Social Influence and Personal Norms in Networks with Strategic Complementarities

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Social Influence and Personal Norms in Networks with Strategic Complementarities

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Contents

Introduction

Social interactions are an essential part of human life. Being born in families, we are already embedded in an existing social environment which nurtures our mindset and mentality. We connect in schools and universities, workplaces, sports and arts clubs. We follow and read opinions of people we might not even know on social media. We form friendships, establish business relations and maintain our family ties. We engage in conversations, arguments, and exchange news and points of view. These diverse relational patterns are best represented by *Social Networks*. The opinions and expectations of people are often influenced by those relations through communicating information and beliefs, and observing behaviors. The social network, alongside with individual characteristics of people, affects their behavior which, in turn, impacts the social and economic outcomes and welfare.

The worldwide COVID-19 pandemic provides vivid examples of how the collective behavior and expectations of individuals may affect the economies. The uncertainties and negative expectations from the social and economic disruptions brought by the pandemic caused panic buying and overconsumption of certain products resulting in their deficit. Such behavior was fostered by the social media interactions and exchange of viral videos of long queues and empty shelves in supermarkets.[1] Mistrusts, worries and conspiracies, combined with unavailable or inaccessible scientific output, affect vaccination choices and thus formation of collective immunity.[2]

Many bank runs occurred during global financial crises of 2007-2008 and the Great Depression as a result of pessimistic public expectations on banks' liquidity or default. The depositors' beliefs about lack of funds led to panic and mass withdrawals of deposits causing the liquidity problem and the bank run. Multiple examples of bank runs resulted from a rumor or fake information spread by means of learning from social

[1] Zheng, Shou, and Yang (2021) analyze the impact of social learning on consumers' panic buying decisions and study the effects on social welfare. Empirical study by Naeem (2021) shows that observing similar views on social media enhances and motivates the "social proof", that is, the tendency of mimicking the panic buying behavior of other people.

[2] Forsyth (2020) study the groupthink approach, that is, a consensus over an issue with lack of critical judgment, and pressure to conform in antiquarantine and antivaccination groups.

contacts.[3] One such example is the Toyokawa Shinkin Bank incident in December 1973 in Japan, when a bank run occurred as a result of a rumor, which allegedly originated from a conversation between three high school students.[4]

The examples above aim to demonstrate that while people are forced to make abundance of decisions in daily life, we may not always be the experts on the matter, or may not always have the right amount and quality of information for better understanding of which would be the best choice to make, thus often relying and learning from opinions of others. It is also safe to assume that people do not always rely on only one source of information, be it a government or other central authority. The *Social Influence* theory put forward by Kelman (1958) suggests that the beliefs and attitudes, and thus the subsequent actions of individuals, are affected by interactions with other people. The way people's beliefs and expectations are formed is highly affected by the exchange of opinions, as well as observation of behavior in their social network, thus, the social learning. Moreover, while we may value some opinions better than others, they may also have higher influence on our expectations, decisions and actions. The observation of behaviors and social norms, communication of beliefs and expectations result in public opinion formation and the corresponding behaviors which impact social and economic outcomes, such as investment decisions, consumption choices, voting, social and political movements, vaccination, climate friendly behavior, to name a few. Understanding the role and the impact of social influence on behavior of individuals is one of the goals of this thesis.

Given the importance of social influence on individual behaviors, understanding of how the opinion of an individuals is affected by the information obtained from her social contacts is of utmost importance. The two main approaches in modeling the social learning are the Bayesian and non-Bayesian learning models. The former approach assumes that agents use Bayes' rule to update their opinions, while the latter suggests less sophisticated assumptions on the ability of agents to update opinions with new information. The seminal work in the literature on opinion formation by DeGroot (1974) suggests a simple non-Bayesian model in which agents update their opinions in every period of repeated communication by taking the weighted average of the opinions of their neighbors. Many non-Bayesian models were introduced later.[5]

[3] Kelly and O Grada (2000) show that the social network, determined by place of origin and current neighborhood, was the prime determinant of bank run behavior in panics of 1854 and 1857 in New York. The empirical study on depositors by Iyer and Puri (2012) shows that the social network, as the neighborhood residence and ethnic group, can play an important role through the contagion effect of bank run behavior in depositors.

[4] See Sekiya (2016) for references.

[5] See e.g. DeMarzo, Vayanos, and Zwiebel (2003); Ellison and Fudenberg (1995).

It has been shown that the communication and learning in social networks affect the opinion dynamics and the emergence of a consensus.[6] Additionally, the structure of the network plays an important role in those matters.[7]

Chapter 1 addresses the question of the impact of social influence on expectations and effort dynamics, and long-run effort equilibria. The dynamic model introduced in the chapter captures the social influence through communication of beliefs in a fixed network of agents repeatedly engaging in a group production task with randomly formed groups. The belief formation takes place in two stages in every period. First, the players observe the outcome of the game in their group and learn the minimum effort causing the outcome, and then they update their belief according to the current observation. Secondly, they communicate these beliefs with their neighbors in the network, and update the belief according to the weighted average of their own beliefs and those of their neighbors.

Chapters 2 and 3 of the thesis incorporate a different aspect of social influence, namely, social norms and conformity. Together with personal characteristics, social norms are an important factor in shaping individual behaviors. People adapt their attitudes, beliefs and behaviors driving them closer to those of others they interact with. Such behavior can be a result of convincing arguments from friends, complexity of making decisions and simplicity of following an example, seeking for social approval, as well as social pressure to conform with norms.[8] Conformism models had been widely studied in *Network Games*, the literature focusing on analysis of interactions between individuals connected in a social network, where interactions are modeled using game theory.[9] Starting from the seminal contribution of Ballester, Calvó-Armengol, and Zenou (2006) the link was established between the pure Nash equilibrium effort of players and their Katz-Bonacich centralities in a large class of network games.[10] An additional important component of individual belief and behavior dynamics are the personal norms. The *Theory of Cognitive Dissonance* introduced by social psychologist Festinger (1957) suggests that people seek for internal consistency aligning their attitudes and behaviors. While the inconsistency of personal norms with one's own actions gives rise to cognitive dissonance.[11] Chapters 2 and 3 study network game

[6] See e.g. the survey by Acemoglu and Ozdaglar (2011b).

[7] E.g. Golub and Jackson (2010a).

[8] See e.g. Cialdini and Goldstein (2004) and Flache, Mäs, Feliciani, Chattoe-Brown, Deffuant, Huet, and Lorenz (2017) for reviews.

[9] See e.g. Bramoullé, Kranton, and D'amours (2014); Jackson and Zenou (2015) for an extensive surveys on games on networks.

[10] The Katz-Bonacich centrality, introduced by Katz (1953) and Bonacich (1987), is a measure of influence of a player embedded in a social network.

[11] See e.g. Stone and Cooper (2001) among others.

models with personal norms and conformity, where on one hand players want to be consistent with their personal norms, on the other, they are punished for not complying with the social norm. The models in the two chapters differ with the types of spillover effects present in the network and the assumptions on the network structure. More specifically, in Chapter 2 a two-layer network structure is assumed, where the sources of spillovers and the pressure to conform are different layers of the network. The model in Chapter 3 does not assume network multidimensionality, thus the local spillovers and the pressure to conform originate from the same relational patterns. Additionally, the model in this chapter incorporates global spillovers in the network suggesting that the collective behavior in the whole populations affects the utility of a single player. The focus of Chapter 2 is the dynamics of personal norms of the players and the emergence of consensus. The evolution of personal norms takes place through updating those with actual behavior, aligning the personal norms and action, in each period. Chapter 3, on the contrary, does not address the problem of opinion dynamics. Along with analysis of spillovers, social and personal conflict in the network game, the main contribution of the chapter is the incorporation of unions in the network game models.

Overview of the Thesis

In Chapter 1, coauthored with Prof. Dr. Herbert Dawid[12] and Prof. Dr. Jasmina Arifovic[13], we explore the role of social influence for the coordination of effort choice in a game with strategic complementarities. Players are repeatedly randomly partitioned in groups to play a minimum effort game and choose their effort based on their beliefs about the minimal effort among the other members of their group. Individual expectations about this minimal effort is influenced by own experience as well as by communication of beliefs within a social network. We show that increasing the importance of social influence in the expectation formation process has positive effects on the emerging (long run) effort level, thereby improving the efficiency of the outcome. Furthermore, a more centralized social network leads to higher average efficiency, but also to increased variance of outcomes. Finally, communication of actual minimum effort cannot replace the communication of beliefs as a device fostering the emergence of high long run effort.

Chapter 2 studies a network game on a fixed multi-layer network. The players are embedded in a network of two types of relationships. One is a network of social interactions with pressure to conform to the social norm, the other provides additional

[12]Department of Business Administration and Economics and Center for Mathematical Economics, Bielefeld University, Germany.

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strategic complementarities from players' interaction. Additionally, players are endowed with personal ideal efforts, a personal norm. They repeatedly choose their effort level in the network game and update the ideal effort based on the new effort choice. We find the pure Nash equilibrium of the game in each period and provide conditions for the convergence of efforts and ideals to a steady state. Furthermore, we provide conditions for emerging long-run consensus about ideals in groups of players and the entire network.

In Chapter 3, based on the joint work with Dr. Simon Schopohl, we study a network game with spillovers, social conflict, and private dissonance. We consider a fixed network where players are heterogeneous in their ideal efforts and returns on spillovers. There exists a global spillover effect between all players and an additional local spillover effect between neighbors. The players suffer disutility from the discrepancy between effort choices of their neighbors and themselves, and inconsistency with their ideal effort. We find the unique Nash equilibrium of the game and the key players in the network. Additionally, we introduce unions in the network as groups of players that choose efforts by maximizing their joint utility. We find the pure Nash equilibrium in the game with unions and analyze how their presence affects the aggregate effort. We define a measure of intercentrality for players in unions and distinguish the union-induced contribution of a player to the aggregate effort. Moreover, we provide conditions for an increase in the effort of the player and the aggregate effort in the network when adding the player to a union. Furthermore, we define the key unions of a fixed size as groups of players that increase the aggregate effort the most when joined in a union.

Each chapter contains a distinct research paper that can be read independently. All references can be found in the common bibliography in the end of the thesis.

Chapter 1

Efficiency Gains through Social Influence in a Minimum Effort Game

1.1 Introduction

Group production plays a central role in many economic contexts. In the presence of strategic complementarities between the actions of group members, the problem of determining individual effort by each member in the group typically gives rise to multiple (Pareto ranked) equilibria and hence to a severe coordination problem. In particular, the optimal effort choice of an agent is crucially determined by her beliefs about the effort the other group members invest into the project. Taking this into account, the way expectations are formed and adjusted over time is a key driver for determining the outcome of (repeated) group production problems. In particular, whether an efficient outcome can be reached in the long run strongly depends on the evolution of the individual expectations of the players.

The agenda of this paper is to study the role of one important aspect of expectation formation, namely social influence, in the dynamics of expectations and actions in a population of agents which repeatedly undertake some joint production task with strategic complementarities in changing groups. As an illustrative example of a situation we have in mind we envision a university which repeatedly encounters calls for interdisciplinary project proposals and for each of these calls identifies a group of faculty members from different departments who, based on their background, are suitable to contribute to the proposal. Each group member then decides how much effort to invest in developing her part of the proposal. After submission the proposal is evaluated and the group receives an outcome (e.g. amount of funding, invitation to resubmit, rejection). We assume that referees tend to focus in their decision to a large extend on potential weaknesses they see, such that the outcome is determined by the lowest effort shown by any group member. Faculty members over time repeatedly participate in different such group proposals and over time build expectations about the minimal effort shown among the other members of their group. We assume that these expectations are not only based on their own experience, i.e. the outcome of the projects they have been previously involved in, but also on communication with their friends, close colleagues and co-authors in the profession about their experience with similar project proposals. In particular, we assume that agents communicate their own beliefs about the level of minimal effort shown in such an interdisciplinary group to their social contacts. We denote such kind of communication as the social influence channel of the expectation formation process. Whereas the groups jointly producing the project proposals differ from call to call, we assume that the set of social contacts of an agent stays constant over time. The main questions we address within such a setting are, whether the quality of the project proposals in the long run is higher if there is communication of the agents' own beliefs in the social network, and, how the outcome is affected by the topology of the social network.

Our research agenda builds on a large body of empirical and theoretical work studying the role of social influence for opinion dynamics and expectation formation. Starting with the seminal contribution of DeGroot (1974) there is by now a rich body of literature highlighting how communication in social networks affects the dynamics of opinion formation, in particular the emergence of consensus in a population (see e.g. the survey Acemoglu and Ozdaglar (2011b)), and under which circumstances there is 'wisdom of the crowd' in the sense that communication in a network allows the agents to learn the true state of the world (e.g. Golub and Jackson (2010a)). As shown e.g. in Golub and Jackson (2010a) or Acemoglu, Ozdaglar, and ParandehGheibi (2010) also the topology of the social network matters with respect to these issues. Recent contributions have also stressed that the impact of social influence on economic expectations can have important implications for actual dynamics in different economic contexts (e.g. Arifovic, Deissenberg, and Kostyshyna (2010); Burnside, Eichenbaum, and Rebelo (2016); Rotemberg (2017)). However, so far a systematic analysis of the implications of social influence for the selection of the outcome emerging in a population faced with a group coordination problem is missing. The main contribution of this paper is to fill this gap.

We address our main research questions by considering a dynamic model of a population of agents, which every period is randomly partitioned into groups of given size. In each group agents interact by playing a minimum effort game with a finite set of effort choices. We use the minimum effort game as the most widely used model in the theoretical (starting with Bryant (1983)) and experimental (starting with Van Huyck, Battalio, and Beil (1990)) literature capturing a group coordination problem with strategic complementarities and multiple Pareto ranked equilibria. Each agent in each period plays the best response to her current expectations. These expectations have the form of a belief distribution over the set of possible effort choices, expressing the probability that a given effort level is the minimum of the effort choices of the other members of the group. In each group only the outcome, i.e. the minimal effort level in the group, is observable for its members. At the end of each period agents update their expectations in two steps. First, in line with standard adaptive expectations models[14], agents build intermediate beliefs as a weighted average of their previous beliefs and the observed outcome in the current period. In a second step, agents might communicate their intermediate beliefs with all their contacts in a social network, which is exogeneously given and constant over time. Following a standard approach in the literature on social influence dynamics introduced in DeGroot (1974) agents put identical weight on the intermediate beliefs of all their contacts and the updated belief is a weighted average of their own intermediate belief and average intermediate beliefs of their contacts.[15] The weight agents put on the average beliefs of their social contacts determines the importance of social influence in the population. A main goal of our analysis is to understand how an increase in this social influence parameter affects the evolution of expectations and the distribution of effort levels chosen in the long run. We explore this question by combining analytical findings with insights from statistical analysis of data obtained through extensive simulations of the model under different assumptions about the size of the social influence parameter and the social network topology.

Since the focus of our analysis is on the role of social influence, we consider a model setup which, apart from the considered communication of beliefs, is as simple

[14]See e.g. Huyck and Stahl (2018) for a recent contribution in the framework of minimum effort games.

[15]This model of belief diffusion is sometimes referred to as 'naive learning' since individuals do not take into account that due to the structure of the social network there might be differences in the correlations between the intermediate beliefs of different pairs of their contacts, which should be reflected in the weights put on their intermediate beliefs. Alternatively a Bayesian Approach could be employed in which agents take into account the network structure in a fully rational way. As pointed out in DeMarzo et al. (2003) however a very high degree of rationality on the agents' side has to be assumed for them to infer the correct weights to be put on all their contacts. Indeed, as is shown in Grimm and Mengel (2019), the naive approach put forward in DeGroot (1974) is better able to explain the data from experiments studying the effect of social influence than a Bayesian model. Hence, we stick in our analysis to the assumption that identical weights are put on the communications from all social contacts.

as possible and directly corresponding to the baseline stag hunt game, for which the experimental results of Van Huyck et al. (1990) show convergence to the least efficient equilibrium under large group sizes. Thereby, we do not incorporate into our model several mechanisms which have been shown in the literature to improve the efficiency of the emerging long run outcome of the game. In particular, our assumption that each agent considers only the own payoff when determining a best response and then $a$lways chooses that best response does neither consider social preferences (see e.g. Chen and Chen (2011)) nor stochastic choice by agents, e.g. based on a the well studied logit model (e.g. Anderson, Goeree, and Holt (2001); Huyck and Stahl (2018)). Furthermore, we assume that an exogenous process stochastically determines the interaction group every period, while it has been shown in Riedel, Rohde, and Strobl (2016) that endogenous partner choice can improve the efficiency in coordination games. Agents in our setup with social influence communicate at the end of each period with their social contacts about their own beliefs, however there is no pre-play communication within each interaction group, which might improve the effort level chosen in the game (see e.g. Blume and Ortmann (2007); Kriss, Blume, and Weber (2016)). Also, in our setting agents do not condition their effort on the fraction of members of their interaction group with whom they have direct ties in the social network. This assumption is due to the fact that we consider large interaction groups where this fraction is typically small.[16] Overall, by abstracting from all these effects and considering a very basic environment we are able to isolate the effect of social influence for the agent's effort choice.

The first main insight from our analysis is that in the absence of social influence, i.e. if the value of the social influence parameter is zero, the effort level in the population converges to the lowest effort level chosen in the entire population in the initial period. In accordance with the experimental evidence of Van Huyck et al. (1990), this implies that for large groups the long run effort level coincides with high probability with the lowest possible value and therefore the least efficient equilibrium is reached.[17] Intuitively, in the absence of communication with their social contacts, agents observe only the outcome of the game in their own group, which corresponds to the lowest effort shown by any group member. Hence, the agent with the most pessimistic belief in the whole population never receives information that could move her best response upwards. Since every period all agents matched with this most pessimistic agent observe

[16] For small group sizes, in particular interaction groups of size two, experimental findings, e.g. by Chen and Chen (2011) show that effort in minimum effort games tends to be higher if a player is matched with another player belonging to the same identity group.

[17] Van Huyck et al. (1990) consider the groups to be large if they consist of more than two players.

the group outcome corresponding to this agent's (low) effort, their beliefs become more pessimistic. Due to this mechanism all agents adjust their efforts downwards over time until it matches the lowest effort level in the population. The picture changes radically if there is social influence in the population. First, due to learning the expectations of her social contacts, an agent's beliefs can become more optimistic even if the observed outcome in the own group was low in the previous period. Second, due to the communication between agents, the beliefs in the population become homogeneous much faster. This induces a fast coordination on an equilibrium and avoids the 'downward drift' of beliefs over time, which occurs in heterogeneous populations due to the fact that in every group the observed outcome corresponds to the lowest effort in the group. We show that an increase of the social influence parameter induces statistically significantly higher long run effort in the population. Furthermore, our analysis establishes that the topology of the social network matters. In particular, simulation results show that the expected long run effort of agents in the population is significantly larger in a centralized star network compared to a random network. However, this increase of average efficiency comes at the cost of less predictability of the outcome in the sense that the variance of the long run outcome across simulation runs is much larger under a centralized network. Intuitively, in a centralized network the population is strongly influenced by the initial beliefs of the agent in the center of the network. This fosters fast coordination, but at the same time introduces a strong dependency of the long run outcome from the (stochastic) initial beliefs of a single agent. We show that our findings are not only robust with respect to variations of the model parameters, but also with respect to a change in the model setup, where agents do not only exchange their beliefs with their social contacts but also are able to observe the outcomes of all the group interactions in which any of their social contacts are involved. In particular, this implies that exchange of information about the outcomes of the group interaction in the social network cannot substitute the communication of beliefs in fostering coordination on more efficient equilibria.

The paper is organized as follows. In Section 1.2 we describe our model setting. In Section 1.3 we derive several analytical findings about the long run outcomes for special cases of our setting, including the scenario without social influence. In Section 1.4 we analyze the general case with social influence and in Section 1.5 we consider a model extension, where not only beliefs but also information about the actual outcomes of the group interactions are communicated in the social network. Concluding remarks are given in Section 1.6. Appendix 1.A provides the proofs of the propositions in Section 1.3. In Appendix 1.B we provide the statistical test results discussed in the paper and in Appendix 1.C the robustness of our results with respect to parameter variations is demonstrated.

1.2 Model

1.2.1 Setting

There is a set of agents N, with $|N| = n$, connected within a social network s. The set of connections of agent i in the social network is fixed over time. We denote the set of social contacts of agent i by $m_i(s) = \{j | ij \in s\}$ and by $\eta_i(s) = |m_i(s)|$ the number of social contacts of agent i.

Every period the set of agents is randomly partitioned into n/k groups of size k, where each partition has equal probability.[18] Thus, each agent faces an equal probability of being a member of each group. We denote by $g(i,t) \subset N$ the set of members of the group to which agent i belongs at t (this implies $i \in g(i,t)$ for all t).

At each period each group plays the following minimum effort game (see Van Huyck et al. (1990)). Each agent i chooses an effort $e_{i,t}$ from a given set of strategies $\mathcal{X} = \{1, ..., \overline{e}\}$. The payoff of the agent from the game is determined by her own effort and the minimal effort chosen by the members of her group in period t, denoted by $\underline{e}_{i,t}$. The payoff of agent i is given by:

$$\pi(e_{i,t}, e_{-i,t}) = \alpha \underline{e}_{i,t} - \beta e_{i,t},$$

with $\alpha > \beta > 0$, $e_{-i,t} = (e_{j,t})_{j \in g(i,t) \setminus \{i\}}$ and $\underline{e}_{i,t} = \min_{j \in g(i,t)} e_{j,t}$ is the minimum effort in the group of agent i.

1.2.2 Beliefs

Each agent at each t has a belief about the distribution of minimal effort $\underline{e}$ in her (randomly generated) group: $b_{i,t} \in \Delta(\mathcal{X}) := \langle b \in \mathbb{R}_+^{\overline{e}} : \sum_{e=1}^{\overline{e}} b(e) = 1 \rangle$. We denote by $b_{i,t}(e)$ the probability that the minimal effort of the other players in the group is e. A belief vector which puts probability one on some effort level $\tilde{e}$ will be referred to as a point belief and formally written as $b = \mathbb{1}_{\tilde{e}}$. The population profile of beliefs at t is denoted by $B_t = (b_{i,t})_{i \in N}$.

An agent has no information about effort level chosen by individual agents and hence their beliefs about the distribution of minimal effort by the other members of

[18] Assuming that k is a divisor of n, the number of possible ways to form such partitions is given by $\left(\prod_{j=0}^{\frac{n}{k}-1} \binom{n-jk}{k} \right) / (n/k)!$.

their current group does not depend on the identities of these members. The expected payoff of an agent with effort $e \in \mathcal{X}$ and belief $b \in \Delta(\mathcal{X})$ is given by:

$$\pi^e(e,b) = \sum_{\underline{e} \in \mathcal{X}} \pi(e,\underline{e})b(\underline{e}).$$

Each agent i chooses her effort by maximizing this expected payoff using her current belief distribution $b_{i,t}$. Hence, an agent with belief b chooses an action $a^*(b)$ such that:

$$a^*(b) = \max\{e \in \mathcal{X} | \pi^e(e,b) \geq \pi^e(\tilde{e},b)) \ \forall \tilde{e} \in \mathcal{X}.\}$$

We denote the action of agent i at t by $a_{i,t} = a^*(b_{i,t})$. Note that the above formulation of $a^*(b)$ implies that in case an agent is indifferent between different levels of efforts she always chooses the largest of these levels.

After the game has been played, all agents update their belief distributions. They utilize both the acquired information about the minimum effort in their group, and the information about the beliefs of their neighbors in the network. Precisely, each agent i forms an *intermediate belief* $\tilde{b}_{i,t}$ as a weighted average of the previous belief and her current observation $\underline{e}_{i,t}$:

$$\tilde{b}_{i,t+1} = (1-\xi)b_{i,t} + \xi \mathbb{1}_{\underline{e}_{i,t}}, \tag{1.1}$$

where $0 \leq \xi \leq 1$ denotes the speed of individual updating. The fact that only the minimum effort, rather than all individual effort choices of $g(i,t)$, is used to update the beliefs is based on the assumption that the individual efforts of the group members are not observable. The only information available is the outcome of the game, which is determined by the minimum effort of all group members.[19]

Furthermore, individuals change beliefs due to social influence. In particular, they learn about intermediate beliefs $\tilde{b}_{j,t+1}(e)$ of their social contacts and use them to form their own final beliefs. Following standard formulations in the literature on opinion formation (e.g. DeGroot (1974)) we assume that the updated belief is a linear combination of the agent's intermediate belief and that of her social contacts. Formally, we have:

$$b_{i,t+1} = \tilde{b}_{i,t+1} + \chi \frac{1}{\eta_i(s)} \sum_{j \in m_i(s)} \big(\tilde{b}_{j,t+1} - \tilde{b}_{i,t+1}\big). \tag{1.2}$$

The parameter $\chi \in [0,1]$ in (1.2) represents the level of confidence or trust in the

[19] This assumption seems realistic in many examples of projects carried out by groups, where it is impossible to identify the individual contributions to the success of the project.

beliefs of social contacts. It determines the importance of *social influence* and is a key parameter in our analysis. This updated belief distribution is then the basis for the agent's effort choice in period $t + 1$.

1.3 Analytical Results

The evolution of the profile of belief vectors B_t constitutes a Markov process[20] on the state space $\Delta(\mathcal{X})^n$. A general analytical characterization of the transient dynamics or the long run distribution of this process seems infeasible and therefore in Section 1.4 we will use simulations to gain insights in this respect. However, it is possible to derive general characterizations of absorbing sets of the process, and for the special case where $\chi = 0$ also a description of the long-run outcome of beliefs and induced effort can be derived.

It is well known that for each effort level $e \in \mathcal{X}$ there exists a symmetric Nash equilibrium of the underlying minimal effort game, in which all players choose the effort e. These Nash equilibria are Pareto ranked and due to $\alpha > \beta$ the outcome is more efficient the higher the equilibrium effort. In our setting each such symmetric Nash equilibrium corresponds to a uniform population profile with $b_{i,t} = \mathbb{1}_e$ for all $i \in N$. Since $a^*(\mathbb{1}_e) = e$ and therefore $\underline{e}_i = e$ for all agents i under such a belief profile, it follows directly that any such uniform profile is an absorbing state of the process B_t. However, as we will show below, the process does not necessarily reach a state with uniform point beliefs, or even with uniform induced actions in the long run, at least as long as the social influence parameter χ is positive.

The following proposition shows that if all agents in the population in some period choose identical effort, they will all continue to choose this effort in all future periods as well.

Proposition 1.1. *If at some $t \geq 0$ there exists an effort level $e \in \mathcal{X}$ such that $a^*(b_{i,t}) = e \ \forall i \in N$ then*

(i) $a_{i,\tau} = e \ \forall i \in N$ for all $\tau \geq t$,

(ii) $b_{i,\tau} \to \mathbb{1}_e$ for $\tau \to \infty$ for all $i \in N$.

The intuition for this result is quite straight-forward. If there is no communication of beliefs, then in a period t, in which all agents observe a minimal effort e, the weight

[20] For each partition of the agents into groups of size k the belief of each agent i is given by (1.1) and (1.2) and is therefore deterministic. Hence, the distribution of B_{t+1} given B_t is determined by the probabilities of all possible group partitions and the beliefs of all agents in $t + 1$ given a certain partition.

of this effort level in the updated belief distribution of every agent becomes larger. Due to the strategic complementarity this increases for all agents their incentive to choose that effort level in period $t+1$ and, since choosing e has already been their optimal choice in t, they all choose effort e also in period $t+1$.

Proposition 1.1 shows that any set of beliefs corresponding to some uniform choice of effort among agents is absorbing. This raises the question whether it is guaranteed that the belief process ends up in one of these absorbing states with uniform effort choice. In the absence of social influence the answer to this question is affirmative. In the following proposition we show that if $\chi = 0$ the population always converges to a state in which all agents have identical point beliefs, i.e. they all expect with probability one that the minimum effort is some $\tilde{e} \in \mathcal{X}$ and choose their own effort level equal to $\tilde{e}$. Moreover, the long run effort is determined by the minimal effort chosen among all agents at $t = 1$.

Proposition 1.2. *Assume that $\chi = 0$ and denote by $\underline{e}_1 = \min_{i \in N}[a^*(b_{i,1})]$. Then,*

(i) actions of an agent never increase over time: $a_{i,t+1} \leq a_{i,t}$ for all $i \in N, \ t \geq 1$.

(ii) for every $\epsilon > 0$ there exists $T > 0$ such that $I\!P[a_{i,\tau} = \underline{e}_1 \ \forall i = 1,..,N$ for all $\tau \geq t] > 1 - \epsilon$,

(iii) $b_{i,t} \to \mathbb{1}_{\underline{e}_1}$ (in probability) for $t \to \infty$ for all $i \in N$.

Proposition 1.2 shows that in the absence of social influence the population in the long run always coordinates on some effort level, which means that the population profile of beliefs always reaches a state corresponding to a Nash equilibrium of the game. However, the proposition also gives a clear indication that the dynamic adjustment of individual's beliefs induces a downward trend in the chosen effort and in the long run all agents adopt the smallest among all effort levels chosen in the initial period. The reason for this downward trend is that, due to the structure of the minimum effort game, no agent ever observes an outcome which is above its own effort level. In the absence of social influence this implies that when an agent updates her beliefs she always increases the weight of an effort level which is below or equal her best response in the current period. Hence, the best response of an agent can never increase over time. This induces contagion-like dynamics of agents switching to beliefs that induce the minimal effort in the population, which constantly remains at $\underline{e}_1$. Hence, in the long-run all agents in the population choose this minimal effort.

For a sufficiently large value of the social influence parameter χ the claim that beliefs and actions become uniform in the long are in general no longer true. Actions might stay heterogeneous forever and, therefore, also effort levels above the initial minimal effort are chosen by some agents in the long run. Defining $\underline{\chi} := \frac{(n-2)\beta}{(n-k)\alpha}$ we obtain the following proposition.

Proposition 1.3. *Assume that* $(n-k)\alpha > (n-2)\beta$ *and* $\chi \geq \underline{\chi} > 0$. *Then for any initial belief profile* B_1 *with* $|\{i \in N : a^*(b_{i,1}) = \underline{e}_1\}| = 1$ *there exists a network s such that* $\max_{i\in N}[a_{i,t}] > \min_{i\in N}[a_{i,t}]$ *for all t with probability one.*

To interpret the proposition it should first be noted that the condition $(n-k)\alpha > (n-2)\beta$ implies that $\underline{\chi} < 1$, such that the interval $(\underline{\chi}, 1]$ of values of the social influence parameter χ leading to heterogeneous long-run beliefs and actions is not empty. Given that we always have $\alpha > \beta$ the condition is quite weak as long as we assume that the size of the interaction group (k) is small compared to the population size (n). Whereas Proposition 1.3 is formulated for initial beliefs inducing that the minimal effort in the first period is chosen by a single agent, analogous results can be obtained for scenarios with a larger number of agents choosing the lowest initial effort $\underline{e}_1$ with an adjusted value of the threshold $\underline{\chi}$.

The intuition for the potential long-run heterogeneity of beliefs and actions in the presence of social influence is that in a situation, in which the beliefs of the social contacts of an agent are more optimistic than the observed outcome of the group interaction of that agent, the social influence, i.e. the direct communication of beliefs, might prevent the downward adjustment of that agent's beliefs. Hence the contagion of low effort choices, which drives the dynamics in the absence of social influence, might be stopped. If the social network is such that agents, which initially choose low effort do no have social ties to more optimistic individuals, which is the type of network on which the proof Proposition 1.3 is based, then neither do these agents increase their actions over time, nor do the more optimistic individuals, who are linked through the social network, adjust their beliefs so strongly downwards to choose the minimal population effort. Whenever these optimistic agents are in the same interaction group with the pessimistic agent, the negative impact of the observed low outcome in their group on their beliefs is outweighed by the social influence from their optimistic social contacts. Hence, in such a scenario long-run heterogeneity of beliefs and actions prevails.

Our analytical findings provide little guidance on the shape of the dynamics and long run distribution of beliefs and actions for $\chi > 0$. In the following section we employ numerical simulations to explore how the evolution of beliefs and efforts are affected by social influence in different settings. In particular, we examine how the degree of confidence in the beliefs of others, and the topology of the social network influence the distribution of minimum efforts and associated payoffs in the game.

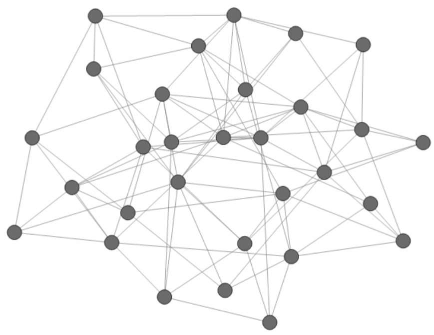


Figure 1.1: Erdős–Rényi random network of $n = 30$ players, with probability $p = 0.2$ of a formed link between each pair.

1.4 Effect of Social Influence

The following analysis relies on a baseline parametrization of the model given by $n = 30, k = 5, \alpha = 2, \beta = 1, \bar{e} = 15, \xi = 0.1$ and $\chi = 0.3$. The game parameters α and β are chosen in line with Van Huyck et al. (1990). A relatively low speed of updating ξ is chosen to prevent overly naive behavior of agents driven entirely by their previous period observation. Variations of the confidence parameter χ will be discussed extensively in the next section. The chosen values of n, k and $\bar{e}$ turn out not to be crucial for the qualitative results we will discuss. Robustness checks showing that our results still hold for alternative specifications of the parameters are provided in Appendix 1.C. Furthermore, we assume that all agents initially have heterogeneous point beliefs of the form $b_{i,1} = \mathbb{1}_{\tilde{e}}$ for $\tilde{e}$ uniformly chosen from $\mathcal{X}$. Our comparison of model outcomes under different parameter and network constellations is based on batches of $Q = 20$ simulation runs carried out for each constellation. In order to avoid spurious effects induced by different set of initial beliefs across the sets of batch runs, we generate a set of Q initial beliefs (one for each single simulation run). We use these same initial beliefs in each set of batch runs carried out under the different considered parameter constellations. Concerning the social network, our benchmark is to consider a random network with linking probability $p = 0.2$ between each pair of agents. Similarly to our approach taken with respect to initial beliefs we generate a set of Q random networks and use this same set in every batch of run across different parameter constellations.

1.4.1 Model Dynamics in the Baseline Scenario

To gain some initial understanding of the mechanisms at work we first illustrate the dynamics of a single simulation run in the case of a random social network displayed in Figure 1.1.

