naval Research Logistics Quarterly*, 6(1):57–61, 1959.