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Bargaining Solutions via Surface Measures

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Abstract

We introduce a generalization of the Maschler–Perles bargaining solution to smooth bargaining problems for n players. We proceed by the construction of measures on the Pareto surface of a convex body.

The MP surface measure is defined for Cephoids, i.e., Minkowski sums of simplices (see [14] for a coherent description); this measure cannot directly be extended to a smooth Pareto surface.

Therefore, we introduce a further extension of the Maschler–Perles idea to Pareto surfaces of convex bodies. This extension is suggested by the density $\sqrt[n]{\mathfrak{n}_1 \cdots \mathfrak{n}_n}$ of normals in coordinate directions – a term generalizing the Maschler–Perles line integral of $\sqrt{-dx_1 dx_2}$ – the "donkey cart" in their interpretation. The "deGua" measure defined this way, is then verified to be the limiting measure of the MP measures along the filter of convergent Cephoids as established in [15].

1 Notations and Definitions

We consider specific compact convex comprehensive polyhedra located within the nonnegative orthant of $\mathbb{R}^n$. The notation is taken from [14], see also [6], [7]. To this end, let $\boldsymbol{I} := \{1, \ldots, n\}$ denote the set of coordinates of $\mathbb{R}^n$, the positive orthant is $\mathbb{R}^n_+ := \{\boldsymbol{x} = (x_1, \ldots, x_n) \mid x_i \geq 0 \, , (i \in \boldsymbol{I})\}$. Let $\boldsymbol{e}^i$ denote the i^{th} unit vector of $\mathbb{R}^n$ and $\boldsymbol{e} := (1, \ldots, 1) = \sum_{i=1}^n \boldsymbol{e}^i \in \mathbb{R}^n$ the "diagonal" vector.

The notation $\boldsymbol{CovH}\, A$ is used to denote the ***convex hull*** of a subset A of $\mathbb{R}^n_+$. The ***comprehensive hull*** of a set $A \subseteq \mathbb{R}^n_+$ is given by

$$\boldsymbol{CmpH}\, A \;:=\; \left\{ \boldsymbol{y} \in \mathbb{R}^n_+ \,\middle|\, \exists \boldsymbol{x} \in A \;:\; \boldsymbol{y} \leq \boldsymbol{x} \right\} \; .$$

Given a vector $\boldsymbol{a} = (a_1, \ldots, a_n) > \boldsymbol{0} \in \mathbb{R}^n_+$, we consider the n multiples $\boldsymbol{a}^i := a_i \boldsymbol{e}^i \;(i \in \boldsymbol{I})$ of the unit vectors. Then

$$\Delta^{\boldsymbol{a}} \;:=\; \boldsymbol{CovH} \left\{ \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n \right\} \tag{1.1}$$

is the ***Simplex*** resulting from $\boldsymbol{a}$ (we use capitals in this context). Figure 1.1 represents a Simplex in $\mathbb{R}^3_+$.

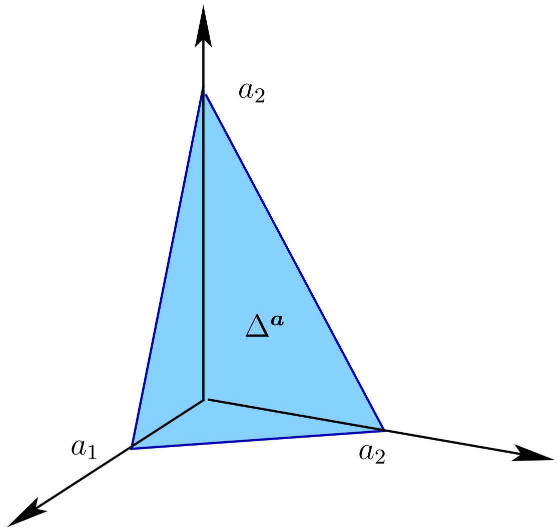

Figure 1.1: The Simplex in $\mathbb{R}^3_+$ generated by $\boldsymbol{a} = (a_1, a_2, a_3)$

Next, for $\boldsymbol{J} \subseteq \boldsymbol{I}$ we write $\mathbb{R}^n_{\boldsymbol{J}} := \{\boldsymbol{x} \in \mathbb{R}^n \,|\, x_i = 0 \; (i \notin \boldsymbol{J})\}$. Accordingly, we obtain the ***Subsimplex***

$$\Delta^{\boldsymbol{a}}_{\boldsymbol{J}} := \{x \in \Delta^{\boldsymbol{a}} \,|\, x_i = 0 \; (i \notin \boldsymbol{J})\} = \Delta^{\boldsymbol{a}} \cap \mathbb{R}^n_{\boldsymbol{J}} = \boldsymbol{CovH} \left\{ \boldsymbol{a}^i \,\middle|\, i \in \boldsymbol{J} \right\} \; . \tag{1.2}$$

There is a second type of simplex we associate with a positive vector $\boldsymbol{a} \in \mathbb{R}^n_+$. This is the one spanned by the vectors $\boldsymbol{a}^i$ plus the vector $\boldsymbol{0} \in \mathbb{R}^n_+$, that is

$$\Pi^{\boldsymbol{a}} \;:=\; \boldsymbol{CovH} \left\{ \boldsymbol{0}, \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n \right\} \;=\; \boldsymbol{CmpH}\, \Delta^{\boldsymbol{a}} \; . \tag{1.3}$$

In order to distinguish both types, we call $\Pi^{\boldsymbol{a}}$ the ***deGua Simplex*** associated to $\boldsymbol{a}$, paying homage to J.P. de Gua de Malves [3] who generalized the Pythagorean theorem for simplices of this type. Consistently, we write, for any $\boldsymbol{J} \subseteq \boldsymbol{I}$ the corresponding ***deGua Subsimplex*** of $\Pi^{\boldsymbol{a}}$ as

$$\begin{aligned} \Pi^{\boldsymbol{a}}_{\boldsymbol{J}} \ &:= \ \{x \in \Pi^{\boldsymbol{a}} \,|\, x_i = 0 \ (i \notin \boldsymbol{J})\} \\ &= \Pi^{\boldsymbol{a}} \cap \mathbb{R}^n_{\boldsymbol{J}} \ = \ \boldsymbol{CovH}\left\{\boldsymbol{0}, \ \boldsymbol{a}^i \,\middle|\, i \in \boldsymbol{J}\right\} \ = \ \boldsymbol{CmpH}\,\Delta^{\boldsymbol{a}}_{\boldsymbol{J}} \ . \end{aligned} \tag{1.4}$$

Figure 1.2 indicates the deGua Simplex $\Pi^{\boldsymbol{a}}$ generated by $\boldsymbol{a}$.

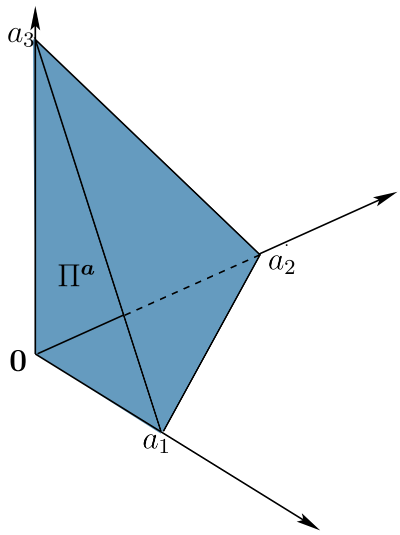

Figure 1.2: The deGua Simplex $\Pi^{\boldsymbol{a}}$; $\boldsymbol{a} = (a_1, a_2, a_3)$

In the terminology of Convex Analysis, $\Delta^{\boldsymbol{a}}$ is the ***maximal (outward) face*** of $\Pi^{\boldsymbol{a}}$. Here we prefer the MathEcon notation, calling $\Delta^{\boldsymbol{a}}$ the ***Pareto face*** of $\Pi^{\boldsymbol{a}}$.

A ***normal*** to some convex set $\boldsymbol{C}$ in some point $\bar{\boldsymbol{x}} \in \partial C$ is a vector that generates a separating hyperplane in $\bar{\boldsymbol{x}}$. A vector that is a normal to some face $\boldsymbol{F}$ of a convex set $\boldsymbol{C}$ in *all* points of $\boldsymbol{F}$ is called normal to $\boldsymbol{F}$. A deGua Simplex $\Pi^{\boldsymbol{a}}$ admits of a normal

$$\mathfrak{n}^{\boldsymbol{a}} \ := \ \left(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\right) \ . \tag{1.5}$$

to $\Delta^{\boldsymbol{a}}$. All other normals to $\Delta^{\boldsymbol{a}}$ are positive multiples of this one, i.e., the ***normal cone*** to $\Delta^{\boldsymbol{a}}$ is

$$\mathcal{N}^{\boldsymbol{a}} \ := \ \{t\mathfrak{n}^{\boldsymbol{a}} \,|\, t > 0\} \ .$$

We refer to this situation saying that the normal of $\Delta^{\boldsymbol{a}}$ is "unique up to a multiple" or "essentially unique" etc.

The projection of $\mathfrak{n}^{\boldsymbol{a}}$ to $\mathbb{R}^n_{\boldsymbol{J}+}$ is denoted by $\mathfrak{n}^{\boldsymbol{a}}_{\boldsymbol{J}} \ := \ \mathfrak{n}^{\boldsymbol{a}}\ _{|\ \mathbb{R}^n_{\boldsymbol{J}+}}$. The subface $\Delta^{\boldsymbol{a}}_{\boldsymbol{J}}$ of the Pareto face admits of a normal cone $\mathcal{N}^{\boldsymbol{a}}_{\boldsymbol{J}}$ generated by the normals

$$\{\mathfrak{n}^{\boldsymbol{a}}_{\boldsymbol{J}'} \,|\, \boldsymbol{J} \subseteq \boldsymbol{J}' \subseteq \boldsymbol{I}\} \ .$$

We use operations on convex sets that are a standard in Convex Geometry, see e.g. PALLASCHKE–URBAŃSKI [8]). E.g., for $A, B \subseteq \mathbb{R}^n_+$, $\lambda \in \mathbb{R}_+$, the algebraic or Minkowski sum $A + B$ and the multiple λA are well defined quantities.

A ***Cephoid*** is a *Minkowski sum of deGua Simplices*, precisely:

Definition 1.1. Let $\boldsymbol{K} = \{1, \ldots, K\}$ and let $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ denote a family of positive vectors. The Minkowski (or algebraic) sum

$$\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}} \tag{1.6}$$

is called a ***Cephoid***.

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The surface of a polyhedron can be described by either a list of extremal points or by maximal faces. We focus on the *Pareto surface* of a Cephoid. For completeness, we provide the following

Definition 1.2.

1. A face $\boldsymbol{F}$ of a Cephoid Π is ***maximal*** if, for any face $\boldsymbol{F}^0$ of Π with $\boldsymbol{F} \subseteq \boldsymbol{F}^0$ it follows that $\boldsymbol{F} = \boldsymbol{F}^0$ is true.
2. The ***(outward)*** or ***Pareto surface*** of a compact convex set (specifically: of a Cephoid Π) is the set

$$\partial\Pi := \{\boldsymbol{x} \in \Pi \mid \nexists \boldsymbol{y} \in \Pi, i \in \boldsymbol{I} \ : \ \boldsymbol{y} \geq \boldsymbol{x},\ y_i > x_i \}. \tag{1.7}$$

3. The points of the Pareto surface are called ***Pareto efficient***.
4. Maximal faces in the Pareto surface are called ***Pareto faces***.
5. Pareto efficient extremal points are called ***vertices***

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The vector $\boldsymbol{0}$ is always an extremal point of a Cephoid in $\mathbb{R}^n$ but it is not Pareto efficient. $\Delta^{\boldsymbol{a}}$ is the only Pareto face of $\Pi^{\boldsymbol{a}}$; similarly for $\Delta^{\boldsymbol{a}}_{\boldsymbol{J}}$ and $\Pi^{\boldsymbol{a}}_{\boldsymbol{J}}$.

Remark 1.3. Generally, the Pareto surface of a Cephoid in n dimensions with K summands consists of a number $\varphi(K, n)$ of Pareto faces. This number is universal, it depends on K and n only (see CHAPTER 5 of [14]). Yet, there is a great variety of orderings of such faces. A most efficient tool to describe the lattice of Pareto faces is provided by representations. These are mappings of $\partial\Pi$ onto a multiple of $\Delta^{\boldsymbol{e}}$.

Recall the ***canonical representation*** of a Cephoid as defined in DEFINITION 2.1 of CHAPTER 1 of [14]. This notion is established via a bijection

$$\boldsymbol{\kappa} : \partial\Pi \mapsto K\Delta^{\boldsymbol{e}} \tag{1.8}$$

of the Pareto surface of Π onto an appropriate multiple of the unit Simplex $\Delta^{\boldsymbol{e}}$. $\boldsymbol{\kappa}$ maps the Pareto faces bijectively on a system of convex polyhedra defined on a grid of $K\Delta^{\boldsymbol{e}}$ such that the lattice structure is preserved bijectively as well (see [14], CHAPTER II).

More detailed, any extremal $\boldsymbol{u} \in \partial\Pi$ is uniquely represented as a sum of vertices of the $\{\Delta^{(k)}\}_{k\in\boldsymbol{K}}$ via a mapping

$$\mathbf{i}_{\bullet} \ := \ \boldsymbol{K} \to \boldsymbol{I} \tag{1.9}$$

i.e., by

$$\boldsymbol{u} = \boldsymbol{a}^{\mathbf{i}_{\bullet}} \ := \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k}. \tag{1.10}$$

$\boldsymbol{u}$ is mapped onto

$$\boldsymbol{u}^0 \ := \ \boldsymbol{\kappa}(\boldsymbol{u}) \ := \ \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{0(k)\mathbf{i}_k} \ . \tag{1.11}$$

Correspondingly, for any Pareto face $\boldsymbol{F}$ with extremals $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^L$, we obtain for a vector $\boldsymbol{x} \in \boldsymbol{F}$ with representation

$$\boldsymbol{x} = \sum_{l}^{L} \alpha_l \boldsymbol{u}^{(l)}$$

the image

$$\boldsymbol{\kappa}(\boldsymbol{x}) \ = \ \sum_{l}^{L} \alpha_l \boldsymbol{\kappa}(\boldsymbol{u}^l) \ . \tag{1.12}$$

This induces an image $\boldsymbol{F}^0 = \boldsymbol{\kappa}(\boldsymbol{F}) \subseteq K\Delta^{\boldsymbol{e}}$. The mapping $\boldsymbol{\kappa}$ is then extended to bijectively map the lattice $\mathcal{V}$ of Pareto faces from $\partial\Pi$ onto the corresponding lattice $\mathcal{V}^0$ of $K\Delta^{\boldsymbol{e}}$.

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Example 1.4. Figure 1.3 (the Cephoid "Odot") is a sum of 4 deGua Simplices in 3 dimensions.

Figure 1.4 shows the canonical representation of "Odot" . The lattice of Pareto efficient faces is exactly copied.

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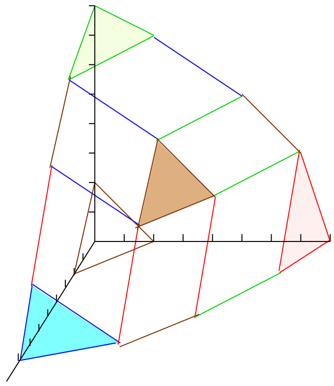

Figure 1.3: “Odot”: a sum of 4 deGua Simplices

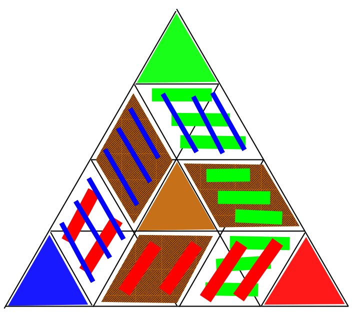

Figure 1.4: The canonical representation of “Odot”

2 Surface Measures: The Machler–Perles Surface Measure

We review the idea of extending the Maschler–Perles solution ([9], see also [14], [10]) via an appropriate measure defined on the Pareto surface of a Cephoid. We follow the presentation provided in [5] and explained in detail in [14].

For $\mathbf{0} < \boldsymbol{a} \in \mathbb{R}^n_+$ we write

$$\boldsymbol{\tau}_{\boldsymbol{a}} := \sqrt[n]{\prod_{i \in \boldsymbol{J}} a_i}. \tag{2.1}$$

For a family $\boldsymbol{a}^{\bullet} = \left\{\boldsymbol{a}^{(k)}\right\}_{k \in \boldsymbol{K}}$ of positive vectors we extend this notion via

$$\boldsymbol{\tau}_{\boldsymbol{a}^{\bullet}} := \sum_{k \in \boldsymbol{K}} \boldsymbol{\tau}_{\boldsymbol{a}^k} . \tag{2.2}$$

Surface measures for a Cephoid are defined on the Pareto surface. We will discuss two versions named in homage to Maschler–Perles and deGua.