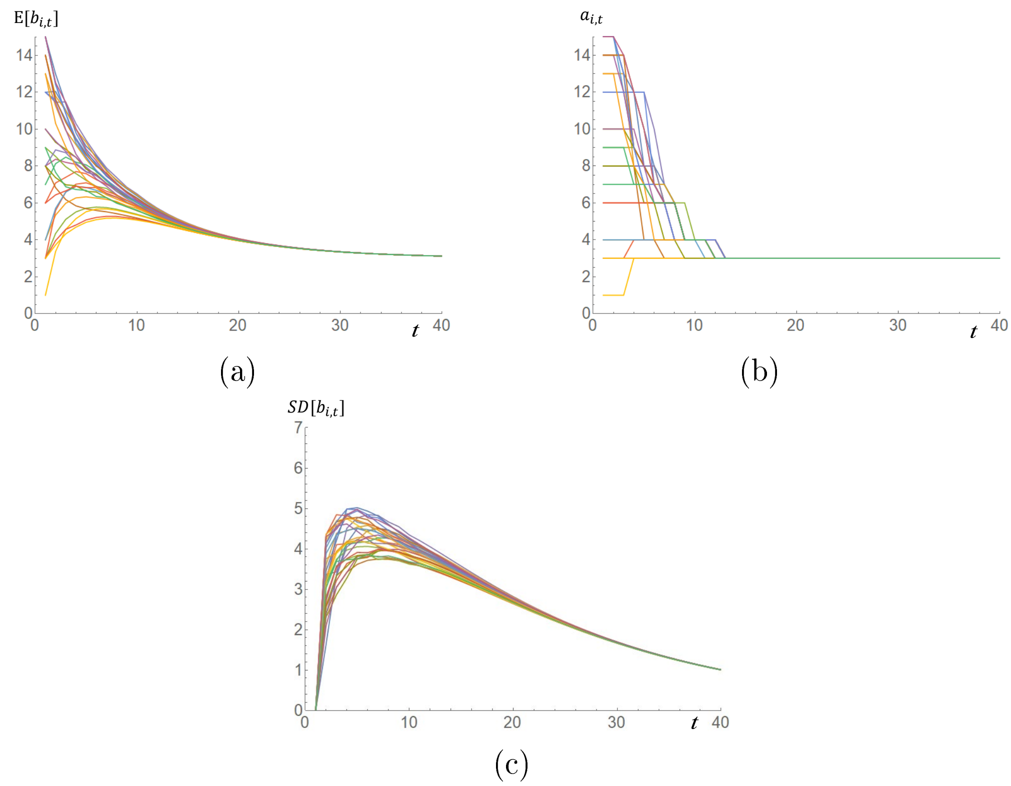


Figure 1.2: Dynamics of the expected values of the belief distributions (a), the chosen actions (b) and the standard deviation of the belief distributions (c) for all agents in a single run in the baseline scenario.

Figure 1.2 shows the dynamics of the action as well as the expectation and the standard deviation of the belief of each agent in the population. The figure illustrates that whereas agents start with heterogeneous point beliefs, initially the belief distributions of individual agents quickly become more dispersed (i.e. the standard deviation of individual belief distributions increases, see panel (c)) and at the same time the expected values of individual beliefs approach each other (panel (a)). This is due to the interplay of observing the actual minimum effort in an agent's own group, which might differ from the agent's expectation, and the communication of beliefs from the agent's social contacts. Actions stay strongly heterogeneous for approximately 10 periods and then quickly converge to a common effort level of $e = 3$ in this run. Once actions have converged to a uniform profile they stay constant over time (as shown in Proposition 1.1). Panel (b) of Figure 1.2 illustrates this result. The standard deviation of individual beliefs goes to zero (see panel(c)) and the expectations of the individual beliefs become uniform across agents slowly converging to the actual effort level observed in all groups (see panel (a)). In other words, the belief profile converges towards a profile of homogeneous point beliefs.

Agents with low point beliefs at $t = 1$, due to social influence, quickly become

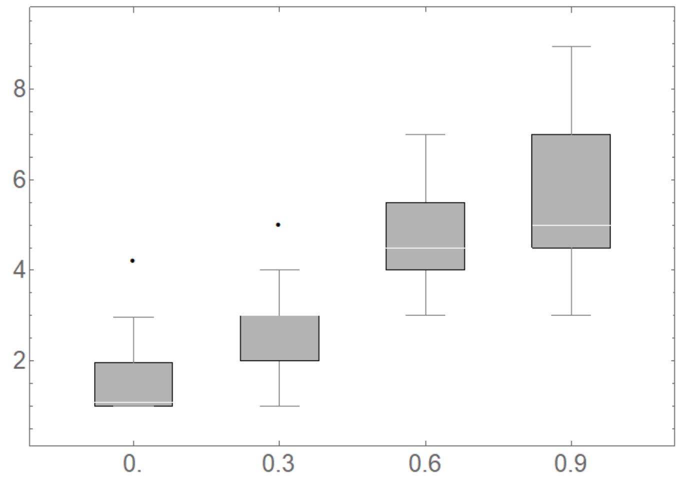


Figure 1.3: Distribution of average efforts at $t = 40$ for $\chi = 0, 0.3, 0.6, 0.9$.

more optimistic about the minimum effort in their group (see panel (a)). For some of them this leads to an increase of the chosen action over time (see panel (b)). In light of Proposition 1.2(i), which shows that in the absence of social influence individual effort levels can never increase over time, it is clear that this effect is driven by the communication of beliefs between agents. The intuition is similar to that already discussed at the end of Section 1.3. In a setting where outcomes are determined by the lowest effort in the group, like the minimum effort game, direct communication is important because it allows agents to realize that other agents in the population have expectations that are much more optimistic than observable outcomes would suggest. Hence, social influence might induce an upward adjustment of individual beliefs which is sufficiently strong to give rise to an increase of the agent's (optimal) action choice.

1.4.2 Effects of Social Influence

The discussion of the single run in our baseline scenario highlights the strong importance of social influence for the dynamics of effort level choices in the population. To examine the role of social influence in more detail we now systematically analyze how the agents' level of confidence in the beliefs of their social contacts affects the emerging level of effort in the population. More precisely, we vary the confidence parameter χ between $\chi = 0$ and $\chi = 0.9$ and for each of the considered values carry out a batch of $Q = 20$ simulation runs of $T = 40$ periods. Figure 1.3 shows boxplots of the distributions of the population average of actions across the batch runs. It should be noted that in all runs at $t = 40$ actions are already uniform such that the population average coincides with the action of every single agent in the population. It can be clearly seen

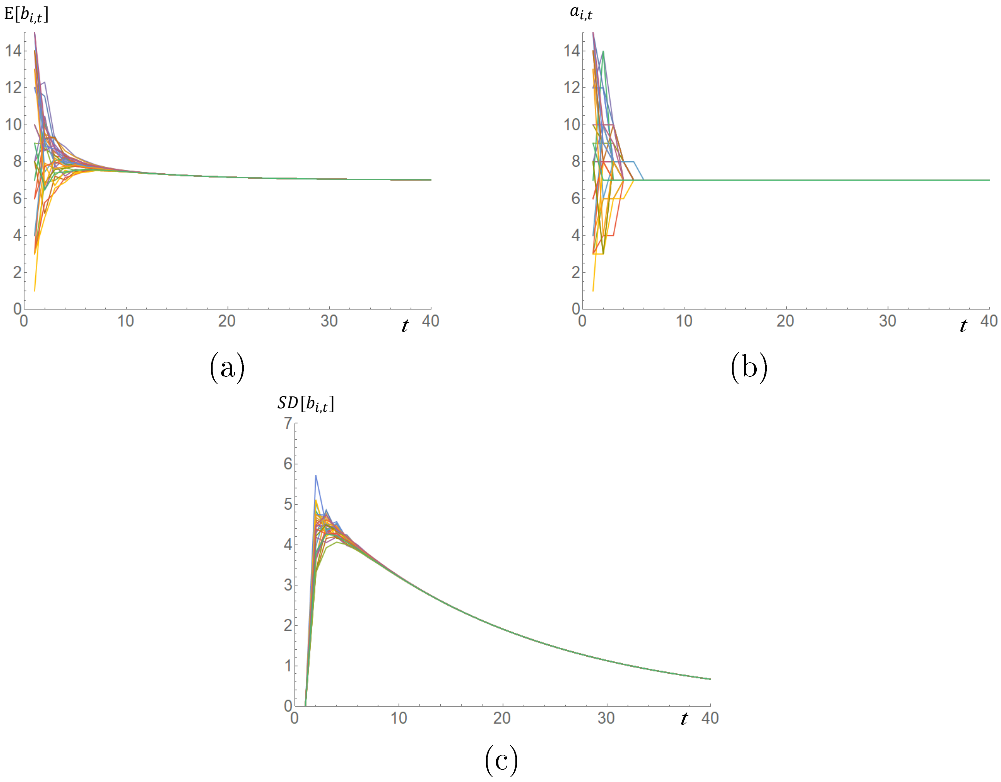


Figure 1.4: Dynamics of expected values of minimal effort (a), chosen actions (b) and standard deviation of the belief distributions (c) for all agents in a single run under strong social influence ($\chi = 0.9$).

that a higher level of social influence, expressed by a larger value of χ, significantly increases the distribution of long run efforts in the population.[21] In Appendix 1.B we provide the results of the Wilcoxon Signed Rank Tests to show that the distributions of efforts change significantly for different values of χ.

Similarly to Figure 1.2 described earlier, Figure 1.4 shows the dynamic of the action (panel (b)), expected value and the standard deviation of the belief of each agent in the population (panels (a) and (c)) where the level of confidence in social contacts' beliefs χ is equal to 0.9 compared to $\chi = 0.3$ in the baseline scenario. In order to see the effect of increased social influence one can first notice that the beliefs of the agents converge much faster with higher confidence level χ. Particularly, the agents with low initial beliefs become more optimistic quicker and the beliefs across all agents converge faster. Based on this, the beliefs stay more optimistic in the long run compared to the benchmark in Figure 1.2 where the population variance of the beliefs decreases much slower. Panel (c) of Figure 1.4 shows that the beliefs of individual agents quickly become more dispersed reaching the peak of the standard deviations earlier compared

[21] Throughout the analysis we consider the difference between two distributions to be statistically significant when the when the p-value of the Wilcoxon signed rank test is below 0.05.

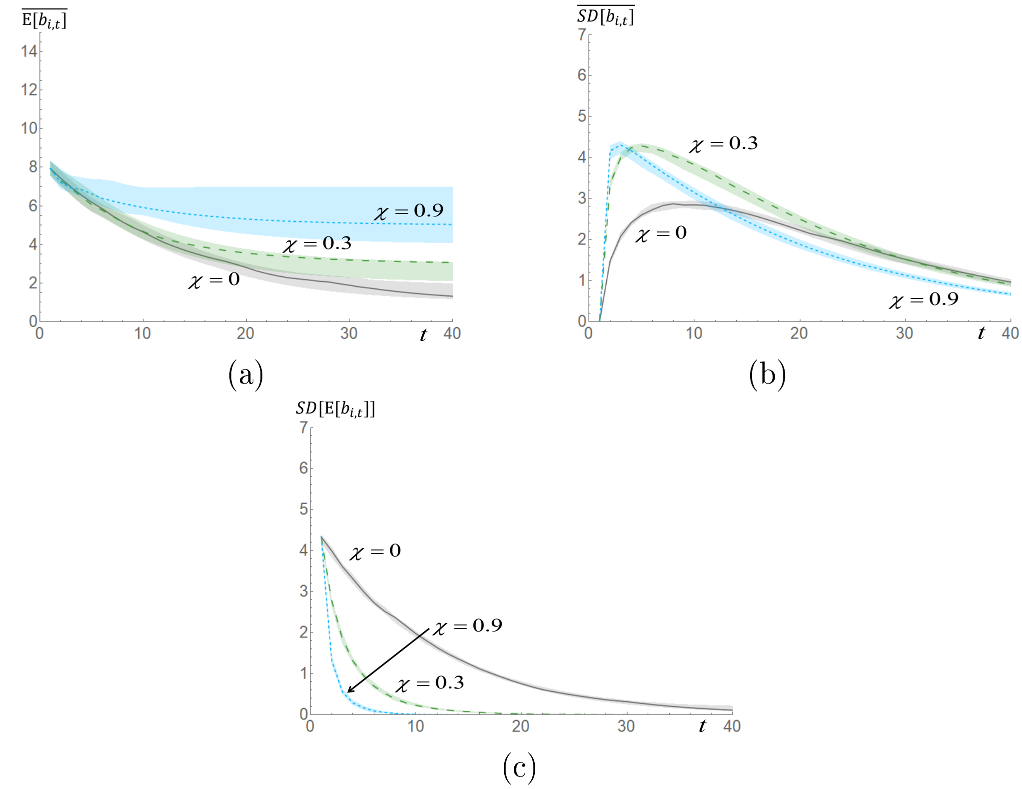


Figure 1.5: Dynamics of distribution of the population mean of expectation (a) and standard deviation of the individual belief distributions in the baseline scenario. Panel (c) shows the standard deviation of expected minimal effort across agents. The confidence bands in all panels illustrate the dynamics of the mean across the set of batch runs and one standard deviation from it for $\chi = 0$ (grey), 0.3 (green) and 0.9 (blue).

to the benchmark, but this dispersion then is reduced more quickly than in the case of a low value of χ.

The positive effect of the social influence on the pace of convergence and effort choice proves to be robust. Figure 1.5 shows the dynamics of distributions of average beliefs about the minimal effort across the agents in the network (panel (a)), average standard deviation of beliefs across the agents (panel (b)) and the standard deviation of the beliefs in the network (panel (c)) for different values of χ. The figure illustrates confidence bands (across batch runs) of these values for baseline scenario (green bands), high level of social influence (blue bands) and the absence of social influence (grey bands). Similar to the single runs discussed above, the confidence bands for the (population) standard deviation of expectations about the minimum effort decreases much faster with higher level of social influence (see panel (c)). The distribution of average standard deviation of beliefs in the network reaches its maximum earlier and decreases much faster with higher value of χ (panel (b)). Finally, the beliefs converge to a significantly higher value of long run efforts when the level of trust in beliefs of the social contacts is high (panel (a)). Considering the initial periods in the dynamics of population mean of the expectations in Figure 1.5(a), it can be seen that the average speed (across batch runs) of its decrease is almost identical for the different values of χ. The crucial difference between the three considered scenarios is that, for large values of χ, the moment in which the population variance is close to zero is much earlier, and the mean expectation hardly decreases further once the population has become almost uniform. This observation reinforces our intuition that a stronger effect of social influence improves the long-run effort level mainly by fostering faster convergence of population beliefs, thereby avoiding a long lasting drift towards a minimal population effort which is substantially below the average population effort.

The grey bands in Figure 1.5 illustrate the results derived in Proposition 1.2(iii) that beliefs of all agents slowly converge to the lowest effort exerted in the first period of the games when there is no social influence in the network.

1.4.3 Effects of Social Network Topology

Previous work on opinion formation (e.g. Golub and Jackson (2009), Acemoglu et al. (2010)) has demonstrated the importance of the social network structure on the emergence of consensus in a population and the ability to learn the true state of the world. In this section we investigate how the ability of a population to coordinate on an effort level in our minimum effort game and the efficiency of the emerging effort level is influenced by different properties of the social network.

Number of Links

First we explore the effect of a changing level of connectedness in the network by varying the parameter p which determines the probability that there is a link between two nodes. As can be seen in Figure 1.6 increasing this parameter has a non-linear effect on the distribution of the emerging long-run effort levels. In particular, the change in p has a significant positive effect only on a short interval from 0.1 to 0.2. Further increase in the number of links doesn't affect the expectations and thus the long-run effort levels in a statistically significant way. Hence, above a certain minimal level a higher degree of connectedness in the network does not foster the emergence of more efficient equilibria with higher effort level.

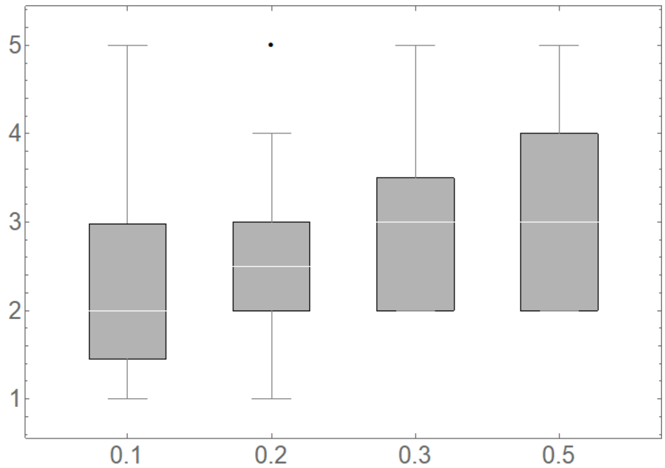


Figure 1.6: Distribution of average efforts at $t = 40$ for linking probability $p = 0.1, 0.2, 0.3, 0.5$.

Network Centralization

So far we have considered random networks with identical linking probabilities between all agents. However, many social networks are characterized by a 'core-periphery' structure where a few central agents are each linked to a large number of individuals (see Borgatti and Everett (1999)). In order to study the effect of such (partially) centralized communication structure, we first consider the extreme case of a star network, in which all nodes are connected to one central agent, and then consider scenarios with several coexisting star networks.

Contrary to the weak effect of increasing connectedness in the random network, changing the type of the network into a star network with a single center boosts the expected minimum effort and the payoffs in the long run. Comparing the individual expectations and variances for random and star networks in Figures 1.2 and 1.7 one can observe that the expectations about the minimum effort converge to much higher level when the network is a star. The significant positive effect from this change in network topology on the final distribution of average efforts is shown in Figure 1.8. However, the variance of this distribution is much higher in the case of a star network. This is implied by the strong dependence of the outcome on the initial belief of the central node in the star. More specifically, each of the periphery nodes in the network learns about the belief of the central node, which over time then is adapted to the average beliefs of all periphery nodes. If an agent with pessimistic initial beliefs is located at the center of the star and therefore the only source of belief communication for periphery nodes, the average beliefs in the network quickly become more pessimistic. While a very optimistic center will push the beliefs in the network up.

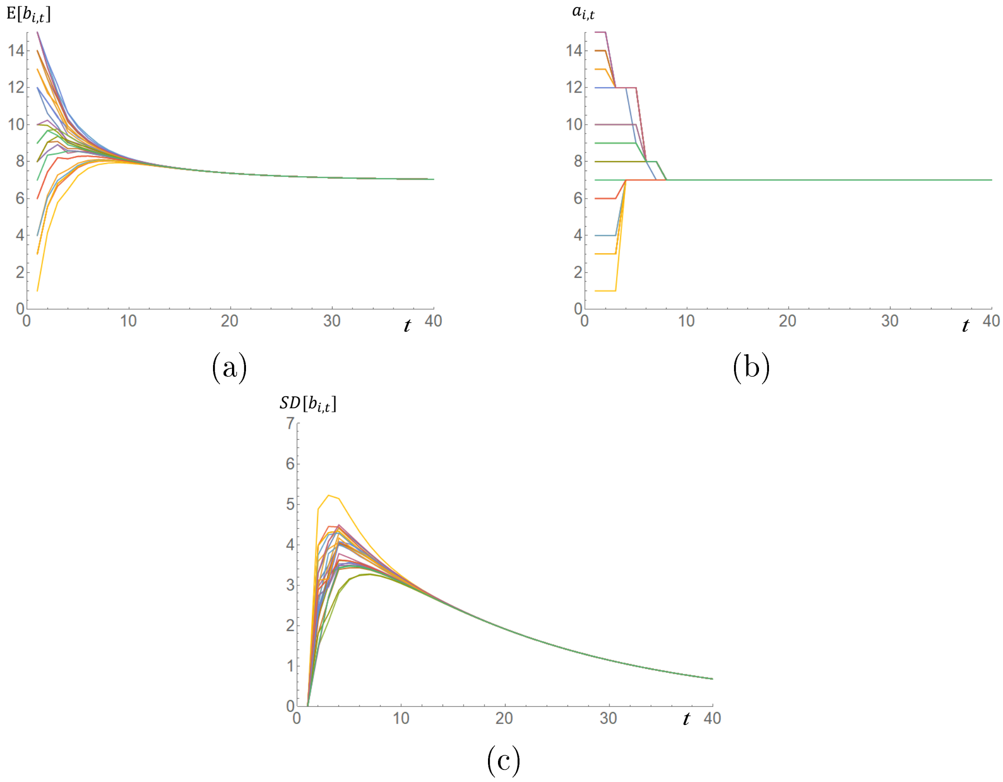


Figure 1.7: Dynamics of expected values of minimal effort (a), chosen actions (b) and variance of the belief distributions for all agents in a single run with a star social network.

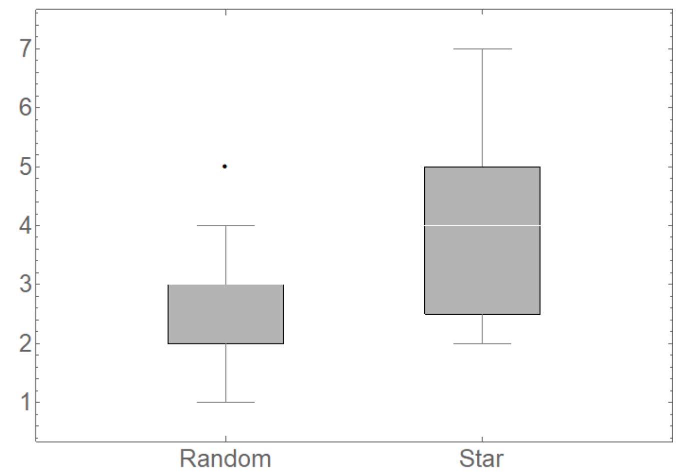


Figure 1.8: Distribution of average efforts at $t = 40$ for a random network (left) and a star network (right)

As becomes apparent from Figure 1.7, the beliefs of the different agents become uniform much faster in a centralized network, which leads to a faster convergence of actions. Since average effort levels exhibit a negative trend as long as there is substantial heterogeneity of beliefs in the population, the fast convergence of beliefs

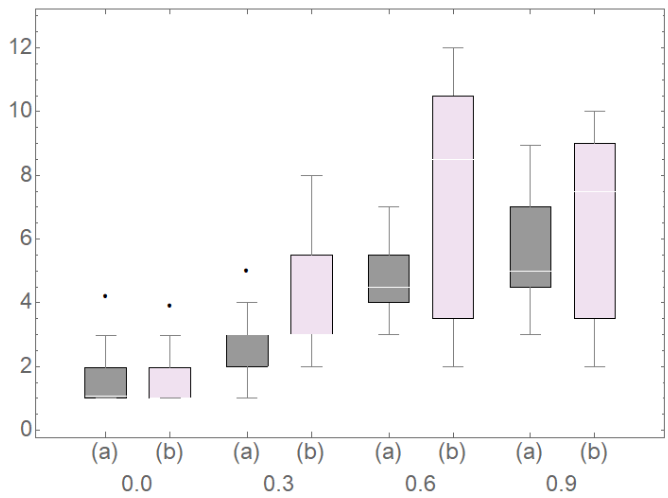


Figure 1.9: Distribution of average efforts at $t = 40$ with a random network (a) and a star network (b) for $\chi = 0, 0.3, 0.6, 0.9$.

leads to systematically higher effort levels compared to the random network. Given this effect of centralization, higher confidence in beliefs of social contacts for a large range of χ has a stronger effect in the star network compared to the random one (Figure 1.9). The increase of χ in a star network however has a significant positive effect only up to some level. Increasing the value of χ from 0.6 to 0.9 affects the long run distribution of beliefs negatively (see Figure 1.9).

If the centralization of the information flow is not global, like in a network with a single star, but rather characterized by the coexistence of several influential 'local stars' the positive effect of the network centralization quickly diminishes. Figure 1.10 shows the distribution of long-run efforts for networks with one to six star components. Increasing the number of components from two to three and then from three to four components each yields a significant reduction in long-run effort, and the distribution of effort in a network with four star components is actually already below the distribution under a random network. Figure 1.11 illustrates the dynamics of individual beliefs and actions for a network with three star components. The mechanism leading to the relatively low long-term effort can be clearly identified in this figure. Individual beliefs of agents in each of the three components converge quickly due to communication of beliefs and social influence, however, agents from each component are repeatedly matched in groups with agents from the component with the lowest beliefs, yielding low effort observations for these agents. Hence, beliefs in all components over time slowly adjust downwards towards the beliefs of the lowest component and actions across these components converge to a level determined by the actions taken by members of the most pessimistic component.

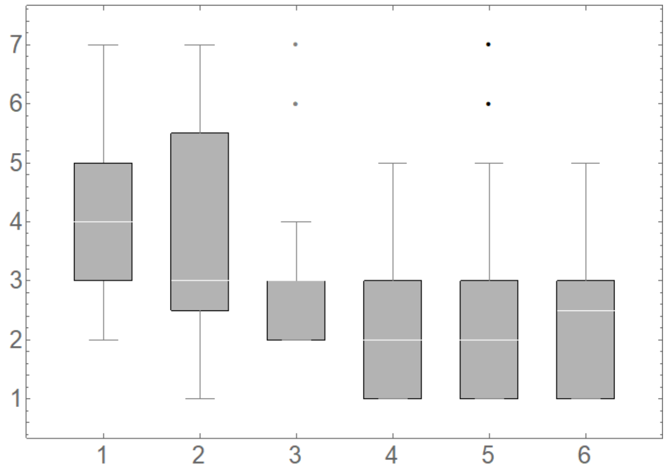
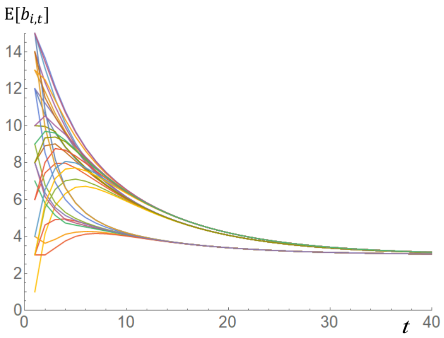
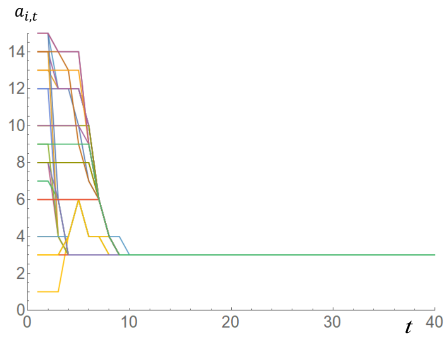


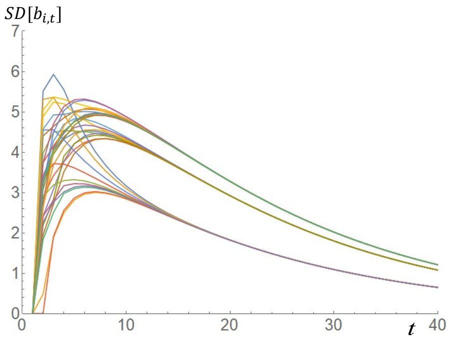
Figure 1.10: Distribution of average efforts at $t = 40$ for star network (left) and segregated networks with 2 to 6 star components.



(a)



(b)



(c)

Figure 1.11: Dynamics of expected values of minimal effort (a), chosen actions (b) and variance of the belief distributions (c) for individual agents in a single run with a segregated social network with 3 star components.

The effect of a segregation of the network into several disconnected components is much weaker if each component is a random network. Simulation results not reported in detail here show that increasing the number of components in general has only insignificant effects in such a setting.

Before analyzing an extension of our benchmark model, we like to point out that in all simulation runs in all settings considered in this section the beliefs and effort choices eventually become uniform. In particular, this observation also holds true for scenarios with high values of the social influence parameter χ. Relating this to Proposition 1.3, which shows that for large values of that parameter there always exist social networks and initial beliefs such that convergence to a uniform profile does not occur, shows that the types of social networks that do not induce uniform long-run profiles are of very special structure. As sketched in Section 1.3, long-run heterogeneity of beliefs requires the existence of separated components in the network where the social influence in the component with optimistic beliefs is sufficiently strong to outweigh the low-effort observations made by the agents in that component who are matched with members of the (smaller) component with pessimistic beliefs. Simulation results not reported here show that adding a single connection between these components typically is sufficient to induce uniform long-run beliefs in the population. Also, the results reported in the previous paragraphs show that even under fully separated components uniform beliefs and effort choices emerge in the long-run, if the components are of equal size.

1.5 Communication of Information

In our benchmark model we assume that an agent receives information about the outcomes of the minimal effort games in groups other than their own only indirectly through the communication of the beliefs of their social contacts. However, in many situations, individuals might not only communicate their beliefs to their social contacts, but also the actual outcome of their own interaction group. In this section we analyze whether the presence of such communication of information about the group outcomes in the social network changes the qualitative effects of belief communication. Furthermore, we explore whether communication of information might act as a substitute for the communication of beliefs or whether it might even reinforce the (positive) effect of belief communication on the long-run effort level emerging in the population.

We extend the model described in section 1.2 by assuming that, when building their intermediate beliefs, agents do not only take into account the outcome of their own interaction group, but also that in all groups of their social contacts. In particular, we replace in equation (1.1) term $\mathbb{1}_{\underline{e}_{i,t}}$ with $\hat{b}_{i,t}(e)$, which is given by

$$\hat{b}_{i,t} = \mathbb{1}_{\underline{e}_{i,t}} + \kappa \frac{1}{\eta_i} \sum_{j \in m_i(s)} \left(\mathbb{1}_{\underline{e}_{j,t}} - \mathbb{1}_{\underline{e}_{i,t}} \right). \tag{1.3}$$

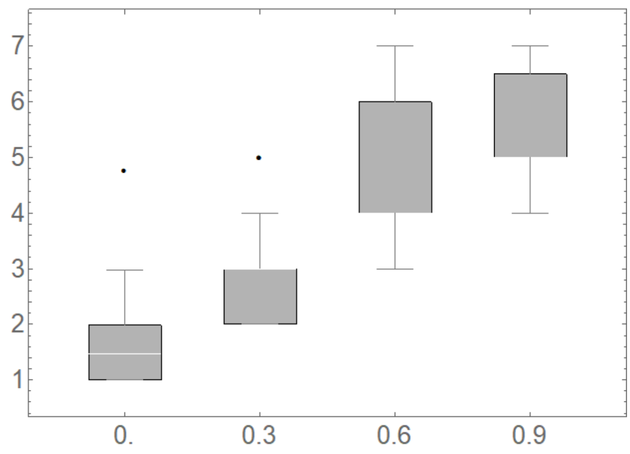


Figure 1.12: Distribution of average efforts at $t = 40$ for $\chi = 0, 0.3, 0.6, 0.9$, with $\kappa = 0.3$.

Here $0 \leq \kappa \leq 1$ is a weight that the agent i assigns to the information about the minimum effort observed and communicated by her social contacts. Thus, we obtain the following generalization of equation (1.1), describing the intermediate belief formation with information communication:

$$\tilde{b}_{i,t+1}(e) = (1 - \xi) b_{i,t}(e) + \xi \hat{b}_{i,t}(e)$$

with $\hat{b}_{i,t}$ given in (1.3). For $\kappa = 0$ this formulation gives our benchmark model studied in the previous section.

We again use our baseline parametrization and additionally set $\kappa = 0.3$ as the baseline value for the weight assigned to the observations of the social contacts. Figure 1.12 shows that the impact of variation of the confidence parameter χ on the long run effort choice stays significant also with communication of information. Comparing the figure with Figure 1.3 also shows that quantitatively the effects of an increase of the social influence parameter χ are hardly affected by the presence of communication of information. Extensive analyses of the extended model for all the scenarios examined in Section 1.4 furthermore show that all earlier results remain qualitatively unchanged also in the presence of information communication. Statistical tests demonstrating this are presented in Appendix 1.B together with the tests for the baseline model.

Concerning the question whether the exchange of information can act as a substitute for the exchange of beliefs, we show in Figure 1.13 how the distribution of long-run effort changes if there is no exchange of beliefs ($\chi = 0$) and the parameter κ determining the weight of the information obtained from an agent's social contacts is increased. The figure clearly demonstrates that exchanging only information without exchanging

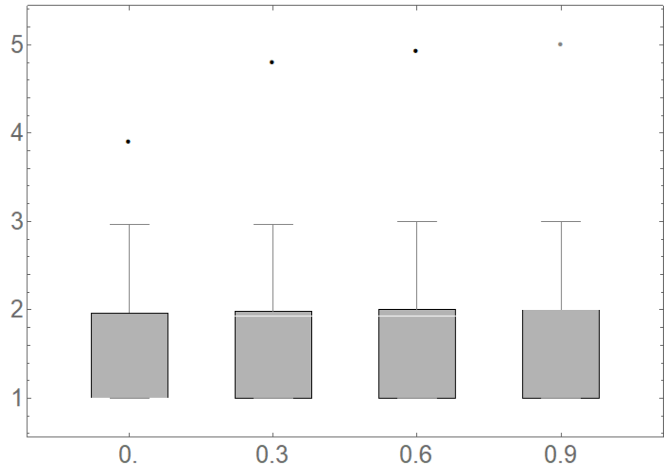


Figure 1.13: Distribution of average efforts at $t = 40$ for $\kappa = 0, 0.3, 0.6, 0.9$, when $\chi = 0$.

beliefs has hardly any positive effect on the level of long-run effort which emerges. This insight is consistent with the intuition developed above that the main role of the exchange of beliefs is that agents in this way get signals about (expected) minimal effort in groups that are more positive than the actual project outcomes which are observed. This role cannot be played by the communication of actual minimal efforts in the groups of the social contacts. Therefore, the communication of information about project outcomes cannot act as a substitute for the communication of beliefs in fostering more efficient outcomes of the minimal effort game. Overall, our results show that the main results discussed in the previous section qualitatively stay intact if apart from beliefs also information is communicated in the social network.

1.6 Conclusions

This paper highlights from a theoretical perspective the potential importance of social influence for improving efficiency of the outcome of group production problems with strategic complementarities. Also, it shows that the topology of the social network has a significant influence on the achieved outcome. Unfortunately, at this point experimental studies exploring the role of social influence in such a setting is missing. Hence, the analysis provided in this paper is a natural basis for designing and carrying out experiments clarifying in how far the effects identified in our study are also observable in the lab. Also from a theoretical perspective it would be interesting to explore the relevance of several of the assumptions that have been made in this paper. This includes the consideration of endogenous updating of the social network as well a generalization of the agents' behavior by incorporating logit best reply or social preferences or a more refined expectation updating process, e.g. like the one developed in Grimm and Mengel (2019) based on its good match with experimental evidence. These issues are left for future research.

