# Term Structure Modeling and the Pricing of Interest Rate Derivatives under Volatility Uncertainty

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# Abstract


In this thesis, we tame the uncertainty about the volatility in interest rate models. We treat the uncertainty about the volatility as model uncertainty or Knightian uncertainty, resulting in robust models. That means, we model interest rates in the presence of a family of probability measures, each corresponding to a different scenario for the volatility, without imposing any assumptions on which is more likely to be the correct one. This setting is naturally connected to the calculus of $G$-Brownian motion, which is the main tool for the mathematical analysis.

First, we investigate the Hull-White model for the term structure of interest rates under volatility uncertainty. The main question in this part is how to find an arbitrage-free term structure, which is crucial since we can show that the classical approach, martingale modeling, does not work in the presence of volatility uncertainty. We therefore need to adjust the model to find an arbitrage-free term structure. Although the adjustment changes the structure of the model, it is still consistent with the traditional Hull-White model after fitting the yield curve.

Next, we examine term structure movements in the spirit of the (more general) Heath-Jarrow-Morton methodology under volatility uncertainty. Within this part, we derive a sufficient condition for the absence of arbitrage, known as the drift condition. The drift condition allows to construct arbitrage-free term structure models that are completely robust with respect to the volatility. In particular, we obtain robust versions of classical term structure models, including the robust version of the Hull-White model from the previous part.

In the end, we study the pricing of interest rate derivatives under volatility uncertainty, starting from an arbitrage-free term structure as determined by the previous part. The uncertainty about the volatility leads to a sublinear pricing measure, which complicates the pricing procedure in several ways. We develop pricing methods for different types of contracts in this framework to derive robust pricing formulas for all major interest rate derivatives. The pricing procedure exhibits many interesting economic phenomena, such as a robust expectations hypothesis and unspanned stochastic volatility.

# Contents

# Introduction

Traditional models in mathematical finance are subject to model uncertainty. The standard assumption in models of mathematical finance is that there is a single, known probability measure determining the behavior of the underlying quantities on the market. This assumption simplifies the modeling of financial markets and the pricing of derivatives written on financial quantities, since it allows to acquire all results from probability theory and stochastic calculus. In many situations, it is, however, not possible to specify the underlying probability measure of the model. The uncertainty about using the correct probability law is called *model uncertainty*. The problem of model uncertainty led to the investigation of financial markets in the presence of a family of possible probability measures or none at all. The overall aim is to make models in mathematical finance robust with respect to misspecifications regarding the probability law.

A prominent example of model uncertainty is volatility uncertainty. The volatility in mathematical finance measures the magnitude of the underlying's short-term fluctuations. Standard models in mathematical finance, such as the famous models of Black and Scholes (1973) and Merton (1973), treat the volatility as a constant. Since there is plenty of empirical evidence that the volatility of financial quantities is not constant, volatility modeling is an actively studied topic. Most alternative models use a stochastic volatility, which makes the model more realistic, but still relies on the assumption that the probabilistic law of the volatility is known. There are several reasons why this is a doubtful assumption, which ultimately leads to the presence of model uncertainty, termed *volatility uncertainty*.

The rapidly growing literature on volatility uncertainty and, more generally, model uncertainty in mathematical finance, referred to as *robust finance*, primarily focuses on asset market models. Avellaneda, Levy, and Parás (1995) and Lyons (1995) were the first to investigate the pricing and hedging of derivatives in asset market models under volatility uncertainty. More recent studies on this topic include the works of Epstein and Ji (2013) and Vorbrink (2014). Furthermore, there are various works investigating asset markets under model uncertainty in general. The objective is to study the basic principles in mathematical finance, i.e., the absence of arbitrage and the pricing and hedging of derivatives, in the presence of a family of probability measures instead of one, called *multiple prior* setting, or without using any reference probability measure at all, called *model-free* setting. The fundamental theorem of asset pricing, determining cause and effect of the absence of arbitrage, was studied (among others) by Bayraktar and Zhou (2017), Biagini, Bouchard, Kardaras, and Nutz (2017), and Bouchard and Nutz (2015) in a multiple prior setting and by Acciaio, Beiglböck, Penkner, and Schachermayer (2016), Burzoni, Frittelli, Hou, Maggis, and Obłój (2019), and Riedel (2015) in a model-free setting. The pricing and hedging of derivatives was studied (among others) by Aksamit, Deng, Obłój, and Tan (2019), Carassus, Obłój, and Wiesel (2019), and Possamaï, Royer, and Touzi (2013) in a mutiple prior setting and by Bartl, Kupper, Prömel, and Tangpi (2019), Beiglböck, Cox, Huesmann, Perkowski, and Prömel (2017), and Schied and Voloshchenko (2016) in a model-free setting.

In addition to asset market models, there is a tremendous amount of interest rate models in the overall literature of mathematical finance, which are also called *term structure models*. Since the path-breaking publications of Black and Scholes (1973) and Merton (1973) the mathematical finance literature on the term structure of interest rates has been rapidly growing. First notable contributions include the works of Cox, Ingersoll Jr., and Ross (1985) and Vasicek (1977), models of the short term interest rate that characterized the term structure of interest rates by equilibrium theory and no-arbitrage arguments, respectively. Afterwards, the trend shifted towards models taking the current term structure as an input—instead of an output—in order to price fixed income derivatives, contracts depending on the term structure of interest rates. Well-known articles on this approach are the ones of Ho and Lee (1986) and Hull and White (1990). The breakthrough of this approach was achieved by the methodology of Heath, Jarrow, and Morton (1992); the methodology is based on directly modeling term structure movements as a diffusion process, starting from an initially observed term structure, instead of a single short term interest rate. Since then the number of articles on term structure models has been growing even further by more sophisticated models.

Although interest rate models are equally exposed to volatility uncertainty, this problem is rarely studied in the literature on robust finance. The number of articles dealing with volatility uncertainty or, more generally, model uncertainty in interest rate models is relatively sparse in comparison to the literature on model uncertainty in asset market models. Early contributions in this regard are due to Avellaneda and Lewicki (1996) and Epstein and Wilmott (1999), relying on intuitive arguments rather than a rigorous formulation. A more recent treatment of interest rates in conjunction with model uncertainty appears in the works of El Karoui and Ravanelli (2009) and Lin and Riedel (2021), which do not focus on the term structure of interest rates and arbitrage. Further related articles of Biagini and Zhang (2019) and Fadina and Schmidt (2019) deal with credit risk and model uncertainty. The most related work is the one of Fadina, Neufeld, and Schmidt (2019), studying affine processes under parameter uncertainty (in the sense of model uncertainty) and corresponding interest rate models. However, the work of Fadina, Neufeld, and Schmidt (2019) is restricted to models of the short term interest rate and a superhedging argument for the pricing of contracts, which does not apply to the term structure of interest rate, since bonds are the fundamentals of fixed income markets and therefore cannot be hedged.

Accompanying the literature on robust finance, there is an increasing share in the mathematics literature dealing with the mathematical problems related to volatility uncertainty. Since the presence of volatility uncertainty is represented by a family of mutually singular probability measures, i.e., measures with different null sets, many concepts from probability theory and stochastic calculus break down. Two classical approaches to meet these issues were introduced by Denis and Martini (2006) and Peng (2007, 2008), respectively. The two approaches are actually different but equivalent, as it was shown by Denis, Hu, and Peng (2011). The difference is that the approach of Denis and Martini (2006) starts from a probabilistic setting and relies on capacity theory, whereas the calculus of $G$-Brownian motion from Peng (2007, 2008) completely relies on nonlinear partial differential equations. In contrast, Soner, Touzi, and Zhang (2011a,b, 2012, 2013) approached the problem of volatility uncertainty by using aggregation and obtained further related results. In fact, volatility uncertainty is closely related to second-order backward stochastic differential equations, introduced by Cheridito, Soner, Touzi, and Victoir (2007). Other extensions and further results were obtained by Nutz (2012, 2013) and Nutz and van Handel (2013). In addition, there are also approaches to a model-free stochastic calculus (Cont and Perkowski, 2019, and references therein).

A very convenient tool to analyze volatility uncertainty is the calculus of $G$-Brownian motion. The literature on $G$-Brownian motion is very extensive and still increasing. The book of Peng (2019), who invented the theory of $G$-Brownian motion, gives a good introduction to the topic with a detailed treatment of the most important results. The calculus of $G$-Brownian motion extends the classical Itô calculus to a Brownian motion with an uncertain volatility—termed *$G$-Brownian motion*. The extension is based on nonlinear expectations and nonlinear partial differential equations. A nonlinear expectation replaces the classical (linear) expectation and leads to a worst-case measure. The distribution of a random variable under a nonlinear expectation is characterized by a nonlinear partial differential equation. The distribution of a $G$-Brownian motion is given by a nonlinear heat equation. The letter $G$ refers to the (nonlinear) generator of the partial differential equation. A $G$-Brownian motion is an extension of a standard Brownian motion, since the latter is normally distributed, i.e., its expectation solves a linear heat equation. Interestingly enough, most results of the Itô calculus still hold in this framework with some minor differences though. Although the definition and the construction of a $G$-Brownian motion completely rely on nonlinear partial differential equations instead of probabilities, there is a probabilistic framework connected to a $G$-Brownian motion. Hu and Peng (2009) and Denis, Hu, and Peng (2011) showed that the $G$-expectation, that is, the nonlinear expectation related to a $G$-Brownian motion, can be represented as an upper expectation of a family of probability measures. The representation of Denis, Hu, and Peng (2011) explicitly shows that a $G$-Brownian motion represents volatility uncertainty in the sense of model uncertainty.

In this thesis, we investigate volatility uncertainty in interest rate models—as opposed to asset market models—by using the calculus of $G$-Brownian motion. The investigation is divided into three different but interconnected steps and works within the same mathematical framework. We elaborate the three steps in Chapters 2, 3, and 4, which resulted in three papers (Hölzermann, 2021a,c,b), respectively. In the preliminary Chapter 1, we motivate the problem of volatility uncertainty and introduce the mathematical framework used to analyze volatility uncertainty. The mathematical framework is a probabilistic setting that allows to acquire the results from the calculus of $G$-Brownian motion; the calculus of $G$-Brownian motion is the main pillar for the mathematical analysis in the succeeding chapters. In Chapter 2, we study the presence of volatility uncertainty in one of the most well-known models of the short term interest rate: the Hull-White model for the term structure of interest rates. The simple nature of the model shows the implications of volatility uncertainty on term structure models and allows to discuss its consequences from an economic point of view. In Chapter 3, we investigate term structure movements in the spirit of the famous Heath-Jarrow-Morton methodology under volatility uncertainty. The mathematically more demanding methodology makes it possible to generalize the results of Chapter 2 to a general class of term structure models. In Chapter 4, we study the effects volatility uncertainty has on the pricing of interest rate derivatives. The aim is to find robust pricing formulas for derivative contracts on the term structure, where the latter is characterized by the results from Chapter 3. Finally, in Chapter 5, we conclude by summarizing the results.

# Chapter 1

# Volatility Uncertainty

In this preliminary chapter, we motivate the problem of volatility uncertainty in mathematical finance, we set up the model framework for representing such uncertainty, and we discuss the mathematical problems resulting from such a framework. First of all, we briefly introduce the concept of volatility in mathematical finance by considering the most common modeling approaches. The starting point is a model driven by Brownian motion à la Black and Scholes (1973). Within this setting, we introduce the concept of volatility and the challenges of modeling it. Afterwards, we explain why the traditional approaches to model the uncertainty about the volatility are problematic and propose an alternative. To show the problematic, we relate the uncertainty about the volatility to the concepts of model uncertainty and Knightian uncertainty, respectively. The alternative approach is a model framework with multiple probability measures instead of one. In the end, we discuss the mathematical issues related to the alternative model framework and show how to deal with them. The issues arise from the fact that the family of probability measures contains mutually singular measures. In order to overcome the issues, we use results from the calculus of $G$-Brownian motion, which we briefly introduce.

## 1.1 Volatility in Mathematical Finance

The traditional way in mathematical finance of representing random short-term fluctuations of financial quantities is to use a Brownian motion. The idea goes back to the early work of Bachelier (1900) and was later on popularized by the famous contributions of Black and Scholes (1973) and Merton (1973). In contrast to models in economics, the behavior of financial quantities in models of mathematical finance is modeled exogenously. Due to many short-term activities in financial markets, the behavior of financial quantities, such as asset prices or interest rates, is affected by white noise. The latter is modeled by a Brownian motion in most models of mathematical finance. That means, we consider the following model framework. Let $\Omega := C_0^d(\mathbb{R}_+)$ for $d \in \mathbb{N}$, where $C_0^d(\mathbb{R}_+)$

denotes the space of all $\mathbb{R}^d$-valued continuous paths on $\mathbb{R}_+$ starting in 0. We equip $\Omega$ with the distance $\delta : \Omega \times \Omega \to \mathbb{R}$, defined by

$$\delta(\omega, \tilde{\omega}) := \sum_{i=1}^{\infty} 2^{-i} \big( (\max_{t \in [0,i]} |\omega_t - \tilde{\omega}_t|) \wedge 1 \big).$$

Furthermore, let $\mathcal{F} := \mathcal{B}(\Omega)$, where $\mathcal{B}(\Omega)$ denotes the Borel $\sigma$-algebra on $\Omega$, let $P_0$ be the Wiener measure, and let $B = (B_t^1, ..., B_t^d)_{t \geq 0}$ be the canonical process on $\Omega$. Then the canonical process $B$ is a $d$-dimensional standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P_0)$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the filtration generated by $B$ and completed by all $P_0$-null sets, which describes the information available at each time. Each component of $B$ represents a risk factor influencing the model in some way.

Then the behavior of financial quantities is modeled as a diffusion process driven by a Brownian motion, which is typically scaled by a constant—the *volatility*, measuring the intensity of short-term fluctuations. For example, we can describe the dynamics of an asset price $S = (S_t)_{t \geq 0}$ by a stochastic differential equation of the form

$$S_t = S_0 + \int_0^t \alpha(u, S_u) du + \int_0^t \beta(u, S_u) dB_u,$$

where $B$ is a one-dimensional Brownian motion in this case. In general, there could be many risk factors driving the asset price dynamics. The functions $\alpha, \beta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are referred to as the drift coefficient and the diffusion coefficient and determine the average behavior and the short-term fluctuations of $S$, respectively. The most well-know example is the one of Black and Scholes (1973), given by

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dB_u$$

for $\mu, \sigma \in \mathbb{R}$ such that $\sigma > 0$. The constant $\sigma$ represents the volatility of $S$ and scales the magnitude of sudden price movements.

One of the main applications of such a model framework is the pricing of derivative contracts written on an underlying. Derivative contracts are actively traded in financial markets and have a future payoff depending on the evolution of the underlying in the future. Since not all derivatives are liquidly traded in the market, the seller faces the problem of finding a suitable price for the contract. The remarkable feature of the model specification of Black and Scholes (1973) from above is that it yields a unique price for derivatives written on the asset price $S$ under the assumption that the market is arbitrage-free. This assumption differs from the economics literature, where prices are usually characterized by an equilibrium. Nevertheless, the absence of arbitrage is a reasonable assumption, since arbitrage opportunities in reality only exist for a short time before they become publicly known and prices adjust to erase them. So if we believe in the remaining assumptions, such as a constant volatility, we can use the model framework from above to trade derivatives in financial markets.

However, there is plenty of empirical evidence, derived from market prices, showing that the volatility of financial quantities is not constant and, in particular, not deterministic. Market data on derivative prices reveal what is known as the *volatility smile* or *volatility skew*; this term refers to a plot of the implied volatility as a function of the remaining model parameters, which are observable. The implied volatility is the value of the constant $\sigma$ such that the theoretical price of Black and Scholes (1973) yields the price observed on the market for a given (liquidly traded) derivative contract. The shape of the plot usually turns out to be a smile or a skew and indicates that the volatility of the underlying cannot be constant, since the latter would imply a straight line in the plot instead of a smile or a skew. Moreover, a statistical analysis of historical asset price movements shows that the historical volatility exhibits random characteristics.

This well-known issue is addressed by stochastic volatility models, in which the volatility of the underlying is modeled as a stochastic process. The approach consists of replacing the constant volatility parameter $\sigma$ by a function $f : \mathbb{R} \to \mathbb{R}_+$ depending on the current realization of a stochastic process $\nu = (\nu_t)_{t \geq 0}$ whose dynamics (in addition to the dynamics of the underlying) are described by a stochastic differential equation. That is,

$$\nu_t = \nu_0 + \int_0^t \tilde{\alpha}(u, \nu_u) du + \int_0^t \tilde{\beta}(u, \nu_u) d\tilde{B}_u,$$

where $\tilde{\alpha}, \tilde{\beta} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\tilde{B} = (\tilde{B}_t)_{t \geq 0}$ is an additional one-dimensional Brownian motion, possibly correlated with $B$. In general, there could be many risk factors influencing the volatility. The famous stochastic volatility model of Heston (1993) uses the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $f(x) := \sqrt{x}$ and the process $\nu$ given by

$$\nu_t = \nu_0 + \int_0^t \theta(\tilde{\mu} - \nu_u) du + \int_0^t \sigma \sqrt{\nu_u} d\tilde{B}_u$$

for suitable parameters $\theta, \tilde{\mu}, \sigma \in \mathbb{R}$. Then the model becomes more sophisticated, but it remains tractable enough to use it for option pricing. The dynamics of the volatility in stochastic volatility models are generally chosen such that the volatility process shares the properties of the historical volatility, such as positivity and mean reversion, and such that the model-implied option prices match the prices observed on the market. For example, we could choose the model specification of Heston (1993), since it ensures that the volatility stays positive and satisfies a mean reverting behavior, and then we would choose the parameters such that the theoretical prices (implied by the model) of liquidly traded contracts match the observed prices on the market.

## 1.2 Model Uncertainty and Robust Finance

From a theoretical perspective, the problem with stochastic volatility models is that they are subject to model uncertainty, which is commonly termed *volatility uncertainty*. Model uncertainty refers to the uncertainty about the probabilistic law of the underlying model. Most models in mathematical finance assume that there is a single, known probability measure. This is a critical assumption since it is not possible to specify the probabilistic law governing the model in many situations. Stochastic volatility models face the same kind of uncertainty. As mentioned above, the dynamics of the volatility, which in turn determine the probabilistic law of the underlying, are chosen to be consistent with the historical volatility and the current option prices available on the market. However, there could be many model specifications performing this task. Moreover, it is not sure if a volatility specification that is consistent with the past is still valid in the future, since the market environment can change drastically. The dynamics of the volatility are therefore far from perfectly known, which ultimately leads to model uncertainty.

Model uncertainty essentially describes Knightian uncertainty about the probability law. The concept of *Knightian uncertainty* is named after the economist Frank H. Knight. In his book *Risk, Uncertainty, and Profit*, published a century ago, Knight (1921) distinguishes risk and uncertainty. He associates risk with something that can be measured by a probability whereas uncertainty cannot. Uncertainty applies to events in reality that are too complex to be assigned a probability or for which the related data are missing to infer the probability of it to happen. Therefore, we can interprete model uncertainty as Knightian uncertainty about the probability law; that is, we agree on the stochastic nature of the model, but it is completely unknown which probability measure describes the randomness best. Translated to volatility uncertainty, we are uncertain about the probabilistic law of the underlying as we are uncertain about its volatility.

As a consequence, we simultaneously consider a family of probability measures in the presence of model uncertainty without any assumptions on which is the right probability measure. Instead of a single probability measure $P_0$ (as in the previous section), we consider a family of possible probability measures $\mathcal{P}$ on the measure space $(\Omega, \mathcal{F})$ in the presence of model uncertainty. We call $\mathcal{P}$ the *set of beliefs*, since it contains all our beliefs about the probabilistic nature of the model. We are completely uncertain about which probability measure is correct; hence, we do not impose any assumptions on which measure is more likely to be correct. The latter reflects the characteristics of Knightian uncertainty, since it is not possible to measure the uncertainty about the probability measure by a probability.

In the presence of volatility uncertainty, each measure in the family of probability laws corresponds to a different belief about the correct volatility. In particular, we consider all probability measures corresponding to a bounded volatility process. The boundedness assumption ensures a sufficient degree of regularity and stands for ruling out all scenarios that are too extreme. We denote the state space for the volatility by $\Sigma$, where $\Sigma$ is a bounded, closed, and convex subset of $\mathbb{R}^{d\times d}$. For example, if $d = 1$, then $\Sigma$ is an interval $[\underline{\sigma}, \overline{\sigma}]$, where $\overline{\sigma}$ and $\underline{\sigma}$ represent worst-case values. We denote by $\mathcal{A}$ the collection of all possible volatility processes; that is, the space $\mathcal{A}$ consists of all $\Sigma$-valued $\mathbb{F}$-adapted processes $\sigma = (\sigma_t)_{t\geq 0}$. We construct the set of beliefs $\mathcal{P}$ in such a way that the canonical process $B$ has a different volatility under each measure $P \in \mathcal{P}$. For each $\sigma \in \mathcal{A}$, we define the process $B^\sigma = (B^\sigma_t)_{t\geq 0}$ by

$$B^\sigma_t := \int_0^t \sigma_u dB_u,$$

and we define the measure $P^\sigma$ to be the law of the process $B^\sigma$, that is,

$$P^\sigma := P_0 \circ (B^\sigma)^{-1}.$$

The set of beliefs $\mathcal{P}$ is the collection of all measures constructed in this way.

Due to the presence of multiple probability measures, we replace the classical (linear) expectation by a sublinear expectation. The notion of expectation is a substantial concept in probability theory and its applications in economics and finance. Obviously, the classical notion of expectation depends on the underlying probability measure. Since we consider many probability measures at the same time, there are several ways to form expectations. A typical approach is to use the upper expectation of the family of probability measures $\mathcal{P}$. We denote the upper expectation of $\mathcal{P}$ by $\hat{\mathbb{E}}$, which is defined by

$$\hat{\mathbb{E}}[X] := \sup_{P\in\mathcal{P}} \mathbb{E}_P[X]$$

for each measurable random variable $X$ such that $\mathbb{E}_P[X]$ exists for all $P \in \mathcal{P}$. We can interprete $\hat{\mathbb{E}}$ as a worst-case measure, since it yields the highest possible expectation among all measures in $\mathcal{P}$. In fact, it is a coherent risk measure if the random variables represent financial losses. The main difference compared to the classical (linear) expectation is that the upper expectation is sublinear.

The probabilistic setting from above is the mathematical framework we use to represent the uncertainty about the volatility and to analyze its implications. Considering all measures in $\mathcal{P}$ simultaneously reflects volatility uncertainty, since diffusion processes driven by the canonical process $B$ (in contrast to Section 1.1) have a different volatility under each measure. It should be noted that the volatility under each measure in the set of beliefs is a stochastic process and not simply a constant that varies among the measures in $\mathcal{P}$. In fact, most parts of the mathematical analysis in the succeeding chapters are based on the calculus of $G$-Brownian motion, which does not require a probabilistic framework at all. We still use the probabilistic setting from above, since it is a more natural approach to model volatility uncertainty from an economic perspective and some notions in mathematical finance, such as the notion of arbitrage, crucially depend on probabilities.

## 1.3 Mathematics of Volatility Uncertainty

The presence of volatility uncertainty leads to mathematical difficulties, since the family of probability measures representing volatility uncertainty contains mutually singular measures. The canonical process $B$ has (by construction) a different volatility under each measure in $\mathcal{P}$. Thus, the quadratic variation process $\langle B\rangle = (\langle B\rangle_t)_{t\geq 0}$ differs among the measures in $\mathcal{P}$. For example, if $d=1$ and $\Sigma = [\underline{\sigma}, \overline{\sigma}]$ and we consider the measures $P^{\overline{\sigma}}$ and $P^{\underline{\sigma}}$, induced by the constant volatilities $\overline{\sigma}$ and $\underline{\sigma}$, respectively, we have

$$P^{\overline{\sigma}}(\langle B\rangle_t = \overline{\sigma}^2 t) = 1 \neq 0 = P^{\underline{\sigma}}(\langle B\rangle_t = \overline{\sigma}^2 t).$$

Therefore, there are measures in the set of beliefs that have different null sets, that is, the set $\mathcal{P}$ contains mutually singular measures. This causes mathematical problems, since many results from probability theory and stochastic calculus only hold up to null sets of the underlying measure. Important examples include the definition of time consistent conditional expectations and stochastic integrals. The former and the latter can be solved by restricting the class of random variables and the space of admissible integrands, respectively, which is explained at the end of this section.

The probabilistic setting from Section 1.2 is naturally connected to the calculus of $G$-Brownian motion. According to Denis, Hu, and Peng (2011, Theorem 54), the upper expectation $\hat{\mathbb{E}}$ corresponds to the $G$-expectation on $L_G^1(\Omega)$, and the canonical process $B$ is a $G$-Brownian motion under $\hat{\mathbb{E}}$, where $G : \mathbb{S}^d \to \mathbb{R}$ is given by

$$G(A) = \tfrac{1}{2} \sup_{\sigma\in\Sigma} \operatorname{tr}(\sigma\sigma' A).$$

The space $\mathbb{S}^d$ denotes the space of all symmetric $d\times d$ matrices, and $'$ denotes the transpose of a matrix. The connection to the calculus of $G$-Brownian motion allows us to acquire all of its results. In particular, we can use the results to overcome the problems mentioned above. The formal definition and the construction of a $G$-Brownian motion can be found in Chapter A of the appendix or in the book of Peng (2019). Below, we state some of the most important results in order to give the reader an intuition about how the mathematical framework differs from the classical Itô calculus, and we introduce all notions that the succeeding chapters require.

The main difference between the definition of a $G$-Brownian motion and the definition of a standard Brownian motion relates to the distribution, apart from which they basically coincide. The function $G$ is the generator of the nonlinear partial differential equation that defines the $G$-expectation and characterizes the distribution and the uncertainty of a $G$-Brownian motion. For example, the function $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}, (t,x) \mapsto \hat{\mathbb{E}}[\varphi(x + B_t)]$, for a sufficiently regular function $\varphi : \mathbb{R}^d \to \mathbb{R}$, is the unique viscosity solution to the nonlinear partial differential equation

$$\partial_t u + G(D^2_{xx} u) = 0, \quad u(0,x) = \varphi(x),$$

called *$G$-heat equation*. The operator $D^2_{xx}$ denotes the Hessian of a function with respect to $x$. The $G$-expectation and the conditional $G$-expectation, denoted by $\hat{\mathbb{E}}_t$ for $t \in \mathbb{R}_+$, of a function depending on finitely many increments of a $G$-Brownian motion are defined in a similar fashion. Moreover, for $p \geq 1$, both can be extended to the completion of functions of this type under the natural norm $\| \cdot \|_p := \hat{\mathbb{E}}[| \cdot |^p]^{\frac{1}{p}}$, which is denoted by $L^p_G(\Omega)$. The space $L^1_G(\Omega)$ represents the space of all random variables for which the $G$-expectation is defined. Due to the probabilistic representation of the $G$-expectation, the equality of random variables in $L^1_G(\Omega)$ is equivalent to random variables being equal *quasi-surely*, i.e., they are equal $P$-almost surely for all $P \in \mathcal{P}$. The same applies to inequalities between elements in $L^1_G(\Omega)$. We also use the terminology *$\mathcal{P}$-quasi-surely* if we need to indicate under which set of measures a statement holds quasi-surely.

In comparison to a standard Brownian motion, a $G$-Brownian motion has many interesting properties. To demonstrate this, we consider the one-dimensional case, that is, the case where $d = 1$ and $\Sigma = [\underline{\sigma}, \overline{\sigma}]$. One can use the fact that the $G$-expectation is defined by the $G$-heat equation to show that

$$\hat{\mathbb{E}}[B_t] = 0 = -\hat{\mathbb{E}}[-B_t],$$
$$\hat{\mathbb{E}}[B_t^2] = \overline{\sigma}^2 t \geq \underline{\sigma}^2 t - \hat{\mathbb{E}}[-B_t^2].$$

We call $-\hat{\mathbb{E}}[-\xi]$ the *lower expectation of* $\xi$ for a random variable $\xi \in L^1_G(\Omega)$, since it has the probabilistic representation

$$-\hat{\mathbb{E}}[-\xi] = \inf_{P \in \mathcal{P}} \mathbb{E}_P[\xi].$$

In contrast to the upper expectation, the lower expectation yields the lowest possible expectation. Hence, a $G$-Brownian motion has no mean uncertainty, but it has an uncertain variance as long as $\overline{\sigma} > \underline{\sigma}$. In addition, one can show that the quadratic variation of a $G$-Brownian motion is an uncertain process (if $\overline{\sigma} > \underline{\sigma}$), satisfying

$$\overline{\sigma}^2 t \geq \langle B \rangle_t \geq \underline{\sigma}^2 t.$$

Despite the differences, a $G$-Brownian motion is a generalization of a standard Brownian motion, since the former corresponds to the latter if the uncertainty about the volatility vanishes. If there is no uncertainty about the volatility, the state space for the volatility $\Sigma$ is a singleton consisting of the identity matrix, denoted by $I_d$. Then the $G$-heat equation becomes a linear heat equation; thus, the $G$-expectation coincides with the expectation of a standard Brownian motion. Apart from that, the construction of the set of beliefs shows that $\mathcal{P}$ is a singleton consisting solely of $P_0$ if $\Sigma = I_d$. Then we remain in the traditional probabilistic framework of Section 1.1, in which the canonical process is a standard Brownian motion instead of a $G$-Brownian motion. In any case, this enables us to check if the results in the succeeding chapters are consistent with traditional models driven by a standard Brownian motion.

It is possible to generalize most parts of the well-known Itô calculus to a $G$-Brownian motion. For example, the definition of a stochastic integral with respect to a $G$-Brownian motion relies on the same procedure used to define the stochastic integral with respect to a standard Brownian motion. First, one defines the integral for simple processes, then, thanks to the isometry property, one extends the integral to the completion of all simple processes with respect to a suitable norm. In the same way, one can construct the integral with respect to the quadratic variation of a $G$-Brownian motion. The space of admissible integrands on $[0, T]$ for integrals related to a $G$-Brownian motion is denoted by $M_G^p(0, T)$ for $T < \infty$ and $p \geq 1$, and it is a Banach space under the norm

$$\| \cdot \|_{M,p} := \hat{\mathbb{E}}\left[ \int_0^T | \cdot |^p dt \right]^{\frac{1}{p}}.$$

The formal construction of stochastic integrals and the space $M_G^p(0, T)$ can be found in Chapter A of the appendix or in the book of Peng (2019). In addition, there are extensions of various results from stochastic calculus to a $G$-Brownian motion, which we use throughout the thesis by referring to the literature.

A further important generalization concerns the concept of martingales in the calculus of $G$-Brownian motion. The concept of martingales is of fundamental importance in mathematical finance, since it is related to the absence of arbitrage, and its definition depends on the expectation under a specific probability measure. In the presence of volatility uncertainty, we consider several probability measures simultaneously, which ultimately leads to a sublinear expectation corresponding to the $G$-expectation. As the $G$-expectation differs from the classical notion of expectation, the notion of martingales has to be suitably adapted. A process $M = (M_t)_{t \geq 0}$ is called a *$G$-martingale* if $M_t \in L_G^1(\Omega_t)$ for all $t \geq 0$ and if it satisfies

$$M_s = \hat{\mathbb{E}}_s[M_t]$$

for $s \leq t$. That means, a $G$-martingale is essentially a martingale in the worst case among all considered scenarios, if we interprete $\hat{\mathbb{E}}$ as a worst-case measure. The important difference compared to the classical definition of martingales is that $-M$ is not necessarily a $G$-martingale if $M$ is a $G$-martingale, which is due to the nonlinearity of the $G$-expectation. We call a process $M$ a *symmetric $G$-martingale* if $M$ and $-M$ are $G$-martingales. The notion of symmetric $G$-martingales is important for the succeeding chapters, since it rules out arbitrage opportunities.

The space of admissible random variables in the calculus of $G$-Brownian motion has a probabilistic representation. For this purpose, we denote by $L^0(\Omega)$ the space of all $\mathcal{B}(\Omega)$-measurable functions, mapping from $\Omega$ into $\mathbb{R}$. By Proposition 6.3.2 of Peng (2019), which was originally shown by Denis, Hu, and Peng (2011), we have

$$L_G^p(\Omega) = \big\{ \xi \in L^0(\Omega) \;\big|\; \xi \text{ has a q.c. version, } \lim_{n\to\infty} \hat{\mathbb{E}}[|\xi|^p 1_{\{|\xi|>n\}}] = 0 \big\}.$$

We say that $\xi : \Omega \to \mathbb{R}$ is *quasi-continuous* (q.c.) if for all $\epsilon > 0$, there exists an open set $O$ with $\sup_{P\in\mathcal{P}} P(O) < \epsilon$ such that $\xi$ is continuous on $O^c$, and we say that $\xi$ has a q.c. version if there exists a q.c. function $\tilde{\xi}$ such that $\xi = \tilde{\xi}$ quasi-surely. In fact, the same holds if we replace $\Omega$ by $\Omega_T := C_0^d([0,T])$ for $T < \infty$. The space $L_G^p(\Omega_T)$ consists, roughly speaking, of all random variables only depending on the trajectory of $B$ up to time $T$. The precise construction of the space $L_G^p(\Omega_T)$ can be found in Chapter A of the appendix or in the book of Peng (2019).

In addition, there is a probabilistic representation of the space of admissible integrands in the calculus of $G$-Brownian motion. For this purpose, we introduce the following notation. We define the capacity $c : \mathcal{B}([0,T]) \otimes \mathcal{F}_T \to \mathbb{R}$ by $c(A) := \frac{1}{T}\|1_A\|_{M,p}^p$, and we define the space

$$M^p(0,T) := \big\{ \eta : [0,T] \times \Omega_T \to \mathbb{R} \;\big|\; \eta \text{ is progressively measurable, } \|\eta\|_{M,p}^p < \infty \big\}.$$

We say that a progressively measurable process $\eta : [0,T] \times \Omega_T \to \mathbb{R}$ is q.c. if for all $\epsilon > 0$, there exists a progressively measurable open set $O \subset [0,T] \times \Omega_T$ such that $c(O) < \epsilon$ and $\eta$ is continuous on $O^c$. We equip $[0,T] \times \Omega_T$ with the distance

$$\tilde{\delta}\big((t,\omega),(\tilde{t},\tilde{\omega})\big) := |t - \tilde{t}| + \max_{s\in[0,T]} |\omega_s - \tilde{\omega}_s|.$$

We say that a progressively measurable process $\eta : [0,T] \times \Omega_T \to \mathbb{R}$ has a q.c. version if there exists a q.c. process $\tilde{\eta}$ such that $c(\{\eta \neq \tilde{\eta}\}) = 0$. Then, by Theorem 6.4.5 from Peng (2019), which was originally shown by Hu, Wang, and Zheng (2016), we have

$$M_G^p(0,T) = \left\{ \eta \in M^p(0,T) \;\middle|\; \eta \text{ has a q.c. version, } \lim_{n\to\infty} \hat{\mathbb{E}}\left[ \int_0^T |\eta|^p 1_{\{|\eta|\geq n\}} dt \right] = 0 \right\}.$$

The restriction to the spaces from above solves the mathematical issues mentioned at the beginning of this section. One can define a time consistent conditional sublinear expectation, given by the conditional $G$-expectation, mapping from $L_G^1(\Omega)$ into $L_G^1(\Omega_t)$. The conditional $G$-expectation basically satisfies the same properties as $\hat{\mathbb{E}}$. In addition, it satisfies the tower property—namely, it holds $\hat{\mathbb{E}}_s[\hat{\mathbb{E}}_t[\xi]] = \hat{\mathbb{E}}_{s\wedge t}[\xi]$ for $s, t \in \mathbb{R}_+$ and $\xi \in L_G^1(\Omega)$—and it holds $\hat{\mathbb{E}}_t[\xi] = \xi$ for $t \in \mathbb{R}_+$ and $\xi \in L_G^1(\Omega_t)$. The regularity of the conditional $G$-expectation is due to the fact that it is defined via the $G$-heat equation. Moreover, one can define the stochastic integral $\int_0^T \eta_t dB_t$, mapping into $L_G^2(\Omega_T)$, for a process $\eta \in M_G^2(0,T)$ for $T \in \mathbb{R}_+$. The definition relies on the isometry property of the stochastic integral with respect to $B$, which, in turn, relies on the time consistency of the conditional $G$-expectation. In any case, we can circumvent the mathematical problems arising in the presence of volatility uncertainty by using the spaces $L_G^p(\Omega)$ and $M_G^p(0,T)$. However, the disadvantage compared to the traditional Itô calculus is that we need to restrict to random variables and stochastic processes that have a q.c. version and satisfy some kind of uniform integrability condition, as the representation of the spaces $L_G^p(\Omega)$ and $M_G^p(0,T)$ shows. As mentioned in the introduction, there are several approaches to model volatility uncertainty and a lot of extensions. In particular, there are extensions to spaces greater than $L_G^p(\Omega)$ and $M_G^p(0,T)$. Yet we stick to the classical spaces to use all of the results from the literature on $G$-Brownian motion.

# Chapter 2

# The Hull-White Model

In the present chapter, we study the Hull-White model for the term structure of interest rates under volatility uncertainty. The Hull-White model is based on modeling the instantaneous spot interest rate, called *short rate*, as a diffusion process. As in the classical Hull-White model, we describe the evolution of the short rate by a diffusion process of Ornstein-Uhlenbeck type. This ensures that the short rate satisfies a mean reverting behavior, which is a typical feature of interest rates. The difference compared to the traditional model is that the volatility is uncertain in the sense of model uncertainty. As in Section 1.2, we represent the uncertainty about the volatility by a family of probability measures, termed *set of beliefs*, and it is completely uncertain which one is correct. In particular, we consider all measures such that the volatility is bounded by two extreme values. Since this setting naturally leads to a $G$-Brownian motion (as we described in Section 1.3), the driver of the short rate dynamics then becomes a $G$-Brownian motion. Thus, the short rate evolves as an Ornstein-Uhlenbeck process driven by a $G$-Brownian motion. Then the variance of the short rate is uncertain while its mean is not.

The main question in this chapter is how to find an arbitrage-free term structure in the presence of volatility uncertainty, that is, how to price zero-coupon bonds such that the related bond market is arbitrage-free when the volatility is uncertain. The crucial characteristic of volatility uncertainty is that it is represented by a nondominated set of beliefs. That means, there is no measure dominating all measures in the set of beliefs. Hence, it is not possible to find a single equivalent martingale measure for the related bond market. The discussion about arbitrage thus becomes a subtle issue in this framework. If we want to follow a martingale modeling approach, we need to choose the bond prices in such a way that the discounted bonds are symmetric $G$-martingales, which means that they are martingales in each possible scenario for the volatility. Martingale modeling is a common approach in short rate models, but we can unfortunately show that this approach does not work under the initially given set of beliefs.