Definition 2.1. 1. For positive $\boldsymbol{a} \in \mathbb{R}^n_+$ the ***MP measure*** assigned to $\Delta^{\boldsymbol{a}}$ is

$$\boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{a}}) := \boldsymbol{\tau}_{\boldsymbol{a}}^{n-1} , \tag{2.3}$$

in particular,

$$\boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{e}}) = 1 . \tag{2.4}$$

2. Let $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be Cephoid and let

$$\boldsymbol{F} = \sum_{k \in \boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{2.5}$$

be a Pareto face with reference system $\mathfrak{J} = \left\{\boldsymbol{J}^{(k)}\right\}_{k \in \boldsymbol{K}}$. Then the ***MP measure*** of $\boldsymbol{F}$ is given by

$$\boldsymbol{\iota}_{\Delta}(\boldsymbol{F}) = c_{\mathfrak{J}} \sqrt[n]{\left[\prod_{i \in \boldsymbol{J}^{(1)}} a_i^1\right]^{j_1 - 1} \cdot \ldots \cdot \left[\prod_{i \in \boldsymbol{J}^{(K)}} a_i^K\right]^{j_K - 1}} \tag{2.6}$$

with certain "normalizing coefficients" $c_{\mathfrak{J}}$.

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For details and motivation see [14].

Example 2.2. In 3 dimensions let $\boldsymbol{a} = (a_1, a_2, a_3) > 0$ and let $\Pi^{\boldsymbol{a}}$ be the Cephoid generated. The MP–measure of $\Delta^{\boldsymbol{a}}$ is

$$\iota_\Delta(\Delta^{\boldsymbol{a}}) \;=\; \sqrt[3]{(a_1a_2a_3)^2}\;. \tag{2.7}$$

This way, the unit Simplex is normalized in measure to $\iota_\Delta(\Delta^{\boldsymbol{e}}) = 1$. Thus, $\iota_\Delta(\bullet)$ on multiples of the unit Simplex is Lebesgue measure up to a constant density.

Consider the sum of two deGua simplices $\Pi \;=\; \Delta^{\boldsymbol{a}} + \Delta^{\boldsymbol{b}}$ as indicated in Figure 2.1. The Pareto surface of this Cephoid consists of two translates of the deGua Simplices involved and the rhombus

$$\Lambda^{\boldsymbol{ab}}_{23\ 13} \;=\; \Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{13}$$

This rhombus is the sum of two Subsimplices of $\Delta^{\boldsymbol{a}}$ and $\Delta^{\boldsymbol{b}}$.

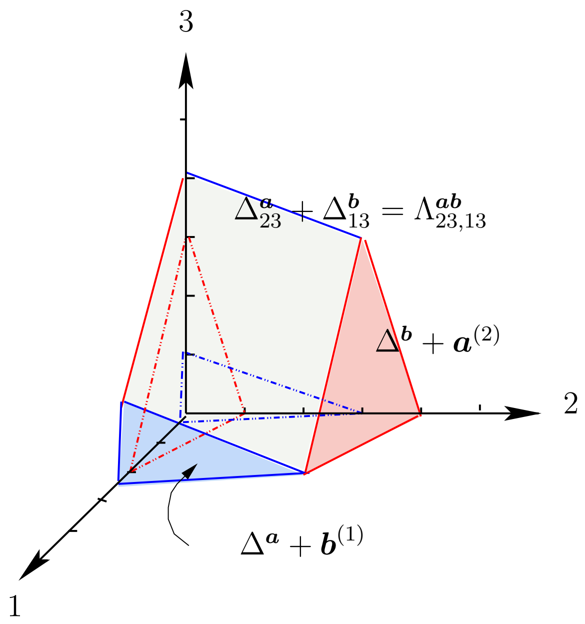

Figure 2.1: A sum of two deGua Simplices

To $\Lambda^{\boldsymbol{ab}}_{23\ 13}$ we assign

$$\iota_\Delta(\Lambda^{\boldsymbol{ab}}) \;:=\; 2\sqrt[3]{(a_1a_2a_3)(b_1b_2b_3)}\;, \tag{2.8}$$

Note that equation (2.8) involves all coordinates of $\boldsymbol{a}$ and $\boldsymbol{b}$. The result is based on the notion of a "measure preserving" representation, which allows to map the Pareto surface of a cephoid consistently onto a multiple $\boldsymbol{\tau_a}\bullet\Delta^{\boldsymbol{e}}$ of the unit Simplex. (see [14], CHAPTER III, SECTION 1 for a detailed motivation).

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Remark 2.3. [Modifications of the canonical representation]

By a slight modification we obtain a normalized version of the canonical representation, which we call the ***Simplex representation***. This version uses the unit Simplex $\Delta^{\boldsymbol{e}}$ simultaneously for all Cephoids. It is defined via

$$\stackrel{\Delta}{\kappa} \;:\; \partial\Pi \to \Delta^{\boldsymbol{e}} \qquad \stackrel{\Delta}{\boldsymbol{\kappa}}(\xi) \;:=\; \frac{\boldsymbol{\kappa}(\boldsymbol{x})}{n} \tag{2.9}$$

In what follows (Section 3, in particular Definition 3.3) we will change our viewpoint and consider the inverse of a representation to be a parametrization of a Pareto surface of a Cephoid. E.g., the mapping

$$\stackrel{\Delta}{\boldsymbol{x}}(\bullet) \;=\; \stackrel{\Delta}{\boldsymbol{\kappa}}{}^{-1} : \Delta^{\boldsymbol{e}} \mapsto \partial\Pi$$

provides a parametrization

$$(\Delta^{\boldsymbol{e}}, \stackrel{\Delta}{\boldsymbol{x}}(\bullet)) \tag{2.10}$$

of $\partial\Pi$ by $\Delta^{\boldsymbol{e}}$. The advantage of the Simplex representation (2.9) is that its inverse $\stackrel{\Delta}{\boldsymbol{x}}(\bullet) \;=\; \stackrel{\Delta}{\boldsymbol{\kappa}}{}^{-1}$ acts simultaneously on a domain for all Cephoids.

Somewhat more generally, for some $\bar{\boldsymbol{a}} > \boldsymbol{0}$ we can choose a multiple of $\Delta^{\bar{\boldsymbol{a}}}$ instead of $K\Delta^{\boldsymbol{e}}$ for the representation. Let $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ and choose a set of positive coefficients $\{\alpha_k\}_{k\in\boldsymbol{K}}$ with sum $\sum_{k\in\boldsymbol{K}} \alpha_k \;=:\; \alpha$. Then the representation will take place on $\bar{\boldsymbol{\Pi}} \;:=\; \sum_{k\in\boldsymbol{K}} \alpha_k \Pi^{\bar{\boldsymbol{a}}} = \alpha\Delta^{\bar{\boldsymbol{a}}}$ with $\alpha \;:=\; \sum_{k\in\boldsymbol{K}} \alpha_k$.

We call this a ***modified canonical representation*** written $\boldsymbol{\kappa}^{\bar{\boldsymbol{a}}}$. A Pareto face $\boldsymbol{F} = \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ of Π is then mapped onto a face

$$\boldsymbol{\kappa}^{\bar{\boldsymbol{a}}}(\boldsymbol{F}) \;=\; \bar{\boldsymbol{F}} \;=\; \sum_{k\in\boldsymbol{K}} \alpha_k \Delta^{\bar{\boldsymbol{a}}^{(k)}}_{\boldsymbol{J}^{(k)}} \tag{2.11}$$

of the Cephoid $\bar{\boldsymbol{\Pi}} = \alpha\Delta^{\bar{\boldsymbol{a}}}$. Also, the lattice $\mathcal{V}$ of Pareto faces of Π is bijectively mapped on the lattice $\boldsymbol{\kappa}^{\bar{\boldsymbol{a}}}(\mathcal{V}) = \overline{\mathcal{V}}$ on $K\Delta^{\bar{\boldsymbol{a}}}$.

Specificylly, we set $\alpha_k \;:=\; \boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{a}^{(k)}})$ $(k \in \boldsymbol{K})$. The inverse mapping $\boldsymbol{x}^{\bar{\boldsymbol{a}}}(\bullet)$ constitutes a parametrization of Π.

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3 Surface Measures: The deGua Measure

Now we turn our focus on more general compact convex sets $\Gamma \subseteq \mathbb{R}^n_+$ with Pareto surface $\partial\Gamma$. We distinguish the following versions.

Generally, we if we speak of a "convex body" we mean a compact convex set, the surface being differentiable within an open and dense subset (the complement of finitely many hyperplanes).

Specifically, let $\partial\Gamma$ be a compact and continuously differentiable surface which, in addition, has not flat areas and no zero derivatives. That is, the normal varies continuously, is always positive and uniquely corresponds to Pareto efficient points. In this case we shall – somewhat sloppily – refer to Γ as a "smooth body". Or else we consider Γ to be a Cephoid – which we indicate by writing Π – with Pareto surface $\partial\Gamma = \partial\Pi$.

We will heavily rely on the use of parametrizations as follows (see also Remark 2.3).

Definition 3.1. Let Γ be a convex body. A ***parametrization*** of $\partial\Gamma$ is a pair $(\boldsymbol{T}, \boldsymbol{x}(\bullet))$, such that $\boldsymbol{T} \in \mathbb{R}^{n-1}$ is compact and convex and $\boldsymbol{x}(\bullet)$ is a bijective mapping

$$\boldsymbol{x}(\bullet) : \boldsymbol{T} \to \partial\Gamma \tag{3.1}$$

which is continuously differentiable within an open and dense set. For a smooth body we require that $\boldsymbol{x}(\bullet)$ is continuously differentiable.

For a Cephoid Π, a ***parametrization*** of $\partial\Pi$ is a pair $(\boldsymbol{T}, \boldsymbol{x}(\bullet))$, such that $\boldsymbol{x}(\bullet) : \boldsymbol{T} \to \partial\Pi$ is continuously differentiable within an open and dense subset (the complement of finitely many hyperplanes) of $\boldsymbol{T}$. We write $\boldsymbol{t} = (t_1, \ldots, t_{n-1}) \in \boldsymbol{T}$ for the generic element of $\boldsymbol{T}$.

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For a convex body Γ the ***canonical parametrization*** is defined as follows. Let

$$\begin{aligned} \Gamma^{(-n)} &= \Gamma_{\boldsymbol{I}-\{n\}} := \Gamma \cap \mathbb{R}_{\boldsymbol{I}\setminus\{n\}} \\ &= \left\{ \boldsymbol{t} \in \mathbb{R}^{n-1}_+ \,\middle|\, \exists t \geq 0 := (\boldsymbol{t}, t) \in \Gamma \right\} \end{aligned} \tag{3.2}$$

and define the (continuous and concave) function

$$\begin{aligned} C &: \Gamma^{(-n)} \to \mathbb{R} \\ C(\boldsymbol{t}) &:= \max\{t \geq 0 \,|\, (\boldsymbol{t}, t) \in \Gamma\} , \end{aligned} \tag{3.3}$$

then the canonical parametrization is $\left(\Gamma^{(-n)}, \boldsymbol{x}^C(\bullet)\right)$ with

$$\begin{aligned} \boldsymbol{x}^C(\bullet) &:= \Gamma^{(-n)} \to \partial\Gamma \\ \boldsymbol{x}^C(\boldsymbol{t}) : &= (\boldsymbol{t}, C(\boldsymbol{t})) \ \ (\boldsymbol{t} \in \Gamma^{(-n)}) . \end{aligned} \tag{3.4}$$

Essentially, describing Γ is tantamount to presenting a function C such that $\partial\Gamma$ (or $\partial\Pi$) is the graph of C. This version is emphasized for the case of 2 dimensions in the context of the Maschler–Perles solution, see [14], CHAPTER XI, SECTION 3.

Example 3.2. If $\Gamma = \Delta^{\boldsymbol{a}}$ is a deGua Simplex, then the canonical parametrization is provided via the deGua unit Simplex in $n-1$ dimensions; i.e.,

$$(3.5)\qquad \boldsymbol{T} = \Delta^{\boldsymbol{a}^-} = \left\{ \boldsymbol{t} \in \mathbb{R}_+^{n-1} \;\middle|\; \sum_{j=1}^{n-1} \frac{t_j}{a_j} \leq 1 \right\} = \boldsymbol{CovH}\{\boldsymbol{a}^1, \ldots, \boldsymbol{a}^{n-1}\} \,.$$

and the function C given by

$$(3.6)\qquad C(\boldsymbol{t}) \;=\; a_n \left(1 - \sum_{j=1}^{n-1} \frac{t_j}{a_j} \right) \,.$$

Alternatively, we may use

$$(3.7)\qquad \boldsymbol{T} \;:=\; \Pi^{\boldsymbol{e}^{n-1}} \;=\; \left\{ \boldsymbol{t} \in \mathbb{R}_+^{n-1} \;\middle|\; \sum_{j=1}^{n-1} t_j \leq 1 \right\} \,.$$

and a parametrization $\boldsymbol{x}(\bullet)$ given by

$$(3.8)\qquad \boldsymbol{x}(\boldsymbol{t}) = (a_1 t_1, \ldots, a_{n-1} t_{n-1}, a_n \left(1 - \sum_{j=1}^{n-1} t_j \right)) \;\; (\boldsymbol{t} \in \Delta^{\boldsymbol{e}^-}) \,.$$

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The following parametrization allows for the simultaneous choice of $\boldsymbol{T}$ for all Cephoids. It will turn out that it can be extended to smooth surfaces as well. Compare also Remark 2.3.

Definition 3.3. Let Π be a Cephoid and let $\boldsymbol{\kappa}$ be the canonical representation. The Simplex parametrization $(\Delta^{\boldsymbol{e}}, \boldsymbol{x}^{\Delta}(\bullet))$ is obtained by shrinking the canonical representation to the unit Simplex; i.e.,

$$(3.9)\qquad \begin{aligned} \boldsymbol{x}^{\Delta}(\bullet) \;&:\; \Delta^{\boldsymbol{e}} \to \partial\Pi \\ \boldsymbol{x}^{\Delta}(\boldsymbol{t}) \;&:=\; \boldsymbol{\kappa}^{-1}(K\boldsymbol{t}) \;\; (\boldsymbol{t} \in \Delta^{\boldsymbol{e}}) \,. \end{aligned}$$

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The Simplex parametrization preserves the lattice structure $\mathcal{V}$ of the Pareto surface $\partial\Pi$. It is just a rescaling of the canonical representation, but yields a simultaneous parametrization basis $\mathbf{T} = \Delta^{\boldsymbol{e}}$. Later on (Lemma 4.4, Definition 4.5) we will show that, by uniform continuity, this choice of $\boldsymbol{T} = \Delta^{\boldsymbol{e}}$ can be introduced for a smooth surface as well.

In what follows we construct surface integrals which are independent on the parametrization chosen. Not always will we explicitely verify this fact.

Let Γ be a convex body with surface $\partial\Gamma$ parametrized by $(\boldsymbol{T}, \boldsymbol{x}(\bullet))$. Let $\bar{\boldsymbol{t}} \in \boldsymbol{T}$ be a point at which we have differentiability. Then, for sufficiently small $\varepsilon > 0$, a tangent through the point $\boldsymbol{x}(\bar{\boldsymbol{t}})$ in direction of the curve

$$\left\{ \boldsymbol{x}(t_j, \bar{\boldsymbol{t}}_{-j}) \,\middle|\, t_j \in [\bar{t}_j - \varepsilon, \bar{t}_j + \varepsilon] \right\} \tag{3.10}$$

is given by the vector

$$\frac{\partial \boldsymbol{x}}{\partial t_j} = \left(\frac{\partial x_1}{\partial t_j}, \frac{\partial x_2}{\partial t_j}, \ldots, \frac{\partial x_n}{\partial t_j} \right) (\bar{\boldsymbol{t}}) \quad (j \in \{1, \ldots, n-1\}). \tag{3.11}$$

Let

$$\boldsymbol{D}(\bar{\boldsymbol{t}}) = (\boldsymbol{D}\boldsymbol{x})(\bar{\boldsymbol{t}}) = \left(\frac{\partial x_i}{\partial t_j} \right)_{i \in \boldsymbol{I}, j \in \boldsymbol{I} \setminus \{n\}} \tag{3.12}$$

and let

$$D_i(\bar{\boldsymbol{t}}) = (D_i \boldsymbol{x})(\bar{\boldsymbol{t}}) = \left| \frac{\partial x_k}{\partial t_j} \right|_{k \in \boldsymbol{I} \setminus \{i\}, j \in \boldsymbol{I} \setminus \{n\}} \quad (i \in \boldsymbol{I}) \tag{3.13}$$

denote the functional determinant of the quadratic submatrix of $\boldsymbol{D}$ obtained by omitting the $i-th$ column. Then the normal at $\partial\Gamma$ in $\boldsymbol{x}(\bar{\boldsymbol{t}})$ is given by

$$\bar{\mathfrak{n}} = \mathfrak{n}^{\boldsymbol{x}(\bar{\boldsymbol{t}})} = (D_1(\bar{\boldsymbol{t}}), \ldots, D_n(\bar{\boldsymbol{t}})) \ . \tag{3.14}$$

Definition 3.4. Let $\Gamma \subseteq \mathbb{R}^n_+$ be a convex body with Pareto surface $\partial\Gamma$ and let $(\boldsymbol{T}, (\boldsymbol{x}(\bullet))$ be a parametrization. The ***deGua measure*** on $\partial\Gamma$ is

(3.15)

$$\begin{aligned}
\boldsymbol{\vartheta}(\partial\Gamma) &= \int_{\partial\Gamma} \sqrt[n]{d\mathfrak{n}_1 \cdots d\mathfrak{n}_n} \\
&= \int_{\boldsymbol{T}} \sqrt[n]{(\mathfrak{n}_1 \cdots \mathfrak{n}_n) \circ \boldsymbol{x}} \; d\boldsymbol{\lambda} = \int_{\boldsymbol{T}} \sqrt[n]{(D_1 \cdots D_n) \circ \boldsymbol{x}} \; d\boldsymbol{\lambda} \\
&= \int_{\boldsymbol{T}} \sqrt[n]{D_1(\boldsymbol{x}(\bullet)) \cdots D_n(\boldsymbol{x}(\bullet))} \; d\boldsymbol{\lambda}(\bullet) \\
&= \int_{\boldsymbol{T}} \sqrt[n]{D_1(\boldsymbol{x}(t_1 \cdots, t_{n-1})) \cdots D_n(\boldsymbol{x}(t_1, \cdots t_{n-1}))} \; dt_1 \cdots dt_{n-1}
\end{aligned}$$

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As we have emphasized above, one has to verify that the deGua measure is independent on the parametrization chosen; a tedious but straightforward procedure well known for the analogues of Lebesgue measure on a smooth surface. We provide a shorthand version as follows.

