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Network Games with Heterogeneous Players

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Abstract

We consider network games in which players simultaneously form partnerships and choose actions. Players are heterogeneous with respect to their action preferences. We characterize pairwise Nash equilibria for a large class of games, including coordination and anti-coordination games, varying the strength of action preferences and the size of the linking cost. We find that, despite the symmetry and simplicity of the setting, quite irregular network structures can arise in equilibrium, implying that heterogeneity in players' action preferences may already explain a large part of observed irregularity in endogenously formed networks.

JEL codes: *C62, C72, D85.*

Keywords: *network games; strategic network formation; preference heterogeneity; efficiency.*

1 Introduction

In social contexts, an individual's choice is often strongly influenced by choices of other related to her individuals. This social influence is frequently modeled as a non-cooperative game played on a fixed network, where each individual plays a common bilateral game with each of her network partners and obtains the sum of these bilateral games' payoffs. Games on networks were first systematically introduced in Galeotti et al. (2010) and have been actively

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studied since then (see a recent overview of Bramoullé and Kranton (2016)). However, quite often individuals also have considerable control over whom they interact with. The first models of strategic network formation date back to Myerson (1977) and are more recently surveyed, for instance, in Goyal (2016) and Mauleon and Vannetelbosch (2016). These two strands of research – network formation and games on networks – have been subsequently combined in models that consider games on endogenous networks. A good overview of the literature that studies the interplay between individual behavior and the formation of an interactional structure is Vega-Redondo (2016). Our paper contributes to this literature.

We investigate one particular aspect – the impact of *ex ante* heterogeneity between players. In particular, we allow players differ in their action preferences. On a fixed network, this would often create a conflict between a player's idiosyncratic action preference and her interactional incentives dictated by action choices of her network partners. If the network is endogenous, however, a player might just choose not to interact with those whose actions do not correspond to her own preferred action.[1] Ellwardt et al. (2016) and Goyal et al. (2021) show experimentally that this is a typical outcome in a two-stage coordination game under complete information, when individuals form their partnerships prior to choosing their actions. Goyal et al. (2021) also check the robustness of these results to non-zero values of the linking cost. Our aim is to derive equilibrium characterizations analytically and for a considerably larger class of games, varying also the strength of individuals' action preferences and the size of the linking cost. For this purpose, we extend the theoretical framework of Orlova (2019) from games with heterogeneous players on a fixed network to games on an endogenous network.

The setting is the following. We consider network games in which players with heterogeneous preferences over actions simultaneously form a network and choose their actions. The action choice is binary and hence there are two types of players. If a player chooses her preferred action, she gets a higher payoff in every bilateral game she plays. Link formation is two-sided, that is, links are formed between those players who have made mutual link proposals. Both link proposals and link maintenance are costly. The same bilateral game is played between all pairs of players who decided to be linked; it can be either a coordination game, an anti-coordination game, or a dominant action game (if individuals' action preferences are very strong). We consider a complete information setting and use a static solution concept – pairwise Nash equilibrium. The implications of alternative equilibium concepts are also discussed.

[1]This concerns two-sided link formation models, in which every link requires an agreement of both involved partners but can be severed unilaterally.

We find that, despite relative simplicity and symmetry of the setting (ex ante there are only two types of players that differ in their action preferences), quite irregular network structures are possible in equilibrium (see Table 1). These are partially connected networks with heterogeneous action profiles such that only a part of players choose their preferred actions. Such irregular equilibrium structures might exist both for coordination and for anti-coordination games, and it can be shown that their existence is robust to some equilibrium refinements – for instance, a bilateral Nash equilibrium.

The rest of the paper is organized as follows. Section 2 describes the model and provides all necessary definitions. Section 3 presents the results, proposes a classification of equilibria with respect to the action profile and the network structure and illustrates them with examples. Section 4 highlights the impact of heterogeneity on equilibrium outcomes, discusses alternative equilibrium concepts and describes planned follow-up research on efficiency of the derived equilibria. Appendix contains the proofs of all the results.

2 The model

2.1 The game

Let $N = \{1, ..., n\}$ be the set of players and $\theta = (\theta_1, ..., \theta_n)$ be the *preference profile* of players, where $\theta_i \in \{0, 1\}$ $\forall i \in N$. For $\pi \in \{0, 1\}$ we call $N^\pi = \{i \in N \mid \theta_i = \pi\}$ a *preference group* with action preference π and denote its cardinality by n^π. We assume that $n^\pi \geq 2$ $\forall \pi \in \{0, 1\}$, that is, we consider games with heterogeneous preference profiles.