In order to find an arbitrage-free term structure, we consider sublinear expectations defined by a linear $G$-backward stochastic differential equation. By standard results on

$G$-backward stochastic differential equations, we can define consistent sublinear expectations by this procedure. Since the $G$-backward stochastic differential equation is linear, there exists an explicit solution. The representation of the solution shows that the resulting sublinear expectation corresponds to the expectation under an equivalent change of measure. We can also formally show that sublinear expectations defined in this way are in some sense equivalent to the initial one. As a consequence, the bond market is arbitrage-free if there exists a sublinear expectation of this particular type under which the discounted bonds are symmetric $G$-martingales.

We show that there exists a sublinear expectation of the above kind under which there is a unique arbitrage-free term structure. If we choose a particular process as a coefficient in the linear $G$-backward stochastic differential equation defining equivalent sublinear expectations, we obtain a sublinear expectation under which there is a unique expression for the bond prices such that the discounted bonds are symmetric $G$-martingales. The choice might seem special, but it can be justified by economic arguments. Due to the Girsanov transformation for $G$-Brownian motion, the process represents an adjustment factor, adjusting the short rate by its uncertain variance. Alternatively, we can interpret the process as the market price of risk. Since the model is not only subject to risk but also subject to uncertainty, we also refer to the process as the *market price of uncertainty*. The resulting bond prices are different—though similar—to the prices from the traditional model without volatility uncertainty. In particular, they have an affine structure with respect to the short rate and the market price of uncertainty.

Even though the structure of the model is different from the traditional one, we are yet consistent with the classical Hull-White model after fitting the yield curve. Since we consider an equivalent sublinear expectation, the Girsanov transformation for $G$-Brownian motion implies that the dynamics of the short rate—as well as the bond prices—differ from the ones of the traditional model. As in the classical model, we use the mean reversion level of the short rate to fit the bond prices of the model to an initial yield curve, observable on the market. Surprisingly, then the short rate dynamics and the bond prices are again consistent with the ones from the classical Hull-White model; they are consistent in the sense that the short rate dynamics and the bond prices coincide with the classical ones if we drop the uncertainty about the volatility.

In addition, we study an extension of the model driven by multiple risk factors. For the sake of simplicity, we derive the results mentioned above in the presence of a single risk factor; that is, the short rate is driven by a single $G$-Brownian motion. Such a framework simplifies the interpretation and the intuition of the results and enables us to compare the results with the classical Hull-White model. Empirical studies, however, show that more factors are needed in order to explain term structure movements (Adrian, Crump, and Moench, 2013; Dai and Singleton, 2003; Joslin, Priebsch, and Singleton, 2014). Therefore, we consider a model extension in which the short rate is affected by several risk factors with uncertain volatilities and uncertain correlations. We are able to extend all of the previous results to the general case.

The chapter is organized as follows. In Section 2.1, we present the framework for modeling volatility uncertainty and the short rate process, and we study the properties of the latter. Section 2.2 introduces the related bond market. In Section 2.3, we adapt the concept of martingale modeling to volatility uncertainty and show that martingale modeling does not work in the presence of volatility uncertainty. Hence, we define equivalent sublinear expectations in Section 2.4, which we can use to find an arbitrage-free term structure. In Section 2.5, we show that there exists an equivalent sublinear expectation under which we obtain a unique arbitrage-free term structure. Section 2.6 demonstrates how to fit the model to an initially observed term structure. In Section 2.7, we extend the model to a version driven by multiple risk factors. Section 2.8 discusses further related investigations.

## 2.1 Short Rate Dynamics

In the traditional Hull-White model—without volatility uncertainty—the behavior of the short rate is described by an Ornstein-Uhlenbeck process driven by a standard Brownian motion. Let us consider the probability space $(\Omega, \mathcal{F}, P_0)$, which we introduced in Section 1.1, with the canonical process $B = (B_t)_{t\geq 0}$ and the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, which is generated by $B$ and completed by all $P_0$-null sets. We assume that $d = 1$ in this chapter (except in Section 2.7); that is, the canonical process $B$ is a one-dimensional standard Brownian motion under $P_0$. The classical Hull-White model, without volatility uncertainty, assumes that the short rate process $r = (r_t)_{t\geq 0}$ satisfies the stochastic differential equation

$$r_t = r_0 + \int_0^t \big(\mu(u) - \theta r_u\big)du + \sigma B_t \tag{2.1}$$

for a suitably integrable function $\mu : \mathbb{R}_+ \to \mathbb{R}$ and constants $\theta, \sigma > 0$. Then the short rate is a mean reverting process with a time dependent mean reversion level $\mu$, a constant mean reversion speed $\theta$, and a constant volatility $\sigma$. The mean reversion level is time dependent to make the model more realistic and to ensure a perfect fit of the model to market data, called *yield curve fitting*. The mean reversion speed and the volatility are kept constant for tractability reasons.

In the presence of volatility uncertainty, we consider a family of probability measures that leads to a $G$-Brownian motion. That means, we consider the set of beliefs $\mathcal{P}$ from Section 1.2, in which each measure represents a different belief about the volatility. Since we assume that $d = 1$, the state space for the (uncertain) volatility $\Sigma$ is given by an interval $[\underline{\sigma}, \overline{\sigma}]$, where we assume that $\overline{\sigma} \geq \underline{\sigma} > 0$. The constants $\overline{\sigma}$ and $\underline{\sigma}$ represent worst-case values for the volatility. So the set of beliefs consists of all measures such that the volatility of the canonical process is a $[\underline{\sigma}, \overline{\sigma}]$-valued $\mathbb{F}$-adapted process. As described in Section 1.3, the upper expectation of the set of beliefs,

$$\hat{\mathbb{E}}[\cdot] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[\cdot],$$

corresponds to the $G$-expectation on $L_G^1(\Omega)$, and the canonical process $B$ is a $G$-Brownian motion under $\hat{\mathbb{E}}$. In this case, the nonlinear generator $G : \mathbb{R} \to \mathbb{R}$ of the $G$-heat equation is given by

$$G(a) = \tfrac{1}{2} \sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \{\sigma^2 a\},$$

and the canonical process $B$ is a one-dimensional $G$-Brownian motion.

We describe the behavior of the short rate by an Ornstein-Uhlenbeck process driven by a $G$-Brownian motion. We choose the same structure as in the classical Hull-White model. The difference is that we include volatility uncertainty by replacing the constant volatility and the standard Brownian motion by a $G$-Brownian motion. Hence, we suppose that the short rate process $r$ is given by the $G$-stochastic differential equation

$$r_t = r_0 + \int_0^t \big(\mu(u) - \theta r_u\big) du + B_t \tag{2.2}$$

for a suitably integrable function $\mu : \mathbb{R}_+ \to \mathbb{R}$ and a constant $\theta > 0$. A $G$-stochastic differential equation refers to a stochastic differential equation driven by a $G$-Brownian motion. Then the short rate has a time dependent mean reversion level, which is deterministic, and a time dependent volatility, which is uncertain. This is desirable since we can use the mean reversion level for yield curve fitting and we do not have to specify any volatility structure. It should be noted that (2.2) corresponds to (2.1)—i.e., the classical case without volatility uncertainty—if $\overline{\sigma} = \sigma = \underline{\sigma}$.

The $G$-stochastic differential equation describing the dynamics of the short rate has a closed-form solution. By a classical result on the existence and the uniqueness of solutions to $G$-stochastic differential equations (Peng, 2019, Theorem 5.1.3), we know that (2.2) has a unique solution in $\bar{M}_G^2(0,T)$ for every $T < \infty$; the space $\bar{M}_G^2(0,T)$ is a subspace of $M_G^2(0,T)$. Therefore, the short rate $r$ is a regular process and, in particular, an admissible integrand. As in the classical case, we can explicitly solve (2.2).

**Proposition 2.1.** *The solution to the $G$-stochastic differential equation* (2.2) *is given by*

$$r_t = e^{-\theta t} r_0 + \int_0^t e^{-\theta(t-u)} \mu(u) du + \int_0^t e^{-\theta(t-u)} dB_u. \tag{2.3}$$

*Proof.* This can be verified by using Itô's formula for $G$-Brownian motion (Li and Peng, 2011, Theorem 5.4). The verification works totally analogous to the classical case with a standard Brownian motion. □

The short rate has no mean uncertainty, but it has an uncertain variance. We can easily show that the upper expectation of the short rate coincides with its lower expectation. Thus, the mean of the short rate is deterministic. In addition, we can show that the upper, respectively lower, expectation of the squared deviation of the short rate from its mean is given by the variance from the classical Hull-White model with the highest, respectively lowest, possible volatility. Hence, the short rate has an uncertain variance, which is bounded by two extreme values.

**Theorem 2.2.** *For all $t$, the short rate $r_t$ satisfies*

$$\hat{\mathbb{E}}[r_t] = e^{-\theta t} r_0 + \int_0^t e^{-\theta(t-u)} \mu(u) du = -\hat{\mathbb{E}}[-r_t], \tag{2.4a}$$

$$\hat{\mathbb{E}}\big[(r_t - \hat{\mathbb{E}}[r_t])^2\big] = \tfrac{\overline{\sigma}^2}{2\theta}(1 - e^{-2\theta t}) \geq \tfrac{\underline{\sigma}^2}{2\theta}(1 - e^{-2\theta t}) = -\hat{\mathbb{E}}\big[-(r_t - \hat{\mathbb{E}}[r_t])^2\big]. \tag{2.4b}$$

*Proof.* First, we sketch how to obtain (2.4a). The first two summands on the right-hand side of (2.3) are deterministic. We know that the upper expectation and the lower expectation of an integral with respect to a $G$-Brownian motion vanish. Therefore, it holds (2.4a).

In order to compute the upper and the lower expectation in (2.4b), we use the nonlinear Feynman-Kac formula from Hu, Ji, Peng, and Song (2014). We define the process $X = (X_t)_{t \geq 0}$ as the deviation of $r$ from its mean; that is,

$$X_t := r_t - \hat{\mathbb{E}}[r_t] = \int_0^t e^{-\theta(t-u)} dB_u.$$

By Proposition 2.1, we know that $X$ solves the $G$-stochastic differential equation

$$X_t = -\int_0^t \theta X_u du + B_t.$$

Then the nonlinear Feynman-Kac formula implies $\hat{\mathbb{E}}_t[X_T^2] = u(t, X_t)$ for $t \leq T$ (Hu, Ji, Peng, and Song, 2014, Theorems 4.4, 4.5), where the function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ is the unique viscosity solution to the nonlinear partial differential equation

$$\partial_t u + G(\partial_{xx}^2 u) - \theta x \partial_x u = 0, \quad u(T,x) = x^2.$$

One can verify that the solution to the nonlinear partial differential equation is given by

$$u(t,x) = \tfrac{\overline{\sigma}^2}{2\theta}(1 - e^{-2\theta(T-t)}) + e^{-2\theta(T-t)} x^2.$$

This proves the first equality in (2.4b); the second follows by the same procedure. □

## 2.2 Related Bond Market

The corresponding bond market consists of a money-market account and zero-coupon bonds for all possible maturities. First of all, we fix a finite time $\tau < \infty$ and suppose that all trading takes place within the finite time horizon $[0, \tau]$. The market consists of the following investment opportunities. The first one is to invest in the money-market account, which grows by the short rate $r$. The money-market account is a process denoted by $M = (M_t)_{0 \le t \le \tau}$, and it is given by

$$M_t := \exp\left( \int_0^t r_s ds \right).$$

In addition to the money-market account, the market offers zero-coupon bonds for all maturities within the time horizon. For $T \le \tau$, the price of a bond with maturity $T$ at time $t$ is denoted by $P_t(T)$ for $t \le T$. The bond has a terminal payoff of 1; that is,

$$P_T(T) = 1$$

for all $T$. Henceforth, we use the money-market account as a numéraire. That means, we restrict to the discounted bonds $\tilde{P}(T) = (\tilde{P}_t(T))_{0 \le t \le T}$ for $T \le \tau$, defined by

$$\tilde{P}_t(T) := M_t^{-1} P_t(T).$$

We assume that the discounted bond $\tilde{P}(T)$, for all $T$, is a diffusion process driven by the $G$-Brownian motion $B$; i.e.,

$$\tilde{P}_t(T) = \tilde{P}_0(T) + \int_0^t \alpha_u(T) du + \int_0^t \beta_u(T) dB_u + \int_0^t \gamma_u(T) d\langle B \rangle_u$$

for processes $\alpha(T) = (\alpha_t(T))_{0 \le t \le T}$, $\beta(T) = (\beta_t(T))_{0 \le t \le T}$, and $\gamma(T) = (\gamma_t(T))_{0 \le t \le T}$ in $M_G^2(0, T)$. This is a technical assumption to make the following definition work. The assumption is satisfied in all of the succeeding scenarios.

The agents can participate in the market by choosing a trading strategy to create a portfolio. Choosing a market strategy means that they can select a finite number of discounted bonds they want to trade and decide on how much of them they want to buy or sell at each time within the time horizon. The value of the related portfolio is the integral of the market strategy with respect to the price processes—that means, we implicitly assume that the trading strategy is self-financing.

**Definition 2.3.** *An admissible market strategy $(\pi, T)$ is a couple consisting of a bounded* *process* $\pi = (\pi_t^1, ..., \pi_t^n)_{0 \leq t \leq \tau}$ *in* $M_G^2(0, \tau; \mathbb{R}^n)$ *and a vector* $T = (T_1, ..., T_n) \in [0, \tau]^n$ *for some* $n \in \mathbb{N}$*. The corresponding portfolio value at terminal time is defined by*

$$\tilde{v}_\tau(\pi, T) := \sum_{i=1}^{n} \int_0^{T_i} \pi_t^i d\tilde{P}_t(T_i).$$

The restriction to trading a finite number of discounted bonds could be generalized by using methods from large financial markets (Klein, Schmidt, and Teichmann, 2016) or by allowing for measure-valued trading strategies (Björk, Di Masi, Kabanov, and Runggaldier, 1997). Here we restrict to trading finitely many discounted bonds, since such a generalization is not the objective of the present chapter.

We use a quasi-sure notion of arbitrage. The classical definition of arbitrage depends on the underlying probability measure of the model. Since we are dealing with more than one measure in the presence of volatility uncertainty, we have to consider a definition slightly different from the classical one. The following definition of arbitrage corresponds to the one that is commonly used in the literature on robust finance (Biagini, Bouchard, Kardaras, and Nutz, 2017; Bouchard and Nutz, 2015).

**Definition 2.4.** *An admissible market strategy* $(\pi, T)$ *is called arbitrage strategy if*

$$\tilde{v}_\tau(\pi, T) \geq 0 \quad \text{quasi-surely,} \quad P\big(\tilde{v}_\tau(\pi, T) > 0\big) > 0 \quad \text{for at least one } P \in \mathcal{P}.$$

*Moreover, we say that the bond market is arbitrage-free if there is no arbitrage strategy.*

This is a weaker version than requiring that the strategy has to be an arbitrage in the classical sense with respect to all measures. The difference is that the probability of a strictly positive win does not have to be strictly positive under each measure.

## 2.3 Martingale Modeling

Most short rate models use a martingale modeling approach to ensure that the related bond market is arbitrage-free. A standard result in mathematical finance is that the market is arbitrage-free if and only if the traded quantities on the market are martingales under a measure equivalent to the real world measure, termed *fundamental theorem of asset pricing*. The common practice in short rate models is martingale modeling, since bond markets are incomplete; incomplete means that there is not a unique martingale measure but many of them. Thus, one usually supposes that the short rate satisfies certain dynamics under a given martingale measure, and then the bond prices are chosen such that the discounted bonds are martingales under the exogenously given martingale measure in order to exclude arbitrage.

In the presence of volatility uncertainty, martingale modeling requires that the discounted bonds are symmetric $G$-martingales under $\hat{\mathbb{E}}$. If there is volatility uncertainty, the set of beliefs contains mutually singular measures. Hence, there is no dominating measure for the set of beliefs, which implies that it is not possible to find a single martingale measure equivalent to all measures in the set of beliefs. A fundamental theorem of asset pricing under a possibly nondominated set of beliefs was established by Bouchard and Nutz (2015), for the discrete-time case, and Biagini, Bouchard, Kardaras, and Nutz (2017), for the continuous-time case. Roughly speaking, the theorem says that the absence of arbitrage is equivalent to the existence of a set of martingale measures that is in some sense equivalent to the set of beliefs—that means, the price processes have to be martingales under each measure in the equivalent set of measures. Therefore, if we want to follow a martingale modeling approach in the presence of volatility uncertainty, we need to assume that our set of beliefs is a set of exogenously given martingale measures. Then we need to choose the bond prices such that the discounted bonds are martingales under each measure in the set of beliefs. Being a martingale under each measure in the set of beliefs is equivalent to being a symmetric $G$-martingale under $\hat{\mathbb{E}}$. The sufficiency of this martingale modeling approach for the absence of arbitrage is shown by the following proposition.

**Proposition 2.5.** *The bond market is arbitrage-free if the discounted bond $\tilde{P}(T)$ is a symmetric $G$-martingale under $\hat{\mathbb{E}}$ for all $T$.*

*Proof.* We suppose that there exists an arbitrage strategy $(\pi, T)$ and show that this leads to a contradiction. By Definition 2.4, it holds $\tilde{v}_\tau(\pi, T) \geq 0$. Hence, we know that $|\tilde{v}_\tau(\pi, T)| = \tilde{v}_\tau(\pi, T)$, which implies

$$\hat{\mathbb{E}}[|\tilde{v}_\tau(\pi, T)|] = \hat{\mathbb{E}}[\tilde{v}_\tau(\pi, T)].$$

Using Definition 2.3 and the sublinearity of $\hat{\mathbb{E}}$, we obtain

$$\hat{\mathbb{E}}[\tilde{v}_\tau(\pi, T)] \leq \sum_{i=1}^{n} \hat{\mathbb{E}}\Big[\int_0^{T_i} \pi_t^i d\tilde{P}_t(T_i)\Big].$$

By the representation theorem for symmetric $G$-martingales (Song, 2011, Theorem 4.8), for all $T$, there exists a process $H(T) = (H_t(T))_{0 \leq t \leq T}$ in $M_G^2(0, T)$ such that

$$\tilde{P}_t(T) = \tilde{P}_0(T) + \int_0^t H_u(T) dB_u.$$

Since $\pi^i$ is a bounded process in $M_G^2(0, \tau)$, we have $\pi^i H(T_i) \in M_G^2(0, T_i)$ for all $i$. Thus,

$$\hat{\mathbb{E}}\Big[\int_0^{T_i} \pi_t^i d\tilde{P}_t(T_i)\Big] = \hat{\mathbb{E}}\Big[\int_0^{T_i} \pi_t^i H_t(T_i) dB_t\Big] = 0$$

for all $i$. Combining the previous steps, we get $\tilde{v}_\tau(\pi, T) = 0$, which is a contradiction to

$$P\big(\tilde{v}_\tau(\pi, T) > 0\big) > 0 \quad \text{for at least one } P \in \mathcal{P}.$$

Therefore, there is no arbitrage strategy. □

Unfortunately, we can show that the martingale modeling approach in the Hull-White model does not work in the presence of volatility uncertainty. Martingale modeling only works in the classical case when there is no uncertainty about the volatility. In that case, the bond prices are obviously given by the bond prices from the classical Hull-White model.

**Theorem 2.6.** *The discounted bond $\tilde{P}(T)$ is a symmetric $G$-martingale under $\hat{\mathbb{E}}$ if and only if $\overline{\sigma} = \underline{\sigma}$ and the bond price is given by*

$$P_t(T) = \exp\big(A^{\overline{\sigma}}(t, T) - B(t, T) r_t\big) \tag{2.5}$$

*for all $t$, where $A^\sigma, B : [0, \tau] \times [0, \tau] \to \mathbb{R}$, for $\sigma > 0$, are defined by*

$$A^\sigma(t, T) := \int_t^T \big(\tfrac{1}{2}\sigma^2 B(s, T)^2 - \mu(s) B(s, T)\big) ds, \tag{2.6a}$$

$$B(t, T) := \tfrac{1}{\theta}(1 - e^{-\theta(T-t)}), \tag{2.6b}$$

*respectively.*

*Proof.* First, let us suppose that the discounted bond $\tilde{P}(T)$ is a symmetric $G$-martingale under $\hat{\mathbb{E}}$. We show that the expectation of the discount factor is the same under each measure in the set of beliefs. The definition of symmetric $G$-martingales and the terminal condition of the bond implies

$$\tilde{P}_t(T) = \hat{\mathbb{E}}_t[\tilde{P}(T, T)] = \hat{\mathbb{E}}_t[M_T^{-1}],$$
$$\tilde{P}_t(T) = -\hat{\mathbb{E}}_t[-\tilde{P}(T, T)] = -\hat{\mathbb{E}}_t[-M_T^{-1}]$$

for all $t$. Combining the previous equations and setting $t = 0$ yields

$$\sup_{P \in \mathcal{P}} \mathbb{E}_P[M_T^{-1}] = \inf_{P \in \mathcal{P}} \mathbb{E}_P[M_T^{-1}], \tag{2.7}$$

which in turn implies that the expectation of $M_T^{-1}$ is the same under each measure.

Now we use the expression for the bond prices from the classical Hull-White model to show that (2.7) implies $\overline{\sigma} = \underline{\sigma}$ and (2.5). Let us consider the measures $P^{\overline{\sigma}}, P^{\underline{\sigma}} \in \mathcal{P}$ induced by the highest and the lowest possible volatility, respectively. The expectations of $M_T^{-1}$ under $P^{\overline{\sigma}}$ and $P^{\underline{\sigma}}$ are given by

$$\mathbb{E}_{P^{\overline{\sigma}}}[M_T^{-1}] = \exp\big(A^{\overline{\sigma}}(0,T) - B(0,T)r_0\big),$$
$$\mathbb{E}_{P^{\underline{\sigma}}}[M_T^{-1}] = \exp\big(A^{\underline{\sigma}}(0,T) - B(0,T)r_0\big),$$

respectively (Björk, 2004, Subsection 22.4.4). By (2.7), the latter expressions are equal. From (2.6a) and (2.6b), we see that this only holds if $\overline{\sigma} = \underline{\sigma}$. Since $\overline{\sigma} = \underline{\sigma}$ implies $\mathcal{P} = \{P^{\overline{\sigma}}\}$, we are back in the classical case without volatility uncertainty. In that case, the bond price is given by (2.5).

Next, let us suppose that $\overline{\sigma} = \underline{\sigma}$ and the bond price is determined by (2.5). Then we are again back in the classical case without volatility uncertainty, and the discounted bond is clearly a martingale under $P^{\overline{\sigma}}$. Since $\mathcal{P} = \{P^{\overline{\sigma}}\}$, the discounted bond is also a symmetric $G$-martingale. □

## 2.4 Equivalent Sublinear Expectations

In order to find an arbitrage-free term structure, we consider the following type of sublinear expectations defined by a $G$-backward stochastic differential equation. Let $\lambda = (\lambda_t)_{0\leq t\leq\tau}$ be a bounded process in $M_G^p(0,\tau)$ for some $p > 1$. For $\xi \in L_G^p(\Omega_\tau)$ with $p > 1$, we define the sublinear expectation $\bar{\mathbb{E}}$ by $\bar{\mathbb{E}}_t[\xi] := Y_t^\xi$, where $Y^\xi = (Y_t^\xi)_{0\leq t\leq\tau}$ solves the $G$-backward stochastic differential equation

$$Y_t^\xi = \xi + \int_t^\tau \lambda_u Z_u du - \int_t^\tau Z_u dB_u - (K_\tau - K_t).$$

Then $\bar{\mathbb{E}}$ is a time consistent sublinear expectation (Hu, Ji, Peng, and Song, 2014, Theorem 5.1). The reader may refer to the paper of Hu, Ji, Peng, and Song (2014) for all details related to $G$-backward stochastic differential equations.

We can show that a sublinear expectation of the above kind is equivalent to the initial sublinear expectation in the sense that the null spaces induced by the natural norms related to both sublinear expectations are the same.

**Lemma 2.7.** *For $\xi \in L_G^p(\Omega_\tau)$ with $p > 1$, it holds $\xi = 0$ if and only if $\bar{\mathbb{E}}[|\xi|] = 0$.*

*Proof.* Before we show the assertion, we explicitly solve the $G$-backward stochastic differential equation defining $\bar{\mathbb{E}}$. For this purpose, we consider the extended $\tilde{G}$-expectation space $(\tilde{\Omega}_\tau, L_{\tilde{G}}^1(\tilde{\Omega}_\tau), \tilde{\mathbb{E}})$ with the canonical process $(B,\tilde{B}) = (B_t,\tilde{B}_t)_{t\geq 0}$, where we set $\tilde{\Omega}_\tau := C_0^2([0,\tau])$ and the generator $\tilde{G} : \mathbb{S}^2 \to \mathbb{R}$ is given by

$$\tilde{G}(A) := \tfrac{1}{2} \sup_{\sigma\in[\underline{\sigma}^2,\overline{\sigma}^2]} \operatorname{tr}\left(\begin{pmatrix} \sigma & 1 \\ 1 & \sigma^{-1} \end{pmatrix} A\right).$$

Then, by Theorem 3.2 of Hu, Ji, Peng, and Song (2014), we know that $Y^\xi$ is given by

$$Y_t^\xi = \mathcal{E}_t^{-1} \tilde{\mathbb{E}}_t[\mathcal{E}_\tau \xi],$$

where the process $\mathcal{E} = (\mathcal{E}_t)_{0 \leq t \leq \tau}$ is defined by

$$\mathcal{E}_t := \exp\biggl( \int_0^t \lambda_u d\tilde{B}_u - \tfrac{1}{2} \int_0^t \lambda_u^2 d\langle \tilde{B} \rangle_u \biggr).$$

Now we show the assertion by representing $\tilde{\mathbb{E}}$ and $\bar{\mathbb{E}}$ as an upper expectation of a family of probability measures. It suffices to show that $\tilde{\mathbb{E}}[|\xi|] = 0$ if and only if $\bar{\mathbb{E}}[|\xi|] = 0$ for $\xi \in L_G^p(\Omega_\tau)$ with $p > 1$, since we know that $\tilde{\mathbb{E}}[\xi] = \hat{\mathbb{E}}[\xi]$ for all $\xi \in L_G^1(\Omega)$. As in Section 1.2, we can construct a family $\tilde{\mathcal{P}}$ of probability measures on $(\tilde{\Omega}_\tau, \mathcal{B}(\tilde{\Omega}_\tau))$ such that

$$\tilde{\mathbb{E}}[\xi] = \sup_{\tilde{P} \in \tilde{\mathcal{P}}} \mathbb{E}_{\tilde{P}}[\xi]$$

for all $\xi \in L_{\tilde{G}}^1(\tilde{\Omega}_\tau)$. Moreover, the process $\mathcal{E}$ solves the $G$-stochastic differential equation

$$\mathcal{E}_t = 1 + \int_0^t \lambda_u \mathcal{E}_u d\tilde{B}_u.$$

This implies that $\mathcal{E}$ is a symmetric $G$-martingale, satisfying $\tilde{\mathbb{E}}[\mathcal{E}_\tau] = 1$. Thus, for $\tilde{P} \in \tilde{\mathcal{P}}$, we can define a probability measure on $(\tilde{\Omega}_\tau, \mathcal{B}(\tilde{\Omega}_\tau))$ by $Q(\tilde{P}) := \mathcal{E}_\tau \cdot \tilde{P}$. Since $\mathcal{E}_\tau > 0$ $\tilde{\mathcal{P}}$-quasi-surely, we know that $Q(\tilde{P}) \sim \tilde{P}$. If we now define $\mathcal{Q} := \{Q(\tilde{P}) | \tilde{P} \in \tilde{\mathcal{P}}\}$, we obtain

$$\bar{\mathbb{E}}[\xi] = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[\xi]$$

for all $\xi \in L_G^p(\Omega_\tau)$. Since $\mathcal{Q}$ consists of equivalent measures, we get $\xi = 0$ $\tilde{\mathcal{P}}$-quasi-surely if and only if $\xi = 0$ $\mathcal{Q}$-quasi-surely. Hence, the proof is complete. □

As a consequence, we can show that there is no abitrage on the bond market if there exists an equivalent sublinear expectation of the above kind under which the discounted bonds are symmetric $G$-martingales.

**Proposition 2.8.** *The bond market is arbitrage-free if the discounted bond $\tilde{P}(T)$ is a symmetric $G$-martingale under $\bar{\mathbb{E}}$ for all $T$.*

*Proof.* We proceed as in the proof of Proposition 2.5 by using Lemma 2.7 and the Girsanov transformation for $G$-Brownian motion from Hu, Ji, Peng, and Song (2014). Let us suppose that there exists an arbitrage strategy $(\pi, T)$. By Definition 2.4, it holds

$\tilde{v}_\tau(\pi, T) \geq 0$, which implies $|\tilde{v}_\tau(\pi, T)| = \tilde{v}_\tau(\pi, T)$. By Lemma 2.7, we have

$$\bar{\mathbb{E}}[|\tilde{v}_\tau(\pi, T)|] = \bar{\mathbb{E}}[\tilde{v}_\tau(\pi, T)].$$

Using Definition 2.3 and the sublinearity of $\bar{\mathbb{E}}$, we obtain

$$\bar{\mathbb{E}}[\tilde{v}_\tau(\pi, T)] \leq \sum_{i=1}^n \bar{\mathbb{E}}\left[\int_0^{T_i} \pi_t^i d\tilde{P}_t(T_i)\right].$$

By the Girsanov transformation for $G$-Brownian motion (Hu, Ji, Peng, and Song, 2014, Theorem 5.2), the process $\bar{B} = (\bar{B}_t)_{0\leq t\leq\tau}$, defined by

$$\bar{B}_t := B_t - \int_0^t \lambda_u du,$$

is a $G$-Brownian motion under $\bar{\mathbb{E}}$. Since $\tilde{P}(T)$ is a symmetric $G$-martingale under $\bar{\mathbb{E}}$, for all $T$, there exists a process $H(T) = (H_t(T))_{0\leq t\leq T}$ in $M_G^2(0, T)$ such that

$$\tilde{P}_t(T) = \tilde{P}_0(T) + \int_0^t H_u(T) d\bar{B}_u.$$

Thus, as in the proof of Proposition 2.5, we have

$$\bar{\mathbb{E}}\left[\int_0^{T_i} \pi_t^i d\tilde{P}_t(T_i)\right] = \bar{\mathbb{E}}\left[\int_0^{T_i} \pi_t^i H_t(T_i) d\bar{B}_t\right] = 0$$

for all $i$. Combining the previous steps, we get $\tilde{v}_\tau(\pi, T) = 0$ by Lemma 2.7, which is a contradiction to

$$P\big(\tilde{v}_\tau(\pi, T) > 0\big) > 0 \quad \text{for at least one } P \in \mathcal{P}.$$

Therefore, there is no arbitrage strategy. □

## 2.5 Arbitrage-Free Term Structure

There exists an equivalent sublinear expectation of the above kind under which the discounted bonds are symmetric $G$-martingales. We define the process $q = (q_t)_{0\leq t\leq\tau}$ by

$$q_t := \int_0^t e^{-2\theta(t-u)} d\langle B\rangle_u.$$

By applying Itô's formula for $G$-Brownian motion, we observe that $q$ satisfies

$$q_t = \langle B \rangle_t - \int_0^t 2\theta q_u du.$$

If we use the process $q$ to define an equivalent sublinear expectation as in Section 2.4, we obtain a sublinear expectation under which there is a unique expression for the bond prices such that the discounted bonds are symmetric $G$-martingales. A justification for choosing the process $q$ follows the proof of the succeeding theorem.

**Theorem 2.9.** *Let $\lambda = q$. Then the discounted bond $\tilde{P}(T)$ is a symmetric $G$-martingale under $\bar{\mathbb{E}}$ if and only if the bond price is given by*

$$P_t(T) = \exp\big(A(t,T) - B(t,T)r_t - \tfrac{1}{2}B(t,T)^2 q_t\big) \tag{2.8}$$

*for all $t$, where $A, B : [0,\tau] \times [0,\tau] \to \mathbb{R}$ are defined by*

$$A(t,T) := -\int_t^T B(s,T)\mu(s)ds \tag{2.9}$$

*and* (2.6b)*, respectively.*

*Proof.* First of all, we show that the process $X = (X_t)_{0 \leq t \leq T}$, defined by

$$X_t := \exp\bigg(A(t,T) - B(t,T)r_t - \tfrac{1}{2}B(t,T)^2 q_t - \int_0^t r_s ds\bigg),$$

is a symmetric $G$-martingale under $\hat{\mathbb{E}}$. Applying Itô's formula for $G$-Brownian motion to $X$ leads to the dynamics

$$X_t = X_0 + \int_0^t \Delta_u X_u du - \int_0^t B(u,T) X_u dB_u + \int_0^t \tilde{\Delta}_u X_u d\langle B \rangle_u,$$

where the drift terms $\Delta = (\Delta_t)_{0 \leq t \leq T}$ and $\tilde{\Delta} = (\tilde{\Delta}_t)_{0 \leq t \leq T}$ are given by

$$\begin{aligned}
\Delta_t &:= \partial_t A(t,T) - \partial_t B(t,T) r_t - B(t,T)\partial_t B(t,T) q_t \\
&\quad - B(t,T)\big(\mu(t) - \theta r_t\big) + B(t,T)^2 \theta q_t - r_t \\
&= \big(\partial_t A(t,T) - \mu(t)B(t,T)\big) - \big(\partial_t B(t,T) - \theta B(t,T) + 1\big) r_t \\
&\quad - B(t,T)\big(\partial_t B(t,T) - \theta B(t,T)\big) q_t, \\
\tilde{\Delta}_t &:= -\tfrac{1}{2}B(t,T)^2 + \tfrac{1}{2}B(t,T)^2 = 0,
\end{aligned}$$

respectively. The functions $A$ and $B$ satisfy

$$\partial_t A(t,T) = \mu(t)B(t,T),$$
$$\partial_t B(t,T) = \theta B(t,T) - 1,$$

respectively. Thus, we get

$$X_t = X_0 + \int_0^t B(u,T)X_u q_u du - \int_0^t B(u,T)X_u dB_u.$$

Since the previous equation is a linear $G$-stochastic differential equation with bounded coefficients, it has a unique solution, which is in $M_G^2(0,T)$. Hence, it holds $X \in M_G^2(0,T)$, which implies $X_t \in L_G^2(\Omega_t)$ for all $t$. By the Girsanov transformation for $G$-Brownian motion, the process $\bar{B} = (\bar{B}_t)_{0\leq t\leq \tau}$, defined by

$$\bar{B}_t := B_t - \int_0^t q_u du,$$

is a $G$-Brownian motion under $\bar{\mathbb{E}}$. Therefore, $X$ is a symmetric $G$-martingale under $\bar{\mathbb{E}}$.

Using the first step, we now prove the assertion. If $\tilde{P}(T)$ is a symmetric $G$-martingale under $\bar{\mathbb{E}}$, for all $t$, it holds

$$\tilde{P}_t(T) = \bar{\mathbb{E}}_t[\tilde{P}_T(T)] = \bar{\mathbb{E}}_t[M_T^{-1}].$$

Since $X$ is also a symmetric $G$-martingale under $\bar{\mathbb{E}}$ and $A(T,T) = 0 = B(T,T)$, we get

$$X_t = \bar{\mathbb{E}}_t[X_T] = \bar{\mathbb{E}}_t[M_T^{-1}]$$

for all $t$. Thus, we have $\tilde{P}_t(T) = X_t$ for all $t$, which is equivalent to (2.8). Conversely, if (2.8) holds, we get $\tilde{P}_t(T) = X_t$ for all $t$; consequently, we know that $\tilde{P}(T)$ is a symmetric $G$-martingale by the first step of the proof. □

We use the process $q$ to obtain an arbitrage-free term structure, since it serves as an adjustment factor for the uncertain volatility. The proof of Theorem 2.6 shows that the discounted bond cannot be a symmetric $G$-martingale under $\hat{\mathbb{E}}$ in the presence of volatility uncertainty, as the expectation of the discount factor is not the same for every measure in the set of beliefs. The expectation differs among the measures in $\mathcal{P}$, since the short rate has a different variance under each of them, as Theorem 2.2 shows. So in order to unify the expectation of the discount factor under each measure, we need to adjust the short rate by the uncertainty about its variance. The following identity shows that the process $q$ is a suitable adjustment factor, since it contains the same information as the variance of the short rate. By Proposition 2.1, Theorem 2.2, and a standard property of integrals with respect to $G$-Brownian motion (see Proposition A.19), we have

$$\hat{\mathbb{E}}[(r_t - \hat{\mathbb{E}}[r_t])^2] = \hat{\mathbb{E}}\left[\left(\int_0^t e^{-\theta(t-u)} dB_u\right)^2\right] = \hat{\mathbb{E}}\left[\int_0^t e^{-2\theta(t-u)} d\langle B\rangle_u\right] = \hat{\mathbb{E}}[q_t].$$

We therefore set $\lambda = q$ and use the Girsanov transformation for $G$-Brownian motion to adjust the short rate by its variance. Then the short rate evolves according to the dynamics

$$r_t = r_0 + \int_0^t \big(\mu(u) - \theta r_u + q_u\big) du + \bar{B}_t.$$

Another important observation, which can be deduced from the dynamics of $q$, is that the process $q$ mean reverts twice as fast as the short rate towards the quadratic variation of the $G$-Brownian motion—that is, towards the quadratic variation of the short rate. So the process always adjusts towards the correct belief about the volatility, which is unknown beforehand.

From an economic point of view, setting $\lambda = q$ is reasonable as well. In the proof of Theorem 2.9, we see that the instantaneous excess return of a zero-coupon bond with maturity $T$ over the money-market account at time $t$ is $B(t,T)q_t$. Dividing by the diffusion coefficient, given by $-B(t,T)$, we obtain the market price of risk, given by $-q_t$. In general, the market price of risk measures how much better we are doing with a bond compared to investing in the money-market account per one unit of risk. Since $q$ is positive, we use a negative market price of risk. This is appropriate because the bonds are not risky in this model. They have a certain payoff of 1 at the maturity; i.e., there is no default risk. On the other hand, investing in the money-market account is risky, since the short rate is stochastic and uncertain. Hence, we use a process representing the variance of the short rate to measure the risk and the uncertainty of the money-market account. So one may also refer to $-q$ as the market price of uncertainty.