Indeed, let $\boldsymbol{\xi}(\bullet)\ :\ \boldsymbol{S} \to \partial\Gamma$ be a second parametrization. Then

$$\boldsymbol{h}(\bullet)\ :=\ \boldsymbol{x}(\bullet)^{(-1)} \circ \boldsymbol{\xi}(\bullet)\ ,\quad \boldsymbol{h}(\bullet)\ :\ \boldsymbol{S} \to \boldsymbol{T}$$

constitutes a bijective mapping. Now for some $\boldsymbol{s} \in \boldsymbol{S}$, $\boldsymbol{t} \in \boldsymbol{T}$ and $\boldsymbol{t} = \boldsymbol{h}(\boldsymbol{s})$ we have

$$\boldsymbol{x}(\boldsymbol{t})\ =\ \boldsymbol{x}(\boldsymbol{h}(\boldsymbol{s})) = \boldsymbol{\xi}(\boldsymbol{s})\ ,$$

hence writing the functional determinant

$$\boldsymbol{Dh}\ =\ \left|\frac{\partial h_i}{\partial t_k}\right|_{i,k\in\boldsymbol{I}} \tag{3.16}$$

we obtain for the deGua measure

$$\begin{aligned}
&\int\limits_{\boldsymbol{T}} \sqrt[n]{D_1\boldsymbol{x}(\boldsymbol{t})\cdots D_n\boldsymbol{x}(\boldsymbol{t})}dt_1,\ldots,dt_n \\
&= \int\limits_{\boldsymbol{S}} \sqrt[n]{D_1(\boldsymbol{x}\circ\boldsymbol{h})(\boldsymbol{s})\cdots D_n(\boldsymbol{x}\circ\boldsymbol{h})(\boldsymbol{s})}(\boldsymbol{Dh})ds_1,\ldots,ds_n \\
&= \int\limits_{\boldsymbol{S}} \sqrt[n]{[D_1(\boldsymbol{x}\circ\boldsymbol{h})(\boldsymbol{s})\boldsymbol{Dh}]\cdots[D_n(\boldsymbol{x}\circ\boldsymbol{h})(\boldsymbol{s})\boldsymbol{Dh}]}ds_1,\ldots,ds_n \\
&= \int\limits_{\boldsymbol{S}} \sqrt[n]{D_1\boldsymbol{\xi}(\boldsymbol{s})\cdots D_n\boldsymbol{\xi}(\boldsymbol{s})}ds_1,\ldots,ds_n
\end{aligned} \tag{3.17}$$

which proves the independence of a particular parametrization.

Example 3.5. Specifically, consider the case that $n = 3$. Let $(\boldsymbol{T}, \boldsymbol{x}(\bullet))$ be a parametrization of $\partial\Gamma$. Then, for fixed $(\bar{t}_1, \bar{t}_2) \in \boldsymbol{T}$, the tangents in direction of the curves $\{(\boldsymbol{x}(\bar{t}_1, t_2)\,|\,(\bar{t}_1, t_2) \in \boldsymbol{T}\}$ and $\{(\boldsymbol{x}(t_1, \bar{t}_2)\,|\,(t_1, \bar{t}_2) \in \boldsymbol{T}\}$ are

$$\frac{\partial \boldsymbol{x}}{\partial t_1}\ =\ (\frac{\partial x_1}{\partial t_1}, \frac{\partial x_2}{\partial t_1}, \frac{\partial x_3}{\partial t_1})\ \text{ and }\ \frac{\partial \boldsymbol{x}}{\partial t_2}\ =\ (\frac{\partial x_1}{\partial t_2}, \frac{\partial x_2}{\partial t_2}, \frac{\partial x_3}{\partial t_2})\ .$$

Thus, the normal is (the "vectorial product")

$$\mathfrak{n}\ =\ \frac{\partial \boldsymbol{x}}{\partial t_1} \times \frac{\partial \boldsymbol{x}}{\partial t_2} \tag{3.18}$$

which is conveniently written

$$\mathfrak{n}\ =\ \begin{vmatrix} \boldsymbol{e}^1 & \boldsymbol{e}^2 & \boldsymbol{e}^3 \\ \frac{\partial x_1}{\partial t_1} & \frac{\partial x_2}{\partial t_1} & \frac{\partial x_3}{\partial t_1} \\ \frac{\partial x_1}{\partial t_2} & \frac{\partial x_2}{\partial t_2} & \frac{\partial x_3}{\partial t_2}) \end{vmatrix}\ =\ D_1\boldsymbol{e}^1 + D_2\boldsymbol{e}^2 + D_3\boldsymbol{e}^3\ ; \tag{3.19}$$

with the above notation D_i for the subdeterminants of the determinant representing $\mathfrak{n}$, i.e.,

$$D_k = \left|\frac{\partial x_i}{\partial t_j}\right|_{i\in\boldsymbol{I}\setminus\{k\},j\in\{1,2\}}\ . \tag{3.20}$$

In particular for the Simplex $\Delta^{\boldsymbol{a}} = \Delta^{(a_1,a_2,a_3)}$ and $\boldsymbol{T} = \boldsymbol{e}^{12} = (1,1)$, consider the parametrization provided in (3.7) and (3.8), i.e.,

$$\begin{aligned} \boldsymbol{x}(\bullet) &:= \Pi^{\boldsymbol{e}^{12}} \to \Delta^{\boldsymbol{a}} \\ (t_1,t_2) &\to (a_1t_1, a_2t_2, a_3(1-(t_1+t_2))) \end{aligned} \tag{3.21}$$

we find the tangents

$$\frac{\partial \boldsymbol{x}}{\partial t_1} = (a_1, 0, -a_3) \quad , \quad \frac{\partial \boldsymbol{x}}{\partial t_2} = (0, a_2, -a_3) \ .$$

Consequently, we obtain

$$\begin{aligned} \mathfrak{n} &= \begin{vmatrix} \boldsymbol{e}^1 & \boldsymbol{e}^2 & \boldsymbol{e}^3 \\ a_1 & 0 & -a_3 \\ 0 & a_2 & -a_3 \end{vmatrix} = D_1\boldsymbol{e}^1 + D_2\boldsymbol{e}^2 + D_3\boldsymbol{e}^3 \\ &= a_2a_3\boldsymbol{e}^1 + a_1a_3\boldsymbol{e}^2 + a_1a_2\boldsymbol{e}^3 = (a_2a_3, a_1a_3, a_1a_2) \end{aligned} \tag{3.22}$$

Hence the Lebesgue surface integral is

$$\begin{aligned} \int_{\Delta^{e^{12}}} \sqrt{D_1^2 + D_2^2 + D_3^2}\, dt_1dt_2 &= \int_{\Delta^{e^{12}}} \sqrt{a_2^2a_3^2 + a_1^2a_3^2 + a_1^2a_2^2}\, dt_1dt_2 \\ &= \frac{1}{2}\sqrt{a_2^2a_3^2 + a_1^2a_3^2 + a_1^2a_2^2}; \end{aligned} \tag{3.23}$$

which is deGua's Theorem.

On the other hand, the deGua measure (Definition 3.15) and formula (3.15) for the Simplex $\Delta^{\boldsymbol{a}}$ is given by.

$$\begin{aligned} \boldsymbol{\vartheta}(\Delta^{\boldsymbol{a}}) &= \int_{\Delta^{\boldsymbol{a}}} \sqrt[3]{d\mathfrak{n}_1 d\mathfrak{n}_2 d\mathfrak{n}_3} \\ &= \int_{\Delta^{e^{12}}} \sqrt[3]{(\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3)\circ \boldsymbol{x}}\; d\boldsymbol{\lambda} = \int_{\Delta^{e^{12}}} \sqrt[3]{(D_1D_2D_3)\circ \boldsymbol{x}}\; d\boldsymbol{\lambda} \\ &= \int_{\Delta^{e^{12}}} \sqrt[3]{D_1(\boldsymbol{x}(\bullet))D_2(\boldsymbol{x}(\bullet))D_3(\boldsymbol{x}(\bullet))}\; d\boldsymbol{\lambda}(\bullet) \\ &= \int_{\Delta^{e^{12}}} \sqrt[3]{D_1(\boldsymbol{x}(t_1,t_2))D_2(\boldsymbol{x}(t_1,t_2))D_3(\boldsymbol{x}(t_1,t_2))}\; dt_1dt_2 \\ &= \int_{\Delta^{e^{12}}} \sqrt[3]{a_2a_3a_1a_3a_1a_2} = \frac{1}{2}\sqrt[3]{(a_1a_2a_3)^2} \end{aligned} \tag{3.24}$$

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Example 3.6. Slightly more generally consider the canonical parametrization $(\Gamma^{(-n}, \boldsymbol{x}^C(\bullet))$ as described in (3.3),(3.4).

The tangent vectors as in (3.11) are then

$$\frac{\partial \boldsymbol{x}}{\partial t_j} = \left(0, \ldots, 1, \ldots, \frac{\partial C}{\partial t_j}\right)(\bar{\boldsymbol{t}}) \quad (j \in \{1, \ldots, n-1\}) \tag{3.25}$$

with 1 at position j. A straightforward computation therefore reveals for the functional determiants

$$D_i(\boldsymbol{x}^C(\bar{\boldsymbol{t}})) = \frac{\partial C}{\partial t_i}(\bar{\boldsymbol{t}}) \quad (i \in \boldsymbol{I} \setminus \{n\}) \quad D_n(\boldsymbol{x}^C(\bar{\boldsymbol{t}})) = 1 \ . \tag{3.26}$$

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Remark 3.7. Definition 3.4 implies a deGua measure and integral calculus defined on $\partial\Gamma$. For any function $F : \partial\Gamma \to \mathbb{R}$ we define the version transported via some parametrization $\boldsymbol{x}(\bullet) : \boldsymbol{T} \to \partial\Gamma$ to be

$$F^\star : \boldsymbol{T} \to \mathbb{R}, \quad F^\star := F \circ \boldsymbol{x}(\bullet) =: \boldsymbol{x}(\bullet)F \ . \tag{3.27}$$

Functions are being transported in a contravariant manner; $F^\star$ is the function transported via $\boldsymbol{x}(\bullet)$ from $\partial\Gamma$ to $\boldsymbol{T}$. On the other hand, measures are being transported covariantly with a mapping. Thus, in the present situation, we want the transport to be carried on by $\boldsymbol{x}(\bullet)^{-1}$. This is a feasible operation as $\boldsymbol{x}(\bullet)$ is bijective. We obtain the transported measure

$$\boldsymbol{\vartheta}^\star := \boldsymbol{\vartheta} \circ \boldsymbol{x}(\bullet) =: \boldsymbol{x}^{-1}(\bullet)\boldsymbol{\vartheta}$$

We emphasize the transport of the measures involved:

$$\begin{aligned} \boldsymbol{x}(\bullet) &: \boldsymbol{\vartheta}^\star \to \boldsymbol{\vartheta} = \boldsymbol{x}(\bullet)\boldsymbol{\vartheta}^\star := \boldsymbol{\vartheta}^\star \circ \boldsymbol{x}^{-1}(\bullet) \\ \boldsymbol{x}^{-1}(\bullet) &: \boldsymbol{\vartheta} \to \boldsymbol{\vartheta}^\star = \boldsymbol{x}^{-1}(\bullet)\boldsymbol{\vartheta} := \boldsymbol{\vartheta} \circ \boldsymbol{x}(\bullet) \ . \end{aligned}$$

Then we obtain for the integral via the formula for transformation of the variables:

$$\begin{aligned} \int_{\partial\Gamma} F d\boldsymbol{\vartheta} &= \int_{\partial\Gamma} F \sqrt[n]{d\mathfrak{n}_1 \cdots d\mathfrak{n}_n} = \int_{\partial\Gamma} F(\boldsymbol{x}) d\boldsymbol{\vartheta}(\boldsymbol{x}) \\ &= \int_{\partial\Gamma} F d(\boldsymbol{x}(\bullet)\boldsymbol{\vartheta}^\star) = \int_{\boldsymbol{T}} (\boldsymbol{x}(\bullet)F) d\boldsymbol{\vartheta}^\star = \int_{\boldsymbol{T}} F^\star d\boldsymbol{\vartheta}^\star(\boldsymbol{t}) \\ &= \int_{\boldsymbol{T}} (F \circ \boldsymbol{x}(\bullet)) d\boldsymbol{\vartheta}^\star = \int_{\boldsymbol{T}} F(\boldsymbol{x}(\boldsymbol{t})) d\boldsymbol{\vartheta}^\star(\boldsymbol{t}) \\ &= \int_{\boldsymbol{T}} F(\boldsymbol{x}(\boldsymbol{t})) \sqrt[n]{D_1 \cdots D_n} dt_1 \cdots dt_n \\ &= \int_{\boldsymbol{T}} F(\boldsymbol{x}(\boldsymbol{t})) \sqrt[n]{D_1(\boldsymbol{x}(\boldsymbol{t})) \cdots D_n(\boldsymbol{x}(\boldsymbol{t}))} dt_1 \cdots dt_n \ . \end{aligned} \tag{3.28}$$

In particular, consider the canonical parametrization $\left(\Gamma^{(-n)}, \boldsymbol{x}^C(\bullet)\right)$. A straightforward computation yields (for $\boldsymbol{x}(\bullet) = \boldsymbol{x}^C(\bullet)$)

$$\begin{aligned} D_1 \circ \boldsymbol{x}(\boldsymbol{t}) &= \frac{\partial G}{\partial t_1}(\boldsymbol{t}), \\ &= \ldots, \\ D_{n-1} \circ \boldsymbol{x}(\boldsymbol{t}) &= \frac{\partial G}{\partial t_{n-1}}(\boldsymbol{t}), \\ D_n \circ \boldsymbol{x}(\boldsymbol{t}) &= 1 \ . \end{aligned} \tag{3.29}$$

Hence, the deGua integral is

$$\int\limits_{\partial\Gamma} F d\boldsymbol{\vartheta} = \int\limits_{\Gamma^{(-n)}} F(\boldsymbol{x}(\boldsymbol{t})) \sqrt[n]{\frac{\partial C}{\partial t_1}(\boldsymbol{x}(\boldsymbol{t})) \cdots \frac{\partial C}{\partial t_{n-1}}(\boldsymbol{x}(\boldsymbol{t})) dt_1 \cdots dt_{n-1}} \ . \tag{3.30}$$

The above formulae (3.17) and (3.30) slightly generalize (3.15), they can be considered as the defining relation for the deGua measure $\boldsymbol{\vartheta}$ in its own right. Thus, for any measurable subset $G \subseteq \partial\Gamma$, the term $\boldsymbol{\vartheta}(G)$ is well defined.

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Let $\boldsymbol{F}$ be a Pareto face of some Cephoid Π. Then $\boldsymbol{\vartheta}$ on $\boldsymbol{F}$ has a constant density w.r.t Lebuesge measure. The same is true for the MP measure $\boldsymbol{\iota}_\Delta$. To compute the detailed factors is not necessary. However, we want to describe the relation between both measures for the special case of a deGua Simplex.

Lemma 3.8. For $n \in \mathbb{N}$ there is a constant $\boldsymbol{o}(n)$ such that for $0 < \boldsymbol{a} \in \mathbb{R}^n_+$

$$\boldsymbol{\iota}_\Delta(\bullet) = \boldsymbol{o}(n)\boldsymbol{\vartheta}(\bullet) \quad \text{on} \quad \Delta^{\boldsymbol{a}} \ , \tag{3.31}$$

holds true.

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Example 3.9. Recall Example 2.2, the sum of two deGua Simplices is repeated in Figure 3.1. The Pareto face

$$\Lambda^{\boldsymbol{ab}}_{23\ 13} = \Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{13} \tag{3.32}$$

is the sum of two Subsimplices of $\Delta^{\boldsymbol{a}}$ and $\Delta^{\boldsymbol{b}}$.

For $\Lambda^{\boldsymbol{ab}}_{23\ 13}$ we computed in (2.8) the value.

$$\boldsymbol{\iota}_\Delta(\Lambda^{\boldsymbol{ab}}) := 2\sqrt[3]{(a_1a_2a_3)(b_1b_2b_3)} \ , \tag{3.33}$$

Note that equation (3.33) involves all coordinates of $\boldsymbol{a}$ and $\boldsymbol{b}$.

