THE ASYMPTOTIC SEMIGROUP OF A SUBSEMIGROUP OF A NILPOTENT LIE GROUP

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Dedicated to the memory of Ernest B. Vinberg

Abstract. Let S be a subsemigroup of a simply connected nilpotent Lie group G. We construct an asymptotic semigroup S_0 in the associated graded Lie group G_0 of G. We can compute the image of S_0 in the abelianization $G_0^{\mathrm{ab}} = G^{\mathrm{ab}}$. This gives useful information about S. As an application, we obtain a transparent proof of the following result of E. B. Vinberg and the author: either there is an epimorphism $f : G \to \mathbb{R}$ such that $f(s) \geq 0$ for every s in S or the closure $\overline{S}$ of S is a subgroup of G and $G/\overline{S}$ is compact.

1. Introduction

Given a nilpotent Lie algebra $\mathfrak{g}$, we endow the underlying vector space V with a family of Lie brackets $[\ ,\ ]_t, t > 0$ and corresponding Lie algebras $\mathfrak{g}_t$. For $t \to 0$ these structures converge to a Lie algebra structure $\mathfrak{g}_0$ that is isomorphic with the associated graded Lie algebra $\mathrm{gr}(\mathfrak{g})$ of $\mathfrak{g}$. We also define a family $\delta_t, t > 0$, of linear automorphisms of V, which give isomorphisms of Lie algebras $\mathfrak{g} \to \mathfrak{g}_t$. The Campbell–Hausdorff multiplication turns this into a family of nilpotent Lie groups $G_t, t \geq 0$, and isomorphisms of Lie groups $\delta_t : G \to G_t$ for $t > 0$. Then every $\delta_t, t > 0$, is an automorphism of $\mathfrak{g}_0$ and of G_0. For a subset M of G, we define the limit $M_0 := \lim_{t\to 0} \delta_t M$. We think of M_0 as an asymptote of M. The set M_0 has pleasant properties: it is closed and δ_t-invariant for every $t > 0$. If S is a subsemigroup of G then its asymptote S_0 is a subsemigroup of G_0. Furthermore, we can compute the image of S_0 under the natural map of G_0 to its abelianization G_0^{ab}. It is the smallest closed convex cone that contains the image $\pi(S)$ of S under the natural map π of G to its abelianization $G^{\mathrm{ab}} \cong G_0^{\mathrm{ab}}$. In particular, if S is not contained in a half space of the vector space G^{ab}, then its asymptote S_0 projects

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onto G_0^{ab} and hence the asymptote S_0 of S is the whole Lie group G_0. These results may be of independent interest. As an application, we give a transparent proof of the main result of [AV], using stability results developed in [AV]: see Section 3.

Acknowledgments. The author profited very much from the cooperation with Ernest Borisovitch Vinberg. The construction of the asymptotic semigroup given here is reminiscent of the construction of an asymptotic semigroup in [V], as Ernest Borisovitch pointed out. To explore this relation and to continue our joint work on semigroups was our plan. But his sudden and unexpected death put a sad end to all our plans. I dedicate this paper to his memory. I thank the various institutions which supported his regular stays in Bielefeld over the years, the Humboldt foundation which honored him with a Humboldt award and subsequent invitations, the DFG under SFB 343 and SFB 701, and the Faculty of Mathematics of Bielefeld University.

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Convention. In this paper, we understand the term "semigroup" as a semigroup with identity element.

2. An approximation procedure

Let $\mathfrak{g}$ be a nilpotent finite dimensional Lie algebra over $\mathbb{R}$. Let $\mathfrak{z}_i, i = 1, \ldots, d$ be its descending central series. So $\mathfrak{z}_1 = \mathfrak{g}$ and $\mathfrak{z}_{i+1} = [\mathfrak{z}_i, \mathfrak{g}]$ for $i \geq 1$. Let $V_1, \ldots, V_d$ be vector subspaces of $\mathfrak{g}$ such that

$$V_i \oplus \mathfrak{z}_{i+1} = \mathfrak{z}_i$$

for every $i = 1, \ldots, d$. We denote the projection to the first summand in this direct sum by $\phi_i : \mathfrak{z}_i \to V_i$.