In order to compare the bond prices from Theorem 2.9 with the prices from the traditional model, we derive an adjustment factor, linking both expressions. We denote the bond price from the traditional Hull-White model with constant volatility $\sigma$ by $P_t^\sigma(T)$, which is defined by

$$P_t^\sigma(T) := \exp\big(A^\sigma(t,T) - B(t,T)r_t\big),$$

where $A^\sigma(t,T)$ and $B(t,T)$ are defined by (2.6a) and (2.6b), respectively. The bond price of the Hull-White model with volatility uncertainty, denoted by $P_t(T)$, is given by (2.8).

Then it holds

$$\frac{P_t(T)}{P_t^\sigma(T)} = \exp\left( - \int_t^T \tfrac{1}{2}\sigma^2 B(s,T)^2 ds - \tfrac{1}{2}B(t,T)^2 q_t \right).$$

The expression on the right-hand side represents an adjustment factor, which we can use to migrate from the traditional model to the model with volatility uncertainty.

Examining the adjustment factor, we note the following differences between the traditional and the present model. Since the adjustment factor is less than one, the bond prices in the present model are less than the prices in the classical model without volatility uncertainty. Moreover, we see that the squared term, depending on the volatility parameter $\sigma$, is missing in $P_t(T)$; instead, we have an additional term in $P_t(T)$, depending on the market price of uncertainty. Thus, the prices are independent of the volatility as well as the bounds for the volatility, which is the case in most models dealing with pricing under volatility uncertainty. It also implies that the bond price at the initial time corresponds to the price in the deterministic version of the Hull-White model without white noise, since the additional part in the exponential vanishes at the initial time. Though, this also applies to the standard Hull-White model after fitting it to the initial yield curve. In contrast to classical affine models, the bond price is now affine with respect to the short rate and the market price of uncertainty. The affine structure is similar to short rate models with a stochastic volatility (Fong and Vasicek, 1991; Longstaff and Schwartz, 1992). However, the additional factor in the bond price is not the volatility but a process adjusting towards the current value of the quadratic variation of the short rate. A surprisingly similar structure can be found in the short rate model from Casassus, Collin-Dufresne, and Goldstein (2005), displaying unspanned stochastic volatility.

The most important implications of the prices in this model are as follows. Primarily, we manage to obtain a term structure that is robust with respect to the volatility itself and the bounds for the volatility. So we do neither have to estimate the future volatility of the short rate nor its bounds. Admittedly, there is a price we have to pay for this. We have to specify the market price of uncertainty, which appears in the bond prices. The market price of uncertainty depends on the past evolution of the quadratic variation of the $G$-Brownian motion, which corresponds to the quadratic variation of the short rate in the Hull-White model. The past evolution of the quadratic variation of the short rate is observable and can be inferred from market data. Alternatively, one could also estimate the variance of the short rate as an approximation for the market price of uncertainty. Moreover, in accordance to the short rate model of Casassus, Collin-Dufresne, and Goldstein (2005), the bond prices are completely unaffected by the structure of the volatility. The bounds for the volatility presumably enter the model when it is used for pricing derivatives on bonds—that is, nonlinear contracts. Therefore, we conjecture that the model, despite its simple structure, displays unspanned stochastic volatility, which was introduced by Collin-Dufresne and Goldstein (2002). The pricing of derivatives and a detailed discussion about unspanned stochastic volatility is, however, postponed to Chapter 4.

## 2.6 Yield Curve Fitting

As in the classical Hull-White model, we can use the time dependent mean reversion level to fit the theoretical bond prices to an initially observable term structure. We introduce the following notions and assumptions, which are common in term structure models. Let us assume that there is an initial forward curve $f_0^* : [0, \tau] \to \mathbb{R}$, which is observed on the market. We assume that the initial forward curve $f_0^*$ is differentiable and satisfies $f_0^*(0) = r_0$. For $T \leq \tau$, the theoretical forward rate of the model is denoted by $f_t(T)$ for $t \leq T$ and defined by

$$f_t(T) := -\partial_T \log P_t(T).$$

The following theorem gives a necessary and a sufficient condition for the theoretical forward curve matching the observable one at inception, which characterizes the mean reversion level of the short rate.

**Theorem 2.10.** *Let the bond price be given by* (2.8). *Then it holds* $f_0^*(T) = f_0(T)$ *for all* $T$ *if and only if the mean reversion level of the short rate, for all* $t$*, satisfies*

$$\mu(t) = \theta f_0^*(t) + \partial_t f_0^*(t). \tag{2.10}$$

*Proof.* First, we derive the initial forward rate $f_0(T)$ for $T \leq \tau$ when the bond price is given by (2.8). Taking the derivative of the logarithm of the bond price at time 0 and changing the sign, for $T \leq \tau$, we obtain

$$f_0(T) = \int_0^T \mu(t) e^{-\theta(T-t)} dt + e^{-\theta T} r_0.$$

Let us suppose that $f_0^*(T) = f_0(T)$ for all $T$. By the equation from above, we have

$$e^{\theta T} f_0^*(T) = \int_0^T \mu(t) e^{\theta t} dt + r_0$$

for all $T$. Differentiating the latter equation with respect to $T$ yields

$$\theta e^{\theta T} f_0^*(T) + e^{\theta T} \partial_T f_0^*(T) = \mu(T) e^{\theta T}$$

for all $T$. Hence, the mean reversion level satisfies (2.10).

If we suppose that the mean reversion level satisfies (2.10), we can plug it into the first equation of the proof and check, by reversing the above calculations, that it holds $f_0^*(T) = f_0(T)$ for all $T$. □

After fitting the yield curve, the model is consistent with the classical Hull-White model. In general, the model is not consistent with the traditional Hull-White model, since the short rate dynamics and the bond prices differ from the ones in the traditional model—even if there is no volatility uncertainty. The adjusted short rate dynamics are given by

$$r_t = r_0 + \int_0^t \big(\mu(u) + q_u - \theta r_u\big) du + \bar{B}_t,$$

which clearly differ from the Hull-White short rate dynamics. The bond prices also differ from the ones obtained in the Hull-White model—as the comparison from the previous section shows. However, the model becomes consistent with the classical one after fitting the model to the yield curve. After inserting (2.10) and the definition of $q$ in the (adjusted) short rate dynamics, one can check that these are the same dynamics as in the fitted Hull-White model if there is no volatility uncertainty, i.e., if $\overline{\sigma} = \sigma = \underline{\sigma}$ (Brigo and Mercurio, 2001, Subsection 3.3.1). Furthermore, inserting (2.10) in $A(t,T)$, which is defined in (2.9), and performing some calculations, yields

$$A(t,T) = -\int_t^T f_0^*(s)ds + f_0^*(t)B(t,T).$$

Plugging the above expression into (2.8) and dividing by the fitted bond price from the traditional Hull-White model with volatility $\sigma$ (Brigo and Mercurio, 2001, Subsection 3.3.2) leads to

$$\frac{P_t(T)}{P_t^\sigma(T)} = \exp\Big(\tfrac{1}{2}B(t,T)^2\big(\tfrac{\sigma^2}{2\theta}(1-e^{-2\theta t}) - q_t\big)\Big).$$

Thus, the adjustment factor is now determined by the difference between the variance of the short rate with constant volatility and the uncertain variance of the short rate with volatility uncertainty. Hence, the adjustment factor is equal to 1 if $\overline{\sigma} = \sigma = \underline{\sigma}$.

The consistency with the classical Hull-White model further justifies the choice of $q$ as the market price of uncertainty. The discussion from above shows that the adjustment factor $q$, appearing in the short rate dynamics, is actually included in the dynamics of the classical model after fitting the theoretical prices to the observed ones. Hence, the process $q$ naturally appears in the risk-neutral dynamics of the short rate, which gives another justification for choosing the market price of uncertainty in this particular way. The interesting thing is, however, that in the classical model, this expression is used for yield curve fitting, whereas in this model, the expression is needed in order to have an arbitrage-free model.

In order to completely calibrate the model, one has to establish a robust estimation procedure for the mean reversion speed. Theorem 2.10 characterizes the mean reversion level in terms of an initially observable term structure. However, the term structure still involves a parameter: the mean reversion speed $\theta$. A typical approach in short rate models to estimate parameters is to use a maximum likelihood approach. The maximum likelihood approach heavily relies on the probabilistic law of the short rate. In the presence of volatility uncertainty, there is a family of possible probabilistic laws for the short rate, and we are uncertain about which one is correct. Therefore, one has to use a robust approach to calibrate the model instead of the classical maximum likelihood approach.

## 2.7 Multifactor Extension

The previous model can be generalized to a model driven by multiple risk factors. For this purpose, we consider the probabilistic setting from Section 1.2 for a general $d \in \mathbb{N}$, and we replace the state space of the volatility, given by $[\underline{\sigma}, \overline{\sigma}]$, with a general bounded, closed, and convex subset $\Sigma \subset \mathbb{R}^{d \times d}$. We need to assume that $\Sigma$ is such that the generator $G$ is non-degenerate, which we formalize in the next chapter, to be able to apply all required results from the calculus of $G$-Brownian motion. In the one-dimensional setting, this assumption is satisfied if $\underline{\sigma} > 0$, which we assume in the previous sections. In contrast to the previous sections, the $d$-dimensional case allows us to consider a possible uncertain correlation between the risk factors. As described in Section 1.3, this setting leads to a $d$-dimensional $G$-Brownian motion $B = (B_t^1, ..., B_t^d)_{t \geq 0}$. In particular, then $B^i = (B_t^i)_{t \geq 0}$ is a one-dimensional $G_i$-Brownian motion for a suitable generator $G_i : \mathbb{R} \to \mathbb{R}$ for all $i = 1, ..., d$. In this setting, the short rate process $r$ is defined by

$$r_t := \mu(t) + \sum_{i=1}^{d} X_t^i,$$

where $\mu : \mathbb{R}_+ \to \mathbb{R}$ is a suitably integrable function and the factor $X^i = (X_t^i)_{t \geq 0}$ satisfies

$$X_t^i = -\int_0^t \theta_i X_u^i du + B_t^i$$

for some constant $\theta_i > 0$ for all $i$. The process $X^i$ is given by

$$X_t^i = \int_0^t e^{-\theta_i (t-u)} dB_u^i$$

for all $i$ and represents a risk factor that affects the short rate.

Such a multifactor extension does not lead to an arbitrage-free term structure. Similar to the discussion in Section 2.3, we can show that the short rate dynamics from above are not suitable for martingale modeling. As in Theorem 2.6, we can prove that the discounted bonds can be symmetric $G$-martingales under $\hat{\mathbb{E}}$ if and only if there is no volatility uncertainty, that is, if and only if $\Sigma$ is a singleton. This can be done, as in the proof of Theorem 2.6, by considering two different beliefs about the volatility, which lead to different bond prices.

We consider sublinear expectations defined by a $G$-backward stochastic differential equation and equivalent to the initial sublinear expectation in order to find an arbitrage-free term structure. Let $\lambda = (\lambda_t^1, ..., \lambda_t^d)_{0 \le t \le \tau}$ be a $d$-dimensional bounded process in $M_G^p(0, \tau; \mathbb{R}^d)$ for some $p > 1$. For $\xi \in L_G^p(\Omega_\tau)$ with $p > 1$, we define the sublinear expectation $\bar{\mathbb{E}}$ by $\bar{\mathbb{E}}_t[\xi] := Y_t^\xi$, where $Y^\xi = (Y_t^\xi)_{0 \le t \le \tau}$ solves the $G$-backward stochastic differential equation

$$Y_t^\xi = \xi + \sum_{i=1}^d \int_t^\tau \lambda_u^i Z_u^i du - \sum_{i=1}^d \int_t^\tau Z_u^i dB_u^i - (K_\tau - K_t).$$

Then we can show, as in Proposition 2.8, that the bond market is arbitrage-free if the discounted bonds are symmetric $G$-martingales under $\bar{\mathbb{E}}$.

We obtain an arbitrage-free term structure by considering a particular sublinear expectation of the above form. Let us define the process $q = (q_t^1, ..., q_t^d)_{0 \le t \le \tau}$ by

$$q_t^i := \sum_{j=1}^d q_t^{ij},$$

where $q^{ij} = (q_t^{ij})_{0 \le t \le \tau}$ is defined by

$$q_t^{ij} := \int_0^t e^{-(\theta_i + \theta_j)(t-u)} d\langle B^i, B^j \rangle_u.$$

By applying Itô's formula for $G$-Brownian motion, we know that $q^{ij}$, for all $i, j$, satisfies

$$q_t^{ij} = \langle B^i, B^j \rangle_t - \int_0^t (\theta_i + \theta_j) q_u^{ij} du.$$

If we use the process $q$ to define a sublinear expectation as above, then there is a unique arbitrage-free term structure.

**Theorem 2.11.** *Let $\lambda = q$. Then the discounted bond $\tilde{P}(T)$ is a symmetric $G$-martingale* *under $\bar{\mathbb{E}}$ if and only if the bond price is given by*

$$P_t(T) := \exp\Big(-\int_t^T \mu(s)ds - \sum_{i=1}^d B_i(t,T)X_t^i - \tfrac{1}{2}\sum_{i,j=1}^d B_i(t,T)B_j(t,T)q_t^{ij}\Big)$$

*for all $t$, where $B_i : [0,\tau] \times [0,\tau] \to \mathbb{R}$, for all $i$, is defined by*

$$B_i(t,T) := \tfrac{1}{\theta_i}(1 - e^{-\theta_i(T-t)}).$$

*Proof.* As in the proof of Theorem 2.9, the assertion follows if we show that the process $X = (X_t)_{0\le t\le T}$, defined by

$$X_t := \exp\Big(-\int_0^T \mu(s)ds - \sum_{i=1}^d B_i(t,T)X_t^i - \tfrac{1}{2}\sum_{i,j=1}^d B_i(t,T)B_j(t,T)q_t^{ij} - \sum_{i=1}^d \int_0^t X_s^i ds\Big),$$

is a symmetric $G$-martingale. Applying Itô's formula for $G$-Brownian motion to $X$ yields

$$X_t = X_0 + \int_0^t \Delta_u X_u du - \sum_{i=1}^d \int_0^t B_i(u,T)X_u dB_u^i + \sum_{i,j=1}^d \int_0^t \Delta_u^{ij} X_u d\langle B^i, B^j\rangle_u,$$

where the drift terms $\Delta = (\Delta_t)_{0\le t\le T}$ and $\Delta^{ij} = (\Delta_t^{ij})_{0\le t\le T}$, for all $i,j$, are given by

$$\begin{aligned}
\Delta_t :=& -\sum_{i=1}^d \partial_t B_i(t,T)X_t^i - \tfrac{1}{2}\sum_{i,j=1}^d \big(\partial_t B_i(t,T)B_j(t,T) + B_i(t,T)\partial_t B_j(t,T)\big)q_t^{ij} \\
& -\sum_{i=1}^d B_i(t,T)(-\theta_i X_t^i) - \tfrac{1}{2}\sum_{i,j=1}^d B_i(t,T)B_j(t,T)\big(-(\theta_i+\theta_j)q_t^{ij}\big) - \sum_{i=1}^d X_t^i \\
=& -\sum_{i=1}^d \big(\partial_t B_i(t,T) - \theta_i B_i(t,T) + 1\big)X_t^i \\
& -\tfrac{1}{2}\sum_{i,j=1}^d B_j(t,T)\big(\partial_t B_i(t,T) - \theta_i B_i(t,T)\big)q_t^{ij} \\
& -\tfrac{1}{2}\sum_{i,j=1}^d B_i(t,T)\big(\partial_t B_j(t,T) - \theta_j B_j(t,T)\big)q_t^{ij}, \\
\Delta_t^{ij} :=& -\tfrac{1}{2}B_i(t,T)B_j(t,T) + \tfrac{1}{2}B_i(t,T)B_j(t,T) = 0,
\end{aligned}$$

respectively. Since the function $B_i$, for all $i$, satisfies

$$\partial_t B_i(t,T) = \theta_i B_i(t,T) - 1$$

and $q^{ij} = q^{ji}$ for all $i, j$, we obtain

$$X_t = X_0 + \sum_{i=1}^{d} \int_0^t B_i(u,T) X_u q_u^i du - \sum_{i=1}^{d} \int_0^t B_i(u,T) X_u dB_u^i.$$

We can use the same argument as in the proof of Theorem 2.9 to show that $X_t \in L_G^2(\Omega_t)$ for all $t$. The Girsanov transformation for $G$-Brownian motion implies that the process $\bar{B} = (\bar{B}_t^1, ..., \bar{B}_t^d)_{0 \le t \le \tau}$, defined by

$$\bar{B}_t^i := B_t^i - \int_0^t q_u^i du,$$

is a $G$-Brownian motion under $\bar{\mathbb{E}}$. Therefore, the process $X$ is a symmetric $G$-martingale under $\bar{\mathbb{E}}$. □

We can use the function $\mu$ to fit the model to an initially observed term structure. Let us assume that there is a sufficiently regular forward curve $f_0^* : [0, \tau] \to \mathbb{R}$, which is currently observed on the market. Then one can check that the theoretical forward curve implied by the model matches the observed one at inception if and only if $\mu(t) = f_0^*(t)$ for all $t$.

## 2.8 Equilibrium and Empirical Analysis

From an economic point of view, a first question for further research (outside the scope of this thesis) is whether the proposed term structure and the particular choice of the market price of uncertainty can be supported by an equilibrium in a representative agent economy. The present approach is purely based on no-arbitrage pricing; instead, one could investigate a structural model in the spirit of Cox, Ingersoll Jr., and Ross (1985) under model uncertainty. Gagliardini, Porchia, and Trojani (2009) examined a structural model with ambiguity—that is, model uncertainty. The representative agent in the model faces ambiguity about the drift of the underlying risk factors. Since she is ambiguity averse, the agent has to solve a max-min expected utility problem. The solution determines the uncertain drift process, which is termed *market price of ambiguity*. In a similar fashion, one could examine a structural model in which the representative agent faces ambiguity about the volatility. For this purpose one could adapt the framework of Epstein and Ji (2013) to find out if there is an equilibrium in a representative agent economy supporting the specific market price of uncertainty used in the present model.

Apart from that, it would be interesting to test the empirical performance of the model—especially in comparison to traditional term structure models. One could, e.g., test if the model is able to explain the violation of the expectations hypothesis, as it was done, for instance, by Dai and Singleton (2003) and Gagliardini, Porchia, and Trojani

(2009), and if it does better than traditional models. Fama and Bliss (1987), Campbell and Shiller (1991), and Cochrane and Piazzesi (2005) empirically tested the expectations hypothesis by regressing changes in the yield curve onto the slope of the yield curve, which shows that the expectations hypothesis is violated. The regression produces negative coefficients, decreasing with respect to the maturity. Dai and Singleton (2003) tested the ability of several term structure models to explain the empirical findings. For this purpose, they fitted the models to data and simulated term structures from the fitted models. Then they ran regressions as above and compared the regression coefficients from the simulated data with the ones from real data. A successful model is supposed to match the coefficients from real data. In order to test the performance of the present model in this regard, one has to use a robust approach to calibrate the model, as it is described at the end of Section 2.6. Moreover, one has to develop a simulation procedure that works in the presence of volatility uncertainty. The volatility is uncertain in the sense of Knightian uncertainty. By its definition, Knightian uncertainty cannot be measured by any probability. Thus, standard simulation procedures cannot be used. Instead, one has to construct a robust simulation procedure.

A further interesting comparison is to study the relation between the present model and regime switching term structure models both from a theoretical point of view and from an empirical perspective. Using regime switching models is a different, though related, approach to overcome the stylized facts about the volatility of financial quantities (as explained in Section 1.1 in the case of asset market models). Regime switching term structure models assume that the volatility of the short rate follows a continuous-time Markov chain, which jumps between a finite number of values. The literature on regime switching term structure models shows that these models offer advantages compared to classical term structure models (Dai, Singleton, and Yang, 2007; Gourieroux, Monfort, Pegoraro, and Renne, 2014; Monfort and Pegoraro, 2007). The comparison of models with volatility uncertainty to regime switching models is particularly well-suited in contrast to other stochastic volatility models, since the volatility process in regime switching models is also bounded (as it has a finite state space). In the present chapter, we consider all volatility processes bounded by two extreme values; thus, we basically also consider trajectories described by regime switching term structure models. This is, however, just a pathwise argument. One needs to enlarge the probability space of the model in order to obtain the same Markov chain considered by regime switching term structure models, since the probability of the Markov chain jumping to a different state is usually independent from the remaining risk factors. Then one can compare the approaches and the corresponding results in detail.

# Chapter 3

# The Heath-Jarrow-Morton Model

In the present chapter, we study term structure movements in the spirit of Heath, Jarrow, and Morton (1992) (HJM) under volatility uncertainty. The HJM methodology is based on modeling the instantaneous forward rate as a diffusion process as opposed to the short rate. As in the classical HJM framework, we model the behavior of the forward rate as a general diffusion process. The forward rate determines all quantities on the related bond market. The difference compared to the classical framework is that we consider the probabilistic setting from Section 1.2 to model the uncertainty about the volatility, where, in contrast to the previous chapter, we consider a general state space for the uncertain volatility, leading to a $d$-dimensional $G$-Brownian motion. As a consequence, the forward rate dynamics are driven by a $d$-dimensional $G$-Brownian motion in the presence of volatility uncertainty. Compared to the classical HJM model, the forward rate has uncertain drift terms in addition to the classical (certain) drift term, since the quadratic covariations of a $G$-Brownian motion are uncertain processes (as mentioned in Section 1.3). Despite the differences, the present model is still consistent with the classical HJM model. We impose some assumptions on the coefficients of the forward rate dynamics in order to get a sufficient degree of regularity.

Similar to the traditional HJM model, the main result of the present chapter is a drift condition that implies that the related bond market is arbitrage-free. The traditional HJM drift condition relates the absence of arbitrage to the existence of a market price of risk and shows that the risk-neutral dynamics of the forward rate are completely characterized by its diffusion coefficient. In order to derive a drift condition in the presence of volatility uncertainty, we set up a suitable market structure for the related bond market in this setting. In contrast to the traditional HJM model, the drift condition in the presence of volatility uncertainty requires the existence of several market prices. We call the additional market prices—in addition to the market price of risk—the *market prices of uncertainty*. As in the traditional HJM model, the risk-neutral dynamics of the forward rate are completely determined by its diffusion term—with the addition that the uncertainty of the diffusion term determines the uncertainty of the drift. If the uncertainty about the volatility vanishes, the drift condition reduces to the traditional one. The proof of the main result is based on deriving the dynamics of the discounted bonds and using a Girsanov transformation for $G$-Brownian motion together with some results on $G$-backward stochastic differential equations.

The drift condition derived in this chapter is a very powerful tool, since it allows to construct arbitrage-free term structure models that are completely robust with respect to the volatility. In the classical case without volatility uncertainty, almost every (arbitrage-free) term structure model corresponds to a specific example in the HJM methodology. Due to the main result of the present chapter, we are able to obtain arbitrage-free term structure models in the presence of volatility uncertainty by considering specific examples. In particular, we recover robust versions of classical term structure models: the examples include the Ho-Lee term structure, the Hull-White term structure—which actually corresponds to the term structure from the previous chapter—and the Vasicek term structure. The examples show that the drift of the risk-neutral short rate dynamics and the bond prices, which still have an affine structure, include an additional uncertain factor when there is uncertainty about the volatility. As in the previous chapter, with this procedure, we obtain term structure models that are robust with respect to the volatility as well as its worst-case values, which differs from most works on pricing under volatility uncertainty.

In order to make the analysis from above work, we construct a space of admissible integrands for the forward rate dynamics. The forward rate is a diffusion process parameterized by its maturity and needs to be integrable with respect to its maturity to compute the bond prices. Therefore, the integrands in the forward rate dynamics need to be regular with respect to the maturity apart from being admissible stochastic processes in a diffusion driven by a $G$-Brownian motion. In order to achieve this, we use the space of Bochner integrable functions; that is, Banach space-valued functions that are sufficiently measurable and integrable, where the functions are mapping from the set of maturities into the space of admissible stochastic processes in this particular case. For such functions, we can define the Bochner integral, mapping into the space of admissible stochastic processes. This ensures that the forward rate is integrable with respect to its maturity. Moreover, we derive further necessary results for the HJM model, including a version of Fubini's theorem for stochastic integrals. We give a sufficient condition for functions to be Bochner integrable, which applies to all considered examples.

In addition, we provide a sufficient condition ensuring that the discounted bonds and the portfolio value related to the bond market are well-posed. Each discounted bond is an exponential of a diffusion process driven by a $G$-Brownian motion, and the portfolio value consists of integrals with respect to the discounted bonds, respectively. First, we have to make sure that the discounted bonds are well-posed; second, we need to ensure that the dynamics of each discounted bond are sufficiently regular to imply that the portfolio value is well-posed. For this purpose, we use a condition similar to $G$-Novikov's condition of Osuka (2013) to obtain the desired regularity. As in the classical case, the advantage of such a condition is that it can be easily verified compared to other conditions implying that the exponential of an Itô diffusion is integrable. One can verify that all examples considered in this article satisfy this condition.

The chapter is organized as follows. Section 3.1 introduces the forward rate, determining all quantities on the related bond market, and the framework representing the uncertainty about the volatility. In Section 3.2, we set up the market structure for the related bond market and derive the drift condition, ensuring the absence of arbitrage. In Section 3.3, we study examples, including the Ho-Lee term structure, the Hull-White term structure, and the Vasicek term structure, and discuss their implications. In Section 3.4, we construct the space of admissible integrands for the forward rate dynamics and derive related results. Section 3.5 provides the sufficient condition for the discounted bonds to be well-posed.

## 3.1 Term Structure Movements

In the traditional HJM framework—without volatility uncertainty—term structure movements are driven by a standard Brownian motion. That means, we consider the canonical process $B = (B_t^1, ..., B_t^d)_{t\geq 0}$, for $d \in \mathbb{N}$, on the probability space $(\Omega, \mathcal{F}, P_0)$, introduced in Section 1.1, and the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, which is generated by $B$ and completed by all $P_0$-null sets. Then the canonical process $B$ is a $d$-dimensional standard Brownian motion under $P_0$. For $T \leq \tau$, where $\tau < \infty$ is a fixed terminal time, we denote the forward rate with maturity $T$ at time $t$ by $f_t(T)$ for $t \leq T$. In the classical HJM model, the dynamics of the forward rate process $f(T) = (f_t(T))_{0\leq t\leq T}$, for all $T$, are given by

$$f_t(T) = f_0(T) + \int_0^t \alpha_u(T)du + \sum_{i=1}^d \int_0^t \beta_u^i(T)dB_u^i$$

for some initial (observable) forward curve $f_0 : [0, \tau] \to \mathbb{R}$, which is integrable, and sufficiently regular processes $\alpha(T) = (\alpha_t(T))_{0\leq t\leq\tau}$ and $\beta(T) = (\beta_t^1(T), ..., \beta_t^d(T))_{0\leq t\leq\tau}$. So instead of modeling a single interest rate and determining the term structure of interest rates endogenously (as in the previous chapter), we directly model the evolution of the whole term structure, represented by the forward rate, starting from an initial term structure observed on the market.

The forward rate determines all remaining quantities on the bond market. The market offers zero-coupon bonds for all maturities, which are discounted by the money-market account. The bond price process, denoted by $P(T) = (P_t(T))_{0\leq t\leq T}$, is defined by

$$P_t(T) := \exp\Big( - \int_t^T f_t(s)ds \Big)$$

for all $T \leq \tau$, and the money-market account, denoted by $M = (M_t)_{0\leq t\leq \tau}$, is defined by

$$M_t := \exp\Big( \int_0^t r_s ds \Big),$$

where $r = (r_t)_{0\leq t\leq \tau}$ is the short rate, defined by $r_t := f_t(t)$. We use the money-market account as a numéraire; i.e., we focus on the discounted bonds, which are denoted by $\tilde{P}(T) = (\tilde{P}_t(T))_{0\leq t\leq T}$ for all $T \leq \tau$ and defined by

$$\tilde{P}_t(T) := M_t^{-1} P_t(T).$$

In the presence of volatility uncertainty, we consider a family of probability measures resulting in a $d$-dimensional $G$-Brownian motion. That is, we consider the set of beliefs $\mathcal{P}$ from Section 1.2, consisting of all beliefs about the volatility. In contrast to the previous chapter, we use a general state space $\Sigma$ without assuming that $d = 1$. Then, as in Section 1.3, the sublinear expectation

$$\hat{\mathbb{E}}[\cdot] = \sup_{P\in\mathcal{P}} \mathbb{E}_P[\cdot]$$

corresponds to the $G$-expectation, and the canonical process $B$ is a $d$-dimensional $G$-Brownian motion. We assume that the generator $G : \mathbb{S}^d \to \mathbb{R}$, which is given by

$$G(A) = \tfrac{1}{2} \sup_{\sigma\in\Sigma} \mathrm{tr}(\sigma\sigma' A),$$

is non-degenerate; namely, there exists a constant $C > 0$ such that

$$G(A) - G(B) \geq C\mathrm{tr}(A - B)$$

for all $A \geq B$. The latter is required by some results from the calculus of $G$-Brownian motion we need for the subsequent analysis.

As a consequence, term structure movements are driven by a $G$-Brownian motion in the presence of volatility uncertainty. That means, for all $T$, the forward rate dynamics are now given by

$$f_t(T) = f_0(T) + \int_0^t \alpha_u(T)du + \sum_{i=1}^d \int_0^t \beta_u^i(T)dB_u^i + \sum_{i,j=1}^d \int_0^t \gamma_u^{i,j}(T)d\langle B^i, B^j\rangle_u$$

for some initial forward curve $f_0 : [0, \tau] \to \mathbb{R}$, which is integrable, and for functions $\alpha, \gamma^{i,j} : [0, \tau] \to M_G^1(0, \tau)$ and $\beta^i : [0, \tau] \to M_G^2(0, \tau)$. Since the space $M_G^p(0, \tau)$ consists of stochastic processes that are admissible integrands in the definition of all stochastic integrals related to a $G$-Brownian motion, the forward rate and the short rate are well-defined in the sense that for all $T$, $f_t(T), r_t \in L_G^1(\Omega_t)$ for all $t$.

In contrast to the traditional HJM model, the forward rate has additional, uncertain drift terms when there is volatility uncertainty. The additional drift terms of the forward rate are uncertain due to the fact that they depend on the quadratic covariation processes of the $G$-Brownian motion. In the presence of volatility uncertainty, the quadratic covariations of the driving process are uncertain processes, since they differ among the measures in the set of beliefs (as it is described in Section 1.3). Moreover, it can be shown that the additional drift terms cannot be included in the first drift term (Song, 2013, Corollary 3.3). Hence, we have to add them to the forward rate dynamics instead of including them in the first drift term. In this way, we can distinguish between the part of the drift that is driven by uncertainty and the part that is not.

When there is no volatility uncertainty, the model corresponds to a classical HJM model. If we drop the uncertainty about the volatility, then $B$ becomes a standard Brownian motion, and its quadratic covariation processes are no longer uncertain. That means, if $\Sigma = \{I_d\}$, then for all $i$, we have $\langle B^i, B^i \rangle_t = t$ and $\langle B^i, B^j \rangle_t = 0$ for all $j \neq i$. In that case, the forward rate dynamics are given by

$$f_t(T) = f_0(T) + \int_0^t \left( \alpha_u(T) + \sum_{i=1}^d \gamma_u^{i,i}(T) \right) du + \sum_{i=1}^d \int_0^t \beta_u^i(T) dB_u^i$$

for all $T$; that is, the model corresponds to a classical HJM model in which the drift is given by the sum of $\alpha$ and $\sum_{i=1}^d \gamma^{i,i}$.

We henceforth impose the following two regularity assumptions. The first assumption ensures that the forward rate and the short rate are integrable and that all succeeding computations are feasible.

**Assumption 3.1.** *There exists a $p > 1$ such that $\alpha, \gamma^{i,j} \in \tilde{M}_G^p(0, \tau)$ and $\beta^i \in \tilde{M}_G^{2p}(0, \tau)$ for all $i, j$.*

The space $\tilde{M}_G^p(0, \tau)$, which we construct in Section 3.4, consists of all functions mapping from $[0, \tau]$ into $M_G^p(0, \tau)$ that are strongly measurable and whose norm on $M_G^p(0, \tau)$ is integrable. A function is called *strongly measurable* if it is Borel measurable and its image is separable. For example, we know that all continuous functions mapping from $[0, \tau]$ into $M_G^p(0, \tau)$ belong to $\tilde{M}_G^p(0, \tau)$ by Proposition 3.14. This implies that all examples in Section 3.3 satisfy Assumption 3.1, as Example 3.15 shows. By Proposition 3.17, Assumption 3.1 implies that the forward rate and, by Proposition 3.19, the short rate are integrable. The second assumption ensures that the discounted bonds and the portfolio value are sufficiently regular.

**Assumption 3.2.** *There exist $\tilde{p} > p^*$ and $\tilde{q} > 2$, where $p^* := \frac{2pq}{p-q}$ for some $q \in (1,p)$, such that for all $T \leq \tau$, it holds*

$$\hat{\mathbb{E}}\Big[\int_0^T \exp\Big(\tfrac{\tilde{p}\tilde{q}}{\tilde{q}-2}\Big(\int_0^t a_u(T)du + \sum_{i,j=1}^d \int_0^t c_u^{i,j}(T)d\langle B^i, B^j\rangle_u\Big)\Big)dt\Big] < \infty,$$
$$\hat{\mathbb{E}}\Big[\int_0^T \exp\Big(\tfrac{1}{2}(\tilde{p}\tilde{q})^2 \sum_{i,j=1}^d \int_0^t b_u^i(T)b_u^j(T)d\langle B^i, B^j\rangle_u\Big)dt\Big] < \infty.$$

The processes $a(T) = (a_t(T))_{0\leq t\leq\tau}$, $b^i(T) = (b_t^i(T))_{0\leq t\leq\tau}$, and $c^{i,j}(T) = (c_t^{i,j}(T))_{0\leq t\leq\tau}$, for $T \leq \tau$, are defined by

$$a_t(T) := \int_t^T \alpha_t(s)ds, \quad b_t^i(T) := \int_t^T \beta_t^i(s)ds, \quad c_t^{i,j}(T) := \int_t^T \gamma_t^{i,j}(s)ds,$$

respectively, for all $i, j$, for which we have $a(T), c^{i,j}(T) \in M_G^p(0,\tau)$ and $b^i(T) \in M_G^{2p}(0,\tau)$ by Assumption 3.1 and Proposition 3.19. One can easily verify that all examples in Section 3.3 satisfy Assumption 3.2. Assumption 3.2 is similar to $G$-Novikov's condition from Osuka (2013) and implies that for every maturity, the discounted bond price is in $L_G^1(\Omega_t)$ at each time $t$. Moreover, Assumption 3.2 ensures that the dynamics of the discounted bonds are regular enough to imply that the portfolio value, which is defined below, is well-posed. We show both of these implications in Section 3.5, which requires Lemma 3.7 from the succeeding section.

## 3.2 Arbitrage-Free Forward Rate Dynamics

In the traditional HJM model, the absence of arbitrage on the related bond market is ensured by the HJM drift condition, which assumes the existence of a market price of risk and characterizes the drift of the forward rate in terms of its diffusion coefficient. More precisely, the market is arbitrage-free if there exists a suitable process $\lambda = (\lambda_t^1, ..., \lambda_t^d)_{0\leq t\leq\tau}$ such that for all $T$,

$$\alpha(T) - \beta(T)b(T)' + \beta(T)\lambda' = 0,$$

where $b(T) = (b^1(T), ..., b^d(T))$. The process $\lambda$ is termed *market price of risk*, since it erases the drift of each discounted bond under an equivalent probability measure, called *risk-neutral measure*, to make it a martingale. Then the forward rate dynamics under the risk-neutral measure are completely determined by its diffusion coefficient; that is,

$$f_t(T) = f_0(T) + \int_0^t \beta_u(T)b_u(T)'du + \sum_{i=1}^d \int_0^t \beta_u^i(T)d\bar{B}_u^i,$$

where $\bar{B} = (\bar{B}_t^1, ..., \bar{B}_t^d)_{0\leq t\leq\tau}$ is a Brownian motion under the risk-neutral measure. This fact is of practical importance since there is no need to specify the drift term $\alpha$ or the market price of risk $\lambda$.

In order to derive a drift condition in the presence of volatility uncertainty, we first define admissible market strategies and a suitable notion of arbitrage. As in the previous chapter, the agents in the market are allowed to select a finite number of bonds they want to trade. The corresponding portfolio value is determined by the gains from trade; i.e., we restrict to self-financing strategies.

**Definition 3.3.** *An admissible market strategy $(\pi, T)$ is a couple consisting of a bounded process $\pi = (\pi_t^1, ..., \pi_t^n)_{0\leq t\leq\tau}$ in $M_G^2(0,\tau;\mathbb{R}^n)$ and a vector $T = (T_1, ..., T_n) \in [0,\tau]^n$ for some $n \in \mathbb{N}$. The corresponding portfolio value at terminal time is defined by*

$$\tilde{v}_\tau(\pi, T) := \sum_{i=1}^n \int_0^{T_i} \pi_t^i d\tilde{P}_t(T_i).$$

The portfolio value is well-posed, since the dynamics of $\tilde{P}(T)$, which are derived in Proposition 3.8 below, are sufficiently regular for each $T$ by Assumption 3.2 and Proposition 3.25. In addition, we use the quasi-sure notion of arbitrage from the preceding chapter, which corresponds to the definition of arbitrage frequently used in the literature on model uncertainty (Biagini, Bouchard, Kardaras, and Nutz, 2017; Bouchard and Nutz, 2015).

**Definition 3.4.** *An admissible market strategy $(\pi, T)$ is called arbitrage strategy if*

$$\tilde{v}_\tau(\pi, T) \geq 0 \quad \textit{quasi-surely}, \quad P\big(\tilde{v}_\tau(\pi, T) > 0\big) > 0 \quad \textit{for at least one } P \in \mathcal{P}.$$

*Moreover, we say that the bond market is arbitrage-free if there is no arbitrage strategy.*

**Remark 3.5.** *As mentioned in the previous chapter, it is possible to generalize the notion of trading strategies and—additionally—the concept of arbitrage. The notion of trading strategies can be generalized by allowing for measure-valued trading strategies (Björk, Di Masi, Kabanov, and Runggaldier, 1997) or by using methods from large financial markets (Klein, Schmidt, and Teichmann, 2016). For example, we could use the techniques from Section 3.4 to introduce measure-valued trading strategies. There are also other no-arbitrage concepts related to bond markets, which are based on the theory of large financial markets (Cuchiero, Klein, and Teichmann, 2016). We stick to the definitions from above, since such a generalization does not alter the results of this thesis.*

In the presence of volatility uncertainty, the absence of arbitrage requires the existence of several market prices. In that case, there is a sublinear expectation under which the discounted bonds are symmetric $G$-martingales, ruling out arbitrage opportunities. Moreover, the drift condition characterizes the dynamics of the forward rate.