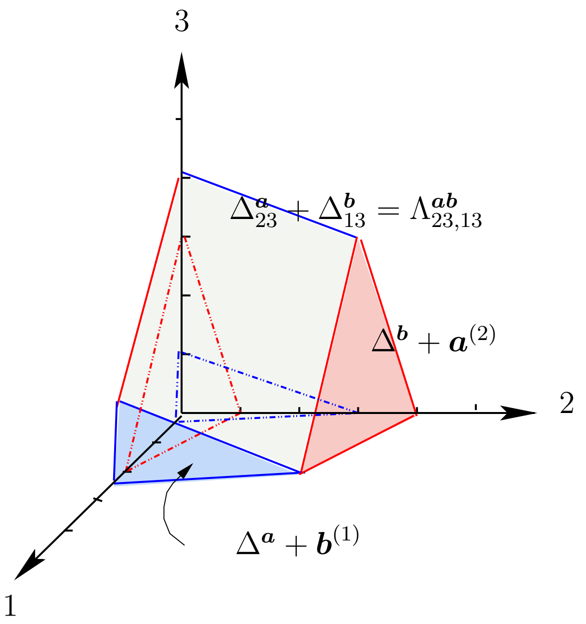

Figure 3.1: The sum of two deGua Simplices

Now we turn to the computation of $\boldsymbol{\vartheta}$. The extremals of the rhombus are

(3.34)
$$\begin{aligned} \boldsymbol{a}^3 + \boldsymbol{b}^1 &= (b_1, 0, a_3)\ , & \boldsymbol{a}^2 + \boldsymbol{b}^1 &= (b_1, a_2, 0)\ , \\ \boldsymbol{a}^2 + \boldsymbol{b}^3 &= (0, a_2, b_3)\ , & \boldsymbol{a}^3 + \boldsymbol{b}^3 &= (0, 0, a_3 + b_3)\ . \end{aligned}$$

For a convenient parametrization we choose the square

(3.35)
$$\square^{\boldsymbol{e}} = \left\{ \boldsymbol{x} \in \mathbb{R}^2_+ \,\middle|\, \boldsymbol{x} \le \boldsymbol{e} = (1,1) \right\}$$

and the parametrization

(3.36)
$$\begin{aligned} \boldsymbol{x} &: \square^{\boldsymbol{e}} \to \Lambda^{\boldsymbol{ab}}_{23\ 13} \\ \boldsymbol{x}(t_1, t_2) &= (b_1 t_1, a_2 t_2, a_3(1 - t_2) + b_3(1 - t_1)) \quad (t_1, t_2) \in \square^{\boldsymbol{e}} \end{aligned}$$

which yields

(3.37)
$$\begin{aligned} &\boldsymbol{x}(0,0) = (0, 0, a_3 + b_3), \quad \boldsymbol{x}(1,0) = (b_1, 0, a_3) \\ &\boldsymbol{x}(0,1) = (0, a_2, b_3), \quad \boldsymbol{x}(1,1) = (b_1, a_2, 0)\ . \end{aligned}$$

In this case the tangents are

$$\frac{\partial \boldsymbol{x}}{\partial t_1} = (b_1, 0, -b_3)\ , \quad \frac{\partial \boldsymbol{x}}{\partial t_2} = (0, a_2, -a_3)\ .$$

Consequently we obtain

(3.38)
$$\mathfrak{n} = \begin{vmatrix} \boldsymbol{e}^1 & \boldsymbol{e}^2 & \boldsymbol{e}^3 \\ b_1 & 0 & -b_3 \\ 0 & a_2 & -a_3 \end{vmatrix} = (a_2 b_3, b_1 a_3, b_1 a_2)\ .$$

Consequently,

(3.39)
$$\begin{aligned}\boldsymbol{\vartheta}(\Lambda^{\boldsymbol{ab}}_{23\ 13}) &= \int\limits_{\Lambda^{\boldsymbol{ab}}_{23\ 13}} \sqrt[3]{d\mathfrak{n}_1 d\mathfrak{n}_2 d\mathfrak{n}_3} \\ &= \int\limits_{\square^e} \sqrt[3]{a_2 b_3 b_1 a_3 b_1 a_2} dt_1 dt_2 \\ &= \sqrt[3]{a_2 b_3 b_1 a_3 b_1 a_2} \int\limits_{\square^e} dt_1 dt_2 \\ &= \sqrt[3]{a_2 a_2 a_3 b_1 b_1 b_3}\ .\end{aligned}$$

which involves only coordinates $a_2, a_3, b_1 b_3$.

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Remark 3.10. The essential difference between the MP measure $\boldsymbol{\iota}_\triangle$ and the deGua measure $\boldsymbol{\vartheta}$ is demonstrated in the rhombus of Example 3.9. Equation (3.33) involves all coordinates of $\boldsymbol{a}$ and $\boldsymbol{b}$. By contrast, (3.39), involves only $a_i \ \ \{i \in \{2,3\}\}$ and $b_i \ \ \{i \in \{1,3\}\}$, the coordinates involved in determining the rhombus.

Both terms would be equal whenever

(3.40)
$$a_2 b_1 \ = \ a_1 b_2 \quad \text{that is} \quad \frac{a_2}{a_1} \ = \ \frac{b_2}{b_1}\ .$$

This would reflect the fact that $\Delta^{\boldsymbol{a}}_{1,2}$ and $\Delta^{\boldsymbol{b}}_{1,2}$ have the same slope. This is not compatible with the Cephoid Π under consideration as shown by inspection of Figure 3.1. Obviously (3.40) violates the non degeneracy condition. However, when Π approaches in some sense a sum of homothetic deGua Simplices, then $\boldsymbol{\iota}_\triangle$ and $\boldsymbol{\vartheta}$ would be approximately equal. This observation paves the ground for the development in SECTION 7. There it will be shown that $\boldsymbol{\iota}_\triangle$ and $\boldsymbol{\vartheta}$ are approximating the same limit when a sequence of Cephoid approaches a smooth body.

The deGua measure is a surface measure respecting the rhombi, cylinders, etc. It obeys certain continuity properties. The MP measure is lacking these properties. It is specified on Cephoids only – justified by the axiomatic treatment of the Maschler–Perles Solution as discussed in [14].

This provides the incentive for extending the Maschler–Perles solution via the deGua measure to smooth surfaces as we shall attempt to do in the sequal.

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4 Approximation

The Main Theorem of [15] (Theorem 4.2 and Corollary 4.3) establishes the approximation of a smooth body by a sequence of Cephoids. For reference, we reformulate the content as follows.

For a smooth surface $\partial\Gamma$ and a finite set of points

$$(4.1) \qquad \mathfrak{X}^{\boldsymbol{Q}} = \left\{ \overset{\{q\}}{\boldsymbol{x}} \right\}_{q \in \boldsymbol{Q}} \subseteq \partial\Gamma , \quad \text{with} \quad \boldsymbol{Q} = \{1, \ldots, Q\} \subseteq \mathbb{N},$$

located on $\partial\Gamma$ one can construct a Cephoid

$$(4.2) \qquad \Pi^{\mathfrak{X}^{\boldsymbol{Q}}} = \Pi^{\boldsymbol{Q}}$$

arbitrarily close to $\partial\Gamma$ (uniformly or in Haussdorff distance). $\Pi^{\boldsymbol{Q}}$ is a sum of Q (Pseudo–) Windmills

$$(4.3) \qquad \Pi^{\boldsymbol{Q}} = \sum_{q \in \boldsymbol{Q}} \overset{\{q\}}{\Pi}$$

which are locally adapted, i.e., result from a “calotte” or segment

$$(4.4) \qquad \Gamma_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}}^{\overset{\{q\}}{\boldsymbol{x}}} = \left\{ \Gamma - \overset{\{q\}}{\widehat{\boldsymbol{x}}} \right\} \cap \mathbb{R}^n_+ .$$

The common point

$$(4.5) \qquad \overset{\{q\}}{\boldsymbol{x}} \in \partial\Gamma \cap \partial\Pi_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}}^{\overset{\{q\}}{\boldsymbol{x}}}$$

is the central vertex of the Windmill $\overset{\{q\}}{\Pi}$. Moreover, for each $q \in \boldsymbol{Q}$ the normal $\overset{\{q\}}{\mathfrak{n}}$ of $\partial\Gamma$ in $\overset{\{q\}}{\boldsymbol{x}}$ is also a normal at (the local windmill $\Gamma_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}}^{\overset{\{q\}}{\boldsymbol{x}}}$ as well as) at $\Pi^{\boldsymbol{Q}}$ in $\overset{\{q\}}{\boldsymbol{x}}$ or its translate respectively. This fact is not explicitly stated in the formulation of Theorem 4.2 of [15], but obviously appears as part of the proof. Combining we obtain

Corollary 4.1. Let $\boldsymbol{Q} \subseteq \mathbb{N}$ and let $\mathfrak{X}^{\boldsymbol{Q}} \subseteq \partial\Gamma$. Then there exists a Cephoid $\Pi^{\boldsymbol{Q}}$ such that

1. For every $\overset{\{q\}}{\boldsymbol{x}} \in \mathfrak{X}^{\boldsymbol{Q}}$

$$(4.6) \qquad \overset{\{q\}}{\boldsymbol{x}} \in \partial\Gamma \bigcap \partial\Pi^{\boldsymbol{Q}} ,$$

2. for every $q \in \boldsymbol{Q}$ the normal $\overset{\{q\}}{\mathfrak{n}}$ in $\overset{\{q\}}{\boldsymbol{x}}$ at $\partial\Gamma$ is as well a normal at $\partial\Pi^{\boldsymbol{Q}}$.

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The collection of finite sets as in (4.1) satisfying Corollary 4.1 is denoted by

$$\mathcal{Q} \;:=\; \left\{\mathfrak{X}^{\boldsymbol{Q}} \subseteq \partial\Gamma \,\middle|\, \boldsymbol{Q} \subseteq \mathbb{N}\right\} \tag{4.7}$$

$\mathcal{Q}$ is ordered by inclusion; we consider the lattice $\mathcal{Q}$ to constitute a filter (with reference to $\partial\Gamma$).

We can choose the approximating Cephoids in a way to ensure uniform convergence or convergence in the Hausdorff metric along the filter $\mathcal{Q}$. Formally, we proceed as follows.

Definition 4.2. Let $\boldsymbol{Q} \subseteq \mathbb{N}$ and $\mathfrak{X}^{\boldsymbol{Q}} \in \mathcal{Q}$.

1. For $\overset{\{q\}}{\boldsymbol{x}} \in \mathfrak{X}^{\boldsymbol{Q}}$ with normal $\overset{\{q\}}{\mathfrak{n}}$ at $\partial\Gamma$ and at $\Pi^{\boldsymbol{Q}}$ let

$$\boldsymbol{H}^q \;:=\; \left\{\boldsymbol{x} \in \mathbb{R}^n_+ \,\middle|\, \boldsymbol{x}\overset{\{q\}}{\mathfrak{n}} \leq \overset{\{q\}}{\boldsymbol{x}}\overset{\{q\}}{\mathfrak{n}}\right\} \tag{4.8}$$

   be the half space generated by the supporting hyperplane to $\partial\Gamma$ in $\overset{\{q\}}{\boldsymbol{x}}$.

2. Also let

$$\boldsymbol{C}^{\boldsymbol{Q}} \;:=\; \bigcap_{q\in\boldsymbol{Q}} \boldsymbol{H}^q \tag{4.9}$$

   denote the convex body generated by these half spaces.

3. Next

$$\boldsymbol{C}_{\boldsymbol{Q}} \;:=\; \boldsymbol{CovH}\,\boldsymbol{CmpH}\{\overset{\{q\}}{\boldsymbol{x}} \,|\, q \in \boldsymbol{Q}\} \tag{4.10}$$

   is the convex hull of the "southwest area" of $\mathfrak{X}^{\boldsymbol{Q}}$.

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Collecting our results we obtain

Lemma 4.3. 1. $\boldsymbol{C}_{\boldsymbol{Q}} \subseteq \Pi^{\boldsymbol{Q}}, \Gamma \subseteq \boldsymbol{C}^{\boldsymbol{Q}}$

2. $\boldsymbol{C}_{\boldsymbol{Q}}, \Pi^{\boldsymbol{Q}}, \boldsymbol{C}^{\boldsymbol{Q}}$ converge uniformly to $\partial\Gamma$ along the filter $\mathcal{Q}$

$$\Pi^{\boldsymbol{Q}} \underset{\mathcal{Q}}{\longrightarrow} \partial\Gamma\ . \tag{4.11}$$

   Here the topology maybe chosen to be the Haussdorff topology or equivalently the uniform topology for the functions C defined by the canonical parametrization (justifying the term "uniform").

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Thus, given a smooth surface Γ we may refer to $\mathfrak{Q}$ as the convergent filter (of Cephoids).

Now we will consider the behavior of the surface measures $\boldsymbol{\iota}_\Delta$ and $\boldsymbol{\vartheta}$ along such a filter. We will show that both measures approximate each other and – finally – the measure $\boldsymbol{\vartheta}$ on $\partial\Gamma$. We start out with extending the Simplex parametrization (Definition 3.3), i.e., the mapping

$$\overset{\Delta}{\boldsymbol{x}}(\bullet) \ : \ \Delta^{\boldsymbol{e}} \to \Pi \ , \quad \overset{\Delta}{\boldsymbol{x}}(\boldsymbol{t}) \ := \ \boldsymbol{\kappa}^{-1}(K\boldsymbol{t}) \ \ (\boldsymbol{t} \in \Delta^{\boldsymbol{e}}) \tag{4.12}$$

to smooth surfaces via uniform continuity.

Lemma 4.4. Let Γ be smooth and let $\left\{\Pi^{\boldsymbol{Q}}\right\}_{\boldsymbol{Q}\subseteq\mathbb{N}}$ be an approximating filter, that is,

$$\Pi^{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\to} \partial\Gamma \tag{4.13}$$

uniformly. Let the Simplex parametrizations be given by $\boldsymbol{x}^{\boldsymbol{Q}}(\bullet)$ $(\boldsymbol{Q} \in \mathfrak{Q})$. Then the Simplex parametrizations converge uniformly to a mapping

$$\boldsymbol{x}^{\mathfrak{Q}}(\bullet) \ : \ \Delta^{\boldsymbol{e}} \to \partial\Gamma \ , \tag{4.14}$$

which is continuous and bijective, hence constitutes a parametrization of $\partial\Gamma$.

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Proof:

The proof is standard, however, one has to establish bijectivity. This follows as $\partial\Gamma$ is smooth, i.e. in our terminology, there is a bijection between points $\boldsymbol{x} \in \partial\Gamma$ and normals $\mathfrak{n}^{\boldsymbol{x}}$.

q.e.d.

Definition 4.5. $\overset{\Delta}{\boldsymbol{x}}(\bullet) \ := \ \lim_{\boldsymbol{Q}\in\mathfrak{Q}} \boldsymbol{x}^{\mathfrak{Q}}(\bullet)$ constitutes the Simplex parametrization $(\Delta^{\boldsymbol{e}}, \overset{\Delta}{\boldsymbol{x}}(\bullet))$ of $\partial\Gamma$.

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Next we derive some continuity properties of both measures on surfaces we are dealing with.

Theorem 4.6. **(Continuity of Surface Measures on Cephoids)** *For any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any two families $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ and $\left\{\overset{\star}{\boldsymbol{a}}^{(k)}\right\}_{k\in\boldsymbol{K}}$ with*

$$\left| \boldsymbol{a}^{(k)} - \overset{\star}{\boldsymbol{a}}^{(k)} \right| < \delta \quad (k \in \boldsymbol{K}) \tag{4.15}$$

such that the corresponding Cephoids $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *and* $\overset{\star}{\Pi} = \sum_{k\in\boldsymbol{K}} \Pi^{\overset{\star}{\boldsymbol{a}}^{(k)}}$ *have Pareto Faces* $\boldsymbol{F} = \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ *and* $\overset{\star}{\boldsymbol{F}} = \sum_{k\in\boldsymbol{K}} \overset{\star}{\Delta}^{(k)}_{\boldsymbol{J}^{(k)}}$ *with the same referecce set* $\mathfrak{J} = \left\{\boldsymbol{J}^{(k)}\right\}_{k\in\boldsymbol{K}}$*, it follows that*

$$\left|\boldsymbol{\iota}_\Delta(\boldsymbol{F}) - \boldsymbol{\iota}_\Delta(\overset{\star}{\boldsymbol{F}})\right| < \varepsilon \quad \textit{and} \quad \left|\boldsymbol{\vartheta}(\boldsymbol{F}) - \boldsymbol{\vartheta}(\overset{\star}{\boldsymbol{F}})\right| < \varepsilon \tag{4.16}$$

holds true.

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Proof: For the MP–measure $\boldsymbol{\iota}_\Delta$ this follows immediately from Definition 2.1 which is continuous in terms of a family $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$; see also *Definition 2.4* in CHAPTER XII of [14]. For the deGua measure $\boldsymbol{\vartheta}$ the proof follows from Formulae (3.28) by inserting the Simplex parametrization $\overset{\Delta}{\boldsymbol{x}}(\bullet)$ simultaneously for all Cephoids (and their limits)

q.e.d.

The deGua measure is defined on all convex bodies, hence we can move one step further with its continuity properties.