Appendix 1.A Proofs

Proof of Proposition 1.1. We first show a Lemma, which will be used in this proof as well as in the proofs of the following propositions.

Lemma 1.1. *Consider beliefs $b^1, b^2 \in \Delta(\mathcal{X})$ such that $a^*(b^1) \leq a^*(b^2)$ then $a^*(\kappa b^1 + (1-\kappa)b^2) \in [a^*(b^1), a^*(b^2)]$ for all $\kappa \in [0,1]$.*

Proof of Lemma 1.1. In light of the form of $\pi(e_i, e_{-i})$ we can write the expected profit of an agent with effort e and belief b as

$$\pi^e(e,b) = \alpha \left(\sum_{\tilde{e} \leq e} b(\tilde{e})\tilde{e} + e \sum_{\tilde{e} > e} b(\tilde{e}) \right) - \beta e.$$

Hence,

$$\Delta\pi^e(e,b) = \pi^e(e+1,b) - \pi^e(e,b) = \alpha \sum_{\tilde{e} > e} b(\tilde{e}) - \beta,$$

which is (weakly) decreasing in e. Now consider some effort $= e^1 < a^*(b^1)$. Since $e^1 < a^*(b^1) \leq a^*(b^2)$ and $\Delta\pi(e,b)$ is decreasing in e, we must have $\Delta\pi^e(e^1, b^1) \geq 0$ and $\Delta\pi^e(e^1, b^2) \geq 0$. Therefore,

$$\Delta\pi^e(e^1, \kappa b^1 + (1-\kappa)b^2) = \kappa\Delta\pi^e(e^1, b^1) + (1-\kappa)\Delta\pi^e(e^1, b^2) \geq 0.$$

Since $a^*(\kappa b^1 + (1-\kappa)b^2)$ is the largest effort among those maximizing the expected profit of the agent, this shows that $e_1 < a^*(\kappa b^1 + (1-\kappa)b^2)$. Hence $a^*(\kappa b^1 + (1-\kappa)b^2) \geq a^*(b_1)$. Analogous arguments show that $a^*(\kappa b^1 + (1-\kappa)b^2) \leq a^*(b_2)$. This completes the proof of the lemma. □

To prove claim (i) of the proposition consider a profile B_t such that $a^*(b_{i,t}) = e$ for all $i \in N$. It should be noted that under such a profile $\underline{e}_{i,t} = e$ for all $i \in N$ regardless of the realization of the group partition. Hence B_{t+1} is deterministic and we show that also for B_{t+1} we have $a^*(b_{i,t+1}) = e$ for all $i \in N$. Claim (i) then follows by induction. To show that $a^*(b_{i,t+1}) = e$ for all $i \in N$, we consider an arbitrary agent i. Since $\underline{e}_{i,t} = e$ and $a^*(\mathbb{1}_e) = e$, it follows from Lemma 1.1 that

$$a^*(\tilde{b}_{i,t+1}) = a^*((1-\xi)b_{i,t} + \xi\mathbb{1}_e) = e.$$

Since the same reasoning also applies to all social contacts of i, repeated application of the lemma establishes that $a^*\left(\frac{1}{\eta_i(s)}\sum_{j\in m_i(s)}\tilde{b}_{j,t+1}\right)=e$, and therefore

$$a^*(b_{i,t+1}) = a^*\left((1-\chi)\tilde{b}_{i,t} + \chi\frac{1}{\eta_i(s)}\sum_{j\in m_i(s)}\tilde{b}_{j,t+1}\right) = e.$$

Hence, we obtain part (i) of the proposition.

To show part (ii) we define for some arbitrary effort $\hat{e}\in\mathcal{X}\setminus\{e\}$ the maximal probability for this effort level in any belief distribution in B_τ as $\bar{b}(\hat{e})_\tau = \max[b_{i,\tau}(\hat{e}) : i\in N]$ for all $\tau\geq t$. Since $\underline{e}_{j,\tau} = e \neq \hat{e}$ for all $j\in N$ and $\tau\geq t$ we have

$$\tilde{b}_{j,\tau+1}(\hat{e}) = (1-\xi)b_{j,\tau}(\hat{e}) \leq (1-\xi)\bar{b}(\hat{e})_\tau,$$

and therefore

$$b_{i,\tau+1}(\hat{e}) = (1-\chi)(1-\xi)b_{i,\tau}(\hat{e}) + \frac{\chi}{\eta_i(s)}\sum_{j\in m_i(s)}\tilde{b}_{j,\tau+1} \leq (1-\xi)\bar{b}(\hat{e})_\tau$$

for all $i\in N$. Hence, $\bar{b}(\hat{e})_{\tau+1}\leq(1-\xi)\bar{b}(\hat{e})_\tau$ and accordingly $\lim_{\tau\to\infty}\bar{b}(\hat{e})_\tau = 0$ for all $\hat{e}\neq e$. This implies that $b_{i,\tau}\to\mathbb{1}_e$ for $\tau\to\infty$ for all $i\in N$ □

Proof of Proposition 1.2. (i) By definition we have $\underline{e}_{i,t}\leq a^*(b_{i,t})$ for all i and all t. Hence, $a^*(\mathbb{1}_{\underline{e}_{i,t}}) = \underline{e}_{i,t}\leq a^*(b_{i,t})$. Using Lemma 1.1 we obtain

$$a_{i,t+1} = a^*(b_{i,t+1}) = a^*((1-\xi)b_{i,t} + \xi\mathbb{1}_{\underline{e}_{i,t}}) \leq a^*(b_{i,t}) = a_{i,t}.$$

(ii) First, we show that for any agent i with $a_{i,1} = \underline{e}_1 := \min_{j\in N} a_{j,1}$ we must have $a_{i,\tau} = \underline{e}_\tau = \underline{e}_1\ \forall\tau\geq 1$. This can be shown by induction. Assume that for all $\tau = 1,..,t$ beliefs $b_{i,\tau}$ are such that $a^*(b_{i,\tau}) = \underline{e}_t = \underline{e}_1$. Then obviously we have $a_{i,t} = \underline{e}_1$ and therefore, under consideration of $\chi = 0$, the updated belief reads

$$b_{i,t+1} = \tilde{b}_{i,t+1} = (1-\xi)b_{i,t} + \xi\mathbb{1}_{\underline{e}_1}.$$

Using Lemma 1.1 it follows from $a^*(b_{i,t}) = a^*(\mathbb{1}_{\underline{e}_1}) = \underline{e}_1$ that $a_{i,t+1} = a^*(b_{i,t+1}) = \underline{e}_1$. For any agent $j\neq i$ we have $a^*(b_{j,t})\geq\underline{e}_1$ and $\underline{e}_{j,t}\geq\underline{e}_1$, which implies that

$$a_{j,t+1} = a^*\left((1-\xi)b_{j,t} + \xi\mathbb{1}_{\underline{e}_{j,t}}\right) \geq \underline{e}_1.$$

From this we conclude that $\underline{e}_{t+1}\geq\underline{e}_1$, which in light of $a_{i,t+1} = \underline{e}_1$ implies $\underline{e}_{t+1} = \underline{e}_1$. This completes the induction and we have shown that $a_{i,t} = \underline{e}_t = \underline{e}_1$ for all $t\geq 1$.

Consider now an agent j with $a_{j,t} > \underline{e}_1$. Taking into account that e is a strictly optimal action under belief $b = \mathbb{1}_e$ and the continuity of π^e with respect to b, it follows that there exists some $\tilde{\lambda}$ such that $a^*(\lambda \mathbb{1}_{\underline{e}_1} + (1-\lambda) b_{j,t}) = \underline{e}_1$ for all $\lambda \geq \tilde{\lambda}$. Choose a $\tilde{T}$ such that $(1-\xi)^{\tilde{T}} < 1 - \tilde{\lambda}$. There is a positive probability that agent j is matched with an agent i with $a_{i,t} = \underline{e}_1$ for $\tilde{T}$ periods in a row. The belief of agent j in period $t + \tilde{T}$ then is given by

$$b_{j,t+\tilde{T}} = (1-\xi)^{\tilde{T}} b_{j,t} + (1 - (1-\xi)^{\tilde{T}}) \mathbb{1}_{\underline{e}_1}$$

and our reasoning above shows that, in case such a matching pattern occurs, this agent j switches to action $a_{j,t+\tilde{T}} = \underline{e}_1$. Following the arguments provided in the first part of the proof, this implies that $a_{j,\tau} = \underline{e}_1$ for all $\tau \geq t + \tilde{T}$. The same reasoning can be applied sequentially to every agent j with $a_{j,t} > \underline{e}_1$. Together, this establishes that from every profile of beliefs B_t, that can be reached with positive probability from B_1, there is a positive probability path to a profile $\tilde{B}$ with the property that $a^*(\tilde{b}_i) = \underline{e}_1$ for all $i \in N$. Denote the set of all such belief profiles $\tilde{B}$ by $\tilde{\mathcal{B}} = \{\tilde{B} | a^*(\tilde{b}_i) = \underline{e}_1 \ \forall i \in N\}$. We know from Proposition 1.1 that this set $\tilde{\mathcal{B}}$ of belief profiles is absorbing. Together with the fact that there is a transition path with positive probability from any reachable state into $\tilde{\mathcal{B}}$, this implies that $\mathbb{P}(B_t \notin \tilde{\mathcal{B}}) \to 0$ for $t \to \infty$. This shows part (ii) of the proposition. Part (iii) follows directly from the observation that if $B_t \in \tilde{\mathcal{B}}$ then $b_{i,t+\tau}(\hat{e}) \leq (1-\xi)^{\tau} b_{i,t}(\hat{e})$ for all $i \in N, \ \hat{e} \neq \underline{e}_1$. □

Proof of Proposition 1.3. Consider any initial belief profile B_1 with $|\{i \in N : a^*(b_{i,1}) = \underline{e}_1\}| = 1$. Without restriction of generality we label the agent with the smallest initial effort as agent 1. Hence, we have $a^*(b_{1,1}) = \underline{e}_1$ and $a^*(b_{i,1}) > \underline{e}_1 \ \forall i = 2, .., n$. Consider the social network s with $m_1(s) = \emptyset, \ m_i(s) = N \setminus \{1, i\}$. Since agent 1 does not have any social contacts under this social network, the beliefs of agent 1 in period t are given by

$$b_{1,t} = (1-\xi)^{t-1} b_{1,1} + (1 - (1-\xi)^{t-1}) \mathbb{1}_{\underline{e}_1}$$

and it follows from Lemma 1.1 that $a_{1,t} = \underline{e}_1$ for all t.

Considering agents $i = 2, .., n$ we proceed by induction. Assume that $a^*(b_{i,\tau}) > \underline{e}_1$ for all $i = 2, ..n$ and $\tau = 1, .., t$. We show that then also $a^*(b_{i,t+1}) > \underline{e}_1$ for all $i = 2, ..n$. Define, as in the proof of Proposition 1.2, the expected profit difference between two adjacent effort levels as $\Delta\pi^e(e, b) = \pi^e(e+1, b) - \pi^e(e, b)$. Since $\Delta\pi^e(e, b)$ is weakly decreasing in e and $a^*(b_{i,t}) > \underline{e}_1$ for all $i = 2, .., n$ it follows that

$$\Delta\pi^e(\underline{e}_1, b_{i,t}) = \alpha \sum_{\tilde{e} > \underline{e}_1} b_{i,t}(\tilde{e}) - \beta \geq 0.$$

Defining $x_{i,t} = \sum_{\tilde{e} > \underline{e}_1} b_{i,t}(\tilde{e})$, it follows that

$$x_{i,t} \geq \frac{\beta}{\alpha} \; \forall i = 2, .., n.$$

In every period $(k-1)$ agents are matched with agent 1 and hence observe $\underline{e}_{i,t} = \underline{e}_1$, whereas for the remaining $n-k$ agents we have $\underline{e}_{i,t} > \underline{e}_1$. Denote by agent j one of the agents matched with agent 1 in t. Furthermore, denote by $y_t = \sum_{i \neq 1,j} \sum_{\tilde{e} > \underline{e}_1} b_{i,t}(\tilde{e})$ the sum of the probabilities that all agents apart from agents 1 and j put on effort choices above $\underline{e}_1$. Clearly, we have $y_t \geq (n-2)\frac{\beta}{\alpha}$. Defining $\tilde{x}_{j,t} = \sum_{\tilde{e} > \underline{e}_1} \tilde{b}_{j,t}(\tilde{e})$ we obtain from (1.1)

$$\tilde{x}_{j,t+1} = (1-\xi)x_{j,t}.$$

Furthermore, we define by $\tilde{y}_t = \sum_{i \neq 1,j} \sum_{\tilde{e} > \underline{e}_1} \tilde{b}_{i,t}(\tilde{e})$ and we get

$$\tilde{y}_{t+1} = (1-\xi)y_t + \xi(n-k).$$

Inserting this into the belief adjustment due to social influence for agent j (see 1.2) we obtain

$$\begin{aligned} x_{j,t+1} &= (1-\chi)\tilde{x}_{j,t+1} + \frac{\chi}{n-2}\tilde{y}_{t+1} \\ &= (1-\chi)(1-\xi)x_{j,t} + \frac{\chi}{n-2}\left((1-\xi)y_t + \xi(n-k)\right) \\ &\geq (1-\chi)(1-\xi)\frac{\beta}{\alpha} + \frac{\chi}{n-2}\left((1-\xi)(n-2)\frac{\beta}{\alpha} + \xi(n-k)\right) \\ &= (1-\xi)\frac{\beta}{\alpha} + \chi\frac{n-k}{n-2}\xi, \end{aligned}$$

where we have used that $x_{j,t} > \frac{\beta}{\alpha}$ and $y_t \geq (n-2)\frac{\beta}{\alpha}$. If $\chi > \underline{\chi}$ this directly implies that $x_{j,t+1} \geq \frac{\beta}{\alpha}$ and therefore $a^*(b_{j,t+1}) > \underline{e}_1$. Clearly, the beliefs of all agents who have not been matched with agent 1 in period t are more optimistic than that of agent j and therefore we have $a^*(b_{i,t+1}) > \underline{e}_1$ for all $i = 2, .., n$. This completes the induction. Overall, we have shown that for all agents $i = 2, .., n$ we have $a_{i,t} > \underline{e}_1$ for all $t \geq 1$. Together with $a_{1,t} = \underline{e}_1$ for all $t \geq 1$ this shows that with probability one $\max_{i \in N}[a_{i,t}] > \underline{e}_1 = \min_{i \in N}[a_{i,t}]$, which completes the proof. □

Appendix 1.B Test Results

We perform Wilcoxon signed-rank test to show the significance of difference between distributions of long run effort choices. Tables 1.1-1.7 in this appendix show the p-values of the tests. We consider the difference of two distributions to be significant when $p < 0.05$.

Table 1.1: Effect of the social influence on long-run efforts.

χ	0 and 0.3	0.3 and 0.6	0.6 and 0.9
$\kappa = 0$	0.0002	0.0001	0.0006
$\kappa = 0.3$	0.0001	0.0001	0.0457

The table shows the effects of increasing social influence on the long-run efforts in the baseline model ($\kappa = 0$) and the extended model with the information communication ($\kappa = 0.3$) with random network.
We compare each pair of long-run effort distributions for scenarios with trust parameter values χ equal to 0 and 0.3, 0.3 and 0.6, 0.6 and 0,9.

Table 1.2: Effect of changing the random network connectedness.

	0.1 and 0.2	0.2 and 0.3	0.3 and 0.5
$\kappa = 0$	0.0419	0.2396	0.4666
$\kappa = 0.3$	0.6677	0.7794	0.5882

The table shows the effect of increasing average degree in the random network. We compare the pairs of long-run effort distributions in random networks with probabilities of link formation p equal to 0.1 and 0.2, 0.2 and 0.3, 0.3 and 0.5. The Wilcoxon signed-rank tests are performed for the baseline model ($\kappa = 0$) and the extended model with the information communication ($\kappa = 0.3$).

Appendix 1.C Robustness

In this appendix we show the robustness of our model with respect to changes in size of the social network (n), randomly generated groups' size (k), the number of possible strategies, that is, the highest effort level ($\bar{e}$), and the speed of beliefs' updating (ξ). In particular, we show that the effect of the increasing level of trust in beliefs of social contacts remains the same when changing the benchmark parametrization.

Table 1.3: Social influence effect in the star network.

χ	0 and 0.3	0.3 and 0.6	0.6 and 0.9
$\kappa = 0$	0.0001	0.0012	0.0178
$\kappa = 0.3$	0.0001	0.0021	0.0021

The table shows the effects of increasing social influence on the long-run efforts in the baseline model ($\kappa = 0$) and the extended model with the information communication ($\kappa = 0.3$) with star network.
We compare each pair of long-run effort distributions for scenarios with trust parameter values χ equal to 0 and 0.3, 0.3 and 0.6, 0.6 and 0,9.

Table 1.4: Effect of network centralization.

	$\chi = 0$	$\chi = 0.3$	$\chi = 0.6$	$\chi = 0.9$
$\kappa = 0$	0.2708	0.0007	0.0076	0.1978
$\kappa = 0.3$	0.0894	0.0003	0.0085	0.1506

The test result in the table indicate the significance of difference between long-run effort distributions of random network with benchmark value of link formation probability 0.2 and star network. The tests are performed for the baseline model ($\kappa = 0$) and the extended model with information communication ($\kappa = 0.3$). Alongside with the benchmark value of trust parameter $\chi = 0.3$ other levels of social influence are tested.

Table 1.5: Effect of increasing segregation level in random networks.

	1 and 2	2 and 3	3 and 4	4 and 5	5 and 6
$\kappa = 0$	0.9553	0.4441	0.2396	0.0119	0.6677
$\kappa = 0.3$	0.9553	0.8373	0.8082	0.0966	0.1850

The table shows Wilcoxon signed-rank tests results of comparing long-run effort distributions of random networks with 1 component and 2 disconnected components, as well as random networks with 2 and 3, 3 and 4, 4 and 5, and 5 and 6 disconnected components. The average degree in all networks is kept equal to the benchmark value.

Network Size

We compare the effect of increasing social influence in networks of sizes $n = 15$ and $n = 50$ to the benchmark of size 30. We find that in smaller network of size 15 increasing the value of parameter χ from 0.6 to 0.9 doesn't have a significant effect

Table 1.6: Effect of increasing segregation level in networks with star components.

	1 and 2	2 and 3	3 and 4	4 and 5	5 and 6
$\kappa = 0$	0.4441	0.0545	0.0239	0.5883	0.7795
$\kappa = 0.3$	0.0160	0.2396	0.0054	0.6407	0.6676

The table shows Wilcoxon signed-rank tests results of comparing long-run effort distributions of a star network and a network with 2 disconnected star components, networks with 2 and 3, 3 and 4, 4 and 5, and 5 and 6 disconnected star components correspondingly.

Table 1.7: Comparing the effects of χ and κ.

χ or κ	0 and 0.3	0.3 and 0.6	0.6 and 0.9
$\kappa = 0$	0.0002	0.0001	0.0006
$\chi = 0$	0.0048	0.0594	0.0117
$\kappa = 0.3$	0.0001	0.0001	0.0457
$\chi = 0.3$	0.2396	0.9256	0.1850

The table shows the effects of increasing social influence (χ) and coefficient of information communication (κ) on the long-run efforts when the κ and χ are set to 0.3 correspondingly.

(Figure 1.14[22], Table 1.8). This results in a significant difference between distributions of long run efforts for network sizes 15 and 30, the "cross-effect" of network size (Table 1.9, $p = 0.016$). In all remaining cases the increase in value of trust parameter χ has a significant effect on the long run effort choice, whereas the "cross-effect" of network size for a given value of χ is insignificant.

Table 1.8: Effect of increasing social influence on long-run efforts in different size networks.

χ	0 and 0.3	0.3 and 0.6	0.6 and 0.9
$n = 15$	0.0038	0.003	0.0545
$n = 30$	0.0002	0.0001	0.0006
$n = 50$	0.0001	0.0001	0.0018

The table shows the effect of increasing social influence on the long-run efforts in the baseline model ($\kappa = 0$). We compare each pair of long-run effort distributions for scenarios with trust parameter values χ equal to 0 and 0.3, 0.3 and 0.6, 0.6 and 0,9 when the size of the network is 15, 30(benchmark) and 50.

[22]In this and further figures, the darker grey coloring of the plots is to represent the benchmark model.

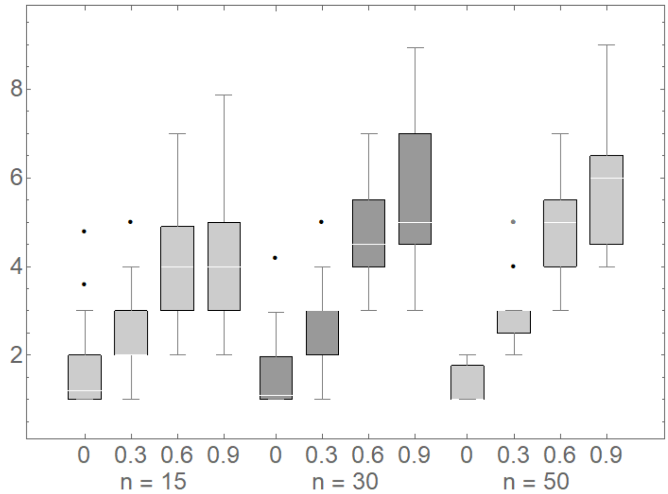


Figure 1.14: Distribution of average efforts at $t = 40$ for $\chi = 0, 0.3, 0.6, 0.9$, for network sizes $n = 15$, $n = 30$ and $n = 50$.

Table 1.9: Cross-effect of the network size.

	15 and 30	30 and 50
$\chi = 0$	0.9256	0.1305
$\chi = 0.3$	0.9256	0.1851
$\chi = 0.6$	0.1213	0.5135
$\chi = 0.9$	0.016	0.6407

The table shows the "cross-effect" of increasing network size for different levels of social influence parameter χ.

Random Group Size

The positive effect of the increasing χ holds when changing the size of random game groups k in the network of size 30. This result is illustrated in Figure 1.15 and confirmed by test results in Table 1.10. We also find that the increasing the number of people in randomly formed groups playing minimum effort game decreases the efforts significantly. Thus, increase in k from 2 to 5, and from 6 to 10 leads to a significant decrease in long-run efforts for all analyzed values of the trust parameter. Moreover, the negative effect increasing the group size from 5 to 6 is significant given there is no social influence in the network (Table 1.11).

Strategy Set

For a given set of strategies $\mathcal{X} = \{1, ..., \overline{e}\}$ the positive effect of trust parameter on effort choice is independent from the value of $\overline{e}$ (Figure 1.16, Table 1.12). Despite of

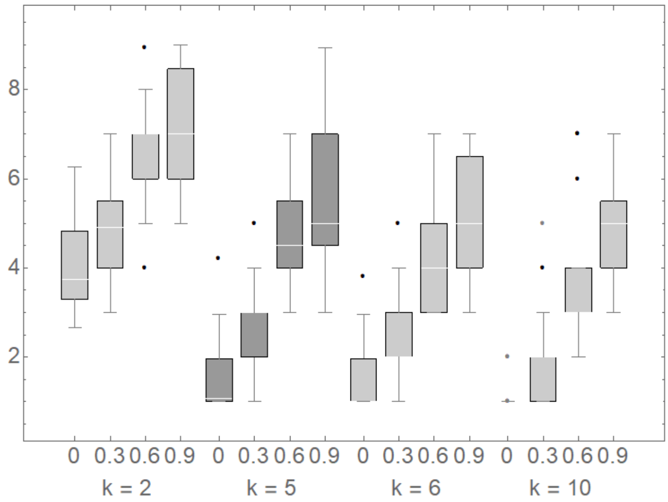


Figure 1.15: Distribution of average efforts at $t = 40$ for random group sizes $k = 2, 5, 6, 10$ and $\chi = 0, 0.3, 0.6, 0.9$.

Table 1.10: Effect of increasing social influence on long-run efforts with different random group sizes.

χ	0 and 0.3	0.3 and 0.6	0.6 and 0.9
$k = 2$	0.0003	0.0001	0.0021
$k = 5$	0.0002	0.0001	0.0006
$k = 6$	0.0002	0.0001	0.0002
$k = 10$	0.0023	0.0001	0.0002

The table shows the effect of increasing social influence on the long-run efforts in the baseline model ($\kappa = 0$). We compare each pair of long-run effort distributions for scenarios with trust parameter values χ equal to 0 and 0.3, 0.3 and 0.6, 0.6 and 0,9 when the size of randomly generated groups k are 2, 5, 6, and 10.

Table 1.11: Cross-effect of increasing the random group size.

	2 and 5	5 and 6	6 and 10
$\chi = 0$	0.0001	0.0458	0.0054
$\chi = 0.3$	0.0001	0.0826	0.0023
$\chi = 0.6$	0.0001	0.0646	0.0003
$\chi = 0.9$	0.0001	0.4009	0.0239

The table shows the "cross-effect" of increasing random group size k for different levels of social influence parameter χ.

the difference in number of possible efforts and the higher average effort, the long-run effort' distribution doesn't differ significantly with absence of social influence. While this difference grows significantly when the trust parameter χ is positive (Table 1.13). This can be explained scaling up with the average of possible strategies.

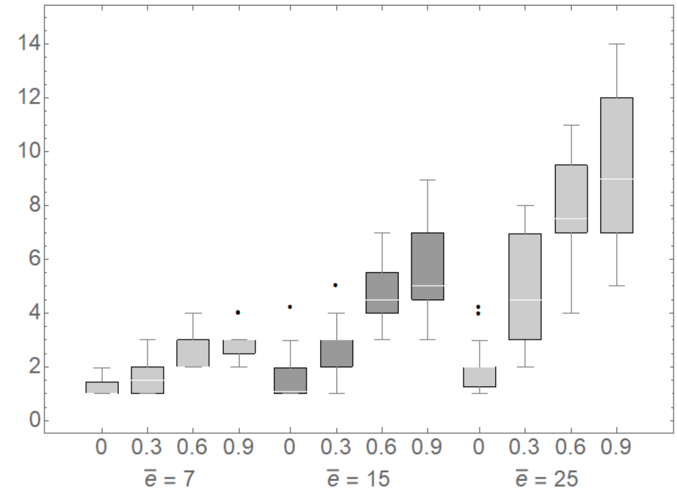


Figure 1.16: Distribution of average efforts at $t = 40$ with highest possible effort and number of possible strategies $\overline{e} = 7, 15, 25$ and $\chi = 0, 0.3, 0.6, 0.9$.

Table 1.12: Effect of increasing social influence on long-run efforts with different values of $\overline{e}$.

χ	0 and 0.3	0.3 and 0.6	0.6 and 0.9
$\overline{e} = 7$	0.0239	0.0002	0.0043
$\overline{e} = 15$	0.0002	0.0001	0.0006
$\overline{e} = 25$	0.0001	0.0001	0.0006

The table shows the effect of increasing social influence on the long-run efforts in the baseline model ($\kappa = 0$). We compare each pair of long-run effort distributions for scenarios with trust parameter values χ equal to 0 and 0.3, 0.3 and 0.6, 0.6 and 0,9 when the highest possible effort is 7, 15, and 25.

Speed of Updating

Increasing the speed for updating old beliefs to new information affects the long-run effort choice for any level of trust in the network. The negative cross-effect of faster updating is significant (Table 1.15). Test results in Table 1.14 show that the positive effect from increasing social influence remains significant when $\xi = 0.1$, and $\xi = 0.3$. While for faster belief updating ($\xi = 0.5$, $\xi = 0.6$) increase in trust parameter from absence of social influence $\chi = 0$ to $\chi = 0.3$ becomes insignificant.

Table 1.13: Cross-effect of changing the highest possible effort.

	7 and 15	15 and 25
$\chi = 0$	0.0826	0.2111
$\chi = 0.3$	0.0068	0.0034
$\chi = 0.6$	0.0002	0.0007
$\chi = 0.9$	0.0002	0.0005

The table shows the "cross-effect" of increasing the highest possible effort $\overline{e}$ for different levels of social influence parameter χ.

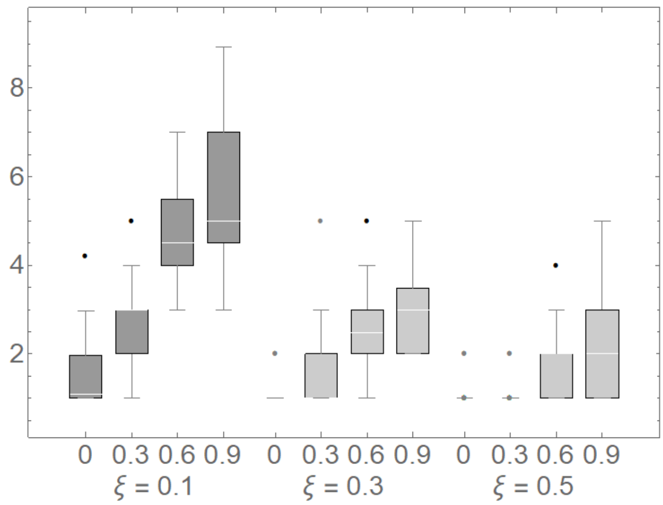


Figure 1.17: Distribution of average efforts at $t = 40$ for $\chi = 0, 0.3, 0.6, 0.9$ with $\xi = 0.1$, $\xi = 0.3$ and $\xi = 0.5$.

Table 1.14: Effect of increasing social influence on long-run efforts with different values of speed of updating ξ.

χ	0 and 0.3	0.3 and 0.6	0.6 and 0.9
$\xi = 0.1$	0.0002	0.0001	0.0006
$\xi = 0.3$	0.0066	0.0002	0.0076
$\xi = 0.5$	1	0.0029	0.0355
$\xi = 0.6$	1	0.012	0.0033

The table shows the effect of increasing social influence on the long-run efforts in the baseline model ($\kappa = 0$). We compare each pair of long-run effort distributions for scenarios with trust parameter values χ equal to 0 and 0.3, 0.3 and 0.6, 0.6 and 0.9 when the speed of belief updating ξ is 0.1, 0.3, 0.5, 0.6.

Table 1.15: Cross-effect of increasing speed of belief updating.

	0.1 and 0.3	0.3 and 0.5
$\chi = 0$	0.0458	0.0144
$\chi = 0.3$	0.0004	0.0048
$\chi = 0.6$	0.0001	0.0012
$\chi = 0.9$	0.0001	0.0004

The table shows the "cross-effect" of increasing the speed of belief updating ξ for different levels of social influence parameter χ.

Chapter 2

Dynamics of Ideal Efforts and Consensus in a Multi-Layer Network Game

2.1 Introduction

Individual choices are often affected by the social environment.[23] The behaviors and the attitudes of social contacts affect their well-being, driving the choices and leading them to adapt, or even adopt, opinions. Such social construct of opinion formation can often lead to conformity.[24] Yet, individual beliefs on many issues vary significantly in society. The willingness and the incentives to change those beliefs together with behaviors differ across people, too.[25] Despite the importance of a consensus in problems where coordination matters, the presence of diverse opinions about the issue is an important part in the evolution of the beliefs, thus the dynamics of individual choices.[26]

Recent research has shown that studying the social and economic relations in the context of network interactions can provide valuable insights about group outcomes and individual choices.[27] The literature on network games provides a game-theoretic

[23] Behavioral spillovers and peer effects through social interaction are present in education, labor markets, environmental choices, crime, etc. For surveys on theory and empirical evidence see Durlauf (2004), Ioannides and Datcher Loury (2004), Ioannides and Ioannides (2012).

[24] See the pioneering work by Asch (1955) on opinions and social pressure.

[25] Bednar and Page (2007) define *behavioral stickiness* as one of the elements of cultural behavior. They show that the individuals may not alter their behavior despite changes in incentives.

[26] "*When consensus comes under the dominance of conformity, the social process is polluted and the individual at the same time surrenders the powers on which his functioning as a feeling and thinking being depends.*" Asch (1955).

[27] There is a wide range of applications of network theory in social and economic problems, such as in

framework for analyzing the behavior of players in a variety of economic settings, with players embedded in a network. The structure of the network, the players' characteristics, along with the type of ties connecting the players and economic incentives, affect the individual outcomes. The topology of the network impacts the efforts dynamics and learning in the population (see e.g Golub and Jackson (2010b) or Acemoglu and Ozdaglar (2011a)). While the games on networks commonly study players embedded in a network structure representing one type of relationships,[28] oftentimes people are engaged in multiple types of relationships that influence individual choices regarding a single issue.[29] Within the same population, the people are connected in a friendship network, a network of professional connections, cultural ties, religious or ethnic groups, and even online communities. The presence of links in each dimension of such a multi-layer network of social interactions may affect the individual differently through spillovers, social norms, peer pressure, etc.

The objective of this paper is to study the evolution of personal norms and efforts in a population of players embedded in a two-layer network. The two layers of the network influence differently the individual's choice of action regarding a single activity. The presence of social interactions in the first layer carries a pressure to conform with the social norm within the layer. One may consider relations within a religious and ethnic communities, family ties, where social norms can be strongly imposed. The second layer represents a network where the interaction with neighbors generates strategic complementaries of efforts, as present in friendship networks.[30] It is natural to assume that same social contacts can generates strategic complementaries of efforts and create pressure to conform with the norms at the same time, thus be represented in both layers of the network. Yet, depending on the nature of relationships, the magnitude of the received complementarities and the level of peer pressure can be different. Additionally, players have personal norms related to the effort choice, or *ideal efforts*,[31] and they are

labor markets (e.g. Calvó-Armengol (2004), Calvo-Armengol and Jackson (2004)), financial markets (e.g. Gale and Kariv (2007), Elliott, Golub, and Jackson (2014)), R&D collaborations (e.g. Goyal and Moraga-Gonzalez (2001), König, Battiston, Napoletano, and Schweitzer (2012), Dawid and Hellmann (2014)), as well as peer effects in networks (see the recent survey by Bramoullé, Djebbari, and Fortin (2020)), opinion diffusion (e.g. Golub and Jackson (2010b), Jackson and Yariv (2011)), adoption of environmentally friendly behavior (Currarini, Marchiori, and Tavoni (2016)) and technology adoption (e.g. Beaman, BenYishay, Magruder, and Mobarak (2021)).

[28]There are extensive surveys on network games literature, e.g. Jackson and Zenou (2015) and Bramoullé and Kranton (2015).