We reserve the symbol $\mathfrak{g}$ for the Lie algebra. If we consider $\mathfrak{g}$ just as a vector space we denote it by V.

For $t > 0$, we consider the linear invertible maps $\delta_t : V \to V$, which are uniquely determined by the property that

$$\delta_t(v) = t^i(v) \text{ if } v \in V_i.$$

These maps are sometimes called dilations and have been used in analysis (see [G]). We note that δ_t depends on the choice of the vector spaces $V_i, i = 1, \ldots, d$.

We define for $t > 0$ a Lie bracket $[\ ,\ ]_t$ on V by transport of structure from $\mathfrak{g}$ via δ_t. So

$$[x, y]_t = \delta_t[\delta_t^{-1}x, \delta_t^{-1}y]$$

for $x, y \in V$. This defines a new Lie algebra structure on V which we denote by $\mathfrak{g}_t$. So $\mathfrak{g}_1 = \mathfrak{g}$. We are interested in the limit structure $\mathfrak{g}_0$, defined by

$$\mathfrak{g}_0 = \lim_{t \to 0} \mathfrak{g}_t.$$

We have to check that the limit exists and we list some of its properties in the following proposition. We denote by $\mathsf{gr}(\mathfrak{g})$ the associated graded Lie algebra corresponding to the filtration $\mathfrak{z}_i, i = 1, \ldots, d$ by the descending central series of $\mathfrak{g}$.

Proposition 1.

a) *For every pair* x, y *of vectors of* V*, the limit*

$$[x, y]_0 = \lim_{t \to 0} [x, y]_t$$

exists and turns V *into a Lie algebra, which we denote by* $\mathfrak{g}_0$*.*

b) $\mathfrak{g}_0$ *is a graded Lie algebra and we have*

$$[V_i, V_j]_0 \subset V_{i+j}.$$

c) *The set of Lie brackets* $[x, y]_0, x \in V_1, y \in V_{i-1}$*, spans* V_i*. In particular,* $\mathfrak{z}_i, i = 1, \ldots, d$*, is the descending central series of* $\mathfrak{g}_0$*.*

d) δ_t *is an automorphism of the Lie algebra* $\mathfrak{g}_0$ *for every* $t > 0$*.*

e) *Let* $\overline{\phi_i} : \mathfrak{z}_i/\mathfrak{z}_{i+1} \to V_i$ *be the linear isomorphism induced by the projection* $\phi_i : \mathfrak{z}_i \to V_i$*. Let* $\overline{\phi} : \bigoplus_i \mathfrak{z}_i/\mathfrak{z}_{i+1} \to V$ *be the linear isomorphism with the property that* $\overline{\phi}|_{\mathfrak{z}_i/\mathfrak{z}_{i+1}} = \overline{\phi_i}$*. Then* $\overline{\phi}$ *is an isomorphism of Lie algebras* $\overline{\phi} : \mathrm{gr}(\mathfrak{g}) \to \mathfrak{g}_0$*.*

So we can think of the family of Lie algebras $\mathfrak{g}_t, t \to 0$, as an approximation of $\mathrm{gr}(\mathfrak{g})$.

Proof. For $x \in V_i$ and $y \in V_j$ we have

$$[x, y] = \sum z_k,$$

where $z_k \in V_k$ and $z_k = 0$ if $k < i + j$. Thus

$$\begin{aligned} [x, y]_t &= \delta_t[\delta_t^{-1} x, \delta_t^{-1} y] \\ &= \delta_t[t^{-i} x, t^{-j} y] \\ &= \sum t^{k-i-j} z_k, \end{aligned}$$

hence

$$[x, y]_t = z_{i+j} + t z_{i+j+1} + t^2 z_{i+j+2} + \cdots \tag{1}$$

So for $x \in V_i$ and $y \in V_j$, the limit $\lim_{t \to 0}[x, y]_t = z_{i+j}$ exists and we have computed the limit

$$[x, y]_0 = \phi_{i+j}[x, y]. \tag{2}$$

It follows that the limit exists for every every pair x, y of vectors of V, by bilinearity of the Lie brackets $[\ ,\ ]_t$.