Theorem 4.7. (Continuity of the deGua Measure)

Let Γ *be a smooth body and let* $\mathfrak{Q}$ *be a filter of a Cephoids such that*

$$\partial\Pi^{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\rightarrow} \partial\Gamma$$

holds true uniformly. Let $\boldsymbol{\vartheta}, \boldsymbol{\vartheta}^{\boldsymbol{Q}}$ $(\boldsymbol{Q} \in \mathfrak{Q})$ *denote the deGua measures on the surfaces* $\partial\Gamma, \partial\Gamma^{\boldsymbol{Q}}$ *respectively. Then, for every continuous function* $F : \partial\Gamma \rightarrow \mathbb{R}$ *we have*

$$\int\limits_{\partial\Pi^{\boldsymbol{Q}}} F d\boldsymbol{\vartheta}^{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\rightarrow} \int\limits_{\partial\Gamma} F d\boldsymbol{\vartheta} \ . \tag{4.17}$$

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Proof:

By Lemma 4.4 we can assume that the Simplex parametrization is employed simultaneously for all surfaces involved, hence $\mathbf{T}$ is the same for all of them. Continuity follows then from Formula 3.28.

q.e.d.

We perceive a Cephoid to be "almost flat" if all deGua Simplices involved are close to each other or, simultaneously, of all the normals are closed. Formally, this is reflected by

Definition 4.8. Let $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$ be a Cephoid. Let $\mathbf{0} < \mathfrak{n}^{\bar{\boldsymbol{a}}} = \bar{\mathfrak{n}} \in \mathbb{R}^n_+$ and let $\mathfrak{n}^{(k)} := \mathfrak{n}^{\boldsymbol{a}^{(k)}}$ $(k \in \boldsymbol{K})$. Let $\delta > 0$. We say that Π is *$(\bar{\mathfrak{n}}, \delta)$–flat* if

$$
(4.18)\qquad \left|\mathfrak{n}^{(k)} - \bar{\mathfrak{n}}\right| < \delta \quad \text{and} \quad \left|\boldsymbol{a}^{(k)} - \bar{\boldsymbol{a}}\right| < \delta \;\; (k \in \boldsymbol{K})
$$

holds true.

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By Remark 2.3 we can represent the Pareto surface of a Cephoid on a suitable multiple of a deGua Simplex $\Delta^{\bar{\boldsymbol{a}}}$. In particular, let a Cephoid $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be $(\bar{\mathfrak{n}}, \delta)$–flat and let $\bar{\boldsymbol{a}}$ correspond to $\bar{\mathfrak{n}}$. Let $\alpha_k := \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}^{(k)}}$ $(k \in \boldsymbol{K})$ and $\alpha := \sum_{k\in\boldsymbol{K}} \alpha_k$ (see Remark 2.3). Consider the representation

$$
\boldsymbol{\kappa}^{\bar{\boldsymbol{a}}} \; : \; \Pi \to \; \alpha\Pi^{\bar{\boldsymbol{a}}} \; .
$$

We expect that the surface measures are approximated by the measures on $\alpha\Delta^{\bar{\boldsymbol{a}}}$.

Theorem 4.9. *Let $\varepsilon > 0$. For any $\mathbf{0} < \bar{\mathfrak{n}} \in \mathbb{R}^n_+$ there exists $\delta > 0$ such that for all $(\bar{\mathfrak{n}}, \delta)$–flat Cephoids Π it follows that (locally on Pareto faces)*

1. $$
(4.19)\qquad \left|\boldsymbol{\iota}_\Delta^{\partial\Pi}(\bullet) - \boldsymbol{\iota}_\Delta^{\alpha\Delta^{\bar{\boldsymbol{a}}}}(\bullet)\right| < \varepsilon \quad \textit{and} \quad \left|\boldsymbol{\vartheta}^{\partial\Pi}(\bullet) - \boldsymbol{\vartheta}^{\alpha\Delta^{\bar{\boldsymbol{a}}}}(\bullet)\right| < \varepsilon \; .
$$

2. *Moreover,*

$$
(4.20)\qquad |\boldsymbol{\iota}_\Delta(\bullet) - \boldsymbol{o}(n)\boldsymbol{\vartheta}(\bullet)| < \varepsilon \; .
$$

That is, $\boldsymbol{\iota}_\Delta$ and $\boldsymbol{\vartheta}$ approach the measure of their representations on $\alpha\Delta^{\bar{\boldsymbol{a}}}$.

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Proof:

1stSTEP : First we consider $\Delta^{\alpha\bar{\boldsymbol{a}}}$. By Lemma 3.8 we have

$$
(4.21)\qquad \boldsymbol{\iota}_\Delta(\Delta^{\bar{\boldsymbol{a}}}) \;=\; \boldsymbol{o}(n)\; \boldsymbol{\vartheta}(\Delta^{\bar{\boldsymbol{a}}}) \; .
$$

Then

$$
(4.22)\qquad \boldsymbol{\iota}_\Delta(\Delta^{\alpha\bar{\boldsymbol{a}}}) \;=\; \sum_{k\in\boldsymbol{K}} \alpha_k \boldsymbol{\iota}_\Delta(\Delta^{\bar{\boldsymbol{a}}^{(k)}})
$$

and hence by (4.21)

$$
(4.23)\qquad \boldsymbol{\vartheta}(\Delta^{\alpha\bar{\boldsymbol{a}}}) \;=\; \sum_{k\in\boldsymbol{K}} \alpha_k \boldsymbol{\vartheta}(\Delta^{\bar{\boldsymbol{a}}^{(k)}})
$$

2$^{\text{nd}}$STEP : We know that $\boldsymbol{\iota}_\Delta$ as well as $\boldsymbol{\vartheta}$ behave continuously with respect to vectors $\boldsymbol{a}$ or corresponding normals $\mathfrak{n}^{\boldsymbol{a}}$ respectively (Theorem 4.6).

Consequently, for any $\varepsilon > 0$ there exists some $\delta > 0$ such that whenever $|\mathfrak{n}^{\boldsymbol{a}} - \overset{\star}{\mathfrak{n}}| < \varepsilon$, it follows that

$$|\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}) - \boldsymbol{\iota}_\Delta(\Delta^{\bar{\boldsymbol{a}}})| < \varepsilon \ , \quad |\boldsymbol{\vartheta}(\Delta^{\boldsymbol{a}}) - \boldsymbol{\vartheta}(\Delta^{\bar{\boldsymbol{a}}})| < \varepsilon \ . \tag{4.24}$$

3$^{\text{rd}}$STEP : Now the analogue for (4.24) holds true for general Pareto faces. Indeed, let

$$\boldsymbol{F} \ = \ \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{4.25}$$

be a Pareto face of Π and let

$$\bar{\boldsymbol{F}} \ = \ \sum_{k\in\boldsymbol{K}} \Delta^{\bar{\boldsymbol{a}}^{(k)}}_{\boldsymbol{J}^{(k)}} \tag{4.26}$$

be the corresponding face of the Cephoid (homothetic) $\alpha\Pi^{\bar{\boldsymbol{a}}}$. Then, analogously to (4.24) we have: for any $\varepsilon > 0$ there exists some $\delta > 0$ such that whenever $|\mathfrak{n}^{\boldsymbol{a}} - \overset{\star}{\mathfrak{n}}| < \varepsilon$, it follows from the Continuity Theorem 4.6 that

$$|\boldsymbol{\iota}_\Delta(\boldsymbol{F}) - \boldsymbol{\iota}_\Delta(\bar{\boldsymbol{F}})| < \varepsilon \ , \quad |\boldsymbol{\vartheta}(\boldsymbol{F}) - \boldsymbol{\vartheta}(\bar{\boldsymbol{F}})| < \varepsilon, \tag{4.27}$$

holds true. This is – not writing the superscripts – Formula (4.19). Also,

$$\boldsymbol{\iota}_\Delta(\bar{\boldsymbol{F}}) = \boldsymbol{o}(n) \ \boldsymbol{\vartheta}(\bar{\boldsymbol{F}}) \ . \tag{4.28}$$

follows immediately, as this refers to the homothetic case.

Consequently, for any face $\boldsymbol{F}$ and the corresponding face $\bar{\boldsymbol{F}}$ we obtain that, whenever $|\mathfrak{n}^{\boldsymbol{a}} - \bar{\mathfrak{n}}| < \delta$, it follows that

(4.29)
$$|\ \boldsymbol{\iota}_\Delta(\boldsymbol{F}) - \boldsymbol{o}(n)\boldsymbol{\vartheta}(\boldsymbol{F})\ | \ < | \ \boldsymbol{\iota}_\Delta(\bar{\boldsymbol{F}}) - \boldsymbol{o}(n)\boldsymbol{\vartheta}(\bar{\boldsymbol{F}})\ | + \varepsilon(\boldsymbol{o}(n) + 1) = \varepsilon(\boldsymbol{o}(n) + 1)$$

which is (4.20).

q.e.d.

Now, whenever we approximate $\partial\Gamma$ by a sum of Local Windmills as described in [15] (see (4.3),(4.4),(4.5)), then, for some common point $\overset{\{q\}}{\boldsymbol{x}}$ of the Pareto-surfaces the family of normals $\overset{(i)}{\mathfrak{n}}$ of the windmill $\overset{\{q\}}{\Pi}$ approximates the normal $\overset{\{q\}}{\mathfrak{n}}$, this can be done uniformly for $i \in \boldsymbol{I}$.

Locally, therefore, both measures will be approximately the same up to the scaling factor $\boldsymbol{o}(n)$. We have

Theorem 4.10. *Let Γ be a smooth body with surface $\partial\Gamma$ and let $\widehat{\boldsymbol{x}} \leq \bar{\boldsymbol{x}} \in \partial\Gamma$. Let $\Pi^{\widetilde{\boldsymbol{x}}}_{\widehat{\boldsymbol{x}}}$ denote the the local windmill generated, i.e., adapted to the calotte $\Gamma^{\bar{\boldsymbol{x}}}_{\widehat{\boldsymbol{x}}}$. Then, for $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever $|\bar{\boldsymbol{x}} - \widehat{\boldsymbol{x}}| < \delta$, it follows that*

$$|\boldsymbol{\iota}_\Delta(\bullet) - \boldsymbol{o}(n)\boldsymbol{\vartheta}(\bullet)| < \varepsilon \tag{4.30}$$

holds true locally on $\Pi^{\widetilde{\boldsymbol{x}}}_{\widehat{\boldsymbol{x}}}$.

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Proof: Both terms are defined to be measures on the local windmill, i.e., explicitly should be written

$$\boldsymbol{\iota}_\Delta \;=\; \boldsymbol{\iota}_\Delta^{\Pi^{\widetilde{\boldsymbol{x}}}_{\widehat{\boldsymbol{x}}}} \quad \text{and} \quad \boldsymbol{\vartheta} \;=\; \boldsymbol{\vartheta}^{\Pi^{\widetilde{\boldsymbol{x}}}_{\widehat{\boldsymbol{x}}}}$$

Both measures have a constant density w.r.t. Lebesgue measure and w.r.t. each other on any Pareto face of the Cephoid $\Pi^{\widetilde{\boldsymbol{x}}}_{\widehat{\boldsymbol{x}}}$. Therefore, it suffices to prove (4.30) with respect to arguments $\boldsymbol{F}$ that are Pareto faces. Consequently, the result follows immediately from Theorem 4.9.

q.e.d.

Now, given $\boldsymbol{Q} \subseteq \mathbb{N}$ and $\mathfrak{X}^{\boldsymbol{Q}} \in \mathfrak{Q}$ with associated (Pseudo) Windmill $\Pi^{\boldsymbol{Q}}$; we also consider the measure $\boldsymbol{\iota}_\Delta^{\boldsymbol{Q}}$ on $\Pi^{\boldsymbol{Q}}$. As the unit sphere of measures on $\partial\Gamma$ is weakly compact, there exist weak accumulation points of the system of measures

$$\left\{ \boldsymbol{\iota}_\Delta{}^{\boldsymbol{Q}} \,\middle|\, \boldsymbol{Q} \subseteq \mathbb{N} \right\} .$$

This notion can be made precise. For every continuous function $\bar{\boldsymbol{F}} : \Delta^{\boldsymbol{e}} \to \mathbb{R}$ with transported versions $F^{\boldsymbol{Q}}$ and $F^{\partial\Gamma}$ integrals

$$\int\limits_{\partial\Pi^{\boldsymbol{Q}}} F^{\boldsymbol{Q}} d\boldsymbol{\iota}_\Delta^{\boldsymbol{Q}}, \quad \int\limits_{\partial\Pi^{\boldsymbol{Q}}} F^{\boldsymbol{Q}} d\boldsymbol{\vartheta}^{\boldsymbol{Q}}, \quad \int\limits_{\partial\Gamma} F^{\partial\Gamma} d\boldsymbol{\vartheta}^{\partial\Gamma} \tag{4.31}$$

etc. can be formulated simultaneously in terms of the Simplex parametrization, see Remark 3.7, Theorem 4.6 and 4.7.

Theorem 4.11. *Let $\boldsymbol{\iota}_\Delta^\Gamma$ be a weak accumulation point of $\left\{ \boldsymbol{\iota}_\Delta{}^{\boldsymbol{Q}} \,\middle|\, \boldsymbol{Q} \subseteq \mathbb{N} \right\}$. Then*

$$\boldsymbol{\iota}_\Delta^\Gamma \;=\; \boldsymbol{o}(n)\boldsymbol{\vartheta} \tag{4.32}$$

holds true.

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Proof:

1^st STEP : First of all we deal with the quantities on $\Pi^{\boldsymbol{Q}}$. We show that, as $\Pi^{\boldsymbol{Q}}$ is sufficiently close to $\partial\Gamma$, the measure $\boldsymbol{\iota}_\Delta$ on $\partial\Pi$ approaches $\boldsymbol{\vartheta}$ weakly.

This happens locally, but as we have uniform convergence, globally as well. As all measures involved have a constant density w.r.t to Lebesgue measure on each Pareto face of some (Pseudo) Windmill involved, we have just to verify the $\boldsymbol{\iota}_\Delta$ and $\boldsymbol{\vartheta}$ are close in measure on any such Pareto face. This is implied by Theorem 4.9, for the normals of an approaching windmill are all close to the one of the center point which equals the normal of this point, at $\partial\Pi^{\boldsymbol{Q}}$ as well s at $\partial\Gamma$.

More precisely, let $\varepsilon > 0$ and choose δ according to Theorem 4.9. Choose $\delta_0 > 0$ and $\mathfrak{X}^{\boldsymbol{Q}}$ such that the Haussdorff distance between $\partial\Pi^{\boldsymbol{Q}}$ and $\partial\Gamma$ is smaller than δ_0. For $\overset{\{q\}}{\boldsymbol{x}} \in \mathfrak{X}^{\boldsymbol{Q}}$ with normal $\mathfrak{n}^q$ (which is the same at $\Pi^{\boldsymbol{Q}}$ and at $\partial\Gamma$) it follows that all normals at the Windmill $\Pi_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}}^{\overset{\{q\}}{\boldsymbol{x}}}$ are close to $\mathfrak{n}^q$. Decrease δ_o such that each $\partial\Pi_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}}^{\overset{\{q\}}{\boldsymbol{x}}}$ $(q \in \boldsymbol{Q})$ is δ–flat. Then, by Theorem 4.9, we know that on $\partial\Pi_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}}^{\overset{\{q\}}{\boldsymbol{x}}}$ $(q \in \boldsymbol{Q})$ we have

$$|\boldsymbol{\iota}_\Delta(\bullet) - \boldsymbol{o}(n)\boldsymbol{\vartheta}(\bullet)| < \varepsilon \tag{4.33}$$

holds true. That is, $\boldsymbol{\iota}_\Delta = \boldsymbol{\iota}_\Delta^{\Pi^{\boldsymbol{Q}}}$ and $\boldsymbol{o}(n) = \boldsymbol{o}(n)\boldsymbol{\vartheta}^{\Pi^{\boldsymbol{Q}}}$ are close in the weak topology ("on $\Pi^{\boldsymbol{Q}}$").

2ndSTEP : Now let $F : \partial\Gamma \to \mathbb{R}$ be a continuous function, the by (4.33) we can find for any $\varepsilon > 0$ some $\mathfrak{X}^{\boldsymbol{Q}} \in \mathfrak{Q}$ such that

$$\left| \int_{\partial\Pi^{\boldsymbol{Q}}} F d\boldsymbol{\iota}_\Delta^{\boldsymbol{Q}} - \boldsymbol{o}(n) \int_{\partial\Gamma} F d\boldsymbol{\vartheta}^{\boldsymbol{Q}} \right| < \varepsilon \tag{4.34}$$

holds true. According to Theorem 4.17 we obtain

$$\int_{\partial\Pi^{\boldsymbol{Q}}} F d\boldsymbol{\vartheta}^{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\mapsto} \int_{\partial\Gamma} F d\boldsymbol{\vartheta} \; . \tag{4.35}$$

Consequently,

$$\int_{\partial\Gamma} F d\boldsymbol{\iota}_\Delta^\Gamma \;=\; \boldsymbol{o}(n) \int_{\partial\Gamma} F d\boldsymbol{\vartheta}^\Gamma \; . \tag{4.36}$$

q.e.d.