[29]The papers by Belhaj and Deroïan (2014) and Chen, Zenou, and Zhou (2018) capture multiple activities on a single-layer network. Walsh (2019) studies a public good game on a two-layer network where the players choose to invest in one of the layers.

[30]See e.g. Calvó-Armengol, Patacchini, and Zenou (2005) and Calvó-Armengol, Patacchini, and Zenou (2009).

[31]The notion of ideal efforts in network games is used in recent works by Olcina, Panebianco, and

heterogeneous in their ideal efforts and productivity. They repeatedly participate in a network game where the deviation from the social norm creates conflict cost from the first network layer. At the same time, the deviation from the ideal effort causes disutility from the cognitive dissonance or inner conflict.[32] The players are myopic, so they do not consider how their effort choice affects those of other players in the network. We find the equilibrium efforts of the players, which are equivalent to their weighted Katz-Bonacich centrality,[33,34] that is, the network centrality measure indicating the weighted number of paths in the network stemming from a given player, additionally weighted by idiosyncratic productivities and ideal efforts of each player on the paths.

We assume that the norms, both social and personal, tend to change. The players update their ideal efforts by adapting those to their actual efforts in the network game. The ideal effort is updated as a linear combination of the ideal in the previous period and the actual effort. This form of updating implies that the players adjust their ideals to justify their actions and to manage their cognitive dissonance.[35] Such linear updating method is most similar to the well-known DeGroot updating introduced by DeGroot (1974) and widely used in opinion formation and social learning models. Compared to DeGroot updating where the beliefs are revised according to the social norm, in our model the ideal is updated according to the actual effort that is, in turn, a linear combination of social norm, the current ideal effort of the player, her productivity, and effort complementarities from the additional network layer. A similar approach to DeGroot updating has been adopted by DeMarzo et al. (2003) who define the notion of persuasion bias as a failure to adjust for repetition of information, whether it is coming from one source throughout the time or multiple sources that are connected in the social network. Such bounded rationality of players is assumed in most non-Bayesian

Zenou (2017) and Galeotti, Golub, Goyal, and Rao (2021).

[32]The *Theory of Cognitive Dissonance* introduced by social psychologists dates back to Festinger (1957). It suggests that people strive for internal consistency continuously aligning the cognitions with their actions in order to minimize the cognitive dissonance. Experimental work by Elliot and Devine (1994) proves this result. Akerlof and Dickens (1982) and Rabin (1994) incorporate the psychological theory of cognitive dissonance into theoretical economic models of rational choice. They show that such approach provides a better explanation of the welfare consequences of some economic phenomena.

[33]This network centrality measure, introduced first by Katz (1953) and reestablished later by Bonacich (1987), captures the power or the status as a relative degree of influence of the player in the network.

[34]Following the seminal work by Ballester et al. (2006), the link between the Katz-Bonacich centrality of a player and her effort is an established result in the literature on network games. Introducing heterogeneity in players' productivities results in equilibrium efforts being equivalent to the weighted Katz-Bonacich centrality measure. E.g. see Remark 1 in Ballester et al. (2006).

[35]An early experimental study by Brehm (1956) shows that people attempt to increase the desirability of their made choices to stay consistent with their decisions. In psychology literature such ex-post rationalization of decisions and choice-supportive bias is present in experimental results (see e.g. Mather and Johnson (2000), Mather, Shafir, and Johnson (2000)).

learning models. We undertake the same approach in the current paper.

Similarly to subjective beliefs, we assume that ideal efforts are a subjective measure idiosyncratic to the players. We study the dynamics of ideal efforts given the multi-layer network structure, the taste for conformity, sensitivity to cognitive dissonance in the network, and players' productivity coefficients. We show that under certain conditions on the network layer of strategic complementarities, the steady-state exists. Moreover, in presence of regular equivalence of players in the given two-layer network, that is, similarity in their relational patterns and neighborhoods, the ideals of players converge to a consensus. Referring to ideal efforts as to an individual subjective matter, we refrain from studying the correctness of it. Thus, the consensus over the ideals in this setting represents the possibility of coordination in the network, rather than the "correctness" of learning and convergence to true action. A number of works have studied the ability of correct aggregation of information and conditions to converge to the truth. Golub and Jackson (2010b) characterize the population and the learning process as wise if, given the existence of consensus, the influence of the most influential player vanishes as the network grows. Similarly to DeMarzo et al. (2003), they show that the social influence of an individual depends on her connectedness and the position in the network. Bala and Goyal (1998) study a non-Bayesian model of social learning in the context of technology adoption. Similarly to Golub and Jackson (2010b), they show that the structure of the network has important implications in the adoption of new technologies, and their diffusion in society. Acemoglu and Ozdaglar (2011a) provide an overview of works on opinion dynamics and social learning.

Finally, this paper is closely related to the work by Olcina et al. (2017) on assimilation dynamics in the population. They study a network game where the players make decisions on the effort to assimilate to the majority norm. This decision is based on the personal norm of the individual and the behavior of her peers. Further, they study the economic incentives for assimilation. Similarly, we study a network game where the economic incentives, as well as the ideal efforts and the social norms, affect the effort choice of the players. The introduction of multidimensionality in the network is the main contribution of the current work. We show that the layer of network complementarities affects the structural similarities of players in the network providing conditions for the consensus between players.

The rest of the paper is organized as follows. We introduce the network game and find the equilibrium efforts in Section 2.2. Convergence to the steady-state and the consensus are characterized in Section 2.3. The dynamics of the ideal efforts are studied on numerical examples in the same section. We conclude in Section 2.4. The proofs not given in the main text are provided in the Appendix 2.A.

2.2 Model

2.2.1 The Game

We have a set of players N with $|N| = n$, embedded in a multi-layer network of social influences. The network consists of two layers, where one layer represents the network $\boldsymbol{g}$ of social connections, and the other is the network of complementarities $\boldsymbol{l}$. The network $\boldsymbol{g}$ is a directed weighted network described by the weighted adjacency matrix $\mathbf{G}^*$ with elements g^*_{ij}. The weight g^*_{ij} on the link $i \to j$ is normalized by the degree of player i on $\boldsymbol{g}$, such that $\sum\limits_{j \in N} g^*_{ij} = 1$ for all $i \in N$. The network $\boldsymbol{l}$ is represented by the matrix $\mathbf{\Lambda}$ with elements $\lambda_{ij} > 0$ as coefficients of effort complementarity from players j given the existence of a directed link $i \to j$ in the network $\boldsymbol{l}$ for all $i, j \in N$. By convention, we assume there are no self-loops in the network, i.e. $g^*_{ii} = 0$ and $\lambda_{ii} = 0$ for all $i \in N$.

Players are initially endowed with *personal ideals*, or *personal norms*, y_i^0. Each period t they choose their efforts x_i^t based on the network game on the layers $\boldsymbol{g}$ and $\boldsymbol{l}$. The choice of effort for each player i is based on her idiosyncratic productivity θ_i, complementarities from the network $\boldsymbol{l}$, the cost from miscoordination in the network $\boldsymbol{g}$, as well as the cost from inconsistency with her *ideal behavior* y_i^t. After each period network game, players update their ideal efforts, adapting those to their actual efforts with a speed of updating $\xi \in [0; 1]$.

The timing of the model is the following for each period t.

1. *Network game and effort selection.* Players $i \in N$ choose the best response efforts x_i^t in the two-layer network game by maximizing the following utility:

$$u_i^t(x) = \theta_i x_i^t + x_i^t \sum_{j \in N} \lambda_{ij} x_j^t - \underbrace{\frac{\omega_1}{2} \sum_{j \in N} g^*_{ij}(x_i^t - x_j^t)^2}_{\text{Conformism or conflict}} - \frac{\omega_2}{2}(x_i^t)^2 - \underbrace{\frac{\omega_3}{2}(x_i^t - y_i^t)^2}_{\substack{\text{Consistency or} \\ \text{cognitive dissonance}}}, \quad (2.1)$$

$$x_i^t = BR_i(\mathbf{x}^t_{-i}; \mathbf{G}^*, \mathbf{\Lambda}, y_i^t, \theta_i, \omega_1, \omega_2, \omega_3),$$

where $\mathbf{x}^t_{-i}$ is the list of efforts of players $N \setminus i$.

2. *Updating ideal efforts.* Players update their ideal efforts based on the actual effort x_i^t and ideal effort y_i^t of the current period:

$$y_i^{t+1} = \xi x_i^t + (1 - \xi) y_i^t. \quad (2.2)$$

The coefficients ω_1, ω_2 and ω_3 in the utility of player i in (2.1) are the cost coefficients of the model. The difference in efforts of neighbors in the network $\boldsymbol{g}$ is amplified by the coefficient ω_1, that is, the cost of conflict. The cost of conflict may also be regarded as a taste for conformity. The cost of unit effort is given by the coefficient $\frac{\omega_2}{2}$. In addition, ω_3 is the cost of moral conflict or sensitivity to cognitive dissonance. When the cost of conflict ω_1 is high, the disutility from miscoordination in the network $\boldsymbol{g}$ is high, forcing the neighbors to coordinate around social norm faster, resulting in conformity. If the sensitivity to cognitive dissonance ω_3 is high, the players remain consistent with their ideal efforts longer.[36]

2.2.2 Equilibrium

Solving for the first-order conditions on the utility (2.1) of player i, we find her best response effort in period t:

$$x_i^t = \frac{1}{\omega_1 + \omega_2 + \omega_3} \left(\theta_i + \omega_3 y_i^t + \sum_{j \in N} \lambda_{ij} x_j^t + \omega_1 \sum_{j \in N} g_{ij}^* x_j^t \right).$$

One can see that the best response of the player is the linear combination of her productivity θ_i, the ideal effort y_i^t, complementarities $\sum_{j \in N} \lambda_{ij} x_j^t$ from the network $\boldsymbol{l}$, and the social norm $\sum_{j \in N} g_{ij}^* x_j^t$ in the network $\boldsymbol{g}$.

Let $\mathbf{x}^t$, $\mathbf{y}^t$ be the vectors of efforts and ideal efforts in period t respectively, and let $\boldsymbol{\theta}$ denote the vector of idiosyncratic elements θ_i, for all $i \in N$. Then, the best response in matrix form will be given by:

$$\mathbf{x}^t = \frac{1}{\omega_1 + \omega_2 + \omega_3} \left(\boldsymbol{\theta} + \omega_3 \mathbf{y}^t + \boldsymbol{\Lambda} \mathbf{x}^t + \omega_1 \mathbf{G}^* \mathbf{x}^t \right). \tag{2.3}$$

We characterize the conditions for the existence of unique Nash equilibrium in the following proposition.

Proposition 2.1. *Assuming the spectral radius of $\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*$ is smaller than $(\omega_1 + \omega_2 + \omega_3)$, the unique Nash equilibrium in pure strategies is then given by:*[37]

[36] The cost of private dissonance indicates the level of flexibility of the players. This leaves room for extending the model by introducing heterogeneity in players' types. Particularly, dividing the players into stubborn (or persistent) and flexible agents, with the first group having higher coefficient ω_3 compered to the flexible agents, and thus *sticky* ideals.

[37] The spectral radius of a matrix is its largest absolute eigenvalue. Debreu and Herstein (1953) show that for a square matrix A, $(sI - A)^{-1}$ is well-defined and non-negative if and only if the spectral radius of A is smaller than s. This result was used and the link between the equilibrium efforts of the network game and the Katz-Bonacich centralities had been first established in the seminal work by Ballester et al. (2006).

$$\mathbf{x}^t = \frac{1}{\omega_1+\omega_2+\omega_3}\left(\mathbf{I} - \frac{1}{\omega_1+\omega_2+\omega_3}(\mathbf{\Lambda} + \omega_1\mathbf{G}^*)\right)^{-1}(\boldsymbol{\theta} + \omega_3\mathbf{y}^t).$$

One can see that the unique Nash equilibrium of the game is proportional to the weighted Katz-Bonacich centrality of the players on a network represented by the matrix $(\mathbf{\Lambda} + \omega_1\mathbf{G}^*)$ and weighted by the linear combination of productivites and ideal efforts at time $(\boldsymbol{\theta} + \omega_3\mathbf{y}^t)$.[38] The matrix $(\mathbf{\Lambda} + \omega_1\mathbf{G}^*)$ represents a network formed by projecting the links of the layer $\boldsymbol{l}$ on $\boldsymbol{g}$, where the adjacency matrix of the layer $\boldsymbol{g}$ is weighted by the coefficient of conflict cost.

To be able to characterize the dynamics of ideal efforts by the multi-layer network representation, analogously to Olcina et al. (2017), we define the following vectors of length $2n$.

$$\hat{\mathbf{x}}^t := \left[\begin{array}{c}\boldsymbol{\theta} \\ \hline \mathbf{x}^t\end{array}\right] \qquad \hat{\mathbf{y}}^t := \left[\begin{array}{c}\boldsymbol{\theta} \\ \hline \mathbf{y}^t\end{array}\right]$$

The vectors $\hat{\mathbf{x}}^t$ and $\hat{\mathbf{y}}^t$ are constructed using the vector $\boldsymbol{\theta}$ as first n elements, and vectors $\mathbf{x}^t$ and $\mathbf{y}^t$ as the second half of the respective vector. Additionally, we define the following matrix.

$$\bar{\mathbf{G}} := \frac{1}{\omega_1+\omega_2}\left[\begin{array}{c|c}(\omega_1+\omega_2)\mathbf{I} & \mathbf{O} \\ \hline \mathbf{I} & (\mathbf{\Lambda} + \omega_1\mathbf{G}^*)\end{array}\right] \tag{2.4}$$

Using the definitions above and the equation (2.3) we can write the augmented vector of efforts $\hat{\mathbf{x}}^t$ as the weighted average of $\hat{\mathbf{y}}^t$ and the social norm in the network described by the augmented adjacency matrix $\bar{\mathbf{G}}$.

$$\hat{\mathbf{x}}^t = \frac{\omega_3}{\omega_1+\omega_2+\omega_3}\hat{\mathbf{y}}^t + \frac{\omega_1+\omega_2}{\omega_1+\omega_2+\omega_3}\bar{\mathbf{G}}\hat{\mathbf{x}}^t$$

One can notice, that the first n elements of the vector $\hat{\mathbf{x}}^t$, the vector $\boldsymbol{\theta}$, does not change over time and does not carry any economic meaning. While the elements from $n+1$ to $2n$ correspond to (2.3). This allows us to rewrite the equilibrium efforts and characterize new conditions for equilibrium existence.

Proposition 2.2. *Let δ_1 be the spectral radius of the matrix $\bar{\mathbf{G}}$. If $\delta_1 < \dfrac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$,*

[38] E.g. see Remark 1 in Ballester et al. (2006).

then the unique Nash equilibrium in pure strategies is given by:

$$\hat{\mathbf{x}}^t = \frac{\omega_3}{\omega_1+\omega_2+\omega_3}\left(\mathbf{I} - \frac{\omega_1+\omega_2}{\omega_1+\omega_2+\omega_3}\bar{\mathbf{G}}\right)^{-1}\hat{\mathbf{y}}^t.$$

One can see that 1 is always an eigenvalue of $\bar{\mathbf{G}}$ by construction, whereas for the parameter values satisfying $\omega_2 \geq 1 + \max_i \sum_{j\in N} \lambda_{ij}$ it is also the spectral radius.

Lemma 2.1 provides conditions on parameter values and combinations that ensure the equilibrium existence in Proposition 2.2.

Let's introduce the following notations:

$$\overline{\omega}_2 \equiv \max_i \sum_{j\in N} \lambda_{ij},$$
$$\overline{\overline{\omega}}_2 \equiv \max_j \sum_{i\in N\backslash j} (\lambda_{ij} + \omega_1 g^*_{ij}) - \omega_1 - 1.$$

Lemma 2.1. *Let δ_1 be the spectral radius of the matrix $\bar{\mathbf{G}}$. The condition $\delta_1 < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ is satisfied if one of the conditions below holds:*

(a) Given $\overline{\omega}_2 > \overline{\overline{\omega}}_2$, $\omega_2 < \overline{\overline{\omega}}_2$ and $\omega_3 > \overline{\overline{\omega}}_2 + 1 - \omega_2$;

(b) Given $\overline{\omega}_2 > \overline{\overline{\omega}}_2$, $\omega_2 \in [\overline{\overline{\omega}}_2; \overline{\omega}_2)$ and $\omega_3 > 1$;

(c) Given $\overline{\omega}_2 \leq \overline{\overline{\omega}}_2$, $\omega_2 < \overline{\overline{\omega}}_2$ and $\omega_3 > 1 + \overline{\omega}_2 - \omega_2$;

(d) $\omega_2 \in [\overline{\omega}_2; 1 + \overline{\omega}_2)$ and $\omega_3 > 1 + \overline{\omega}_2 - \omega_2$;

(e) $\omega_2 \geq 1 + \overline{\omega}_2$ and $\omega_3 > 0$.

Figure 2.1 illustrates the conditions provided in Lemma 2.1. The combination of parameter values of ω_2 and ω_3 in the shaded area above the black line in Figure 2.1 ensures the existence of equilibrium given that $\max_i \sum_{j\in N} \lambda_{ij} > \max_j \sum_{i\in N\backslash j} (\lambda_{ij} + \omega_1 g_{ij}) - \omega_1 - 1$. The shaded area above the green line in the figure displays the satisfying parameter combinations, when the opposite is true.

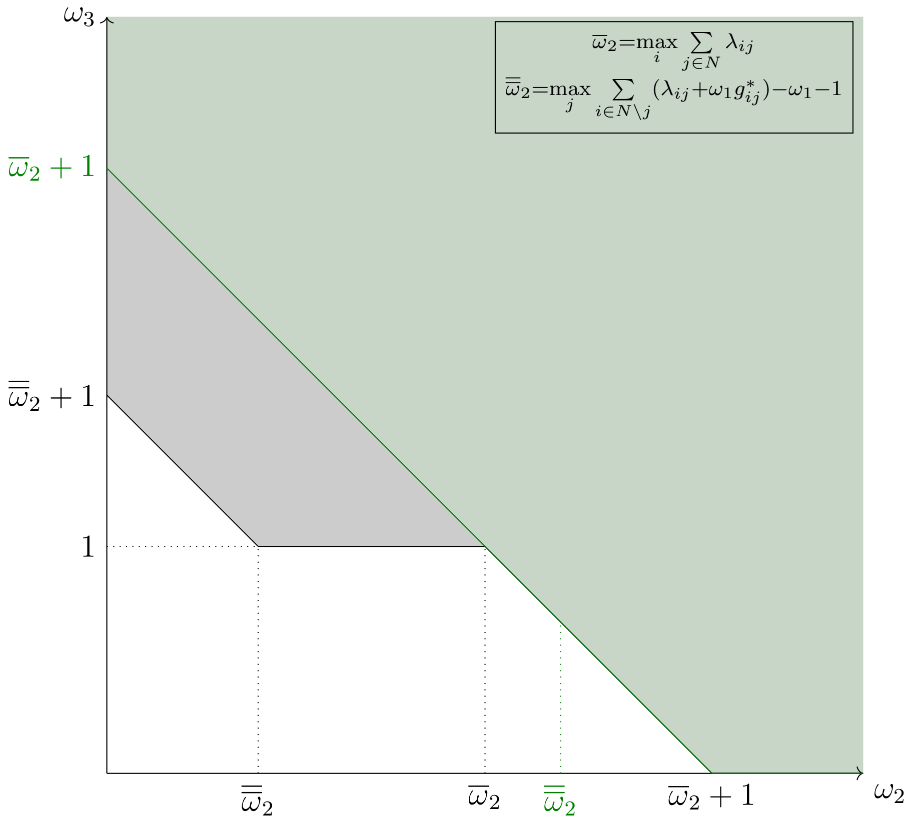


Figure 2.1: The shaded area covers the combination of parameter values according to the conditions on $\bar{\mathbf{G}}$ provided by Lemma 2.1.

When the coefficients of cost and cognitive dissonance are too low relative to complementarities in the network, the positive feedback loops in players' efforts escalate infinitely, so the Nash equilibrium does not exist. Thus, the cost coefficients have to be high enough to curb the complementarities in the network measured by the spectral radius of the matrix $\bar{\mathbf{G}}$. The combinations of parameters in Figure 2.1 ensure this requirement and the existence of Nash equilibria.

2.3 Dynamics and Convergence of Ideal Efforts

2.3.1 Updating Ideal Efforts

Let $\phi = \dfrac{\omega_1+\omega_2}{\omega_1+\omega_2+\omega_3}$. We can then rewrite the augmented vector of equilibrium efforts in Proposition 2.2 as follows:

$$\hat{\mathbf{x}}^t = (1-\phi)\left(\mathbf{I}-\phi\bar{\mathbf{G}}\right)^{-1}\hat{\mathbf{y}}^t. \tag{2.5}$$

Similarly to $\hat{\mathbf{x}}^t$, in the stage of updating the ideal efforts, we can apply the updating rule on the augmented vector $\hat{\mathbf{y}}^t$. The first n elements of $\hat{\mathbf{y}}^t$ remain unchanged and equal to $\boldsymbol{\theta}$ over time, while the elements $n+1$ to n correspond to ideal efforts $\mathbf{y}^t$ in time period t.

$$\hat{\mathbf{y}}^{t+1} = \xi\hat{\mathbf{x}}^t + (1-\xi)\hat{\mathbf{y}}^t \tag{2.6}$$

Plugging $\hat{\mathbf{x}}^t$ in (2.5) into the equation (2.6) we have:

$$\hat{\mathbf{y}}^{t+1} =[\xi(1-\phi)\left(\mathbf{I}-\phi\bar{\mathbf{G}}\right)^{-1} + (1-\xi)\mathbf{I}]\hat{\mathbf{y}}^t.$$

The transition to a new ideal effort depends on player's current ideal effort and the equilibrium effort. The latter is determined by her location in the network, that is, her weighted Katz-Bonacich centrality.

2.3.2 Convergence and Consensus

Let us define the following matrices:

$$\begin{aligned}\mathbf{M} &:= \left(\mathbf{I}-\phi\bar{\mathbf{G}}\right)^{-1}\\ \mathbf{T} &:= \xi(1-\phi)\mathbf{M} + (1-\xi)\mathbf{I}\end{aligned} \tag{2.7}$$

We can rewrite the ideal efforts using the notations in (2.7).

$$\hat{\mathbf{y}}^{t+1} = \mathbf{T}\hat{\mathbf{y}}^t = \mathbf{T}^{t+1}\hat{\mathbf{y}}^0 \tag{2.8}$$

Given the initial ideal efforts $\hat{\mathbf{y}}^0$ and the matrix $\mathbf{T}$, we can find the ideal efforts at any time period t, as well as the limit beliefs $\hat{\mathbf{y}}^\infty$. Proposition 2.4 provides the conditions for existence of the limit ideal efforts. Moreover, it shows that in order to find the ideal and equilibrium efforts in the limit it is sufficient to use the augmented matrix of weights $\bar{\mathbf{G}}$ instead of $\mathbf{T}$. To do so, we first define the relationship between the corresponding eigenvalues of the two matrices.

Proposition 2.3. *Consider the distinct and ordered sets of eigenvalues δ_i of $\bar{\mathbf{G}}$, and τ_i of $\mathbf{T}$. Then for all $i \in [1,n]$:*

$$\tau_i = \xi\frac{1-\phi}{1-\phi\delta_i} + (1-\xi).$$

One can notice that if $\sum\limits_{j\in N}\lambda_{ij} = \omega_2 - 1$ for all $i \in N$, then the matrix of weights

$\bar{\mathbf{G}}$, and thus, the matrix $(1-\phi)\mathbf{M}$ are row-normalized matrices.[39] As a result, the matrix $\mathbf{T}$ is row-normalized as a convex combination of a row-stochastic matrix and an identity. With this setting, the dynamics described in (2.8) is a time-homogeneous Markov process with transition matrix $\mathbf{T}$.

Proposition 2.4. *For a given network with layers $\boldsymbol{g}$ and $\boldsymbol{l}$ such that* $\max_i \sum_{j\in N} \lambda_{ij} \leq \omega_2 - 1$, *and $\bar{\mathbf{G}}$ is diagonalizable, there exist* $\bar{\mathbf{G}}^\infty = \lim_{t\to\infty} \bar{\mathbf{G}}^t$ *and* $\mathbf{T}^\infty = \lim_{t\to\infty} \mathbf{T}^t$ *such that:*

$$\hat{\mathbf{y}}^\infty = \mathbf{T}^\infty \hat{\mathbf{y}}^0 = \bar{\mathbf{G}}^\infty \hat{\mathbf{y}}^0$$

and

$$\hat{\mathbf{x}}^\infty = (1-\phi)\left(\mathbf{I} - \phi\bar{\mathbf{G}}\right)^{-1} \bar{\mathbf{G}}^\infty \hat{\mathbf{y}}^0.$$

A special case fulfilling the conditions Proposition 2.4 are the networks with undirected layers $\boldsymbol{g}$ and $\boldsymbol{l}$ described by symmetric matrices $\mathbf{G}^*$ and Λ. This makes the augmented matrix $\bar{\mathbf{G}}$ symmetric and thus diagonalizable. Therefore, for symmetric matrices $\mathbf{G}^*$ and Λ, the existence of limit ideal efforts and actions depends on the maximum row-sum norm of the matrix of weights Λ.

It is easy to show that matrices $\bar{\mathbf{G}}$ and $\mathbf{T}$ are commuting matrices, that is $\bar{\mathbf{G}}\mathbf{T} = \mathbf{T}\bar{\mathbf{G}}$.[40] It is known that commuting matrices share the same set of eigenvectors. Using this fact, together with the assumption on diagonalizability of $\bar{\mathbf{G}}$, we can rewrite the two matrices as follows.

$$\bar{\mathbf{G}} = \mathbf{E}\boldsymbol{\Delta}\mathbf{E}^{-1} \qquad \text{and} \qquad \mathbf{T} = \mathbf{E}\boldsymbol{\Sigma}\mathbf{E}^{-1},$$

where $\mathbf{E}$ is the matrix of eigenvectors of $\bar{\mathbf{G}}$ and $\mathbf{T}$ as columns, $\boldsymbol{\Delta}$ is the diagonal matrix of eigenvalues of $\bar{\mathbf{G}}$, and $\boldsymbol{\Sigma}$ is the diagonal matrix of eigenvalues of $\mathbf{T}$. Thus, there exists a transformation $P(.)$ such that $P(\boldsymbol{\Delta}) = \boldsymbol{\Sigma}$ [41] given which, we can find $\hat{\mathbf{y}}^\infty$:

$$\boldsymbol{\Sigma}^\infty = \lim_{t\to\infty} \boldsymbol{\Sigma}^t = \lim_{t\to\infty} P(\boldsymbol{\Delta})^t$$

$$\hat{\mathbf{y}}^\infty = \mathbf{E}\boldsymbol{\Sigma}^\infty\mathbf{E}^{-1}\hat{\mathbf{y}}^0.$$

[39] For a row-normalized $\bar{\mathbf{G}}$ and $\phi < 1$, the matrix $\mathbf{M} = \sum_{k=0}^{\infty} \phi^k \bar{\mathbf{G}}^k$ is converging Neumann series with sum of elements in each row equal to $\frac{1}{1-\phi}$. Therefore, the matrix $(1-\phi)\mathbf{M}$ is row-normalized.

[40] See the proof in Step 1 of the proof of Proposition 2.4 in the Appendix.

[41] E.g. see Zhang (2011), Chapter 3, Theorem 3.1.

Given the convergence of $\bar{\mathbf{G}}$ in Proposition 2.4 we can find the steady-state equilibrium efforts of the players.

$$\mathbf{x}^{\infty} = \mathbf{y}^{\infty} = \frac{1}{\omega_1 + \omega_2}\left(\mathbf{I} - \frac{1}{\omega_1 + \omega_2}(\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*)\right)^{-1} \boldsymbol{\theta} \tag{2.9}$$

Indeed, when $\hat{\mathbf{y}}^{t+1} = \hat{\mathbf{y}}^{t} = \hat{\mathbf{y}}^{\infty}$, the equilibrium efforts are identical to the ideals, $\hat{\mathbf{y}}^{\infty} = \hat{\mathbf{x}}^{\infty}$. From equilibrium efforts in (2.5) in the steady state we have

$$\hat{\mathbf{x}}^{\infty} = (1-\phi)\left(\mathbf{I} - \phi\bar{\mathbf{G}}\right)^{-1}\hat{\mathbf{x}}^{\infty} \quad \Longrightarrow \quad \hat{\mathbf{x}}^{\infty} = \bar{\mathbf{G}}\hat{\mathbf{x}}^{\infty}.$$

Recalling the definition of $\bar{\mathbf{G}}$ in (2.4) we find the steady-state equilibrium efforts of players.

$$\mathbf{x}^{\infty} = \frac{1}{\omega_1 + \omega_2}\left(\boldsymbol{\theta} + (\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*)\,\mathbf{x}^{\infty}\right)$$

Assume there exists a row vector $\mathbf{e}^{\intercal}$, such that $y_i^{\infty} = \mathbf{e}^{\intercal}\mathbf{y}^{\infty}$ for all players $i \in N$. Denote $y^{\infty} := \mathbf{e}^{\intercal}\mathbf{y}^{\infty}$. This implies, that in the steady state the ideal efforts of the players converge to a consensus y^{∞}, in which case we have:

$$\mathbf{y}^{\infty} = \frac{1}{\omega_1 + \omega_2}\left(\boldsymbol{\theta} + (\boldsymbol{\Lambda} + \omega_1)\,\mathbf{y}^{\infty}\right),$$

$$y_i^{\infty} = \frac{\theta_i}{\omega_2 - \sum\limits_{j \in N} \lambda_{ij}} \quad \text{s.t.} \quad y_i^{\infty} = y^{\infty} \quad \text{for all} \quad i \in N. \tag{2.10}$$

A consensus over ideal efforts occurs among *structurally equivalent players* (Lorrain and White, 1971), that is, the players that share the same neighborhood and similar relationship patterns in the network, and have the same productivity. Such strict form of players' equivalence ensures the consensus. A more general definition of similarity of players in the network is the *regular equivalence* (Sailer, 1978; White and Reitz, 1983).

Definition 2.1. *Let $C \subset N$ be a subset of players in the network. Given the layers $\boldsymbol{g}$ and $\boldsymbol{l}$, and productivity coefficients $\boldsymbol{\theta}$, we say that the players in C are regularly equivalent if their neighborhoods are equivalent to each other, and for all $i \in C$ there exists a consensus such that $y_i^{\infty} = y_C^{\infty}$.*

Unlike structurally equivalent players, regularly equivalent players do not necessarily share the same neighbors in the network, but they have similar connection patterns and are connected to similar neighbors. The neighbors, in turn, are also regularly equivalent. The consensus ideal effort of regularly equivalent players is given by:

$$y_C^{\infty} = \frac{\theta_i + \sum\limits_{j \in N \setminus C} (\lambda_{ij} + \omega_1 g^*_{ij}) y_j^{\infty}}{\omega_2 - \sum\limits_{j \in C} \lambda_{ij} + \omega_1 (1 - \sum\limits_{j \in C} g^*_{ij})} \tag{2.11}$$

Proof. Using the definition of ideal efforts in the steady state in (2.9) and assuming there exists consensus among structually equivalent players inn C, such that $y_i^{\infty} = y_C^{\infty} =$ for all $i \in C$, we can find the consensus ideal for players in C.

$$\left(\mathbf{I} - \frac{1}{\omega_1 + \omega_2}(\mathbf{\Lambda} + \omega_1 \mathbf{G}^*)\right) \mathbf{y}^{\infty} = \frac{1}{\omega_1 + \omega_2} \boldsymbol{\theta}$$

For player $i \in C$ with $y_i^{\infty} = y_C^{\infty}$ we have:

$$y_C^{\infty} - \frac{1}{\omega_1 + \omega_2} \left(\sum_{j \in N \setminus C} \lambda_{ij} y_j^{\infty} + \underbrace{\sum_{j \in C} \lambda_{ij} y_j^{\infty}}_{y_C^{\infty} \sum_{j \in C} \lambda_{ij}} + \omega_1 \sum_{j \in N \setminus C} g^*_{ij} y_j^{\infty} + \omega_1 \underbrace{\sum_{j \in C} g^*_{ij} y_j^{\infty}}_{y_C^{\infty} \sum_{j \in C} g^*_{ij}} \right) = \frac{1}{\omega_1 + \omega_2} \theta_i$$

$$y_C^{\infty} - \frac{1}{\omega_1 + \omega_2} \left(y_C^{\infty} (\sum_{j \in C} \lambda_{ij} + \omega_1 \sum_{j \in C} g^*_{ij}) + \sum_{j \in N \setminus C} (\lambda_{ij} + \omega_1 g^*_{ij}) y_j^{\infty} \right) = \frac{1}{\omega_1 + \omega_2} \theta_i$$

$$y_C^{\infty}(\omega_1 + \omega_2) - y_C^{\infty} (\sum_{j \in C} \lambda_{ij} + \omega_1 \sum_{j \in C} g^*_{ij}) - \sum_{j \in N \setminus C} (\lambda_{ij} + \omega_1 g^*_{ij}) y_j^{\infty} = \theta_i$$

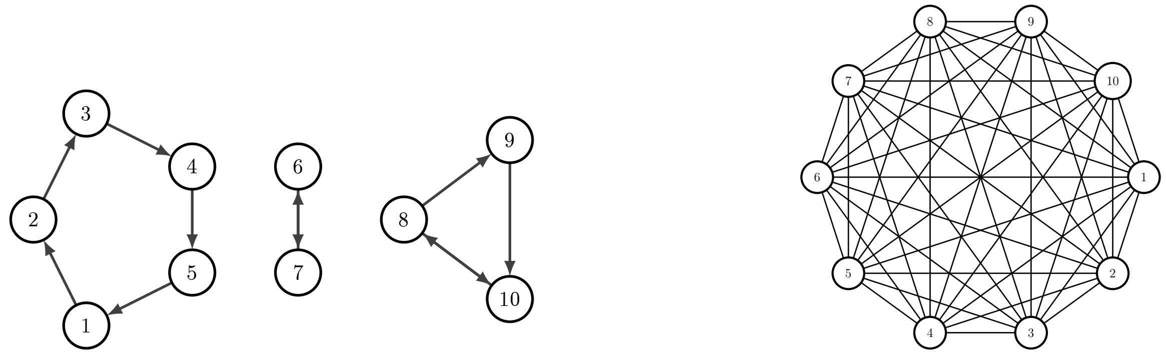
$$y_C^{\infty} \left(\omega_2 - \sum_{j \in C} \lambda_{ij} + \omega_1 (1 - \sum_{j \in C} g^*_{ij}) \right) = \sum_{j \in N \setminus C} (\lambda_{ij} + \omega_1 g^*_{ij}) y_j^{\infty} + \theta_i$$

Thus, the consensus effort of regularly equivalent group of players is given by (2.11). □

The regular equivalence is associated with similarities in networks characteristics of the players, such as the neighborhood and the weights on all layers of the network, the idiosyncratic productivity parameters of the neighbors, as well as the productivity of the player herself.