Each player simultaneously chooses a (pure) strategy $s_i = (x_i, p_i) \in S_i = \{0, 1\}^n$ consisting of an action $x_i \in \{0, 1\}$ and a vector of link proposals to other players $p_i = (p_{i1}, ..., p_{i\,i-1}, p_{i\,i+1}, ..., p_{in}) \in \{0, 1\}^{n-1}$. Any strategy profile $s \in S = S_1 \times ... \times S_n$ induces a directed graph of proposals P, which can be represented by an adjacency matrix: $P_{ij} = p_{ij}$ $\forall i \neq j$ and $P_{ii} = 0$ $\forall i \in N$.[2] The links are formed between those players who made mutual proposals, inducing an undirected graph (network) G with $G_{ij} = P_{ij} \cdot P_{ji}$ $\forall i, j \in N$.[3]

We denote by $\bar{S}$ the subset of strategy profiles that do not contain unreciprocated proposals: $\bar{S} = \{s \in S \mid p_{ij} = p_{ji}\ \forall i, j \in N\}$. In what follows s_{-i} designates the strategy vector of all players except for i and s_{-i-j} the strategy vector of all players except for i and j. For a given s_{-i}, we denote by $\bar{S}_i(s_{-i})$ all i's strategies that do not contain i's unreciprocated

[2]By convention, players do not make link proposals to themselves. Note that link proposals p_{ij} are defined only for such $i, j \in N$ that $i \neq j$.

[3]The terms *network* and (undirected) *graph* are used interchangeably in this paper.

proposals: $\bar{S}_i(s_{-i}) = \{s_i \in S_i \mid p_{ij} = 1 \Rightarrow p_{ji} = 1 \ \forall j \in N\}$. Obviously, for $s \in \bar{S}$ it holds that $s_i \in \bar{S}_i(s_{-i}) \ \forall i \in N$.

The payoff for a player i with action preference θ_i is

$$u_i(s) = \sum_{j \in N} p_{ij} p_{ji} \left(\delta \cdot \mathbb{1}_{\{x_i = x_j\}} + (1-\delta) \cdot \mathbb{1}_{\{x_i \neq x_j\}} + \lambda \cdot \mathbb{1}_{\{x_i = \theta_i\}} - (c - \varepsilon)\right) - \varepsilon \cdot \sum_{j \in N} p_{ij},$$

where $\delta \in [0;1]$, $\lambda \in [0;+\infty)$ and $c > \varepsilon > 0$.

Hence, a player enjoys network benefits from her connections in the induced network G, while she has to pay a positive cost ε for each link proposal and a positive link maintenance cost $c - \varepsilon$ for each link. Note that i's network benefits consist of two parts: interactional benefits, that depend on the actions chosen by i's network neighbors (parameter δ determines relative advantage of matching versus mismatching actions), and idiosyncratic benefits, that arise if i chooses her preferred action θ_i (parameter λ determines the strength of action preferences). We analyze a class of games $\Gamma = \{\Gamma_{\delta,\lambda} \mid 0 \leq \delta \leq 1, \lambda \geq 0\}$, where every specific game is determined by two parameters (with a slight abuse of terminology, we will sometimes refer to a pair (δ, λ) as "a game", implying the corresponding $\Gamma_{\delta,\lambda}$). Depending on the relative values of these parameters, any game $\Gamma_{\delta,\lambda}$ can be classified into one of the following subclasses: coordination games, anti-coordination games or dominant action games (see Figure 1).

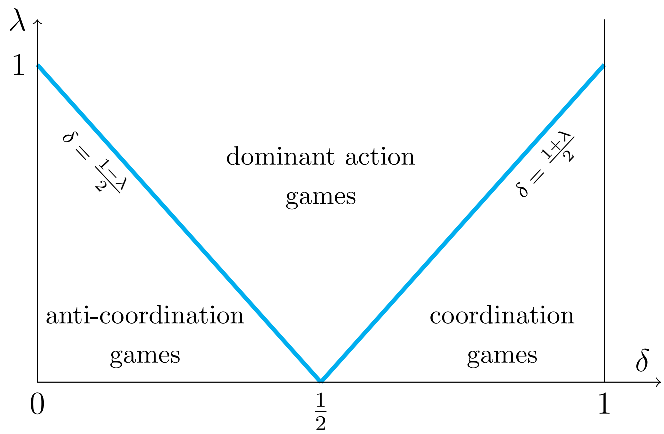


Figure 1: Parameter regions, representing three subclasses of games.

