Nonuniqueness of Laws on State and Path Space: Flow Selections and Superposition for Fokker–Planck–Kolmogorov Equations and Convex Integration for Stochastic Hypodissipative Navier–Stokes Equations

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Summary

This thesis is divided into three parts, which are essentially independent of each other, although the equations studied in the first two parts overlap to some degree. In Parts I and II, we study parabolic Fokker–Planck–Kolmogorov equations (FPK equations), which are second-order differential equations for measures. In both parts, we prove structural results, which are of particular interest in cases of nonuniqueness of solutions.

More precisely, in Part I, we select *solution flows* for FPK equations, i.e. under suitable assumptions on the coefficients, we choose a particular solution for each initial condition, such that the selected family fulfills the flow property. The selection is made either in the whole class of solutions or in suitable subclasses. Moreover, we show that such a solution flow is unique if and only if the equation is well-posed in the respective solution classes. Our results blend into results of Markovian selections for stochastic problems and, in a lose sense, are parallel to Markovian selections to martingale problems by Stroock and Varadhan. We prove our results in the case of linear and nonlinear equations for measures on $\mathbb{R}^d$, as well as for linear equations for measures on $\mathbb{R}^\infty$.

In Part II, we study deterministic and stochastic nonlinear FPK equations on $\mathbb{R}^d$. In spirit of the recent work [190], we use and extend the *linearization* of such equations. More precisely, it is known that deterministic nonlinear FPK equations admit a naturally associated linear first-order continuity equation for curves in the space of measures $\mathcal{P}(\mathcal{P})$. In this case, we prove a *superposition principle* between solutions to these equations, without imposing any regularity on the coefficients. This result is in the spirit of well-known superposition principles for ordinary and stochastic differential equations and their corresponding first- and second-order linear FPK equations. In our case, the nonlinear FPK equation replaces the ordinary differential equation, and the continuity equation for curves in $\mathcal{P}(\mathcal{P})$ replaces the linear FPK equation for measures on $\mathbb{R}^d$. Moreover, we extend the linearization to the case of *stochastic* nonlinear FPK equations by showing that such equations are associated to deterministic second-order equations for curves in $\mathcal{P}(\mathcal{P})$. Also in this case, we prove a corresponding superposition principle.

In Part III, which can be considered entirely independent of the previous parts, we apply the method of *convex integration* to the incompressible fractional Navier–Stokes equations on the 3D torus, with the exponent α of the fractional Laplacian in the range $0 < \alpha < 1/2$, perturbed by an additive Brownian noise. Similar to comparable existing results for other stochastic equations, we prove nonuniqueness in law for analytically and probabilistically weak solutions. In comparison with the existing literature for stochastic equations, we obtain our new result by a use of simpler building blocks for the construction of a solution with anomalous energy behavior. Notably, we construct a solution, which is even probabilistically strong up to a strictly positive stopping time.

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Chapter 0

General notation and basic facts

Symbols. We use the conventions $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}_+ = [0, \infty)$, and the symbols $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ for the fields of rational, real and complex numbers, respectively. For $z \in \mathbb{C}$, $\bar{z}$ denotes its complex conjugate. The lower and upper Gauss brackets are denoted by $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$, respectively. For $x, y \in \mathbb{R}$, we write $x \wedge y$ and $x \vee y$ for their minimum and maximum, respectively.

Euclidean space, analysis and matrices. In $\mathbb{R}^d$, e_i, $1 \le i \le d$, denotes the canonical i-th unit vector, and for $a = (a_1, \dots, a_d)$ and $b = (b_1, \dots, b_d) \in \mathbb{R}^d$, we write $a \cdot b := \langle a, b\rangle_{\mathbb{R}^d}$ for the usual Euclidean inner product with induced norm $|a|^2 := a \cdot a$. We use the same notation $A \cdot b := \big((A_{ij})_{1\le i\le d} \cdot b\big)_{1\le j\le d_1} \in \mathbb{R}^{d_1}$ for a $d \times d_1$-matrix A, and also write

$$x \cdot y := \sum_{k=1}^{\infty} x_k y_k, \quad x, y \in \ell^2,$$

for the usual inner product in the space of square-summable real sequences ℓ^2. $\mathbb{S}^2 \subseteq \mathbb{R}^3$ denotes the unit sphere, id the identity on $\mathbb{R}^d$, $a \otimes b := (a_k b_l)_{1\le k,l\le d}$ the outer product of $a, b \in \mathbb{R}^d$, and $a \times b \in \mathbb{R}^3$ the usual cross product. The linear span of vectors $v_1, \dots, v_n \in \mathbb{R}^d$ is denoted by $\langle v_1, \dots, v_n\rangle$. $B_r(x)$ is the ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$.

Analysis. For once and twice differentiable functions $\varphi : \mathbb{R}^d \to \mathbb{R}$, we use the standard notation $\partial_i\varphi$, $\partial_{ij}\varphi$ for the first- and second-order partial derivatives in direction e_i, and e_i and e_j, respectively. For $i = j$, we also write ∂_i^2 instead of ∂_{ij}. If φ is partially differentiable in a distinguished variable $t \in \mathbb{R}$, we denote the partial derivative by $\partial_t\varphi$. In the case $d = 1$, we write φ' for the derivative of φ. We use the standard symbols $\nabla\varphi$ and $\Delta\varphi$ for the Euclidean gradient and Laplace operator, respectively. For vector fields $F : \mathbb{R}^d \to \mathbb{R}^d$, we denote the divergence of F by $\operatorname{div} F$ and, in the case $d = 3$, the curl-operator by $\operatorname{curl} F$.

Matrices. We write Id for the $d \times d$-identity matrix in any dimension d. For real $d \times d$-matrices A, B, we write A^T for the transpose of A, and $A : B := \sum_{i,j=1}^d A_{ij} B_{ij}$. We use the same notation for $A = (A_{ij})_{i,j\ge1}$, $B = (B_{ij})_{i,j\ge1}$, if either A or B contain only finitely many nontrivial entries. $\mathbb{S}_d^+$ is the space of symmetric positive semidefinite $d \times d$-matrices. We denote the *trace-free* part of a matrix A by $\mathring{A}$, and write $a \mathring{\otimes} b$ in the case $A = a \otimes b$.

Topology and measure theory. Let X, Y be topological spaces. We denote the Borel σ-algebra of X by $\mathcal{B}(X)$, the space of continuous, bounded real functions on X by $C_b(X)$, and the space of continuous functions from X to Y by $C(X,Y)$. For $x \in X$, we generally write $\pi_x : C(X,Y) \to Y$, $\pi_x : f \mapsto f(x)$ for the canonical projection on $C(X,Y)$. If $X = [t,T]$, we also write $C_{t,T}Y$. If $t = 0$ or if $X = \mathbb{R}_+$, we write $C_T Y$ and CY, respectively. For a subset $A \subseteq X$, we write $\mathcal{B}(X)_{|A}$ for the trace σ-algebra of A with respect to X. We say $A \subseteq X$ is *precompact*, if each sequence in A has a limit point in X. If Y is embedded into X, we write $Y \hookrightarrow X$. A metric d_1 is *compatible* with the topological space X, if d_1 induces the prescribed topology on X. A second metric d_2 on X is *weakly equivalent* to d_1, if both metrics induce the same topology on X. The topological closure of a set A is denoted by $\bar{A}$. The *support* of a function $f : X \to \mathbb{R}$ is $\operatorname{supp} f := \overline{\{x \in X : f(x) \neq 0\}}$.

Measure theory. If I is some index set and $\mathcal{C}$ is a fixed σ-algebra on Y, for mappings $f_i : X \to Y$, $i \in I$, we denote the σ-algebra on X generated by $\{f_i, i \in I\}$ by $\sigma(f_i, i \in I)$, which is the smallest σ-algebra $\mathcal{A}$ on X such that each f_i is $\mathcal{A}/\mathcal{C}$-measurable. By $\mathcal{A}_1 \bigvee \mathcal{A}_2$, we denote the *$\sigma$-algebra generated by $\mathcal{A}_1$ and $\mathcal{A}_2$*, i.e. the smallest σ-algebra containing $\mathcal{A}_1$ and $\mathcal{A}_2$. By a *measure* μ we mean a nonnegative, σ-additive function on the σ-algebra $\mathcal{F}$ of a measurable space $(\mathcal{X}, \mathcal{F})$, and say μ is *bounded* or *finite*, provided $\mu(\mathcal{X}) < \infty$. If μ also assumes negative values, we call it a *signed measure*. If $\mathcal{X}$ is a topological space and $\mathcal{F} = \mathcal{B}(\mathcal{X})$, μ is a *Borel measure*. In this case, we denote by $\operatorname{supp} \mu$ the *support* of μ, i.e. the set of all points $x \in \mathcal{X}$ such that $\mu(N_x) > 0$ for every open neighborhood N_x of x. For $x \in \mathcal{X}$, δ_x is the usual *Dirac measure* in x. A set $N \subseteq \mathcal{X}$ is *μ-negligible*, if there is a set $M \in \mathcal{F}$ such that $N \subseteq M$ and $\mu(M) = 0$. The Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$ is denoted by dx (we also write dt in the case $d = 1$, if $\mathbb{R}$ is considered as the axis of time). A set of functions $\mathcal{G} \subseteq C_b(\mathcal{X})$ is *measure separating*, if $\mu(g) = \nu(g)$ for each $g \in \mathcal{G}$ implies $\mu = \nu$ for each pair of bounded Borel measures μ, ν on $\mathcal{X}$. For an interval $I \subseteq \mathbb{R}$, a family $(\mu_t)_{t \in I}$ of finite Borel measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is a *Borel curve*, if $t \mapsto \mu_t(A)$ is Borel measurable for each $A \in \mathcal{B}(\mathcal{X})$. Two such curves are *versions* of each other, if there is a dt-negligible set $N \subseteq I$ such that both curves coincide for each $t \in N^c$.

Operators. For normed spaces U, H, $L(U,H)$ is the vector space of linear continuous (equivalently: bounded) operators $T : U \to H$, with the usual operator norm $||\cdot||_{L(U,H)}$. If U, H are Hilbert spaces, we denote by $L_2(U,H)$ the subspace of $L(U,H)$ of *Hilbert–Schmidt operators* $T : U \to H$, i.e. the space of operators T such that $\sum_{k=1}^{\infty} ||Te_k||_H^2 < \infty$ for some (equivalently: any) orthonormal basis (ONB) $\{e_k\}_{k \in \mathbb{N}}$ of U. If $U = H$, we shortly write $L(H)$ and $L_2(H)$, respectively. The *trace* of an operator $T : H \to H$ on a Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$ is $\operatorname{Tr} T := \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle_H$ for a fixed ONB $\{e_k\}_{k \in \mathbb{N}}$ on H. If $T : U \to H$ is an unbounded operator, we denote its domain by $\mathcal{D}(T) \subseteq U$.

Function spaces. If X, Y are metric spaces and X is not compact, we also write $C_{\text{loc}}(X,Y)$ instead of $C(X,Y)$ to stress that the space is endowed with the topology of locally uniform convergence. If $X = \mathbb{R}_+$, we also write CY or $C_{\text{loc}}Y$. If Y carries a norm $||\cdot||_Y$ and the spaces are given from the context, we write $||\varphi||_\infty := \sup_{x \in X} ||\varphi(x)||_Y$.

Hölder functions. For a normed space $(Y, ||\cdot||_Y)$, $a < b$ and $\alpha \in (0,1]$, we denote the space of α-Hölder continuous functions $\varphi : [a,b] \mapsto Y$ by $C^\alpha([a,b],Y)$, with norm $||\varphi||_{C^\alpha([a,b],Y)} := ||\varphi||_\infty + [\varphi]_{C^\alpha}$, where

$$[\varphi]_{C^\alpha} := [\varphi]_{C^\alpha([a,b],Y)} := \sup_{\substack{t\neq r,\\ t,r\in[a,b]}} \frac{||\varphi(t)-\varphi(r)||_Y}{|t-r|^\alpha}.$$

If $[a,b]$ is replaced by $\mathbb{R}_+$, we write $C^\alpha_{\text{loc}}Y$ for the space of locally Hölder continuous functions with the usual topology of local convergence with respect to the norms $||\cdot||_{C^\alpha([a,b],Y)}$ on any compact interval $[a,b]$.

Differentiable functions. For an open set $A \subseteq \mathbb{R}^d$, we denote by $C^k_b(A)$ and $C^k_c(A)$ the spaces of functions $\varphi : A \to \mathbb{R}$ with continuous (bounded and compactly supported, respectively) partial derivatives up to order k. We write $C_b(\mathbb{R}^d)$ and $C_c(\mathbb{R}^d)$ in the case $k=0$, and $C^\infty_b(A)$ and $C^\infty_c(A)$ for $\bigcap_{k\geq 1} C^k_b(A)$ and $\bigcap_{k\geq 1} C^k_c(A)$, respectively. In both cases, for $k<\infty$, these spaces are normed with $||\cdot||_{C^k}$,

$$||\varphi||_{C^k} := ||\varphi||_\infty + \sum_{l=1}^k [\varphi]_{C^l},$$

where, using the usual notation $\partial^\alpha\varphi := \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}\varphi$ for a multiindex $\alpha = (\alpha_1,\dots,\alpha_d)$ of length $|\alpha| = \alpha_1+\cdots+\alpha_d$, we define the semi norm on $C^k_b(A)$ for $0 \leq l \leq k$ as

$$[\varphi]_{C^l} := \sum_{|\alpha|=l} ||\partial^\alpha\varphi||_\infty.$$

For $(a,b) \subseteq \mathbb{R}$, we denote by $C^{k,l}_b\big((a,b)\times\mathbb{R}^d\big)$ and $C^{k,l}_c\big((a,b)\times\mathbb{R}^d\big)$ the spaces of functions $\varphi : (a,b)\times\mathbb{R}^d \to \mathbb{R}$ with k continuous (bounded and compactly supported, respectively) derivatives with respect to $t \in (a,b)$ and l continuous (bounded and compactly supported, respectively) partial derivatives with respect to $x \in \mathbb{R}^d$. If $k=l$, we write $C^k_b\big((a,b)\times\mathbb{R}^d\big)$ and $C^k_c\big((a,b)\times\mathbb{R}^d\big)$. We also set $C^\infty_b\big((a,b)\times\mathbb{R}^d\big) := \bigcap_{k\geq 1} C^k_b\big((a,b)\times\mathbb{R}^d\big)$ and $C^\infty_c\big((a,b)\times\mathbb{R}^d\big) := \bigcap_{k\geq 1} C^k_c\big((a,b)\times\mathbb{R}^d\big)$.

L^p-spaces. For $n \geq 1$, $p \in [1,\infty)$, and a Borel measure μ on $\mathbb{R}^d$, we denote by $L^p(\mathbb{R}^d,\mathbb{R}^n;\mu)$ the space of equivalence classes (with respect to equality μ-a.e.) of Borel functions $\varphi : \mathbb{R}^d \to \mathbb{R}^n$ such that

$$||\varphi||^p_{L^p} := ||\varphi||^p_{L^p(\mathbb{R}^d,\mathbb{R}^n;\mu)} := \int_{\mathbb{R}^d} |\varphi(x)|^p d\mu(x) < \infty.$$

If $p=2$, this norm is induced by the scalar product

$$\langle f,g\rangle_{L^2} := \langle f,g\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^n;\mu)} := \int_{\mathbb{R}^d} f\cdot g d\mu.$$

For an interval $I \subseteq \mathbb{R}$ and a normed space $(Y, ||\cdot||_Y)$, we denote by $L^p(I,Y) := L^p(I,Y;dt)$ the space of equivalence classes (with respect to equality dt-a.e.) of Borel functions $\varphi : I \to Y$ such that

$$||\varphi||^p_{L^p(I,Y)} := \int_I ||\varphi||^p_Y dt < \infty.$$

If $Y = \mathbb{R}$, we simply write $L^p(I)$. For $p = \infty$, we denote by $L^\infty(\mathbb{R}^d, \mathbb{R}^n; \mu)$ and $L^\infty(I, Y)$ the usual spaces of equivalence classes of Borel functions, which are bounded up to a negligible set, and we use the same notation $||\cdot||_\infty$ for the essential supremum norm, as in the case of differentiable functions above. For $p \in [1, \infty]$, the usual local L^p-spaces are denoted by $L^\infty_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^n; \mu)$ and $L^p_{\text{loc}}(I, Y)$, with abbreviations as in the global case above.

Probability theory. A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ is *complete*, if both $\mathcal{F}$ and $\mathcal{F}_0$ contain all $\mathbb{P}$-negligible sets $N \subseteq \Omega$. We write $\mathbb{E}_{\mathbb{P}}[X] = \int_\Omega X d\mathbb{P}$ for the *expectation* of a random variable $X : \Omega \to \mathbb{R}$, and simply $\mathbb{E}[X]$, if the measure $\mathbb{P}$ is given from the context. A finite-dimensional Brownian motion $B = (B_t)_{0\le t\le T}$ on such a probability space is an $(\mathcal{F}_t)_{0\le t\le T}$-*Brownian motion*, if B_t is $(\mathcal{F}_t)_{0\le t\le T}$-adapted and $B_u - B_t$ is independent of $\mathcal{F}_t$ for each $0 \le t \le u \le T$. The *quadratic variation* of a stochastic process $t \mapsto X_t$ is denoted by $t \mapsto \langle\langle X\rangle\rangle_t$. Likewise, if Y is a second stochastic process, we write $t \mapsto \langle\langle X, Y\rangle\rangle_t$ for the *covariation* of X and Y.

Probability measures and weak topology. For a topological space X, $\mathcal{P}(X)$ is the space of Borel probability measures on X. We endow $\mathcal{P}(X)$ with the *topology of weak convergence of measures*, i.e. with the initial topology of the maps $\mu \mapsto \int \varphi d\mu$, $\varphi \in C_b(X)$. If X is Polish, then so is $\mathcal{P}(X)$. In particular, with this topology, the map $x \mapsto \delta_x$ is continuous. If $X = \mathbb{R}^d$, we write $\mathcal{P} = \mathcal{P}(\mathbb{R}^d)$, if no confusion on the dimension d can arise. A family of probability measures $(\mu_i)_{i\in I}$ on the Borel σ-algebra $\mathcal{B}(X)$ of a Polish space X is *tight*, if for each $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq X$ such that $\mu_i(K_\varepsilon^c) \le \varepsilon$ for all $i \in I$. Tightness of such a family is equivalent to its precompactness with respect to the weak topology. A sufficient condition for tightness is the existence of a *compact function* $V : X \to \mathbb{R}_+$ (i.e. V has compact sublevel sets $\{V \le c\}$, $c > 0$) such that $\sup_{i\in I} \int_X V d\mu_i < \infty$.

Subprobability measures and vague topology. $\mathcal{SP}$ is the set of Borel subprobability measures on $\mathbb{R}^d$, i.e. the set of Borel measures μ with $\mu(\mathbb{R}^d) \le 1$. We consider $\mathcal{SP}$ with the *vague topology*, i.e. the initial topology of the maps $\mu \mapsto \int \varphi d\mu$, $\varphi \in C_c(\mathbb{R}^d)$. Its Borel σ-algebra is denoted by $\mathcal{B}(\mathcal{SP})$. $\mathcal{SP}$ with the vague topology is Polish and compact. In particular, $\mathcal{P}(\mathcal{SP})$, the space of Borel probability measures on $\mathcal{SP}$, is a Polish space with the weak topology of probability measures on $(\mathcal{SP}, \mathcal{B}(\mathcal{SP}))$. In case of vague convergence $\mu_n \longrightarrow \mu$ in $\mathcal{SP}$ as $n \to \infty$ with $\mu_n(\mathbb{R}^d) \le c$ for some $c \in [0, 1]$ for all but finitely many $n \ge 1$, $\mu(\mathbb{R}^d) \le c$ follows. In particular, for each $c \in [0, 1]$, the set $\mathcal{SP}_c := \{\mu \in \mathcal{SP} : \mu(\mathbb{R}^d) \le c\}$ are closed in $\mathcal{SP}$, so that we obtain

$$\mathcal{P} = \Bigg(\bigcup_{q\in[0,1)\cap\mathbb{Q}} \mathcal{SP}_q\Bigg)^c \in \mathcal{B}(\mathcal{SP}).$$

We have $\mathcal{B}(\mathcal{P}) = \mathcal{B}(\mathcal{SP})_{|\mathcal{P}}$. Hence, we can consider measures $\Gamma \in \mathcal{P}(\mathcal{P})$ as elements in $\mathcal{P}(\mathcal{SP})$ with mass on $\mathcal{P}$. Standard references for these and further basics on (sub)probability measures can be found in [27, 17, 35].

Part I

Solution flows for Fokker–Planck–Kolmogorov equations

Abstract. We study Fokker–Planck–Kolmogorov equations and prove the existence of a solution flow $\{\mu^{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ for such equations, i.e. $\mu^{s,\nu} = (\mu_t^{s,\nu})_{t\in[s,T]}$ solves the equation with initial condition $(s,\nu) \in [0,T] \times \mathcal{SP}$ and the flow property $\mu_t^{s,\nu} = \mu_t^{r,\mu_r^{s,\nu}}$ prevails. Moreover, we show that the well-posedness of such equations is equivalent to the uniqueness of the flow. We obtain these results for linear and nonlinear Fokker–Planck–Kolmogorov equations for measures on $\mathbb{R}^d$, as well as for linear equations for measures on $\mathbb{R}^\infty$. The results of this part of the thesis contain and extend the contents of our paper [187].

Chapter 1

Introduction

1.1 Introduction to Fokker–Planck–Kolmogorov equations

In the present part, as well as in Part II of this thesis, we study parabolic Fokker–Planck–Kolmogorov equations (FPK equations), which are second-order equations for measures. We are mostly concerned with equations for measures on $\mathbb{R}^d$, but study equations for measures on $\mathbb{R}^\infty$ in Chapter 4 as well. In the former case, we are interested in linear and nonlinear equations.

In the linear, finite-dimensional case, the principal object of interest is the time-dependent second-order differential operator $\mathcal{L}$,

$$\mathcal{L}_t\varphi(x) = \sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{ij}\varphi(x) + \sum_{i=1}^{d} b_i(t,x)\partial_i\varphi(x), \quad \varphi \in C^2(\mathbb{R}^d), \tag{1.1}$$

for coefficients $a = (a_{ij})_{1\le i,j\le d} \in \mathbb{R}^{d\times d}, b = (b_i)_{1\le i\le d} \in \mathbb{R}^d$ on $[0,T] \times \mathbb{R}^d$. The linear FPK equation associated to these coefficients, to be solved for curves of Borel measures $t \mapsto \mu_t$ on $\mathbb{R}^d$, is

$$\partial_t\mu_t = \mathcal{L}_t^*\mu_t, \quad t \in [0,T], \tag{FPK}$$

where $T \in (0,\infty)$, $\mathcal{L}^*$ denotes the formal dual operator of $\mathcal{L}$, and the equation is understood in weak ("distributional") form, see Definition 2.1.1. In general, such equations make sense for Borel curves of bounded, signed measures, but for all considerations of this thesis, we restrict our attention to curves of (sub)probability measures. Consideration of such equations dates back at least to works by physicists Fokker [104] and Planck [185] from 1914 and 1917, respectively, and to Kolmogorov [138, 139, 140] about twenty years later. See also [208] and [63]. Since then, FPK equations have become an extensively studied field in physics, quantum mechanics, partial differential equations and stochastic analysis, and there is an enormous literature on historic and recent progress in the field. A full account on the available literature is beyond the scope of this thesis, but we particularly refer the reader to the extensive monograph [38] for a systematic introduction and for a vast list of references, see for example the comment section starting on p.283 in [38]. The book contains large parts of the available results on existence, uniqueness and regularity of solutions to (FPK), as well as the interesting topic of elliptic FPK equations, which we do not touch in this thesis.

Stochastic analytic background. Among several directions from which one can motivate equations of type (FPK), the approach via stochastic analysis and the theory of diffusion processes and stochastic differential equations is of particular interest to us. Here, we briefly outline this connection. Suppose $t \mapsto X_t$ is a stochastic process governed by the stochastic equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \in [0, T], \tag{1.2}$$

where B is a d_1-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $b = (b_i)_{1 \leq i \leq d}$ is as above and $\sigma(t, x) \in \mathbb{R}^{d \times d_1}$ is a diffusion coefficient. Then, by Itô's formula, the curve of one-dimensional marginals of X_t, $\mu_t := \mathbb{P} \circ X_t^{-1}$, is a solution to (FPK) with coefficients b and $a = 1/2\sigma\sigma^T$, see Appendix B for more details. More generally, the same is true for the transition probabilities $P(0, x, t, dy)$, $(t, x) \in [0, T] \times \mathbb{R}^d$, for any *diffusion process* with moderate assumptions on its drift and diffusion coefficient b and a, respectively, see [38, Prop.1.3.1] and also [230, 144] for background on the topic. Recall that for $B \in \mathcal{B}(\mathbb{R}^d)$, $P(s, x, t, B)$ intuitively gives the probability that $X_t \in B$, conditioned on the event $X_s = x$. In the special case $P(s, x, t, dy) = \rho(s, x, t, y)dy$, i.e. that the transitions are absolutely continuous with respect to Lebesgue measure, one rewrites (FPK) as a second-order partial differential equation for the densities $\rho(s, x, t, y)$ in (t, y) as

$$\partial_t \rho(s, x, t, y) = \partial_{ij}\big(a_{ij}(t, y)\rho(s, x, t, y)\big) - \partial_i\big(b_i(t, y)\rho(s, x, t, y)\big),$$

which is also called the *Kolmogorov forward equation* (in this case, (s, x) is considered a fixed initial condition). However, we stress that in general equations of type (FPK) need to be understood for curves of measures and only in special cases these measures admit densities with respect to Lebesgue measure.

The connection between FPK equations and diffusion processes and stochastic analysis is not only fruitful due to recent results, such as various highly interesting *superposition principles* (on which we comment in more detail in the introduction in Chapter 5), but it can also be considered one of the original motivations for the development of the field. Indeed, in the fundamental work [138], Kolmogorov posed the question of existence and uniqueness of solutions $\mu^{s,x} = (\mu_t^{s,x})_{t \in [s,T]}$ to equations of type (FPK) with initial condition $(s, \delta_x) \in \mathbb{R}_+ \times \mathcal{P}$, and asked whether such solutions fulfill the *Chapman-Kolmogorov equations*

$$\mu_t^{s,x} = \int_{\mathbb{R}^d} \mu_t^{r,y} d\mu_r^{s,x}(y), \quad s \leq r \leq t \leq T. \tag{CK}$$

Roughly, this amounts to the question whether solutions to (FPK) can be understood as the marginal distributions of a Markov process on $\mathbb{R}^d$. Related questions will be the main objective of this first part of the thesis, see in particular Section 2.5.

1.1.1 Nonlinear FPK equations

The theory of linear equations of type (FPK) is complemented by that of *nonlinear* FPK equations, which are governed by coefficients a_{ij} and b_i, $1 \leq i, j \leq d$, defined on $[0,T] \times \mathcal{M} \times \mathbb{R}^d$, for some set of bounded, signed Borel measures $\mathcal{M}$ on $\mathbb{R}^d$. More precisely, similar to the linear case, one considers the associated second-order operator $\mathcal{L}_{t,\mu}$, $(t,\mu) \in [0,T] \times \mathcal{M}$,

$$\mathcal{L}_{t,\mu}\varphi(x) = \sum_{i,j=1}^{d} a_{ij}(t,\mu,x)\partial_{ij}\varphi(x) + \sum_{i=1}^{d} b_i(t,\mu,x)\partial_i\varphi(x), \quad \varphi \in C^2(\mathbb{R}^d), \tag{1.3}$$

and the corresponding nonlinear FPK equation

$$\partial_t \mu_t = \mathcal{L}^*_{t,\mu_t}\mu_t, \quad t \in [0,T], \tag{NL-FPK}$$

in weak ("distributional") sense, to be solved for Borel curves of signed, bounded measures $(\mu_t)_{t\in[s,T]}$, as before. We study such type of equations in Chapter 3 and in Part II of this thesis, and we restrict attention to solution curves of (sub)probability measures. In particular, we are only interested in cases with $\mathcal{M} \subseteq \mathcal{SP}$.

The nonlinearity of equations of type (NL-FPK) arises from the dependence of the coefficients on the solution, which, comparable in spirit to the field of PDEs, renders the theory of well-posedness of such equations a more delicate issue compared to the linear case. See [105] for an introduction and overview of the field, as well as the content on nonlinear equations in [38]. Applications of such equations range from description of porous media and neurophysics to population dynamics and computational science, as explained in [105]. A typical nonlinear dependence of the coefficients is through convolution-type kernels, e.g. $b(t,\mu,x) = \int K(t,x,y)d\mu(y)$ for a Borel function K, which often fulfills $|K(t,x,y)| \longrightarrow \infty$ as $|y| \to \infty$, see [45, 95, 145, 146] for the important and historic example of Vlasov equations.

Here, we give a brief account on the close and very natural connection to interacting particle systems and McKean–Vlasov equations. As in the deterministic case, it is straightforward to check that the time-marginals of solution processes to certain stochastic differential equations fulfill an equation of type (NL-FPK). More precisely, consider the distribution-dependent SDE (*McKean–Vlasov equation*)

$$dX_t = b(t,\mathcal{L}_{X_t},X_t)dt + \sigma(t,\mathcal{L}_{X_t},X_t)dB_t, \quad t \in [0,T], \tag{1.4}$$

where $\mathcal{L}_X$ denotes the distribution of a random variable X, and b and σ are coefficients similar to the case of equation (1.2), which here additionally depend on $\mu \in \mathcal{M}$. For example, such equations describe particle movement in stochastic regimes, where the particle's behavior is governed both by its local position and its global distribution. Such equations arise in the theory of filtering [79] and multi-agent systems [44, 26], and as the governing equation in the particle limit of weakly interacting particle systems. Among the huge literature on such distribution-dependent stochastic equations and their connection to interacting particle systems, we refer to the classical works [106, 172, 227, 95, 216], as well as to the more recent papers [175, 114, 118, 173, 229, 126]. Moreover, we point out the recent comprehensive presentation in [61].

Similarly to the linear case, it is readily seen by Itô's formula that the curve of one-dimensional distributions $\mu_t := \mathcal{L}_{X_t}$ of any solution to (1.4) solves (NL-FPK) with coefficients b and $a = 1/2\sigma\sigma^T$. Indeed, by fixing $\mathcal{L}_{X_t}$ in b and σ in (1.4), $t \mapsto X_t$ is, of course, in particular a solution to a linear stochastic equation of type (1.2). Hence, by the observation for the linear case in the previous paragraph, it follows that $t \mapsto \mu_t$ solves (NL-FPK). This connection to linear equations was exploited in [21, 22] by Barbu and Röckner in order to solve equations of type (1.4) via the corresponding nonlinear FPK equations (NL-FPK): the authors establish a nonlinear superposition principle in order to lift solutions to the latter to a solution to the former equation (see Chapter 5 for a review and references). For further recent progress on nonlinear FPK equations, we also mention the works [119, 19, 20].

Connection to PDEs. Let us temporarily restrict attention to absolutely continuous solutions $t \mapsto \mu_t = \rho_t(x)dx$ to (NL-FPK), and assume the following special type of dependence of the coefficients $a_{ij}(t,\mu,x)$, $b_i(t,\mu,x)$ on $\mu \in \mathcal{M}$, with $\mathcal{M}$ equal to the set of measures absolutely continuous with respect to Lebesgue measure:

$$(t, \rho(y)dy, x) \mapsto a_{ij}(t,\rho(x),x), b_i(t,\rho(x),x), \quad 1 \leq i,j \leq d,$$

i.e. in contrast to the general situation, the dependence of the coefficients in the measure component is not global, but local in the argument of the density ρ of μ. To ensure measurability of the coefficients as a function in (t,x), one usually considers the Lebesgue version in the dx-equivalence class of ρ. Such coefficients are often called *Nemytskii-type* coefficients. Then, one rewrites the nonlinear FPK equation as a nonlinear PDE as

$$\partial_t \rho_t = \partial_{ij}\big(a_{ij}(t,\rho_t,x)\rho_t\big) - \partial_i\big(b_i(t,\rho_t,x)\rho_t\big). \tag{1.5}$$

Still, the equation is understood in weak sense, except for the special case that a_{ij} and b_i are sufficiently regular to make sense of their respective first- and second-order derivatives. This way, one can study the well-posedness in the class of absolutely continuous solutions to equations with Nemytskii-type coefficients by methods from the field of partial differential equations or, vice versa, one can aim to transfer existence and uniqueness results for FPK equations to PDEs of type (1.5). We point out that since equation (1.5) may still be considered a FPK equation, one can study its Cauchy problem with general measures as initial conditions. There are very interesting results on the existence of function-valued solutions to PDEs of type (1.5) with a completely degenerate initial condition, e.g. a Dirac measure δ_x, $x \in \mathbb{R}^d$, which are bounded after an infinitesimally short time, see [20] for such a result in the case of a generalized porous medium equation with a first-order perturbation term. There are also results on the uniqueness of such function-valued solutions to (1.5) in the case of a general bounded measures as initial data, at least in the case of a general porous medium equation without first-order perturbation, i.e. $b \equiv 0$, cf. [48, 184].

Nemytskii-type coefficients pose several delicate problems, in particular since even a continuity assumption in the measure argument on a_{ij}, b_i with respect to the weak topology does not imply continuity of $(t,x) \mapsto a_{ij}(t, \rho_t(x), x)$ and $(t,x) \mapsto b_i(t, \rho_t(x), x)$, because weak convergence of a sequence of absolutely continuous measures (with respect to Lebesgue measure) does not imply the pointwise convergence of their densities. For the remainder of the thesis, we will not consider such Nemytskii-type coefficients explicitly, but we point out already now that our results in Part II contain the case of such irregular coefficients.

1.1.2 FPK equations for measures on infinite-dimensional spaces

FPK equations for measures on infinite-dimensional spaces arise naturally in the context of stochastic *partial* differential equations, which are usually studied on infinite-dimensional state spaces. Similarly to the relation of finite-dimensional SDEs and the corresponding FPK equations for measures on $\mathbb{R}^d$, also in the infinite-dimensional case, solutions to SPDEs $t \mapsto X(t)$ induce solution curves to FPK equations via their one-dimensional marginals. Often, such processes X assume values in a separable Hilbert space $H \cong \ell^2 \subseteq \mathbb{R}^\infty$. In this case, one can consider the marginals of X as Borel probability measures on $\mathbb{R}^\infty$. If the drift and diffusion coefficients of the SPDE, $b(t,x)$ and $\sigma(t,x)$, are defined on $[0,T] \times H$ (or, in the framework of a Gelfand triple, on some Banach space $V \hookrightarrow H$), one extends these coefficients to $\mathbb{R}^\infty$ (possible by the value ∞) and considers the operator L, defined via

$$L_t\varphi(x) := \sum_{i,j=1}^{\infty} a_{ij}(t,x)\partial_{ij}\varphi(t,x) + \sum_{i=1}^{\infty} b_i(t,x)\partial_i\varphi(t,x), \tag{1.6}$$

for $\varphi : x \mapsto \Phi(x_1, \dots, x_d), \Phi \in C^2(\mathbb{R}^d), d \in \mathbb{N}$, and $(a_{ij})_{i,j\geq 1} = 1/2\sigma\sigma^T$. Hence, both appearing sums contain only finitely many nontrivial summands. L is the infinite-dimensional analogue to the infinitesimal generator $\mathcal{L}$ in the finite-dimensional setting in (1.1). We study the corresponding linear FPK equation for Borel measures on $\mathbb{R}^\infty$

$$\partial_t \mu_t = L_t^* \mu_t, \quad t \in [0,T], \tag{FPK$_\infty$}$$

where, as before, L^* denotes the formal dual operator to L, and the equation is understood in the weak sense in duality with test functions of type $\varphi = \Phi \circ P_d, \Phi \in C^2(\mathbb{R}^d)$, with $P_d : \mathbb{R}^\infty \to \mathbb{R}^d, P_d : x \mapsto (x_1, \dots, x_d)$. See Section 4.1 for details.

Besides this connection to infinite-dimensional stochastic differential equations, it should be noted that infinite-dimensional FPK equations arise from the study of finite-dimensional equations. In this direction, we refer to [1, 128, 130, 131, 129, 132, 134, 133, 136, 135].

The literature on such infinite-dimensional equations is vast, although equations of type (FPK$_\infty$) are less studied than the finite-dimensional linear and nonlinear equations (FPK) and (NL-FPK). Since the infinite-dimensional case is not in the center of our attention, we do not aim to provide a complete list of references. For an introduction to the field, we refer the reader to [38, Ch.10], and to the works [197, 81, 33, 31, 34] and the references therein for recent results on existence and uniqueness of solutions to such equations. We only briefly mention that one can, of course, also study *nonlinear* FPK equations for measures on $\mathbb{R}^\infty$. We do not pursue this direction in this work, but mention the references listed in [38, Sect.10.5(ii)].

1.2 Selection theorems for stochastic systems

Before we turn our attention to the main objective of this first part of the thesis, which is to study *solution flows* to the Cauchy problem of equations of type (FPK), (NL-FPK) and (FPK_∞), we briefly shed light on some well-known results, which are in a similar spirit.

1.2.1 From Markov processes to FPK equations, SDEs, and the martingale problem

For the moment, the central point of our considerations is the Markov process X (whose existence we assume for the moment) in $\mathbb{R}^d$, whose evolution in time $t \in [0,T]$ is governed by the infinitesimal drift vector field $b(t,x) = \big(b_i(t,x)\big)_{1\leq i\leq d}$ and covariance matrix $a(t,x) = \big(a_{ij}(t,x)\big)_{1\leq i,j\leq d}$, where the latter takes its values in the space of symmetric, nonnegative definite matrices. In other words, we would like to study the Markov process $t \mapsto X(t)$, such that if $X(t) = x_0$ and $h > 0$ is small, the increment $X(t+h) - X(t)$ is approximately normally distributed with mean $b(t,x_0)$ and covariance matrix $a(t,x_0)$. Often, one is primarily interested in the path distribution of X, i.e. in the family of probability measures $P_{s,x} \in \mathcal{P}(C_{s,T}\mathbb{R}^d)$ such that for $\mathcal{C} \in \mathcal{B}(C_{s,T}\mathbb{R}^d)$, $P_{s,x}(\mathcal{C})$ is the probability that $[s,T] \ni t \mapsto X_t$ is a path in $\mathcal{C}$, conditioned on the event $X_s = x$. For the existence of such coefficients for Markov processes in quite general situations, see for example [215, Ch.0]. Among the enormous list of introductory and advanced books on Markov processes, we mention [97, 192, 214, 24]. From such a Markov process X, natural connections to the area of PDEs, as well as to stochastic analysis arise as follows.

FPK equations. On the one hand, under broad assumptions on the coefficients, the transition probabilities $P(s,x,t,dy)$ of X fulfill the FPK equation (FPK) with coefficients b and a. Assume temporarily that solutions to (FPK) are unique, so that $P(s,x,t,dy)$, $0 \leq s \leq t \leq T, x \in \mathbb{R}^d$, are uniquely determined by (the Cauchy problem of) (FPK). Since $P_{s,x}$ as the law of a Markov process is uniquely determined by its one-dimensional marginals $P(s,x,t,dy)$, $s \leq t \leq T$, in this case the (Cauchy problem of the) FPK equation (FPK) characterizes X in terms of its family of path laws $\{P_{s,x}\}_{(s,x)\in[0,T]\times\mathbb{R}^d}$. However, obtaining well-posedness of (FPK) is usually a difficult analytic task, even in the case when the transition probabilities are absolutely continuous with respect to Lebesgue measure, in which case one rewrites (FPK) as a PDE, as discussed above.

Stochastic differential equations. On the other hand, a more probabilistic approach towards a description of X is offered by the aforementioned intuition that $X(t+h) - X(t)$ is approximately normally distributed with mean $b(t,X(t))$ and covariance $a(t,X(t))$, i.e. in differential form, one suggestively writes

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, \quad t \in [0,T], \tag{1.7}$$

where $\sigma = (2a)^{1/2}$ and B is a Brownian motion. More precisely, $\sigma \in \mathbb{R}^{d\times d_1}$ such that $a = 1/2\sigma\sigma^T$, and B is d_1-dimensional, where d_1 is, in principle, arbitrary. Thanks to the celebrated work by Itô [123], (1.7) is meaningful as a stochastic differential equation, with the second summand on the right-hand side interpreted as a stochastic Itô integral. Hence, if (1.7) is well-posed (for initial conditions $X_s = x$), such stochastic differential equations characterize the Markov process X.

While this approach is very natural from a probabilistic viewpoint and has proven to be a powerful method in the area of Markov processes and beyond, the rather strong conditions on the coefficients necessary to obtain well-posedness of (1.7) by probabilistic methods pose a certain restriction to this direction.

Martingale problem. Another way to approach the process X is the Stroock-Varadhan theory of the associated *martingale problem*. Recall that we write π_t for the canonical projection $\pi_t(f) = f(t)$, $s \leq t \leq T$, on $C_{s,T}\mathbb{R}^d$. Both from the connection of the process X to (FPK) and to (1.7), by the Markov property of $\{P_{s,x}\}_{(s,x)\in[0,T]\times\mathbb{R}^d}$, it is easily seen that for each $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have that for the operator $\mathcal{L}$ as in (1.1), the process

$$\varphi \circ \pi_t - \int_s^t \mathcal{L}_u\varphi(\pi_u)du, \quad t \in [s, T], \tag{1.8}$$

is a $P_{s,x}$-martingale with respect to the canonical filtration on $C_{s,T}\mathbb{R}^d$ for each $0 \leq s \leq T$. See [215, Ch.0,Ch.6] for details. As our objective is to characterize the Markov process X in terms of b and a, the martingale approach seems advantageous, as it does not contain intervening quantities between the path laws $P_{s,x}$ and the coefficients (in contrast to the approaches via (FPK) or (1.7), which are based on the marginals $P(s, x, t, dy)$, and the additional coefficient σ and the Brownian motion B, respectively). Again, it is clear that in the case that for each (s, x), $P_{s,x}$ is the only measure $P \in \mathcal{P}(C_{s,T}\mathbb{R}^d)$ such that the processes (1.8) are P-martingales, the martingale problem characterizes X in terms of its laws $P_{s,x}$. A fundamental introduction to the theory of martingale problems can be found in the classical book [215].

Summarizing the above repetition of these well-known connections, under suitable assumptions on the coefficients b and a (that is, conditions sufficient for the well-posedness of either of the above corresponding equations), the Markovian dynamics with respect to b and a are characterized via its marginals by a FPK equation, via its diffusive evolution as a solution to an SDE, or via its path laws as the solution to a martingale problem.

1.2.2 Markovian selections in nonuniqueness regimes

So far, we assumed the existence of a unique Markov process X associated to the coefficients b and a. One may also go in the opposite direction, i.e. start from either the corresponding FPK equation (FPK), the SDE (1.7) or the processes (1.8) of the martingale problem with respect to b and a, assume one of these problems to be well-posed, and ask whether the corresponding solutions give rise to a unique Markov process. In all three cases, the answer is affirmative. More precisely, if solutions $(P(s, x, t, dy))_{t\in[s,T]}$, $(s, x) \in [0, T] \times \mathbb{R}^d$, are unique for the Cauchy problem of (FPK), then by the stability of solutions to (FPK) under convex combinations, it is straightforward to check that these solutions fulfill the Chapman-Kolmogorov equations (CK) (with $P(s, x, t, dy)$ replacing $\mu_t^{s,x}$). Consequently, by a classical result, the existence of a unique Markovian family of path laws $\{P_{s,x}\}_{(s,x)\in[0,T]\times\mathbb{R}^d}$ such that $P_{s,x}$ has one-dimensional marginals $P(s, x, t, dy)$ follows, see [24].

Similarly, under broad assumptions on b and a, (strong) well-posed of the stochastic equation (1.7) implies that the family of solutions is a Markov process. Also in the case of well-posedness of the martingale problem (1.8), the unique martingale solutions $P_{s,x}$ such that $P_{s,x}(\pi_s = x) = 1$ form a Markov process, see [215, Thm.6.2.2].

As mentioned before, well-posedness of either (FPK), (1.7) or the martingale problem (1.8) holds only under rather strong assumptions on the coefficients b and a. As an example, we mention that in the case of (FPK), uniqueness results are obtained under the assumption of local Lipschitz continuity of a_{ij} and locally uniform ellipticity of $a = (a_{ij})_{1\leq i,j\leq d}$ (together with further local and global integrability conditions on b and a, see [38, Thm.9.4.3]). For the martingale problem, the assumption of nondegeneracy of a is not necessary, if one assumes, for example, $a_{ij}, b_i \in C^2(\mathbb{R}^d)$ [215, Cor.6.3.3]. Despite gradual improvements of such results, there are inevitable examples of ill-posedness of these problems, even for very nice coefficients, see for example [38, Exe.9.8.48] for the case of $a = \mathrm{Id}$ and $b \in C^\infty(\mathbb{R})$.

In such ill-posedness situations, there is no a priori natural notion of (*the*) *Markovian dynamics with respect to* b *and* a, since it is neither clear of which solutions to the Cauchy problem such a Markov process with respect to b and a should consist, nor how to select these solutions among the possibly large class of solutions for each initial condition such that the Markov property prevails. Consequently, one poses the interesting question:

If for either (FPK), (1.7) *or* (1.8) *several solutions exist, can one select a family of solutions, which gives rise to a Markov process?* (Q1)

Moreover, if the answer is affirmative, it is natural to ask whether, and in which sense, such a selected Markov process is unique, i.e. we are also interested in the question:

Is it possible to characterize the well-posedness of (FPK), (1.7) *or* (1.8) *in terms of uniqueness of the selected Markov process?* (Q2)

Since the martingale problem (1.8) and (weak solutions to) the stochastic equation (1.7) are equivalent (at least for bounded coefficients, see [213]), we restrict these questions to the FPK equation and the martingale problem.

Markovian selections for the martingale problem. Needless to say, we are by no means the first to raise these questions. The topic of Markovian selections from nonunique solutions to stochastic equations has been considered in several past and present works. To the best of our knowledge, the first major contribution in this direction is due to Krylov [147], who established a general Markovian selection procedure for a large class of stochastic equations. In a somewhat similar spirit, in their classical book [215, Ch.12], Stroock and Varadhan provide the following positive answer to question (Q1): in the case of time-independent, continuous and bounded coefficients b and a, one can select a strong Markov family $\{P_{s,x}\}_{(s,x)\in[0,T]\times\mathbb{R}^d}$ such that $P_{s,x} \in \mathcal{P}(C_{s,T}\mathbb{R}^d)$ is a solution to the martingale problem (1.8) with initial condition (s,x) (i.e. $P_{s,x}(\pi_s = x) = 1$). From [215, Ch.6,Ch.7], it is clear that for such coefficients, solutions to the martingale problem exist, but need not be unique. Their idea essentially amounts to realizing that under the above assumptions on the coefficients, the set of solutions to the martingale problem with a common initial condition (s,x) is weakly compact in $\mathcal{P}(C_{s,T}\mathbb{R}^d)$. This allows them to select a unique element among each of these solution sets, which they characterize by iteratively maximizing a suitable sequence of continuous functionals, chosen such that the Markov property holds for the family of the selected extremal elements. Moreover, as a direct consequence of their maximizing procedure, they prove that such a Markovian selection is unique if and only if the martingale problem is well-posed, which also settles question (Q2) in this case.

Further results in this direction are known for martingale problems with respect to Lévy-type operators [149, 97] and for martingale solutions to stochastic *partial* differential equations, including but not limited to the Navier–Stokes equations [110, 103, 47], see also [30, 80, 91].

Selections for FPK equations. Naturally, the question arises whether a similar selection is possible for the FPK equation (FPK). On this level, one is interested in a family of solutions to the Cauchy problem of (FPK) with initial condition (s,δ_x), now denoted by $\mu^{s,x} = (\mu^{s,x}_t)_{t\in[s,T]} \in C_{s,T}\mathcal{P}$, $(s,x)\in[0,T]\times\mathbb{R}^d$, such that the Chapman-Kolmogorov equations (CK) are fulfilled. As mentioned at the beginning of the present subsection, to such $\{\mu^{s,x}\}_{(s,x)\in[0,T]\times\mathbb{R}^d}$, one associates a unique family of path laws $\{P_{s,x}\}_{(s,x)\in[0,T]\times\mathbb{R}^d}$, which is then considered a Markovian evolution emerging from (FPK). It is not difficult to see that any Markovian selection $\{P_{s,x}\}_{(s,x)\in[0,T]\times\mathbb{R}^d}$ for the corresponding martingale problem induces a family of solutions to (FPK) $\{\mu^{s,x}\}_{(s,x)\in[0,T]\times\mathbb{R}^d}$, which fulfills (CK), via its one-dimensional marginals $\mu^{s,x}_t := P_{s,x}\circ\pi_t^{-1}$. However, one may ask whether a selection is possible without the detour via the corresponding martingale problem, in particular in order to obtain results under weaker assumptions on the coefficients and via analytic methods, which do not apply to the setting of the martingale problem.

FPK equations can be considered differential equations for measures. For such equations, there is another notion of dynamical behavior, namely the usual *flow property* for solutions to differential equations: one may additionally ask whether there is a way to select a solution family $\{\mu^{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}}$, which is a flow in the sense that

$$\mu^{s,\nu}_t = \mu^{r,\mu^{s,\nu}_r}_t, \quad 0\le s\le r\le t\le T, \nu\in\mathcal{P}. \tag{1.9}$$

Then, it is natural to ask whether the flow property (1.9) and the Chapman-Kolmogorov equations are different notions of dynamical regularity for (FPK), or whether one implies the other. In this regard, it is easy to see that any family fulfilling (CK) induces a solution flow, see Proposition 2.5.4 for details. Moreover, in the case of uniqueness, clearly the family of solutions to the Cauchy problem of (FPK) fulfills the flow property (1.9). Hence, our questions of interest arise only in the case of ill-posed equations. For quite general recent results on flow selections to a large class of ordinary and partial differential equations, we refer to [60, 59].

1.3 Main results: linear case

As mentioned above, any solution family satisfying the Chapman-Kolmogorov equations gives a flow of solutions to (FPK) in the sense of (1.9). This raises hopes that selecting a solution flow to (FPK) might be possible without appealing to a Markovian selection to the martingale problem in the sense of Stroock and Varadhan. Moreover, in analogy to question (Q2), it is interesting to characterize the well-posedness of (FPK) in terms of the selected flow. In this part of the thesis, we provide affirmative results to both questions. In a more general context than described so far, we establish a selection method for a solution flow in suitable *flow-admissible* systems $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ of subclasses of subprobability solutions to the Cauchy problem of (FPK). Furthermore, we call an initial condition (s,ν) *admissible*, if $\mathcal{A}_{s,\nu} \neq \emptyset$. See Definition 2.2.1 for the definition of such flow-admissible families and Remark 6.3.4 for an explanation why the more general setting of subprobability solution curves is technically more suitable for our goal. More precisely, our first main result is the following.

Theorem 1.3.1. *Let $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ be a flow-admissible family of sets of solutions to (FPK) such that $\mathcal{A}_{s,\nu}$ is compact in $C_{s,T}\mathcal{SP}$ for each admissible initial condition (s,ν). Then, there exists a solution flow to (FPK) with respect to $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$.*

We also investigate the measurability of the selected solution flow in Subsection 2.3.1. The assumption of compactness is crucial to our approach, and we will present several examples and applications. In particular, our results apply in the case of locally bounded and continuous (in $x \in \mathbb{R}^d$) coefficients, which are integrable in time, see Proposition 2.4.3, as well as in the case of the presence of a suitable Lyapunov function, see Corollary 2.4.8.

As it turns out, the selection method used in the proof of the previous theorem allows to obtain a characterization of the well-posedness of the Cauchy problem of (FPK) in terms of the uniqueness of the selected flow. This is contained in the following result, which is our second main theorem.

Theorem 1.3.2. *Under the assumptions of Theorem 1.3.1, the following are equivalent.*

(i) There exists at most one solution flow with respect to $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$.

(ii) For each $(s,\nu) \in [0,T]\times\mathcal{SP}$, solutions in $\mathcal{A}_{s,\nu}$ are unique.

The observation that the selection method for the proof of Theorem 1.3.1 allows to easily deduce the characterization of Theorem 1.3.2 is similar to the case of the somewhat analogue results in [215, Ch.12], which we explained in the previous section. Concerning the question whether any solution flow fulfills the Chapman-Kolmogorov equations, we obtain a positive answer in Proposition 2.5.5 under rather strong assumptions on the coefficients, which coincide with the assumptions in [215, Ch.12]. To the best of our knowledge, the above results and considerations mentioned above have not been considered in the literature before. We note that to date, to our knowledge, it is open whether the same results are true under significantly weaker assumptions on the coefficients. We plan to investigate this question in the future.

1.3.1 Idea of proof

The reader familiar with the selection proof for the martingale problem in [215, Ch.12] will immediately note the (in spirit) parallels to our proof. We aim to select a unique solution curve $\mu^{s,\nu} = (\mu_t^{s,\nu})_{t\in[s,T]}$ among the elements in $\mathcal{A}_{s,\nu}$, and we do so by characterizing each $\mu^{s,\nu}$ as the unique element, which iteratively maximizes a prescribed sequence of linear, continuous functionals on $C_{s,T}\mathcal{SP} \supseteq \mathcal{A}_{s,\nu}$. Roughly, these functionals $\{G_k\}_{k\in\mathbb{N}}$ (the notation in the proof in Section 2.3 is more involved) are chosen as

$$G_k : \mu \mapsto \int_{\mathbb{R}^d} h_k \mu_{q_k}, \quad \mu = (\mu_t)_{t\in[s,T]} \in C_{s,T}\mathcal{SP},$$

where $h_k \in C_c(\mathbb{R}^d)$, and $\{q_k\}_{k\in\mathbb{N}} \subseteq \mathbb{Q} \cap [s,T]$ is chosen to be dense in $[s,T]$. Compactness of $\mathcal{A}_{s,\nu} \subseteq C_{s,T}\mathcal{SP}$ implies that the supremum of G_1 on $\mathcal{A}_{s,\nu}$ is attained on some nonempty, compact set $M_1 \subseteq \mathcal{A}_{s,\nu}$. The same is true for G_2, with $\mathcal{A}_{s,\nu}$ replaced by M_1. Iterating this procedure, we obtain the nonempty set $\bigcap_{k\in\mathbb{N}} M_k \subseteq \mathcal{A}_{s,\nu}$. If the family $\{h_k\}_{k\in\mathbb{N}}$ is chosen such that it separates measures on $\mathbb{R}^d$, it is easy to note that this intersection is a singleton, whose element we denote by $\mu^{s,\nu}$. From here, the flow-admissibility of the sets $\mathcal{A}_{s,\nu}$ yields the flow property (1.9).

Concerning the proof of Theorem 1.3.2, the main observation is that one may freely choose the separating functions h_k and the order of the functionals G_k. This way, assuming the existence of two solutions μ^1, μ^2 to (FPK) for a common initial condition such that μ^1 is part of the solution flow (which is assumed to be unique), we choose an ordering of the functionals G_k, $k \in \mathbb{N}$, such that the supremum of G_1 is attained in μ^2, but not in μ^1. Consequently, the flow with respect to this ordering, constructed as in the proof of Theorem 1.3.1, cannot comprise μ^1 and, hence, does not coincide with the initially given (unique) flow. Thus, we arrive at a contradiction.

1.4 Nonlinear and infinite-dimensional equations

1.4.1 Nonlinear FPK equations

Inspecting the proofs of Theorems 1.3.1 and 1.3.2, it is clear that neither of them depends on the linearity of the equation (FPK). Indeed, it is readily seen that once compactness of the sets $\mathcal{A}_{s,\nu}$ holds, the proofs remain valid, if one replaces the linear equation with nonlinear ones of type (NL-FPK). Thereby, in Chapter 3, we obtain similar results on the existence of solutions flows to such nonlinear equations, and we also provide a characterization of the well-posedness of (NL-FPK) similar to the linear case. However, the nonlinearity renders compactness of subclasses of solutions $\mathcal{A}_{s,\nu}$ a delicate issue, for which we have to assume additional regularity assumptions on the nonlinearity argument of the coefficients. Compare the assumptions B1-B3 with the assumptions A1-A3 from the linear case.

1.4.2 FPK equations for measures on $\mathbb{R}^\infty$

Similarly, also in the case of infinite-dimensional equations (FPK_∞), the proofs apply without changes, and yield the main results Theorems 4.2.1 and 4.2.2, stated in Chapter 4. Again, there are no a priori assumptions on the coefficients, but the proof is entirely based on the compactness of the solution classes $\mathcal{A}_{s,\nu}$ one considers. However, verifying compactness of solutions, which are curves of measures on $\mathbb{R}^\infty$, turns out to be difficult. After the formulation of the main results in Chapter 4, we present a few examples. Since the infinite-dimensional case is not in the center of our attention, most of these examples are preliminary in the sense that additional assumptions need to be verified for the flow-admissible family $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$, as it is the case in Propositions 4.3.2 and 4.3.4. A full example is provided in Corollary 4.3.7.

Organization of Part I. The remaining contents of Part I of this thesis are organized as follows. In Chapter 2, we treat the linear finite-dimensional case, to which we devoted the majority of this introduction. The first section contains the notion of solution to the Cauchy problem of (FPK). In Section 2.2, we introduce the exact notion of flow-admissible solution systems and solution flows with respect to such systems, and present several examples. In Section 2.3, we prove both main results of this chapter and also discuss measurability of the selected flow. Afterwards, in Section 2.4, we present several examples and applications to our results. Finally, in the last section of the chapter, we investigate the relation to Markovian selections, i.e. families of solutions to (FPK), which fulfill the Chapman-Kolmogorov equations.

In Chapter 3, we treat the case of nonlinear (still finite-dimensional) equations. After introducing the notion of solution in this case in the first section, we state the main results (Theorems 3.2.2 and 3.2.3), and briefly discuss how their proofs follow as in the linear case in Section 3.2. Section 3.3 contains examples of situations to which our results apply.

We close this part with a presentation of our results in the case of infinite-dimensional (linear) FPK equations in Chapter 4. After presenting the framework of solutions to such equations in Section 4.1, we formulate the main results in Section 4.2. The chapter finishes with a few examples in Section 4.3. Finally, Appendix A contains auxiliary results on FPK equations.

Chapter 2

Solution flows for linear equations

2.1 Linear FPK equations

Let $T > 0$, consider $a = (a_{ij})_{1\leq i,j\leq d}$ and $b = (b_i)_{1\leq i\leq d}$ with $\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}^d)$-measurable coefficients

$$a_{ij}, b_i : [0,T]\times\mathbb{R}^d \to \mathbb{R},$$

and let $\mathcal{L}$ denote the second-order differential operator as in (1.1). Since it is common in the literature and natural from, e.g., a stochastic analysis point of view, where one often has $a = 1/2\sigma\sigma^T$ for some diffusion coefficient $\sigma \in \mathbb{R}^{d\times d_1}$, we also assume that a takes values in $\mathbb{S}_d^+$, although large parts of the subsequent presentation also work without this assumption. We study the Cauchy problem of the linear FPK equation (FPK) with initial condition $(s,\nu) \in [0,T] \times \mathcal{SP}$, i.e.

$$\begin{cases} \partial_t \mu_t &= \mathcal{L}_t^* \mu_t, \\ \mu_s &= \nu, \end{cases} \tag{FPK}$$

in the sense of the following definition.

Definition 2.1.1. (i) A vaguely continuous curve $t \mapsto \mu_t \in \mathcal{SP}$ on $[s,T]$ is a *solution to (FPK)*, if

$$\int_s^T \int_K |a_{ij}(t,x)| + |b_i(t,x)| d\mu_t(x) dt < \infty \tag{2.1}$$

holds for all $1 \leq i,j \leq d$ and any compact set $K \subseteq \mathbb{R}^d$, and for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, it holds

$$\int_{\mathbb{R}^d} \varphi(x) d\mu_t - \int_{\mathbb{R}^d} \varphi(x) d\nu = \int_s^t \int_{\mathbb{R}^d} \mathcal{L}_u \varphi(x) d\mu_u(x) du, \quad t \in [s,T]. \tag{2.2}$$

(ii) A solution $t \mapsto \mu_t$ to (FPK) is a *probability solution*, if $\mu_t \in \mathcal{P}$ for each $t \in [s,T]$.

Of course, any probability solution is weakly continuous as a vaguely continuous curve of measures with constant mass. In the case of an initial condition (T,ν), the notion of solution reduces to the single measure ν.

Remark 2.1.2. *In general, one can consider solutions to FPK equations as discontinuous Borel curves of (signed) bounded measures. In the absence of (vague) continuity, one requires (2.2) to hold for each $\varphi \in C_c^\infty(\mathbb{R}^d)$ for $t \in J_\varphi \subseteq (s,T)$ such that J_φ^c is dt-negligible. Equivalently, $t \mapsto \mu_t$ fulfills*

$$\int_s^T \int_{\mathbb{R}^d} \partial_t \phi(t,x) + \mathcal{L}_t \phi(t,x) d\mu_t(x) dt = 0 \tag{2.3}$$

for each $\phi \in C_c^\infty((s,T) \times \mathbb{R}^d)$, and for each $\varphi \in C_c^\infty(\mathbb{R}^d)$, there exists a set $I_\varphi \subseteq (s,T)$ of full dt-measure such that

$$\int_{\mathbb{R}^d} \varphi(x) d\nu(x) = \lim_{t \to s, t \in I_\varphi} \int_{\mathbb{R}^d} \varphi(x) d\mu_t(x), \tag{2.4}$$

see [38, Prop.6.1.2].

However, we will exclusively study vaguely continuous subprobability solutions as in the previous definition.

We state the next lemma to convince the reader that confining to vaguely continuous solution curves is not restrictive.

Lemma 2.1.3. *Let $(\mu_t)_{t \in (s,T)}$ be a Borel curve of nonnegative bounded measures with $\mu_t \in \mathcal{SP}$ for dt-a.e. $t \in (s,T)$, such that for each compact set $K \subseteq \mathbb{R}^d$ and $1 \leq i,j \leq d$, we have*

$$\int_s^T \int_K |a_{ij}(t,x)| + |b_i(t,x)| d\mu_t(x) dt < \infty. \tag{2.5}$$

If $(\mu_t)_{t\in(s,T)}$ fulfills (2.3), then there exists a unique vaguely continuous version $(\tilde\mu_t)_{t\in[s,T]}$, which fulfills (2.2). Furthermore, if $(\mu_t)_{t\in(s,T)}$ satisfies (2.4), then $\tilde\mu_s=\nu$, i.e. $(\tilde\mu_t)_{t\in[s,T]}$ solves the Cauchy problem (FPK).

Proof. Choosing $\phi(t,x)=\zeta(t)\varphi(x)$ with $\zeta\in C_c^\infty((s,T)),\varphi\in C_c^\infty(\mathbb{R}^d)$, (2.3) gives

$$\int_s^T\zeta'(t)\Big(\int_{\mathbb{R}^d}\varphi(x)d\mu_t(x)\Big)dt=-\int_s^T\zeta(t)\Big(\int_{\mathbb{R}^d}\mathcal{L}_t\varphi(x)d\mu_t(x)\Big)dt. \tag{2.6}$$

Therefore, and since $t\mapsto\int_{\mathbb{R}^d}\mathcal{L}_t\varphi d\mu_t\in L^1((s,T))$ by assumption, the map $t\mapsto\int_{\mathbb{R}^d}\varphi d\mu_t$ belongs to the usual Sobolev space $W^{1,1}((s,T))$ with weak derivative $t\mapsto\int_{\mathbb{R}^d}\mathcal{L}_t\varphi d\mu_t$ dt-a.s. Hence, choosing a countable set $\mathcal{F}\subseteq C_c^\infty(\mathbb{R}^d)$, which is dense in $C_c(\mathbb{R}^d)$, there exists a real-valued map $(\varphi,t)\mapsto F(\varphi,t)$ on $\mathcal{F}\times[s,T]$ such that for each $\varphi\in\mathcal{F}$, $t\mapsto F(\varphi,t)$ is an absolutely continuous version of $t\mapsto\int\varphi d\mu_t$. Let $\mathcal{T}$ denote the set of all $t\in[s,T]$ such that $\mu_t\in\mathcal{SP}$ and $F(\varphi,t)=\int\varphi d\mu_t$ for all $\varphi\in\mathcal{F}$. By assumption, $\mathcal{T}^c$ is dt-negligible. If $t\in\mathcal{T}$, then

$$|F(\varphi,t)-F(\varphi',t)|\le\int_{\mathbb{R}^d}|\varphi-\varphi'|d\mu_t\le||\varphi-\varphi'||_\infty, \tag{2.7}$$

whereby $\varphi\mapsto F(\varphi,t)$ is uniformly continuous on $\mathcal{F}$ and hence uniquely extends to a continuous linear map on all of $C_c(\mathbb{R}^d)$ (again denoted $F(\cdot,t)$) via

$$F(\varphi,t):=\lim_{n\to\infty}F(\varphi_n,t)=\lim_{n\to\infty}\int\varphi_nd\mu_t=\int\varphi d\mu_t.$$

Here and for the rest of the proof, $(\varphi_n)_{n\ge1}\subseteq\mathcal{F}$ denotes any sequence converging to φ in $C_c(\mathbb{R}^d)$. Thus, $F(\cdot,t)=\mu_t$ for each $t\in\mathcal{T}$ as elements in the dual space of $C_c(\mathbb{R}^d)$ (with the identification $\mu_t(f)=\int fd\mu_t$). For $t\in\mathcal{T}^c$, we have for $\varphi\in\mathcal{F}$

$$F(\varphi,t)=\lim_{n\to\infty}F(\varphi,t_n)=\lim_{n\to\infty}\int_{\mathbb{R}^d}\varphi(x)d\mu_{t_n}(x). \tag{2.8}$$

Here and below, $(t_n)_{n\ge1}\subseteq(s,T)\cap\mathcal{T}$ is any sequence converging to t. In particular, $F(\cdot,t)$ is linear and uniformly continuous on $\mathcal{F}$. For $\varphi\in C_c(\mathbb{R}^d)\backslash\mathcal{F}$, we set $F(\varphi,t):=\lim\limits_{l\to\infty}F(\varphi_l,t)$ (with $(\varphi_l)_{l\ge1}$ as above), which is well-defined due to the uniform continuity of $\varphi\mapsto F(\varphi,t)$ on $\mathcal{F}$ and is hence a linear, positive functional on $C_c(\mathbb{R}^d)$ with $||F(\cdot,t)||_{L(C_c(\mathbb{R}^d),\mathbb{R})}\le1$ (the inequality holds, since μ_t is a subprobability measure for each $t\in\mathcal{T}$). Therefore, the Riesz-Markov-Kakutani representation theorem implies the existence of a unique element $\mu_t'\in\mathcal{SP}$ such that $F(\cdot,t)=\mu_t'$, see Theorem D.0.3. For $t\in[s,T]$, define

$$\tilde\mu_t:=\begin{cases}\mu_t, & t\in\mathcal{T}\\ \mu_t', & t\in\mathcal{T}^c.\end{cases} \tag{2.9}$$

By definition, $[s,T]\ni t\mapsto\int\varphi d\tilde\mu_t$ is continuous for each $\varphi\in\mathcal{F}$. Since for each $\varphi\in C_c(\mathbb{R}^d)\backslash\mathcal{F}$, we have

$$\int\varphi d\tilde\mu_t=\lim_{l\to\infty}\int\varphi_ld\tilde\mu_t=\lim_{l\to\infty}\lim_{n\to\infty}\int\varphi_ld\tilde\mu_{t_n}=\lim_{n\to\infty}\lim_{l\to\infty}\int\varphi_ld\tilde\mu_{t_n}=\lim_{n\to\infty}\int\varphi d\tilde\mu_{t_n}, \tag{2.10}$$

it follows that $t \mapsto \tilde{\mu}_t$ is vaguely continuous and, by construction, coincides with $(\mu_t)_{t\in(s,T)}$ dt-a.s. Since for vaguely continuous curves, (2.3) is equivalent to (2.2), $(\tilde{\mu}_t)_{t\in[s,T]}$ fulfills (2.2). Since the final assertion concerning the initial condition is obvious, the proof is complete. □

Remark 2.1.4. *(i) The proof of Lemma 2.1.3 still works on each (r_1, r_2) instead of (s, T) with $s < r_1 < r_2 < T$, if one replaces (2.5) by the weaker assumption*

$$\int_{r_1}^{r_2}\int_K |a_{ij}(t,x)| + |b_i(t,x)| d\mu_t(x)dt < \infty, \quad \forall s < r_1 < r_2 < T,\ K \subseteq \mathbb{R}^d \text{ compact.}$$

Considering a sequence of intervals (r_1^n, r_2^n) with $r_1^n \searrow s$ and $r_2^n \nearrow T$ as $n \to \infty$, one obtains unique vaguely continuous versions $(\tilde{\mu}_t^n)_{t\in[r_1^n, r_2^n]}$ of $(\mu_t)_{t\in(r_1^n, r_2^n)}$, which are consistent in the sense that for each $n \geq 1$ we have $\tilde{\mu}_t^{n+1} = \tilde{\mu}_t^n$ whenever both curves are defined in $t \in (s,T)$. Hence, there exists a unique vaguely continuous version $(\tilde{\mu}_t)_{t\in(s,T)}$ of $(\mu_t)_{t\in(s,T)}$. If $\mu_t \longrightarrow \nu$ vaguely as $t \searrow s$, then also $\tilde{\mu}_t \longrightarrow \nu$ vaguely and we set $\tilde{\mu}_s := \nu$ and write $(\tilde{\mu}_t)_{t\in[s,T)}$. However, in this situation, this version might not extend to $t = T$ and (2.2) does not necessarily hold for $t = T$, since in this case, the right-hand side of (2.2) need not be integrable.

(ii) If $\mu_t \in \mathcal{P}$ for each t, the unique vaguely continuous version $(\tilde{\mu}_t)_{t\in[s,T]}$ need not consist of probability measures for each t (but clearly for dt-a.e. t). To have $\mu_t \in \mathcal{P}$ for each $t \in [s,T]$, one needs, for instance, either a global integrability assumption or tightness of $(\mu_t)_{t\in(s,T)}$.

For the remainder of this part of the thesis, Remarks 2.1.2 and 2.1.4 and Lemma 2.1.3 are of no further importance, since we exclusively consider vaguely continuous solutions on $[s,T]$ as in Definition 2.1.1. The purpose of these statements was to present a glimpse to possible more general situations.

2.2 Solution flows

Our main objective is to construct *solution flows* to the Cauchy problem (FPK). In this section, we introduce the notion of such flows and discuss several examples. Throughout, we use the following notation, where by *solution*, we always mean a vaguely continuous curve $t \mapsto \mu_t \in \mathcal{SP}$ as in Definition 2.1.1.

$$\mathcal{M}_{s,\nu} := \{\mu = (\mu_t)_{t\in[s,T]} : \mu \text{ solution to (FPK) with initial condition } (s,\nu)\},$$
$$\mathcal{M}^1_{s,\nu} := \{\mu = (\mu_t)_{t\in[s,T]} : \mu \text{ probability solution to (FPK) with initial condition } (s,\nu)\},$$
$$\mathcal{M}_s := \bigcup_{\nu\in\mathcal{SP}} \mathcal{M}_{s,\nu}, \quad \mathcal{M}^1_s := \bigcup_{\nu\in\mathcal{P}} \mathcal{M}^1_{s,\nu}.$$

Our principal objective is to select a family $\{\mu^{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ such that $\mu^{s,\nu} \in \mathcal{M}_{s,\nu}$, which fulfills (1.9). However, since the iterative selection method we apply to find such a selection only works on compact sets of solutions (see the proof of Theorem 1.3.1), we possibly need to restrict to proper subsets $\mathcal{A}_{s,\nu} \subseteq \mathcal{M}_{s,\nu}$. For instance, this is the case when selecting probability solutions $\mu^{s,\nu} \in \mathcal{M}^1_{s,\nu}$ in regimes with unbounded coefficients a and b.

In such cases, it is necessary to impose natural compatibility-conditions on the sets $\mathcal{A}_{s,\nu}$. To this end, we introduce the following notion of *flow-admissibility*.

Definition 2.2.1. (i) A family $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ with $\mathcal{A}_{s,\nu} \subseteq \mathcal{M}_{s,\nu}$ is *flow-admissible*, if it fulfills the following properties for any $0 \leq s \leq r \leq T$ and $\nu \in \mathcal{SP}$.

(a) If $(\mu_t)_{t\in[s,T]} \in \mathcal{A}_{s,\nu}$, then $(\mu_t)_{t\in[r,T]} \in \mathcal{A}_{r,\mu_r}$.

(b) If $(\mu_t)_{t\in[s,T]} \in \mathcal{A}_{s,\nu}$ and $(\eta_t)_{t\in[r,T]} \in \mathcal{A}_{r,\mu_r}$, then $\mu \circ_r \eta \in \mathcal{A}_{s,\nu}$, where we define the curve $\mu \circ_r \eta$ by

$$(\mu \circ_r \eta)_t := \begin{cases} \mu_t, & t \in [s, r] \\ \eta_t, & t \in (r, T]. \end{cases} \tag{2.11}$$

(ii) For $s \in [0, T]$, the set of $\nu \in \mathcal{SP}$ such that $\mathcal{A}_{s,\nu} \neq \emptyset$ is denoted by A_s. Initial conditions (s, ν) with $\nu \in A_s$ are called *admissible*.

Note that a flow-admissible family consists of a set $\mathcal{A}_{s,\nu} \subseteq \mathcal{M}_{s,\nu}$ for each initial condition (s, ν), but in general A_s may be a strict subset of $\mathcal{SP}$, i.e. there may exist initial conditions (s, ν) with $\mathcal{A}_{s,\nu} = \emptyset$. If $\nu \in A_0$, i.e. for $\nu \in \mathcal{SP}$ there exists a solution $\eta \in \mathcal{A}_{0,\nu}$, then $\eta_s \in A_s$ for each $s \in [0, T]$ by (a) of the above definition, and thus each A_s is nonempty. We consider $A_s \subseteq \mathcal{SP}$ as a topological space with the induced subspace topology from $\mathcal{SP}$, i.e. the corresponding Borel σ-algebra $\mathcal{B}(A_s) = \mathcal{B}(\mathcal{SP})_{|A_s}$ is the trace σ-algebra of A_s on $\mathcal{SP}$.

Definition 2.2.2. (i) Let $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ be flow-admissible. A *solution flow* to (FPK) (with respect to $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$) is a family of solutions $\{\mu^{s,\nu}\}$ to (FPK) such that $\mu^{s,\nu} \in \mathcal{A}_{s,\nu}$, which fulfill (1.9) for all $0 \leq s \leq r \leq T$ and $\nu \in A_s$.

(ii) A flow is called *measurable*, if for each $0 \leq s \leq t \leq T$, the *transition map* of the flow

$$U^s_t : A_s \to \mathcal{SP},\, U^s_t : \nu \mapsto \mu^{s,\nu}_t,$$

is $\mathcal{B}(A_s)/\mathcal{B}(\mathcal{SP})$-measurable.

Remark 2.2.3. *(i) By virtue of the transition maps U^s_t, an alternative formulation of the flow-property* (1.9) *is*

$$U^r_t \circ U^s_r(\nu) = U^s_t(\nu),\ 0 \leq s \leq r \leq t \leq T,\, \nu \in A_s. \tag{2.12}$$

(ii) If $(\mu_t)_{t\in[s,T]}$ solves (FPK) *with initial condition (s, ν), then for any $s \leq r \leq T$, $(\mu_t)_{t\in[r,T]}$ solves* (FPK) *with initial condition (r, μ_r). Moreover, if $(\mu')_{t\in[r,T]}$ is another solution with initial condition (r, μ_r), then $\mu \circ_r \mu'$ is a solution on $[s, T]$ with initial condition (s, ν). Therefore, the family $(\mathcal{M}_{s,\nu})_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ is flow-admissible.*

We proceed with the presentation of several examples to the terms introduced above.

Example 2.2.4. *(i) Entire subprobability-flow.* As mentioned in Remark 2.2.3 (ii), the family $\{\mathcal{M}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ is flow-admissible. If $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ is such that all initial conditions are admissible, i.e. $A_s = \mathcal{SP}$ for each $s \in [0, T]$, then a corresponding solution flow consists of a selected solution curve $\mu^{s,\nu}$ for each initial condition (s,ν). We call such a subprobability flow *entire*.

In this case, the transition maps U^s_t are transformations $U^s_t : \mathcal{SP} \to \mathcal{SP}$, which describe the transport of $\mathcal{SP}$ along the flow from time s to time t.

(ii) Entire probability-flow. Similarly to the previous case, the family given by

$$\mathcal{A}_{s,\nu} := \begin{cases} \mathcal{M}^1_{s,\nu} & , \text{ if } \nu \in \mathcal{P} \\ \emptyset & , \text{ else} \end{cases} \tag{2.13}$$

is flow-admissible. If $A_s = \mathcal{P}$ for each $s \in [0,T]$, a flow with respect to this family consists of probability solutions starting from each $(s,\nu) \in [0,T] \times \mathcal{P}$. We call such a flow an *entire probability flow.* If the flow is measurable, each U^s_t is $\mathcal{B}(\mathcal{P})/\mathcal{B}(\mathcal{P})$-measurable, since the Borel σ-algebras of $\mathcal{P}$ with respect to the vague and weak topology, respectively, coincide, and the former is equal to $\mathcal{B}(\mathcal{SP})_{|\mathcal{P}}$.

(iii) Flow subject to a Lyapunov function. Let $V : \mathbb{R}^d \mapsto \mathbb{R}_+$ be lower semicontinuous, $\alpha : [0,T] \to \mathbb{R}$ a nonnegative map and set

$$\mathcal{A}_s := \left\{ (\mu_t)_{t\in[s,T]} : \mu_t \in \mathcal{P}, \int V d\mu_t \le \alpha(t)\, \forall t \in [s,T] \right\}.$$

Then, the sets $\mathcal{A}_{s,\nu} := \mathcal{A}_s \cap \mathcal{M}_{s,\nu}$ form a flow-admissible family and the set of admissible initial conditions is a subset of $\{(s,\nu) : \nu \in \mathcal{P}, \int V d\nu \le \alpha(s)\}$.

2.3 Proofs of main results

In this section, we prove the main results of this chapter, Theorems 1.3.1 and 1.3.2. Before we do so, we introduce the following notation and topological prerequisites. Throughout, we consider $C_{s,T}\mathcal{SP}$ as a normed space with the usual supremum-norm, i.e. the induced topology is the topology of uniform convergence with respect to the vague topology on $\mathcal{SP}$. Recall that for $t \in [s,T]$, we denote by π_t the natural continuous projection $\pi_t : (\mu_t)_{t\in[s,T]} \mapsto \mu_t$ on $C_{s,T}\mathcal{SP}$.

Remark 2.3.1. *The topology of uniform convergence on $C_{s,T}\mathcal{SP}$ coincides with the compact-open topology, i.e. the topology with the subbase consisting of the sets $V(K,U) := \{f \in C_{s,T}\mathcal{SP} : f(K) \subseteq U\}$, for any compact $K \subseteq [s,T]$ and open $U \subseteq \mathcal{SP}$. In particular, this topology is independent of the metric used to metrize the vague topology on $\mathcal{SP}$. Moreover, $C_{s,T}\mathcal{SP}$ is separable and the map*

$$(\mu_t)_{t\in[s,T]} \mapsto \int_{\mathbb{R}^d} h(x) d\mu_t(x) \tag{2.14}$$

is continuous on $C_{s,T}\mathcal{SP}$ for any $t \in [s,T]$ and $h \in C_c(\mathbb{R}^d)$.

The idea for the proof of Theorem 1.3.1 is to select a particular element from each $\mathcal{A}_{s,\nu}$ by iteratively maximizing a sequence of suitable functionals on $C_{s,T}\mathcal{SP}$ of type (2.14). Next, we introduce the necessary terms concerning this iteration and subsequently present the proof of the theorem. Below, we set $\mathbb{Q}^T_s := \mathbb{Q} \cap [s,T]$ for $0 \le s \le T$.

Definition 2.3.2. (i) We call a bijection $\xi : \mathbb{N} \times \mathbb{Q}_0^T \to \mathbb{N}_0$ an *enumeration*. For a given enumeration and $k \in \mathbb{N}_0$, we write $(n_k, q_k) := \xi^{-1}(k)$.

(ii) For $s \in [0, T]$, denote by $(m_k^s)_{k \in \mathbb{N}_0}$ the enumerating sequence of $\mathbb{N} \times \mathbb{Q}_s^T$ with respect to a given enumeration ξ, i.e. there exist exactly k elements (n, q) in $\mathbb{N} \times \mathbb{Q}_s^T$ with $\xi(n, q) < m_k^s$. Put differently, $\xi^{-1}(m_k^s)$ is the k-th element in $\mathbb{N} \times \mathbb{Q}_s^T$ according to ξ.

Note that for $0 \leq s \leq r \leq T$, the sequence $(m_l^r)_{l \in \mathbb{N}_0}$ is a subsequence of $(m_l^s)_{l \in \mathbb{N}_0}$.

Proof of Theorem 1.3.1. Let $\mathcal{H} = \{h_n, n \in \mathbb{N}\} \subseteq C_c(\mathbb{R}^d)$ be measure separating and let ξ be an enumeration, for which we use the notation introduced in Definition 2.3.2. Let $(s, \nu) \in [0, T] \times \mathcal{SP}$ be an arbitrary, fixed admissible initial condition and consider

$$\begin{aligned} G_0^{s,\nu} : C_{s,T}\mathcal{SP} \to \mathbb{R},\ \mu = (\mu_t)_{t \in [s,T]} \mapsto \int_{\mathbb{R}^d} h_{n_{m_0^s}} d\mu_{q_{m_0^s}}, \\ u_0^{s,\nu} := \sup_{\mu \in \mathcal{A}_{s,\nu}} G_0^{s,\nu}(\mu), \\ M_0^{s,\nu} := (G_0^{s,\nu})^{-1}(u_0^{s,\nu}) \cap \mathcal{A}_{s,\nu}, \end{aligned}$$

and iteratively, for $k \in \mathbb{N}_0$,

$$\begin{aligned} G_{k+1}^{s,\nu} : C_{s,T}\mathcal{SP} \to \mathbb{R},\ (\mu_t)_{t \in [s,T]} \mapsto \int_{\mathbb{R}^d} h_{n_{m_{k+1}^s}} d\mu_{q_{m_{k+1}^s}}, \\ u_{k+1}^{s,\nu} := \sup_{\mu \in M_k^{s,\nu}} G_{k+1}^{s,\nu}(\mu), \\ M_{k+1}^{s,\nu} := {G_{k+1}^{s,\nu}}^{-1}(u_{k+1}^{s,\nu}) \cap M_k^{s,\nu}. \end{aligned}$$

By Remark 2.3.1, and since each h_n belongs to $C_c(\mathbb{R}^d)$, $G_0^{s,\nu}$ is continuous. Furthermore, since (s, ν) is admissible, $\mathcal{A}_{s,\nu}$ is nonempty and compact by assumption, so that $M_0^{s,\nu}$ is nonempty and compact as well. The same is iteratively true for $G_{k+1}^{s,\nu}$ and $M_{k+1}^{s,\nu}$ for each $k \in \mathbb{N}_0$. Since by construction $M_k^{s,\nu}$ is decreasing in k and $C_{s,T}\mathcal{SP}$ is Hausdorff, this implies

$$M^{s,\nu} := \bigcap_{k \geq 0} M_k^{s,\nu} \neq \emptyset.$$

Now assume $\mu^i = (\mu_t^i)_{t \in [s,T]} \in M^{s,\nu}$ for $i \in \{1, 2\}$. By construction, this implies

$$\int_{\mathbb{R}^d} h_{n_{m_k^s}} d\mu^1_{q_{m_k^s}} = \int_{\mathbb{R}^d} h_{n_{m_k^s}} d\mu^2_{q_{m_k^s}}, \quad k \in \mathbb{N}_0. \tag{2.15}$$

Since $\{(n_{m_k^s}, q_{m_k^s}),\ k \in \mathbb{N}_0\} = \mathbb{N} \times \mathbb{Q}_s^T$, this yields $\int h_n d\mu_q^1 = \int h_n d\mu_q^2$ for all $(n, q) \in \mathbb{N} \times \mathbb{Q}_s^T$ and hence $\mu_q^1 = \mu_q^2$ for all $q \in \mathbb{Q}_s^T$, because $\{h_n\}_{n \geq 1}$ is measure separating. Since both μ^1 and μ^2 are vaguely continuous, $\mu^1 = \mu^2$ follows. Consequently, $M^{s,\nu} \subseteq \mathcal{A}_{s,\nu}$ is a singleton, i.e. $\mathcal{M}^{s,\nu} = \{\mu^{s,\nu}\}$.

It remains to show that the family of all such $\mu^{s,\nu}$ forms a solution flow. To this end, let (s, ν) be admissible, and fix $0 \leq s \leq r \leq t \leq T$. Consider the admissible initial condition

$(r, \mu_r^{s,\nu})$ and let $\gamma = (\gamma_t)_{t\in[r,T]} \in M^{r,\mu_r^{s,\nu}}$ be the corresponding unique solution selected in the first part of the proof, i.e. in our notation $\gamma = \mu^{r,\mu_r^{s,\nu}}$. We need to show

$$\gamma_t = \mu_t^{s,\nu}, \quad t \in [r, T]. \tag{2.16}$$

To this end, set $\zeta := \mu^{s,\nu} \circ_r \gamma \in \mathcal{A}_{s,\nu}$. Due to the iteratively maximizing selection procedure of the first part of the proof, we have

$$\int_{\mathbb{R}^d} h_{n_{m_0^s}} d\mu^{s,\nu}_{q_{m_0^s}} \geq \int_{\mathbb{R}^d} h_{n_{m_0^s}} d\zeta_{q_{m_0^s}}. \tag{2.17}$$

If $q_{m_0^s} \in [s, r)$, then $\zeta_{q_{m_0^s}} = \mu^{s,\nu}_{q_{m_0^s}}$ and we have equality in (2.17). If $q_{m_0^s} \in [r, T]$, then $q_{m_0^s} = q_{m_0^r}$ and by the characterizing property of γ in $\mathcal{A}_{r,\mu_r^{s,\nu}}$, and since $(\mu_t^{s,\nu})_{t\in[r,T]} \in \mathcal{A}_{r,\mu_r^{s,\nu}}$, we obtain

$$\int_{\mathbb{R}^d} h_{n_{m_0^s}} d\mu^{s,\nu}_{q_{m_0^s}} \leq \int_{\mathbb{R}^d} h_{n_{m_0^s}} d\gamma_{q_{m_0^s}} = \int_{\mathbb{R}^d} h_{n_{m_0^s}} d\zeta_{q_{m_0^s}},$$

and hence we have equality in (2.17) in any case. Next, consider m_1^s: since (2.17) is an equality, both $(\mu_t^{s,\nu})_{t\in[s,T]}$ and $(\zeta_t)_{t\in[s,T]}$ belong to $M_0^{s,\nu}$. Hence, using the characterization of $\mu^{s,\nu}$ again, we obtain

$$\int_{\mathbb{R}^d} h_{n_{m_1^s}} d\mu^{s,\nu}_{q_{m_1^s}} \geq \int_{\mathbb{R}^d} h_{n_{m_1^s}} d\zeta_{q_{m_1^s}}, \tag{2.18}$$

clearly with equality if $q_{m_1^s} \in [s, r)$. If $q_{m_1^s} \in [r, T]$ and $q_{m_0^s} \in [s, r)$, then $m_1^s = m_0^r$, i.e.

$$\int_{\mathbb{R}^d} h_{n_{m_1^s}} d\mu^{s,\nu}_{q_{m_1^s}} \leq \int_{\mathbb{R}^d} h_{n_{m_1^s}} d\gamma_{q_{m_1^s}} = \int_{\mathbb{R}^d} h_{n_{m_1^s}} d\zeta_{q_{m_1^s}} \tag{2.19}$$

by the characterizing property of γ, which gives equality in (2.18). If $q_{m_0^s}, q_{m_1^s} \in [r, T]$, then $m_0^s = m_0^r$, $m_1^s = m_1^r$ and both $\mu^{s,\nu}$ and γ are in $M_0^{r,\mu_r^{s,\nu}}$, which also gives (2.19). Hence, equality in (2.18) holds in any case. Iterating this procedure yields

$$\int_{\mathbb{R}^d} h_{n_{m_k^s}} d\mu^{s,\nu}_{q_{m_k^s}} = \int_{\mathbb{R}^d} h_{n_{m_k^s}} d\zeta_{q_{m_k^s}}, \quad k \in \mathbb{N}_0,$$

and hence, since $\mathcal{H}$ is measure separating,

$$\mu_q^{s,\nu} = \zeta_q, \quad q \in \mathbb{Q}_s^T,$$

so in particular $\mu_q^{s,\nu} = \zeta_q = \gamma_q$ for all $q \in \mathbb{Q}_r^T$. Since both curves are vaguely continuous, we obtain (2.16). □

Remark 2.3.3. *We point out that one may freely choose the measure separating family $\mathcal{H}$ and the corresponding enumeration ξ, as well as the dense, countable subset of $[s, T]$ (which is $\mathbb{Q}_s^T$ in the above setting). Clearly, the selected solution flow may depend on these choices.*

Let us now derive the proof of Theorem 1.3.2 as a simple consequence of the selection procedure of the previous proof.