2.3.3 Numerical Examples

Consider a network of $n = 10$ players connected through a social network $\boldsymbol{g}$ (Figure 2.2) with normalized equal weights on the directed links, and a network of additional



(a) (b)

Figure 2.2: Social network $\boldsymbol{g}$

complementarities $\boldsymbol{l}$ (Figure 2.3). In each period the players engage in the two-layer network game in (2.1) where the productivity of effort is $\theta_i = 1$ for all players, and the effort cost parameter is $\omega_2 = 2$. In the initial period 0, the players are endowed with ideal efforts $y_i = 0.2i$:

$$\mathbf{y}^{0\intercal} = [0.2\ 0.4\ 0.6\ 0.8\ 1\ 1.2\ 1.4\ 1.6\ 1.8\ 2].$$

After each period game the players update their ideal efforts according to equation (2.2) with the speed $\xi = 0.2$. We consider an arbitrary set of combinations for parameters of conflict cost and sensitivity to cognitive dissonance with $\omega_1, \omega_3 \in \{2, 8, 15\}$.

Consider the network of social interactions in Figure 2.2a. In the absence of complementarity layer $\boldsymbol{l}$, the ideal efforts in all three components of the network $\boldsymbol{g}$ converge to the consensus $y^\infty = 0.5$ by expression (2.10). The layer of complementarities in the network affects the steady-state level of efforts and ideals, as well as the existence of the consensus. Let us now add the layer $\boldsymbol{l}$ in Figure 2.3a described by the following adjacency matrix.

$$\boldsymbol{\Lambda}_{(a)} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

For this and further examples, we distinguish the following sets of players that form disconnected components in both networks in Figures 2.2a and 2.3a:

$$C_1 = \{1, 2, 3, 4, 5\}, \quad C_2 = \{6, 7\}, \quad C_3 = \{8, 9, 10\}.$$

Moreover, we assume that the players in each set C_i are homogeneous in their aggregate

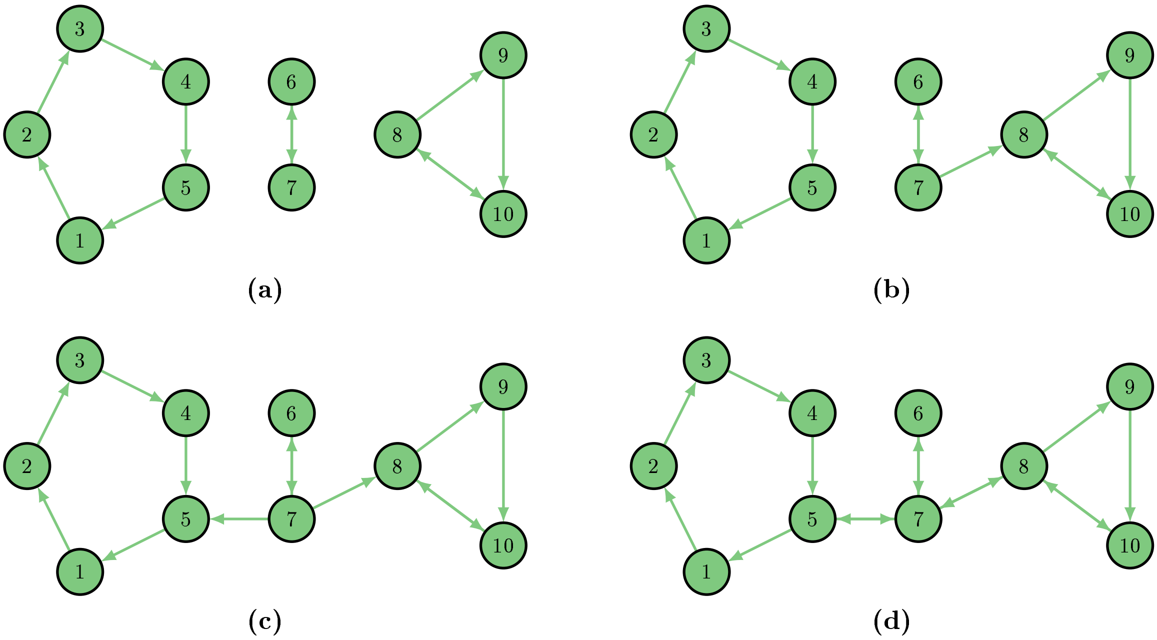


Figure 2.3: Network of complementarities $\boldsymbol{l}$

complementarity coefficients such that

$$\sum_{j\in N}\lambda_{ij} = 0.5\ \forall i \in C_1, \quad \sum_{j\in N}\lambda_{ij} = 0.7\ \forall i \in C_2 \quad \text{and} \quad \sum_{j\in N}\lambda_{ij} = 1\ \forall i \in C_3. \tag{2.12}$$

It is easy to see that the players within each component $C_{i,i\in\{1,2\}}$ are regularly equivalent. The players in C_3 are regularly equivalent since player 8 receives the same aggregate complementarity from the regularly equivalent players 9 and 10. Moreover, the components are disconnected in both layers of the network, thus the steady-state efforts and ideals can be found using the equation (2.10). The following vector shows the ideal efforts of players in the steady state:

$$\mathbf{y}^{\infty\intercal} = [0.667\ 0.667\ 0.667\ 0.667\ 0.667\ 0.769\ 0.769\ 1\ 1\ 1].$$

Unsurprisingly, we can see that the complementarities of efforts from $\boldsymbol{l}$ increase the ideal efforts of players in the steady state. Figure 2.4 shows the dynamics of the ideal efforts of the players. One can observe that with a higher cost of conflict ω_1 the ideal efforts within each component C_i converge faster. While the increase in the cost of cognitive dissonance ω_3 delays the convergence to the steady-state.

Assume now a complete network of social interactions $\boldsymbol{g}$ in Figure 2.2b. The regular equivalence of players within each set C_i remains. Thus, the consensus over ideal efforts among players in C_i satisfies the expression (2.11). E.g. for any player $i \in C_3$, given

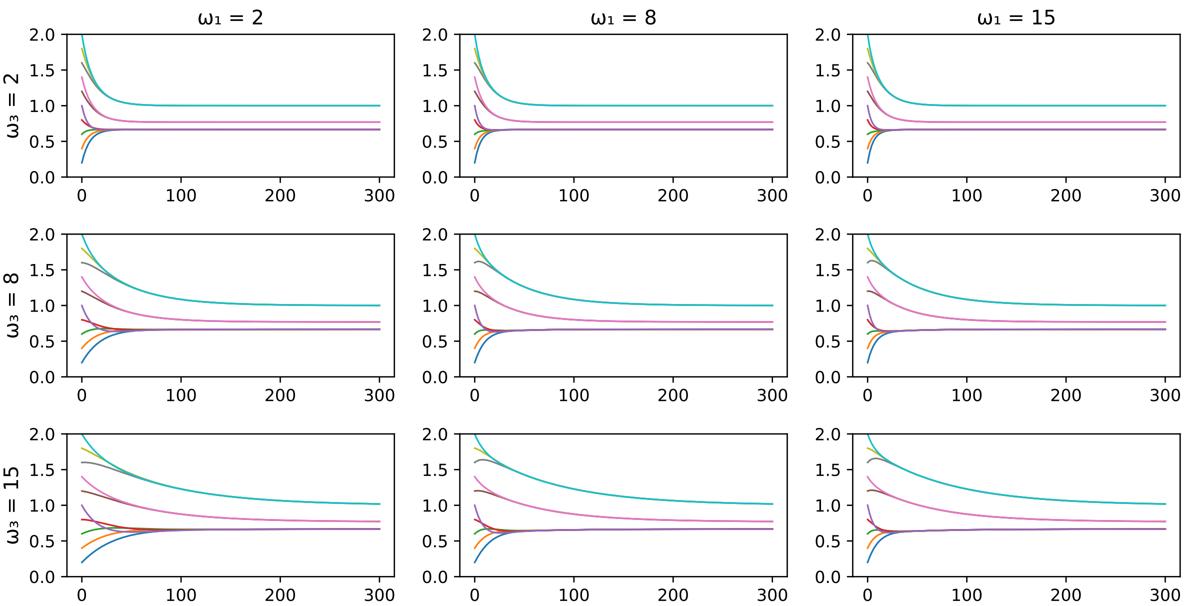


Figure 2.4: Dynamics of ideal efforts in the network of complementarities in Figure 2.3a and the network of social interactions in Figure 2.2a

$\omega_1 = 15$, we find:

$$y_{C_3}^{\infty} = \frac{\theta_i + \omega_1(\frac{5}{9}y_{C_1}^{\infty} + \frac{2}{9}y_{C_2}^{\infty})}{\omega_2 - \sum\limits_{j \in C_3} \lambda_{ij} + \omega_1 \frac{7}{9}} = 0.778.$$

The ideal efforts of players in the steady states, given the cost of conflict $\omega_1 = 15$, are the following:

$$\mathbf{y}^{\infty\intercal} = [0.756\ 0.756\ 0.756\ 0.756\ 0.756\ 0.765\ 0.765\ 0.778\ 0.778\ 0.778].$$

The dynamics of ideal effort with the complete network $\boldsymbol{g}$ are displayed in Figure 2.5. Comparing the two examples above and the dynamics of ideal efforts in Figures 2.4 and 2.5, we can see that the variance between the steady-state ideals of the components is lower in the network with densely connected layer $\boldsymbol{g}$,

Let us further consider only the social network $\boldsymbol{g}$ depicted in Figure 2.2a for the rest of the examples below. Further, let us consider the layer of effort complementarities $\boldsymbol{l}$ in Figure 2.3b. We assume that the aggregate of the complementarity coefficients of players do not change by adding a new link. Thus, by adding a connection in the network layer $\boldsymbol{l}$, we redistribute the complementarity coefficients of outgoing links maintaining the characteristics described in (2.12). By doing so, we can analyze the effect of the change in the structure of the layer $\boldsymbol{l}$ without increasing the complemen-

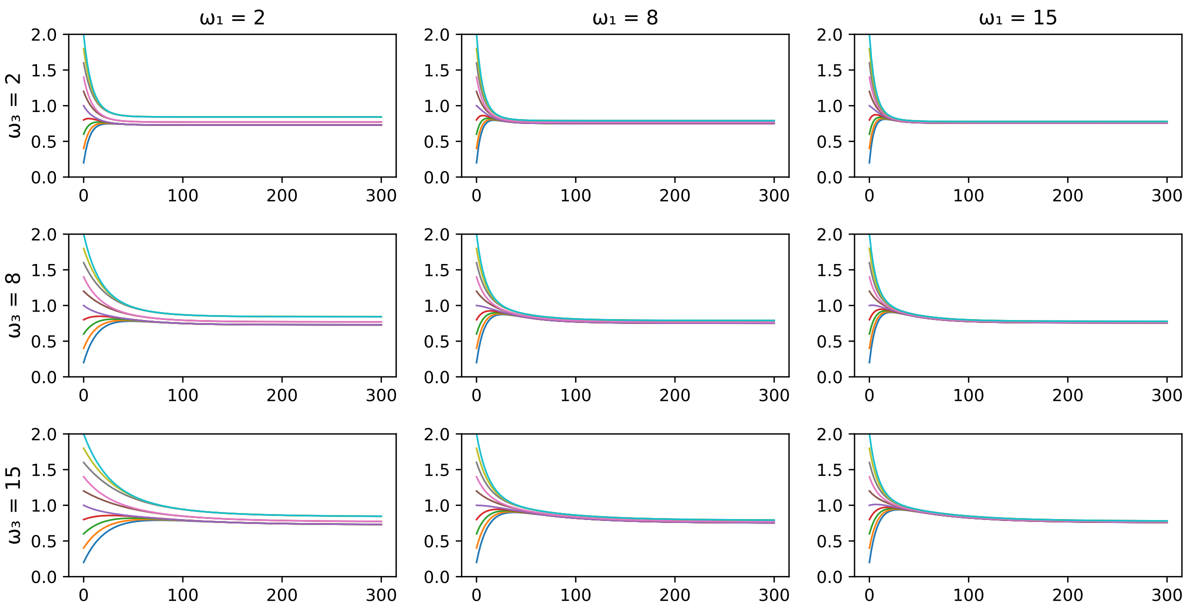


Figure 2.5: Dynamics of ideal efforts in the network of complementarities in Figure 2.3a and the complete network of social interactions in Figure 2.2b.

tarity levels in the network. Assume, the layer $\boldsymbol{l}$ is described by the following matrix of complementarity coefficients.

$$\boldsymbol{\Lambda}_{(b)} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The additional link $7 \to 8$ in $\boldsymbol{l}$ makes the player 7 influenced by the component C_3 driving the ideal efforts of players 6 and 7 closer to $y^{\infty}_{C_3}$. Unlike the case with $\boldsymbol{l}$ in Figure 2.3a, players 6 and 7 are not equivalent anymore, so the consensus over ideal effort in C_2 is not achieved.

Further, we add a complementarity link connecting player 7 to player 5 as in Figure 2.3c, the component C_2 is then additionally influenced by the efforts in the component C_1. Assume the link $7 \to 5$ with $\lambda_{75} = 0.2$ in the matrix of weights $\boldsymbol{\Lambda}_{(c)}$ describing the layer in Figure 2.3c.

$$\mathbf{\Lambda}_{(c)} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & {\color{red}0.2} & 0.3 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

As shown in Table 2.1, the steady-state consensus effort in the component C_1 is 0.667 and is lower than that of player 6 and component C_3 in the previous example. Receiving complementarity from the lower-effort component C_1, the ideal efforts of player 7 decreases. At the same time, the pressure to conform in layer $\boldsymbol{g}$ forces player 6 to adjust her effort according to the new social norm.

Consider now the layer $\boldsymbol{l}$ in Figure 2.3d where players 5 and 8 reciprocate player 7 with complementarity links. The matrix of weights is the following.

$$\mathbf{\Lambda}_{(d)} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0 & 0 & {\color{red}0.2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.3 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0.2} & 0 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The new links make the network connected. There is no regular equivalence between players anymore, which results in all players having different steady-state efforts. The ideal efforts of the players in C_1 are positively affected by the efforts in the rest of the network. This positive effect is weaker for players that are further from player 5.

The steady-state ideal efforts in previous examples are summarized in Table 2.1.

Table 2.1: Ideal efforts of players in the steady-state.

		y_1^∞	y_2^∞	y_3^∞	y_4^∞	y_5^∞	y_6^∞	y_7^∞	y_8^∞	y_9^∞	y_{10}^∞
	$\mathbf{\Lambda}_{(a)}$	0.667	0.667	0.667	0.667	0.667	0.769	0.769	1	1	1
$\omega_1 = 2$	$\mathbf{\Lambda}_{(b)}$	0.667	0.667	0.667	0.667	0.667	0.783	0.79	1	1	1
	$\mathbf{\Lambda}_{(c)}$	0.667	0.667	0.667	0.667	0.667	0.777	0.78	1	1	1
	$\mathbf{\Lambda}_{(d)}$	0.668	0.669	0.67	0.671	0.673	0.776	0.779	0.98	0.989	0.985
$\omega_1 = 8$	$\mathbf{\Lambda}_{(b)}$	0.667	0.667	0.667	0.667	0.667	0.785	0.787	1	1	1
	$\mathbf{\Lambda}_{(c)}$	0.667	0.667	0.667	0.667	0.667	0.778	0.779	1	1	1
	$\mathbf{\Lambda}_{(d)}$	0.6687	0.669	0.6694	0.667	0.671	0.777	0.778	0.983	0.986	0.984
$\omega_1 = 15$	$\mathbf{\Lambda}_{(b)}$	0.667	0.667	0.667	0.667	0.667	0.786	0.787	1	1	1
	$\mathbf{\Lambda}_{(c)}$	0.667	0.667	0.667	0.667	0.667	0.778	0.779	1	1	1
	$\mathbf{\Lambda}_{(d)}$	0.669	0.6692	0.6695	0.6698	0.67	0.777	0.778	0.983	0.985	0.984

One can also observe that while the sensitivity to cognitive dissonance ω_3 does not affect the steady-state levels of ideals efforts, increasing the said parameter slows down the emergence of the consensus among players and the steady state. The higher conflict cost ω_1, on the other hand, accelerates the agreement between players connected in the layer of social influences $\boldsymbol{g}$.

2.4 Conclusions

In this paper, we have proposed a network game model with multidimensionality of network relations, where people repeatedly choose their efforts and update their ideal beliefs based on their current choices. Specifically, we study a model with a two-layer network where the first layer dictates the social norm, and the second is a network of interactions with strategic complementarities in efforts. The players in the network are initially endowed with heterogeneous ideal efforts and differ in their productivity. In line with the network game literature, we find that the pure Nash equilibrium of the game is proportional to Katz-Bonacich centrality in the combined network of the two layers. Additionally, we find the steady-state equilibrium efforts and ideals. We derive the conditions for the existence of consensus about ideal efforts in the whole population and among structurally and regularly equivalent players.

The dynamics of ideal efforts in our model show that the sensitivity to cognitive dissonance and the taste for conformity have opposing effects on the speed of converges to a consensus and the steady state. The sensitivity to cognitive dissonance captures the behavioral stickiness of the players affecting the speed of convergence to a steady-state, but does not change the long-run efforts. The taste for conformity, on the other hand, affects the steady-state levels of efforts. Consideration of possible heterogeneity in the cost of the cognitive dissonance, as well as the taste for conformity, is left for future research.

We show that the multidimensionality of network interactions may change the overall effort levels and the beliefs in society. While focusing on the network layer with strategic complementarities we leave the consideration of other types of relations for future research. In particular, the direct extension of the model provided in this paper is the consideration of strategic substitutability along with complementarities on the same network layer.

Appendix 2.A Proofs

Proof of Proposition 2.1. Given x_i^* is the optimal effort choice for i

$$\frac{\partial u_i^t}{\partial x_i^t}(x_i^{t*}) \equiv 0$$

$$\begin{aligned}
\frac{\partial u_i^t(x^t)}{\partial x_i^t} &= \theta_i + \sum_{j\in N\setminus i} \lambda_{ij} x_j^t - \omega_1 \sum_{j\in N} g_{ij}^*(x_i^t - x_j^t) - \omega_2 x_i^t - \omega_3(x_i^t - y_i^t) = \\
&= \theta_i + \sum_{j\in N\setminus i} \lambda_{ij} x_j^t - \omega_1 x_i^t \underbrace{\sum_{j\in N} g_{ij}^*}_{=1} + \omega_1 \sum_{j\in N} g_{ij}^* x_j^t - \omega_2 x_i^t - \omega_3 x_i^t + \omega_3 y_i^t = \\
&= (\theta_i + \omega_3 y_i^t) + \sum_{j\in N\setminus i} \lambda_{ij} x_j^t + \omega_1 \sum_{j\in N} g_{ij}^* x_j^t - (\omega_1 + \omega_2 + \omega_3) x_i^t \\
&= (\theta_i + \omega_3 y_i^t) + \sum_{j\in N\setminus i} (\lambda_{ij} + \omega_1 g_{ij}^*) x_j^t - (\omega_1 + \omega_2 + \omega_3) x_i^t
\end{aligned}$$

Thus the equilibrium effort choice of i is:

$$x_i^{t*} = \frac{\theta_i + \omega_3 y_i^t}{(\omega_1 + \omega_2 + \omega_3)} + \sum_{j\in N\setminus i} \frac{\lambda_{ij} + \omega_1 g_{ij}^*}{(\omega_1 + \omega_2 + \omega_3)} x_j^t,$$

for all $i \in N$, or:

$$x_i^{t*} = \frac{\theta_i + \omega_3 y_i^t}{(\omega_1 + \omega_2 + \omega_3)} + \frac{1}{(\omega_1 + \omega_2 + \omega_3)} \sum_{j\in N\setminus i} \lambda_{ij} x_j^t + \frac{\omega_1}{(\omega_1 + \omega_2 + \omega_3)} \sum_{j\in N} g_{ij}^* x_j^t,$$

We can rewrite it in a matrix form:

$$\mathbf{x}^t = \frac{1}{(\omega_1 + \omega_2 + \omega_3)} \boldsymbol{\theta} + \frac{\omega_3}{(\omega_1 + \omega_2 + \omega_3)} \mathbf{y}^t + \frac{1}{(\omega_1 + \omega_2 + \omega_3)} \boldsymbol{\Lambda} \mathbf{x}^t + \frac{\omega_1}{(\omega_1 + \omega_2 + \omega_3)} \mathbf{G}^* \mathbf{x}^t,$$

So the, matrix form solution of the equilibrium efforts is the following:

$$\left(\mathbf{I} - \frac{1}{(\omega_1 + \omega_2 + \omega_3)} (\boldsymbol{\Lambda} + \omega_1 \mathbf{G}^*)\right) \mathbf{x}^t = \frac{1}{(\omega_1 + \omega_2 + \omega_3)} (\boldsymbol{\theta} + \omega_3 \mathbf{y}^t)$$

Ans thus the vector of equilibrium efforts is:

$$\mathbf{x}^t = \frac{1}{(\omega_1+\omega_2+\omega_3)}\left(\mathbf{I} - \frac{1}{(\omega_1+\omega_2+\omega_3)}(\mathbf{\Lambda}+\omega_1\mathbf{G}^*)\right)^{-1}(\boldsymbol{\theta}+\omega_3\mathbf{y}^t)$$

□

Proof of Lemma 2.1. In order to ensure the condition of Proposition 2.2 on the spectral radius δ_1 of the matrix $\bar{\mathbf{G}}$ in (2.4), we can use maximum row-sum and column-sum norms of the matrix, $|||\bar{\mathbf{G}}|||_\infty$ and $|||\bar{\mathbf{G}}|||_1$, to find an upper bound for the spectral radius[42].

One can see that the row-sum for each of the first n rows of the matrix $\bar{\mathbf{G}}$ is equal to 1. While for the consecutive n rows, the sum of row elements of $(n+i)$th row is equal to $\frac{1+\omega_1+\sum_{j\in N}\lambda_{ij}}{\omega_1+\omega_2}$, for each $i \in N$. As a result, the maximum row-sum norm of the matrix is the maximum of given values:

$$|||\bar{\mathbf{G}}|||_\infty = \max\left\{1, \frac{1}{\omega_1+\omega_2}(1+\omega_1+\max_i \sum_{j\in N}\lambda_{ij}\right\}$$

Moreover, Gershgorin circle theorem allows us to see from the first n rows of the matrix $\bar{\mathbf{G}}$ that 1 is one of its' eigenvalues.

Similarly, the sum for each of the first n column elements of the matrix $\bar{\mathbf{G}}$ is $\frac{1+\omega_1+\omega_2}{\omega_1+\omega_2}$. And the sum of column elements for columns $n+j$ for each $j \in N$ is $\frac{\sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij})}{\omega_1+\omega_2}$. Thus the maximum column-sum norm of the matrix is the following:

$$|||\bar{\mathbf{G}}|||_1 = \max\left\{1+\frac{1}{\omega_1+\omega_2}, \frac{1}{\omega_1+\omega_2}\max_j \sum_{i\in N\setminus j}(\lambda_{ij}+\omega_1 g_{ij})\right\}$$

Now let's find the better proxy for the upper bound $UB(\delta_1)$ of the spectral radius by finding $\min\{|||\bar{\mathbf{G}}|||_\infty, |||\bar{\mathbf{G}}|||_1\}$ for all parameter values that satisfy $\omega_2 \geq \max_i \sum_{j\in N}\lambda_{ij}$. And $|||\bar{\mathbf{G}}|||_1$ can refine the upper bound for δ_1 when $\omega_2 < \max_i \sum_{j\in N}\lambda_{ij}$.

Case 1: Assume $|||\bar{\mathbf{G}}|||_\infty = 1$. Thus $1 \geq \frac{1+\omega_1+\max_i \sum_{j\in N}\lambda_{ij}}{\omega_1+\omega_2}$, which follows by $\omega_2 \geq 1 + \max_i \sum_{j\in N}\lambda_{ij}$.

(i) If $|||\bar{\mathbf{G}}|||_1 = 1 + \frac{1}{\omega_1+\omega_2} > 1$.

[42] This follows common linear algebra results, e.g one can see the Theorem 5.6.9 in Horn and Johnson (2012).

(ii) If $|||\bar{\mathbf{G}}|||_1 = \frac{\max_j \sum_{i \in N \setminus j} (\lambda_{ij} + \omega_1 g_{ij})}{\omega_1 + \omega_2} \geq 1 + \frac{1}{\omega_1 + \omega_2} > 1$.

Case 2: Assume $|||\bar{\mathbf{G}}|||_\infty = \frac{1 + \omega_1 + \max_i \sum_{j \in N} \lambda_{ij}}{\omega_1 + \omega_2}$. This means that $\frac{1 + \omega_1 + \max_i \sum_{j \in N} \lambda_{ij}}{\omega_1 + \omega_2} > 1$. Then we have $\omega_2 < 1 + \max_i \sum_{j \in N} \lambda_{ij}$.

(i) If $|||\bar{\mathbf{G}}|||_1 = 1 + \frac{1}{\omega_1 + \omega_2}$. Then $\omega_2 \geq \max_j \sum_{i \in N \setminus j} (\lambda_{ij} + \omega_1 g_{ij}) - \omega_1 - 1$

And $1 + \frac{1}{\omega_1 + \omega_2} \geq \frac{1 + \omega_1 + \max_i \sum_{j \in N} \lambda_{ij}}{\omega_1 + \omega_2}$ only when $\omega_2 \geq \max_i \sum_{j \in N} \lambda_{ij}$.

(ii) If $|||\bar{\mathbf{G}}|||_1 = \frac{\max_j \sum_{i \in N \setminus j} (\lambda_{ij} + \omega_1 g_{ij})}{\omega_1 + \omega_2} \geq 1 + \frac{1}{\omega_1 + \omega_2}$. Then $\omega_2 < \max_j \sum_{i \in N \setminus j} (\lambda_{ij} + \omega_1 g_{ij}) - \omega_1 - 1$.

Thus $|||\bar{\mathbf{G}}|||_\infty$ is the upper bound for the spectral radius if $\frac{\max_j \sum_{i \in N \setminus j} (\lambda_{ij} + \omega_1 g_{ij})}{\omega_1 + \omega_2} \geq \frac{1 + \omega_1 + \max_i \sum_{j \in N} \lambda_{ij}}{\omega_1 + \omega_2}$.

Let's introduce the following notations:

$$\overline{\omega}_2 \equiv \max_i \sum_{j \in N} \lambda_{ij}$$
$$\overline{\overline{\omega}}_2 \equiv \max_j \sum_{i \in N \setminus j} (\lambda_{ij} + \omega_1 g^*_{ij}) - \omega_1 - 1$$

To sum up

$$\begin{array}{llllll}
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_\infty & = 1 & \text{when} & \omega_2 \geq \overline{\omega}_2 + 1. \\
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_\infty & = \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2} & \text{when} & \omega_2 \in [\overline{\omega}_2; \overline{\omega}_2 + 1). \\
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_1 & = 1 + \frac{1}{\omega_1+\omega_2} & \text{when} & \omega_2 \in [\overline{\overline{\omega}}_2; \overline{\omega}_2) \\
 & & & \text{and} & \overline{\overline{\omega}}_2 < \overline{\omega}_2. \\
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_1 & = \frac{\overline{\overline{\omega}}_2+\omega_1+1}{\omega_1+\omega_2} & \text{when} & \omega_2 < \overline{\overline{\omega}}_2 \\
 & & & \text{and} & \overline{\overline{\omega}}_2 < \overline{\omega}_2. \\
UB(\delta_1) & = |||\bar{\mathbf{G}}|||_\infty & = \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2} & \text{when} & \omega_2 < \overline{\overline{\omega}}_2 \\
 & & & \text{and} & \overline{\overline{\omega}}_2 \geq \overline{\omega}_2.
\end{array}$$

We can now say that for $\delta_1 < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ it is sufficient to have $UB(\delta_1) < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$. Thus,

- When $UB(\delta_1) = |||\bar{\mathbf{G}}|||_\infty = 1$, then $1 < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ when $\omega_3 > 0$.
- When $UB(\delta_1) = |||\bar{\mathbf{G}}|||_\infty = \frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2}$, then $\frac{1+\omega_1+\overline{\omega}_2}{\omega_1+\omega_2} < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ when $\omega_3 > 1 + \overline{\omega}_2 - \omega_2$.

- When $UB(\delta_1) = |||\bar{\mathbf{G}}|||_1 = 1 + \frac{1}{\omega_1+\omega_2}$, then $1 + \frac{1}{\omega_1+\omega_2} < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ when $\omega_3 > 1$.
- When $UB(\delta_1) = |||\bar{\mathbf{G}}|||_1 = \frac{\overline{\overline{\omega}}_2+\omega_1+1}{\omega_1+\omega_2}$, then $\frac{\overline{\overline{\omega}}_2+\omega_1+1}{\omega_1+\omega_2} < \frac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ when $\omega_3 > \overline{\overline{\omega}}_2 + 1 - \omega_2$.

These results are illustrated in Figure 2.1. □

Proof of Proposition 2.3. Notice the following:

$$\bar{\mathbf{G}} = \mathbf{E}\boldsymbol{\Delta}\mathbf{E}^{-1} \implies \boldsymbol{\Delta} = \mathbf{E}^{-1}\bar{\mathbf{G}}\mathbf{E}$$
$$\mathbf{T} = \mathbf{E}\boldsymbol{\Sigma}\mathbf{E}^{-1} \implies \boldsymbol{\Sigma} = \mathbf{E}^{-1}\mathbf{T}\mathbf{E}$$

multiplying $\mathbf{T}$ in (2.7) by $\mathbf{E}$ from the left and by $\mathbf{E}^{-1}$ from the right we get the following:

$$\begin{aligned}
\mathbf{E}^{-1}\mathbf{T}\mathbf{E} &= \mathbf{E}^{-1}\left(\xi(1-\phi)\mathbf{M} + (1-\xi)\mathbf{I}\right)\mathbf{E} \\
\boldsymbol{\Sigma} &= \xi(1-\phi)\mathbf{E}^{-1}\mathbf{M}\mathbf{E} + (1-\xi)\mathbf{I} \\
&= \xi(1-\phi)\mathbf{E}^{-1}\left(\mathbf{I}-\phi\bar{\mathbf{G}}\right)^{-1}\mathbf{E} + (1-\xi)\mathbf{I} \\
&= \xi(1-\phi)\left(\mathbf{E}^{-1}(\mathbf{I}-\phi\bar{\mathbf{G}})\mathbf{E}\right)^{-1} + (1-\xi)\mathbf{I} \\
&= \xi(1-\phi)\left(\mathbf{I}-\phi\mathbf{E}^{-1}\bar{\mathbf{G}}\mathbf{E}\right)^{-1} + (1-\xi)\mathbf{I} \\
&= \xi(1-\phi)\left(\mathbf{I}-\phi\boldsymbol{\Delta}\right)^{-1} + (1-\xi)\mathbf{I}
\end{aligned}$$

So we have:

$$\boldsymbol{\Sigma} = \xi(1-\phi)\left(\mathbf{I}-\phi\boldsymbol{\Delta}\right)^{-1} + (1-\xi)\mathbf{I}$$

We can find the vector of eigenvalues $\boldsymbol{\tau}$ of $\mathbf{T}$ by multiplying the above expression by a vector of ones, $\mathbf{1}$, such that $\boldsymbol{\tau} = P(\boldsymbol{\delta})$.

Notice that:

$$\left(\mathbf{I}-\phi\boldsymbol{\Delta}\right)^{-1} = \sum_{t=0}^{\infty}\phi^t\boldsymbol{\Delta}^t$$

$$\begin{aligned}
\boldsymbol{\Sigma}\mathbf{1} &= \xi(1-\phi)\left(\mathbf{I}-\phi\boldsymbol{\Delta}\right)^{-1}\mathbf{1} + (1-\xi)\mathbf{I}\mathbf{1} \\
\boldsymbol{\tau} &= \xi(1-\phi)\sum_{t=0}^{\infty}\phi^t\boldsymbol{\Delta}^t\mathbf{1} + (1-\xi)\mathbf{1}
\end{aligned}$$

And for each distinct eigenvalue we have:

$$\begin{aligned}\tau_i &= \xi(1-\phi)\sum_{t=0}^{\infty}\phi^t\delta_i^t + (1-\xi)\\ &= \xi\frac{1-\phi}{1-\phi\delta_i} + (1-\xi)\end{aligned}$$

Recall that $\phi = \dfrac{\omega_1+\omega_2}{\omega_1+\omega_2+\omega_3}$ and from the condition in Proposition 2.2 we have that $\delta_1 < \dfrac{\omega_1+\omega_2+\omega_3}{\omega_1+\omega_2}$ which implies that $\phi\delta_i < 1$. One can notice that when $\delta_i < 1$ then $\tau_i < 1$, as a convex combination of 1 and a value below 1, for all $i \in [1, n]$. Additionally, this is always true when $\omega_3 = 0$.

So when $\mathbf{\Delta}^\infty$ converge $\mathbf{\Sigma}^\infty$ converge too.

□

Proof of Proposition 2.4. Similarly to Proposition 1 in Olcina et al. (2017) we prove the proposition by following steps.

Step 1. $\bar{\mathbf{G}}$ and $\mathbf{T}$ are commuting matrices: $\bar{\mathbf{G}}\mathbf{T} = \mathbf{T}\bar{\mathbf{G}}$.
Recalling the notations in (2.7) we have that

$$\bar{\mathbf{G}}\mathbf{T} = \xi(1-\phi)\bar{\mathbf{G}}\mathbf{M} + (1-\xi)\bar{\mathbf{G}}.$$

To show that $\bar{\mathbf{G}}$ and $\mathbf{M}$ are commuting matrices we first show that $\bar{\mathbf{G}}$ and $\mathbf{M}^{-1}$ commute.

$$\bar{\mathbf{G}}\mathbf{M}^{-1} = \bar{\mathbf{G}}\left(\mathbf{I}-\phi\bar{\mathbf{G}}\right) = \left(\mathbf{I}-\phi\bar{\mathbf{G}}\right)\bar{\mathbf{G}} = \mathbf{M}^{-1}\bar{\mathbf{G}}.$$

Multiplying the above equality by $\mathbf{M}$ from left and right we have that $\bar{\mathbf{G}}\mathbf{M} = \mathbf{M}\bar{\mathbf{G}}$. From which it directly follows that $\bar{\mathbf{G}}\mathbf{T} = \mathbf{T}\bar{\mathbf{G}}$.