**Theorem 3.6.** *The bond market is arbitrage-free if for some $p > 1$, there exist bounded processes $\kappa = (\kappa_t^1, ..., \kappa_t^d)_{0\le t\le\tau}$ and $\lambda^{i,j} = (\lambda_t^{i,j,1}, ..., \lambda_t^{i,j,d})_{0\le t\le\tau}$ in $M_G^p(0,\tau;\mathbb{R}^d)$ such that*

$$\begin{aligned} \alpha(T) + \beta(T)\kappa' = 0, \\ \gamma^{i,j}(T) - \tfrac{1}{2}\big(\beta^i(T)b^j(T) + b^i(T)\beta^j(T)\big) + \beta(T)(\lambda^{i,j})' = 0 \end{aligned} \tag{3.1}$$

*for almost all $T$ for all $i,j$. In particular, then there exists a sublinear expectation $\bar{\mathbb{E}}$ under which $\tilde{P}(T)$ is a symmetric $G$-martingale for all $T$ and*

$$f_t(T) = f_0(T) + \sum_{i=1}^{d}\int_0^t \beta_u^i(T)d\bar{B}_u^i + \sum_{i,j=1}^{d}\int_0^t \tfrac{1}{2}\big(\beta_u^i(T)b_u^j(T) + b_u^i(T)\beta_u^j(T)\big)d\langle\bar{B}^i,\bar{B}^j\rangle_u$$

*for almost all $T$, where $\bar{B} = (\bar{B}_t^1, ..., \bar{B}_t^d)_{0\le t\le\tau}$ is a $G$-Brownian motion under $\bar{\mathbb{E}}$.*

The additional market prices occuring in the drift condition represent the market prices of uncertainty. In comparison to the classical case without volatility uncertainty, the forward rate and (hence) the discounted bonds have additional drift terms, which are uncertain (as explained in Section 3.1). As a consequence, we need additional market prices in order to make the discounted bonds symmetric $G$-martingales, which ultimately rules out arbitrage. Since the additional market prices relate to the uncertain drift terms of the discounted bonds, they are termed *market prices of uncertainty.*

The risk-neutral dynamics of the forward rate are fully characterized by its diffusion term, which—in contrast to the classical HJM model—does not only apply to the coefficients but also to the uncertainty. We call the dynamics of the forward rate under $\bar{\mathbb{E}}$, given in Theorem 3.6, *risk-neutral dynamics*, since the discounted bonds are symmetric $G$-martingales under $\bar{\mathbb{E}}$. As in the classical HJM model, the diffusion coefficient $\beta$ determines the drift coefficient of the risk-neutral forward rate dynamics. In addition, the uncertain volatility, included in the $G$-Brownian motion $\bar{B}$, determines the uncertainty of the drift, represented by the quadratic covariation processes of $\bar{B}$. That means, arbitrage-free term structure models also exhibit drift uncertainty in the presence of volatility uncertainty.

Despite the differences, the drift condition is still consistent with the classical HJM drift condition. If there is no uncertainty about the volatility—that is, if $\Sigma = \{I_d\}$—then the forward rate satisfies for all $T$,

$$f_t(T) = f_0(T) + \int_0^t \Big(\alpha_u(T) + \sum_{i=1}^{d}\gamma_u^{i,i}(T)\Big)du + \sum_{i=1}^{d}\int_0^t \beta_u^i(T)dB_u^i$$

as it is described in Section 3.1. The drift condition in Theorem 3.6 implies that

$$\Big(\alpha(T) + \sum_{i=1}^{d} \gamma^{i,i}(T)\Big) - \beta(T)b(T)' + \beta(T)\Big(\kappa + \sum_{i=1}^{d} \lambda^{i,i}\Big)' = 0$$

for almost all $T$. The latter corresponds to the classical HJM drift condition for a market price of risk given by the process $\kappa + \sum_{i=1}^{d} \lambda^{i,i}$.

In order to prove Theorem 3.6, we first of all derive the dynamics of the discounted bond for each maturity. This is based on the following lemma.

**Lemma 3.7.** *For all $T$, the logarithm of the discounted bond satisfies the dynamics*

$$\log\big(\tilde{P}_t(T)\big) = \log\big(\tilde{P}_0(T)\big) - \int_0^t a_u(T)du - \sum_{i=1}^{d} \int_0^t b_u^i(T)dB_u^i - \sum_{i,j=1}^{d} \int_0^t c_u^{i,j}(T)d\langle B^i, B^j\rangle_u.$$

*Proof.* We obtain the dynamics by applying Fubini's theorem, which can be found in Section 3.4. By its definition, the logarithm of the discounted bond is given by

$$\log\big(\tilde{P}_t(T)\big) = -\int_t^T f_t(s)ds - \int_0^t r_s ds$$

for all $T$. Inserting the definition of the forward rate and the short rate, we obtain

$$\begin{aligned}\log\big(\tilde{P}_t(T)\big) = \ & \log\big(\tilde{P}_0(T)\big) - \int_t^T \int_0^t \alpha_u(s)duds - \int_0^t \int_0^s \alpha_u(s)duds \\ & - \sum_{i=1}^{d} \int_t^T \int_0^t \beta_u^i(s)dB_u^i ds - \sum_{i=1}^{d} \int_0^t \int_0^s \beta_u^i(s)dB_u^i ds \\ & - \sum_{i,j=1}^{d} \int_t^T \int_0^t \gamma_u^{i,j}(s)d\langle B^i, B^j\rangle_u ds - \sum_{i,j=1}^{d} \int_0^t \int_0^s \gamma_u^{i,j}(s)d\langle B^i, B^j\rangle_u ds\end{aligned}$$

for all $T$. Then an application of Corollary 3.18 yields for all $i, j$,

$$\begin{aligned}\int_t^T \int_0^t \alpha_u(s)duds + \int_0^t \int_0^s \alpha_u(s)duds &= \int_0^t a_u(T)du, \\ \int_t^T \int_0^t \beta_u^i(s)dB_u^i ds + \int_0^t \int_0^s \beta_u^i(s)dB_u^i ds &= \int_0^t b_u^i(T)dB_u^i, \\ \int_t^T \int_0^t \gamma_u^{i,j}(s)d\langle B^i, B^j\rangle_u ds + \int_0^t \int_0^s \gamma_u^{i,j}(s)d\langle B^i, B^j\rangle_u ds &= \int_0^t c_u^{i,j}(T)d\langle B^i, B^j\rangle_u\end{aligned}$$

for all $T$, which proves the assertion. □

Since for all $i, j$, the processes $a(T)$, $b^i(T)$, and $c^{i,j}(T)$ are sufficiently regular for each $T$, we can use Itô's formula for $G$-Brownian motion from Li and Peng (2011) to derive the dynamics of the discounted bond for each $T$.

**Proposition 3.8.** *For all $T$, the discounted bond satisfies the dynamics*

$$\tilde{P}_t(T) = \tilde{P}_0(T) - \int_0^t a_u(T)\tilde{P}_u(T)du - \sum_{i=1}^d \int_0^t b_u^i(T)\tilde{P}_u(T)dB_u^i$$
$$- \sum_{i,j=1}^d \int_0^t \left(c_u^{i,j}(T) - \tfrac{1}{2}b_u^i(T)b_u^j(T)\right)\tilde{P}_u(T)d\langle B^i, B^j\rangle_u.$$

*Proof.* The assertion follows by Lemma 3.7 and an application of Itô's formula for $G$-Brownian motion (Li and Peng, 2011, Theorem 5.4). We are able to apply Itô's formula for $G$-Brownian motion, since $a(T), c^{i,j}(T) \in M_G^1(0,T)$ and $b^i(T) \in M_G^2(0,T)$ for all $i, j$ by the contruction of the Bochner integral in Section 3.4 and Proposition 3.19. □

Next, we prove Theorem 3.6 by using results on $G$-backward stochastic differential equations of Hu, Ji, Peng, and Song (2014), including a Girsanov transformation for $G$-Brownian motion. All details regarding $G$-backward stochastic differential equations can be found in the paper of Hu, Ji, Peng, and Song (2014).

*Proof of Theorem 3.6.* First, we rewrite the dynamics of the forward rate and the discounted bond for each maturity by using the Girsanov transformation for $G$-Brownian motion from Hu, Ji, Peng, and Song (2014). For this purpose, we consider the following sublinear expectation: for $\xi \in L_G^p(\Omega_\tau)$ with $p > 1$, we define the sublinear expectation $\bar{\mathbb{E}}$ by $\bar{\mathbb{E}}_t[\xi] := Y_t^\xi$, where $Y^\xi = (Y_t^\xi)_{0\leq t\leq\tau}$ solves the $G$-backward stochastic differential equation

$$Y_t^\xi = \xi + \int_t^\tau \kappa_u Z_u' du + \sum_{i,j=1}^d \int_t^\tau \lambda_u^{i,j} Z_u' d\langle B^i, B^j\rangle_u - \sum_{i=1}^d \int_t^\tau Z_u^i dB_u^i - (K_\tau - K_t).$$

Then $\bar{\mathbb{E}}$ is a time consistent sublinear expectation (Hu, Ji, Peng, and Song, 2014, Theorem 5.1), and the Girsanov transformation for $G$-Brownian motion implies that the process $\bar{B} = (\bar{B}_t^1, ..., \bar{B}_t^d)_{0\leq t\leq\tau}$, defined by

$$\bar{B}_t := B_t - \int_0^t \kappa_u du - \sum_{i,j=1}^d \int_0^t \lambda_u^{i,j} d\langle B^i, B^j\rangle_u,$$

is a $G$-Brownian motion under $\bar{\mathbb{E}}$ (Hu, Ji, Peng, and Song, 2014, Theorems 5.2, 5.4). The quadratic covariations of $B$ and $\bar{B}$, respectively, are the same, since the drift terms of $\bar{B}$ are of bounded variation. Consequently, for each $T$, we can rewrite the dynamics of the forward rate as

$$f_t(T) = f_0(T) + \int_0^t \big(\alpha_u(T) + \beta_u(T)\kappa_u'\big)du + \sum_{i=1}^d \int_0^t \beta_u^i(T)d\bar{B}_u^i$$
$$+ \sum_{i,j=1}^d \int_0^t \big(\gamma_u^{i,j}(T) + \beta_u(T)(\lambda_u^{i,j})'\big)d\langle \bar{B}^i, \bar{B}^j\rangle_u$$

and the dynamics of the discounted bond as

$$\tilde{P}_t(T) = \tilde{P}_0(T) - \int_0^t \big(a_u(T) + b_u(T)\kappa_u'\big)\tilde{P}_u(T)du - \sum_{i=1}^d \int_0^t b_u^i(T)\tilde{P}_u(T)d\bar{B}_u^i$$
$$- \sum_{i,j=1}^d \int_0^t \big(c_u^{i,j}(T) - \tfrac{1}{2}b_u^i(T)b_u^j(T) + b_u(T)(\lambda_u^{i,j})'\big)\tilde{P}_u(T)d\langle \bar{B}^i, \bar{B}^j\rangle_u.$$

Next, we deduce the forward rate dynamics and the dynamics of the discounted bond under $\bar{\mathbb{E}}$ from the drift condition. As $\kappa$ and $\lambda^{i,j}$, for all $i,j$, satisfy (3.1) for almost all $T$,

$$f_t(T) = f_0(T) + \sum_{i=1}^d \int_0^t \beta_u^i(T)d\bar{B}_u^i + \sum_{i,j=1}^d \int_0^t \tfrac{1}{2}\big(\beta_u^i(T)b_u^j(T) + b_u^i(T)\beta_u^j(T)\big)d\langle \bar{B}^i, \bar{B}^j\rangle_u$$

for almost all $T$. Additionally, we can integrate the terms in (3.1) to get for all $i,j$,

$$\int_\cdot^T \big(\alpha(s) + \beta(s)\kappa'\big)ds = a(T) + b(T)\kappa',$$
$$\int_\cdot^T \big(\gamma^{i,j}(s) + \beta(s)(\lambda^{i,j})'\big)ds = c^{i,j}(T) + b(T)(\lambda^{i,j})',$$
$$\int_\cdot^T \big(\beta^i(s)b^j(s) + b^i(s)\beta^j(s)\big)ds = b^i(T)b^j(T)$$

for all $T$, where the latter follows from Corollary 3.24. Thus, by (3.3), for all $i,j$, we have

$$a(T) + b(T)\kappa' = 0,$$
$$c^{i,j}(T) - \tfrac{1}{2}b^i(T)b^j(T) + b(T)(\lambda^{i,j})' = 0$$

for all $T$, which implies

$$\tilde{P}_t(T) = \tilde{P}_0(T) - \sum_{i=1}^d \int_0^t b_u^i(T)\tilde{P}_u(T)d\bar{B}_u^i.$$

In the end, we conclude that the market is arbitrage-free, since the discounted bonds are symmetric $G$-martingales. From Assumption 3.2 and Proposition 3.25, we deduce that for each $T$, we have $\tilde{P}_t(T) \in L_G^2(\Omega_t)$ for all $t$. In addition, the dynamics of the discounted bond from above imply that $\tilde{P}(T)$ is a symmetric $G$-martingale under $\bar{\mathbb{E}}$ for all $T$. Then—as in the proof of Proposition 2.8—we can show that the bond market is arbitrage-free. This relies on the fact that $\bar{\mathbb{E}}$ is equivalent to the initial sublinear expectation $\hat{\mathbb{E}}$ in the sense that it holds $\xi = 0$ if and only if $\bar{\mathbb{E}}[|\xi|] = 0$ for all $\xi \in L_G^p(\Omega_\tau)$ with $p > 1$ (see Lemma 2.7). □

## 3.3 Robust Versions of Classical Term Structures

The main reason the HJM methodology is so popular is that essentially every term structure model corresponds to a specific example in the HJM model. One can verify that the forward rate implied by any arbitrage-free term structure satisfies the HJM drift condition for a particular diffusion coefficient. Conversely, the diffusion coefficient fully characterizes the risk-neutral dynamics of the forward rate, and the forward rate, in turn, determines all other quantities of the model. Thus, one is able to construct arbitrage-free term structure models by simply specifying the diffusion term of the forward rate.

We investigate what kind of term structure models we obtain when we consider specific examples in the present setting. Theorem 3.6 shows that the risk-neutral dynamics of the forward rate are also determined by its diffusion term when there is volatility uncertainty. Therefore, the drift condition derived in the previous section enables us to construct arbitrage-free term structure models in the presence of volatility uncertainty by specifying the diffusion term of the forward rate, which we demonstrate in the succeeding examples. In particular, our aim is to recover robust versions of classical term structure models by considering the corresponding diffusion coefficients in the present framework, respectively.

Throughout the section, we impose the following assumptions. First, we consider a one-dimensional $G$-Brownian motion—i.e., we have $d = 1$ and $\Sigma = [\underline{\sigma}, \overline{\sigma}]$ for $\overline{\sigma} \geq \underline{\sigma} > 0$. Second, we suppose that the initial forward curve is differentiable; this is necessary for the derivation of the related short rate dynamics. Third, we assume that the drift condition is satisfied; this assumption ensures that the model is arbitrage-free and allows us to directly compute the risk-neutral dynamics of the forward rate. It should also be noted that the following examples are feasible in the sense that the respective diffusion coefficients satisfy the regularity assumptions from Section 3.1.

### 3.3.1 The Ho-Lee Term Structure

If we consider the diffusion coefficient of the forward rate implied by the Ho-Lee term structure, we obtain a robust version of the Ho-Lee model.

**Example 3.9.** *If we define $\beta$ by $\beta_t(T) := 1$, then the short rate satisfies*

$$r_t = r_0 + \int_0^t \big(\partial_u f_0(u) + q_u\big)du + \bar{B}_t,$$

*and the bond prices are of the form*

$$P_t(T) = \exp\big(A(t,T) - \tfrac{1}{2}B(t,T)^2 q_t - B(t,T)r_t\big),$$

*where the process $q = (q_t)_{0\le t\le \tau}$ is defined by*

$$q_t := \langle \bar{B}\rangle_t$$

*and the functions $A, B : [0,\tau]\times[0,\tau] \to \mathbb{R}$ are defined by*

$$\begin{aligned} A(t,T) &:= -\int_t^T f_0(s)ds + B(t,T)f_0(t),\\ B(t,T) &:= (T-t), \end{aligned}$$

*respectively.*

*The risk-neutral short rate dynamics are determined by the risk-neutral forward rate dynamics. According to Theorem 3.6, the latter are given by*

$$f_t(T) = f_0(T) + \bar{B}_t + \int_0^t (T-u)d\langle \bar{B}\rangle_u.$$

*By the definition of the short rate, we have*

$$r_t = f_0(t) + \bar{B}_t + \int_0^t (t-u)d\langle \bar{B}\rangle_u.$$

*Applying Itô's formula for G-Brownian motion then yields*

$$r_t = r_0 + \int_0^t \big(\partial_u f_0(u) + q_u\big)du + \bar{B}_t.$$

*The bond prices follow from integrating the risk-neutral forward rate dynamics.*

$$\int_t^T f_t(s)ds = \int_t^T f_0(s)ds + B(t,T)\bar{B}_t + \int_t^T\int_0^t (s-u)d\langle \bar{B}\rangle_u ds.$$

*If we perform some calculations on the last term, we get*

$$\int_t^T\int_0^t (s-u)d\langle \bar{B}\rangle_u ds = B(t,T)\int_0^t (t-u)d\langle \bar{B}\rangle_u + \tfrac{1}{2}B(t,T)^2\langle \bar{B}\rangle_t.$$

*Substituting the latter in the previous equation, we obtain*

$$\int_t^T f_t(s)ds = -A(t,T) + \tfrac{1}{2}B(t,T)^2 q_t + B(t,T)r_t,$$

*which yields the bond prices from above.*

### 3.3.2 The Hull-White Term Structure

If we use the diffusion coefficient of the forward rate implied by the Hull-White term structure, we get a robust version of the Hull-White model.

**Example 3.10.** *If we define $\beta$ by $\beta_t(T) := e^{-\theta(T-t)}$ for $\theta > 0$, then the short rate satisfies*

$$r_t = r_0 + \int_0^t \big(\partial_u f_0(u) + \theta f_0(u) + q_u - \theta r_u\big)du + \bar{B}_t,$$

*and the bond prices are of the form*

$$P_t(T) = \exp\big(A(t,T) - \tfrac{1}{2}B(t,T)^2 q_t - B(t,T)r_t\big),$$

*where the process $q = (q_t)_{0\le t\le\tau}$ is defined by*

$$q_t := \int_0^t e^{-2\theta(t-u)} d\langle \bar{B}\rangle_u$$

*and the functions $A, B : [0,\tau]\times[0,\tau] \to \mathbb{R}$ are defined by*

$$\begin{aligned} A(t,T) &:= -\int_t^T f_0(s)ds + B(t,T)f_0(t), \\ B(t,T) &:= \tfrac{1}{\theta}(1 - e^{-\theta(T-t)}), \end{aligned}$$

*respectively.*

*Again, the risk-neutral short rate dynamics are determined by the risk-neutral forward rate dynamics. By Theorem 3.6, the latter are given by*

$$f_t(T) = f_0(T) + \int_0^t e^{-\theta(T-u)} d\bar{B}_u + \int_0^t e^{-\theta(T-u)} \tfrac{1}{\theta}(1 - e^{-\theta(T-u)}) d\langle \bar{B}\rangle_u.$$

*The definition of the short rate then implies*

$$r_t = f_0(t) + \int_0^t e^{-\theta(t-u)} d\bar{B}_u + \int_0^t e^{-\theta(t-u)} \tfrac{1}{\theta}(1 - e^{-\theta(t-u)}) d\langle \bar{B}\rangle_u.$$

*Applying Itô's formula for G-Brownian motion yields*

$$r_t = r_0 + \int_0^t \big(\partial_u f_0(u) + \theta f_0(u) + q_u - \theta r_u\big) du + \bar{B}_t.$$

*We obtain the bond prices by integrating the risk-neutral dynamics of the forward rate.*

$$\begin{aligned}\int_t^T f_t(s)ds = &\int_t^T f_0(s)ds + \int_t^T \int_0^t e^{-\theta(s-u)} d\bar{B}_u ds \\ &+ \int_t^T \int_0^t e^{-\theta(s-u)} \tfrac{1}{\theta}(1 - e^{-\theta(s-u)}) d\langle \bar{B}\rangle_u ds.\end{aligned}$$

*The first double integral can be written as*

$$\int_t^T \int_0^t e^{-\theta(s-u)} d\bar{B}_u ds = B(t,T) \int_0^t e^{-\theta(t-u)} d\bar{B}_u.$$

*After some calculations, the second double integral becomes*

$$\begin{aligned}\int_t^T \int_0^t e^{-\theta(s-u)} \tfrac{1}{\theta}(1 - e^{-\theta(s-u)}) d\langle \bar{B}\rangle_u ds = B(t,T) &\int_0^t e^{-\theta(t-u)} \tfrac{1}{\theta}(1 - e^{-\theta(t-u)}) d\langle \bar{B}\rangle_u \\ &+ \tfrac{1}{2} B(t,T)^2 \int_0^t e^{-2\theta(t-u)} d\langle \bar{B}\rangle_u.\end{aligned}$$

*Thus, we obtain*

$$\int_t^T f_t(s)ds = -A(t,T) + \tfrac{1}{2} B(t,T)^2 q_t + B(t,T) r_t,$$

*which leads to the bond prices given above.*

**Remark 3.11.** *The Hull-White model under volatility uncertainty is also analyzed in Chapter 2. In that chapter, we show how to obtain an arbitrage-free term structure in the Hull-White model when there is uncertainty about the volatility. In order to achieve this, the structure of the short rate dynamics has to be suitably modified. Here we get exactly the same structure.*

### 3.3.3 The Vasicek Term Structure

The previous example shows that the Vasicek model needs to be adjusted in order to fit into the HJM methodology when there is volatility uncertainty.

**Example 3.12.** *If we use the same diffusion coefficient as in the previous example, we see that it is not possible to exactly replicate the Vasicek model in the presence of volatility uncertainty. The forward rates implied by the Vasicek term structure and the Hull-White term structure, respectively, have the same diffusion coefficient. If we define* $\beta$ *as in*

*Example 3.10, then the short rate dynamics are given by*

$$r_t = r_0 + \int_0^t \big(\partial_u f_0(u) + \theta f_0(u) + q_u - \theta r_u\big)du + \bar{B}_t,$$

*where the process $q$ is defined as in Example 3.10. In order to obtain the short rate dynamics of the Vasicek model, we need to make sure that for all $t$,*

$$\partial_t f_0(t) + \theta f_0(t) + q_t = \mu \tag{3.2}$$

*for a constant $\mu > 0$, since the mean reversion level of the short rate is constant in the Vasicek model. As equation (3.2) does not hold for any initial forward curve, the equation imposes a condition on $f_0$ that ensures a constant mean reversion level. If there is no volatility uncertainty, one can check that the initial forward curve of the Vasicek term structure satisfies (3.2). In the presence of volatility uncertainty, there is no initial forward curve $f_0$ satisfying (3.2), since then the process $q$ is uncertain—i.e., its realization $q_t$ is only known after time $t$—while $f_0$ is observable at inception.*

*We can circumvent the problem by modifying the Vasicek model. Let us suppose that the initial forward curve satisfies*

$$\partial_t f_0(t) + \theta f_0(t) = \mu$$

*for all $t$; that means, the function $f_0$ solves a simple ordinary differential equation with initial condition $f_0(0) = r_0$. This yields*

$$f_0(t) = e^{-\theta t} r_0 + \mu B(0,t)$$

*for all $t$, where the function $B$ is defined as in Example 3.10. Then the short rate satisfies*

$$r_t = r_0 + \int_0^t (\mu + q_u - \theta r_u)du + \bar{B}_t,$$

*and (as in Example 3.10) the bond prices are of the form*

$$P_t(T) = \exp\big(A(t,T) - \tfrac{1}{2}B(t,T)^2 q_t - B(t,T)r_t\big),$$

*where the function $A$, for all $t$ and $T$, now satisfies*

$$A(t,T) = -\mu \int_t^T B(s,T)ds.$$

*So instead of exactly replicating the Vasicek model, we can obtain a version of the Vasicek model in which the mean reversion level of the short rate is adjusted by the process $q$.*

**Remark 3.13.** *The problem mentioned in Example 3.12 does not occur in Example 3.10 due to the time dependent mean reversion level in the Hull-White model. Since the mean reversion level can be time dependent and possibly uncertain, equation* (3.2) *only imposes a condition on the mean reversion level but not on the initial forward curve.*

*In fact, the problematic is related to the ability of term structure models to match arbitrary forward curves observed on the market, since the HJM methodology is based on modeling the forward rate starting from an arbitrary initial forward curve. The Hull-White model—as well as the Ho-Lee model—involves a time dependent parameter; hence, it offers enough flexibility to fit the model-implied term structure to any term structure obtained from data. On the other hand, the Vasicek model has only three (constant) parameters, restricting the model to a small class of term structures it can fit. Therefore, one has to impose further assumptions on* $f_0$ *to reproduce the Vasicek model in the HJM methodology in general, which, however, does not work when there is volatility uncertainty.*

### 3.3.4 Economic Consequences

The examples show that arbitrage-free term structure models in the presence of volatility uncertainty exhibit an additional uncertain factor. In all examples we consider, there is an uncertain process, always denoted by $q$, which enters the short rate dynamics and the bond prices. The process $q$ is uncertain, since it depends on the quadratic variation process. It emerges due to the fact that the risk-neutral forward rate dynamics display drift uncertainty (as Theorem 3.6 shows). That means, the additional factor is required in order to make the model arbitrage-free when the volatility is uncertain. Despite this difference, the short rate dynamics and the bond prices still have an affine form. In fact, the resulting term structure models (except the one in Example 3.12) are consistent with the classical ones. If there is no volatility uncertainty—that is, if $\overline{\sigma} = \underline{\sigma}$—then Examples 3.9 and 3.10 correspond to the traditional Ho-Lee model (Filipović, 2009, Subsection 5.4.4) and the traditional Hull-White model (Brigo and Mercurio, 2001, Subsections 3.3.1, 3.3.2), respectively. In that case, the process $q$ is no longer uncertain. Hence, the process $q$ is actually included in the Ho-Lee model and in the Hull-White model, but it is hidden as there is no volatility uncertainty in the traditional models. The Vasicek model does not include such a factor; thus, it needs to be adjusted in order to be arbitrage-free in the presence of volatility, as demonstrated in Example 3.12.

The term structure models resulting from the examples are completely robust with respect to the volatility. The bond prices in the examples are completely independent of the volatility. They do not even depend on the extreme values of the volatility (given by $\overline{\sigma}$ and $\underline{\sigma}$), which usually happens in option pricing under volatility uncertainty. Of course, such a degree of robustness has its price: instead of the volatility, the bond prices depend on the additional factor $q$, which is determined by the quadratic variation of the driving risk factor. That means, the term structure models we construct do not require any knowledge about how the volatility evolves in the future; all necessary information is included in the quadratic variation of the driver—that is, in the historical volatility. From a theoretical point of view—especially regarding the motivation of volatility uncertainty given in Chapter 1—it is desirable to have a term structure model that does not impose any assumptions on the future evolution of the volatility. The need to specify the evaluations of the process $q$ instead is not a problem, since this (in principle) can be done by inferring the evolution of the historical volatility from data. How these models perform in practice is, however, a challenging question for future research, as many applications of term structure models involve estimation procedures and simulations (Dai and Singleton, 2003), which is a nontrivial task in the presence of a family of probability measures (see Section 2.8).

## 3.4 Admissible Integrands for the Forward Rate

We construct the space of admissible integrands for the forward rate dynamics as follows. For $T < \infty$, let us consider the measure space $([0,T], \mathcal{B}([0,T]), \lambda)$, where $\mathcal{B}([0,T])$ denotes the Borel $\sigma$-algebra on $[0,T]$ and $\lambda$ is the Lebesgue measure on $[0,T]$, and the space of admissible stochastic processes $M_G^p(0,T)$ for $p \geq 1$, which is a Banach space with respect to the norm $\|\cdot\|_p$, defined by

$$\|\eta\|_p := \hat{\mathbb{E}}\Big[\int_0^T |\eta_t|^p dt\Big]^{\frac{1}{p}}$$

for a process $\eta = (\eta_t)_{0 \leq t \leq T}$ in $M_G^p(0,T)$. Then we define by $\tilde{M}_G^{p,0}(0,T)$ the space of all simple functions mapping from $[0,T]$ into $M_G^p(0,T)$—i.e., functions $\phi : [0,T] \to M_G^p(0,T)$ such that

$$\phi(s) = \sum_{i=1}^n \varphi^i 1_{A_i}(s),$$

where $(\varphi^i)_{i=1}^n$ is a finite sequence of processes $\varphi^i = (\varphi_t^i)_{0 \leq t \leq T}$ in $M_G^p(0,T)$ and $(A_i)_{i=1}^n$ is a finite sequence of pairwise disjoint sets $A_i$ in $\mathcal{B}([0,T])$. On the space $\tilde{M}_G^{p,0}(0,T)$, we introduce the seminorm $\|\cdot\|_{\sim,p}$, defined by

$$\|\phi\|_{\sim,p} := \int_0^T \|\phi(s)\|_p ds.$$

By considering the quotient space with respect to the null space

$$\bar{M}_G^p(0,T) := \big\{\phi \in \tilde{M}_G^{p,0}(0,T) \;\big|\; \|\phi\|_{\sim,p} = 0\big\},$$

still denoted by $\tilde{M}_G^{p,0}(0,T)$, we get a normed space. The completion of $\tilde{M}_G^{p,0}(0,T)$ under the norm $\|\cdot\|_{\sim,p}$ is denoted by $\tilde{M}_G^p(0,T)$, being the space of admissible integrands.

There is an explicit representation of the space of admissible integrands. It can be shown that the abstract completion of $\tilde{M}_G^{p,0}(0,T)$ is given by the following space of functions (Prévôt and Röckner, 2007, Section A.1):

$$\tilde{M}_G^p(0,T) = \big\{\phi : [0,T] \to M_G^p(0,T) \;\big|\; \phi \text{ is strongly measurable, } \|\phi\|_{\sim,p} < \infty\big\},$$

where we say that a function $\phi : [0,T] \to M_G^p(0,T)$ is *strongly measurable* if it is $\mathcal{B}([0,T])/\mathcal{B}(M_G^p(0,T))$-measurable and $\phi([0,T])$ is separable. Therefore, we know that $\phi \in \tilde{M}_G^p(0,T)$ is a regular stochastic process for a fixed $s$; that is, we have $\phi(s) \in M_G^p(0,T)$ for each $s$.

Due to the explicit representation of $\tilde{M}_G^p(0,T)$, we can give a sufficient condition for functions to lie in this space.

**Proposition 3.14.** *Let $\phi : [0,T] \to M_G^p(0,T)$ be continuous. Then $\phi \in \tilde{M}_G^p(0,T)$.*

*Proof.* First, we show that $\phi$ is strongly measurable. Since $\phi$ is continuous, it is clearly $\mathcal{B}([0,T])/\mathcal{B}(M_G^p(0,T))$-measurable. Furthermore, the interval $[0,T]$ is separable, and the image of a continuous function with separable domain is separable. Hence, the image $\phi([0,T])$ is separable, which proves the strong measurability.

It is left to show that the norm of $\phi$ is finite. The norm $\|\cdot\|_p$ is obviously a continuous function; thus, the function $f : \mathbb{R} \to \mathbb{R}, s \mapsto \|\phi(s)\|_p$ is continuous, since $\phi$ is continuous. Therefore, we have $\|\phi\|_{\sim,p} < \infty$. □

By Proposition 3.14, we have the following examples of functions in $\tilde{M}_G^p(0,T)$. First, we know that continuous real-valued functions on $[0,T] \times [0,T]$ belong to $\tilde{M}_G^p(0,T)$.

**Example 3.15.** *The function $\phi : [0,T] \to M_G^p(0,T), s \mapsto f(\cdot, s)$ belongs to $\tilde{M}_G^p(0,T)$, where $f : [0,T] \times [0,T] \to \mathbb{R}$ is continuous. The function $\phi$ maps into $M_G^p(0,T)$, since $f(\cdot, s)$ is a continuous function for all $s$; this can be deduced from the representation of the space $M_G^p(0,T)$ (Hu, Wang, and Zheng, 2016, Theorem 4.7). The continuity of $\phi$ follows from*

$$\|\phi(s) - \phi(\tilde{s})\|_p = \left(\int_0^T |f(t,s) - f(t,\tilde{s})|^p dt\right)^{\frac{1}{p}}$$

*and dominated convergence. Thus, by Proposition 3.14, it holds $\phi \in \tilde{M}_G^p(0,T)$.*

Second, the product of a continuous real-valued function on $[0,T]$ and an admissible stochastic process lies in the space $\tilde{M}_G^p(0,T)$.

**Example 3.16.** *The function* $\phi : [0,T] \to M_G^p(0,T), s \mapsto f(s)\eta$ *belongs to* $\tilde{M}_G^p(0,T)$*, where* $f : [0,T] \to \mathbb{R}$ *is continuous and* $\eta = (\eta_t)_{0\le t\le T}$ *belongs to* $M_G^p(0,T)$*. Clearly, the function* $\phi$ *maps into* $M_G^p(0,T)$*. The continuity of* $\phi$ *follows from*

$$\|\phi(s) - \phi(\tilde{s})\|_p = \|\eta\|_p |f(s) - f(\tilde{s})|.$$

*Hence, Proposition 3.14 implies that* $\phi \in \tilde{M}_G^p(0,T)$*.*

We are able to define integrals and—more importantly—double integrals for functions in $\tilde{M}_G^p(0,T)$. First, we define the Bochner integral for simple functions. For a function $\phi \in \tilde{M}_G^{p,0}(0,T)$ with a representation as introduced at the beginning of this section, we define

$$\int_0^T \phi(s)ds := \sum_{i=1}^n \varphi^i \lambda(A_i).$$

The Bochner integral is a linear operator mapping from $\tilde{M}_G^{p,0}(0,T)$ into $M_G^p(0,T)$. In addition, the operator is continuous, since we have the inequality

$$\left\| \int_0^T \phi(s)ds \right\|_p \le \int_0^T \|\phi(s)\|_p ds. \tag{3.3}$$

Thus, we can extend the operator to the completion $\tilde{M}_G^p(0,T)$, still satisfying (3.3). For $A \in \mathcal{B}([0,T])$, we define $\int_A \phi(s)ds := \int_0^T 1_A(s)\phi(s)ds$. Since the integral maps into $M_G^p(0,T)$, we can define the double integral $\int_0^T \int_A \phi_t(s)dsdB_t^i$ for $\phi \in \tilde{M}_G^2(0,T)$, mapping into $L_G^2(\Omega_T)$, and the double integrals $\int_0^T \int_A \psi_t(s)dsdt$ and $\int_0^T \int_A \psi_t(s)dsd\langle B^i, B^j\rangle_t$ for $\psi \in \tilde{M}_G^1(0,T)$, mapping into $L_G^1(\Omega_T)$, for all $i,j = 1,...,d$.

We can also define double integrals for the reversed order of integration and interchange the order of integration. For this purpose, we use a classical result from the theory of Bochner integration (Prévôt and Röckner, 2007, Proposition A.2.2).

**Proposition 3.17.** *Let* $\phi \in \tilde{M}_G^p(0,T)$ *and let* $F : M_G^p(0,T) \to X$ *be a continuous linear operator, where* $X$ *is a Banach space. Then we can define the integral* $\int_A F(\phi(s))ds$*, mapping into* $X$*, and it holds*

$$\int_A (F \circ \phi)(s)ds = F\Big(\int_A \phi(s)ds\Big).$$

All stochastic integrals related to a $G$-Brownian motion are continuous linear operators. Thus, Proposition 3.17 allows us to define the integral $\int_A \int_0^T \phi_t(s)dB_t^i ds$ for $\phi \in \tilde{M}_G^2(0,T)$, mapping into $L_G^2(\Omega_T)$, and the integrals $\int_A \int_0^T \psi_t(s)dtds$ and $\int_A \int_0^T \psi_t(s)d\langle B^i, B^j\rangle_t ds$ for $\psi \in \tilde{M}_G^1(0,T)$, mapping into $L_G^1(\Omega_T)$, for all $i,j = 1,...,d$. Moreover, we obtain a version of Fubini's theorem, which is an essential tool in the HJM model.

**Corollary 3.18.** *Let $\phi \in \tilde{M}_G^2(0,T)$ and $\psi \in \tilde{M}_G^1(0,T)$. Then for all $i,j$, it holds*

$$\int_A \int_0^T \phi_t(s) dB_t^i ds = \int_0^T \int_A \phi_t(s) ds dB_t^i,$$
$$\int_A \int_0^T \psi_t(s) dt ds = \int_0^T \int_A \psi_t(s) ds dt,$$
$$\int_A \int_0^T \psi_t(s) d\langle B^i, B^j\rangle_t ds = \int_0^T \int_A \psi_t(s) ds d\langle B^i, B^j\rangle_t.$$

In addition to the double integrals from above, we need to define more complex integrals. In order to achieve this, we need the following proposition:

**Proposition 3.19.** *Let $\phi \in \tilde{M}_G^p(0,T)$ and $\psi : [0,T] \to M_G^p(0,T), s \mapsto 1_{[0,s]}\phi(s)$. Then we have $\psi \in \tilde{M}_G^p(0,T)$.*

*Proof.* First, we decompose $\psi$ into several functions to show that it is strongly measurable. Let us define the subspace

$$\mathcal{M}^p := \left\{\eta \in M_G^p(0,T) \;\middle|\; \eta = 1_{[0,a]} \text{ for some } a \in [0,T]\right\} \subset M_G^p(0,T).$$

Furthermore, we define the functions $g : [0,T] \to \mathcal{M}^p \times M_G^p(0,T), s \mapsto (f(s), \phi(s))$, where $f : [0,T] \to \mathcal{M}^p, s \mapsto 1_{[0,s]}$, and $h : \mathcal{M}^p \times M_G^p(0,T) \to M_G^p(0,T), (\eta, \zeta) \mapsto \eta\zeta$. Then we have $\psi = h \circ g$.