Note that if a player's strategy does not contain unreciprocated proposals, i.e. $s_i \in \bar{S}_i(s_{-i})$, then her payoff function can be simplified:

$$u_i(s) = \sum_{j\in N} p_{ij}p_{ji}\left(\delta\cdot\mathbb{1}_{\{x_i=x_j\}} + (1-\delta)\cdot\mathbb{1}_{\{x_i\neq x_j\}} + \lambda\cdot\mathbb{1}_{\{x_i=\theta_i\}} - c\right)$$
$$= \sum_{j\in N} p_{ij}p_{ji}u_{ij}(x_i,x_j), \tag{1}$$

where $u_{ij}(x_i, x_j)$ denotes i's payoff component due to her link with j.[4] The total linking cost c, that a player pays for each of her links, combines the cost of link proposal and the cost of link maintenance.

We consider a complete information setting, that is, the players' preference profile and their payoff functions are common knowledge prior to the game. Players choose their strategies simultaneously, aiming to maximize their respective payoffs.

2.2 Equilibrium concept and some graph theory notions

Consider a game $\Gamma_{\delta,\lambda}$ and fix some linking cost $c > 0$. A strategy profile s is a Nash equilibrium (NE) of the game if and only if $\forall i \in N\ \forall s'_i \in S_i\ u_i(s'_i, s_{-i}) \leq u_i(s)$.[5] In the spirit of the networks literature, we refine the set of Nash equilibria by introducing pairwise Nash equilibria. We allow pairs of unlinked players to deviate cooperatively by creating a mutual link with a possibility to simultaneously adjust their action choices. Formally, for a strategy profile $s \in S$ and a pair of players $i, j \in N$ s.t. $p_{ij}p_{ji} = 0$, a *pairwise deviation* $((x'_i, p'_i), (x'_j, p'_j))$ is such a deviation that $p'_{ij}p'_{ji} = 1$ and $p'_{kl} = p_{kl}\ \forall k \in \{i, j\}\ \forall l \notin \{i, j\}$. A pairwise Nash equilibrium is a Nash equilibrium proof against such pairwise deviations.[6]

Definition 1. A strategy profile $s = (x, p)$ is a *pairwise Nash equilibrium (PNE)* of the above game if it is a Nash equilibrium and for any pair $i, j \in N$ s.t. $p_{ij}p_{ji} = 0$ and any $x'_i, x'_j \in \{0, 1\}$,

$$u_i((x'_i, p'_i), (x'_j, p'_j), s_{-i-j}) > u_i(s) \Rightarrow u_j((x'_i, p'_i), (x'_j, p'_j), s_{-i-j}) < u_j(s),$$

where $p'_{ij}p'_{ji} = 1$ and $p'_{kl} = p_{kl}\ \forall k \in \{i, j\}\ \forall l \notin \{i, j\}$.

For a given game we denote by $S^{PNE} \subseteq S^{NE}$ the sets of pairwise Nash equilibria and Nash equilibria respectively. In the following section, we analyze pairwise Nash equilibria for different games $\Gamma_{\delta,\lambda} \in \Gamma$ and different sizes of the linking cost c.

[4]Note that for any pair of connected players $u_{ij}(x_i, x_j)$ has only four possible values: $\delta - c$, $1-\delta-c$, $\delta+\lambda-c$ or $1-\delta+\lambda-c$.

[5]Since only pure strategies are admissible, all equilibria in this paper are pure strategy equilibria.

[6]The same definition appears in Hiller (2017).

Before we move to equilibrium characterizations, let us remind several definitions from the graph theory that will appear useful in our analysis.[7]

A graph G in which each pair of distinct nodes is linked, $G_{ij} = 1\ \forall i \neq j$, is called a *complete* graph. An *empty* graph, on the other hand, is one with no links: $G_{ij} = 0\ \forall i, j \in N$. A *bipartite* graph is one that admits a partition of its set of nodes N into two subsets N' and N'' in such a way that every link of G connects a node of N' and a node of N'': $G_{ij} = 1 \Rightarrow (i \in N' \wedge j \in N'') \vee (i \in N'' \wedge j \in N')$. In a *complete bipartite* graph every node of N' is linked to every node of N'': $G_{ij} = 1 \Leftrightarrow (i \in N' \wedge j \in N'') \vee (i \in N'' \wedge j \in N')$.