Proof of Theorem 1.3.2. The implication (ii) $\Longrightarrow$ (i) is obvious and we focus on (i) $\Longrightarrow$ (ii). Assume there is an admissible initial condition $(s', \nu') \in [0, T] \times \mathcal{SP}$ with

$|\mathcal{A}_{s',\nu'}| \geq 2$. As mentioned in Remark (2.3.3), we may choose an enumeration ξ and a family of measure separating functions $\mathcal{H} = \{h_n, n \in \mathbb{N}\} \subseteq C_c(\mathbb{R}^d)$ such that $h \in \mathcal{H}$ implies $-h \in \mathcal{H}$. Consider the flow $\{\mu^{s,\nu}\}$ with (s,ν) running through all admissible initial conditions, constructed as in the proof of Theorem 1.3.1 subject to these choices. By assumption, there exists $\gamma \in \mathcal{A}_{s',\nu'}$ with $\mu^{s',\nu'} \neq \gamma$, and since both solution curves are vaguely continuous, there is $q \in \mathbb{Q}^T_{s'}$ such that $\mu^{s',\nu'}_q \neq \gamma_q$. Thus, considering $-h$ instead of h if necessary, we may assume that there is $h \in \mathcal{H}$ such that

$$\int_{\mathbb{R}^d} h d\gamma_q > \int_{\mathbb{R}^d} h d\mu^{s',\nu'}_q. \tag{2.20}$$

Now consider a new enumeration ξ' such that according to ξ' we have $(h_{n_0}, q_{n_0}) = (h, q)$ and denote the flow subject to $\mathcal{H}$ and ξ' by $\{\eta^{s,\nu}\}$ (again, with (s,ν) running through all admissible initial conditions, which remain the same as before). Comparing with the beginning of the proof of Theorem 1.3.1, we have by construction

$$\int_{\mathbb{R}^d} h d\eta^{s',\nu'}_q = \sup_{\mu \in \mathcal{A}_{s',\nu'}} \left(\int_{\mathbb{R}^d} h d\mu_q \right)$$

and $\gamma \in \mathcal{A}_{s',\nu'}$. Therefore, taking into account (2.20), we conclude

$$\int_{\mathbb{R}^d} h d\eta^{s',\nu'}_q \geq \int_{\mathbb{R}^d} h d\gamma_q > \int_{\mathbb{R}^d} h d\mu^{s',\nu'}_q.$$

Hence, $\eta^{s',\nu'} \neq \mu^{s',\nu'}$, which contradicts (i) and finishes the proof. □

2.3.1 Measurability of the selected solution flow

Furthermore, under the additional Assumption A1 stated below, we have the following information concerning the measurability of the selected solution flow. For the notation of this subsection, we refer to Appendix E.

Assumption A1.

(A1.i) $\int_0^T \sup_{x \in K} \big(|a_{ij}(t,x)| + |b_i(t,x)|\big) dt < \infty \; \forall\, K \subseteq \mathbb{R}^d$ compact and $1 \leq i,j \leq d$.

(A1.ii) $x \mapsto a_{ij}(t,x)$ and $x \mapsto b_i(t,x)$ are continuous for dt-a.e. $t \in (0,T)$ and each $1 \leq i,j \leq d$.

Note that in the case of time-homogeneous coefficients $a(t,x) = \bar{a}(x)$, $b(t,x) = \bar{b}(x)$, (A1.ii) implies (A1.i). Of course, (A1.i) is fulfilled, if a and b are bounded on $(0,T) \times B$ for each ball $B \subseteq \mathbb{R}^d$. Moreover, we assume the following stability assumption for $\{\mathcal{A}_{s,\nu}\}_{(s,\nu) \in [0,T] \times \mathcal{SP}}$:

$$\text{If } \nu_n \longrightarrow \nu \text{ in } A_s \subseteq \mathcal{SP} \text{ and } \mu^{\nu_n} \in \mathcal{A}_{s,\nu_n} \text{ converges to } \mu \in \mathcal{M}_{s,\nu} \text{ in } C_{s,T}\mathcal{SP}, \text{ then } \mu \in \mathcal{A}_{s,\nu}. \tag{2.21}$$

In this situation, we obtain the following auxiliary result.

Lemma 2.3.4. *If the coefficients a and b satisfy Assumption A1 and (2.21) holds, then, the map $A_s \ni \nu \mapsto \mathcal{A}_{s,\nu} \in \mathrm{comp}(C_{s,T}\mathcal{SP})$ is Borel measurable.*

Proof. We want to apply Lemma E.0.2 to deduce the assertion. Hence, in view of the assumption (2.21), it remains to prove the following: whenever $(\nu_n)_{n\in\mathbb{N}}$ converges to ν vaguely in A_s and $\mu^n \in \mathcal{A}_{s,\nu_n}$, then there exists a subsequence $(\mu^{n_k})_{k\in\mathbb{N}}$ converging in $C_{s,T}\mathcal{SP}$ with a limit point $\mu \in \mathcal{M}_{s,\nu}$. To this end, it suffices to prove precompactness of $\bigcup_{\nu\in A_s}\mathcal{A}_{s,\nu}$ in $C_{s,T}\mathcal{SP}$ and closedness of $\mathcal{M}_s \subseteq C_{s,T}\mathcal{SP}$. Indeed, it is clear that then μ has initial condition (s,ν), i.e. μ belongs to $\mathcal{M}_{s,\nu}$. Concerning the latter, let $(\mu^n)_{n\geq 1}\subseteq \mathcal{M}_s$ be a converging sequence with limit $\mu \in C_{s,T}\mathcal{SP}$. Then, by Assumption A1, μ fulfills (2.1), and we have for each $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t\in[s,T]$

$$\int_{\mathbb{R}^d}\varphi(x)d\mu_t^n(x) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d}\varphi(x)d\mu_t(x)$$

as well as, using Lebesgue's dominated convergence theorem,

$$\int_s^t\int_{\mathbb{R}^d}\mathcal{L}_u\varphi(x)d\mu_u^n(x)du \underset{n\to\infty}{\longrightarrow} \int_s^t\int_{\mathbb{R}^d}\mathcal{L}_u\varphi(x)d\mu_u(x)du.$$

Indeed, the latter convergence follows, since $\mathcal{L}_t\varphi \in C_c(\mathbb{R}^d)$ for each $t\in[s,T]$ due to assumption (A1.ii). Consequently, $\mu = (\mu_t)_{t\in[s,T]} \in \mathcal{M}_{s,\nu}$.

Concerning precompactness of $\bigcup_{\nu\in A_s}\mathcal{A}_{s,\nu}\subseteq C_{s,T}\mathcal{SP}$, note that for each $t\in[s,T]$ the set $\pi_t(\bigcup_\nu \mathcal{A}_{s,\nu})\subseteq \mathcal{SP}$ is precompact, since the latter is a compact space. Moreover, in Section 2.4, even equicontinuity of $\mathcal{M}_s\subseteq C_{s,T}\mathcal{SP}$ is shown under the same assumptions on a and b. By Proposition 2.4.1, this yields the desired precompactness and the assertion follows. □

Proposition 2.3.5. *Consider the situation of Theorem 1.3.1 and let $\{\mu^{s,\nu}\}$ be the solution flow with respect to $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ constructed in the proof of that theorem. If the coefficients a and b satisfy Assumption A1 and (2.21) holds, then this flow is measurable in the sense of Definition 2.2.2 (ii).*

Proof. In fact, we prove the following stronger measurability property for each $s\in[0,T]$, where by $\mu^{s,\nu}$ we denote the selected solution from $\mathcal{A}_{s,\nu}$ according to the proof of Theorem 1.3.1:

$$\nu \mapsto \mu^{s,\nu} \text{ is a Borel map from } A_s \text{ to } C_{s,T}\mathcal{SP}. \tag{2.22}$$

From here, the measurability of each U_t^s follows from the measurability of the projections $\pi_t : C_{s,T}\mathcal{SP}\to\mathcal{SP}$. In order to verify (2.22), it suffices to prove measurability of

$$A_s \ni \nu \mapsto \{\mu^{s,\nu}\} \in \mathrm{comp}(C_{s,T}\mathcal{SP}). \tag{2.23}$$

Indeed, from here (2.22) follows, since $C_{s,T}\mathcal{SP}$ is a separable metric space, so that we can apply Lemma E.0.3. Concerning (2.23), for $N\geq 0$ consider (using the notation of the proof of Theorem 1.3.1) the maps

$$X_N : A_s \to \mathrm{comp}(C_{s,T}\mathcal{SP}),\ X_N : \nu \mapsto \bigcap_{k=0}^N M_k^{s,\nu} = M_N^{s,\nu}.$$

Since each function $G_k^{s,\nu}$ of the proof of Theorem 1.3.1 is continuous on $C_{s,T}\mathcal{SP}$, finitely many iterative applications of Lemma E.0.1 together with Lemma 2.3.4 yield the measurability of X_N. Finally, the map (2.23) is the pointwise limit of X_N as $N \to \infty$ with respect to the Hausdorff distance on $\mathrm{comp}(C_{s,T}\mathcal{SP})$, see Appendix E, which implies its measurability. This concludes the proof. □

2.4 Applications and examples

Here, we apply Theorems 1.3.1 and 1.3.2 to several examples. We start by discussing some topological prerequisites. In particular, we need the theorem by Arzela-Ascoli in the following formulation, see [178, Thm.47.1,p.290]. We recall that we consider $C_{s,T}\mathcal{SP}$ with the compact-open topology, c.f. Remark 2.3.1.

Proposition 2.4.1 (Arzela-Ascoli). *Let (X, d_X) be a compact metric space, let Y be metrizable and d_Y a compatible metric on Y. A subset $\mathcal{F} \subseteq C(X, Y)$ is precompact with respect to the compact-open topology if and only if*

(i) $\mathcal{F}$ is pointwise precompact, i.e. $\pi_x(\mathcal{F}) \subseteq Y$ is precompact for any $x \in X$.

(ii) $\mathcal{F}$ is equicontinuous with respect to d_Y, i.e. for any $x_0 \in X$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x_0) > 0$ such that

$$d_X(x_0, x) \leq \delta \implies \sup_{f\in\mathcal{F}} d_Y(f(x_0), f(x)) \leq \varepsilon.$$

Remark 2.4.2. *Below, we apply this result to subsets of $C_{s,T}\mathcal{SP}$. The following observation is crucial to our approach: in general, the notion of equicontinuity depends on the metric of the state space. On the other hand, the compact-open topology on $C(X, Y)$ is defined in terms of the topologies on X and Y only and does not depend on the specific compatible metric on Y. Therefore, (pre)compactness of $\mathcal{F} \subseteq C(X, Y)$ is invariant under a change of compatible metrics on Y. Of course, this is also true for (i) of the above proposition. Hence, for X and Y as above, it turns out that equicontinuity of $\mathcal{F} \subseteq C(X, Y)$ is independent of the compatible metric on Y, provided $\mathcal{F}$ fulfills (i). Therefore, in order to show equicontinuity of $\mathcal{F} \subseteq C_{s,T}\mathcal{SP}$, we may consider any metric on $\mathcal{SP}$, which is compatible with the vague topology.*

Due to the previous remark, we decide to consider the following type of metric on $\mathcal{SP}$. For $\nu_1, \nu_2 \in \mathcal{SP}$, set

$$d(\nu_1, \nu_2) := \sum_{l\geq 1} 2^{-l} C_l^{-1} \left[\left(\int_{\mathbb{R}^d} f_l d\nu_1 - \int_{\mathbb{R}^d} f_l d\nu_2 \right) \wedge 1 \right], \tag{2.24}$$

where $C_l \geq 1$ and $\{f_l, l \in \mathbb{N}\} \subseteq C_c^2(\mathbb{R}^d)$ is dense in $C_c(\mathbb{R}^d)$ with respect to uniform convergence. Due to this choice of $\{f_l, l \in \mathbb{N}\}$, it is clear that such metrics are compatible with the vague topology on $\mathcal{SP}$. From here on, we fix such a family $\{f_l, l \in \mathbb{N}\}$ and assume without loss of generality that each f_l has nontrivial support. Particular choices for C_l will be made below as needed.

Entire subprobability flow. Here, we consider the case that for each initial condition $(s, \nu) \in [0, T] \times \mathcal{SP}$, there exists at least one vaguely continuous subprobability solution to the Cauchy problem (FPK) and we construct a solution flow $\{\mu^{s,\nu}\}$, which comprises a solution $\mu^{s,\nu} \in \mathcal{M}_{s,\nu}$ for every initial condition $(s,\nu) \in [0,T] \times \mathcal{SP}$, i.e we set $\mathcal{A}_{s,\nu} = \mathcal{M}_{s,\nu}$ and $A_s = \mathcal{SP}$. We call such a solution flow *entire*.

We point out that we do not assume nondegeneracy of $a = (a_{ij})_{1\leq i,j\leq d}$, which is a source for possible nonuniqueness of solutions. Consider Assumption A1 introduced in the previous section.

Proposition 2.4.3. *Suppose Assumption A1 is fulfilled for the Borel coefficients a_{ij}, b_i, $1 \leq i,j \leq d$, and suppose $\mathcal{M}_{s,\nu}$ is nonempty for each $(s,\nu) \in [0,T] \times \mathcal{SP}$. Then, there exists an entire solution flow $\{\mu^{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$. Furthermore, this flow is measurable.*

Proof. By assumption, each initial condition $(s,\nu) \in [0,T] \times \mathcal{SP}$ is admissible. In light of Theorem 1.3.1, it suffices to prove compactness of $\mathcal{M}_{s,\nu} \subseteq C_{s,T}\mathcal{SP}$, for which we evoke the result by Arzela-Ascoli as stated in Proposition 2.4.1. Since $\mathcal{SP}$ with the vague topology is compact, precompactness of $\pi_t(\mathcal{M}_{s,\nu}) \subseteq \mathcal{SP}$ for each $t \in [s,T]$ follows immediately. Concerning equicontinuity, consider the metric d as in (2.24), with constants $C_l := 1 + D_l$, where

$$D_l := \max_{1\leq i,j\leq d} \{||\partial_i f_l||_\infty, ||\partial_{ij} f_l||_\infty\} > 0.$$

By (2.2), for each solution $\mu = (\mu_t)_{t\in[s,T]} \in \mathcal{M}_{s,\nu}$, we have for $t_1, t_2 \in [s,T]$

$$\begin{aligned} d(\mu_{t_1}, \mu_{t_2}) &\leq \sum_{l\geq 1} 2^{-l} C_l^{-1} \int_{t_1\wedge t_2}^{t_1\vee t_2} \int_{\mathbb{R}^d} |\mathcal{L}_t f_l(x)| d\mu_t(x) dt \\ &\leq \sum_{l\geq 1} 2^{-l} C_l^{-1} D_l \int_{t_1\wedge t_2}^{t_1\vee t_2} \max_{1\leq i,j\leq d} \sup_{x\in K_l} \big(|a_{ij}(t,x)| + |b_i(t,x)|\big) dt \\ &\leq \sum_{l\geq 1} 2^{-l} \int_{t_1\wedge t_2}^{t_1\vee t_2} \max_{1\leq i,j\leq d} \sup_{x\in K_l} \big(|a_{ij}(t,x)| + |b_i(t,x)|\big) dt, \end{aligned} \tag{2.25}$$

with $K_l := \text{supp}\, f_l$. For any fixed $t_1 \in [s,T]$ and $\varepsilon > 0$, by (A1.i) there is $\delta = \delta(t_1, \varepsilon) > 0$ such that

$$t_2 \in [s,T],\ |t_1 - t_2| \leq \delta \implies \int_{t_1\wedge t_2}^{t_1\vee t_2} \max_{1\leq i,j\leq d} \sup_{x\in K_l} \big(|a_{ij}(t,x)| + |b_i(t,x)|\big) dt < \varepsilon,$$

which implies equicontinuity of $\mathcal{M}_{s,\nu} \subseteq C_{s,T}\mathcal{SP}$ with respect to d by (2.25). Finally, concerning closedness of $\mathcal{M}_{s,\nu}$, let $\mu^n, n \geq 1, \mu^n = (\mu^n_t)_{t\in[s,T]}$ be a converging sequence in $\mathcal{M}_{s,\nu}$ with limit $\mu = (\mu_t)_{t\in[s,T]} \in C_{s,T}\mathcal{SP}$ and let $\varphi \in C_c^\infty(\mathbb{R}^d)$. Clearly,

$$\int_{\mathbb{R}^d} \varphi(x) d\mu^n_t(x) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \varphi(x) d\mu_t(x), \quad t \in [s,T],$$

and $\mu_s = \nu$. Furthermore, $\mathcal{L}_t\varphi(x) d\mu_t(x) \in L^1([0,T])$ by (A1.i) and, due to (A1.ii), we have $\mathcal{L}_t\varphi \in C_c(\mathbb{R}^d)$ dt-a.s. Consequently,

$$\int_{\mathbb{R}^d} \mathcal{L}_t\varphi(x) d\mu^n_t(x) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \mathcal{L}_t\varphi(x) d\mu_t(x) \quad dt\text{-a.s.},$$

and Lebesgue's dominate convergence theorem gives

$$\int_s^t \int_{\mathbb{R}^d} \mathcal{L}_u\varphi(x)d\mu_u^n(x)du \underset{n\to\infty}{\longrightarrow} \int_s^t \int_{\mathbb{R}^d} \mathcal{L}_u\varphi(x)d\mu_u(x)du.$$

Therefore, $\mu \in \mathcal{M}_{s,\nu}$. Altogether, we have shown compactness of $\mathcal{M}_{s,\nu} \subseteq C_{s,T}\mathcal{SP}$ and may therefore apply Theorem 1.3.1 to obtain the first claim.

Concerning the measurability of $\{\mu^{s,\nu}\}$, it suffices to note that Proposition 2.3.5 applies to the present situation, because (2.21) is trivially fulfilled for the choice $\mathcal{A}_{s,\nu} = \mathcal{M}_{s,\nu}$. □

Remark 2.4.4. *Proposition 2.4.3 applies in the situation of [38, Thm.6.7.3]. Indeed, the assumptions therein imply Assumption A1 with $c \equiv 0$, and in this case, elements $\mu \in \mathcal{M}_\nu$ as in [38, Thm.6.7.3] are curves of subprobability measures dt-a.s. for $\nu \in \mathcal{SP}$ (note that the existence results of [38, Thm.6.7.3] is stated for $\nu \in \mathcal{P}$ only, but obviously extends to $\nu \in \mathcal{SP}$). For each such solution curve μ, Lemma 2.1.3 yields the existence of a vaguely continuous version $\tilde{\mu} \in \mathcal{M}_{s,\nu}$, which shows that indeed in this situation each initial condition is admissible. Of course, also our second main result, Theorem 1.3.2, applies in this situation.*

Entire probability-flow. Due to the fruitful connection to stochastic analysis and Markov processes as outlined in Chapter 1, often one is primarily interested in probability solutions, see also Section 2.5. In this paragraph, we apply Theorem 1.3.1 to select an *entire probability flow*, i.e. a solution flow $\{\mu^{s,\nu}\}$ consisting of a solution $\mu^{s,\nu} \in \mathcal{M}^1_{s,\nu}$ for each $\nu \in \mathcal{P}$. We consider the following refined version of Assumption A1.

Assumption A2.

(A2.i) $\int_0^T \sup_{x\in\mathbb{R}^d} \big(|a_{ij}(t,x)| + |b_i(t,x)|\big)dt < \infty$.

(A2.ii) $x \mapsto a_{ij}(t,x)$ and $x \mapsto b_i(t,x)$ are continuous for dt-a.e. $t \in (0,T)$ and each $1 \le i,j \le d$.

In other words, (A2.ii) coincides with (A1.ii), and (A1.i) is a global (in space) version of (A1.i).

Remark 2.4.5. *In this situation, any solution $(\mu_t)_{t\in[s,T]} \in \mathcal{M}_{s,\nu}$ is a probability solution, provided $\nu \in \mathcal{P}$. Indeed, due to the global integrability assumption (A2.i), considering (2.2) for $(\mu_t)_{t\in[s,T]}$ and choosing a nonnegative sequence $\{\varphi_n\}_{n\ge1} \subseteq C_c^2(\mathbb{R}^d)$, which increases pointwise to 1 such that $\sup_{n\ge1} ||\varphi_n||_{C^2} < \infty$, we obtain $\mu_t(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ for any $t \in [s,T]$. In particular, any such $t \mapsto \mu_t$ is weakly continuous.*

We select the solution flow from the sets

$$\mathcal{A}_{s,\nu} = \begin{cases} \mathcal{M}_{s,\nu} & , \text{ if } \nu \in \mathcal{P} \\ \emptyset & , \text{ if } \nu \in \mathcal{SP}\backslash\mathcal{P}. \end{cases} \tag{2.26}$$

More precisely, we have the following result.

Corollary 2.4.6. *Suppose the Borel coefficients a_{ij}, b_i, $1 \leq i,j \leq d$, satisfy Assumption A2 and the set $\mathcal{M}_{s,\nu}$ is nonempty for each $(s,\nu) \in [0,T] \times \mathcal{P}$. Then, there exists an entire probability flow, consisting of weakly continuous probability solutions $\{\mu^{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}}$. Moreover, this flow is measurable.*

Proof. The family (2.26) is flow-admissible and, as explained above, any element in $\mathcal{M}_{s,\nu}$, $\nu \in \mathcal{P}$, is a weakly continuous probability solution. Compactness of $\mathcal{A}_{s,\nu} \subseteq C_{s,T}\mathcal{SP}$ is proven proven exactly as in the proof of Proposition 2.4.3. Hence, by Theorem 1.3.1, there exists an entire probability solution flow. Since Assumption A2 implies Assumption A1 and the choices of $\mathcal{A}_{s,\nu}$ in (2.26) clearly satisfy (2.21), this flow is measurable, which completes the proof. □

We point out that without the global in space integrability assumption (A2.i), we cannot prove that each solution with initial condition $\nu \in \mathcal{P}$ is a probability solution, which is, however, crucial to obtain flow-admissibility of the family (2.26). Also, without global integrability, we cannot replace $\mathcal{M}_{s,\nu}$ by $\mathcal{M}^1_{s,\nu}$ in (2.26), since for these sets we are not able to prove closedness in $\mathcal{M}_{s,\nu}$ (again, due to the lack of global boundedness). Moreover, considering $C_{s,T}\mathcal{P}$ with the weak topology on $\mathcal{P}$ seems to be of no advantage, since in this case, in general we do not know how to prove pointwise precompactness of families $\mathcal{F} \subseteq C_{s,T}\mathcal{P}$.

Remark 2.4.7. *An alternative proof for the case of bounded coefficients on $[0,T]\times\mathbb{R}^d$, which satisfy (A2.ii) is given in [187]. The proof is based on the close connection between probability solutions to (FPK) and the corresponding martingale problem via the superposition principle, as explained in Chapter 1. However, the approach presented in this part of the thesis seems to be more general in the sense that it also applies to subprobability flows and nonlinear equations, as well as to infinite-dimensional version of (FPK) and under weaker assumptions on the coefficients a and b.*

Moreover, note that even global boundedness and spatial continuity of the coefficients does not imply uniquenss of probability solutions to (FPK) (nonuniqueness may, for example, be caused by degeneracy of $a = (a_{ij})_{1\leq i,j\leq d}$). Hence, under Assumption A1 or A2, the selection of a solution flow is a nontrivial problem.

Flow with respect to a Lyapunov function. Here, we temporarily consider situations of unbounded coefficients. We are interested in cases where the lack of global boundedness is compensated by a suitable control on the growth of the coefficients at infinity in terms of a so-called Lyapunov function.

We call a nonnegative function $\psi \in C^2(\mathbb{R}^d)$ with compact sublevel sets $\{\psi \leq c\} \subseteq \mathbb{R}^d$, $c \geq 0$, a *compact function*. It is clear that $\psi \in C^2(\mathbb{R}^d)$ is compact if and only if $\lim \psi(x) = \infty$ if $|x| \to \infty$. For such ψ, we denote by $\mathcal{P}_\psi$ the set of all $\nu \in \mathcal{P}$ such that $\psi \in L^1(\nu)$. Assume there is such ψ, for which it holds

$$\mathcal{L}_t\psi(x) \leq C + C\psi(x) \quad dxdt\text{-a.s. in } [0,T] \times \mathbb{R}^d \tag{2.27}$$

for some $C \geq 0$. Such functions are usually called *Lyapunov function*. Due to the spatial continuity of a and b, (2.27) then holds pointwise in $x \in \mathbb{R}^d$ for dt-a.e. $t \in [0,T]$. Let

$\nu \in \mathcal{P}_\psi$. By Lemma 3.3.3, any probability solution $(\mu_t)_{t\in[s,T]}$ (which, by our definition is weakly continuous) to (FPK) with initial condition (s,ν) fulfills, for each $t \in [s,T]$,

$$\int_{\mathbb{R}^d} \psi d\mu_t \leq \int_{\mathbb{R}^d} \psi d\nu + C(t-s) + C\exp(Ct)\int_s^t \exp(-Cr)\Big[\int_{\mathbb{R}^d}\psi d\nu + C(t-r)\Big]dr. \quad (2.28)$$

We aim to select a solution flow from the sets

$$\mathcal{A}_{s,\nu} = \begin{cases} \mathcal{M}^1_{s,\nu} & , \text{ if } \nu \in \mathcal{P}_\psi \\ \emptyset & , \text{ else.} \end{cases} \quad (2.29)$$

Since (2.28) is valid for each element of any $\mathcal{A}_{s,\nu}$, and since the right-hand side of (2.28) is finite, these sets form a flow-admissible family.

Corollary 2.4.8. *Suppose the Borel coefficients a_{ij}, b_i, $1 \leq i,j \leq d$, fulfill Assumption A1 and there exists a compact function ψ such that (2.27) holds. If the sets $\mathcal{M}^1_{s,\nu}$ are nonempty for each initial condition $(s,\nu) \in [0,T]\times\mathcal{P}$ such that $\nu \in \mathcal{P}_\psi$, then there exists a solution flow of probability solutions with respect to the family $\mathcal{A}_{s,\nu}$ as in (2.29).*

In this situation, the flow evolves in the subset $\mathcal{P}_\psi \subseteq \mathcal{P}$. Any measure with bounded support belongs to $\mathcal{P}_\psi$, i.e. in particular each Dirac measure. It is clear that $\mathcal{P}_\psi \neq \mathcal{P}$ and that $\mathcal{P}_\psi$ is not closed in $\mathcal{P}$.

Proof. Since the sets introduced in (2.29) are flow-admissible, due to Theorem 1.3.1, it is sufficient to prove compactness of $\mathcal{A}_{s,\nu} \subseteq C_{s,T}\mathcal{SP}$ for each admissible initial condition (s,ν) (that is, for each $s \in [0,T]$ and $\nu \in \mathcal{P}_\psi$). For any such admissible condition (s,ν) and any $t \in [s,T]$, precompactness of $\pi_t(\mathcal{A}_{s,\nu}) \subseteq \mathcal{P}$ holds even with respect to the weak topology, because (2.28) holds for any $(\mu_t)_{t\in[s,T]} \in \mathcal{A}_{s,\nu}$ and the right-hand side of (2.28) is independent of $(\mu_t)_{t\in[s,T]}$ and finite, which yields

$$\sup_{\mu_t\in\pi_t(\mathcal{A}_{s,\nu})} \int_{\mathbb{R}^d} \psi(x)d\mu_t(x) < \infty. \quad (2.30)$$

Thus, $\pi_t(\mathcal{A}_{s,\nu}) \subseteq \mathcal{P}$ is tight and hence precompact with respect to the weak topology. Equicontinuity of $\mathcal{A}_{s,\nu} \subseteq C_{s,T}\mathcal{SP}$ follows as in the proof of Proposition 2.4.3. Concerning closedness of $\mathcal{A}_{s,\nu} \subseteq C_{s,T}\mathcal{SP}$, it suffices to observe that the validity of (2.28) for each solution $\mu \in \mathcal{M}^1_{s,\nu}$, with the right-hand side of (2.28) being independent of μ, implies that the limit of any converging sequence $(\mu^n)_{n\in\mathbb{N}}$ in $C_{s,T}\mathcal{SP}$ is a curve of probability measures and solves (FPK). The latter follows exactly as in the proof of Proposition 2.4.3. This completes the proof. □

Consider the situation of Theorem 3.1 in [36] and assume additionally that b_i, $1 \leq i \leq d$, satisfies (A1.i) and (A1.ii). Note that in this case, a_{ij}, $1 \leq i,j \leq d$ has a version, which fulfills these assumptions due to (C1) and (C2) of [36]. Indeed, by the Sobolev embedding theorem, each $a_{ij}(t,\cdot)$ has a (Hölder) continuous version, which is bounded on balls in $\mathbb{R}^d$, independently of t. Then, for any $(s,\nu) \in [0,T]\times\mathcal{P}$, [36, Thm.3.1] yields the existence of a weakly continuous probability solution $(\mu^{s,\nu}_t)_{t\in[s,T)}$, so that Corollary 2.4.8 applies. However, in this situation, solutions possibly do not extend to the endpoint $t = T$ and hence the flow is only constructed on $[0,T)$. Corollary 3.4 of [36] gives another application of Proposition 2.4.8, if one assumes boundedness of the coefficients on $[0,T]\times B$ instead of $[0,T)\times B$ for any bounded set $B\subseteq\mathbb{R}^d$.

2.5 Comparison to Markovian semigroups

In this section, we study the comparison and connection of solutions flows to *Markovian semigroups* in the case of coefficients a and b such that

Assumption A3. For each $1\leq i,j\leq d$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}|a_{ij}(t,x)|+|b_i(t,x)|<\infty \text{ and } x\mapsto a_{ij}(t,x),b_i(t,x) \text{ is continuous for } t\in[0,T]$$

is fulfilled, which is the strongest set of assumptions we have imposed so far. Therefore, all results established under the validity of assumptions A1 and A2 hold, e.g. we have $\mathcal{M}_{s,\nu}=\mathcal{M}^1_{s,\nu}$ for each $(s,\nu)\in[0,T]\times\mathcal{P}$. In particular, the proof of the following lemma follows exactly as in Corollary 2.4.6 and Proposition 2.4.3. We also note that Assumption A3 yields that any solution set $\mathcal{M}^1_{s,\nu}$, $\nu\in\mathcal{P}$, is nonempty, see for example [38, Prop.6.7.3] and Remark 2.4.4.

Lemma 2.5.1. *Let Assumption A3 be fulfilled. Then, for any $(s,\nu)\in[0,T]\times\mathcal{P}$, $\mathcal{M}^1_{s,\nu}$ is compact in $C_{s,T}\mathcal{SP}$.*

Below, we write μ or η for curves $(\mu_t)_{t\in I}$ or $(\eta_t)_{t\in I}$, respectively, for a time interval I. In contrast, single measures are denoted ν and γ. Furthermore, for any measurable family of probability measures $\{\gamma_x\}_{x\in\mathbb{R}^d}$ and $\nu\in\mathcal{P}$, the Borel probability measure $\int_{\mathbb{R}^d}\gamma_x d\nu(x)$ is defined via $\big(\int_{\mathbb{R}^d}\gamma_x d\nu(x)\big)(A):=\int_{\mathbb{R}^d}\gamma_x(A)d\nu(x)$ for $A\in\mathcal{B}(\mathbb{R}^d)$.

Definition 2.5.2. A family of solutions $\{\mu^{s,x}\}_{(s,\delta_x)\in[0,T]\times\mathcal{P}}$, $\mu^{s,x}\in\mathcal{M}^1_{s,x}$, to (FPK) is called *Markovian semigroup* (on $[0,T]$), if for any $0\leq s\leq r\leq t\leq T$, $x\mapsto\mu^{s,x}_t$ is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathcal{P})$-measurable and the *Chapman-Kolmogorov equations*

$$\mu^{s,x}_t=\int_{\mathbb{R}^d}\mu^{r,y}_t d\mu^{s,x}_r(y) \tag{2.31}$$

are fulfilled for any $x\in\mathbb{R}^d$. For a Markovian semigroup, define $\mu^{s,\nu}:=\int_{\mathbb{R}^d}\mu^{s,y}d\nu(y)$ for any non-Dirac initial condition $\nu\in\mathcal{P}$ and call the extended family $\{\mu^{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}}$ the *convex extension* of $\{\mu^{s,x}\}_{(s,\delta_x)\in[0,T]\times\mathcal{P}}$.

Note that the assumed measurability ensures the well-definedness of the appearing integrals in the above definition.

Remark 2.5.3. *Let a and b fulfill the boundedness part of Assumption A3. In this case, for any measurable family of solutions $x\mapsto\mu^{s,x}\in\mathcal{M}^1_{s,\delta_x}$ and any $\nu\in\mathcal{P}$, the weakly continuous curve of probability measures $t\mapsto\int\mu^{s,y}_t d\nu(y)$ belongs to $\mathcal{M}^1_{s,\nu}$. Indeed, this is readily seen from the linearity of equation (FPK) and the boundedness of a_{ij} and b_i. Hence, the convex extension of any Markovian semigroup as in the above definition is a family of solutions to (FPK).*

We want to compare the Chapman-Kolmogorov equations (2.31) to the flow property (1.9). Such a comparison requires any initial condition (s,x) to be admissible for the flow, because the Chapman-Kolmogorov equations (2.31) require the existence of solutions $\mu^{s,x} \in \mathcal{M}^1_{s,x}$.

It turns out that at least under Assumption A3, both notions are equivalent. The following simple proposition establishes the first part of the equivalence of Markovian semigroups and a (entire probability) solution flow.

Proposition 2.5.4. *The convex extension of any Markovian semigroup* $\{\mu^{s,x}\}_{(s,\delta_x)\in[0,T]\times\mathcal{P}}$ *is an entire probability flow in the sense of Example 2.2.4.*

Proof. For any $0 \le s \le r \le t \le T$ and $\nu \in \mathcal{P}$, we have

$$\mu^{s,\nu}_t = \int_{\mathbb{R}^d} \mu^{s,y}_t d\nu(y) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mu^{r,z}_t d\mu^{s,y}_r(z) \right) d\nu(y) = \int_{\mathbb{R}^d} \mu^{r,z}_t d\mu^{s,\nu}_r(z) = \mu^{r,\mu^{s,\nu}_r}_t,$$

where we used the definition of the convex extension for the first, third and last equality and the Chapman-Kolmogorov equations (2.31) for the second one. □

Note that the above result does not require any assumptions on the coefficients a_{ij} and b_i. The following proposition, which establishes the converse relation to Proposition 2.5.4, is more involved, both in terms of the required assumptions and the method of proof.

Proposition 2.5.5. *Assume the coefficients a_{ij}, b_i fulfill Assumption A3. Then, any entire probability flow* $\{\mu^{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}}$ *selected as in the proof of Theorem 1.3.1 is the convex extension of a Markovian semigroup. In particular, the family* $\{\mu^{s,x}\}_{(s,\delta_x)\in[0,T]\times\mathcal{P}}$ *fulfills the Chapman-Kolmogorov equations (2.31).*

In combination with Lemma 2.5.1 and our main result, Theorem 1.3.1, we obtain the following result.

Corollary 2.5.6. *Let the coefficients a_{ij}, b_i satisfy Assumption A3. Then, there exists an entire probability flow to (FPK), which is the convex extension of a Markovian semigroup. In particular, there exists a Markovian semigroup of solutions to (FPK).*

Proof of Proposition 2.5.5. Consider an entire probability flow $\{\mu^{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}}$ selected as in the proof of Theorem 1.3.1. It suffices to prove $\mu^{s,\nu}_t = \int \mu^{s,y}_t d\nu(y)$ for all $0 \le s \le t \le T$ and $\nu \in \mathcal{P}$. Indeed, then the flow is the convex extension of $\{\mu^{s,x}\}_{(s,\delta_x)\in[0,T]\times\mathcal{P}}$ and (2.31) holds, since in this case, for all $0 \le s \le r \le t \le T$ and $\nu \in \mathcal{P}$, we have

$$\mu^{s,x}_t = \mu^{r,\mu^{s,x}_r}_t = \int_{\mathbb{R}^d} \mu^{r,y}_t d\mu^{s,x}_r(y).$$

By Lemma 2.5.9 below, the weakly continuous curve $\eta^{s,\nu}$, $\eta^{s,\nu}_t := \int_{\mathbb{R}^d} \mu^{s,y}_t d\nu(y)$, is an element of $\mathcal{M}^1_{s,\nu}$ (indeed, the measurability of $y \mapsto \mu^{s,y}_t$ for any t follows by Lemma 2.5.8). Using the notation of the proof of Theorem 1.3.1, we have

$$\int_{\mathbb{R}^d} h_{n_{m^s_0}} d\mu^{s,\nu}_{q_{m^s_0}} \ge \int_{\mathbb{R}^d} h_{n_{m^s_0}} d\eta^{s,\nu}_{q_{m^s_0}}, \tag{2.32}$$

where the notation is with respect to the specific measure separating family $\{h_n, n \in \mathbb{N}\}$ and the enumeration used within the selection of $\mu^{s,\nu}$ as in the proof of Theorem 1.3.1.

Since Lemma 2.5.9 yields $\mu^{s,\nu} = \int_{\mathbb{R}^d} \mu^y d\nu(y)$ for some measurable family $\{\mu^y\}_{y\in\mathbb{R}^d}$ with $\mu^y \in \mathcal{M}^1_{s,\delta_y}$ ν-a.s. (we suppress the dependence of μ^y on s in the notation) and the selection process of any $\mu^{s,y}$ in $\mathcal{M}^1_{s,\delta_y}$ implies

$$\int_{\mathbb{R}^d} h_{n_{m_0^s}} d\mu^{s,y}_{q_{m_0^s}} \geq \int_{\mathbb{R}^d} h_{n_{m_0^s}} d\mu^{y}_{q_{m_0^s}}, \quad \nu\text{-a.s.}, \tag{2.33}$$

we deduce

$$\begin{aligned}\int_{\mathbb{R}^d} h_{n_{m_0^s}} d\eta^{s,\nu}_{q_{m_0^s}} &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} h_{n_{m_0^s}} d\mu^{s,y}_{q_{m_0^s}} \right) d\nu(y) \\ &\geq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} h_{n_{m_0^s}} d\mu^{y}_{q_{m_0^s}} \right) d\nu(y) = \int_{\mathbb{R}^d} h_{n_{m_0^s}} d\mu^{s,\nu}_{q_{m_0^s}}. \end{aligned} \tag{2.34}$$

Hence, equality holds in (2.32). In turn, also (2.34) is a chain of equalities, which particularly gives that (2.33) is an equality. Iterating these arguments, (2.32), (2.34) and (2.33) hold with equality for any index pair $(n_{m_k^s}, q_{m_k^s})$, $k \in \mathbb{N}_0$. Using continuity of $t \mapsto \mu^{s,\nu}_t$ and $t \mapsto \eta^{s,\nu}_t$, we conclude $\mu^{s,\nu} = \eta^{s,\nu}$, i.e. $\mu^{s,\nu}_t = \int \mu^{s,y}_t d\nu(y)$ for each $t \in [s,T]$, and the claim follows. □

For the above proof, we used the following lemmas. We use the notation of the proof of Theorem 1.3.1 as follows. For a measure separating family $\{h_n, n \in \mathbb{N}\} \subset C_c(\mathbb{R}^d)$ and an enumeration ξ, the sets $M^{s,x}_k = M^{s,\delta_x}_k$ and maps $G^{s,x}_k = G^{s,\delta_x}_k$ are as defined in the proof of Theorem 1.3.1 with $\mathcal{A}_{s,x} = \mathcal{M}^1_{s,\delta_x} = \mathcal{M}_{s,\delta_x}$ (the second equality holds due to the boundedness part of Assumption A3).

Lemma 2.5.7. *The map $\mathcal{P} \ni \nu \mapsto \mathcal{M}_{s,\nu}$ is $\mathcal{B}(\mathcal{P})/\mathcal{B}(\mathrm{comp}(C_{s,T}\mathcal{SP}))$-measurable.*

Proof. We want to apply Lemma E.0.2. To this end, we prove: if $(\nu_n)_{n\in\mathbb{N}}$ converges to ν weakly and for any $n \in \mathbb{N}$ there is $\mu^n \in \mathcal{M}_{s,\nu_n}$, then there is a limit $\mu \in \mathcal{M}_{s,\nu}$ of a subsequence $(\mu^{n_k})_{k\in\mathbb{N}}$ in $C_{s,T}\mathcal{SP}$. However, this can be proven as in Lemma 2.3.4. Since in the present case, we have $\mathcal{A}_{s,\nu} = \mathcal{M}_{s,\nu}$, the additional assumption (2.21) is fulfilled. □

Lemma 2.5.8. *For each $(s,x) \in [0,T] \times \mathbb{R}^d$, let $\mu^{s,x} \in \mathcal{M}^1_{s,\delta_x}$ denote the solution curve selected by the iterative selection method presented within the proof of Theorem 1.3.1, i.e.*

$$\{\mu^{s,x}\} = \bigcap_{k\geq 0} M^{s,x}_k.$$

Then, the mapping $x \mapsto \mu^{s,x}$ is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(C_{s,T}\mathcal{P})$-measurable. In particular, for each $t \in [s,T]$, $x \mapsto \mu^{s,x}_t$ is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathcal{P})$-measurable.

Proof. It suffices to prove Borel measurability of

$$x \mapsto \bigcap_{k\geq 0} M^{s,x}_k \tag{2.35}$$

from $\mathbb{R}^d$ to $\mathrm{comp}(C_{s,T}\mathcal{SP})$, since then Lemma E.0.3 implies the measurability of $x \mapsto \mu^{s,x} \in C_{s,T}\mathcal{SP}$ and the claim follows, since $\mu^{s,x} \in C_{s,T}\mathcal{P}$ and $\mathcal{B}(\mathcal{SP})$ restricted to $\mathcal{P}$ coincides with the Borel σ-algebra of $\mathcal{P}$ with respect to the weak topology. Since the mapping (2.35) can be rewritten as $x \mapsto \lim_{N\to\infty} X_N(x)$ (the limit is taken in $\mathrm{comp}(C_{s,T}\mathcal{SP})$, see Appendix E), with

$$X_N : x \mapsto \bigcap_{10\leq k\leq N} M_k^{s,x} = M_N^{s,x} \in \mathrm{comp}(C_{s,T}\mathcal{SP}),$$

it suffices to prove measurability of each X_N. By definition of $M_0^{s,x}$ and lemmas 2.5.7 and E.0.1, X_0 is measurable (indeed, the map $G_0^{s,y}$ is continuous on $C_{s,T}\mathcal{SP}$). Iteratively applying Lemma E.0.1 to the maps $M_k^{s,x} \mapsto M_{k+1}^{s,x}$ and using the continuity of each $G_k^{s,x}$, the measurability of each X_N follows. The final assertion follows immediately by the measurability of the projection maps $\pi_t : C_{s,T}\mathcal{SP} \mapsto \mathcal{SP}$. □

Lemma 2.5.9. *Let $(s,\nu) \in [0,T) \times \mathcal{P}$ and assume Assumption A3 is fulfilled. Then, $\mu \in \mathcal{M}^1_{s,\nu}$ if and only if there exists a family $\{\mu^x\}_{x\in\mathbb{R}^d}$ such that $\mu^x \in \mathcal{M}^1_{s,x}$ ν-a.s. and $x \mapsto \mu^x_t$ is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathcal{P})$-measurable for each $t \in [s,T]$ such that $\mu_t = \int_{\mathbb{R}^d} \mu^x_t d\nu(x)$.*

Proof. Clearly, for any family $\{\mu^x\}_{x\in\mathbb{R}^d}$ as in the assertion, the curve $t \mapsto \mu_t := \int_{\mathbb{R}^d} \mu^x_t d\nu(x)$ is weakly continuous and a solution to (FPK) with initial condition $\mu_s = \nu$, i.e. $\mu \in \mathcal{M}^1_{s,\nu}$.

Conversely, for any solution curve $\mu \in \mathcal{M}^1_{s,\nu}$, there exists a probability measure $P = P(\mu) \in \mathcal{P}(C_{s,T}\mathbb{R}^d)$, which is a solution to the martingale problem associated to the coefficients a, b with $P \circ \pi_t^{-1} = \mu_t$ for each $t \in [s,T]$, see [100] as well as the introduction in Chapter 5 for details. Disintegrating P with respect to $P \circ \pi_s^{-1}$, we obtain a $\nu-$a.s. unique family $\{P_x\}_{x\in\mathbb{R}^d}$ of probability measures on $C_{s,T}\mathbb{R}^d$, such that $x \mapsto P_x$ is measurable and $P = \int_{\mathbb{R}^d} P_x d\nu(x)$. Furthermore, P_x is a solution to the associated martingale problem with initial condition (s,δ_x) for ν-a.e. $x \in \mathbb{R}^d$, see [223, Prop.2.8]. Since the curve of one-dimensional marginals of any solution to the martingale problem with initial condition (s,ν) is a weakly continuous probability solution to the corresponding FPK equation with initial condition (s,ν), we have $(P_x \circ \pi_t^{-1})_{t\in[s,T]} \in \mathcal{M}^1_{s,\delta_x}$ for $\nu-$a.e. $x \in \mathbb{R}^d$ and $\mu_t = \int_{\mathbb{R}^d} P_x \circ \pi_t^{-1} d\nu(x)$, i.e. the claim follows with $\mu^x_t := P_x \circ \pi_t^{-1}$. □

To us, it is unclear whether under weaker assumptions on the coefficients, any entire probability flow as in the proof of Theorem 1.3.1 gives rise to a Markovian semigroup.

Chapter 3

Solution flows for nonlinear equations

In this chapter, we study *nonlinear* Fokker–Planck–Kolmogorov equations (FPK equations) of type (NL-FPK). Our goal is to prove results on the existence and uniqueness of solution flows, comparable to the linear case of the previous chapter. To this end, we observe that the proofs of Theorems 1.3.1 and 1.3.2 do not rely on the linearity of the equations of type (FPK), but carry over to a nonlinear setting as well. It turns out that the nonlinearity of the coefficients renders compactness of (subclasses of) solutions a more delicate issue, which in comparison with the linear case requires additional regularity assumptions, see Section 3.3. We recall that nonlinear FPK equations of type (NL-FPK) naturally arise as the equations fulfilled by the marginal curves of solutions to McKean–Vlasov stochastic differential equations, in particular in connection to interacting particle systems. Moreover, such parabolic equations for measures are widely used in the area of statistical mechanics and physics, see Chapter 1 for more details and references for these interesting connections and origins.

3.1 Nonlinear FPK equations

As before, we study equations on the finite time interval $[0,T]$ for some $T>0$. Let $\mathcal{S}_0 \subseteq \mathcal{SP}$ and suppose that for each $\mu \in \mathcal{S}_0$, the coefficients

$$a_{ij}(\cdot,\mu,\cdot), b_i(\cdot,\mu,\cdot) : [0,T]\times\mathbb{R}^d \to \mathbb{R}, \quad 1 \le i,j \le d,$$

are $\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}^d)$-measurable such that $a(t,\mu,x) := (a_{ij}(t,\mu,x))_{1\le i,j\le d} \in \mathbb{S}_d^+$ for each $(t,\mu,x) \in [0,T]\times\mathcal{S}_0\times\mathbb{R}^d$. Via the (strict) subset $\mathcal{S}_0 \subseteq \mathcal{SP}$, we take into account the possibility that the coefficients may only be defined for a certain subclass of subprobability measures. More generally, one could allow dependence of $\mathcal{S}_0$ on t. Similar to the notation in the previous chapter, for $(t,\mu) \in [0,T]\times\mathcal{S}_0$, we use the notation $\mathcal{L}_{t,\mu}$ for the second-order differential operator as introduced in (1.3).

Given such coefficients, we study the Cauchy problem for nonlinear second-order parabolic equations for measures with initial condition $(s,\nu) \in [0,T]\times\mathcal{SP}$ of type

$$\begin{cases} \partial_t\mu_t &= \mathcal{L}^*_{t,\mu_t}\mu_t, \\ \mu_s &= \nu, \end{cases} \tag{NL-FPK}$$

according to the following definition. In contrast to the previous chapter, here the dependence of the coefficients on the solution renders (NL-FPK) a nonlinear equation.

Definition 3.1.1. A vaguely continuous curve $t \mapsto \mu_t \in \mathcal{SP}$ is a *solution to* (NL-FPK), if $(t,x) \mapsto a_{ij}(t,\mu_t,x)$ and $(t,x) \mapsto b_i(t,\mu_t,x)$ are $\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}^d)$-measurable and

(i) $\int_s^T \int_K |a_{ij}(t,\mu_t,x)| + |b_i(t,\mu_t,x)| d\mu_t(x)dt < \infty, \quad \forall K \subseteq \mathbb{R}^d$ compact $, 1 \le i,j \le d.$

(ii) For all $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \varphi(x)d\mu_t - \int_{\mathbb{R}^d} \varphi(x)d\nu = \int_s^t \int_{\mathbb{R}^d} \mathcal{L}_{u,\mu_u}\varphi(x)d\mu_u(x)du, \quad t \in [s,T]. \tag{3.1}$$

In particular, the definition requires $\mu_t \in \mathcal{S}_0$ for each $t \in [0,T]$. Similarly to Definition 2.1.1, a solution $t \mapsto \mu_t$ is a *probability solution*, provided μ_t is a probability measure for each $t \in [s,T]$. In this case, the solution is weakly continuous.

Remark 3.1.2. *More generally, one may consider discontinuous curves of signed, bounded measures $t \mapsto \mu_t$ as solutions to (NL-FPK). Furthermore, the dependence of a_{ij} and b_i on $(\mu_t)_{t\in[s,T]}$ may be nonlocal in time, i.e. writing $\bar{\mu} := \mu_t dt$ as a measure on $\mathbb{R}^d \times [0,T]$, coefficients may be of type $a_{ij}(t,\bar{\mu},x)$ and $b_i(t,\bar{\mu},x)$ instead of $a_{ij}(t,\mu_t,x)$ and $b_i(t,\mu_t,x)$. An example in dimension $d=1$ is the typical case of a convolution kernel*

$$b(t,\bar{\mu},x) = \int_0^t \int_{\mathbb{R}^d} K(x,y,s)d\mu_s(y)ds$$

for some Borel function $K : \mathbb{R}^d \times \mathbb{R}^d \times [0,T] \mapsto \mathbb{R}$. For classical works on equations with convolution-type coefficients, see [45, 146, 145, 146], and also the more recent work [170], whose more general approach covers the former classical works.

Solutions to (NL-FPK) as solutions to linear equations. We would briefly like to stress the following immediate, yet very important observation. Any solution $t \mapsto \mu_t$ to (NL-FPK) becomes a solution to a linear FPK equation of type (FPK) with coefficients $(t,x) \mapsto a_{ij}(t,\mu_t,x), b_i(t,\mu_t,x)$ by *freezing* the curve $t \mapsto \mu_t$ in the argument of the measure-component of a_{ij} and b_i. This way, many results are recovered from the linear situation, for example one proves results parallel to Lemma 2.1.3 and Remark 2.1.4. To this end, note that Definition 3.1.1 (ii) is not sensitive to changes of $t \mapsto \mu_t$ on dt-negligible sets, as long as these changes occur in the "domain" $\mathcal{S}_0$.

The method of freezing the nonlinearity-coefficient in (NL-FPK) is frequently employed for proofs of existence and uniqueness via fixed point theorems, see for example [69, 170], as well as in order to establish a connection between the existence of solutions to nonlinear FPK equations and the corresponding distribution-dependent stochastic differential equation via superposition of solutions, see [21, 22] and the introduction to Part II of this thesis in Chapter 5 for details.

3.2 Nonlinear solution flows and main results

For the remainder of this chapter, we use the same notation as introduced in the linear case at the beginning of Section 2.2, i.e. we denote by $\mathcal{M}_{s,\nu}$ the set of all solutions to the Cauchy problem (NL-FPK) with initial condition $(s,\nu) \in [0,T] \times \mathcal{SP}$, by $\mathcal{M}^1_{s,\nu}$ its subset of probability solutions, and by $\mathcal{M}_s$ and $\mathcal{M}^1_s$ the union of $\mathcal{M}_{s,\nu}$ and $\mathcal{M}^1_{s,\nu}$ over $\nu \in \mathcal{SP}$ and $\nu \in \mathcal{P}$, respectively.

3.2.1 Nonlinear solution flows

As in the linear case, it might be necessary to restrict the selection of a solution flow to subclasses $\mathcal{A}_{s,\nu} \subseteq \mathcal{M}_{s,\nu}$, $(s,\nu) \in [0,T] \times \mathcal{SP}$. To this end, we use the notion of a *flow-admissible* family $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ and a *solution flow* $\{\mu^{s,\nu}\}$ *(with respect to* $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$*)* exactly as in Definitions 2.2.1 and 2.2.2, respectively. Again, we denote by $A_s \subseteq \mathcal{SP}$ the set of $\nu \in \mathcal{SP}$ such that $\mathcal{A}_{s,\nu} \neq \emptyset$ and call such (s,ν) *admissible*. We also use the notion of *entire (sub)probability flow* as in the previous chapter.

Remark 3.2.1. *Since we only consider cases in which the dependence of $t \mapsto \mu_t$ on the coefficients is local in time in the sense $a_{ij}(t,\mu_t,x)$ and $b_i(t,\mu_t,x)$, i.e. we do not consider* *the more general case of coefficients $a_{ij}(t,\bar{\mu},x)$ and $b_i(t,\bar{\mu},x)$ with $\bar{\mu}=\mu_t dt$ as in Remark 3.1.2, it is straightforward to see that the family $\{\mathcal{M}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ is flow-admissible.*

3.2.2 Nonlinear case: Main results.

Now, we formulate the main results for the case of nonlinear equations, i.e. Theorems 3.2.2 and 3.2.3 below. Both are in complete analogy to our main results for linear equations, Theorems 1.3.1 and 1.3.2.

Theorem 3.2.2. *Let $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ be a flow-admissible family of sets of solutions to (NL-FPK) such that $\mathcal{A}_{s,\nu}$ is compact in $C_{s,T}\mathcal{SP}$ for each admissible initial condition (s,ν). Then, there exists a solution flow to (NL-FPK) with respect to $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$.*

Theorem 3.2.3. *In the situation of Theorem 3.2.2, the following are equivalent.*

(i) There exists at most one solution flow to (NL-FPK) with respect to $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$.

(ii) For each $(s,\nu)\in[0,T]\times\mathcal{SP}$, solutions to (NL-FPK) in $\mathcal{A}_{s,\nu}$ are unique.

Concerning the proof of Theorem 3.2.2, it suffices to note that the entire proof of Theorem 1.3.1 does not use the definition of solution to the linear equation (FPK), but is solely based on the compactness of $\mathcal{A}_{s,\nu}\subseteq C_{s,T}\mathcal{SP}$ and the specific iterative selection procedure. Hence, under the assumptions of Theorem 3.2.2, the proof follows in the exact same way.

Similarly, it is apparent that one can mimic the proof of Theorem 1.3.2 to prove Theorem 3.2.3. Indeed, the former does not take into account the linear equation (FPK), but is only based on the specific selection method employed in the proof of Theorem 1.3.1.

3.3 Applications and examples

For the following applications and examples to the main results in the case of nonlinear equations, Theorems 3.2.2 and 3.2.3, we recall and use Proposition 2.4.1, Remark 2.4.2 and the prototype of metric d on $\mathcal{SP}$, as introduced in (2.24).

Entire subprobability flow. The first rather straightforward application to the above results arises under the following assumptions on a and b. Suppose that $\mathcal{S}_0\subseteq\mathcal{SP}$ is closed with respect to the vague topology.

Assumption B1.

(B1.i) $\int_0^T \sup_{(\mu,x)\in\mathcal{S}_0\times K}\big(|a_{ij}(t,\mu,x)|+|b_i(t,\mu,x)|\big)dt<\infty$ holds for all compact $K\subseteq\mathbb{R}^d$ and $1\leq i,j\leq d$.

(B1.ii) $x\mapsto a_{ij}(t,\mu,x), b_i(t,\mu,x)$ is continuous for each $1\leq i,j\leq d, t\in[0,T]$ and $\mu\in\mathcal{S}_0$.

(B1.iii) If $\mu_n\longrightarrow\mu$ vaguely in $\mathcal{S}_0$ for $n\to\infty$, then $a_{ij}(t,\mu_n,x)\longrightarrow a_{ij}(t,\mu,x)$ and $b_i(t,\mu_n,x)\longrightarrow b_i(t,\mu,x)$ locally uniformly in $x\in\mathbb{R}^d$ for each $t\in[0,T]$.

The first two assumptions are comparable to Assumption A1 from the previous chapter, and the rather strong additional assumption (B1.iii) is necessary in order to show closedness of $\mathcal{M}_{s,\nu}$, which is more delicate than in the linear case (see the proof of Proposition 3.3.1).

Proposition 3.3.1. *Suppose Assumption B1 is fulfilled and suppose $\mathcal{M}_{s,\nu}$ is nonempty for each $(s,\nu) \in [0,T] \times \mathcal{SP}$. Then, there exists an entire solution flow $\{\mu^{s,\nu}\}$ of subprobability solutions to* (NL-FPK).

Proof. Comparable to the proof of Proposition 2.4.3, we evoke the Arzela-Ascoli theorem in order to prove precompactness of $\mathcal{M}_{s,\nu} \subseteq \mathcal{C}_{s,T}\mathcal{SP}$. Of course, the range of each projection $\pi_t(\mathcal{M}_{s,\nu}) \subseteq \mathcal{SP}$, $t \in [0,T]$, is precompact as a subset of the compact space $\mathcal{SP}$. Replacing assumption (A1.i) by (B1.i), equicontinuity of $\mathcal{M}_{s,\nu}$ can be proven exactly as in the proof of Proposition 2.4.3. Therefore, $\mathcal{M}_{s,\nu} \subseteq C_{s,T}\mathcal{SP}$ is precompact.

Concerning closedness, assume $\mu^n = (\mu^n_t)_{t\in[s,T]}$ converges to $\mu = (\mu_t)_{t\in[s,T]}$ in $C_{s,T}\mathcal{SP}$ as $n \to \infty$. Then, $(\mu_t)_{t\in[s,T]} \subseteq \mathcal{S}_0$, since we assume $\mathcal{S}_0$ to be closed in $\mathcal{SP}$. Since $a_{ij}(t,\mu^n_t,x) \longrightarrow a_{ij}(t,\mu_t,x)$ for each $(t,x) \in [0,T] \times \mathbb{R}^d$ as $n \to \infty$ by (B3.iii), it follows that $(t,x) \mapsto a_{ij}(t,\mu_t,x)$, $1 \le i,j \le d$, is Borel measurable. The same is true for b_i, $1 \le i \le d$. Of course, (B1.i) gives the integrability condition (i) of Definition 3.1.1. Therefore, it remains to prove

$$\int_s^t \int_{\mathbb{R}^d} \mathcal{L}_{u,\mu^n_u}\varphi(x)d\mu^n_u(x)du \underset{n\to\infty}{\longrightarrow} \int_s^t \int_{\mathbb{R}^d} \mathcal{L}_{u,\mu_u}\varphi(x)d\mu_u(x)du \tag{3.2}$$

for each $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t \in [s,T]$. This can, for example, be realized by rewriting

$$\int_{\mathbb{R}^d} \mathcal{L}_{u,\mu^n_u}\varphi(x)d\mu^n_u(x) = {}_{C_c^*}\langle \mu^n_u, \mathcal{L}_{u,\mu^n_u}\varphi\rangle_{C_c},$$

where ${}_{C_c^*}\langle \mu, f\rangle_{C_c}$ denotes the dual pairing of $f \in C_c(\mathbb{R}^d)$ and a bounded Borel measure μ, considered as an element in the dual space of $C_c(\mathbb{R}^d)$. Since the vague topology on $\mathcal{SP}$ coincides with the weak-* topology in the dual space of $C_c(\mathbb{R}^d)$, and since assumptions (B1.i) and (B1.iii) yield that $\mathcal{L}_{u,\mu^n_u}\varphi \longrightarrow \mathcal{L}_{u,\mu_u}\varphi$ in $C_c(\mathbb{R}^d)$ as $n \to \infty$ (i.e. the convergence is uniform in $x \in \mathbb{R}^d$), it follows that

$$\int_{\mathbb{R}^d} \mathcal{L}_{u,\mu^n_u}\varphi(x)d\mu^n_u(x) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \mathcal{L}_{u,\mu_u}\varphi(x)d\mu_u(x), \quad u \in [s,T].$$

From here, (3.2) follows by (B1.i) and Lebesgue's dominated convergence theorem. □

We point out that in general it is a hard task to solve (NL-FPK) for every initial condition $\nu \in \mathcal{SP}$. Existence results for subclasses of initial conditions are, for example, obtained in [175] via the corresponding McKean–Vlasov equation, by semigroup methods in [21, 22], and by a fixed point argument in [170]. In the recent paper [20], existence of a solution for any probability measure as initial value is shown. However, in the situation of that paper, our Assumption B1 is not fulfilled.

Entire solution flow under relaxed assumptions of [69]. Another existence and uniqueness result for solutions to (NL-FPK) for arbitrary initial conditions $\nu \in \mathcal{SP}$ via the corresponding McKean–Vlasov equation is [69, Thm.5.3]. Indeed, set $\sigma \equiv 0$ in [69, Eq.(1.1)] to recover equation (NL-FPK). In this case, existence of solutions holds, if the Borel coefficients $a_{ij}, b_i : [0,T] \times \mathcal{SP} \times \mathbb{R}^d \to \mathbb{R}$ are uniformly bounded, uniformly (in (t,x)) Lipschitz continuous in μ with respect to the Kantorovich-Rubinstein metric (which metrizes weak convergence in $\mathcal{SP}$), and uniformly (in (t,μ)) Lipschitz continuous in x. Additionally, uniqueness of the solution is obtained in [69, Thm.5.4] under the additional rather strong assumption

$$\sup_{(t,\mu)\in[0,T]\times\mathcal{SP}} ||a_{ij}(t,\mu,\cdot)||_{C^m} < \infty, \qquad \sup_{(t,\mu)\in[0,T]\times\mathcal{SP}} ||b_i(t,\mu,\cdot)||_{C^m} < \infty,$$

for some $m > d/2+2$. Our next aim is to apply our selection result Theorem 3.2.2 to equations with coefficients a and b fulfilling an intermediate set of conditions. For the present situation, we have $\mathcal{S}_0 = \mathcal{SP}$. More precisely, suppose the Borel coefficients a_{ij} and b_i are defined on $[0,T] \times \mathcal{SP} \times \mathbb{R}^d$ and fulfill

Assumption B2.

(B2.i) $(t,\mu,x) \mapsto a_{ij}(t,\mu,x)$ and $(t,\mu,x) \mapsto b_i(t,\mu,x)$ are bounded on $[0,T] \times \mathcal{SP} \times \mathbb{R}^d$.

(B2.ii) $x \mapsto a_{ij}(t,\mu,x), b_i(t,\mu,x)$ are continuous for each $1 \le i,j \le d, t \in [0,T]$ and $\mu \in \mathcal{SP}$.

(B2.iii) If $\mu_n \longrightarrow \mu$ weakly in $\mathcal{SP}$ for $n \to \infty$, then $a_{ij}(t,\mu_n,x) \longrightarrow a_{ij}(t,\mu,x)$ and $b_i(t,\mu_n,x) \longrightarrow b_i(t,\mu,x)$ locally uniformly in $x \in \mathbb{R}^d$ for each $t \in [0,T]$.

Of course, these assumptions are fulfilled in the situation of [69, Thm.5.3] as described at the beginning of this paragraph. At the same time, Assumption B2 is considerably weaker than the assumptions in [69, Thm.5.4], since we do not impose Lipschitz continuity in either x or μ and also no C^m-regularity in x for $m > 0$. Hence, under Assumption B2, we have existence of solutions for each initial condition $(s,\nu) \in [0,T] \times \mathcal{SP}$, but not necessarily uniqueness. We claim that in this ill-posedness situation, we can select an entire subprobability flow. Note that (B2.iii) is weaker than (B1.iii).

Proposition 3.3.2. *Suppose the Borel coefficients a_{ij}, b_i, $1 \le i,j \le d$, fulfill Assumption B2. Then, there exists an entire subprobability solution flow to* (NL-FPK).

Proof. In view of the assertion and the proof of Proposition 3.3.1, precompactness of $\mathcal{M}_{s,\nu} \subseteq C_{s,T}\mathcal{SP}$ holds, and it only remains to prove closedness of $\mathcal{M}_{s,\nu}$. To this end, and in view of (B1.iii), inspecting the proof of Proposition 3.3.1, it is clear that (B1.iii) is only needed for the case $\mu_n = \mu^n_t$ and $\mu = \mu_t$ such that $(\mu^n_t)_{t\in[s,T]} \in \mathcal{M}_{s,\nu}$ converges to $(\mu_t)_{t\in[s,T]}$ in $C_{s,T}\mathcal{SP}$ as $n \to \infty$. But in this situation, Lemma A.0.2 yields tightness of $\{\mu^n_t\}_{n\in\mathbb{N}}$, so that the vague convergence $\mu^n_t \longrightarrow \mu_t$ is actually weak. From here, (B2.iii) gives (B1.iii) and the assertion follows as in the proof of Proposition 3.3.1. □

In the presence of Assumption B2, one can, of course, also select an entire probability flow, i.e. a solution flow with respect to the solution classes $\mathcal{A}_{s,\nu}$ as in (2.26). Indeed, this family is flow-admissible, by a reasoning similar to Remark 2.4.5.

Entire solution flow for nonuniqueness examples from [203]. Several examples of nonuniqueness for McKean–Vlasov equations of type (1.4) with $\sigma = 0$ are presented in the classical work [203]. In particular, [203, Counterex.3] states the existence of several solutions in a one-dimensional case with coefficient $b(t,\mu,x) := \int h d\mu$, for some $h \in C_c(\mathbb{R})$. Clearly, in this case, Assumption B2 is fulfilled. That this equation has a solution for each initial distribution $\nu \in \mathcal{P}$ can, for example, be observed by applying Theorem 5.3. from [69]. Since any solution to a McKean–Vlasov equation induces a probability solution to the corresponding nonlinear FPK equation of type (NL-FPK) (with coefficient b as above and $a = 0$), it is clear that there exists a solution to the Cauchy problem (NL-FPK) for each $(s,\nu) \in [0,T] \times \mathcal{SP}$ (because any subprobability measure is, of course, a probability measure up to a normalizing constant). Consequently, Proposition 3.3.1 applies and yields the existence of an entire solution flow of subprobability measures. As described at the end of the previous paragraph, one can also construct an entire probability flow.

Restricted flow subject to a Lyapunov function. Here, we consider the case of unbounded coefficients in presence of a *Lyapunov function*. We use the notation $\mathcal{P}_m := \{\mu \in \mathcal{P} : |\cdot|^m \in L^1(\mu)\}$ for $m \in \mathbb{N}$. Suppose we are given coefficients

$$a_{ij}, b_i : [0,T] \times \mathcal{P}_4 \times \mathbb{R}^d \to \mathbb{R}, \quad 1 \leq i,j \leq d,$$

such that $t \mapsto a_{ij}(t,\mu_t,x)$ and $t \mapsto b_i(t,\mu_t,x)$ are Borel measurable for each $x \in \mathbb{R}^d$ and each Borel curve $t \mapsto \mu_t$ from $[0,T]$ to $\mathcal{P}_4$. As usual, we set $b = (b_i)_{1\leq i\leq d}$ and assume $a(t,\mu,x) = (a_{ij}(t,\mu,x))_{1\leq i,j\leq d} \in \mathbb{S}_d^+$ for each $(t,\mu,x) \in [0,T] \times \mathcal{P}_4 \times \mathbb{R}^d$. Assume there is a continuous, nonnegative function C^* on $[0,T]$ such that the coefficients satisfy

Assumption B3.

(B3.i) $\sup_{\mu\in\mathcal{P}_4} \sum_{i,j=1}^d |a_{ij}(t,\mu,x)| + \sup_{\mu\in\mathcal{P}_4} \sum_{i=1}^d |b_i(t,\mu,x)| \leq C^*(t) + C^*(t)|x|, \quad x \in \mathbb{R}^d.$

(B3.ii) $x \mapsto a_{ij}(t,\mu,x), b_i(t,\mu,x)$ is continuous for $1 \leq i,j \leq d$ and each $(t,\mu) \in [0,T] \times \mathcal{P}_4$.

(B3.iii) If $\mu_n \longrightarrow \mu$ weakly in $\mathcal{P}_4$ and the second moments of μ^n converge to the second moment of μ for $n \to \infty$ (i.e. $d_2(\mu_n,\mu) \longrightarrow 0$, with d_2 as introduced below), then $a_{ij}(t,\mu_n,x) \longrightarrow a_{ij}(t,\mu,x)$ and $b_i(t,\mu_n,x) \longrightarrow b_i(t,\mu,x)$ locally uniformly in $x \in \mathbb{R}^d$ for each $t \in [0,T]$.

Then, setting $V : x \mapsto 1 + |x|^4$, it is straightforward to check that for each curve $t \mapsto \mu_t \in \mathcal{P}_4$, we have

$$\mathcal{L}_{t,\mu_t} V(x) \leq C(t) + C(t)V(x), \quad (t,x) \in [0,T] \times \mathbb{R}^d \tag{3.3}$$

for a continuous, nonnegative function C on $[0,T]$, which is easily calculated from C^* and the definition of V, but its exact calculation is not needed in the sequel.

In this situation, we apply the following lemma, which is a slightly simplified version of Lemma 2.2. of [36], to obtain a bound on $\int V d\mu_t$ uniformly in $\mu = (\mu_t)_{t\in[s,T]} \in \mathcal{M}^1_{s,\nu}$ and t.

Lemma 3.3.3. *Let $t \mapsto \mu_t$ be a probability solution to (NL-FPK) with initial condition $(s, \nu) \in [0, T] \times \mathcal{P}_4$. Then, for coefficients a_{ij} and b_i as in the beginning of this paragraph, which fulfill (B3.i), we have for each $t \in [s, T]$*

$$\int_{\mathbb{R}^d} V d\mu_t \leq \int_{\mathbb{R}^d} V d\nu + \int_s^t C(u)du + c^+ \exp(c^+ t) \int_s^t \exp(-c^- u)\left[\int_{\mathbb{R}^d} V d\nu + \int_s^u C(r)dr\right]du, \tag{3.4}$$

where we set $c^+ := \max_{t \in [0,T]} C(t)$ and $c^- := \min_{t \in [0,T]} C(t)$.

Since the right-hand side of (3.4) is bounded independently of $\mu \in \mathcal{M}^1_{s,\nu}$ and $t \in [s, T]$ by some $D = D(\nu, V, T) > 0$, in our present situation the previous lemma yields

$$\sup_{\mu \in \mathcal{M}^1_{s,\nu}} \int_{\mathbb{R}^d} V d\mu_t \leq D. \tag{3.5}$$

Choosing a sequence $(\varphi_k)_{k \in \mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d)$ such that $\varphi_k(x) = V(x)$ for $|x| \leq k$ and $\sup_{k \in \mathbb{N}} ||\partial_i \varphi_k||_\infty \leq c_0 ||\partial_i V||_\infty + c_0$, $\sup_{k \in \mathbb{N}} ||\partial_{ij} \varphi_k||_\infty \leq c_0 ||\partial_{ij} V||_\infty + c_0$, $1 \leq i, j \leq d$, for some $c_0 > 0$, and considering (3.1) with φ_k instead of φ in the limit $k \to \infty$, it follows by (B3.i), Lemma 3.3.3 and Lebesgue's dominated convergence theorem that $\mathcal{L}_{u,\mu_u} V \in L^1(\mu_u du)$ and

$$\int_{\mathbb{R}^d} |x|^4 d\mu_t(x) - \int_{\mathbb{R}^d} |x|^4 d\nu(x) = \int_s^t \int_{\mathbb{R}^d} \mathcal{L}_{u,\mu_u} V(x) d\mu_u(x) du. \tag{3.6}$$

It is clear that a similar calculation holds when $|\cdot|^4$ is replaced by $|\cdot|^m$ for $1 \leq m < 4$ and V on the right-hand side is replaced by $|\cdot|^m$. In particular, $t \mapsto \int |\cdot|^m d\mu_t$ is continuous for $1 \leq m \leq 4$, i.e. in this situation, we have $\mathcal{M}^1_{s,\nu} \subseteq C_{s,T}\mathcal{P}_2$, where we equip $\mathcal{P}_2$ with the metric

$$d_2 : (\mu, \tilde{\mu}) \mapsto d(\mu, \tilde{\mu}) + \left| \int_{\mathbb{R}^d} |x|^2 d\mu(x) - \int_{\mathbb{R}^d} |x|^2 d\tilde{\mu}(x) \right|,$$

where d is the same metric as fixed in the beginning of the proof of Proposition 2.4.3. From the definition of d and since convergence with respect to d_2 is equivalent to weak convergence plus convergence of the second moments, it is clear that d_2 is weakly equivalent to the usual Wasserstein distance on $\mathcal{P}_2$. Our goal is to apply Theorem 3.2.2 to the flow-admissible family $\mathcal{A}_{s,\nu} \subseteq C_{s,T}\mathcal{P}_2$,

$$\mathcal{A}_{s,\nu} := \begin{cases} \mathcal{M}^1_{s,\nu} & , \text{ if } \nu \in \mathcal{P}_4, \\ \emptyset & , \text{ if } \nu \in \mathcal{SP} \backslash \mathcal{P}_4. \end{cases} \tag{3.7}$$

Indeed, the admissibility of the above family follows, since we have shown by Lemma 3.3.3 that $\nu \in \mathcal{P}_4$ gives $(\mu_t)_{t \in [s,T]} \subseteq \mathcal{P}_4$ for each $(\mu_t)_{t \in [s,T]} \in \mathcal{M}^1_{s,\nu}$. Before we continue, we pause for the following observation.

Remark 3.3.4. *Although we have not mentioned it before, it is evident that the proof of Theorem 3.2.2 (as well as that of Theorem 1.3.1 in the linear case) is not specific to the space $C_{s,T}\mathcal{SP}$. Indeed, if we are given a flow-admissible family $\{\mathcal{A}_{s,\nu}\}$ such that $\mathcal{A}_{s,\nu}$ is compact in $C_{s,T}\mathfrak{P}$, where $\mathfrak{P}$ is some metric space of (sub)probability measures with a* *topology finer or equal to the vague topology, then the proof works in the exact same way. The reason we exclusively used $\mathfrak{P} = \mathcal{SP}$ with the vague topology until now is that in general it is, of course, potentially much harder to prove compactness $\mathcal{A}_{s,\nu} \subseteq C_{s,T}\mathfrak{P}$ for spaces $\mathfrak{P}$ with a finer topology. However, in the case of nonlinear FPK equations, we typically need to assume continuity of the coefficients in their measure argument with respect to the topology on $\mathfrak{P}$ in order to prove closedness of $\mathcal{A}_{s,\nu} \subseteq C_{s,T}\mathfrak{P}$, see assumptions (B1.iii), (B2.iii) and (B3.iii). This kind of assumption becomes weaker for finer topologies on $\mathfrak{P}$ (which, on the other hand, renders the task of proving precompactness of $\mathcal{A}_{s,\nu} \subseteq C_{s,T}\mathfrak{P}$ more difficult). In the present situation, which we continue after this remark, the Lyapunov function V allows to prove precompactness of $\mathcal{A}_{s,\nu}$ in $C_{s,T}\mathcal{P}_2$, i.e. we choose $\mathfrak{P} = \mathcal{P}_2$. This allows to obtain compactness of $\mathcal{A}_{s,\nu}$ under Assumption B3, which comprises (B3.iii), which is a weaker assumption than (B1.iii) or (B2.iii).*

Using the observations of the previous remark, we arrive at the following result.

Proposition 3.3.5. *Suppose the coefficients a_{ij}, b_i, $1 \leq i, j \leq d$, are as specified at the beginning of this paragraph and fulfill Assumption B3. Moreover, assume the sets $\mathcal{M}^1_{s,\nu}$ are nonempty for each $(s,\nu) \in [0,T] \times \mathcal{P}_4$. Then, there exists a solution flow to (NL-FPK) with respect to $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{SP}}$ as in (3.7), i.e. the flow arises in $\mathcal{P}_4$.*

Proof. Again, we evoke the Arzela-Ascoli theorem, see Proposition 2.4.1. Fix $(s,\nu) \in [0,T] \times \mathcal{P}_4$. For each $t \in [s,T]$, (3.5) implies tightness of $\pi_t(\mathcal{A}_{s,\nu})$. Since a uniform bound on the q-th moments of a weakly converging sequence of Borel probability measures implies the convergence of their p-th moments for any $1 \leq p < q$, we obtain that the limit of any weakly converging sequence in $\pi_t(\mathcal{A}_{s,\nu})$ is even a limit with respect to d_2, which proves precompactness of $\pi_t(\mathcal{A}_{s,\nu}) \subseteq (\mathcal{P}_2, d_2)$.

Concerning equicontinuity, note that (3.6), (3.3) and (3.5) yield

$$\sup_{(\mu_t)_{t\in[s,T]}\in\mathcal{A}_{s,\nu}} \left| \int_{\mathbb{R}^d} |x|^2 d\mu_{t_2}(x) - \int_{\mathbb{R}^d} |x|^2 d\mu_{t_1}(x) \right| \leq c(1+D)(t_2-t_1), \quad t_1, t_2 \in [s,T], \tag{3.8}$$

for each $(s,\nu) \in [0,T] \times \mathcal{P}_4$. From here, and in view of the definition of d_2 and assumption (B3.i), equicontinuity follows as in the proof of Proposition 3.3.1.

Finally, closedness of $\mathcal{A}_{s,\nu} \subseteq C_{s,T}\mathcal{P}_2$ can be proven as in the proof of Proposition 3.3.1. Indeed, in the present case, assumption (B3.iii) is sufficient, since we want to prove closedness in $C_{s,T}\mathcal{P}_2$ instead of $C_{s,T}\mathcal{SP}$ as in the proof of Proposition 3.3.1. Taking into account Remark 3.3.4, this concludes the proof. □

Concerning existence of solutions in $\mathcal{M}^1_{s,\nu}$ in the present situation, one can, for example, evoke [170, Thm.1.1.(ii)], which in comparison to Assumption B3 additionally requires the continuous dependence of the coefficients on x to be uniform in (t, μ), compare with (H1)-(H3) of the above source.

Chapter 4

Solution flows for FPK equations for measures on infinite-dimensional spaces

We conclude this first part of the thesis with a brief account of flow selections and the characterization of well-posedness for linear FPK equations for measures on infinite-dimensional spaces: for equations of type (FPK$_\infty$), we prove results similar to the finite-dimensional cases presented in the previous two chapters. The main results in this chapter are Theorems 4.2.1 and 4.2.2. Again, both proofs readily follow as in the basic case discussed in Chapter 2. It is not surprising that the infinite-dimensional setting renders the compactness of solutions, as needed in the assertion of both main theorems, a difficult issue, and the findings in this chapter should only be considered a first step towards further investigations of similar questions in infinite-dimensional cases. We restrict the considerations in this chapter to the case of probability-valued solutions.

4.1 FPK equations for measures on $\mathbb{R}^\infty$

We start by presenting notation specific to the infinite-dimensional case, which is only needed in the present chapter. Afterwards, we give the definition of a solution to (FPK$_\infty$) and briefly discuss the continuity assumption of solutions, comparable to the finite-dimensional case in Lemma 2.1.3.

4.1.1 Notation

For $n \in \mathbb{N}$, e_n denotes the n-th unit vector in $\mathbb{R}^\infty$, i.e. the sequence with value 1 in its n-th entry and 0 otherwise. Let P_d be the projection from $\mathbb{R}^\infty$ to the linear span of $\{e_1, \dots, e_d\}$, i.e.

$$P_d : \mathbb{R}^\infty \to \langle e_1, \dots, e_d \rangle \cong \mathbb{R}^d, \quad P_d : (x_i)_{i \in \mathbb{N}} \mapsto (x_1, \dots, x_d).$$

The set of probability measures on $\mathcal{B}(\mathbb{R}^\infty)$ is denoted by $\mathcal{P}(\mathbb{R}^\infty)$, and we still use the notation $\mathcal{P}$ for the space $\mathcal{P}(\mathbb{R}^d)$, if the dimension d is given from the context. Furthermore, we need the following spaces of *cylindrical functions* on $\mathbb{R}^\infty$. For $T>0$, we set

$$\mathcal{F}_c^\infty(t,x) := \bigcup_{d\in\mathbb{N}} \Big\{ \varphi : (0,T)\times\mathbb{R}^\infty \to \mathbb{R} \,\big|\, \varphi(t,x) = \Phi(t, P_d(x)), \Phi \in C_c^\infty\big((0,T)\times\mathbb{R}^d\big) \Big\}$$

and

$$\mathcal{F}_c^\infty(x) := \bigcup_{d\in\mathbb{N}} \Big\{ \varphi : \mathbb{R}^\infty \to \mathbb{R} \,\big|\, \varphi = \Phi \circ P_d, \Phi \in C_c^\infty\big(\mathbb{R}^d\big) \Big\}.$$

Note that any nontrivial function $\phi \in \mathcal{F}_c^\infty(t,x) \cup \mathcal{F}_c^\infty(x)$ is not compactly supported in $\mathbb{R}^\infty$.

4.1.2 Solutions to FPK equations for measures on $\mathbb{R}^\infty$

Let $T>0$ and consider the compact time interval $[0,T]$. We endow $\mathbb{R}^\infty$ with the product topology, i.e. a sequence $\{x^{(n)}\}_{n\in\mathbb{N}}$ in $\mathbb{R}^\infty$ converges to $x=(x_i)_{i\in\mathbb{N}}$ if and only if $x_i^{(n)} \longrightarrow x_i$ in the Euclidean topology on $\mathbb{R}$ for each $i\in\mathbb{N}$ as $n\to\infty$. Recall that $\mathbb{R}^\infty$ with this topology is Polish. In particular, the facts about spaces of probability measure $\mathcal{P}(X)$ recalled in Chapter 0 apply in the case $X=\mathbb{R}^\infty$. Let

$$a_{ij}, b_i : [0,T]\times\mathbb{R}^\infty \to \mathbb{R}, \quad i,j\in\mathbb{N},$$

be Borel measurable and set $a := (a_{ij})_{i,j\geq 1}$ and $b := (b_i)_{i\in\mathbb{N}}$. At this point, it is not necessary to assume symmetry or any kind of nonnegative definiteness for a.

We consider the differential operator L as in (1.6) and study the Cauchy problem for the corresponding linear FPK equation (FPK_∞), i.e.

$$\begin{cases} \partial_t\mu_t &= L_t^*\mu_t, \\ \mu_s &= \nu \end{cases} \tag{4.1}$$

for an initial condition $(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)$.

Definition 4.1.1. A *probability solution* to (4.1) with initial condition $(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)$ is a weakly continuous curve $\mu=(\mu_t)_{t\in[s,T]}$ in $\mathcal{P}(\mathbb{R}^\infty)$, such that for each $i,j\geq 1$, the coefficients a_{ij} and b_i satisfy

$$\int_s^T \int_{\mathbb{R}^\infty} |a_{ij}(t,x)| + |b_i(t,x)| d\mu_t(x)dt < \infty, \tag{4.2}$$

and for each $\varphi\in\mathcal{F}_c^\infty(x)$ and $t\in[s,T]$, we have

$$\int_{\mathbb{R}^\infty} \varphi(x)d\mu_t(x) - \int_{\mathbb{R}^\infty} \varphi(x)d\nu(x) = \int_s^t \int_{\mathbb{R}^\infty} L_r\varphi(x)d\mu_r(x)dr. \tag{4.3}$$

Remark 4.1.2. *(i) Since in an infinite-dimensional framework the global integrability condition* (4.2) *seems more natural than an analogue of the local in space condition* (2.1)*, here we restrict our considerations to probability solutions instead of subprobability solutions, as in the finite-dimensional case. Under the local integra-*

bility assumption on solutions in the previous chapter, the more general setting of subprobability measures turned out helpful, since the vague topology on $\mathcal{SP}$ is better suited in the case of such local conditions. Nevertheless, also in this situation, the results in the present chapter can immediately be extended to subprobability valued solutions.

(ii) In general, (4.1) makes sense for discontinuous Borel curves of signed, bounded measures on $\mathbb{R}^\infty$. In this case, the general notion of solution is an analogue to Remark 2.1.2, i.e. a bounded Borel curve $(\mu_t)_{t\in(s,T)}$ of measures on $\mathbb{R}^\infty$ is a solution to (FPK$_\infty$) on (s,T), if it consists of probability measures dt-a.s., fulfills the global condition (4.2), and satisfies

$$\int_s^T \int_{\mathbb{R}^\infty} (\partial_t + L_t)\varphi(t,x)d\mu_t(x)dt = 0 \tag{4.4}$$

for each $\varphi \in \mathcal{F}_c^\infty(t,x)$. If in addition, for each $\varphi \in \mathcal{F}_c^\infty(x)$ such that $\varphi(t,\cdot) = 0$ for $t \in [0, C_\varphi]$ for some $C_\varphi > s$, also

$$\lim_{t\to s} \int_{\mathbb{R}^\infty} \varphi(x)d\mu_t(x) = \int_{\mathbb{R}^\infty} \varphi(x)d\nu(x) \tag{4.5}$$

holds, then $(\mu_t)_{t\in(s,T)}$ is a solution to the Cauchy problem (4.1), see [38, Ch.10] for details. Similarly to Lemma 2.1.3, in the subsequent proposition we briefly discuss that the assumption of weak continuity for solutions to (4.1) is not restrictive in the presence of the global condition (4.2).

Proposition 4.1.3. *Let $t \mapsto \mu_t$ be a Borel curve of bounded measures on $\mathbb{R}^\infty$ on (s,T), which is a solution to (4.1) in the general sense of part (ii) of the previous remark. Then, there exists a unique weakly continuous version $t \mapsto \tilde{\mu}_t$ on $[s,T]$, which is a solution to (4.1) in the sense of Definition (4.1.1).*

Concerning the proof, we first show the existence of a unique version, which is continuous in duality with the function class $\mathcal{F}_c^\infty(x)$. From here, the assertion is obtained by the following lemma, whose proof we have shifted to Appendix A.

Lemma 4.1.4. *Let $(\eta_n)_{n\in\mathbb{N}}, \eta \in \mathcal{P}(\mathbb{R}^\infty)$. If*

$$\int_{\mathbb{R}^\infty} \varphi(x)d\eta_n(x) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^\infty} \varphi(x)d\eta(x), \quad \forall \varphi \in \mathcal{F}_c^\infty(x), \tag{4.6}$$

then $(\eta_n)_{n\in\mathbb{N}}$ converges to η weakly.

Proof of Proposition 4.1.3. First of all, it is clear that any Borel measurable version $t \mapsto \tilde{\mu}_t$ of $(\mu_t)_{t\in(s,T)}$ still solves the Cauchy problem in the sense of Remark 4.1.2 (ii). Moreover, for weakly continuous curves, this notion of solution is equivalent to Definition 4.1.1, see for example [32, Lem.1.1]. Hence, it remains to construct a version as in the assertion.