We deduce the measurability of $\psi$ from the measurability of the decomposition. The function $f$ is continuous, since

$$\|1_{[0,a]} - 1_{[0,\tilde{a}]}\|_p = |a - \tilde{a}|$$

for $a, \tilde{a} \in [0,T]$; thus, we know that $f$ is $\mathcal{B}([0,T])/\mathcal{B}(\mathcal{M}^p)$-measurable. By assumption, the function $\phi$ is $\mathcal{B}([0,T])/\mathcal{B}(M_G^p(0,T))$-measurable, and so we know that the function $g$ is $\mathcal{B}([0,T])/\mathcal{B}(\mathcal{M}^p) \otimes \mathcal{B}(M_G^p(0,T))$-measurable. Now it is left to show that the function $h$ is $\mathcal{B}(\mathcal{M}^p) \otimes \mathcal{B}(M_G^p(0,T))/\mathcal{B}(M_G^p(0,T))$-measurable to deduce the measurability of $\psi$. We equip $\mathcal{M}^p \times M_G^p(0,T)$ with the norm $\|\cdot\|$, defined by

$$\|(\eta, \zeta)\| := \max\{\|\eta\|_p, \|\zeta\|_p\}.$$

For $a, \tilde{a} \in [0,T]$ and $\zeta = (\zeta_t)_{0\le t\le T}$ and $\tilde{\zeta} = (\tilde{\zeta}_t)_{0\le t\le T}$ in $M_G^p(0,T)$, we have

$$\|1_{[0,a]}\zeta - 1_{[0,\tilde{a}]}\tilde{\zeta}\|_p \le \|\zeta - \tilde{\zeta}\|_p + \|1_{[a,\tilde{a}]}\tilde{\zeta}\|_p,$$

where the last term converges to 0 as $a$ converges to $\tilde{a}$. The function $h$ is therefore continuous. So we need to show that $M \in \mathcal{B}(\mathcal{M}^p) \otimes \mathcal{B}(M_G^p(0,T))$ for an arbitrary open set $M \subset \mathcal{M}^p \times M_G^p(0,T)$ to obtain the measurability of $h$. If $M$ is open, we can represent it as a union of open balls in $\mathcal{M}^p \times M_G^p(0,T)$. By the definition of $\|\cdot\|$, every open ball in $\mathcal{M}^p \times M_G^p(0,T)$ can be written as a rectangle whose sides are open balls in $\mathcal{M}^p$ and $M_G^p(0,T)$, respectively. Hence, we can represent $M$ as a rectangle whose sides are open sets in $\mathcal{M}^p$ and $M_G^p(0,T)$, respectively. Therefore, we get $M \in \mathcal{B}(\mathcal{M}^p) \otimes \mathcal{B}(M_G^p(0,T))$.

We show that the image of $\psi$ is separable by using the continuity from the second step. By assumption, the image of $\phi$ is separable. Moreover, the interval $[0,T]$ is separable, and $f$ is continuous; thus, the image of $f$ is separable. This implies that the image of $g$ is separable. We also have that $h$ is continuous. Hence, the image $h(g([0,T]))$ is separable. Since $\psi = h \circ g$, it follows that $\psi([0,T])$ is separable.

It is left to show that the norm of $\psi$ is finite. We have

$$\int_0^T \|\psi(s)\|_p ds \leq \int_0^T \|\phi(s)\|_p ds < \infty,$$

which completes the proof. □

Due to Proposition 3.19, we are able to define integrals of the form $\int_0^T \int_t^T \phi_t(s) ds dB_t^i$ and, by Proposition 3.17, integrals of the form $\int_0^T \int_0^s \phi_t(s) dB_t^i ds$ for $\phi \in \tilde{M}_G^2(0,T)$, mapping into $L_G^2(\Omega_T)$, for all $i$. Moreover, Corollary 3.18 implies that for all $i$,

$$\int_0^T \int_t^T \phi_t(s) ds dB_t^i = \int_0^T \int_0^s \phi_t(s) dB_t^i ds.$$

The same holds if we replace $dB_t^i$ and the spaces $\tilde{M}_G^2(0,T)$ and $L_G^2(\Omega_T)$ by $dt$—as well as $d\langle B^i, B^j \rangle_t$ for all $j$—and the spaces $\tilde{M}_G^1(0,T)$ and $L_G^1(\Omega_T)$, respectively.

In the end, we deal with the differentiability of integrals and (especially) the differentiability of products of two integrals, used for the calculations on the diffusion coefficient of the forward rate. As the classical Lebesgue integral, integrals of functions in $\tilde{M}_G^p(0,T)$ are, loosely speaking, differentiable and absolutely continuous.

**Proposition 3.20.** *Let $\phi \in \tilde{M}_G^p(0,T)$ and $\Phi : [0,T] \to M_G^p(0,T), s \mapsto \int_0^s \phi(u) du$. Then the following properties hold.*

*(i) The function $\Phi$ is almost everywhere differentiable and $\Phi' = \phi$.*

*(ii) For all $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\sum_{i=1}^n \|\Phi(s_i) - \Phi(\tilde{s}_i)\|_p < \epsilon$$

*for every sequence of disjoint open intervals $((\tilde{s}_i, s_i))_{i=1}^n$ such that $\sum_{i=1}^n (s_i - \tilde{s}_i) < \delta$.*

*Proof.* We deduce all statements from the inequality (3.3) and the properties of the

Lebesgue integral. By (3.3), we have

$$\left\| \tfrac{1}{s-\tilde{s}} \big( \Phi(s) - \Phi(\tilde{s}) \big) - \phi(\tilde{s}) \right\|_p \le \left| \tfrac{1}{s-\tilde{s}} \big( f(s) - f(\tilde{s}) \big) \right|,$$

where $f : [0,T] \to \mathbb{R}$ is defined by $f(s) := \int_0^s \|\phi(u) - \phi(\tilde{s})\|_p du$. Due to the differentiability of the Lebesgue integral, the expression on the right-hand side of the previous inequality converges to 0 as $s$ converges to $\tilde{s}$ for almost all $\tilde{s}$. Therefore, the function $\Phi$ is almost everywhere differentiable and $\Phi' = \phi$. Furthermore, we can use (3.3) to obtain

$$\|\Phi(s) - \Phi(\tilde{s})\|_p \le |g(s) - g(\tilde{s})|,$$

where $g : [0,T] \to \mathbb{R}$ is defined by $g(s) := \int_0^s \|\phi(u)\|_p du$. Since the Lebesgue integral is absolutely continuous, for all $\epsilon > 0$, we can find a $\delta > 0$ such that

$$\sum_{i=1}^{n} \|\Phi(s_i) - \Phi(\tilde{s}_i)\|_p \le \sum_{i=1}^{n} |g(s_i) - g(\tilde{s}_i)| < \epsilon$$

for every sequence of disjoint open intervals $((\tilde{s}_i, s_i))_{i=1}^n$ such that $\sum_{i=1}^n (s_i - \tilde{s}_i) < \delta$. □

Conversely, we have a version of the fundamental theorem of calculus for functions in $\tilde{M}_G^p(0,T)$ that are differentiable and absolutely continuous.

**Proposition 3.21.** *Let $\Phi : [0,T] \to M_G^p(0,T)$ be almost everywhere differentiable and $\Phi' = \phi$, where $\phi \in \tilde{M}_G^p(0,T)$. If for all $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\sum_{i=1}^{n} \|\Phi(s_i) - \Phi(\tilde{s}_i)\|_p < \epsilon$$

*for every sequence of disjoint open intervals $((\tilde{s}_i, s_i))_{i=1}^n$ such that $\sum_{i=1}^n (s_i - \tilde{s}_i) < \delta$, then it holds*

$$\Phi(s) - \Phi(0) = \int_0^s \phi(u) du.$$

*Proof.* We prove the assertion by using a consequence of the Hahn-Banach theorem and the fundamental theorem of calculus for Lebesgue integrals. Let $F : M_G^p(0,T) \to \mathbb{R}$ be a continuous linear functional. Then we have

$$|(F \circ \Phi)(s) - (F \circ \Phi)(\tilde{s})| \le C \|\Phi(s) - \Phi(\tilde{s})\|_p$$

for some constant $C > 0$. Due to the last assumption on $\Phi$, we deduce that $F \circ \Phi$ is absolutely continuous. The fundamental theorem of calculus for Lebesgue integrals implies that $F \circ \Phi$ is almost everywhere differentiable, its derivative is integrable, and

$$(F \circ \Phi)(s) - (F \circ \Phi)(0) = \int_0^s (F \circ \Phi)'(u)du.$$

Furthermore, it holds $(F \circ \Phi)' = F \circ \phi$, since the continuity and the linearity of $F$ imply

$$\begin{aligned}|(F \circ \Phi)'(\tilde{s}) - (F \circ \phi)(\tilde{s})| \leq \left|(F \circ \Phi)'(\tilde{s}) - \tfrac{1}{s-\tilde{s}}\big((F \circ \Phi)(s) - (F \circ \Phi)(\tilde{s})\big)\right| \\ + C\left\|\tfrac{1}{s-\tilde{s}}\big(\Phi(s) - \Phi(\tilde{s})\big) - \phi(\tilde{s})\right\|_p\end{aligned}$$

for some constant $C > 0$, where the terms on the right-hand side converge to 0 as $s$ converges to $\tilde{s}$ for almost all $\tilde{s}$. Hence, we obtain

$$(F \circ \Phi)(s) - (F \circ \Phi)(0) = \int_0^s (F \circ \phi)(u)du.$$

By Proposition 3.17 and the linearity of $F$, it holds

$$F\left(\Phi(s) - \Phi(0) - \int_0^s \phi(u)du\right) = 0.$$

Since this holds for every continuous linear functional, the Hahn-Banach theorem implies the assertion. □

In order to derive a product rule for differentiable and absolutely continuous functions, we use the following lemma:

**Lemma 3.22.** *Let $\phi, \psi \in \tilde{M}_G^{2p}(0,T)$ and $\upsilon : [0,T] \to M_G^p(0,T), s \mapsto \phi(s)\psi(s)$, where $\phi$ is continuous. Then $\upsilon \in \tilde{M}_G^p(0,T)$.*

*Proof.* First of all, we know that $\upsilon$ maps into $M_G^p(0,T)$, since one can show (by Hölder's inequality) that the product of two processes in $M_G^{2p}(0,T)$ belongs to $M_G^p(0,T)$.

Next, we show that $\upsilon$ is strongly measurable. The strong measurability follows as in the proof of Proposition 3.19 if the function $f : M_G^{2p}(0,T) \times M_G^{2p}(0,T) \to M_G^p(0,T)$, defined by $f(\eta, \zeta) := \eta\zeta$, is continuous, where we equip $M_G^{2p}(0,T) \times M_G^{2p}(0,T)$ with the norm $\|\cdot\|$, defined by

$$\|(\eta, \zeta)\| := \max\{\|\eta\|_{2p}, \|\zeta\|_{2p}\}.$$

By applying Hölder's inequality, for $(\eta, \zeta) = (\eta_t, \zeta_t)_{0 \leq t \leq T}$ and $(\tilde{\eta}, \tilde{\zeta}) = (\tilde{\eta}_t, \tilde{\zeta}_t)_{0 \leq t \leq T}$ in $M_G^{2p}(0,T) \times M_G^{2p}(0,T)$, we have

$$\|\eta\zeta - \tilde{\eta}\tilde{\zeta}\|_p \leq \|\eta\|_{2p}\|\zeta - \tilde{\zeta}\|_{2p} + \|\eta - \tilde{\eta}\|_{2p}\|\tilde{\zeta}\|_{2p}.$$

Therefore, the function $f$ is continuous, since $\|\eta\|_{2p}$ and $\|\tilde{\zeta}\|_{2p}$ are finite and $\|\eta-\tilde{\eta}\|_{2p}$ and $\|\zeta-\tilde{\zeta}\|_{2p}$ converge to 0 as $(\eta,\zeta)$ converges to $(\tilde{\eta},\tilde{\zeta})$.

Finally, we deduce that the norm of $\upsilon$ is finite. The function $s \mapsto \|\phi(s)\|_{2p}$ is continuous due to the continuity of $\phi$ and the norm; hence, it is bounded. Thus,

$$\int_0^T \|\upsilon(s)\|_p ds \leq \int_0^T \|\phi(s)\|_{2p}\|\psi(s)\|_{2p} ds \leq C\int_0^T \|\psi(s)\|_{2p} ds < \infty$$

for some constant $C>0$, which completes the proof. □

With the help of Lemma 3.22, we obtain the desired product rule.

**Proposition 3.23.** *Let $\Phi,\Psi:[0,T]\to M_G^{2p}(0,T)$ be almost everywhere differentiable and $\Phi'=\phi$ and $\Psi'=\psi$, where $\phi,\psi\in\tilde{M}_G^{2p}(0,T)$. If for all $\epsilon>0$, there exists a $\delta>0$ such that*

$$\sum_{i=1}^n \|\Phi(s_i)-\Phi(\tilde{s}_i)\|_p, \sum_{i=1}^n \|\Psi(s_i)-\Psi(\tilde{s}_i)\|_p < \epsilon$$

*for every sequence of disjoint open intervals $((\tilde{s}_i,s_i))_{i=1}^n$ such that $\sum_{i=1}^n (s_i-\tilde{s}_i)<\delta$, then it holds*

$$\Phi(s)\Psi(s)-\Phi(0)\Psi(0) = \int_0^s \phi(u)\Psi(u)+\Phi(u)\psi(u)du.$$

*Proof.* We show that $\Upsilon:[0,T]\to M_G^p(0,T), s\mapsto \Phi(s)\Psi(s)$ satisfies the assumptions of Proposition 3.21 to deduce the assertion. Using Hölder's inequality, we have

$$\|\Upsilon(s)-\Upsilon(\tilde{s})\|_p \leq \|\Phi(s)\|_{2p}\|\Psi(s)-\Psi(\tilde{s})\|_{2p} + \|\Phi(s)-\Phi(\tilde{s})\|_{2p}\|\Psi(\tilde{s})\|_{2p}.$$

The last assumption implies that $s\mapsto \|\Phi(s)\|_{2p}$ and $\tilde{s}\mapsto\|\Psi(\tilde{s})\|_{2p}$ are continuous; thus, they are bounded. Hence, for all $\epsilon>0$, we can find a $\delta>0$ such that

$$\sum_{i=1}^n \|\Upsilon(s_i)-\Upsilon(\tilde{s}_i)\|_p < \epsilon$$

for every sequence of disjoint open intervals $((\tilde{s}_i,s_i))_{i=1}^n$ such that $\sum_{i=1}^n (s_i-\tilde{s}_i)<\delta$. Let $\upsilon:[0,T]\to M_G^p(0,T), s\mapsto \phi(s)\Psi(s)+\Phi(s)\psi(s)$. By Lemma 3.22, we have $\upsilon\in\tilde{M}_G^p(0,T)$, since $\phi,\psi\in\tilde{M}_G^{2p}(0,T)$ and $\Phi$ and $\Psi$ are continuous and belong to $\tilde{M}_G^{2p}(0,T)$ by the assumptions. Furthermore, we have

$$\begin{aligned}\left\|\tfrac{1}{s-\tilde{s}}\big(\Upsilon(s)-\Upsilon(\tilde{s})\big)-\upsilon(\tilde{s})\right\|_p \leq & \left\|\tfrac{1}{s-\tilde{s}}\big(\Upsilon(s)-\Phi(\tilde{s})\Psi(s)\big)-\phi(\tilde{s})\Psi(s)\right\|_p \\ &+\|\phi(\tilde{s})\Psi(s)-\phi(\tilde{s})\Psi(\tilde{s})\|_p \\ &+\left\|\tfrac{1}{s-\tilde{s}}\big(\Phi(\tilde{s})\Psi(s)-\Upsilon(\tilde{s})\big)-\Phi(\tilde{s})\psi(\tilde{s})\right\|_p.\end{aligned}$$

The three terms on the right-hand side converge to 0 as $s$ converges to $\tilde{s}$ for almost all $\tilde{s}$. The first term converges almost everywhere to 0, since

$$\Big\|\tfrac{1}{s-\tilde{s}}\big(\Upsilon(s)-\Phi(\tilde{s})\Psi(s)\big)-\phi(\tilde{s})\Psi(s)\Big\|_p \leq \Big\|\tfrac{1}{s-\tilde{s}}\big(\Phi(s)-\Phi(\tilde{s})\big)-\phi(\tilde{s})\Big\|_{2p}\|\Psi(s)\|_{2p},$$

the function $\Phi$ is almost everywhere differentiable, it holds $\Phi'=\phi$, and $s\mapsto\|\Psi(s)\|_{2p}$ is continuous. The second term converges to 0, since

$$\|\phi(\tilde{s})\Psi(s)-\phi(\tilde{s})\Psi(\tilde{s})\|_p \leq \|\phi(\tilde{s})\|_{2p}\|\Psi(s)-\Psi(\tilde{s})\|_{2p},$$

we have $\phi(\tilde{s})\in M_G^{2p}(0,T)$, and $s\mapsto\|\Psi(s)\|_{2p}$ is continuous. The third term converges to 0 almost everywhere, since

$$\Big\|\tfrac{1}{s-\tilde{s}}\big(\Phi(\tilde{s})\Psi(s)-\Upsilon(\tilde{s})\big)-\Phi(\tilde{s})\psi(\tilde{s})\Big\|_p \leq \|\Phi(\tilde{s})\|_{2p}\Big\|\tfrac{1}{s-\tilde{s}}\big(\Psi(s)-\Psi(\tilde{s})\big)-\psi(\tilde{s})\Big\|_{2p},$$

we have $\Phi(\tilde{s})\in M_G^{2p}(0,T)$, the function $\Psi$ is almost everywhere differentiable, and it holds $\Psi'=\psi$. Therefore, the function $\Upsilon$ is almost everywhere differentiable and $\Upsilon'=\upsilon$. This completes the proof. □

Combining Proposition 3.20 and Proposition 3.23, we have the following result, which we apply to the diffusion coefficient of the forward rate.

**Corollary 3.24.** *Let $\phi,\psi\in\tilde{M}_G^{2p}(0,T)$ and let $\Phi,\Psi:[0,T]\to M_G^p(0,T)$ be defined by $\Phi(s):=\int_0^s\phi(u)du$ and $\Psi(s):=\int_0^s\psi(u)du$, respectively. Then it holds*

$$\Phi(s)\Psi(s)=\int_0^s \phi(u)\Psi(u)+\Phi(u)\psi(u)du.$$

## 3.5 Regularity of the Discounted Bonds

In order to show that the discounted bonds are sufficiently regular, we consider the exponential of a diffusion process driven by a $G$-Brownian motion. Let us define the process $X=(X_t)_{0\leq t\leq T}$ by

$$X_t:=\exp\bigg(\int_0^t a_u du+\sum_{i=1}^d\int_0^t b_u^i dB_u^i+\sum_{i,j=1}^d\int_0^t c_u^{i,j}d\langle B^i,B^j\rangle_u\bigg),$$

where $a=(a_t)_{0\leq t\leq T}$ and $c^{i,j}=(c_t^{i,j})_{0\leq t\leq T}$ belong to $M_G^1(0,T)$ and $b^i=(b_t^i)_{0\leq t\leq T}$ belongs to $M_G^2(0,T)$ for all $i,j=1,...,d$. Then the dynamics of $X$ are given by

$$X_t=1+\int_0^t a_u X_u du+\sum_{i=1}^d\int_0^t b_u^i X_u dB_u^i+\sum_{i,j=1}^d\int_0^t (c_u^{i,j}+\tfrac{1}{2}b_u^i b_u^j)X_u d\langle B^i,B^j\rangle_u$$

by Itô's formula for $G$-Brownian motion.

The following result provides a sufficient condition ensuring that the dynamics of the process $X$ are regular. As a consequence, we obtain that $X$ itself is well-posed.

**Proposition 3.25.** *If there exists a $p>1$ such that $a, c^{i,j} \in M_G^p(0,T)$ and $b^i \in M_G^{2p}(0,T)$ for all $i,j=1,...,d$ and there exist $\tilde{p}>p^*$ and $\tilde{q}>2$, where $p^* := \frac{2pq}{p-q}$ for some $q \in (1,p)$ such that*

$$\hat{\mathbb{E}}\bigg[\int_0^T \exp\bigg(\tfrac{\tilde{p}\tilde{q}}{\tilde{q}-2}\bigg(\int_0^t a_u du + \sum_{i,j=1}^d \int_0^t c_u^{i,j} d\langle B^i, B^j\rangle_u\bigg)\bigg)dt\bigg] < \infty,$$
$$\hat{\mathbb{E}}\bigg[\int_0^T \exp\bigg(\tfrac{1}{2}(\tilde{p}\tilde{q})^2 \sum_{i,j=1}^d \int_0^t b_u^i b_u^j d\langle B^i, B^j\rangle_u\bigg)dt\bigg] < \infty,$$

*then we have $X \in M_G^{p^*}(0,T)$. In particular, this implies $aX, (c^{i,j}+\frac{1}{2}b^i b^j)X \in M_G^1(0,T)$ and $b^i X \in M_G^2(0,T)$ for all $i,j=1,...,d$.*

*Proof.* In order to show that $X \in M_G^{p^*}(0,T)$, we use the characterization of the space $M_G^{p^*}(0,T)$ from Hu, Wang, and Zheng (2016). The space $M_G^{p^*}(0,T)$ consists of all progressively measurable processes $\eta=(\eta_t)_{0\le t\le T}$ that have a quasi-continuous version and satisfy $\|\eta\|_{p^*}^{p^*}<\infty$ and

$$\lim_{n\to\infty} \hat{\mathbb{E}}\bigg[\int_0^T |\eta_t|^{p^*} 1_{\{|\eta_t|\ge n\}} dt\bigg] = 0$$

(Hu, Wang, and Zheng, 2016, Theorem 4.7). Since $a, c^{i,j} \in M_G^p(0,T)$ and $b^i \in M_G^{2p}(0,T)$ for all $i,j$, we know that $X$ is progressively measurable and has a quasi-continuous version. Therefore, since

$$\hat{\mathbb{E}}\bigg[\int_0^T |X_t|^{p^*} 1_{\{|X_t|\ge n\}} dt\bigg] \le \tfrac{1}{n^{\tilde{p}-p^*}} \hat{\mathbb{E}}\bigg[\int_0^T |X_t|^{\tilde{p}} dt\bigg],$$

it is left to show that $\|X\|_{\tilde{p}}^{\tilde{p}}<\infty$ in order to deduce that $X \in M_G^{p^*}(0,T)$. We have

$$\begin{aligned}\hat{\mathbb{E}}\bigg[\int_0^T |X_t|^{\tilde{p}} dt\bigg] = \hat{\mathbb{E}}\bigg[\int_0^T &\exp\bigg(\tilde{p}\sum_{i=1}^d \int_0^t b_u^i dB_u^i - \tfrac{1}{2}\tilde{p}^2\tilde{q}\sum_{i,j=1}^d \int_0^t b_u^i b_u^j d\langle B^i, B^j\rangle_u\bigg) \\ &\times \exp\bigg(\tfrac{1}{2}\tilde{p}^2\tilde{q}\sum_{i,j=1}^d \int_0^t b_u^i b_u^j d\langle B^i, B^j\rangle_u\bigg) \\ &\times \exp\bigg(\tilde{p}\bigg(\int_0^t a_u du + \sum_{i,j=1}^d \int_0^t c_u^{i,j} d\langle B^i, B^j\rangle_u\bigg)\bigg)dt\bigg].\end{aligned}$$

By Hölder's inequality, we get

$$\hat{\mathbb{E}}\left[\int_0^T |X_t|^{\tilde{p}} dt\right] \leq \hat{\mathbb{E}}\left[\int_0^T \exp\left(\tilde{p}\tilde{q}\sum_{i=1}^d \int_0^t b_u^i dB_u^i - \tfrac{1}{2}(\tilde{p}\tilde{q})^2 \sum_{i,j=1}^d \int_0^t b_u^i b_u^j d\langle B^i, B^j\rangle_u\right) dt\right]^{\frac{1}{\tilde{q}}}$$
$$\times \hat{\mathbb{E}}\left[\int_0^T \exp\left(\tfrac{1}{2}(\tilde{p}\tilde{q})^2 \sum_{i,j=1}^d \int_0^t b_u^i b_u^j d\langle B^i, B^j\rangle_u\right) dt\right]^{\frac{1}{\tilde{q}}}$$
$$\times \hat{\mathbb{E}}\left[\int_0^T \exp\left(\tfrac{\tilde{p}\tilde{q}}{\tilde{q}-2}\left(\int_0^t a_u du + \sum_{i,j=1}^d \int_0^t c_u^{i,j} d\langle B^i, B^j\rangle_u\right)\right) dt\right]^{\frac{\tilde{q}-2}{\tilde{q}}}.$$

By assumption, we know that the second and the third term on the right-hand side are finite; hence, it is left to show that the first term is also finite. We can use the classical Fubini theorem and the last assumption to get

$$\mathbb{E}_P\left[\exp\left(\tfrac{1}{2}(\tilde{p}\tilde{q})^2 \sum_{i,j=1}^d \int_0^t b_u^i b_u^j d\langle B^i, B^j\rangle_u\right)\right] < \infty$$

for almost all $t$ for all $P \in \mathcal{P}$. Thus, we know that Novikov's condition is satisfied, which implies that

$$\mathbb{E}_P\left[\exp\left(\tilde{p}\tilde{q}\sum_{i=1}^d \int_0^t b_u^i dB_u^i - \tfrac{1}{2}(\tilde{p}\tilde{q})^2 \sum_{i,j=1}^d \int_0^t b_u^i b_u^j d\langle B^i, B^j\rangle_u\right)\right] = 1$$

for almost all $t$ for all $P \in \mathcal{P}$. Integrating and using Fubini's theorem once more, we obtain

$$\mathbb{E}_P\left[\int_0^T \exp\left(\tilde{p}\tilde{q}\sum_{i=1}^d \int_0^t b_u^i dB_u^i - \tfrac{1}{2}(\tilde{p}\tilde{q})^2 \sum_{i,j=1}^d \int_0^t b_u^i b_u^j d\langle B^i, B^j\rangle_u\right) dt\right] = T$$

for all $P \in \mathcal{P}$, which implies

$$\hat{\mathbb{E}}\left[\int_0^T \exp\left(\tilde{p}\tilde{q}\sum_{i=1}^d \int_0^t b_u^i dB_u^i - \tfrac{1}{2}(\tilde{p}\tilde{q})^2 \sum_{i,j=1}^d \int_0^t b_u^i b_u^j d\langle B^i, B^j\rangle_u\right) dt\right] < \infty.$$

We are left to show that $aX, (c^{i,j} + \frac{1}{2}b^i b^j)X \in M_G^1(0,T)$ and $b^i X \in M_G^2(0,T)$ for all $i,j$. By the argument from the first step, we need to show that $\|aX\|_q^q < \infty$ in order to deduce that $aX \in M_G^1(0,T)$. By Hölder's inequality, it holds

$$\hat{\mathbb{E}}\left[\int_0^T |a_t X_t|^q dt\right] \leq \hat{\mathbb{E}}\left[\int_0^T |a_t|^p dt\right]^{\frac{q}{p}} \hat{\mathbb{E}}\left[\int_0^T |X_t|^{\frac{1}{2}p^*} dt\right]^{\frac{p-q}{p}}.$$

The two terms on the right-hand side are finite, since $a \in M_G^p(0,T)$ and $X \in M_G^{p^*}(0,T)$.

Thus, we obtain $\|aX\|_q^q < \infty$. We can show that $(c^{i,j} + \frac{1}{2}b^i b^j)X \in M_G^1(0,T)$ and $b^i X \in M_G^2(0,T)$ for all $i, j$ in the same way. $\square$

If $a$, $b^i$, and $c^{i,j}$, for all $i, j$, satisfy the assumptions of Proposition 3.25, we get $X_t \in L_G^1(\Omega_t)$ for all $t$ by Itô's formula. If—in addition—it holds $a = 0 = c^{i,j} + \frac{1}{2}b^i b^j$ for all $i, j$, we even have $X_t \in L_G^2(\Omega_t)$ for all $t$.

# Chapter 4

# Pricing Interest Rate Derivatives

Starting from an arbitrage-free term structure, the present chapter deals with the pricing of interest rate derivatives in the presence of volatility uncertainty. As in Chapters 2 and 3, we represent the uncertainty about the volatility by the family of probability measures introduced in Section 1.2, where we assume that the state space for the volatility is such that the resulting $d$-dimensional $G$-Brownian motion is uncorrelated and its generator is non-degenerate. The framework for modeling the term structure of interest rates is the one of Chapter 3—i.e., we model the forward rate as a diffusion process driven by a $G$-Brownian motion—since this is a general framework that covers the model considered in Chapter 2 (as shown in Example 3.3.2). The remaining quantities on the bond market are defined in terms of the forward rate in accordance with the HJM methodology. We model the forward rate in such a way that it satisfies the drift condition from the previous chapter, ensuring the absence of arbitrage on the bond market. Additionally, we assume that the diffusion coefficient of the forward rate is deterministic, which results in analytical pricing formulas and corresponds to an HJM model in which the foward rate is normally distributed.

In the presence of volatility uncertainty, we obtain a sublinear pricing measure for additional contracts we add to the bond market. Within the framework described above, we consider additional contracts, which we want to price without admitting arbitrage. The pricing of contracts under volatility uncertainty is different from the classical approach, since the expectation—which corresponds to the pricing measure in the classical case without volatility uncertainty—is sublinear in this setting. In contrast to the classical case, we use the sublinear expectation to determine the price of a contract or its bounds; hence, we refer to it as the *risk-neutral sublinear expectation.* To show that this approach indeed yields arbitrage-free prices, we extend the notions of trading strategies and arbitrage, respectively, to the bond market extended by the additional contract. Then we show that the extended bond market is arbitrage-free, meaning that we can use this approach to price contracts as desired.

To simplify the pricing of cashflows, we introduce a counterpart of the forward measure, called *forward sublinear expectation*. The forward sublinear expectation is defined by a $G$-backward stochastic differential equation and corresponds to the expectation under the forward measure. The forward measure, invented by Geman (1989), is used for pricing contracts in classical models without volatility uncertainty (Brace and Musiela, 1994; Geman, El Karoui, and Rochet, 1995; Jamshidian, 1989). Similar to the forward measure, the forward sublinear expectation has the advantage that computing the sublinear expectation of a discounted payoff reduces to computing the forward sublinear expectation of the payoff, discounted with the bond price. Under the forward sublinear expectation, we obtain several results needed for pricing cashflows of typical fixed income products. Moreover, we obtain a robust version of the expectations hypothesis under the forward sublinear expectation and show how to price bond options. In many typical cases, prices of bond options are characterized by pricing formulas of models without volatility uncertainty.

In addition, we develop pricing methods for contracts consisting of several cashflows. In traditional models without volatility uncertainty, there is no distinction between pricing single cashflows and pricing a stream of cashflows, since the pricing measure is linear. However, when there is uncertainty about the volatility, the nonlinearity of the pricing measure implies that we cannot generally price a stream of cashflows by pricing each cashflow separately. Therefore, we provide different schemes for pricing a family of cashflows. If the cashflows of a contract are sufficiently simple, we can price the contract as in the classical case. In general, we use a backward induction procedure to find the price of a contract, which we can use to price—for example—a stream of bond options. In typical situations, the price of the latter is determined by the pricing formulas from models without volatility uncertainty.

With the tools mentioned above, we derive robust pricing formulas for all major interest rate derivatives. We consider typical linear contracts, such as fixed coupon bonds, floating rate notes, and interest rate swaps, and nonlinear contracts, such as swaptions, caps and floors, and in-arrears contracts. Due to the linearity of the payoff, we obtain a single price for fixed coupon bonds, floating rate notes, and interest rate swaps; the pricing formula is the same as the one from classical models without volatility uncertainty. Due to the nonlinearity of the payoff, we obtain a range of prices for swaptions, caps and floors, and in-arrears contracts; the range is bounded from above, respectively below, by the price from classical models with the highest, respectively lowest, possible volatility. Therefore, the pricing of common interest rate derivatives under volatility uncertainty reduces to computing prices in models without volatility uncertainty.

The pricing formulas show that volatility uncertainty naturally leads to unspanned stochastic volatility. According to empirical evidence, volatility risk in fixed income markets cannot be hedged by trading solely bonds, which is termed *unspanned stochastic volatility*. Collin-Dufresne and Goldstein (2002) empirically showed that interest rate derivatives exposed to volatility risk are driven by factors that do not affect bond prices. These findings contradict traditional term structure models. The empirical investigation has led to the development of models that are able to exhibit unspanned stochastic volatility (Casassus, Collin-Dufresne, and Goldstein, 2005; Filipović, Larsson, and Statti, 2019; Filipović, Larsson, and Trolle, 2017). Since the presence of volatility uncertainty naturally leads to market incompleteness, the pricing formulas mentioned above show that it is no longer possible to hedge volatility risk in fixed income markets with a portfolio consisting solely of bonds when there is uncertainty about the volatility. Moreover, the pricing formulas are in line with the empirical findings of Collin-Dufresne and Goldstein (2002).

The remainder of this chapter is organized as follows. Section 4.1 introduces the overall setting of the model: an arbitrage-free bond market under volatility uncertainty. In Section 4.2, we show that we can use the risk-neutral sublinear expectation as a pricing measure for additional contracts. In Section 4.3, we define the forward sublinear expectation and derive related results for the pricing of single cashflows. Section 4.4 provides schemes for pricing contracts consisting of a stream of cashflows. In Section 4.5, we derive pricing formulas for the most common interest rate derivatives. Section 4.6 discusses market incompleteness and shows that volatility uncertainty leads to unspanned stochastic volatility. Some minor technical parts are deferred to Section 4.7.

## 4.1 Arbitrage-Free Bond Market

We represent the uncertainty about the volatility by a familiy of probability measures leading to a $d$-dimensional $G$-Brownian motion without correlation and with a non-degenerate generator. As in Chapter 3, we consider the set of beliefs $\mathcal{P}$ from Section 1.2, in which each measure corresponds to a specific belief about the volatility. Thus, we obtain a $d$-dimensional $G$-Brownian motion $B = (B_t^1, ..., B_t^d)_{t \geq 0}$, since the sublinear expectation

$$\hat{\mathbb{E}}[\cdot] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[\cdot]$$

corresponds to the $G$-expectation. In addition, we assume that the state space of the uncertain volatility is given by

$$\Sigma = \left\{ \sigma \in \mathbb{R}^{d \times d} \;\middle|\; \sigma = \operatorname{diag}(\sigma_1, ..., \sigma_d),\ \sigma_i \in [\underline{\sigma}_i, \overline{\sigma}_i] \text{ for all } i = 1, ..., d \right\},$$

where $\overline{\sigma}_i \geq \underline{\sigma}_i > 0$ for all $i$; that means, we consider all scenarios in which there is no correlation between the risk factors and the volatility is bounded by two extremes: the matrices $\overline{\sigma} = \operatorname{diag}(\overline{\sigma}_1, ..., \overline{\sigma}_d)$ and $\underline{\sigma} = \operatorname{diag}(\underline{\sigma}_1, ..., \underline{\sigma}_d)$. For a vector $(x_1, ..., x_d)$, the notation diag$(x_1, ..., x_d)$ refers to a diagonal matrix with entries $x_1, ..., x_d$ on the diagonal. The additional assumption on the state space $\Sigma$ allows us to derive explicit pricing formulas for typical interest rate derivatives. It should also be noted that the assumption implies that the generator $G$ is non-degenerate, which we assume in Chapter 3.

As in the previous chapter, we model the forward rate as a diffusion process in the spirit of the HJM methodology. We denote by $f_t(T)$ the forward rate with maturity $T$ at time $t$ for $t \leq T \leq \tau$, where $\tau < \infty$ is a fixed terminal time. We assume that the forward rate process $f(T) = (f_t(T))_{0 \leq t \leq T}$, for all $T \leq \tau$, evolves according to the dynamics

$$f_t(T) = f_0(T) + \int_0^t \alpha_u(T) du + \sum_{i=1}^d \int_0^t \beta_u^i(T) dB_u^i + \sum_{i=1}^d \int_0^t \gamma_u^i(T) d\langle B^i \rangle_u$$

for some initial integrable forward curve $f_0 : [0, \tau] \to \mathbb{R}$ and sufficiently regular processes $\alpha(T) = (\alpha_t(T))_{0 \leq t \leq \tau}$, $\beta^i(T) = (\beta_t^i(T))_{0 \leq t \leq \tau}$, and $\gamma^i(T) = (\gamma_t^i(T))_{0 \leq t \leq \tau}$ to be specified. As described in the previous chapter, the difference compared to the classical HJM model without volatility uncertainty is that there are additional drift terms depending on the quadratic variation processes of the $G$-Brownian motion. We need the additional drift terms in order to obtain an arbitrage-free model, which can be inferred from the drift condition in Chapter 3. Since the uncertainty about the volatility implies that the quadratic variation processes are uncertain and cannot be included in the first drift term, we add additional drift terms to the dynamics of the forward rate (see Section 3.1).

The forward rate determines the remaining quantities on the related bond market. The bond market consists of zero-coupon bonds for all maturities in the time horizon and the money-market account. The zero-coupon bonds, denoted by $P(T) = (P_t(T))_{0 \leq t \leq T}$ for $T \leq \tau$, are defined by

$$P_t(T) := \exp\left( - \int_t^T f_t(s) ds \right)$$

and the money-market account, denoted by $M = (M_t)_{0 \leq t \leq \tau}$, is given by

$$M_t := \exp\left( \int_0^t r_s ds \right),$$

where $r = (r_t)_{0 \leq t \leq \tau}$ denotes the short rate process, defined by $r_t := f_t(t)$. We use the money-market account as a numéraire—that is, we focus on the discounted bonds, which are denoted by $\tilde{P}(T) = (\tilde{P}_t(T))_{0 \leq t \leq T}$ for $T \leq \tau$ and given by

$$\tilde{P}_t(T) := M_t^{-1} P_t(T).$$

We model the forward rate in such a way that the related bond market is arbitrage-free. That means, we assume that the forward rate satisfies the drift condition from Theorem 3.6, which implies the absence of arbitrage. In particular, we directly model the forward rate in a risk-neutral way in order to avoid technical difficulties due to a migration to a risk-neutral framework. More specifically, for all $T$, we assume that the drift terms $\alpha(T)$ and $\gamma^i(T)$, for all $i$, are defined by

$$\begin{aligned}\alpha_t(T) &:= 0,\\ \gamma_t^i(T) &:= \beta_t^i(T)b_t^i(T),\end{aligned}$$

respectively, where (as in Chapter 3) the process $b^i(T) = (b_t^i(T))_{0\leq t\leq\tau}$ is defined by

$$b_t^i(T) := \int_t^T \beta_t^i(s)ds.$$

Under suitable regularity assumptions on $T \mapsto \beta^i(T)$, by Theorem 3.6, we then know that the discounted bonds are symmetric $G$-martingales under $\hat{\mathbb{E}}$, which implies that the bond market is arbitrage-free. As mentioned above, this shows that we need the additional drift terms in the forward rate dynamics to obtain an arbitrage-free model.