The bracket $[\ ,\ ]_0$ turns V into a Lie algebra, since bilinearity, anticommutativity, and the Jacobi identity are preserved under limits. This shows a). Claim b) follows from Equation (2), and b) implies d). Equation (2) also implies e). To see c), note that the set of commutators $[x, y], x \in V_1, y \in V_{i-1}$, spans $\mathfrak{z}_i$ modulo $\mathfrak{z}_{i+1}$. So c) also follows from Equation (2). $\square$

We remark that the explicit formula Equation (2) for the Lie bracket of $\mathfrak{g}_t, t \geq 0$ shows that the family $\mathfrak{g}_t$ of Lie algebras is a polynomial family: i.e., the mapping

$$V \times V \times [0, \infty) \to V, \ (x, y, t) \mapsto [x, y]_t \tag{3}$$

is given by polynomial functions.

For every $t \geq 0$, the Campbell–Hausdorff multiplication turns every $\mathfrak{g}_t$ into a nilpotent Lie group, denoted G_t. We denote the multiplication in G_t by $\cdot_t$. The family of Lie groups $G_t, t \geq 0$ is also a polynomial family, since all our Lie algebras are nilpotent of the same degree. The groups G_t are all isomorphic for $t > 0$; in fact δ_t is an isomorphism of $G := G_1$ to G_t.

Let M be a subset of V. We define the limit set $M_0 = \lim_{t\to 0} \delta_t M$ as follows. M_0 is the set of points $x \in V$ with the following property. For every neighborhood U of x there is a positive number ϵ such that $U \cap \delta_t M \neq \varnothing$ for every $t \in (0, \epsilon)$. We take this strict definition of the limit set since we want the limit set to be a subsemigroup of G_0 if M is a subsemigroup of G.

Lemma 2. *Let M be a subset of V and let M_0 be the limit set as defined above. Then*

a) *M_0 is a closed subset of V,*
b) *M_0 is invariant under δ_t for every $t > 0$,*
c) *let M and N be subsets of G. Then we have for their limit sets $M_0 \cdot_0 N_0 \subset (M \cdot_1 N)_0$.*

Note that the multiplication $\cdot_1$ in G_1 is just the original multiplication $\cdot$ in G. So c) establishes a relation between the original multiplication and the limit multiplication of limit sets. In particular, we have the following corollary.

Corollary 3. *Let S be a subsemigroup of S and let S_0 be its limit set. Then S_0 is a closed subsemigroup of G_0, which is invariant under δ_t for every $t > 0$.*

Proof. a) The proof of a) is straightforward.

b) Let x be a point of M_0, let s be a positive number and let U be a neighborhood of $\delta_s x$. Then $\delta_s^{-1} U$ is a neighborhood of x. So $\delta_s^{-1} U \cap \delta_t M \neq \varnothing$ for $t \in (0, \epsilon)$ for some $\epsilon > 0$. Note that $\delta_s \delta_t = \delta_{st}$, hence $U \cap \delta_{st} M \neq \varnothing$ whenever $st \in (0, s\epsilon)$. Thus $\delta_s x \in M_0$ whenever $x \in M_0$ and $s > 0$.

c) Let x and y be elements of M_0 and N_0, respectively. We claim that $x \cdot_0 y \in (M \cdot_1 N)_0$. Let U be a neighborhood of $x \cdot_0 y$. The joint continuity of the family of Lie groups $G_t, t \geq 0$, implies that there are neighborhoods V of x and W of y and a $\delta > 0$ such that $x' \cdot_t y' \in U$ whenever $x' \in V$, $y' \in W$ and $t \in [0, \delta)$. Now there is a number $\epsilon > 0$, which we may assume to be less than δ, such that $V \cap \delta_t M \neq \varnothing$ and $W \cap \delta_t N \neq \varnothing$ whenever $t \in (0, \epsilon)$. For such t let us take $x' \in V \cap \delta_t M$ and $y' \in W \cap \delta_t N$. Then $x' \cdot_t y' \in U \cap \delta_t M \cdot_t \delta_t N$. But $\delta_t M \cdot_t \delta_t N = \delta_t (M \cdot_1 N)$. Thus $U \cap \delta_t (M \cdot_1 N) \neq \varnothing$ for $t \in (0, \epsilon)$ and hence $x \cdot_0 y \in (M \cdot_1 N)_0$. $\square$