For fixed $d \in \mathbb{N}$, choose $\varphi(t,x) = f(t)g\big(P_d(x)\big) \in \mathcal{F}_c^\infty(t,x)$ in (4.4), where $f \in C_c^\infty\big((s,T)\big)$ and $g \in C_c^\infty(\mathbb{R}^d)$, to observe that $t \mapsto \int_{\mathbb{R}^d} g(x)d\mu_t \circ P_d^{-1}(x)$ belongs to the Sobolev space $W^{1,1}\big((s,T)\big)$ with weak derivative

$$t \mapsto \int_{\mathbb{R}^\infty} a_{ij}(t,x)\partial_{ij}g\big(P_d(x)\big) + b_i(t,x)\partial_i g\big(P_d(x)\big)d\mu_t(x).$$

From here, we employ the same arguments as in the proof of Lemma 2.1.3 to obtain a dt-version $t \mapsto \mu_t^d \in \mathcal{P}(\mathbb{R}^d)$ of $(\mu_t \circ P_d)_{t\in(s,T)}$ on $[s,T]$, which is even weakly continuous, due to the global integrability (4.2) of $(\mu_t)_{t\in(s,T)}$ with respect to the coefficients. Hence, repeating these steps for each $d \in \mathbb{N}$, we obtain a set $A \subseteq [s,T]$ such that A^c is dt-negligible with

$$t \in A \implies \mu_t^d = \mu_t \circ P_d^{-1} \;\; \forall d \in \mathbb{N},$$

and for $t \in A$ we set $\tilde{\mu}_t := \mu_t$.

Now let $t \notin A$ and let us show that the family $\{\mu_t^d\}_{d\in\mathbb{N}}$ is consistent: For $d \geq 2$, let $g_d \in C_b(\mathbb{R}^d)$ be arbitrary such that $g_d = g_{d-1} \circ P_{d-1}^d$ for some $g_{d-1} \in C_b(\mathbb{R}^{d-1})$, where we set

$$P_{d-1}^d : \langle e_1, ..., e_d\rangle \to \langle e_1, ..., e_{d-1}\rangle,\; P_{d-1}^d(x_1, .., x_d) := (x_1, ..., x_{d-1}).$$

Choosing a sequence $t_n \to t$ as $n \to \infty$ with $(t_n)_{n\in\mathbb{N}} \subseteq A$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} g_d d\mu_t^d &= \lim_{n\to\infty}\int_{\mathbb{R}^d} g_d d\mu_{t_n}^d = \lim_{n\to\infty}\int_{\mathbb{R}^d} g_d d\mu_{t_n} \circ P_d^{-1} = \lim_{n\to\infty}\int_{\mathbb{R}^d} g_{d-1} \circ P_{d-1}^d d\mu_{t_n} \circ P_d^{-1}\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^{d-1}} g_{d-1} d\mu_{t_n} \circ P_{d-1}^{-1} = \lim_{n\to\infty}\int_{\mathbb{R}^{d-1}} g_{d-1} d\mu_{t_n}^{d-1} = \int_{\mathbb{R}^{d-1}} g_{d-1} d\mu_t^{d-1}.\end{aligned}$$

For arbitrary nonempty open sets $B_1, ..., B_{d-1} \subseteq \mathbb{R}$, approximate the function $\mathbb{1}_{B_1\times...\times B_{d-1}}$ pointwise from below by a nondecreasing sequence of nonnegative functions $(g_n^B)_{n\in\mathbb{N}} \subseteq C_b(\mathbb{R}^{d-1})$. Then, clearly also $g_n^B \circ P_{d-1}^d \nearrow \mathbb{1}_{B_1\times...\times B_{d-1}\times\mathbb{R}}$ and each $g_n^B \circ P_{d-1}^d$ is nonnegative, continuous and bounded on $\mathbb{R}^d$. Thus, applying the above chain of equalities to each pair of functions g_n^B and $g_n^B \circ P_{d-1}^d$ and letting $n \to \infty$ on both sides, yields

$$\mu_t^d(B_1 \times ... \times B_{d-1} \times \mathbb{R}) = \mu_t^{d-1}(B_1 \times ... \times B_{d-1}),$$

which gives the desired consistency. Therefore, by Kolmogorov's extension theorem (see Theorem D.0.5), there exists a unique element $\tilde{\mu}_t \in \mathcal{P}(\mathbb{R}^\infty)$ such that $\tilde{\mu}_t \circ P_d^{-1} = \mu_t^d$ for all $d \in \mathbb{N}$.

Altogether, we obtain a family of probability measures $\tilde{\mu} = (\tilde{\mu}_t)_{t\in[s,T]}$ on $\mathcal{B}(\mathbb{R}^\infty)$ such that $\mu_t = \tilde{\mu}_t$ dt-a.s. (namely for each $t \in A$) and $(\tilde{\mu}_t \circ P_d^{-1})_{t\in[s,T]}$ is weakly continuous on $\mathcal{B}(\mathbb{R}^d)$ for each $d \in \mathbb{N}$. In particular, $(\tilde{\mu}_t)_{t\in[s,T]}$ fulfills (4.6), with $(\eta_n)_{n\in\mathbb{N}}$ and η in (4.6) replaced by any sequence $(\tilde{\mu}_{t_n})_{n\in\mathbb{N}}$ and $\tilde{\mu}_t$ such that $t_n \to t$ as $n \to \infty$. By Lemma 4.1.4, $t \mapsto \tilde{\mu}_t$ is weakly continuous. This completes the proof. □

Metrizing the weak topology on $\mathcal{P}(\mathbb{R}^\infty)$. We close this section by introducing a metric d_∞ on $\mathcal{P}(\mathbb{R}^\infty)$, which induces the weak topology of measures on $\mathcal{P}(\mathbb{R}^\infty)$ and only comprises functions from the class $\mathcal{F}_c^\infty(x)$. Compare with the definition of the metric d in (2.24) from the finite-dimensional situation.

For $d \in \mathbb{N}$, fix a countable set $G_d \subseteq C_c^\infty(\mathbb{R}^d)$, which is dense with respect to the topology of uniform convergence, assume without loss of generality that no element in G_d is constantly 0, and let

$$G := \bigcup_{d\in\mathbb{N}} G_d.$$

Due to the density of each G_d in $C_c^\infty(\mathbb{R}^d)$ and Lemma 4.1.4, weak convergence of a sequence of probability measures $(\mu^n)_{n\in\mathbb{N}}$ to $\mu \in \mathcal{P}(\mathbb{R}^\infty)$ is characterized by the convergence

$$\int_{\mathbb{R}^\infty} \varphi d\mu^n \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^\infty} \varphi d\mu \ \ \forall \varphi \in \{\Phi \circ P_d \,|\, \Phi \in G_d, d \in \mathbb{N}\}.$$

With this in mind, we introduce the metric

$$d_\infty(\mu^1, \mu^2) := \sum_{k=1}^\infty 2^{-k} \left[\frac{\left| \int_{\mathbb{R}^\infty} \varphi_k d\mu^1 - \int_{\mathbb{R}^\infty} \varphi_k d\mu^2 \right|}{||\varphi_k||_{C^2}} \wedge 1 \right] \tag{4.7}$$

on $\mathcal{P}(\mathbb{R}^\infty)$, with $\{\varphi_k, k \in \mathbb{N}\} = \{\Phi \circ P_d \,|\, \Phi \in G_d, d \in \mathbb{N}\}$, and note that d_∞ induces the topology of weak convergence on $\mathcal{P}(\mathbb{R}^\infty)$. Here, with slight abuse of notation, for $\varphi = \Phi \circ P_d, \Phi \in G_d$, we write $||\varphi||_{C^2} := ||\Phi||_{C^2}$. Clearly, the specific choices of G_d as well as the numbering of elements of G is not relevant, in the sense that two different numberings lead to weakly equivalent metrics on $\mathcal{P}(\mathbb{R}^\infty)$. Whenever we refer to d_∞ in the sequel, we refer to a fixed choice of G and a fixed numbering of its elements as above. We make the following observation.

Remark 4.1.5. *(i) If we replace $(2^{-k})_{k\in\mathbb{N}}$ in the definition of d_∞ by another summable and strictly positive sequence $(\alpha_k)_{k\in\mathbb{N}}$, we obtain a metric weakly equivalent to d_∞.*

(ii) The first part of Remark 2.3.1 applies in this context as well, i.e. the topology of uniform convergence on $C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$ coincides with the compact-open topology on $C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$, which is independent of a change of weakly equivalent metrics on $\mathcal{P}(\mathbb{R}^\infty)$.

4.2 Main results

Since no confusion can appear, we use the notation from the previous two chapters by writing $\mathcal{M}^1_{s,\nu}$ for the set of all weakly continuous probability solutions to the Cauchy problem (4.1) with initial condition (s, ν). We also use the notion of flow-admissible families of solutions $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$, $\mathcal{A}_{s,\nu} \subseteq \mathcal{M}^1_{s,\nu}$, and the sets of admissible initial conditions A_s as in Definition 2.2.1. We aim to select solution flows to (4.1) from such flow-admissible families $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$, as introduced in (1.9).

With this notation, our main results in the case of FPK equations for measures on infinite-dimensional spaces are the following theorems. Let $a = (a_{ij})_{i,j\geq 1}$ and $b = (b_i)_{i\geq 1}$ be $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^\infty)$-measurable.

Theorem 4.2.1. *Let $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$ be a flow-admissible family of sets of weakly continuous probability solutions to (4.1) such that $\mathcal{A}_{s,\nu}$ is compact in $C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$ for each admissible initial condition (s, ν). Then, there exists a solution flow to (4.1) with respect to $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$.*

Theorem 4.2.2. *In the situation of Theorem 4.2.1, the following are equivalent.*

(i) There exists at most one solution flow to (4.1) *with respect to* $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$.

(ii) For each $(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)$, *solutions to* (4.1) *in* $\mathcal{A}_{s,\nu}$ *are unique.*

Proofs of the main results. Concerning the proof of Theorem 4.2.1, it suffices to note that the proof of the analogue result in the finite-dimensional case from Chapter 2, Theorem 1.3.1, also applies in this situation in the same way. Indeed, concerning a countable measure separating family of functions on $\mathbb{R}^\infty$, one may, for example, consider a countable set of measure separating functions on $\mathbb{R}^d$, say $\mathcal{H}_d = \{h^d_n, n\in\mathbb{N}\}\subseteq C^\infty_c(\mathbb{R}^d)$, and then choose $\mathcal{H}\subseteq\mathcal{F}C^\infty_c(x)$ as the set of functions $\varphi = h^d_n\circ P_d$, $n,d\in\mathbb{N}$.

Likewise, the proof of Theorem 4.2.2 is a copy of the proof of Theorem 1.3.2, without any mandatory changes due to the present infinite-dimensional setting.

4.3 Examples

As in the previous chapters, finding conditions for solution families $\mathcal{A}_{s,\nu}$ and the coefficients a_{ij} and b_i, $i,j\geq 1$, under which the main theorems apply, amounts to finding sufficient conditions on these objects in order to have compactness of $\mathcal{A}_{s,\nu}\subseteq C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$. As in the finite-dimensional case, we stress once more that no a priori assumptions on the Borel coefficients are imposed. As before, the Arzela-Ascoli theorem, as stated in Proposition 2.4.1, is our main tool to obtain the necessary compactness of $\mathcal{A}_{s,\nu}$.

We use the notion of an *entire probability flow* as in the previous chapters, i.e. this term refers to a solution flow with respect to the solution classes $\mathcal{A}_{s,\nu}=\mathcal{M}^1_{s,\nu}$ for each $(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)$.

Remark 4.3.1. *As in the finite-dimensional situation in Chapter 2, we point out that the contents of Remark 2.4.2 apply also in the present case: the topology of uniform convergence on* $C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$ *is the same for any metric weakly equivalent to* d_∞, *with* d_∞ *as in* (4.7), *and the Arzela-Ascoli theorem characterizes the topological property of precompactness of sets* $A\subseteq C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$ *in terms of equicontinuity and the topological property of precompactness of* $\pi_t(A)\subseteq\mathcal{P}(\mathbb{R}^\infty)$, $t\in[s,T]$. *Hence, if the latter property holds, equicontinuity of* A *is independent under a change of weakly equivalent metrics to* d_∞.

4.3.1 Bounded and continuous coefficients

Here, we investigate our main theorems in the case of bounded Borel coefficients a_{ij} and b_i, $i,j\geq 1$. The main proposition in this context is

Proposition 4.3.2. *Let the Borel coefficients* a_{ij}, b_i, $i,j\geq 1$, *be globally bounded in* $(t,x)\in[0,T]\times\mathbb{R}^\infty$ *and continuous in* $x\in\mathbb{R}^\infty$. *Suppose the sets* $\mathcal{M}^1_{s,\nu}$ *are nonempty for each* $(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)$. *Then, there exists a full probability flow for* (4.1), *provided the sets* $\pi_t(\mathcal{M}^1_{s,\nu})$ *are tight for all* $0\leq s\leq t\leq T$ *and* $v\in\mathcal{P}(\mathbb{R}^\infty)$.

For the proof, we need the following auxiliary result.

Lemma 4.3.3. *Suppose the Borel coefficients a_{ij} and b_i, $i,j \geq 1$, are continuous in $x \in \mathbb{R}^\infty$ and bounded on $[0,T] \times \mathbb{R}^\infty$. Then, $\mathcal{M}^1_{s,\nu} \subseteq C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$ is closed for each $(s,\nu) \in [0,T] \times \mathcal{P}(\mathbb{R}^\infty)$.*

Proof. Let $\{\mu^{(n)}\}_{n\in\mathbb{N}} \subseteq \mathcal{M}^1_{s,\nu}$ converge to $\mu = (\mu_t)_{t\in[s,T]} \in C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$. In particular, $\mu_s = \lim_{n\to\infty} \mu_s^{(n)} = \nu$. Since the coefficients are bounded, it is clear that μ fulfills the global integrability condition (4.2). Concerning (4.3), note that for each $t \in [s,T]$ and $\varphi \in \mathcal{F}C^\infty_c(x)$, the function

$$x \mapsto L_t\varphi(x)$$

is continuous and bounded on $\mathbb{R}^\infty$. Therefore, Lebesgue's dominated convergence theorem gives

$$\int_s^t \int_{\mathbb{R}^\infty} L_r\varphi(x)d\mu_r(x)dr = \lim_{n\to\infty} \int_s^t \int_{\mathbb{R}^\infty} L_r\varphi(x)d\mu_r^{(n)}(x)dr.$$

Since each $\mu^{(n)}$ fulfills (4.3), the above equality implies that this is also valid for μ in place of $\mu^{(n)}$. Consequently, μ fulfills Definition 4.1.1. □

From here, we can prove Proposition 4.3.2.

Proof of Proposition 4.3.2. In view of Lemma 4.3.3 and the Arzela-Ascoli theorem, it remains to show equicontinuity of $\mathcal{M}^1_{s,\nu} \subseteq C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$, i.e. for each $t, (t_n)_{n\in\mathbb{N}} \subseteq [s,T]$ such that $t_n \to t$ as $n \to \infty$, we need to show

$$\sup_{\mu\in\mathcal{M}^1_{s,\nu}} d_\infty(\mu_{t_n}, \mu_t) \underset{n\to\infty}{\longrightarrow} 0$$

for d_∞ as introduced in (4.7) or (c.f. Remark 4.3.1) any metric weakly equivalent to d_∞. By the boundedness assumption on a_{ij} and b_i, we have

$$\max_{1\leq i,j\leq k} \{||a_{ij}||_\infty, ||b_i||_\infty\} < \infty$$

for every $k \in \mathbb{N}$, where $||\cdot||_\infty$ is taken with respect to $(t,x) \in [s,T] \times \mathbb{R}^\infty$. Recall that each member of the fixed family $\{\varphi_k\}_{k\in\mathbb{N}}$ used in (4.7) is of type $\varphi_k = \Phi_k \circ P_d$ with $d = d(\varphi_k) \in \mathbb{N}$. Set

$$C_k := [d(\varphi_k)^2 + d(\varphi_k)] \cdot \max_{1\leq i,j\leq d(\varphi_k)} \{||a_{ij}||_\infty, ||b_i||_\infty\} < \infty.$$

According to Remark 4.1.5, we can replace $(2^{-k})_{k\in\mathbb{N}}$ in (4.7) by

$$\alpha_k := \frac{1}{2^k(C_k+1)}$$

and denote the corresponding metric d_α. Using this metric, we have

$$\begin{aligned}\sup_{\mu\in\mathcal{M}^1_{s,\nu}} d_\alpha(\mu_{t_n}, \mu_t) &= \sup_{\mu\in\mathcal{M}^1_{s,\nu}} \left[\sum_{k=1}^\infty \alpha_k \left(\frac{|\int \varphi_k d\mu_{t_n} - \int \varphi_k d\mu_t|}{||\varphi_k||_{C^2}} \wedge 1\right)\right] \\ &= \sup_{\mu\in\mathcal{M}^1_{s,\nu}} \left[\sum_{k=1}^\infty \alpha_k \left(\frac{|\int_{t\wedge t_n}^{t\vee t_n} \int L_r\varphi_k(x)d\mu_r(x)dr|}{||\varphi_k||_{C^2}} \wedge 1\right)\right]\end{aligned}$$

$$\leq \sup_{\mu \in \mathcal{M}^1_{s,\nu}} \left[\sum_{k=1}^{\infty} \alpha_k \left(\frac{|t_n - t| \cdot C_k ||\varphi_k||_{C^2}}{||\varphi_k||_{C^2}} \wedge 1 \right) \right]$$
$$\leq |t_n - t| \cdot \sum_{k=1}^{\infty} \frac{C_k}{2^k (C_k + 1)} \underset{n \to \infty}{\longrightarrow} 0.$$

Here, the second equality follows from (4.3), which holds for each element in $\mathcal{M}^1_{s,\nu}$. Hence, equicontinuity of $\mathcal{M}^1_{s,\nu}$ follows, so that $\mathcal{M}^1_{s,\nu} \subseteq C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$ is compact by the Arzela-Ascoli-theorem, and Theorem 4.2.1 applies. □

Of course, the above proof implies equicontinuity of any subclass of solutions $\mathcal{A}_{s,\nu} \subseteq \mathcal{M}^1_{s,\nu}$. Therefore, under the present assumptions on the coefficients, Theorem 4.2.1 also applies to a flow-admissible family $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$, such that $\mathcal{A}_{s,\nu} \subseteq C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$ is closed and the sets $\pi_t(\mathcal{A}_{s,\nu}) \subseteq \mathcal{P}(\mathbb{R}^\infty)$ are precompact. The latter can, for example, be obtained in the presence of a Lyapunov function, see the existence results in [32].

4.3.2 Coefficients on an embedded Hilbert space in the presence of Lyapunov functions

In many cases, the assumptions of global boundedness in $(t, x) \in [0, T] \times \mathbb{R}^\infty$ and spatial continuity on the whole space $\mathbb{R}^\infty$ are a too strong set of conditions on a_{ij} and b_i. Instead, one may encounter situations in which solution curves are concentrated on an embedded Hilbert space, on which a and b can be estimated from above by a Lyapunov function V. Below, we present this setting and apply our main theorems to it.

Consider the space of square-summable real sequences ℓ^2 with its usual Hilbert space topology, and denote the set of Borel probability measures on $(\ell^2, \mathcal{B}(\ell^2))$ by $\mathcal{P}(\ell^2)$. The setting we investigate is the following.

Assume there exists a Borel function $V : \mathbb{R}^\infty \to [1, \infty]$ such that $V(x) < \infty$ if and only if $x \in \ell^2$, with compact sublevel sets $\{V \leq R\} \subseteq \mathbb{R}^\infty, R > 0$, and assume there exist constants $C_{ij} > 0$ and $m_{ij} \in \mathbb{N}$ for all $i, j \in \mathbb{N}$ with

$$|a_{ij}(t, x)| + |b_i(t, x)| \leq C_{ij} V^{m_{ij}}(x) \ \ \forall (t, x) \in [0, T] \times \mathbb{R}^\infty. \tag{4.8}$$

In particular, this does not impose any bound on the coefficients on $\mathbb{R}^\infty \backslash \ell^2$. In this situation, we consider flow-admissible families $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$ such that $\mathcal{A}_{s,\nu} = \emptyset$, if ν is not concentrated on ℓ^2, and assume that for each admissible initial condition $(s, \nu) \in [0, T] \times \mathcal{P}(\ell^2)$ and each $m \in \mathbb{N}$, there is a measurable, bounded function $F_{m,s,\nu} : [s, T] \to \mathbb{R}_+$ such that for each $t \in [s, T]$, we have

$$\sup_{\mu \in \mathcal{A}_{s,\nu}} \int_{\mathbb{R}^\infty} V^m(x) d\mu_t(x) \leq F_{m,s,\nu}(t). \tag{4.9}$$

In particular, since $V \equiv \infty$ on $\mathbb{R}^\infty \backslash \ell^2$, each curve $\mu \in A_{s,\nu}$ then consists of elements in $\mathcal{P}(\ell^2)$.

For a general existence theorem to which our above framework applies, see [32, Thm.3.1]. The main result for this case is the following proposition.

Proposition 4.3.4. *Suppose (4.8) holds. Assume* $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$ *is flow-admissible such that each* $\mathcal{A}_{s,\nu}$ *is nonempty and closed in* $C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$*, and such that for any admissible initial condition* (s,ν)*, (4.9) holds. Then, there exists a solution flow for (4.1) with respect to* $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$*.*

Proof. Closedness of $\mathcal{A}_{s,\nu}\subseteq C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$ holds by assumption. For $0\leq s\leq t\leq T$ and any admissible initial condition (s,ν), tightness of $\pi_t(\mathcal{A}_{s,\nu})$ follows, because V has compact sublevel sets and since (4.9) yields

$$\sup_{\mu\in\mathcal{A}_{s,\nu}}\int_{\mathbb{R}^\infty}V(x)d\mu_t(x)<\infty.$$

Concerning equicontinuity, we exploit the same idea as in the proof of Proposition 4.3.2, i.e. we consider a suitable metric d_β on $\mathcal{P}(\mathbb{R}^\infty)$, for which we show

$$\sup_{\mu\in\mathcal{A}_{s,\nu}} d_\beta(\mu_{t_n},\mu_t)\underset{n\to\infty}{\longrightarrow}0$$

for each $(t_n)_{n\in\mathbb{N}}, t\in[s,T]$ such that $t_n\to t$ as $n\to\infty$. More precisely, we consider d_∞ as in (4.7) and replace $(2^{-k})_{k\in\mathbb{N}}$ by $(\beta_k)_{k\in\mathbb{N}}$ with (using the notation of the proof of Proposition 4.3.2 and of (4.8) and (4.9))

$$\beta_k:=\frac{1}{2^k(D_{k,s,\nu}+1)},\quad m_k:=\max_{1\leq i,j\leq d(\varphi_k)}m_{ij},$$

$$D_k:=[d(\varphi_k)^2+d(\varphi_k)]\cdot\max_{1\leq i,j\leq d(\varphi_k)}C_{ij},\quad D_{k,s,\nu}:=D_k||F_{m_k,s,\nu}||_\infty,$$

and denote the corresponding metric β. Here, $F_{m_k,s,\nu}$ is as in (4.9). Then, we calculate

$$\begin{aligned}\sup_{\mu\in\mathcal{A}_{s,\nu}} d_\beta(\mu_{t_n},\mu_t)&=\sup_{\mu\in\mathcal{A}_{s,\nu}}\left[\sum_{k=1}^\infty\beta_k\left(\frac{|\int_{\mathbb{R}^\infty}\varphi_k d\mu_{t_n}-\int_{\mathbb{R}^\infty}\varphi_k d\mu_t|}{||\varphi_k||_{C^2}}\wedge 1\right)\right]\\&=\sup_{\mu\in\mathcal{A}_{s,\nu}}\left[\sum_{k=1}^\infty\beta_k\left(\frac{|\int_{t\wedge t_n}^{t\vee t_n}\int_{\mathbb{R}^\infty}L_r\varphi_k d\mu_r dr|}{||\varphi_k||_{C^2}}\wedge 1\right)\right]\\&\leq\sup_{\mu\in\mathcal{A}_{s,\nu}}\left[\sum_{k=1}^\infty\beta_k\left(\frac{D_k||\varphi_k||_{C^2}\int_{t\wedge t_n}^{t\vee t_n}\int_{\mathbb{R}^\infty}V^{m_k}d\mu_r dr}{||\varphi_k||_{C^2}}\right)\right]\\&\leq\sum_{k=1}^\infty\beta_k D_k\int_{t\wedge t_n}^{t\vee t_n}F_{m_k,s,\nu}(r)dr\\&\leq\sum_{k=1}^\infty|t_n-t|\beta_k D_k||F_{m_k,s,\nu}||_\infty\\&\leq|t_n-t|\sum_{k=1}^\infty\frac{D_{k,s,\nu}}{2^k(D_{k,s,\nu}+1)}\underset{n\to\infty}{\longrightarrow}0.\end{aligned}$$

Here, the second equality follows from (4.3). The first inequality is due to (4.8), and the second one follows from (4.9). Thereby, each $\mathcal{A}_{s,\nu}\subseteq C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$ is compact and the result follows by Theorem 4.2.1. □

Remark 4.3.5. *We point out that this situation does not require any a priori continuity assumption on a and b. However, it is clear that certain regularity of the coefficients helps to find flow-admissible families, which are closed in* $C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$.

4.3.3 Finitely based coefficients

Here, we assume that a_{ij} and b_i, $i,j \geq 1$, are globally bounded on $[0,T]\times\mathbb{R}^\infty$ and continuous in $x\in\mathbb{R}^\infty$. Moreover, we assume the coefficients depend only on a finite number of spatial variables. More precisely, we consider the case that

$$a_{ij}(t,x) = \bar{a}_{ij}\big(t, P_{max(i,j)}(x)\big), b_i = \bar{b}_i\big(t, P_i(x)\big), \quad i,j\geq 1, \tag{4.10}$$

for Borel functions $\bar{a}_{ij}:[0,T]\times\mathbb{R}^{max(i,j)}\to\mathbb{R}$, $\bar{b}_i:[0,T]\times\mathbb{R}^i\to\mathbb{R}$, which are continuous in their spatial arguments and bounded in (t,x). In this situation, the tightness assumption of Proposition 4.3.2 is fulfilled and we obtain the following result.

Proposition 4.3.6. *Let* $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$ *be such that each* $\mathcal{A}_{s,\nu}\subseteq C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$ *is closed and nonempty. If the coefficients* a_{ij} *and* b_i *satisfy (4.10), then there is a solution flow for* (4.1) *with respect to* $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$.

Recalling Lemma 4.3.3, we immediately obtain the following corollary.

Corollary 4.3.7. *If in the situation of (4.10) for each* $(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)$ *there exists a probability solution to* (4.1)*, then there exists a full probability flow for* (4.1).

Let us prove Proposition 4.3.6. The main idea is to use assumption (4.10) to apply a precompactness result for finite-dimensional FPK equations, as investigated in Chapter 2. Recall that for natural numbers $l\leq d$, P^d_l denotes the canonical projection from $\langle e_1,\dots,e_d\rangle$ to $\langle e_1,\dots e_l\rangle$.

Proof of Proposition 4.3.6. In view of the Arzela-Ascoli theorem, it remains to prove equicontinuity of $\mathcal{A}_{s,\nu}$ and tightness of $\pi_t\big(\mathcal{A}_{s,\nu}\big)\subseteq\mathcal{P}(\mathbb{R}^\infty)$ for each $s\leq t\leq T$ and any admissible initial condition (s,ν). Concerning tightness, in view of Lemma A.0.4, it suffices to prove tightness of $\{\mu_r\circ P_d^{-1}|(\mu_t)_{t\in[s,T]}\in\mathcal{A}_{s,\nu}\}$ as Borel probability measures on $\mathbb{R}^d$ for each $d\in\mathbb{N}$. For each $d\in\mathbb{N}$, considering (4.3) for $\varphi=\Phi\circ P_d^{-1}$ from $\mathcal{F}C_c^\infty(x)$, every element $(\mu_t)_{t\in[s,T]}\in\mathcal{A}_{s,\nu}$ fulfills

$$\begin{aligned}\int_{\mathbb{R}^d}\Phi d\mu_t\circ P_d^{-1} - \int_{\mathbb{R}^d}\Phi d\nu\circ P_d^{-1}\\ = \int_s^t\int_{\mathbb{R}^d}\bar{a}_{ij}\big(u,P^d_{max(i,j)}x\big)\partial_{ij}\Phi(x)+\bar{b}_i(u,P^d_i x)\partial_i\Phi(x)d\mu_u\circ P_d^{-1}(x)du\end{aligned}$$

for each $\Phi\in C_c^\infty(\mathbb{R}^d)$, where the suppressed sums on the right-hand side comprise $i,j\leq d$ and $i\leq d$, respectively. Hence, $t\mapsto\mu_t\circ P_d^{-1}$ solves the d-dimensional FPK equation with space-continuous and bounded coefficients $\bar{a}_{ij}(\cdot,P^d_{max(i,j)})$, $\bar{b}_i(\cdot,P^d_i)$, $1\leq i,j\leq d$, and initial condition $(s,\nu\circ P_d^{-1})\in[0,T]\times\mathcal{P}$. Clearly, this solution curve is also weakly continuous. By Section 2.4, the set of all weakly continuous probability solutions to a finite-dimensional equation with such coefficients for a common initial condition is compact in $C_{s,T}\mathcal{P}$. In particular, $\{\mu_t\circ P_d^{-1}\,|\,(\mu_t)_{t\in[s,T]}\in\mathcal{A}_{s,\nu}\}$ is tight, so that Lemma A.0.4 applies.

Of course, equicontinuity of $\mathcal{A}_{s,\nu}$ follows exactly as in the proof of Proposition 4.3.2. $\square$

We close this chapter (and with it the first part of the thesis) with the following remark, which might be interesting for future considerations.

Remark 4.3.8. *In principle, there is a certain degree of freedom to the techniques we use to apply our main results. In particular, a careful analysis of the proof of Theorem 4.2.1 shows that in the present situation, it is at no point necessary to equip $C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$ with the topology of uniform convergence, since all we ever use is pointwise convergence. Of course, we choose this approach to have the Arzela-Ascoli theorem at hand in order to characterize compactness of subsets of $C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$. However, in principle it is much easier to find compact subsets in $C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$ when this space is endowed with the coarser topology of pointwise convergence. Thus, if one has a family $\{\mathcal{A}_{s,\nu}\}_{(s,\nu)\in[0,T]\times\mathcal{P}(\mathbb{R}^\infty)}$ of compact subsets with respect to the pointwise topology on $C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$, then our results apply as well. In fact, for the proofs of our main results, there is no natural topology on $C_{s,T}\mathcal{P}(\mathbb{R}^\infty)$, which should be used for intrinsic reasons. Of course, this remark also prevails accordingly in the finite-dimensional situation of the previous chapters.*

Appendix A

Auxiliary results on FPK equations

Lemma A.0.2.

The objective of this part of the appendix is to prove Lemma A.0.2, which we use in the proof of Proposition 3.3.2. Below, we denote by B_r the Euclidean ball with radius $r > 0$ centered at 0, and by $\overline{B_r}$ its closure. First of all, we need the following auxiliary result. Recall that we call a function *compact*, if it has compact sublevel sets.

Lemma A.0.1. *For any $\nu \in \mathcal{SP}$, there exists a nonnegative compact function $V = V_\nu \in C^2(\mathbb{R}^d)$ such that $\max_{1\leq i,j\leq d}(||\partial_i V||_\infty, ||\partial_{ij}V||_\infty) < \infty$ and $\int_{\mathbb{R}^d} V d\nu < \infty$.*

Proof. Since every single Borel probability measure on $\mathbb{R}^d$ is tight, there exists a sequence of strictly increasing radii $R_n > 0$, $n \geq 1$, such that $\nu(\overline{B_{R_n}}^c) \leq n^{-3}$. Without loss of generality, we may assume $R_{n+1} \geq R_n + 1$. Set $W_\nu(x) := 1$ on $\overline{B_{R_1}}$ and $W_\nu(x) := n$ on $\overline{B_{R_{n+1}}}\backslash\overline{B_{R_n}}$.

Clearly, W_ν is nonnegative, radial and has compact sublevel sets $\{W_\nu \leq c\} = \overline{B_{R_{\lfloor c \rfloor +1}}}$ for $c \geq 1$ and $\{W_\nu \leq c\} = \emptyset$ for $c < 1$. Furthermore, note that

$$\int_{\mathbb{R}^d} W_\nu d\nu \leq \sum_{n\geq 1} n^{-2} < \infty.$$

Now consider, for each $n \geq 1$, a function h_n, defined via

$$h_n(r) := \begin{cases} n & , r \in [R_{n+1}, R_{n+1} + \frac{1}{4}], \\ g_n(r) & , r \in (R_{n+1} + \frac{1}{4}, R_{n+1} + \frac{3}{4}), \\ n+1 & , r \in [R_{n+1} + \frac{3}{4}, R_{n+2}], \end{cases}$$

for a suitable increasing C^2-function $g_n : \mathbb{R} \to \mathbb{R}$ with bounded first- and second-order derivatives such that $h_n \in C^2([R_n, R_{n+1}])$. Note that we used the assumption $R_{n+1} \geq R_n + 1$ for the definition of h_n. Clearly, the family $\{g_n\}_{n\geq 1}$ can be chosen with uniformly (in n) bounded first- and second-order derivatives. Compounding these functions, we note that $V_\nu : \mathbb{R}^d \to \mathbb{R}$,

$$V_\nu(x) := \begin{cases} 1 & , x \in \overline{B_{R_2}}, \\ h_n(|x|) & , x \in \overline{B_{R_{n+2}}} \backslash \overline{B_{R_{n+1}}}, n \geq 1, \end{cases}$$

is a nonnegative function in $C^2(\mathbb{R}^d)$ such that $V_\nu(x) \longrightarrow \infty$ as $|x| \to \infty$ with uniformly bounded first- and second-order partial derivatives and compact sublevel sets, i.e. it is a compact function as in the assertion. Finally, since $V_\nu \leq W_\nu$ by construction, $\int V_\nu d\nu < \infty$ follows, which completes the proof. □

Lemma A.0.2. *Let the Borel coefficients a_{ij}, b_i, $1 \leq i, j \leq d$, be defined on $[0,T] \times \mathcal{SP} \times \mathbb{R}^d$ and suppose they fulfill (B2.i) and (B2.ii) of Assumption B2 in Section 3.3. Then, for each $0 \leq s \leq t \leq T$ and $\nu \in \mathcal{SP}$, the set $\pi_t(\mathcal{M}_{s,\nu})$ is tight.*

Proof. Fix $(s,\nu) \in [0,T] \times \mathcal{SP}$ and $t \in [s,T]$. Consider a function $V = V_\nu \in C^2(\mathbb{R}^d)$ with the properties stated in Lemma A.0.1 and let $\{\varphi_l\}_{l\geq 1} \subseteq C_c^2(\mathbb{R}^d)$ have the following properties: φ_l is nonnegative, increases pointwise to V as $l \to \infty$ such that $\varphi_l = V$ on B_l and such that $\partial_i \varphi_l, \partial_{ij}\varphi_l$ are bounded uniformly in $1 \leq i,j \leq d$ and $l > 1$ by some number $0 < D < \infty$. Then, for any $(\mu_t)_{t\in[s,T]} \in \mathcal{M}_{s,\nu}$, (B2.i) entails

$$\sup_{l\geq 1} \left| \int_s^t \int_{\mathbb{R}^d} a_{ij}(u,\mu_u,x)\partial_{ij}\varphi_l(x) + b_i(u,\mu_u,x)\partial_i\varphi_l(x) d\mu_u(x) du \right| < C, \tag{A.1}$$

with $C = C(D) > 0$ independent of the particular solution $(\mu_t)_{t\in[s,T]} \in \mathcal{M}_{s,\nu}$, and hence, using (3.1),

$$\sup_{l\geq 1} \left| \int_{\mathbb{R}^d} \varphi_l d\mu_t - \int_{\mathbb{R}^d} \varphi_l d\nu \right| < \infty,$$

which, together with $\sup_{l\geq 1} \int \varphi_l d\nu = \int V_\nu d\nu < \infty$, entails a uniform in $\mathcal{M}_{s,\nu}$ bound on $\int V_\nu d\mu_t$. Therefore, $\pi_t(\mathcal{M}^1_{s,\nu})$ is a tight family of subprobability measures, as claimed. □

Proof of Lemma 4.1.4.

We begin by stating and proving two preliminary results. Denote by $\mathcal{F}_b(x)$ the set of bounded continuous cylindrical functions, i.e. the functions $\varphi : \mathbb{R}^\infty \to \mathbb{R}$ of type $\varphi = \Phi \circ P_d$ for $\Phi \in C_b(\mathbb{R}^d)$ and $d \in \mathbb{N}$.

Lemma A.0.3. *Let $K \subseteq \mathbb{R}^\infty$ be compact and $f \in C_b(\mathbb{R}^\infty)$. For every $\varepsilon \in (0,1)$, there exists a function $\psi = \psi(\varepsilon) \in \mathcal{F}_b(x)$ such that $\sup_{x\in K} |f(x) - \psi(x)| \leq \varepsilon$. Furthermore, for $\varepsilon \in (0,1)$, $\psi(\varepsilon)$ may be chosen such that*

$$\sup_{x\in\mathbb{R}^\infty} |\psi_\varepsilon(x)| \leq ||f||_\infty + 1.$$

Proof. Recall that $\mathbb{R}^\infty$ with the topology of pointwise convergence is Polish. Moreover, $\mathcal{F}_b(x)_{|K}$ (the set of all $\mathbb{R}$-valued functions on K, which are restrictions of elements of $\mathcal{F}_b(x)$) is a subalgebra of $C(K,\mathbb{R})$, which contains the constant functions and separates points in K. Hence, the first claim follows by the Stone-Weierstraß theorem, see Theorem D.0.4.

Concerning the second claim, note that $\psi = \Psi \circ P_d$ for some $d \in \mathbb{N}$, $\Psi \in C_b(\mathbb{R}^d)$, and hence

$$\sup_{x\in P_d(K)} |\Psi(x)| \leq ||f||_\infty + 1.$$

Since $P_d(K) \subseteq \mathbb{R}^d$ is closed, we may change Ψ on $P_d(K)^c$ such that the function remains continuous and attains its supremum on $\mathbb{R}^\infty$ on $P_d(K)$, and such that the approximation of f by ψ on K remains true. □

Lemma A.0.4. *Let I be some index set and $\{\mu_i\}_{i\in I}$ a family in $\mathcal{P}(\mathbb{R}^\infty)$ such that for each $d \in \mathbb{N}$, the family $\{\mu_i \circ P_d^{-1}\}_{i\in I}$ is tight as Borel probability measures in $\mathbb{R}^d$. Then, $\{\mu_i\}_{i\in I}$ is tight in $\mathcal{P}(\mathbb{R}^\infty)$ as well.*

Proof. Fix $\varepsilon > 0$. By tightness of $\{\mu_i \circ P_d^{-1}\}_{i\in I}$, we find $Z_d^\varepsilon \in \mathbb{R}_+$ such that

$$\mu_i\big(P_d \in \{\underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{d-1 \text{ times}} \times [-Z_d^\varepsilon, Z_d^\varepsilon]^c\}\big) \leq \varepsilon \cdot 2^{-d} \tag{A.2}$$

for all $i \in I$ and $d \in \mathbb{N}$. Set $K_\varepsilon := \prod_{d=1}^\infty [-Z_d^\varepsilon, Z_d^\varepsilon]$, which is compact in $\mathbb{R}^\infty$. We have

$$\begin{aligned}\mu_i(K_\varepsilon^c) &= \mu_i\Bigg(\bigcup_{d\in\mathbb{N}} P_d^{-1}\big(\mathbb{R} \times \ldots \times \mathbb{R} \times [-Z_d^\varepsilon, Z_d^\varepsilon]^c\big)\Bigg) \\ &\leq \sum_{d=1}^\infty \mu_i\Bigg(P_d^{-1}\big(\mathbb{R} \times \ldots \times \mathbb{R} \times [-Z_d^\varepsilon, Z_d^\varepsilon]^c\big)\Bigg) \leq \varepsilon \sum_{d=1}^\infty 2^{-d} = \varepsilon\end{aligned}$$

uniformly in $i \in I$, which yields tightness of $\{\mu_i\}_{i\in I}$ as elements in $\mathcal{P}(\mathbb{R}^\infty)$. □

Proof of Lemma 4.1.4. Let $(\eta_n)_{n\in\mathbb{N}}$ converge to η in the sense of (4.6). First, recall that weak convergence of Borel probability measures on $\mathbb{R}^d$ is characterized by convergence of integrals against all $C_c^\infty(\mathbb{R}^d)$-functions (if the limit is a priori known to be a probability measure as well). Hence, weak convergence

$$\eta_n \circ P_d^{-1} \underset{n\to\infty}{\longrightarrow} \eta \circ P_d^{-1} \tag{A.3}$$

holds for each $d \in \mathbb{N}$ by assumption. In particular, $(\eta_n \circ P_d^{-1})_{n\in\mathbb{N}}$ is tight for each $d \in \mathbb{N}$. By Lemma A.0.4, we obtain tightness of $(\eta_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(\mathbb{R}^\infty)$.
Now, let $f \in C_b(\mathbb{R}^\infty)$, $\varepsilon \in (0,1)$ and $K_\varepsilon \subseteq \mathbb{R}^\infty$ compact such that

$$\max\big(\sup_{n\in\mathbb{N}} \eta_n(K_\varepsilon^c), \eta(K_\varepsilon^c)\big) \leq \varepsilon. \tag{A.4}$$

Such a set exists, since any single probability measure on a Polish space is tight. Furthermore, due to Lemma A.0.3 we may choose $\psi_\varepsilon \in \mathcal{F}_b(x)$ with $\sup_{x\in K_\varepsilon} |\psi_\varepsilon(x) - f(x)| \leq \varepsilon$ and $\sup_{x\in\mathbb{R}^\infty} |\psi_\varepsilon(x)| \leq ||f||_\infty + 1$. Then, setting $C_f := ||f||_\infty + 1 < \infty$, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^\infty} f d\eta - \int_{\mathbb{R}^\infty} f d\eta_n \right| \\
&\leq \int_{\mathbb{R}^\infty} |f - \psi_\varepsilon| d\eta + \left| \int_{\mathbb{R}^\infty} \psi_\varepsilon d\eta - \int_{\mathbb{R}^\infty} \psi_\varepsilon d\eta_n \right| + \int_{\mathbb{R}^\infty} |\psi_\varepsilon - f| d\eta_n \\
&\leq 2(1+2C_f)\varepsilon + \left| \int_{\mathbb{R}^\infty} \psi_\varepsilon d\eta - \int_{\mathbb{R}^\infty} \psi_\varepsilon d\eta_n \right| \underset{n\to\infty}{\longrightarrow} 2(1+2C_f)\varepsilon.
\end{aligned}$$

Here, the second inequality follows by splitting the first and third summand into integrals over K_ε and K_ε^c, together with the approximation of f by ψ_ε on K_ε and (A.4). The final convergence holds due to (A.3), and since $\psi_\varepsilon \in \mathcal{F}_b(x)$. Since C_f is independent of ε, the claim follows. □

Part II

Superposition principle for nonlinear Fokker–Planck–Kolmogorov equations

Abstract. We prove a superposition principle for solutions to nonlinear Fokker–Planck–Kolmogorov equations (FPK equations) and the associated linear continuity equations for curves in $\mathcal{P}(\mathcal{SP})$: Under rather mild global integrability assumptions, any solution curve to the latter arises as the curve of one-dimensional marginals of a superposition of solutions to the former. To this end, we use the well-known linearization of nonlinear FPK equations, which is based on a geometric manifold-like structure on $\mathcal{SP}$. In the second half, we derive a similar superposition result for stochastic nonlinear FPK equations. To do so, we extend the geometric structure on $\mathcal{SP}$ in order to associate a linear equation of second order for curves in $\mathcal{P}(\mathcal{SP})$ to such stochastic equations. As a consequence, in both cases, we can transfer existence and uniqueness results between the nonlinear equation and its associated linearized equation. The contents of this part are a slightly extended version of the recent preprint [186].

Chapter 5

Introduction

For a general short introduction to (nonlinear) Fokker–Planck–Kolmogorov equations, we refer to Section 1.1.

5.1 Superposition principle for finite-dimensional equations

Before we turn to the framework and the main results of this part of the thesis, let us first describe the underlying classical situation in finite dimensions, i.e. the case of (stochastic) differential equations on $\mathbb{R}^d$. Later on, we replace these equations by (stochastic) Fokker–Planck–Kolmogorov equations (FPK equations) for curves in $\mathcal{SP}$. The principal ideas for the geometric approach to our main results in Chapters 6 and 7 stem from a comparison to this finite-dimensional case.

5.1.1 The deterministic case

Well-posed equations. Let $d \in \mathbb{N}$, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field and consider the ordinary differential equation on $[0,T]$

$$\begin{cases} \dot{\gamma}_t = b_t(\gamma_t), \\ \gamma_0 = x \end{cases} \tag{ODE}$$

for an initial value $x \in \mathbb{R}^d$. Under suitable assumptions, e.g. in the classical Cauchy-Lipschitz situation, where b is continuous in (t,x) and globally Lipschitz continuous in its spatial variable, the set of solutions $AC_T(b,x)$ to (ODE) with initial value x (i.e. the absolutely continuous curves $t \mapsto \gamma_t$ such that $\dot{\gamma}_t = b_t(\gamma_t)$ dt-a.s. and $\gamma_0 = x$) is a singleton for each x and the unique solutions $t \mapsto \gamma_t^x$ depend continuously on the initial value. To the differential equation (ODE), one naturally associates an equation for curves of measures $t \mapsto \mu_t$ on the state space of (ODE), namely the *continuity equation* (in distributional sense)

$$\partial_t \mu_t = -\nabla \cdot (b_t \mu_t). \tag{CE}$$

Note that this is a *linear* equation (for measures), while (ODE) is a nonlinear equation (for curves in $\mathbb{R}^d$). However, in contrast to the finite-dimensional ODE, (CE) is an equation on an infinite-dimensional space of measures. This equation makes sense for curves of signed Borel measures on $\mathbb{R}^d$, but with regard to our subsequent considerations, we temporarily restrict attention to weakly continuous solutions $t \mapsto \mu_t \in \mathcal{P}$. In this situation, the connection between (ODE) and (CE) is given by the following well-known observation, see for example [12, Lem.8.1.6].

Proposition 5.1.1. *Let $\mu_0 \in \mathcal{P}$ and let $\gamma : (t,x) \mapsto \gamma_t^x$ denote the flow map of* (ODE). *Define $\mu_t := \mu_0 \circ \gamma_t^{-1} \in \mathcal{P}$. If*

$$\int_0^T \int_{\mathbb{R}^d} |b_t(x)| d\mu_t(x) dt < \infty, \tag{5.1}$$

then $t \mapsto \mu_t$ is a weakly continuous solution to (CE).

In particular, the proposition applies in the case $\mu_0 = \delta_{x_0}$, and then $\mu_t = \delta_{\gamma_t^{x_0}}$. Under the present assumptions on b, the continuity equation (CE) is well-posed in the space of weakly continuous curves $t \mapsto \mu_t \in \mathcal{P}$ such that (5.1) holds [12, Prop.8.1.7]. It is not difficult, yet very interesting to observe, that in this situation, the uniqueness of (ODE) and (CE) together with the above proposition yields the following representation result for the solution to (CE), see [12, Prop.8.1.8].

Proposition 5.1.2. *Let $t \mapsto \mu_t \in \mathcal{P}$ be a weakly continuous solution to* (CE) *such that* (5.1) *holds. Then, $\mu_t = \mu_0 \circ \gamma_t^{-1}$ for each $t \in [0,T]$.*

This representation result may be considered a first instance of the so-called *superposition principle*: Solutions $t \mapsto \mu_t$ to (CE) are obtained by mixing, or *superposing*, solution curves to the corresponding (ODE) in a compatible way with respect to the initial distribution μ_0. However, in the present situation, for each x in the support of μ_0, this mixing involves at most *one* curve with initial value x (the unique ODE solution $t \mapsto \gamma_t^x$) and, hence, does not constitute a true superposition of paths with common initial value.

Superposition principle (CE) $\Longrightarrow$ (ODE). An extensively studied question is that of existence and uniqueness of solutions to (ODE) and (CE) under weaker assumptions on b, and whether such results can be transferred between these equations. On the one hand, the celebrated DiPerna-Lions theory establishes existence and uniqueness of a so-called *regular Lagrangian flow* of solutions to (ODE) via the corresponding continuity equation for the case of Sobolev vector fields [94]. Roughly, a Lagrangian flow is a selection of solutions γ_t^x to (ODE), which is compatible with the usual flow property of differential equations up to

dx-negligible sets. Subsequently, these results where extended to the case of BV vector fields b by Ambrosio [6]. For (an overview of) further results in this direction, see also [8, 78, 10, 77] and the references therein.

On the other hand, and closer to the results of the present part of this thesis, the question arises whether a representation formula for solutions to (CE) similar to Proposition 5.1.2 holds in the case that (ODE) does not have unique solutions. In this case, the simple representation as in Proposition 5.1.2 cannot hold in general. Indeed, it is easily seen that any measure $\eta \in \mathcal{P}(C_T\mathbb{R}^d)$ concentrated on solutions to (ODE) with the integrability property

$$\int_{C_T\mathbb{R}^d}\int_0^T |b_t(\pi_t)| dt d\eta < \infty \tag{5.2}$$

induces a weakly continuous solution $t \mapsto \mu_t \in \mathcal{P}$ to (CE) via

$$\mu_t := \eta \circ \pi_t^{-1}, \quad t \in [0,T] \tag{5.3}$$

(recall that $\pi_t : C_T\mathbb{R}^d \to \mathbb{R}^d$ denotes the projection $\pi_t : f \mapsto f(t)$, $t \in [0,T]$). In particular, $\operatorname{supp}\eta \cap AC_T(b,x)$ need not be a singleton, i.e. the solution curve $t \mapsto \mu_t$ arises by a possibly nontrivial mixing of (ODE)-solutions with initial value x, according to a probability measure μ_0 on these initial values. Such solutions μ_t (and, sometimes, the corresponding measure η) are usually called *superposition solutions*. In the previously mentioned case of a Lipschitz vector field b, any such measure η is necessarily given by $\eta(\cdot) = \int_{\mathbb{R}^d} \delta_{\gamma^x}(\cdot) d\mu_0(x)$ for some $\mu_0 \in \mathcal{P}$, i.e. in this case, the curve in (5.3) takes the form $\mu_t = \mu_0 \circ \gamma_t^{-1}$, where γ denotes the unique flow of (ODE). By Proposition 5.1.2, it follows that in this Lipschitz case, *any* weakly continuous solution to (CE) with (5.1) is of this type.

The natural question which arises is the following: *Under which low regularity and integrability assumption on $b_t(x)$ is a weakly continuous (CE)-solution $t \mapsto \mu_t$ a superposition solution?*

Remarkably, it turns out that no regularity assumption on b is needed at all. Indeed, the following well-known superposition principle holds.

Superposition principle (CE) $\implies$ (ODE). *Let $b_t(x)$ be a Borel vector field. If $t \mapsto \mu_t \in \mathcal{P}$ is a weakly continuous solution to (CE), which fulfills the global integrability condition (I), then there exists a measure $\eta \in \mathcal{P}(C_T\mathbb{R}^d)$, concentrated on solutions to (ODE) such that $\mu_t = \eta \circ \pi_t^{-1}$ for each $t \in [0,T]$.*

The first result in this direction appears to be [12, Thm.8.2.1], with the condition (I) being

$$\int_0^T \int_{\mathbb{R}^d} |b_t(x)|^p d\mu_t(x) dt < \infty, \quad p > 1. \tag{I.1}$$

In [169], this result is extended to the case $p = 1$, and a inhomogeneous version of the continuity equation is treated as well. To the best of the author's knowledge, the most general form of the global integrability (I) is given in [7], where the above conditions are replaced by the substantially weaker assumption

$$\int_0^T \int_{\mathbb{R}^d} \frac{|b_t(x)|}{1+|x|} d\mu_t(x)dt < \infty. \tag{I.2}$$

As shown in [7], this superposition principle may be used to deduce existence and uniqueness of a regular Lagrangian flow for the corresponding (ODE). Moreover, it follows that uniqueness of the ODE yields uniqueness of the continuity equation.

In a nutshell, proofs of such superposition results proceed as follows. First, one approximates b by regular (say, Lipschitz) vector fields $b^\varepsilon, \varepsilon > 0$, and uses the representation of Proposition 5.1.2 for the unique solutions $t \mapsto \mu_t^\varepsilon$ of the corresponding equations (CE^ε), hence obtaining superposition measures $(\eta^\varepsilon)_{\varepsilon>0} \subseteq \mathcal{P}(C_T\mathbb{R}^d)$. Under a suitable choice of b^ε, one obtains tightness of this family, and hence the existence of a weak limit point η. The specific choice of the approximations b^ε and μ_t^ε (usually obtained via convolution of b and μ_t with suitable mollifiers) yields that any such limit point η is a superposition measure as in the assertion.

For further results and recent surveys of the field, we refer the reader to [7, 11, 13, 14, 211] and the references therein. In particular, the last three sources study the question addressed above in the general framework of metric measures spaces, a direction which we do not pursue in this thesis. For a partial result and interesting counterexamples in the case of signed bounded measure-valued solutions to the continuity equation, see [9, 41].

5.1.2 The stochastic case

From a probabilistic viewpoint, it is natural to ask to which extent these interesting results for ODEs and their corresponding continuity equations are valid for the respective stochastic counterparts as well. Consider the *stochastic* differential equation

$$\begin{cases} dX_t = b_t(X_t)dt + \sigma_t(X_t)dB_t \\ \mathcal{L}(X_0) = \mu, \end{cases} \tag{SDE}$$

where b is a time-dependent Borel vector field as before, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times l}$ is a Borel diffusion coefficient, formally multiplied by the increments of an l-dimensional Brownian motion $B = (B_t)_{t\in[0,T]}$, and the initial condition $\mathcal{L}(X_0) = \mu \in \mathcal{P}$ means that X_0 should have distribution μ. The second term on the right-hand side of (SDE) is understood as the usual finite-dimensional stochastic Itô integral.

Let $t \mapsto X_t$ be a solution on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ with Brownian motion B, and denote by $\mu_t := \mathbb{P} \circ X_t^{-1}$ its one-dimensional marginals. By Itô's formula, it is straightforward to check that $t \mapsto \mu_t$ solves the FPK equation

$$\begin{cases} \partial_t \mu_t = \mathcal{L}_t^* \mu_t, \\ \mu_0 = \mu, \end{cases} \tag{FPK}$$

in distributional sense, where $\mathcal{L}_t^*$ denotes the formal dual of the second-order differential operator $\mathcal{L}_t\varphi(x) = b_t^i(x)\partial_i\varphi(x) + a_t^{ij}(x)\partial_{ij}\varphi(x)$ with coefficients $b_t = (b_t^i)_{1\le i\le d}$, $a_t =$

$(a_t^{ij})_{1\leq i,j\leq d} = 1/2\sigma_t\sigma_t^T$. We refer the reader to the general introduction to equations of type (FPK) in Section 1.1. Hence, in analogy to the deterministic case, a solution to the nonlinear differential equation (SDE) induces a solution curve for the linear equation of measures (FPK) on the state space of (SDE).

Superposition of solutions to (FPK). The question arises what one can say about a converse result under low regularity and integrability assumptions on b and σ or a, respectively.

To this end, one usually studies the connection between the FPK equation and the corresponding *martingale problem*. A solution to the martingale problem with respect to b and a with initial value μ (at time $t = 0$) is a probability measure $\eta \in \mathcal{P}(C_T\mathbb{R}^d)$ such that

(i) $\eta \circ \pi_0^{-1} = \mu$ and $\int_{C_T\mathbb{R}^d}\int_0^T |b_t(\pi_t)| + |a_t(\pi_t)|dtd\eta < \infty$.

(ii) The process

$$t \mapsto \varphi(\pi_t) - \int_0^t \mathcal{L}_s\varphi(\pi_s)ds$$

is a η-martingale on $C_T\mathbb{R}^d$ with respect to the canonical filtration for each $\varphi \in C_c^\infty(\mathbb{R}^d)$.

The martingale formulation and the differential equation (SDE) are essentially equivalent, see [213, Thm.2.6] for the case of bounded coefficients (without any regularity assumption). The enormous and diverse interest in the martingale problem in connection with the theory of diffusion processes dates back at least to Stroock and Varadhan's celebrated book [215], to which we refer for a complete introduction to the area. It is readily seen be integration with respect to $\int_{C_T\mathbb{R}^d} d\eta$ and Fubini's theorem that any martingale solution η induces a weakly continuous solution $t \mapsto \mu_t \in \mathcal{P}$ to (FPK) via

$$\mu_t := \eta \circ \pi_t^{-1}, \quad t \in [0,T], \tag{5.4}$$

and, in analogy to the deterministic setting, we call such a solution μ_t a *superposition solution*. Now, the question proposed above can be stated as follows: *Under which low regularity and integrability assumption on b and σ is a weakly continuous* (FPK)*-solution $t \mapsto \mu_t$ a superposition solution?*

As in the deterministic case, an affirmative answer holds under moderate global integrability assumptions without any regularity on b and σ:

Superposition principle (FPK) $\implies$ (SDE). *Let $b_t(x)$ and $a_t(x)$ be Borel maps. If $t \mapsto \mu_t \in \mathcal{P}$ is a weakly continuous solution to* (FPK)*, which fulfills the global integrability condition (I'), then there exists a solution $\eta \in \mathcal{P}(C_T\mathbb{R}^d)$ to the corresponding martingale problem such that $\mu_t = \eta \circ \pi_t^{-1}$ for each $t \in [0,T]$.*

The first result in this direction is a remarkable theorem by Figalli [100], who proved the above superposition principle under the assumption of bounded coefficients b and a. The boundedness assumptions on b and a were notably replaced by Trevisan [223] by the weaker condition

$$\int_0^T \int_{\mathbb{R}^d} |b_t(x)| + |a_t(x)|d\mu_t(x)dt < \infty. \tag{I'.1}$$

Finally, Bogachev, Röckner and Shaposhnikov [37] extended the result to the (to date) optimal assumption

$$\int_0^T \int_{\mathbb{R}^d} \frac{|b_t(x) \cdot x| + |a_t(x)|}{1 + |x|^2} d\mu_t(x) dt < \infty. \tag{I'.2}$$

It is easy to see that in the deterministic case, i.e. for $a = 0$, martingale solutions are exactly the path measures on $C_T\mathbb{R}^d$ with mass on solution curves to (ODE). Therefore, the deterministic results are contained in the stochastic case. In this sense, for $a = 0$, (I'.2) contains the best known assumption for the deterministic case. We point out that a pure *local* integrability assumption is not sufficient for a superposition principle, since even for $a = \mathrm{Id}$, situations are known in which the martingale problem has a unique solution, while the corresponding FPK equation has multiple probability solutions, compare [215, Cor.10.1.2] and [38, Sect.9.2].

Very roughly, and leaving aside delicate technical difficulties, the idea of proof is similar to the deterministic case: For smooth and bounded coefficients, the superposition principle is a simple consequence of the classical well-posedness results for martingale problems and FPK equations. Starting from this base case, one considers solutions μ_t to equations with less regular coefficients and uses suitable approximations, similar to (but possibly distinct from) the deterministic case, in order to obtain a sequence of approximating coefficients and corresponding solutions $(b^\varepsilon, a^\varepsilon, \mu_t^\varepsilon)$. The main task is to choose these approximations such that the corresponding superposition measures $(\eta^\varepsilon)_{\varepsilon>0} \subseteq \mathcal{P}(C_T\mathbb{R}^d)$ are tight, and hence, have a limit point η, for which one shows (5.4). For a thorough presentation of this procedure, see [223, App.A].

Similar to the deterministic case, the superposition principle allows to transfer existence of solutions to FPK equations to existence of solutions to the martingale problem for irregular coefficients. Dual to this, uniqueness for the martingale problem (for a fixed initial value $\mu \in \mathcal{P}$) implies uniqueness for the FPK equation (with initial value μ). This transfer between the martingale (equivalently: SDE) and FPK level renders such superposition results a significant tool for the study of stochastic differential equations and FPK equations with low regularity coefficients, see (among others) [199, 235]. We point out that beyond this transfer of existence and uniqueness, the superposition principle is also used to develop a notion of *regular Lagrangian flow*, see [223]. However, we do not pursue this interesting direction in this thesis.

5.1.3 Further results

Here, we briefly mention further results in the spirit of the above mentioned superposition principles. In [198], the authors prove a superposition result for *nonlocal* FPK equations. Concerning the connection of nonlinear FPK equations and distribution-dependent SDEs (McKean–Vlasov equations), superposition-type results are obtained in [21, 22] (see also [200]). Interestingly, these results build on the aforementioned superposition results for linear FPK equations. We also mention the PhD-thesis [93], in which the author studies a superposition principle for linear and nonlinear equations on Hilbert spaces.

Finally, we point out that Ambrosio and Trevisan established a superposition principle for equations on $\mathbb{R}^\infty$ and on general metric measures spaces [13, Thm.7.1, Thm.7.6]. The former result is also obtained in [222], where also a result for stochastic equations on $\mathbb{R}^\infty$ is included, see Theorem 7.1. therein. The superposition principle on $\mathbb{R}^\infty$ will be of great importance within the proof of our main results, both in the deterministic and stochastic case.

5.2 Superposition principle for (stochastic) nonlinear FPK equations

Summarizing, the results recalled in the Euclidean situation is twofold, i.e. it consists of a *linearization* and *superposition*, which leads to the (somewhat formal) well-known equivalence (ODE) $\iff$ (CE):

- (ODE) $\implies$ (CE): The nonlinear differential equation is *linearized* to a linear equation for curves of measures on its state space in the sense that any solution to the former induces a solution to the latter.
- (CE) $\implies$ (ODE): Any solution to this linear equation for measures with suitable moderate global integrability is actually a *superposition* of solution curves to the differential equation.

The corresponding results in the stochastic case are summarized by a similar equivalence between (SDE) and (FPK).

5.2.1 Deterministic nonlinear FPK equations

The first objective of this part of the thesis is to replace the finite-dimensional equation (ODE) and the corresponding continuity equation (CE) by the nonlinear FPK equation

$$\partial_t \mu_t = \mathcal{L}^*_{t,\mu_t}\mu_t,\ \ t \in [0,T], \tag{NLFPK}$$

and its corresponding linear equation for measures on the space of measures $\mathcal{SP}$

$$\partial_t \Gamma_t = \mathbf{L}^*_t \Gamma_t,\ \ t \in [0,T], \tag{$\mathcal{SP}$-CE}$$

and to establish a superposition principle similar to the one in Subsection 5.1.1. Here, as already in Part I of this thesis, $\mathcal{L}^*_{t,\mu}$ denotes the formal dual of the second-order nonlinear differential operator

$$\mathcal{L}_{t,\mu}\varphi(x) = \sum_{i,j=1}^{d} a_{ij}(t,\mu,x)\partial_{ij}\varphi(x) + \sum_{i=1}^{d} b_i(t,\mu,x)\partial_i\varphi(x) \tag{5.5}$$

for Borel coefficients b and a on $[0,T]\times\mathcal{SP}\times\mathbb{R}^d$. We refer to Section 1.1 for a brief account of nonlinear FPK equations and relevant references. For reasons to be explained later, we consider vaguely continuous subprobability solutions to (NLFPK), i.e. the associated solutions to ($\mathcal{SP}$-CE) are curves in $\mathcal{P}(\mathcal{SP})$, see Definitions 6.1.1 and 6.2.4. In [190], the authors introduce the first-order linear operator $\mathbf{L}$, acting on suitable test functions $F : \mathcal{SP} \to \mathbb{R}$ via

$$\mathbf{L}_t F : \mu \mapsto \left\langle \nabla^{\mathcal{SP}} F, b_t + a_t \nabla \right\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)} \tag{5.6}$$

and show that any solution $t \mapsto \mu_t$ to (NLFPK) induces a curve of measures $t \mapsto \Gamma_t := \delta_{\mu_t} \in \mathcal{P}(\mathcal{SP})$, which solves ($\mathcal{SP}$-CE) in distributional sense. Here, $\nabla^{\mathcal{SP}}$ is a natural gradient on $\mathcal{SP}$, which is derived in [190, App.A] as the natural operator arising in the derivation of the equation solved by $t \mapsto \delta_{\mu_t}$, see Section 6.2 for details.

The analogy to the classical Euclidean case is that the ansatz for the derivation of ($\mathcal{SP}$-FPK) stems from considering (ODE) as an equation on the manifold $\mathbb{R}^d$, compare Appendix B and Subsection 6.2.2. Treating $\mathcal{SP}$ as a manifold-like space as well, one can mimic the derivation of the linearized equation associated to (ODE) in order to derive the linear equation associated to (NLFPK). With this viewpoint, one considers (NLFPK) as a nonlinear differential equation on $\mathcal{SP}$ (in distributional sense). This derivation is a nowadays well-known technique, see the pioneering works [5, 4] as well as [183], due to which ∇^{SP} is often called *Otto-gradient*, and also [2, 3, 168, 196].

Main result: Deterministic case. In view of the linearization (ODE) $\implies$ (CE) and the superposition principle (CE) $\implies$ (ODE) in the Euclidean case, and the linearization (NLFPK) $\implies$ ($\mathcal{SP}$-CE), it is a natural next step to develop a superposition principle ($\mathcal{SP}$-CE) $\implies$ (NLFPK). This is achieved by our first main result of the present part of the thesis, which states that each weakly continuous solution $(\Gamma_t)_{0\leq t\leq T}$ to ($\mathcal{SP}$-CE) with a natural global integrability property is a *superposition* of solutions to (NLFPK), i.e. (now denoting by π_t the canonical projection $\pi_t : (\mu_t)_{0\leq t\leq T} \mapsto \mu_t$ on $C_T\mathcal{SP}$)

$$\Gamma_t = \eta \circ \pi_t^{-1}, \quad t \in [0,T], \tag{5.7}$$

for some probability measure η concentrated on solution curves to (NLFPK). More precisely, we obtain the following analogue to the superposition principle of Subsection 5.1.1.

Theorem 5.2.1. *Let b and a be Borel coefficients on $[0,T]\times\mathcal{SP}\times\mathbb{R}^d$. Then, for any weakly continuous solution $t \mapsto \Gamma_t$ to ($\mathcal{SP}$-CE) with (6.10), there exists a probability measure $\eta \in \mathcal{P}(C_T\mathcal{SP})$, which is concentrated on vaguely continuous subprobability solutions to (NLFPK) such that (5.7) holds. Moreover, if $\Gamma_0 \in \mathcal{P}(\mathcal{P})$, then η is concentrated on weakly continuous probability solutions to (NLFPK). In particular, in this situation, we have $\Gamma_t \in \mathcal{P}(\mathcal{P})$ for each $t \in [0,T]$.*

We stress that no regularity of the coefficients is needed. As immediate corollaries, we obtain the transfer of existence and uniqueness results from ($\mathcal{SP}$-CE) to (NLFPK) and vice versa, respectively, see Subsection 6.2.2. Moreover, as a further application, in Proposition 6.4.3, we prove an open conjecture from [190]. We stress that the probability part of the result is important, because in connection to diffusion processes and stochastic analysis, one is typically interested in solution curves of probability measures to (NLFPK). Nevertheless, to us it seemed indispensable to develop our results for vaguely continuous subprobability solutions. We comment on the advantages of this approach in Remark 6.3.4 for the deterministic case and note that similar arguments apply also in the stochastic case.

Idea of proof. The proof of Theorem 5.2.1 proceeds along three steps. First, one introduces a differential equation and a continuity equation, which resemble (NLFPK) on $\mathbb{R}^\infty$ and ($\mathcal{SP}$-CE) on $\mathcal{P}(\mathbb{R}^\infty)$, respectively. To do so, we use a map $G : \mathcal{SP} \to \mathbb{R}^\infty$, which is a homeomorphism onto its image, see (6.5), in order to define a suitable vector field on $\mathbb{R}^\infty$, which gives rise to the continuity equation ($\mathbb{R}^\infty$-CE) for curves in $\mathcal{P}(\mathbb{R}^\infty)$. Heuristically, G may be considered a global chart for the manifold-like space $\mathcal{SP}$. Then, it is easy to show that any solution $t \mapsto \Gamma_t$ to ($\mathcal{SP}$-CE) induces a solution to the continuity equation on $\mathbb{R}^\infty$, via $\bar{\Gamma}_t := \Gamma_t \circ G^{-1}$.

Secondly, in this $\mathbb{R}^\infty$-framework, we use the superposition principle result [13, Thm.7.1] in order to lift $t \mapsto \bar{\Gamma}_t$ to a superposition solution $\bar{\eta} \in \mathcal{P}(C_T\mathbb{R}^\infty)$ of solution curves to ($\mathbb{R}^\infty$-ODE). It is crucial to observe that the compactness of $\mathcal{SP}$ with the vague topology yields closedness of $G(\mathcal{SP}) \subseteq \mathbb{R}^\infty$ and that hence $\bar{\eta}$ is concentrated on curves in $C_T G(\mathcal{SP})$.

Finally, we pull $\bar{\eta}$ back to a measure $\eta \in \mathcal{P}(C_T\mathcal{SP})$ and show that η is concentrated on solutions to (NLFPK) and fulfills (5.7), which concludes the proof. The final assertion for probability solutions is a simple consequence of the global integrability assumption (6.10).

5.2.2 Stochastic nonlinear FPK equations

In Chapter 7, we treat the case of stochastic nonlinear FPK equations of type

$$\partial_t \mu_t = \mathcal{L}^*_{t,\mu_t}\mu_t + \operatorname{div}(\sigma(t,\mu_t))dW_t, \ t \in [0,T], \tag{SNLFPK}$$

i.e. equations of type (NLFPK) perturbed by a bounded first-order noise coefficient $\sigma : [0,T] \times \mathcal{SP} \times \mathbb{R}^d \to \mathbb{R}^{d\times d_1}$ driven by a d_1-dimensional Brownian motion W. We consider solutions to (SNLFPK) as vaguely continuous stochastic processes with values in $\mathcal{SP}$.

Such equations appear naturally in the study of interacting particle systems with common noise and the corresponding McKean–Vlasov-equations. Here, we only give a very brief account on these interesting connections and refer to [69] for a more detailed presentation (see also [151]).

Let coefficients b, a and σ be given such that $\alpha = (2a - \sigma\sigma^T)^{1/2}$ is defined. Consider the system of N weakly interacting particles $X^{1,N}_t, \dots, X^{d,N}_t$ in $\mathbb{R}^d$, governed by the equations

$$\begin{cases} dX^{i,N}_t = b(t, X^{i,N}_t, L^N_t) + \alpha(t, X^{i,N}_t, L^N_t)dB^i_t + \sigma(t, X^{i,N}_t, L^N_t)dW_t, \\ X^{i,N}_0 = X^i_0, \end{cases} \tag{5.8}$$

where $(B^i)_{i\geq 1}$ and W are independent Brownian motions on the underlying probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, $(X^i_0)_{i\geq 1}$ is a sequence of iid random variables on Ω, and $L^N_t := \frac{1}{N}\sum_{1\leq i\leq N}\delta_{X^{i,N}_t}$ denotes the empirical (random) measure of the system. This set of stochastic equations describes particles, which weakly interact via their joint empirical measure L^N_t and are experiencing an individual stochastic perturbation B^i as well as a common noise W. On the one hand, under suitable assumptions on the coefficients, this system is related to the McKean–Vlasov equation

$$\begin{cases} dX_t = b\big(t, X_t, \mathcal{L}(X_t|W)\big)dt + \alpha\big(t, X_t, \mathcal{L}(X_t|W)\big)dB^i_t + \sigma\big(t, X_t, \mathcal{L}(X_t|W)\big)dW_t, \\ X_0 = X^i_0, \end{cases} \tag{5.9}$$

in the sense that for each $i \geq 1$, $X^{i,N}$ converges to a solution X^i_t to this equation as $N \to \infty$ [151]. Here, $\mathcal{L}(X_t|W)$ denotes the conditional distribution of X_t conditioned on the common noise W. At the same time, on the level of marginals, L^N_t converges to the conditional law $\mu_t := \mathcal{L}(X_t^1|W)$ [151], and, at least under suitable assumptions on the coefficients, $t \mapsto \mu_t$ solves (SNLFPK), see [69]. Hence, equations of type (SNLFPK) describe the time evolution of the particle limit distribution (conditioned on the common noise W) of the weakly interacting particle system (5.8) with common noise.

Linearization of (SNLFPK). In comparison to the stochastic Euclidean situation, (SNLFPK) replaces (SDE). In a first step, we linearize this equation, i.e. we find the suitable analogue to (FPK), which naturally turns out to be a second-order equation for curves in $\mathcal{P}(\mathcal{SP})$. Similarly to the deterministic case, one considers the derivation of the linearization (SDE) $\Longrightarrow$ (FPK) in manifold language and then mimics this procedure for solutions to (SNLFPK), compare Appendix B and Section 7.2. In contrast to the deterministic situation, here the linearized equation comprises a (deterministic) second-order term, which is of course due to the stochastic perturbation $\operatorname{div}(\sigma(t,\mu_t))dW_t$. To this end, in Section 7.2, we extend the heuristic consideration of $\mathcal{SP}$ as a manifold-like space by a natural (partial) notion of a Levi-Civita-type connection and a Hessian-like 0-2 tensor.

In this way, we find that the marginal curve $t \mapsto \Gamma_t = \mathbb{P} \circ \mu_t^{-1}$ of any vaguely continuous solution process $\mu : [0,T] \times \Omega \to \mu_t(\omega) \in \mathcal{SP}$ to (SNLFPK) on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ solves in distributional sense the linear second-order Fokker–Planck–Kolmogorov-type equation for curves in $\mathcal{P}(\mathcal{SP})$

$$\partial_t \Gamma_t = (\mathbf{L}_t^{(2)})^* \Gamma_t, \; t \in [0,T], \qquad (\mathcal{SP}\text{-FPK})$$

where the operator $\mathbf{L}_t^{(2)}$ is given by

$$(\mathbf{L}_t^{(2)} F)(\mu) := \mathbf{L}_t F(\mu) + \frac{1}{2}(\operatorname{Hess} F)\big(\sigma(t,\mu), \sigma(t,\mu)\big),$$

see (7.12), with $\mathbf{L}$ as in the deterministic case, i.e. as in ($\mathcal{SP}$-CE).

Main result: Superposition principle in the stochastic case. Our second main result is the following superposition principle ($\mathcal{SP}$-FPK) $\Longrightarrow$ (SNLFPK), which, in spirit, is comparable to (FPK) $\Longrightarrow$ (SDE) in the Euclidean situation. We assume σ to be bounded.

Theorem 5.2.2. *Let $t \mapsto \Gamma_t$ be a weakly continuous solution to ($\mathcal{SP}$-FPK) such that (7.15) holds. Then, there exists a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$, an d_1-dimensional $(\mathcal{F}_t)_{0 \le t \le T}$-Brownian motion $W = (W_t)_{0 \le t \le T}$ and an $\mathcal{SP}$-valued $(\mathcal{F}_t)_{0 \le t \le T}$-adapted vaguely continuous process $t \mapsto \mu_t$ such that (μ, W) solves (SNLFPK) and*

$$\mathbb{P} \circ \mu_t^{-1} = \Gamma_t, \quad t \in [0,T].$$

Moreover, if Γ_0 is concentrated on $\mathcal{P}$, i.e. $\Gamma_0(\mathcal{P}) = 1$, then the paths $t \mapsto \mu_t(\omega)$ are $\mathcal{P}$-valued and hence even weakly continuous.

Again, we point out that no regularity on b, a or σ is assumed, and that under our assumption of boundedness of σ, the global integrability condition on b, a is identical to the deterministic case. The formulation immediately gives the existence transfer ($\mathcal{SP}$-FPK) $\Longrightarrow$ (SNLFPK). Moreover, it also implies the transfer of (probabilistic) weak uniqueness of (SNLFPK) to uniqueness for ($\mathcal{SP}$-FPK), see Corollary 7.3.12.

In view of the results of [69], we make the following comparing remark. In [69], the authors prove existence and uniqueness to (5.9) under Lipschitz assumptions on the coefficients, see [69, Thm.3.3]. From these results, they deduce existence and uniqueness of solutions to (SNLFPK) [69, Thm.5.3,Thm.5.4]. In particular, the authors use the McKean–Vlasov equation (5.9) in order to solve (SNLFPK). In contrast, our main stochastic result, Theorem 5.2.2, is concerned with the relation between (SNLFPK) and its corresponding linearized second-order equation for curves in $\mathcal{P}(\mathcal{SP})$, which does not require any regularity assumptions on the coefficients. In other words, we approach the stochastic nonlinear FPK equation not from its associated McKean–Vlasov equation (which may be considered an equation with two degrees of nonlinearity), but from its associated linearized equation. This way, we transfer the equation to a much more complicated state space, namely $\mathcal{P}(\mathcal{SP})$, instead of $\mathbb{R}^d$ as for the McKean–Vlasov equation (5.9).

Remark on a similar result in [153]. A result similar to Theorem 5.2.2 was obtained completely independently of this work in [153, Thm.1.5] under the more restrictive assumption of L^p-integrability, $p > 1$, instead of L^1-integrability in (7.15) in our result. To us, there seems to be a gap in the proof of [153]. To the best of our understanding, in Step 4 of the proof of [153, Thm.1.5], it is not clear why one can map back the measure $Q \in \mathcal{P}(C_T\mathbb{R}^\infty)$ to a probability measure on $C_T\mathcal{P}$ without the detour via the space of subprobability measures. This becomes apparent even in the deterministic case, compare also with our Remark 6.3.4. In our proof of Theorems 5.2.1 and 5.2.2, this point is one of the crucial technical steps, see Step 3 of the proof of both theorems. Another point of distinction is the geometric approach to $\mathcal{SP}$ and (SNLFPK) as a second-order differential equation on $\mathcal{SP}$, which was initiated in [190] and extended in this part of the thesis, and which is not considered in [153]. From our perspective, this geometric approach helps to understand the true connection to the classical superposition principles for Euclidean ordinary and stochastic differential equations, as summarized in Chapter 5.

Idea of proof. The proof follows a pattern similar to the deterministic case. In particular, we proceed along a similar three step procedure. First, in order to handle the stochastic integral term in (SNLFPK), we consider a homeomorphism $H : \mathcal{SP} \to \ell^2$ instead of $G : \mathcal{SP} \to \mathbb{R}^\infty$ as in the deterministic case. By means of H, we derive suitable first- and second-order coefficients $\bar{B}$, $\bar{\Sigma}$ and $\bar{A}$ on ℓ^2, which give rise to a FPK-type equation (ℓ^2-FPK) on ℓ^2 and its corresponding martingale problem, see Definitions 7.3.1 and 7.3.3, respectively. The choice of $\bar{B}$ and $\bar{A}$ is made such that any solution $t \mapsto \Gamma_t$ to ($\mathcal{SP}$-FPK) induces a solution $t \mapsto \bar{\Gamma}_t$ to (ℓ^2-FPK) via $\bar{\Gamma}_t := \Gamma_t \circ H^{-1}$.

Secondly, we apply the superposition principle [222, Thm.7.1] in order to lift $t \mapsto \bar{\Gamma}_t$ to a solution $\bar{Q} \in \mathcal{P}(C_T\ell^2)$ to the corresponding martingale problem such that

$$\bar{Q} \circ (\pi_t^\infty)^{-1} = \bar{\Gamma}_t, \quad t \in [0, T],$$

where π_t^∞, $t \in [0, T]$, denote the canonical projections on $C_T\ell^2$. Comparable to the deterministic case, it follows that $\bar{Q}$ is concentrated on $C_TH(\mathcal{SP})$.

Thirdly, applying the representation result [181, Thm.2], we deduce the existence of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq T}, \mathbb{P})$, a d_1-dimensional $(\mathcal{F}_t)_{0\leq t\leq T}$-Brownian motion

W and a process $t \mapsto Y_t$ on Ω with law $\bar{Q}$, which fulfills (7.23). Consequently, Y has paths in $C_T H(\mathcal{SP})$, i.e. $Y = H \circ \mu$ for some vaguely continuous process $\mu : [0, T] \times \Omega \mapsto \mu_t(\omega) \in \mathcal{SP}$. From here, the definitions of $\bar{B}$ and $\bar{\Sigma}$ imply that (μ, W) is a solution to (SNLFPK) and

$$\mathbb{P} \circ \mu_t^{-1} = \Gamma_t, \quad t \in [0, T].$$

Organization of Part II. In Chapter 6, we consider the case of deterministic nonlinear equations. After discussing the setting and the notion of solution to such equations in Section 6.1, we repeat the geometric approach to the space $\mathcal{SP}$ as presented in [190] with minor changes in Section 6.2, and we transfer the equations of interest to associated equations on $\mathbb{R}^\infty$. These considerations enable us to prove the main result of Chapter 6, Theorem 5.2.1, in Section 6.3. We close this chapter with the discussion of immediate corollaries on the existence and uniqueness of solutions to (NLFPK) and ($\mathcal{SP}$-CE), and prove an open conjecture of [190] in Section 6.4.

In Chapter 7, we proceed similarly for the case of stochastic equations of type (SNLFPK). In the first section, we present the setting and the notion of solution. Then, in Section 7.2, we extend the geometric approach to $\mathcal{SP}$ from the deterministic case to second-order equations on $\mathcal{P}(\mathcal{SP})$, which we consider one of our central contributions of this part of the thesis. This way, in parallel to the deterministic case, we obtain a linearized equation for (SNLFPK) of second order on $\mathcal{P}(\mathcal{SP})$. Afterwards, we prove the main result of the stochastic case, i.e. Theorem 5.2.2, in Section 7.3.

Appendix B contains a brief repetition of the well-known derivation of the corresponding continuity and FPK equation from its ordinary and stochastic differential equation, respectively.

Chapter 6

Superposition principle for deterministic nonlinear FPK equations

In this and the next chapter, for a measurable function $f : \mathcal{X} \to \mathbb{R}$ on a measure space $(\mathcal{X}, \mathcal{F}, \mu)$, we write $\mu(f) := \int f d\mu$, whenever the integral is defined.

6.1 Nonlinear FPK equations

Throughout, we consider the compact time interval $[0, T]$ for $T > 0$. The case $T = \infty$ can be treated analogously. Consider coefficients $a = (a_{ij})_{1 \le i,j \le d}$ and $b = (b_i)_{1 \le i \le d}$ with $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathcal{SP}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable coefficients

$$a_{ij} : [0, T] \times \mathcal{SP} \times \mathbb{R}^d \to \mathbb{R}, \quad b_i : [0, T] \times \mathcal{SP} \times \mathbb{R}^d \to \mathbb{R},$$

such that a takes values in $\mathbb{S}_d^+$. For the operator $\mathcal{L}_{t,\mu}$ defined in (5.5), we study the deterministic nonlinear FPK equation (NLFPK) in the following sense.

Definition 6.1.1. (i) A vaguely continuous curve $t \mapsto \mu_t \in \mathcal{SP}$ is a *subprobability solution to* (NLFPK), if for each $1 \le i, j \le d$ and each compact set $K \subseteq \mathbb{R}^d$, the local integrability condition

$$\int_0^T \int_K |a_{ij}(t, \mu_t, x)| + |b_i(t, \mu_t, x)| d\mu_t(x) dt < \infty \tag{6.1}$$

holds and for each $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t \in [0, T]$, we have

$$\int_{\mathbb{R}^d} \varphi(x) d\mu_t(x) - \int_{\mathbb{R}^d} \varphi(x) d\mu_0(x) = \int_0^t \int_{\mathbb{R}^d} \mathcal{L}_{s,\mu_s} \varphi(x) d\mu_s(x) ds. \tag{6.2}$$

(ii) A *probability solution* to (NLFPK) is a curve $t \mapsto \mu_t$ as in (i) with $\mu_t \in \mathcal{P}$ for each $t \in [0, T]$. In this case, μ_t is weakly continuous.

The appearing integrals in the above definition are well-defined, since vaguely continuous curves of measures are in particular Borel curves. Clearly, by approximation, the validity of (6.2) immediately extends to each $\varphi \in C_c^2(\mathbb{R}^d)$.

As already mentioned in Part I, there are more general notions of solutions to (NLFPK), such as (discontinuous) curves of signed, bounded measures [38]. However, here we restrict our attention to continuous (sub-)probability solutions.

For large parts of the following presentation, we will consider the global in space integrability condition for subprobability solutions $t \mapsto \mu_t$

$$\int_0^T \int_{\mathbb{R}^d} |a_{ij}(t, \mu_t, x)| + |b_i(t, \mu_t, x)| d\mu_t(x) dt < \infty, \quad 1 \le i, j \le d. \tag{6.3}$$

Remark 6.1.2. *(i) Under the global assumption* (6.3)*, any subprobability solution $t \mapsto \mu_t$ to* (NLFPK) *with $\mu_0 \in \mathcal{P}$ is a probability solution. Indeed, to prove this, it suffices to show $\mu_t(\mathbb{R}^d) = 1$ for each $0 \le t \le T$. In view of* (6.2)*, it suffices to choose a sequence φ_l, $l \ge 1$, from $C_c^2(\mathbb{R}^d)$ with the following properties: $0 \le \varphi_l \nearrow 1$ pointwise such that $\partial_i \varphi_l \longrightarrow 0$, $\partial_{ij} \varphi_l \longrightarrow 0$ pointwise as $l \to \infty$ with all first and second-order derivatives bounded by some $M < \infty$ uniformly in $l \ge 1$ and $x \in \mathbb{R}^d$. Considering (6.2) for the limit $l \to \infty$, we obtain, by (6.3) and dominated convergence, for each $t \in [0, T]$*

$$\int_{\mathbb{R}^d} 1 d\mu_t - \int_{\mathbb{R}^d} 1 d\mu_0 = 0,$$

which gives the claim.

(ii) By such approximations, the validity of (6.2) *can be extended to each* $\varphi \in C_b^2(\mathbb{R}^d)$.

6.2 Geometry on $\mathcal{SP}$

As mentioned in the introduction in Chapter 5, we consider $\mathcal{SP}$ as a manifold-like infinite-dimensional space. For the present case of the deterministic equation (NLFPK), we essentially follow the approach of [190, App.A], where the authors derive a natural gradient for suitable test functions $F : \mathcal{P} \to \mathbb{R}$ as a section in the tangent bundle $\bigsqcup_{\mu \in \mathcal{P}} T_\mu \mathcal{P}$ with tangent spaces $T_\mu \mathcal{P} = L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$. Such a program is well-known, for example in the case when $\mathcal{P}$ is replaced by the $\mathbb{N}$-valued Radon measures on a Riemannian manifold, see [5, 4, 2, 3, 168, 196, 183]. In this section, we repeat the derivation in [190, App.A] with minor adjustments to our case of subprobability measures and the transfer of (NLFPK) to $\mathbb{R}^\infty$, which we discuss later. Let

$$\mathcal{G} = \{g_i, i \in \mathbb{N}\} \subseteq C_c^2(\mathbb{R}^d)$$

be dense with respect to $||\cdot||_{C^2}$ such that $g_i \neq 0$ for any $i \in \mathbb{N}$. Clearly, any such set of functions is measure separating and dense in $C_c(\mathbb{R}^d)$ with respect to uniform convergence. We point out the following simple, but important properties of such sets $\mathcal{G}$.

Lemma 6.2.1. *Let* $\mathcal{G} \subseteq C_c^2(\mathbb{R}^d)$ *be as above. Then,*

(i) $(\mu_n)_{n \geq 1} \subseteq \mathcal{SP}$ *converges vaguely to* $\mu \in \mathcal{SP}$ *if and only if* $\mu_n(g_i) \underset{n\to\infty}{\longrightarrow} \mu(g_i)$ *for each* $g_i \in \mathcal{G}$.

(ii) A vaguely continuous curve $t \mapsto \mu_t$*, which fulfills (6.1), is a subprobability solution to (NLFPK) if and only if* (6.2) *holds for each* $g_i \in \mathcal{G}$ *in place of* φ.