Step 2. From diagonalizability of $\bar{\mathbf{G}}$ we can infer that $\mathbf{T}$ is also diagonalizable given that $\mathbf{M}$ is a polynomial of $\bar{\mathbf{G}}$, and $\mathbf{T}$ is a linear convex combination of $\mathbf{M}$ and $\mathbf{I}$.
As commuting matrices, $\bar{\mathbf{G}}$ and $\mathbf{T}$ share the same set of eigenvectors, thus also the eigenvector associated with the eigenvalue equal to 1. We can rewrite the two matrices as

$$\bar{\mathbf{G}} = \mathbf{E}\mathbf{\Delta}\mathbf{E}^{-1} \qquad \text{and} \qquad \mathbf{T} = \mathbf{E}\mathbf{\Sigma}\mathbf{E}^{-1},$$

where $\mathbf{E}$ is the matrix of eigenvectors of $\bar{\mathbf{G}}$ and $\mathbf{T}$ as columns. $\mathbf{\Delta}$ is the diagonal matrix of eigenvalues of $\bar{\mathbf{G}}$, and $\boldsymbol{\Sigma}$ is the diagonal matrix of eigenvalues of $\mathbf{T}$.[43] The eigenvalues in $\boldsymbol{\Delta}$ and $\boldsymbol{\Sigma}$ are ordered in the decreasing manner, with the order of eigenvectors in $\mathbf{E}$ corresponding to associated eigenvalues. From the above-mentioned it directly follows that:

$$\bar{\mathbf{G}}^t = \mathbf{E}\boldsymbol{\Delta}^t\mathbf{E}^{-1} \qquad \text{thus} \qquad \bar{\mathbf{G}}^\infty = \mathbf{E}\boldsymbol{\Delta}^\infty\mathbf{E}^{-1} \qquad \text{with} \qquad \boldsymbol{\Delta}^\infty = \lim_{t\to\infty} \boldsymbol{\Delta}^t$$

and

$$\mathbf{T}^t = \mathbf{E}\boldsymbol{\Sigma}^t\mathbf{E}^{-1} \qquad \text{thus} \qquad \mathbf{T}^\infty = \mathbf{E}\boldsymbol{\Sigma}^\infty\mathbf{E}^{-1} \qquad \text{with} \qquad \boldsymbol{\Sigma}^\infty = \lim_{t\to\infty} \boldsymbol{\Sigma}^t$$

The condition $\max_i \sum_{j\in N} \lambda_{ij} \leq \omega_2 - 1$ ensures that $\delta_1 = 1$.[44] Which, as it is easy to see from Proposition 2.3, implies that $\tau_1 = 1$. While for any other eigenvalue $\delta_i < 1$ of $\bar{\mathbf{G}}$, we have that $\tau_i < 1$. This follows that

$$\boldsymbol{\Delta}^\infty = \boldsymbol{\Sigma}^\infty \qquad \text{and} \qquad \bar{\mathbf{G}}^\infty = \mathbf{T}^\infty.$$

Using the results above in (2.8) we find the $\hat{\mathbf{y}}^\infty$. And plugging it into the equilibrium efforts in equation (2.5) we prove the second part of the proposition. □

[43] Notice that given the diagonalizability of $\bar{\mathbf{G}}$, it has $2n$ eigenvectors.
[44] Recall that, by construction of the matrix in (2.4), 1 is always an eigenvalue of $\bar{\mathbf{G}}$.

Chapter 3

Unions in Network Games With Conflicts and Spillovers

3.1 Introduction

Cooperation and collective acts are often key in achieving a common goal or improving welfare in many social and economic contexts. Collaborative behavior is known to be present in human populations from ancient times.[45] Shared intentions are a basis for formation of economic unions, climate treaties, military alliances, cooperatives, labor unions, etc. The interactions between people involving the exchange of ideas, opinions, or beliefs, affect their behaviors that influence the well-being of the society. Whether as individuals, households, firms, or countries, the decisions of agents affect each other through the network of connections.[46,47] The composition and the structure of such networks, in turn, influence the evolution of cooperative behavior in the population.[48]

In this paper, we study a network game of spillover effects, conflicts, and private

[45]Smith (2003) provide a survey of literature on evolution of human cooperation.

[46]Jackson (2009) roughly categorizes two settings where the network interactions impact the behavior. In one setting the network structure is the primary source of information transmission, in the other, the network determines the structure of interdependence of individual outcomes on the actions of their neighbors or, in other words, *spillovers*. In many circumstances, networks embody both roles. Learning and information communication through social networks can result in the adoption of green products and innovative technologies, while the decisions about product or technology adoption are affected by potential spillovers from neighbors.

[47]Behavioral spillovers and peer effects in networks had been studied in adoption of education, green behavior and technology adoption, criminal behavior. See the recent survey by Bramoullé et al. (2020).

[48]Fowler and Christakis (2010) survey a set of experiments showing that the cooperative behavior spreads in social network.

dissonance in a fixed network.[49] We later extend the game by allowing cooperative behavior in groups of players, assuming a shared intention to maximize the joint utility of the group. We consider a global spillover effect between all players and an additional local spillover effect between neighbors in the network. Players receive utility from network spillovers with heterogeneous returns. The players in the network suffer disutility when their efforts are different from those of their neighbors. This disutility can be described as social dissonance or conflict from non-compliance with the social norm. The second source of the conflict we incorporate in our network game is the individual cognitive dissonance which occurs when the individual observes a discrepancy between her belief about the ideal behavior and the factual effort. The idiosyncratic *ideal effort* of the player, together with her return coefficient of global and local spillovers, compile the type of the player. We show that the type of the player increases her intercentrality, and along with her position in the network of a given structure, affects the contribution of the player to the aggregate effort. Additionally, the efforts of the players in the game are strategic complements. Thus, the increase in the efforts of network neighbors fosters an increase in the effort of the given player. Further, we find the unique Nash equilibrium of the game. The equilibrium efforts of players are proportional to the weighted Katz-Bonacich centralities, allowing us to follow the literature on key players going back to Ballester et al. (2006). In their seminal paper, Ballester et al. (2006) provide a geometric characterization of the *key player* in the network with peer effects by defining a new measure of centrality, *intercentrality*. The key player is the player in the network the removal of which entails the highest impact on the aggregate effort in the network. The intercentrality measure captures the externalities that players have from each other.[50] They find that the Nash equilibrium of the game with peer effects and effort complemetarities is proportional to the Katz-Bonacich centrality put forward by Bonacich (1987) and Katz (1953). We find the positive key player in our model, i.e. the player that once removed from the network has the highest impact on the total effort level in the network, and the negative key player, i.e. the player with the lowest impact, respectively.

Assuming a collaborative behavior in groups of players, we allow for unions where the members choose their efforts by maximizing their joint utility in the network.

[49]For network games see e.g. Jackson and Zenou (2015). Another survey on games on fixed networks by Bramoullé and Kranton (2015) covers a variety of settings as peer effects, public goods, and technology adoption. An important contribution in the literature on games on networks is the paper by Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010) where authors develop a general framework to study network games.

[50]The peer effects with the implementation of key player policies had been studies empirically in education (Liu, Patacchini, and Zenou (2014)) and criminal networks (Lee, Liu, Patacchini, and Zenou (2020)).

We show that apart from individual efforts influencing the decisions of other players through spillovers, the efforts of all players in a connected network are additionally affected by the unions. We consider a set of unions as a partition of players in the network where links between members of a union are not required. We provide the generalized characterization of the unique Nash equilibrium for the game with unions and analyze how the presence of those affects the total effort in the population of players. We define a *union intercentrality* measure to find the key players in the network with unions. Given a set of initial parameters and network structure, we show that the key players in the setting with unions may differ from the ones of the original network of single players as a result of additional *union effect* in union intercentrality. Furthermore, the *key addition* to the existing union is defined. That is the player that increases the aggregate effort in the network by joining a union. We define the *key union* as the union that, once created, increases the aggregate effort in the game the most.

There are various examples of social and economic issues that can be described by games of spillovers and conflicts. When talking about global health issues, it comes naturally to think of the possible global effect of collective efforts in the development of a vaccine against a fast-spreading virus on one individual that will enjoy the positive spillovers from these efforts. While the vigilance of friends, the small community of direct contacts, may help to keep a low transmission level of the disease in our network, helping to avoid subsequent costs by avoiding the infection. This can be seen as local spillovers from direct contacts. At the same time, the behavior of the direct contacts may affect the health of the individual also negatively, causing conflict and disutility. Furthermore, the cost restrictions due to required quarantines and lockdowns may differ among individuals due to their personal needs and preferences, giving rise to private dissonance.

Another example of a network game with spillovers and conflicts is the adoption of technology when compatibility matters. The choice of technology such as the choice of messaging applications, software for video conferences, cloud computing services, or file transfer services is strongly affected by the choice of peers. While failure to coordinate creates incompatibility. One reason for such miscoordination can be personal preferences or ideal choices of individuals which can be based on trust or mistrust in the specific service provider, data protection issues, or simply personal habits.

Environmentally friendly behavior is another example where spillovers and miscoordination, together with private dissonance, affect individual choices. Studies show that social norms and private values are strong determinants of collective pro-environmental behaviors.[51] Examples of collective pro-environmental behaviors are group activities, educational programs that aim to preserve the environment through recycling, collective clean-ups and beautification organized by communities. The collective action with a shared intention to promote green behavior is put through organizations and movements such as Greenpeace, Fridays for Future, or country-level treaties, such as the Paris agreement. Other factors behind the reasoning and pro-environmental decisions of individuals are the social influence and private values. The discomfort from acting differently from the neighbors, or the willingness to fit the social norms can drive the attitudes of individuals regardless of any monetary benefit.[52] Meanwhile, private values, or private norms, are another driving force of individual behavior.[53,54]

We contribute to the literature on network games by studying the local and global spillovers, conflict with neighbors and taste for consistency with ideal behavior in one game. Moreover, we introduce unions, groups of collaborative players maximizing the joint utility, in the network and extend the definition of the key player to the network game with unions. The welfare analysis and the study of utility outcomes are outside of the scope of this paper. This opens ground for future research on the study of union formation in networks with spillovers and social and private dissonance, and study of the union stability using fair allocation rules in network-dependent coalitions.

The rest of the paper is organized as follows. In Section 3.2 we describe the network spillover game with social and private dissonance. We find the unique Nash equilibrium of the game and study different settings of the game in examples. In Section 3.3 we introduce unions in the network game. We show the effect of such unions on equilibrium efforts. Concluding remarks are provided in Section 3.4 and the proofs are in the Appendix 3.A.

[51]See e.g. Viscusi, Huber, and Bell (2011), Gifford and Nilsson (2014).

[52]An empirical study conducted by Ando, Ohnuma, Blöbaum, Matthies, and Sugiura (2010) identifies and compares the determinants of individual and collective environmentally conscious behaviors. They show that the subjective norms from the network impact the collective pro-environmental behaviors, which indicates the importance of the social influence on the collective behavior regarding environmental issues.

[53]Gifford and Nilsson (2014) provide an extensive survey of studies of personal and social determinants of pro-environmental behavior.

[54]Viscusi et al. (2011) investigate empirically the importance of the social norms by characterizing the latter with the normatively appropriate behavior. They capture the influence of personal values on the recycling decision with data on personal attitude towards the environment. Moreover, the authors show that the private values reflected in becoming upset at neighbors not recycling are far more predictive of a person's behavior than the external norms reflected in their beliefs about what their neighbors might think of them. Meanwhile, the people that consider themselves environmentalists are more likely to be concerned about "private and external" norms experiencing *green guilt* from acting less environmentally friendly compared to their neighbors.

3.2 The Network Spillover Game

3.2.1 The Game

We study a network game with continuous action space and strategic complementaries between the actions of the neighbors. We have a set of $N \in \{1, 2, ..., n\}$ players connected through a directed network $\boldsymbol{g}$ where g_{ij} is the positive weight on the directed link from player i to j for any neighbors $i, j \in N$ in the network. The weight is zero, $g_{ij} = 0$, when such directed link is absent. By convention we assume there are no self-loops in the network, thus $g_{ii} = 0$ for all $i \in N$. We use the weighted adjacency matrix $\mathbf{G}^*$ associated with the network $\boldsymbol{g}$, with elements $g^*_{ij} \in [0, 1]$ corresponding to the original weights g_{ij} normalized by out-degrees $d^+_i(\boldsymbol{g})$ of player i, that is, her weighted number of outgoing links. For simplicity we assume that the network does not contain disconnected nodes, so that $d^+_i(\boldsymbol{g}) > 0$ for all players. Thus, the matrix $\mathbf{G}^*$ is row-normalized $\sum_{j \in N} g^*_{ij} = 1$, for all $i \in N$.

The players are endowed with personal *ideals* regarding their own behavior. We denote the *ideal efforts* of the players with $y_i \in \mathbb{R}^+$.[55] They choose their efforts $x_i \in \mathbb{R}^+$ based on spillovers and conflicts from the neighbors in the network, while trying to remain consistent with their personal ideal effort levels to avoid disutility from private *cognitive dissonance*. We define this private dissonance as the discrepancy between the actual and ideal efforts. We distinguish two types of spillovers in the model: global and local spillovers. Similarly to public goods games, the players benefit from the efforts exerted by all players in the network. Yet, unlike public goods in networks studied by Bramoullé and Kranton (2007) or local-aggregate or -average models,[56] we assume that these spillovers are not restricted to direct network neighbors, thus we refer to those as *global spillovers*. Conversely, the local interactions between neighbors cause *local spillovers*. These are the spillovers occurring from complementarities in player's effort x_i and her descriptive social norm $\sum_{j \in N} g^*_{ij} x_j$,[57] which in network games is commonly

[55] The *ideal efforts* can be perceived as a *personal norm* of a player, personal standard, an attitude towards the specific issue, feeling of moral obligation, or self-expectations (disregarding the possibility that these expectations may derive from socially shared norms) (Ajzen and Fishbein, 1970; Schwartz, 1973; Schwartz and Howard, 1980). While non-compliance with, or the violation of the personal norm may results in feeling of guilt, loss of self-esteem, self-depreciation (Schwartz, 1973), this discrepancy between the self-standard, or *personal norm*, and actual behavior causes aroused *cognitive dissonance* (Stone and Cooper, 2001).

[56] See e.g. Liu et al. (2014) and Ushchev and Zenou (2020).

[57] While players enjoy the collective effort of the whole population (the public good), their behavior is directly affected by those of their direct social contacts. Therefore, we assume no effort complementarities through global spillovers.

referred to simply as social norm of player i. While the descriptive social norms is the representation of social behavior, the injunctive norm is the perception of social approval. We assume that the disapproval is costly and players suffer disutility from *social dissonance*, or *conflict*, caused by the difference in their efforts and those of each of their neighbors. In other words, the deviation from injunctive norm, that is, the difference in behavior with each of direct neighbors, results in disutility. Minimization of this cost of conflict leads to conformism similarly to conventional local-average models with descriptive social norms. As also noted by Ushchev and Zenou (2020), this form of representation is still a conformist model and does not affect the choice of equilibrium efforts compared to the more common representation. However, it results in different welfare outcomes due to different utilities in equilibrium.

Let $\mathbf{x} = (x_1, ..., x_n)$ be the vector of efforts of player in the network $\boldsymbol{g}$, and $\mathbf{x}_{-i}$ the list of efforts of players $N \setminus i$. The utility of player i with effort x_i and idiosyncratic ideal y_i is then given by the linear-quadratic function

$$u_i(x_i, \mathbf{x}_{-i}, y_i, \theta_i, \boldsymbol{g}) = \underbrace{\lambda_1 \theta_i \sum_{j \in N} x_j}_{\substack{\text{global spillover} \\ \text{effect}}} + \underbrace{\lambda_2 \theta_i x_i \sum_{j \in N} g^*_{ij} x_j}_{\substack{\text{local spillover} \\ \text{effect}}} - \underbrace{\frac{\omega_1}{2} \sum_{j \in N} g^*_{ij} (x_i - x_j)^2}_{\substack{\text{cost of} \\ \text{social dissonance}}} - \underbrace{\frac{\omega_2}{2} x_i^2}_{\text{effort cost}} - \underbrace{\frac{\omega_3}{2} (x_i - y_i)^2}_{\substack{\text{cost of} \\ \text{private dissonance}}}, \tag{3.1}$$

where λ_1, λ_2, ω_1, ω_2 and ω_3 are positive coefficients of the respective source of splillovers and costs. The players benefit from spillovers and suffer quadratic cost from their social dissonance, effort, and private dissonance. The coefficients λ_1 and λ_2 capture the effects of global and local spillovers, ω_1 is the sensitivity to conflict or taste for conformity, and ω_2 is the coefficient of effort cost. The sensitivity to cognitive dissonance or the taste for consistency with ideal behavior is represented by ω_3. Additionally, the players are heterogeneous in their return on spillovers captured by parameters $\theta_i \in \mathbb{R}^+$, for all $i \in N$. One can see that given the positive coefficients and parameters θ_i, the efforts of network neighbors are strategic complements:

$$\frac{\partial^2 u_i}{\partial x_i \partial x_j} = (\lambda_2 \theta_i + \omega_1) g^*_{ij} \geq 0.$$

Utility representation in (3.1) captures the benefits of players through positive spillovers from aggregate effort in a given network $\boldsymbol{g}$ and their direct neighbors, as well as the peer pressure to exert efforts close to those of the neighbors as an incentive to minimize the social dissonance and avoid miscoordination and conflict. In addition, the players have an incentive to be consistent with one's personal ideal y_i.[58] The efforts incur quadratic costs independently from the network structure and personal ideal effort.

Idiosyncratic ideal y_i together with parameter θ_i can be interpreted as the type of player i in the network. By setting either of the two parameters to be homogeneous across the network, we let the other indicate the player's type.

3.2.2 Nash Equilibrium

Consider the following restructured form of the utility in (3.1)

$$u_i = (\lambda_1\theta_i + \omega_3 y_i)x_i + \lambda_1\theta_i \sum_{j\in N\setminus i} x_j + \sum_{j\in N\setminus i} g^*_{ij}\left((\lambda_2\theta_i + \omega_1)x_i - \frac{\omega_1}{2}x_j\right)x_j$$
$$- \frac{1}{2}(\omega_1 + \omega_2 + \omega_3)x_i^2 - \frac{\omega_3}{2}y_i^2.$$

Using this representation, we can observe the direct effect of the player's effort on her utility, with type (y_i, θ_i) defining the return $(\lambda_1\theta_i + \omega_3 y_i)$ on effort, and $\frac{1}{2}(\omega_1+\omega_2+\omega_3)$ being the coefficient of its' quadratic cost. We can also observe a quadratic cost inflicted by player's personal ideal, $\frac{\omega_3}{2}y_i^2$. The component exhibiting the utility from interaction with neighbors shows the positive complemetarities in their efforts with coefficient $(\lambda_2\theta_i + \omega_1)$, as already noted above, and a cost that player i bears from the effort x_j of her neighbors with coefficient $\frac{\omega_1}{2}$. It directly follows that the overall impact of direct interaction with the neighbor j of player i on her utility is positive if [59]

$$x_j < 2\left(\frac{\lambda_2\theta_i + \omega_1}{\omega_1}\right)x_i.$$

We introduce the following notations. Let $\alpha_i(\theta_i, y_i)$ and $\phi_i(\theta_i)$ be the coefficients of player i's utility from her own effort and that of her network neighbors weighted by the overall cost coefficient associated with player's effort, respectively:

$$\alpha_i(\theta_i, y_i) = \frac{\lambda_1\theta_i + \omega_3 y_i}{(\omega_1 + \omega_2 + \omega_3)} \quad \text{and} \quad \phi_i(\theta_i) = \frac{\lambda_2\theta_i + \omega_1}{(\omega_1 + \omega_2 + \omega_3)}. \tag{3.2}$$

[58] In network games literature, the notion of ideal efforts and consistency is present in works by Olcina et al. (2017) and Galeotti et al. (2021).

[59] We consider a fixed network structure. One may argue that severing the link with a neighbor j where the given condition of positive impact doesn't hold is a possible step in the strategy of utility maximization. Similarly to Arifovic, Eaton, and Walker (2015) one can study the dynamics of network structure and personal ideals. Yet, the study of network evolution is outside the scope of this paper.

Note that α_i is an increasing function of player's type (θ_i, y_i), and ϕ_i increases with parameter θ_i. Using the notations in (3.2) we find the best response $BR_i(\mathbf{x}_{-i}; \mathbf{G}^*, y_i, \theta_i) := BR_i(\mathbf{x}_{-i}; \mathbf{G}^*, \phi_i, \alpha_i)$ of player i from first order conditions on (3.1):

$$BR_i(\mathbf{x}_{-i}; \mathbf{G}^*, \phi_i, \alpha_i) = \alpha_i + \phi_i \sum_{j \in N} g^*_{ij} x_j. \tag{3.3}$$

Thus, we see that the best response of player i is linear in the efforts of her neighbors.

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ be n-dimensional vectors of α_i and ϕ_i respectively, and $\mathbf{I}$ be n-identity matrix. Further, let $[\boldsymbol{\phi} \otimes \mathbf{G}^*]$ be a matrix formed by coordinate-by-coordinate multiplication of the vector $\boldsymbol{\phi}$ and the matrix $\mathbf{G}^*$.[60]

Proposition 3.1. *Assume that the spectral radius of the matrix $[\boldsymbol{\phi} \otimes \mathbf{G}^*]$ is smaller than 1. Then, the unique Nash equilibrium in pure strategies is given by*

$$\mathbf{x} = (\mathbf{I} - \boldsymbol{\phi} \otimes \mathbf{G}^*)^{-1} \boldsymbol{\alpha}. \tag{3.4}$$

Lemma 3.1. *Let ρ be the spectral radius of the matrix $[\boldsymbol{\phi} \otimes \mathbf{G}^*]$. For any network $\boldsymbol{g}$ and types θ_i of players $i \in N$, if $\max_i \theta_i < \frac{\omega_2 + \omega_3}{\lambda_2}$ then the condition $\rho < 1$ is satisfied.*

Proof. The proof of Lemma 3.1 follows directly from the fact that the weighted adjacency matrix $\mathbf{G}^*$ is row-normalized, and the application of Gershgorin circle theorem. Indeed the spectral radius ρ lies within the unit circle if the maximum row-sum norm of the matrix $[\boldsymbol{\phi} \otimes \mathbf{G}^*]$ is smaller than one. This is satisfied if the maximum of ϕ_i for all $i \in N$ is smaller than one. Using the notations in (3.2) it follows that the condition $\rho < 1$ in Proposition 3.1 is satisfied when $\max_i \theta_i < \frac{\omega_2 + \omega_3}{\lambda_2}$. □

Proposition 3.1 provides an equilibrium solution to the general problem of the network game with spillovers and social and private dissonance. We can derive the equilibrium efforts for a special case such as the class of homogeneous games where the players are identical in their ideal efforts and return parameters, that is, $(y_i, \theta_i) = (y, \theta)$ for all $i \in N$. Let $\alpha = \alpha_i(y, \theta)$ and $\phi = \phi_i(\theta)$ be the corresponding homogeneous parameters α_i and ϕ_i in (3.2), for all $i \in N$. The following corollary shows that for any network structure where each node has at least one outgoing link, the equilibrium effort choices in the network game will be homogeneous when the players' types are homogeneous across the players.

[60] Throughout the paper we use the symbol $\otimes$ to define matrix formed by coordinate-by-coordinate multiplication of a vector and a matrix, i.e. by each i-th row of the matrix multiplied by corresponding i-th element of the vector. Thus the matrix $[\boldsymbol{\phi} \otimes \mathbf{G}^*]$ is an $(n \times n)$ matrix resulting from multiplying each i-th element of the vector $\boldsymbol{\phi}$ with i-th row of the matrix $\mathbf{G}^*$ for each $i \in N$.

Corollary 3.1. *For a network* $\boldsymbol{g}$ *represented by row-normalized weighted adjacency matrix* $\mathbf{G}^*$*, and homogeneous player types* (y, θ) *across the network, such that* $\theta < \dfrac{\omega_2 + \omega_3}{\lambda_2}$*, the unique Nash equilibrium in pure strategies is given by*

$$x = \frac{\alpha}{1 - \phi}.$$

with $x_i = x$ *for all* $i \in N$.

Proof. Following Proposition 3.1, to ensure the existence of equilibrium the spectral radius of the matrix $\phi \mathbf{G}^*$ has to be smaller than 1. Similarly to the proof of Lemma 3.1, it is easy to show that this condition is satisfied when $\theta < \dfrac{\omega_2 + \omega_3}{\lambda_2}$. Then, the equilibrium efforts in the game (3.4) with homogeneous types (y, θ) are

$$\mathbf{x}^* = \alpha(\mathbf{I} - \phi \mathbf{G}^*)^{-1}\mathbf{1},$$

with $\mathbf{1}$ being a unit vector of size n. Additionally, given row-normalized adjacency matrix $\mathbf{G}^*$, we have that $(\mathbf{I} - \phi \mathbf{G}^*)\mathbf{1} = (1 - \phi)\mathbf{1}$. Using the properties of matrix inverse we have

$$\mathbf{1} = (\mathbf{I} - \phi \mathbf{G}^*)^{-1}(\mathbf{I} - \phi \mathbf{G}^*)\mathbf{1} = (\mathbf{I} - \phi \mathbf{G}^*)^{-1}(1 - \phi)\mathbf{1}.$$

Thus $(\mathbf{I} - \phi \mathbf{G}^*)^{-1}\mathbf{1} = \dfrac{1}{1 - \phi}\mathbf{1}$. From which follows that

$$\alpha(\mathbf{I} - \phi \mathbf{G}^*)^{-1}\mathbf{1} = \frac{\alpha}{1 - \phi}\mathbf{1}$$

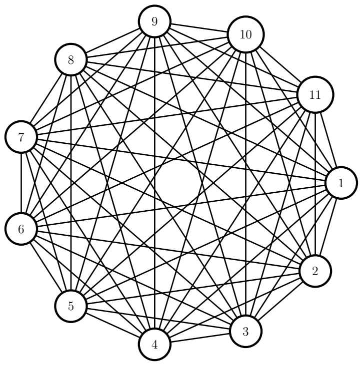
and the equilibrium efforts in the game with homogeneous types are given by

$$\mathbf{x}^* = \frac{\alpha}{1 - \phi}\mathbf{1}.$$

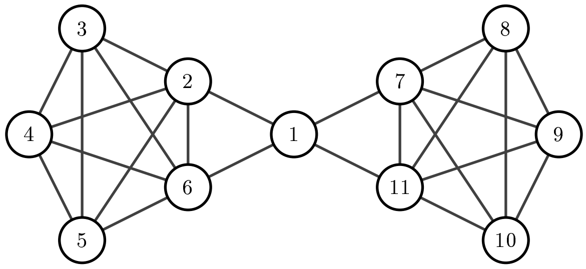
□

Using the original notations in (3.2) we find that $x^* = \dfrac{\lambda_1 \theta + \omega_3 y}{\omega_2 + \omega_3 - \lambda_2 \theta}$ for each player. As expected, in the homogeneous setup of Corollary 3.1 the coefficient of social dissonance ω_1 does not affect the equilibrium outcomes due to absence of conflict when all neighbors are identical in their types.

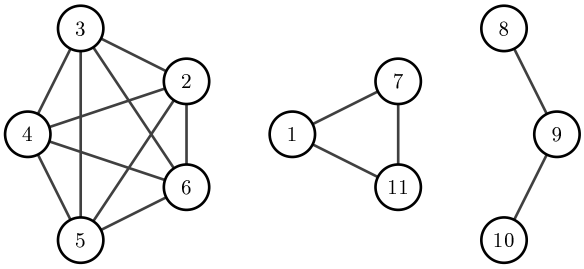
The most basic example to satisfy Corollary 3.1 are complete networks. Given equal weights on each link, every node in such network is equivalent to another. With homogeneous parameters of the model, the spillover game on the complete network is symmetric, and thus equilibrium efforts are equal for all players.



(a) Complete network



(b) "Bridge"



(c) Network with 3 components

Figure 3.1

More interestingly, regardless of the network structure $\boldsymbol{g}$ and its' size, the equilibrium efforts of all players in the network are still identical and depend only on the coefficients of the game, ideal efforts and type parameter values. This result is mainly driven by the fact that we associate any network with its' row-normalized weighted adjacency matrix. Figure 3.1 illustrates three networks of the same size,[61] yet with different structures. The three networks have players exerting the same effort in equilibrium. Which also means that the aggregate utilities and efforts in three networks are also the same. This property of symmetry in players' effort choices regardless of the network structure, and given that the type parameters are homogeneous, allows us to distinguish the effects of model coefficients for a given network when heterogeneity is introduced.

Using the results above we can distinguish a subset of players in the network for which homogeneity in ideal efforts and θ imply homogeneous effort choices in equilibrium. Players forming a *strongly connected component* in a given network *with no path connecting the component with rest of the network* are such a subset. Assume there is a subset $C \subset N$ of players in the directed network $\boldsymbol{g}$ that comprise a strongly connected component in the network. In addition to strong connectivity, for any $i \in C$ and any

[61] In all network figures illustrated in the paper we purposely omit the arrows indicating the direction of the link whenever the weights are non-zero on both directions.

$j \in N \setminus C$ the weight $g^*_{ij} = 0$. Corollary 3.2 below provides the pure Nash equilibria for players in C, given that the ideal efforts and θ types within the component are homogeneous.

Corollary 3.2. *For any strongly connected component of players $C \subset N$ with no outgoing path towards players in $N \setminus C$ on the directed network $\boldsymbol{g}$, given that all players in C are homogeneous in ideal efforts y^c and type parameters θ^c, $(y_k, \theta_k)_{\forall k \in C} = (y^c, \theta^c)$, such that $\theta^c < \dfrac{\omega_2 + \omega_3}{\lambda_2}$, the unique Nash equilibrium in pure strategies for each player in C is given by*

$$x^{c*} = \frac{\lambda_1 \theta^c + \omega_3 y^c}{\omega_2 + \omega_3 - \lambda_2 \theta^c}.$$

Proof. By definition, the subset of players C is disconnected from the rest of the network due to absence of outgoing links from the nodes in the component to those in $N \setminus C$. This implies that the weighted adjacency matrix associated with any strongly connected component without outgoing paths on $\boldsymbol{g}$ is the submatrix $\mathbf{G}^*_c$ of $\mathbf{G}^*$ formed by the subset of rows and columns corresponding to players in C. Using the properties of the best response function (3.3) the results in Corollary 3.1 can be applied. Thus the equilibrium effort choice of any player in C with homogeneous types (y^c, θ^c) is equal to

$$x = \frac{\alpha^c}{1 - \phi^c},$$

where $\alpha^c = \dfrac{\lambda_1 \theta^c + \omega_3 y^c}{(\omega_1 + \omega_2 + \omega_3)}$ and $\phi^c = \dfrac{\lambda_2 \theta^c + \omega_1}{(\omega_1 + \omega_2 + \omega_3)}$. □

3.2.3 Key Players

In this section we use the notion of *key players* introduced by Ballester et al. (2006) to find the most influential players in the network. To identify the key players we define the intercentrality (or key player centrality) measure for directed networks analogous to the one of Ballester et al. (2006) for our network game on the directed network $\boldsymbol{g}$, represented by row-normalized weighted adjacency matrix $\mathbf{G}^*$. We use the Katz-Bonacich centrality as an established measure of player's contribution in the aggregate outcomes of the network.

Introduced by Katz (1953) and redefined later by Bonacich (1987), the Katz-Bonacich centrality measures the centrality of player i as the total number of all possible paths that stem from i, with each path weighted inversely to its length. Thus, the higher number of shorter paths connecting the player to others in the network increases her Katz-Bonacich centrality. The vector of Katz-Bonacich centralities associated with

$\mathbf{G}^*$, with scalar values of ϕ and α is given by

$$\boldsymbol{b}(\mathbf{G}^*, \phi, \alpha) = \alpha \sum_{t=0}^{\infty} \phi^t \mathbf{G}^{*t} \mathbf{1},$$

with $\boldsymbol{b}(\mathbf{G}^*, \phi, \alpha) = \alpha(\mathbf{I} - \phi\mathbf{G}^*)^{-1}\mathbf{1}$ for $\phi < 1$.

Let $\mathbf{M}(\mathbf{G}^*, \boldsymbol{\phi}) := (\mathbf{I} - \boldsymbol{\phi} \otimes \mathbf{G}^*)^{-1}$ be well defined and non-negative. And let's define $\boldsymbol{b}(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha}) := \mathbf{M}(\mathbf{G}^*, \boldsymbol{\phi})\boldsymbol{\alpha}$ for vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$, then

$$\boldsymbol{b}(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \sum_{t=0}^{\infty} [\boldsymbol{\phi} \otimes \mathbf{G}^*]^t \boldsymbol{\alpha}$$

is the vector of weighted Bonacich centralities in the network represented by the adjacency matrix $[\boldsymbol{\phi} \otimes \mathbf{G}^*]$. This measure corresponds to equilibrium efforts $\mathbf{x}^*$ of the game $(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha})$ defined in Proposition 3.1. We can then say that the equilibrium effort of player i in the game is given by her weighted Katz-Bonacich centrality $b_i(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha})$. And $\mathcal{B} := \sum_{i \in N} b_i$ is equivalent to the aggregate effort in the network. Next, we define the contribution of player i as a difference in the aggregate effort and that of the network without player i.

Definition 3.1. *The contribution δ_i of player i to the aggregate effort in the network is the difference between the aggregate effort exerted in the initial game $(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha})$ and the new game $(\mathbf{G}^{*-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i})$ on the network $\boldsymbol{g}^{-i}$ obtained by removing node i from the initial network $\boldsymbol{g}$:*

$$\delta_i(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \mathcal{B}(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha}) - \mathcal{B}(\mathbf{G}^{*-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i}).$$

$\mathbf{G}^{*-i}$ is the row-normalized weighted adjacency matrix of the new network $\boldsymbol{g}^{-i}$ formed by removing player i from the original network $\boldsymbol{g}$. While the vectors $\boldsymbol{\phi}^{-i}$ and $\boldsymbol{\alpha}^{-i}$ are formed by simply removing the i-th element from the corresponding initial vector.