Lemma 4. *Let S be a subsemigroup of G and let S_0 be its limit semigroup. Then*

a) $S_0 \cap V_i = \phi_i(S_0 \cap \mathfrak{z}_i)$
b) *$S_0 \cap V_i$ is a closed convex cone in V_i,*
c) *$S_0 \cap V_i$ contains $\phi_i(S \cap \mathfrak{z}_i)$.*

Proof. a) Trivially $S_0 \cap V_i \subset \phi_i(S_0 \cap \mathfrak{z}_i)$. The opposite inclusion follows from the fact that if $s \in S_0 \cap \mathfrak{z}_i$, say $s = s_i + s_{i+1} + \cdots$ with $s_k \in V_k$, then $\delta_{1/n} s^{n^i} = s_i + \frac{1}{n} s_{i+1} + \frac{1}{n^2} s_{i+2} + \cdots$ is an element of S_0 for $n \in \mathbb{N}$ and hence so is its limit $s_i = \phi_i(s)$.

b) First of all, $S_0 \cap V_i$ is an additive subsemigroup of V_i. For, suppose x and y are elements of $S_0 \cap V_i$, then $x \cdot_0 y \in S_0 \cap \mathfrak{z}_i$. But $x \cdot_0 y \equiv x + y \mod \mathfrak{z}_{2i}$ by the Campbell–Hausdorff formula and hence $\phi_i(x \cdot_0 y) = x + y \in \phi_i(S_0 \cap \mathfrak{z}_i) = S_0 \cap V_i$. Now claim b) follows since $S_0 \cap V_i$ is closed and δ_t-invariant.

c) Suppose $s \in S \cap \mathfrak{z}_i$, say $s = s_i + s_{i+1} + \cdots$ with $s_j \in V_j$ for $j \geq i$. We claim that $s_i \in S_0$. By our convention a semigroup contains the identity element. It follows that S_0 contains the identity element. We thus may assume that $s_i \neq 0$. For $t \in (0, 1]$ let $n_t \in \mathbb{N}$ be such that $t^i n_t \leq 1 < t^i(n_t + 1)$ and for $t > 1$ we set $n_t = 1$. Then $\delta_t s^{n_t} = t^i n_t s_i + t^{i+1} n_t s_{i+1} + \cdots \in \delta_t S$ converges to s_i when t tends to 0, since $\lim_{t\to 0} t^i n_t = 1$ and $\lim_{t\to 0} t^j n_t = 0$ for $j > i$. So $s_i \in S_0 \cap V_i$. □

For the case $i = 1$, we have in particular $\phi_1(S) \subset S_0$. In this case, we have the following precise information.

Proposition 5. *Let S be a subsemigroup of G and let S_0 be its limit subsemigroup of G_0. Then the set $S_0 \cap V_1$ is the smallest closed convex cone in V_1 that contains $\phi_1(S)$.*

Proof. Let C be the smallest closed convex cone in V_1 which contains $\phi_1(S)$. We know that C is contained in the closed convex cone $S_0 \cap V_1$, by the preceding Lemma 4. To show the inverse inclusion, it suffices by the separating hyperplane theorem to show that every linear map $l : V_1 \to \mathbb{R}$ which has non-negative values on $\phi_1(S)$ also has non-negative values on $S_0 \cap V_1$. For such l consider the linear map $\psi = l \circ \phi_1 : V \to \mathbb{R}$. We have $\psi(S) \geq 0$ and $\psi(\delta_t x) = t\psi(x)$ for $t > 0$ and $x \in V$, hence $\psi(\delta_t S) \geq 0$ for all $t > 0$ and hence $\psi(S_0) \geq 0$. Thus $l(S_0 \cap V_1) \geq 0$ as was to be shown. □

3. An application

As an application, we obtain a new proof of the main result of [AV].

Let me recall the main result of [AV]. A subsemigroup S of a topological group G is called *cocompact* if there is a compact subset K of G such that $G = SK$. Let now G be a simply connected nilpotent Lie group and let $G^{\mathrm{ab}} = G/(G, G)$ be its abelianization, a vector group. Let $\pi : G \to G^{\mathrm{ab}}$ be the natural projection. By a half space in a real vector space V we mean a subset of the form $\{v \in V;\, l(v) \geq 0\}$ for some nonzero linear function l on V. Let S be a subsemigroup of G.