Proof. (i) From $\mu_n(g_i) \underset{n\to\infty}{\longrightarrow} \mu(g_i)$ for each $g_i \in \mathcal{G}$, one obtains for each $f \in C_c(\mathbb{R}^d)$ and $\varepsilon > 0$, by choosing $g_i \in \mathcal{G}$ with $||f - g_i||_\infty < \frac{\varepsilon}{3}$,

$$|\mu_n(f) - \mu(f)| \leq |\mu_n(f) - \mu_n(g_i)| + |\mu_n(g_i) - \mu(g_i)| + |\mu(g_i) - \mu(f)| \leq \varepsilon \qquad (6.4)$$

for all sufficiently large $n \geq 1$, where we used $\mu_n(\mathbb{R}^d), \mu(\mathbb{R}^d) \leq 1$.

(ii) Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be approximated uniformly up to second-order derivatives by a sequence $\{g_{i_k}\}_{k \geq 1}$ from $\mathcal{G}$ with $\operatorname{supp}\varphi, \operatorname{supp} g_{i_k} \subseteq K$ for some compact $K \subseteq \mathbb{R}^d$. Considering (6.2) for such g_{i_k} and letting $k \to \infty$, the result follows by dominated convergence, which applies due to (6.1) and the choice of g_{i_k}.

□

We endow $\mathbb{R}^\infty$ with its usual product topology, i.e. with the topology of pointwise convergence, which renders $\mathbb{R}^\infty$ a Polish space. For $\mathcal{G}$ as above, we consider the map

$$G : \mathcal{SP} \to \mathbb{R}^\infty, \quad G : \mu \mapsto \big(\mu(g_i)\big)_{i \in \mathbb{N}}. \qquad (6.5)$$

In view of the approach to $\mathcal{SP}$ as a manifold-like space, the forthcoming lemma yields that G may formally be considered a global chart for $\mathcal{SP}$. We consider $G(\mathcal{SP}) \subseteq \mathbb{R}^\infty$ with the natural subspace topology inherited from $\mathbb{R}^\infty$ and, in accordance with our general notation, write $C_TG(\mathcal{SP})$ for the set of elements in $C_T\mathbb{R}^\infty$ with values in $G(\mathcal{SP})$.

We use the notation π_t and π_t^∞ for the canonical projections at time t on $C_T\mathcal{SP}$ and $C_T\mathbb{R}^\infty$, respectively, and endow both spaces with the respective topology of uniform convergence. In particular, their Borel σ-algebras are given by

$$\mathcal{B}(C_T\mathcal{SP}) = \sigma(\pi_t, t \in [0,T]) \text{ and } \mathcal{B}(C_T\mathbb{R}^\infty) = \sigma(\pi_t^\infty, t \in [0,T]),$$

respectively. Indeed, the Borel σ-algebra of C_TX with respect to the topology of uniform convergence coincides with the σ-algebra generated by the canonical projections whenever the metric space X is separable. Moreover, we endow $C_TG(\mathcal{SP})$ with the natural subspace σ-algebra of $\mathcal{B}(C_T\mathbb{R}^\infty)$.

Lemma 6.2.2. *Let $\mathcal{G} = \{g_i\}_{i\geq 1} \subseteq C_c^2(\mathbb{R}^d)$ and G be as above.*

(i) G is a homeomorphism between $\mathcal{SP}$ and $G(\mathcal{SP})$ (hence, formally, a global chart *for $\mathcal{SP}$). In particular, $G(\mathcal{SP}) \subseteq \mathbb{R}^\infty$ is compact. Moreover, if $\mathcal{G}' = \{g_i', i \geq 1\} \subseteq C_c^2(\mathbb{R}^d)$ is another set with the same properties as $\mathcal{G}$ with corresponding map $G' : \mathcal{SP} \to \mathbb{R}^\infty$, then $G' = \mathcal{V} \circ G$ for a unique homeomorphism $\mathcal{V} : G(\mathcal{SP}) \to G'(\mathcal{SP})$.*

(ii) The map

$$J : C_T\mathcal{SP} \to C_T\mathbb{R}^\infty,\ J : (\mu_t)_{0\leq t\leq T} \mapsto \big(G(\mu_t)\big)_{0\leq t\leq T}$$

is measurable and one-to-one with measurable inverse $J^{-1} : C_TG(\mathcal{SP}) \to C_T\mathcal{SP}$. Furthermore, $C_TG(\mathcal{SP}) \subseteq C_T\mathbb{R}^\infty$ is a Borel measurable set.

Proof. (i) The continuity of G is obvious by definition of the vague topology on $\mathcal{SP}$ and since $\mathcal{G} \subseteq C_c(\mathbb{R}^d)$. Since $\mathcal{SP}$ with the vague topology is compact, compactness of $G(\mathcal{SP}) \subseteq \mathbb{R}^\infty$ follows. $\mathcal{G}$ is measure separating on $\mathbb{R}^d$, which implies that G is one-to-one. Since by definition

$$G(\mu_n) \underset{n\to\infty}{\longrightarrow} G(\mu) \text{ in } \mathbb{R}^\infty \iff \mu_n(g_i) \underset{n\to\infty}{\longrightarrow} \mu(g_i) \text{ for each } g_i \in \mathcal{G},$$

continuity of G^{-1} holds due to Lemma 6.2.1 (i). The final claim follows, since for G' as in the assertion, $\mathcal{V} : (\mu(g_i))_{i\in\mathbb{N}} \mapsto (\mu(g_i'))_{i\in\mathbb{N}}$ is a homeomorphism.

(ii) Since G is one-to-one and measurable, so is J. Clearly, $C_TG(\mathcal{SP})$ is the range of J and hence $J : C_T\mathcal{SP} \to C_TG(\mathcal{SP})$ is a bijection between standard Borel spaces (because $\mathcal{SP}$ and $G(\mathcal{SP})$ with the respective topologies are Polish). This yields the measurability of J^{-1}. Finally, closedness of $G(\mathcal{SP}) \subseteq \mathbb{R}^\infty$ implies that $C_TG(\mathcal{SP}) \subseteq C_T\mathbb{R}^\infty$ is a measurable set, because $G(\mathcal{SP})$ carries the subspace topology inherited from $\mathbb{R}^\infty$.

□

In particular, due to the last assertion of part (i) of the above lemma, it is justified from now on to consider a fixed set $\mathcal{G}$ as above with its corresponding homeomorphism G as in (6.5).

6.2.1 Tangent spaces, test functions and gradient on $\mathcal{SP}$

The motivation for the definition of $\nabla^{\mathcal{SP}}$ stems from the classical case of a d-dimensional Riemannian manifold (M,g). Recall that in this case, the tangent space at $x \in M$, T_xM, consists of the "directions" of smooth curves $\gamma^x : [0,1] \to M$ passing through x, $\gamma^x(0) = x$. Moreover, for a smooth function $F : M \to \mathbb{R}$, the gradient $\nabla F : M \to \bigsqcup_{x\in M} T_xM$ is the unique smooth section in the tangent bundle $\bigsqcup_{x\in M} T_xM$ such that for each $x \in M$, $\nabla F(x)$ represents the cotangent element dF_x with respect to the scalar product $\langle \cdot,\cdot\rangle_{g(x)}$ on T_xM, i.e. $\langle \nabla F(x), \xi\rangle_{g(x)} = dF_x(\xi)$ for each $\xi \in T_xM$. See Appendix F for more details.

In the present case, we do not consider $\mathcal{SP}$ as a (Fréchet) manifold in a rigorous way, i.e. no tangent spaces $T_\mu\mathcal{SP}$ are given a priori. Instead, one first of all chooses a suitable set of curves $\gamma : [0,1] \to \mathcal{SP}$, which represent directions on $\mathcal{SP}$. It turns out that a suitable class is given by

$$\gamma^\mu_\Phi(t) := \mu \circ \big(\mathrm{id} + t\Phi\big)^{-1}, \quad \mu \in \mathcal{SP},\ \Phi \in L^2(\mathbb{R}^d, \mathbb{R}^d; \mu),\ t \in [0,1],$$

i.e. the $\mathcal{SP}$-valued curve γ^μ_Φ starts at μ in the direction of the vector field Φ. In other words, the set of directions at the point μ is parametrized by $L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$. Consequently, we define for each $\mu \in \mathcal{SP}$

$$T_\mu\mathcal{SP} := L^2(\mathbb{R}^d, \mathbb{R}^d; \mu).$$

Before we proceed to the definition of $\nabla^{\mathcal{SP}}$, it is necessary to select a suitably large class of test functions $F : \mathcal{SP} \to \mathbb{R}$, which are differentiable along γ^μ_Φ. Similarly to [190], we choose the test function class

$$\mathcal{F}C^2_b(\mathcal{G}) := \{F : \mathcal{SP} \to \mathbb{R} |\, F : \mu \mapsto f\big(\mu(g_1), \dots, \mu(g_n)\big), f \in C^2_b(\mathbb{R}^n), n \geq 1\}.$$

In comparison with [190], here we restrict the set of inner test functions from $C^2_c(\mathbb{R}^d)$ to $\{g_i, i \in \mathbb{N}\} = \mathcal{G} \subseteq C^2_c(\mathbb{R}^d)$ and the set of outer functions from $C^1_b(\mathbb{R}^d)$ to $C^2_b(\mathbb{R}^d)$. The former restriction is necessary in order to transfer test functions F from $\mathcal{F}C^2_b(\mathcal{G})$ to test functions $\bar{F} : \mathbb{R}^\infty \to \mathbb{R}$ via the homeomorphism G later on, while the latter choice is only made for consistency with the stochastic case in Chapter 7, where we need test functions with second-order differentiability. However, since $C^2_b(\mathbb{R}^d)$ is dense in $C^1_b(\mathbb{R}^d)$ with respect to locally uniform convergence, and due to the choice of $\mathcal{G}$, neither of these choices yields an essential restriction of the class of test functions, see Remark 6.2.5 (ii).

Based on this choice of tangent bundle $\bigsqcup_{\mu\in\mathcal{SP}} T_\mu\mathcal{SP}$ and test function class $\mathcal{F}C^2_b(\mathcal{G})$, analogously to [190], the ansatz for a suitable notion of the gradient $\nabla^{\mathcal{SP}}$ is that it should fulfill the characterizing equality

$$\Big(\text{"}dF_\mu(\Phi)\text{"} = \Big)\frac{\mathrm{d}}{\mathrm{d}t}F\big(\gamma^\mu_\Phi(t)\big)_{|t=0} = \langle \nabla^{\mathcal{SP}}F(\mu), \Phi\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)} \tag{6.6}$$

for any $\Phi \in T_\mu\mathcal{SP} = L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$, $F \in \mathcal{F}C^2_b(\mathcal{G})$ and $\mu \in \mathcal{SP}$. To this end, let $F \in \mathcal{F}C^2_b(\mathcal{G})$ have the representation

$$F(\mu) = f\big(\mu(g_1), \dots, \mu(g_n)\big) \tag{6.7}$$

and observe

$$\frac{\mathrm{d}}{\mathrm{d}t}F\big(\gamma^{\mu}_{\Phi}(t)\big)_{|t=0} = \sum_{k=1}^{n}(\partial_k f)\big(\mu(g_1),\dots,\mu(g_n)\big)\mu(\nabla g_k\cdot\Phi)$$
$$= \big\langle \sum_{i=1}^{n}(\partial_k f)\big(\mu(g_1),\dots,\mu(g_n)\big)\nabla g_k, \Phi\big\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)},$$

where we used $\frac{\mathrm{d}}{\mathrm{d}t}\gamma^{\mu}_{\Phi}(t)(g)_{|t=0} = \mu\big(\nabla g(\mathrm{id}+t\Phi)\cdot\Phi\big)_{|t=0} = \mu(\nabla g\cdot\Phi)$ for the first equality. Consequently, defining $\nabla^{\mathcal{SP}}F(\mu)\in T_\mu\mathcal{SP}$ as

$$\nabla^{\mathcal{SP}}F(\mu) := \sum_{k=1}^{n}(\partial_k f)\big(\mu(g_1),\dots,\mu(g_n)\big)\nabla g_k,$$

we have (6.6). In particular, $\nabla^{\mathcal{SP}}F$ is independent of the particular representation of F in (6.7). For later use, we state the following observation.

Remark 6.2.3. *Since $\nabla^{\mathcal{SP}}F$ is bounded on $\mathcal{SP}\times\mathbb{R}^d$, the map $\Phi\mapsto\big\langle\nabla^{\mathcal{SP}}F(\mu),\Phi\big\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)}$ extends to $L^1(\mathbb{R}^d,\mathbb{R}^d;\mu)$ for any $\mu\in\mathcal{SP}$. Hence, in the sequel, for $\Phi\in L^1(\mathbb{R}^d,\mathbb{R}^d;\mu)$ we slightly abuse notation and write*

$$\int_{\mathbb{R}^d}\nabla^{\mathcal{SP}}F(\mu)\cdot\Phi d\mu =: \langle\nabla^{\mathcal{SP}}F(\mu),\Phi\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)}.$$

6.2.2 Linearization of (NLFPK)

Having derived a natural definition of the gradient $\nabla^{\mathcal{SP}}$ for test functions $F\in\mathcal{F}C^2_b(\mathcal{G})$ on $\mathcal{SP}$, we recall the derivation of the continuity equation associated to (NLFPK) obtained in [190]. Interpreting (NLFPK) as a differential equation on $\mathcal{SP}$, the procedure resembles the finite-dimensional Euclidean case, see Appendix B. If $t\mapsto\mu_t$ is a subprobability solution to (NLFPK), we obtain for any $F\in\mathcal{F}C^2_b(\mathcal{G}), F(\mu) = f\big(\mu(g_1),\dots,\mu(g_n)\big)$ the calculation

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\mu_t) = \sum_{k=1}^{n}(\partial_k f)\big(\mu(g_1),\dots,\mu(g_n)\big)\partial_t\mu_t(g_k)$$
$$= \sum_{k=1}^{n}(\partial_k f)\big(\mu(g_1),\dots,\mu(g_n)\big)\int_{\mathbb{R}^d}\mathcal{L}_{t,\mu_t}g_k d\mu_t$$
$$= \big\langle\sum_{k=1}^{n}(\partial_k f)\big(\mu(g_1),\dots,\mu(g_n)\big)\nabla g_k, a(t,\mu_t,\cdot)\nabla + b(t,\mu_t,\cdot)\big\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu_t)}$$
$$= \big\langle\nabla^{\mathcal{SP}}F(\mu_t), a(t,\mu_t,\cdot)\nabla + b(t,\mu_t,\cdot)\big\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu_t)},$$

where the meaning of the formal vector field $a(t,\mu,\cdot)\nabla + b(t,\mu,\cdot)$ is given by the third equality above. Setting $\Gamma_t := \delta_{\mu_t}\in\mathcal{P}(\mathcal{SP})$ and integrating dt over $[0,t]$ with dt leads to

$$\int_{\mathcal{SP}}F(\mu)d\Gamma_t(\mu) - \int_{\mathcal{SP}}F(\mu)\Gamma_0(\mu)$$
$$= \int_0^t\int_{\mathcal{SP}}\langle\nabla^{\mathcal{SP}}F(\mu_s), a(s,\mu_s,\cdot)\nabla + b(s,\mu_s,\cdot)\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)}d\Gamma_s(\mu)ds.$$

Consequently, the weakly continuous curve $t \mapsto \Gamma_t \in \mathcal{P}(\mathcal{SP})$ is a distributional solution to the linear first-order continuity-type equation

$$\partial_t \Gamma_t = -\nabla^{\mathcal{SP}} \cdot ([b(t) + a(t)\nabla]\Gamma_t), \quad 0 \leq t \leq T, \tag{6.8}$$

in duality with test functions $F \in \mathcal{F}C_b^2(\mathcal{G})$. Here, we abbreviated $b(t) := \mu \mapsto b(t,\mu) := b(t,\mu,\cdot)$ and, analogously, write $a(t)$. We introduce the time-dependent operator $\mathbf{L}$, acting on $F \in \mathcal{F}C_b^2(\mathcal{G})$ via

$$(\mathbf{L}_t F)(\mu) := \big\langle \nabla^{\mathcal{SP}} F(\mu), a(t,\mu)\nabla + b(t,\mu) \big\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)}, \quad \mu \in \mathcal{SP}. \tag{6.9}$$

With this notation, the continuity equation (6.8) is the desired linearized equation ($\mathcal{SP}$-CE) of (NLFPK) as mentioned in the introduction in Chapter 5.

In terms of the interpretation of $\mathcal{SP}$ as manifold-like space, the formal vector field $a(t)\nabla + b(t)$ (which has only rigorous meaning in its own right, if the spatial regularity of $a(t)$ allows to put the first-order term ∇ on $a(t)$ via integration by parts, which we do not assume at any point) can be considered a time-dependent section in the tangent bundle $\bigsqcup_{\mu \in \mathcal{SP}} T_\mu \mathcal{SP}$.

More generally, we define solutions to (6.8) (that is, to ($\mathcal{SP}$-CE)) as follows.

Definition 6.2.4. A weakly continuous curve $t \mapsto \Gamma_t \in \mathcal{P}(\mathcal{SP})$ is a *solution to* (6.8), if the global integrability condition

$$\int_0^T \int_{\mathcal{SP}} ||b(t,\mu,\cdot)||_{L^1(\mathbb{R}^d,\mathbb{R}^d;\mu)} + ||a(t,\mu,\cdot)||_{L^1(\mathbb{R}^d,\mathbb{R}^{d^2};\mu)} d\Gamma_t(\mu)dt < \infty \tag{6.10}$$

is fulfilled and for each $F \in \mathcal{F}C_b^2(\mathcal{G})$ and $t \in [0,T]$, we have

$$\int_{\mathcal{SP}} F(\mu)d\Gamma_t(\mu) - \int_{\mathcal{SP}} F(\mu)d\Gamma_0(\mu) = \int_0^t \int_{\mathcal{SP}} \mathbf{L}_s F(\mu) d\Gamma_s(\mu)ds. \tag{6.11}$$

We make the following observations.

Remark 6.2.5. *(i) In order to make sense of the integrals in* (6.11)*, it is sufficient to require the local (in space) condition*

$$\int_0^T \int_{\mathcal{SP}} ||b(t,\mu,\cdot)||_{L^1(K,\mathbb{R}^d;\mu)} + ||a(t,\mu,\cdot)||_{L^1(K,\mathbb{R}^{d^2};\mu)} d\Gamma_t(\mu)dt < \infty$$

for each compact set $K \subseteq \mathbb{R}^d$. Indeed, each inner test function g_k in the representation of $F : \mu \mapsto f\big(\mu(g_1),\dots,\mu(g_n)\big)$ is compactly supported. However, we need the global condition (6.10) *in our main result Theorem 5.2.1 and therefore include it directly in Definition 6.2.4.*

(ii) The choice of $\mathcal{G}$ implies that the validity of (6.11) *can be extended to each test function F of type $F : \mu \mapsto f\big(\mu(h_1),\dots,\mu(h_n)\big)$ for $n \in \mathbb{N}$, arbitrary $h_k \in C_c^2(\mathbb{R}^d)$ and $f \in C_b^1(\mathbb{R}^n)$. Indeed, let $\{g_l^k\} \subseteq \mathcal{G}$ be such that $g_l^k \longrightarrow h_k$ with respect to $||\cdot||_{C^2}$*

as $l \to \infty$ and consider $\mathcal{F}C_b^2(\mathcal{G}) \ni F_l : \mu \mapsto f\big(\mu(g_l^1), \dots, \mu(g_l^n)\big)$. Clearly, we have for each $0 \leq t \leq T$

$$\int_{\mathcal{SP}} F_l \, d\Gamma_t \underset{l\to\infty}{\longrightarrow} \int_{\mathcal{SP}} F \, d\Gamma_t$$

and

$$\mathbf{L}_t F_l(\mu) \underset{l\to\infty}{\longrightarrow} \mathbf{L}_t F(\mu)$$

pointwise in (t, μ). Since $\{g_l^k : k \leq n, l \geq 1\}$ is uniformly bounded with respect to $||\cdot||_{C^2}$, considering (6.11) for F_l, in the limit $l \to \infty$ we obtain the validity of (6.11) for F. In particular, the solution notion of Definition (6.2.4) is independent of the choice of $\mathcal{G}$.

In terms of the transfer of existence and uniqueness statements, we summarize the considerations up to this point as follows. If there exists a subprobability solution $t \mapsto \mu_t$ to (NLFPK) with initial value $\mu \in \mathcal{SP}$, which fulfills the global assumption (6.3), then there also exists a solution $t \mapsto \Gamma_t$ to ($\mathcal{SP}$-CE) (i.e. to (6.8)) in the sense of Definition 6.2.4 with initial value $\Gamma_0 = \delta_\mu$, given by $\Gamma_t := \delta_{\mu_t}$. Vice versa, if solutions to ($\mathcal{SP}$-CE) with initial value $\Gamma_0 = \delta_\mu$ are unique, then also solutions to subprobability solutions to (NLFPK) with initial condition μ (and the global integrability condition (6.3)) are unique.

The respective reversed existence and uniqueness results follow as corollaries of the main result of this chapter, see Subsection 6.4.1.

6.3 Proof of main result

We turn to the proof of Theorem 5.2.1. As a preparation, we transfer the equations (NLFPK) and ($\mathcal{SP}$-CE) to corresponding equations on $\mathbb{R}^\infty$.

6.3.1 Transfer to $\mathbb{R}^\infty$

We use the (formal) global chart $G : \mathcal{SP} \to \mathbb{R}^\infty$ and the map J of Lemma 6.2.2 to obtain a differential equation and a continuity equation on $\mathbb{R}^\infty$, which are closely related to (NLFPK) and ($\mathcal{SP}$-CE), respectively. To this end, we define a Borel vector field $\bar{B} = (\bar{B}_k)_{k\in\mathbb{N}}$ on $\mathbb{R}^\infty$ component-wise as follows. For $t \in [0, T]$, consider the Borel set $A_t \in \mathcal{B}(\mathcal{SP})$,

$$A_t := \left\{\mu \in \mathcal{SP} : \int_{\mathbb{R}^d} |a_{ij}(t, \mu, x)| + |b_i(t, \mu, x)| d\mu(x) < \infty \; \forall i, j \leq d\right\}$$

and set $B := (B_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty$ via

$$B_k(t, \mu) := \int_{\mathbb{R}^d} \mathcal{L}_{t,\mu} g_k(x) d\mu(x), \quad (t, \mu) \in [0, T] \times A_t. \tag{6.12}$$

Now define $\bar{B} : [0, T] \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ via

$$\bar{B}(t, z) := \begin{cases} B(t, G^{-1}(z)) & \text{, if } z \in G(A_t) \\ 0 & \text{, else} \end{cases},$$

which is Borel measurable by Lemma 6.2.2. Consider the differential equation for curves $t \mapsto z_t \in \mathbb{R}^\infty$ and the continuity equation for curves $t \mapsto \bar{\Gamma}_t$ in $\mathcal{P}(\mathbb{R}^\infty)$ induced by this vector field, i.e.

$$\dot{z}_t = \bar{B}(t, z_t),\ t \in [0, T], \tag{$\mathbb{R}^\infty$-ODE}$$

and

$$\partial_t \bar{\Gamma}_t = -\bar{\nabla} \cdot (\bar{B}\bar{\Gamma}_t),\ t \in [0, T], \tag{$\mathbb{R}^\infty$-CE}$$

with $\bar{\nabla}$ as in (6.13). These equations can roughly be understood as (NLFPK) and ($\mathcal{SP}$-CE) transferred from $\mathcal{SP}$ to $\mathbb{R}^\infty$ via the chart G. Let

$$p_i : \mathbb{R}^\infty \to \mathbb{R}, \quad p_i\big((z_i)_{i\in\mathbb{N}}\big) := z_i$$

denote the canonical projection to the i-th component and set $\pi^{(n)} := (p_1, \dots, p_n)$. In analogy to the test function class $\mathcal{F}C_b^2(\mathcal{G})$ on $\mathcal{SP}$, here we consider the class

$$\mathcal{F}C_b^2(\mathbb{R}^\infty) := \{\bar{F} : \mathbb{R}^\infty \to \mathbb{R} | \bar{F} = f \circ \pi^{(n)}, f \in C_b^2(\mathbb{R}^n), n \geq 1\}.$$

As for $\mathcal{F}C_b^2(\mathcal{G})$, the restriction to test functions possessing second-order derivatives is only made for consistency with the stochastic (second-order) case in Chapter 7. By $\bar{\nabla}$ we denote the gradient-type operator on $\mathbb{R}^\infty$, which acts on $\bar{F} = f \circ \pi^{(n)} \in \mathcal{F}C_b^2(\mathbb{R}^\infty)$ via

$$\bar{\nabla}\bar{F}(z) := \big((\partial_1 f)(\pi^{(n)}z), \dots, (\partial_n f)(\pi^{(n)}z), 0, 0, \dots\big). \tag{6.13}$$

The notion of solution to equations ($\mathbb{R}^\infty$-ODE) and ($\mathbb{R}^\infty$-CE) is as follows. Recall that we abuse notation for the standard Euclidean inner product and write $x \cdot y = \sum_{k\geq 1} x_k y_k$ also in the case $x, y \in \mathbb{R}^\infty$, if either x or y contain only finitely many nontrivial summands.

Definition 6.3.1. (i) A continuous curve $t \mapsto z_t \in \mathbb{R}^\infty$ is a *solution to* ($\mathbb{R}^\infty$-ODE), if for each $i \in \mathbb{N}$, the real curve $t \mapsto p_i \circ z_t$ is absolutely continuous with weak derivative $t \mapsto p_i \circ \bar{B}(t, z_t)$ dt-a.s.

(ii) A weakly continuous curve $t \mapsto \bar{\Gamma}_t \in \mathcal{P}(\mathbb{R}^\infty)$ is a *solution to* ($\mathbb{R}^\infty$-CE), if it fulfills the global integrability condition

$$\int_0^T \int_{\mathbb{R}^\infty} |\bar{B}_k(t, z)| d\bar{\Gamma}_t(z) dt < \infty, \quad k \in \mathbb{N}, \tag{6.14}$$

and for each $\bar{F} \in \mathcal{F}C_b^2(\mathbb{R}^\infty)$ the identity

$$\int_{\mathbb{R}^\infty} \bar{F}(z) d\bar{\Gamma}_t(z) - \int_{\mathbb{R}^\infty} \bar{F}(z) d\bar{\Gamma}_0(z) = \int_0^t \int_{\mathbb{R}^\infty} \bar{\nabla}\bar{F}(z) \cdot \bar{B}(s, z) d\bar{\Gamma}_s(z) ds \tag{6.15}$$

holds for all $t \in [0, T]$.

Remark 6.3.2. *It is clear that the validity of* (6.15) *extends to each* $\bar{F} \in \mathcal{F}C_b^1(\mathbb{R}^\infty)$*, i.e. to functions of type* $\bar{F} = f \circ \pi^{(n)}$ *for* $f \in C_b^1(\mathbb{R}^n)$*. Indeed, for this it suffices to approximate* $f \in C_b^1(\mathbb{R}^n)$ *pointwise by a sequence* $f_n \in C_b^2(\mathbb{R}^n)$ *and to consider* (6.15) *for* $\bar{F}_n = f_n \circ \pi^{(n)}$ *as* $n \to \infty$*.*

6.3.2 Proof of Theorem 5.2.1

The proof follows the three step procedure outlined in the introduction in Chapter 5. First, we transfer $(\Gamma_t)_{0\le t\le T}$ to a solution $(\bar\Gamma_t)_{0\le t\le T}$ to ($\mathbb{R}^\infty$-CE). Secondly, by a superposition principle on $\mathbb{R}^\infty$ (cf. Proposition 6.3.3), we obtain a measure $\bar\eta \in \mathcal{P}(C_T\mathbb{R}^\infty)$ with $\bar\eta \circ (\pi_t^\infty)^{-1} = \bar\Gamma_t$ concentrated on solution curves to ($\mathbb{R}^\infty$-ODE). Finally, we transfer $\bar\eta$ back to a measure $\eta \in \mathcal{P}(C_T\mathcal{SP})$ with all properties claimed in Theorem 5.2.1.

We will use the following superposition principle for equations of type ($\mathbb{R}^\infty$-ODE) and ($\mathbb{R}^\infty$-CE), cf. [13, Thm.7.1]. Due to Remark 6.3.2, it is no additional assumption to consider the larger class of test functions $\mathcal{F}C_b^1(\mathbb{R}^\infty)$ in the next proposition.

Proposition 6.3.3. *Let $t \mapsto \bar\Gamma_t$ be a solution to ($\mathbb{R}^\infty$-CE) in the sense of Definition 6.3.1 (ii) with test functions $\mathcal{F}C_b^1(\mathbb{R}^\infty)$ instead of $\mathcal{F}C_b^2(\mathbb{R}^\infty)$. Then, there exists a Borel measure $\bar\eta \in \mathcal{P}(C_T\mathbb{R}^\infty)$ concentrated on solutions to ($\mathbb{R}^\infty$-ODE) in the sense of Definition 6.3.1 (i) such that*

$$\bar\eta \circ (\pi_t^\infty)^{-1} = \bar\Gamma_t,\ 0 \le t \le T.$$

We now proceed to the three step procedure in order to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. Let $\Gamma = (\Gamma_t)_{0\le t\le T}$ be a weakly continuous solution to ($\mathcal{SP}$-CE) as in Definition 6.2.4.

Step 1: From ($\mathcal{SP}$-CE) to ($\mathbb{R}^\infty$-CE). Set

$$\bar\Gamma_t := \Gamma_t \circ G^{-1}, \quad 0 \le t \le T,$$

with G as in Lemma 6.2.2. Since G is continuous, $t \mapsto \bar\Gamma_t$ is a weakly continuous curve of Borel subprobability measures on $\mathbb{R}^\infty$, and it solves ($\mathbb{R}^\infty$-CE). Indeed, the integrability condition (6.14) is fulfilled, since Γ fulfills Definition 6.2.4. Furthermore, since Γ solves ($\mathcal{SP}$-CE), we have for any $\mathcal{F}C_b^2(\mathcal{G}) \ni F : \mu \mapsto f\big(\mu(g_1), \dots, \mu(g_n)\big)$ and $t \in [0,T]$

$$\int_{\mathcal{SP}} F(\mu)d\Gamma_t(\mu) - \int_{\mathcal{SP}} F(\mu)d\Gamma_0(\mu) = \int_0^t \int_{\mathcal{SP}} \mathbf{L}_s F(\mu)d\Gamma_s(\mu)ds \tag{6.16}$$

and hence, abbreviating $B_k(t,\cdot)$ and $\bar B_k(t,\cdot)$ by $B_{k,t}$ and $\bar B_{k,t}$, respectively, and setting $\bar F = f \circ \pi^{(n)}$ for f as above, we have

$$\begin{aligned}\int_0^t \int_{\mathcal{SP}} \mathbf{L}_s F(\mu)d\Gamma_s(\mu)ds &= \int_0^t \int_{\mathcal{SP}} \sum_{k=1}^n (\partial_k f)\big(\mu(g_1),\dots,\mu(g_n)\big)\Big(\int_{\mathbb{R}^d} \mathcal{L}_{s,\mu} g_k(x)d\mu(x)\Big)\Gamma_s(\mu)ds \\ &= \int_0^t \int_{\mathcal{SP}} \sum_{k=1}^n (\partial_k f)\big(\mu(g_1),\dots,\mu(g_n)\big)B_{k,s}(\mu)d\Gamma_s(\mu)ds \\ &= \int_0^t \int_{\mathcal{SP}} \sum_{k=1}^n (\partial_k f)\big(p_1\circ G(\mu),\dots,p_n\circ G(\mu)\big)\bar B_{k,s}\circ G(\mu)d\Gamma_s(\mu)ds \\ &= \int_0^t \int_{\mathbb{R}^\infty} \bar\nabla \bar F(z)\cdot \bar B_s(z)\bar\Gamma_s(z)ds.\end{aligned}$$

Moreover, for each $t \in [0,T]$, we have

$$\int_{\mathcal{SP}} F(\mu)d\Gamma_t(\mu) = \int_{\mathcal{SP}} f\big(p_1 \circ G(\mu), \dots, p_n \circ G(\mu)\big)d\Gamma_t(\mu) = \int_{\mathbb{R}^\infty} \bar{F}(z)d\bar{\Gamma}_t(z).$$

Comparing with (6.16), it follows that $(\bar{\Gamma}_t)_{0\le t\le T}$ is a solution to ($\mathbb{R}^\infty$-CE) as claimed, because the above calculation holds for arbitrary $F \in \mathcal{F}C^2_b(\mathcal{G})$ and hence for arbitrary $\bar{F} \in \mathcal{F}C^2_b(\mathbb{R}^\infty)$.

Step 2: From ($\mathbb{R}^\infty$-CE) to ($\mathbb{R}^\infty$-ODE). Proposition 6.3.3 implies the existence of a measure $\bar{\eta} \in \mathcal{P}(C_T\mathbb{R}^\infty)$ such that

(i) $\bar{\eta} \circ (\pi^\infty_t)^{-1} = \bar{\Gamma}_t, \quad 0 \le t \le T.$

(ii) $\bar{\eta}$ is concentrated on solution paths to ($\mathbb{R}^\infty$-ODE).

Step 3: From ($\mathbb{R}^\infty$-ODE) to (NLFPK). We show that the measure $\eta \in \mathcal{P}(C_T\mathcal{SP})$ defined as

$$\eta := \bar{\eta} \circ (J^{-1})^{-1}, \tag{6.17}$$

with J as in Lemma 6.2.2, fulfills all properties of the assertion. Indeed, since

$$\bar{\eta} \circ (\pi^\infty_t)^{-1} = \bar{\Gamma}_t = \Gamma_t \circ G^{-1}$$

for each $t \in [0,T]$, we deduce that $\bar{\eta} \circ (\pi^\infty_t)^{-1}$ is concentrated on $G(\mathcal{SP})$. By Lemma 6.2.2, $G(\mathcal{SP}) \subseteq \mathbb{R}^\infty$ is closed. Since by construction $\bar{\eta}$ is concentrated on continuous curves in $\mathbb{R}^\infty$, $\bar{\eta}$ is concentrated on $C_TG(\mathcal{SP})$. Furthermore, $C_TG(\mathcal{SP}) \subseteq C_T\mathbb{R}^\infty$ is a measurable set and $J^{-1} : C_TG(\mathcal{SP}) \to C_T\mathcal{SP}$ is measurable by Lemma 6.2.2. Therefore, we may define $\eta \in \mathcal{P}(C_T\mathcal{SP})$ as in (6.17).

It remains to verify $\eta \circ \pi_t^{-1} = \Gamma_t$ for all $t \in [0,T]$ and that η is concentrated on subprobability solutions to (NLFPK). Concerning the former property, we note

$$\eta \circ \pi_t^{-1} = \bar{\eta} \circ (J^{-1})^{-1} \circ \pi_t^{-1} = \bar{\eta} \circ (\pi_t \circ J^{-1})^{-1}$$

and

$$\Gamma_t = \Gamma_t \circ (G^{-1} \circ G)^{-1} = \bar{\Gamma}_t \circ (G^{-1})^{-1} = \bar{\eta} \circ (G^{-1} \circ \pi^\infty_t)^{-1}.$$

Since $\pi_t \circ J^{-1}$ and $G^{-1} \circ \pi^\infty_t$ coincide on $C_TG(\mathcal{SP})$, and we have shown above that $\bar{\eta}$ is concentrated on $C_TG(\mathcal{SP})$, we obtain $\eta \circ \pi_t^{-1} = \Gamma_t$ for each $t \in [0,T]$, as desired.

Concerning the second aspect, note that by definition of η and $\bar{\Gamma}_t$ and the equality $\pi_t \circ J^{-1} = G^{-1} \circ \pi^\infty_t$ on $C_TG(\mathcal{SP})$, assumption (6.10) for $(\Gamma_t)_{0\le t\le T}$ implies

$$\begin{aligned}
&\int_{C_T\mathcal{SP}} \int_0^T ||b(t,\pi_t)||_{L^1(\mathbb{R}^d,\mathbb{R}^d;\pi_t)} + ||a(t,\pi_t)||_{L^1(\mathbb{R}^d,\mathbb{R}^{d^2};\pi_t)} dtd\eta \\
&= \int_0^T \int_{C_TG(\mathcal{SP})} ||b(t,G^{-1}\circ\pi^\infty_t)||_{L^1(\mathbb{R}^d,\mathbb{R}^d;G^{-1}\circ\pi^\infty_t)} + ||a(t,G^{-1}\circ\pi^\infty_t)||_{L^1(\mathbb{R}^d,\mathbb{R}^{d^2};G^{-1}\circ\pi^\infty_t)} d\bar{\eta}dt \\
&= \int_0^T \int_{\mathcal{SP}} ||b(t,\mu)||_{L^1(\mathbb{R}^d,\mathbb{R}^d;\mu)} + ||a(t,\mu)||_{L^1(\mathbb{R}^d,\mathbb{R}^{d^2};\mu)} d\Gamma_t(\mu)dt < \infty.
\end{aligned} \tag{6.18}$$

Consequently, η is concentrated on vaguely continuous curves $t \mapsto \mu_t$ in $\mathcal{SP}$ with the global integrability property (6.3) such that $t \mapsto G(\mu_t)$ is a solution path to ($\mathbb{R}^\infty$-ODE), i.e.

$$\frac{d}{dt} p_k \circ G(\mu_t) = p_k \circ \bar{B}(t, G(\mu_t))\, dt\text{-a.s.}, \quad k \in \mathbb{N}. \tag{6.19}$$

Each such curve $t \mapsto \mu_t$ is a subprobability solution to (NLFPK). Indeed, by definition of the vector field $\bar{B}$ and since the previous calculation in particular gives $\mu_t \in A_t$ dt-a.s., (6.19) is equivalent to the following equality for each $k \in \mathbb{N}$:

$$\frac{d}{dt}\int_{\mathbb{R}^d} g_k d\mu_t = \int_{\mathbb{R}^d} \mathcal{L}_{t,\mu_t} g_k d\mu_t \, dt\text{-a.s.}$$
$$\iff \int_{\mathbb{R}^d} g_k d\mu_t - \int_{\mathbb{R}^d} g_k \mu_0 = \int_0^t \int_{\mathbb{R}^d} \mathcal{L}_{s,\mu_s} g_k d\mu_s ds, \quad t \in [0,T].$$

Hence, Lemma 6.2.1 (ii) implies that $t \mapsto \mu_t$ is a subprobability solution to (NLFPK).

It remains to prove the additional assertion about probability solutions. To this end, assume Γ_0 is concentrated on $\mathcal{P}$. Then, $\eta(\pi_0 \in \mathcal{P}) = \Gamma_0(\mathcal{P}) = 1$, and hence the claim follows by Remark 6.1.2 and the aforementioned observation that (6.18) implies that η is concentrated on vaguely continuous curves $t \mapsto \mu_t$ which fulfill (6.3). □

Remark 6.3.4. *Here, we explain why our basic space of measures in the entire chapter is $\mathcal{SP}$ with the vague topology instead of $\mathcal{P}$ (with either the vague or weak topology), even though we are mainly interested in (weakly continuous) probability solutions $t \mapsto \mu_t$ to* (NLFPK)*. If we had restricted the entire approach to probability solutions, and hence to solutions $(\Gamma_t)_{0 \le t \le T}$ to ($\mathcal{SP}$-CE) with $\Gamma_t \in \mathcal{P}(\mathcal{P})$, we could not have proven that $\bar{\eta}$ as in the above proof is concentrated on $C_T G(\mathcal{P})$ (in fact, we could not even show $\bar{\eta}(C_T G(\mathcal{P})) > 0$). Indeed, inspecting the proof, we only could have proven that $\bar{\eta} \circ \pi_t^{-1}$ is concentrated on $G(\mathcal{P})$ for each $t \le T$. But since $\mathcal{P}$ with the vague topology is not closed, this does not imply that a measure on $\mathbb{R}^\infty$ with one-dimensional marginals $\bar{\Gamma}_t$ is concentrated on $C_T G(\mathcal{P})$. Therefore, in this situation it seems not possible to pull $\bar{\eta}$ back to a measure η on $C_T\mathcal{P}$ as in* (6.17)*. Considering $\mathcal{P}$ with the (more natural) weak topology yields similar obstacles, since the lack of separability of the corresponding test function class $C_b(\mathbb{R}^d)$ yields that a map of type G cannot have closed range in $\mathbb{R}^\infty$.*

These issues are resolved by considering $\mathcal{SP}$ with the vague topology, which is compact, and the corresponding test function class $C_c(\mathbb{R}^d)$, which is separable. In order to recover a result for probability measures as in the second part of the assertion of Theorem 5.2.1, we need to assume the global integrability assumption (6.10)*. We point out that this global condition can be weakened to a local assumption in the sense of Remark 6.2.5 (i), if one omits the final assertion of Theorem 5.2.1.*

6.4 Consequences: Existence, uniqueness and an application

6.4.1 Transfer of existence and uniqueness

The following observations follow readily from the superposition principle Theorem 5.2.1 and allow to transfer existence and uniqueness results between the nonlinear equation (NLFPK) and its linearized continuity equation ($\mathcal{SP}$-CE).

Corollary 6.4.1. *Let $\Gamma \in \mathcal{P}(\mathcal{SP})$ and assume there exists a solution $t \mapsto \Gamma_t$ to (SP-CE) with initial condition $\Gamma_0 = \Gamma$. Then, for Γ-a.e. $\mu \in \mathcal{SP}$, there exists a subprobability solution $t \mapsto \mu_t$ to (NLFPK) with initial condition $\mu_0 = \mu$. Moreover, if $\Gamma \in \mathcal{P}(\mathcal{P})$, then for Γ-a.e. μ, there exists a probability solution $t \mapsto \mu_t$ to (NLFPK) with initial condition $\mu_0 = \mu$.*

Proof. Under the above assumption, Theorem 5.2.1 yields the existence of a probability measure η concentrated on subprobability solutions to (NLFPK) with $\eta \circ \pi_0^{-1} = \Gamma$. Hence, the set of $\mu \in \mathcal{SP}$ for which there is no solution $t \mapsto \mu_t$ to (NLFPK) with $\mu_0 = \mu$ is Γ-negligible. Concerning the second assertion, up to a η-negligible set, each solution curve to (NLFPK) with a probability initial condition in the support of η is a probability solution due to the global integrability condition (6.10). This implies the claim. □

Corollary 6.4.2. *Let $\nu \in \mathcal{SP}$ and assume there exists at most one vaguely continuous subprobability solution $t \mapsto \mu_t$ to (NLFPK) with $\mu_0 = \nu$. Then, there exists also at most one weakly continuous solution $t \mapsto \Gamma_t$ to (SP-CE) with initial condition δ_ν. If $\nu \in \mathcal{P}$, then, in the case of existence, we have $\Gamma_t \in \mathcal{P}(\mathcal{P})$ for each $t \in [0, T]$.*

Proof. Let $\Gamma^{(1)}$ and $\Gamma^{(2)}$ be weakly continuous solutions to (SP-CE) with $\Gamma_0^{(i)} = \delta_\nu$ for $i \in \{1, 2\}$. By Theorem 5.2.1, there exist probability measures $\eta^{(i)}$, $i \in \{1, 2\}$, concentrated on subprobability solutions to (NLFPK) with initial condition ν such that $\eta^{(i)} \circ \pi_t^{-1} = \Gamma_t^{(i)}$ for all $t \in [0, T]$ and $i \in \{1, 2\}$. By assumption, we obtain $\eta^{(1)} = \delta_\mu = \eta^{(2)}$ for a unique curve $\mu : t \mapsto \mu_t$ in $C_T\mathcal{SP}$ and thus also $\Gamma^{(1)} = \Gamma^{(2)}$. If $\nu \in \mathcal{P}$, then $\mu \in C_T\mathcal{P}$ by Remark 6.1.2, which gives the second assertion. □

6.4.2 Application: Coupled nonlinear-linear FPK equations

We close this chapter by applying our result to an open conjecture posed in [190]. Let us shortly recapitulate the necessary framework. For more details, we refer to [190]. Therein, for an operator $\mathcal{L}$ as in (5.5), the authors consider a coupled nonlinear-linear FPK equation of type

$$\begin{cases} \partial_t \mu_t = \mathcal{L}^*_{t,\mu_t} \mu_t \\ \partial_t \nu_t = \mathcal{L}^*_{t,\mu_t} \nu_t, \end{cases} \tag{6.20}$$

i.e. comparing to our situation, the first nonlinear equation is of type (NLFPK) and the second (linear) equation is obtained by "freezing" a solution $(\mu_t)_{0 \le t \le T}$ to the first equation in the measure argument of $\mathcal{L}$. For an initial condition $(\mu, \nu) \in \mathcal{P} \times \mathcal{P}$, (6.20) has a *unique solution*, if there exists a unique probability solution $(\mu_t)_{0 \le t \le T}$ to the first equation in the sense of Definition 6.1.1 with the global integrability condition (6.3) such that $\mu_0 = \mu$, and a unique weakly continuous curve $(\nu_t)_{0 \le t \le T} \subseteq \mathcal{P}$ such that

$$\int_0^T \int_{\mathbb{R}^d} |a(t, \mu_t, x)| + |b(t, \mu_t, x)| d\nu_t(x) dt < \infty,$$

which solves the second equation of (6.20) with fixed coefficient μ_t with $\nu_0 = \nu$ in distributional sense as in (6.2).

In [190], the authors associate a linear continuity equation on $\mathbb{R}^d \times \mathcal{P}$ to (6.20) in the following sense. Let $\mathbb{L}$ be the operator acting on functions

$$\mathcal{C} := \big\{\Phi : (x,\mu) \mapsto \varphi(x)F(\mu) : \varphi \in C_c^2(\mathbb{R}^d), F \in \mathcal{F}C_b^2(\mathcal{P})\big\},$$

via

$$\mathbb{L}_t\Phi(x,\mu) := \mathcal{L}_{t,\mu}\Phi(\cdot,\mu)(x) + \mathbf{L}_t\Phi(x,\cdot)(\mu),$$

with $\mathcal{L}$ as in (5.5) and $\mathbf{L}$ as in (6.9). Here, adapting the notation of [190], we denote by $\mathcal{F}C_b^2(\mathcal{P})$ the set of test functions $F : \mathcal{P} \to \mathbb{R}$, $F : \mu \mapsto f\big(\mu(h_1),\dots,\mu(h_n)\big)$, $f \in C_b^2(\mathbb{R}^n)$, $n \geq 1$, $h_i \in C_c^2(\mathbb{R}^d)$. Consider the linear continuity equation

$$\partial_t\Lambda_t = \mathbb{L}_t^*\Lambda_t, \ t \in [0,T], \tag{6.21}$$

for curves of probability measures on $\mathbb{R}^d \times \mathcal{P}$. A weakly continuous curve $t \mapsto \Lambda_t \in \mathcal{P}(\mathbb{R}^d \times \mathcal{P})$ is a called *solution* to (6.21), provided

$$\int_0^T \int_{\mathbb{R}^d\times\mathcal{P}} ||b(s,\mu,\cdot)||_{L^1(\mathbb{R}^d,\mathbb{R}^d;\mu)} + ||a(s,\mu,\cdot)||_{L^1(\mathbb{R}^d,\mathbb{R}^{d^2};\mu)} + (|b|+|a|)(s,\mu,x)\Lambda_s(dxd\mu)ds < \infty,$$

and for any $G \in \mathcal{C}$

$$\int_{\mathbb{R}^d\times\mathcal{P}} Gd\Lambda_t - \int_{\mathbb{R}^d\times\mathcal{P}} Gd\Lambda_0 = \int_0^t \int_{\mathbb{R}^d\times\mathcal{P}} \mathbb{L}_s G d\Lambda_s ds, \quad t \in [0,T],$$

compare [190, Def.2.2]. One readily observes that a weakly continuous curve $t \mapsto (\mu_t,\nu_t) \in \mathcal{P}\times\mathcal{P}$ solves (6.20) if and only if $\Lambda_t := \nu_t \times \delta_{\mu_t}$ solves (6.21).

The open question of [190, Rem.4.4] can be stated as follows: If (6.20) has a unique solution for some initial pair $(\mu,\nu) \in \mathcal{P}\times\mathcal{P}$, does it follow that $t \mapsto \nu_t \times \delta_{\mu_t} \in \mathcal{P}(\mathbb{R}^d \times \mathcal{P})$ is the only solution to (6.21)? By our main result of this chapter, Theorem 5.2.1, the answer is affirmative:

Proposition 6.4.3. *If $(\mu_t,\nu_t)_{0\leq t\leq T}$ is the unique solution to (6.20) with initial condition $(\mu,\nu) \in \mathcal{P}\times\mathcal{P}$, then $\Lambda_t := (\nu_t \times \delta_{\mu_t})_{0\leq t\leq T}$ is the unique solution to (6.21) with initial condition $\nu \times \delta_\mu$.*

Proof. The notion of uniqueness to (6.20) in particular implies that $t \mapsto \mu_t$ is the unique probability solution to the first equation of (6.20) with initial condition $\mu_0 = \mu$, i.e. an equation of type (NLFPK). By Corollary (6.4.2), the unique solution to the corresponding continuity equation ($\mathcal{SP}$-CE) for curves in $\mathcal{P}(\mathcal{P})$ with initial condition δ_μ is $t \mapsto \delta_{\mu_t}$. Let $(\Lambda_t^{(1)})_{0\leq t\leq T}$ and $(\Lambda_t^{(2)})_{0\leq t\leq T}$ be two solutions to (6.21) with initial condition $\nu \times \delta_\mu$. It is straightforward to check that the curves of second marginals $(\Lambda_t^{(1)} \circ \Pi_2^{-1})_{0\leq t\leq T}$ and $(\Lambda_t^{(2)} \circ \Pi_2^{-1})_{0\leq t\leq T}$ are probability solutions to ($\mathcal{SP}$-CE) with initial condition δ_μ (where we denote by $\Pi_2 : \mathbb{R}^d \times \mathcal{P} \to \mathcal{P}$ the projection from $\mathbb{R}^d \times \mathcal{P}$ to the second coordinate). Hence, we have

$$\Lambda_t^{(1)} \circ \Pi_2^{-1} = \delta_{\mu_t} = \Lambda_t^{(2)} \circ \Pi_2^{-1}, \quad t \in [0,T].$$

Since any probability measure on a product space with one one-dimensional marginal being a Dirac-measure is of product type, it follows that $\Lambda_t^{(i)} = \sigma_t^{(i)} \times \delta_{\mu_t}$ for weakly continuous curves $(\sigma_t^{(i)})_{0\leq t\leq T} \subseteq \mathcal{P}$, $i\in\{1,2\}$. It is immediate to show that $t\mapsto \sigma_t^{(i)}$ solves the second equation of (6.20) with fixed μ_t and initial condition ν. Hence, we obtain $\sigma_t^{(i)} = \nu_t$ for each $t\in[0,T]$ and $i\in\{1,2\}$, which implies $\Lambda_t^{(1)} = \Lambda_t^{(2)}$. Consequently, the unique solution to (6.21) with initial condition $\nu\times\delta_\mu$ is given by $(\nu_t\times\delta_{\mu_t})_{0\leq t\leq T}$. □

Chapter 7

Superposition principle for stochastic nonlinear FPK equations

7.1 Preliminaries

7.1.1 Notation and conventions

On the space of square-summable real sequences $\ell^2 := \{(x_i)_{i\in\mathbb{N}} : \sum_{i\geq 1} x_i^2 < \infty\}$, we consider the usual inner product $\langle x,y\rangle_{\ell^2} = \sum_{i\geq 1} x_i y_i$ (also abbreviated $x\cdot y$, if no confusion can appear) and the induced norm $||\cdot||_{\ell^2}$. On ℓ^2 and the space $C_T\ell^2$ of continuous functions $f:[0,T]\to\ell^2$, we unambiguously use the same notation p_i, $\pi^{(n)}$ and π_t^∞ as on $\mathbb{R}^\infty$ and $C_T\mathbb{R}^\infty$ from the previous chapter. In particular, we have $\mathcal{B}(C_T\ell^2) = \sigma(\pi_t^\infty, t\in[0,T])$.

Pathwise properties of stochastic processes, such as continuity, are always understood up to a negligible exception set with respect to the underlying measure, e.g. for a process $\mu:[0,T]\times\Omega\to\mathcal{SP}$ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, *the paths of μ are vaguely continuous* means that there is a $\mathbb{P}$-negligible set $N\subseteq\Omega$ such that for each $\omega\in N^c$, the path $t\mapsto\mu_t(\omega)$ is vaguely continuous.

As in the previous chapter, we consider $\mathcal{SP}$ as a compact Polish space with the vague topology.

7.1.2 Stochastic nonlinear FPK-equations

In addition to $\mathcal{B}([0,T])\otimes\mathcal{B}(\mathcal{SP})\otimes\mathcal{B}(\mathbb{R}^d)$-measurable coefficients

$$a_{ij}:[0,T]\times\mathcal{SP}\times\mathbb{R}^d\to\mathbb{R},\quad b_i:[0,T]\times\mathcal{SP}\times\mathbb{R}^d\to\mathbb{R},\quad 1\leq i,j\leq d,$$

as in the previous chapter, also let $d_1\in\mathbb{N}$ and consider $\sigma(t,\mu,x) = (\sigma_{ij}(t,\mu,x))_{i\leq d,j\leq d_1}$ with $\mathcal{B}([0,T])\otimes\mathcal{B}(\mathcal{SP})\otimes\mathcal{B}(\mathbb{R}^d)$-measurable coefficients σ_{ij}. We still use the notation $\mathcal{L}$ for the second-order differential operator with coefficients a_{ij}, b_i as in (5.5). Throughout, we assume σ to be bounded. In contrast to the deterministic framework of the previous chapter, now we consider nonlinear *stochastic* FPK-equations of type (SNLFPK) on $[0,T]$.

Definition 7.1.1. (i) A pair (μ, W), consisting of an $(\mathcal{F}_t)_{0\le t\le T}$-adapted, $\mathcal{SP}$-valued stochastic process $\mu = (\mu_t)_{0\le t\le T}$ with vaguely continuous paths, and a d_1-dimensional $(\mathcal{F}_t)_{0\le t\le T}$-Brownian motion $W = (W_t)_{0\le t\le T}$ on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ is a *subprobability solution to* (SNLFPK), provided

$$\int_0^T \int_{\mathbb{R}^d} |b_i(t,\mu_t,x)| + |a_{ij}(t,\mu_t,x)| + |\sigma_{ik}(t,\mu_t,x)|^2 d\mu_t(x)dt < \infty \quad \mathbb{P}\text{-a.s.} \tag{7.1}$$

for each $i, j \le d$, $k \le d_1$, and

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x)d\mu_t(x) &- \int_{\mathbb{R}^d} \varphi(x)d\mu_0(x) \\ &= \int_0^t \int_{\mathbb{R}^d} \mathcal{L}_{s,\mu_s}\varphi(x)d\mu_s(x)ds + \int_0^t \int_{\mathbb{R}^d} \sigma(s,\mu_s,x)\cdot\nabla\varphi(x)d\mu_s(x)dW_s\end{aligned} \tag{7.2}$$

holds $\mathbb{P}$-a.s. for each $t \in [0,T]$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$.

(ii) A *probability solution to* (SNLFPK) is a pair (μ, W) as above such that the paths $t \mapsto \mu_t$ are $\mathcal{P}$-valued (and hence weakly continuous).

Of course, under the present assumption of boundedness of σ, the integrability assumption on the last summand in (7.1) is fulfilled for any curve $t \mapsto \mu_t \in \mathcal{SP}$. By approximation, the validity of (7.2) extends to $\varphi \in C_c^2(\mathbb{R}^d)$. Note that this notion of solution is probabilistically weak, i.e. the probability space is part of the solution and the process μ is not necessarily adapted to the canonical Brownian filtration.

Remark 7.1.2. *(i) Since the paths $t \mapsto \mu_t(\omega)$ are vaguely continuous and the stochastic integral $t \mapsto \int_0^t \int_{\mathbb{R}^d} \sigma(s,\mu_s)\cdot\nabla\varphi d\mu_s dW_s$ has continuous paths, the exceptional set in the above definition can be chosen independently of t.*

(ii) The first integral on the right-hand side of (7.2) is a pathwise integral (that is, for individual fixed $\omega \in \Omega$) with respect to the finite measure $\mu_s(\omega)ds$ on $[0,T]\times\mathbb{R}^d$. The second integral is a stochastic integral, which is defined since the integrand

$$(t,\omega) \mapsto \int_{\mathbb{R}^d} \sigma(t,\mu_t(\omega),x)\cdot\nabla\varphi(x)d\mu_t(\omega)(x)$$

is $\mathbb{R}^{d_1}$-valued, bounded, product-measurable and $\mathcal{F}_t$-adapted [68, Thm.3.8]. More precisely, it reads

$$\int_0^t \int_{\mathbb{R}^d} \sigma(s,\mu_s,x)\cdot\nabla\varphi(x)d\mu_s(x)dW_s = \sum_{\alpha=1}^{d_1} \int_0^t \int_{\mathbb{R}^d} \sigma^\alpha(s,\mu_s,x)\cdot\nabla\varphi(x)d\mu_s(x)dW_s^\alpha,$$

where $\sigma^\alpha = (\sigma^{i\alpha})_{1\le i\le d}$ denotes the α-th column of σ, and the components W^α, $1 \le \alpha \le d_1$, of W are real, independent Brownian motions.

Due to (7.1), and since σ is bounded, we obtain (in analogy to Remark 6.1.2) the following conservation of mass, which we use to prove the final assertion of the main result Theorem 5.2.2.

Lemma 7.1.3. *Let $(\mu_t)_{0\leq t\leq T}$ be a subprobability solution to (SNLFPK). If $\mu_0 \in \mathcal{P}$ $\mathbb{P}$-a.s., then the paths $t \mapsto \mu_t$ are $\mathcal{P}$-valued and, in particular, weakly continuous.*

Proof. Let $(\varphi_k)_{k\geq 1} \subseteq C_c^\infty(\mathbb{R}^d)$ approximate the constant function 1 as in Remark 6.1.2. For the continuous $(\mathcal{F}_t)_{0\leq t\leq T}$-martingales

$$X_t^k := \int_0^t \int_{\mathbb{R}^d} \sigma(s,\mu_s,x)\cdot\nabla\varphi_k(x)d\mu_s(x)dW_s, \quad k\in\mathbb{N},$$

we obtain by Burkholder-Davies-Gundy inequality, Itô's isometry, and boundedness of σ

$$\mathbb{E}\big[\sup_{t\leq T}|X_t^k|\big] \leq C\mathbb{E}\Bigg[\int_0^T \left|\int_{\mathbb{R}^d}\sigma(s,\mu_s,x)\cdot\nabla\varphi_k(x)d\mu_s(x)\right|^{1/2} ds\Bigg] \underset{k\to\infty}{\longrightarrow} 0. \tag{7.3}$$

Consequently, along a subsequence $\{k_l\}_{l\in\mathbb{N}}$, we have $\sup_{t\leq T}|X_t^{k_l}| \underset{l\to\infty}{\longrightarrow} 0$ for each $\omega\in N_1^c$, for a $\mathbb{P}$-negligible set N_1. By (7.1), it is clear that the left-hand side and the first integral on the right-hand side of (7.2) converge ω-wise to $\mu_t(\mathbb{R}^d)-\mu_0(\mathbb{R}^d)$ and 0, respectively, as $l\to\infty$, i.e. in the limit of (7.2) for $l\to\infty$, we obtain the existence of a $\mathbb{P}$-negligible set $N_2\subseteq\Omega$ such that for each $\omega\in N_1^c\cap N_2^c$, we have

$$\mu_t(\omega)(\mathbb{R}^d) = \mu_0(\omega)(\mathbb{R}^d), \quad t\in[0,T],$$

which gives the claim. □

7.2 Geometry on $\mathcal{SP}$ revisited: Second-order equations

As in the deterministic case in Section 6.2, we consider $\mathcal{SP}$ as a manifold-like space with tangent spaces $T_\mu\mathcal{SP} = L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)$. However, instead of $G:\mathcal{SP}\to\mathbb{R}^\infty$ as in 6.2.2, now we consider a homeomorphism

$$H:\mathcal{SP}\to\ell^2$$

in order to handle the stochastic integral term in (7.2). To this end, we replace the set of functions $\mathcal{G}=\{g_i, i\geq 1\}$ of the deterministic case by

$$\mathcal{H} := \{h_i,\, i\geq 1\},\, h_i := i^{-1}\frac{g_i}{||g_i||_{C^2}} \tag{7.4}$$

and consider the map

$$H:\mathcal{SP}\to\ell^2,\; H:\mu\mapsto(\mu(h_i))_{i\geq 1}.$$

The following lemma collects useful properties of $\mathcal{H}$ and H, which are in the spirit of Lemma 6.2.1 and 6.2.2. We point out that we could have used the function class $\mathcal{H}$ instead of $\mathcal{G}$ already in Chapter 6, but we decided to pass from $\mathcal{G}$ to $\mathcal{H}$ at this point in order to stress the technical adjustments necessary due to the stochastic case.

Lemma 7.2.1. *(i) The set $\mathcal{H}$ is measure separating. Furthermore, a process $(\mu_t)_{0\leq t\leq T}$ as in Definition 7.1.1 is a solution to (SNLFPK) if and only if (7.2) holds for each $h_i \in \mathcal{H}$ in place of φ.*

(ii) H is a homeomorphism between $\mathcal{SP}$ and its range $H(\mathcal{SP}) \subseteq \ell^2$, endowed with the ℓ^2-subspace topology. In particular, $H(\mathcal{SP}) \subseteq \ell^2$ is compact.

Proof. (i) The first claim is obvious, since $\mathcal{G}$ is measure separating. Concerning the second claim, it is clear that (7.2) for each $h_i \in \mathcal{H}$ is equivalent to the validity of (7.2) for each $g_i \in \mathcal{G}$. Since the latter set is dense in $C^2_c(\mathbb{R}^d)$, for any $\varphi \in C^2_c(\mathbb{R}^d)$, there is a sequence $(g_{i_k})_{k\in\mathbb{N}}$ such that $||g_{i_l} - \varphi||_{C^2} \longrightarrow 0$ as $k \to \infty$. Then, by Itô's isometry, we have for each $t \in [0, T]$

$$\mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}^d}\sigma(s,\mu_s)\cdot\nabla(g_{i_k}-\varphi)d\mu_s dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t\left|\int_{\mathbb{R}^d}\sigma(s,\mu_s)\cdot\nabla(g_{i_k}-\varphi)d\mu_s\right|^2 ds\right]$$

and the right-hand side converges to 0 as $k \to \infty$ due to the boundedness of σ. Hence, along a further subsequence, for simplicity again denoted by $(i_k)_{k\in\mathbb{N}}$, we have $\mathbb{P}$-a.s.

$$\int_0^t\int_{\mathbb{R}^d}\sigma(s,\mu_s,x)\cdot\nabla g_{i_k}(x)d\mu_s(x)dW_s \underset{k\to\infty}{\longrightarrow} \int_0^t\int_{\mathbb{R}^d}\sigma(s,\mu_s,x)\cdot\nabla\varphi(x)d\mu_s(x)dW_s.$$

The ω-wise convergence of all other terms in (7.2) is clear, so that for each $(t, \varphi) \in [0, T] \times C^2_c(\mathbb{R}^d)$, (7.2) holds on the complement of a $\mathbb{P}$-negligible set. This gives the claim.

(ii) By definition, H maps into ℓ^2. Since $\mathcal{H}$ is measure separating, it follows that H is one-to-one, hence bijective onto its range. If $(\mu_n)_{n\in\mathbb{N}}$ converges vaguely to μ in $\mathcal{SP}$, then $H(\mu_n)$ converges to $H(\mu)$ in the product topology. Since for any $i \geq 1$, we have

$$\sup_{n\geq 1}|H(\mu_n)_i| \leq i^{-1},$$

the convergence holds in the ℓ^2-topology as well, which implies continuity of H. In particular, $H(\mathcal{SP}) \subseteq \ell^2$ is compact. Conversely, if $H(\mu_n)$ converges to some z in ℓ^2, then, by closedness of $H(\mathcal{SP}) \subseteq \ell^2$, we have $z = H(\mu)$ for a unique element $\mu \in \mathcal{SP}$ and $\mu_n \underset{n\to\infty}{\longrightarrow} \mu$ vaguely. Indeed, the latter follows as in Lemma 6.2.2 (i).

□

In this chapter, we use the test function class $\mathcal{F}C^2_b(\mathcal{H})$, where $F \in \mathcal{F}C^2_b(\mathcal{H})$ if and only if $F : \mathcal{SP} \to \mathbb{R}$, $F : \mu \mapsto f\big(\mu(h_1), \dots \mu(h_n)\big)$, $f \in C^2_b(\mathbb{R}^n)$, $n \in \mathbb{N}$ and $h_i \in \mathcal{H}$. Note that $\mathcal{F}C^2_b(\mathcal{H})$ coincides with $\mathcal{F}C^2_b(\mathcal{G})$ from Chapter 6, since the transition from the inner test functions g_i to h_i can be incorporated in the choice of f. Nevertheless, we use the new notation $\mathcal{F}C^2_b(\mathcal{H})$ in order to stress the change of test functions from $\mathcal{G}$ to $\mathcal{H}$ in comparison to the deterministic case.

7.2.1 A natural Hessian-type operator on $\mathcal{SP}$

As in the deterministic case, one can associate a linear equation for curves in $\mathcal{P}(\mathcal{SP})$ to the stochastic nonlinear equation (SNLFPK). Of course, the basic idea stems from the deterministic case presented in Chapter 6. From Itô's, formula one expects this linearized equation to be of second order. In the finite-dimensional case, the second-order term of the linearization comprises the Hessian operator on $\mathbb{R}^d$ (i.e., the Hessian matrix $\operatorname{Hess} f$ for test functions $f : \mathbb{R}^d \to \mathbb{R}$), cf. Appendix B. In this section, we introduce a natural notion of an analogous Hessian-type operator on the manifold-like space $\mathcal{SP}$, which will be used later on to derive the linear second-order equation for curves in $\mathcal{P}(\mathcal{SP})$ associated to (SNLFPK). To the best of our knowledge, such a geometric approach towards the linearization of stochastic FPK equations has not been considered in the literature before.

Recall that for a smooth function $f : M \to \mathbb{R}$ on a Riemannian manifold (M, g) with tangent bundle TM, the *Hessian* $\operatorname{Hess} f$ is the 0-2 tensor, which acts on smooth vector fields $X, Y : M \to \bigsqcup_{x\in M} T_xM$ as a bilinear form via

$$\operatorname{Hess} F(X, Y) = \left\langle \nabla^L_X \nabla F, Y \right\rangle_g. \tag{7.5}$$

Here, $\nabla^L : \bigsqcup_{x\in M} T_xM \times \bigsqcup_{x\in M} T_xM \to \bigsqcup_{x\in M} T_xM$ denotes the Levi-Civita connection on M, the unique torsion-free affine connection compatible with g, and ∇ denotes the gradient on (M, g). We refer to Appendix F for more details. Intuitively, for another smooth vector field Z on M, $\big(\nabla^L_X Z\big)(x) \in T_xM$ denotes the change in direction $X(x)$ of the vector field Z at x. With this intuition in mind, in the case $M = \mathcal{SP}$, it is reasonable to set

$$(\nabla^{L,\mathcal{SP}}_X Z)(\mu) = \left\langle (\nabla^{\mathcal{SP}} Z)(\mu), X(\mu) \right\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)}, \tag{7.6}$$

provided we can make sense of the above right-hand side. This ansatz is also motivated by the finite-dimensional case $M = \mathbb{R}^d$ with the standard Euclidean metric, since in this case, for vector fields $X, Z : \mathbb{R}^d \to \mathbb{R}^d$, the Levi-Civita connection is given by

$$(\nabla^L_X Z)(x) = (\nabla_{X(x)} Z)(x) = \langle (\nabla Z)(x), X(x) \rangle_{\mathbb{R}^d} \in \mathbb{R}^d.$$

For our particular case of interest, i.e. $Z = \nabla^{\mathcal{SP}} F$ for $F \in \mathcal{F}C^2_b(\mathcal{H})$, $F : \mu \mapsto f\big(\mu(h_1), \dots, \mu(h_n)\big)$, we can indeed make sense of $\nabla^{\mathcal{SP}}\nabla^{\mathcal{SP}} F$, and hence of the right-hand side of (7.6), because the gradient

$$\mu \mapsto \nabla^{\mathcal{SP}} F(\mu) = \sum_{k=1}^n (\partial_k f)\big(\mu(h_1), \dots, \mu(h_n)\big) \nabla h_k$$

is a linear combination of the functions $\mu \mapsto \partial_k f\big(\mu(h_1), \dots, \mu(h_n)\big)$, which are of $\mathcal{F}C^2_b(\mathcal{H})$-type up to a missing second derivative of the outer functions $\partial_k f \in C^1_b(\mathbb{R}^n)$. The linear combination has to be understood x-wise in the functions ∇h_k, which do not depend on the variable of interest μ. Denoting $F_k(\mu) := (\partial_k f)\big(\mu(h_1), \dots, \mu(h_n)\big)$, we then define

$$(\nabla^{\mathcal{SP}})^2 F(\mu)(x, y) := \sum_{k=1}^n \big(\nabla^{\mathcal{SP}} F_k(\mu)\big)(y) \nabla h_k(x), \ \ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{7.7}$$

From here, for $\sigma \in T_\mu\mathcal{SP}$, we consider the natural Levi-Civita connection-type operator applied to $\nabla^{\mathcal{SP}} F$, given as

$$\begin{aligned}\nabla^{L,\mathcal{SP}}_\sigma \nabla^{\mathcal{SP}} F(\mu) &:= \big\langle (\nabla^{\mathcal{SP}})^2 F(\mu), \sigma \big\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)} \\ &= \sum_{k,l=1}^n (\partial_{kl} f)\big(\mu(h_1),\dots,\mu(h_n)\big) \nabla h_k \bigg(\int_{\mathbb{R}^d} \sigma(y)\cdot \nabla h_l(y) d\mu(y) \bigg) \in L^2(\mathbb{R}^d,\mathbb{R}^d;\mu).\end{aligned}$$

The definition of $(\nabla^{\mathcal{SP}})^2 F$ (as well as $\nabla^{L,\mathcal{SP}}_\sigma \nabla^{\mathcal{SP}} F$ and $\operatorname{Hess} F$ below) is independent of the particular representation of F. Indeed, we have (c.f. [190, App.A]) for

$$\gamma^\mu_\sigma(t) := \mu \circ (\operatorname{id} + t\sigma)^{-1}, \quad \mu \in \mathcal{SP},\, t \in [0,T],$$

the following pointwise (in $x \in \mathbb{R}^d$) equality for each $\mu \in \mathcal{SP}, \sigma \in L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)$

$$\begin{aligned}\frac{d}{dt} \nabla^{\mathcal{SP}} F\big(\gamma^\sigma_\mu(t)\big)_{|t=0} &= \sum_{k,l=1}^n (\partial_{kl} f)\big(\mu(h_1),\dots,\mu(h_n)\big) \big\langle \nabla h_l, \sigma \big\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)} \nabla h_k \\ &= \big\langle (\nabla^{\mathcal{SP}})^2 F(\mu), \sigma \big\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)}.\end{aligned}$$

Since the gradient $\nabla^{\mathcal{SP}} F$ is independent of the particular representation of F (see Chapter 6) and $\sigma \in L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)$ is arbitrary, also $(\nabla^{\mathcal{SP}})^2 F$ is independent of the representation of F.

In the spirit of (7.5), now we set for $F \in \mathcal{F}C^2_b(\mathcal{H})$ and $\sigma, \tilde\sigma \in L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)$

$$\operatorname{Hess} F(\mu) : (\sigma,\tilde\sigma) \mapsto \big\langle \langle (\nabla^{\mathcal{SP}})^2 F(\mu), \sigma \rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)}, \tilde\sigma \big\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)}, \tag{7.8}$$

i.e. explicitly we have

$$\operatorname{Hess} F(\mu) : (\sigma,\tilde\sigma) \mapsto \sum_{k,l=1}^n (\partial_{kl} f)\big(\mu(h_1),\dots,\mu(h_n)\big) \bigg(\int_{\mathbb{R}^d} \sigma \cdot \nabla h_l d\mu \bigg) \bigg(\int_{\mathbb{R}^d} \tilde\sigma \cdot \nabla h_k d\mu \bigg), \tag{7.9}$$

which is a symmetric bilinear form on $T_\mu \mathcal{SP} = L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)$ for each $\mu \in \mathcal{SP}$.

7.2.2 Linearization and second-order equations on $\mathcal{SP}$

Let $\big((\mu_t)_{0\le t\le T}, (W_t)_{0\le t\le T}\big)$ be a subprobability solution to (SNLFPK) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, and let $F : \mu \mapsto f\big(\mu(h_1),\dots,\mu(h_n)\big)$ be from $\mathcal{F}C^2_b(\mathcal{H})$. As before, we abbreviate $b(t,\mu) := b(t,\mu,\cdot)$ and similarly for a and $\sigma = (\sigma^\alpha)_{1\le\alpha\le d_1}$, and we denote the components of W by W^α, $1 \le \alpha \le d_1$. By Itô's formula, we have $\mathbb{P}$-a.s.

$$\begin{aligned}F(\mu_t) - F(\mu_0) = &\int_0^t \big\langle \nabla^{\mathcal{SP}} F(\mu_s), b(s,\mu_s) + a(s,\mu_s)\nabla \big\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu_s)} ds \\ &+ \frac12 \sum_{\alpha=1}^{d_1} \int_0^t \sum_{k,l=1}^n (\partial_{kl} f)\big(\mu_s(h_1),\dots,\mu_s(h_n)\big) \\ &\qquad \cdot \bigg(\int_{\mathbb{R}^d} \sigma^\alpha(s,\mu_s)\cdot \nabla h_k d\mu_s \bigg) \bigg(\int_{\mathbb{R}^d} \sigma^\alpha(s,\mu_s)\cdot \nabla h_l d\mu_s \bigg) ds \\ &+ M^F_t,\end{aligned}$$

where the meaning of the first-order term $\int_0^t \langle \nabla^{\mathcal{SP}} F(\mu_s), b(s,\mu_s) + a(s,\mu_s)\nabla \rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu_s)} ds$ is rigorously given as in Subsection 6.2.2, and the continuous $(\mathcal{F}_t)_{0\le t\le T}$-martingale $t \mapsto M_t^F$ is

$$M_t^F = \sum_{\alpha=1}^{d_1} \int_0^t \left[\sum_{l=1}^n (\partial_l f)\big(\mu_s(h_1),\dots,\mu_s(h_n)\big) \int_{\mathbb{R}^d} \sigma^\alpha \cdot \nabla h_l d\mu_s \right] dW_s^\alpha.$$

Since $M_0^F = 0$ $\mathbb{P}$-a.s., integrating with respect to $\mathbb{P}$ and defining the curve of measures in $\mathcal{P}(\mathcal{SP})$ $t \mapsto \Gamma_t$ by

$$\Gamma_t := \mathbb{P} \circ \mu_t^{-1}, \ 0 \le t \le T,$$

yields

$$\begin{aligned}\int_{\mathcal{SP}} F d\Gamma_t - \int_{\mathcal{SP}} F d\Gamma_0 = &\int_0^t \int_{\mathcal{SP}} \langle \nabla^{\mathcal{SP}} F(\mu), b(s,\mu) + a(s,\mu)\nabla \rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)} d\Gamma_s(\mu) ds \\ &+ \frac{1}{2} \sum_{\alpha=1}^{d_1} \int_0^t \int_{\mathcal{SP}} \sum_{k,l=1}^n (\partial_{kl} f)\big(\mu(h_1),\dots,\mu(h_n)\big) \\ &\cdot \left(\int_{\mathbb{R}^d} \sigma^\alpha(s,\mu) \cdot \nabla h_k d\mu \right) \left(\int_{\mathbb{R}^d} \sigma^\alpha(s,\mu) \cdot \nabla h_l d\mu \right) d\Gamma_s(\mu) ds.\end{aligned} \tag{7.10}$$

Using the operator $\mathbf{L}$ as defined in (5.5) and the notion of Hess F derived in the preceding subsection, we rewrite (7.10) as

$$\int_{\mathcal{SP}} F d\Gamma_t - \int_{\mathcal{SP}} F d\Gamma_0 = \int_0^t \int_{\mathcal{SP}} \mathbf{L}_t F(\mu) + \frac{1}{2} \sum_{\alpha=1}^{d_1} \mathrm{Hess}\, F(\mu)\big(\sigma^\alpha(s,\mu), \sigma^\alpha(s,\mu)\big) d\Gamma_s(\mu) ds. \tag{7.11}$$

Introducing the second-order operator $\mathbf{L}^{(2)}$, acting on $F \in \mathcal{F}C_b^2(\mathcal{H})$ via

$$(\mathbf{L}_t^{(2)} F)(\mu) := \langle \nabla^{\mathcal{SP}} F(\mu), b(t,\mu) + a(t,\mu)\nabla \rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)} + \frac{1}{2} \sum_{\alpha=1}^{d_1} \mathrm{Hess}\, F\big(\sigma^\alpha(t,\mu), \sigma^\alpha(t,\mu)\big), \tag{7.12}$$

we infer from (7.10) that $t \mapsto \Gamma_t = \mathbb{P} \circ \mu_t^{-1}$ solves the linear second-order FPK-type equation ($\mathcal{SP}$-FPK) for curves in $\mathcal{P}(\mathcal{SP})$

$$\partial_t \Gamma_t = (\mathbf{L}_t^{(2)})^* \Gamma_t, \quad 0 \le t \le T, \tag{7.13}$$

which is just equation ($\mathcal{SP}$-CE) of the introduction in Chapter 5.

Remark 7.2.2. *Equation ($\mathcal{SP}$-CE) is the natural analogue to second-order FPK-equations on $\mathbb{R}^d$. Indeed, for the stochastic equation on $\mathbb{R}^d$*

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t, \tag{7.14}$$

by Itô's formula, the corresponding linear second-order equation for measures in distributional form is

$$\partial_t \mu_t = \big(\mathcal{L}_t^{(2)}\big)^* \mu_t,$$

with

$$\mathcal{L}_t^{(2)} f = \nabla f \cdot b_t + \frac{1}{2}\langle \sigma_t, \operatorname{Hess} f \cdot \sigma_t \rangle,$$

where Hess f *denotes the usual Euclidean Hessian matrix of* $f \in C^2(\mathbb{R}^d)$*, see Appendix B for more details. In this spirit, in comparison with (7.14), it appears natural to consider (SNLFPK) as a stochastic equation with the state space* $\mathcal{SP}$ *replacing* $\mathbb{R}^d$*, and (*$\mathcal{SP}$*-CE) as the corresponding linear FPK-type equation on* $\mathcal{SP}$*.*

By the above derivation, any subprobability solution $(t,\omega) \mapsto \mu_t(\omega)$ to (SNLFPK) induces a solution curve in $\mathcal{P}(\mathcal{SP})$ to ($\mathcal{SP}$-FPK) via $\Gamma_t := \mathbb{P} \circ \mu_t^{-1}$. Of course, this is in close analogy to the deterministic case, where we have seen that any (deterministic) solution path $t \mapsto \mu_t$ to (NLFPK) induces a solution $t \mapsto \Gamma_t := \delta_{\mu_t}$ to ($\mathcal{SP}$-CE). More generally, we introduce the following notion of solution to ($\mathcal{SP}$-FPK).

Definition 7.2.3. A weakly continuous curve $t \mapsto \Gamma_t \in \mathcal{P}(\mathcal{SP})$ is a *solution to (*$\mathcal{SP}$*-FPK)*, if the integrability condition

$$\int_0^T \int_{\mathcal{SP}} ||b(t,\mu)||_{L^1(\mathbb{R}^d,\mathbb{R}^d;\mu)} + ||a(t,\mu)||_{L^1(\mathbb{R}^d,\mathbb{R}^{d^2};\mu)} + ||\sigma(t,\mu)||^2_{L^2(\mathbb{R}^d,\mathbb{R}^{d\times d_1};\mu)} d\Gamma_t(\mu)dt < \infty \tag{7.15}$$

holds, and for each $F \in \mathcal{F}C_b^2(\mathcal{H})$, (7.11) holds for each $t \in [0,T]$.

7.3 Proof of main result

7.3.1 Transfer to ℓ^2

Reminiscent to the deterministic case, we use the homeomorphism $H : \mathcal{SP} \to H(\mathcal{SP}) \subseteq \ell^2$ to introduce auxiliary equations on ℓ^2 and $\mathcal{P}(\ell^2)$ as follows. Again, we use the notation

$$A_t := \left\{ \mu \in \mathcal{SP} : \int_{\mathbb{R}^d} |a_{ij}(t,\mu,x)| + |b_i(t,\mu,x)| d\mu(x) < \infty \;\forall\, 1 \le i,j \le d \right\}, \quad t \in [0,T].$$

For $i,j \ge 1$ and $1 \le \alpha \le d_1$, define the $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathcal{SP})$-measurable coefficients B_i for (t,μ) such that $\mu \in A_t$, and Σ_i^α and A_{ij} on $[0,T] \times \mathcal{SP}$, by

$$\begin{aligned}
B_i(t,\mu) &:= \int_{\mathbb{R}^d} \mathcal{L}_{t,\mu} h_i(x) d\mu(x), \quad (t,\mu) \in [0,T] \times A_t,\\
\Sigma_i^\alpha(t,\mu) &:= \int_{\mathbb{R}^d} \sigma^\alpha(t,\mu,x) \cdot \nabla h_i(x) d\mu(x),\\
\Sigma_i(t,\mu) &:= \left(\Sigma_i^\alpha(t,\mu)\right)_{1 \le \alpha \le d_1},\\
A_{ij}(t,\mu) &:= (\Sigma_i \cdot \Sigma_j)(t,\mu),
\end{aligned}$$

and set

$$B := (B_i)_{i \ge 1},\; \Sigma := (\Sigma_i^\alpha)_{1 \le \alpha \le d_1, i \ge 1},\; A := (A_{ij})_{i,j \ge 1}.$$

Now, transferring to ℓ^2, define $\bar{B}, \bar{\Sigma}$ and $\bar{A}_{ij}$ on $[0,T] \times \ell^2$ component-wise via

$$\bar{B}_i(t,z) := \begin{cases} B_i(t, H^{-1}(z)) & , z \in H(A_t) \\ 0 & , \text{else} \end{cases},$$

and

$$\bar{\Sigma}_i^\alpha(t,z) := \begin{cases} \Sigma_i^\alpha(t, H^{-1}(z)) & , z \in H(\mathcal{SP}) \\ 0 & , z \in \ell^2 \backslash H(\mathcal{SP}) \end{cases},$$

$$\bar{\Sigma}_i(t,z) := \left(\bar{\Sigma}_i^\alpha(t,z)\right)_{1 \le \alpha \le d_1},$$
$$\bar{A}_{ij}(t,z) := (\bar{\Sigma}_i \cdot \bar{\Sigma}_j)(t,z).$$

$\bar{B}$ and $\bar{\Sigma}^\alpha$ take values in ℓ^2, since for $z = H(\mu)$ with $\mu \in A_t$, we have

$$|\bar{B}_i(t,z)| \le \int_{\mathbb{R}^d} |\mathcal{L}_{t,\mu} h_i(x)| d\mu(x) \le C i^{-1},$$

where $C = C(a,b,d)$ is a finite constant independent of $i \ge 1$. A similar argument is valid for $\bar{\Sigma}^\alpha$. Each $\bar{B}_i$ and $\bar{\Sigma}_i^\alpha$ is product-measurable with respect to the ℓ^2-topology due to the measurability of B and Σ^α. Reminiscent to ($\mathbb{R}^\infty$-CE) in the previous chapter, we associate to ($\mathcal{SP}$-FPK) the Fokker–Planck–Kolmogorov-type equation on ℓ^2

$$\partial_t \bar{\Gamma}_t = -\bar{\nabla} \cdot (\bar{B}(t,z)\bar{\Gamma}_t) + \partial_{ij}(\bar{A}_{ij}(t,z)\bar{\Gamma}_t), \qquad (\ell^2\text{-FPK})$$

which we understand in the sense of the following definition, with $\bar{\nabla}$ as in (6.13). Subsequently, we denote by $\mathcal{F}C_b^2(\ell^2)$ the set of functions $\bar{F} : \ell^2 \to \mathbb{R}$ of type $\bar{F} = f \circ \pi^{(n)}$ for $n \ge 1$ and $f \in C_b^2(\mathbb{R}^n)$. Moreover, for such $\bar{F}$, set

$$D^2 \bar{F} := \begin{cases} (\partial_{ij} f) \circ \pi^{(n)} & , 1 \le i,j \le n \\ 0 & , \text{ else.} \end{cases}$$

Consequently, both terms of the right-hand side in (7.17) contain only finitely many nontrivial summands.

Definition 7.3.1. A weakly continuous curve $t \mapsto \bar{\Gamma}_t \in \mathcal{P}(\ell^2)$ is a *solution to* (ℓ^2-FPK), if it fulfills the integrability condition

$$\int_0^T \int_{\ell^2} |\bar{B}_i(t,z)| + |\bar{A}_{ij}(t,z)| d\bar{\Gamma}_t dt < \infty, \quad \forall\, i,j \ge 1, \qquad (7.16)$$

and for any $\bar{F} \in \mathcal{F}C_b^2(\ell^2)$, $\bar{F} := f \circ \pi^{(n)}$, we have

$$\int_{\ell^2} \bar{F}(z) d\bar{\Gamma}_t(z) - \int_{\ell^2} \bar{F}(z) d\bar{\Gamma}_0(z) = \int_0^t \int_{\ell^2} \bar{\nabla}\bar{F}(z) \cdot \bar{B}(s,z) + \frac{1}{2} D^2 \bar{F}(z) : \bar{A}(s,z) d\bar{\Gamma}_s(z) ds. \qquad (7.17)$$

for each $0 \le t \le T$.

Due to the boundedness of σ, the integrability condition (7.16) is the same as in (6.14).

7.3.2 Proof of Theorem 5.2.2

We turn to the proof of the main result of this chapter, Theorem 5.2.2. As in the proof of Theorem 5.2.1, we proceed in three steps. Since parts of the proof are technically more involved than in the deterministic case, we first present the ingredients of each step and then state the proof as a corollary.

Step 1: From ($\mathcal{SP}$-FPK) to (ℓ^2-FPK).

Lemma 7.3.2. *For any solution $(\Gamma_t)_{0\leq t\leq T}$ to ($\mathcal{SP}$-FPK), the curve $\bar\Gamma_t = \Gamma_t \circ H^{-1}$ is a solution to (ℓ^2-FPK).*

Proof. Clearly, $t \mapsto \bar\Gamma_t$ is a weakly continuous curve in $\mathcal{P}(\ell^2)$ due to the continuity of $H : \mathcal{SP} \to \ell^2$. In view of Definition 7.3.1, the integrability condition (7.16) holds, since $t \mapsto \Gamma_t$ fulfills (7.15). Moreover, for $t \in [0,T]$, $\bar F = f \circ \pi^{(n)} \in \mathcal{F}C_b^2(\ell^2)$ and $F : \mu \mapsto f\big(\mu(h_1),\dots,\mu(h_n)\big)$, we have

$$
\begin{aligned}
&\int_0^t \int_{\ell^2} \bar\nabla \bar F(z)\cdot \bar B(s,z) + \frac{1}{2} D^2\bar F(z) : \bar A(s,z) d\bar\Gamma_s(z) ds \\
&= \int_0^t \int_{\mathcal{SP}} \sum_{k=1}^n (\partial_k f)\big(\mu(h_1),\dots,\mu(h_n)\big) B_k(s,\mu) \\
&\qquad + \frac{1}{2}\sum_{\alpha=1}^{d_1}\sum_{k,l=1}^n (\partial_{kl} f)\big(\mu(h_1),\dots,\mu(h_n)\big)\Sigma_k^\alpha(s,\mu)\Sigma_l^\alpha(s,\mu) d\Gamma_s(\mu) ds \\
&= \int_0^t \int_{\mathcal{SP}} \big\langle \nabla^{\mathcal{SP}} F(\mu), b(s,\mu) + a(s,\mu)\nabla\big\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)} + \frac{1}{2}\sum_{\alpha=1}^{d_1} \operatorname{Hess} F\big(\sigma^\alpha(s,\mu),\sigma^\alpha(s,\mu)\big) d\Gamma_s(\mu) ds
\end{aligned}
$$

and

$$
\int_{\ell^2} \bar F(z) d\bar\Gamma_t = \int_{\mathcal{SP}} F(\mu) d\Gamma_t.
$$

Since $t \mapsto \Gamma_t$ fulfills (7.11), the claim follows. □

Step 2: From (ℓ^2-FPK) to the martingale problem (ℓ^2-MGP). Next, we introduce a martingale problem on ℓ^2, which is related to (ℓ^2-FPK) in the sense of Remark 7.3.4 below and is, roughly speaking, the stochastic analogue to ($\mathbb{R}^\infty$-ODE) from the previous chapter.