A graph G' is called a subgraph of a graph G if every node and link of G' is a node and link, respectively, of G. A graph G is a subgraph of itself; all other subgraphs are proper subgraphs of G. If a proper subgraph $G' \subset G$ is complete, it is called a *clique*.[8] Let G' and G'' be proper subgraphs of G with corresponding sets of nodes N' and N''. We say that G' and G'' are *disjoint* if they have no nodes in common: $N' \cap N'' = \emptyset$. We say that disjoint $G', G'' \subset G$ are *connected*, if $\exists i \in N'\ \exists j \in N''$ s.t. $G_{ij} = 1$, otherwise they are *disconnected*.

Finally, let G' and G'' be two graphs. A *union* of G' and G'' is a graph with the set of nodes $N = N' \cup N''$ and links such that $G_{ij} = 1 \Leftrightarrow G'_{ij} = 1 \vee G''_{ij} = 1\ \forall i, j \in N$.

3 Equilibrium analysis

3.1 Preliminaries

This subsection establishes important relations between certain sets of strategy profiles and then formulates necessary and sufficient conditions for a strategy profile to be a pairwise Nash equilibrium.

First, fix any game $\Gamma_{\delta,\lambda} \in \Gamma$. Without loss of generality, let us make a technical assumption about admissible values of the linking cost.

Assumption 1. Given a game $\Gamma_{\delta,\lambda}$, a linking cost c can take any values in $C_{\delta,\lambda} := \mathbb{R}_{++} \setminus \{\delta,\ 1-\delta,\ \delta+\lambda,\ 1-\delta+\lambda\}$.

That is, a linking cost c can take any positive real values, except for four specific ones.[9]

[7] The following definitions are based on Bondy and Murty (1977), Diestel (2017) and Benjamin et al. (2015). The terms *vertex* and *edge* are substituted by more common in the networks literature terms *node* and *link* respectively.

[8] Note that this definition is different from another common one that appears, for instance, in Jackson (2008) and defines a clique as a *maximal* completely connected subgraph of G.

[9] This assumption guarantees that players are never indifferent to any of their links, that is, $u_{ij}(x_i, x_j) \neq 0\ \forall i, j \in N\ \forall x_i, x_j \in \{0, 1\}$ (see footnote 4). If this assumption does not hold, more equi-

Taking into account this assumption, fix any linking cost c. The first lemma relates Nash equilibria of a game to the set $\bar{S}$ of strategy profiles without unreciprocated proposals.

Lemma 1. $S^{NE} \subseteq \bar{S}$.

In other words, a Nash equilibrium cannot contain unreciprocated proposals. This directly follows from the fact that every link proposal carries a strictly positive cost. Formal proofs of this and the following lemmas are moved to the appendix.

Next, we notice that not only Nash equilibria do not contain unreciprocated proposals, but also profitable unilateral strategy deviations cannot contain unreciprocated proposals of the deviating player. This leads to an alternative characterization of the Nash equilibrium set.

Lemma 2. $s \in S^{NE}$ *if and only if* $\forall i \in N\ \forall s'_i \in \bar{S}_i(s_{-i})\ u_i(s'_i, s_{-i}) \leq u_i(s)$.

Compared to the original definition, this one narrows down the set of relevant deviations and thus simplifies the search of equilibria.

Consider now the following set: $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow (u_{ij}(x_i, x_j) > 0 \wedge u_{ji}(x_j, x_i) > 0)\}$. It is the subset of strategy profiles without unreciprocated proposals in which two players are linked if and only if they both benefit from the link. Lemma 3 states that all pairwise Nash equilibria must be in this set.

Lemma 3. $S^{PNE} \subseteq \bar{S}^+$.

This allows us to consider $\bar{S}^+$ as a pool of candidate equilibium profiles. Note, however, that $S^{NE} \subseteq \bar{S}^+$ does not have to hold. The next lemma establishes necessary and sufficient conditions for $s \in \bar{S}^+$ to be a Nash equilibrium. In what follows we denote by $\tilde{s}_i = (\tilde{x}_i, \tilde{p}_i)$ a particular unilateral deviation of player i from her strategy in the strategy profile $s = (x, p)$: $\tilde{x}_i \neq x_i$ and $\tilde{p}_{ij} = \begin{cases} 0 & \text{if } u_{ij}(\tilde{x}_i, x_j) < 0 \\ p_{ij} & \text{otherwise} \end{cases}$. In this deviation, a player i changes her action and withdraws all her proposals for those links that are no longer profitable for i.