In order to obtain a sufficient degree of regularity and to derive pricing formulas for derivative contracts, we use a deterministic diffusion coefficient. We assume that $\beta^i$, for all $i$, is a continuous function mapping from $[0,\tau]\times[0,\tau]$ into $\mathbb{R}$. Then for each $T$, $\beta^i(T)$ and $b^i(T)$, for all $i$, are bounded processes in the space of admissible stochastic processes $M_G^p(0,\tau)$ for all $p<\infty$. Therefore, the assumption on the diffusion coefficient ensures that the forward rate is sufficiently regular to apply the results from the previous chapter. In addition, it enables us to obtain specific pricing formulas for common interest rate derivatives. This is similar to the classical case without volatility uncertainty, in which it is possible to obtain analytical pricing formulas by assuming that the diffusion coefficient is deterministic. So the present model corresponds to an HJM model with a normally distributed forward rate.

## 4.2 Risk-Neutral Valuation

Now we extend the bond market to an additional contract, for which we want to find a price. A typical contract in fixed income markets consists of a stream of cashflows; so we consider a contract, denoted by $X$, which has a payoff of $\xi_i$ at each time $\tau_i$ for all $i=0,1,...,N$, where $0<\tau_0<\tau_1<...<\tau_N=\tau$ is the tenor structure. The price at time $t$ of such a contract is denoted by $X_t$ for all $t\leq\tau$. As for the bonds, we consider the discounted payoff $\tilde{X}$, defined by

$$\tilde{X} := \sum_{i=0}^{N} M_{\tau_i}^{-1} \xi_i,$$

and the discounted price $\tilde{X}_t$ for $t \leq \tau$, which is defined by

$$\tilde{X}_t := M_t^{-1} X_t.$$

We assume that $M_{\tau_i}^{-1} \xi_i \in L_G^2(\Omega_{\tau_i})$ for all $i = 0, 1, ..., N$ for $X$ to be regular enough.

The pricing of contracts in the presence of volatility uncertainty differs from the traditional approach. Classical arbitrage pricing theory suggests that prices of contracts are determined by computing the expected discounted payoff under the risk-neutral measure. In the presence of volatility uncertainty, we call $\hat{\mathbb{E}}$ the *risk-neutral sublinear expectation*, corresponding to the expectation under the risk-neutral measure in the classical case, since the discounted bonds are symmetric $G$-martingales under $\hat{\mathbb{E}}$. Compared to the classical case, the important difference in the case of volatility uncertainty is that the risk-neutral sublinear expectation is nonlinear. In particular, it holds

$$\hat{\mathbb{E}}[\tilde{X}] \geq -\hat{\mathbb{E}}[-\tilde{X}], \tag{4.1}$$

that is, the upper expectation does not necessarily coincide with the lower expectation. Thus, we distinguish between symmetric and asymmetric contracts; we consider two contracts: a contract $X^S$, which has a symmetric payoff, and a contract $X^A$, which has an asymmetric payoff. Strictly speaking, this means that $\tilde{X}^S$ satisfies (4.1) with equality and for $\tilde{X}^A$ the inequality (4.1) is strict. Of course, the discounted payoffs $\tilde{X}^S$ and $\tilde{X}^A$ are defined as above by considering different payoffs $\xi_i^S$ and $\xi_i^A$ for all $i$, respectively. The related prices are denoted by $X_t^S$ and $\tilde{X}_t^S$ and $X_t^A$ and $\tilde{X}_t^A$ for all $t$, respectively.

We determine the prices of contracts by using the risk-neutral sublinear expectation to either obtain the price of a contract or the upper and the lower bound for the price. In the case of a symmetric payoff, we proceed as in the classical case without volatility uncertainty and choose the expected discounted payoff as the price for the contract. In the case of an asymmetric payoff, we use the upper and the lower expectation as bounds for the price, which is a typical approach in the literature on model uncertainty. Hence, we assume that

$$\tilde{X}_t^S = \hat{\mathbb{E}}_t[\tilde{X}^S]$$

for all $t$, where we recall that $\hat{\mathbb{E}}_t$ denotes the conditional $G$-expectation, and

$$\hat{\mathbb{E}}[\tilde{X}^A] > \tilde{X}_0^A > -\hat{\mathbb{E}}[-\tilde{X}^A].$$

Since $X^S$ has a symmetric payoff, by the martingale representation theorem for symmetric $G$-martingales (Song, 2011, Theorem 4.8), there exists a process $H = (H_t^1, ..., H_t^d)_{0\leq t\leq\tau}$ in $M_G^2(0,\tau;\mathbb{R}^d)$ such that for all $t$,

$$\tilde{X}_t^S = \tilde{X}_0^S + \sum_{i=1}^d \int_0^t H_u^i dB_u^i.$$

The latter ensures that the portfolio value (defined below) is well-posed. The reason why we only impose assumptions on the price of the asymmetric contract at time 0 is described below.

In order to show that this pricing procedure yields no-arbitrage prices, we extend the notions of trading strategies and arbitrage, respectively, to the extended bond market. As in Chapters 2 and 3, we allow the agents in the market to trade a finite number of bonds. The symmetric contract can be traded dynamically, but we only allow static trading strategies for the asymmetric contract. Therefore, we do not impose assumptions on $\tilde{X}_t^A$ for $t > 0$. The assumption that the asymmetric contract can only be traded statically might seem restrictive. This is a common assumption in the literature on robust finance, since it is important for excluding arbitrage. In this case, the assumption is also reasonable, since most contracts in fixed income markets are traded over-the-counter.

**Definition 4.1.** *An admissible market strategy is a quadruple $(\pi, \pi^S, \pi^A, T)$ consisting of a bounded process $\pi = (\pi_t^1, ..., \pi_t^n)_{0\leq t\leq\tau}$ in $M_G^2(0,\tau;\mathbb{R}^n)$, a bounded process $\pi^S = (\pi_t^S)_{0\leq t\leq\tau}$ in $M_G^2(0,\tau)$, a constant $\pi^A \in \mathbb{R}$, and a vector $T = (T_1, ..., T_n) \in [0,\tau]^n$ for some $n \in \mathbb{N}$. The corresponding portfolio value at terminal time is given by*

$$\tilde{v}_\tau(\pi, \pi^S, \pi^A, T) := \sum_{i=1}^n \int_0^{T_i} \pi_t^i d\tilde{P}_t(T_i) + \int_0^\tau \pi_t^S d\tilde{X}_t^S + \pi^A(\tilde{X}^A - \tilde{X}_0^A). \tag{4.2}$$

The three terms on the right-hand side of (4.2) correspond to the gains from trading a finite number of bonds, the symmetric contract, and the asymmetric contract, respectively. The assumptions on the processes ensure that the integrals in (4.2) are well-defined. Similar to the previous chapters, we use the quasi-sure definition of arbitrage, which is commonly used in the literature on model uncertainty (Biagini, Bouchard, Kardaras, and Nutz, 2017; Bouchard and Nutz, 2015).

**Definition 4.2.** *An admissible market strategy $(\pi, \pi^S, \pi^A, T)$ is an arbitrage strategy if*

$$\tilde{v}_\tau(\pi, \pi^S, \pi^A, T) \geq 0 \quad \textit{quasi-surely},$$
$$P\big(\tilde{v}_\tau(\pi, \pi^S, \pi^A, T) > 0\big) > 0 \quad \textit{for at least one } P \in \mathcal{P}.$$

*We say that the extended bond market is arbitrage-free if there is no arbitrage strategy.*

The following proposition shows that we can use the risk-neutral sublinear expectation as a pricing measure as described above, since the extended bond market is arbitrage-free under the assumptions stated in this section. As a consequence, we can reduce the problem of pricing a contract to evaluating the upper and the lower expectation of the corresponding discounted payoff. The proof—apart from the asymmetric contract—is similar to the proof of Proposition 2.5, since all other quantities on the market are symmetric $G$-martingales under $\hat{\mathbb{E}}$.

**Proposition 4.3.** *The extended bond market is arbitrage-free.*

*Proof.* We assume that there exists an arbitrage strategy $(\pi, \pi^S, \pi^A, T)$ and show that this yields a contradiction. We only examine the case in which $X^A$ is traded, i.e., it holds $\pi^A \neq 0$; if $\pi^A = 0$, the proof is similar to the proof of Proposition 2.5. By the definition of arbitrage, it holds $\tilde{v}_\tau(\pi, \pi^S, \pi^A, T) \geq 0$. Then the monotonicity of $\hat{\mathbb{E}}$ implies

$$\hat{\mathbb{E}}\bigg[\sum_{i=1}^n \int_0^{T_i} \pi_t^i d\tilde{P}_t(T_i) + \int_0^\tau \pi_t^S d\tilde{X}_t^S\bigg] \geq \hat{\mathbb{E}}[-\pi^A(\tilde{X}^A - \tilde{X}_0^A)].$$

Due to the sublinearity of $\hat{\mathbb{E}}$ and the fact that the discounted bonds are symmetric $G$-martingales under $\hat{\mathbb{E}}$ and $X^S$ has a symmetric payoff, we have

$$\hat{\mathbb{E}}\bigg[\sum_{i=1}^n \int_0^{T_i} \pi_t^i d\tilde{P}_t(T_i) + \int_0^\tau \pi_t^S d\tilde{X}_t^S\bigg] \leq 0$$

by the martingale representation theorem for symmetric $G$-martingales (Song, 2011, Theorem 4.8). Furthermore, if we use the properties of the $G$-expectation and the assumption on $\tilde{X}_0^A$, we get

$$\hat{\mathbb{E}}[-\pi^A(\tilde{X}^A - \tilde{X}_0^A)] = (\pi^A)^+(\hat{\mathbb{E}}[-\tilde{X}^A] + \tilde{X}_0^A) + (\pi^A)^-(\hat{\mathbb{E}}[\tilde{X}^A] - \tilde{X}_0^A) > 0.$$

Combining the previous steps, we obtain a contradiction. □

## 4.3 Pricing Single Cashflows

In the classical case without volatility uncertainty, discounted cashflows are priced under the forward measure. Evaluating the expectation of a discounted cashflow related to an interest rate derivative can be very elaborate; this is due to the fact that the discount factor—in addition to the cashflows—is stochastic. The common way to avoid this issue is the forward measure approach. The forward measure, which was introduced by Geman (1989), is equivalent to the pricing measure and defined by choosing a particular density process. The density process is defined in such a way that the expectation of a discounted cashflow under the risk-neutral measure can be rewritten as the expectation of the cashflow under the forward measure, discounted by a zero-coupon bond. Thus, by changing the measure, we can replace the stochastic discount factor by the current bond price (which is already determined by the model).

In the presence of volatility uncertainty, we define a counterpart of the forward measure, termed *forward sublinear expectation*, to simplify the pricing of discounted cashflows. In contrast to the forward measure approach, we define the forward sublinear expectation by a $G$-backward stochastic differential equation (similar to the definition of the sublinear expectations considered in Section 2.4 and in the proof of Theorem 3.6, respectively).

**Definition 4.4.** *For $\xi \in L_G^p(\Omega_T)$ with $p > 1$ and $T \leq \tau$, we define the $T$-forward sublinear expectation $\hat{\mathbb{E}}^T$ by $\hat{\mathbb{E}}_t^T[\xi] := Y_t^{T,\xi}$, where $Y^{T,\xi} = (Y_t^{T,\xi})_{0 \leq t \leq T}$ solves the $G$-backward stochastic differential equation*

$$Y_t^{T,\xi} = \xi - \sum_{i=1}^{d} \int_t^T b_u^i(T) Z_u^i d\langle B^i \rangle_u - \sum_{i=1}^{d} \int_t^T Z_u^i dB_u^i - (K_T - K_t).$$

By Theorem 5.1 of Hu, Ji, Peng, and Song (2014), the forward sublinear expectation is a time consistent sublinear expectation. Again, we refer to the paper of Hu, Ji, Peng, and Song (2014) for further details related to $G$-backward stochastic differential equations.

The forward sublinear expectation corresponds to the expectation under the forward measure. This can be deduced from the explicit solution to the $G$-backward stochastic differential equation defining the forward sublinear expectation. For $T \leq \tau$, we define the process $X^T = (X_t^T)_{0 \leq t \leq T}$ by

$$X_t^T := \tfrac{\tilde{P}_t(T)}{P_0(T)}.$$

The process $X^T$ is the density used to define the forward measure. As in Proposition 3.8, one can verify that $X^T$ satisfies the $G$-stochastic differential equation

$$X_t^T = 1 - \sum_{i=1}^{d} \int_0^t b_u^i(T) X_u^T dB_u^i.$$

By Theorem 3.2 of Hu, Ji, Peng, and Song (2014), the process $Y^{T,\xi}$ is given by

$$Y_t^{T,\xi} = (X_t^T)^{-1} \hat{\mathbb{E}}_t[X_T^T \xi].$$

Thus, we basically arrive at the same expression as in the classical definition of the forward measure.

We obtain the following preliminary results related to the forward sublinear expectation, which simplify the pricing of discounted cashflows. Similar to the classical case, we find that the valuation of a discounted cashflow reduces to determining the forward sublinear expectation of the cashflow, which is then discounted with the bond price. Furthermore, there is a relation between forward sublinear expectations with different maturities, and the forward rate process $f(T)$ and the forward price process $X^{T,\tilde{T}} = (X_t^{T,\tilde{T}})_{0\leq t\leq T\wedge\tilde{T}}$, defined by

$$X_t^{T,\tilde{T}} := \tfrac{P_t(\tilde{T})}{P_t(T)},$$

for $T, \tilde{T} \leq \tau$ are symmetric $G$-martingales under the $T$-forward sublinear expectation.

**Proposition 4.5.** *Let $\xi \in L_G^p(\Omega_T)$ with $p > 1$ and $t \leq T, \tilde{T} \leq \tau$. Then we have the following properties.*

*(i) It holds*

$$M_t\hat{\mathbb{E}}_t[M_T^{-1}\xi] = P_t(T)\hat{\mathbb{E}}_t^T[\xi].$$

*(ii) For $T \leq \tilde{T}$, it holds*

$$P_t(\tilde{T})\hat{\mathbb{E}}_t^{\tilde{T}}[\xi] = P_t(T)\hat{\mathbb{E}}_t^T[P_T(\tilde{T})\xi].$$

*(iii) The forward rate process $f(T)$ is a symmetric $G$-martingale under $\hat{\mathbb{E}}^T$.*

*(iv) The forward price process $X^{T,\tilde{T}}$ satisfies $X_t^{T,\tilde{T}} \in L_G^p(\Omega_t)$ for all $p < \infty$ and*

$$X_t^{T,\tilde{T}} = X_0^{T,\tilde{T}} - \sum_{i=1}^d \int_0^t \sigma_u^i(T,\tilde{T})X_u^{T,\tilde{T}}dB_u^i - \sum_{i=1}^d \int_0^t \sigma_u^i(T,\tilde{T})X_u^{T,\tilde{T}}b_u^i(T)d\langle B^i\rangle_u,$$

*where $\sigma^i(T,\tilde{T}) = (\sigma_t^i(T,\tilde{T}))_{0\leq t\leq T\wedge\tilde{T}}$, for all $i$, is defined by*

$$\sigma_t^i(T,\tilde{T}) := b_t^i(\tilde{T}) - b_t^i(T),$$

*and it is a symmetric $G$-martingale under $\hat{\mathbb{E}}^T$.*

*Proof.* Part $(i)$ follows by a simple calculation; we have

$$M_t\hat{\mathbb{E}}_t[M_T^{-1}\xi] = P_t(T)M_t\tfrac{P_0(T)}{P_t(T)}\hat{\mathbb{E}}_t[M_T^{-1}\tfrac{P_T(T)}{P_0(T)}\xi] = P_t(T)(X_t^T)^{-1}\hat{\mathbb{E}}_t[X_T^T\xi] = P_t(T)\hat{\mathbb{E}}_t^T[\xi].$$

To show part $(ii)$, we use some properties of $G$-backward stochastic differential equations. By Definition 4.4, we have $\hat{\mathbb{E}}_t^{\tilde{T}}[\xi] = Y_t^{\tilde{T},\xi}$, where $Y^{\tilde{T},\xi}$ solves

$$Y_t^{\tilde{T},\xi} = \xi - \sum_{i=1}^{d} \int_t^{\tilde{T}} b_u^i(\tilde{T}) Z_u^i d\langle B^i \rangle_u - \sum_{i=1}^{d} \int_t^{\tilde{T}} Z_u^i dB_u^i - (K_{\tilde{T}} - K_t).$$

Since $\xi \in L_G^p(\Omega_T)$, the process $Y^{\tilde{T},\xi}$ also solves the $G$-backward stochastic differential equation

$$Y_t^{\tilde{T},\xi} = \xi - \sum_{i=1}^{d} \int_t^{T} b_u^i(\tilde{T}) Z_u^i d\langle B^i \rangle_u - \sum_{i=1}^{d} \int_t^{T} Z_u^i dB_u^i - (K_T - K_t).$$

By Theorem 3.2 of Hu, Ji, Peng, and Song (2014), the solution to the latter is given by

$$Y_t^{\tilde{T},\xi} = (X_t^{\tilde{T}})^{-1} \hat{\mathbb{E}}_t[X_T^{\tilde{T}} \xi].$$

Moreover, for each $t \leq T$, we have $X_t^{\tilde{T}} = X_t^{T,\tilde{T}} X_0^{\tilde{T},T} X_t^T$. Hence, we obtain

$$\hat{\mathbb{E}}_t^{\tilde{T}}[\xi] = X_t^{\tilde{T},T} X_0^{T,\tilde{T}} (X_t^T)^{-1} \hat{\mathbb{E}}_t[X_T^{T,\tilde{T}} X_0^{\tilde{T},T} X_T^T \xi] = X_t^{\tilde{T},T} \hat{\mathbb{E}}_t^T[X_T^{T,\tilde{T}} \xi],$$

which proves part $(ii)$.

For part $(iii)$, we use the Girsanov transformation for $G$-Brownian motion from Hu, Ji, Peng, and Song (2014). We define the process $B^T = (B_t^{1,T}, ..., B_t^{d,T})_{0 \leq t \leq T}$ by

$$B_t^{i,T} := B_t^i + \int_0^t b_u^i(T) d\langle B^i \rangle_u.$$

Then $B^T$ is a $G$-Brownian motion under $\hat{\mathbb{E}}^T$ (Hu, Ji, Peng, and Song, 2014, Theorems 5.2, 5.4). Since the dynamics of the forward rate are given by

$$f_t(T) = f_0(T) + \sum_{i=1}^{d} \int_0^t \beta_u^i(T) dB_u^i + \sum_{i=1}^{d} \int_0^t \beta_u^i(T) b_t^i(T) d\langle B^i \rangle_u,$$

the forward rate is a symmetric $G$-martingale under $\hat{\mathbb{E}}^T$.

To obtain part $(iv)$, we first show that $X_t^{T,\tilde{T}} \in L_G^p(\Omega_t)$ for all $p < \infty$ by using the representation of the space $L_G^p(\Omega_t)$ from Denis, Hu, and Peng (2011) and a proof similar to the proof of Proposition 5.10 from Osuka (2013). The space $L_G^p(\Omega_t)$ consists of all Borel measurable random variables $X$ that have a quasi-continuous version and satisfy $\lim_{n \to \infty} \hat{\mathbb{E}}[|X|^p 1_{\{|X| > n\}}] = 0$ (Peng, 2019, Proposition 6.3.2). As in Lemma 3.7, one can show that

$$X_t^{T,\tilde{T}} = X_0^{T,\tilde{T}} \exp\Big(-\sum_{i=1}^d \int_0^t \sigma_u^i(T,\tilde{T})dB_u^i - \sum_{i=1}^d \int_0^t \big(\tfrac{1}{2}\sigma_u^i(T,\tilde{T})^2 + \sigma_u^i(T,\tilde{T})b_u^i(T)\big)d\langle B^i\rangle_u\Big).$$

Since $\sigma^i(T,\tilde{T})$ and $b^i(T)$, for all $i$, are bounded processes in $M_G^p(0,\tau)$ for all $p<\infty$, we already know that $X_t^{T,\tilde{T}}$ is measurable and has a quasi-continuous version. Now we show that $\hat{\mathbb{E}}[|X_t^{T,\tilde{T}}|^{\tilde{p}}] < \infty$ for $\tilde{p} > p$, which implies $\lim_{n\to\infty}\hat{\mathbb{E}}[|X|^p 1_{\{|X|>n\}}] = 0$. By Hölder's inequality, for $\tilde{p} > p$ and $\tilde{q} > 1$, we have

$$\begin{aligned}\hat{\mathbb{E}}[|X_t^{T,\tilde{T}}|^{\tilde{p}}] \leq X_0^{T,\tilde{T}}\hat{\mathbb{E}}\Big[\exp\Big(-\tilde{p}\tilde{q}\sum_{i=1}^d \int_0^t \sigma_u^i(T,\tilde{T})dB_u^i - \tfrac{1}{2}(\tilde{p}\tilde{q})^2\sum_{i=1}^d \int_0^t \sigma_u^i(T,\tilde{T})^2 d\langle B^i\rangle_u\Big)\Big]^{\frac{1}{\tilde{q}}} \\ \times \hat{\mathbb{E}}\Big[\exp\Big(\tfrac{\tilde{p}\tilde{q}}{\tilde{q}-1}\sum_{i=1}^d \int_0^t \big(\tfrac{1}{2}(\tilde{p}\tilde{q}-1)\sigma_u^i(T,\tilde{T})^2 - \sigma_u^i(T,\tilde{T})b_u^i(T)\big)d\langle B^i\rangle_u\Big)\Big]^{\frac{\tilde{q}-1}{\tilde{q}}}.\end{aligned}$$

The two terms on the right-hand side are finite. The second term is finite since $\sigma^i(T,\tilde{T})$ and $b^i(T)$ are bounded for all $i$. By the same argument, we have

$$\hat{\mathbb{E}}\Big[\exp\Big(\tfrac{1}{2}(\tilde{p}\tilde{q})^2\sum_{i=1}^d \int_0^t \sigma_u^i(T,\tilde{T})^2 d\langle B^i\rangle_u\Big)\Big] < \infty.$$

Then we can use Novikov's condition to show that the first term is finite, since the exponential inside the sublinear expectation is a martingale under each $P \in \mathcal{P}$.

Using Itô's formula for $G$-Brownian motion from Li and Peng (2011) and the Girsanov transformation of Hu, Ji, Peng, and Song (2014) completes the proof. We have

$$X_t^{T,\tilde{T}} = X_0^{T,\tilde{T}} - \sum_{i=1}^d \int_0^t \sigma_u^i(T,\tilde{T})X_u^{T,\tilde{T}}dB_u^i - \sum_{i=1}^d \int_0^t \sigma_u^i(T,\tilde{T})X_u^{T,\tilde{T}}b_u^i(T)d\langle B^i\rangle_u$$

by Itô's formula (Li and Peng, 2011, Theorem 5.4). Moreover, since $\sigma^i(T,\tilde{T})$ and $b^i(T)$, for all $i$, are bounded processes in $M_G^p(0,\tau)$ for all $p<\infty$, one can then show that $X^{T,\tilde{T}}$ belongs to $M_G^p(0,\tau)$ for all $p<\infty$ (see Proposition 3.25). As in the proof of part (*iii*), the Girsanov transformation for $G$-Brownian motion then implies that $X^{T,\tilde{T}}$ is a symmetric $G$-martingale under $\hat{\mathbb{E}}^T$. □

Due to Proposition 4.5 (*iii*), we obtain a robust version of the expectations hypothesis. The traditional expectations hypothesis states that forward rates reflect the expectation of future short rates. In the classical case without volatility uncertainty, we know that the forward rate is a martingale under the forward measure; therefore, the expectations hypothesis holds true under the forward measure. In our case, we obtain a much stronger version—called *robust expectations hypothesis*. This is because the forward rate is a symmetric $G$-martingale under the forward sublinear expectation. Thus, the forward rate reflects the upper expectation of the short rate and the lower expectation of the short rate. In particular, it implies that the forward rate reflects the expectation of the short rate in each possible scenario for the volatility.

**Corollary 4.6.** *The forward rate satisfies the robust expectations hypothesis under the forward sublinear expectation—that is, for $t \leq T \leq \tau$, it holds*

$$\hat{\mathbb{E}}_t^T[r_T] = f_t(T) = -\hat{\mathbb{E}}_t^T[-r_T].$$

For convex bond options, the upper, respectively lower, bound for the price is given by the price in the corresponding HJM model without volatility uncertainty with the highest, respectively lowest, possible volatility. If we consider a bond option, the payoff is a function depending on a selection of bond prices for different maturities. We consider the more general case when the payoff is a function depending on a selection of forward prices, since we can express every bond option as an option on forward prices. If the payoff function is convex and satisfies a suitable growth condition, we can use the nonlinear Feynman-Kac formula from Hu, Ji, Peng, and Song (2014) to show that the range of prices is bounded from above, respectively below, by the price from the classical model when the dynamics of the forward price are driven by a standard Brownian motion with constant volatility $\overline{\sigma}$, respectively $\underline{\sigma}$.

**Proposition 4.7.** *For $n \in \mathbb{N}$, let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that*

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \tag{4.3}$$

*for a positive integer $m$ and a constant $C > 0$ and let $0 < t_1 < ... < t_n \leq \tau$. Then*

$$\hat{\mathbb{E}}_t^T[\varphi(X_{t_1}^{T,t_1}, ..., X_{t_1}^{T,t_n})] = u_{\overline{\sigma}}(t, X_t^{T,t_1}, ..., X_t^{T,t_n}),$$
$$-\hat{\mathbb{E}}_t^T[-\varphi(X_{t_1}^{T,t_1}, ..., X_{t_1}^{T,t_n})] = u_{\underline{\sigma}}(t, X_t^{T,t_1}, ..., X_t^{T,t_n})$$

*for $t \leq t_1 \leq T \leq \tau$, where the function $u_\sigma : [0, t_1] \times \mathbb{R}^n \to \mathbb{R}$, for $\sigma \in \Sigma$, is defined by*

$$u_\sigma(t, x_1, ..., x_n) := \mathbb{E}_{P_0}[\varphi(X_{t_1}^1, ..., X_{t_1}^n)]$$

*and the process $X^i = (X_s^i)_{t \leq s \leq t_1}$, for all $i = 1, ..., n$, is given by*

$$X_s^i = x_i - \sum_{j=1}^d \int_t^s \sigma_u^j(T, t_i) X_u^i \sigma_j dB_u^j.$$

*Proof.* First, we characterize the forward sublinear expectation in the first equation as the solution to a nonlinear partial differential equation by using the nonlinear Feynman-Kac formula of Hu, Ji, Peng, and Song (2014). With Proposition 4.5 ($iv$) and inequality (4.3), one can show that $\xi := \varphi(X_{t_1}^{T,t_1}, ..., X_{t_1}^{T,t_n})$ belongs to $L_G^p(\Omega_{t_1}) \subset L_G^p(\Omega_T)$ with $p > 1$. By Definition 4.4, we have $\hat{\mathbb{E}}_t^T[\xi] = Y_t^{T,\xi}$, where $Y^{T,\xi} = (Y_t^{T,\xi})_{0 \leq t \leq T}$ solves the $G$-backward stochastic differential equation

$$Y_t^{T,\xi} = \varphi(X_{t_1}^{T,t_1}, ..., X_{t_1}^{T,t_n}) - \sum_{i=1}^{d} \int_t^T b_u^i(T) Z_u^i d\langle B^i\rangle_u - \sum_{i=1}^{d} \int_t^T Z_u^i dB_u^i - (K_T - K_t).$$

Since $\xi \in L_G^p(\Omega_{t_1})$, the process $Y^{T,\xi}$ also solves the $G$-backward stochastic differential equation

$$Y_t^{T,\xi} = \varphi(X_{t_1}^{T,t_1}, ..., X_{t_1}^{T,t_n}) - \sum_{i=1}^{d} \int_t^{t_1} b_u^i(T) Z_u^i d\langle B^i\rangle_u - \sum_{i=1}^{d} \int_t^{t_1} Z_u^i dB_u^i - (K_{t_1} - K_t),$$

where $\varphi$ satisfies (4.3). From Proposition 4.5 ($iv$), we deduce the dynamics and the regularity of $X^{T,t_i}$ for all $i = 1, ..., n$. Then, by Theorems 4.4 and 4.5 of Hu, Ji, Peng, and Song (2014), we have $Y_t^{T,\xi} = u(t, X_t^{T,t_1}, ..., X_t^{T,t_n})$, where $u : [0, t_1] \times \mathbb{R}^n \to \mathbb{R}$ is the unique viscosity solution to the nonlinear partial differential equation

$$\partial_t u + G\bigg(\Big(\sum_{k,l=1}^{n} \sigma_t^i(T, t_k) x_k \sigma_t^j(T, t_l) x_l \partial_{x_k x_l}^2 u\Big)_{i,j=1,...,d}\bigg) = 0,$$
$$u(t_1, x_1, ..., x_n) = \varphi(x_1, ..., x_n).$$

Now we show that $u_{\overline{\sigma}}$ solves the nonlinear partial differential equation. By the classical Feynman-Kac formula, we know that $u_\sigma$, for $\sigma \in \Sigma$, satisfies

$$\partial_t u_\sigma + \tfrac{1}{2}\mathrm{tr}\bigg(\sigma\sigma'\Big(\sum_{k,l=1}^{n} \sigma_t^i(T, t_k) x_k \sigma_t^j(T, t_l) x_l \partial_{x_k x_l}^2 u_\sigma\Big)_{i,j=1,...,d}\bigg) = 0,$$
$$u_\sigma(t_1, x_1, ..., x_n) = \varphi(x_1, ..., x_n).$$

In addition, the convexity of $\varphi$ implies that $u_\sigma(t, \cdot)$ is convex for each $\sigma$ and $t$; thus,

$$\sum_{k,l=1}^{n} \sigma_t^i(T, t_k) x_k \sigma_t^j(T, t_l) x_l \partial_{x_k x_l}^2 u_\sigma \geq 0$$

for all $i, j = 1, ..., d$. Therefore, one can verify that $u_{\overline{\sigma}}$ solves the nonlinear partial differential equation from above, which proves the first assertion.

In order to prove the second assertion, we repeat the procedure from above. Due to the nonlinear Feynman-Kac formula, we have $\hat{\mathbb{E}}_t^T[-\xi] = u(t, X_t^{T,t_1}, ..., X_t^{T,t_n})$, where $u : [0, t_1] \times \mathbb{R}^n \to \mathbb{R}$ is the unique viscosity solution to the nonlinear partial differential equation

$$\partial_t u + G\Bigg(\bigg(\sum_{k,l=1}^{n} \sigma_t^i(T,t_k)x_k\sigma_t^j(T,t_l)x_l\partial^2_{x_kx_l}u\bigg)_{i,j=1,...,d}\Bigg) = 0,$$

$$u(t_1,x_1,...,x_n) = -\varphi(x_1,...,x_n).$$

Then we can use the concavity of $-u_\sigma(t,\cdot)$ to show that $-u_{\underline{\sigma}}$ solves the nonlinear partial differential equation from above. □

For bond options that are neither convex nor concave, we generally need to use numerical procedures to obtain the pricing bounds. If we deal with a bond option having a concave (instead of a convex) payoff function, we can use the same approach as in Proposition 4.7 to find the pricing bounds by simply interchanging $\overline{\sigma}$ and $\underline{\sigma}$. The convexity or the concavity of the payoff function reduces the nonlinear partial differential equation that results from the nonlinear Feynman-Kac formula of Hu, Ji, Peng, and Song (2014) and determines the pricing bounds to a linear partial differential equation. Then the pricing bounds coincide with the prices of traditional models when the underlying is driven by a standard Brownian motion with volatility $\overline{\sigma}$ and $\underline{\sigma}$, respectively. When the payoff function is neither convex nor concave, we can still use the nonlinear Feynman-Kac formula to obtain the pricing bounds, but then we need to solve the nonlinear partial differential equation, since it does not reduce to a linear one. One can find the solution, for example, by using numerical schemes similar to the ones of Nendel (2021, Section 5).

## 4.4 Pricing a Stream of Cashflows

Due to the nonlinearity of the pricing measure, in general, we cannot price interest rate derivatives by pricing each cashflow separately. As in Section 4.2, we consider a contract consisting of a stream of cashflows, which we denote by $X$. Then the discounted payoff is given by

$$\tilde{X} = \sum_{i=0}^{N} M_{T_i}^{-1}\xi_i$$

for a tenor structure $0 < T_0 < T_1 < ... < T_N = \tau$ and $\xi_i \in L_G^p(\Omega_{T_i})$ with $p > 1$ for all $i$. In order to price the contract, we are interested in $\hat{\mathbb{E}}[\tilde{X}]$ and $-\hat{\mathbb{E}}[-\tilde{X}]$. When there is no volatility uncertainty, we can simply price the contract by pricing each cashflow individually, since the pricing measure is linear in that case. However, in the presence of volatility uncertainty, the pricing measure $\hat{\mathbb{E}}$ is sublinear, which implies

$$\begin{aligned} \hat{\mathbb{E}}[\tilde{X}] &\leq \sum_{i=0}^{N} \hat{\mathbb{E}}[M_{T_i}^{-1}\xi_i], \\ -\hat{\mathbb{E}}[-\tilde{X}] &\geq \sum_{i=0}^{N} -\hat{\mathbb{E}}[-M_{T_i}^{-1}\xi_i]. \end{aligned}$$

Therefore, if we price each cashflow separately, we possibly only obtain an upper, respectively lower, bound for the upper, respectively lower, bound of the price—which does not yield much information about the price of the contract.

For contracts with symmetric cashflows, we can still determine the price of the contract by pricing each of its cashflows individually. If each cashflow has a symmetric payoff under the forward sublinear expectation, there is a single price for the contract, which coincides with the sum of the prices of the cashflows. Hence, the pricing measure is linear on the subspace of contracts with symmetric cashflows.

**Lemma 4.8.** *If $\xi_i$, for all $i$, satisfies $\hat{\mathbb{E}}_t^{T_i}[\xi_i] = -\hat{\mathbb{E}}_t^{T_i}[-\xi_i]$ for $t \leq T_0$, then it holds*

$$M_t\hat{\mathbb{E}}_t[\tilde{X}] = \sum_{i=0}^{N} P_t(T_i)\hat{\mathbb{E}}_t^{T_i}[\xi_i] = -M_t\hat{\mathbb{E}}_t[-\tilde{X}].$$

*Proof.* We derive an upper, respectively lower, bound for the upper, respectively lower, expectation of $\tilde{X}$ and show that they coincide. Using the sublinearity of $\hat{\mathbb{E}}$ and Proposition 4.5 $(i)$, for $t \leq T_0$, we get

$$M_t\hat{\mathbb{E}}_t[\tilde{X}] \leq \sum_{i=0}^{N} M_t\hat{\mathbb{E}}_t[M_{T_i}^{-1}\xi_i] = \sum_{i=0}^{N} P_t(T_i)\hat{\mathbb{E}}_t^{T_i}[\xi_i].$$

By the same arguments, for $t \leq T_0$, we obtain

$$-M_t\hat{\mathbb{E}}_t[-\tilde{X}] \geq \sum_{i=0}^{N} -M_t\hat{\mathbb{E}}_t[-M_{T_i}^{-1}\xi_i] = \sum_{i=0}^{N} -P_t(T_i)\hat{\mathbb{E}}_t^{T_i}[-\xi_i].$$

For $t \leq T_0$, it holds $\hat{\mathbb{E}}_t[\tilde{X}] \geq -\hat{\mathbb{E}}_t[-\tilde{X}]$ and $\hat{\mathbb{E}}_t^{T_i}[\xi_i] = -\hat{\mathbb{E}}_t^{T_i}[-\xi_i]$ for all $i$. Therefore, all expressions from above are equal. □

For general contracts, we can use a backward induction procedure to obtain the upper and the lower expectation of the discounted payoff. The procedure works as follows. First, we compute the forward sublinear expectation of the last cashflow conditioned on the second last payoff time and discount it with the bond price. Next, we compute the forward sublinear expectation of the second last cashflow and the previous expression conditioned on the third last payoff time and discount it with the bond price. Then we recursively repeat this procedure until we arrive at the first payoff. This gives us eventually the upper expectation of the discounted payoff. The procedure for the lower expectation is similar.

**Lemma 4.9.** *It holds* $\hat{\mathbb{E}}[\tilde{X}] = \tilde{Y}_0$ *and* $-\hat{\mathbb{E}}[-\tilde{X}] = -\tilde{Z}_0$*, where* $\tilde{Y}_i$ *and* $\tilde{Z}_i$ *are defined by*

$$\tilde{Y}_i := P_{T_{i-1}}(T_i)\hat{\mathbb{E}}^{T_i}_{T_{i-1}}[\xi_i + \tilde{Y}_{i+1}],$$
$$\tilde{Z}_i := P_{T_{i-1}}(T_i)\hat{\mathbb{E}}^{T_i}_{T_{i-1}}[-\xi_i + \tilde{Z}_{i+1}],$$

*respectively, for all* $i = 0, 1, ..., N$ *and* $T_{-1} := 0$ *and* $\tilde{Y}_{N+1} := 0$ *and* $\tilde{Z}_{N+1} := 0$.