Definition 7.3.3. A measure $\bar Q \in \mathcal{P}(C_T\ell^2)$ is a *solution to the ℓ^2-martingale problem (ℓ^2-MGP)*, provided

$$
\int_{C_T\ell^2} \int_0^T |\bar B_i(t,\pi_t^\infty)| + |\bar A_{ij}(t,\pi_t^\infty)| dt d\bar Q < \infty, \quad i,j \geq 1, \tag{7.18}
$$

and

$$
\bar F \circ \pi_t^\infty - \int_0^t \bar\nabla \bar F \circ \pi_s^\infty \cdot \bar B(s,\pi_s^\infty) + \frac{1}{2} D^2 \bar F \circ \pi_s^\infty : \bar A(s,\pi_s^\infty) ds \tag{7.19}
$$

is a $\bar Q$-martingale on $C_T\ell^2$ with respect to the natural filtration on $C_T\ell^2$ for any $\bar F \in \mathcal{F}C_b^2(\ell^2)$.

Remark 7.3.4. *By construction, any such solution $\bar{Q}$ induces a weakly continuous solution $(\bar{\Gamma}_t)_{0\leq t\leq T}$ to (ℓ^2-FPK) via $\bar{\Gamma}_t := \bar{Q}\circ(\pi_t^\infty)^{-1}$. Indeed, this is readily seen by integrating (7.19) with respect to $\bar{Q}$ and an application of Fubini's theorem.*

In view of Proposition 7.3.6 below, we extend the coefficients $\bar{B}_i, \bar{\Sigma}_i^\alpha$ (and hence also $\bar{A}_{ij}$) from ℓ^2 to $\mathbb{R}^\infty$ via

$$\bar{B}_i := 0 =: \bar{\Sigma}_i^\alpha \text{ on } [0,T]\times\mathbb{R}^\infty\backslash\ell^2.$$

We continue to use the notation $\bar{B}$, $\bar{\Sigma}^\alpha$ and $\bar{A}$ and note that these maps are $\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}^\infty)/\mathcal{B}(\mathbb{R}^\infty)$-measurable, due to Remark 7.3.5 below. Due to the same remark, we may regard any solution $(\bar{\Gamma}_t)_{0\leq t\leq T}$ to (ℓ^2-FPK) as a solution to a FPK-type equation on $\mathbb{R}^\infty$ by considering (ℓ^2-FPK) with the extended coefficients and test functions $\bar{F}\in\mathcal{F}C_b^2(\ell^2)$ extended to $\mathbb{R}^\infty$ by considering $\pi^{(n)}$ on $\mathbb{R}^\infty$ instead of ℓ^2. Similarly, the formulation of the martingale problem (ℓ^2-MGP) as in Definition 7.3.3 extends to $\mathbb{R}^\infty$ in the sense that a measure $\bar{Q}\in\mathcal{P}(C_T\mathbb{R}^\infty)$ is understood as a solution, provided the process (7.19) is a $\bar{Q}$-martingale on $C_T\mathbb{R}^\infty$ with respect to the natural filtration for each $\bar{F} = f\circ\pi^{(n)}:\mathbb{R}^\infty\to\mathbb{R}$, $f\in C_b^2(\mathbb{R}^n)$ as above.

Remark 7.3.5. *We recall that $\ell^2\in\mathcal{B}(\mathbb{R}^\infty)$ and $\mathcal{B}(\ell^2)=\mathcal{B}(\mathbb{R}^\infty)_{|\ell^2}$. In particular, any probability measure $\bar{\Gamma}\in\mathcal{P}(\ell^2)$ uniquely extends to an element in $\mathcal{P}(\mathbb{R}^\infty)$ via $\bar{\Gamma}(A):=\bar{\Gamma}(A\cap\ell^2)$, $A\in\mathcal{B}(\mathbb{R}^\infty)$.*

We will need the following superposition principle [222, Thm.7.1], which lifts a solution to a FPK equation on $\mathbb{R}^\infty$ to a solution to the associated martingale problem. Note that in [222], the author assumes an integrability condition of order $p>1$ instead of $p=1$ as in (7.16) in order to essentially reduce the proof to the corresponding finite-dimensional result, see [222, Thm.2.14], which requires such a higher order integrability. However, since the latter result was later extended to the case of an L^1-integrability condition by the same author [223, Thm.2.5], it is easy to see that also the infinite-dimensional result [222, Thm.7.1] holds for solutions with L^1-integrability as in Definition 7.3.1.

Proposition 7.3.6. *[Superposition principle on $\mathbb{R}^\infty$ [222, Thm.7.1]] For any weakly continuous solution $(\bar{\Gamma}_t)_{0\leq t\leq T}\subseteq\mathcal{P}(\mathbb{R}^\infty)$ to the $\mathbb{R}^\infty$-extended version of (ℓ^2-FPK), there exists $\bar{Q}\in\mathcal{P}(C_T\mathbb{R}^\infty)$, which solves the $\mathbb{R}^\infty$-extended version of (ℓ^2-MGP) such that $\bar{Q}\circ(\pi_t^\infty)^{-1}=\bar{\Gamma}_t$ for each $t\in[0,T]$.*

Note that a path $t\mapsto z_t\in H(\mathcal{SP})$ is continuous with respect to the product topology if and only if it is continuous with respect to the ℓ^2-topology. Hence, we may use the notation $C_TH(\mathcal{SP})$ unambiguously and consider it as a subset of either $C_T\mathbb{R}^\infty$ or $C_t\ell^2$. Since $H(\mathcal{SP})\subseteq\ell^2$ is closed even with respect to the product topology, $C_TH(\mathcal{SP})$ belongs to $\mathcal{B}(C_T\ell^2)$ and $\mathcal{B}(C_T\mathbb{R}^\infty)$.

Lemma 7.3.7. *If in the situation of the previous proposition, each $\bar{\Gamma}_t$ is concentrated on the Borel set $H(\mathcal{SP})\subseteq\mathbb{R}^\infty$, then $\bar{Q}$ is concentrated on $C_TH(\mathcal{SP})$. In particular, in this case, $\bar{Q}$ may be considered an element of $\mathcal{P}(C_T\ell^2)$ and as a solution to the martingale problem (ℓ^2-MGP) as in Definition 7.3.3.*

Proof. The closedness of $H(\mathcal{SP})\subseteq\mathbb{R}^\infty$ yields

$$\bar{Q}(C_TH(\mathcal{SP})) = \bar{Q}\Big(\bigcap_{q\in[0,T]\cap\mathbb{Q}}\{\pi_q^\infty\in H(\mathcal{SP})\}\Big) = 1,$$

where the second equality is due to $\bar{Q} \circ (\pi_t^\infty)^{-1} = \bar{\Gamma}_t$ for each $0 \leq t \leq T$. In particular, we have $\bar{Q} \in \mathcal{P}(C_T\ell^2)$. It is clear that this measure fulfills Definition 7.3.3. $\square$

Hence, subsequently we may consider $\bar{Q}$ as in Proposition 7.3.6 as a solution to the martingale problem on either $\mathbb{R}^\infty$ or ℓ^2 without differing the notation. Recall that we write $p_i : \ell^2 \to \mathbb{R}$, $p_i(z) = z_i$.

Lemma 7.3.8. *Let $\bar{Q}$ be a solution to the martingale problem (ℓ^2-MGP) on ℓ^2. Then, for any $i \geq 1$, the process*

$$M_i(t) := p_i \circ \pi_t^\infty - \int_0^t \bar{B}_i(s, \pi_s^\infty)ds, \quad t \in [0,T], \tag{7.20}$$

is a real-valued, continuous $\bar{Q}$-martingale on $C_T\ell^2$ with respect to the canonical filtration. Furthermore, the covariation process $t \mapsto \langle\langle M_i, M_j\rangle\rangle_t$ of M_i and M_j is $\bar{Q}$-a.s. given by

$$\langle\langle M_i, M_j\rangle\rangle_t = \int_0^t \bar{A}_{ij}(s, \pi_s^\infty)ds, \ t \in [0,T]. \tag{7.21}$$

Proof. For $i, j \geq 1$, let $n \geq \max\{i, j\}$, consider $p_i^n : \mathbb{R}^n \to \mathbb{R}$, $p_i^n(x) = x_i$, and let

$$\bar{F}_i^n : \ell^2 \to \mathbb{R}, \ \bar{F}_i^n(z) = p_i^n \circ \pi^{(n)}(z).$$

Note that $\bar{F}_i^n = p_i$ on ℓ^2, independent of $n \geq \max(i,j)$. For $k \geq 1$, introduce the stopping time $\tau_k := \inf\{t \in [0,T] : ||\pi_t^\infty||_{\ell^2} \geq k\}$ with respect to the canonical filtration on $C_T\ell^2$. Clearly, $\tau_k \nearrow T$ pointwise. Consider $\eta_k \in C_c^2(\mathbb{R}^n)$ such that $\eta_k(x) = 1$ for $|x| \leq k+1$. Since $\partial_k p_i^n = \delta_{ki}$ and $\partial_{kl} p_i^n = 0$ for $1 \leq k, l \leq n$, we have

$$M_i(t) = \bar{F}_i^n \circ \pi_t^\infty - \int_0^t \bar{\nabla}\bar{F}_i^n \circ \pi_s^\infty \cdot \bar{B}(s, \pi_s^\infty) + \frac{1}{2}D^2\bar{F}_i^n \circ \pi_s^\infty : \bar{A}(s, \pi_s^\infty)ds$$

and, setting $\bar{F}_i^{n,k} := (\eta_k p_i^n) \circ \pi^{(n)} \in \mathcal{F}C_b^2(\ell^2)$,

$$M_i(\tau_k \wedge t) = \bar{F}_i^{n,k} \circ \pi_{t\wedge\tau_k}^\infty - \int_0^{t\wedge\tau_k} \bar{\nabla}\bar{F}_i^{n,k} \circ \pi_s^\infty \cdot \bar{B}(s, \pi_s^\infty) + \frac{1}{2}D^2\bar{F}_i^{n,k} \circ \pi_s^\infty : \bar{A}(s, \pi_s^\infty)ds.$$

Since the latter is a continuous $\bar{Q}$-martingale for each $k \geq 1$, it follows that M_i is a continuous local $\bar{Q}$-martingale up to T. Concerning (7.21), it suffices to prove that for any $\bar{F} \in \mathcal{F}C_b^2(\ell^2)$, $\bar{F} = f \circ \pi^{(n)}$, we have

$$\langle\langle M^{\bar{F}}\rangle\rangle_t = \int_0^t \left\langle \bar{\nabla}\bar{F}(\pi_s^\infty), \bar{A}(s, \pi_s^\infty)\bar{\nabla}\bar{F}(\pi_s^\infty)\right\rangle_{\ell^2}, \ t \in [0,T], \quad \bar{Q}\text{-a.s.,} \tag{7.22}$$

with

$$M_t^{\bar{F}} := \bar{F} \circ \pi_t^\infty - \int_0^t \bar{\nabla}\bar{F}(\pi_s^\infty) \cdot \bar{B}(s, \pi_s^\infty) + \frac{1}{2}D^2\bar{F}(\pi_s^\infty) : \bar{A}(s, \pi_s^\infty)ds.$$

Indeed, from here, (7.21) follows by considering (7.22) for $\bar{F}_i^{n,k}$ as $k \to \infty$ and polarization for the quadratic variation. (7.22) follows from the subsequent calculation. We abbreviate the integrand of the integral term in the definition of $M^{\bar{F}}$ by $\bar{\mathbf{L}}_t^{(2)}\bar{F}(\pi_s^\infty)$.

$$\begin{aligned}
(M_t^{\bar{F}})^2 &= \bar{F}^2(\pi_t^\infty) - 2\bar{F}(\pi_t^\infty)\int_0^t \bar{\mathbf{L}}_s^{(2)}\bar{F}(\pi_s^\infty)ds + \Big(\int_0^t \bar{\mathbf{L}}_s^{(2)}\bar{F}(\pi_s^\infty)ds\Big)^2 \\
&= M_t^{\bar{F}^2} + \int_0^t \bar{\mathbf{L}}_s^{(2)}\bar{F}^2(\pi_s^\infty)ds - 2\int_0^t M_s^{\bar{F}} d\Big(\int_0^s \bar{\mathbf{L}}_u^{(2)}\bar{F}(\pi_u^\infty)du\Big) \\
&\qquad - 2\int_0^t \bar{\mathbf{L}}_s^{(2)}\bar{F}(\pi_s^\infty)d\big(M_s^{\bar{F}}\big) - \Big(\int_0^t \bar{\mathbf{L}}_s^{(2)}\bar{F}(\pi_s^\infty)ds\Big)^2 \\
&= N_t^{\bar{F}} + \int_0^t \bar{\mathbf{L}}_s^{(2)}\bar{F}^2(\pi_s^\infty)ds - 2\int_0^t \bar{F}(\pi_s^\infty)d\Big(\int_0^s \bar{\mathbf{L}}_u^{(2)}\bar{F}(\pi_u^\infty)du\Big) \\
&\qquad + 2\int_0^t \Big[\int_0^s \bar{\mathbf{L}}_u^{(2)}\bar{F}(\pi_u^\infty)du\Big]d\Big(\int_0^s \bar{\mathbf{L}}_r^{(2)}\bar{F}(\pi_r^\infty)dr\Big) - \Big(\int_0^t \bar{\mathbf{L}}_s^{(2)}\bar{F}(\pi_s^\infty)ds\Big)^2 \\
&= N_t^{\bar{F}} + \int_0^t \bar{\mathbf{L}}_s^{(2)}\bar{F}^2(\pi_s^\infty)ds - 2\int_0^t \bar{F}(\pi_s^\infty)\bar{\mathbf{L}}_s^{(2)}\bar{F}(\pi_s^\infty)ds \\
&= N_t^{\bar{F}} + \int_0^t \big\langle \bar{\nabla}\bar{F}(\pi_s^\infty), \bar{A}(s,\pi_s^\infty)\bar{\nabla}\bar{F}(\pi_s^\infty)\big\rangle_{\ell^2} ds,
\end{aligned}$$

where

$$N_t^{\bar{F}} := M_t^{\bar{F}^2} - 2\int_0^t \bar{\mathbf{L}}_s^{(2)}\bar{F}(\pi_s^\infty)d\big(M_s^{\bar{F}}\big)$$

is a continuous $\bar{Q}$-martingale on $C_T\ell^2$. Since the martingale solution $\bar{Q}$ particularly fulfills

$$\int_{C_T\ell^2}\int_0^T |\bar{A}_{ii}(t,\pi_t^\infty)|dt\bar{Q} < \infty, \quad i \geq 1,$$

(7.21) implies that M_i is a martingale on $[0,T]$ (which is even square-integrable). □

We summarize the results of this step in the following proposition.

Proposition 7.3.9. *Let $(\bar{\Gamma}_t)_{0\leq t\leq T}$ be a weakly continuous solution to (ℓ^2-FPK) such that $\bar{\Gamma}_t(H(\mathcal{SP})) = 1$ for each $t \in [0,T]$. Then, there exists a solution $\bar{Q} \in \mathcal{P}(C_T\ell^2)$ to the martingale problem (ℓ^2-MGP) such that $\bar{Q}$ is concentrated on $C_TH(\mathcal{SP})$ with $\bar{Q}\circ(\pi_t^\infty)^{-1} = \bar{\Gamma}_t$ for each $t \in [0,T]$. Furthermore, the results of Lemma 7.3.8 apply to $\bar{Q}$.*

Step 3: From (ℓ^2-MGP) to (SNLFPK): If $\bar{Q} \in \mathcal{P}(C_T\ell^2)$ is a solution to (ℓ^2-MGP), set, for $t \in [0,T]$,

$$\mathcal{C} := \mathcal{B}(C_T\ell^2)\bigvee \mathcal{N}_{\bar{Q}}, \quad \mathcal{C}_t := \sigma(\pi_s^\infty, 0 \leq s \leq t)\bigvee \mathcal{N}_{\bar{Q}},$$

where $\mathcal{N}_{\bar{Q}}$ denotes the set of $\bar{Q}$-negligible sets $N \in \mathcal{B}(C_T\ell^2)$. Of course, $\mathcal{C}$ and $\mathcal{C}_t$ depend on $\bar{Q}$, but we suppress this dependence in the notation. We extend $\bar{Q}$ to $\mathcal{C}$ in the canonical way, and denote this extension again by $\bar{Q}$. Then, $(C_T\ell^2, \mathcal{C}, (\mathcal{C}_t)_{0\leq t\leq T}, \bar{Q})$ is a complete filtered probability space. Clearly, $(t,\gamma) \mapsto \bar{\Sigma}(t,\pi_t^\infty(\gamma))$ as in Subsection 7.3.1 is $\mathcal{C}_t$-progressively measurable from $[0,T]\times C_T\ell^2$ to $L(\mathbb{R}^{d_1},\ell^2)$, the space of bounded linear operators from $\mathbb{R}^{d_1}$ to ℓ^2, with $\bar{\Sigma}(t,z) \in L(\mathbb{R}^{d_1},\ell^2)$ given by $\bar{\Sigma}(t,z): x \mapsto (x\cdot\bar{\Sigma}_i(t,z))_{i\geq 1} \in \ell^2$ for $x \in \mathbb{R}^{d_1}$.

Remark 7.3.10. *We extend $(C_T\ell^2, \mathcal{C}, (\mathcal{C}_t)_{0\le t\le T}, \bar{Q})$ as follows. Let $(\Omega', \mathcal{F}'', (\mathcal{F}''_t)_{0\le t\le T}, P)$ be a complete filtered probability space with a real-valued $(\mathcal{F}''_t)_{0\le t\le T}$-Brownian motion β on it. Define*

$$\Omega := C_T\ell^2 \otimes \bigotimes_{l\ge 1}\Omega',\ \mathcal{F}' := \mathcal{C} \otimes \bigotimes_{l\ge 1}\mathcal{F}'',\ \mathcal{F}'_t := \mathcal{C}_t \otimes \bigotimes_{l\ge 1}\mathcal{F}''_t,\ \mathbb{P}' := \bar{Q} \otimes \bigotimes_{l\ge 1}P,$$

let $\mathcal{F}$ and $\mathcal{F}_t$ be the $\mathbb{P}'$-completion of $\mathcal{F}'$ and $\mathcal{F}'_t$, respectively, and denote the canonical extension of $\mathbb{P}'$ to $\mathcal{F}$ by $\mathbb{P}$. Furthermore, we denote the Brownian motion β on the i-th copy of Ω' by β_i and extend each β_i to Ω by $\beta_i(\omega) := \beta_i(\omega_i)$ for $\omega = \big(\gamma, (\omega_i)_{i\ge 1}\big) \in \Omega$. Similarly, we extend each projection π^∞_t from $C_T\ell^2$ to Ω via $\pi^\infty_t(\omega) := \pi^\infty_t(\gamma)$ for ω as above, but keep the same notation for this extended process. Obviously, $(\pi^\infty_t)_{0\le t\le T}$ is continuous and $(\mathcal{F}_t)_{0\le t\le T}$-adapted on Ω, and each β_i is an $(\mathcal{F}_t)_{0\le t\le T}$-Brownian motion on Ω under $\mathbb{P}$. Moreover, by construction, $(\pi^\infty_t)_{0\le t\le T}$ and $(\beta_i)_{i\ge 1}$ are independent on Ω with respect to $\mathbb{P}$. Furthermore, it is clear that the canonical extensions of the processes M_i as in (7.20) to Ω are $\mathbb{P}$-martingales with respect to $(\mathcal{F}_t)_{0\le t\le T}$ for each $i \ge 1$ with covariation as in (7.21), and that $(t, \omega) \mapsto \bar{\Sigma}(t, \pi^\infty_t(\omega)) \in L(\mathbb{R}^{d_1}, \ell^2)$ is $(\mathcal{F}_t)_{0\le t\le T}$-progressively measurable on $[0, T] \times \Omega$.

Finally, we need the following result, which is a special case of [181, Thm.2].

Proposition 7.3.11. *Let $\bar{Q} \in \mathcal{P}(C_T\ell^2)$ be a solution to the martingale problem (ℓ^2-MGP). Then, there exists a complete filtered probability space with a d_1-dimensional Brownian motion $W = (W^\alpha)_{1\le\alpha\le d_1}$ with respect to the filtration of that space, and an ℓ^2-valued adapted continuous process $t \mapsto Y_t$ such that the law of Y on $C_T\ell^2$ is $\bar{Q}$, and for $i \ge 1$ and $t \in [0, T]$, we have almost surely*

$$p_i \circ Y_t - p_i \circ X_0 - \int_0^t \bar{B}_i(s, Y_s)ds = \sum_{\alpha=1}^{d_1} \int_0^t \bar{\Sigma}^\alpha_i(s, Y_s)dW^\alpha_s, \tag{7.23}$$

and the exceptional set can be chosen independent of t and i.

To see this, consider [181, Thm.2] with $X = \ell^2$, $U_0 = \mathbb{R}^{d_1}$, $D = \{p_i, i \ge 1\}$, the processes $M(p_i)$ given by M_i as in (7.20) on the probability space Ω of Remark 7.3.10 and

$$g_s = \bar{\Sigma}(s, \pi^\infty_s) \in L(\mathbb{R}^{d_1}, \ell^2).$$

These choices fulfill all requirements of [181], and the ℓ^2-valued process Y on Ω is given by $Y_t = \pi^\infty_t$. Since all terms in (7.23) are continuous in t, the exceptional set may indeed be chosen independently of $t \in [0, T]$ and $i \ge 1$.

The proof of Theorem 5.2.2 now follows from the above three-step scheme as follows.

Proof of Theorem 5.2.2. Let $t \mapsto \Gamma_t \in \mathcal{P}(\mathcal{SP})$ be a weakly continuous solution to ($\mathcal{SP}$-FPK). By Lemma 7.3.2 of Step 1, the weakly continuous curve of Borel probability measures on ℓ^2

$$\bar{\Gamma}_t := \Gamma_t \circ H^{-1},\ t \in [0, T],$$

solves (ℓ^2-FPK), and each $\bar{\Gamma}_t$ is concentrated on $H(\mathcal{SP})$. By Proposition 7.3.9 of Step 2, there exists a solution $\bar{Q} \in \mathcal{P}(C_T\ell^2)$ to the martingale problem (ℓ^2-MGP), which is concentrated on $C_T H(\mathcal{SP})$ such that

$$\bar{Q} \circ (\pi_t^\infty)^{-1} = \bar{\Gamma}_t,\ t \in [0,T].$$

Furthermore, Lemma 7.3.8 applies to $\bar{Q}$. By Lemma 7.3.8 and Proposition 7.3.11 of Step 3, there is a d_1-dimensional $(\mathcal{F}_t)_{0\leq t\leq T}$-Brownian motion $W = (W^\alpha)_{1\leq\alpha\leq d_1}$ and an $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted process Y on some complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, which fulfill (7.23) and $Y \in C_T H(\mathcal{SP})$ $\mathbb{P}$-a.s., such that $\bar{Q}$ is the law of Y under $\mathbb{P}$.

Possibly redefining $Y(\omega)$ on a $\mathbb{P}$-negligible set $N \subseteq \Omega$ (which preserves (7.23) and the adaptedness, the latter due to the completeness of the underlying filtered probability space), we may assume $Y_t(\omega) = H(\mu_t(\omega))$ for some $\mu_t(\omega) \in \mathcal{SP}$ for each $(t,\omega) \in [0,T]\times\Omega$. The continuity of $H^{-1} : H(\mathcal{SP}) \to \mathcal{SP}$ and $t \mapsto Y_t(\omega)$ implies vague continuity of

$$t \mapsto \mu_t(\omega) = H^{-1} \circ Y_t(\omega) \tag{7.24}$$

for each $\omega \in \Omega$, and also gives $(\mathcal{F}_t)_{0\leq t\leq T}$-adaptedness of the $\mathcal{SP}$-valued process $t \mapsto \mu_t$. Considering (7.23), $Y_t = H(\mu_t)$ and the definition of $\bar{B}$ and $\bar{\Sigma}_i^\alpha$, we obtain, recalling $p_i(H(\nu)) = \nu(h_i)$ for each $\nu \in \mathcal{SP}$,

$$\mu_t(h_i) - \mu_0(h_i) - \int_0^t B_i(s,\mu_s)ds = \sum_{\alpha=1}^{d_1}\int_0^t \Sigma_i^\alpha(s,\mu_s)dW_s^\alpha,\ 0 \leq t \leq T,$$

$\mathbb{P}$-a.s. for each $i \geq 1$. From here, it follows by Lemma 7.2.1 (i) that $t \mapsto \mu_t$ is a solution to (SNLFPK) as in Definition 7.1.1. Furthermore, we note

$$\mathbb{P}\circ\mu_t^{-1} = (\mathbb{P}\circ Y_t^{-1})\circ(H^{-1})^{-1} = \bar{\Gamma}_t\circ(H^{-1})^{-1} = \Gamma_t\circ H^{-1}\circ(H^{-1})^{-1} = \Gamma_t.$$

It remains to prove the final assertion of the theorem. To this end, note that $\Gamma_0(\mathcal{P}) = 1$ implies $\mu_0 \in \mathcal{P}$ $\mathbb{P}$-a.s., with μ_0 as in (7.24) for $t = 0$. From here, the assertion follows by Lemma 7.1.3. □

Besides the transfer of existence ($\mathcal{SP}$-FPK) $\Longrightarrow$ (SNLFPK), which follows immediately from the formulation of Theorem 5.2.2, one also obtains the following uniqueness transfer (SNLFPK) $\Longrightarrow$ ($\mathcal{SP}$-FPK), which is in a similar spirit to Corollary 6.4.2.

Corollary 7.3.12. *Let $\Gamma \in \mathcal{P}(\mathcal{SP})$. Assume any two solutions (μ, W) and $(\bar{\mu}, \bar{W})$ to (SNLFPK) on probability spaces $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ and $(\bar{\Omega}, \mathcal{F}, (\bar{\mathcal{F}}_t)_{0\leq t\leq T}, \bar{\mathbb{P}})$ with initial distribution $\mathbb{P}\circ\mu_0^{-1} = \Gamma = \bar{\mathbb{P}}\circ\bar{\mu}_0^{-1}$ are equal in law, i.e. $\mathbb{P}\circ\mu^{-1} = \bar{\mathbb{P}}\circ\bar{\mu}^{-1}$ on $\mathcal{B}(C_T\mathcal{SP})$. Then, solutions $t \mapsto \Gamma_t$ to ($\mathcal{SP}$-FPK) with $\Gamma_0 = \Gamma$ are unique. If $\Gamma \in \mathcal{P}(\mathcal{P})$, then under the above assumption, solutions $t \mapsto \Gamma_t \in \mathcal{P}(\mathcal{P})$ to ($\mathcal{SP}$-FPK) are unique.*

Appendix B

From finite-dimensional differential to continuity equations

For comparison with the geometric approach to the derivation of the linear equations ($\mathcal{SP}$-CE) and ($\mathcal{SP}$-FPK) on $\mathcal{P}(\mathcal{SP})$, here we briefly recall the corresponding classical Euclidean situation.

Linearization (ODE) $\implies$ (CE)**.** We repeat the well-known steps to show that any solution $t \mapsto \gamma_t$ to (ODE) induces a solution $t \mapsto \delta_{\gamma_t} \in \mathcal{P}$.

If $t \mapsto \gamma_t \in \mathbb{R}^d$ is absolutely continuous with $\dot{\gamma}_t = b_t(\gamma_t)$ dt-a.s., then for any $f \in C^1(\mathbb{R}^d)$, the chain rule and integration with respect to $\int_0^t ds$ for $t \in [0,T]$ yield

$$f(\gamma_t) - f(\gamma_0) = \int_0^t (\nabla f)(\gamma_s) \cdot \dot{\gamma}_s ds = \int_0^t (\nabla f)(\gamma_t) \cdot b_s(\gamma_t) ds.$$

Setting $\mu_t := \delta_{\gamma_t}$, we obtain

$$\int_{\mathbb{R}^d} f(x) d\mu_t(x) - \int_{\mathbb{R}^d} f(x) d\mu_0(x) = \int_0^t \int_{\mathbb{R}^d} \nabla f(x) \cdot b_s(x) d\mu_s(x) ds, \tag{B.1}$$

which yields that $t \mapsto \mu_t$ solves the continuity equation (CE) in distributional sense. It is clear that the above calculation still makes sense when the state space $\mathbb{R}^d$ is replaced by a Riemannian manifold (M,g), in which case the Euclidean gradient is replaced by the gradient on (M,g) and the integrand $\nabla f \cdot b_s$ is replaced by the scalar product $\langle \nabla f(x), b_s(x) \rangle_{g(x)}$, where $\langle \cdot, \cdot \rangle_{g(x)}$ denotes the scalar product in the tangent space $T_x M$ at $x \in M$.

Linearization (SDE) $\implies$ (FPK)**.** Let $t \mapsto X_t$ be a solution to (SDE) on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ and let $f \in C^2(\mathbb{R}^d)$. By Itô's formula, we obtain

$$f(X_t) - f(X_0) = \int_0^t (\nabla f)(X_s) \cdot b_s(X_s) ds + \frac{1}{2} \int_0^t (\partial_{ij} f)(X_s) a_s^{ij}(X_s) ds + M(t) \quad \mathbb{P}\text{-a.s.}$$

where $t \mapsto M(t)$ is a $(\mathcal{F}_t)_{0\le t\le T}$-martingale with respect to $\mathbb{P}$ with $M(0) = 0$, and we have set $a_s(x) = (a_s^{ij}(x))_{1\le i,j\le d} = \sigma_s(x)\sigma_s(x)^T$. Hence, integrating with respect to $\int_{\mathbb{R}^d} d\mathbb{P}$, we have $\mathbb{E}[M(t)] = 0$ and therefore, setting $\mu_t := \mathbb{P} \circ X_t^{-1}$, we obtain

$$\int_{\mathbb{R}^d} f(x)d\mu_t(x) - \int_{\mathbb{R}^d} f(x)d\mu_0(x) = \int_0^t \int_{\mathbb{R}^d} \nabla f(x) \cdot b_s(x) + \frac{1}{2}\partial_{ij} f(x)a_s^{ij}(x)d\mu_s(x)ds.$$

Consequently, the weakly continuous curve $t \mapsto \mu_t$ in $\mathcal{P}$ solves (FPK). In manifold language, the second-order term $\partial_{ij} f(x)a_s^{ij}(x)$ may be rewritten as

$$\sum_{i,j=1}^{d} \partial_{ij} f(x)a_s^{ij}(x) = \big\langle \operatorname{Hess} f \cdot \sigma_s(x), \sigma_s(x)\big\rangle_{\mathbb{R}^d},$$

where $\operatorname{Hess} f$ denotes the usual Hessian matrix of f.

Part III

Nonuniqueness in law for stochastic hypodissipative Navier–Stokes equations

The contents of the following third part of the thesis are based on the joint work [189] with Andre Schenke, former student in the IRTG 2235 Bielefeld–Seoul, who is now a postdoc at Bielefeld University. I acknowledge his valuable contributions to our preprint. Throughout, from first ideas to the technical execution, the project was a true and equal collaboration, with valuable contributions coming from both of us. The contents of [189] and Part III of this thesis have not been used in a thesis or any further publication by Andre Schenke.

Abstract. We study the incompressible hypodissipative Navier–Stokes equations with dissipation exponent $0 < \alpha < 1/2$ on the 3D torus perturbed by an additive Wiener noise term, and we prove the existence of an initial condition for which distinct probabilistically weak solutions exist. To this end, we employ convex integration methods to construct even a probabilistically strong solution, which violates a pathwise energy inequality, up to a suitable stopping time. This work seems to be the first to construct solutions via simple Beltrami waves instead of intermittent jets or flows in a stochastic setting. The contents of this part of the thesis are an extended version of the preprint [189], which is joint work with Andre Schenke (Bielefeld University).

Chapter 8

Introduction

8.1 (Fractional) Navier–Stokes equations

8.1.1 Navier–Stokes equations: An overview

The *Navier–Stokes equations (NSE)* are one of the fundamental sets of equations in the area of fluid dynamics, and arguably its most prominent one. Introduced nearly two centuries ago by French physicist Navier and Irish mathematician Stokes [180, 212], generations of scientists not only from mathematics have been fascinated by the intriguing, but desperately difficult questions of its solvability. In the incompressible case, most often considered in two or three spatial dimensions, the NSE

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p - \nu \Delta v &= 0, \\ \operatorname{div} v &= 0 \end{cases} \tag{NSE}$$

predict the evolution of the velocity field v, describing the incompressible viscous flow of a fluid in a region $\Omega \subseteq \mathbb{R}^d$. Here, $\nu > 0$ denotes the kinematic viscosity parameter of the fluid, and the scalar pressure term p is unknown, but may be computed from v. From now on, we always set $\nu = 1$ for simplicity.

The NSE may be derived from the conservation principles for momentum and mass, respectively. For a thorough derivation based on physical principles and a concise introduction to the field, we refer to the textbooks [191, 221, 164, 209, 73] and the references therein. Needless to say, this list is certainly not complete. The NSE can also be derived rigorously from the Boltzmann equation, cf. [111, 23, 165]. From now on, we restrict attention to three-dimensional cases, but note that there is a vast literature on other dimensions, in particular for the case $d = 2$, cf. the above sources.

The NSE are usually complemented by an initial condition $v(0,\cdot) = x_0$ and, depending on its underlying domain $\Omega \subseteq \mathbb{R}^3$, by a boundary condition. On the one hand, while being of ample physical relevance, boundary conditions add further mathematically difficulties to the question of solvability. At the same time, it is advantageous to work on a bounded domain in order to avoid solutions to grow as $|x| \to \infty$. To avoid both these difficulties, it is common to study the equations on the three-dimensional torus $\mathbb{R}^3/(2\pi\mathbb{Z}^3)$ subject to periodic boundary conditions (cf. Section 9.1).

The fight for regularity and the energy balance. The Laplacian $-\Delta v$ and the nonlinear term $\operatorname{div}(v \otimes v)$ are contrary in their effect on the regularity of solutions. On the one hand, by definition, the Laplacian *harmonizes* the value at a point $v(t,x)$ and its surroundings: locally extreme values are conformed to the average value in a neighborhood, thereby providing a smoothing effect for solutions v.

On the other hand, the nonlinear term disrupts this regularizing effect and is, in fact, a main source for the enormous complexity of the question of existence of global smooth solutions. For an in-depth discussion, also from a physical point of view, we again refer to the textbooks [191, 221, 164, 209, 73].

Here, we would like to give simple evidence of the conflictive effects of $-\Delta v$ and $\operatorname{div}(v \otimes v)$, see [191, Introduction]. To this end, let us separate both terms from each other for a moment and consider their regularization effects individually. The following observations are based on the *kinetic energy profile* of solutions v, i.e. on

$$t \mapsto e_{kin}(v)(t) := \frac{1}{2}\int_{\mathbb{T}^3} |v(t,x)|^2 dx = \frac{1}{2}||v(t,\cdot)||_{L^2}^2,$$

a quantity which will be of consistent interest to us.

Diffusion: the Laplacian. Focusing on the diffusive term $-\Delta v$ (and disregarding the pressure term ∇p and the incompressibility condition for a moment), the NSE boils down to the classical *heat equation*

$$\partial_t v - \Delta v = 0. \tag{8.1}$$

By standard theory [98], solutions to this equation exist globally in time and are smooth in (t,x), even for possibly nonsmooth initial data $v(0,\cdot) = x_0 \in L^2 = L^2(\mathbb{T}^3, \mathbb{R}^3)$. Multiplying (8.1) by v, integration by parts and integrating over $\int_0^t \int_{\mathbb{T}^3} dxdt$ yields

$$\frac{1}{2}\int_{\mathbb{T}^3} |v(t,x)|^2 dx = \frac{1}{2}\int_{\mathbb{R}^3} |v(0,x)|^2 dx - \int_0^t ||\nabla v(t)||_{L^2}^2 dt, \quad t \geq 0. \tag{8.2}$$

Consequently, the kinetic energy profile $t \mapsto e_{kin}(v)(t)$ of any solution v to (8.1) is decreasing. In other words, kinetic energy is *dissipated*, and the term $\int_0^t ||\nabla v(t)||_{L^2}^2 dx$ quantifies the total dissipation up to time t. The connection of this energy dissipation and the smoothing effect of the heat equation can be seen as follows. Assume the initial condition $v(0,\cdot) = x_0$ has the Fourier series

$$x_0(x) = \sum_{\xi \in \mathbb{Z}^3} \hat{x}_0(\xi) e^{i\xi\cdot x}.$$

Then, it is standard to derive the Fourier series of the solution v to (8.1) as

$$v(t,x) = \sum_{\xi \in \mathbb{Z}^3} \hat{x}_0(\xi) e^{-|\xi|^2 t} e^{i\xi \cdot x}.$$

Recall that Fourier modes $\hat{x}_0(\xi)e^{i\xi\cdot x}$ for larger values of $|\xi|$ correspond to smaller length scales and more rapid oscillations. Intuitively, the more active such small scales are, the more irregular we expect the corresponding function to be. With this in mind, comparing the Fourier series of $v(t,\cdot)$ and x_0, we note that the effect of the damping term $e^{-t|\xi|^2}$ is stronger for larger modes, i.e. the impact of small scales with rapid oscillations is reduced significantly, which provides intuitive evidence for the strong smoothing effect for solutions to the heat equation. Put another way, the loss of kinetic energy observed through (8.1) essentially happens on small scales.

Advection: the nonlinear term. Concerning the nonlinear transport term $\operatorname{div}(v \otimes v) = (v \cdot \nabla)v$ (the equality holds under the incompressibility assumption $\operatorname{div} v = 0$), we observe quite the opposite effect. Indeed, still following [191, Introduction], assume the Fourier series of a function $u : \mathbb{T}^3 \to \mathbb{R}^3$ to be a finite sum

$$u(x) = \sum_{|\xi| \le N} \hat{u}(\xi) e^{i\xi \cdot x}$$

for some $N \ge 1$. A direct calculation gives

$$(u \cdot \nabla)u(x) = \sum_{|\xi| \le 2N} \hat{w}(\xi) e^{i\xi \cdot x},$$

where $\hat{w}(\xi) = \sum_{|\zeta| \le N} [\hat{u}(\xi - \zeta) \cdot \zeta] i\hat{u}(\zeta)$. Consequently, compared to u, the nonlinear term $(u \cdot \nabla)u$ activates higher oscillating modes and pumps energy into these smaller scales, an effect which potentially may lead to a blow-up of the gradient of solutions in finite time. Comparing with the damping effect for solutions to the heat equation, it is clear that these effects conflict each other.

Existence and (non)uniqueness results. It is one of the remarkable oddities of mathematics that despite enormous past and ongoing efforts of brilliant mathematicians, the following natural question remains unanswered in the case $d = 3$: *Is there a global in time smooth solution for any divergence-free smooth initial vector field to the NSE?* This notoriously difficult problem has nurtured enormous interest in the NSE, in particular after becoming one of the famous *Millenium problems*, postulated by the Clay Mathematics Institute in 2000. The unbroken ambitions to tackle this million-dollar question led to significant progress towards a comprehensive understanding of the NSE, which we would like to briefly survey here.

While the existence of (necessarily unique) strong solutions (i.e. smooth vector fields $v : [0,\infty) \times \mathbb{T}^3 \to \mathbb{R}^3$ solving the NSE pointwise) remains to be resolved, global existence of *weak* solutions is known. Taking inner product of the NSE with a strong solution v, integration by parts and integrating over $[0,T] \times \mathbb{T}^3$ gives the same *a priori energy balance* as in (8.2). Inspired by this necessary energy relation for strong solutions, Leray [162] and Hopf [117] famously proved the existence of a global in time weak solution

$v \in C_{\text{weak}}(\mathbb{R}_+, L^2) \cap L^2([0,\infty), H^1)$ with respect to any initial value $x_0 \in L^2$, such that (8.2) holds with an inequality, i.e.

$$\frac{1}{2}\int_{\mathbb{T}^3} |v(T,x)|^2 dx + \int_0^T ||\nabla v(t)||_{L^2}^2 dt \leq \frac{1}{2}\int_{\mathbb{T}^3} |v(0,x)|^2 dx, \quad T \geq 0. \tag{8.3}$$

A *weak solution* is a vector field v, which fulfills the equation distributionally, i.e. $\operatorname{div} v = 0$ holds in distribution and

$$\int_{[0,\infty)\times\mathbb{T}^3} v \cdot \big(\partial_t \varphi + (v \cdot \nabla)\varphi + \Delta\varphi\big) dx dt + \int_{\mathbb{T}^3} v(0) \cdot \varphi(0) dx = 0 \tag{8.4}$$

holds for any divergence-free test vector field φ. To date, it remains to be proven whether such *Leray–Hopf solutions* are unique. In fact, the *non*-uniqueness in this class of solutions was conjectured by Ladyzhenskaya [154] in 1967 but, while there is reasonable supporting numerical evidence [124], a mathematical proof of his conjecture is yet to be found.

We do not dwell here on very interesting further topics such as *local* Leray–Hopf solutions, solutions with small sets of nonsmooth times and the existence of strong solutions locally in time, but refer to [53, Ch.1-3], [54], and the references therein.

In this regard, it is interesting that quite recently Buckmaster and Vicol [54] proved a severe nonuniqueness result in a (possibly) strictly larger class of solutions, namely for weak solutions as above *without* the energy inequality (8.3). More precisely, their result is the following striking theorem. Here, H^β denotes the usual fractional Sobolev space of vector fields on $\mathbb{T}^3$ with integrability parameter $p = 2$.

Theorem 8.1.1. *There is $\beta > 0$ such that for any smooth function $e : [0,T] \to [0,\infty)$, there is a weak solution $v \in C([0,T], H^\beta)$ such that $e_{kin}(v)(t) = e(t)$ for all $t \in [0,T]$.*

In particular, there exist arbitrarily many weak solutions with trivial initial condition. The construction is based on the methods of *convex integration*, which provide a powerful tool to construct weak solutions to (fluid dynamical) PDEs with *wild* energy behavior. This result shatters hopes to tackle the famous question of global well-posedness of strong solutions by showing that the NSE are well-posed in this class of weak solutions and that any such weak solution is strong. The question whether a similar nonuniqueness result holds in the class of Leray–Hopf solutions remains an open problem of significant relevance.

The method of convex integration is the main technical ingredient for the present part of the thesis. Before we review its history and basic principles, in the next section, we turn our attention to the equations we are going to treat with them.

8.1.2 Fractional NSE

The equations of interest for this part of the thesis are not the classical NSE, but (a stochastically perturbed version of) the *fractional Navier–Stokes equations (FNSE)*. As before, we assume incompressibility of the flow and work on the three-dimensional torus, i.e. subject to periodic boundary conditions. More precisely, we study (a stochastic version of)

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p + (-\Delta)^\alpha v &= 0, \\ \operatorname{div} v &= 0, \end{cases} \tag{FNSE}$$

and our focus will be on the *hypodissipative* parameter range $0 < \alpha < 1/2$. For example, in [174], these equations model fluid motion subject to internal friction. For details on the operator $(-\Delta)^\alpha$ and its definition via Fourier series, see Section 9.1. The following comparison to the classical Laplacian (in the whole space) might be illuminating. First, recall that for sufficiently regular functions $f : \mathbb{R}^3 \to \mathbb{R}$, we have

$$-\Delta f(x) = C \lim_{r\to 0} \fint_{B_r(x)} \frac{u(x) - u(y)}{r^2} dy, \quad x \in \mathbb{R}^3.$$

On the other hand, it is standard to obtain for $0 < \alpha < 1$ [152] that

$$(-\Delta)^\alpha f(x) = C(\alpha) \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3\setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{3+2\alpha}} dy, \quad x \in \mathbb{R}^3,$$

which allows for the following comparison. While for the classical Laplacian, $-\Delta f(x)$ is reverted to the average of f in an infinitesimally small neighborhood around x, the averaging effect of $(-\Delta)^\alpha$ comprises the whole space in a weighted sense: Values far away from x contribute less to the average than those in proximity to x, and this long-term effect diminishes as $\alpha \nearrow 1$. In particular, in contrast to $-\Delta$, the fractional Laplace is a nonlocal operator and, in fact, gives rise to a Lévy process with long-distance interactions, see [193, 152, 16, 92].

There is a large number of models from, e.g., physics, biology and finance for which the fractional Laplace arguably provides a more fitting description of the setting than the classical Laplacian, or offers at least an interesting variant of the particular model. Hence, it is not surprising that the literature in this direction is extensive. Prominent examples include, but go beyond fractional heat equations [40, 39, 65, 224] and Schrödinger equations [156, 204, 155], quasi-geostrophic equations [58, 137, 84], and obstacle problems [57, 207]. For a further in-depth survey of the field of fractional diffusion equations, we refer to [29, 15, 228, 18, 56]. Concerning the FNSE, known results include the existence of smooth solutions in the case $\alpha \geq 5/4$ [163, 218], and partial regularity results for the cases $\alpha \in [3/4, 1)$ and $(1, 5/4)$, cf. [71, 217].

In the light of the resolved and unresolved (non)uniqueness questions for the classical NSE, it is interesting to ask whether the known results carry over to the case of the FNSE and whether any of the open questions for the NSE can be answered for the FNSE. In this direction, Colombo, De Lellis and De Rosa, and De Rosa in a second paper obtained the following very interesting result [70, 194]: For $0 < \alpha < 1/3$, there is an initial condition $x_0 \in L^2$ subject to which there exist infinitely many global in time α-Hölder continuous weak solutions to (FNSE), which fulfill even a *local* Leray-inequality (i.e. (8.3), with ∇v replaced by $(-\Delta)^{\alpha/2}$, in a local in time sense) up to some common time $T > 0$. Moreover, for any $0 < \alpha < 1/2$, Hölder continuous weak solutions (not necessarily fulfilling an energy inequality) are nonunique. We see that for the range $0 < \alpha < 1/3$, the ill-posedness result is more severe than in the classical case $\alpha = 1$, where uniqueness of Leray–Hopf solutions remains open. Roughly, the above explanation of the contradicting effects of the Laplacian and the nonlinear term for the classical NSE offers the following intuition to this result. Replacing $-\Delta$ by the weaker diffusive term $(-\Delta)^\alpha$, $0 \leq \alpha < 1$, the advective nonlinear term is more dominant, hampers the regularizing effect for solutions to (FNSE), and hence gives rise to more severe nonuniqueness results. As already mentioned for the classical NSE, also in this case, this shatters hopes to prove strong well-posedness via weak solutions, here even in the class of Leray–Hopf solutions.

As for the classical NSE, the results of [70, 194] rely on the techniques of convex integration, which we review in the next subsection.

8.2 A brief history of convex integration

8.2.1 Origins: Nash's C^1 isometric embedding problem

Nowadays famously used as a method to construct weak solutions to (fluid dynamical) PDEs, the origins of convex integration date back to a famous differential geometry result by John Nash [179], which answers the following questions that had intrigued geometers for decades: *Can any smooth d-dimensional Riemannian manifold M be isometrically embedded in ambient space $\mathbb{R}^N$ and, if so, which restriction on the ambient dimension N and which kind of regularity for the isometry can be expected?* It came with general astonishment that Nash provided the following affirmative answer.

Theorem 8.2.1 (C^1 isometric embedding theorem)**.** *Let (M, g) be a closed d-dimensional Riemannian manifold, $v : M \to \mathbb{R}^N$ a strictly short embedding with $N \geq d + 2$ and $\varepsilon > 0$. Then, there is a C^1 isometric embedding $u : M \to \mathbb{R}^N$ such that $||v - u||_{C^0} < \varepsilon$.*

For a short review of the necessary differential geometric concepts, see Appendix F. Although it does not directly contribute to the geometric ideas of convex integration, we point out that under additional technical conditions the closedness assumption can be dropped, and that Kuiper [150] improved Nash's result to the optimal threshold $N \geq d+1$. Moreover, a version of the result with "immersion" instead of "embedding" at all places holds true. For an excellent presentation of Nash's result (as well as other masterpieces of him), we refer to the review [87].

For example, it is standard to embed the torus $\mathbb{T}^2$ in $\mathbb{R}^3$; however, the usual embedding is not isometric, but can be made strictly short. Thanks to Nash's ingenious theorem, there must exist an isometric C^1 embedding $u : \mathbb{T}^2 \to \mathbb{R}^3$ as well.

Idea of proof. As in Appendix F, denote by e the standard Euclidean metric tensor on $\mathbb{R}^N$, by $g_{ij}, 1 \leq i, j \leq d$, the components of the metric tensor g on M, and by $u^\sharp e$ the pullback metric on M for an embedding $u : M \to \mathbb{R}^N$. The mapping u is an isometry, provided

$$g_{ij} = \partial_i u \cdot \partial_j u, \quad 1 \leq i, j \leq d. \tag{8.5}$$

By assumption, there is a *strictly short* embedding v, i.e. it holds

$$g_{ij} > \partial_i v \cdot \partial_j v, \quad 1 \leq i, j \leq d. \tag{8.6}$$

In spirit, the relation (8.6) is easier to fulfill than (8.5) and the task is to convey the solution v to the *flexible* relation (8.6) to a solution u to the *rigid* one (8.5), such that $||u - v||_{C^0} < \varepsilon$. In other words, one needs to stretch out curves on $v(M) \subseteq \mathbb{R}^N$ in order to increase their lengths while staying in a uniform small ε-neighborhood around $v(M)$.

The principal idea of Nash is to iteratively construct a sequence of still strictly short embeddings $(v_q)_{q\in\mathbb{N}_0}$ (with $v_0 = v$) such that the *defect* $g - v_q^\sharp e$, i.e. the gap between the inequality (8.6) and the equality (8.5), decreases in each iteration stage $q \to q+1$ and vanishes as $q \to \infty$. In addition, the iteration is performed such that the differences $||v_{q+1} - v_q||_{C^1}$ are suitably small in terms of ε, such that the limit $\lim_{q\to\infty} v_q := u$ exists in C^1. From here, it follows that

$$g - \partial_i u \cdot \partial_j u = g - \lim_q (\partial_i v_q \cdot \partial_j v_q) = 0,$$

and consequently u is an isometry.

The iteration $v_q \to v_{q+1}$ is set up by adding a suitable perturbation w_{q+1} to v_q, i.e. Nash defines

$$v_{q+1} := v_q + w_{q+1}.$$

The construction of these perturbations is the most delicate part of the proof, since $w_{q+1} = v_{q+1} - v_q$ needs to be sufficiently small in C^1-norm while reducing the gap between (8.5) and (8.6) at stage q substantially. Nash's integral idea was to construct w_{q+1} via (sums of locally supported) rapidly oscillating waves with high frequencies $\lambda_{q+1} \gg \lambda_q \gg 1$ and small amplitudes of order λ_{q+1}^{-1} in such a way that $w_{q+1}^\sharp e$ comprises a large portion of the remaining defect $g - v_q^\sharp e$. A suggestive picture is that one pervades $v_q(M) \subseteq \mathbb{R}^N$ with small scale waves with small amplitudes in order to ripple curves on $v_q(M)$.

The perturbation w_{q+1} is constructed as a sum of locally supported waves of type

$$w_{q+1}^{(j)}(x) = \frac{c_{q+1}^{(j)}}{\lambda_{q+1}} \Big(\nu(x) \cos\left(\lambda_{q+1}\psi_{q+1}^{(j)}(x)\right) + b(x) \sin\left(\lambda_{q+1}\psi_{q+1}^{(j)}(x)\right) \Big), \tag{8.7}$$

where $\nu(x), b(x) \in \mathbb{R}^N$ are of unit length such that $\nu(x) \perp b(x)$ and $\nu(x)$ and $b(x)$ are (roughly) normal to the tangent space $T_{v_q(x)} v_q(M) \subseteq \mathbb{R}^N$. On the one hand, suitable bounds for $||w_{q+1}||_{C^0}$ and $||w_{q+1}||_{C^1}$ follow via a sufficiently large choice of $\lambda_{q+1} \gg 1$. On the other hand, $c_{q+1}^{(j)} > 0$ and $\psi_{q+1}^{(j)} \in C^\infty(M)$ stem from a geometric result by Nash [87, Prop.2.3.1], which allows to (locally) represent the defect metric tensor $g - v_q^\sharp e$ on M as a locally finite sum of *primitive metrics*

$$g - v_q^\sharp e = \sum_j c_{q+1}^{(j)} d\psi_{q+1}^{(j)} \otimes d\psi_{q+1}^{(j)}. \tag{8.8}$$

As we shall see later, a loosely related kind of geometric lemma plays a major role in convex integration techniques in general, as well as for our main result of this part of the thesis, cf. Lemma 11.3.4. From here, a detailed calculation, which is beyond the scope of this introduction, shows that the definition of $w_{q+1} = \sum_j w_{q+1}^{(j)}$ entails the pivotal approximate cancellation

$$w_{q+1}^\sharp e \approx Dw_{q+1} Dw_{q+1}^T \approx g - v_q^\sharp e, \tag{8.9}$$

i.e. a major part of the remaining isometry gap $g - v_q^\sharp e$ is compensated by w_{q+1}, rendering the remaining gap $g - v_{q+1}^\sharp e = (g - v_q^\sharp e) - w_{q+1}^\sharp e$ at stage $q+1$ much smaller than the previous one.

Finally, we point out that the rapidly oscillating waves in (8.7) actually allow for convergence of $(v_q)_{q\in\mathbb{N}}$ in $C^{1,\alpha}$ for suitably low $\alpha \in (0,1)$ [43, 74, 160], but obstruct C^2-regularity of $u = \lim_{q\to\infty} v_q = v_0 + \sum_{q\geq 0} w_{q+1}$. Indeed, it is readily seen that $||w_{q+1}||_{C^2} \sim \lambda_{q+1} \longrightarrow \infty$ as $q \to \infty$.

Let us summarize the guiding principles of Nash's geometric construction, which passes over to the techniques of convex integration, as follows. In order to solve the rigid relation (8.5), one first of all iteratively constructs a sequence $(v_q)_{q\in\mathbb{N}}$ of solutions to the flexible relation (8.6). The iteration proceeds via perturbations w_{q+1}, consisting of waves with increasing frequency $\lambda_{q+1} \gg 1$ and decreasing amplitudes $a_{q+1} \ll 1$. Striking the right balance between the scales of λ_{q+1} and a_{q+1} yields convergence of $(v_q)_{q\in\mathbb{N}}$ in C^1 (even in $C^{1,\alpha}$ for small $\alpha < 1$). At the same time, the perturbations may be constructed in a geometric way in order to approximately cancel the remaining isometry gap $g - v_q^\sharp e$, which implies that $u = \lim_{q\to\infty} v_q$ is the desired isometry, i.e. a solution to the rigid relation (8.5).

8.2.2 Convex integration and the Onsager conjecture

In 1973, Gromov realized that the techniques of Nash's proof of Theorem 8.2.1 are actually an instance of the so-called *h-principle* [112], a deep and general method to obtain solutions to differential relations, on which we do not comment here. Thirty years later, this direction was spurred further by Müller and Šverak by linking these methods to the theory of Lipschitz solutions to differential inclusions [177]. Nash's intriguing geometric approach experienced enormous additional interest once De Lellis and Székelyhidi Jr. in 2009 understood the second fundamental set of equations of fluid dynamics as such a differential inclusion, namely the incompressible Euler equations (EE) on the torus $\mathbb{T}^d$, $d \in \{2,3\}$,

$$\begin{cases} \partial_t v + \operatorname{div}(v\otimes v) + \nabla p &= 0, \\ \operatorname{div} v &= 0. \end{cases}$$

This way, they obtained alternative proofs for the interesting results of Scheffer and Shnirelman [202, 205, 206], in which the authors construct rough weak solutions v to the EE with dissipative kinetic energy $t \mapsto e_{kin}(v)(t)$. From now on, we restrict attention to the case $d = 3$ on $\mathbb{T}^3$.

These results met considerable interest in the community since they were believed to offer a starting point towards a proof of the famous *Onsager conjecture*. In 1949, Norwegian-American physicist Lars Onsager conjectured the threshold of Hölder regularity below which weak solutions to the EE can dissipate kinetic energy to be $\beta = 1/3$ [182]. In this regard, note that smooth solutions v to the EE have constant kinetic energy profile $t \mapsto e_{kin}(v)(t)$, which follows by multiplication of the EE with v and integration by parts. The possible dissipation of kinetic energy for solutions with low regularity, also known as *anomalous dissipation of energy*, is intimately connected to the transfer of energy from larger to smaller scales by the nonlinearity $\operatorname{div}(v\otimes v)$, as suggested in Subsection 8.1.1, and the theory of turbulence, in particular when considering the EE as the inviscid limit of the NSE for vanishing viscosity parameter $\nu \longrightarrow 0$, cf. the fundamental works of Kolmogorov [142, 143, 141] and [53, Sect.2] and the references therein.

More precisely, the conjecture can be stated as follows.

Onsager's conjecture.
For any $0 < \beta < 1/3$, there exists a weak solution $v_\beta \in C\big([0,T], C^\beta(\mathbb{T}^3, \mathbb{R}^3)\big)$ to the EE with decreasing kinetic energy $t \mapsto e_{kin}(v_\beta)(t)$. In contrast, for each $\beta > 1/3$, any weak solution $v \in C\big([0,T]C^\beta(\mathbb{T}^3\mathbb{R}^3)\big)$ conserves kinetic energy.

We remark that the positive part of the assertion concerning th range $\beta > 1/3$ was proven in [99, 72] in 1994. Concerning the negative part, the solutions in [202, 205, 206], while allowing for dissipative energy profiles, do not have any continuity property and hence do not provide evidence for Onsager's conjecture.

Continuous weak solutions to the EE. A further step towards the conjecture was the remarkable work of De Lellis and Székelyhidi [89] from 2013, in which the authors use techniques from the aforementioned geometric and analytic foundations [179, 112, 177], coined as *convex integration*, to construct for the first time *continuous* weak solutions to the EE with decreasing energy profiles $t \mapsto e_{kin}(t)$. See also [161] for the close connection to Gromov's h-principle. More precisely, their solutions can even be constructed such that they obey any prescribed smooth, strictly positive energy profile.

In a way, [89] may be considered the technical starting point of modern convex integration methods. In fact, previous results as in [88] make explicit use of the results of [177] by considering the EE as a differential inclusion and follow a Baire category approach. As pointed out in [89], such techniques are not suited for the construction of (Hölder) continuous weak solutions to the EE. Let us briefly point out the main new techniques of [89].

Common to all convex integration results, and much in the spirit of the geometric origin by Nash, is the iterative construction of a sequence of solutions $(v_q)_{q\in\mathbb{N}_0}$ to a flexible version of the particular equation, which allows for an error term. For the EE (and analogously for the (F)NSE), these flexible equations are the *Euler–Reynolds equations*

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q &= \operatorname{div} \mathring{R}_q, \\ \operatorname{div} v_q &= 0, \end{cases} \tag{ER}$$

where $\mathring{R}_q$ is a symmetric trace-free 3×3-matri,x and the name is due to the form of the error $\operatorname{div} \mathring{R}_q$, which is similar to the so-called *Reynolds stress* in fluid dynamical models, see [161]. We will usually simply refer to $\operatorname{div} \mathring{R}_q$ and to $\mathring{R}_q$, as the *error (term)*. Similarly as in [179], given a smooth solution $(v_q, p_q, \mathring{R}_q)$ to (ER) at stage q, the next solution $(v_{q+1}, p_q, \mathring{R}_{q+1})$ is constructed by means of a perturbation w_{q+1}, i.e.

$$v_{q+1} = v_q + w_{q+1}.$$

The goal is to construct $(v_q)_{q\in\mathbb{N}_0}$ in such a way that the sequence converges in $C^0([0,T] \times \mathbb{T}^3, \mathbb{R}^3)$ while $||\mathring{R}_q||_{C^0}$ vanishes as $q \longrightarrow \infty$. To this end, in analogy to Nash's construction, two of the main features of the perturbation w_{q+1} are its small amplitudes a_{q+1} such that

$\sum_{q\geq 1}||w_{q+1}||_{C^0}$ is finite, and its geometric construction, which allows for a cancellation of type

$$\operatorname{div}(w_{q+1}\otimes w_{q+1})\approx -\operatorname{div}\mathring{R}_q, \tag{8.10}$$

in order to reduce the error term $\operatorname{div}\mathring{R}_q$ (compare with (8.9)). In comparison to [179], the iterative solutions v_q to (ER) assume the role of the derivatives Dv_q of Nash's short embeddings, and the error $\mathring{R}_q$ corresponds to the defect $g - v_q^\sharp e$. Provided $v := \lim_{q\to\infty} v_q$ exists in C^0 and $\lim_{q\to\infty}\mathring{R}_q = 0$, it is readily seen that v is a weak solution to the EE. Note that by subtracting (ER) at stage $q+1$ and q, one can calculate the unknowns $\mathring{R}_{q+1}$ and p_{q+1} from $v_q, \mathring{R}_q$ and w_{q+1} as

$$\operatorname{div}\mathring{R}_{q+1}-\nabla p_{q+1} = \operatorname{div}(w_{q+1}\otimes w_{q+1}+\mathring{R}_q)+[\partial_t+v_q\cdot\nabla]w_{q+1}+(w_{q+1}\cdot\nabla)v_q-\nabla p_q. \tag{8.11}$$

In order to handle both these central aspects at once, in [89] the authors use for the first time three-dimensional building blocks, the so-called *Beltrami waves*, which are vector fields of type $B_\xi e^{i\lambda_{q+1}\xi\cdot x}$, with $\xi, B_\xi\in\mathbb{R}^3$ such that $|\xi| = 1 = |B_\xi|$. On the one hand, each Beltrami wave may be multiplied by a suitably small amplitude in order to obtain any desired C^0-bound for w_{q+1}. Hence, it is plausible to design the perturbation as

$$w_{q+1}(t,x)=\sum_\xi a_{q+1,\xi}(t,x)B_\xi e^{i\lambda_{q+1}\xi\cdot x} \tag{8.12}$$

for suitable amplitudes $a_{q+1,\xi}\sim\delta_{q+1}$, where the decay of $\delta_{q+1}\longrightarrow 0$ needs to scale similarly (but weaker) than the growth of λ_{q+1}. On the other hand, the Beltrami waves allow for the approximate cancellation (8.10), cf. [89, Lem.7.2]. This is sufficient to obtain a C^0-limit of $(v_q)_{q\in\mathbb{N}_0}$ and the convergence $\mathring{R}_q\to 0$ as desired. In fact, in [159], by more careful estimates of essentially the same procedure, the same authors obtained β-Hölder dissipative continuous solutions with $0<\beta<1/10$. However, it turned out that these methods are ill-suited to advance further in the direction of the conjectured threshold $\beta = 1/3$.

From $1/10$ to $1/3$: Proof of Onsager's conjecture. The main impediment to higher regularity of the limit v to solutions of (ER) in [159] is in fact the delicate *transport error* $[\partial_t + v_q\cdot\nabla]w_{q+1}$, which obstructs a sharper estimate for $||\mathring{R}_{q+1}||_{C^0}$. This issue was overcome in a remarkable way by Isett [120], who pushed the threshold of the modulus of Hölder continuity up to which solutions with a *compactly supported* energy profile exist to $\beta = 1/5$. Simplifying and adapting these novel techniques allowed the authors of [50] to construct also dissipative solutions with this regularity.

The main new ideas of [120] can be described as follows. The linear phase $\xi\cdot x$ in the summands of (8.12) is replaced by the nonlinear one $\xi\cdot\Phi(t,x)$, where Φ is the solution to the transport equation

$$\begin{cases}\partial_t\Phi+(v_{q+1}\cdot\nabla)\Phi &= 0,\\ \Phi(0) &= x.\end{cases} \tag{8.13}$$

This way, the material derivative $\partial_t + v_{q+1}\cdot\nabla$ does not fall on the high frequency term $e^{i\lambda_{q+1}\xi\cdot x}$. Moreover, since the amplitudes $a_{q+1,\xi} = a_{q+1,\xi}(\mathring{R}_q)$ are functions of $\mathring{R}_q$ (this necessity arises from the desired cancellation (8.10)), a more precise estimate of the material derivative $[\partial_t+v_{q+1}\cdot\nabla]\mathring{R}_q$ of the previous error is crucial to improve the bound on $||\mathring{R}_{q+1}||_{C^0}$.

Roughly speaking, sharpening this estimate allows for slightly smaller frequency parameters λ_{q+1} in [120, 50], which leads to improved Hölder regularity of solutions. Since in comparison to the linear phase functions, in this situation one additionally has to control the deviation $\Phi - x$, it is important to introduce a localization in time: instead of Φ, one actually considers Φ_j, $j \in \{1, \ldots, \mu\}$, which is the solution to (8.13) on $I_j = (j/\mu - 1/\mu, j\mu + 1/\mu)$. Here, $\mu \in \mathbb{N}$ is a large parameter, which is determined by its relation to the scale of λ_{q+1} and δ_{q+1}. Augmenting (8.12) by suitable cutoff functions χ_j with $\operatorname{supp} \chi_j \subseteq I_j$, the perturbation w_{q+1} in [50] essentially becomes

$$w_{q+1}(t,x) = \sum_j \sum_\xi \chi_j a_{q+1,\xi}(t,x) B_\xi e^{i\lambda_{q+1}\xi\cdot\Phi(t,x)}. \tag{8.14}$$

The remaining step towards the conjectured threshold $\beta = 1/3$ proceeds via several gradual improvements. In [49], solutions with Hölder regularity up to $\beta = 1/3$ *almost everywhere* in time are constructed by establishing more careful almost everywhere *local* estimates on $||w_{q+1}||_{C^1}$ and $||\mathring{R}_{q+1}||_{C^0}$. In [51], such estimates are employed in order to prove the existence of solutions in $L^1_t C^{1/3-}$, resolving Onsager's conjecture up to sufficient regularity in time.

The final proof of the conjecture was given by Isett [122] and Buckmaster, De Lellis, Székelyhidi and Vicol [52] in 2018 and 2019, respectively. More precisely, Isett again constructs solutions with compactly supported energy profiles, which are, strictly speaking, not dissipative. However, his new techniques for the first time led to solutions up to Onsager critical regularity and were adapted by the authors of [52] in order to construct $C^{1/3-}$-solutions, which obey any prescribed positive smooth energy profile. The main novelty of [122] is that *Mikado flows* instead of simple Beltrami waves are used as the oscillating building blocks of w_{q+1}. Compared to Beltrami waves, Mikado flows have a better self-interaction behavior, which leads to improved bounds for the oscillation error. Essentially, the reason is that they are advected by a mean flow. We point out that Mikado flows are introduced already in [86]. Furthermore, Isett employs a new gluing procedure within the construction of the perturbations w_{q+1}. However, since we will not rely on these improvements, we do not comment on these techniques in detail here.

To conclude this survey part, we mention that the schemes of [122, 52] were further optimized by Isett in [121] to obtain solutions in $\cap_{\varepsilon>0} C^{1/3-\varepsilon}$. However, to date the endpoint case $\beta = 1/3$ remains an interesting open question.

8.2.3 Further results and applications

It is no surprise that the convex integration techniques leading to the resolution of Onsager's conjecture have successfully been applied to further models in the area of fluid dynamics and beyond. A prominent example is [54], where Buckmaster and Vicol prove the existence of finite energy weak solutions to the NSE with low Sobolev regularity. To this end, they use yet another type of oscillatory building blocks, namely so-called *intermittent Beltrami flows*. Very roughly, these are approximately Beltrami waves, which, however, have different scaling properties in different L^p-spaces, which renders them more suitable in relation with the diffusive term of the NSE. However, the constructed solutions are *not* Leray solutions. The ill-posedness of weak solutions in the Leray class remains an intriguing problem, and to date it is only conjectured that convex integration techniques can be amended to address this question.

Further applications of convex integration techniques include the isentropic Euler equations as a model of gas dynamics [64], the Hall-MHD equations [85] and ideal MHD equations [25], compressible EE [171], linear transport equations [176] and porous medium equations [76]. For a more extensive list of models and results, there are several excellent survey articles on historical and recent progress in the field, see for example [53, 55] and the references therein. A list of intriguing open questions is included in [53] as well.

Concerning this part of the thesis, of particular interest to us are the convex integration results for the FNSE in [70], which we already mentioned at the end of Section 8.1.1. The techniques therein are very close to those in [50], including the use of rather simple Beltrami waves instead of Mikado or intermittent Beltrami flows. The weak diffusive term $(-\Delta)^\alpha$ can be incorporated into the iterative scheme by classical Schauder estimates. Most interestingly, by a careful estimate of the Hölder norms, the authors were able to prove the local Leray inequality of weak solutions obtained via convex integration up to some time $T > 0$, thereby proving the ill-posedness of the FNSE in the physically relevant class of Leray solutions. However, they obtained such estimates only for small fractional Laplace exponents $0 < \alpha < 1/5$. Despite the remarkable extension of De Rosa to exponents $0 < \alpha < 1/3$ [194], to date it remains open whether these techniques will eventually lead to nonuniqueness of Leray–Hopf solutions to the classical NSE, i.e. $\alpha = 1$.

8.3 Stochastic PDEs

There is a variety of reasons to study (partial) differential equations under the additional influence of random external forces. Typically, differential equations from physics, biology, finance and other areas of applied sciences are used to model the evolution in time of dynamical systems and processes. These systems are influenced by its ambient surroundings on small scale levels via effects which are too complex to be accurately captured by deterministic models. A simple, yet famous example is the apparently random movement of a dust particle suspended in water, triggered by molecule collisions.

It turns out that the bulk of such microscopic effects to deterministic models and additional model uncertainties can be captured via a stochastic perturbation of the corresponding PDE, which gives rise to the *stochastic* partial differential equations (SPDEs). The area of SPDEs is comparably young and mainly emerged in the past fifty years, see [234] for a historical review of the developments in the field.

8.3.1 Regularization by noise

Remarkably, including an external noise term not only takes into account small scale effects to the underlying model in a reasonable way, but is also extremely interesting from a purely mathematical point of view. It turns out that random external forces can lead to a regularization of deterministic PDE. More precisely, ill-posed deterministic equations may turn into well-posed SPDEs. This phenomenon, known as *regularization by noise*, is the central mathematical reason for the enormous recent interest in the theory of SPDEs. A rough intuitive picture is that a typical source for ill-posedness of a differential equation is a singularity in the corresponding vector field, in which a solution can linger for an arbitrary amount of time. In this situation, a sufficiently nondegenerate random external force may move the solution out of the singularity immediately, hence preventing the emergence of several solutions at this point. There is a large number of works in this direction, see [101, 102, 225, 148, 201, 113, 62, 109, 107], as well as the survey article [108].

8.3.2 Convex integration for SPDEs

In this light, one is led to the intriguing question whether the nonuniqueness results obtained by convex integration carry over to the stochastic case. In other words, one may ask whether convex integration techniques can be applied to SPDEs in order to construct solutions with wild energy behavior. First impressive results in this direction are given in the remarkable works by Hofmanová, Zhu and Zhu [115, 116], in which the authors prove ill-posedness of analytically weak martingale solutions to the *stochastic* NSE and EE, respectively, perturbed by an additive or multiplicative Wiener noise. To do so, they adapt convex integration methods in order to construct *pathwise* solutions with wild energy behavior after splitting off the stochastic term to obtain a PDE with random coefficients to which convex integration methods apply pathwise. The dependence of this equation on random coefficients restricts the construction of convex integration solutions to a bounded stopping time. From there, a nontrivial measure theoretic gluing technique allows to construct probabilistically strong solutions (that is, the solution is defined on a prescribed probability space and is adapted to the canonical Wiener filtration). We do not comment in detail on the precise convex integration methods in the stochastic case here, but refer to Section 11.1, where we outline the techniques for our main result, which are much in the spirit of [115].

Further nonuniqueness results for SPDEs obtained via convex integration include the full Euler system [67, 47], the three-dimensional FNSE in the hyperdissipative case $1 \leq \alpha < 5/4$ [232], and the two-dimensional FNSE in the case $0 < \alpha < 1$ [231].

By the classical Engelbert-Cherny theory, pathwise (i.e. strong) uniqueness for SPDEs is equivalent to the existence of a probabilistically strong solution and uniqueness in law (i.e. weak uniqueness), see [96, 66] for the finite-dimensional case, and [188] for a generalization to variational solutions to SPDEs. Therefore, there was a general hope that proving weak uniqueness of, e.g., stochastic EE and NSE might eventually lead to strong uniqueness of these equations. The aforementioned convex integration nonuniqueness results prove these hopes wrong. However, we stress once more that in the case of the NSE, nonuniqueness is not known in the physically important class of Leray–Hopf solutions.

8.4 Main result

In the light of these ill-posedness results for stochastic PDEs, the objective of this part of the thesis is to study the *stochastic hypodissipative Navier–Stokes equations* on the $3D$ torus $\mathbb{T}^3$, i.e.

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p + (-\Delta)^\alpha v &= dB, \\ \operatorname{div} v &= 0, \end{cases} \tag{$\mathrm{HNSE_{sto}}$}$$

with fractional exponent $0 < \alpha < 1/2$. Here, B is a GG^*-Wiener process on a prescribed probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $G : U \to L^2_\sigma$ is a Hilbert–Schmidt operator on some auxiliary Hilbert space U with values in L^2_σ, the solenoidal L^2-vector fields on $\mathbb{T}^3$. We prove nonuniqueness of martingale solutions to this equation, compare [115, Thm.1.2]. To do so, we employ convex integrations methods as described above to construct even a probabilistically strong solution up to a strictly positive stopping time, which violates a natural energy inequality. We impose the following regularity for G.

Assumption on G. The operator $G : U \to L^2_\sigma$ is Hilbert–Schmidt, and there is $\sigma > 0$ such that

$$\mathrm{Tr}((-\Delta)^{\frac{5+2\sigma}{2}-\alpha} GG^*) < \infty. \tag{8.15}$$

Under this standing assumption, we prove the following main result of this part of the thesis.

Theorem 8.4.1. *Let $0 < \alpha < 1/2$ and suppose G fulfills (8.15). Then, for any $T > 0$, there exist two martingale solutions to* ($\mathrm{HNSE}_{\mathrm{sto}}$) *on $[0, \infty)$ subject to a common deterministic initial condition $x_0 \in L^2_\sigma$, which are distinct on $[0, T]$.*

This result fills a gap in the existing literature on nonuniqueness results for stochastic fluid dynamical equations via convex integration techniques. We would like to point out that in contrast to the aforementioned remarkable results in this direction, in particular [115, 116], the present work is the first in which simple Beltrami waves are used as oscillatory building blocks in the convex integration scheme. For a detailed outline of the proof, we refer to Section 10.1, and to Section 11.1 for an explanation of the particular convex integration methods. We would like to mention that completely independent to our preprint [189], a result very similar to the above theorem appeared in [233] at the exact same time.

At this point, we mention that the range $1/2 < \alpha < 1$ seems to be still open, at least in the three-dimensional case, although a nonuniqueness result comparable to Theorem 8.4.1 is strongly expected to hold. However, we do not know whether simple Beltrami waves as used in the present case are applicable to the situation of a higher dissipation term $1/2 < \alpha$.

Organization of Part III. In Chapter 9, in the first two sections, we introduce basics on fractional Sobolev spaces and the fractional Laplacian, and give additional notation conventions. In Sections 9.3 and 9.4, we discuss martingale solutions to ($\mathrm{HNSE}_{\mathrm{sto}}$) and the extensions of local martingale solutions to global ones.

In Chapter 10, we present the proof of the main result, Theorem 8.4.1. First, we outline the idea in Section 10.1. Afterwards, we decompose ($\mathrm{HNSE}_{\mathrm{sto}}$) into equations ($\mathrm{SL}_\alpha$) and (NL-SHNSE), and discuss regularity for the solution to the former equation. Then, in Section 10.3, we use the results of the previous section and the pathwise weak solution to (NL-SHNSE) (which we will construct in Chapter 11) to construct an analytically weak solution u to ($\mathrm{HNSE}_{\mathrm{sto}}$) up to a stopping time T_L, and we show that u gives rise to a local martingale solution in Section 10.4. Finally, in Section 10.5, we use the extension method discussed in Section 9.4 to extend this local martingale solution to a global one in order to complete the proof.

In Chapter 11, we focus on the construction of a pathwise analytically weak solution to (NL-SHNSE) with anomalous energy behavior, which we used in Chapter 10 to complete the proof of the main result. The construction is based on the method of convex integration, which we outline in Section 11.1. The core result of this chapter, Corollary 11.2.3, follows from the iterative proposition 11.2.2, as outlined in Section 11.2. Finally, this iteration is proven in Section 11.3. This final section on the proof of Proposition 11.2.2 consists of the implementation of convex integration techniques to our setting.

Chapter 9

Preliminaries

We begin this chapter by recalling basics about Fourier analysis on the torus $\mathbb{T}^3$ and introduce (fractional) Sobolev spaces and the fractional Laplace operator. Moreover, we fix the notation specific to this part of the thesis. Afterwards, we introduce the notion of global and local martingale solutions, state a crucial existence and stability result for such solutions in Proposition 9.3.4, and present the measure theoretic techniques by which local solutions may be extended to global ones.

9.1 Fourier analysis on $\mathbb{T}^3$, Sobolev spaces, fractional Laplacian

Functions on $\mathbb{T}^3$. A function $f : \mathbb{R}^3 \to \mathbb{C}$ is called 2π-*periodic*, if $f(x + k2\pi e_i) = f(x)$ for each $x \in \mathbb{R}^3$, $k \in \mathbb{N}$ and $i \in \{1, 2, 3\}$. For $S^1 := \{e^{i\theta}, \theta \in \mathbb{R}\}$, we denote by $\mathbb{T}^3 := \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ the *three-dimensional torus*. It is clear that any 2π-periodic function f determines a unique function $\tilde{f} : \mathbb{T}^3 \to \mathbb{C}$ and vice versa, and that any such f is uniquely determined by its values on $[-\pi, \pi]^3$. In comparison to the general notation introduced in Chapter 0, in the present periodic setting, we write, for $p \in [1, \infty]$ and $l \in \mathbb{N}$,

$$L^p(\mathbb{T}^3, \mathbb{C}) := L^p([-\pi, \pi]^3, \mathbb{C}), \quad L^p(\mathbb{T}^3, \mathbb{C}^l) := \big\{f = (f_1, \dots, f_l) : f_i \in L^p(\mathbb{T}^3, \mathbb{C}), i \leq l\big\}$$

and endow these spaces with the following usual norms, suppressing the dimension of the state space,

$$||f||^p_{L^p} := \frac{1}{(2\pi)^3} \int_{[-\pi,\pi]^3} |f|^p dx \text{ for } p < \infty, \text{ and } ||f||_{L^\infty} := \text{ess sup}_{x\in[-\pi,\pi]^3} |f(x)|.$$

In the case $p = 2$, these are Hilbert spaces with inner product

$$\langle f, g\rangle_{L^2} := \frac{1}{(2\pi)^3} \int_{[-\pi,\pi]^3} f \cdot \bar{g}\, dx. \tag{9.1}$$

We introduce the subspaces of *solenoidal* elements as

$$L^2_\sigma(\mathbb{T}^3, \mathbb{C}^l) := L^2(\mathbb{T}^3, \mathbb{C}^l) \cap \{f : \operatorname{div} f = 0\},$$

where the equality $\operatorname{div} f = 0$ is understood in distributional sense. The usual solenoidal orthogonal projection is denoted by $\mathbb{P} : L^2(\mathbb{T}^3, \mathbb{C}^l) \to L^2_\sigma(\mathbb{T}^3, \mathbb{C}^l)$. Moreover, we write

$$C^0(\mathbb{T}^3, \mathbb{C}^l) := C(\mathbb{T}^3, \mathbb{C}^l) := \big\{f = (f_1, \dots, f_l) \in C(\mathbb{R}^3, \mathbb{C}^l) : f_i \ 2\pi\text{-periodic}, \ i \le l\big\},$$

and similarly for the spaces $C^k(\mathbb{T}^3, \mathbb{C}^l)$, $k \in \mathbb{N} \cup \{\infty\}$, and $C^\gamma(\mathbb{T}^3, \mathbb{C}^l)$, $\gamma \in (0, 1)$. In the case $l = 3$, for any $p \in [1, \infty]$, $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\gamma \in (0, 1)$, we use the abbreviations L^p, L^2_σ, C^k and C^γ.

Fourier analysis on $\mathbb{T}^3$. It is readily seen that $\{e^{i\xi\cdot x}, \xi \in \mathbb{Z}^3\}$ is an orthonormal system in $L^2(\mathbb{T}^3, \mathbb{C})$. In fact, it is well-known that this system is also complete and hence an orthonormal basis. Define the *ξ-th Fourier coefficient of* $f \in L^1(\mathbb{T}^3, \mathbb{C}) \supseteq L^2(\mathbb{T}^3, \mathbb{C})$ as

$$\hat{f}(\xi) := \frac{1}{(2\pi)^3} \int_{[-\pi,\pi]^3} e^{-i\xi\cdot x} f dx = \langle f, e^{i\xi\cdot x}\rangle_{L^2}.$$

In particular, we have

$$f = \sum_{\xi\in\mathbb{Z}^3} \hat{f}(\xi) e^{i\xi\cdot x}, \quad f \in L^2(\mathbb{T}^3, \mathbb{C}),$$

and *Parseval's identity* holds:

$$||f||^2_{L^2} = \sum_{\xi\in\mathbb{Z}^3} |\hat{f}(\xi)|^2, \quad f \in L^2(\mathbb{T}^3, \mathbb{C}). \tag{9.2}$$

The latter even implies that

$$\mathcal{F} : L^2(\mathbb{T}^3, \mathbb{C}) \to \ell^2(\mathbb{Z}^3, \mathbb{C}), \quad \mathcal{F}(f) := \big(\hat{f}(\xi)\big)_{\xi\in\mathbb{Z}^3},$$

is a unitary isomorphism. The above observations clearly remain valid component wise for multidimensional-valued functions.

Sobolev spaces and fractional Laplace operator. For $s \ge 0$, we set

$$\begin{aligned} H^s := H^s(\mathbb{T}^3, \mathbb{C}^3) &:= \{f \in L^2_\sigma : ||(1 - \Delta)^{s/2} f||_{L^2} < \infty\} \\ &= \Big\{f = (f_1, f_2, f_3) \in L^2_\sigma : \sum_{\xi\in\mathbb{Z}^3} (1 + |\xi|^2)^s \hat{f}_i(\xi)^2 < \infty, i \le 3\Big\}. \end{aligned}$$

H^s is a $\mathbb{C}$-Hilbert space with scalar product

$$\langle f,g\rangle_{H^s} := \langle (1-\Delta)^{s/2} f, (1-\Delta)^{s/2} g\rangle_{L^2} = \sum_{1\le i\le 3}\sum_{\xi\in\mathbb{Z}^3} (1+|\xi|^2)^s \hat{f}_i(\xi)\overline{\hat{g}_i(\xi)},$$

where the equality stems from the symbol $(1+|\xi|^2)^{s/2}$ of the operator $(1-\Delta)^{s/2}$ as a Fourier multiplier. The corresponding norm is denoted by $||\cdot||_{H^s}$. In particular, we have $H^0 = L^2_\sigma$.

For $s>0$, let H^{-s} denote the topological dual space of H^s with the standard dual norm and consider the dual pairing

$$H^{-s}\times H^s \to \mathbb{C},\ (f,g)\mapsto \langle f,g\rangle_{(-s,s)} := f(g).$$

Furthermore, we recall that the embedding $H^s \hookrightarrow H^r$ is dense and compact for any $-\infty < r < s < \infty$, cf. [220, Eq.(3.12)]. We also recall that for $\gamma\in[0,1)$, the embeddings

$$H^s \hookrightarrow C^\gamma, \quad s\ge \frac{3}{2}+\gamma, \tag{9.3}$$

are continuous.

For $\alpha\in(0,1)$, the *fractional Laplace operator* $(-\Delta)^\alpha$ is the operator with symbol $|\xi|^{2\alpha}$ as a Fourier multiplier, i.e. for $f\in H^s$, $s\in\mathbb{R}$, it has the (formal) Fourier series

$$(-\Delta)^\alpha f(x) = \sum_{\xi\in\mathbb{Z}^3} |\xi|^{2\alpha}\hat{f}(\xi)e^{i\xi\cdot x}, \tag{9.4}$$

which is convergent if and only if $f\in H^s$, $s\ge 2\alpha$. The map $(-\Delta)^\alpha : H^s\to H^{s-2\alpha}$, $f\mapsto(-\Delta)^\alpha f$, is continuous for each $s\in\mathbb{R}$ and $\alpha\in(0,1)$. We recall the following relation between $(-\Delta)^{s/2}$ and the norm $||\cdot||_{H^s}$: There is a constant $C=C_S>1$ such that

$$C^{-1}\Big(||f||^2_{L^2} + ||(-\Delta)^{s/2}f||^2_{L^2}\Big) \le ||f||^2_{H^s} \le C\Big(||f||^2_{L^2} + ||(-\Delta)^{s/2}f||^2_{L^2}\Big), \quad f\in H^s. \tag{9.5}$$

9.2 Notation

For this part of the thesis, in addition to the general notation, we use the following conventions and abbreviations. For an interval $I\subseteq\mathbb{R}$ and $r\in[0,1)\cup\mathbb{N}$, we write $C^r_{I,x} := C^r(I\times\mathbb{T}^3,\mathbb{R}^3)$, and $||\cdot||_{C^r_{I,x}}$ and $[\cdot]_{C^r_{I,x}}$ for the corresponding norm and seminorm, respectively, as introduced in Chapter 0. In the special case $I=[0,t]$, we simply write $C^r_{t,x}$, $||\cdot||_{C^r_{t,x}}$ and $[\cdot]_{C^r_{t,x}}$. Similarly, we write $C^0_I C^r_x := C\big(I, C^r(\mathbb{T}^3,\mathbb{R}^3)\big)$ with norm and seminorm $||f||_{C^0_I C^r_x} = \sup_{t\in I}||f(t,\cdot)||_{C^r}$ and $[f]_{C^0_I C^r_x} = \sup_{t\in I}[f(t,\cdot)]_{C^r}$, respectively, and $C^0_t C^r_x$, $||\cdot||_{C^0_t C^r_x}$ and $[\cdot]_{C^0_I C^r_x}$ in the case $I=[0,t]$. For spaces of smooth functions, we use the notation $C^\infty_{t,x}\mathbb{R}^l := C^\infty([0,t]\times\mathbb{T}^3,\mathbb{R}^l)$, and $C^\infty_{t,x}$ in the case $l=3$. For a normed space $(X,||\cdot||_X)$ and $f:[0,t]\to X$, we also use the standard notation $||f||_{L^\infty_t X} = \operatorname{ess\,sup}_{s\in[0,t]}||f(s)||_X$. For $\varphi = (\varphi^1,\varphi^2,\varphi^3)\in C^1(\mathbb{R}^3,\mathbb{R}^3)$, we write $D\varphi := (\nabla\varphi^1,\dots,\nabla\varphi^3)\in\mathbb{R}^{3\times3}$. In the case of an inequality $a\le Cb$ for real numbers $a,b\in\mathbb{R}$ and an absolute constant $C>0$, we also write $a\lesssim b$.

Probabilistic elements. The main path space for this part of the thesis is $\Omega_0 := C(\mathbb{R}_+, H^{-3})$, equipped with the topology of locally uniform convergence, which renders this a Polish space. We denote a generic element in Ω_0 by $x = (x_t)_{t\geq 0}$ and write $\pi_t : \Omega_0 \to H^{-3}$, $\pi_t(x) := x_t$, for the canonical projection at $t \geq 0$. We introduce the σ-algebras

$$\mathcal{B} := \sigma(\pi_s, s \geq 0) = \mathcal{B}(\Omega_0), \quad \mathcal{B}_t^0 := \sigma(\pi_s, 0 \leq s \leq t), \quad \mathcal{B}^t := \sigma(\pi_s, s \geq t), \quad t \geq 0,$$

and we denote by $(\mathcal{B}_t)_{t\geq 0}$ the right-continuous filtration associated to $(\mathcal{B}_t^0)_{t\geq 0}$. Since Ω_0 is Polish, $\mathcal{P}(\Omega_0)$ is a separable metric space.