Corollary 4.12. Let Γ be a smooth body. Let $\{\Pi^{\boldsymbol{Q}}\}_{\boldsymbol{Q}\in\mathfrak{Q}}$ be an approximating Filter. Then, $\{\boldsymbol{\iota}_\Delta^{\boldsymbol{Q}}\}_{\boldsymbol{Q}\in\mathfrak{Q}}$ has a weak limit $\boldsymbol{\iota}_\Delta^\Gamma$ satisfying

$$\boldsymbol{\iota}_\Delta{}^{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\longrightarrow} \boldsymbol{\iota}_\Delta^\Gamma \;=\; \boldsymbol{o}(n)\boldsymbol{\vartheta}^\Gamma \; . \tag{4.37}$$

holds true. For every continuous function $\bar{\boldsymbol{F}} \;:\; \Delta^{\boldsymbol{e}} \to \mathbb{R}$ with transported Versions $F^{\boldsymbol{Q}}$ and $F^{\boldsymbol{G}}$ one has

$$\int_{\partial\Pi^{\boldsymbol{Q}}} F^{\boldsymbol{Q}} d\boldsymbol{\iota}_\Delta^{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\longrightarrow} \int_{\partial\Gamma} F^{\partial\Gamma} d\boldsymbol{\iota}_\Delta^\Gamma \;=\; \boldsymbol{o}(n) \int_{\partial\Gamma} F^\Gamma d\boldsymbol{\vartheta}^\Gamma \; . \tag{4.38}$$

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5 The deGua Solution

Let Γ be a convex body. Choosing a parametrization $(\boldsymbol{T}, \boldsymbol{x}(\bullet))$ we compute for a measurable $\boldsymbol{F} \subseteq \partial\Gamma$

$$(5.1) \qquad \boldsymbol{\vartheta}(\boldsymbol{F}) = \int_{\boldsymbol{F}} d\boldsymbol{\vartheta} = \int_{\boldsymbol{x}^{-1}(\boldsymbol{F})} \sqrt[n]{D_1 \cdots D_n} dt_1 \cdots dt_{n-1} ,$$

cf. (3.28). In particular, for the canonical parametrization $\left(\Gamma^{(-n)}, \boldsymbol{x}^C(\bullet)\right)$ we obtain by (3.30)

$$(5.2) \qquad \boldsymbol{\vartheta}(\boldsymbol{F}) = \int_{\boldsymbol{x}^{-1}(\boldsymbol{F})} \sqrt[n]{\frac{\partial C}{\partial t_1}(\boldsymbol{t}) \cdots \frac{\partial C}{\partial t_{n-1}}(\boldsymbol{t})} dt_1 \ldots dt_{n-1}$$

and if $\boldsymbol{F}^\star \subseteq \boldsymbol{T}$ is a rectangle

$$\boldsymbol{F}^\star = [\alpha_1, \beta_1] \times \ldots \times [\alpha_{n-1}, \beta_{n-1}] = [\boldsymbol{\alpha}, \boldsymbol{\beta}],$$

such that

$$\boldsymbol{F} = \boldsymbol{x}^C(\boldsymbol{F}^\star) = \{(\boldsymbol{t}, C(\boldsymbol{t})) \,|\, \boldsymbol{t} \in \boldsymbol{F}^\star\} ,$$

then (5.2) changes to

$$(5.3) \qquad \boldsymbol{\vartheta}(\boldsymbol{F}) = \int_{[\boldsymbol{\alpha}, \boldsymbol{\beta}]} \sqrt[n]{\frac{\partial C}{\partial t_1}(\boldsymbol{t}) \cdots \frac{\partial C}{\partial t_{n-1}}(\boldsymbol{t})} dt_1 \ldots dt_{n-1} .$$

Applying this, we define our version of the generalized MASCHLER–PERLES solution via the deGua measure $\boldsymbol{\vartheta}$.

Definition 5.1. Let Γ be a smooth body and let $(\boldsymbol{T}, \boldsymbol{x}(\bullet))$ be a parametrization of $\partial\Gamma$.

1. The image measure of $\boldsymbol{\vartheta}$ transported to $\boldsymbol{T}$ via $\boldsymbol{x}^{-1}$ is denoted

   $$(5.4) \qquad \boldsymbol{\vartheta}^\star = \boldsymbol{\vartheta}^\star_{\boldsymbol{x}(\bullet)} := \boldsymbol{\vartheta} \circ \boldsymbol{x}(\bullet) = \boldsymbol{x}^{-1}(\bullet)\boldsymbol{\vartheta}$$

   (measures are being transported contravariantly, see Remark 3.7).

2. The $(n-1)$ dimensional vector $\boldsymbol{\beta}^\star$

   $$(5.5) \qquad \boldsymbol{\beta}^\star = \boldsymbol{\beta}^\star(\boldsymbol{T}) = \boldsymbol{\beta}^\star(\boldsymbol{T}, \boldsymbol{x}(\bullet)) := \frac{1}{\boldsymbol{\vartheta}^\star(\boldsymbol{T})} \int_{\boldsymbol{T}} \boldsymbol{t}\boldsymbol{\vartheta}^\star(d\boldsymbol{t})$$

   is the ***barycenter*** or ***center of gravity*** of $\boldsymbol{T}$ under $\boldsymbol{\vartheta}^\star$.

3. The ***DeGua solution*** $\boldsymbol{\delta}(\Gamma) \in \partial\Gamma$ of Γ is the inverse barycenter of $\boldsymbol{T}$, that is,

$$\boldsymbol{\delta}(\Gamma) \;:=\; \boldsymbol{x}(\boldsymbol{\beta}^\star) \;=\; \boldsymbol{x}\left(\frac{1}{\boldsymbol{\vartheta}^\star(\boldsymbol{T})}\int\limits_{\boldsymbol{T}} \boldsymbol{t}\boldsymbol{\vartheta}^\star(d\boldsymbol{t})\right) \tag{5.6}$$

• ~~~~~~~ •

Remark 5.2. 1. The intuition behind this idea is the analogue definition of the generalized MASCHLER–PERLES solution solution for Cephoids ([9], see also [14], [10]). Within that context, we obtain the solution as the (transported) barycenter of the Pareto surface, with weights assigned according to $\boldsymbol{\iota}_\Delta$. The axiomatic justification – via superadditivity in 2 dimensions and the appropriate generalizations in higher dimensions – has been provided by Maschler and Perles and extended in [5] and [13]. For a detailed discussion and justification of bargaining solutions via surface measures we refer to the presentation in [14].

2. Necessarily, one should verify the independence of all concepts from the choice of the parametrization. We omit the lengthy and tedious proofs.

3. The standard formula for the change of variables yields

$$\boldsymbol{\delta}(\Gamma) \;=\; \boldsymbol{x}\left(\frac{1}{\boldsymbol{\vartheta}(\partial\Gamma)}\int\limits_{\partial\Gamma} (\boldsymbol{x}(\bullet))^{-1} d\boldsymbol{\vartheta}\right) \tag{5.7}$$

4. Specifying this to the canonical parametrization $\left(\Gamma^{(-n)}, \boldsymbol{x}^C(\bullet)\right)$ we have

$$\boldsymbol{\vartheta}(\partial\Gamma) \;=\; \int\limits_{\Gamma^{(-n)}} \sqrt[n]{\frac{\partial C}{\partial t_1}(\boldsymbol{t}) \cdots \frac{\partial C}{\partial t_{n-1}}(\boldsymbol{t})} dt_1 \dots dt_{n-1} \tag{5.8}$$

such that

$$\boldsymbol{\vartheta}^\star \;=\; \boldsymbol{\vartheta}^\star_{\boldsymbol{x}^C(\bullet)} \;:=\; \boldsymbol{\vartheta} \circ \boldsymbol{x}^C(\bullet) \;=\; \boldsymbol{x}^{C^{-1}}(\bullet)\boldsymbol{\vartheta} \tag{5.9}$$

is the image measure of $\boldsymbol{\vartheta}$ transported to $\Gamma^{(-n)}$ via $\boldsymbol{x}(\bullet)^{C^{-1}}$. We obtain for the barycenter

$$\begin{aligned} \boldsymbol{\beta}^\star \;&=\; \frac{1}{\boldsymbol{\vartheta}^\star(\mathbf{T})}\int\limits_{\mathbf{T}} \boldsymbol{t}\boldsymbol{\vartheta}^\star(d\boldsymbol{t}) \;=\; \frac{1}{\boldsymbol{\vartheta}^\star(\Gamma^{(-n)})}\int\limits_{\Gamma^{(-n)}} \boldsymbol{t}\boldsymbol{\vartheta}^\star(d\boldsymbol{t}) \\ &=\; \frac{1}{\boldsymbol{\vartheta}(\partial\Gamma)}\int\limits_{\partial\Gamma} (x_1, \dots, x_{n-1})\boldsymbol{\vartheta}(dx_1, \dots, dx_n) \end{aligned} \tag{5.10}$$

and for the deGua solution

$$(5.11) \qquad \boldsymbol{\delta}(\Gamma) \;=\; \boldsymbol{x}^C(\boldsymbol{\beta}^\star) \;=\; \boldsymbol{x}^C\left(\frac{1}{\boldsymbol{\vartheta}(\partial\Gamma)}\int_{\partial\Gamma}(x_1,\ldots,x_{n-1})d\boldsymbol{\vartheta}\right)$$

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Example 5.3. For computational purposes, we change the parametrization. For example, to obtain the barycenter for a Simplex $\Delta^{\boldsymbol{a}}$ in 3 dimensions, we choose the parametrization via the unit Simplex; see (3.7) and (3.8), also (3.21). Then (3.24) suggests for any function f on $\Delta^{\boldsymbol{e}^{12}}$

$$(5.12) \qquad \begin{aligned}\int_{\Delta^{e^{12}}} f(\boldsymbol{t})\boldsymbol{\vartheta}^\star(d\boldsymbol{t}) \;&=\; \int_{\Delta^{\boldsymbol{a}}} f(\boldsymbol{t})\sqrt[3]{(D_1D_2D_3)\circ\boldsymbol{x}}\;dt_1dt_2 \\ &=\; \frac{1}{2}\sqrt[3]{(a_1a_2a_3)^2}\int_{\Delta^{e^{12}}} f(\boldsymbol{t})\;dt_1dt_2.\end{aligned}$$

Thus, in particular

$$(5.13) \qquad \int_{\Delta^{e^{12}}}(t_1,t_2)\boldsymbol{\vartheta}^\star(d\boldsymbol{t}) \;=\; \frac{1}{2}\sqrt[3]{(a_1a_2a_3)^2}\int_{\Delta^{e^{12}}}(t_1,t_2)\;dt_1dt_2.$$

Now

$$(5.14) \qquad \begin{aligned}\int_{\Delta^{e^{12}}} t_1dt_1dt_2 \;&=\; \int_0^1 t_1\left(\int_0^{1-t_1} dt_2\right)dt_1 \\ &=\; \int_0^1 t_1(1-t_1)dt_1 \;=\; \left|\frac{t_1^2}{2}-\frac{t_1^3}{3}\right|_0^1 \\ &=\; \frac{1}{2}-\frac{1}{3} \;=\; \frac{1}{6}\end{aligned}$$

Also

$$(5.15) \qquad \begin{aligned}\int_{\Delta^{e^{12}}} dt_1dt_2 \;&=\; \int_0^1\left(\int_0^{1-t_1} dt_2\right)dt_1 \\ &=\; \int_0^1 (1-t_1)dt_1 \;=\; \left|t_1-\frac{t_1^2}{2}\right|_0^1 \\ &=\; 1-\frac{1}{2} \;=\; \frac{1}{2}\,.\end{aligned}$$

For to determine $\boldsymbol{\beta}^\star$ the common factor $\frac{1}{2}\sqrt[3]{(a_1a_2a_3)^2}$ in (5.12) cancels out, thus we obtain

$$(5.16) \qquad \boldsymbol{\beta}^\star \;=\; \frac{\int_{\Delta^{e^{12}}} t_1dt_1dt_2}{\int_{\Delta^{e^{12}}} dt_1dt_2} \;=\; \frac{(\frac{1}{6},\frac{1}{6})}{\frac{1}{2}} \;=\; (\frac{1}{3},\frac{1}{3})\,.$$

From this we derive the deGua solution to be

$$(5.17)\quad \boldsymbol{\delta}(\Delta^{\boldsymbol{a}}) \;=\; \boldsymbol{x}(\boldsymbol{\beta}^\star) \;=\; (a_1 t_1^\star, a_2 t_2^\star, a_3(1-(t_1^\star+t_2^\star))) \;=\; \left(\frac{a_1}{3},\frac{a_2}{3},\frac{a_3}{3}\right) .$$

As expected, this is the barycenter of $\Delta^{\boldsymbol{a}}$.

Similarly, for to compute the barycenter of the Pareto face $\Lambda^{\boldsymbol{ab}}_{23\ 13}$ we use the parametrization indicated in (3.36), that is $\square^{\boldsymbol{e}} \;=\; \left\{\boldsymbol{x}\in\mathbb{R}^2_+ \,\middle|\, \boldsymbol{x}\leq\boldsymbol{e}=(1,1)\right\}$ and

$$(5.18)\qquad \boldsymbol{x}(t_1,t_2) \;=\; (b_1t_1, a_2t_2, a_3(1-t_2)+b_3(1-t_1)) \quad (t_1,t_2)\in\square^{\boldsymbol{e}} .$$

According to (3.39) the common factor $\sqrt[3]{a_2a_2a_3b_1b_1b_3}$ will cancle out, so the barycenter of $\square^{\boldsymbol{e}}$ is

$$t^\star = (\frac{1}{2},\frac{1}{2})$$

and hence

(5.19)

$$\begin{aligned}\boldsymbol{\delta}(\Lambda^{\boldsymbol{ab}}_{23\ 13}) \;=\; \boldsymbol{x}(\boldsymbol{\beta}^\star) \;&=\; \left(\frac{b_1}{2},\frac{a_2}{2},\frac{a_3+b_3}{2}\right) \;=\; \left(0,\frac{a_2}{2},\frac{a_3}{2}\right)+\left(\frac{b_1}{2},0,\frac{b_3}{2}\right)\\ &=\; \frac{1}{2}(\boldsymbol{a}^2+\boldsymbol{a}^3)+\frac{1}{2}(\boldsymbol{b}^1+\boldsymbol{b}^3)\end{aligned}$$

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The results of the above example confirm the obvious: for simple geometrical objects (DeGua Simplices, rhombi, cylinders, etc.) the barycenter is easily computed via symmetry properties. We combine these facts as follows.

Lemma 5.4. 1. Let $\boldsymbol{a} > \boldsymbol{0}$ and $\boldsymbol{J}\subseteq\boldsymbol{I}$. Then

$$(5.20)\qquad \boldsymbol{\delta}(\Delta^{\boldsymbol{a}}_{\boldsymbol{J}}) = \frac{1}{|\boldsymbol{J}|}\sum_{i\in\boldsymbol{J}}\boldsymbol{a}^i .$$

2. Let $\Pi \;=\; \sum_{k\in\boldsymbol{K}}\Pi^{\boldsymbol{a}^{(k)}}$ be a Cephoid and let $\boldsymbol{F} \;=\; \sum_{k\in\boldsymbol{K}}\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ be a Pareto face of Π. Then

$$(5.21)\qquad \boldsymbol{\delta}(\boldsymbol{F}) \;=\; \sum_{k\in\boldsymbol{K}}\boldsymbol{\delta}(\Delta^{(k)}_{\boldsymbol{J}^{(k)}}) = \sum_{k\in\boldsymbol{K}}\frac{1}{|\boldsymbol{J}^{(k)}|}\sum_{i\in\boldsymbol{J}^{(k)}}\boldsymbol{a}^{(k)i} .$$

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Proof: A formal proof just has to refer to the normal that is constant on any Pareto face as treated. Hence, the deGua measure has a constant density w.r.t. to the Lebesgue measure and in computing the barycenter we experience that the common factors (generated by the density) do cancle out as in our above examples.

q.e.d.

For comparison we list the barycenter and the deGua solution for general Cephoids. In the context of the MP–measure and the $\boldsymbol{\mu\pi}$–solution the measure preserving mapping already incorporated the measure $\boldsymbol{\iota}_\Delta$ by constructing the appropriate multiple of $\Delta^{\boldsymbol{e}}$.

The results of Lemma (5.4) immediately carry over to the barycenter in $K\Delta^{\boldsymbol{e}}$.