We define i^* as a *positive (negative) key player* if removing her from the initial network $\boldsymbol{g}$ has the highest (lowest) overall impact on the aggregate equilibrium level.

Definition 3.2. *The player is the positive key player i^*_+ of the game if she has the highest contribution δ_i:*

$$i^*_+ = \arg\max_i \delta_i(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha}).$$

The negative key player i^*_- is the player with lowest contribution δ_i:

$$i^*_- = \arg\min_i \delta_i(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha}).$$

The problem of finding the key players defined above is equivalent to finding the $\arg\min_i \mathcal{B}(\mathbf{G}^{*-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i})$ and $\arg\max_i \mathcal{B}(\mathbf{G}^{*-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i})$ for positive and negative key players respectively. In other words, the *key player* is the player which, once removed from the network, has the highest or lowest impact on aggregate effort exerted in the population of players. The following proposition suggests that the contribution δ_i of the player is equivalent to an intercentrality measure proportional to its' weighted Katz-Bonacich centrality.

Proposition 3.2. *The contribution of player i to the aggregated effort $\mathcal{B}$ of the game is given by her intercentrality:*

$$\delta_i(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \frac{b_i(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha})}{m_{ii}} \sum_{j \in N} m_{ji},$$

where m_{ij}, $i, j \in N$ are the elements of matrix $\mathbf{M}(\mathbf{G}^*, \boldsymbol{\phi})$.

The weighted Katz-Bonacich centrality b_i of player i measures all the paths in $\boldsymbol{g}$ that start from i and are weighted according to parameters in $\boldsymbol{\phi}$ and $\boldsymbol{\alpha}$. The *intercentrality* δ_i measures the total number of paths that pass *through* i, that is, i's Katz-Bonacich centrality *and* her contribution to every other player's centrality.

3.2.4 Spillovers, Conflicts and Private Dissonance

In the rest of the section we try to disentangle the effects of spillovers and social and private dissonance of players in games on a fixed networks by studying specific examples. We find the key players and discuss the resulting equilibrium outcomes given parameter settings.

Example 3.1. ***Game of spillovers***
Consider the networks in Figure 3.1a and 3.1b, with players $N = \{1, \dots, 11\}$. For simplicity, we assume there is no social and private dissonance in this example, $\omega_1 = \omega_3 = 0$, and the effort cost coefficient $\omega_2 = 1$. The global spillovers have a coefficient of $\lambda_1 = 1$ and the local spillovers' coefficient is $\lambda_2 = 0.1$.

Given the initial parameter values we need the types θ_i to satisfy the condition in Lemma 3.1. As a reference point we use the game with homogeneous players. Recalling

Corollary 3.1 we choose a unique $\theta < \frac{\omega_2}{\lambda_2}$ for all players. Setting $\theta = 1$ we find the unique Nash equilibrium effort equal to 1.11. The intercentrality δ_i for all players is the same and is equal to their efforts. Thus, in the homogeneous game there are no distinguished key players. We introduce heterogeneity by assigning higher type $\theta_i = 2$ to one of the players in given networks. In Table 3.1 we show how the total effort and key players in the network change with the introduction of a higher type player. As all players in the homogeneous game on the complete network (Figure 3.1a) are structurally equivalent, it does not matter which player is assigned a higher type. We see that the presence of a higher type player in the network increases the total effort in the network making the player herself the positive key player with higher intercetrality compared to that of the others'. Unlike complete networks, in the bridge network in Figure 3.1b the choice of the player matters due to structural differences. Exploiting the symmetries in this network we identify equivalent nodes which reduces the number of players we need to study down to players 1, 2 and 3. We see in Table 3.1 that while the positive key player is the one with higher type in each of the three settings, the highest aggregate effort in the network among the three is reached when the higher type is assigned to the player in position 2. When player 1 is the higher type player, the negative key players are those that are the furthest from 1 in the network. In case of player 2, when $\theta_2 = 2$, the negative key player is her direct neighbor, player 6, which can be explained by lack of spillovers from 6 given the absence of social links other than those of player 2. Player 3 when assigned $\theta_3 = 2$ has the highest intercentrality of all. Yet this setting makes player 1 a negative key player, resulting in a lower level of aggregate effort compared to the case of $\theta_2 = 2$.

Example 3.2. ***Social dissonance***
We change the setup in Example 3.1 by adding the social conflict. We set the conflict cost coefficient to $\omega_1 = 1$ in the games on networks 3.1a and 3.1b.

In the homogeneous game, where the players are identical in their types, the introduction of social conflict does not cause any change in outcomes. With heterogeneity in the player types the conflict with neighbors can affect equilibrium efforts. Assume all players but player i have their return parameter equal to 1, $\theta_j = 1$ for all $j \in N \setminus i$, and assign $\theta_i = 2$.

In the game on the complete network the conflict with neighbors, or the social dissonance, increases the aggregate effort in the game. As it is displayed in Table 3.1, when player 1 is the higher type player her effort in the game with social dissonance is lower than that of the game with $\omega_1 = 0$. While the higher type player has the highest intercentrality, thus is the positive key player, she has a lower effort due to peer pressure. The negative key players, thus the rest of the players in the complete network, increase their efforts when social dissonance is introduced. As a result, the presence of disutility from deviation from the social norm increases the efforts of the players.

Like in the previous example, the bridge network in 3.1b requires to study the effects of the social dissonance in three different cases by assuming players 1, 2 or 3 as the higher type player. While the efforts of negative key players i^*_- are increasing in all three cases, their intercentralities δ_i^- are decreasing. As in the game on the complete network, the efforts of the positive key players are decreasing in all three games on the bridge network when the conflict is introduced. While only the intercetrality of positive key player 2 increases with her having a higher type. The total effort is decreasing with the presence of conflict with neighbors when player 1 or 3 have higher type, while in case of $\theta_2 = 2$ the total effort in the game is increasing. The results are displayed in Table 3.1.

Table 3.1: Aggregate effort, equilibrium efforts and contributions of positive (+) and negative (−) keyplayers in games on Complete and Bridge networks in Figure 3.1, with spillovers and social dissonance, Examples 3.1 and 3.2.

Conflict	**Type**	**Total effort**	**Positive key player**			**Negative key player**		
ω_1	$\theta_{j\in N\setminus i} = 1$	$\mathcal{B}$	$+$	b_+	δ_+	$-$	b_-	δ_-
	$\theta_i = 1$	12.2223	N	1.1112	1.1112	N	1.1112	1.1112
Complete Network								
$\omega_1 = 0$	$\theta_{i=1} = 2$	13.4582	$\{1\}$	2.2247	2.348	$N \setminus \{1\}$	1.1234	1.111
$\omega_1 = 1$	$\theta_{i=1} = 2$	13.464	$\{1\}$	1.7056	2.353	$N \setminus \{1\}$	1.1759	1.1104
Bridge Network								
$\omega_1 = 0$	$\theta_{i=1} = 2$	13.4365	$\{1\}$	2.2268	2.326	$\{3\},\{4\},\{5\},\{8\},\{9\},\{10\}$	1.1124	1.0984
	$\theta_{i=2} = 2$	13.4838	$\{2\}$	2.228	2.373	$\{6\}$	1.1359	1.0948
	$\theta_{i=3} = 2$	13.4478	$\{3\}$	2.2277	2.337	$\{1\}$	1.1124	1.0988
$\omega_1 = 1$	$\theta_{i=1} = 2$	13.3791	$\{1\}$	1.7192	2.268	$\{3\},\{4\},\{5\},\{8\},\{9\},\{10\}$	1.1443	1.0407
	$\theta_{i=2} = 2$	13.5847	$\{2\}$	1.7479	2.474	$\{6\}$	1.2413	1.0533
	$\theta_{i=3} = 2$	13.4104	$\{3\}$	1.7429	2.3	$\{1\}$	1.1444	1.0479

Example 3.3. ***Cognitive dissonance***
In addition to the conflict with neighbors in Example 3.2 we add the personal conflict, the private dissonance arising from difference in the ideal y_i and the actual efforts. To do so we set the coefficient $\omega_3 = 1$.

To study the effect of the private dissonance we assume the type parameter θ to be homogeneous across the network and equal to 1. Endowing all players with same ideal efforts $y = 1$ satisfies Corollary 3.1 and results in identical equilibrium efforts of 1.0527. The aggregate effort is then equal to 11.579 for a network with $n = 11$ players. Note, that the ideal effort $y = 1$ and the equilibrium effort 1.0527 are lower than the equilibrium effort of 1.11 in the homogeneous game without private dissonance. This reflects the fact that the ideal effort incurs an additional cost on the utilities of players through the private dissonance.

Assuming one of the players in the network has higher ideal effort, $y_i = 2$ while $y_j = 1$ for all $j \in N \setminus i$, we find the key players and their contributions. Similarly to Examples 3.1 and 3.2 we study the complete and the bridge networks in Figure 3.1a and 3.1b, distinguishing three cases on the bridge network with players 1, 2 and 3 having a higher ideal effort. In presence of private dissonance, the introduction of a player with higher ideal effort increases the efforts of players in the network. The higher ideal effort increases the intercentrality of the player making her the key player on the network. Table 3.2 displays the results.

Table 3.2: Aggregate effort, equilibrium efforts and contributions of positive (+) and negative (−) keyplayers in games on Complete and Bridge networks in Figure 3.1, with spillovers, social and private dissonance, Example 3.3.

Type	**Total effort**	**Positive key player**			**Negative key player**		
$y_{j \in N \setminus i} = 1$	$\mathcal{B}$	$+$	b_+	δ_+	$-$	b_-	δ_-
$y_i = 1$	11.579	N	1.0527	1.0527	N	1.0527	1.0527
Complete Network							
$y_{i=1} = 2$	12.1053	$\{1\}$	1.3928	1.579	$N \setminus \{1\}$	1.0713	1.0527
Bridge Network							
$y_{i=1} = 2$	12.0766	$\{1\}$	1.3966	1.551	$\{3\},\{4\},\{5\},$ $\{8\},\{9\},\{10\}$	1.0591	1.0332
$y_{i=2} = 2$	12.1389	$\{2\}$	1.4009	1.613	$\{6\}$	1.0904	1.0343
$y_{i=3} = 2$	12.0877	$\{3\}$	1.4001	1.562	$\{1\}$	1.0591	1.0351

3.3 Unions in Networks

In this section we introduce unions as groups of players maximizing their joint utility. These unions are a coordination device that can potentially have beneficial impact on collective efforts in the network. We study how the presence of such unions affects the aggregate effort in the network, and we provide conditions based on which the participation in the union increases the effort of a player. In addition, we distinguish the role of the union in player's contribution to the aggregate effort by defining *union intercentrality* and *union-induced intercentrality* measures. We also find the key players in the network with unions and define *key unions* as unions that, once formed, incur the highest increase in the aggregate effort.

3.3.1 Equilibrium Efforts In the Network Game with Unions

In this paper we refer as unions to the sets of players in the network that maximize their joint utility. We denote by $p(i)$ the *union* of players to which player i belongs to. That is, for all $i \in N$, $p(i) \subseteq N$ is the set of players in the network $\boldsymbol{g}$ that the union of player i consists of. We assume that each player can be a member of only one union. Thus for any $k \in N$ such that $k \notin p(i)$, $p(i) \cap p(k) = \emptyset$. And for any players $i, j \in N$ that belong to the same union the sets $p(i)$ and $p(j)$ are equivalent, $p(i) = p(j)$. This notation allows us to be consistent with cases where the players do not belong to any union, and generalize definitions and results in Section 3.2 accounting for presence of unions. We say that player $k \in N$ is a *single player* if she is not a member of any union and the set $p(k)$ consists of only player k herself, $p(k) = \{k\}$.

Let P be the set of all unions and single players in the network. Namely, P is a partition of players N into groups with $p(i) \in P$, for any $i \in N$, being the union with player i. We define $S \subseteq N$ as the set of all single players, such that $p(i) = \{i\}$ for any $i \in S$. The set of *union players*, or team players, in the network is then the set $N \setminus S$. For any union $p(i) \in P$ in the network $\boldsymbol{g}$ the utility of the union, that is, the joint utility of all union members, is the following:

$$U_{p(i)} = \sum_{j \in p(i)} u_j.$$

Using the utility of a single player defined in equation (3.1) we have:

$$U_{p(i)} = \sum_{j \in p(i)} \Big(\lambda_1 \theta_j \sum_{k \in N} x_k + \lambda_2 \theta_j x_j \sum_{k \in N} g^*_{jk} x_k - \frac{\omega_1}{2} \sum_{k \in N} g^*_{jk} (x_j - x_k)^2 - \frac{\omega_2}{2} x_j^2 - \frac{\omega_3}{2} (x_j - y_j)^2 \Big). \tag{3.5}$$

Note, when $p(i)$ is a singleton, $i \in S$ and $p(i) = \{i\}$, players i's utility is then $U_{\{i\}} = \sum_{j \in \{i\}} u_j = u_i$. Thus, for any single player $i \in S$ in the network with unions the definition of the player's utility in (3.5) is unchanged with respect to the one of the game without unions (3.1) in Section 3.2. This property allows us to redefine the parameters α_i and ϕ_i in 3.2 without loss of generality. Equation (3.6) provides a general definition of these parameters extending their application to the network game with unions. In addition, a new parameter γ_{ij} is defined.

$$\alpha_i = \frac{\lambda_1 \Theta_i + \omega_3 y_i}{\Omega_i}, \qquad \phi_i = \frac{\lambda_2 \theta_i + \omega_1}{\Omega_i}, \qquad \gamma_{ij} = \begin{cases} \dfrac{\lambda_2 \theta_j + \omega_1}{\Omega_i} & \text{if } p(i) = p(j) \\ 0 & \text{otherwise} \end{cases}, \tag{3.6}$$

$$\text{where} \qquad \Theta_i = \sum_{j \in p(i)} \theta_j \qquad \text{and} \qquad \Omega_i = \omega_1 (1 + \sum_{j \in p(i) \setminus i} g^*_{ji}) + \omega_2 + \omega_3.$$

Notice that for any single player $i \in S$, $\gamma_{ij} = 0$ for all $j \in N$.

As in Section 3.2, let $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ be n-dimensional vectors of α_i and ϕ_i, respectively. Let $\mathbf{H}$ be a matrix with elements h_{ij} that represent the union relation of any pair of players:

$$h_{ij} = \gamma_{ij} g^*_{ji} \tag{3.7}$$

for all $j \in N$. So we have $h_{ij} > 0$ for all $j \in p(i)$ with $g^*_{ji} > 0$.[62]

Using the notations above and given the definition of union utility in (3.5) we can now find the best response for each player $i \in N$ in the network with unions. We find that the best response functions $BR_i(\mathbf{x}_{-i}; \mathbf{G}^*, \mathbf{H}, \phi_i, \alpha_i)$ are linear in players' efforts $\mathbf{x}_{-i}$:

$$\begin{aligned} BR_i(\mathbf{x}_{-i}; \mathbf{G}^*, \mathbf{H}, \phi_i, \alpha_i) &= \alpha_i + \phi_i \sum_{k \in N} g^*_{ik} x_k + \sum_{k \in N} \gamma_{ik} g^*_{ki} x_k \\ &= \alpha_i + \sum_{k \in N} (\phi_i g^*_{ik} + h_{ik}) x_k. \end{aligned}$$

The additional term $\sum_{k \in N} \gamma_{ik} g^*_{ki} x_k$ in the best response of the player, compared to the best response in (3.3) of the game without unions, is the additional influence from the union peers of player i on her effort. In particular, this influence is non-zero when the union peers have directed links towards the player.

The following proposition provides the Nash equilibrium efforts of players in the network spillover game with unions.

Proposition 3.3. *Assume that the spectral radius of* $[\boldsymbol{\phi} \otimes \mathbf{G}^* + \mathbf{H}]$ *is smaller than 1. Then, the unique Nash equilibrium in pure strategies is given by*

$$\mathbf{x} = (\mathbf{I} - [\boldsymbol{\phi} \otimes \mathbf{G}^* + \mathbf{H}])^{-1} \boldsymbol{\alpha}. \tag{3.8}$$

[62] In matrix terms $\mathbf{H} = \boldsymbol{\Gamma} \otimes \mathbf{G}^{*T}$ with $\boldsymbol{\Gamma}$ being a matrix of parameters γ_{ij} and $\otimes$ an element-by-element multiplication of the two matrices.

We can consider Proposition 3.3 as a generalization of the equilibrium efforts of the game without unions provided in Proposition 3.1. In other words, Proposition 3.1 provides conditions and equilibrium efforts for the special case of Proposition 3.3 where $p(i) = \{i\}$ for all $i \in N$, thus $S = N$, in which case equation (3.4) is equivalent to (3.8).

Given the strategic complementarities in efforts of players in the network game it is easy to see that an increase in efforts of players through formation of a union positively affects the effort of the rest in the network. The following proposition establishes the presence of strategic complementarity in efforts in the game with unions.

Proposition 3.4. *An increase in effort x_i of any player $i \in N$ in the network with unions P, weakly increases the efforts of all players in the network.*

While the efforts of players in the network game with unions are strategic complements, the effort of a single player does not always increase by joining a union. The following lemma provides a sufficient condition for the efforts of players and the aggregate effort in the network with unions being higher than those of the network with only single players.

Lemma 3.2. *Consider a network $\boldsymbol{g}$ and it's weighted adjacency representation $\mathbf{G}^*$, with ideal efforts y_i and type coefficients θ_i for all players $i \in N$. Let $l \in p(i) \setminus i$ be such that*

$$l = \arg\min_{k \in p(i) \setminus i} (\lambda_2 \theta_k + \omega_1) \frac{g^*_{ki}}{g^*_{ik}}.$$

If the condition below holds, it is then sufficient to say that the union $p(i)$ increases the equilibrium effort of player i.

$$\frac{\omega_1 \sum\limits_{j \in p(i) \setminus i} g^*_{ji}}{\omega_1 + \omega_2 + \omega_3} \leq \min \left\{ \frac{(\lambda_2 \theta_l + \omega_1) g^*_{li}}{(\lambda_2 \theta_i + \omega_1) g^*_{il}}, \frac{\lambda_1 \sum\limits_{j \in p(i) \setminus i} \theta_j}{\lambda_1 \theta_i + \omega_3 y_i} \right\}$$

Moreover, if the condition above is satisfied for all $i \in N \setminus S$, then the union set P increases the aggregate effort in the network with respect to the game without unions.

Being in a union increases player i's utility through global spillovers. At the same time, the local spillovers and the social dissonance of union peers that are influences by player i (all $j \in p(i) \setminus i$ such that $g^*_{ji} > 0$) now affect player i's choice of effort directly. Thus, the union $p(i)$ increases the effort of player i if the positive spillovers received from the union outweigh the cost of conflict created by her participation.

As a direct consequence of the condition in Lemma 3.2, we can say that the intercentrality of player i in Proposition 3.2 (for the vector $\boldsymbol{b}$ and the matrix $\mathbf{M}$ redefined according to network game *with* unions and presented in upcoming section) is increasing with union $p(i)$ with respect to being a singleton.

3.3.2 Key Players in the Network with Unions

We search for the positive and negative key players in the network with unions, i.e. the players that have the highest and the lowest impact on the aggregate effort in the game regardless of their affiliation to any union in the network. Using the results of Proposition 3.3 we generalize the definition of the matrix $\mathbf{M}$ introduced earlier. Let $\mathbf{M}(\mathbf{G}^*, \boldsymbol{\phi}, \mathbf{H}) := (\mathbf{I} - [\boldsymbol{\phi} \otimes \mathbf{G}^* + \mathbf{H}])^{-1}$ be well defined and non-negative with entries m_{ij}, $i, j \in N$. Then the vector of weighted Katz-Bonacich centralities of players is $\boldsymbol{b}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) := \boldsymbol{M}(\mathbf{G}^*, \boldsymbol{\phi}, \mathbf{H})\boldsymbol{\alpha}$. The effort of each player i is then given by $b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \sum_{k\in N} m_{ik}\alpha_k$.

Similarly to the game without unions let $\mathcal{B}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})$ denote the total effort of all players $\mathcal{B}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \sum_{j\in N} b_j(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})$. Next, we redefine the contribution of player i to the total effort in the network game with unions.

Definition 3.3. *The contribution $\delta_{i,P}$ of player i to the total effort in the network with set of unions P is the difference between the aggregate effort exerted in the initial game $(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})$ and the game $(\mathbf{G}^{*-i}, \mathbf{H}^{-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i})$ after removing player i from the network:*

$$\delta_{i,P}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \mathcal{B}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) - \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i}).$$

By removing player i from the network $\boldsymbol{g}$ we automatically assume her being removed from the union $p(i)$. Given that, the matrix $\mathbf{H}^{-i}$ and the vectors $\boldsymbol{\alpha}^{-i}$ and $\boldsymbol{\phi}^{-i}$ are constructed by reevaluating the respective entries of $\mathbf{H}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ through the change in parameters Θ_j and Ω_j in (3.6) and the weighted adjacency matrix $\mathbf{G}^{*-i}$ of the network $\boldsymbol{g}^{-i}$ for each $j \in N \setminus i$. Accordingly, we define $\mathbf{M}^{-i} = (\mathbf{I}_{n-1} - [\boldsymbol{\phi}^{-i} \otimes \mathbf{G}^{*-i}] - \mathbf{H}^{-i})^{-1}$ with new entries m_{ij}^{-i}. Using the above mentioned, we can find the relation of the player's contribution to the aggregate effort with her intercentrality and the additional effect resulting from her membership in a union. Player's *union intercentrality* is the combination of her intercentrality and a union effect.

Proposition 3.5. *The contribution of player i to the aggregate effort in the network* *with set of unions P is given by her union intercentrality*

$$\delta_{i,P}(\mathbf{G}^*,\mathbf{H},\boldsymbol{\phi},\boldsymbol{\alpha}) = \underbrace{\sum_{k\in p(i)\setminus i}\sum_{j\in N} m_{jk}^{-i}(\alpha_k-\alpha_k^{-i})}_{\textit{Union effect}} + \underbrace{\frac{b_i(\mathbf{G}^*,\mathbf{H},\boldsymbol{\phi},\boldsymbol{\alpha})}{m_{ii}}\sum_{j\in N} m_{ji}}_{\textit{Intercentrality}}$$

or

$$\delta_{i,P}(\mathbf{G}^*,\mathbf{H},\boldsymbol{\phi},\boldsymbol{\alpha}) = \sum_{k\in p(i)\setminus i}\sum_{j\in N} m_{jk}^{-i}(\alpha_k-\alpha_k^{-i}) + \delta_i(\mathbf{G}^*,\mathbf{H},\boldsymbol{\phi},\boldsymbol{\alpha}). \tag{3.9}$$

As part of her union intercentrality, the membership of player i in union $p(i)$ induces a *union effect* on her union peers through additional splillovers.[63] Given a non-negative matrix $\mathbf{M}^{-i}$, if $\alpha_k \geq \alpha_k^{-i}$ for all $k \in p(i)$, it is sufficient to say that the union effect in player's intercentrality is positive. Using the definition of α_k in (3.6) we can find a condition for positive union effect in player's intercentrality. The following lemma provides this condition.

Lemma 3.3. *For the union effect in union intercentrality of player i to be positive, it is sufficient to have either of the following conditions satisfied for all $i \in N \setminus S$ and $k \in p(i) \setminus i$:*

$$g_{ik}^* = 0 \qquad \textit{or} \qquad \frac{\lambda_1\theta_k}{\omega_1 g_{ik}^*} \geq \frac{\lambda_1\Theta_k + \omega_3 y_k - \lambda_1\theta_i}{\Omega_k - \omega_1 g_{ik}^*}.$$

One may notice that the union effect in $\delta_{i,P}(\mathbf{G}^*,\mathbf{H},\boldsymbol{\phi},\boldsymbol{\alpha})$ disappears if player i is a single player, $i \in S$. Which means that the definition of union intercentrality for the single player in the network with unions is equivalent to her intercentrality (or key player centrality) in the network without unions.

Corollary 3.3. *The contribution of single player $j \in S$ to the aggregate effort in the network game with unions P is given by her (union) intercentrality $\delta_{j,P} = \delta_j$,*

$$\delta_{j,P}(\mathbf{G}^*,\mathbf{H},\boldsymbol{\phi},\boldsymbol{\alpha}) = \frac{b_j(\mathbf{G}^*,\mathbf{H},\boldsymbol{\phi},\boldsymbol{\alpha})}{m_{jj}}\sum_{k\in N} m_{kj}$$

[63] Throughout the paper we assume the network $\boldsymbol{g}$ to be represented by weighted adjacency matrix such that the weights on outgoing links of each player i sum up to 1, i.e. $d_i^+(\boldsymbol{g}) = \sum_{j=1}^{n} g_{ij} = 1$. Changing this assumption affects the parameters $\alpha_i, \phi_i, \gamma_{ij}$ through the cost Ω_i which will then be as follows: $\Omega_i = \omega_1(d_i^+(\boldsymbol{g}) + \sum_{j\in p(i)\setminus i} g_{ji}) + \omega_2 + \omega_3$.
The change in Ω_i entails a change in the definition of union intercentrality and with an additional "neighborhood effect" from the set of neighbors $N(i)$ of player i along with the union effect:

$$\delta_{i,P}(\mathbf{G}^*,\mathbf{H},\boldsymbol{\phi},\boldsymbol{\alpha}) = \sum_{\substack{k\in p(i)\cup N(i)\\ k\neq i}}\sum_{j\in N} m_{jk}^{-i}(\alpha_k-\alpha_k^{-i}) + \delta_i(\mathbf{G}^*,\mathbf{H},\boldsymbol{\phi},\boldsymbol{\alpha}).$$

for all $j \in S$.

Using the definitions of the union intercentrality for players in the game with unions, we can now define the key players in the game.

Definition 3.4. *The positive key player i^*_+ is the player with the highest union intercentrality in the network:*

$$i^*_+ = \arg\max_{j \in N} \delta_{j,P}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}).$$

The negative key player i^*_- is the player with the lowest union intercentrality:

$$i^*_- = \arg\min_{j \in N} \delta_{j,P}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}).$$

Example 3.4. ***No unions***
Consider a game on the bridge network in Figure 3.1b. Let the coefficients of the game be as follows: $\lambda_1 = 0.5$, $\lambda_2 = 0.125$, $\omega_1 = 0.125$, $\omega_2 = 0.5$, $\omega_3 = 0.125$. We assume the ideal efforts y_i, $i \in N$ to be homogeneous and equal to 1*. The return coefficients θ_i are randomly drawn from the range* $[0.5; 1.5]$ *and displayed in Figure 3.2.*

The aggregate effort $\mathcal{B}$ of the game where all players are single is 14.8. The positive key player in the network is player 2 with type $\theta_2 = 1.5$, effort $b_2 = 1.78$ and intercentrality $\delta_2 = 2.07$. The negative key player of the game is player 7 with type $\theta_7 = 0.5$, effort $b_7 = 0.82$ and intercentrality $\delta_7 = 0.57$. The list of all efforts and contributions of each player is displayed on Table 3.3.

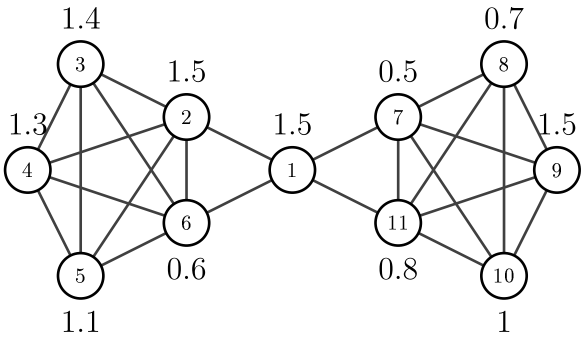


Figure 3.2: Bridge network with idiosyncratic coefficients θ_i of players $i \in \{1, ..., 11\}$ discussed in Examples 3.4-3.7.

Example 3.5. ***Union*** $\{1, 2\}$
Following the setting in the Example 3.4 we introduces the union $p(1)$ consisting of

*players 1 and 2 of the network **g**. The rest of the players in the network remain singletons, $S = \{3, 4, \ldots, 11\}$. So the set of unions is $P = \{\{1, 2\}, \{3\}, \{4\}, \ldots, \{11\}\}$.*

The total effort exerted in the network game with the union $\{1, 2\}$ increases to 18.41 compared to 14.8 in the no union case. The contribution of the union members to the total effort is much higher, with player 2 remaining the positive key player of the game with higher effort $b_2 = 3.13$ and union intercentrality $\delta_{2,P} = 5.68$. The negative key player of the game with union $\{1, 2\}$ is now player 6 with effort $b_6 = 1.17$ and union intercentrality $\delta_{6,P} = 0.3$. The list of efforts and contributions for each player is displayed on Table 3.3.

Table 3.3: Equilibrium efforts and contributions of players in the network in Figure 3.2, with no unions (Example 3.4), and union $\{1, 2\}$ (Example 3.5) and union $\{1, 2, 3, 4, 5\}$ (Example 3.7). The contributions of players in Examples 3.4 with no unions are measured by intercentralities δ_i, and by union intercentralities $\delta_{i,P}$ when a union exists in the network.

		No Unions		**Union {1,2}**		**Union {1,2,3,4,5}**	
Player	**Type**	**Effort**	**Contrib.**	**Effort**	**Contrib.**	**Effort**	**Contrib.**
i	θ_i	b_i	δ_i	b_i	$\delta_{i,P}$	b_i	$\delta_{i,P}$
1	1.5	1.65	1.76	2.99	5.37	6.81	15.51
2	1.5	1.78	2.07	3.13	5.68	9.44	20.43
3	1.4	1.68	1.78	1.87	1.72	9.09	18.24
4	1.3	1.59	1.65	1.78	1.57	9.02	17.44
5	1.1	1.43	1.38	1.6	1.27	8.89	15.77
6	0.6	1	0.59	1.17	0.3	2.87	-7.74
7	0.5	0.82	0.57	0.9	0.43	1.11	0.03
8	0.7	0.96	0.86	0.98	0.8	1.02	0.63
9	1.5	1.59	1.78	1.61	1.74	1.67	1.6
10	1	1.2	1.22	1.22	1.17	1.27	1.02
11	0.8	1.07	1.02	1.16	0.93	1.41	0.62
$\mathcal{B}$		**14.8**		**18.41**		**52.6**	

3.3.3 Key Unions

In social and economic problems seeking partnership, and looking for best options in forming unions in a given network is a commonly addressed issue. When addressing the issue of increasing the aggregate efforts in the population, finding the key player in the network helps to identify the most influential player affecting the total effort the most. Given a network structure, our aim is to find the unions of players, as groups that maximize their joint utility, that affect the total efforts in the network game the most.

Assume a set of unions $\bar{P}$ in the network $\boldsymbol{g}$ formed by joining a single players i to the union p of initial union structure P, such that $\bar{p} = p \cup \{i\}$.

Proposition 3.6. *The change in aggregate effort in the network through the addition of a single player $i \in S$ to the union p, such that $\bar{p} = p \cup \{i\}$, is given by the difference of player's union intercentralities for union structures $\bar{P}$ and P in the network:*

$$\sigma_i^p = \underbrace{\delta_{i,\bar{P}}(\mathbf{G}^*, \mathbf{H}_{i\in\bar{p}}, \boldsymbol{\phi}_{i\in\bar{p}}, \boldsymbol{\alpha}_{i\in\bar{p}})}_{\substack{\textit{union intercentrality} \\ \textit{when } i \in \bar{p}}} - \underbrace{\delta_{i,P}(\mathbf{G}^*, \mathbf{H}_{i\in S}, \boldsymbol{\phi}_{i\in S}, \boldsymbol{\alpha}_{i\in S})}_{\substack{\textit{(union) intercentrality} \\ \textit{when } i \textit{ is a single player}}}$$

We call σ_i^p the *union-induced intercentrality* of player i from joining the union p.

Using the union intercentrality definition in (3.9) and that of a single player in the network with unions in Corollary 3.3 we can rewrite σ_i^p in the following way.

$$\sigma_i^p = \underbrace{\sum_{k\in\bar{p}\setminus i}\sum_{j\in N} m_{jk}^{-i}(\alpha_k - \alpha_k^{-i})}_{\text{Union effect}} + \underbrace{\delta_i(\mathbf{G}^*, \mathbf{H}_{i\in\bar{p}}, \boldsymbol{\phi}_{i\in\bar{p}}, \boldsymbol{\alpha}_{i\in\bar{p}})}_{\substack{\text{intercentrality of } i \\ \text{when } i \in \bar{p}}} - \underbrace{\delta_i(\mathbf{G}^*, \mathbf{H}_{i\in S}, \boldsymbol{\phi}_{i\in S}, \boldsymbol{\alpha}_{i\in S})}_{\substack{\text{(union) intercentrality of } i \\ \text{when } i \in S}}$$

If the condition of Lemma 3.2 holds the difference of player's intercentrality when it belongs to a union is greater than that of the same player when she is single. So we have $\delta_i(\text{when } i \in \bar{p}) - \delta_i(\text{when } i \in S) \geq 0$ for $\bar{p} = p \cup \{i\}$, such that $\{i\}, p \in P$ and $\bar{p} \in \bar{P}$. In addition, Lemma 3.3 provides conditions for the union effect of player i to be positive. It is then straightforward to see that $\sigma_i^p \geq 0$ for all players that satisfy the conditions in Lemma 3.2 and Lemma 3.3. In other words, if the union $p \in P$ increases the intercentrality of player i, and union effect of player i is positive, then the *union-induced intercentrality* σ_i^p of the player is positive.

Using the definition of the union-induced intercentrality we can find the *key addition*, the player that has the highest impact on the aggregate effort in the network when added to a given union.

Definition 3.5. *The key addition to a given union $p \in P$ is player $i \notin p$ that has the highest union-induced intercentrality when added to the union,*

$$i = \arg\max_{j\notin p} \sigma_j^p.$$

Given a network with set of unions P, the key addition will induce the highest increase in aggregate effort in the game when added to a given union. The key addition to a single player $i \in S$ is player $j \in S$ with highest impact on the aggregate effort when the union $\{i, j\}$ is formed. We call the union of these initially single players a *key union* of two, when it induces the highest increase in total effort in the network among all other possible pairs.

Definition 3.6. *Assume a network* $\mathbf{g}$ *with set of unions* P*, such that there exists a set of single players* $|S| > 2$*. The key union with two players* i, j *is the one that solves*

$$\max_{i,j \in S} \sigma_i^{\{j\}}.$$

To generalize Definition 3.6, we can find the key union of any size $z < |S|$, given a network with initial set of unions P including a set of single players S.