If τ is a finite $(\mathcal{B}_t)_{t\geq 0}$-stopping time, we denote by $\Omega_{0,\tau}$ the space of paths stopped at τ, i.e. $\Omega_{0,\tau} := \{x(\cdot \wedge \tau) : x \in \Omega_0\} = \{x \in \Omega_0 : x = x(\cdot \wedge \tau)\}$. Note that $\Omega_{0,\tau} \in \mathcal{B}(\Omega_0)$ and hence $\mathcal{P}(\Omega_{0,\tau}) \subseteq \mathcal{P}(\Omega_0)$.

9.3 Martingale solutions

In this section, we introduce the notion of global and local martingale solutions to (HNSE_{sto}), which we consider throughout this part of the thesis. Moreover, Proposition 9.3.4 contains an existence and stability result, which we will employ for the conclusion of the proof of our main theorem as well, as in order to extend the local martingale solution, which we will obtain in Proposition 10.4.3 to a global one, cf. Lemma 9.4.2 and Proposition 10.5.1.

For this entire part of the thesis, we fix $0 < \alpha < 1/2$ and use the notation

$$F_\alpha : y \mapsto \operatorname{div}(y \otimes y) + (-\Delta)^\alpha y, \quad y \in \mathcal{D}(F_\alpha) \subseteq L^2_\sigma.$$

The domain of F_α is discussed in Lemma 9.3.6.

9.3.1 Global and local martingale solutions

The subsequent definition of global martingale solutions is in order with the framework of [110], which is important to note, since for the proof of Proposition 9.3.4 (i) below, we want to evoke a general existence result of [110] in order to obtain martingale solutions to (HNSE_{sto}).

Definition 9.3.1. Let $\gamma \in (0, 1)$ and $(s, x_0) \in \mathbb{R}_+ \times L^2_\sigma$. A probability measure $P \in \mathcal{P}(\Omega_0)$ is a *martingale solution to* (HNSE_{sto}) *on* $[s, \infty)$ *with initial condition* (s, x_0), if

(M1) $P\big(x \in \Omega_0 : x(t) = x_0,\, t \in [0, s]\big) = 1$.

(M2) For each $e \in H^3$, the process

$$M_s^e(t) := \langle \pi_t - x_0, e\rangle_{(-3,3)} + \int_s^t \langle F_\alpha\big(\pi_r\big), e\rangle_{(-3,3)} dr, \quad t \geq s,$$

is a continuous real-valued, square-integrable $(\mathcal{B}_t)_{t\geq s}$-martingale on Ω_0 with respect to P with quadratic variation

$$t \mapsto \langle\langle M_s^e \rangle\rangle_t = (t - s)||G^* e||_U^2 \quad P\text{-a.s.}$$

(M3) For each $q \in \mathbb{N}$, there is a nonnegative continuous function $t \mapsto C_{t,q} = C_{t,q}(s, x_0, P)$, such that for every $t \geq s$

$$\mathbb{E}_P \left[\sup_{r \in [0,t]} ||\pi_r||_{L^2}^{2q} + \int_s^t ||\pi_r||_{L^2}^{2(q-1)} ||\pi_r||_{H^\gamma}^2 dr \right] \leq C_{t,q}\big(||x_0||_{L^2}^{2q} + 1\big). \tag{9.6}$$

Remark 9.3.2. *(i) By (9.6), it follows that the complement of $L^\infty_{\text{loc}}(\mathbb{R}_+, L^2_\sigma)$ in $C(\mathbb{R}_+, H^{-3})$ is P-negligible for any martingale solution P. Since the embedding $H^{-3} \hookrightarrow L^2_\sigma$ is continuous, it follows from [103, Lem.2.1] that P is concentrated on weakly continuous paths in L^2_σ and that $L^\infty_{\text{loc}}(\mathbb{R}_+, L^2_\sigma) \cap \Omega_0 \in \mathcal{B}$. In particular, π_t maps into $\in L^2_\sigma$ for all $t \geq 0$ P-a.s., which, concerning (M2), yields*

$$\langle \pi_t, e \rangle_{(-3,3)} = \langle \pi_t, e \rangle_{L^2},\ t \geq 0,\ P\text{-a.s.}$$

and, in view of (9.13), that the integral term in the definition of M^e is well-defined.

(ii) Since G is Hilbert–Schmidt, $GG^* \in L(L^2_\sigma)$ is symmetric, nonnegative and has finite trace. By the regularity assumption $\operatorname{Tr}[(-\Delta)^{\rho_0 \alpha} GG^*] < \infty$, there exists an orthonormal basis $\{e_j\}_{j \geq 1}$ of L^2_σ in H^3, consisting of eigenvectors of GG^*. Denote by $\{\lambda_j\}_{j \geq 1}$ the corresponding sequence of eigenvalues with $\lambda_j > 0$. By (M2), in the context of the above definition, $\lambda_j^{-1/2} M_s^{e_j}$ has quadratic variation

$$\langle\langle \lambda_j^{-1/2} M^{e_j} \rangle\rangle_t = t - s, \quad t \geq s,$$

i.e. $\lambda_j^{-1/2} M^{e_j}$ is a real-valued $(\mathcal{B}_t)_{t \geq s}$-Brownian motion on Ω_0 under P. Consequently,

$$M_s(t) := \sum_{j \geq 1} M_s^{e_j}(t) e_j, \quad t \geq s,$$

is an L^2_σ-valued GG^*-Wiener process starting from s on $(\Omega_0, \mathcal{B}, (\mathcal{B}_t)_{t \geq s}, P)$.

As we shall see in Chapter 11, our convex integration method does not yield a global martingale solution in the sense of the above definition. Instead, we construct an analytically weak solution u to (HNSE_{sto}) up to a suitable bounded stopping time τ on Ω_0 and consider its law as a probability measure on $\Omega_{0,\tau}$. Therefore, we introduce the following definition of local martingale solutions to (HNSE_{sto}) up to a stopping time, which is similar to the global case introduced above.

Definition 9.3.3. Let $\gamma \in (0,1)$, $(s, x_0) \in \mathbb{R}_+ \times L^2_\sigma$ and $\tau \geq s$ be a finite $(\mathcal{B}_t)_{t \geq s}$-stopping time. A probability measure $P \in \mathcal{P}(\Omega_{0,\tau})$ is a *martingale solution to* (HNSE_{sto}) *on* $[s, \tau]$ *with initial condition* (s, x_0), if

(M1) $P\big(x \in \Omega_0 : x(t) = x_0,\ t \in [0, s]\big) = 1.$

(M2) For each $e \in H^3$, the process

$$M_s^e(t \wedge \tau) = \langle \pi_{t \wedge \tau} - x_0, e \rangle_{(-3,3)} - \int_s^{t \wedge \tau} \langle F_\alpha(\pi_r), e \rangle_{(-3,3)} dr, \quad t \geq s,$$

is a continuous real-valued, square-integrable $(\mathcal{B}_t)_{t\geq s}$-martingale on Ω_0 with respect to P with quadratic variation

$$t \mapsto \langle\langle M_s^e(\cdot \wedge \tau)\rangle\rangle_t = (t\wedge\tau - s)||G^* e||_U^2 \quad P\text{-a.s.}$$

(M3) For each $q \in \mathbb{N}$, there is a nonnegative continuous function $t \mapsto C_{t,q} = C_{t,q}(s, x_0, P)$ such that for every $t \geq s$

$$\mathbb{E}_P\left[\sup_{r\in[0,t\wedge\tau]} ||\pi_r||_{L^2}^{2q} + \int_s^{t\wedge\tau} ||\pi_r||_{L^2}^{2(q-1)} ||\pi_r||_{H^\gamma}^2 dr\right] \leq C_{t,q}\big(||x_0||_{L^2}^{2q} + 1\big). \tag{9.7}$$

Similarly to Remark 9.3.2 (i) for the global case, for any local martingale solution P up to a stopping time τ, we have $P\big(L^\infty_{\text{loc}}(\mathbb{R}_+, L^2_\sigma) \cap \Omega_0\big) = 1$ and $\pi_{t\wedge\tau} \in L^2_\sigma$ for all $t \geq 0$ P-a.s.

9.3.2 General existence and stability for martingale solutions to (HNSE$_{\text{sto}}$)

As mentioned before, we need an existence result for global martingale solutions for two reasons. Firstly, as outlined in Section 10.1, concerning our main result Theorem 8.4.1, we need a martingale solution which we can distinguish from the solution constructed via convex integration methods in Chapter 11 and Sections 10.4 and 10.5. Secondly, in order to extend local martingale solutions (which is all we are able to construct via our convex integration methods) to global ones, we follow the procedure described in Section 9.4, which requires the existence of martingale solutions with an arbitrary initial condition $(s, x_0) \in \mathbb{R}_+ \times L^2_\sigma$.

Furthermore, since in Section 9.4 we will select *measurable* families of martingale solutions $R = (R_{\tau(x),x(\tau(x))})_{x\in\Omega_0}$, we also need a stability result for the class of martingale solutions with respect to the initial data. The subsequent proposition contains everything we need in this direction. Below, for $\gamma > 0$, we denote by $\mathcal{C}_\gamma\big(s, x_0, C_{t,q}\big)$ the class of all global martingale solutions which fulfill (M3) with γ and the family of functions $t \mapsto C_{t,q}$, $q \in \mathbb{N}$, and we stress that in the second part of the proposition, $\tilde{C}_{t,q}$ is assumed to be independent of $n \in \mathbb{N}$.

Proposition 9.3.4. *(i) There is a family of continuous nonnegative functions $\mathbb{R}_+ \ni t \mapsto C_{t,q}$, $q \in \mathbb{N}$, such that for each $(s, x_0) \in \mathbb{R}_+ \times L^2_\sigma$ there exists a global martingale solution $P = P_{s,x_0} \in \mathcal{C}_\alpha\big(s, x_0, C_{t,q}\big)$ to (HNSE$_{\text{sto}}$).*

(ii) If $P_n \in \mathcal{C}_\gamma\big(s_n, x_n, \tilde{C}_{t,q}\big)$ for each $n \in \mathbb{N}$ and $(s_n, x_n) \underset{n\to\infty}{\longrightarrow} (s, x_0)$ in $\mathbb{R}_+ \times L^2_\sigma$, then there is a subsequence $(P_{n_k})_{k\in\mathbb{N}}$ which converges weakly in $\mathcal{P}(\Omega_0)$ to some $P \in \mathcal{C}_\gamma\big(s, x_0, \tilde{C}_{t,q}\big)$.

For later use, we state the following important observation concerning the global martingale solutions constructed in [110].

Remark 9.3.5. *The construction of the measure P_{s,x_0} within the proof of [110, Thm.4.6] via Galerkin approximations implies that P_{s,x_0} obeys the energy estimate*

$$\mathbb{E}_{P_{s,x_0}}\big[||\pi_t||_{L^2}^2\big] \leq ||x_0||_{L^2}^2 + (t-s)\,\mathrm{Tr}(GG^*), \quad t \geq s. \tag{9.8}$$

Indeed, it is readily seen that the Galerkin approximations P_n in the proof of [110, Thm.4.6] satisfy (9.8) uniformly in $n \in \mathbb{N}$. Hence, the claim follows from the weak convergence $P_n \underset{n\to\infty}{\longrightarrow} P_{s,x_0}$ in $\mathcal{P}(\Omega_0)$ and the lower semicontinuity of $x \mapsto ||x(t)||^2_{L^2}$ on Ω_0 for $t \geq 0$.

Concerning the proof of the above proposition, for (i) one shows that the general result of [110, Thm.4.6] applies to our setting. The second part is a close adaption of an analogous result for the case $\alpha = 1$, i.e. the classical stochastic Navier–Stokes equations, see [115, Thm.3.1]. Since both parts of the proposition are pivotal for the proof of our main result, we provide a full proof.

First of all, we state the following identities, which readily follow from the definition of $(-\Delta)^\alpha$ as a Fourier multiplier, see (9.4), and the dense continuous embedding $H^3 \hookrightarrow H^\alpha$, $0 < \alpha < \frac{1}{2}$.

$$||y||^2_{H^\alpha} = \sup_{z\in H^3,\, ||z||_{H^\alpha}\leq 1} \langle y, z\rangle_{H^\alpha} = \sup_{z\in H^3, ||z||_{H^\alpha}\leq 1} \langle y, (1-\Delta)^\alpha z\rangle_{L^2}, \quad y \in H^\alpha, \tag{9.9}$$

$$\langle (-\Delta)^\alpha y, z\rangle_{(-3,3)} = \langle y, (-\Delta)^\alpha z\rangle_{L^2}, \quad y \in L^2_\sigma,\, z \in H^3, \tag{9.10}$$

and

$$\langle -(-\Delta)^\alpha y, y\rangle_{(-3,3)} = -||(-\Delta)^{\alpha/2} y||^2_{L^2}, \quad y \in H^3. \tag{9.11}$$

Furthermore, we need the following lemma.

Lemma 9.3.6. *For $0 < \alpha < 1/2$, the mapping $F_\alpha : y \mapsto \operatorname{div}(y \otimes y) + (-\Delta)^\alpha y$ extends continuously from H^1 to an operator $F_\alpha : L^2_\sigma \to H^{-3}$ with*

$$\langle F_\alpha(y), z\rangle_{(-3,3)} = -\langle y \otimes y, Dz\rangle_{L^2} + \langle y, (-\Delta)^\alpha z\rangle_{L^2}, \quad y \in L^2_\sigma,\, z \in H^3.$$

Proof. Considering (9.10), we only need to extend $y \mapsto \operatorname{div}(y \otimes y)$ from H^1 to L^2_σ. For $y \in H^1$, using integration by parts, we have for each $z \in H^3$ with $||z||_{H^3} \leq 1$

$$|\langle \operatorname{div}(y\otimes y), z\rangle_{(-3,3)}| = |\langle (y\cdot\nabla)z, y\rangle_{L^2}| = |\langle y\otimes y, Dz\rangle_{L^2}| \leq ||Dz||_{L^\infty}||y||^2_{L^2} \leq C||y||^2_{L^2},$$

where $C > 0$ is independent of y and z and comes from the Sobolev embeddings $H^3 \hookrightarrow H^2 \hookrightarrow L^\infty$. We conclude $||\operatorname{div}(y\otimes y)||_{H^{-3}} \leq C||y||^2_{L^2}$, which by density of H^1 in L^2_σ allows us to extend $y \mapsto \operatorname{div}(y\otimes y)$, and hence F_α as claimed. □

With these preparation at hand, we proceed to the proof of Proposition 9.3.4.

Proof of Proposition 9.3.4 (i). We aim to apply the existence result [110, Thm.4.6]. To this end, we claim that the following choices fulfill all assumptions of [110], where the symbols on the left-hand sides follow the notation of that paper.

$$Y = L^2_\sigma, \quad H = L^2_\sigma, \quad X = H^{-3}, \quad \mathcal{N}_q(y) = \begin{cases} ||y||^{2(q-1)}_{L^2} \cdot ||y||^2_{H^\alpha}, & \text{if } y \in H^\alpha, \\ \infty, & \text{if } y \in L^2_\sigma\backslash H^\alpha, \end{cases}$$

$$A = -F_\alpha : L^2_\sigma \to H^{-3}, \quad B \equiv G \in L_2(U, H).$$

Indeed, the embeddings $H^3 \hookrightarrow L^2_\sigma \hookrightarrow L^2_\sigma \hookrightarrow H^{-3}$ are continuous and dense between separable Hilbert spaces, and the first embedding is in addition compact. In particular, there is an orthonormal basis of L^2_σ in H^3. We choose the linear span of such basis as the countable set of test vector fields $\mathcal{E}$ of [110]. Clearly, the required $\mathcal{B}(L^2_\sigma)/\mathcal{B}(H^{-3})$- and $\mathcal{B}(L^2_\sigma)/\mathcal{B}(L_2(U, L^2_\sigma))$-measurability of our choices for A and B are fulfilled since F_α is continuous by Lemma 9.3.6, and since G is constant as a map from L^2_σ to $L_2(U, L^2_\sigma)$.

Moreover, $\mathcal{N}_1$ belongs to the class $\mathcal{U}^2$ of [110], since $\mathcal{N}_1 \geq 0$, $\mathcal{N}_1(cy) \leq c^2\mathcal{N}(y)$ for all $c \geq 0$, $y \in L^2_\sigma$, it has the precompact level set

$$\{y \in L^2_\sigma : \mathcal{N}_1(y) \leq 1\} \subseteq L^2_\sigma, \tag{9.12}$$

and is lower semicontinuous on L^2_σ. More precisely, (9.12) follows from the compact embedding $H^\alpha \hookrightarrow L^2_\sigma$ and the lower semicontinuity can be realized as follows. Assume $y_n \longrightarrow y$ in L^2_σ as $n \to \infty$ and, without loss of generality, $\sup_{n\geq 1} ||y_n||^2_{H^\alpha} < \infty$, which implies the existence of a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ and an element $y' \in H^\alpha$ such that y_{n_k} converges weakly to y' in H^α as $k \to \infty$. This gives $y = y'$ and hence $y \in H^\alpha$. Now the lower semicontinuity follows from (9.9) and the density of $H^3 \subseteq H^\alpha$ via

$$\begin{aligned} ||y||^2_{H^\alpha} &= \sup_{\substack{z\in H^3,\\ ||z||_{H^\alpha}\leq 1}} |\langle y, (1-\Delta)^\alpha z\rangle_{L^2}| = \sup_{\substack{z\in H^3,\\ ||z||_{H^\alpha}\leq 1}} \lim_{n\to\infty} |\langle y_n, (1-\Delta)^\alpha z\rangle_{L^2}| \\ &\leq \liminf_{n\to\infty} \sup_{\substack{z\in H^3,\\ ||z||_{H^\alpha}\leq 1}} |\langle y_n, (1-\Delta)^\alpha z\rangle_{L^2}| = \liminf_{n\to\infty} ||y_n||^2_{H^\alpha}. \end{aligned}$$

From here, it is clear that each $\mathcal{N}_q$ is lower semicontinuous on L^2_σ, as well as the product of $\mathcal{N}_1$ with a nonnegative continuous function.

To conclude the proof of (i), it remains to verify conditions (C1)-(C3) of [110, Sect.4]. Note that all conditions for the constant $L_2(U, L^2_\sigma)$-valued map $B \equiv G$ are fulfilled. Turning to $A = -F_\alpha$, concerning (C1), let $y_n \longrightarrow y$ in L^2_σ as n $\to \infty$ and $z \in H^3$. By Lemma 9.3.6, we have

$$|\langle F_\alpha(y_n) - F_\alpha(y), z\rangle_{(-3,3)}| \leq |\langle y_n \otimes y_n - y \otimes y, Dz\rangle_{L^2}| + |\langle y_n - y, (-\Delta)^\alpha z\rangle_{L^2}|,$$

where the convergence to 0 as $n \to \infty$ of the second summand is clear. Hence, the demicontinuity (C1) follows from

$$\begin{aligned} |\langle y_n \otimes y_n - y \otimes y, Dz\rangle_{L^2}| &\leq ||Dz||_{L^\infty} ||y_n \otimes y_n - y \otimes y||_{L^1} \\ &\leq ||Dz||_{L^\infty} \big(||y_n||_{L^2} + ||y||_{L^2}\big) \cdot ||y_n - y||_{L^2} \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Next, for $z \in H^3$, due to $\mathrm{div}(z) = 0$, (9.11) and (9.5), we find

$$\langle -F_\alpha(z), z\rangle_{(-3,3)} = -||(-\Delta)^{\alpha/2} z||^2_{L^2} \leq -C\mathcal{N}_1(z) + ||z||^2_{L^2},$$

which gives the required coercivity (C2). Note that, to be even more precise, comparing with [110] shows that one should include the constant C, which comes from (9.5), in the definition of $\mathcal{N}_1$, which would cause its appearance in (9.6). However, since there is no further restriction on the maps $C_{t,q}$, in this case C can be incorporated in $C_{t,q}$.

Finally, concerning the boundedness (C3), for $y \in L^2_\sigma$ we bound $F_\alpha(y)$ in H^{-3} via

$$\begin{aligned}
||F_\alpha(y)||_{H^{-3}} &\le ||\operatorname{div}(y \otimes y)||_{H^{-3}} + ||(-\Delta)^\alpha(y)||_{H^{-3}} \\
&\le \sup_{\substack{z \in H^3, \\ ||z||_{H^3} \le 1}} |\langle y \otimes y, Dz\rangle_{L^2}| + \sup_{\substack{z \in H^3, \\ ||z||_{H^3} \le 1}} |\langle y, (-\Delta)^\alpha z\rangle_{L^2}| \\
&\le \Bigg(\sup_{\substack{z \in H^3, \\ ||z||_{H^3} \le 1}} ||Dz||_{L^\infty} \Bigg) ||y||^2_{L^2} + \Bigg(\sup_{\substack{z \in H^3, \\ ||z||_{H^3} \le 1}} ||(-\Delta)^\alpha z||_{L^2} \Bigg) ||y||_{L^2} \\
&\le C\big(1 + ||y||^2_{L^2}\big),
\end{aligned} \tag{9.13}$$

where we used Lemma 9.3.6, and the constant C, which is independent of y and z, comes from the Sobolev embeddings $H^3 \hookrightarrow H^2 \hookrightarrow L^\infty$ and $H^3 \hookrightarrow H^{2\alpha}$. Consequently, still denoting the possibly changing constant by C, we have

$$||F_\alpha(y)||^2_{H^{-3}} \le C(1 + ||y||^2_{L^2})^2 \le C(1 + ||y||^4_{L^2}), \tag{9.14}$$

which gives the desired growth condition and thereby concludes the verification of all requirements of [110, Thm.4.6]. Therefore, noting that with our choices made at the beginning of the proof, any solution in the sense of [110, Def.3.1] is a global martingale solution in the sense of our Definition 9.3.1, the first part of Proposition 9.3.4 follows. □

For part (ii), we start with the following lemma, which is identical to [116, Lem.A.1]. For completeness, we give a slightly rewritten proof. Recall that the theorem of Arzela-Ascoli implies that for $-\infty < a < b < \infty$ and any index set I, any bounded family of elements $\{f_i\}_{i \in I}$ in $L^\infty([a,b], L^2_\sigma)$, which also obeys a bound

$$\sup_{i \in I} \sup_{r \neq t \in [a,b]} \frac{||f_i(t) - f_i(r)||_{H^{-3}}}{|t - r|^\gamma} < \infty$$

for some $\gamma \in (0,1)$, is precompact in $C([a,b], H^{-3})$.

Lemma 9.3.7. *Let $\{(s_n, y_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+ \times L^2_\sigma$ converge to some $(s, y_0) \in \mathbb{R}_+ \times L^2_\sigma$ and let $P_n \in \mathcal{P}(\Omega_0)$ satisfy*

$$P_n(\pi_t = y_n, t \in [0, s_n]) = 1 \tag{9.15}$$

for each $n \in \mathbb{N}$, and suppose that for some $\gamma, \kappa > 0$ and any $T > 0$

$$\sup_{n \in \mathbb{N}} \mathbb{E}_{P_n} \left[\sup_{t \in [0,T]} ||\pi_t||_{L^2} + \sup_{r \neq t \in [0,T]} \frac{||\pi_t - \pi_r||_{H^{-3}}}{|t - r|^\kappa} + \int_{s_n}^T ||\pi_r||^2_{H^\gamma} dr \right] < \infty. \tag{9.16}$$

Then, $\{P_n\}_{n \in \mathbb{N}}$ is tight as a family of measures on $C_{\mathrm{loc}}(\mathbb{R}_+, H^{-3}) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, L^2_\sigma)$.

Proof. Let $\varepsilon > 0$ and set $k_0 := \sup_{n \in \mathbb{N}} s_n < \infty$. Due to (9.16), for each $k \in \mathbb{N}$ with $k \ge k_0$, there is $R_k > 0$ such that

$$\sup_{n \in \mathbb{N}} P_n \left(x \in \Omega_0 : \sup_{t \in [0,k]} ||x(t)||_{L^2} + \sup_{r \neq t \in [0,k]} \frac{||x(t) - x(r)||_{H^{-3}}}{|t - r|^\kappa} + \int_{s_n}^k ||x(r)||^2_{H^\gamma} dr > R_k \right) \le \frac{\varepsilon}{2^k}. \tag{9.17}$$

Set $\Omega_n := \{x \in \Omega_0 : x(t) = y_n, t \in [0, s_n]\}$ and let $K = K(\varepsilon) \subseteq C_{\text{loc}}(\mathbb{R}_+, H^{-3}) \cap L^2_{\text{loc}}(\mathbb{R}_+, L^2_\sigma)$ be defined as

$$K := \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq k_0} K_n^k,$$

where for abbreviation we set

$$K_n^k := \left\{ x \in \Omega_n : \sup_{t\in[0,k]} ||x(t)||_{L^2} + \sup_{r\neq t\in[0,k]} \frac{||x(t) - x(r)||_{H^{-3}}}{|t-r|^\kappa} + \int_{s_n}^k ||x(r)||^2_{H^\gamma} dr \leq R_k \right\}.$$

In order to prove tightness of $\{P_n\}_{n\in\mathbb{N}}$, our goal is to prove that $\overline{K} \subseteq C_{\text{loc}}(\mathbb{R}_+, H^{-3}) \cap L^2_{\text{loc}}(\mathbb{R}_+, L^2_\sigma)$ is compact and that we have

$$\sup_{n\in\mathbb{N}} P_n(K^c) \leq \varepsilon. \tag{9.18}$$

First, (9.18) follows from

$$P_n(K^c) \leq P_n\Big(\bigcup_{k\geq k_0} (K_n^k)^c \Big) \leq \sum_{k\geq k_0} P_n\big((K_n^k)^c\big) \leq \varepsilon,$$

where the final estimate is due to (9.17). Secondly, concerning compactness of $\overline{K}$, by definition of the topology of the local spaces $C_{\text{loc}}(\mathbb{R}_+, H^{-3})$ and $L^2_{\text{loc}}(\mathbb{R}_+, L^2_\sigma)$, it suffices to show that for each $L > k_0$, the set $K_{[0,L]}$ of elements of K restricted to $[0, L]$ is precompact in $C([0, L], H^{-3}) \cap L^2([0, L], L^2_\sigma)$. To this end, let $\{x_m\}_{m\in\mathbb{N}}$ be a sequence in K and let $L > k_0$. If there is $n \in \mathbb{N}$ such that there are infinitely many m with

$$x_m \in \bigcap_{k\geq k_0} K_n^k, \tag{9.19}$$

then the definition of K_n^k implies that $\{x_m\}_{m\in\mathbb{N}}$ contains a subsequence $\{x_{m_l}\}_{l\in\mathbb{N}}$ which is uniformly bounded in $L^\infty([s_n, L], L^2_\sigma) \cap L^2([s_n, L], H^\gamma)$ and, additionally,

$$\sup_{l\in\mathbb{N}} \sup_{r\neq t\in[s_n,L]} \frac{||x_{m_l}(t) - x_{m_l}(r)||_{H^{-3}}}{|t-r|^\kappa} < \infty.$$

Consequently, since the embedding

$$L^\infty([s_n, L], L^2_\sigma) \cap C^\kappa([s_n, L], H^{-3}) \cap L^2([s_n, L], H^\gamma) \hookrightarrow L^2([s_n, L], L^2_\sigma) \cap C([s_n, L], H^{-3})$$

is compact (cf. [46, Sect.1.8.2]), we conclude the existence of a further subsequence which converges in $C([s_n, L], H^{-3}) \cap L^2([s_n, L], L^2_\sigma)$. Since $x_{m_l} \in \Omega_n$ for each l by assumption, this convergence also holds in $C([0, L], H^{-3}) \cap L^2([0, L], L^2_\sigma)$, which concludes this case.

If for each $n \in \mathbb{N}$, there exist only finitely many m such that (9.19) holds, we may, up to a possible relabeling of the sequence, in particular assume $x_m \in \Omega_m$ for each m. In order to show the desired precompactness in this case, we first find a convergent subsequence in

$C([0,L],H^{-3})$ and then show that this subsequence is Cauchy also in $L^2([0,L],L^2_\sigma)$. This works as follows. The definition of K gives the boundedness of $\{x_m\}_{m\in\mathbb{N}}$

$$\sup_{m\in\mathbb{N}}\left(\sup_{t\in[0,L]}||x_m(t)||_{L^2}+\sup_{r\neq t\in[0,L]}\frac{||x_m(t)-x_m(r)||_{H^{-3}}}{|t-r|^\kappa}\right)<\infty,$$

from which the Arzela-Ascoli theorem, as recalled above the assertion of the present lemma, yields the existence of a convergent subsequence $\{x_{m_l}\}_{l\in\mathbb{N}}$ in $C([0,L],H^{-3})$. Concerning the Cauchy property of this subsequence in $L^2([0,L],L^2_\sigma)$, we obtain

$$\int_0^L ||x_{m_l}(t)-x_{m_j}(t)||^2_{L^2}dt = I_1+I_2+I_3,$$

with

$$I_1 := \int_0^{s_{m_l}\wedge s_{m_j}} ||x_{m_l}(t)-x_{m_j}(t)||^2_{L^2}dt, \quad I_2 := \int_{s_{m_l}\wedge s_{m_j}}^{s_{m_l}\vee s_{m_j}} ||x_{m_l}(t)-x_{m_j}(t)||^2_{L^2}dt,$$
$$I_3 := \int_{s_{m_l}\vee s_{m_j}}^{L} ||x_{m_l}(t)-x_{m_j}(t)||^2_{L^2}dt.$$

Since $x_{m_l}\in\Omega_{m_l}$ and $x_{m_j}\in\Omega_{m_j}$, and because $\{y_n\}_{n\in\mathbb{N}}$ is in particular Cauchy in L^2_σ, we have

$$I_1 \le k_0||y_{m_l}-y_{m_j}||^2_{L^2} \underset{l,j\to\infty}{\longrightarrow} 0. \tag{9.20}$$

Furthermore, we find, using the convergence $s_n \underset{n\to\infty}{\longrightarrow} s$,

$$I_2 \le 2\left(\int_{s_{m_l}\wedge s_{m_j}}^{s_{m_l}\vee s_{m_j}} ||x_{m_l}||^2_{L^2}+||x_{m_j}||^2_{L^2}dt\right) \le 4R_L^2\left(s_{m_l}\vee s_{m_j}-s_{m_l}\wedge s_{m_j}\right) \underset{l,j\to\infty}{\longrightarrow} 0.$$

Finally, we interpolate L^2_σ between H^{-3} and H^γ and use Young's inequality to obtain for arbitrary $\delta>0$ and a constant $C_\delta>0$ only dependent on δ and γ

$$\begin{aligned} I_3 &\le \delta\int_{s_{m_l}\vee s_{m_j}}^{L} ||x_{m_l}(t)-x_{m_k}(t)||^2_{H^\gamma}dt + C_\delta\int_{s_{m_l}\vee s_{m_j}}^{L} ||x_{m_l}(t)-x_{m_k}(t)||^2_{H^{-3}}dt \\ &\le 2\delta R_L + C_\delta L\sup_{t\in[0,L]}||x_{m_l}(t)-x_{m_j}(t)||^2_{H^{-3}}. \end{aligned}$$

Choosing $\delta>0$ arbitrarily small and noting that the second summand converges to 0 as $l,j\longrightarrow\infty$ since $(x_{m_l})_{l\in\mathbb{N}}$ converges in $C([0,L],H^{-3})$, the convergence of I_3 to 0 follows.

Summarizing, any sequence $\{x_m\}_{m\in\mathbb{N}}\subseteq K$ contains a convergent subsequence in $C_{\text{loc}}(\mathbb{R}_+,H^{-3})\cap L^2_{\text{loc}}(\mathbb{R}_+,L^2_\sigma)$, which yields the necessary precompactness of K and the proof is complete. □

We are now prepared to prove part (ii) of Proposition 9.3.4.

Proof of 9.3.4 (ii). For $\{P_n\}_{n\in\mathbb{N}}$ as in the assertion, we first of all prove that this is a tight family of Borel probability measures on $\mathbb{S}:=C_{\text{loc}}(\mathbb{R}_+,H^{-3})\cap L^2_{\text{loc}}(\mathbb{R}_+,L^2_\sigma)$, for which we aim to use the previous lemma. Note that (M3) of Definition 9.3.1 for $q = 1$ and $\sup_{n\in\mathbb{N}} ||x_n||_{L^2} < \infty$ implies the bound

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{P_n}\left[\sup_{t\in[0,T]} ||\pi_t||_{L^2} + \int_{s_n}^T ||\pi_t||^2_{H^\gamma} dr\right] < \infty, \quad T > 0, \tag{9.21}$$

so that tightness of $\{P_n\}_{n\in\mathbb{N}}$ follows from Lemma 9.3.7, if we can show for some $\kappa \in (0,1)$

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{P_n}\left[\sup_{r\neq t\in[0,T]} \frac{||\pi_t - \pi_r||_{H^{-3}}}{|t-r|^\kappa}\right] < \infty, \quad T > 0. \tag{9.22}$$

It follows from (M2) that for each $n \in \mathbb{N}$ and $e \in H^3$, we have

$$\langle x(t), e\rangle_{(-3,3)} = \langle x_n, e\rangle_{(-3,3)} - \int_{s_n}^t \langle F_\alpha(x(r)), e\rangle_{(-3,3)} dr + M^e_{s_n}(t), \quad t \geq s_n, \quad P_n\text{-a.e. } x \in \Omega_0,$$

and, consequently, the P_n-a.s. equality in H^{-3}

$$x(t) = x_n - \int_{s_n}^t F_\alpha(x(r))dr + M_{s_n}(t), \quad t \geq s_n, \tag{9.23}$$

holds, where M_{s_n} is the GG^*-Brownian motion in L^2_σ with respect to P_n defined in Remark 9.3.2 (ii). Concerning (9.22), we treat the summands of (9.23) individually as follows. On the one hand, applying a Burkholder-Davies-Gundy inequality (cf. [166, Thm.6.1.2]) for $p > 2$ gives

$$\begin{aligned}\mathbb{E}_{P_n}\left[||M_{s_n}(t) - M_{s_n}(r)||^{2p}_{L^2}\right] &\leq C_p \mathbb{E}_{P_n}\left[\left(\int_r^t ||G||^2_{L_2(U,L^2_\sigma)} ds\right)^p\right] \\ &\leq C_p|t-r|^{p-1}\mathbb{E}_{P_n}\left[\int_r^t ||G||^{2p}_{L_2(U,L^2_\sigma)} ds\right] \\ &\leq C(p,T,G)|t-r|^{p-1}, \quad s_n \leq r \leq t \leq T,\end{aligned}$$

where we used Hölder inequality for the second estimate and we note that the constant $C(p,T,G)$ is independent of n. From here, Kolmogorov's continuity criterion Theorem D.0.2 allows to conclude the existence of a locally κ-Hölder continuous version of M_{s_n} for $\kappa \in (0, \frac{p-2}{2p})$, again denoted by M_{s_n}, such that

$$\mathbb{E}_{P_n}\left[||M_{s_n}||_{C^\kappa([0,T],L^2_\sigma)}\right] \leq C, \quad T > 0, \tag{9.24}$$

for a finite constant C, which depends on p, G and T, but not on n. On the other hand, for any $p > 1$, using Hölder inequality for $p_1 = \frac{p}{p-1}$ and $p_2 = p$, we find

$$\begin{aligned}\mathbb{E}_{P_n}\left[\sup_{r\neq t\in[s_n,T]} \frac{||\int_r^t F_\alpha(\pi_u)du||^p_{H^{-3}}}{|t-r|^{p-1}}\right] &\leq \mathbb{E}_{P_n}\left[\int_{s_n}^T ||F_\alpha(\pi_u)||^p_{H^{-3}}\right] \\ &\leq C_p\mathbb{E}_{P_n}\left[\int_{s_n}^T 1 + ||\pi_u||^{2p}_{L^2} du\right] \\ &\leq C(||x_n||^{2p}_{L^2} + 1).\end{aligned}$$

Here, the second inequality follows from the growth bound (9.14) and the finite constant $C > 0$ in the last line depends on p, T and the value of $\tilde{C}_{T,p}$ from the statement of the present proposition. In particular, since each P_n is assumed to obey (M3) of Definition 9.3.1 with the same family $(\tilde{C}_{t,q})_q$, this constant is independent of n. Since $\sup_{n\in\mathbb{N}} ||x_n||_{L^2} < \infty$, we obtain

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{P_n}\left[\sup_{r\neq t\in[s_n,T]} \frac{||\int_r^t F_\alpha(\pi_u)du||_{H^{-3}}}{|t-r|^{\frac{p-1}{p}}}\right] < \infty. \tag{9.25}$$

Combining with (9.23), (9.24) and the fact that $x(t) = x_n$, $t \in [0, s_n]$, P_n-a.s., we conclude that (9.22) is fulfilled for any $0 < \kappa < 1/2$ (note that we may choose different values for p in (9.24) and (9.25)). Consequently, we evoke Lemma 9.3.7 to conclude tightness of $\{P_n\}_{n\in\mathbb{N}}$ as Borel probability measures on $\mathbb{S} = C_{\text{loc}}(\mathbb{R}_+, H^{-3}) \cap L^2_{\text{loc}}(\mathbb{R}_+, L^2_\sigma)$ and extract a subsequence, again denoted by $(P_n)_{n\in\mathbb{N}}$, which converges weakly to some $P \in \mathcal{P}(\mathbb{S})$. Since for each $k \in \mathbb{N}$ the map

$$x \mapsto ||x||_{L^\infty((0,k),L^2_\sigma)} \in \mathbb{R}_+ \cup \{\infty\}$$

is lower semicontinuous on $\mathbb{S}$ and each P_n is concentrated on $\Omega_0 \cap L^\infty_{\text{loc}}(\mathbb{R}_+, L^2_\sigma)$, we may consider P as a probability measure on $\Omega_0 \cap L^\infty_{\text{loc}}(\mathbb{R}_+, L^2_\sigma)$.

To conclude the proof it remains to show $P \in \mathcal{C}_\gamma(s, x_0, \tilde{C}_{t,q})$. To this end, we employ the Skorohod representation, cf. Theorem D.0.1, to obtain a probability space $(\tilde{\Omega}, \tilde{F}, \tilde{P})$ and $\mathbb{S}$-valued random variables $\tilde{y}_n$ and $\tilde{y}$ such that

(i) $\tilde{P} \circ \tilde{y}_n^{-1} = P_n$ for each $n \in \mathbb{N}$,

(ii) $\tilde{y}_n \underset{n\to\infty}{\longrightarrow} \tilde{y}$ $\tilde{P}$-a.s. and $\tilde{P} \circ \tilde{y}^{-1} = P$.

From here, (M1) of Definition 9.3.1 follows via

$$\begin{aligned} P\big(x \in \Omega_0 : x(t) = x_0,\, t \in [0, s]\big) &= \tilde{P}\big(\tilde{y}(t) = x_0,\, t \in [0, s]\big) \\ &= \tilde{P}\big(\lim_{n\to\infty} \tilde{y}_n(t) = x_0,\, t \in [0, s]\big) \\ &\geq \tilde{P}\Big(\bigcap_{n\in\mathbb{N}} \{\tilde{y}_n(t) = x_n,\, t \in [0, s_n]\}\Big) = 1, \end{aligned}$$

where the final equality holds since $\{\tilde{y}_n(t) = x_n,\, t \in [0, s_n]\}^c$ is $\tilde{P}$-negligible for each $n \in \mathbb{N}$ due to (M1) for P_n and (i) above.

Concerning (M2), the almost sure convergence $\tilde{y}_n \underset{n\to\infty}{\longrightarrow} \tilde{y}$ in $\mathbb{S}$ under $\tilde{P}$ gives for each $e \in H^3$ and $t \geq s$

$$\langle \tilde{y}_n(t), e\rangle_{(-3,3)} \underset{n\to\infty}{\longrightarrow} \langle \tilde{y}(t), e\rangle_{(-3,3)}, \qquad \int_{s_n}^t \langle F_\alpha(\tilde{y}_n(r)), e\rangle_{(-3,3)} dr \underset{n\to\infty}{\longrightarrow} \int_s^t \langle F_\alpha(\tilde{y}(r)), e\rangle_{(-3,3)} dr$$

$\tilde{P}$-a.s., and therefore, with the notation of Definition 9.3.1, $M^e_{s_n}(\tilde{y}_n, t) \underset{n\to\infty}{\longrightarrow} M^e_s(\tilde{y}, t)$ for all $t \geq s$, $\tilde{P}$-a.s. Writing $M^{e,\tilde{y}_n}_{s_n} = M^e_{s_n}(\tilde{y}_n)$ and similarly for $\tilde{y}$ instead of $\tilde{y}_n$, evoking the Burkholder-Davies-Gundy inequality [166, Thm.6.1.2] implies

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{\tilde{P}}\left[\big|M^{e,\tilde{y}_n}_{s_n}(t)\big|^{2p}\right] \leq C \sup_{n\in\mathbb{N}} \mathbb{E}_{P_n}\left[\left(\int_{s_n}^t ||G||^2_{L_2(U,L^2_\sigma)} ds\right)^p\right] < \infty, \quad p > 1, \tag{9.26}$$

which shows that under $\tilde{P}$, the random variables $M_{s_n}^{e,\tilde{y}_n}(t)$, $n \in \mathbb{N}$, are uniformly integrable, which allows to apply Lebesgue dominated convergence theorem to obtain

$$\lim_{n\to\infty} \mathbb{E}_{\tilde{P}}\big[\big|M_{s_n}^{e,\tilde{y}_n}(t) - M_s^{e,\tilde{y}}(t)\big|\big] = 0. \tag{9.27}$$

From here, we deduce the martingale property of M_s^e stated in (M2) as follows. For $t \geq r \geq s$ and $g : \Omega_0 \to \mathbb{R}$ continuous $\mathcal{B}_r$-measurable, using (9.27) we know by the martingale property of M^e under each P_n that

$$\begin{aligned}\mathbb{E}_P\Big[\big(M_s^e(t) - M_s^e(r)\big)g\Big] &= \lim_{n\to\infty} \mathbb{E}_{\tilde{P}}\Big[\big(M_{s_n}^{e,\tilde{y}_n}(t) - M_{s_n}^{e,\tilde{y}_n}(r)\big)g(\tilde{y})\Big] \\ &= \lim_{n\to\infty} \mathbb{E}_{P_n}\Big[\big(M_{s_n}^e(t) - M_{s_n}^e(r)\big)g\Big] = 0,\end{aligned}$$

which implies that $t \mapsto M_s^e(t)$ is a continuous $(\mathcal{B}_t)_{t\geq s}$-martingale on Ω_0 with respect to P for each $e \in H^3$. Using (9.26) for higher p, we similarly obtain

$$\lim_{n\to\infty} \mathbb{E}_{\tilde{P}}\big[\big|M_{s_n}^{e,\tilde{y}_n}(t) - M_s^{e,\tilde{y}}(t)\big|^2\big] = 0,$$

and from there the martingale property of

$$t \mapsto M_s^e(t) - (t-s)||G^*e||_U^2,$$

i.e. M_s^e has quadratic variation $t \mapsto \langle\langle M_s^e\rangle\rangle_t = (t-s)||G^*e||_U^2$. Consequently, M_s^e is in particular P-square integrable, so that everything concerning (M2) for P is proven. Finally, we verify (M3). For $q \in \mathbb{N}$, set

$$x \mapsto S_q(t,s,x) := \sup_{r\in[0,t]} ||x(r)||_{L^2}^{2q} + \int_s^t ||x(u)||_{L^2}^{2(q-1)} ||x(u)||_{H^\gamma}^2 du,$$

which is lower semicontinuous on $\mathbb{S}$. Therefore, the weak convergence $P_n \longrightarrow P$ in $\mathcal{P}(\mathbb{S})$ as $n \to \infty$ gives for each $s < s^* < t$

$$\mathbb{E}_P\big[S_q(t,s^*,x)\big] \leq \liminf_{n\to\infty} \mathbb{E}_{P_n}\big[S_q(t,s_n,x)\big] \leq \tilde{C}_{t,q} \liminf_{n\to\infty} \big(||x_n||_{L^2}^{2q} + 1\big) = \tilde{C}_{t,q}\big(||x_0||_{L^2}^{2q} + 1\big), \tag{9.28}$$

where we used the assumption that $P_n \in \mathcal{C}_\gamma(s_n, x_n, \tilde{C}_{t,q})$ for some γ with $(\tilde{C}_{t,q})_q$ independent of n, and the fact that due to the convergence $s_n \longrightarrow s$ as $n \to \infty$, we have $s_n \leq s^*$ for all but finitely many n and $S_q(t,s^*,x) \leq S_q(t,s_n,x)$ for all such n. Since the right-hand side in (9.28) is independent of s^*, we finally obtain

$$\mathbb{E}_P\big[S_q(t,s,x)\big] \leq \tilde{C}_{t,q}\big(||x_0||_{L^2}^{2q} + 1\big),$$

which shows $P \in \mathcal{C}_\gamma(s, x_0, \tilde{C}_{t,q})$, and thereby completes the proof of Proposition 9.3.4. □

9.4 Extension of local martingale solutions

As mentioned at the beginning of the previous subsection, in view of our main result, we need to extend local martingale solutions obtained via the convex integration methods of Chapter 11 to global ones. To do so, we would like to make use of a classical measure theoretic extension technique, see [215, Thm.6.1.2]: If we are given a local solution P up to a finite stopping time τ, we would like to concatenate P in a pathwise sense at each end point $\big(\tau(x), x(\tau(x))\big) \in \mathbb{R}_+ \times L^2_\sigma$ with a global martingale solution $R_{\tau(x),x(\tau(x))}$, whose initial condition is the indexed end point. The existence of such a family of martingale solutions follows from Proposition 9.3.4 (i). However, it turns out that the stopping times τ_L up to which we can construct local martingale solutions by convex integration methods are stopping times only with respect to the right-continuous filtration $(\mathcal{B}_t)_{t\geq 0}$ on Ω_0 instead of the natural filtration $(\mathcal{B}^0_t)_{t\geq 0}$. This rules out a direct application of the classical result of [215]. Instead, we use an extension of these techniques, which was already used in [116]. Hereafter, we state and discuss the necessary results in this direction.

Lemma 9.4.1. *Let τ be a bounded $(\mathcal{B}_t)_{t\geq 0}$-stopping time. Then, for every $x \in \Omega_0 \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^2_\sigma)$, there exists $Q_x \in \mathcal{P}(\Omega_0)$ such that*

$$Q_x\big(x' \in \Omega_0 : x(t) = x'(t),\, t \in [0, \tau(x)]\big) = 1, \tag{9.29}$$

and we have

$$Q_x(A) = R_{\tau(x),x(\tau(x))}(A), \quad A \in \mathcal{B}^{\tau(x)}, \tag{9.30}$$

where $R_{\tau(x),x(\tau(x))} \in \mathcal{P}(\Omega_0)$ is a global martingale solution to (HNSE$_{\text{sto}}$) with initial condition $x(\tau(x)) \in L^2_\sigma$ at time $\tau(x)$. Furthermore, $x \mapsto Q_x(B)$ is $\mathcal{B}_\tau$-measurable for each $B \in \mathcal{B}$.

Proof. In order to select a measurable family of martingale solutions, we apply the general framework of [215, Ch.12]. Note that, as a consequence of the stability part (ii) of Proposition 9.3.4, the sets $\mathcal{C}_\gamma\big(s, x_0, \tilde{C}_{t,q}\big) \subseteq \mathcal{P}(\Omega_0)$ are compact with respect to the topology of weak convergence of measures, i.e. if such a set is nonempty, it is an element of $\mathrm{Comp}(\mathcal{P}(\Omega_0))$, the space of all nonempty compact subsets of $\mathcal{P}(\Omega_0)$.

Since Proposition 9.3.4 yields a family $(C_{t,q})_q$ such that $\mathcal{C}_\alpha(s, x_0, C_{t,q})$ is nonempty for each $(s, x_0) \in \mathbb{R}_+ \times L^2_\sigma$, and since $\mathcal{P}(\Omega_0)$ is a separable metric space, Lemma E.0.2 and E.0.3 imply the existence of a Borel map

$$\mathbb{R}_+ \times L^2_\sigma \ni (s, x_0) \mapsto R_{s,x_0} \in \mathcal{C}_\alpha(s, x_0, C_{t,q}). \tag{9.31}$$

Since the canonical process $x : \mathbb{R}_+ \times \Omega_0 \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^2_\sigma) \to H^{-3}$ is $(\mathcal{B}^0_t)_{t\geq 0}$-progressively measurable and the embedding $H^{-3} \hookrightarrow L^2_\sigma$ is dense and continuous, it follows via Kuratowski's theorem that $L^2_\sigma \in \mathcal{B}(H^{-3})$, $\mathcal{B}(L^2_\sigma) = \mathcal{B}(H^{-3}) \cap L^2_\sigma$ and that this process is also $(\mathcal{B}^0_t)_{t\geq 0}$-progressively measurable with respect to $\mathcal{B}(L^2_\sigma)$. In particular, it is $(\mathcal{B}_t)_{t\geq 0}$-progressively measurable. Since τ is in particular finite, it follows from [215, Lem.1.2.4] that the map on $\mathbb{R}_+ \times \Omega_0 \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^2_\sigma)$

$$(t, x) \mapsto \big(\tau(x), x(\tau(x))\big) \in \mathbb{R}_+ \times L^2_\sigma$$

is $\mathcal{B}_\tau$-measurable. Combining with the measurable selection (9.31), we obtain a $\mathcal{B}_\tau$-measurable map

$$\Omega_0 \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^2_\sigma) \to \mathcal{P}(\Omega_0), \quad x \mapsto R_{\tau(x),x(\tau(x))}, \tag{9.32}$$

with $R_{\tau(x),x(\tau(x))}$ being a global martingale solution to (HNSE$_{\mathrm{sto}}$) with initial condition $(\tau(x), x(\tau(x)))$. Consequently, noting that [215, Lem.6.1.1] still holds if one replaces the state space $\mathbb{R}^d$ by H^{-3}, we may apply [215, Lem.6.1.1] to construct a unique element $Q_x \in \mathcal{P}(\Omega_0)$ via

$$Q_x := \delta_x \otimes_{\tau(x)} R_{\tau(x),x(\tau(x))},$$

which is uniquely characterized by (9.29) and (9.30). To conclude the proof, we turn to the measurability part of the assertion. To this end, it is sufficient to consider cylinder sets of type $A = \{\pi_{t_1} \in B_1, \dots, \pi_{t_n} \in B_n\}$ for $n \in \mathbb{N}$, $0 \le t_1 \cdots \le t_n$, and $B_i \in \mathcal{B}(H^{-3})$, $1 \le i \le n$. Using the definition of Q_x, we have

$$\begin{aligned} Q_x(A) = &\mathbb{1}_{[0,t_1)}(\tau(x)) R_{\tau(x),x(\tau(x))}(A) \\ &+ \sum_{k=1}^{n-1} \Big(\mathbb{1}_{[t_k,t_{k+1})}(\tau(x)) \mathbb{1}_{B_1}(x(t_1)) \cdots \mathbb{1}_{B_k}(x(t_k)) \\ &\qquad \cdot R_{\tau(x),x(\tau(x))}\big(x(t_{k+1}) \in B_{k+1}, \dots, x(t_n) \in B_n\big) \Big) \\ &+ \mathbb{1}_{[t_n,\infty)}(\tau(x)) \mathbb{1}_{B_1}(x(t_1)) \cdots \mathbb{1}_{B_n}(x(t_n)), \end{aligned}$$

and each summand of the above right-hand side is $\mathcal{B}_\tau$-measurable due to the corresponding measurability of τ and (9.32), which completes the proof. □

We want to extend local martingale solutions P defined up to a stopping time τ to global solution in such a way that the extended solution coincides with P up to τ. To this end, the principal idea is to consider the family $R = (R_{\tau(x),x(\tau(x))})_x$ obtained through Proposition 9.4.1 to define the extension of P past τ as

$$P \otimes_\tau R(A) := \int_{\Omega_0} Q_x(A) dP(x), \quad A \in \mathcal{B}. \tag{9.33}$$

Note that by the local version of Remark 9.3.2 (i), any local martingale solution up to a bounded $(\mathcal{B}_t)_{t\ge 0}$-stopping time τ is concentrated on paths in $\Omega_0 \cap L^\infty(\mathbb{R}_+, L^2_\sigma)$, which are weakly continuous in L^2_σ. Consequently, Lemma 9.4.1 provides a unique measure Q_x for P-a.e. $x \in \Omega_0$, and the measurability of $x \mapsto Q_x$ yields that $P \otimes_\tau R$ is well-defined as an element of $\mathcal{P}(\Omega_0)$. We remark that this is the only instant where we use measurability of $x \mapsto Q_x$, and therefore mere $\mathcal{B}$-measurability would be suffice as well.

Lemma 9.4.1 implies $Q_x = \delta_x$ on $\mathcal{B}^0_{\tau(x)}$ P-a.s. If τ was a $(\mathcal{B}^0_t)_{t\ge 0}$-stopping time, we could infer

$$Q_x\big(x' \in \Omega_0 : \tau(x') = \tau(x)\big) = 1, \quad P\text{-a.e. } x \in \Omega_0, \tag{9.34}$$

which would imply $P = P \otimes_\tau R$ on $\mathcal{B}^0_\tau$ as follows:

$$P \otimes_\tau R(A) = \int_{\Omega_0} Q_x(A \cap \{\tau = \tau(x)\}) dP(x) = \int_{\Omega_0} \delta_x(A) dP(x) = P(A), \quad A \in \mathcal{B}^0_\tau,$$

where we used $A \cap \{\tau = \tau(x)\} \in \mathcal{B}^0_{\tau(x)}$, which clearly holds under the assumption that τ is an $(\mathcal{B}^0_t)_{t\geq 0}$-stopping time. However, as mentioned before and as we will see in Section 10.4, the stopping times we will use, namely $\tau = \tau_L$ with τ_L as in (10.19), are only stopping times with respect to $(\mathcal{B}_t)_{t\geq 0}$, i.e. with respect to the larger right-continuous filtration. In this case, (9.29) is not sufficient to obtain (9.34). Roughly speaking, this is due to the observation that since τ_L is only a stopping time with respect to a right-continuous filtration, in order to decide whether $\tau_L(x') = \tau_L(x)$, in general one needs information about the path x' not only up to time $\tau_L(x)$, but also infinitesimally beyond, which is not granted by (9.29).

Consequently, it seems indispensable to *assume* the family $\{Q_x\}_x$ to fulfill (9.34), which we will do in Lemma 9.4.2. However, note that even under this additional assumption, it seems out of reach to obtain $P = P \otimes_\tau R$ on $\mathcal{B}_\tau$, since this would require $Q_x = \delta_x$ on the right-continuous σ-algebras $\mathcal{B}_{\tau(x)}$, which we cannot infer. However, by means of (9.34), we are able to conclude the equality $P = P \otimes_\tau R$ on $[0, \tau]$ in the sense that

$$P(A) = P \otimes_\tau R(A), \quad A \in \sigma(\pi_{t\wedge\tau}, t \geq 0), \tag{9.35}$$

see Lemma 9.4.2. We shall see that it is sufficient for P and $P \otimes_{\tau_L} R$ to coincide on this σ-algebra. The preceding discussion leads to the following

Lemma 9.4.2. *For τ as in Lemma 9.4.1 and $x_0 \in L^2_\sigma$, let $P \in \mathcal{P}(\Omega_{0,\tau}) \subseteq \mathcal{P}(\Omega_0)$ be a local martingale solution on $[0, \tau]$ with initial condition $(0, x_0)$. In addition to the situation in Lemma 9.4.1, assume there is a P-negligible set $\mathcal{N} \in \Omega_{0,\tau}$ such that for every $x \in \mathcal{N}^c$ (9.34) holds. Then, the probability measure $P \otimes_\tau R \in \mathcal{P}(\Omega_0)$, defined as in (9.33), is a global martingale solution to ($\mathrm{HNSE_{sto}}$) with initial condition $(0, x_0)$, and it satisfies (9.35).*

Proof. The final claim follows, if we show (9.35) for any $A \in \sigma(\pi_{t\wedge\tau}, t \geq 0)$ of type $A = \{\pi_{t_1\wedge\tau} \in B_1, \dots, \pi_{t_n\wedge\tau} \in B_n\}, B_i \in \mathcal{B}(H^{-3})$, $0 \leq t_1 \leq \dots \leq t_n$, $n \in \mathbb{N}$, since the set of such A is a $\cap$-stable generator of $\sigma(\pi_{t\wedge\tau}, t \geq 0)$. We find

$$\begin{aligned} P \otimes_\tau R(A) &= \int_{\Omega_0} Q_x(A) dP(x) = \int_{\Omega_0} Q_x(\{\pi_{t_1\wedge\tau(x)} \in B_1, \dots, \pi_{t_n\wedge\tau(x)} \in B_n\}) dP(x) \\ &= \int_{\Omega_0} \delta_x(A) dP(x) = P(A), \end{aligned}$$

where we used (9.34) for the second, and $Q_x = \delta_x$ on $\mathcal{B}^0_{\tau(x)}$ (which we observed on the previous page) for the third equality. Hence, it remains to prove that $P \otimes_\tau R$ fulfills Definition 9.3.1. Concerning (M1), since $\{\pi_0 = x_0\} \in \sigma(\pi_{t\wedge\tau}, t \geq 0)$, we have

$$P \otimes_\tau R(x \in \Omega_0 : x(0) = x_0) = P(x \in \Omega_0 : x(0) = x_0) = 1.$$

Here, the second equality holds, since P is a local martingale solution with initial value x_0 at time $s = 0$. Turning to (M3), first recall that due to Proposition 9.3.4, the martingale solutions $R_{\tau(x),x(\tau(x))}$, used for the construction of the measures Q_x in Lemma 9.4.1, may be chosen such that they fulfill Definition 9.3.1 with the common value $\gamma = \alpha$ and common functions $t \mapsto C''_{t,q}$. Assume without loss of generality that $\gamma' \leq \alpha$, where γ' denotes the value for γ for the local solution P in Definition 9.3.3. Choose γ in Definition 9.3.1 for $P \otimes_\tau R$ as $\gamma = \gamma'$. Then, we have for each $t \geq 0$ and $q \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}_{P\otimes_\tau R}\Big[\sup_{r\in[0,t]} ||\pi_r||_{L^2}^{2q} + \int_s^t ||\pi_r||_{L^2}^{2(q-1)}||\pi_r||_{H^\gamma}^2 dr\Big] \\
&\leq \mathbb{E}_{P\otimes_\tau R}\Big[\sup_{r\in[0,t\wedge\tau]} ||\pi_r||_{L^2}^{2q} + \int_0^{t\wedge\tau} ||\pi_r||_{L^2}^{2(q-1)}||\pi_r||_{H^\gamma}^2 dr\Big] \\
&\quad + \mathbb{E}_{P\otimes_\tau R}\Big[\sup_{r\in[t\wedge\tau,t]} ||\pi_r||_{L^2}^{2q} + \int_{t\wedge\tau}^{t} ||\pi_r||_{L^2}^{2(q-1)}||\pi_r||_{H^\gamma}^2 dr\Big] \\
&= \mathbb{E}_{P}\Big[\sup_{r\in[0,t\wedge\tau]} ||\pi_r||_{L^2}^{2q} + \int_0^{t\wedge\tau} ||\pi_r||_{L^2}^{2(q-1)}||\pi_r||_{H^\gamma}^2 dr\Big] \\
&\quad + \int_{\Omega_0} \mathbb{E}_{Q_x}\Big[\sup_{r\in[t\wedge\tau(x),t]} ||\pi_r||_{L^2}^{2q} + \int_{t\wedge\tau(x)}^{t} ||\pi_r||_{L^2}^{2(q-1)}||\pi_r||_{H^\gamma}^2 dr\Big] dP(x) \\
&\leq C'_{t,q}(||x_0||_{L^2}^{2q} + 1) + CC''_{t,q}\int_{\Omega_0} ||\pi_{\tau(x)}||_{L^2}^{2q} + 1\, dP(x) \\
&\leq C_{t,q}(||x_0||_{L^2}^{2q} + 1).
\end{aligned}$$

Here, we used (9.34) for the first equality. Moreover, by $C'_{t,q}$ we denote the function for P in (M3) of Definition 9.3.3, and the additional constant $C > 0$ is due to the Sobolev embedding $H^\alpha \hookrightarrow H^\gamma$. For the final inequality, we used (M3) for P together with the boundedness of τ, and set $C_{t,q} = C'_{t,q} + CC''_{t,q}$. Finally, let us show (M2) for $P \otimes_\tau R$. For each M_0^e as in the notation of (M2) of Definition 9.3.1, we show that $t \mapsto M_0^e(t)$ is a $P \otimes_\tau R$-martingale as follows. First, let $x \in \Omega_0 \cap L^\infty_{\text{loc}}(\mathbb{R}_+, L^2_\sigma)$, $0 \leq s \leq t$, $A = \{\pi_{t_1} \in B_1, \ldots, \pi_{t_n} \in B_n\}$ for $0 \leq t_1 \leq \cdots \leq t_n \leq s$, $B_i \in \mathcal{B}(H^{-3})$, and let us show

$$\mathbb{E}_{Q_x}\big[M_0^e(t)\mathbb{1}_A\big] = \mathbb{E}_{Q_x}\big[M_0^e((t\wedge\tau(x))\vee s)\mathbb{1}_A\big]. \tag{9.36}$$

By definition of Q_x, and the martingale property of each $R_{\tau(x),x(\tau(x))}$ from time $\tau(x)$ on, we find

$$\begin{aligned}
&\mathbb{E}_{Q_x}\Big[(M_0^e(t) - M_0^e((t\wedge\tau(x))\vee s)\mathbb{1}_A\Big] \\
&= \mathbb{1}_{[0,t_1]}(\tau(x))\mathbb{E}_{R_{\tau(x),x(\tau(x))}}\Big[(M_0^e(t) - M_0^e(s))\mathbb{1}_A\Big] \\
&\quad + \sum_{k=1}^{n-1} \mathbb{1}_{[t_k,t_{k+1}]}(\tau(x))\mathbb{1}_{B_1}(\pi_{t_1}(x))\cdots\mathbb{1}_{B_k}(\pi_{t_k}(x)) \\
&\quad \cdot \mathbb{E}_{R_{\tau(x),x(\tau(x))}}\Big[(M_0^e(t) - M_0^e(s))\mathbb{1}_{\pi_{t_{k+1}}\in B_{k+1},\ldots,\pi_{t_n}\in B_n}\Big] \\
&\quad + \mathbb{1}_{[t_n,\infty)}(\tau(x))\mathbb{1}_{B_1}(\pi_{t_1}(x))\cdots\mathbb{1}_{B_n}(\pi_{t_n}(x))\cdot\mathbb{E}_{R_{\tau(x),x(\tau(x))}}\Big[M_0^e(t) - M_0^e((t\wedge\tau(x))\vee s)\Big] \\
&= 0.
\end{aligned}$$

Since sets of type A as above generate $\mathcal{B}_s^0$, and since M_0^e is continuous, it follows that (9.36) holds for every $A \in \mathcal{B}_s$.

Now, it follows by definition of Q_x and $P \otimes_\tau R$, and by (9.36) and (9.34) that

$$\mathbb{E}_{P\otimes\tau R}\Big[M_0^e(t)\mathbb{1}_A\Big] = \int_{\Omega_0} \mathbb{E}_{Q_x}\big[M_0^e(t)\mathbb{1}_A\big]dP(x) = \int_{\Omega_0} \mathbb{E}_{Q_x}\big[M_0^e((t\wedge\tau(x))\vee s)\mathbb{1}_A\big]dP(x)$$
$$= \mathbb{E}_{P\otimes_\tau R}\Big[M_0^e((t\wedge\tau)\vee s)\mathbb{1}_A\Big]$$
$$= \mathbb{E}_{P\otimes_\tau R}\Big[M_0^e(t\wedge\tau)\mathbb{1}_{A\cap\{\tau>s\}}\Big] + \mathbb{E}_{P\otimes_\tau R}\Big[M_0^e(s)\mathbb{1}_{A\cap\{\tau\le s\}}\Big].$$

The integrand of the first summand in the last line can be handled as follows.

$$\mathbb{E}_{P\otimes_\tau R}\Big[M_0^e(t\wedge\tau)\mathbb{1}_{A\cap\{\tau>s\}}\Big] = \int_{\Omega_0} \mathbb{E}_{Q_x}\Big[M_0^e(t\wedge\tau(x))\mathbb{1}_{A\cap\{\tau(x)>s\}}\Big]dP(x)$$
$$= \mathbb{E}_P\Big[M_0^e(t\wedge\tau)\mathbb{1}_{A\cap\{\tau>s\}}\Big] = \mathbb{E}_P\Big[M_0^e(s)\mathbb{1}_{A\cap\{\tau>s\}}\Big] = \mathbb{E}_{P\otimes_\tau R}\Big[M_0^e(s)\mathbb{1}_{A\cap\{\tau>s\}}\Big],$$

where we used the martingale property of $M_0^e(\cdot\wedge\tau)$ with respect to P for the second last equality. Combining with the previous chain of equalities, altogether we obtain

$$\mathbb{E}_{P\otimes\tau R}\Big[M_0^e(t)\mathbb{1}_A\Big] = \mathbb{E}_{P\otimes_\tau R}\Big[M_0^e(s)\mathbb{1}_{A\cap\{\tau>s\}}\Big] + \mathbb{E}_{P\otimes_\tau R}\Big[M_0^e(s)\mathbb{1}_{A\cap\{\tau\le s\}}\Big]$$
$$= \mathbb{E}_{P\otimes_\tau R}\Big[M_0^e(s)\mathbb{1}_A\Big].$$

This implies the martingale property of M_0^e with respect to P and the filtration $(\mathcal{B}_t)_{t\ge0}$. Concerning the quadratic variation of M_0^e, we similarly obtain for any cylindrical set $A \in \mathcal{B}_s$ as above

$$\mathbb{E}_{P\otimes_\tau R}\Big[\big(M_0^e(t)^2 - t||G^*e||_U^2\big)\mathbb{1}_A\Big]$$
$$= \int_{\Omega_0} \mathbb{E}_{Q_x}\bigg[\Big(\big(M_0^e(t) - M_0^e(t\wedge\tau(x))\big)^2 - \big(t-(t\wedge\tau(x))\big)||G^*e||_U^2\Big)\mathbb{1}_A\bigg]dP(x)$$
$$+ \int_{\Omega_0} \mathbb{E}_{Q_x}\bigg[\Big(M_0^e(t\wedge\tau(x))^2 - (t\wedge\tau(x))||G^*e||_U^2\Big)\mathbb{1}_A\bigg]dP(x)$$
$$+ 2\int_{\Omega_0} \mathbb{E}_{Q_x}\bigg[\Big(M_0^e(t\wedge\tau(x))\big(M_0^e(t) - M_0^e(t\wedge\tau(x))\big)\Big)\mathbb{1}_A\bigg]dP(x)$$
$$=: J_1 + J_2 + J_3.$$

Using the martingale property of the process $M_0^e(t) - M_0^e(t\wedge\tau(x)) = M_{t\wedge\tau(x)}^e(t)$ with respect to $R_{\tau(x),x(\tau(x))}$ from the deterministic time $\tau(x)$ on, (9.34), and the martingale property of $M_0^e(\cdot\wedge\tau)$ with respect to P, we have

$$J_1 = \int_{\Omega_0} \mathbb{E}_{Q_x}\bigg[\Big(\big(M_0^e((t\wedge\tau(x))\vee s) - M_0^e(t\wedge\tau(x))\big)^2$$
$$- \big((t\wedge\tau(x))\vee s - t\wedge\tau(x)\big)||G^*e||_U^2\Big)\mathbb{1}_A\bigg]dP(x),$$
$$J_2 = \int_{\Omega_0} \mathbb{E}_{Q_x}\bigg[\Big(M_0^e(s\wedge\tau(x))^2 - (s\wedge\tau(x))||G^*e||_U^2\Big)\mathbb{1}_A\bigg]dP(x),$$

$$J_3 = 2\int_{\Omega_0} \mathbb{E}_{Q_x}\Big[M_0^e(t\wedge\tau(x))\Big(M_0^e((t\wedge\tau(x))\vee s) - M_0^e(t\wedge\tau(x))\Big)\mathbb{1}_A\Big]dP(x).$$

Using (9.34) and the definition of $P\otimes_\tau R$ once more, we finally obtain

$$\begin{aligned}
&\mathbb{E}_{P\otimes_\tau R}\Big[\big(M_0^e(t)\big)^2 - t||G^*e||_U^2\mathbb{1}_A\Big] \\
&= \mathbb{E}_{P\otimes_\tau R}\Big[\Big(M_0^e(s\wedge\tau)^2 - (s\wedge\tau)||G^*e||_U^2\Big)\mathbb{1}_A\Big] \\
&\quad + \mathbb{E}_{P\otimes\tau R}\Big[\Big(\big(M_0^e(s) - M_0^e(\tau)\big)^2 - (s-\tau)||G^*e||_U^2\Big)\mathbb{1}_{A\cap\{\tau\le s\}}\Big] \\
&\quad + 2\mathbb{E}_{P\otimes_\tau R}\Big[\Big(M_0^e(\tau)\big(M_0^e(s) - M_0^e(\tau)\big)\Big)\mathbb{1}_{A\cap\{\tau\le s\}}\Big] \\
&= \mathbb{E}_{P\otimes_\tau R}\Big[\big(M_0^e(s)\big)^2 - s||G^*e||_U^2\mathbb{1}_A\Big].
\end{aligned}$$

In particular, $M_0^e(t)$ is square-integrable, which completes the proof. □

We summarize the benefit of the above measure theoretic considerations as follows: Once we have constructed a local martingale solution P up to a stopping time τ_L in Section 10.4, by virtue of Lemmas 9.4.1 and 9.4.2, we can extend it to a global martingale solution $P\otimes_{\tau_L} R$ such that (9.35) holds, provided we can verify (9.34).

Chapter 10

Proof of the main result

In the present chapter we prove the main result of this part of the thesis, which is Theorem 8.4.1. As we will outline in Section 10.1, one major technical task within the proof is resolved by the method of convex integration, which we will elaborate on in Chapter 11. In this chapter, for the time being, we use Corollary 11.2.3 of Chapter 11 in order to complete the proof of our main result.

10.1 Outline of the proof

In this section, we outline the structure of the proof of the main result, which shares many features of the proof of [116, Thm.1.2]. Let $0 < \alpha < 1/2$ and $T > 0$. The principal idea to obtain two global martingale solutions P_1, P_2 to ($\mathrm{HNSE}_{\mathrm{sto}}$) with a common deterministic initial condition $x_0 \in L^2_\sigma$ which are distinct on $[0,T]$ is to observe their distinctness via the energy inequality

$$\mathbb{E}_{\mathfrak{P}}\big[||\pi_T||^2_{L^2}\big] \leq ||x_0||^2_{L^2} + T\,\mathrm{Tr}(GG^*). \tag{10.1}$$

Here, $\mathfrak{P}$ is a place holder for a probability measure on Ω_0. More precisely, on the one hand we construct a solution P_1 with some deterministic initial value $x_0 \in L^2_\sigma$, which *violates* (10.1) in the sense that

$$\mathbb{E}_{P_1}\big[||\pi_T||^2_{L^2}\big] > 2\big(||x_0||^2_{L^2} + T\,\mathrm{Tr}(GG^*)\big). \tag{10.2}$$

On the other hand, Proposition 9.3.4 (i) and Remark 9.3.5 guarantee the existence of a second global solution P_2 with the same initial value x_0 such that (10.1) holds with P_2 in place of $\mathfrak{P}$. From here, the assertion of the main result follows.

We remark that the initial condition x_0 of P_1 cannot be prescribed, but is an outcome of the construction via the method of convex integration in Chapter 11, as we will see later. In other words, the approach via convex integration does not lead to martingale solutions to ($\mathrm{HNSE_{sto}}$) with a general Cauchy-type initial condition. Hence, at this point, we benefit from the general existence result Proposition 9.3.4 (i), which holds for *any* initial value in L^2_σ. Furthermore, we point out that the solution P_2 emerging from Proposition 9.3.4 (i) fulfills (M3) of Definition 9.3.1 with $\gamma = \alpha$, whereas for P_1, with the present techniques of Chapter 11, we cannot attain a prescribed value for γ. Instead, similarly to the initial condition, the values of γ permitted for P_1 are restricted by our convex integration approach, cf. Corollary 11.2.3.

Consequently, the proof reduces to the construction of a solution P_1 with the energy violation (10.2). To this end, we first construct a local solution P up to a suitable stopping time τ_L on Ω_0, which violates an energy inequality in the sense that

$$\mathbb{E}_P\big[\mathbb{1}_{\{\tau_L \geq T\}}||\pi_T||^2_{L^2}\big] > 2\big(||x_0||^2_{L^2} + T\,\mathrm{Tr}(GG^*)\big). \tag{10.3}$$

As we shall see later, the parameter $L > 1$ needs to obey several lower bounds, one of them in order to guarantee $P(\tau_L \geq T) > 1/2$. Having proceeded up to this point, we extend P to a global martingale solution $P \otimes_{\tau_L} R$ by Lemma 9.4.2 and derive

$$\mathbb{E}_{P\otimes_{\tau_L}R}\big[||\pi_T||^2_{L^2}\big] \geq \mathbb{E}_{P\otimes_{\tau_L}R}\big[\mathbb{1}_{\{\tau_L \geq T\}}||\pi_T||^2_{L^2}\big] = \mathbb{E}_P\big[\mathbb{1}_{\{\tau_L \geq T\}}||\pi_T||^2_{L^2}\big],$$

which combined with (10.3) shows that $P_1 := P \otimes_{\tau_L} R$ fulfills (10.2).

Therefore, our goal is to construct suitable stopping times τ_L and a corresponding local martingale solution P up to τ_L such that (10.3) holds and Lemma 9.4.2 applies. It turns out that our techniques actually yield a pathwise analytically weak, but probabilistically strong solution, i.e. for a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}, B)$ with a GG^*-Wiener process B on L^2_σ and the augmented Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$, we can construct an $(\mathcal{F}_t)_{t\geq 0}$-adapted process u on Ω, which analytically weakly solves ($\mathrm{HNSE_{sto}}$), see Theorem 10.3.2. This solution has paths with low positive Sobolev regularity and is constructed up to an $(\mathcal{F}_t)_{t\geq 0}$-stopping time T_L, which should not be confused with τ_L (the former is defined on Ω, the latter on Ω_0). However, the definitions of both stopping times are related. Roughly speaking, one introduces T_L first and then aims to define τ_L (on Ω_0) in the same spirit as T_L (on Ω), compare (10.5) and (10.19). As mentioned before, the large parameter

$L > 1$ is particularly chosen in order to guarantee a certain largeness condition for T_L and, in turn, for τ_L. Moreover, u is constructed such that it violates an energy inequality, which one may roughly consider as a *pathwise* version of (10.3), see (10.11) for the precise formulation.

Once such a solution u is constructed together with a suitable pair of stopping times T_L and τ_L, we continue as follows. We define

$$P := \mathbf{P} \circ u^{-1} \in \mathcal{P}(\Omega_{0,\tau_L}),$$

verify that P is a local martingale solution up to τ_L and that P fulfills the energy violation (10.3). We also need to prove that the definiton of τ_L allows to apply the extension result Lemma 9.4.2 in order to extend P to a global solution P_1 with (10.2) as desired. These steps will be tackled in Propositions 10.4.3 and 10.5.1.

To conclude this overview section, we turn our attention to the construction of the analytically weak and probabilistically strong local solution u. For this purpose, we *split* ($\mathrm{HNSE}_{\mathrm{sto}}$) into two equations in order to separate the two challenging terms, namely the nonlinearity $\operatorname{div}(u \otimes u)$ and the stochastic perturbation B. This splitting results in the linear stochastic PDE

$$\begin{cases} \partial_t z + (-\Delta)^\alpha z + \nabla p_1 &= dB, \\ z(0) &= 0, \\ \operatorname{div} z &= 0, \end{cases}$$

and the PDE with random coefficients

$$\begin{cases} \partial_t v + \operatorname{div}([v+z] \otimes [v+z]) + (-\Delta)^\alpha v + \nabla p_2 &= 0, \\ \operatorname{div} v &= 0. \end{cases}$$

By classical SPDE techniques, the former equation has a unique analytically weak $(\mathcal{F}_t)_{t\geq 0}$-adapted solution z. In Section 10.2, we obtain this solution as a stochastic convolution in terms of the semigroup generated by $\mathbb{P}(-\Delta)^\alpha$ and derive additional crucial regularity of z, see Proposition 10.2.4.

Fixing this unique solution z in the second equation above, this turns into a PDE with a random nonlinear coefficient. It remains to construct a pathwise analytically weak local solution v to this equation up to a bounded stopping time T_L, which is the exclusive aim of Chapter 11. The reason for the restriction in terms of the stopping time is the necessity to bound the random term z in the otherwise deterministic equation for v.

Chapter 11 culminates in the delicate construction of such v in Corollary 11.2.3, from where we obtain an analytically weak local solution u to ($\mathrm{HNSE}_{\mathrm{sto}}$) by setting

$$u(t) := z(t) + v(t), \quad t \in [0, T_L].$$

In order to achieve a pathwise energy violation for u on $\{T_L \geq T\}$, we also need to prescribe, roughly speaking, a sufficiently increasing energy profile for v in the sense that the energy violation must not only hold for v, but also after addition of z. Note that we do not have any kind of monotonicity for the energy profile of z, but only the growth bounds (10.8) up to T_L. Precise calculations in this direction are given in the proof of Theorem 10.3.2.

10.2 Decomposition of the equation

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}, B)$, where B is a GG^*-Wiener process on Ω, with G as in the assertion of Theorem 8.4.1. In this context we recall that $G : U \to L^2_\sigma$ is a Hilbert–Schmidt operator $G \in L_2(U, L^2_\sigma)$, where U is some auxiliary separable Hilbert space, and that we impose the regularity assumption (8.15). Moreover, $(\mathcal{F}_t)_{t\geq 0}$ denotes the augmented Brownian filtration, i.e. the canonical filtration generated by B augmented by all $\mathbf{P}$-negligible subsets of Ω. We recall that this filtration is right-continuous. For the remainder of the chapter, and also for Chapter 11, we fix this $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}, B)$.

As mentioned in the outline of the proof in the preceding section, we split ($\mathrm{HNSE}_{\mathrm{sto}}$) in order to treat the difficulties arising from the nonlinearity and the stochastic perturbation separately. Following the decomposition used in [116], we obtain the linear SPDE on $\mathbb{R}_+ \times \Omega$ as

$$\begin{cases} \partial_t z + (-\Delta)^\alpha z + \nabla p_1 &= dB, \\ z(0) &= 0, \\ \operatorname{div} z &= 0, \end{cases} \tag{SL_α}$$

and the nonlinear PDE with random coefficients

$$\begin{cases} \partial_t v + \operatorname{div}([v+z] \otimes [v+z]) + (-\Delta)^\alpha v + \nabla p_2 &= 0, \\ \operatorname{div} v &= 0. \end{cases} \tag{NL-SHNSE}$$

In order to make sense of (NL-SHNSE), we first show that (SL_α) admits a unique solution z, which we then fix in the formulation of (NL-SHNSE). The remaining aim of this section is to obtain a unique probabilistically strong solution z to (SL_α) with additional Sobolev regularity, which is crucial in order to control the size of z in (NL-SHNSE) up to a suitable $(\mathcal{F}_t)_{t\geq 0}$-stopping time T_L on Ω.

10.2.1 The linear equation (SL_α)

Definition 10.2.1. An *analytically weak solution with initial condition* $x_0 \in L^2_\sigma$ to (SL_α) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}, B)$ is an $(\mathcal{F}_t)_{t\geq 0}$-predictable process $z : \mathbb{R}_+ \times \Omega \to L^2_\sigma$ such that for each $e \in H^{2\alpha}$ and $t \geq 0$ we have

$$\langle z(t), e\rangle_{L^2} - \langle x_0, e\rangle_{L^2} + \int_0^t \langle z(s), (-\Delta)^\alpha e\rangle_{L^2} ds = \langle B(t), e\rangle_{L^2} \quad \mathbf{P}\text{-a.s.}, \tag{10.4}$$

where the exceptional set may depend on e and t. In particular, the appearing integral has to be well-defined, which is equivalent to the $\mathbf{P}$-a.s. local Bochner integrability of $t \mapsto z(t) \in L^2_\sigma$.

Remark 10.2.2. *The pressure term* p_1 *of* (SL_α) *is not present in the weak formulation* (10.4) *since each test vector field* $e \in H^{2\alpha} \subseteq L^2_\sigma$ *is solenoidal, i.e. it fulfills* $\operatorname{div} e = 0$*, which gives*

$$\langle \nabla p_1, e\rangle_{(-2\alpha,2\alpha)} = \langle p_1, \operatorname{div} e\rangle_{L^2} = 0.$$

We also remark that the operators $\big((-\Delta)^\alpha, \mathcal{D}((-\Delta)^\alpha)\big)$ *and* $\big(\mathbb{P}(-\Delta)^\alpha, \mathcal{D}(\mathbb{P}(-\Delta)^\alpha)\big)$ *coincide on* L^2_σ *since* $\mathbb{P}$ *and* $(-\Delta)^\alpha$ *commute on* $\mathcal{D}\big((-\Delta)^\alpha\big)$.

Since an analytically weak solution is $(\mathcal{F}_t)_{t\geq 0}$-predictable by definition, which in particular implies its adaptedness with respect to the normal Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$, and since the underlying probability space is given in advance (hence its construction is not part of the notion of solution), such solutions are *probabilistically strong*.

The existence of such a unique probabilistically strong, analytically weak solution to (SL_α) follows from the following proposition from [83, Thm.5.4].

Proposition 10.2.3. *There exists a unique analytically weak solution $z : \mathbb{R}_+ \times \Omega \to L^2_\sigma$ to the linear SPDE* (SL_α).

Concerning the proof, we refer to [83, Thm.5.4] and stress that in this respect all assumptions are fulfilled. Indeed, using the notation of this source, we have $H = L^2_\sigma$, $f = 0$, the initial condition $\xi = 0$, and $B = G \in L_2(U, L^2_\sigma)$. That $A = \mathbb{P}(-\Delta)^\alpha$ generates a C_0-semigroup $\big(S_\alpha(t)\big)_{t\geq 0}$ on L^2_σ, which is even a contraction semigroup, follows from Lemma C.0.1. From here, we obtain

$$\int_0^T ||S_\alpha(r)G||_{L_2(U,L^2_\sigma)}dr \leq ||G||_{L_2(U,L^2_\sigma)} \int_0^T ||S_\alpha(r)||_{L(L^2_\sigma)}dr \leq T||G||_{L_2(U,L^2_\sigma)} < \infty, \quad \forall T > 0,$$

which yields that [83, Thm.5.4] applies to our setting.

As we mentioned above, it is crucial to derive additional regularity for the solution z to (SL_α). This is achieved in the following proposition.

Proposition 10.2.4. *Assume G fulfills the regularity assumption* (8.15). *Then, for sufficiently small $\delta > 0$, we have for the unique solution z to* (SL_α)

$$z \in C_{\mathrm{loc}} H^{\frac{5+\sigma}{2}} \cap C^{1/2-2\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}} \quad \mathbf{P}\text{-a.s.},$$

and z satisfies

$$\mathbb{E}_{\mathbf{P}}\left[||z||_{C_T H^{\frac{5+\sigma}{2}}} + ||z||_{C_T^{1/2-2\delta} H^{\frac{3+\sigma}{2}}}\right] < \infty, \quad \forall T > 0.$$

The proof uses the explicit formula of z as a stochastic convolution with respect to the semigroup $(S_\alpha(t))_{t\geq 0}$ and is given in Proposition C.0.2. In particular, by the continuous embedding (9.3), for almost every ω and each $t \geq 0$, $z(t,\omega)$ has a continuous representative in C^0. Consequently, we may consider $z(\omega) \in C_T C_x$ for such ω, i.e. in particular $z \in C(\mathbb{R}_+ \times \mathbb{T}^3, \mathbb{R}^3)$ $\mathbf{P}$-a.s.

We conclude this section with the introduction of the stopping times T_L. The purpose of T_L is to control the size of the random coefficient z in (NL-SHNSE) within the convex integration method applied in Chapter 11. The parameter $L > 1$ needs to satisfy several lower bounds, which we introduce in due time. For $\delta > 0$ as in the previous proposition, set

$$T_L := \inf\{t \geq 0 : ||z(t)||_{H^{\frac{5+\sigma}{2}}} \geq L^{1/4} C_S^{-1}\} \wedge \inf\{t \geq 0 : ||z||_{C_t^{1/2-2\delta} H^{\frac{3+\sigma}{2}}} \geq L^{1/2} C_S^{-1}\} \wedge L, \tag{10.5}$$

where

$$C_S = \max\{C_1 C_2, C_2\}, \tag{10.6}$$

and C_1 and C_2 come from the Sobolev inequalities

$$||f||_{H^{\frac{3+\sigma}{2}}} \leq C_1 ||f||_{H^{\frac{5+\sigma}{2}}} \text{ and } ||f||_{L^\infty} \leq C_2 ||f||_{H^{\frac{3+\sigma}{2}}}, \tag{10.7}$$

respectively. Due to Proposition 10.2.4, T_L is a bounded $(\mathcal{F}_t)_{t\geq 0}$-stopping time with $T_L > 0$ $\mathbf{P}$-a.s. and $T_L \nearrow \infty$ $\mathbf{P}$-a.s. as $L \to \infty$. Moreover, the choice of C_S and the Sobolev inequalities (10.7) imply the pathwise estimates

$$||z(t)||_{L^\infty} \leq L^{1/4}, \quad ||Dz(t)||_{L^\infty} \leq L^{1/4}, \quad ||z||_{C_t^{1/2-2\delta} L^\infty} \leq L^{1/2}, \quad t \in [0, T_L]. \tag{10.8}$$

10.3 Analytically weak local solutions

As the next step towards the proof of Theorem 8.4.1, in this section we combine the result on the existence and regularity of the weak solution z to (SL$_\alpha$) up to the stopping time T_L with the weak solution of (NL-SHNSE), which we will construct in the forthcoming Chapter 11. More precisely, we use Propositions 10.2.3, 10.2.4 and Corollary 11.2 to obtain Theorem 10.3.2. Besides being essential for the proof of our main theorem, the existence of probabilistically strong local solutions to (HNSE$_{\text{sto}}$) which fulfill the energy violation (10.11), might also be of independent interest. In this regard, we mention again that the stopping times T_L up to which these solutions are defined increase pointwise to ∞ as $L \to \infty$. To start with, we recall the definition of an analytically weak local solution to (HNSE$_{\text{sto}}$).

Definition 10.3.1. Let $\mathfrak{t}$ be a bounded $(\mathcal{F}_t)_{t\geq 0}$-stopping time on Ω. An $(\mathcal{F}_t)_{t\geq 0}$-predictable process $u : [0, \mathfrak{t}] \times \Omega \to L^2_\sigma$ is an *analytically weak local solution to* (HNSE$_{\text{sto}}$) *on* $[0, \mathfrak{t}]$ *with initial value* $x_0 \in L^2_\sigma$, if for any $e \in H^3$ and $t \geq 0$, it holds

$$\langle u(t\wedge\mathfrak{t}) - x_0, e\rangle_{L^2} + \int_0^{t\wedge\mathfrak{t}} \langle u(r), (-\Delta)^\alpha e\rangle_{L^2} - \langle u(r)\otimes u(r), \nabla e\rangle_{L^2} dr = \langle B(t\wedge\mathfrak{t}), e\rangle_{L^2}, \quad \mathbf{P}\text{-a.s.}$$

Concerning the next theorem, we note that the analogous result for the classical stochastic Navier–Stokes equations (i.e. $\alpha = 1$) is [116, Thm.1.1]. Here, adaptedness is understood with respect to the state space σ-algebra $\mathcal{B}(L^2)$.