Lemma 5.5. Let $\boldsymbol{F} = \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ be a Pareto face of Π and let $\boldsymbol{F}^\star = \boldsymbol{\kappa}(\boldsymbol{F})$. Then

$$\boldsymbol{\beta}^\star(\boldsymbol{F}^\star) = \boldsymbol{\kappa}(\boldsymbol{\delta}(\boldsymbol{F})) = \sum_{k\in\boldsymbol{K}} \frac{1}{|\boldsymbol{J}^{(k)}|} \sum_{i\in\boldsymbol{J}^{(k)}} \boldsymbol{e}^{(k)i} . \tag{5.22}$$

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Proof: Obvious. **q.e.d.**

Theorem 5.6. *Let Π be a Cephoid and let $(K\Delta^{\boldsymbol{e}}, \boldsymbol{\kappa}^{-1})$ be the standard parametrization. Let $\mathcal{P}$ denote the collection of maximal Pareto faces and denote $\boldsymbol{F}^\star := \boldsymbol{\kappa}(\boldsymbol{F})$. Then*

$$\begin{aligned} \boldsymbol{\beta}^\star(K\Delta^{\boldsymbol{e}}) &= \boldsymbol{\beta}^\star(\boldsymbol{\kappa}(\partial\Pi)) \\ &= \sum_{\boldsymbol{F}\in\mathcal{P}} \frac{\boldsymbol{\vartheta}^\star(\boldsymbol{\kappa}(\boldsymbol{F}))}{\boldsymbol{\vartheta}^\star(\boldsymbol{\kappa}(\partial\Pi))} \boldsymbol{\beta}^\star(\boldsymbol{\kappa}(\boldsymbol{F})) = \sum_{\boldsymbol{F}\in\mathcal{P}} \frac{\boldsymbol{\vartheta}^\star(\boldsymbol{F}^\star)}{\boldsymbol{\vartheta}^\star(K\Delta^{\boldsymbol{e}})} \boldsymbol{\beta}^\star(\boldsymbol{F}) \end{aligned} \tag{5.23}$$

is the barycenter of $K\Delta^{\boldsymbol{e}}$. Consequently, the deGua solution is

$$\boldsymbol{\delta}(\Pi) = \boldsymbol{\kappa}^{-1}(\boldsymbol{\beta}^\star)(\boldsymbol{\kappa}(\partial\Pi)) = \boldsymbol{\kappa}^{-1}\left(\sum_{\boldsymbol{F}\in\mathcal{P}} \frac{\boldsymbol{\vartheta}^\star(\boldsymbol{\kappa}(\boldsymbol{F}))}{\boldsymbol{\vartheta}^\star(\boldsymbol{\kappa}(\partial\Pi))} \boldsymbol{\beta}^\star(\boldsymbol{\kappa}(\boldsymbol{F})) \right) \tag{5.24}$$

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Proof: Formula (5.23) is a standard proposition regarding the barycenter in Physical context. As $\boldsymbol{\kappa}$ is bijective and preserving the lattice structure of the Pareto faces we have

$$K\Delta^{\boldsymbol{e}} = \bigcup_{\boldsymbol{F}\in\mathcal{P}} \boldsymbol{\kappa}(\boldsymbol{F}) \tag{5.25}$$

and hence

$$\begin{aligned} \boldsymbol{\beta}^\star(K(\Delta^{\boldsymbol{e}})) &= \frac{1}{\boldsymbol{\vartheta}^\star(K\Delta^{\boldsymbol{e}})} \int_{K\Delta^{\boldsymbol{e}}} \boldsymbol{t}\boldsymbol{\vartheta}^\star(d\boldsymbol{t}) \\ &= \sum_{\boldsymbol{F}\in\mathcal{P}} \frac{\boldsymbol{\vartheta}^\star(\boldsymbol{\kappa}(\boldsymbol{F}))}{\boldsymbol{\vartheta}^\star(K\Delta^{\boldsymbol{e}})} \frac{1}{\boldsymbol{\vartheta}^\star(\boldsymbol{\kappa}(\boldsymbol{F}))} \int_{\boldsymbol{\kappa}(\boldsymbol{F})} \boldsymbol{t}\boldsymbol{\vartheta}^\star(d\boldsymbol{t}) \\ &= \sum_{\boldsymbol{F}\in\mathcal{P}} \frac{\boldsymbol{\vartheta}^\star(\boldsymbol{\kappa}(\boldsymbol{F}))}{\boldsymbol{\vartheta}^\star(K\Delta^{\boldsymbol{e}})} \boldsymbol{\beta}^\star(\boldsymbol{\kappa}(\boldsymbol{F})) \end{aligned} \tag{5.26}$$

q.e.d.

6 Standard Axioms for $\boldsymbol{\delta}$

As a routine exercise we verify that $\boldsymbol{\delta}$ is a bargaining solution, that is,

Theorem 6.1. *The mapping $\boldsymbol{\delta}$ respects anonymity, affine transformation of utility, and is Pareto efficient.*

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Proof: 1stSTEP : Pareto efficiency is obvious by definition.

2ndSTEP : In order to check anonymity, consider a permutation $\pi : \boldsymbol{I} \to \boldsymbol{I}$. A permutation constitutes a linear mapping $\pi : \mathbb{R}^n \to \mathbb{R}^n$ via

$$(\pi(\boldsymbol{x}))_i := x_{\pi^{-1}(i)} \quad (\boldsymbol{x} \in \mathbb{R}^n, \ i \in \boldsymbol{I}).$$

Accordingly, for $\boldsymbol{F} \subseteq \mathbb{R}^n_+$ the permuted version is

$$\pi(\boldsymbol{F}) := \{\pi(\boldsymbol{x}) \,|\, \boldsymbol{x} \in \boldsymbol{F}\} \tag{6.1}$$

and if $\boldsymbol{x}(\bullet)$ parametrizes $\partial\Gamma$ then

$$(\pi\boldsymbol{x})(\bullet) \ := \ \pi(\boldsymbol{x}(\bullet))$$

parametrizes $\pi(\partial\Gamma) = \partial(\pi\Gamma)$. Therefore we obtain

$$\begin{aligned}
\boldsymbol{\delta}(\boldsymbol{\pi}\Gamma) &= (\boldsymbol{\pi x})\left(\frac{1}{\boldsymbol{\vartheta}^{\star}_{(\boldsymbol{\pi x})(\bullet)}(\mathbf{T})}\int\limits_{\boldsymbol{\pi}\Gamma} \boldsymbol{t}\left(\boldsymbol{\vartheta}^{\star}_{(\boldsymbol{\pi x})(\bullet)}\right)(d\boldsymbol{t})\right)\\
&= (\boldsymbol{\pi x})\left(\frac{1}{(\boldsymbol{\pi x}(\bullet))^{-1}\boldsymbol{\vartheta}(\mathbf{T})}\int\limits_{\boldsymbol{\pi}\Gamma} \boldsymbol{t}\left(((\boldsymbol{\pi x})(\bullet))^{-1}\boldsymbol{\vartheta}\right)(d\boldsymbol{t})\right)\\
&= (\boldsymbol{\pi x})\left(\frac{1}{\boldsymbol{\vartheta}(\boldsymbol{\pi}(\boldsymbol{x}(\mathbf{T})))}\int\limits_{\boldsymbol{\pi}\Gamma} \boldsymbol{t}\left(\boldsymbol{\vartheta}(\boldsymbol{\pi}(\boldsymbol{x}(\bullet)))(d\boldsymbol{t})\right)\right)\\
&= (\boldsymbol{\pi x})\left(\frac{1}{\boldsymbol{\vartheta}(\boldsymbol{x}(\mathbf{T}))}\int\limits_{\Gamma} \boldsymbol{t}\left(\boldsymbol{\vartheta}(\boldsymbol{x}(\bullet))(d\boldsymbol{t})\right)\right)\\
&= \boldsymbol{\pi}(\boldsymbol{\delta}(\Gamma))
\end{aligned} \tag{6.2}$$

Hence, $\boldsymbol{\pi}(\boldsymbol{\delta})$ is the deGua solution to $\pi\Gamma$, i.e.,

$$\boldsymbol{\delta}(\boldsymbol{\pi}\Gamma) \ = \ \boldsymbol{\pi}(\boldsymbol{\delta}(\Gamma)) \ , \tag{6.3}$$

3rdSTEP :

Covariance with a.t.u. is verified similarly. For positive $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ let

$$L \ : \ \mathbb{R}^n \to \mathbb{R}^n \ , \qquad L(\boldsymbol{x}) \ = \ (\alpha_1 x_1, \ldots, \alpha_n x_n) \ \ (\boldsymbol{x} \in \mathbb{R}^n)$$

be a (positive) linear mapping and let Γ be a a bargaining problem with smooth surface $\partial\Gamma$. The translated version is

$$L\Gamma \;:=\; \{L(\boldsymbol{x}) \,|\, \boldsymbol{x} \in \Gamma\} \tag{6.4}$$

and if $\boldsymbol{x}(\bullet) \;:=\; \mathbf{T} \to \partial\Gamma$ parametrizes the surface $\partial\Gamma$, then

$$\begin{aligned} (L\boldsymbol{x})(\bullet):\; &\mathbf{T} \to L\Gamma \\ (L\boldsymbol{x})(t_1,\ldots,t_n) =\; &L(\boldsymbol{x}(t_1,\ldots,t_n)) \;\; (t \in \mathbf{T}) \end{aligned} \tag{6.5}$$

parametrizes $L\Gamma$. In view of (3.11) we obtain for the tangent vector

$$\frac{\partial(L\boldsymbol{x})}{\partial t_j} \;=\; \left(\frac{\alpha_1 \partial x_1}{\partial t_j}, \frac{\alpha_2 \partial x_2}{\partial t_j}, \ldots, \frac{\alpha_n \partial x_n}{\partial t_j}\right)(\bar{\boldsymbol{t}}) \;\; (j \in \boldsymbol{I}) \;, \tag{6.6}$$

and observing (3.13) we obtain

$$(D_i(L\boldsymbol{x}))(\bar{\boldsymbol{t}}) = \left(\prod_{j \in \boldsymbol{I}\setminus\{i\}} \alpha_j\right) \; (D_i\boldsymbol{x})(\bar{\boldsymbol{t}}) \;\; (i \in \boldsymbol{I}). \tag{6.7}$$

This yields the coordinates of the normal vector

$$\mathfrak{n}^{(L\boldsymbol{x})(\bar{\boldsymbol{t}})} \;=\; ((D_1(L\boldsymbol{x}))(\bar{\boldsymbol{t}}), \ldots, (D_n(L\boldsymbol{x}))(\bar{\boldsymbol{t}})) \;. \tag{6.8}$$

Now, if $\boldsymbol{F} \subseteq \partial\Gamma$ then the translated $L\boldsymbol{F}$ yields a deGua measure

$$\begin{aligned} \boldsymbol{\vartheta}(L\boldsymbol{F}) \;&=\; \int_{L\boldsymbol{F}} \sqrt[n]{d(L\mathfrak{n})_1 \cdots d(L\mathfrak{n})_n} \\ &=\; \int_{\mathbf{T}} \sqrt[n]{D_1(L\boldsymbol{x})(\bullet) \cdots D_n(L\boldsymbol{x})(\bullet)} \; d\boldsymbol{\lambda}(\bullet) \\ &=\; \left(\prod_{i \in \boldsymbol{I}} \alpha_i\right)^{n-1} \int_{\mathbf{T}} \sqrt[n]{D_1(\boldsymbol{x})(\bullet) \cdots D_n(\boldsymbol{x})(\bullet)} \; d\boldsymbol{\lambda}(\bullet) \\ &=\; \left(\prod_{i \in \boldsymbol{I}} \alpha_i\right)^{n-1} \boldsymbol{\vartheta}(\boldsymbol{F}) \;. \end{aligned} \tag{6.9}$$

and in particular

$$\boldsymbol{\vartheta}\,(L\Gamma) \;=\; \left(\prod_{i \in \boldsymbol{I}} \alpha_i\right)^{n-1} \boldsymbol{\vartheta}\,(\Gamma) \;. \tag{6.10}$$

Therefore, we obtain

(6.11)

$$\begin{aligned}
\boldsymbol{\delta}(L\Gamma) &= (L\boldsymbol{x})\left(\frac{1}{\boldsymbol{\vartheta}^{\star}_{(L\boldsymbol{x})(\bullet)}(\mathbf{T})}\int\limits_{L\Gamma}\boldsymbol{t}\left(\boldsymbol{\vartheta}^{\star}_{(L\boldsymbol{x})(\bullet)}\right)(d\boldsymbol{t})\right)\\
&= (L\boldsymbol{x})\left(\frac{1}{(L\boldsymbol{x}(\bullet))^{-1}\boldsymbol{\vartheta}(\mathbf{T})}\int\limits_{L\Gamma}\boldsymbol{t}\left((L\boldsymbol{x})(\bullet))^{-1}\boldsymbol{\vartheta}(d\boldsymbol{t})\right)\right)\\
&= (L\boldsymbol{x})\left(\frac{1}{\boldsymbol{\vartheta}(L(\boldsymbol{x}(\mathbf{T})))}\int\limits_{L\Gamma}\boldsymbol{t}\boldsymbol{\vartheta}(L(\boldsymbol{x}(\bullet)))(d\boldsymbol{t})\right)\\
&= (L\boldsymbol{x})\left(\frac{1}{\left(\prod_{i\in\boldsymbol{I}}\alpha_i\right)^{n-1}\boldsymbol{\vartheta}(\boldsymbol{x}(\mathbf{T}))}\int\limits_{L\Gamma}\boldsymbol{t}\left(\prod_{i\in\boldsymbol{I}}\alpha_i\right)^{n-1}\boldsymbol{\vartheta}(\boldsymbol{x}(\bullet))(d\boldsymbol{t})\right)\\
&= \boldsymbol{\pi}(\boldsymbol{\delta}(\Gamma))
\end{aligned}$$

Hence, $L(\boldsymbol{\delta})$ is the deGua solution to $L\Gamma$, i.e.,

(6.12) $$\boldsymbol{\delta}(L\Gamma) = L(\boldsymbol{\delta}(\Gamma)) ,$$

q.e.d.

7 Coincidence: The MP–Solution on Smooth Bodies

Our two concepts of a bargaining solution via a surface measure coincide on smooth bodies. Therefore, the deGua Solution on smooth bodies can be regarded as an extension of the Maschler–Perles solution.

Formally, the result is based on the existence of a unique positive normal. Therefore, the approximating Cephoids essentially have locally a flat region of the surface dictated by the limiting normal. It is important to reconsider Example 3.10 in this context.

That example also shows that $\boldsymbol{\vartheta}$ and $\boldsymbol{\iota}_{\Delta}$ may differ and hence $\boldsymbol{\delta}$ and $\boldsymbol{\mu}$ cannot be expected to generally coincide on Cephoids..

Theorem 7.1. *Let Γ be a smooth body. Let $\left\{\Pi^{\boldsymbol{Q}}\right\}_{\boldsymbol{Q}\in\mathfrak{Q}}$ be the filter of Windmills according to Lemma 4.3 such that*

$$\Pi^{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\rightarrow} \partial\Gamma.$$

Then

$$\lim_{\boldsymbol{Q}\in\mathfrak{Q}} \boldsymbol{\delta}(\Pi^{\boldsymbol{Q}}) \;=\; \boldsymbol{\delta}(\Gamma) \tag{7.1}$$

holds true.

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Proof: This follows from the continuity properties of the deGua measure $\boldsymbol{\vartheta}$.

1^st^STEP : According to Theorem 4.7 we know that

$$\boldsymbol{\vartheta}(\Pi^{\boldsymbol{Q}}) \underset{\mathfrak{Q}}{\longrightarrow} \boldsymbol{\vartheta}(\partial\Gamma) \tag{7.2}$$

holds true. Also, as the Simplex parametrization

$$\left(\Delta^{\boldsymbol{e}}, \overset{\Delta}{\boldsymbol{x}}{}^{\boldsymbol{Q}}(\bullet)\right)_{\boldsymbol{Q}\in\ \mathfrak{Q}} \quad \text{and} \quad \left(\Delta^{\boldsymbol{e}}, \overset{\Delta}{\boldsymbol{x}}{}^{\partial\Gamma}(\bullet)\right)$$

is chosen simultaneously for all surfaces involved, we know by Lemma 4.4 that

$$\overset{\Delta}{\boldsymbol{x}}{}^{\boldsymbol{Q}}(\bullet) \underset{\mathfrak{Q}}{\longrightarrow} \overset{\Delta}{\boldsymbol{x}}{}^{\partial\Gamma}(\bullet) \quad \text{as well as} \quad \left(\overset{\Delta}{\boldsymbol{x}}{}^{\boldsymbol{Q}}(\bullet)\right)^{-1} \underset{\mathfrak{Q}}{\longrightarrow} \left(\overset{\Delta}{\boldsymbol{x}}{}^{\partial\Gamma}(\bullet)\right)^{-1} \tag{7.3}$$

uniformly, as the inverse of a continuous bijection is continuous.

2^nd^STEP :

Consequently, in view of Remark 5.2, and in particular Formula (5.7), we obtain

$$(7.4)\quad \overset{\wedge}{\boldsymbol{x}}^{\boldsymbol{Q}}\left(\frac{1}{\boldsymbol{\vartheta}(\partial\Gamma^{\boldsymbol{Q}})}\int\limits_{\mathbf{T}}(\boldsymbol{x}^{\boldsymbol{Q}}(\bullet))^{-1}d\boldsymbol{\vartheta}\right)\underset{\mathfrak{Q}}{\longrightarrow}\overset{\wedge}{\boldsymbol{x}}^{\partial\Gamma}\left(\frac{1}{\boldsymbol{\vartheta}(\partial\Gamma)}\int\limits_{\mathbf{T}}(\boldsymbol{x}^{\partial\Gamma}(\bullet))^{-1}d\boldsymbol{\vartheta}\right),$$

which is (7.1).

q.e.d.