*Proof.* First, we exclude the last cashflow from the sum and write it in terms of $\tilde{Y}_N$. Due to the time consistency of the $G$-expectation, we have

$$\hat{\mathbb{E}}[\tilde{X}] = \hat{\mathbb{E}}\Bigg[\sum_{i=0}^{N-1} M^{-1}_{T_i}\xi_i + \hat{\mathbb{E}}_{T_{N-1}}[M^{-1}_{T_N}\xi_N]\Bigg].$$

By Proposition 4.5 $(i)$, we obtain

$$\hat{\mathbb{E}}_{T_{N-1}}[M^{-1}_{T_N}\xi_N] = M^{-1}_{T_{N-1}} P_{T_{N-1}}(T_N)\hat{\mathbb{E}}^{T_N}_{T_{N-1}}[\xi_N] = M^{-1}_{T_{N-1}}\tilde{Y}_N.$$

Second, we exclude the second last cashflow from the sum and repeat the calculation from above. Using the time consistency of $\hat{\mathbb{E}}$, we get

$$\hat{\mathbb{E}}[\tilde{X}] = \hat{\mathbb{E}}\Bigg[\sum_{i=0}^{N-2} M^{-1}_{T_i}\xi_i + \hat{\mathbb{E}}_{T_{N-2}}[M^{-1}_{T_{N-1}}(\xi_{N-1} + \tilde{Y}_N)]\Bigg].$$

Due to Proposition 4.5 $(i)$, we have

$$\hat{\mathbb{E}}_{T_{N-2}}[M^{-1}_{T_{N-1}}(\xi_{N-1} + \tilde{Y}_N)] = M^{-1}_{T_{N-2}} P_{T_{N-2}}(T_{N-1})\hat{\mathbb{E}}^{T_{N-1}}_{T_{N-2}}[\xi_{N-1} + \tilde{Y}_N] = M^{-1}_{T_{N-2}}\tilde{Y}_{N-1}.$$

Now we work recursively backwards until we arrive at the last cashflow. Repeating the step from above, we finally obtain

$$\hat{\mathbb{E}}[\tilde{X}] = \hat{\mathbb{E}}[M^{-1}_{T_0}\xi_0 + M^{-1}_{T_0}\tilde{Y}_1] = P_0(T_0)\hat{\mathbb{E}}^{T_0}[\xi_0 + \tilde{Y}_1] = \tilde{Y}_0.$$

By replacing $\tilde{X}$, $\xi_i$, and $\tilde{Y}_i$, for all $i = 0, 1, ..., N$, by $-\tilde{X}$, $-\xi_i$, and $\tilde{Z}_i$, respectively, we get $\hat{\mathbb{E}}[-\tilde{X}] = \tilde{Z}_0$, which completes the proof. □

If the contract can be written as a stream of convex bond options, the upper, respectively lower, bound for the price is given by the price from the classical model without volatility uncertainty with the highest, respectively lowest, possible volatility. Similar to Proposition 4.7, if the cashflows can be written as convex functions of forward prices satisfying a suitable growth condition, we can show that the upper, respectively lower, expectation of the discounted payoff is given by its linear expectation when the dynamics of the forward price are driven by a standard Brownian motion with constant volatility $\overline{\sigma}$, respectively $\underline{\sigma}$. We show this by using the backward induction procedure of Lemma 4.9 and recursively applying the nonlinear Feynman-Kac formula of Hu, Ji, Peng, and Song (2014).

**Proposition 4.10.** *For $m, n \in \mathbb{N}$ such that $m \neq n$, let $\bar{Y}_i$ and $\bar{Z}_i$ be defined by*

$$\begin{aligned}\bar{Y}_i &:= X_{t_i}^{t_{i-1+n}, t_{i+n}} \hat{\mathbb{E}}_{t_i}^{t_{i+n}}[\varphi_i(X_{t_{i+1}}^{t_{i+n}, t_{i+m}}) + \bar{Y}_{i+1}], \\ \bar{Z}_i &:= X_{t_i}^{t_{i-1+n}, t_{i+n}} \hat{\mathbb{E}}_{t_i}^{t_{i+n}}[-\varphi_i(X_{t_{i+1}}^{t_{i+n}, t_{i+m}}) + \bar{Z}_{i+1}],\end{aligned}$$

*respectively, for all $i = 1, ..., N$, where $\varphi_i : \mathbb{R} \to \mathbb{R}$ is a convex function such that* (4.3) *holds and $0 = t_1 < ... < t_{N+(m \vee n)} \leq \tau$, and $\bar{Y}_{N+1} := 0$ and $\bar{Z}_{N+1} := 0$. Then*

$$\begin{aligned}\bar{Y}_1 &= \sum_{i=1}^{N} X_0^{t_n, t_{i+n}} u_{\overline{\sigma}}^i(0, X_0^{t_{i+n}, t_{i+m}}), \\ -\bar{Z}_1 &= \sum_{i=1}^{N} X_0^{t_n, t_{i+n}} u_{\underline{\sigma}}^i(0, X_0^{t_{i+n}, t_{i+m}}),\end{aligned}$$

*where the function $u_\sigma^i : [0, t_{i+1}] \times \mathbb{R} \to \mathbb{R}$, for all $i = 1, ..., N$ and $\sigma \in \Sigma$, is defined by*

$$u_\sigma^i(t, x_i) := \mathbb{E}_{P_0}[\varphi_i(X_{t_{i+1}}^i)]$$

*and the process $X^i = (X_s^i)_{t \leq s \leq t_{i+1}}$ is given by*

$$X_s^i = x_i - \sum_{j=1}^{d} \int_t^s \sigma_u^j(t_{i+n}, t_{i+m}) X_u^i \sigma_j dB_u^j.$$

*Proof.* We only compute $\bar{Y}_1$; the derivation of $\bar{Z}_1$ can be carried out in the same way, which is similar to the proof of Proposition 4.7.

In order to determine $\bar{Y}_1$, we show, by induction, that

$$\bar{Y}_i = \sum_{j=i}^{N} X_{t_i}^{t_{i-1+n}, t_{j+n}} u_{\overline{\sigma}}^j(t_i, X_{t_i}^{t_{j+n}, t_{j+m}})$$

for all $i = 1, ..., N$. We start by computing $\bar{Y}_N$. We have

$$\bar{Y}_N = X_{t_N}^{t_{N-1+n}, t_{N+n}} \hat{\mathbb{E}}_{t_N}^{t_{N+n}}[\varphi_N(X_{t_{N+1}}^{t_{N+n}, t_{N+m}})].$$

Since $\varphi_N$ is convex and satisfies (4.3), by Proposition 4.7, we obtain

$$\bar{Y}_N = X_{t_N}^{t_{N-1+n},t_{N+n}} u_{\bar{\sigma}}^N(t_N, X_{t_N}^{t_{N+n},t_{N+m}})].$$

Next, we do the inductive step. For $1 \leq i \leq N-1$, let us suppose that

$$\bar{Y}_{i+1} = \sum_{j=i+1}^{N} X_{t_{i+1}}^{t_{i+n},t_{j+n}} u_{\bar{\sigma}}^j(t_{i+1}, X_{t_{i+1}}^{t_{j+n},t_{j+m}}).$$

Then we have

$$\bar{Y}_i = X_{t_i}^{t_{i-1+n},t_{i+n}} \hat{\mathbb{E}}_{t_i}^{t_{i+n}} \left[ \varphi_i(X_{t_{i+1}}^{t_{i+n},t_{i+m}}) + \sum_{j=i+1}^{N} X_{t_{i+1}}^{t_{i+n},t_{j+n}} u_{\bar{\sigma}}^j(t_{i+1}, X_{t_{i+1}}^{t_{j+n},t_{j+m}}) \right].$$

We use the nonlinear Feynman-Kac formula of Hu, Ji, Peng, and Song (2014) to compute the expectation, since we cannot apply Proposition 4.7. Thus, we obtain

$$\bar{Y}_i = X_{t_i}^{t_{i-1+n},t_{i+n}} u(t_i, X_{t_i}^{t_{i+n},t_{i+m}}, ..., X_{t_i}^{t_{N+n},t_{N+m}}, X_{t_i}^{t_{i+n},t_{i+1+n}}, ..., X_{t_i}^{t_{i+n},t_{N+n}}).$$

The function $u : [0, t_{i+1}] \times \mathbb{R}^{2(N-i)+1} \to \mathbb{R}$ is the unique viscosity solution to the nonlinear partial differential equation

$$\begin{aligned} \partial_t u + G\Big(\big(H_{\kappa,\lambda}(t, x, D_x u, D_x^2 u)\big)_{\kappa,\lambda=1,\ldots,d}\Big) &= 0, \\ u(t_{i+1}, x) &= \varphi(x), \end{aligned} \tag{4.4}$$

where $x = (\hat{x}_i, ..., \hat{x}_N, \tilde{x}_{i+1}, ..., \tilde{x}_N) \in \mathbb{R}^{2(N-i)+1}$, the operator $D_x$, respectively $D_{xx}^2$, denotes the gradient, respectively Hessian, with respect to $x$, and

$$\begin{aligned} H_{\kappa,\lambda}(t, x, D_x u, D_x^2 u) :=& \sum_{j,k=i}^{N} \sigma_t^\kappa(t_{j+n}, t_{j+m})\hat{x}_j \sigma_t^\lambda(t_{k+n}, t_{k+m})\hat{x}_k \partial_{\hat{x}_j\hat{x}_k}^2 u \\ &+ 2\sum_{j=i}^{N} \sum_{k=i+1}^{N} \sigma_t^\kappa(t_{j+n}, t_{j+m})\hat{x}_j \sigma_t^\lambda(t_{i+n}, t_{k+n})\tilde{x}_k \partial_{\hat{x}_j\tilde{x}_k}^2 u \\ &+ \sum_{j,k=i+1}^{N} \sigma_t^\kappa(t_{i+n}, t_{j+n})\tilde{x}_j \sigma_t^\lambda(t_{i+n}, t_{k+n})\tilde{x}_k \partial_{\tilde{x}_j\tilde{x}_k}^2 u \\ &+ 1_{\{\kappa=\lambda\}}(\kappa, \lambda) 2 \sum_{j=i+1}^{N} \sigma_t^\kappa(t_{j+n}, t_{j+m})\hat{x}_j \sigma_t^\kappa(t_{j+n}, t_{i+n}) \partial_{\hat{x}_j} u, \\ \varphi(x) :=& \varphi_i(\hat{x}_i) + \sum_{j=i+1}^{N} \tilde{x}_j u_{\bar{\sigma}}^j(t_{i+1}, \hat{x}_j). \end{aligned}$$

It is feasible to apply the nonlinear Feynman-Kac formula, since $\varphi$ satisfies (4.3), which is shown in Subsection 4.7.1. To show that $\varphi$ satisfies (4.3), we use estimates for $u_{\overline{\sigma}}^j$ for all $j = i+1, ..., N$, which follow from the nonlinear Feynman-Kac formula (Hu, Ji, Peng, and Song, 2014, Proposition 4.2). In order to solve (4.4), we define the function $u^* : [0, t_{i+1}] \times \mathbb{R}^{2(N-i)+1} \to \mathbb{R}$ by

$$u^*(t, x) := u_{\overline{\sigma}}^i(t, \hat{x}_i) + \sum_{j=i+1}^{N} \tilde{x}_j u_{\overline{\sigma}}^j(t, \hat{x}_j).$$

Then one can check that it holds $\partial^2_{\hat{x}_j \hat{x}_k} u^* = 0$ and $\partial^2_{\hat{x}_j \tilde{x}_k} u^* = 0$ for all $j, k$ such that $j \neq k$, it holds $\tilde{x}_k \partial^2_{\hat{x}_j \tilde{x}_k} u^* = \partial_{\hat{x}_j} u^*$ for all $j, k$ such that $j = k$, and it holds $\partial^2_{\tilde{x}_j \tilde{x}_k} u^* = 0$ for all $j, k$. Since $u_{\overline{\sigma}}^j(t, \cdot)$ is convex for each $t$ and $u_{\overline{\sigma}}^j$ satisfies

$$\partial_t u_{\overline{\sigma}}^j + \tfrac{1}{2} \mathrm{tr}\Big(\sigma\sigma' \big(\sigma_t^k(t_{j+n}, t_{j+m}) \sigma_t^l(t_{j+n}, t_{j+m}) \hat{x}_j^2 \partial^2_{\hat{x}_j \hat{x}_j} u_{\overline{\sigma}}^j\big)_{k,l=1,\dots,d}\Big) = 0,$$
$$u_{\overline{\sigma}}^j(t_{j+1}, \hat{x}_j) = \varphi_j(\hat{x}_j)$$

for all $j = i, ..., N$ and $\sigma \in \Sigma$, one can verify that $u^*$ solves (4.4) on $[0, t_{i+1}] \times \mathbb{R}_+^{2(N-i)+1}$. We are only interested in a solution for positive $x$, since the forward prices are positive. Hence, we obtain

$$\bar{Y}_i = \sum_{j=i}^{N} X_{t_i}^{t_{i-1+n}, t_{j+n}} u_{\overline{\sigma}}^j(t_i, X_{t_i}^{t_{j+n}, t_{j+m}})$$

and the proof is complete. □

If the stream of cashflows consists of bond options that are neither convex nor concave, we need to use a numerical scheme to apply the backward induction procedure from Lemma 4.9. As in the previous section, we can price a stream of concave bond options in the same way as in Proposition 4.10 by interchanging $\overline{\sigma}$ and $\underline{\sigma}$. When the bond options are neither convex nor concave, we can use the general backward induction procedure from Lemma 4.9 and recursively solve the nonlinear partial differential equations arising due to the nonlinear Feynman-Kac formula of Hu, Ji, Peng, and Song (2014) by numerical procedures (as mentioned at the end of Section 4.3).

## 4.5 Common Interest Rate Derivatives

With the tools from the preceding sections, we can price all major derivatives traded in fixed income markets. We consider typical linear contracts, such as fixed coupon bonds, floating rate notes, and interest rate swaps, and nonlinear contracts, such as swaptions, caps and floors, and in-arrears contracts. Using the general pricing techniques for whole contracts from Section 4.4 and the valuation methods for single cashflows from Section

4.3, we show how to derive robust pricing formulas for all these contracts. That means, we consider a contract with discounted payoff

$$\tilde{X} = \sum_{i=0}^{N} M_{T_i}^{-1} \xi_i$$

for $0 < T_0 < T_1 < ... < T_N = \tau$ and specifically given cashflows, and then we show how to find $\hat{\mathbb{E}}[\tilde{X}]$ and $-\hat{\mathbb{E}}[-\tilde{X}]$ or $M_t\hat{\mathbb{E}}_t[\tilde{X}]$ and $-M_t\hat{\mathbb{E}}_t[-\tilde{X}]$ for $t \leq T_0$ if the contract has a symmetric payoff.

### 4.5.1 Fixed Coupon Bonds

We can price fixed coupon bonds as in the classical case without volatility uncertainty. A fixed coupon bond is a contract that pays a fixed rate of interest, given by $K > 0$, on a nominal value, which is normalized to 1, at each payment date and the nominal value at the last payment date. Hence, the cashflows are given by

$$\xi_i = 1_{\{N\}}(i) + 1_{\{1,\dots,N\}}(i)(T_i - T_{i-1})K \tag{4.5}$$

for all $i = 0, 1, ..., N$. Due to its simple payoff structure, the contract has a symmetric payoff, and its price is given by the same expression as the one obtained in traditional term structure models.

**Proposition 4.11.** *Let $\xi_i$ be given by* (4.5) *for all $i = 0, 1, ..., N$. Then for $t \leq T_0$,*

$$M_t\hat{\mathbb{E}}_t[\tilde{X}] = P_t(T_N) + \sum_{i=1}^{N} P_t(T_i)(T_i - T_{i-1})K = -M_t\hat{\mathbb{E}}_t[-\tilde{X}].$$

*Proof.* Since the cashflows are constants, the assertion follows by Lemma 4.8. □

### 4.5.2 Floating Rate Notes

We can also price floating rate notes as in the classical case without volatility uncertainty. A floating rate note is a fixed coupon bond in which the fixed rate is replaced by a floating rate: the simply compounded spot rate; for $t \leq T \leq \tau$, the simply compounded spot rate with maturity $T$ at time $t$ is defined by

$$L_t(T) := \tfrac{1}{T-t}\left(\tfrac{1}{P_t(T)} - 1\right).$$

The cashflows are then given by

$$\xi_i = 1_{\{N\}}(i) + 1_{\{1,\dots,N\}}(i)(T_i - T_{i-1})L_{T_{i-1}}(T_i) \tag{4.6}$$

for all $i = 0, 1, ..., N$. Although the cashflows are not constant, the contract yet has a symmetric payoff. As in the classical case, the price is simply given by the price of a zero-coupon bond with maturity $T_0$.

**Proposition 4.12.** *Let $\xi_i$ be given by* (4.6) *for all $i = 0, 1, ..., N$. Then for $t \leq T_0$,*

$$M_t \hat{\mathbb{E}}_t[\tilde{X}] = P_t(T_0) = -M_t \hat{\mathbb{E}}_t[-\tilde{X}].$$

*Proof.* We show that the cashflows have a symmetric payoff and apply Lemma 4.8. Due to Proposition 4.5 (*ii*) and (*iv*), we have

$$P_t(T_i)\hat{\mathbb{E}}_t^{T_i}[(T_i - T_{i-1})L_{T_{i-1}}(T_i)] = P_t(T_{i-1})\hat{\mathbb{E}}_t^{T_{i-1}}[1 - P_{T_{i-1}}(T_i)] = P_t(T_{i-1}) - P_t(T_i)$$

for all $i = 1, ..., N$. In a similar fashion we can show that

$$-P_t(T_i)\hat{\mathbb{E}}_t^{T_i}[-(T_i - T_{i-1})L_{T_{i-1}}(T_i)] = P_t(T_{i-1}) - P_t(T_i)$$

for all $i = 1, ..., N$. The result follows by Lemma 4.8 and summation. □

### 4.5.3 Interest Rate Swaps

The pricing formula for interest rate swaps is the same as in traditional models. An interest rate swap exchanges the floating rate with a fixed rate at each payment date. Without loss of generality we consider a payer interest rate swap; that is, we pay the fixed rate and receive the floating rate. Hence, the cashflows are given by

$$\xi_i = 1_{\{1,...,N\}}(i)(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big) \tag{4.7}$$

for all $i = 0, 1, ..., N$. Since the payoff is the difference of a zero-coupon bond and a floating rate note, the contract is symmetric. As in traditional term structure models, the price is given by a linear combination of zero-coupon bonds with different maturities. In particular, this implies that the swap rate—i.e., the value of the fixed rate that makes the value of the contract zero—is uniquely determined and does not differ from the expression obtained by standard models.

**Proposition 4.13.** *Let $\xi_i$ be given by* (4.7) *for all $i = 0, 1, ..., N$. Then for $t \leq T_0$,*

$$M_t \hat{\mathbb{E}}_t[\tilde{X}] = P_t(T_0) - P_t(T_N) - \sum_{i=1}^{N} P_t(T_i)(T_i - T_{i-1})K = -M_t \hat{\mathbb{E}}_t[-\tilde{X}].$$

*Proof.* Again, we show that the cashflows have a symmetric payoff and use Lemma 4.8

to obtain the result. As in the proof of Proposition 4.12, we can show that

$$P_t(T_i)\hat{\mathbb{E}}_t^{T_i}\big[(T_i-T_{i-1})\big(L_{T_{i-1}}(T_i)-K\big)\big] = P_t(T_{i-1}) - P_t(T_i) - P_t(T_i)(T_i-T_{i-1})K,$$
$$-P_t(T_i)\hat{\mathbb{E}}_t^{T_i}\big[-(T_i-T_{i-1})\big(L_{T_{i-1}}(T_i)-K\big)\big] = P_t(T_{i-1}) - P_t(T_i) - P_t(T_i)(T_i-T_{i-1})K$$

for all $i = 1, ..., N$. Then the assertion follows by Lemma 4.8 and summation. □

### 4.5.4 Swaptions

We can price swaptions by using the pricing formulas from traditional models to compute the upper and the lower bound for the price. A swaption gives the buyer the right to enter an interest rate swap at the first payment date. Hence, there is only one cashflow, which is determined by Proposition 4.13—i.e.,

$$\xi_i = 1_{\{0\}}(i)\Bigg(1 - P_{T_0}(T_n) - \sum_{j=1}^{N} P_{T_0}(T_j)(T_j - T_{j-1})K\Bigg)^+ \tag{4.8}$$

for all $i = 0, 1, ..., N$. Due to the nonlinearity of the payoff function, the upper and the lower expectation of the discounted payoff do not necessarily coincide; thus, the contract has an asymmetric payoff. The related pricing bounds are given by the prices from the classical case with the highest and the lowest possible volatility, respectively.

**Theorem 4.14.** *Let $\xi_i$ be given by (4.8) for all $i = 0, 1, ..., N$. Then it holds*

$$\hat{\mathbb{E}}[\tilde{X}] = P_0(T_0)u_{\overline{\sigma}}\big(0, \tfrac{P_0(T_1)}{P_0(T_0)}, ..., \tfrac{P_0(T_N)}{P_0(T_0)}\big),$$
$$-\hat{\mathbb{E}}[-\tilde{X}] = P_0(T_0)u_{\underline{\sigma}}\big(0, \tfrac{P_0(T_1)}{P_0(T_0)}, ..., \tfrac{P_0(T_N)}{P_0(T_0)}\big),$$

*where the function $u_\sigma : [0, T_0] \times \mathbb{R}^N \to \mathbb{R}$, for $\sigma \in \Sigma$, is defined by*

$$u_\sigma(t, x_1, ..., x_N) := \mathbb{E}_{P_0}\Bigg[\Bigg(1 - X_{T_0}^N - \sum_{i=1}^{N} X_{T_0}^i(T_i - T_{i-1})K\Bigg)^+\Bigg]$$

*and the process $X^i = (X_s^i)_{t\le s\le T_0}$, for all $i = 1, ..., N$, is given by*

$$X_s^i = x_i - \sum_{j=1}^{d}\int_t^s \sigma_u^j(T_0, T_i)X_u^i\sigma_j dB_u^j.$$

*Proof.* We prove the claim by using Proposition 4.7. By Proposition 4.5 $(i)$, we have

$$\hat{\mathbb{E}}[\tilde{X}] = P_0(T_0)\hat{\mathbb{E}}^{T_0}\left[\left(1 - X_{T_0}^{T_0,T_N} - \sum_{i=1}^{N} X_{T_0}^{T_0,T_i}(T_i - T_{i-1})K\right)^+\right],$$
$$-\hat{\mathbb{E}}[-\tilde{X}] = \; - P_0(T_0)\hat{\mathbb{E}}^{T_0}\left[-\left(1 - X_{T_0}^{T_0,T_N} - \sum_{i=1}^{N} X_{T_0}^{T_0,T_i}(T_i - T_{i-1})K\right)^+\right].$$

Hence, the assertion follows by Proposition 4.7, since the payoff of a swaption is convex and satisfies (4.3), which is shown in Subsection 4.7.2. □

### 4.5.5 Caps and Floors

Similar to swaptions, we can compute the upper and the lower bound for the price of a cap by using the pricing formulas from traditional models. A cap gives the buyer the right to exchange the floating rate with a fixed rate at each payment date. The cashflows are called caplets and are given by

$$\xi_i = 1_{\{1,...,N\}}(i)(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+ \tag{4.9}$$

for all $i = 0, 1, ..., N$. The upper and the lower bound for the price of the contract are given by the prices from the classical case without volatility uncertainty with the highest and the lowest possible volatility, respectively. We obtain the latter by computing prices of put options on the forward price.

**Theorem 4.15.** *Let $\xi_i$ be given by* (4.9) *for all $i = 0, 1, ..., N$. Then it holds*

$$\hat{\mathbb{E}}[\tilde{X}] = \sum_{i=1}^{N} P_0(T_{i-1})u^i_{\overline{\sigma}}\big(0, \tfrac{P_0(T_i)}{P_0(T_{i-1})}\big),$$
$$-\hat{\mathbb{E}}[-\tilde{X}] = \sum_{i=1}^{N} P_0(T_{i-1})u^i_{\underline{\sigma}}\big(0, \tfrac{P_0(T_i)}{P_0(T_{i-1})}\big),$$

*where the function $u^i_\sigma : [0, T_{i-1}] \times \mathbb{R} \to \mathbb{R}$, for all $i = 1, ..., N$ and $\sigma \in \Sigma$, is defined by*

$$u^i_\sigma(t, x_i) := \tfrac{1}{K_i}\mathbb{E}_{P_0}[(K_i - X^i_{T_{i-1}})^+]$$

*for $K_i := \frac{1}{1+(T_i-T_{i-1})K}$ and the process $X^i = (X^i_s)_{t\leq s\leq T_{i-1}}$ is given by*

$$X^i_s = x_i - \sum_{j=1}^{d}\int_t^s \sigma^j_u(T_{i-1}, T_i)X^i_u\sigma_j dB^j_u.$$

*Proof.* According to Lemma 4.9, we need to determine $\tilde{Y}_0$ and $\tilde{Z}_0$ in order to obtain $\hat{\mathbb{E}}[\tilde{X}]$

and $\hat{\mathbb{E}}[-\tilde{X}]$, respectively. We only show how to obtain $\tilde{Y}_0$; we can compute $\tilde{Z}_0$ in the same way.

We compute $\tilde{Y}_0$ by using Proposition 4.10. For this purpose, we need to rewrite $\tilde{Y}_i$ for all $i = 0, 1, ..., N$ and define a sequence of random variables to which we can apply Proposition 4.10. For all $i = 0, 1, ..., N$, we have

$$\tilde{Y}_i = P_{T_{i-1}}(T_i)\hat{\mathbb{E}}^{T_i}_{T_{i-1}}[\xi_i + \tilde{Y}_{i+1}],$$

where $\xi_i$ is given by (4.9), and $\tilde{Y}_{N+1} = 0$. Since $\xi_i \in L^1_G(\Omega_{T_{i-1}})$ for all $i = 1, ..., N$ and $\xi_0 = 0$, we can show that

$$\tilde{Y}_i = \tfrac{1}{K_i}(K_i - X^{T_{i-1},T_i}_{T_{i-1}})^+ + X^{T_{i-1},T_i}_{T_{i-1}}\hat{\mathbb{E}}^{T_i}_{T_{i-1}}[\tilde{Y}_{i+1}]$$

for all $i = 1, ..., N$ and $\tilde{Y}_0 = X^{0,T_0}_0\hat{\mathbb{E}}^{T_0}[\tilde{Y}_1]$. Now we define $\bar{Y}_i := X^{T_{i-2},T_{i-1}}_{T_{i-2}}\hat{\mathbb{E}}^{T_{i-1}}_{T_{i-2}}[\tilde{Y}_i]$ for all $i = 1, ..., N + 1$. Then we have $\tilde{Y}_0 = \bar{Y}_1$ and

$$\bar{Y}_i = X^{T_{i-2},T_{i-1}}_{T_{i-2}}\hat{\mathbb{E}}^{T_{i-1}}_{T_{i-2}}[\tfrac{1}{K_i}(K_i - X^{T_{i-1},T_i}_{T_{i-1}})^+ + \bar{Y}_{i+1}]$$

for all $i = 1, ..., N$, where $\bar{Y}_{N+1} = 0$. Moreover, we define $t_i := T_{i-2}$ for all $i = 1, ..., N+2$. Then it holds $0 = t_1 < ... < t_{N+2} \leq \tau$ and

$$\bar{Y}_i = X^{t_i,t_{i+1}}_{t_i}\hat{\mathbb{E}}^{t_{i+1}}_{t_i}[\tfrac{1}{K_i}(K_i - X^{t_{i+1},t_{i+2}}_{t_{i+1}})^+ + \bar{Y}_{i+1}]$$

for all $i = 1, ..., N$. Thus, we can apply Proposition 4.10 to obtain

$$\bar{Y}_1 = \sum_{i=1}^{N} X^{0,t_{i+1}}_0 u^i_{\overline{\sigma}}(0, X^{t_{i+1},t_{i+2}}_0),$$

which proves the assertion. □

Floors can be priced in the same manner as caps. A floor gives the buyer the right to exchange a fixed rate with the floating rate at each payment date. The cashflows are called floorlets and are given by

$$\xi_i = 1_{\{1,...,N\}}(i)(T_i - T_{i-1})\big(K - L_{T_{i-1}}(T_i)\big)^+ \tag{4.10}$$

for all $i = 0, 1, ..., N$. Since the cashflows are very similar to caplets, we obtain similar pricing bounds compared to Theorem 4.15; the only difference is that we need to compute prices of call options on the forward price instead of put options to obtain the pricing bounds. It is remarkable that we can show this with the put-call parity, since the non-linearity of the pricing measure implies that the put-call parity, in general, does not hold in the presence of volatility uncertainty.

**Theorem 4.16.** *Let $\xi_i$ be given by* (4.10) *for all $i = 0, 1, ..., N$. Then it holds*

$$\hat{\mathbb{E}}[\tilde{X}] = \sum_{i=1}^{N} P_0(T_{i-1}) u^i_{\overline{\sigma}}\big(0, \tfrac{P_0(T_i)}{P_0(T_{i-1})}\big),$$
$$-\hat{\mathbb{E}}[-\tilde{X}] = \sum_{i=1}^{N} P_0(T_{i-1}) u^i_{\underline{\sigma}}\big(0, \tfrac{P_0(T_i)}{P_0(T_{i-1})}\big),$$

*where the function $u^i_\sigma : [0, T_{i-1}] \times \mathbb{R} \to \mathbb{R}$, for all $i = 1, ..., N$ and $\sigma \in \Sigma$, is defined by*

$$u^i_\sigma(t, x_i) := \tfrac{1}{K_i} \mathbb{E}_P[(X^i_{T_{i-1}} - K_i)^+]$$

*and $K_i$ and the process $X^i = (X^i_s)_{t \leq s \leq T_{i-1}}$ are given as in Theorem 4.15.*

*Proof.* Although $\hat{\mathbb{E}}$ is sublinear, we can still use the put-call parity to prove the claim, since interest rate swaps have a symmetric payoff. For all $i = 1, ..., N$, we have

$$\xi_i = (T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+ - (T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big).$$

Thus, we get $\tilde{X} = \tilde{Y} - \tilde{Z}$, where $\tilde{Y}$, respectively $\tilde{Z}$, denotes the discounted payoff of a cap, respectively interest rate swap; that is,

$$\tilde{Y} := \sum_{i=1}^{N} M^{-1}_{T_i}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big)^+,$$
$$\tilde{Z} := \sum_{i=1}^{N} M^{-1}_{T_i}(T_i - T_{i-1})\big(L_{T_{i-1}}(T_i) - K\big).$$

Due to the sublinearity of $\hat{\mathbb{E}}$, we get $\hat{\mathbb{E}}[\tilde{X}] \leq \hat{\mathbb{E}}[\tilde{Y}] + \hat{\mathbb{E}}[-\tilde{Z}]$ and $\hat{\mathbb{E}}[\tilde{X}] \geq \hat{\mathbb{E}}[\tilde{Y}] - \hat{\mathbb{E}}[\tilde{Z}]$. Hence, by Proposition 4.13, we obtain $\hat{\mathbb{E}}[\tilde{X}] = \hat{\mathbb{E}}[\tilde{Y}] - \hat{\mathbb{E}}[\tilde{Z}]$. In a similar fashion, we can show that $-\hat{\mathbb{E}}[-\tilde{X}] = -\hat{\mathbb{E}}[-\tilde{Y}] - \hat{\mathbb{E}}[\tilde{Z}]$. Therefore, the assertion follows by the classical put-call parity. □

### 4.5.6 In-Arrears Contracts

The pricing procedure from the previous subsection also works for contracts in which the floating rate is settled in arrears. The difference between the contracts from above and in-arrears contracts is that the simply compounded spot rate is reset each time when the contract pays off. As a representative contract, we show how to price in-arrears swaps; other contracts, such as in-arrears caps and floors, can be priced in a similar way. In contrast to a plain vanilla interest rate swap, the cashflows are now given by

$$\xi_i = 1_{\{0,1,...,N-1\}}(i)(T_{i+1} - T_i)\big(L_{T_i}(T_{i+1}) - K\big) \tag{4.11}$$

for all $i = 0, 1, ..., N$. Then the contract is not necessarily symmetric, and the pricing bounds are given by the prices from traditional models with the highest and the lowest possible volatility, respectively. As a consequence, there is not a unique swap rate for in-arrears swaps.

**Theorem 4.17.** *Let $\xi_i$ be given by (4.11) for all $i = 0, 1, ..., N$. Then it holds*

$$\hat{\mathbb{E}}[\tilde{X}] = \sum_{i=1}^{N} P_0(T_i) u^i_{\overline{\sigma}}\big(0, \tfrac{P_0(T_{i-1})}{P_0(T_i)}\big),$$

$$-\hat{\mathbb{E}}[-\tilde{X}] = \sum_{i=1}^{N} P_0(T_i) u^i_{\underline{\sigma}}\big(0, \tfrac{P_0(T_{i-1})}{P_0(T_i)}\big),$$

*where the function $u^i_\sigma : [0, T_{i-1}] \times \mathbb{R} \to \mathbb{R}$, for all $i = 1, ..., N$ and $\sigma \in \Sigma$, is defined by*

$$u^i_\sigma(t, x_i) := \mathbb{E}_{P_0}[X^i_{T_{i-1}}(X^i_{T_{i-1}} - \tfrac{1}{K_i})],$$

*for $K_i$ as in Theorem 4.15 and the process $X^i = (X^i_s)_{t \leq s \leq T_{i-1}}$ is given by*

$$X^i_s = x_i - \sum_{j=1}^{d} \int_t^s \sigma^j_u(T_i, T_{i-1}) X^i_u \sigma_j dB^j_u.$$

*Proof.* As in the proof of Theorem 4.15, by Lemma 4.9, we need to compute $\tilde{Y}_0$ and $\tilde{Z}_0$ to find $\hat{\mathbb{E}}[\tilde{X}]$ and $\hat{\mathbb{E}}[-\tilde{X}]$, respectively. We only show how to obtain $\tilde{Y}_0$; we can find $\tilde{Z}_0$ in the same way.

In order to find $\tilde{Y}_0$, we rewrite $\tilde{Y}_i$ for all $i = 0, 1, ..., N$ and define a sequence of random variables to which we can apply Proposition 4.10. For all $i = 0, 1, ..., N$, we have

$$\tilde{Y}_i = P_{T_{i-1}}(T_i) \hat{\mathbb{E}}^{T_i}_{T_{i-1}}[\xi_i + \tilde{Y}_{i+1}],$$

where $\xi_i$ is given by (4.11) and $\tilde{Y}_{N+1} = 0$. Since $\xi_N = 0$, we get $\tilde{Y}_N = 0$. For all $i = 0, 1, ..., N-1$, we obtain, by Proposition 4.5 (*ii*),

$$\tilde{Y}_i = X^{T_{i-1}, T_{i+1}}_{T_{i-1}} \hat{\mathbb{E}}^{T_{i+1}}_{T_{i-1}}[X^{T_{i+1}, T_i}_{T_i}(X^{T_{i+1}, T_i}_{T_i} - \tfrac{1}{K_{i+1}}) + X^{T_{i+1}, T_i}_{T_i} \tilde{Y}_{i+1}].$$

We define $\bar{Y}_i := X^{T_{i-1}, T_{i-2}}_{T_{i-2}} \tilde{Y}_{i-1}$ for all $i = 1, ..., N+1$. Then it holds $\tilde{Y}_0 = X^{0, T_0}_0 \bar{Y}_1$ and

$$\bar{Y}_i = X^{T_{i-1}, T_i}_{T_{i-2}} \hat{\mathbb{E}}^{T_i}_{T_{i-2}}[X^{T_i, T_{i-1}}_{T_{i-1}}(X^{T_i, T_{i-1}}_{T_{i-1}} - \tfrac{1}{K_i}) + \bar{Y}_{i+1}]$$

for all $i = 1, ..., N$, where $\bar{Y}_{N+1} = 0$. Furthermore, we set $t_i := T_{i-2}$ for all $i = 1, ..., N+2$.

Then we get $0 = t_1 < ... < t_{N+2} \leq \tau$ and

$$\bar{Y}_i = X_{t_i}^{t_{i+1},t_{i+2}} \hat{\mathbb{E}}_{t_i}^{t_{i+2}} [X_{t_{i+1}}^{t_{i+2},t_{i+1}} (X_{t_{i+1}}^{t_{i+2},t_{i+1}} - \tfrac{1}{K_i}) + \bar{Y}_{i+1}]$$

for all $i = 1, ..., N$. Therefore, by Proposition 4.10, it holds

$$\bar{Y}_1 = \sum_{i=1}^{N} X_0^{t_2,t_{i+2}} u_{\overline{\sigma}}^i (0, X_0^{t_{i+2},t_{i+1}}),$$

which proves the assertion. □

## 4.6 Market Incompleteness

Empirical evidence shows that volatility risk in fixed income markets cannot be hedged by trading solely bonds, which is referred to as *unspanned stochastic volatility* and contradicts many traditional term structure models. By using data on interest rate swaps, caps, and floors, Collin-Dufresne and Goldstein (2002) showed that interest rate derivatives exposed to volatility risk are driven by factors that do not affect the term structure. Therefore, derivatives exposed to volatility risk, such as caps and floors, cannot be replicated by a portfolio consisting solely of bonds, which implies that it is not possible to hedge volatility risk in fixed income markets. The empirical findings of Collin-Dufresne and Goldstein (2002) contradict many traditional term structure models, since bond prices are typically functions depending on all risk factors driving the model and bonds can typically be used to hedge caps and floors. As a consequence, Collin-Dufresne and Goldstein (2002) examined which term structure models exhibit unspanned stochastic volatility; this led to the development of new models displaying unspanned stochastic volatility (Casassus, Collin-Dufresne, and Goldstein, 2005; Filipović, Larsson, and Statti, 2019; Filipović, Larsson, and Trolle, 2017).

In the presence of volatility uncertainty, term structure models naturally exhibit unspanned stochastic volatility, since volatility uncertainty naturally leads to market incompleteness. A classical result in the literature on robust finance is that model uncertainty leads to market incompleteness: instead of perfectly hedging derivatives, one has to superhedge the payoff of most derivatives, which can be inferred from the pricing-hedging duality. Similar to the pricing-hedging duality in the presence of volatility uncertainty (Vorbrink, 2014, Theorem 3.6), we can show that it is not possible to hedge a contract with an asymmetric payoff with a portfolio of bonds. From Theorem 4.15 and Theorem 4.16, we can deduce that caps and floors have an asymmetric payoff if $\overline{\sigma} > \underline{\sigma}$. Therefore, derivatives exposed to volatility risk cannot be hedged by trading solely bonds when there is volatility uncertainty.

Moreover, the uncertain volatility affects prices of nonlinear contracts, while prices of linear contracts and the term structure are robust with respect to the volatility—confirming the empirical findings of Collin-Dufresne and Goldstein (2002). In simple model specifications, bond prices have an affine structure with respect to the short rate and an additional factor—which can be inferred from the examples in Section 3.3. They are, however, completely unaffected by the uncertain volatility and its bounds. The same holds for the swap rate, since the price of an interest rate swap (by Proposition 4.13) is a linear combination of bond prices, as in the classical case without volatility uncertainty. On the other hand, the uncertain volatility influences prices of caps and floors, since they depend on the bounds for the volatility (by Theorems 4.15 and 4.16). Therefore, the prices of caps and floors are driven by an additional factor that does not influence term structure movements and (thus) changes in swap rates.

## 4.7 Estimates for the Proofs

In the end, we derive the estimates used in the proofs of Proposition 4.10 and Theorem 4.14, respectively.