Theorem 10.3.2. *Assume G satisfies* (8.15) *and let $T > 0$, $K > 1$ and $\kappa \in (0, 1)$. Then, there is $\gamma \in (0, 1)$, an $(\mathcal{F}_t)_{t\geq 0}$-stopping time $\mathfrak{t} = \mathfrak{t}(T, K, \kappa)$ satisfying $\mathfrak{t} > 0$ $\mathbf{P}$-a.s. and*

$$\mathbf{P}(\mathfrak{t} \geq T) > \kappa, \tag{10.9}$$

and an $(\mathcal{F}_t)_{t\geq 0}$-adapted analytically weak solution u to (HNSE$_{\text{sto}}$) *on $[0, \mathfrak{t}]$ with some deterministic initial condition $x_0 \in L^2_\sigma$. Moreover, $u \in C([0, \mathfrak{t}], H^\gamma)$ $\mathbf{P}$-a.s. for some $\gamma \in (0, 1)$, and u can be constructed such that*

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{t\geq 0} ||u(t\wedge\mathfrak{t})||_{H^\gamma} < \infty \tag{10.10}$$

and

$$||u(T)||_{L^2} > K\big[||u(0)||_{L^2} + \big(T\,\mathrm{Tr}(GG^*)\big)^{1/2}\big] \quad \text{on } \{\mathfrak{t} \geq T\}. \tag{10.11}$$

Remark 10.3.3. *It turns out that we will choose the stopping time* $\mathfrak{t} = T_L$ *as defined in* (10.5)*. Its definition and the regularity result Proposition 10.2.4 for* z *show that* (10.9) *for* T_L *in place of* $\mathfrak{t}$ *can be fulfilled for any choice of* $T > 0$ *and* $\kappa \in (0,1)$ *if one chooses* $L \geq L_0 = L_0(T,\kappa)$*. Similarly, we show that* L *can be chosen sufficiently large in order to obtain* (10.11) *for any choice of* $K > 1$ *and* $T > 0$ *and that such lower bounds determine the suitable values for* L*.*

For the proof, we need the following result of Chapter 11, which we state here for the convenience of the reader. Again, adaptedness is understood with respect to $\mathcal{B}(L^2)$.

Corollary 11.2.3 of Section 11.2: *For any* $T > 0$*, there is* $L_0 = L_0(T) > 1$ *such that for any* $L \geq L_0$ *there exists an analytically weak solution* $v = v(T,L)$ *to* (NL-SHNSE) *on* $[0,T_L]$ *with the following properties.* v *is* $(\mathcal{F}_t)_{t\geq 0}$*-adapted, has some deterministic initial value* $v(0) = x_0 \in L^2_\sigma$ *and its paths belongs to* $C([0,T_L],H^\gamma)$ *for some* $\gamma \in (0,1)$*. Moreover,* v *fulfills*

$$||v(T)||_{L^2} > \big(||v(0)||_{L^2} + L\big)e^{LT} \tag{10.12}$$

on $\{T_L \geq T\}$ and

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{t\in[0,T_L]} ||v(t)||_{H^\gamma} < \infty \tag{10.13}$$

for some $\gamma \in (0,1)$.

As the subsequent proof shows, combining the above corollary with the solution z to the linear part (SL_α), we obtain the solution u of Theorem 10.3.2 via

$$u(t) := z(t) + v(t), \quad t \in [0,T_L]. \tag{10.14}$$

Using the pathwise bound (10.8) for z and the energy violation (11.12) for sufficiently large $L > 1$, we obtain the desired energy inequality for (10.11).

Proof of Theorem 10.3.2. Let $T > 0$, $K > 1$, and $\kappa \in (0,1)$. First of all, by the splitting of ($\mathrm{HNSE}_{\mathrm{sto}}$) in ($\mathrm{SL}_\alpha$) and (NL-SHNSE), and since any adapted process with continuous paths is predictable, it is clear that u as in (10.14) is an $(\mathcal{F}_t)_{t\geq 0}$-predictable analytically weak solution to ($\mathrm{HNSE}_{\mathrm{sto}}$) on $[0,T_L]$. The initial condition $u(0) = z(0) + v(0) = v(0) \in L^2_\sigma$ is deterministic by Corollary 11.2.3. We already know that T_L is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time such that $T_L > 0$ **P**-a.s., see the passage between (10.7) and (10.8). The **P**-a.s. pointwise divergence of T_L to ∞ as $L \to \infty$ implies that (10.9) holds for any given K and κ for all sufficiently large $L \geq L_1 = L_1(K,\kappa)$. Secondly, (10.10) with T_L in place of $\mathfrak{t}$ follows by definition of T_L and by (10.13) for $L \geq L_0(T)$ and γ as in Corollary 11.2.3. Thirdly, $u \in C([0,T_L],H^\gamma)$ holds due to the regularity result for z and due to $v \in C([0,T_L],H^\gamma)$. Finally, it remains to verify (10.11). To this end, we observe that on $\{T_L \geq T\}$ we have

$$\begin{aligned}
||u(T)||_{L^2} &\geq ||v(T)||_{L^2} - ||z(T)||_{L^2} > \big(||v(0)||_{L^2} + L\big)e^{LT} - (2\pi)^{3/2}L^{1/4} \\
&= \big(||u(0)||_{L^2} + L\big)e^{LT} - (2\pi)^{3/2}L^{1/4} \\
&> K\big[||u(0)||_{L^2} + \big(T\,\mathrm{Tr}(GG^*)\big)^{1/2}\big].
\end{aligned}$$

Here, we employed (10.12) and (10.8) for the first strict inequality and have chosen L larger than a suitable lower bound $L_2 = L_2(K, T, ||u_0||_{L^2}, G) > 0$ for the final estimate. Consequently, all assertions of the proposition are satisfied for each $L \geq \max(L_0, L_1, L_2)$ and $\mathrm{t} = T_L$. □

10.4 From analytically weak to local martingale solutions

Having established Theorem 10.3.2, we advance to the construction of a local martingale solution P to (HNSE_{sto}), which obeys the energy violation (10.3). In particular, we make use of T_L in order to define a suitable stopping time τ_L on Ω_0 up to which we construct P in Proposition 10.4.3.

For the analytically weak solution u of Theorem 10.3.2 up to T_L, we set

$$u_L : \mathbb{R}_+ \times \Omega \to H^\gamma, \;\; u_L(t) := u(t \wedge T_L). \tag{10.15}$$

In particular it holds that $u_L \in \Omega_0$ pathwise. Our next aim is to show that the probability measure $P \in \mathcal{P}(\Omega_0)$, defined as

$$P := \mathbf{P} \circ u_L^{-1}, \tag{10.16}$$

is a local martingale solution to (HNSE_{sto}) up to a suitable stopping time τ_L on Ω_0. To this end, we begin with the construction of τ_L. Of course, by definition, P is concentrated on the space of paths $\Omega_{T_L,0}$ stopped at T_L, but it will be advantageous to consider P on the bigger space Ω_0.

10.4.1 The stopping times τ_L

In principle, we would like to define τ_L on Ω_0 similarly to T_L on Ω as in (10.5). However, to this end we need a probability measure $\mathfrak{P}$ on $(\Omega_0, \mathcal{B}, (\mathcal{B}_t)_{t\geq 0})$ together with a $(\mathcal{B}_t)_{t\geq 0}$-Brownian motion $\mathfrak{B}$ and a solution Z to (SL_α) on the probability space $(\Omega_0, \mathcal{B}, (\mathcal{B}_t)_{t\geq 0}, \mathfrak{P}, \mathfrak{B})$. We choose $\mathfrak{P} = P$ as defined in (10.16), and for $x \in \Omega_0 \cap L^\infty_{\text{loc}}(\mathbb{R}_+, L^2_\sigma)$, we consider the following processes with paths in $C(\mathbb{R}_+, H^{-3})$.

$$M^x_{t,s} := x(t) - x(s) + \int_s^t F_\alpha\big(x(r)\big)dr, \quad 0 \leq s \leq t, \tag{10.17}$$

and

$$Z^x(t) := M^x_{t,0} + \int_0^t \mathbb{P}(-\Delta)^\alpha S_\alpha(r) M^x_{r,0}\, dr, \quad t \geq 0. \tag{10.18}$$

The idea behind these definition is that we will show that under P, $M_{t,0}$ is a GG^*-Brownian motion on Ω_0 and Z is the unique solution to (SL_α) on Ω_0. In other words, this holds for the processes $M^u_{t,0}$ and Z^u under $\mathbf{P}$ on $[0, T_L] \times \Omega$. Consequently, Z^u coincides with z up to T_L, see the proof of Lemma 10.4.2. We use the process Z on Ω_0 for the definition of the stopping time τ_L as follows. For $n \in \mathbb{N}$, $L > 1$ and $\delta > 0$ as in (10.5), set

$$\begin{aligned}\tau^n_L(x) := &\inf\big\{t \geq 0 : ||Z^x(t)||_{H^{\frac{5+\sigma}{2}}} > \big(L - 1/n\big)^{1/4} C_S^{-1}\big\} \\ &\wedge \inf\big\{t \geq 0 : ||Z^x||_{C_t^{1/2-2\delta} H^{\frac{3+\sigma}{2}}} > \big(L - 1/n\big)^{1/2} C_S^{-1}\big\} \wedge L,\end{aligned}$$

and

$$\tau_L := \lim_{n\to\infty} \tau_L^n, \tag{10.19}$$

with C_S as in (10.6). By Lemma 10.4.1, each τ_L^n is a $(\mathcal{B}_t)_{t\geq 0}$-stopping time on $\Omega_0 \cap L^\infty_{\text{loc}}(\mathbb{R}_+, L^2_\sigma)$ and, in turn, also τ_L is a $(\mathcal{B}_t)_{t\geq 0}$-stopping time as the pointwise increasing limit of stopping times, and we have $\tau_L \leq L$.

For a generic path $x \in \Omega_0 \cap L^\infty_{\text{loc}}(\mathbb{R}, L^2_\sigma)$, we have $\tau_L(x) = 0$, since in general x is not in $C(\mathbb{R}_+, H^{\frac{5+\sigma}{2}})$. In Lemma 10.4.2 we will use the regularity of z and the equality $Z^u = z$ $\mathbf{P}$-a.s. in order to obtain that τ_L is actually strictly positive P-a.s. and can be constructed such that $\tau_L \geq T$ holds with P-probability arbitrarily close to 1. Since the assertion of the subsequent auxiliary lemma is identical to [116, Lem.4.1], we omit its proof at this point.

Lemma 10.4.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered measurable space and let H_1, H_2 be separable Hilbert spaces such that the embedding $H_1 \hookrightarrow H_2$ is continuous. Suppose that there exist dual elements $\{h_k\}_{k\in\mathbb{N}} \subseteq H_2' \subseteq H_1'$ such that for each $f \in H^1$*

$$||f||_{H^1} = \sup_{k\in\mathbb{N}} h_k(f).$$

Suppose further that X is an $(\mathcal{F}_t)_{t\geq 0}$-adapted process on Ω with trajectories in $C(\mathbb{R}_+, H_2)$. Then, for $L > 1$ and $\gamma \in (0,1)$, both

$$\tau_1 := \inf\{t \geq 0 : ||X(t)||_{H^1} > L\} \quad \text{and} \quad \tau_2 := \{t \geq 0 : ||X(t)||_{C_t^\gamma H_1} > L\}$$

are $(\mathcal{F}_t^+)_{t\geq 0}$-stopping times, where $(\mathcal{F}_t^+)_{t\geq 0}$ denotes the right-continuous filtration of $(\mathcal{F}_t)_{t\geq 0}$.

Clearly, considering the definition of τ_L^n, we may apply this result for the choice $H_1 = H^{\frac{5+\sigma}{2}}$ and $H^{\frac{3+\sigma}{2}}$, respectively, and $H_2 = H^{-3}$ on $\big(\Omega_0 \cap L^\infty_{\text{loc}}(\mathbb{R}_+, L^2_\sigma), \mathcal{B}, (\mathcal{B}_t^0)_{t\geq 0}\big)$.

Lemma 10.4.2. *For $\mathbf{P}$-a.e. $\omega \in \Omega$, we have*

$$\tau_L(u_L(\omega)) = T_L(\omega). \tag{10.20}$$

In particular, we have $\tau_L > 0$ P-a.s., and for any $T > 0$ and $\kappa \in (0,1)$, there is $L_0 = L_0(T,\kappa) > 1$ such that for any $L \geq L_0$

$$P(\tau_L \geq T) \geq \kappa. \tag{10.21}$$

Proof. First of all, we show

$$Z^u = z \text{ on } [0, T_L] \quad \mathbf{P}\text{-a.s.} \tag{10.22}$$

To this end, we observe that by definition of the process $M^x_{t,s}$ from (11.3) and since u is an analytically weak solution to (HNSE_{sto}) on $[0, T_L]$, we have for each $e \in H^3$

$$\langle M^u_{t,0}, e\rangle_{(-3,3)} = \langle u(t) - u(0), e\rangle + \int_0^t \langle F_\alpha(u(s)), e\rangle_{(-3,3)} ds = \langle B(t), e\rangle_{(-3,3)}, \quad t \in [0, T_L],$$

$\mathbf{P}$-a.s. on Ω. Due to the continuity of the paths of $u : [0, T_L] \to H^\gamma$, the exceptional set is independent of t (but it might depend on e). Consequently, $t \mapsto M^u_{t,0}$ coincides pathwise

P-a.s. with the given GG^*-Wiener process B on Ω. From here, it follows readily from an integration by parts formula that Z^u solves (SL_α) on $[0, T_L]$. Now (10.22) follows from (a local version of) Proposition 10.2.3.

Since the maps

$$t \mapsto ||z(t)||_{H^{\frac{5+\sigma}{2}}} \text{ and } t \mapsto ||z||_{C_t^{1/2-2\sigma} H^{\frac{3+\sigma}{2}}}$$

are **P**-a.s. continuous on $[0, T_L]$ due to Proposition 10.2.4, we have **P**-a.s. one of the three cases

$$T_L = L \quad \text{or} \quad ||z(T_L)||_{H^{\frac{5+\sigma}{2}}} \geq L^{1/4} C_S^{-1} \quad \text{or} \quad ||z||_{C_{T_L}^{1/2-2\delta} H^{\frac{3+\sigma}{2}}} \geq L^{1/2} C_S^{-1}. \tag{10.23}$$

From here, (10.22) immediately gives $\tau_L(u) \leq T_L$ **P**-a.s. in each of these cases. Now assume $\tau_L(u) < T_L$ on a set of strictly positive **P**-measure, i.e., in particular, we have $\tau_L(u) < L$ on this set. By (10.22), it follows that on this set we have **P**-a.s.

$$||z(\tau_L(u))||_{H^{\frac{5+\sigma}{2}}} = ||Z^u(\tau_L(u))||_{H^{\frac{5+\sigma}{2}}} \geq L^{1/4} C_S^{-1} \tag{10.24}$$

or

$$||z||_{C_{\tau_L(u)}^{1/2-2\delta} H^{\frac{3+\sigma}{2}}} = ||Z^u||_{C_{\tau_L(u)}^{1/2-2\delta} H^{\frac{3+\sigma}{2}}} \geq L^{1/2} C_S^{-1}, \tag{10.25}$$

which, however, contradicts the definition of T_L. Consequently, we conclude (10.20). The rest of the assertion now follows from the definition of P as the law $\mathbf{P} \circ u^{-1}$ and since we know already the **P**-a.s. pointwise divergence $T_L \nearrow \infty$ as $L \to \infty$. □

10.4.2 A local martingale solution

Having introduced the bounded $(\mathcal{B}_t)_{t \geq 0}$-stopping time τ_L, we use Lemma 10.4.2 to proceed to the construction of a local martingale solution as follows.

Proposition 10.4.3. *The probability measure P as in* (10.16) *is a local martingale solution to* ($\mathrm{HNSE}_{\mathrm{sto}}$) *on $[0, \tau_L]$ with the deterministic initial condition $x_0 \in L^2_\sigma$ of Corollary 11.2.3.*

Proof. Clearly, $P(\pi_0 = x_0) = \mathbf{P}(u(0) = x_0) = 1$ holds by definition of P and the construction of u as $u = v + z$ with $z(0) = 0$ and v as in Corollary 11.2.3. Concerning (M3), since u fulfills (10.10) with $\mathsf{t} = T_L$ and (10.20), we obtain

$$\mathbb{E}_P \left[\sup_{r \leq \tau_L} ||\pi_r||_{H^\gamma} \right] = \mathbb{E}_{\mathbf{P}} \left[\sup_{r \leq T_L} ||u(r)||_{H^\gamma} \right] < \infty \tag{10.26}$$

for some $\gamma \in (0, 1)$ and, consequently, for each fixed $q \in \mathbb{N}$, the left-hand side of (M3) in Definition 9.3.3 is uniformly bounded in $t \geq 0$. In particular, it is possible to find a family of continuous positive functions $t \mapsto C_{t,q}$, for which P satisfies (M3).

Finally, considering (M2), let $0 \leq r \leq t$, $e \in H^3$ and $g : \Omega_0 \to \mathbb{R}$ be continuous, bounded and $\mathcal{B}_r$-measurable. Below we write $M_0^e(t, u_L) := M_0^e(t) \circ u_L$ for M_0^e as in Definition 9.3.3. Note the equality

$$\mathbb{E}_{\mathbf{P}} \Big[M_0^e\big(t \wedge \tau_L(u_L), u_L\big) g(u_L) \Big] = \mathbb{E}_{\mathbf{P}} \Big[M_0^e\big(r \wedge \tau_L(u_L), u_L\big) g(u_L) \Big], \tag{10.27}$$

which can be obtained as follows. Using Lemma 10.4.2, we have

$$\mathbb{E}_{\mathbf{P}}\Big[M_0^e\big(t \wedge \tau_L(u_L), u_L\big)g(u_L)\Big] = \mathbb{E}_{\mathbf{P}}\Big[M_0^e\big(t \wedge T_L, u_L\big)g(u_L)\Big]. \tag{10.28}$$

Since

$$M_0^e(t \wedge T_L, u_L) = \langle M_{t,0}^{u_L}, e\rangle_{(-3,3)}, \tag{10.29}$$

recalling that we showed $M_{t,0}^{u_L} = B_{t\wedge T_L}$, $t \geq 0$ $\mathbf{P}$-a.s. at the beginning of the proof of Lemma 10.4.1, we obtain that $M_0^e(\cdot \wedge T_L, u_L)$ is a $(\mathcal{B}_t)_{t\geq 0}$-martingale with respect to $\mathbf{P}$ with quadratic variation

$$t \mapsto (t \wedge T_L)||G^*e||_U^2.$$

This yields (10.27), since $g(u_L)$ is $\mathcal{F}_r$-measurable as a concatenation of the $\mathcal{B}_r$-measurable function g with the $(\mathcal{F}_t)_{t\geq 0}$-adapted process u_L. From here, the martingale property of $M_0^e(\cdot \wedge \tau_L)$ with respect to P follows from

$$\begin{aligned}\mathbb{E}_P\Big[M_0^e(t \wedge \tau_L)g\Big] = \mathbb{E}_{\mathbf{P}}\Big[M_0^e(t \wedge \tau_L(u_L), u_L)g(u_L)\Big] &= \mathbb{E}_{\mathbf{P}}\Big[M_0^e(r \wedge \tau_L(u_L), u_L)g(u_L)\Big] \\ &= \mathbb{E}_P\Big[M_0^e(r \wedge \tau_L)g\Big].\end{aligned}$$

Since $t \mapsto M_0^e(\cdot \wedge T_L, u_L)$ has quadratic variation $t \mapsto (t \wedge \tau_L)||G^*e||_U^2$ under $\mathbf{P}$, it follows via (10.20) that $t \mapsto M_0^e(\cdot \wedge \tau_L)$ has the same quadratic variation on Ω_0 under P, which completes the proof. □

10.5 Conclusion of the proof

Up to this point, we have constructed a stopping time τ_L on Ω_0 and a local martingale solution P up to τ_L as the law under $\mathbf{P}$ of the analytically weak local solution $u_L = u(\cdot \wedge T_L)$. We could now use the pathwise energy violation (10.11) to obtain the local energy violation (10.3) for P and, from there, obtain nonuniqueness of local martingale solutions up to τ_L. However, we aim for the stronger global result of Theorem 8.4.1. In the next subsection, we first extend P to a global solution. Afterwards, we complete the proof of Theorem 8.4.1.

10.5.1 Extension of the local solution

The extension of P to a global martingale solution proceeds along the steps presented in Section 9.4 and is given by the following result.

Proposition 10.5.1. *For the stopping time τ_L and the martingale solution P up to τ_L of Proposition 10.4.3, all assumptions of Lemma 9.4.2 are fulfilled. Consequently, P extends to a global martingale solution $P \otimes_{\tau_L} R$ with initial condition $x_0 \in L^2$ at time 0 such that (9.35) holds.*

Proof. In view of Lemma 9.4.2, we only need to verify the existence of a P-negligible set $\mathcal{N} \subseteq \Omega_0$ such that for each $x \in \mathcal{N}^c$ the equality of (9.34) holds. The idea is the following. For each $x \in \mathcal{N}^c$, we find a Q_x-negligible set $\mathcal{N}_x \subseteq \Omega_0$ such that

$$\{x' \in \Omega_0 : \tau_L(x') = \tau_L(x)\} \in \mathcal{N}_x^c \cap \mathcal{B}^0_{\tau_L(x)}. \tag{10.30}$$

Clearly, this follows, if we can show

$$\{x' \in \mathcal{N}_x^c : \tau_L(x') \leq t\} \in \mathcal{N}_x^c \cap \mathcal{B}^0_t \tag{10.31}$$

for each $t < L$. From here, we obtain (9.34) via

$$\begin{aligned} Q_x\big(x' \in \Omega_0 : \tau_L(x') = \tau_L(x)\big) &= Q_x\big(x' \in \mathcal{N}_x^c : x'(t) = x(t),\, t \in [0, \tau_L(x)],\, \tau_L(x') = \tau_L(x)\big) \\ &= Q_x\big(x' \in \mathcal{N}_x^c : x'(t) = x(t),\, t \in [0, \tau_L(x)]\big) = 1, \end{aligned}$$

where Q_x is as in Lemma 9.4.1, and we used (9.29) and (10.30) together with the observation that we have

$$\{x' \in \mathcal{N}_x^c \cap A : x'(t) = x(t),\, t \in [0, \tau_L(x)]\} = \{x' \in \mathcal{N}_x^c : x'(t) = x(t),\, t \in [0, \tau_L(x)]\}$$

for any A such that $\mathcal{N}_x^c \cap A \in \mathcal{N}_x^c \cap \mathcal{B}^0_{\tau_L(x)}$ with $x \in A$.

It is easy to see that (10.31) holds whenever x is such that there is a Q_x-negligible set $\mathcal{N}_x \subseteq \Omega_0$ such that for each $x' \in \mathcal{N}_x^c$, the trajectory $t \mapsto Z^{x'}(t)$ belongs to $C(\mathbb{R}_+, H^{\frac{5+\sigma}{2}}) \cap C^{1/2-2\delta}_{\mathrm{loc}}(\mathbb{R}_+, H^{\frac{3+\sigma}{2}})$. Indeed, in this case, for all $x' \in \mathcal{N}_x^c$ we have $\tilde{\tau}_L(x') = \tau_L(x')$, where we set

$$\begin{aligned} \tilde{\tau}_L(x') := \inf\{t \geq 0 : ||Z^{x'}(t)||_{H^{\frac{5+\sigma}{2}}} &\geq L^{1/4} C_S^{-1}\} \\ &\wedge \inf\{t \geq 0 : ||Z^{x'}||_{C_t^{1/2-2\delta} H^{\frac{3+\sigma}{2}}} \geq L^{1/2} C_S^{-1}\} \wedge L, \end{aligned} \tag{10.32}$$

which gives for $0 < L$

$$\begin{aligned} \{x' \in \mathcal{N}_x^c : \tau_L(x') \leq t\} = &\Big\{x' \in \mathcal{N}_x^c : \sup_{s \in \mathbb{Q}, s \leq t} ||Z^{x'}(s)||_{H^{\frac{5+\sigma}{2}}} \geq L^{1/4} C_S^{-1}\Big\} \\ &\bigcup \Big\{x' \in \mathcal{N}_x^c : \sup_{s_1 \neq s_2 \in \mathbb{Q} \cap [0,t]} \frac{||Z^{x'}(s_1) - Z^{x'}(s_2)||_{H^{\frac{3+\sigma}{2}}}}{|s_1 - s_2|^{1/2-2\delta}} \geq L^{1/2} C_S^{-1}\Big\}, \end{aligned}$$

and the right-hand side is in $\mathcal{N}_x^c \cap \mathcal{B}^0_t$, since the process $(t, x) \mapsto Z^x_t$ is $(\mathcal{B}^0_t)_{t \geq 0}$-adapted on $\Omega_0 \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^2_\sigma)$. Therefore, it remains to prove that for P-a.e. $x \in \Omega_0$, we have

$$Q_x\bigg(x' \in \Omega_0 : Z^{x'} \in C\big(\mathbb{R}_+, H^{\frac{5+\sigma}{2}}\big) \cap C^{1/2-2\delta}_{\mathrm{loc}}\big(\mathbb{R}_+, H^{\frac{3+\sigma}{2}}\big)\bigg) = 1. \tag{10.33}$$

To this end, we first of all note that (10.20), (10.22) and Proposition 10.2.4 entail

$$P\bigg(x \in \Omega_0 : Z^x(\cdot \wedge \tau_L(x)) \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-2\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\bigg)$$

$$= \mathbf{P}\Big(z(\cdot \wedge T_L) \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-2\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\Big) = 1,$$

i.e. there is a measurable P-negligible set $\mathcal{N}$ such that for $x \in \mathcal{N}^c$ we have

$$Z^x_{\cdot\wedge\tau_L(x)} \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-2\delta}_{\mathrm{loc}} CH^{\frac{3+\sigma}{2}}. \tag{10.34}$$

Hence, it remains to prove regularity of Z beyond τ_L. To this end, a direct calculation shows

$$Z^{x'}(t) - Z^{x'}(t \wedge \tau_L(x)) = \mathbb{Z}^{x'}_{\tau_L(x)}(t) + \big(S_\alpha\big(t - t\wedge\tau_L(x)\big) - 1\big)Z^{x'}(t\wedge\tau_L(x)),$$

where we set

$$\mathbb{Z}^{x'}_{\tau_L(x)}(t) := M^{x'}_{t,0} - M^{x'}_{t\wedge\tau_L(x),0} + \int_{t\wedge\tau_L(x)}^t \mathbb{P}(-\Delta)^\alpha S_\alpha(t-s)\big(M^{x'}_{s,0} - M^{x'}_{s\wedge\tau_L(x),0}\big)ds.$$

Note that $x' \mapsto M^{x'}_{r,0} - M^{x'}_{r\wedge\tau_L(x),0}$ is $\mathcal{B}^{\tau_L(x)}$-measurable for each $r \geq 0$, which implies $\mathcal{B}^{\tau_L(x)}$-measurability of $x' \mapsto \mathbb{Z}^{x'}_{\tau_L(x)}(t)$ for each $t \geq 0$. Considering the construction of Q_x from Lemma 9.4.1, it holds that for all $x \in \Omega_0 \cap L^\infty_{\mathrm{loc}} L^2_\sigma$

$$\begin{aligned}
&Q_x\Big(x' \in \Omega_0 : Z^{x'} \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-2\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\Big)\\
&= Q_x\Big(x' \in \Omega_0 : Z^{x'}_{\cdot\wedge\tau_L(x)} \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-2\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}},\, \mathbb{Z}^{x'}_{\tau_L(x)} \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-2\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\Big)\\
&= \delta_x\Big(x' \in \Omega_0 : Z^{x'}_{\cdot\wedge\tau_L(x)} \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-2\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\Big)\\
&\qquad \times R_{\tau_L(x),x(\tau_L(x))}\Big(x' \in \Omega_0 : \mathbb{Z}^{x'}_{\tau_L(x)} \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-2\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\Big).
\end{aligned}$$

For each $x \in \mathcal{N}^c$, the first factor on the above right-hand side equals to 1 due to (10.34). Concerning the second factor, since $R_{\tau_L(x),x(\tau_L(x))}$ is a martingale solution to ($\mathrm{HNSE}_{\mathrm{sto}}$) starting from time $\tau_L(x)$ with the deterministic initial condition $x(\tau_L(x)) \in L^2_\sigma$, it follows by the local version of Remark 9.3.2 that $M_{\cdot,0} - M_{\cdot\wedge\tau_L(x),0}$ is a GG^*-Wiener process with respect to $R_{\tau_L(x),x(\tau_L(x))}$ and $(\mathcal{B}_t)_{t\geq 0}$. Therefore, $(t, x') \mapsto \mathbb{Z}^{x'}_{\tau_L(x)}(t)$ is the unique analytically weak solution to (SL_α) on $\big(\Omega_0, \mathcal{B}, (\mathcal{B}_t)_{t\geq 0}, M_{\cdot,0} - M_{\cdot\wedge\tau_L(x),0}\big)$. Similarly as for z, Proposition 10.2.4 gives the regularity

$$R_{\tau_L(x),x(\tau_L(x))}\Big(x' \in \Omega_0 : \mathbb{Z}^{x'}_{\tau_L(x)} \in CH^{\frac{5+\sigma}{2}} \cap C^{1/2-2\delta}_{\mathrm{loc}} H^{\frac{3+\sigma}{2}}\Big) = 1,$$

which in view of the above chain of equalities entails (10.33). As argued at the beginning of the proof, this allows to apply Lemma 9.4.2, from which the assertion follows. □

10.5.2 Conclusion of the proof

Finally, we can complete the proof of the main result of this part of the thesis, i.e. Theorem 8.4.1. We have constructed a global martingale solution $P \otimes_{\tau_L} R$, which coincides with the law of an analytically weak solution u up to the stopping time τ_L. u fulfills the pathwise energy violation (10.11). From here, we only need to observe that $P \otimes_{\tau_L} R$ is certainly distinct form a martingale solution constructed via classical Galerkin methods as in Proposition 9.3.4 and Remark 9.3.5.

Conclusion of proof of Theorem 8.4.1. Let $T > 0$, $K = 2$ and $\kappa = 1/2$. By Theorem 10.3.2 and Propositions 10.4.3 and 10.5.1, there is $L > 1$ such that there exists a global martingale solution $P_1 = P \otimes_{\tau_L} R$ to (HNSE$_{\text{sto}}$) with the following properties. P_1 coincides with P on $\sigma\big(\pi_{t\wedge\tau_L}, t \geq 0\big)$, where P is the law of $u_L = u(\cdot \wedge T_L)$ on Ω_0 under $\mathbf{P}$, with u_L and $\mathbf{P}$ as in Theorem 10.3.2 and (10.15), respectively. (10.9) holds with $\mathfrak{t} = T_L$ and κ and T as above. $\mathbb{1}_{\{\tau_L \geq T\}}||\pi_T||^2_{L^2} = \mathbb{1}_{\{\tau_L \geq T\}}||\pi_{T\wedge\tau_L}||^2_{L^2}$ is $\sigma(\pi_{t\wedge\tau_L}, t \geq 0)$-measurable, so that

$$\mathbb{E}_{P_1}\left[\mathbb{1}_{\{\tau_L \geq T\}}||\pi_T||^2_{L^2}\right] = \mathbb{E}_P\left[\mathbb{1}_{\{\tau_L \geq T\}}||\pi_T||^2_{L^2}\right].$$

Combining with (10.20) and (10.11) with $K = 2$, this entails

$$\begin{aligned}
\mathbb{E}_{P_1}\left[||\pi_T||^2_{L^2}\right] &= \mathbb{E}_P\left[\mathbb{1}_{\{\tau_L \geq T\}}||\pi_T||^2_{L^2}\right] + \mathbb{E}_{P_1}\left[\mathbb{1}_{\{\tau_L < T\}}||\pi_T||^2_{L^2}\right] \\
&\geq \mathbb{E}_{\mathbf{P}}\left[\mathbb{1}_{\{T_L \geq T\}}||u(T)||^2_{L^2}\right] \\
&> \mathbb{E}_{\mathbf{P}}\left[\mathbb{1}_{\{T_L \geq T\}}4\big(||u(0)||^2_{L^2} + T\,\mathrm{Tr}(GG^*)\big)\right] \\
&> 2\big(||x_0||^2_{L^2} + T\,\mathrm{Tr}(GG^*)\big),
\end{aligned}$$

where $x_0 \in L^2_\sigma$ is the deterministic initial condition of u from Theorem 10.3.2. On the other hand, we have already deduced the existence of a gobal martingale solution P_2, which fulfills

$$\mathbb{E}_{P_2}\big[||\pi_T||^2_{L^2}\big] \leq ||x_0||^2_{L^2} + T\,\mathrm{Tr}(GG^*),$$

see Proposition 9.3.4 and Remark 9.3.5. Comparing with the above chain of inequalities, it is clear that P_1 and P_2 are distinct at time T, which concludes the proof. □

Chapter 11

Convex integration for stochastic hypodissipative NSE

This chapter contains the technical core of the present part of the thesis, namely the method of convex integration to construct pathwise analytically weak local solutions to the

PDE with random coefficients (NL-SHNSE). We fix the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ and the same GG^*-Wiener process with respect to $\mathbf{P}$ and $(\mathcal{F}_t)_{t\geq 0}$ as at the beginning of Section 10.2. In particular, $(\mathcal{F}_t)_{t\geq 0}$ is the Brownian filtration augmented by the $\mathbf{P}$-negligible sets. As already mentioned in the previous chapter, we fix the unique probabilistically strong solution z of (SL_α) on this probability space in (NL-SHNSE), and we repeat that for each $L > 1$ we have introduced the bounded $(\mathcal{F}_t)_{t\geq 0}$-stopping time T_L such that on $[0, T_L]$, z obeys the pathwise bounds (10.8). We obtained these bounds from the regularity properties of z stated in Proposition 10.2.4, which also implies $T_L > 0$ and $T_L \nearrow \infty$ $\mathbf{P}$-a.s. as $L \to \infty$.

11.1 Outline

Fractional Navier–Stokes–Reynolds equations. In order to construct a pathwise analytically weak local solution $v \in C^\gamma_{T_L,x}$ to (NL-SHNSE), we use the general idea of convex integration as explained in Section 8.2. That is, for each $q \in \mathbb{N}_0$, we construct a pathwise solution triple $(v_q, p_q, \mathring{R}_q)$ to the following fractional Navier–Stokes–Reynolds system on $[0, T_L]$:

$$\begin{cases} \partial_t v_q + \operatorname{div}\big((v_q + z) \otimes (v_q + z)\big) + \nabla p_q + (-\Delta)^\alpha v_q &= \operatorname{div}(\mathring{R}_q), \\ \operatorname{div}(v_q) &= 0. \end{cases} \tag{11.1}$$

Here, $v_q \in C^\infty_{T_L,x}$ is a x-periodic velocity vector field, $p_q \in C^\infty([0, T_L] \times \mathbb{T}^3, \mathbb{R})$ denotes the scalar pressure, and $\mathring{R}_q$ and takes values in the space of symmetric, trace-free 3×3 real matrices. Our specific construction of $\mathring{R}_q$ entails that $\mathring{R}_q$ has the same regularity properties as z. By a solution to (11.1), we mean a solution in the strong pointwise sense. As discussed in the survey in Section 8.2, the matrix $\mathring{R}_q$ and the vector field $\operatorname{div} \mathring{R}_q$ may be considered an error term, which renders the equation more flexible in the sense that for given v_q one can calculate $\mathring{R}_q$ such that (11.1) is fulfilled.

We set up an iteration, which from a given stage $(v_q, \mathring{R}_q)$ produces the next stage $(v_{q+1}, \mathring{R}_{q+1})$ in such a way that $(v_q)_{q\in\mathbb{N}_0}$ converges pathwise in $C^\gamma_{T_L,x}$ for some (small) $\gamma \in (0,1)$ and such that the error terms $\mathring{R}_q$ converge to 0 as $q \to \infty$. This will imply that the limit v is a solution as desired. Since in Chapter 10 we use v for the construction of the $(\mathcal{F}_t)_{t\geq 0}$-adapted solution u (see (10.14)), it is crucial that each v_q has a deterministic initial condition $v_q(0)$ and is $(\mathcal{F}_t)_{t\geq 0}$-adapted with respect to $\mathcal{B}(L^2)$ as the σ-algebra on the state space.

The main iteration. The iterative construction of $(v_q, \mathring{R}_q)$ shares many features with the already existing deterministic and stochastic convex integration techniques, as surveyed in Chapter 8: If $(v_q, \mathring{R}_q)$ has already been constructed, we define the velocity at stage $q+1$ via

$$v_{q+1} := v_\ell + w_{q+1},$$

where v_ℓ is a mollified version of v_q with respect to a suitable mollification length scale $\ell = \ell(q)$, cf. (11.20), and w_{q+1} is a perturbation, consisting of rapidly oscillating vector fields. To be more precise, w_{q+1} approximately equals to a sum of finitely many *Beltrami waves* $W_{q+1,\xi}$ with small amplitude $a_{q+1,\xi}$, i.e.

$$w_{q+1} \sim \sum_{\xi} a_{q+1,\xi} W_{\xi,q+1}. \tag{11.2}$$

Precise definitions of these objects will be presented later on. With regard to the introduction in Chapter 8, we would like to mention that our construction is based on simple Beltrami waves instead of the more complex Mikado or Beltrami jets or flows. In this spirit, our techniques are closer to the deterministic results in [53, Sect.5] and [70] than to [116]. However, several changes are necessary due to the random coefficient z in (NL-SHNSE). To take into account the size of z up to T_L and in order to capture the desired energy violation (11.12) of the solution v, we introduce

$$M(t) := M_L(t) := L^4 e^{4Lt}, \quad t \geq 0, \tag{11.3}$$

and we will choose $L > 1$ sufficiently large later on. Roughly, one may consider $M(t)$ as a rapidly increasing energy profile for v.

As outlined in the survey of Section 8.2, it is crucial to establish a suitable balance between the scales of the small amplitudes $a_{q+1,\xi}$ and the high frequency of $W_{\xi,q+1}$. To this end, we set

$$\lambda_q := 2a^{-b/2} a^{cb^{q+1}}, \quad \delta_q := \frac{a^b}{4(2\pi)^3} a^{-b^q}, \quad q \in \mathbb{N}_0, \tag{11.4}$$

for parameters $a \gg 1$, $b > 1$ and $c > \frac{5}{2}$ to be specified later on. Clearly, we have $\lambda_q \nearrow \infty$, $\delta_q \searrow 0$, and also $\lambda_q \delta_q^{1/2} \nearrow \infty$ as $q \to \infty$. Since by definition, it holds

$$|W_{q+1,\xi}| = 1 \text{ and } W_{q+1,\xi} \sim e^{i\lambda_{q+1}\xi\cdot x},$$

for certain geometric vectors $\xi \in \mathbb{R}^3$, constructing the amplitudes $a_{q+1,\xi}$ at stage $q+1$ such that

$$||a_{q+1,\xi}||_{C^0_{t,x}} \sim M^{1/2}(t)\delta_{q+1}^{1/2}$$

will lead to the crucial pathwise estimates for $0 \leq t \leq T_L$

$$||v_{q+1} - v_q||_{C^0_{t,x}} \leq M^{1/2}(t)\delta_{q+1}^{1/2}, \tag{A.1}$$

$$||v_{q+1} - v_q||_{C^1_{t,x}} \leq C_L M^{1/2}(t)\delta_{q+1}^{1/2}\lambda_{q+1}, \tag{A.2}$$

where for $C_{\mathbb{T}^3} := ||\mathrm{id}||_{C^0} = \sqrt{3}\pi$, we set

$$C_L := (1 + C_{\mathbb{T}^3})(1 + 2M(L)^{1/2} + L^{1/4}). \tag{11.5}$$

We remark that such a constant C_L does not appear in the corresponding iterative estimates in any of the stochastic convex integration papers to date. An inspection of the proof in Section 11.3 shows that C_L appears due to the derivative of the nonlinear phase Φ_j, which is transported along $v_\ell + z_\ell$, cf. (11.34) and Lemma 11.3.2. In contrast to deterministic cases, where such a transport of the phase is standard (cf. [50, 70] and the references therein) and where no random perturbation z is present, this has not been used for stochastic cases so far. Hence, the appearance of C_L may be considered a result of our novel approach to use simple Beltrami waves in a stochastic convex integration setting.

We point out that, in general, such additional constants may be troublesome for the iteration scheme, since it is possible that such constants grow (super-)exponentially in q, i.e. in principle, one could have a dependence in the above estimates of C_L in the form of $C_L^{2^q}$. However, our calculations in Section 11.3 show that this is not the case. Instead, we merely reproduce C_L in each iterative step, but are able to absorb it in various estimates, thereby avoiding exponents strictly larger than 1.

Nevertheless, since C_L is not absolute, but depends on $L > 1$, additional care in several estimates throughout Section 11.3 is necessary, see for example (11.28).

Iteration scheme for the error term $\mathring{R}_{q+1}$. Having constructed v_{q+1}, it remains to infer the new error term $\mathring{R}_{q+1}$, which together with v_{q+1} and a suitable pressure p_{q+1} (which will not be of further interest to us) solves (11.1), and obeys for $0 \leq t \leq T_L$

$$||\mathring{R}_q||_{C^0_{t,x}} \leq M(t)\delta_{q+2}c_R. \tag{A.3}$$

Here, c_R is a small geometric constant specified in (11.30). We already mentioned that the flexibility of the fractional Navier–Stokes–Reynolds equations (11.1) allows to simply calculate $\mathring{R}_{q+1}$ based on v_{q+1} and the previous stage $(v_q, \mathring{R}_q)$ by subtracting (11.1) at stage q from stage $q+1$. In doing so, we approximately obtain, using mollified versions $\mathring{R}_\ell$ and p_ℓ of $\mathring{R}_q$ and p_q, respectively,

$$\begin{aligned} \operatorname{div} \mathring{R}_{q+1} - \nabla p_{q+1} = [\partial_t &+ (v_\ell + z_\ell) \cdot \nabla] w_{q+1} \\ &+ \operatorname{div}(w_{q+1} \otimes w_{q+1} + \mathring{R}_\ell) \\ &+ \mathfrak{R}_{q+1} - \nabla p_\ell. \end{aligned}$$

Here, the pivotal terms are the *transport error* $[\partial_t(v_\ell + z_\ell) \cdot \nabla] w_{q+1}$ and the *oscillation error* $\operatorname{div}(w_{q+1} \otimes w_{q+1} + \mathring{R}_\ell)$, while $\mathfrak{R}_{q+1}$ collects all remaining error terms, see (11.59) for more details. In order to bound $\mathring{R}_{q+1}$ as in (A.3), we now benefit from the structure of w_{q+1} as in (11.2). Indeed, the main feature of the Beltrami building blocks $W_{\xi,q+1}$ is the pivotal approximate cancellation of the oscillation error, namely

$$w_{q+1} \otimes w_{q+1} \approx -\mathring{R}_\ell,$$

cf. Lemma 11.3.5 for the precise result.

Moreover, concerning the transport error, we now use that the precise definition of $W_{q+1,\xi}$ contains the nonlinear phase Φ_j, which is transported along $v_\ell + z_\ell$. Consequently, the material derivative $\partial_t + (v_\ell + z_\ell) \cdot \nabla$ does not fall on the large frequency term $e^{i\lambda_{q+1}\xi \cdot \Phi_j}$, which otherwise would prevent us from achieving a small bound as in (A.3).

On the other hand, in order to control the deviation of the solution to the transport equation (11.34) from its initial condition, we use a localization in time in the sense that we solve (11.34) on suitably small time intervals of length approximately μ^{-1} before starting afresh with the original initial condition. To this end, we include the time cutoffs χ_j as in (11.33) in the definition of the amplitudes $a_{q+1,\xi}$ in (11.46) and the corresponding local solutions Φ_j to (11.34) in the Beltrami building blocks $B_\xi e^{i\lambda_{q+1}\xi\cdot\Phi_j}$. See also the first part of Subsection 11.3.3 for further explanations.

With these considerations, it is possible to obtain the error estimate (A.3). We point out that the estimates (A.1)-(A.3) can only be obtained up to time T_L, since beyond this stopping time, we cannot control the size of z, which is crucial for the estimates of the amplitude as well as the error term.

Conclusion. Having constructed a sequence of solutions $(v_q, \mathring{R}_q)_{q\in\mathbb{N}_0}$ which fulfills the iterative estimates (A.1)-(A.3), it is easy to use interpolation to obtain pathwise convergence of $(v_q)_{q\in\mathbb{N}_0}$ in $C^\gamma_{T_L,x}$ for sufficiently small values of $\gamma \in (0,1)$. Since (A.3) yields convergence of $\mathring{R}_q$ to 0 as $q \to \infty$, it readily follows that the limit $v \in C^\gamma_{T_L,x}$ is an analytically weak solution to (NL-SHNSE) up to T_L. The precise Hölder regularity of v is determined by the interpolation between the contrary terms $\delta^{1/2}_{q+1} \ll 1$ and $\delta^{1/2}_{q+1}\lambda_{q+1} \gg 1$ on the right-hand side of (A.1) and (A.2), respectively, and cannot be prescribed in advance.

Finally, it is important to note that the rapidly increasing energy profile $M(t) = L^4 e^{4tL}$ is part of the definition of the amplitudes $a_{q+1,\xi}$. Choosing L sufficiently large in terms of T, this allows to ensure the energy violation (11.12) for v. This is in spirit of deterministic results as in [53, 70, 54, 50], in which the authors construct solutions with any prescribed smooth and strictly positive energy profile. To do so, the construction of such solutions naturally includes energy-dependent amplitudes of the iterative perturbations in these cases as well. In our stochastic situation, we cannot hope to attain an exact energy profile due to the perturbation caused by the random coefficient z in (NL-SHNSE). On the other hand, all we need is to ensure that we start the iteration with a sufficiently large energy profile, which is then essentially preserved throughout the iteration and finally leads to (11.12). This explains the definition of the initial velocity field v_0 in (11.6). In this regard, our approach is in the same spirit as [116].

11.2 Main iteration

11.2.1 Starting triple

Here, we explicitly give the initial solution triple $(v_0, p_0, \mathring{R}_0)$ from which we start the iterative construction of solutions to (11.1). The definition of v_0 is tailored such that it obeys an increasing energy profile, and $\mathring{R}_0$ and p_0 are defined such that the initial triple fulfills (11.1). The parameter $L > 1$ dictates the growth scale of the energy profile of v_0 and, moreover, is chosen such that the estimates (11.8)-(11.10) hold.

For $L > 1$ and $M(t)$ as in (11.3), define

$$v_0(t,x) := \frac{M(t)^{1/2}}{(2\pi)^{3/2}} \left(\cos(x_3), \sin(x_3), 0\right), \tag{11.6}$$

$$R_0(t,x) := \frac{(2L+1)M(t)^{1/2}}{(2\pi)^{3/2}} \begin{pmatrix} 0 & 0 & \sin(x_3) \\ 0 & 0 & -\cos(x_3) \\ \sin(x_3) & \cos(x_3) & 0 \end{pmatrix} + v_0 \otimes z + z \otimes v_0 + z \otimes z,$$

and $\mathring{R}_0$ as the trace-free part of R_0. Furthermore, consider the scalar function p_0 such that

$$\operatorname{div}(R_0) = \operatorname{div}(\mathring{R}_0) - \nabla p_0,$$

i.e.

$$p_0(t,x) := \frac{1}{3}\big(2v_0(t,x)\cdot z(t,x) + |z(t,x)|^2\big).$$

Clearly, $v_0(t,\cdot)$ is 2π-periodic for each $t \geq 0$, v_0 is smooth in $(t,x) \in \mathbb{R}_+ \times \mathbb{T}^3$, and $\mathring{R}_q$ is 2π-periodic in its spatial argument and continuous in (t,x), since by the results of Subsection 10.2.1, we already know that we have $z \in C(\mathbb{R}_+ \times \mathbb{T}^3, \mathbb{R}^3)$. Moreover, the Sobolev regularity of z also gives $\mathring{R}_q(t,\cdot) \in H^{\frac{5+\sigma}{2}}$ for each $t \geq 0$. The small constant $c_R > 0$ is introduced in (11.30), where it is explained that it can be chosen independently of any other parameters which are only chosen later, and is hence to be considered an absolute geometric constant, which we fix from now on.

Lemma 11.2.1. *Let $a > 1, b > 1, c > 5/2$. For $L > 1$, the triple $(v_0, p_0, \mathring{R}_0)$ solves the Euler–Reynolds equations (11.1) on $[0, T_L] \times \mathbb{T}^3$. Moreover, if we additionally assume*

$$20(2\pi)^3 c_R^{-1} \leq L \leq \frac{a^2-2}{2}, \tag{11.7}$$

then the following estimates hold for all $t \in [0, T_L]$.

$$||v_0||_{C^0_{t,x}} \leq M(t)^{1/2}, \tag{11.8}$$

$$||v_0||_{C^1_{t,x}} \leq M(t)^{1/2}\delta_0^{1/2}\lambda_0, \tag{11.9}$$

$$||\mathring{R}_0||_{C^0_{t,x}} \leq M(t) c_R \delta_1. \tag{11.10}$$

Furthermore, v_0 and $\mathring{R}_q$ are $(\mathcal{F}_t)_{t\geq 0}$-adapted, and both $v_0(0)$ and $\mathring{R}_q(0)$ are deterministic.

Proof. By a straightforward calculation, it follows that $(v_0, p_0, \mathring{R}_0)$ solves (11.1) pointwise. (11.8) follows by definition of v_0. Concerning (11.9), we have for $t \in [0, T_L]$

$$||v_0||_{C^1_{t,x}} \leq \frac{(2+2L)M(t)^{1/2}}{(2\pi)^{3/2}}, \tag{11.11}$$

and the right-hand side can be estimated by

$$M(t)^{1/2}\delta_0^{1/2}\lambda_0 = \frac{M(t)^{1/2}}{(2\pi)^{3/2}} a^{cb-1/2},$$

provided $2 + 2L \leq a^{cb-1/2}$. Since $cb > 5/2$, this follows from (11.7). Concerning (11.10), we obtain

$$||\mathring{R}_0||_{C^0_{t,x}} \leq (2L+1)M(t)^{1/2} + M(t)^{1/2}L^{1/4} + L^{1/2} \leq 5\frac{M(t)}{L},$$

which is bounded from above by $M(t)\delta_1 c_R$ due to (11.7), since $\delta_1^{-1} = 4(2\pi)^3$.

Finally, the adaptedness of v_0 and the observation that $v_0(0)$ is deterministic are obvious since v_0 is deterministic. Concerning $\mathring{R}_q$, its adaptedness follows from the adaptedness of

z, which is the only ω-dependent term in the definition of $\mathring{R}_q$. Since $z(0) = 0$, also $\mathring{R}_q(0)$ is deterministic, which concludes the proof. □

11.2.2 Main iteration and conclusion

The following proposition contains the main technical part of the method of convex integration in our setting. From now on, we always consider L to be a natural number.

Proposition 11.2.2 (Main iteration)**.** *Let $L > 1$ satisfy the first estimate of (11.7). Then, there is a choice of parameters $a_0 \gg 1$, $b > 1$ and $c > 5/2$ such that for arbitrary large $a \geq a_0$, there exists a sequence of triples $(v_q, p_q, \mathring{R}_q)_{q\geq 0}$ with the following properties. $(v_0, p_0, \mathring{R}_0)$ is as in Lemma 11.2.1, for each $q \geq 1$ the triple $(v_q, p_q, \mathring{R}_q)$ solves (11.1) pointwise on $[0, T_L] \times \mathbb{T}^3$, v_q and $\mathring{R}_q$ are $(\mathcal{F}_t)_{t\geq 0}$-adapted, and (A.1)-(A.3) hold with respect to a, b and c. Furthermore, for each $q \geq 0$, $v_q(0)$ and $\mathring{R}_q(0)$ are deterministic.*

The proof of the preceding result is contained in the following section. Before we proceed in this direction, we state and prove the following important corollary, which we already used in Chapter 10 in order to construct the analytically weak solution u to (HNSE_{sto}).

Corollary 11.2.3. *For any $T > 0$, there is $L_0 = L_0(T) > 1$ such that for any $L \geq L_0$ there exists an analytically weak solution $v = v(L)$ to (NL-SHNSE) on $[0, T_L]$ with the following properties. v is $(\mathcal{F}_t)_{t\geq 0}$-adapted, has some deterministic initial value $v(0) = x_0 \in L^2_\sigma$, and its paths belongs to $C([0, T_L], H^\gamma)$ for some $\gamma \in (0, 1)$. Moreover, v fulfills*

$$||v(T)||_{L^2} > \big(||v(0)||_{L^2} + L\big)e^{LT} \tag{11.12}$$

on $\{T_L \geq T\}$, and we have

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{t\in[0,T_L]} ||v(t)||_{H^\gamma} < \infty \tag{11.13}$$

for some $\gamma \in (0, 1)$.

Proof. Fix $T > 0$, let L_0 be the smallest natural number such that the first part of (11.7) and

$$L_0 > \frac{\ln(11)}{T} \tag{11.14}$$

are fulfilled, and let $L \geq L_0$ be natural. Then, for a suitable choice of $b > 1$, $c > 5/2$ and $a_0 \gg 1$, for arbitrary large $a \geq a_0$, by Proposition 11.2.2 there exists a sequence of solution triples $(v_q, p_q, \mathring{R}_q)_{q\in\mathbb{N}_0}$ to the fractional Navier–Stokes–Reynolds equations (11.1) defined on $[0, T_L] \times \mathbb{T}^3$, which satisfies (A.1)-(A.3) subject to b, c, and a. Precise conditions on b and c are presented in Subsection 11.3.1. Interpolating the Hölder space $C^\gamma_{t,x}$ between $C^0_{t,x}$ and $C^1_{t,x}$, we obtain for $t \in [0, T_L]$

$$\begin{aligned}\sum_{q\geq 0} ||v_{q+1} - v_q||_{C^\gamma_{t,x}} \leq \sum_{q\geq 0} ||v_{q+1} - v_q||^\gamma_{C^1_{t,x}} ||v_{q+1} - v_q||^{1-\gamma}_{C^0_{t,x}} \leq C^\gamma_L M(t)^{1/2} \sum_{q\geq 0} \delta^{1/2}_{q+1}\lambda^\gamma_{q+1} \\ \lesssim M(t)^{1/2} \sum_{q\geq 0} a^{b^{q+1}(cb\gamma - 1/2)}. \end{aligned} \tag{11.15}$$

Hence, $(v_q)_{q\in\mathbb{N}_0}$ converges pathwise to a limit v in $C^\gamma_{T_L,x}$ if and only if $\gamma\in(0,\frac{1}{2bc})$. Moreover, (A.3) gives the pathwise convergence $\mathring{R}_q \longrightarrow 0$ in $C^0_{T_L,x}$ as $q\to\infty$. Considering the weak formulation of (11.1) for $(v_q,\mathring{R}_q)$, i.e. the dualization with any smooth, x-periodic vector field $\varphi:[0,T_L]\times\mathbb{T}^3\to\mathbb{R}^3$ such that $\operatorname{div}\varphi=0$ and $\varphi(t,\cdot)=0$ for $t\in K^c$ for some compact set $K\subseteq(0,T_L)$, given as

$$\int_0^{T_L}\int_{\mathbb{T}^3} v_q\cdot\left[\partial_t\varphi-(-\Delta)^\alpha\varphi\right]+\big([(v_q+z)\cdot\nabla]\varphi\big)\cdot(v_q+z)\,dxdt=\int_0^{T_L}\int_{\mathbb{T}^3}\mathring{R}_q:D\varphi\,dxdt,$$

it is clear that in the limit for $q\to\infty$ we have

$$\int_0^{T_L}\int_{\mathbb{T}^3} v\cdot\left[\partial_t\varphi-(-\Delta)^\alpha\varphi\right]+\big([(v+z)\cdot\nabla]\varphi\big)\cdot(v+z)\,dxdt=0.$$

Consequently, v is an analytically weak solution to (NL-SHNSE). Furthermore, as C_L and $M(t)$ are independent of ω, (11.15) particularly yields

$$\operatorname*{ess\,sup}_{\omega\in\Omega}||v||_{C^0_{T_L}C^\gamma_x}<\infty,\quad 0<\gamma<\frac{1}{2bc}.$$

In particular, we obtain (11.13) for sufficiently small $\gamma\in(0,1)$. Since each v_q is $(\mathcal{F}_t)_{t\geq0}$-adapted, the convergence in $C([0,T_L],C^\gamma)$ yields that v is $(\mathcal{F}_t)_{t\geq0}$-adapted as well. As each $v_q(0)$ is deterministic, also $v(0)$ is deterministic, i.e. we have $v(0)=x_0$ for some $x_0\in L^2_\sigma$.

To conclude the proof, it remains to verify (11.12). To this end, we note that for sufficiently large $a>1$ and any $t\geq0$, the corresponding solution v fulfills

$$\begin{aligned}||v(t)-v_0(t)||_{L^2}&\leq\sum_{q\geq0}||(v_{q+1}-v_q)(t)||_{L^2}\leq(2\pi)^{3/2}M(t)^{1/2}\sum_{q\geq0}\delta_{q+1}^{1/2}\\&\leq\frac{a^{b/2}}{2}M(t)^{1/2}\sum_{q\geq0}a^{-b^{q+1}/2}\leq\frac{3}{4}M(t)^{1/2}\quad\text{on }\{T_L\geq t\},\end{aligned}\tag{11.16}$$

where we used (A.1) for the second inequality, and $b\geq2$, which implies $b^q\geq qb$ for $q\geq1$, for the last inequality. From here, on the one hand, taking into account the definition of v_0, we obtain on $\{T_L\geq t\}$

$$||v(t)||_{L^2}\geq||v_0(t)||_{L^2}-||v(t)-v_0(t)||_{L^2}=M(t)^{1/2}-||v(t)-v_0(t)||_{L^2}\geq\frac{M(t)^{1/2}}{4}.\tag{11.17}$$

On the other hand, (11.16) also yields

$$\begin{aligned}(||v(0)||_{L^2}+L)e^{LT}&\leq\Big(||v_0(0)||_{L^2}+||v(0)-v_0(0)||_{L^2}+L\Big)e^{LT}\\&\leq\Big(\frac{7}{4}M(0)^{1/2}+L\Big)e^{LT}=\Big(\frac{7}{4}L^2+L\Big)e^{LT}<\frac{M(t)^{1/2}}{4},\end{aligned}$$

where the last inequality holds, if we choose $L\geq L_0$ and L_0 is chosen to satisfy (11.14). Combining with (11.17) for $t=T$, (11.12) follows and the proof of the corollary is complete. □

11.3 Proof of the main iteration

We now turn to the proof of Proposition 11.2.2. To this end, fix $L \in \mathbb{N}$ as in the assertion, $q \in \mathbb{N}_0$, and let $a_1(L) = a_1 > 1$ such that for any $a \geq a_1$ (11.7) is fulfilled. Then, for any $b > 1$ and $c > 5/2$, the initial triple of Subsection 11.2.1 fulfills all assertions of Lemma 11.2.1. Assume for some $a_2 \geq a_1$ that for any $a \geq a_2$ triples up to $(v_q, p_q, \mathring{R}_q)$, fulfilling the iterative bounds (A.1)-(A.3) with a, and with b and c as in (11.24) and (11.26) have already been constructed such that each v_q and $\mathring{R}_q$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted and $v_q(0)$ and $\mathring{R}_q(0)$ are deterministic. Our proof amounts to showing that there is a q-independent number $a_* \geq a_2$ such that for arbitrary large $a \geq a_*$, the next solution triple $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ can be constructed with all the above properties, in particular obeying (A.1)-(A.3) with a and with the same choice of b and c as above. Then, the assertion follows with $a_0 = a_*$, and b and c as in (11.24) and (11.26).

In all that follows, for functions f depending on $(t, x, \omega) \in \mathbb{R}_+ \times \mathbb{R}^d \times \Omega$, we suppress the explicit dependence on ω and simply write $f(t, x)$. In order to construct $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ with all desired properties, we proceed as follows.

Let $a \geq a_2$. In view of Proposition 11.3.3 and Lemma 11.3.4, it is important that λ_q is a multiple of the geometric number n_0 introduced right after Lemma 11.3.4. To this end, since we will particularly choose $b/2, c \in \mathbb{N}$, it suffices to have $a = kn_0$, $k \in \mathbb{N}$, which we consistently assume from now on and which is clearly consistent with what we claimed about a in the assertion of Proposition 11.2.2.

From the definition of λ_q and δ_q in (11.4) and the iterative construction

$$v_q = v_0 + \sum_{1\leq p\leq q} (v_p - v_{p-1}),$$

it follows that once we increase a sufficiently in terms of b and c, we obtain

$$\sum_{p\geq 1} \delta_p^{1/2} \leq 1, \qquad \sum_{0\leq p\leq q} \delta_p^{1/2}\lambda_p \leq 2\delta_q^{1/2}\lambda_q, \tag{11.18}$$

and hence, via (A.1), (A.2), (11.8) and (11.9), the bounds

$$||v_q||_{C^0_{t,x}} \leq 2M(t)^{1/2}, \quad ||v_q||_{C^1_{t,x}} \leq 2C_L M(t)^{1/2}\delta_q^{1/2}\lambda_q, \quad t \in [0, T_L] \tag{11.19}$$

follow.

11.3.1 Choice of parameters

One has to choose carefully the scales of the parameters $\ell = \ell_q$ and $\mu = \mu_q$ in comparison to the high frequency and low amplitude terms λ_q and δ_q. For $q \in \mathbb{N}_0$, set

$$\ell = \ell_q := \delta_{q+1}^{-1/8}\delta_q^{1/8}\lambda_q^{-1/4}\lambda_{q+1}^{-3/4} \tag{11.20}$$

and

$$\mu = \mu_q := (2\pi)^{3/2}\delta_{q+1}^{1/4}\delta_q^{1/4}\lambda_q^{1/2}\lambda_{q+1}^{1/2}. \tag{11.21}$$

These choices lead to the relation

$$\mu \ll \ell^{-1}, \tag{11.22}$$

which can be seen as follows: by definition and since $q \geq 0$, we have that

$$\frac{1}{\mu\ell} = 2a^{-b/2}a^{b^q\left(\frac{1}{8}b+\frac{3}{8}-\frac{1}{4}bc+\frac{1}{4}b^2c\right)} \geq 2a^{-b/2}a^{\frac{1}{8}b+\frac{3}{8}-\frac{1}{4}bc+\frac{1}{4}b^2c} \gg 1, \tag{11.23}$$

which holds since $b > 1, c > \frac{5}{2}$. Choosing a such that $a^{3/4} \in \mathbb{N}$, and b with $\frac{b}{4} \in \mathbb{N}$, it follows that μ is natural, which we will need in the sequel. Moreover, for a fixed sufficiently small choice of $\delta > 0$ as in (10.5), we fix $b \in \mathbb{N}$ such that

$$b > \frac{8}{3}(\frac{1}{2} - 2\delta)^{-1} > 5 \quad \text{and} \quad \frac{b}{4} \in \mathbb{N}, \tag{11.24}$$

set

$$\beta := \frac{b-1}{5b+5} < \frac{1}{5},$$

and fix $\varepsilon < \min\{\frac{1}{4} - \frac{\alpha}{2}, \frac{\beta}{2}\}$. We also introduce

$$N_0 := \left\lceil \frac{1+\varepsilon}{\beta} \right\rceil + 1, \tag{11.25}$$

and choose c such that

$$c > \max\left(\frac{2}{\beta}, \frac{8}{3}\left(\frac{1}{2} - 2\delta\right)^{-1}, \left(\frac{1}{2} - \alpha\right)^{-1}\right), \quad c \in \mathbb{N}. \tag{11.26}$$

At this point, all parameters and length scales are fixed for the remainder of the proof, with the exception of $a \geq a_2$, for which several additional lower bounds will appear throughout the iteration. In this regard, we point out that we may increase the value of a as needed with respect to all parameters fixed above and also with respect to L, as long as the number of imposed lower bounds remains finite. In view of Proposition 11.2.2 it is important to note that neither the lower bounds for a nor the choices for $\delta, b, \varepsilon, N_0$ or c depend on the stage index q. We will not explicitly collect all lower bounds for a, but merely mention any instance where we potentially further increase a. Carefully keeping track of these additional requirements on a, the maximum a_* of all lower bounds imposed along the subsequent proof leads to the lower bound $a_0 = a_*$ of the assertion.

These parameter choices yield the crucial estimates

$$\frac{\delta_q^{1/2}\lambda_q\ell}{\delta_{q+1}^{1/2}} \ll 1, \qquad \frac{\delta_q^{1/2}\lambda_q}{\mu} + \frac{1}{\ell\lambda_{q+1}} \leq \lambda_{q+1}^{-\beta} \ll \frac{\delta_{q+2}}{\delta_{q+1}}, \qquad \lambda_{q+1}^{-1} \leq \frac{\delta_{q+1}^{1/2}}{\mu}, \tag{11.27}$$

which can be verified by a calculation similar to (11.23). From here, we may further increase a in terms of L to obtain

$$C_L M(L)^{1/2}\frac{\delta_q^{1/2}\lambda_q\ell}{\delta_{q+1}^{1/2}} \ll 1, \quad \frac{C_L}{\ell\lambda_{q+1}} + \frac{C_L M(L)^{1/2}\delta_q^{1/2}\lambda_q + L^{1/4}}{\mu} \leq \lambda_{q+1}^{-\beta} \ll \frac{\delta_{q+2}}{\delta_{q+1}}. \tag{11.28}$$

Furthermore, choosing a sufficiently large, the lower bounds for b and c stated above yield

$$C_L \ell^{1/2-2\delta} \delta_q^{1/2} \lambda_q \ll \delta_{q+2}. \tag{11.29}$$

Moreover, the necessary relations for the small absolute parameter c_R are summarized by the condition

$$0 < c_R \leq \min\big(r_0^2, (4Dc_0)^{-4}\big), \tag{11.30}$$

where D is introduced in (11.48), and we denote by $c_0 > 0$ the maximum of all implicit constants appearing in Lemmas 11.3.1, 11.3.2 and 11.3.6. We remark that all these strictly positive implicit constants depend only on $N \in \mathbb{N}$. It is important to note that while the above mentioned lemmas are valid for arbitrary $N \in \mathbb{N}$, we shall employ them for $N \leq N_0 \leq 18$, with N_0 as in (11.25) only. Here, the absolute upper bound for N_0 stems from the relations $\varepsilon < 1$ and $\frac{1}{\beta} < 8$. Therefore, the set of c_R as in (11.30) is nonempty and c_R can even be chosen only in terms of the absolute geometric number r_0 and the implicit constants mentioned above for $N \leq 18$.

11.3.2 Mollification

In order to avoid a loss of derivative for v_q and to improve the regularity of $\mathring{R}_q$ and z, we mollify in space and time. The time mollification needs to be non-anticipating in order to preserve $(\mathcal{F}_t)_{t\geq 0}$-adaptedness, i.e. instead of symmetric time mollifiers on $\mathbb{R}$ centered at 0, we consider mollifiers with support on $\mathbb{R}_+$. Precisely, let $\{\phi_\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers on $\mathbb{R}^3$ and $\{\varphi_\varepsilon\}_{\varepsilon>0}$ a family of standard mollifiers with support on $\mathbb{R}_+$. For technical reasons we replace v_q and z by $v_q(\cdot \wedge T_L)$ and $z(\cdot \wedge T_L)$, i.e. we consider their constant extensions beyond T_L on $[0, L]$. In addition, for the mollification on $\mathbb{R}^3$, we consider the 2π-periodic extensions of v_q, $\mathring{R}_q$ and z to $\mathbb{R}^3$. We still denote these extended maps by v_q, $\mathring{R}_q$ and z and remark that everything stated above in this section remains true on $[0, T_L]$ for these extensions. For the mollification length scale $\ell = \ell_q$ defined in (11.20), set

$$v_\ell := (v_q *_x \phi_\ell) *_t \varphi_\ell, \quad \mathring{R}_\ell := (\mathring{R}_q *_x \phi_\ell) *_t \varphi_\ell, \quad z_\ell := (z *_x \phi_\ell) *_t \varphi_\ell.$$

Note that $v_\ell, \mathring{R}_\ell$ and z_ℓ are $(\mathcal{F}_t)_{t\geq 0}$-adapted, $z_\ell = 0$, $v_\ell(0)$, $\partial_t v_\ell(0)$, $\mathring{R}_\ell(0)$ and $\partial_t \mathring{R}_\ell(0)$ are deterministic, and v_ℓ, z_ℓ and $\mathring{R}_\ell$ are divergence-free. Moreover, it is clear that $v_\ell, \mathring{R}_\ell$ and z_ℓ are 2π-periodic in x, since so are $v_q, \mathring{R}_q$ and z. Therefore, we may consider v_ℓ and z_ℓ as elements in $C^\infty_{L,x}$ and $\mathring{R}_\ell$ in $C^\infty_{T_L,x}\mathbb{R}^{3\times 3}$. It is straightforward to check that on $[0, T_L]$ the pair $(v_\ell, \mathring{R}_\ell)$ solves

$$\begin{cases} \partial_t v_\ell + \operatorname{div}\big((v_\ell + z_\ell) \otimes (v_\ell + z_\ell)\big) + \nabla p_\ell + (-\Delta)^\alpha v_\ell & = \operatorname{div}(\mathring{R}_\ell + R_{\text{com}}) \\ \operatorname{div}(v_\ell) & = 0, \end{cases} \tag{11.31}$$

with the trace-free matrix-valued map

$$R_{\text{com}} := (v_\ell + z_\ell) \mathring{\otimes} (v_\ell + z_\ell) - \big((v_q + z_q) \mathring{\otimes} (v_q + z_q)\big) *_x \phi_\ell *_t \varphi_\ell$$

and

$$p_\ell := (p_q *_x \phi_\ell) *_t \varphi_\ell - \frac{1}{3}\big(|v_\ell + z_\ell|^2 - (|v_q + z_q|^2 *_x \phi_\ell) *_t \varphi_\ell\big).$$

From what we mentioned above, it follows that R_{com} is $(\mathcal{F}_t)_{t\geq 0}$-adapted and that $R_{\text{com}}(0)$ is deterministic.

In the following lemma we collect important estimates for the mollifications introduced above.

Lemma 11.3.1. *The following estimates hold for $N \in \mathbb{N}_0$ and $t \in [0, T_L]$. For v_ℓ, we have*

$$\begin{aligned} ||v_q - v_\ell||_{C^0_{t,x}} &\leq \ell ||v_q||_{C^1_{t,x}} \leq 2C_L M(t)^{1/2} \delta_q^{1/2} \lambda_q \ell, \\ ||v_\ell||_{C^{N+1}_{t,x}} &\lesssim \ell^{-N} ||v_q||_{C^1_{t,x}} \leq C_L M(t)^{1/2} \delta_q^{1/2} \lambda_q \ell^{-N}, \\ ||v_\ell||_{C^0_{t,x}} &\leq ||v_q||_{C^0_{t,x}} \leq 2M(t)^{1/2}. \end{aligned}$$

Similarly, for $\mathring{R}_\ell$ we have

$$||\mathring{R}_\ell||_{C^N_{t,x}} \lesssim \ell^{-N} ||\mathring{R}_q||_{C^0_{t,x}} \leq \ell^{-N} M(t) \delta_{q+1} c_R,$$

and, likewise, for z_ℓ

$$||z_\ell||_{C^0_{t,x}} \leq ||z||_{C^0_{t,x}} \leq L^{1/4}, \quad ||z_\ell||_{C^0_t C^{N+1}_x} \lesssim \ell^{-N} L^{1/4}, \quad ||z - z_\ell||_{C^0_{t,x}} \lesssim \ell^{1/2-2\delta} L^{1/2},$$

where in each line but the last all implicit constants stem from the mollifiers ϕ_ℓ and φ_ℓ and hence only depend on N. The implicit constant in the last estimate is due to Sobolev embeddings and is hence absolute.

Proof. The second inequality of each estimate is due to either (11.19) or (10.8). The first estimate of the third and fifth line follow immediately, since the standard mollifiers φ_ℓ and ϕ_ℓ have mass 1, which also implies the estimate for $||\mathring{R}_\ell||_{C^0_{t,x}}$. The estimate for $||\mathring{R}_q||_{C^N_{t,x}}$ for $N \geq 1$, as well as the first estimate of the second line, are obtained in the usual way: Putting all derivatives (all but one in the case of the second line) to the mollifiers ϕ_ℓ and φ_ℓ, the estimates follow from the fact that ϕ_ℓ and φ_ℓ can be chosen to fulfill

$$||\partial^\beta \phi_\ell||_{L^1(\mathbb{R}^3)} \lesssim \ell^{-|\beta|} \quad \text{and} \quad ||\partial_t^n \varphi_\ell||_{L^1(\mathbb{R})} \lesssim \ell^{-n},$$

for $n \in \mathbb{N}_0$ and any multiindex $\beta \in \mathbb{N}_0^3$, respectively. The sixth estimate follows in the same manner. Indeed, one derivative may be put on z, and the claim follows, since $||Dz||_{L^\infty_{T_L,x}} \leq L^{1/4}$ by (10.8). Concerning the first line, the mean value theorem yields

$$\begin{aligned} v_q(t,x) - v_\ell(t,x) &= \int_0^\ell \int_{B_\ell(0)} \big(v_q(t,x) - v_q(t-s, x-y)\big) \phi_\ell(y) \varphi_\ell(s) dy ds \\ &= \int_0^\ell \int_{B_\ell(0)} \big(v_q(t,x) - v_q(t-s, x)\big) \phi_\ell(y) \varphi_\ell(s) dy ds \\ &\quad + \int_0^\ell \int_{B_\ell(0)} \big(v_q(t-s,x) - v_q(t-s, x-y)\big) \phi_\ell(y) \varphi_\ell(s) dy ds \\ &= \int_0^\ell s \partial_t v_q(\zeta_s, x) \varphi_\ell(s) ds \int_{B_\ell(0)} \phi_\ell(y) dy \\ &\quad + \int_0^\ell \int_{B_\ell(0)} D v_q(t-s, \xi_y) \cdot y \, \phi_\ell(y) \varphi_\ell(s) dy ds, \end{aligned}$$

where $\zeta_s \in (t-s,t)$ and $\xi_y = x - uy$ for some $u = u_y \in (0,1)$ arise from the application of the mean value theorem. Consequently, we deduce

$$||v_q - v_\ell||_{C^0_{t,x}} \leq \ell\big(||\partial_t v_q||_{C^0_{t,x}} + ||Dv_q||_{C^0_{t,x}}\big) \leq \ell ||v_q||_{C^1_{t,x}},$$

where we used $\int_{\mathbb{R}^3} \phi_\ell \, dx = 1 = \int_{\mathbb{R}} \varphi_\ell \, dt$ again. Finally, similarly approach the last estimate:

$$\begin{aligned} |z_\ell(t,x) - z(t,x)| \leq & \int_{B_\ell(0)} \frac{|z(t,x-y) - z(t,x)|}{|y|^\beta} \phi_\ell(y)|y|^\beta dy + \\ & + \int_0^\ell \int_{B_\ell(0)} \frac{|z(t-s,x-y) - z(t,x-y)|}{|s|^\kappa} |s|^\kappa \phi_\ell(y)\varphi_\ell(s) dy ds \\ & \leq \ell^\beta ||z||_{C^0_t C^\beta_x} + \ell^\kappa ||z||_{C^\kappa_t L^\infty_x}. \end{aligned}$$

Choosing $\kappa = 1/2 - 2\delta$ with δ as in (10.5), it follows from (10.8) that

$$\ell^\kappa ||z||_{C^\kappa_t L^\infty_x} = \ell^{1/2-2\delta} ||z||_{C^{1/2-2\delta}_t L^\infty_x} \leq \ell^{1/2-2\delta} L^{1/2}$$

for $t \leq T_L$. Since by the general Sobolev embedding (9.3) we have the continuous embedding $H^{\frac{5+\sigma}{2}} \hookrightarrow C^{1/2}$, the definition of T_L yields for $\beta = 1/2$ the estimate

$$\ell^\beta ||z||_{C^0_t C^\beta_x} \lesssim \ell^{1/2} ||z||_{C^0_t H^{\frac{5+\sigma}{2}}} \lesssim \ell^{1/2} L^{1/4}.$$

Combining with the previous estimate, we conclude by obtaining

$$||z_\ell - z||_{C^0_{t,x}} \lesssim \ell^{1/2-2\delta} L^{1/2}, \quad 0 \leq t \leq T_L.$$

□

In order to estimate the solutions Φ_j to the transport equation (11.34), we also need estimates on v_ℓ and z_ℓ beyond T_L. More precisely, for $t \in [0,L]$ and $N \geq 0$, we need

$$\begin{aligned} &||v_\ell||_{C^0_{t,x}} \leq ||v_q||_{C^0_{T_L \wedge t,x}} \leq 2M(T_L \wedge t)^{1/2}, \qquad ||z_\ell||_{C^0_{t,x}} \leq ||z||_{C^0_{T_L \wedge t,x}} \leq L^{1/4}, \\ &||v_\ell||_{C^{N+1}_{t,x}} \lesssim \ell^{-N} ||v_q||_{C^1_{T_L \wedge t,x}} \leq C_L M (T_L \wedge t)^{1/2} \delta_q^{1/2} \lambda_q \ell^{-N}, \\ &||z_\ell||_{C^0_t C^{N+1}_x} \lesssim \ell^{-N} L^{1/4}. \end{aligned} \tag{11.32}$$

These estimates are obtained similarly as in the preceding lemma. Indeed, $v_q(\cdot \wedge T_L)$ is weakly differentiable on $[0,L] \times \mathbb{R}^3$ and smooth in (t,x) for $t \neq T_L$, and $z(\cdot \wedge T_L)$ has the same spatial regularity on $[0,L]$ as on $[0,T_L]$. Hence, one spatial (and temporal, in the case of v_q) derivative may still be put on $v_q(\cdot \wedge T_L)$, and the claimed estimates follow, since $v_q(t) = v_q(T_L)$ for $t \geq T_L$ and similarly for z.

11.3.3 Time localization and phase transport

Purpose. We have seen in the survey of this proof in Section 11.1 that the calculation of the new error $\mathring{R}_{q+1}$ naturally leads to the material derivative term

$$[\partial_t + (v_\ell + z_\ell) \cdot \nabla] w_{q+1}.$$

Since our primary intention is to define w_{q+1} as a sum of terms of type

$$a_{j,\xi}W_\xi = a_{j,\xi}B_\xi e^{i\lambda_{q+1}\xi\cdot x},$$

where $a_{j,\xi}$ are amplitudes of scale $M(t)^{1/2}\delta_{q+1}^{1/2}$, by the product rule the material derivative would particularly fall on the high frequency term $e^{i\lambda_{q+1}\xi\cdot x}$, resulting in a large factor $\lambda_{q+1} \gg \delta_{q+1}^{-1/2} \gg 1$, which hinders the verification of the small bound (A.3) at stage $q+1$.

In order to prevent this, the idea is to replace the linear phase $\xi \cdot x$ by a nonlinear phase $\xi\cdot\Phi$, where Φ is tailored in such a way that its material derivative vanishes. In other words, Φ should be transported pathwise along $v_\ell + z_\ell$ with initial condition being the identity. On the other hand, the *linear* phase in the Beltrami building blocks $B_\xi e^{i\lambda_{q+1}\xi\cdot x}$ is indispensable in order to retain its main geometric feature. That is, the approximate cancellation of Lemma 11.3.5, which stems from the geometric properties of Beltrami vector fields, cf. Proposition 11.3.3 and Lemma 11.3.4, does not prevail if we replace the linear phase function by a nonlinear one.

Hence, in order to retain a linear phase term for Lemma 11.3.5, we will write

$$e^{i\lambda_{q+1}\xi\cdot\Phi} = e^{i\lambda_{q+1}\xi\cdot x}e^{i\lambda_{q+1}(\Phi-x)},$$

which requires careful estimates on $\Phi - x$, i.e. on the deviation of the solution of a transport equation from its initial value. It is known that a suitable control of such a difference can be obtained on a suitable local time scale only. This leads us to introduce the time cutoffs χ_j as given below. We solve the transport equation (11.34) on suitably small time intervals of diameter approximately μ^{-1} before starting afresh with the original initial condition x. As mentioned before, this approach of transporting the phase function along the vector field appearing in the transport error is standard for deterministic convex integration techniques. In the context of stochastic equations, the respective preprint to this part of the thesis [189] seems to be the first one in which such a transport is considered. We point out again that this is the reason why we obtain the additional constant C_L in (A.2), cf. (11.38).

Cutoffs, transport and estimates. Let $\chi \in C_c^\infty\big((-\frac{3}{4}, \frac{3}{4})\big)$ be a cutoff function such that $0 \le \chi \le 1$ and

$$\sum_{l\in\mathbb{Z}} \chi^2(t-l) = 1, \quad t \in \mathbb{R}.$$

The following considerations are the reason we choose L and μ to be natural numbers. Let $L \in \mathbb{N}$ be as in Proposition 11.2.2, $\mu = \mu_q \in \mathbb{N}$ as in (11.21) and set, for $j \in \{0, \dots, L\mu\}$, $\chi_j(t) := \chi(\mu t - j)$, which yields

$$\sum_j \chi_j^2(t) = 1, \quad t \in [0, L]. \tag{11.33}$$

Here and throughout, the summation in j ranges over $\{0, \dots, L\mu\}$. Since $\operatorname{supp}\chi_j \subseteq B_{\frac{3}{4}\mu^{-1}}(j\mu^{-1})$, at each time t at most two cutoffs χ_j are nontrivial. We recall that by v_ℓ and z_ℓ we always mean the mollification of the extended vector fields $v_q(\cdot \wedge T_L)$ and $z(\cdot \wedge T_L)$, respectively.

Consider $v_\ell + z_\ell$ as a smooth vector field on $[0, L] \times \mathbb{R}^3$, which is 2π-periodic in its spatial argument. For $j \in \{0, \dots, L\mu\}$, let Φ_j: $[0, L] \times \mathbb{R}^3 \times \Omega \to \mathbb{R}^3$ be the pathwise unique solution to the transport equation

$$\begin{cases} [\partial_t + (v_\ell + z_\ell) \cdot \nabla]\Phi_j &= 0, \\ \Phi_j(j\mu^{-1}, x) &= x. \end{cases} \tag{11.34}$$

Note that Φ_j is the inverse flow of the ordinary differential equation with vector field $v_\ell + z_\ell$ with start at time $t = j\mu^{-1}$ as the identity. Thus, since $v_\ell + z_\ell$ is 2π-periodic in its spatial argument, for each $t \in [0, L]$ and $x \in \mathbb{R}^3$, we have $\Phi_j(t, x) - \Phi_j(t, x + y) \in (2\pi\mathbb{Z})^3$ for any $y \in (2\pi\mathbb{Z})^3$. Consequently, $x \mapsto e^{i\lambda_{q+1}\xi\cdot\Phi_j(t,x)}$ is $(2\pi\mathbb{Z})^3$-periodic for each t and may hence be considered an element in $C^\infty(\mathbb{T}^3, \mathbb{C})$. Clearly, $\Phi_0(0)$ and

$$\partial_t\Phi_0(0) = -[(v_\ell(0) + z_\ell(0)) \cdot \nabla]\Phi_0(0) = -[v_\ell(0) \cdot \nabla]\Phi_0(0)$$

are deterministic, and $(t, \omega) \mapsto \Phi_j(t, \omega) \in L^2$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted.

To verify the inductive estimates (A.1)-(A.3) later on, we need the estimates on Φ_j contained in the following lemma, for which we recall the constant C_L introduced in (11.5) and $C_{\mathbb{T}^3} = \sqrt{3}\pi$. We point out that we only need local in time estimates for Φ_j on the support of the respective cutoff χ_j, which is contained in $B_{\frac{3}{4}\mu^{-1}}(j\mu^{-1})$.

Lemma 11.3.2. *For $j \in \{0, \dots, L\mu\}$, the unique solution Φ_j to (11.34) satisfies the following estimates.*

$$\|D\Phi_j\|_{C^0_{\operatorname{supp}\chi_j,x}} \leq 1 + C_{\mathbb{T}^3}, \tag{11.35}$$

$$||D\Phi_j - \operatorname{Id}||_{C^0_{\operatorname{supp}\chi_j,x}} \leq \frac{C_L M(L)^{1/2}\delta_q^{1/2}\lambda_q + L^{1/4}}{\mu} \ll 1, \tag{11.36}$$

$$\|D\Phi_j\|_{C^0_{\operatorname{supp}\chi_j}C^N_x} \lesssim \frac{C_L M(L)^{1/2}\delta_q^{1/2}\lambda_q + L^{1/4}}{\mu}\ell^{-N} \ll \ell^{-N}, \quad N \geq 1, \tag{11.37}$$

$$[\Phi_j]_{C^1_{\operatorname{supp}\chi_j,x}} \leq C_L, \tag{11.38}$$

with the implicit constants only depending on N.

Prior to the proof, we mention that by methods very similar to [50, (135),(136) Prop.D.1], one obtains the following estimates for Φ_j on $\operatorname{supp}\chi_j \subseteq B_{\frac{3}{4}\mu^{-1}}(j\mu^{-1})$.

$$||D\Phi_j - \operatorname{Id}||_{C^0_{\operatorname{supp}\chi_j,x}} \leq \frac{||D(v_\ell + z_\ell)||_{C^0_{\operatorname{supp}\chi_j,x}}}{\mu} \exp\left(\frac{1}{\mu}\|D(v_\ell + z_\ell)\|_{C^0_{\operatorname{supp}\chi_j,x}}\right) \tag{11.39}$$

and, for each $N \geq 2$ and a constant $C_N > 0$ only depending on N, we have

$$[\Phi_j]_{C^0_{\operatorname{supp}\chi_j}C^N_x} \leq C_N \frac{[v_\ell + z_\ell]_{C^0_{\operatorname{supp}\chi_j}C^N_x}}{\mu} \exp\left(\frac{C_N}{\mu}\|D(v_\ell + z_\ell)\|_{C^0_{\operatorname{supp}\chi_j,x}}\right). \tag{11.40}$$

Proof. (11.35) follows immediately from (11.36). The estimate (11.36), in turn, is a simple consequence of (11.39):

$$\begin{aligned}
\|D\Phi_j - \mathrm{Id}\,\|_{C^0_{\operatorname{supp}\chi_j,x}} &\le \frac{1}{\mu}\|D(v_\ell + z_\ell)\|_{C^0_{\operatorname{supp}\chi_j,x}} \exp\left(\frac{1}{\mu}\|D(v_\ell + z_\ell)\|_{C^0_{\operatorname{supp}\chi_j,x}}\right) \\
&\le \frac{1}{\mu}\left(\|v_q\|_{C^1_{L,x}} + L^{1/4}\right)\exp\left(\frac{1}{\mu}\left(\|v_q\|_{C^1_{L,x}} + L^{1/4}\right)\right) \\
&\le \frac{1}{\mu}\left(2C_L M(L)^{1/2}\delta_q^{1/2}\lambda_q + L^{1/4}\right)\exp\left(\frac{1}{\mu}\left(2C_L M(L)^{1/2}\delta_q^{1/2}\lambda_q + L^{1/4}\right)\right) \\
&\le \frac{2C_L M(L)^{1/2}\delta_q^{1/2}\lambda_q + L^{1/4}}{\mu} \ll 1,
\end{aligned}$$

where we used the extended mollification estimates (11.32), and (11.28) twice. Likewise, for (11.37), we employ (11.40) and Lemma 11.3.1 to obtain

$$\begin{aligned}
\|D\Phi_j\|_{C^0_{\operatorname{supp}\chi_j}C^N_x} &\le C_N \ell^{-N}\frac{1}{\mu}(\|v_q\|_{C^1_{L,x}} + L^{1/4})\exp\left(\frac{C_N}{\mu}\left(2M(L)^{1/2}\delta_q^{1/2}\lambda_q + L^{1/4}\right)\right) \\
&\lesssim \frac{C_L M(L)^{1/2}\delta_q^{1/2}\lambda_q + L^{1/4}}{\mu}\ell^{-N} \ll \ell^{-N}.
\end{aligned}$$

We remark that the implicit constant only depends on C_N from (11.40), which in turn only depends on N. Finally, (11.38) follows via

$$\begin{aligned}
[\Phi_j]_{C^1_{\operatorname{supp}\chi_j,x}} &= ||D\Phi_j||_{C^0_{\operatorname{supp}\chi_j,x}} + ||\partial_t\Phi_j||_{C^0_{\operatorname{supp}\chi_j,x}} \le (1 + C_{\mathbb{T}^3})(1 + ||v_\ell + z_\ell||_{C^0_{L,x}}) \\
&\le (1 + C_{\mathbb{T}^3})\big(1 + 2M(L)^{1/2} + L^{1/4}\big) = C_L,
\end{aligned}$$

where we used (11.35), (11.32) and the definition of C_L. □

11.3.4 Beltrami waves

Here, we introduce the geometric backbone of our iteration scheme. As explained before, the central feature of the *Beltrami waves* introduced below is the geometric lemma 11.3.4. Let $\Lambda \subseteq \mathbb{S}^2 \cap \mathbb{Q}^3$ be finite such that $\Lambda = -\Lambda$. For each $\xi \in \Lambda$, choose $A_\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$ such that

$$A_\xi \cdot \xi = 0, \quad A_\xi = A_{-\xi},$$

and define the complex vector

$$B_\xi := \frac{1}{\sqrt{2}}\big(A_\xi + i\xi \times A_\xi\big).$$

By construction, $B_\xi \in \mathbb{C}^3$ has the properties

$$|B_\xi| = 1, \quad B_\xi \cdot \xi = 0, \quad i\xi \times B_\xi = B_\xi, \quad B_{-\xi} = \overline{B_\xi}.$$

Hence, if $\lambda \in \mathbb{Z}$ is such that $\lambda\xi \in \mathbb{Z}^3$, a direct calculation shows that for each $\xi \in \Lambda$ the vector field

$$W_\xi(x) := W_{\xi,\lambda}(x) := B_\xi e^{i\lambda\xi\cdot x}$$

is 2π-periodic, divergence-free and an eigenfunction of the curl-operator with eigenvalue λ. Such vector fields are called *complex Beltrami waves* and are particularly useful due to the following two results, cf. [53, Prop.5.5, Prop.5.6].