Lemma 4. *Let* $s \in \bar{S}^+$. *Then* $s \in S^{NE}$ *if and only if* $\forall i \in N\ u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$.

Hence, for every player i, $\tilde{s}_i$ is the most successful of all possible unilateral deviations. If $s \in \bar{S}^+$ is proof against such deviations, it is also proof against all other unilateral deviations.

Building upon this result, the next lemma provides necessary and sufficient conditions for $s \in \bar{S}^+$ to be a pairwise Nash equilibrium. These conditions include proofness against three additional, pairwise deviations.

libria are possible, but none of them is robust to small changes in parameter values.

Lemma 5. *$s \in S^{PNE}$ if and only if $s \in \bar{S}^+$ and the following conditions hold for all $i \in N$ and all $j \in N$ s.t. $p_{ij}p_{ji} = 0$:*

(1) $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$,

(2) $u_i((x'_i, p'_i), (x'_j, p'_j), s_{-i-j}) > u_i(s) \Rightarrow u_j((x'_i, p'_i), (x'_j, p'_j), s_{-i-j}) < u_j(s)$,

where either $x'_i \neq x_i$ or $x'_j \neq x_j$, $p'_{ij}p'_{ji} = 1$ and $p'_{kl} = p_{kl}$ $\forall k \in \{i, j\}$ $\forall l \notin \{i, j\}$.

Hence, we derived necessary and sufficient conditions for a strategy profile to be a PNE: a strategy profile must contain only reciprocated proposals, must induce a link if and only if both linked players benefit from it, and must be proof against four specific (one unilateral and three pairwise) strategy deviations. Note that since not only original strategies s_i but also all relevant strategy deviations do not contain i's unreciprocated proposals, we can use the utility function (1).

Finally, let us further simplify necessary and sufficient conditions for PNE for a specific (actually, very broad) range of parameter values. Denote by $C^h_{\delta,\lambda}$ the subset of $C_{\delta,\lambda}$ that corresponds to high values of the linking cost (see Figure 2 in the following subsection): $C^h_{\delta,\lambda} = \{c \in C_{\delta,\lambda} \mid \max\{\delta, 1-\delta+\lambda\} < c < \delta+\lambda \ \vee \ \max\{1-\delta, \delta+\lambda\} < c < 1-\delta+\lambda\}$. The final lemma characterizes pairwise Nash equilibria when the linking cost is not high.

Lemma 6. *Let $c \notin C^h_{\delta,\lambda}$. Then $s \in S^{PNE}$ if and only if $s \in \bar{S}^+$ and the following conditions hold for all $i \in N$ and for all $j \in N$ s.t. $p_{ij}p_{ji} = 0$:*

(1) $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$,

(2) $u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) < u_i(s)$,

where $\hat{x}_i \neq x_i$, $\hat{p}_{ij}\hat{p}_{ji} = 1$ and $\hat{p}_{kl} = p_{kl}$ $\forall k \in \{i, j\}$ $\forall l \notin \{i, j\}$.[10]

This characterization differs from the one in Lemma 5 by requiring to consider yet fewer pairwise deviations (two for each unlinked pair of players). In these deviations only one of the players changes her action, and this same player must bear utility loss from such a deviation. Lemmas 5 and 6 will be used extensively to prove the results of the next subsection.

3.2 Classes of equilibria

The following figure depicts ten regions of parameter values – a game (δ, λ) and a linking cost c – that correspond to qualitatively different equilibrium sets. In each region only specific classes of equilibria are possible.

[10] It can be shown that if $c \in C^h_{\delta,\lambda}$ then these conditions are necessary but not sufficient for $s \in S^{PNE}$.