Definition 3.7. *Assume a network* $\mathbf{g}$ *with set of unions* P*, such that there exists a set of single players with* $|S| > z$*. The key union of* z *players* $i_1, \ldots, i_z$ *is the one that solves*

$$\max_{i_1,\ldots,i_z \in S} \sigma_{i_2}^{\{i_1\}} + \sigma_{i_3}^{\{i_1,i_2\}} + \ldots + \sigma_{i_z}^{\{i_1,\ldots,i_{z-1}\}}.$$

The above Definitions 3.5, 3.6, 3.7 together suggest that the key addition to a given union, which in turn is the key union of its' size $z < n$, does not necessarily result in key union of size $z + 1$. This result is illustrated in the following example.

Example 3.6. ***Key Addition and Key Union***
Consider the network in Figure 3.2 with one union in the set P*. Given an opportunity to expand the union by one member we find the best member to add, the key addition. Moreover, we can find the key union of a given size, that is, the union that provides the highest total effort compared to other same-size unions.*

In Example 3.5 we discussed the union $p(1)$ consisting of players 1 and 2 in the network. Following Definition 3.5 we find player i with the highest union-induced intercentrality $\sigma_i^{\{1,2\}}$, which is player 3 with $\sigma_3^{\{1,2\}} = 6.91$. By definition, this player assures the highest possible increase in the aggregate effort when added to the union $\{1, 2\}$, which is then equal to 25.32. Table 3.4 provides the aggregate efforts and union-induced intercentralities of players in S from joining the union $\{1, 2\}$.

It is worth noting that the union of 1 and 2 is the key union of size two, since it satisfies Definition 3.6 with $\sigma_1^{\{2\}} = \sigma_2^{\{1\}} = 3.60$. We can acknowledge that when gradually expanding the union size up to three members by finding the key additions, players 1, 2 and 3 satisfy the following condition:

$$\max_{i_1,i_2} \sigma_{i_2}^{\{i_1\}} + \max_{i_3} \sigma_{i_3}^{\{i_1^*,i_2^*\}} = \sigma_2^{\{1\}} + \sigma_3^{\{1,2\}} = 10.51.$$

On the other hand, given an opportunity to form a union of size three directly, one can find the key union of size three that satisfies Definition 3.7. By doing so, we find that players 2, 3 and 4 form the union of the size with highest aggregate effort:

$$\max_{i_1,i_2,i_3} (\sigma_{i_2}^{\{i_1\}} + \sigma_{i_3}^{\{i_1,i_2\}}) = \sigma_2^{\{3\}} + \sigma_4^{\{2,3\}} = \sigma_2^{\{4\}} + \sigma_3^{\{2,4\}} = \sigma_3^{\{4\}} + \sigma_2^{\{3,4\}} = 11.19.$$

Indeed, the aggregate effort in the network game with union $\{2, 3, 4\}$ is 25.99 which exceeds the $\mathcal{B} = 25.32$ of the game with the union $\{1, 2, 3\}$. Thus, the key union of size 3 is the union $\{2, 3, 4\}$.

Table 3.4: Union-induced intercentralities of players $i \in \{3, ..., 11\}$ and the aggregate efforts in the network with a union $\{1, 2, i\}$ in Figure 3.2, Example 3.6.

Player i	**Union**	$\mathcal{B}$	$\sigma_i^{\{1,2\}}$
	{1, 2}	18.41	
3	{1, 2, 3}	25.32	6.91
4	{1, 2, 4}	25.05	6.65
5	{1, 2, 5}	24.54	6.14
6	{1, 2, 6}	24.22	5.81
7	{1, 2, 7}	23.21	4.81
8	{1, 2, 8}	22.82	4.41
9	{1, 2, 9}	24.49	6.08
10	{1, 2, 10}	23.45	5.04
11	{1, 2, 11}	23.93	5.52

Example 3.7. ***Union*** $\{1, 2, 3, 4, 5\}$
Consider now the union of players $\{1, 2, 3, 4, 5\}$*, with the rest being single players in the network.*

As shown in Table 3.3, the negative key player of the game is player 6. The positive externalities from the neighboring union increase the incentives of player 6 to free-ride, increasing the conflict between the neighbors. Moreover, the union intercentrality of player 6 is negative. That is, removing player 6 from the network will increase the aggregate effort in the population. On the contrary, the union-induced intercentrality of player 6 is the highest among other single players (see Table 3.5). The latter means that, while removing the player from the network would increase the aggregate effort to 60.34, adding the player to the union $\{1, 2, 3, 4, 5\}$ will increase it even more.

Table 3.5: Union-induced intercentralities of players $i \in \{6, ..., 11\}$ and the aggregate efforts in the network with a union $\{1, 2, 3, 4, 5, i\}$ in Figure 3.2, Example 3.7.

Player i	**Union**	$\mathcal{B}$	$\sigma_i^{\{1,2,3,4,5\}}$
	{1, 2, 3, 4, 5}	52.6	
6	{1, 2, 3, 4, 5, 6}	72.95	20.35
7	{1, 2, 3, 4, 5, 7}	64.26	11.66
8	{1, 2, 3, 4, 5, 8}	63.69	11.09
9	{1, 2, 3, 4, 5, 9}	68.7	16.1
10	{1, 2, 3, 4, 5, 10}	65.57	12.97
11	{1, 2, 3, 4, 5, 11}	66.28	13.68

In the example of virus transmission, with the efforts in vaccination and precautionary measures, such as social distancing or wearing masks, one may think of a group of individuals connected in a network similar to one in Figure 3.2. People benefit from being surrounded by others that put high efforts in preventive measures and vaccination. This positive spillovers result in relative safety against the infection and may reduce the incentives of an individual to vaccinate when considering the individualistic approach of player's utility maximization, such as in case of player 6 in Example 3.7. Moreover, such behavior can affect the aggregated outcomes in the network negatively. On the contrary, consideration of the joint benefit in fighting the virus, mitigates the free-riding incentives, increasing the efforts in the network through spillovers.

3.4 Conclusions

Social interactions and the structure of social networks have an important role in economic outcomes. The effect of local interactions had been studied in various settings, such as education, criminal networks or pro-environmental behavior. Recent works on network games have also studied the interplay of conformism or miscoordination with the social norm, and preference for consistency with personal ideal behavior (Olcina et al. (2017), Galeotti et al. (2021)). In this paper, we study a network game where the social influence and the preference for consistency affect the effort choice of the players through spillovers from the network, conflict with neighbors, and a private dissonance from inconsistency with own ideals. We further generalize the model by defining unions in the network.

In the network of single players, the players choose their efforts based on the return on their own effort, as well as the efforts of their network neighbors and the overall performance in the game, the global spillovers. They experience disutility from the differences in behavior with their neighbors and when the effort is inconsistent with their ideal effort resulting private dissonance. We find that the equilibrium efforts of the network game are proportional to the weighted Katz-Bonacich centralities of the players. We show that the personal characteristics of the player, such as idiosyncratic return and ideal effort, affect her intercentrality, together with the position in the network. Therefore, we redefine the problem of finding the key players in the network.

Using the idea of interest group formation and collective action we extend the network game, allowing for unions in the network that work towards achieving a common objective. Similar to cartels, the players in a union maximize their joint utility. We characterize the Nash equilibrium of the network game with unions and find the key players. The contribution of the player to the aggregate effort in the network with unions is complemented by the union effect. Accordingly, we introduce union intercentrality as a measure capturing the key players in the network with unions. We show that the characterized Nash equilibria, union intercentrality, and the problem of finding the key players in the network with unions are generalizations of the corresponding definitions of the game with single players only discussed in Section 3.2. We discuss policies to increase the total effort in the population of players by forming unions of a fixed size, and by adding players to a given union. To do so, union-induced intercentrality is defined in order to find the key addition to a union, and the key union of a given size is defined as the union ensuring the highest increase in the aggregate effort. We show on an example that while the key-player policies target the removal of the negative key player from the network, the union formation approach suggests that adding such player to a given union may have a stronger effect on the collective outcomes in the population. The study of the unions in the network allows us to find the best outcomes for the aggregate effort in the games where cooperation is possible. Yet, the union stability and welfare analysis with consideration of a fair utility allocation rule is left for future research.

Appendix 3.A Proofs

Proof of Proposition 3.1. Given x_i^* is the optimal effort choice for i

$$\frac{\partial u_i}{\partial x_i}(x_i^*) \equiv 0$$

$$\begin{aligned}
\frac{\partial u_i(x)}{\partial x_i} &= \lambda_1\theta_i + \lambda_2\theta_i \sum_{j\in N} g_{ij}^* x_j - \omega_1 \sum_{j\in N} g_{ij}^*(x_i - x_j) - \omega_2 x_i - \omega_3(x_i - y_i) = \\
&= \lambda_1\theta_i + \lambda_2\theta_i \sum_{j\in N} g_{ij}^* x_j - \omega_1 x_i \underbrace{\sum_{j\in N} g_{ij}^*}_{=1} + \omega_1 \sum_{j\in N} g_{ij}^* x_j - \omega_2 x_i - \omega_3 x_i + \omega_3 y_i = \\
&= \lambda_1\theta_i + \omega_3 y_i + (\lambda_2\theta_i + \omega_1) \sum_{j\in N} g_{ij}^* x_j - (\omega_1 + \omega_2 + \omega_3) x_i
\end{aligned}$$

Thus the equilibrium effort choice of i is:

$$x_i^* = \frac{\lambda_1\theta_i + \omega_3 y_i}{(\omega_1 + \omega_2 + \omega_3)} + \frac{\lambda_2\theta_i + \omega_1}{(\omega_1 + \omega_2 + \omega_3)} \sum_{j\in N} g_{ij}^* x_j,$$

for all $i \in N$. We can rewrite it as:

$$x_i^* = \alpha_i + \phi_i \sum_{j\in N} g_{ij}^* x_j, \tag{3.10}$$

where α_i and ϕ_i are defined in (3.2).

Matrix form solution of the equilibrium efforts is the following:

$$\mathbf{x} = \boldsymbol{\alpha} + \boldsymbol{\phi} \otimes \mathbf{G}^*\mathbf{x} \quad \Leftrightarrow \quad (\mathbf{I} - \boldsymbol{\phi} \otimes \mathbf{G}^*)\mathbf{x} = \boldsymbol{\alpha} \quad \Leftrightarrow$$

$$\mathbf{x} = (\mathbf{I} - \boldsymbol{\phi} \otimes \mathbf{G}^*)^{-1}\boldsymbol{\alpha}$$

□

Proof of Proposition 3.2.

$$\begin{aligned}
&\mathcal{B}(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha}) - \mathcal{B}(\mathbf{G}^{*-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i}) \\
&= \sum_{j\in N}\sum_{k\in N}(m_{jk}\alpha_k - m_{jk}^{-i}\alpha_k^{-i}) \\
&= \sum_{j\in N}\sum_{k\in N}(m_{jk}\alpha_k - m_{jk}^{-i}\alpha_k + m_{jk}^{-i}\alpha_k - m_{jk}^{-i}\alpha_k^{-i}) \\
&= \sum_{j\in N}\sum_{k\in N}\underbrace{(m_{jk} - m_{jk}^{-i})}_{*\text{ Lemma 3.4}}\alpha_k + \underbrace{\sum_{j\in N}\sum_{k\in N} m_{jk}^{-i}(\alpha_k - \alpha_k^{-i})}_{=0} \\
&= \sum_{j\in N}\sum_{k\in N}\frac{m_{ji}m_{ik}}{m_{ii}}\alpha_k = \sum_{j\in N}\frac{m_{ji}}{m_{ii}}\sum_{k\in N} m_{ik}\alpha_k \\
&= \frac{b_i(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha})}{m_{ii}}\sum_{j\in N} m_{ji} = \delta_i(\mathbf{G}^*, \boldsymbol{\phi}, \boldsymbol{\alpha})
\end{aligned}$$

* with $\boldsymbol{\Phi}(\boldsymbol{g}) = [\boldsymbol{\phi} \otimes \mathbf{G}^*]$ in Lemma 3.4. □

Lemma 3.4 (Adaptation of Lemma 1 in Ballester et al. (2006))**.** *Let* $\boldsymbol{M}(\boldsymbol{\Phi}(\boldsymbol{g})) = [\mathbf{I} - \Phi(\boldsymbol{g})]^{-1}$ *be well defined and nonnegative. Then* $m_{ji}(\boldsymbol{\Phi}(\boldsymbol{g}))m_{ik}(\boldsymbol{\Phi}(\boldsymbol{g})) = m_{ii}(\boldsymbol{\Phi}(\boldsymbol{g}))[m_{jk}(\boldsymbol{\Phi}(\boldsymbol{g})) - m_{jk}(\boldsymbol{\Phi}(\boldsymbol{g}^{-i}))]$ *for all* $k \neq i \neq j$*.*

Proof. Let f_{ij} with $i, j \in N$, be the elements of the matrix $\boldsymbol{\Phi}(\boldsymbol{g})$ defined on the weighted adjacency matrix of the network $\boldsymbol{g}$, and similarly f_{kj}^{-i} the elements of the matrix $\boldsymbol{\Phi}(\boldsymbol{g}^{-i})$ defined on the weighted adjacency matrix of the network $\boldsymbol{g}^{-i}$ formed by removing player i from $\boldsymbol{g}$.

$$\begin{aligned}
m_{ii}(\boldsymbol{\Phi}(\boldsymbol{g}))[m_{jk}(\boldsymbol{\Phi}(\boldsymbol{g})) - m_{jk}(\boldsymbol{\Phi}(\boldsymbol{g}^{-i}))] &= \sum_{s=1}^{\infty} f_{ii}^{[s]} \sum_{r=0}^{\infty}\left(f_{jk}^{[r]} - f_{jk}^{-i[r]}\right) = \\
&= \sum_{s=1}^{\infty} f_{ii}^{[s]} \sum_{r=0}^{\infty} f_{j(i)k}^{[r]} = \sum_{s=1}^{\infty}\sum_{r=0}^{\infty} f_{ii}^{[s]} f_{j(i)k}^{[r]} \\
&= \sum_{\substack{s=1\\ s'\geq 1}}^{\infty}\sum_{\substack{r=0\\ r'\geq 1}}^{\infty} f_{ji}^{[r-r']} f_{ii}^{[s-s']} f_{ii}^{[s']} f_{ik}^{[r']} \\
&= \sum_{p=1}^{\infty}\sum_{p'\geq 1} f_{ji}^{[p-p']} f_{ik}^{[p']} = m_{ji}m_{ik}
\end{aligned}$$

where $f_{j(i)k}^{[r]}$ is the weighted number of the r-step paths from j to k passing throught the node i, it is the (i, j)th element of the r-power of matrix $\boldsymbol{\Phi}(\boldsymbol{g})$ associated to the Bonacich centrality of player j. □

Proof of Proposition 3.3. We first find the best response of player $i \in p(i)$ for any $p(i) \in P$ from the first order conditions on the joint utility in (3.5).

$$\begin{aligned}\frac{\partial U_{p(i)}(x)}{\partial x_i} =&\lambda_1\theta_i + \omega_3 y_i + (\lambda_2\theta_i + \omega_1)\sum_{k\in N} g^*_{ik}x_k - (\omega_1+\omega_2+\omega_3)x_i + \\ &+ \sum_{j\in p(i)\setminus i}\Big(\lambda_1\theta_j + (\lambda_2\theta_j+\omega_1)g^*_{ji}x_j + \omega_1 g^*_{ji}x_i\Big) = \\ =&\lambda_1\sum_{j\in p(i)}\theta_j + \omega_3 y_i + (\lambda_2\theta_i+\omega_1)\sum_{k\in N} g^*_{ik}x_k + \sum_{j\in p(i)\setminus i}(\lambda_2\theta_j+\omega_1)g^*_{ji}x_j + \\ &-(\omega_1(1+\sum_{j\in p(i)\setminus i} g^*_{ji}) + \omega_2+\omega_3)x_i\end{aligned}$$

$$\frac{\partial U_{p(i)}(x)}{\partial x_i} \equiv 0$$

$$(\omega_1(1+\sum_{j\in p(i)\setminus i} g^*_{ji})+\omega_2+\omega_3)x^*_i = \lambda_1\sum_{j\in p(i)}\theta_j+\omega_3 y_i+(\lambda_2\theta_i+\omega_1)\sum_{k\in N} g^*_{ik}x_k+\sum_{j\in p(i)\setminus i}(\lambda_2\theta_j+\omega_1)g^*_{ji}x_j$$

Using the notations introduced in equations (3.6) and (3.7), x^*_i becomes:

$$x^*_i = \frac{\lambda_1\Theta_i + \omega_3 y_i}{\Omega_i} + \frac{(\lambda_2\theta_i+\omega_1)}{\Omega_i}\sum_{k\in N} g^*_{ik}x_k + \sum_{j\in p(i)\setminus i}\frac{\lambda_2\theta_j+\omega_1}{\Omega_i}g^*_{ji}x_j$$

$$x^*_i = \alpha_i + \phi_i\sum_{k\in N} g^*_{ik}x_k + \sum_{j\in p(i)\setminus i}\gamma_{ij}g^*_{ji}x_j$$

$$x^*_i = \alpha_i + \phi_i\sum_{k\in N} g^*_{ik}x_k + \sum_{k\in N} h_{ik}x_k \tag{3.11}$$

With the following matrix form representation.

$$\mathbf{x} = \boldsymbol{\alpha} + [\boldsymbol{\phi}\otimes\mathbf{G}^*]\mathbf{x} + \mathbf{H}\mathbf{x} \quad\Leftrightarrow\quad (\mathbf{I}-[\boldsymbol{\phi}\otimes\mathbf{G}^*]-\mathbf{H})\mathbf{x} = \boldsymbol{\alpha} \quad\Leftrightarrow$$

$$\mathbf{x} = (\mathbf{I}-[\boldsymbol{\phi}\otimes\mathbf{G}^*]-\mathbf{H})^{-1}\boldsymbol{\alpha}$$

Note, that when $p(i)=\{i\}$ equation (3.11) is equivalent to (3.10), and as a consequence equation (3.4) is equivalent to (3.8).

□

Proof of Proposition 3.4. Let $v = \left(\frac{\partial x_1}{\partial x_i},\ldots,\frac{\partial x_n}{\partial x_i}\right)$ denote the vector of derivatives of the effort of the players with respect to x_i. From $\mathbf{x} = \boldsymbol{\alpha} + [\boldsymbol{\phi}\otimes\mathbf{G}^* + \mathbf{H}]\mathbf{x}$ and the fact that α and ϕ are independent from x_i, it follows $\mathbf{v} = [\boldsymbol{\phi}\otimes\mathbf{G}^* + \mathbf{H}]\mathbf{v}$. Solving for v

yields: $\mathbf{v} = (\mathbf{I} - [\boldsymbol{\phi} \otimes \mathbf{G}^*] - \mathbf{H})^{-1}\mathbf{1}$ where $\mathbf{1}$ is $n \times 1$ vector of 1's.
Under our assumption that the spectral radius of $[\boldsymbol{\phi} \otimes \mathbf{G}^*] + \mathbf{H}$ is smaller than 1, the matrix $(\mathbf{I} - [\boldsymbol{\phi} \otimes \mathbf{G}^*] - \mathbf{H})^{-1}$ is non-negative. It then follows that the entries of v are also non-negative.

□

Proof of Lemma 3.2. To avoid confusion in notations let's name the parameters in (3.2) as α_i^S and ϕ_i^S, to indicate the respective parameters of a single player $i \in S$. The best response of the player in the network without unions is then:

$$x_i^S = \alpha_i^S + \phi_i^S \sum_{j \in N} g_{ij}^* x_j.$$

Recall the best response of the player in the network with set of unions P with $i \in p(i)$:

$$x_i = \alpha_i + \phi_i \sum_{k \in N} g_{ik}^* x_k + \sum_{k \in N} \gamma_{ik} g_{ki}^* x_k.$$

We can derive sufficient conditions to ensure $x_i \geq x_i^S$ using the two steps below.

Step 1: Find the conditions for $\phi_i^S g_{ik}^* \leq \phi_i g_{ik}^* + \gamma_{ik} g_{ki}^*$ for all $k \in p(i) \setminus i$.

$$\frac{\lambda_2\theta_i + \omega_1}{\omega_1 + \omega_2 + \omega_3} g_{ik}^* \leq \frac{\lambda_2\theta_i + \omega_1}{\omega_1(1 + \sum\limits_{j \in p(i)\setminus i} g_{ji}^*) + \omega_2 + \omega_3} g_{ik}^* + \frac{\lambda_2\theta_k + \omega_1}{\omega_1(1 + \sum\limits_{j \in p(i)\setminus i} g_{ji}^*) + \omega_2 + \omega_3} g_{ki}^*$$

$$\frac{\lambda_2\theta_i + \omega_1}{\omega_1 + \omega_2 + \omega_3} g_{ik}^* - \frac{\lambda_2\theta_i + \omega_1}{\omega_1(1 + \sum\limits_{j \in p(i)\setminus i} g_{ji}^*) + \omega_2 + \omega_3} g_{ik}^* \leq \frac{\lambda_2\theta_k + \omega_1}{\omega_1(1 + \sum\limits_{j \in p(i)\setminus i} g_{ji}^*) + \omega_2 + \omega_3} g_{ki}^*$$

$$\frac{(\lambda_2\theta_i + \omega_1) g_{ik}^*}{(\omega_1 + \omega_2 + \omega_3)} \frac{\omega_1 \sum\limits_{j \in p(i)\setminus i} g_{ji}^*}{(\omega_1(1 + \sum\limits_{j \in p(i)\setminus i} g_{ji}^*) + \omega_2 + \omega_3)} \leq \frac{\lambda_2\theta_k + \omega_1}{\omega_1(1 + \sum\limits_{j \in p(i)\setminus i} g_{ji}^*) + \omega_2 + \omega_3} g_{ki}^*$$

$$\frac{\lambda_2\theta_i + \omega_1}{\omega_1 + \omega_2 + \omega_3} g_{ik}^* \omega_1 \sum_{j \in p(i)\setminus i} g_{ji}^* \leq (\lambda_2\theta_k + \omega_1) g_{ki}^*$$

$$\frac{\lambda_2\theta_i + \omega_1}{\omega_1 + \omega_2 + \omega_3} \leq \frac{\lambda_2\theta_k + \omega_1}{\omega_1 \sum\limits_{j\in p(i)\setminus i} g^*_{ji}} \frac{g^*_{ki}}{g^*_{ik}}$$

$$\phi^S_i \leq \frac{\lambda_2\theta_j + \omega_1}{\omega_1 \sum\limits_{j\in p(i)\setminus i} g^*_{ji}} \frac{g^*_{ki}}{g^*_{ik}}$$

Restructuring the condition above we get:

$$\frac{\omega_1 \sum\limits_{j\in p(i)\setminus i} g^*_{ji}}{\omega_1 + \omega_2 + \omega_3} \leq \frac{(\lambda_2\theta_k + \omega_1)g^*_{ki}}{(\lambda_2\theta_i + \omega_1)g^*_{ik}}$$

Step 2: Find the condition for $\alpha^S_i \leq \alpha_i$ for all $i \in N \setminus S$

$$\alpha^S_i \leq \alpha_i$$

$$\frac{\lambda_1\theta_i + \omega_3 y_i}{\omega_1 + \omega_2 + \omega_3} \leq \frac{\lambda_1 \sum\limits_{j\in p(i)} \theta_j + \omega_3 y_i}{\omega_1(1 + \sum\limits_{j\in p(i)\setminus i} g^*_{ji}) + \omega_2 + \omega_3}$$

$$\frac{\lambda_1\theta_i + \omega_3 y_i}{\omega_1 + \omega_2 + \omega_3} \leq \frac{\lambda_1\theta_i + \omega_3 y_i + \lambda_1 \sum\limits_{j\in p(i)\setminus i} \theta_j}{\omega_1(1 + \sum\limits_{j\in p(i)\setminus i} g^*_{ji}) + \omega_2 + \omega_3}$$

$$\frac{\lambda_1\theta_i + \omega_3 y_i}{\omega_1 + \omega_2 + \omega_3} - \frac{\lambda_1\theta_i + \omega_3 y_i}{\omega_1(1 + \sum\limits_{j\in p(i)\setminus i} g^*_{ji}) + \omega_2 + \omega_3} \leq \frac{\lambda_1 \sum\limits_{j\in p(i)\setminus i} \theta_j}{\omega_1(1 + \sum\limits_{j\in p(i)\setminus i} g^*_{ji}) + \omega_2 + \omega_3}$$

$$\frac{(\lambda_1\theta_i + \omega_3 y_i)}{(\omega_1 + \omega_2 + \omega_3)} \frac{\omega_1 \sum\limits_{j\in p(i)\setminus i} g^*_{ji}}{(\omega_1(1 + \sum\limits_{j\in p(i)\setminus i} g^*_{ji}) + \omega_2 + \omega_3)} \leq \frac{\lambda_1 \sum\limits_{j\in p(i)\setminus i} \theta_j}{\omega_1(1 + \sum\limits_{j\in p(i)\setminus i} g^*_{ji}) + \omega_2 + \omega_3}$$

$$\frac{\lambda_1\theta_i + \omega_3 y_i}{\omega_1 + \omega_2 + \omega_3} \leq \frac{\lambda_1 \sum\limits_{j\in p(i)\setminus i} \theta_j}{\omega_1 \sum\limits_{j\in p(i)\setminus i} g^*_{ji}}$$

$$\alpha^S_i \leq \frac{\lambda_1 \sum\limits_{j\in p(i)\setminus i} \theta_j}{\omega_1 \sum\limits_{j\in p(i)\setminus i} g^*_{ji}}$$

Restructuring this condition we have the following:

$$\frac{\omega_1 \sum\limits_{j \in p(i) \setminus i} g^*_{ji}}{\omega_1 + \omega_2 + \omega_3} \leq \frac{\lambda_1 \sum\limits_{j \in p(i) \setminus i} \theta_j}{\lambda_1 \theta_i + \omega_3 y_i}$$

Using the conditions derived in the steps above we can conclude that the participation in the union $p(i)$ increases the effort of player i with respect to her effort as a single player if:

$$\frac{\omega_1 \sum\limits_{j \in p(i) \setminus i} g^*_{ji}}{\omega_1 + \omega_2 + \omega_3} \leq \frac{(\lambda_2 \theta_k + \omega_1) g^*_{ki}}{(\lambda_2 \theta_i + \omega_1) g^*_{ik}}, \text{ for all } k \in p(i) \setminus i, \text{ and } \frac{\omega_1 \sum\limits_{j \in p(i) \setminus i} g^*_{ji}}{\omega_1 + \omega_2 + \omega_3} \leq \frac{\lambda_1 \sum\limits_{j \in p(i) \setminus i} \theta_j}{\lambda_1 \theta_i + \omega_3 y_i}.$$

To simplify, assume player $l \in p(i) \setminus i$ is the player that solves the following problem:

$$l = \arg \min_{k \in p(i) \setminus i} (\lambda_2 \theta_k + \omega_1) \frac{g^*_{ki}}{g^*_{ik}}$$

Combining the two conditions we have that, for a given set of unions P and a player $i \in p(i)$, for $x_i \geq x_i^S$ it is sufficient to have the following condition hold:

$$\frac{\omega_1 \sum\limits_{j \in p(i) \setminus i} g^*_{ji}}{\omega_1 + \omega_2 + \omega_3} \leq \min \left\{ \frac{(\lambda_2 \theta_l + \omega_1) g^*_{li}}{(\lambda_2 \theta_i + \omega_1) g^*_{il}}, \frac{\lambda_1 \sum\limits_{j \in p(i) \setminus i} \theta_j}{\lambda_1 \theta_i + \omega_3 y_i} \right\}$$

The aggregate effort in the network increases with the formation of unions in P if the sufficient condition above is satisfied for all $i \in N \setminus S$. □

Proof of Proposition 3.5.

$$\delta_{i,P}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) = \mathcal{B}(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) - \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha}^{-i})$$

$$= \sum_{j\in N}\sum_{k\in N}(m_{jk}\alpha_k - m_{jk}^{-i}\alpha_k^{-i})$$

$$\stackrel{(*1)}{=} \sum_{j\in N}\sum_{k\in N\setminus p(i)}(m_{jk} - m_{jk}^{-i})\alpha_k + \sum_{j\in N}\sum_{\substack{k\in p(i)\\k\neq i}}(m_{jk}\alpha_k - m_{jk}^{-i}\alpha_k + m_{jk}^{-i}\alpha_k - m_{jk}^{-i}\alpha_k^{-i}) + \sum_{j\in N} m_{ji}\alpha_i$$

$$= \sum_{j\in N}\sum_{k\in N\setminus i}(m_{jk} - m_{jk}^{-i})\alpha_k + \sum_{j\in N} m_{ji}\alpha_i + \sum_{j\in N}\sum_{\substack{k\in p(i)\\k\neq i}} m_{jk}^{-i}(\alpha_k - \alpha_k^{-i})$$

$$\stackrel{(*2)}{=} \frac{b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})}{m_{ii}}\sum_{j\in N} m_{ji} + \sum_{j\in N}\sum_{\substack{k\in p(i)\\k\neq i}} m_{jk}^{-i}(\alpha_k - \alpha_k^{-i})$$

With:

$$(*1) : \alpha_k = \alpha_k^{-i}\ \forall\ k \notin p(i)$$

$$(*2) : \sum_{j\in N}\sum_{k\in N\setminus i}(m_{jk} - m_{jk}^{-i})\alpha_k + \sum_{j\in N} m_{ji}\alpha_i$$

$$= \sum_{j\in N} b_j(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) - \sum_{j\in N\setminus i} b_j(\mathbf{G}^{*-i}, \mathbf{H}^{-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha})$$

$$= b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) + \sum_{j\in N\setminus i} b_j(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) - \sum_{j\in N\setminus i} b_j(\mathbf{G}^{*-i}, \mathbf{H}^{-i}, \boldsymbol{\phi}^{-i}, \boldsymbol{\alpha})$$

$$= b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) + \sum_{j\in N\setminus i}\sum_{k\in N}(m_{jk}\alpha_k - m_{jk}^{-i}\alpha_k)$$

$$\stackrel{(*2A)}{=} b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) + \sum_{j\in N\setminus i}\sum_{k\in N}\frac{m_{ji}m_{ik}}{m_{ii}}\alpha_k$$

$$\stackrel{(*2B)}{=} b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha}) + b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})\sum_{j\in N\setminus i}\frac{m_{ji}}{m_{ii}}$$

$$= b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})(1 + \sum_{j\in N\setminus i}\frac{m_{ji}}{m_{ii}})$$

$$= b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})(\frac{m_{ii}}{m_{ii}} + \sum_{j\in N\setminus i}\frac{m_{ji}}{m_{ii}})$$

$$= b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})\sum_{j\in N}\frac{m_{ji}}{m_{ii}}$$

$$
\begin{aligned}
(*2A) &: \text{Lemma 3.4 with} \boldsymbol{\Phi}(\boldsymbol{g}) = [\boldsymbol{\phi} \otimes \mathbf{G}^*] + \mathbf{H} \\
(*2B) &: \sum_{j \in N \setminus i} \sum_{k \in N} \frac{m_{ji} m_{ik}}{m_{ii}} \alpha_k \\
&= \sum_{j \in N \setminus i} \left(\frac{m_{ji} m_{ii}}{m_{ii}} \alpha_i + \sum_{k \in N \setminus i} \frac{m_{ji} m_{ik}}{m_{ii}} \alpha_k \right) \\
&= \sum_{j \in N \setminus i} \frac{m_{ji}}{m_{ii}} \left(m_{ii} \alpha_i + \sum_{k \in N \setminus i} m_{ik} \alpha_k \right) \\
&= \sum_{j \in N \setminus i} \frac{m_{ji}}{m_{ii}} b_i(\mathbf{G}^*, \mathbf{H}, \boldsymbol{\phi}, \boldsymbol{\alpha})
\end{aligned}
$$

□

Proof of Proposition 3.6.

$$
\begin{aligned}
&\mathcal{B}(\mathbf{G}^*, \mathbf{H}_{i \in \bar{p}}, \boldsymbol{\phi}_{i \in \bar{p}}, \boldsymbol{\alpha}_{i \in \bar{p}}) - \mathcal{B}(\mathbf{G}^*, \mathbf{H}_{i \in S}, \boldsymbol{\phi}_{i \in S}, \boldsymbol{\alpha}_{i \in S}) \\
&= \mathcal{B}(\mathbf{G}^*, \mathbf{H}_{i \in \bar{p}}, \boldsymbol{\phi}_{i \in \bar{p}}, \boldsymbol{\alpha}_{i \in \bar{p}}) - \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{-i}_{i \in \bar{p}}, \boldsymbol{\phi}^{-i}_{i \in \bar{p}}, \boldsymbol{\alpha}^{-i}_{i \in \bar{p}}) \\
&\quad + \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{-i}_{i \in \bar{p}}, \boldsymbol{\phi}^{-i}_{i \in \bar{p}}, \boldsymbol{\alpha}^{-i}_{i \in \bar{p}}) - \mathcal{B}(\mathbf{G}^*, \mathbf{H}_{i \in S}, \boldsymbol{\phi}_{i \in S}, \boldsymbol{\alpha}_{i \in S}) \\
&= \mathcal{B}(\mathbf{G}^*, \mathbf{H}_{i \in \bar{p}}, \boldsymbol{\phi}_{i \in \bar{p}}, \boldsymbol{\alpha}_{i \in \bar{p}}) - \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{-i}_{i \in \bar{p}}, \boldsymbol{\phi}^{-i}_{i \in \bar{p}}, \boldsymbol{\alpha}^{-i}_{i \in \bar{p}}) \\
&\quad - \big(\mathcal{B}(\mathbf{G}^*, \mathbf{H}_{i \in S}, \boldsymbol{\phi}_{i \in S}, \boldsymbol{\alpha}_{i \in S}) - \mathcal{B}(\mathbf{G}^{*-i}, \mathbf{H}^{-i}_{i \in \bar{p}}, \boldsymbol{\phi}^{-i}_{i \in \bar{p}}, \boldsymbol{\alpha}^{-i}_{i \in \bar{p}})\big) \\
&= \delta_{i,\bar{P}}(\mathbf{G}^*, \mathbf{H}_{i \in \bar{p}}, \boldsymbol{\phi}_{i \in \bar{p}}, \boldsymbol{\alpha}_{i \in \bar{p}}) - \delta_{i,P}(\mathbf{G}^*, \mathbf{H}_{i \in S}, \boldsymbol{\phi}_{i \in S}, \boldsymbol{\alpha}_{i \in S})
\end{aligned}
$$

□

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