Proposition 6. *The following statements are equivalent.*

(1) *There is a surjective homomorphism $f : G \to \mathbb{R}$ of Lie groups such that $f(s) \geq 0$ for every $s \in S$.*
(2) *The image $\pi(S)$ of S in G^{ab} is contained in a half space.*
(3) *$\pi(S)$ is not cocompact in G^{ab}.*

Note that all these statements depend only on the image $\pi(S)$ of S in G^{ab}. The proof of this proposition is an application of elementary facts about convex cones in vector spaces: see [AV, Prop. 2.1].

The main result of [AV] is the following theorem, [AV, Thm. 1.2].

Theorem 7. *If the image $\pi(S)$ of S in G^{ab} is cocompact in G^{ab} then S is cocompact in G and the closure $\overline{S}$ of S is a subgroup of G.*

Another formulation, equivalent by Proposition 6, is the following theorem.

Theorem 8. *Either there is a surjective homomorphism $f : G \to \mathbb{R}$ of Lie groups such that $f(s) \geq 0$ for every $s \in S$ or S is cocompact in G and the closure $\overline{S}$ of S is a subgroup of G.*

Thus our result is a common generalization of the following two results. One is the theorem of Maltsev [M], where S in Theorem 7 is supposed to be a subgroup of G. The other one is the theorem of Lawson [L] (see [HHL, V.5]), where S in Theorem 8 is supposed to have interior points, and then one can conclude that actually $S = G$ in the second alternative: see [AV, Cor. 3.4]. So our result has two aspects. One of them is the cocompactness aspect, stated in Theorem 7. The other one is the dichotomy aspect, stated in Theorem 8. It was interesting to see that the two referees of this paper did not agree on which aspect should be considered as the main result of [AV]. Also, Ernest Borisovich Vinberg and the present author may not have agreed on this point. This reminds me of the insight of Adorno: Der Künstler ist nicht gehalten, das eigene Werk zu verstehen [Ad] (The artist cannot be held responsible for understanding his own work; I thank David Gordon for help with the translation).

Outline of proof of Theorem 7. In the proof, we use some of the tools we developed in [AV]. We may assume that G is of the form considered in Section 2. We choose a family of vector subspaces V_i as above. Let S be a subsemigroup of G and suppose that its image $\pi(S)$ in G^{ab} is cocompact. Then $\phi_1(S)$ is cocompact in V_1, since $\phi_1 = \overline{\phi_1} \circ \pi$ if we identify $\mathfrak{g} \cong G$ and $G^{\mathrm{ab}} \cong \mathfrak{g}^{\mathrm{ab}} = \mathfrak{z}_1/\mathfrak{z}_2$ and use $\overline{\phi_1} : \mathfrak{z}_1/\mathfrak{z}_2 \to V_1$ of Proposition 1e). But the only closed convex cone in a vector space which contains a cocompact subsemigroup is the vector space itself: see [AV, Prop. 2.1]. So $S_0 \cap V_1 = V_1$. Thus S_0 is a subsemigroup of G_0 which contains V_1. It follows that $S_0 = G_0$. This can be proved by induction on $\mathsf{dim}(G_0)$. A quick reference would be [AV, Thm. 4.4] since S_0 contains a family of one-parameter sub*groups* of G_0, whose images in $\mathfrak{g}_0^{\mathrm{ab}}$ span $\mathfrak{g}_0^{\mathrm{ab}}$. Then by the approximation argument of [AV, Sect. 3], (which holds also here for the continuous family of $G_t, t \geq 0$) we have that $\delta_t S$ is cocompact in G_t for t small. But δ_t is an isomorphism from G to G_t which maps S to S_t. So S is cocompact in G.

Here are some more details of the approximation argument. Every cocompact subsemigroup of a connected Lie group contains a finitely generated subsemigroup, which is also cocompact: see [AV, Prop. 3.6]. Varying this finite set of generators slightly — and even varying the group law slightly — does not destroy cocompactness. This can be proved as [AV, Thm. 3.7]. It follows that in our case $\delta_t S$ is cocompact in G_t for t small, since S_0 is cocompact in G_0.

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