### 4.7.1 Estimate for the Proof of Proposition 4.10

Let us define the function $\varphi : \mathbb{R}^{2n+1} \to \mathbb{R}$ by

$$\varphi(x) := f(\hat{x}_0) + \sum_{i=1}^{n} \tilde{x}_i u_i(\hat{x}_i),$$

where $f : \mathbb{R} \to \mathbb{R}$, for a positive integer $m$ and a constant $C > 0$, satisfies

$$|f(\hat{x}_0) - f(\hat{y}_0)| \leq C(1 + |\hat{x}_0|^m + |\hat{y}_0|^m)|\hat{x}_0 - \hat{y}_0|$$

and $u_i : \mathbb{R} \to \mathbb{R}$, for all $i = 1, ..., n$, satisfies

$$|u_i(\hat{x}_i)| \leq C(1 + |\hat{x}_i|^{m+1}),$$
$$|u_i(\hat{x}_i) - u_i(\hat{y}_i)| \leq C(1 + |\hat{x}_i|^m + |\hat{y}_i|^m)|\hat{x}_i - \hat{y}_i|.$$

Then we have

$$
\begin{aligned}
|\varphi(x)-\varphi(y)| &= \left| f(\hat{x}_0) - f(\hat{y}_0) + \sum_{i=1}^{n} \tilde{x}_i u_i(\hat{x}_i) - \tilde{y}_i u_i(\hat{y}_i) \right| \\
&\leq |f(\hat{x}_0) - f(\hat{y}_0)| + \sum_{i=1}^{n} |u_i(\hat{x}_i)||\tilde{x}_i - \tilde{y}_i| + |\tilde{y}_i||u_i(\hat{x}_i) - u_i(\hat{y}_i)| \\
&\leq C(1 + |\hat{x}_0|^m + |\hat{y}_0|^m)|\hat{x}_0 - \hat{y}_0| + \sum_{i=1}^{n} C(1 + |\hat{x}_i|^{m+1})|\tilde{x}_i - \tilde{y}_i| \\
&\quad + |\tilde{y}_i| C(1 + |\hat{x}_i|^m + |\hat{y}_i|^m)|\hat{x}_i - \hat{y}_i| \\
&\leq C(2 + |\hat{x}_0|^{m+1} + |\hat{y}_0|^{m+1})|\hat{x}_0 - \hat{y}_0| + \sum_{i=1}^{n} C(1 + |\hat{x}_i|^{m+1})|\tilde{x}_i - \tilde{y}_i| \\
&\quad + 2C(1 + |\tilde{y}_i|^{m+1} + |\hat{x}_i|^{m+1} + |\hat{y}_i|^{m+1})|\hat{x}_i - \hat{y}_i| \\
&\leq 2C\left(1 + |\hat{x}_0|^{m+1} + |\hat{y}_0|^{m+1} + \sum_{i=1}^{n} |\hat{x}_i|^{m+1} + |\tilde{x}_i|^{m+1} + |\hat{y}_i|^{m+1} + |\tilde{y}_i|^{m+1}\right) \\
&\quad \times \left(|\hat{x}_0 - \hat{y}_0| + \sum_{i=1}^{n} |\hat{x}_i - \hat{y}_i| + |\tilde{x}_i - \tilde{y}_i|\right) \\
&\leq 2C\tilde{C}(1 + |x|^{m+1} + |y|^{m+1})|x - y|.
\end{aligned}
$$

### 4.7.2 Estimates for the Proof of Theorem 4.14

Let us define the function $\varphi : \mathbb{R}^N \to \mathbb{R}$ by

$$
\varphi(x) := \left(1 - x_N - \sum_{i=1}^{N} x_i (T_i - T_{i-1}) K\right)^+.
$$

Then for $\lambda \in (0,1)$, we have

$$
\begin{aligned}
\varphi\big(\lambda x + (1-\lambda) y\big) &= \left(1 - \big(\lambda x_N + (1-\lambda) y_N\big) - \sum_{i=1}^{N} \big(\lambda x_i + (1-\lambda) y_i\big)(T_i - T_{i-1}) K\right)^+ \\
&\leq \lambda \left(1 - x_N - \sum_{i=1}^{N} x_i (T_i - T_{i-1}) K\right)^+ \\
&\quad + (1-\lambda)\left(1 - y_N - \sum_{i=1}^{N} y_i (T_i - T_{i-1}) K\right)^+ \\
&= \lambda \varphi(x) + (1-\lambda)\varphi(y).
\end{aligned}
$$

Moreover, it holds

$$\begin{aligned}
|\varphi(x)-\varphi(y)| &= \left|\left(1-x_N-\sum_{i=1}^{N} x_i(T_i-T_{i-1})K\right)^+ - \left(1-y_N-\sum_{i=1}^{N} y_i(T_i-T_{i-1})K\right)^+\right| \\
&\leq \left|(x_N-y_N)+\sum_{i=1}^{N}(x_i-y_i)(T_i-T_{i-1})K\right| \\
&\leq \sum_{i=1}^{N}\big(1+(T_i-T_{i-1})K\big)|x_i-y_i| \\
&\leq \left(\sum_{i=1}^{N}\big(1+(T_i-T_{i-1})K\big)^2\right)^{\frac{1}{2}}|x-y|.
\end{aligned}$$

# Chapter 5

# Conclusion

In this thesis, we tame the uncertainty about the volatility in term structure models by using methods from robust finance and the calculus of $G$-Brownian motion. As opposed to stochastic volatility models, we consider a collection of probabilistic laws for the volatility process instead of one. This framework is naturally connected to a $G$-Brownian motion and enables us to acquire the results from the literature on $G$-Brownian motion. With the tools from the calculus of $G$-Brownian motion, we study classical models in mathematical finance for the term structure of interest rates and the pricing of interest rate derivatives in the presence of volatility uncertainty. As a result, we obtain arbitrage-free term structures that are completely robust with respect to misspecifications regarding the probabilistic law of the volatility, and we derive robust pricing formulas for derivative contracts in fixed income markets, depending on the term structure of interest rates.

In a first step, we investigate the traditional Hull-White model when there is uncertainty about the volatility. We show that the common approach to pricing zero-coupon bonds, martingale modeling, does not work in the presence of volatility uncertainty; hence, we follow a different approach: by introducing a market price of uncertainty, we adjust the short rate by its uncertain variance to obtain an arbitrage-free term structure. The resulting term structure is completely robust with respect to the volatility: the bond prices do neither depend on the future evolution of the volatility nor on its bounds; instead, they depend on the current value of the market price of uncertainty. In particular, the bonds are exponentially affine with respect to the short rate and the market price of uncertainty. Due to the adjustment of the short rate, the model is inconsistent with the traditional Hull-White model. However, the model becomes consistent with the traditional one after fitting the model prices to the yield curve. All of these results hold true if the short rate is driven by multiple risk factors.

In order to generalize the results from the first step, we study the famous HJM model in the presence of volatility uncertainty. The main result is a sufficient condition, called drift condition, for the absence of arbitrage on the related bond market. In the presence of volatility uncertainty, the absence of arbitrage requires additional market prices, which are referred to as the market prices of uncertainty. The drift condition fully characterizes the risk-neutral dynamics of the forward rate in terms of its diffusion term. Since the latter also includes the uncertain volatility, the risk-neutral forward rate dynamics exhibit drift uncertainty. Using the drift condition, it is possible to construct arbitrage-free term structure models in the presence of volatility uncertainty, which we demonstrate in examples. In particular, we obtain robust versions of the Ho-Lee term structure and the Hull-White term structure, respectively. In examples where this is not possible, the drift condition shows how to adjust the model in order to be arbitrage-free when the volatility is uncertain, which we demonstrate in an example corresponding to the Vasicek term structure. The resulting term structures do not rely on any assumptions how the volatility evolves in the future; instead, the term structure is determined by the historical volatility. As a consequence, the resulting term structure models are completely robust with respect to the volatility.

In a last step, we deal with the pricing of contracts in fixed income markets under volatility uncertainty. The starting point is an arbitrage-free bond market under volatility uncertainty as determined by the previous results. Such a framework leads to a sublinear pricing measure, which we can use to determine either the price of a contract or its pricing bounds. To simplify the pricing of cashflows, we introduce the forward sublinear expectation, under which the expectations hypothesis holds in a robust sense. We can use the forward sublinear expectation to price bond options. Due to the nonlinearity of the pricing measure, we additionally derive methods to price contracts consisting of a collection of cashflows, which differs from the case without volatility uncertainty. We can price contracts with a simple payoff structure as in the classical case; for more general contracts, we need to use a backward induction procedure to find the price. We can use this procedure to price contracts consisting of a stream of bond options. These results enable us to price all major interest rate derivatives—including linear contracts, such as fixed coupon bonds, floating rate notes, and interest rate swaps, and nonlinear contracts, such as swaptions, caps and floors, and in-arrears contracts. We obtain a single price for linear contracts, which is the same as the one obtained by traditional term structure models, and a range of prices for nonlinear contracts, which is bounded by the prices from traditional models with the highest and the lowest possible volatility, respectively. Therefore, the pricing of typical interest rate derivatives reduces to computing prices in the corresponding model without volatility uncertainty. Since volatility uncertainty leads to market incompleteness, we can show that term structure models in the presence of volatility uncertainty naturally display unspanned stochastic volatility.

# Appendix A

# $G$-Brownian Motion Calculus

This chapter gives a brief introduction to the calculus of $G$-Brownian motion, which was invented by Peng (2007, 2008). We start by introducing sublinear expectation spaces, which can be seen as a generalization of probability spaces. Then we define distributional properties of random variables on such spaces. These notions allow us to state the definition of a $G$-Brownian motion and to construct a $G$-Brownian motion, proving its existence. In addition, we discuss the most important properties of a $G$-Brownian motion, and we do the first step in stochastic calculus with $G$-Brownian motion by defining stochastic integrals. Further results can be found in the book of Peng (2019) and in the related references given in the previous chapters. The exposition in this chapter closely follows the respective parts in the book of Peng (2019), while the proofs are omitted.

## A.1 Sublinear Expectation Spaces

In order to define a sublinear expectation space, we consider a set of possible states and a particular space of random variables. Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real-valued functions defined on $\Omega$. The set $\Omega$ represents the set of possible states in the future, which are presently unknown, and the space $\mathcal{H}$ (roughly speaking) consists of all random variables, which yield a certain outcome for each possible state in the future. We assume that $\mathcal{H}$ satisfies $c \in \mathcal{H}$ for $c \in \mathbb{R}$ and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. This is the minimal assumption on $\mathcal{H}$ that enables us to define a sublinear expectation space. However, the remaining definitions and results require an additional assumption on $\mathcal{H}$: we assume that $\varphi(X_1, ..., X_d) \in \mathcal{H}$ if $X_1, ..., X_d \in \mathcal{H}$ for $d \in \mathbb{N}$ and $\varphi \in C_{l,Lip}(\mathbb{R}^d)$, where $C_{l,Lip}(\mathbb{R}^d)$ denotes the linear space of functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^n + |y|^n)|x - y|$$

for some $C > 0$ and $n \in \mathbb{N}$, both depending on $\varphi$. We call $X = (X_1, ..., X_d)$ a *$d$-dimensional random vector* if $X_i \in \mathcal{H}$ for all $i = 1, ..., d$.

We obtain a sublinear expectation space if we equip the set of states and the space of random variables with a sublinear expectation, which is a sublinear functional defined on the space of random variables.

**Definition A.1.** *We call a functional* $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ *a sublinear expectation if it satisfies the following properties for* $X, Y \in \mathcal{H}$.

*(i) Monotonicity: If* $X \leq Y$*, then* $\hat{\mathbb{E}}[X] \leq \hat{\mathbb{E}}[Y]$.

*(ii) Constant Preserving: For* $c \in \mathbb{R}$*, it holds* $\hat{\mathbb{E}}[c] = c$.

*(iii) Subadditivity: It holds* $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$.

*(iv) Positive Homogeneity: For* $\lambda \geq 0$*, it holds* $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$.

The tripel $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is then called a *sublinear expectation space*. It should be noted that the positive homogeneity of a sublinear expectation $\hat{\mathbb{E}}$ is equivalent to

$$\hat{\mathbb{E}}[\lambda X] = \lambda^+ \hat{\mathbb{E}}[X] + \lambda^- \hat{\mathbb{E}}[-X]$$

for $\lambda \in \mathbb{R}$ and $X \in \mathcal{H}$.

Next, we construct the completion of a sublinear expectation space, which is needed in Sections A.3 and A.4. We have the following useful inequalities.

**Proposition A.2.** *For* $X, Y \in \mathcal{H}$ *and* $1 < p, q < \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*, it holds*

$$\hat{\mathbb{E}}[|XY|] \leq \hat{\mathbb{E}}[|X|^p]^{\frac{1}{p}} \hat{\mathbb{E}}[|Y|^q]^{\frac{1}{q}},$$
$$\hat{\mathbb{E}}[|X + Y|^p]^{\frac{1}{p}} \leq \hat{\mathbb{E}}[|X|^p]^{\frac{1}{p}} + \hat{\mathbb{E}}[|Y|^p]^{\frac{1}{p}}.$$

*In particular, for* $1 \leq p < \tilde{p}$*, it holds*

$$\hat{\mathbb{E}}[|X|^p]^{\frac{1}{p}} \leq \hat{\mathbb{E}}[|X|^{\tilde{p}}]^{\frac{1}{\tilde{p}}}.$$

For $p \geq 1$, we define $\mathcal{H}_0^p := \{X \in \mathcal{H} \mid \hat{\mathbb{E}}[|X|^p] = 0\}$, which is a linear subspace of $\mathcal{H}$. Then we take $\mathcal{H}_0^p$ as our null-space in order to introduce the quotient space $\mathcal{H}/\mathcal{H}_0^p$. For each equivalence class $\{X\} \in \mathcal{H}/\mathcal{H}_0^p$ with a representation $X \in \mathcal{H}$, we can define a sublinear expectation on the quotient space by $\hat{\mathbb{E}}[\{X\}] := \hat{\mathbb{E}}[X]$. Defining $\| \cdot \|_p := \hat{\mathbb{E}}[| \cdot |^p]^{\frac{1}{p}}$, we obtain a norm on $\mathcal{H}/\mathcal{H}_0^p$ by Proposition A.2. Then we extend $\mathcal{H}/\mathcal{H}_0^p$ to its completion under the norm $\| \cdot \|_p$, which we denote by $\hat{\mathcal{H}}_p$.

Since we can define a partial order on the completion of a sublinear expectation space, we are able to extend the sublinear expectation to a sublinear expectation on the completion. We define the mapping $^+ : \mathcal{H} \to \mathcal{H}$ by $X^+ := \max\{X, 0\}$, which can be continuously extended to $\hat{\mathcal{H}}_p$, since it is a contraction mapping by the inequality

$$|X^+ - Y^+| \leq |X - Y|.$$

Therefore, we can define a partial order, denoted by $\geq$, on the completion $\hat{\mathcal{H}}_p$—that is, we write $X \geq Y$ or $Y \leq X$ if $X - Y = (X - Y)^+$. Since it holds

$$|\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y]| \leq \hat{\mathbb{E}}[|X - Y|] \leq ||X - Y||_p,$$

the sublinear expectation $\hat{\mathbb{E}}$ can be continuously extended to a mapping on $\hat{\mathcal{H}}_p$ as well, on which it is still a sublinear expectation.

## A.2 $G$-Normal Distribution

To introduce $G$-normally distributed random vectors, we first need to define some notions related to distributions of random vectors on sublinear expectation spaces. We denote by $C_{b,Lip}(\mathbb{R}^d)$ the space of real-valued functions on $\mathbb{R}^d$ that are bounded and Lipschitz continuous, serving as the space of test functions in the following definitions.

**Definition A.3.** *Let $X$ and $Y$ be two $d$-dimensional random vectors on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. We say that $X$ and $Y$ are identically distributed, denoted by $X \overset{d}{=} Y$, if for all $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, it holds*

$$\hat{\mathbb{E}}[\varphi(X)] = \hat{\mathbb{E}}[\varphi(Y)].$$

**Definition A.4.** *We say that a $\tilde{d}$-dimensional random vector $Y$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is independent of a $d$-dimensional random vector $X$ on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ if for each $\varphi \in C_{b,Lip}(\mathbb{R}^{d+\tilde{d}})$, it holds*

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}].$$

**Definition A.5.** *Let $X$ and $Y$ be two $d$-dimensional random vectors on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. We call $Y$ an independent copy of $X$ if $X \overset{d}{=} Y$ and $Y$ is independent of $X$.*

The previous definitions enable us to define $G$-normally distributed random vectors. The definition generalizes the notion of normally distributed random vectors (with zero mean) from probability theory.

**Definition A.6.** *We say that a $d$-dimensional random vector $X$ on a sublinear expecta-*

*tion space* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *is* $G$*-normally distributed if*

$$aX + bY \overset{d}{=} \sqrt{a^2 + b^2}X$$

*for* $a, b \in \mathbb{R}_+$*, where* $Y$ *is an independent copy of* $X$*.*

The letter $G$ in the definition of a $G$-normally distributed random vector refers to a function that characterizes its distribution. For a $G$-normally distributed $d$-dimensional random vector $X$, we define the function $G : \mathbb{S}^d \to \mathbb{R}$ by

$$G(A) := \tfrac{1}{2}\hat{\mathbb{E}}[XAX'].$$

By using the properties of $\hat{\mathbb{E}}$, one can check that $G$ is a monotone, sublinear function; that is, for $A, B \in \mathbb{S}^d$, it satisfies

$$\begin{aligned} G(A) &\leq G(B) \quad \text{if } A \leq B, \\ G(\lambda A) &= \lambda G(A) \quad \text{for } \lambda \in \mathbb{R}_+, \\ G(A+B) &\leq G(A) + G(B). \end{aligned} \tag{A.1}$$

Since $G$ is also continuous, one can show that there exists a bounded, closed, and convex subset $\Sigma \subset \mathbb{S}^d_+$ such that

$$G(A) = \tfrac{1}{2} \sup_{\sigma \in \Sigma} \operatorname{tr}(\sigma A).$$

The function $G$ characterizes the distribution of a $G$-normally distributed random vector in the sense that its expectation (which satisfies additional properties) is the solution to a nonlinear partial differential equation with generator $G$.

**Proposition A.7.** *Let* $X$ *be a* $G$*-normally distributed* $d$*-dimensional random vector* $X$ *on a sublinear expectation space* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$*. Then the function* $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$*, which is defined by*

$$u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$$

*for* $\varphi \in C_{l,Lip}(\mathbb{R}^d)$*, satisfies for* $s, t \in \mathbb{R}_+$ *and* $x \in \mathbb{R}^d$*,*

$$u(t + s, x) = \hat{\mathbb{E}}[u(t, x + \sqrt{s}X)]$$

*and for* $T > 0$*, there exist constants* $C, k > 0$ *such that*

$$\begin{aligned} |u(t,x) - u(t,y)| &\leq C(1 + |x|^k + |y|^k)|x - y|, \\ |u(t,x) - u(t+s,x)| &\leq C(1 + |x|^k)\sqrt{s} \end{aligned}$$

*for $s, t \in [0, T]$ and $x, y \in \mathbb{R}^d$. Moreover, the function $u$ is the unique viscosity solution of the nonlinear partial differential equation*

$$\partial_t u + G(D_{xx}^2 u) = 0, \quad u(0, x) = \varphi(x).$$

Conversely, one can show that for an arbitrary monotone, sublinear, and continuous function $G$ there exists a $G$-normally distributed random variable, which proves the existence of $G$-normally distributed random variables.

**Proposition A.8.** *Let $G : \mathbb{S}^d \to \mathbb{R}$ be a continuous function that satisfies* (A.1)*. Then there exists a $G$-normally distributed $d$-dimensional random vector $X$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ satisfying*

$$G(A) = \tfrac{1}{2}\hat{\mathbb{E}}[XAX'].$$

## A.3 $G$-Brownian Motion

First of all, we introduce the definition of a $G$-Brownian motion, which is defined as a special type of stochastic process on a sublinear expectation space.

**Definition A.9.** *Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space. We call $(X_t)_{t\geq 0}$ a $d$-dimensional stochastic process if $X_t$ is a $d$-dimensional random vector in $\mathcal{H}$ for all $t$.*

**Definition A.10.** *We call a $d$-dimensional stochastic process $B = (B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ a $G$-Brownian motion if it satisfies the following properties.*

*(i) It holds $B_0 = 0$.*

*(ii) For $t, s \in \mathbb{R}_+$, the increments $B_{t+s} - B_t$ and $B_s$ are identically distributed, and $B_{t+s} - B_t$ is independent of $(B_{t_1}, ..., B_{t_n})$ for $n \in \mathbb{N}$ and $0 \leq t_1 \leq ... \leq t_n \leq t$.*

*(iii) It holds $\lim_{t\to 0} \frac{1}{t}\hat{\mathbb{E}}[|B_t|^3] = 0$ and $\hat{\mathbb{E}}[B_t] = 0 = \hat{\mathbb{E}}[-B_t]$.*

Similar to the $G$-normal distribution, the letter $G$ in the previous definition refers to a function that characterizes the distribution of a $G$-Brownian motion. For a $G$-Brownian motion $B$, we define the function $G : \mathbb{S}^d \to \mathbb{R}$ by

$$G(A) := \tfrac{1}{2}\hat{\mathbb{E}}[B_1 A B_1'].$$

It is the generator of the partial differential equation that characterizes the distribution of $B$, which shows that a $G$-Brownian motion is $G$-normally distributed.

**Theorem A.11.** *Let $B$ be a $d$-dimensional $G$-Brownian motion on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Then the function $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, defined by*

$$u(t,x) := \hat{\mathbb{E}}[\varphi(x + B_t)]$$

*for $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, is the viscosity solution of the nonlinear partial differential equation*

$$\partial_t u - G(D^2_{xx}u) = 0, \quad u(0,x) = \varphi(x).$$

*In particular, then $B_1$ is $G$-normally distributed and $B_t \overset{d}{=} \sqrt{t}B_1$.*

**Remark A.12.** *If $B = (B^1, ..., B^d)$ is a $d$-dimensional $G$-Brownian motion on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, then for all $i = 1, ..., d$, we know that $B^i$ is a one-dimensional $G_i$-Brownian motion, where $G_i : \mathbb{R} \to \mathbb{R}$ is given by*

$$G_i(a) = \tfrac{1}{2}\hat{\mathbb{E}}[(B^i_1)^2 a].$$

In order to construct a $G$-Brownian motion, which ultimately proves its existence, we consider the canonical process on the following spaces. Let $\Omega := C^d_0(\mathbb{R}_+)$, equipped with the distance $\delta : \Omega \times \Omega \to \mathbb{R}$, defined by

$$\delta(\omega, \tilde{\omega}) := \sum_{i=1}^{\infty} 2^{-i}\big((\max_{t\in[0,i]} |\omega_t - \tilde{\omega}_t|) \wedge 1\big).$$

For $T \in \mathbb{R}_+$, let $\Omega_T := C^d_0([0,T])$ and let $B = (B_t)_{t\geq 0}$ be the canonical process on $\Omega$. Then we define the spaces

$$Lip(\Omega_T) := \big\{\varphi(B_{t_1\wedge T}, ..., B_{t_n\wedge T}) \bigm| n \in \mathbb{N},\ t_1, ..., t_n \in \mathbb{R}_+,\ \varphi \in C_{l,Lip}(\mathbb{R}^{d\times n})\big\}$$

for $T \in \mathbb{R}_+$ and $Lip(\Omega) := \bigcup_{i=1}^{\infty} Lip(\Omega_i)$. In particular, we have $B_t \in Lip(\Omega)$ for all $t$.

Next, we construct a sublinear expectation on $(\Omega, Lip(\Omega))$ such that the canonical process $B$ is a $G$-Brownian motion. For a given monotone, sublinear function $G : \mathbb{S}^d \to \mathbb{R}$, let $(\xi_i)_{i=1}^{\infty}$ be a sequence of $G$-normally distributed $d$-dimensional random vectors on a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ (which exist by Proposition A.8) such that $\xi_{i+1}$ is independent of $(\xi_1, ..., \xi_i)$ for all $i$. For $X \in Lip(\Omega)$ such that

$$X = \varphi(B_{t_1} - B_{t_0}, ..., B_{t_n} - B_{t_{n-1}})$$

for $\varphi \in C_{l,Lip}(\mathbb{R}^{d\times n})$ and $0 = t_0 < t_1 < ... < t_n < \infty$, where $n \in \mathbb{N}$, we define the sublinear expectation $\hat{\mathbb{E}}$ by

$$\hat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, ..., B_{t_n} - B_{t_{n-1}})] := \tilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, ..., \sqrt{t_n - t_{n-1}}\xi_n)],$$

and for all $i = 1, ..., n-1$, we define the related conditional sublinear expectation $\hat{\mathbb{E}}_{t_i}$, mapping into $Lip(\Omega_{t_i})$, by

$$\hat{\mathbb{E}}_{t_i}[\varphi(B_{t_1} - B_{t_0}, ..., B_{t_n} - B_{t_{n-1}})] := \psi(B_{t_1} - B_{t_0}, ..., B_{t_i} - B_{t_{i-1}}),$$

where the function $\psi : \mathbb{R}^{d \times i} \to \mathbb{R}$ is defined by

$$\psi(x_1, ..., x_i) := \tilde{\mathbb{E}}[\varphi(x_1, ..., x_i, \sqrt{t_{i+1} - t_i}\xi_{i+1}, ..., \sqrt{t_n - t_{n-1}}\xi_n)].$$

Then the canonical process $B$ is a $G$-Brownian motion under $\hat{\mathbb{E}}$. We call the sublinear expectation $\hat{\mathbb{E}}$, defined by this procedure, *$G$-expectation.*

The $G$-expectation can be extended to the completions of the spaces introduced above, respectively. For $p \geq 1$, we denote by $L_G^p(\Omega)$ and $L_G^p(\Omega_T)$ the completions of $Lip(\Omega)$ and $Lip(\Omega_T)$ under the norm $\|\cdot\|_p = \hat{\mathbb{E}}[|\cdot|^p]^{\frac{1}{p}}$, respectively, which can be constructed as described in Section A.1. Then we can continuously extend the $G$-expectation to a sublinear expectation on $(\Omega, L_G^1(\Omega))$, which we still denote by $\hat{\mathbb{E}}$. In addition, the conditional $G$-expectation satisfies

$$\left\|\hat{\mathbb{E}}_t[X] - \hat{\mathbb{E}}_t[Y]\right\|_1 \leq \|X - Y\|_1;$$

thus, we can extend it to a continuous mapping $\hat{\mathbb{E}}_t : L_G^1(\Omega) \to L_G^1(\Omega_t)$. One can show that the conditional $G$-expectation satisfies the following properties—including the tower property, which shows that the $G$-expectation is time consistent.

**Proposition A.13.** *For $X, Y \in L_G^1(\Omega)$ and $s, t \in \mathbb{R}_+$, the following properties hold.*

*(i) If $X \leq Y$, then $\hat{\mathbb{E}}_t[X] \leq \hat{\mathbb{E}}_t[Y]$.*

*(ii) For $\xi \in L_G^1(\Omega_t)$, it holds $\hat{\mathbb{E}}_t[\xi] = \xi$.*

*(iii) It holds $\hat{\mathbb{E}}_t[X + Y] \leq \hat{\mathbb{E}}_t[X] + \hat{\mathbb{E}}_t[Y]$.*

*(iv) For $\xi \in L_G^1(\Omega_t)$ bounded, it holds $\hat{\mathbb{E}}_t[\xi X] = \xi^+\hat{\mathbb{E}}_t[X] + \xi^-\hat{\mathbb{E}}_t[-X]$.*

*(v) It holds $\hat{\mathbb{E}}_s[\hat{\mathbb{E}}_t[X]] = \hat{\mathbb{E}}_{s\wedge t}[X]$ and, in particular, it holds $\hat{\mathbb{E}}[\hat{\mathbb{E}}_t[X]] = \hat{\mathbb{E}}[X]$.*

## A.4 Stochastic Integrals

We construct the space of stochastic processes that are admissible for stochastic integration as the completion of the space of simple processes. For $p \geq 1$ and $T > 0$, we denote by $M_G^{p,0}(0,T)$ the collection of all processes $\eta = (\eta_t)_{0\le t\le T}$ that are of the form

$$\eta_t = \sum_{i=1}^{N} \xi_i 1_{[t_{i-1},t_i)}(t),$$

where $\xi_i \in L_G^p(\Omega_{t_{i-1}})$ for all $i = 1, ..., N$ and $0 = t_0 < t_1 < ... < t_N = T$. For each simple process $\eta \in M_G^{p,0}(0,T)$ with a representation from above, we define the related Bochner integral by

$$\int_0^T \eta_t dt := \sum_{i=1}^{N} \xi_i (t_i - t_{i-1}).$$

Then we can define the sublinear expectation $\tilde{\mathbb{E}} : M_G^{p,0}(0,T) \to \mathbb{R}$ by

$$\tilde{\mathbb{E}}[\eta] := \tfrac{1}{T}\hat{\mathbb{E}}\left[\int_0^T \eta_t dt\right].$$

Therefore, as in Section A.1, we can take the completion of $M_G^{p,0}(0,T)$, which we denote by $M_G^p(0,T)$, under the norm $\|\cdot\|_{M,p}$, defined by

$$\|\eta\|_{M,p} := \hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p dt\right]^{\frac{1}{p}}.$$

We denote by $M_G^p(0,T;\mathbb{R}^d)$ the space of all $d$-dimensional processes $\eta = (\eta^1, ..., \eta^d)$ such that $\eta^i \in M_G^p(0,T)$ for all $i = 1, ..., d$.

Next, we define stochastic integrals for integrands in the space of admissible stochastic processes by using the isometry property of stochastic integrals. For this purpose, we consider a one-dimensional $G$-Brownian motion $B = (B_t)_{t\ge 0}$ with

$$G(a) = \tfrac{1}{2} \sup_{\sigma\in[\underline{\sigma}^2,\overline{\sigma}^2]} \{\sigma a\}$$

for $a \in \mathbb{R}$; we can define the stochastic integral of a $d$-dimensional stochastic process with respect to a $d$-dimensional $G$-Brownian motion for each component separately. For each simple process $\eta \in M_G^{2,0}(0,T)$ with a representation as introduced at the beginning of this section, we define the stochastic integral

$$\int_0^T \eta_t dB_t := \sum_{i=1}^{N} \xi_i (B_{t_i} - B_{t_{i-1}}),$$

mapping into $L_G^2(\Omega_T)$. Then we have the following result, which allows us to extend the stochastic integral to the completion of its domain.

**Lemma A.14.** *For $\eta \in M_G^{2,0}(0,T)$, it holds*

$$\hat{\mathbb{E}}\left[\int_0^T \eta_t dB_t\right] = 0,$$
$$\hat{\mathbb{E}}\left[\left(\int_0^T \eta_t dB_t\right)^2\right] \leq \overline{\sigma}^2 \hat{\mathbb{E}}\left[\int_0^T \eta_t^2 dt\right].$$

Hence, the stochastic integral is a continuous linear mapping, and it can be continuously extended to a mapping from $M_G^2(0,T)$ into $L_G^2(\Omega_T)$, still satisfying the properties from Lemma A.14. Moreover, the stochastic integral satisfies the following properties.

**Proposition A.15.** *Let $\eta, \theta \in M_G^2(0,T)$ and $0 \leq r \leq s \leq t \leq T$. Then we have the following properties.*

*(i) It holds*

$$\int_r^t \eta_u dB_u = \int_r^s \eta_u dB_u + \int_s^t \eta_u dB_u.$$

*(ii) For $X \in L_G^1(\Omega_s)$ bounded, it holds*

$$\int_s^t (X\eta_u + \theta_u) dB_u = X \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u.$$

*(iii) For $X \in L_G^1(\Omega)$, it holds*

$$\hat{\mathbb{E}}_r\left[X + \int_s^t \eta_u dB_u\right] = \hat{\mathbb{E}}_r[X].$$

## A.5 Quadratic Variation

First of all, we introduce the quadratic variation process of a one-dimensional $G$-Brownian motion. For a one-dimensional $G$-Brownian motion $B = (B_t)_{t \geq 0}$ with

$$G(a) = \tfrac{1}{2} \sup_{\sigma \in [\underline{\sigma}^2, \overline{\sigma}^2]} \{\sigma a\}$$

for $a \in \mathbb{R}$, we define the quadratic variation process $\langle B \rangle = (\langle B \rangle_t)_{t \geq 0}$ by

$$\langle B \rangle_t := \lim_{n \to \infty} \sum_{i=1}^N (B_{t_i^n} - B_{t_{i-1}^n})^2,$$

where $0 = t_0^n < t_1^n < ... < t_N^n = t$ for each $n \in \mathbb{N}$ such that

$$\lim_{n \to \infty} \max_{i=1,...,N} \{t_i^n - t_{i-1}^n\} = 0.$$

We know that the limit in the definition of $\langle B \rangle$ is well-defined, since we have the identity

$$\sum_{i=1}^{N}(B_{t_i^n} - B_{t_{i-1}^n})^2 = B_t^2 - 2\sum_{i=1}^{N} B_{t_{i-1}^n}(B_{t_i^n} - B_{t_{i-1}^n}),$$

and the sum on the right-hand side converges in $L_G^2(\Omega)$ as $n \to \infty$.

In contrast to the quadratic variation of a standard Brownian motion, the quadratic variation of a $G$-Brownian motion is an uncertain process. The process $\langle B \rangle$ is an increasing process with $\langle B \rangle_0 = 0$. Moreover, for $t, s \in \mathbb{R}_+$, the increments $\langle B \rangle_{t+s} - \langle B \rangle_t$ and $\langle B \rangle_s$ are identically distributed, the increment $\langle B \rangle_{t+s} - \langle B \rangle_t$ is independent of $(\langle B \rangle_{t_1}, ..., \langle B \rangle_{t_n})$ for $n \in \mathbb{N}$ and $0 \leq t_1 \leq ... \leq t_n \leq t$, and it holds $\lim_{t \to 0} \frac{1}{t}\hat{\mathbb{E}}[|\langle B \rangle_t|^2] = 0$. Then one can show that the quadratic variation is maximally distributed, which means that it satisfies the following property.

**Theorem A.16.** *For $\varphi \in C_{l,Lip}(\mathbb{R})$, it holds*

$$\hat{\mathbb{E}}[\varphi(\langle B \rangle_t)] = \sup_{\sigma \in [\underline{\sigma}^2, \overline{\sigma}^2]} \varphi(\sigma t).$$

Therefore, the quadratic variation of a $G$-Brownian motion is not deterministic (unless it holds $\overline{\sigma} = \underline{\sigma}$). Due to Theorem A.16, the process $\langle B \rangle$ is bounded from above, respectively below, by the quadratic variation process corresponding to a standard Brownian motion with volatility $\overline{\sigma}$, respectively $\underline{\sigma}$.

**Corollary A.17.** *It holds*

$$\overline{\sigma}^2 t \geq \langle B \rangle_t \geq \underline{\sigma}^2 t.$$

We can define integrals with respect to the quadratic variation process by the same procedure as in the previous section. For a simple process $\eta \in M_G^{1,0}(0,T)$ with a representation as introduced at the beginning of Section A.4, we define

$$\int_0^T \eta_t d\langle B \rangle_t = \sum_{i=1}^{N} \xi_i(\langle B \rangle_{t_i} - \langle B \rangle_{t_{i-1}}),$$

mapping into $L_G^1(\Omega_T)$. Similar to the integral with respect to a $G$-Brownian motion, we have the following isometry.

**Lemma A.18.** *For $\eta \in M_G^{1,0}(0,T)$, it holds*

$$\hat{\mathbb{E}}\Big[\Big|\int_0^T \eta_t d\langle B \rangle_t\Big|\Big] \leq \overline{\sigma}^2 \hat{\mathbb{E}}\Big[\int_0^T |\eta_t| dt\Big].$$

Thus, the integral can be extended to a mapping from $M_G^1(0,T)$ into $L_G^1(\Omega_T)$, which still satisfies the inequality from Lemma A.18. We also have the following identity.

**Proposition A.19.** *For $\eta \in M_G^2(0,T)$, it holds*

$$\hat{\mathbb{E}}\Big[\Big(\int_0^T \eta_t dB_t\Big)^2\Big] = \hat{\mathbb{E}}\Big[\int_0^T \eta_t^2 d\langle B\rangle_t\Big].$$

We treat the $d$-dimensional case by introducing the quadratic covariation process. For a $d$-dimensional $G$-Brownian motion $B = (B_t^1, ..., B_t^d)_{t\geq 0}$, we can define the quadratic variation process for each component separately. The only important case missing is the quadratic covariation of two different components. For this purpose, we consider two one-dimensional $G$-Brownian motions $B = (B_t)_{t\geq 0}$ and $\tilde{B} = (\tilde{B}_t)_{t\geq 0}$. Then we define the quadratic covariation process $\langle B, \tilde{B}\rangle = (\langle B, \tilde{B}\rangle_t)_{t\geq 0}$ by

$$\langle B, \tilde{B}\rangle_t := \lim_{n\to\infty} \sum_{i=1}^N (B_{t_i^n} - B_{t_{i-1}^n})(\tilde{B}_{t_i^n} - \tilde{B}_{t_{i-1}^n}),$$

where $0 = t_0^n < t_1^n < ... < t_N^n = t$ for each $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \max_{i=1,...,N} \{t_i^n - t_{i-1}^n\} = 0.$$

This definition is meaningful, since it holds

$$\begin{aligned}\sum_{i=1}^N (B_{t_i^n} - B_{t_{i-1}^n})(\tilde{B}_{t_i^n} - \tilde{B}_{t_{i-1}^n}) &= \tfrac{1}{4}\sum_{i=1}^N \big((B_{t_i^n} + \tilde{B}_{t_i^n}) - (B_{t_{i-1}^n} + \tilde{B}_{t_{i-1}^n})\big)^2 \\ &\quad + \tfrac{1}{4}\sum_{i=1}^N \big((B_{t_i^n} - \tilde{B}_{t_i^n}) - (B_{t_{i-1}^n} - \tilde{B}_{t_{i-1}^n})\big)^2\end{aligned}$$

and the two sums on the right-hand side converge to the quadratic variations of $B + \tilde{B}$ and $B - \tilde{B}$, respectively, which are both one-dimensional $G$-Brownian motions. Thus,

$$\langle B, \tilde{B}\rangle_t = \tfrac{1}{4}\langle B + \tilde{B}\rangle_t + \tfrac{1}{4}\langle B - \tilde{B}\rangle_t.$$

For $\eta \in M_G^1(0,T)$, we then define the integral

$$\int_0^T \eta_t d\langle B, \tilde{B}\rangle_t := \tfrac{1}{4}\int_0^T \eta_t d\langle B + \tilde{B}\rangle_t + \tfrac{1}{4}\int_0^T \eta_t d\langle B - \tilde{B}\rangle_t.$$

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