Theorem 7.2. *Let Γ be a smooth body. Let $\left\{\Pi^{\boldsymbol{Q}}\right\}_{\boldsymbol{Q}\in\mathfrak{Q}}$ be the filter of Cephoids according to Lemma 4.3 such that*

$$\Pi^{\boldsymbol{Q}}\underset{\mathfrak{Q}}{\rightarrow}\partial\Gamma$$

holds true. Then there exists

$$(7.5)\qquad \lim_{\boldsymbol{Q}\in\mathfrak{Q}}\ \boldsymbol{\mu}(\Pi^{\boldsymbol{Q}})\ =:\ \boldsymbol{\mu}(\Gamma)\ .$$

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Proof: By Corollary 4.12

$$(7.6)\qquad \boldsymbol{\iota}_{\triangle}{}^{\boldsymbol{Q}}\underset{\mathfrak{Q}}{\longrightarrow}\boldsymbol{\iota}_{\triangle}^{\Gamma}$$

holds true. The Maschler–Perles solution for a Cephoid Π is

$$(7.7)\qquad \boldsymbol{\mu}(\Pi)\ =\ \overset{\wedge}{\boldsymbol{x}}\left(\frac{1}{\boldsymbol{\iota}_{\triangle}(\partial\Gamma)}\int\limits_{\mathbf{T}}(\boldsymbol{x}(\bullet))^{-1}d\boldsymbol{\iota}_{\triangle}\right)\ .$$

The limiting equation

$$(7.8)\qquad \begin{aligned}\boldsymbol{\mu}(\Pi^{\boldsymbol{Q}})\ &=\ \overset{\wedge}{\boldsymbol{x}}^{\boldsymbol{Q}}\left(\frac{1}{\boldsymbol{\iota}_{\triangle}^{\boldsymbol{Q}}(\partial\Gamma^{\boldsymbol{Q}})}\int\limits_{\mathbf{T}}(\boldsymbol{x}^{\boldsymbol{Q}}(\bullet))^{-1}d\boldsymbol{\iota}_{\triangle}^{\boldsymbol{Q}}\right)\\ &\underset{\mathfrak{Q}}{\longrightarrow}\ \overset{\wedge}{\boldsymbol{x}}^{\partial\Gamma}\left(\frac{1}{\boldsymbol{\iota}_{\triangle}^{\Gamma}(\partial\Gamma)}\int\limits_{\mathbf{T}}(\boldsymbol{x}^{\partial\Gamma}(\bullet))^{-1}d\boldsymbol{\iota}_{\triangle}^{\Gamma}\right)\ =:\ \boldsymbol{\mu}(\Gamma)\end{aligned}$$

holds true by exactly the same argments as in Theorem 7.1. **q.e.d.**

Definition 7.3.

$$(7.9)\qquad \boldsymbol{\mu}(\Gamma)\ :=\ \lim_{\boldsymbol{Q}\in\mathfrak{Q}}\ \boldsymbol{\mu}(\Pi^{\boldsymbol{Q}})$$

is the ***Maschler–Perles solution*** of Γ.

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Theorem 7.4. [Main Theorem of MP–Solution Theory] *Let Γ be a smooth body. Then the Maschler–Perles Solution and the deGua solution coincide, i.e.,*

$$\boldsymbol{\mu}(\Gamma) \;=\; \boldsymbol{\delta}(\Gamma)\ . \tag{7.10}$$

Consequently, for any parametrization $(\boldsymbol{T}, \boldsymbol{x}(\bullet))$, the Maschler–Perles solution is the image of the barycenter under the deGua measure $\boldsymbol{\vartheta}$:

$$\boldsymbol{\delta}(\Gamma) \;=\; \boldsymbol{x}\left(\frac{1}{\boldsymbol{\vartheta}(\partial\Gamma)}\int\limits_{\partial\Gamma}(\boldsymbol{x}(\bullet))^{-1}d\boldsymbol{\vartheta}\right)\ . \tag{7.11}$$

• ~~~∘~~~ •

Proof: By Corollary 4.12

$$\boldsymbol{\iota}_{\Delta}^{\Gamma} \;=\; \boldsymbol{o}(n)\boldsymbol{\vartheta}\Gamma\ . \tag{7.12}$$

holds true. Hence,

$$\begin{aligned}
\boldsymbol{\mu}(\Gamma) \;&=\; \left(\frac{1}{\boldsymbol{\iota}_{\Delta}^{\Gamma}(\partial\Gamma)}\int\limits_{\Delta^e}(\boldsymbol{x}^{\partial\Gamma}(\bullet))^{-1}d\boldsymbol{\iota}_{\Delta}^{\Gamma}\right)\\
&=\; \left(\frac{1}{\boldsymbol{o}(n)\boldsymbol{\vartheta}^{\Gamma}(\partial\Gamma)}\int\limits_{\Delta^e}(\boldsymbol{x}^{\partial\Gamma}(\bullet))^{-1}d(\boldsymbol{o}(n)\boldsymbol{\vartheta}^{\Gamma})\right)\\
&=\; \left(\frac{1}{\boldsymbol{\vartheta}^{\Gamma}(\partial\Gamma)}\int\limits_{\Delta^e}(\boldsymbol{x}^{\partial\Gamma}(\bullet))^{-1}d(\boldsymbol{\vartheta}^{\Gamma})\right) \;=\; \boldsymbol{\delta}(\Gamma)\ ,
\end{aligned} \tag{7.13}$$

as the term $\boldsymbol{o}(n)$ cancels out. Formula (7.11) follow then from (5.7).

q.e.d.

8 Conditional Additivity

The axiomatic characterization for the (generalized) Maschler–Perles solution is at length presented in [14]. To some extent, this justifies the solution for smooth bodies, so far as one is willing to accept the limiting procedure as a mere technicality. See AUMANN'S remarks regarding smooth bargaining solutions in [1]. In this context, Aumann provides an axiomatization of SHAPLEY'S NTU value [17].

Analogously, we will now discuss the concept of *conditional additivity* in our context, which is exactly based on AUMANN [1]. However, in Aumanns version (as well as in SHAPLEY'S and others, see HART [4], DE CLIPPEL[2]), authors consider values or solutions as correspondences, that is, set–valued mappings. In our context, a solution is a (point valued) function, see also [14] CHAPTER *XIII*. We recall the basic definition.

Definition 8.1. Let $\boldsymbol{\varphi}$ be a mapping from convex bodies into $\mathbb{R}^n_+$ such that $\boldsymbol{\varphi}(\Gamma) \in \partial\Gamma$ holds true for all G. $\boldsymbol{\varphi}$ is ***conditionally additive*** if, for any two convex bodies Γ and Θ such that $\boldsymbol{\varphi}(\Gamma) + \boldsymbol{\varphi}(\Theta) \in \partial(\Gamma + \Theta)$, it follows that

$$\boldsymbol{\varphi}(\Gamma) + \boldsymbol{\varphi}(\Theta) = \boldsymbol{\varphi}(\Gamma + \Theta) \tag{8.1}$$

holds true. Equivalently, one requires that for any family of convex bodies $\Gamma^{\bullet} = \{\Gamma^q\}_{q\in\boldsymbol{Q}}$ and any probability p on Q such that $\mathbb{E}_{\boldsymbol{p}}\boldsymbol{\varphi}(\Gamma^{\bullet})$ is Pareto efficient in $\mathbb{E}_{\boldsymbol{p}}\Gamma^{\bullet}$ it follows that

$$\boldsymbol{\varphi}(\mathbb{E}_{\boldsymbol{p}}\Gamma^{\bullet}) \;=\; \mathbb{E}_{\boldsymbol{p}}\boldsymbol{\varphi}(\Gamma^{\bullet}) \;. \tag{8.2}$$

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For *two players* conditional additivity is equivalent to superadditivity in order to characterize the Maschler–Perles solution . See CHAPTER XII of [14].

AUMANN'S concept refers to a smooth surfaces. The MASCHLER–PERLES solution $\boldsymbol{\mu}$ and its generalizations as treated in [14] are defined on Cephoids. Indeed $\boldsymbol{\mu}$ (and the derived version of the Shapley value) are conditionally superadditive (CHAPTER XIV, Theorem 3.2., [14])

The main goal of this section is Theorem 8.5 which establishes conditional superadditivity of the MP (= deGua) solution on smooth bodies. The resulat is based on the one for Cephoids (see [14], CHAPTER XIV, SECTION 3). The serious obstacle is that, in a sense, conditional superadditivity is not an l.h.c. property. That is, limiting Cephoids approaching a smooth body may lack limiting correct normals.

Hence, some prerequisites are necessary. We will approximate smooth convex bodies Γ in a way such that the normal at $\boldsymbol{\vartheta}(\Gamma) = \boldsymbol{\delta}(\Gamma)$ is achieved by the normal at $\boldsymbol{\delta}(\Pi^{\boldsymbol{Q}})$ when $\boldsymbol{Q}$ is approaching Γ. This property is not provided by the construction in [15] but follows from a series of Lemmas as below.

Lemma 8.2. Let Γ be a smooth body and let $\bar{\boldsymbol{x}} \in \partial\Gamma$ admit the normal $\bar{\mathfrak{n}} = (\frac{1}{\bar{a}_1}, \ldots, \frac{1}{\bar{a}_n}) > \mathbf{0}$. Let $\boldsymbol{Q} \in \mathfrak{Q}$ be such that $\bar{\boldsymbol{x}} \in \partial\Pi^{\boldsymbol{Q}}$. For small $\varepsilon >$ let

$$\Pi^{\boldsymbol{Q},\varepsilon} := (1-\varepsilon)\Pi^{\boldsymbol{Q}} + \varepsilon\Delta^{\bar{\boldsymbol{a}}}. \tag{8.3}$$

Then the vectors

$$(1-\varepsilon)\bar{\boldsymbol{x}} + \varepsilon\bar{\boldsymbol{a}}^i \quad (i \in \boldsymbol{I})$$

are extremals in $\Pi^{\boldsymbol{Q},\varepsilon}$. Hence

$$\partial\Pi^{\boldsymbol{a},\varepsilon} := (1-\varepsilon)\bar{\boldsymbol{x}} + \Delta^{\bar{\boldsymbol{a}}} \tag{8.4}$$

is Pareto efficient (and “flat”) in $\partial\Pi^{\boldsymbol{Q},\varepsilon}$.

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Proof: *A priori* we have

$$\Gamma \subseteq \Delta^{\bar{\boldsymbol{a}}}\ . \tag{8.5}$$

Choose $\boldsymbol{Q}$ such that $\bar{\boldsymbol{x}} \in \partial\Pi^{\boldsymbol{Q}}$ with $\bar{\mathfrak{n}}$ as normal as well (Corollary 4.1). Then necessarily we have as well

$$\Pi^{\boldsymbol{Q}} \subseteq \Delta^{\bar{\boldsymbol{a}}}\ . \tag{8.6}$$

Therefore, with sufficiently small $\varepsilon > 0$ it follows that $\Pi^{\boldsymbol{Q},\varepsilon} \subseteq \Delta^{\bar{\boldsymbol{a}}}$. The extremals $\bar{\boldsymbol{a}}^i$ $i \in \boldsymbol{I}$ of $\Delta^{\bar{\boldsymbol{a}}}$ yield extremals

$$\bar{\boldsymbol{a}}^{i,\varepsilon} := (1-\varepsilon)\bar{\boldsymbol{x}} + \varepsilon\bar{\boldsymbol{a}}^i \in \Pi^{\boldsymbol{Q},\varepsilon}\ , \tag{8.7}$$

this follows from (8.6). Hence, $\partial\Pi^{\boldsymbol{a},\varepsilon}$ is indeed a Pareto efficient part of $\partial\Pi^{\boldsymbol{Q},\varepsilon}$.

q.e.d.

Lemma 8.3. Let Γ be a smooth body and let $\boldsymbol{Q} \in \mathfrak{Q}$ be sufficiently close to Γ in the Hausdorff metric. Let $\bar{\boldsymbol{x}} = \boldsymbol{\delta}(\Gamma) \in \partial\Gamma$. For sufficiently small $\varepsilon > 0$ let $\Pi^{\boldsymbol{Q},\varepsilon}$ be defined by (8.3). Then

$$(1-\varepsilon)\boldsymbol{\delta}(\Pi^{\boldsymbol{Q}}) + \varepsilon\boldsymbol{\delta}(\Delta^{\bar{\boldsymbol{a}}}) = (1-\varepsilon)\bar{\boldsymbol{x}} + \varepsilon\boldsymbol{\delta}(\Delta^{\bar{\boldsymbol{a}}}) \in \partial\Pi^{\bar{\boldsymbol{a}},\varepsilon}\ . \tag{8.8}$$

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Proof: Instead of the Simplex parametrization $\overset{\Delta}{\boldsymbol{\kappa}}$ we choose the parametrization $\boldsymbol{x}^{\bar{\boldsymbol{a}}}(\bullet)$ suggested by $\Delta^{\bar{\boldsymbol{a}}}$. This is the inverse of the modified canonical representation $\boldsymbol{\kappa}^{\bar{\boldsymbol{a}}}$ as treated in Remark 2.3. It is seen that $\boldsymbol{x}^{\bar{\boldsymbol{a}}}(\bullet)$ behaves in a linear affine manner. Using formula (5.7) for the deGua Solution, we obtain indeed

$$(1-\varepsilon)\boldsymbol{\delta}(\Pi^{\boldsymbol{Q}}) + \varepsilon\boldsymbol{\delta}(\Delta^{\bar{\boldsymbol{a}}}) = (1-\varepsilon)\bar{\boldsymbol{x}} + \varepsilon\boldsymbol{\delta}(\Delta^{\bar{\boldsymbol{a}}}) \in \partial\Pi^{\bar{\boldsymbol{a}},\varepsilon}\ . \tag{8.9}$$

q.e.d.

Corollary 8.4. Let Γ be a smooth body. Then, for $\varepsilon > 0$, there exists a Cephoid Π^{ε} such that

1. the Hausdorff distance between Γ and Π^{ε} is smaller than 2ε.
2. $\boldsymbol{\delta}(\Gamma)$ and $\boldsymbol{\delta}(\Pi^{\varepsilon})$ admit of the same normal.

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Proof: Choose the filter $\Pi^{\boldsymbol{Q}}$ such that

$$\Pi^{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\longrightarrow} \partial\Gamma\ .$$

to Γ according to Lemma 4.3 in a way that $\bar{\boldsymbol{x}} = \boldsymbol{\delta}(\Gamma) \in \Pi^{\boldsymbol{Q}}$ is satisfied for all $\boldsymbol{Q} \in \mathfrak{Q}$. Then $\boldsymbol{\delta}(\Gamma) \underset{\mathfrak{Q}}{\longrightarrow} \boldsymbol{\delta}(\Pi^{\boldsymbol{Q}})$ by the continuity Theorem 7.2.

Consequently

$$\begin{aligned} \boldsymbol{\delta}(\Pi^{\boldsymbol{Q},\varepsilon}) &= (1-\varepsilon)\boldsymbol{\delta}(\Pi^{\boldsymbol{Q}}) + \varepsilon\boldsymbol{\delta}(\Delta^{\bar{\boldsymbol{a}}}) \\ &\underset{\mathfrak{Q}}{\longrightarrow} \ (1-\varepsilon)\boldsymbol{\delta}(\Gamma) + \varepsilon\boldsymbol{\delta}(\Delta^{\bar{\boldsymbol{a}}}) \\ &= \bar{\boldsymbol{x}} + \varepsilon\boldsymbol{\delta}(\Delta^{\bar{\boldsymbol{a}}}) \in \partial\Pi^{\boldsymbol{a},\varepsilon}. \end{aligned} \tag{8.10}$$

Hence, eventually

$$\boldsymbol{\delta}(\Pi^{\boldsymbol{Q},\varepsilon}) \in \partial\Pi^{\boldsymbol{a},\varepsilon} \tag{8.11}$$

holds true. Therefore, eventually $\boldsymbol{\delta}(\Pi^{\boldsymbol{Q},\varepsilon})$ admits of the normal $\bar{\mathfrak{n}}$ which is the normal of $\bar{\boldsymbol{x}} = \boldsymbol{\delta}(\Gamma)$ by construction. **q.e.d.**

Theorem 8.5. *The Maschler–Perles (deGua) solution* $\boldsymbol{\mu}$ $(= \boldsymbol{\delta})$ *is conditionally superadditive on smooth bodies.*

• ~~~∘~~~ •

Proof:

1stSTEP : Let Γ and $\Gamma^{\star}$ be smooth bodies and let $\boldsymbol{\delta}(\Gamma) + \boldsymbol{\delta}(\Gamma^{\star})$ be Pareto efficient in $\Gamma + \Gamma^{\star}$. Then necessarily $\boldsymbol{\delta}(\Gamma), \boldsymbol{\delta}(\Gamma^{\star})$ and $\boldsymbol{\delta}(\Gamma + \Gamma^{\star})$ admit of the same normal $\bar{\mathfrak{n}}$. According to Corollary 8.4 we can approximate Γ and $\Gamma^{\star}$ by Cephoids $\Pi^{\boldsymbol{Q}}$ and $\Pi^{\boldsymbol{Q},\star}$ such that all deGua solutions have the same normal $\bar{\mathfrak{n}}$. The solutions follow the approximations by the Continuity Theorem.

Also, it follows from the common normal property that $\boldsymbol{\delta}(\Pi^{\boldsymbol{Q}}) + \boldsymbol{\delta}(\Pi^{\boldsymbol{Q},\star})$ is Pareto efficient in $\Pi^{\boldsymbol{Q}} + \Pi^{\boldsymbol{Q},\star}$. Hence, as the MP–solution is conditionally superadditive on Cephoids (Theorem 3.2 CHAPTER XIV of [14]) we know that

$$\begin{aligned} \boldsymbol{\delta}(\Gamma + \Gamma^{\star}) &= \lim_{\mathfrak{Q}} \boldsymbol{\delta}(\Pi^{\boldsymbol{Q}} + \Pi^{\boldsymbol{Q},\star}) \\ &= \lim_{\mathfrak{Q}} \boldsymbol{\delta}(\Pi^{\boldsymbol{Q}}) + \lim_{\mathfrak{Q}} \boldsymbol{\delta}(\Pi^{\boldsymbol{Q},\star}) \\ &= \boldsymbol{\delta}(\Gamma) + \boldsymbol{\delta}(\Gamma^{\star}) \end{aligned} \tag{8.12}$$

q.e.d.

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