Proposition 11.3.3. *Let Λ and λ be as above and let $a_\xi \in \mathbb{C}$, $\xi \in \Lambda$, be a family of coefficients such that $a_{-\xi} = \overline{a_\xi}$. Then, the vector field*

$$W(x) := \sum_{\xi\in\Lambda} a_\xi B_\xi e^{i\lambda\xi\cdot x}$$

is $\mathbb{R}^3$-valued and divergence-free with $\operatorname{curl} W = \lambda W$. Hence, it is a stationary solution to the Euler equation

$$\operatorname{div}\big(W\otimes W\big) = \nabla\frac{|W|^2}{2}. \tag{11.41}$$

Furthermore, we have for all $\xi, \xi' \in \Lambda$

$$B_\xi \otimes B_{-\xi} + B_{-\xi}\otimes B_\xi = \operatorname{Id} - \xi\otimes\xi, \tag{11.42}$$

and

$$\operatorname{div}\big(W_\xi\otimes W_{\xi'} + W_{\xi'}\otimes W_\xi\big) = \nabla\big(W_\xi\cdot W_{\xi'}\big). \tag{11.43}$$

The following geometric lemma is the reason we can use Beltrami waves as the building blocks for the perturbation w_{q+1} in the forthcoming construction (see (11.47)) in order to obtain a cancellation for the oscillation error, see (11.59) and Lemma 11.3.5. Below, for a symmetric 3×3-matrix A, we denote the ball of radius $r>0$ centered at A in the space of symmetric real 3×3 matrices by $B_r(A)$ and its closure by $\overline{B_r(A)}$.

Lemma 11.3.4. *There is a small number $r_0 > 0$ such that there exist pairwise disjoint finite sets $\Lambda_\alpha \subseteq \mathbb{S}^2\cap\mathbb{Q}^3$, $\alpha\in\{0,1\}$, with the same cardinality and smooth positive functions $\gamma_\xi^{(\alpha)} \in C^\infty\big(\overline{B_{r_0}(\mathrm{Id})}\big)$ with the following properties. For $\alpha\in\{0,1\}$, it is $\Lambda_\alpha = -\Lambda_\alpha$ and $\gamma_\xi^{(\alpha)} = \gamma_{-\xi}^{(\alpha)}$ for each $\xi\in\Lambda_\alpha$. Moreover, for each $R\in\overline{B_{r_0}(\mathrm{Id})}$, we have the identity*

$$R = \frac{1}{2}\sum_{\xi\in\Lambda_\alpha}\Big(\gamma_\xi^{(\alpha)}(R)\Big)^2(\operatorname{Id}-\xi\otimes\xi). \tag{11.44}$$

It is useful to denote by n_0 the smallest natural number such that $n_0\Lambda_\alpha \subseteq \mathbb{Z}$ for $\alpha\in\{0,1\}$.

11.3.5 Construction of $\mathbf{w_{q+1}}$ and $\mathbf{v_{q+1}}$

After having provided the geometric structure of Beltrami waves as the main building blocks for the perturbation w_{q+1}, we now proceed with the construction of w_{q+1} and v_{q+1}. The velocity at stage $q+1$ is defined as

$$v_{q+1} := v_\ell + w_{q+1}^{(p)} + w_{q+1}^{(c)}, \tag{11.45}$$

i.e. the perturbation consists of a *principal term* $w_{q+1}^{(p)}$ and a *corrector term* $w_{q+1}^{(c)}$. The former is constructed as a sum of Beltrami waves with suitably low amplitude, while the latter is necessary in order to ensure $\operatorname{div} w_{q+1} = 0$.

Let Λ_0, Λ_1 and $\gamma_\xi^{(0)}, \gamma_\xi^{(1)}$ for $\xi \in \Lambda_0, \Lambda_1$ be as in the geometric lemma 11.3.4. We set

$$\Lambda_j := \Lambda_0 \text{ for } j \in 2\mathbb{N}_0 \quad \text{and} \quad \Lambda_j := \Lambda_1 \text{ for } j \in 2\mathbb{N}_0 + 1$$

and, likewise,

$$\gamma_\xi^{(j)} := \gamma_\xi^{(0)} \quad \text{and} \quad \gamma_\xi^{(j)} := \gamma_\xi^{(1)} \text{ for } j \in 2\mathbb{N}_0 \text{ and } j \in 2\mathbb{N}_0 + 1,$$

respectively. On $[0, T_L] \times \mathbb{T}^3$, for (j, ξ) with $\xi \in \Lambda_j$, we define the amplitude

$$a_{j,\xi}(t,x) := a_{q+1,j,\xi}(t,x) := \chi_j(t) M(t)^{1/2} \delta_{q+1}^{1/2} c_R^{1/4} \gamma_\xi^{(j)} \left(\mathrm{Id} - \frac{\mathring{R}_\ell(t,x)}{M(t)\delta_{q+1} c_R^{1/2}} \right) \tag{11.46}$$

and introduce the principal perturbation as

$$w_{q+1}^{(p)}(t,x) := \sum_j \sum_{\xi \in \Lambda_j} w_{q+1,j,\xi}^{(p)}(t,x) := \sum_j \sum_{\xi \in \Lambda_j} a_{j,\xi}(t,x) B_\xi e^{i\lambda_{q+1}\xi \cdot \Phi_j(t,x)}. \tag{11.47}$$

In order for $a_{j,\xi}$ to be well-defined, in view of Lemma 11.3.4, we need

$$\sup_{t \in [0,T_L]} \frac{||\mathring{R}_\ell(t)||_{C^0}}{M(t)\delta_{q+1} c_R^{1/2}} < r_0,$$

which, considering (A.3), holds due to (11.30). Since $\mathring{R}_\ell(0)$ and $\partial_t \mathring{R}_\ell(0)$ as well as $\Phi_0(0)$ and $\partial_t \Phi_0(0)$ are deterministic, and since $\chi_j(0, \cdot) = 0$ for $j \neq 0$, it follows that $w_{q+1}^{(p)}(0)$ and $\partial_t w_{q+1}^{(p)}(0)$ are deterministic as well. Moreover, the $(\mathcal{F}_t)_{t \geq 0}$-adaptedness of $\mathring{R}_\ell$ and each Φ_j yields $(\mathcal{F}_t)_{t \geq 0}$-adaptedness of $w_{q+1}^{(p)}$.

For future reference, it is useful to introduce the notation

$$\phi_{j,\xi}(t,x) := \phi_{q+1,j,\xi}(t,x) := e^{i\lambda_{q+1}\xi \cdot (\Phi_j(t,x) - x)}$$

and

$$W_\xi(x) := W_{q+1,\xi}(x) := B_\xi e^{i\lambda_{q+1}\xi \cdot x},$$

which we use to rewrite

$$w_{q+1}^{(p)}(t,x) = \sum_j \sum_{\xi \in \Lambda_j} a_{j,\xi}(t,x) \phi_{j,\xi}(t,x) W_\xi(x) = \sum_j \sum_{\xi \in \Lambda_j} a_{j,\xi}(t,x) W_\xi(\Phi_j(t,x)).$$

Moreover, we set $|\Lambda| := |\Lambda_j|$, which is independent of $j \in \mathbb{N}_0$ and, for $N_0 \in \mathbb{N}$ as in (11.25), introduce the absolute constant

$$D := 2|\Lambda| \sup_{j,\xi} ||\gamma_\xi^{(j)}||_{C^{N_0}(\overline{B_{r_0}(\mathrm{Id})})}. \tag{11.48}$$

As mentioned before, the definition of $w_{q+1}^{(p)}$ is tailored in order to obtain a cancellation of $w_{q+1}^{(p)} \otimes w_{q+1}^{(p)}$ with $\mathring{R}_\ell$. This cancellation turns out to be pivotal within the verification of (A.3) for the error term at stage $q+1$, cf. (11.65). The precise result is contained in the following statement.

Lemma 11.3.5. *On* $[0,T_L]\times\mathbb{T}^3$*, we have*

$$w_{q+1}^{(p)}\otimes w_{q+1}^{(p)}+\mathring{R}_\ell=M(t)\delta_{q+1}c_R^{1/2}\,\mathrm{Id}+\sum_{j,j',\xi+\xi'\neq 0}a_{j,\xi}a_{j',\xi'}\phi_{j,\xi}\phi_{j',\xi'}W_\xi\otimes W_{\xi'},$$

where the summation ranges over pairs (j,ξ) *and* (j',ξ') *with* $\xi\in\Lambda_j$ *and* $\xi'\in\Lambda_{j'}$ *such that* $\xi\neq-\xi'$*.*

Proof. For abbreviation, we denote the second summand of the right-hand side of the assertion by (II). By definition of $w_{q+1}^{(p)}$, since $\chi_j\chi_{j'}\equiv 0$ if $|j-j'|\geq 2$ and since $\xi+\xi'=0$ for $\xi\in\Lambda_j$ and $\xi'\in\Lambda_{j'}$ implies $|j-j'|\in 2\mathbb{N}_0$, we have

$$\begin{aligned}
w_{q+1}^{(p)}\otimes w_{q+1}^{(p)}(t,x)&=M(t)\delta_{q+1}c_R^{1/2}\sum_j\chi_j^2(t)\sum_{\xi\in\Lambda_j}\gamma_\xi^{(j)}\left(\mathrm{Id}-\frac{\mathring{R}_\ell(t,x)}{M(t)\delta_{q+1}c_R^{1/2}}\right)^2B_\xi\otimes B_{-\xi}+(\mathrm{II})\\
&=M(t)\delta_{q+1}c_R^{1/2}\sum_j\chi_j^2(t)\frac{1}{2}\sum_{\xi\in\Lambda_j}\gamma_\xi^{(j)}\left(\mathrm{Id}-\frac{\mathring{R}_\ell(t,x)}{M(t)\delta_{q+1}c_R^{1/2}}\right)^2(\mathrm{Id}-\xi\otimes\xi)+(\mathrm{II})\\
&=M(t)\delta_{q+1}c_R^{1/2}\sum_j\chi_j^2(t)\left(\mathrm{Id}-\frac{\mathring{R}_\ell(t,x)}{M(t)\delta_{q+1}c_R^{1/2}}\right)+(\mathrm{II})\\
&=M(t)\delta_{q+1}c_R^{1/2}\,\mathrm{Id}-\mathring{R}_\ell+(\mathrm{II}).
\end{aligned}$$

Here, we used (11.42), (11.44) and $\sum_j\chi_j^2=1$ for the second, third and final equation, respectively. ☐

Next, we introduce the corrector part $w_{q+1}^{(c)}$, which accounts for the fact that the principal part $w_{q+1}^{(p)}$ itself is not divergence-free. For $(t,x)\in[0,T_L]\times\mathbb{T}^3$, setting

$$w_{q+1}^{(c)}(t,x):=\sum_j\sum_{\xi\in\Lambda_j}\left[\frac{i}{\lambda_{q+1}}\nabla a_{j,\xi}(t,x)-a_{j,\xi}(t,x)\big(D\Phi_j(t,x)-\mathrm{Id}\big)\xi\right]\times W_\xi(\Phi_j(t,x)),$$

a direct calculation shows that

$$w_{q+1}^{(p)}+w_{q+1}^{(c)}=\frac{1}{\lambda_{q+1}}\sum_j\sum_{\xi\in\Lambda_j}\mathrm{curl}\,\big(a_{j,\xi}W_\xi(\Phi_j)\big)$$

is a perfect curl, and hence the total perturbation

$$w_{q+1}:=w_{q+1}^{(p)}+w_{q+1}^{(c)}$$

is divergence-free. Since $w_{q+1}^{(c)}(0)$ and $\partial_t w_{q+1}^{(c)}(0)$ are deterministic and that $w_{q+1}^{(c)}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted, together with the analogous observations for $w_{q+1}^{(p)}$ from above, it follows that $w_{q+1}(0)$ and $\partial_t w_{q+1}(0)$ are deterministic and w_{q+1} is $(\mathcal{F}_t)_{t\geq 0}$-adapted.

Finally, define v_{q+1} as in (11.45) and note that $v_{q+1}(0)$ and $\partial_t v_{q+1}(0)$ are deterministic and that v_{q+1} is $(\mathcal{F}_t)_{t\geq 0}$-adapted. Moreover, since by construction w_{q+1} is smooth in (t,x) and 2π-periodic in x, so is v_{q+1}.

11.3.6 Estimates for $\mathbf{v_{q+1}} - \mathbf{v_q}$

Having defined the perturbation w_{q+1} in the previous subsection, we now aim to verify the iterative estimates (A.1) and (A.2) at stage $q+1$. Before we do so, we collect several useful estimates in Lemma 11.3.6, which we will also employ for the iterative estimate (A.3) on the error term at stage $q+1$ later on. It is useful to introduce the notation for the coefficients of the full perturbation as

$$w_{q+1} = w_{q+1}^{(p)} + w_{q+1}^{(c)} =: \sum_{j,\xi} L_{j,\xi} e^{i\lambda_{q+1}\xi\cdot\Phi_j(t,x)} = \sum_{j,\xi} L_{j,\xi}\phi_{j,\xi} e^{i\lambda_{q+1}\xi\cdot x},$$

i.e. we set

$$L_{j,\xi} := a_{j,\xi}B_\xi + \left(\frac{i}{\lambda_{q+1}}\nabla a_{j,\xi} - a_{j,\xi}(D\Phi_j - \mathrm{Id})\xi\right) \times B_\xi.$$

Lemma 11.3.6. *For each $N \in \mathbb{N}_0$, (j,ξ) with $\xi \in \Lambda_j$, and $t \in [0, T_L]$, we have the following estimates for the coefficients of the perturbation w_{q+1}, where all implicit constants only depend on N and the fixed functions χ and $\gamma_\xi^{(j)}$.*

$$\|a_{j,\xi}\|_{C_t^0C_x^N} + \|L_{j,\xi}\|_{C_t^0C_x^N} \lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-N}, \tag{11.49}$$

$$\|\phi_{j,\xi}\|_{C^0_{\mathrm{supp}\,\chi_j}C_x^N} \lesssim \lambda_{q+1}^{(1-\beta)N}, \tag{11.50}$$

$$\|\partial_t a_{j,\xi}\|_{C_t^0C_x^N} + \|\partial_t L_{j,\xi}\|_{C_t^0C_x^N} \lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-(N+1)}, \tag{11.51}$$

$$\|(\partial_t + (v_\ell + z_\ell)\cdot\nabla)a_{j,\xi}\|_{C_t^0C_x^N} \lesssim M(t)\delta_{q+1}^{1/2}\ell^{-(N+1)}, \tag{11.52}$$

$$\|(\partial_t + (v_\ell + z_\ell)\cdot\nabla)L_{j,\xi}\|_{C_t^0C_x^N} \lesssim M(t)\delta_{q+1}^{1/2}\ell^{-(N+1)}. \tag{11.53}$$

Proof. For (11.49), by Lemma 11.3.1, the chain rule (D.1) and since $|\chi_j| \le 1$, we find

$$\begin{aligned}
\|a_{j,\xi}\|_{C_t^0C_x^N} &\le c_R^{1/4}M(t)^{1/2}\delta_{q+1}^{1/2}\left\|\gamma_\xi^{(j)}\left(\mathrm{Id} - \frac{\mathring{R}_\ell}{c_R^{1/2}M(t)\delta_{q+1}}\right)\right\|_{C_t^0C_x^N} \\
&\lesssim c_R^{1/4}M(t)^{1/2}\delta_{q+1}^{1/2}\left(\|\gamma_\xi^{(j)}\|_{C^1}\frac{\left\|\mathring{R}_\ell\right\|_{C_t^0C_x^N}}{c_R^{1/2}M(t)\delta_{q+1}} + \|\gamma_\xi^{(j)}\|_{C^N}\left(\frac{\left\|\mathring{R}_\ell\right\|_{C_t^0C_x^1}}{c_R^{1/2}M(t)\delta_{q+1}}\right)^N\right) \\
&\lesssim c_R^{1/4}M(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-N}.
\end{aligned} \tag{11.54}$$

Similarly, using the previous estimate together with the product rule (D.2), (11.27) and Lemma 11.3.2 gives

$$\begin{aligned}
\|L_{j,\xi}\|_{C_t^0C_x^N} &\lesssim \|a_{j,\xi}\|_{C_t^0C_x^N} + \frac{1}{\lambda_{q+1}}\|\nabla a_{j,\xi}\|_{C_t^0C_x^N} + \|a_{j,\xi}\|_{C_t^0C_x^N}\|D\Phi_j - \mathrm{Id}\,\|_{C^0_{\mathrm{supp}\,\chi_j,x}} \\
&\quad + \|a_{j,\xi}\|_{C^0_{t,x}}\|D\Phi_j - \mathrm{Id}\,\|_{C^0_{\mathrm{supp}\,\chi_j}C_x^N} \\
&\lesssim c_R^{1/4}M(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-N}\left(2 + \frac{1}{\lambda_{q+1}\ell} + \frac{M(L)^{1/2}C_L\delta_q^{1/2}\lambda_q + L^{1/4}}{\mu}\right) \\
&\lesssim c_R^{1/4}M(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-N},
\end{aligned} \tag{11.55}$$

which, combining with (11.54), gives (11.49). The aforementioned product and chain rule is used frequently for the forthcoming estimates, but we omit to point this out at every instance from now on.

The estimate for the phase functions $\phi_{j,\xi}$ is trivial for $N = 0$, since $|\phi_{j,\xi}| = 1$. For $N \geq 1$, it follows by Lemma 11.3.2, (11.28) and (11.27) as follows.

$$\begin{aligned}\|\phi_{j,\xi}\|_{C^0_{\operatorname{supp}\chi_j} C^N_x} &\lesssim \left(\lambda_{q+1}\|D\Phi_j - \operatorname{Id}\|_{C^0_{\operatorname{supp}\chi_j} C^{N-1}_x} + \lambda^N_{q+1}\|D\Phi_j - \operatorname{Id}\|^N_{C^0_{\operatorname{supp}\chi_j,x}}\right)\\ &\lesssim \ell^{1-N}\lambda^{1-\beta}_{q+1} + \lambda^{(1-\beta)N}_{q+1} \leq \lambda^{N(1-\beta)}_{q+1}.\end{aligned}$$

Now we turn to the estimates containing temporal derivatives. For brevity we suppress in our notation the argument of $\gamma^{(j)}_\xi$ and χ_j. First, applying (11.22), Lemma 11.3.1, choosing a sufficiently large to have $L \leq \ell^{-1}$, and using $\chi'_j(t) = \mu\chi'(\mu t - j)$, we have

$$\begin{aligned}\|\partial_t a_{j,\xi}\|_{C^0_{t,x}} &= c_R^{1/4} M(t)^{1/2}\delta^{1/2}_{q+1}\left\|2L\chi_j\gamma^{(j)}_\xi + \chi_j(D\gamma^{(j)}_\xi)\left(\frac{\partial_t\mathring{R}_\ell - 4L\mathring{R}_\ell}{c_R^{1/2}M(t)\delta_{q+1}}\right) + \chi'_j\gamma^{(j)}_\xi\right\|_{C^0_{t,x}}\\ &\lesssim M(t)^{1/2}\delta^{1/2}_{q+1}\left(\ell^{-1} + \mu\right) \lesssim M(t)^{1/2}\delta^{1/2}_{q+1}\ell^{-1}.\end{aligned}$$

In a similar way, we find for higher spatial derivatives

$$[\partial_t a_{j,\xi}]_{C^0_t C^N_x} = c_R^{1/4} M(t)^{1/2}\delta^{1/2}_{q+1}\left\|\chi_j D^N\left(2L\gamma^{(j)}_\xi + (D\gamma^{(j)}_\xi)\left(\frac{\partial_t\mathring{R}_\ell - 4L\mathring{R}_\ell}{c_R^{1/2}M(t)\delta_{q+1}}\right)\right) + \chi'_j D^N\gamma^{(j)}_\xi\right\|_{C^0_{t,x}}.$$

Estimating as for the case $N = 0$, the claimed inequality for $a_{j,\xi}$ follows. Concerning $L_{j,\xi}$, for $N \geq 0$, we obtain, using $\partial_t\Phi_j = -[(v_\ell + z_\ell)\cdot\nabla]\Phi_j$ (which holds since Φ_j solves (11.34)),

$$\begin{aligned}&\|\partial_t L_{j,\xi}\|_{C^0_t C^N_x}\\ &\quad\lesssim \|\partial_t a_{j,\xi}\|_{C^0_t C^N_x} + \frac{1}{\lambda_{q+1}}\|\nabla\partial_t a_{j,\xi}\|_{C^0_t C^N_x} + \|\partial_t a_{j,\xi}\|_{C^0_t C^N_x}\|D\Phi_j - \operatorname{Id}\|_{C^0_{\operatorname{supp}\chi_j,x}}\\ &\qquad + \|\partial_t a_{j,\xi}\|_{C^0_{t,x}}\|D\Phi_j\|_{C^0_{\operatorname{supp}\chi_j} C^N_x} + \|a_{j,\xi}\|_{C^0_t C^N_x}\|D\left[\left((v_\ell + z_\ell)\cdot\nabla\right)\Phi_j\right]\|_{C^0_{\operatorname{supp}\chi_j,x}}\\ &\qquad + \|a_{j,\xi}\|_{C^0_{t,x}}\|D\left[\left((v_\ell + z_\ell)\cdot\nabla\right)\Phi_j\right]\|_{C^0_{\operatorname{supp}\chi_j} C^N_x}\\ &\quad\lesssim \|\partial_t a_{j,\xi}\|_{C^0_t C^N_x} + \frac{1}{\lambda_{q+1}}\|\partial_t a_{j,\xi}\|_{C^0_t C^{N+1}_x} + \|\partial_t a_{j,\xi}\|_{C^0_t C^N_x}\|D\Phi_j - \operatorname{Id}\|_{C^0_{\operatorname{supp}\chi_j,x}}\\ &\qquad + \|\partial_t a_{j,\xi}\|_{C^0_{t,x}}\|D\Phi_j\|_{C^0_{\operatorname{supp}\chi_j} C^N_x}\\ &\qquad + \|a_{j,\xi}\|_{C^0_t C^N_x}\left(\|v_\ell + z_\ell\|_{C^0_t C^1_x}\|D\Phi_j\|_{C^0_{\operatorname{supp}\chi_j,x}} + \|v_\ell + z_\ell\|_{C^0_{t,x}}\|D\Phi_j\|_{C^0_{\operatorname{supp}\chi_j} C^1_x}\right)\\ &\qquad + \|a_{j,\xi}\|_{C^0_{t,x}}\left(\|v_\ell + z_\ell\|_{C^0_t C^{N+1}_x}\|D\Phi_j\|_{C^0_{\operatorname{supp}\chi_j,x}} + \|v_\ell + z_\ell\|_{C^0_{t,x}}\|D\Phi_j\|_{C^0_{\operatorname{supp}\chi_j} C^{N+1}_x}\right).\end{aligned}$$

We will show how to further estimate the terms in brackets of the penultimate line. The ones from the last line can be estimated in the same way with an additional factor ℓ^{-N}. An application of Lemma 11.3.2, Lemma 11.3.1, (11.28) and (11.22) yields

$$\|v_\ell + z_\ell\|_{C^0_t C^1_x}\|D\Phi_j\|_{C^0_{\operatorname{supp}\chi_j,x}} + \|v_\ell + z_\ell\|_{C^0_{t,x}}\|D\Phi_j\|_{C^0_{\operatorname{supp}\chi_j} C^1_x}$$

$$\lesssim \frac{C_L M(L)^{1/2}\delta_q^{1/2}\lambda_q + L^{1/4}}{\mu}\mu + (2M(L)^{1/2}+L^{1/4})\frac{C_L M(L)^{1/2}\delta_q^{1/2}\lambda_q + L^{1/4}}{\mu}\ell^{-1}$$
$$\leq \mu + (2M(L)^{1/2}+L^{1/4})\lambda_{q+1}^{-\beta}\ell^{-1} \leq \mu + \ell^{-1} \lesssim \ell^{-1},$$

for $a \geq a_0(L,\beta)$ sufficiently large to absorb the L-dependent constant in the penultimate estimate. Combining with (11.49), (11.51), (11.27) and Lemma 11.3.2, we find

$$\|\partial_t L_{j,\xi}\|_{C_t^0 C_x^N} \lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-(N+1)}\left(1+\frac{1}{\lambda_{q+1}\ell}\right) \lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-(N+1)}.$$

Finally, (11.52) can be obtained by combining the previous findings with the following estimate, which follows by Lemma 11.3.1 and (11.19):

$$\begin{aligned}
&\|(\partial_t + (v_\ell + z_\ell)\cdot\nabla)a_{j,\xi}\|_{C_t^0 C_x^N} \\
&\quad\lesssim \|\partial_t a_{j,\xi}\|_{C_t^0 C_x^N} + \|v_\ell + z_\ell\|_{C_t^0 C_x^N}\|\nabla a_{j,\xi}\|_{C^0_{t,x}} + \|v_\ell + z_\ell\|_{C^0_{t,x}}\|\nabla a_{j,\xi}\|_{C_t^0 C_x^N} \\
&\quad\lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-(N+1)}\left(1+\|v_q\|_{C^0_{t,x}} + \|z\|_{C^0_{t,x}}\right) \\
&\quad\leq M(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-(N+1)}\left(1+2M(t)^{1/2}+L^{1/4}\right) \lesssim M(t)\delta_{q+1}^{1/2}\ell^{-(N+1)},
\end{aligned}$$

since $L^{1/4} \leq M(t)^{1/2}$. Using the previous estimate together with (11.49) and Lemma 11.3.1, we conclude

$$\begin{aligned}
&\|(\partial_t + (v_\ell + z_\ell)\cdot\nabla)L_{j,\xi}\|_{C_t^0 C_x^N} \\
&\quad\lesssim \|\partial_t L_{j,\xi}\|_{C_t^0 C_x^N} + \|v_\ell + z_\ell\|_{C_t^0 C_x^N}\|\nabla L_{j,\xi}\|_{C^0_{t,x}} + \|v_\ell + z_\ell\|_{C^0_{t,x}}\|\nabla L_{j,\xi}\|_{C_t^0 C_x^N} \\
&\quad\lesssim M(t)\delta_{q+1}^{1/2}\ell^{-(N+1)},
\end{aligned}$$

where we once more used $L^{1/4} \leq M(t)^{1/2}$. The proof is complete. □

From here, we can proceed to the main objective of this subsection, namely to the verification of the first two main iterative estimates, (A.1) and (A.2), at stage $q+1$. Using

$$v_{q+1} - v_q = w_{q+1} - (v_q - v_\ell), \tag{11.56}$$

(A.1) follows by (11.55) via

$$\|w_{q+1}\|_{C^0_{t,x}} \leq D\|L_{j,\xi}\|_{C^0_{t,x}} \lesssim Dc_R^{1/4}M(t)^{1/2}\delta_{q+1}^{1/2} \leq \frac{1}{2}M(t)^{1/2}\delta_{q+1}^{1/2}, \tag{11.57}$$

(with D as in (11.48)) and, employing Lemma 11.3.1 and (11.28), via

$$\|v_q - v_\ell\|_{C^0_{t,x}} \lesssim C_L M(t)^{1/2}\delta_q^{1/2}\lambda_q\ell \ll M(t)^{1/2}\delta_{q+1}^{1/2}.$$

For (11.57), we used (11.30) to absorb the geometric absolute constant D introduced in (11.48) and the appearing implicit absolute constants. For future reference, we also state the additional estimate

$$\|w_{q+1}^{(c)}\|_{C^0_{t,x}} \lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\left(\frac{1}{\ell\lambda_{q+1}} + \|D\Phi_j - \mathrm{Id}\|_{C^0_{\operatorname{supp}\chi_j,x}}\right), \tag{11.58}$$

for which we employed (11.54). In a similar manner, (A.2) at stage $q+1$ follows from

$$\begin{aligned}||w_{q+1}||_{C^1_{t,x}} &\leq D\Big(||L_{j,\xi}||_{C^1_{t,x}} + \lambda_{q+1}[\Phi_j]_{C^1_{\operatorname{supp}\chi_j,x}}||L_{j,\xi}||_{C^0_t,x}\Big)\\ &\lesssim DM(t)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}\left(\frac{\ell^{-1}}{\lambda_{q+1}} + c_R^{1/4}C_L\right)\\ &\leq \frac{1}{2}C_L M(t)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}\end{aligned}$$

and from

$$||v_q - v_\ell||_{C^1_{t,x}} \lesssim ||v_q||_{C^1_{t,x}} \leq C_L M(t)^{1/2}\delta_q^{1/2}\lambda_q \ll M(t)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}.$$

For the former chain of estimates, we used (11.49), (11.38), (11.55) as well as (11.30) and (11.27), and we have chosen a sufficiently large in terms of D in order to absorb D and the appearing implicit constants into $(\lambda_{q+1}\ell)^{-1}$. For the latter inequalities, we employed Lemma 11.3.1, (11.19) and chose a sufficiently large in terms of L to absorb C_L and the appearing implicit constant into the inequality $\delta_q^{1/2}\lambda_q \ll \delta_{q+1}^{1/2}\lambda_{q+1}$, which obviously holds by definition of δ_q and λ_q. This concludes the verification of (A.1) and (A.2) at stage $q+1$.

11.3.7 Definition of $\mathring{\mathbf{R}}_{\mathbf{q+1}}$

We now turn our attention to the definition of the new error term $\mathring{R}_{q+1}$. As explained in Section 11.1, we can calculate $\mathring{R}_{q+1}$ based on the definition of v_{q+1} and by using that the triple $(v_q, p_q, \mathring{R}_q)$ at stage q solves (11.1). More precisely, subtracting (11.1) at stage q from (11.1) at stage $q+1$ and solving for the unknown terms $\operatorname{div}(\mathring{R}_{q+1})$ and ∇p_{q+1} yields

$$\begin{aligned}\operatorname{div}(\mathring{R}_{q+1}) - \nabla p_{q+1} = &[\partial_t + (v_\ell + z_\ell)\cdot\nabla]w_{q+1}\\ &+ \operatorname{div}\big(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)} + \mathring{R}_\ell\big)\\ &+ \operatorname{div}\big(w_{q+1}^{(p)}\otimes w_{q+1}^{(c)} + w_{q+1}^{(c)}\otimes w_{q+1}\big)\\ &+ [w_{q+1}\cdot\nabla](v_\ell + z_\ell)\\ &+ (-\Delta)^\alpha w_{q+1}\\ &+ \operatorname{div}\big(v_{q+1}\otimes(z - z_\ell) + (z - z_\ell)\otimes v_{q+1} + z\otimes z - z_\ell\otimes z_\ell\big)\\ &+ \operatorname{div}\big(R_{\text{com}}\big) - \nabla p_\ell.\end{aligned} \tag{11.59}$$

We call the error terms on the right-hand side *transport-, oscillation-, corrector-, Nash-, dissipation-, z-* and *commutator-error* in their order of appearance.

At this point, in order to define the new error $\mathring{R}_{q+1}$, we introduce the operator $\mathcal{R}$ as follows, cf. [50, Sect.1.2]. Recall the notation $\mathbb{P}$ for the orthogonal projection $\mathbb{P}: L^2 \to L^2_\sigma$. $\mathcal{R}$ maps any $v \in C^\infty$ to a 3×3-matrix-valued periodic function via

$$\mathcal{R}v = \frac{1}{4}\big(D\mathbb{P}u + (D\mathbb{P}u)^T\big) + \frac{3}{4}\big(Du + (Du)^T\big) - \frac{1}{2}(\operatorname{div} u)\,\mathrm{Id},$$

where u denotes the solution to

$$\Delta u = v - \fint_{\mathbb{T}^3} v\,dx, \quad \fint_{\mathbb{T}^3} u\,dx = 0 \quad \text{on } \mathbb{T}^3.$$

We observe the following properties of $\mathcal{R}$, which in particular imply that $\mathcal{R}$ is a right-inverse of the div-operator on smooth vector fields with zero average.

Lemma 11.3.7. *For any $v \in C^\infty$, we have*

(i) $\mathcal{R}v(x)$ *is a symmetric trace-free matrix for each* $x \in \mathbb{T}^3$*,*

(ii) $\operatorname{div} \mathcal{R}v = v - \fint_{\mathbb{T}^3} v \, dx$*.*

In particular, $\mathcal{R}$ *is a right-inverse to the* div*-operator on the set of all* $v \in C^\infty$ *with* $\fint_{\mathbb{T}^3} v \, dx = 0$*.*

Proof. Let $v \in C^\infty$. It is clear by definition that $\mathcal{R}v(x)$ is symmetric for each $x \in \mathbb{T}^3$. Since $\operatorname{Tr}(A) = \operatorname{Tr}(A^T)$ and $\operatorname{Tr} D\mathbb{P}u = \operatorname{div} \mathbb{P}u = 0$ by definition of $\mathbb{P}$, we have

$$\operatorname{Tr}(\mathcal{R}v) = \frac{3}{2} \operatorname{Tr}(Du) - \frac{3}{2} \operatorname{div} u = 0$$

pointwise in $x \in \mathbb{T}^3$. Moreover, since $\operatorname{div}(Dh)^T = \nabla \operatorname{div} h$ for any $h \in C^\infty$ and $\operatorname{div} \mathbb{P}u = 0$, we have

$$\begin{aligned} \operatorname{div} \mathcal{R}v &= \frac{1}{4}\big(\Delta \mathbb{P}u + \nabla \operatorname{div} \mathbb{P}u\big) + \frac{3}{4}\big(\Delta u + \nabla \operatorname{div} u\big) - \frac{1}{2} \nabla \operatorname{div} u \\ &= \frac{1}{4}\big(\Delta u - \Delta \nabla g + \nabla \operatorname{div} u\big) + \frac{3}{4} \Delta u \\ &= \Delta u = v - \fint_{\mathbb{T}^3} v \, dx, \end{aligned}$$

where we used that by definition $\mathbb{P}u = u - \nabla g$ for $g \in C^\infty(\mathbb{T}^3, \mathbb{R})$ with $\Delta g = \operatorname{div} u$. Consequently, we deduce

$$\operatorname{div} \mathcal{R}v = v$$

in case $\fint_{\mathbb{T}^3} v \, dx = 0$, which completes the proof. $\square$

With $\mathcal{R}$ at hand, we consider the oscillation-error first. By Lemmas 11.3.5 and (11.43), we have

$$\begin{aligned} \operatorname{div}\big(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell\big) &= \operatorname{div}\Big(\sum_{j,j',\xi+\xi' \neq 0} a_{j,\xi} a_{j',\xi'} \phi_{j,\xi} \phi_{j',\xi'} W_\xi \otimes W_{\xi'}\Big) \\ &= \frac{1}{2} \sum_{j,j',\xi+\xi' \neq 0} a_{j,\xi} a_{j',\xi'} \phi_{j,\xi} \phi_{j',\xi'} \operatorname{div}\big(W_\xi \otimes W_{\xi'} + W_{\xi'} \otimes W_\xi\big) \\ &\quad + \sum_{j,j',\xi+\xi' \neq 0} \big(W_\xi \otimes W_{\xi'}\big) \nabla \big(a_{j,\xi} a_{j',\xi'} \phi_{j,\xi} \phi_{j',\xi'}\big) \\ &= \frac{1}{2} \sum_{j,j',\xi+\xi' \neq 0} a_{j,\xi} a_{j',\xi'} \phi_{j,\xi} \phi_{j',\xi'} \nabla\big(W_\xi \cdot W_{\xi'}\big) \\ &\quad + \sum_{j,j',\xi+\xi' \neq 0} \big(W_\xi \otimes W_{\xi'}\big) \nabla \big(a_{j,\xi} a_{j',\xi'} \phi_{j,\xi} \phi_{j',\xi'}\big) \\ &= \operatorname{div}(R_{\mathrm{osc}}) + \nabla p_{\mathrm{osc}}, \end{aligned}$$

where we set

$$R_{\mathrm{osc}} := \mathcal{R}\Big(\Big(W_\xi \otimes W_{\xi'} - \frac{W_\xi \cdot W_{\xi'}}{2}\mathrm{Id}\Big)\nabla\big(a_{j,\xi}a_{j',\xi'}\phi_{j,\xi}\phi_{j',\xi'}\big)\Big)$$

and

$$p_{\mathrm{osc}} := \frac{1}{2}\sum_{j,j',\xi+\xi'\neq 0} a_{j,\xi}a_{j',\xi'}\phi_{j,\xi}\phi_{j',\xi'}\big(W_\xi \cdot W_{\xi'}\big).$$

The final equality in the above chain of equalities in particularly uses the observation that

$$\sum_{j,j',\xi+\xi'\neq 0}\Big(W_\xi\otimes W_{\xi'} - \frac{W_\xi \cdot W_{\xi'}}{2}\mathrm{Id}\Big)\nabla\big(a_{j,\xi}a_{j',\xi'}\phi_{j,\xi}\phi_{j',\xi'}\big)$$
$$= \operatorname{div}\Big(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell - \frac{1}{2}\sum_{j,j',\xi+\xi'\neq 0}\big(w_{q+1,j,\xi}^{(p)} \cdot w_{q+1,j',\xi'}^{(p)}\big)\,\mathrm{Id}\Big),$$

which yields $\operatorname{div}\mathcal{R}(R_{\mathrm{osc}}) = R_{\mathrm{osc}}$ by Lemma 11.3.7. Note that $R_{\mathrm{osc}}(0)$ is deterministic and R_{osc} is $(\mathcal{F}_t)_{t\geq 0}$-adapted and smooth.

Concerning the further terms of (11.59), we set

$$\begin{aligned}
R_{\mathrm{tra}} &:= \mathcal{R}\big([\partial_t + (v_\ell + z_\ell)\cdot\nabla]w_{q+1}\big),\\
R_{\mathrm{corr}} &:= w_{q+1}^{(p)}\mathring{\otimes}w_{q+1}^{(c)} + w_{q+1}^{(c)}\mathring{\otimes}w_{q+1},\\
R_{\mathrm{Nash}} &:= \mathcal{R}\big([w_{q+1}\cdot\nabla](v_\ell + z_\ell)\big),\\
R_{\mathrm{diss}} &:= \mathcal{R}\big((-\Delta)^\alpha w_{q+1}\big),\\
R_{\mathrm{z}} &:= v_{q+1}\mathring{\otimes}(z - z_\ell) + (z - z_\ell)\mathring{\otimes}v_{q+1} + z\mathring{\otimes}z - z_\ell\mathring{\otimes}z_\ell,
\end{aligned}$$

and $p_{\mathrm{corr}} := \frac{1}{3}\big(2w_{q+1}^{(c)}\cdot w_{q+1}^{(p)} + |w_{q+1}^{(c)}|^2\big)$ and $p_z := \frac{1}{3}\big(2v_{q+1}\cdot(z - z_\ell) + |z|^2 - |z_\ell|^2\big)$. In view of (11.59), now define

$$\mathring{R}_{q+1} := R_{\mathrm{tra}} + R_{\mathrm{osc}} + R_{\mathrm{corr}} + R_{\mathrm{Nash}} + R_{\mathrm{diss}} + R_{\mathrm{z}} + R_{\mathrm{com}} \tag{11.60}$$

and

$$p_{q+1} := p_\ell - p_{\mathrm{osc}} - p_{\mathrm{corr}} - p_z.$$

Clearly, $\mathring{R}_{q+1}$ is trace-free. Moreover, inspecting each stress term defined above, it follows that $\mathring{R}_{q+1}(0)$ is deterministic and that $\mathring{R}_{q+1}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted. Since all terms in the definition of $\mathring{R}_{q+1}$ but z are smooth and since z has a version in $C^0_{T_L,x}$, we conclude that $\mathring{R}_{q+1}$ maps to $C^0\big([0,T_L]\times\mathbb{T}^3,\mathbb{R}^{3\times 3}\big)$. Moreover, by definition of T_L, we note that $\mathring{R}_{q+1}$ has bounded weak first order spatial derivatives.

11.3.8 Estimates for $\mathring{\mathbf{R}}_{\mathbf{q+1}}$

We proceed to the proof of (A.3) for $\mathring{R}_{q+1}$ by considering the summands in the definition (11.60) of $\mathring{R}_{q+1}$ separately. We make repeated use of the following *stationary phase lemma*, which allows to estimate $\mathcal{R}(F)$ in C^0 (even in the Hölder spaces C^ε) for $F(x) = a(x)e^{i\lambda\xi\cdot x}$ in terms of the amplitude a and the high frequency term λ, yielding negative powers of λ on the right-hand side of the estimate, which are comparably smaller than the derivative terms of the amplitude, hence implying sufficiently precise estimates for the verification of (A.3). For the proof of this result, we refer to [50, Prop.G.1].

Lemma 11.3.8 (Stationary phase estimates). *Let $\xi \in \mathbb{S}^2$ and $\lambda \in \mathbb{N}$ be fixed. For a smooth vector field $a \in C^\infty$, let $F(x) := a(x)e^{i\lambda\xi\cdot x}$. Then, we have for any $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$*

$$\|\mathcal{R}(F)\|_{C^\varepsilon} \lesssim \frac{\|a\|_{C^0}}{\lambda^{1-\varepsilon}} + \frac{[a]_{C^N}}{\lambda^{N-\varepsilon}} + \frac{[a]_{C^{N+\varepsilon}}}{\lambda^N},$$

where the implicit constant depends only on ε and N.

However, the first estimates needed for (A.3) do not require an application of the preceding lemma, but follow by the previously obtained estimates on w_{q+1}, z, the length scale hierarchies (11.27) and Lemma 11.3.1 as follows. Let $t \in [0, T_L]$.

Estimate on R_z. By the respective mollification estimate of Lemma 11.3.1, the definition of T_L, (11.19) and (11.29), we obtain, choosing a sufficiently large in terms of c_R and L,

$$||R_z||_{C^0_{t,x}} \le \big(2||v_{q+1}||_{C^0_{t,x}} + ||z||_{C^0_{t,x}} + ||z_\ell||_{C^0_{t,x}}\big)||z - z_\ell||_{C^0_{t,x}} \lesssim M(t)\ell^{1/2-2\delta} \ll M(t)\delta_{q+2}c_R. \tag{11.61}$$

Estimate on R_{corr}. By (11.57)-(11.58) and (11.36), we obtain

$$\begin{aligned} ||R_{\text{corr}}||_{C^0_{t,x}} &\le ||w^{(p)}_{q+1}||_{C^0_{t,x}}||w^{(c)}_{q+1}||_{C^0_{t,x}} + ||w_{q+1}||_{C^0_{t,x}}||w^{(c)}_{q+1}||_{C^0_{t,x}} \\ &\lesssim M(t)\delta_{q+1}\cdot\left(\frac{1}{\ell\lambda_{q+1}} + ||D\Phi_j - \mathrm{Id}\,||_{C^0_{\operatorname{supp}\chi_j,x}}\right) \ll M(t)\delta_{q+2}c_R, \end{aligned} \tag{11.62}$$

where we have used (11.27) and (11.28), and possibly increased a in terms of c_R.

Estimate on R_{com}. By definition of T_L, the mollification lemma 11.3.1 and (11.19), we have

$$\begin{aligned} ||R_{\text{com}}||_{C^0_{t,x}} &\lesssim \ell||v_q + z||_{C^0_{t,x}}\big(||v_q||_{C^1_{t,x}} + ||z||_{L^\infty_t W^{1,\infty}_x}\big) \\ &\quad + \ell^{1/2-2\delta}||v_q + z||_{C^0_{t,x}}\big(||v_q||_{C^1_{t,x}} + ||z||_{C^{1/2-2\delta}_t L^\infty_x}\big) \\ &\lesssim \ell^{1/2-2\delta}\big(2M(t)^{1/2} + L^{1/4}\big)\big(2C_L M(t)^{1/2}\delta_q^{1/2}\lambda_q + L^{1/2}\big) \\ &\lesssim \ell^{1/2-2\delta}C_L M(t)\delta_q^{1/2}\lambda_q \ll M(t)\delta_{q+2}c_R, \end{aligned} \tag{11.63}$$

where we used (11.29) for the final inequality for a sufficiently large in terms of c_R.

For the remaining estimates, we use the stationary phase lemma 11.3.8.

Estimate on $R_{\text{tra}} + R_{\text{Nash}}$. Setting $D_t := \partial_t + (v_\ell + z_\ell)\cdot\nabla$, and using that the phase Φ_j is transported along $v_\ell + z_\ell$, we write

$$\begin{aligned} [\partial_t + (v_\ell + z_\ell)\cdot\nabla]w_{q+1} + [w_{q+1}\cdot\nabla](v_\ell + z_\ell) &= \sum_{j,\xi}\big(D_t L_{j,\xi} + [L_{j,\xi}\cdot\nabla](v_\ell + z_\ell)\big)\phi_{j,\xi}e^{i\lambda_{q+1}\xi\cdot x} \\ &=: \sum_{j,\xi}\Omega_{j,\xi}e^{i\lambda_{q+1}\xi\cdot x}. \end{aligned}$$

We then employ (11.49), (11.53), (11.22), (11.28) and Lemma 11.3.1 to estimate

$$\begin{aligned}\|\Omega_{j,\xi}\|_{C^0_{t,x}} &\le \|D_t L_{j,\xi}\|_{C^0_{t,x}} + \|L_{j,\xi}\|_{C^0_{t,x}}\|v_\ell + z_\ell\|_{C^0_t C^1_x}\\ &\lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\left(M(t)^{1/2}\ell^{-1} + \|v_\ell + z_\ell\|_{C^0_t C^1_x}\right) \lesssim M(t)\delta_{q+1}^{1/2}\ell^{-1} \le M(t)\delta_{q+1}^{1/2}\lambda_{q+1}^{1-\beta}.\end{aligned}$$

Similarly, we get, taking into account also derivatives of the phase function $\phi_{j,\xi}$, via the product rule (D.2) and (11.50)

$$\begin{aligned}\|\Omega_{j,\xi}\|_{C^0_t C^N_x} &\lesssim \|D_t L_{j,\xi}\|_{C^0_t C^N_x} + \|L_{j,\xi}\|_{C^0_t C^N_x}\|v_\ell + z_\ell\|_{C^0_t C^1_x} + \|L_{j,\xi}\|_{C^0_{t,x}}\|v_\ell + z_\ell\|_{C^0_t C^{N+1}_x}\\ &\quad + \left(\|D_t L_{j,\xi}\|_{C^0_{t,x}} + \|L_{j,\xi}\|_{C^0_{t,x}}\|v_\ell + z_\ell\|_{C^0_t C^1_x}\right)\|\phi_{j,\xi}\|_{C^0_{\operatorname{supp}\chi_j} C^N_x}\\ &\lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-N}\left(M(t)^{1/2}\ell^{-1} + \|v_\ell + z_\ell\|_{C^0_t C^1_x}\right) + M(t)\delta_{q+1}^{1/2}\ell^{-1}\lambda_{q+1}^{N(1-\beta)}\\ &\lesssim M(t)\delta_{q+1}^{1/2}\left(\ell^{-(N+1)} + \ell^{-1}\lambda_{q+1}^{N(1-\beta)}\right) \lesssim M(t)\delta_{q+1}^{1/2}\lambda_{q+1}^{(N+1)(1-\beta)}.\end{aligned}$$

It is readily seen by interpolation that the preceding estimate also holds when N is replaced by $N+\varepsilon$ for $\varepsilon \in (0,1)$ at all places. With these preparations, and recalling that for each $t \in [0, T_L]$ at most $2|\Lambda|$ many terms in the sum $\sum_{j,\xi}\Omega_{j,\xi}e^{i\lambda_{q+1}\xi\cdot x}$ are nontrivial, an application of the stationary phase lemma 11.3.8 yields, using the above estimates on $\Omega_{j,\xi}$,

$$\begin{aligned}\|R_{\text{tra}} + R_{\text{Nash}}\|_{C^0_t C^0_x} &= \Big\|\sum_{j,\xi}\mathcal{R}\left(\Omega_{j,\xi}e^{i\xi\cdot x}\right)\Big\|_{C^0_{t,x}} \le \sum_{j,\xi}\Big\|\mathcal{R}\left(\Omega_{j,\xi}e^{i\xi\cdot x}\right)\Big\|_{C^0_t C^\varepsilon_x}\\ &\lesssim \sum_{j,\xi}\left(\frac{\|\Omega_{j,\xi}\|_{C^0_{t,x}}}{\lambda_{q+1}^{1-\varepsilon}} + \frac{\|\Omega_{j,\xi}\|_{C^0_t C^{N_0}_x}}{\lambda_{q+1}^{N_0-\varepsilon}} + \frac{\|\Omega_{j,\xi}\|_{C^0_t C^{N_0+\varepsilon}_x}}{\lambda_{q+1}^{N_0}}\right)\\ &\lesssim M(t)\delta_{q+1}^{1/2}\left(\lambda_{q+1}^{\varepsilon-\beta} + \lambda_{q+1}^{(N_0+1)(1-\beta)-N_0+\varepsilon} + \lambda_{q+1}^{(N_0+1+\varepsilon)(1-\beta)-N_0}\right)\\ &\le M(t)\delta_{q+1}^{1/2}\left(\lambda_{q+1}^{\varepsilon-\beta} + 2\lambda_{q+1}^{1-\beta N_0-\beta+\varepsilon}\right) \ll M(t)\delta_{q+2}c_R, \end{aligned} \tag{11.64}$$

where the final inequality is equivalent to the following two conditions up to a sufficiently large choice of a.

$$(\varepsilon-\beta)b^2c - \frac{1}{2}b + b^2 = b\big[((\varepsilon-\beta)c+1)b - \frac{1}{2}\big] < 0,$$

$$(1-\beta N_0+\varepsilon)b^2c - \beta b^2 c - \frac{1}{2}b + b^2 = b\left(b\left[(1-\beta N_0+\varepsilon)c - \beta c + 1\right] - \frac{1}{2}\right) < 0.$$

These conditions are fulfilled, since by the choice of ε, c and N_0 in Subsection 11.3.1, it follows that $(\varepsilon-\beta)c < -1$ and $(1-\beta N_0+\varepsilon)c + 1 \le 1 - \beta c < -1$. Hence, (11.64) holds.

Estimate on R_{osc}. We set $f_{j,\xi,j',\xi'} := \nabla\big(a_{j,\xi}a_{j',\xi'}\phi_{j,\xi}\phi_{j',\xi'}\big)$. Then, we have by (11.49), (11.50) and (11.27) for $N \ge 0$

$$\|f_{j,\xi,j',\xi'}\|_{C^0_t C^N_x} \lesssim \|a_{j,\xi}\|_{C^0_t C^N_x}\|a_{j,\xi}\|_{C^0_{t,x}} + \|a_{j,\xi}\|^2_{C^0_{t,x}}\|\phi_{j,\xi}\|_{C^0_{\operatorname{supp}\chi_j} C^N_x} \lesssim M(t)\delta_{q+1}\lambda_{q+1}^{(1-\beta)N}.$$

The stationary phase lemma 11.3.8 then yields

$$\begin{aligned}\|R_{\text{osc}}\|_{C^0_{t,x}} &\lesssim \frac{\|f_{j,\xi,j',\xi'}\|_{C^0_{t,x}}}{\lambda_{q+1}^{1-\varepsilon}} + \frac{\|f_{j,\xi,j',\xi'}\|_{C^0_t C^{N_0}_x}}{\lambda_{q+1}^{N_0-\varepsilon}} + \frac{\|f_{j,\xi,j',\xi'}\|_{C^0_t C^{N_0+\varepsilon}_x}}{\lambda_{q+1}^{N_0}} \\ &\lesssim M(t)\delta_{q+1}\left(\lambda_{q+1}^{\varepsilon-\beta} + \lambda_{q+1}^{1-\beta N_0-\beta+\varepsilon}\right) \ll c_R M(t)\delta_{q+2},\end{aligned} \tag{11.65}$$

by the same argument and assumptions as for the previous stress terms.

Estimate on R_{diss}. Following the argument of [70], we use the commutativity of $(-\Delta)^\alpha$ and $\mathcal{R}$, apply Schauder estimates, cf. [70, Thm.B.1], and interpolation to estimate

$$\|R_{\text{diss}}\|_{C^0_{t,x}} = \|(-\Delta)^\alpha \mathcal{R} w_{q+1}\|_{C^0_{t,x}} \le C(\varepsilon)[\mathcal{R}w_{q+1}]_{C^0_t C^{2\alpha+\varepsilon}_x} \lesssim \|\mathcal{R}w_{q+1}\|^{1-2\alpha-\varepsilon}_{C^0_{t,x}} \|\mathcal{R}w_{q+1}\|^{2\alpha+\varepsilon}_{C^0_t C^1_x}.$$

By definition, we have $w_{q+1} = \sum_{j,\xi} L_{j,\xi}\phi_{j,\xi} e^{i\lambda_{q+1}\xi\cdot x} =: \sum_{j,\xi} O_{j,\xi} e^{i\lambda_{q+1}\xi\cdot x}$. By (D.2), (11.49), (11.50) and (11.27), we have for $N \in \mathbb{N}_0$

$$\|O_{j,\xi}\|_{C^0_t C^N_x} \lesssim \|L_{j,\xi}\|_{C^0_t C^N_x} + \|L_{j,\xi}\|_{C^0_{t,x}} \|\phi_{j,\xi}\|_{C^0_{\text{supp}\,\chi_j} C^N_x} \lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}^{(1-\beta)N}.$$

As already mentioned in the calculation for the term $R_{\text{tra}} + R_{\text{Nash}}$, this estimate generalizes to values $N+\varepsilon$, $\varepsilon \in (0,1)$, in place of N. By the stationary phase lemma, we find

$$\begin{aligned}\|\mathcal{R}w_{q+1}\|_{C^0_{t,x}} &\lesssim \sum_{j,\xi}\left(\frac{\|O_{j,\xi}\|_{C^0_{t,x}}}{\lambda_{q+1}^{1-\varepsilon}} + \frac{\|O_{j,\xi}\|_{C^0_t C^{N_0}_x}}{\lambda_{q+1}^{N_0-\varepsilon}} + \frac{\|O_{j,\xi}\|_{C^0_t C^{N_0+\varepsilon}_x}}{\lambda_{q+1}^{N_0}}\right) \\ &\lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\left(\lambda_{q+1}^{\varepsilon-1} + \lambda_{q+1}^{-\beta N_0+\varepsilon}\right) \lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}^{\varepsilon-1},\end{aligned}$$

since $\beta N_0 > 1$. Similarly, we can estimate

$$\begin{aligned}[\mathcal{R}w_{q+1}]_{C^0_t C^1_x} = \|\mathcal{R}Dw_{q+1}\|_{C^0_{t,x}} &\lesssim \sum_{j,\xi}\left(\frac{\|DO_{j,\xi}\|_{C^0_{t,x}}}{\lambda_{q+1}^{1-\varepsilon}} + \frac{\|DO_{j,\xi}\|_{C^0_t C^{N_0}_x}}{\lambda_{q+1}^{N_0-\varepsilon}} + \frac{\|DO_{j,\xi}\|_{C^0_t C^{N_0+\varepsilon}_x}}{\lambda_{q+1}^{N_0}}\right) \\ &\lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}^{\varepsilon}\left(\lambda_{q+1}^{-\beta} + \lambda_{q+1}^{1-\beta N_0-\beta}\right) \lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}^{\varepsilon}.\end{aligned}$$

Both estimates put together imply

$$\|R_{\text{diss}}\|_{C^0_{t,x}} \lesssim \|\mathcal{R}w_{q+1}\|^{1-2\alpha-\varepsilon}_{C^0_{t,x}} \|\mathcal{R}w_{q+1}\|^{2\alpha+\varepsilon}_{C^0_t C^1_x} \lesssim M(t)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}^{2\alpha+2\varepsilon-1} \ll M(t)\delta_{q+2}c_R, \tag{11.66}$$

if we choose a sufficiently large and if we have the relation

$$(2\alpha + 2\varepsilon - 1)c + 1 < 0,$$

which holds by our choice of $\varepsilon < \frac{1}{4} - \frac{\alpha}{2}$ and $c > \frac{1}{1/2-\alpha}$.

Finally, combining (11.61)-(11.66), we obtain (A.3) at stage $q+1$, which completes the verification of the inductive estimates (A.1)-(A.3) at stage $q+1$.

11.3.9 Conclusion of the proof

Summarizing, for fixed $L \in \mathbb{N}$ as in the assertion of Proposition 11.2.2, we have proven the following. If $q \in \mathbb{N}_0$ and for some number $a_2 \geq a_1(L) > 1$ as mentioned in the beginning of the present section, for arbitrary large $a \geq a_2$, and b and c as in (11.24) and (11.26), there exist triples $(v_0, p_0, \mathring{R}_0)$ to $(v_q, p_q, \mathring{R}_q)$ as in the assertion subject to this a, b and c, then for arbitrary large $a \geq a_* \geq a_2$ and the same b and c as before, there exists a further triple $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ as in the assertion such that (A.1) – (A.3) are fulfilled subject to this a, b and c at stage $q+1$. Here, a_* is the maximum of all lower bounds we imposed on the value of a along the above proof and a needs to be chosen as a multiple of the geometric number n_0 of Subsection 11.3.4. Since each of these lower bounds only depends on the fixed, q-independent parameters $b, c, \beta, \delta, \varepsilon, N_0, \alpha$ and on finitely many implicit absolute constants, a_* may be chosen independent of q. Therefore, starting the above iteration with $q = 0$, $a_* = a_2$ and b, c as in (11.24) and (11.26), the assertion of Proposition 11.2.2 follows with $a_0 = a_*$.

This concludes the proof of the main iteration proposition 11.2.2 and hence the entire objective of this section.

Appendix C

Regularity for the stochastic linear equation

Here, we provide the necessary a priori estimates for the solution to the linear part (SL_α) of $\mathrm{HNSE}_{\mathrm{sto}}$ on the fixed probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}, B)$ as in Chapter 10, i.e. our objective is to prove Proposition 10.2.4.

For the remainder of this appendix, for $0 < \alpha < 1/2$, we abbreviate the L^2_σ-based fractional Stokes-Laplacian $\big(\mathbb{P}(-\Delta)^\alpha, \mathcal{D}(\mathbb{P}(-\Delta)^\alpha)\big)$ by $A = A_\alpha$ with domain $\mathcal{D}(A^\alpha) = H^{2\alpha}$. The following lemma collects important properties for the semigroup $S_\alpha(t)$ generated by A_α. For the convenience of the reader, we include a simple proof.

Lemma C.0.1. *Let $(S_\alpha(t))_{t\geq 0}$ be the semigroup of linear operators in $L(L^2)$ generated by $A_\alpha = \mathbb{P}(-\Delta)^\alpha$. Then, $(S_\alpha(t))_{t\geq 0}$ is an analytic, strongly continuous contraction semigroup. In particular, we have the estimates*

$$\|S_\alpha(t)\|_{L(L^2_\sigma)} \leq 1, \quad t \geq 0,$$
$$\|A^\gamma_\alpha S_\alpha(t)\|_{L(L^2_\sigma)} = \|(-\Delta)^{\alpha\gamma} S_\alpha(t)\|_{L(L^2_\sigma)} \leq C_{T,\gamma}(t^{-\gamma} + 1), \quad \forall \gamma > 0, t \in (0, T], T > 0, \tag{C.1}$$

for $C_{T,\gamma} > 0$.

Proof. Since the operator A_α has the explicit Fourier series representation

$$A_\alpha u(x) = \sum_{k\in\mathbb{Z}^3} |k|^{2\alpha} \hat{u}_k e^{ik\cdot x},$$

we infer the corresponding Fourier series representation of the semigroup as

$$S_\alpha(t)u(x) := e^{tA}u(x) := \sum_{k\in\mathbb{Z}^3} e^{-|k|^{2\alpha}t} \hat{u}_k e^{ik\cdot x}, \quad u \in L^2_\sigma.$$

That this is a strongly continuous semigroup can easily be checked. Furthermore, we have the following simple contraction bound, using Parseval's identity (9.2) and estimating the exponential by 1

$$\|S_\alpha(t)u\|^2_{L^2} = \sum_{k\in\mathbb{Z}^3} e^{-2|k|^{2\alpha}t}|\hat{u}_k|^2 \le \sum_{k\in\mathbb{Z}^3} |\hat{u}_k|^2 = \|u\|^2_{L^2} \implies \|S_\alpha(t)\|_{L(L^2)} \le 1.$$

We are left to prove the analyticity. To this end, let $t > 0$ and consider the t-derivative of $t \mapsto S_\alpha(t)$. We find

$$S'_\alpha(t)u(x) = \sum_{k\in\mathbb{Z}^3} -|k|^{2\alpha} e^{-|k|^{2\alpha}t}\hat{u}_k e^{ik\cdot x} = -\frac{1}{t}\sum_{k\in\mathbb{Z}^3} t|k|^{2\alpha}e^{-|k|^{2\alpha}t}\hat{u}_k e^{ik\cdot x}.$$

This implies, using again (9.2), and since the function $z \mapsto z^2 e^{-2z}$ has global maximum e^{-2} on $\mathbb{R}_+$, that

$$\|S'_\alpha(t)u\|^2_{L^2} = \frac{1}{t^2}\sum_{k\in\mathbb{Z}^3} t^2|k|^{4\alpha}e^{-2|k|^{2\alpha}t}|\hat{u}_k|^2 \le \frac{1}{t^2}e^{-2}\|u\|^2_{L^2} \implies \|S'_\alpha(t)\|_{L(L^2)} \le \frac{1}{t}e^{-2}.$$

From here, the assertion follows from [167, Prop.2.1.9] with $M_0 = 1$, $M_1 = e^{-2}$ and $\omega = 0$. ☐

We can now turn to the main objective of this appendix, which is the following regularity result for the unique analytically weak solution z to the linear stochastic equation (SL_α).

Proposition C.0.2. *Assume that the regularity assumption* (8.15) *for* G *holds, i.e. for some* $\sigma > 0$*, we have*

$$\mathrm{Tr}\left[A_\alpha^{\rho_0}GG^*\right] = \mathrm{Tr}\left[(-\Delta)^{-\rho_0\alpha}GG^*\right] < \infty$$

for $\rho_0 = \frac{5+2\sigma-2\alpha}{2\alpha}$. *Then, for sufficiently small* $\delta > 0$*, we have for any* $T > 0$

$$\mathbb{E}\left[\|z\|_{C_T H^{\frac{5+\sigma}{2}}} + \|z\|_{C_T^{\frac{1}{2}-2\delta} H^{\frac{3+\sigma}{2}}}\right] < \infty. \tag{C.2}$$

Proof. The proof proceeds in a similar fashion to that of [90, Prop.34, p. 83]. We use the *factorization method* (cf. [83, Section 5.3.1]) to write, for suitable $\theta \in (0, 1)$,

$$z(t) = \int_0^t (t-s)^{\theta-1} S_\alpha(t-s) Y_\theta(s) ds, \quad \mathbf{P} - \text{a.s.}$$

for each $t \geq 0$, where $(s,\omega) \mapsto Y_\theta(s)$ is a measurable version of

$$(s,\omega) \mapsto \frac{\sin(\pi\theta)}{\pi}\Big(\int_0^s (s-r)^{-\theta} S_\alpha(s-r) G dW(r)\Big)(\omega).$$

Define

$$j(\rho) = \begin{cases} \frac{\rho_0}{2}, & \rho = \frac{5+\sigma}{4\alpha} \\ \rho, & \rho = \frac{3+\sigma}{4\alpha} \\ 0, & \rho = 0 \end{cases}$$

and fix $T > 0$. We first prove that for suitable θ and any $k \in \mathbb{N}$, $\psi = A_\alpha^\rho Y_\theta$ is in $L^{2k}\big(\Omega \times [0,T], L^2\big)$ for any of the three choices for ρ above. Since Y_θ is Gaussian, we can estimate its higher moments by the second moment. Combining this with Itô's isometry and the estimate (C.1), we find, denoting by $C = C(k,\gamma,T,\theta) > 0$ a constant possibly changing form line to line,

$$\begin{aligned} \mathbb{E}\Big[|A_\alpha^\rho Y_\theta(s)|_{L^2}^{2k}\Big] &\leq c_k \left(\mathbb{E}\left[|A_\alpha^\rho Y_\theta(s)|_{L^2}^2\right]\right)^k = c_k \left(\int_0^s (s-r)^{-2\theta} |A_\alpha^\rho S_\alpha(s-r) G|_{L_2}^2 dr\right)^k \\ &\leq c_k \|A_\alpha^{j(\rho)} G\|_{L_2}^{2k} \left(\int_0^s (s-r)^{-2\theta} \|A_\alpha^{\rho - j(\rho)} S_\alpha(s-r)\|_{L(L^2)}^2 dr\right)^k \\ &\leq c_{k,\gamma} \|A_\alpha^{j(\rho)} G\|_{L_2}^{2k} \left(\int_0^s (s-r)^{-2(\theta+\rho-j(\rho))} + (s-r)^{-2\theta} dr\right)^k. \end{aligned}$$

The first factor is finite in all three cases, since $\|A_\alpha^{\frac{\rho_0}{2}} G\|_{L_2}^2 = \mathrm{Tr}\,[A_\alpha^{\rho_0} G G^*] < \infty$ by the regularity assumption (8.15). For the integral terms to be finite, in any of the three cases it is necessary and sufficient to choose

$$\theta < \min\Big(\frac{1}{2}, \frac{1}{2} - \rho + j(\rho)\Big).$$

Note that the range of suitable $0 < \theta < \min\big(\frac{1}{2}, \frac{1}{2} - \rho + j(\rho)\big)$ is nonempty in each case, since by definition $\rho - j(\rho) < 1/2$. By Fubini's theorem, this readily implies $A_\alpha^\rho Y_\theta \in L^{2k}\big(\Omega \times [0,T], L^2\big)$ for any $k \in \mathbb{N}$, for θ independent of k, and thereby also $Y_\theta \in L^{2k}(\Omega \times [0,T], \mathcal{D}(A_\alpha^\rho))$. In particular, we infer $Y_\theta \in L^{2k}([0,T], \mathcal{D}(A_\alpha^\rho))$ $\mathbf{P}$-a.s. for each of the three cases for ρ mentioned above. Following [82], for $\theta > 0$ as above, we define the deterministic convolution operator

$$R_{\theta,0}(\psi) := \int_0^t (t-s)^{\theta-1} S_\alpha(t-s)\psi(s) ds, \quad \psi \in L^{2k}(0,T,L^2).$$

We note that $z(t) = R_{\theta,0}(Y_\theta)(t)$ and $A_\alpha^\rho z(t) = R_{\theta,0}(A_\alpha^\rho Y_\theta)(t)$ $\mathbf{P}$-a.s. for each $t \geq 0$. By [82, Prop.A.1.1], for any $\delta \in (0, \theta - \frac{1}{2k})$, $R_{\theta,0}$ is a bounded linear operator

$$R_{\theta,0} \colon L^{2k}([0,T], L^2) \to C^\delta([0,T], L^2).$$

From here, it already follows that z as well as $A_\alpha^\rho z$ have a Hölder-continuous version in L^2_σ for each choice for ρ as above. Moreover, since $\frac{5+\sigma}{4\alpha} - \frac{\rho_0}{2} = \frac{1}{2} - \frac{\sigma}{4\alpha}$, in view of the above restriction for δ, we can estimate for $0 < \delta < \min\{\frac{1}{2}, \frac{\sigma}{4\alpha}\} - \frac{1}{2k}$ and k sufficiently large:

$$\begin{aligned}\mathbb{E}\left[\|z\|_{C_T H^{\frac{5+\sigma}{2}}}\right] &\leq C_\sigma \mathbb{E}\left[\|z\|_{C_T L^2} + \|A_\alpha^{\frac{5+\sigma}{4\alpha}} z\|_{C_T L^2}\right] \leq C_\sigma \mathbb{E}\left[\|z\|_{C_T^\delta L^2} + \|A_\alpha^{\frac{5+\sigma}{4\alpha}} z\|_{C_T^\delta L^2}\right] \\ &\leq C_{\sigma,k} \mathbb{E}\left[\|Y_\theta\|_{L^{2k}(0,T,L^2)}\right] + C_{\sigma,k} \mathbb{E}\left[\|A_\alpha^{\frac{5+\sigma}{4\alpha}} Y_\theta\|_{L^{2k}(0,T,L^2)}\right] < \infty.\end{aligned}$$

In a similar way we find for any $\delta > 0$ with $0 < \frac{1}{2} - 2\delta < \frac{1}{2} - \frac{1}{k}$ for sufficiently large $k > 2$:

$$\begin{aligned}\mathbb{E}\left[\|z\|_{C_T^{\frac{1}{2}-2\delta} H^{\frac{3+\sigma}{2}}}\right] &\leq C_\sigma \mathbb{E}\left[\|z\|_{C_T^{\frac{1}{2}-2\delta} L^2} + \|A_\alpha^{\frac{3+\sigma}{4\alpha}} z\|_{C_T^{\frac{1}{2}-2\delta} L^2}\right] \\ &\leq C_{\sigma,k} \mathbb{E}\left[\|Y\|_{L^{2k}(0,T,L^2)}\right] + C_{\sigma,k} \mathbb{E}\left[\|A_\alpha^{\frac{3+\sigma}{4\alpha}} Y\|_{L^{2k}(0,T,L^2)}\right] < \infty.\end{aligned}$$

□

Part IV

General appendices

Appendix D

Collected results

We list several well-known results, which we use throughout the thesis.

Product and chain rule. Let $n, m \in \mathbb{N}$, $E \subseteq \mathbb{R}^m$, and $u : \mathbb{R}^n \to E$ and $g : E \to \mathbb{R}$ be smooth. Then, for every $N \in \mathbb{N}$, there is a constant C only depending on n, m, N such that

$$[g \circ u]_{C^N} \leq C\big([g]_{C^1}[u]_{C^m} + ||\nabla g||_{C^{N-1}}[u]_{C^1}^N\big). \tag{D.1}$$

Moreover, for smooth functions $f, g : \mathbb{R}^n \to \mathbb{R}$ and $N \in \mathbb{N}$, there is a constant C, only depending on n, N such that

$$[fg]_{C^N} \leq C\big([f]_{C^N}||g||_{C^0} + ||f||_{C^0}[g]_{C^N}\big). \tag{D.2}$$

Theorem D.0.1 (Skorohod representation, Thm.6.7 [28]). *Let P_n, $n \geq 1$, and P be probability measures on the Borel σ-algebra of a metric space S. If $P_n \underset{n\to\infty}{\longrightarrow} P$ weakly and P has separable support, then there exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and S-valued random variables X_n, $n \geq 1$, and X such that*

(i) $P_n = \mathbf{P} \circ X_n^{-1}$ *and* $P = \mathbf{P} \circ X^{-1}$.

(ii) $X_n(\omega) \underset{n\to\infty}{\longrightarrow} X(\omega)$ *for every* $\omega \in \Omega$.

Theorem D.0.2 (Kolmogorov continuity criterion, Thm.3.23 [127]). *Let $X : \mathbb{R}_+ \times \Omega \to S$ be a stochastic process with values in a complete metric space (S, d) and assume there are $a, b > 0$, and for any $T > 0$ some $C = C(T) > 0$ such that*

$$\mathbb{E}[d(X_s, X_t)^a] \leq C|t - s|^{1+b}, \quad 0 \leq s, t \leq T.$$

Then, X has a continuous version, which is a.s. locally Hölder continuous with any Hölder exponent $c \in (0, \frac{b}{a})$.

Theorem D.0.3 (Riesz-Markov-Kakutani representation theorem, Thm.2.14 [195]). *Let (X, τ) be a locally compact Hausdorff space. Then, for any positive, linear functional $I : C_c(X) \to \mathbb{R}$, there exists a unique Borel measure μ such that*

$$I(f) = \int_X f(x) d\mu(x), \quad f \in C_c(X),$$

and

$$||I||_{L(X,\mathbb{R})} = \mu(X).$$

Theorem D.0.4 (Stone-Weierstraß approximation, Thm.8.1 [75])**.** *Let X be a compact metric space and $\mathcal{A}$ a subalgebra of the space of continuous real functions $C(X,\mathbb{R})$. If $\mathcal{A}$ separates points in X (i.e. for each pair $(x,y) \in X \times X$ such that $x \neq y$, there is $f \in \mathcal{A}$ such that $f(x) \neq f(y)$) and contains the constant function 1, then $\mathcal{A}$ is dense in $C(X,\mathbb{R})$ with respect to uniform convergence on X.*

Theorem D.0.5 (Kolmogorov extension theorem, Thm.2.4.3 [219])**.** *Let A be some index set and suppose for each $a \in A$, X_a is a metric space with Borel sigma algebra $\mathcal{B}(X_a)$. Suppose μ_B is a Borel probability measure on the product $\Pi_{b\in B}X_b$ for each finite $B \subseteq A$, such that whenever $C \subseteq B \subseteq A$ for for finite subsets C, B, we have*

$$\mu_C = \mu_B \circ (P_C^B)^{-1},$$

where P_C^B denotes the canonical projection from $\Pi_{b\in B}X_b$ to $\Pi_{c\in C}X_c$. Then, there exists a unique Borel probability measure μ on $\Pi_{a\in A}$ such that $\mu \circ (P_B^A)^{-1} = \mu_B$ for all finite $B \subseteq A$.

Appendix E

Measurable selections

Here, we present basics on measurable selections in a concise manner. The contents of this appendix are taken from [215, Sect.12.1].

Let (X,d) be a separable metric space and denote by $\mathrm{comp}(X)$ the space of all nonempty compact subsets of X. For $K \in \mathrm{comp}(X)$ and $\varepsilon > 0$ let $K_\varepsilon := \{x \in X : \mathrm{dist}(K,x) < \varepsilon\}$. It is readily seen that the *Hausdorff distance* d_H,

$$d_H(K,J) := \inf\{\varepsilon > 0 : K \subseteq J_\varepsilon \text{ and } J \subseteq K_\varepsilon\}, \quad K,J \in \mathrm{comp}(X),$$

is a metric on $\mathrm{comp}(X)$. If $x, y \in X$, then

$$d_H(\{x\},\{y\}) = d(x,y),$$

i.e. X is isometrically embedded in $\mathrm{comp}(X)$.

Lemma E.0.1. *Let $f : X \to \mathbb{R}$ be upper semicontinuous, set $f_K := \sup_{x\in K} f(x)$ for $K \in \mathrm{comp}(X)$ and define $F : \mathrm{comp}(X) \to \mathrm{comp}(X)$ by*

$$F : K \mapsto \{x \in K : f(x) = f_K\}.$$

Then, the maps $K \mapsto f_K$ and $K \mapsto F(K)$ are Borel maps from $\mathrm{comp}(X)$ *to* $\mathbb{R}$ *and* $\mathrm{comp}(X)$*, respectively.*

Lemma E.0.2. *Let Y be a further metric space and $y \mapsto K_y$ a map from Y to* $\mathrm{comp}(X)$*. Suppose for any $(y_n)_{n\in\mathbb{N}}$, $y \in Y$ with $y_n \longrightarrow y$ as $n \to \infty$ and for any $x_n \in K_{y_n}$, there exists a limit point x of $(x_n)_{n\in\mathbb{N}}$ such that $x \in K_y$. Then, the map $y \mapsto K_y$ is Borel measurable with respect to the metric topologies on Y and* $\mathrm{comp}(X)$*.*

Lemma E.0.3. *Let $(E, \mathcal{F})$ be a measurable space and $q \mapsto K_q$ a measurable map from E to* $\mathrm{comp}(X)$*. Then, there is a $\mathcal{F}/\mathcal{B}(X)$-measurable map $h : E \to X$ such that $h(q) \in K_q$ for every $q \in E$.*

Appendix F

Basics of differential geometry

We review the basic concepts of differential geometry, which are used for the description of the formal manifold-approach to equations (NLFPK) and (SNLFPK) in Part II, as well as for Nash's C^1 isometric embedding theorem as the geometric origin of the convex integration methods used in Part III. The presented material is absolutely standard. For a thorough introduction to the field of differential geometry and topology, we refer for example to the classical texts [158, 157, 210, 226, 42, 125].

Manifolds and (co)tangent spaces. Let M be a smooth d-dimensional manifold, i.e. a second-countable topological space (M, τ) such that there exists a countable collection of *charts* (U_i, φ_i), $i \geq 1$, consisting of open sets $U_i \in \tau$ and homeomorphisms $\varphi_i : U_i \to \mathbb{R}^d$, such that $\{U_i, i \geq 1\}$ covers M and the *coordinate changes* $\varphi_i \circ \varphi_j^{-1}$ are smooth on $\varphi_j(U_i \cap U_j)$ whenever $U_i \cap U_j$ is nonempty. Such a collection is called a smooth *atlas*. For $k \in \mathbb{N} \cup \{\infty\}$, the space $C^k(M)$ consists of all function $f : M \to \mathbb{R}^\infty$ such that $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{R}$ is k times continuously differentiable.

At each point $x \in M$, the *tangent space* T_xM is the linear space of $\mathbb{R}$-differentiations at x, that is T_xM consists of linear maps $\xi : C^\infty(M) \to \mathbb{R}$ such that $\xi(fg) = \xi(f)g(x) + f(x)\xi(g)$ for each $f, g \in C^\infty(M)$. Examples are the *directional derivatives* at x, $\frac{\partial}{\partial x_i}_{|x}$, which in local coordinates (U_i, φ_i) with $x \in U_i$ take the form $\frac{\partial}{\partial x_i}_{|x}(f) = \partial_i(f \circ \varphi_i^{-1})(\varphi_i(x))$. In fact, $\{\frac{\partial}{\partial x_i}_{|x}, i \leq d\}$ is a basis of T_xM. Thus, T_xM is d-dimensional and for each $\xi \in T_xM$, in local coordinates, we have $\xi = \sum_{1\leq i\leq d} \frac{\partial}{\partial x_i}_{|x}\xi_i$ for unique $\xi_i \in \mathbb{R}$. Intuitively, one thinks of $\xi \in T_xM$ as the direction $(\xi_i)_{1\leq i\leq d} \in \mathbb{R}^d$ at the point x, and of $\xi(f) = \sum_{1\leq i\leq d} \xi_i \frac{\partial}{\partial x_i}_{|x} f$ as the derivative of f in direction ξ at x. Equivalently, T_xM can be defined as the space of tangential directions at x of curves passing through x. More precisely, for a chart (U_i, φ_i) with $x \in U_i$, let $\gamma : (-1,1) \to M$ be such that $\varphi_i \circ \gamma$ is smooth and $\gamma(0) = x$. Then, T_xM may be considered as the space of equivalence classes of such curves subject to the equivalence relation $\gamma_1 \sim \gamma_2 :\iff (\frac{d}{dt}[\varphi_i \circ \gamma_1])(0) = (\frac{d}{dt}[\varphi_i \circ \gamma_2])(0)$. The equivalence class T_xM of a curve γ is denoted by $\gamma'(0)$. This definition is independent of the choice of (U_i, φ_i). In the case $M = \mathbb{R}^d$, one has $T_xM = \mathbb{R}^d$ for each $x \in \mathbb{R}^d$.

As any linear space, T_xM possesses a dual space T_x^*M of same dimension d, i.e. the space of all linear, continuous real functions on T_xM. Elements in T_x^*M are called *cotangent vectors*. For $f \in C^\infty(M)$, the *differential of f (at x) $df_x : T_xM \to \mathbb{R}$*, $df_x(\xi) := \xi(f)$ is a cotangent vector. In local coordinates, the dual basis of T_x^*M with respect to $\{\frac{\partial}{\partial x_i}_{|x}, i \le d\}$ consists of the differentials dx_i, $i \le d$, of the coordinate maps $x_i \in C^\infty(M)$, $x_i(x) := \varphi(x)_i$.

Riemannian metric, curve lengths, gradient, connections and Hessian. If $g = \{g(x)\}_{x\in M}$ is a *metric tensor* on M, i.e. for each $x \in M$, $g(x)$ is a scalar product on T_xM such that $\big(T_xM, g(x)\big)$ is a Hilbert space, (M, g) is called *Riemannian manifold*. In local coordinates, g can be written as $g(x) = g_{ij}(x)dx_i \otimes dx_j$ with $g_{ij}(x) \in \mathbb{R}$, using Einstein summation convention for $i, j \le d$. An example is the usual Euclidean metric tensor e on $\mathbb{R}^d$, in local coordinates $e = \delta_{ij}dx_i \otimes dx_j$.

If $\gamma : [0,1] \to M$ is a smooth curve, the metric tensor g and the derivative $t \mapsto \gamma'(t) \in T_{\gamma(t)}M$ allow to measure the *length* of γ as $\ell_g(\gamma) := \int_0^1 |\gamma'(t)|_{g(\gamma(t))}dt$. It is clear how this notion generalizes to piecewise C^1 curves on any time interval I.

Let $f \in C^\infty(M)$. For df, as a section in the cotangent bundle $T^*M := \bigsqcup_{x\in M} T_x^*M := \bigcup_{x\in M}\{x\} \times T_x^*M$, by Riesz isomorphism, there exists a unique section ∇f in the tangent bundle $TM := \bigsqcup_{x\in M} T_xM := \bigcup_{x\in M}\{x\} \times T_xM$ such that $df_x(\xi) = g(x)\big(\nabla f(x), \xi\big)$ for each $x \in M$ and $\xi \in T_xM$. ∇f is called *gradient* of f. In the case $M = \mathbb{R}^d$, the gradient ∇ is the usual first-order differential operator $\nabla f := (\partial_1 f, \dots, \partial_d f)$.

For $M = \mathbb{R}^d$, given smooth vector fields $X, Y : \mathbb{R}^d \to \bigsqcup_{x\in\mathbb{R}^d} T_x\mathbb{R}^d \cong \mathbb{R}^d$ with $X = (X^1, \dots, X^d)$, one can calculate the infinitesimal change of X in direction Y at x as $(\nabla_Y X)(x) := (\partial_{Y(x)}X^1, \dots, \partial_{Y(x)}X^d)(x)$, because the directions $X(x)$, $x \in \mathbb{R}^d$, can be considered in the common linear space $\mathbb{R}^d$. Here, $\partial_\nu f$ denotes the classical directional derivative of a function $f : \mathbb{R}^d \to \mathbb{R}$ in direction $\nu \in \mathbb{R}^d$. In the general case of a Riemannian manifold (M, g), elements of different tangent spaces cannot be compared directly to each other, since $\bigcup_{x\in M} T_xM$ is generally not a linear space. In order to generalize the notion of the change of direction $\nabla_Y X$ of X in direction Y, one considers *connections* on M. An *affine connection* on M is a map $\nabla : \bigsqcup_{x\in M} T_xM \times \bigsqcup_{x\in M} T_xM \to \bigsqcup_{x\in M} T_xM$ such that ∇ is bilinear and for any two vector fields X, Y on M (i.e.: smooth sections in $\bigsqcup_{x\in M} T_xM$), and any $f \in C^\infty(M)$, the following identities hold pointwise on M.

(i) $f\nabla_Y X = \nabla_{fY} X$,

(ii) $\nabla_Y fX = f\nabla_Y X + df(Y)X$.

For a Riemannian manifold (M, g), it turns out that there exists an affine connection ∇^L, which is naturally related to the metric tensor g in the following sense. For vector fields X, Y, Z on M, considering $x \mapsto g_x(X(x), Y(x))$ as a smooth function on M, it holds

$$\partial_Z g(X, Y) = g(\nabla_Z X, Y) + g(\nabla_Z Y, X),$$

i.e. the geometry induced by g is compatible with the connection ∇^L. If one also demands such ∇^L to be torsion-free (see, for example, [42]), then such a connection is unique and called *Levi-Civita connection*. For the case $(M, g) = (\mathbb{R}^d, e)$, the Levi-Civita connection is given via the usual gradient, i.e. $\nabla^L_Y X = \nabla_Y X : x \mapsto (\nabla_{Y(x)} X)(x) \in \mathbb{R}^d$.

From the Levi-Civita connection on (M, g), for $f \in C^\infty(M)$, one defines the *Hessian* $0{-}2$ tensor $\operatorname{Hess} f$ for smooth vector fields X, Y as $\operatorname{Hess} f(X,Y)(x) := \langle (\nabla^L_X \nabla f)(x), Y(x)\rangle_{g(x)}$, where ∇ denotes the gradient on (M, g). Intuitively, $\operatorname{Hess} f(X, Y)(x)$ measures the change of the directional derivative of f along X in direction Y at x. In the case $(M, g) = (\mathbb{R}^d, e)$, we have

$$\operatorname{Hess} f(X,Y) = \langle \nabla_X \nabla f, Y\rangle_e = \sum_{1\le i,j\le d} X^i Y^j \partial_{ij} f = H(f)X \cdot Y,$$

where $H(f) = (\partial_{ij} f)_{1\le i,j\le d}$ denotes the usual Hessian matrix of $f : \mathbb{R}^d \to \mathbb{R}$.

Embeddings and pullback metric. Let (M, g) be a d-dimensional Riemannian manifold and $N \in \mathbb{N}$. For a map $F = (F^1, \dots, F^N)$ with $F^i \in C^\infty(M)$, the *differential* at $x \in M$ is the linear map $dF_x : T_x M \to T_{F(x)}\mathbb{R}^N = \mathbb{R}^N$, $\xi \mapsto dF_x(\xi)$, with $dF_x(\xi) : C^\infty(\mathbb{R}^N) \to \mathbb{R}$ defined as $dF_x(\xi)(\eta) := \xi(\eta \circ F)$. One says that dF_x pushes the tangent space $T_x M$ through to $T_{F(x)}\mathbb{R}^N$. If dF_x is one-to-one for each $x \in M$, F is an *immersion*. If, additionally, F is a topological embedding, F is called *embedding*. While embeddings are one-to-one, immersions are generally not. Clearly, the existence of an immersion $F : M \to \mathbb{R}^N$ implies $N \ge d$.

If M is a d-dimensional manifold (not necessarily Riemannian) and $F : M \to \mathbb{R}^N$ is an embedding, F and the Euclidean metric e on $\mathbb{R}^N$ induce a metric tensor on M, the *pullback metric* $F^\sharp e$. In local coordinates, the components of $F^\sharp e$ are given by $\partial_i F \cdot \partial_j F$, $i, j \le d$. It is straightforward to show that for a smooth curve $\gamma : [0,1] \to M$, the identity $\ell_{F^\sharp e}(\gamma) = \ell_e(F \circ \gamma)$ for the curve length with respect to $F^\sharp e$ holds. In the case that M carries a metric tensor g, one may compare curve lengths with respect to g and $F^\sharp e$ on M. An embedding $F : M \to \mathbb{R}^N$ is *(strictly) short*, if

$$\ell_{F^\sharp e}(\gamma) \underset{(<)}{\le} \ell_g(\gamma)$$

holds for all smooth curves γ on M. Similarly, F is an *isometric* embedding, provided $\ell_{F^\sharp e}(\gamma) = \ell_g(\gamma)$ for all γ. In local coordinates, these relations transform to the pointwise relations

$$\partial_i F \cdot \partial_j F \underset{\substack{(<)\\(=)}}{\le} g_{ij}, \quad i, j \le d.$$

In these cases, one says that F (strictly) shrinks or preserves the length of curves on M. Everything of the above passage remains valid when "immersion" is replaced by "embedding".

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