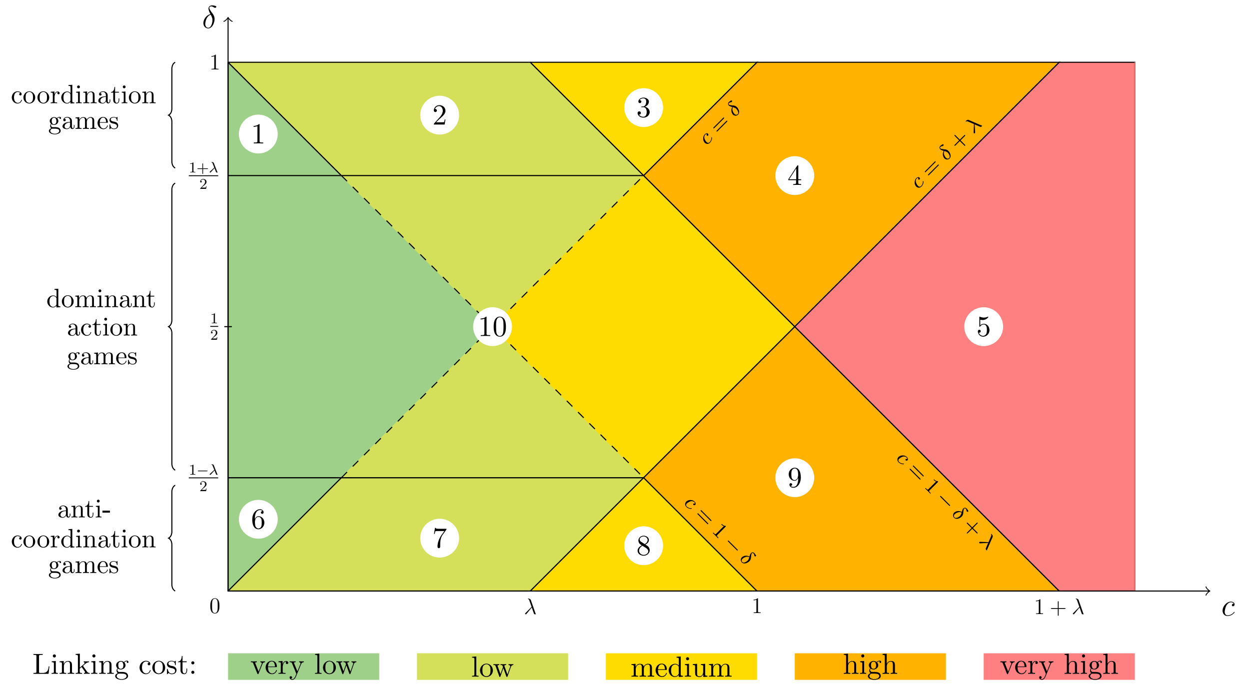


Figure 2: Regions of the linking cost values that correspond to different equilibrium sets. Each horizontal section defines a game (δ, λ).

With respect to the equilibrium action profile, we differentiate between PNE with a homogeneous action profile ($x_i = x_j \ \forall i, j \in N$), PNE with a so-called fully satisfying action profile ($x_i = \theta_i \ \forall i \in N$) and the remaining ones – PNE with a heterogeneous not fully satisfying action profile.

With respect to the equilibrium network structure, all PNE appear to fall into one of the following six classes: such that induce an empty network ($G_{ij} = 0 \ \forall i, j \in N$), a complete network ($G_{ij} = 1 \ \forall i, j \in N$), a network consisting of two disconnected disjoint cliques (more specifically, $G_{ij} = 1 \Leftrightarrow x_i = x_j$), a network consisting of two connected disjoint cliques ($G_{ij} = 1 \Leftrightarrow x_i = x_j \vee \theta_i = x_i \neq x_j = \theta_j$), a complete bipartite network ($G_{ij} = 1 \Leftrightarrow x_i \neq x_j$) or a union of a complete bipartite network and a clique ($G_{ij} = 1 \Leftrightarrow x_i \neq x_j \vee \theta_i = x_i = x_j = \theta_j$).[11]

We provide existence and uniqueness results for different classes of equilibria in different parameter regions. Table 1 summarizes the results. Its first column corresponds to the numbered regions in Figure 2, the next two columns provide qualitative descriptions of the respective regions, the fourth column describes PNE and the last one illustrates them with

[11]Note that the equilibrium networks described in brackets are more specific and relate equilibrium network structures to corresponding equilibrium action profiles.

an example. To facilitate comparisons between the regions, the same simple example is analyzed: six players, four of whom (referred to as "the majority") prefer action 1 and two ("the minority") prefer action 0. Equilibria in brackets exist under additional conditions. The depicted sets of equilibria are not intended to be exhaustive for this particular example, but rather illustrative of possible classes of equilibria in each region.

Turning to equilibrium analysis, the first thing to note is that if the linking cost is very high (region 5), then S^{PNE} consists of all strategy profiles in $\bar{S}$ that induce an empty network.

Proposition 1 [Empty network]
Let $c > \max\{\delta + \lambda, 1 - \delta + \lambda\}$. A strategy profile $s \in S^{PNE}$ if and only if $p_{ij} = 0 \ \forall i, j \in N$.

Such a high linking cost makes any link unprofitable. At the same time, an isolated player (not linked to anyone) gets zero utility regardless of the action she chooses, that is why any action profile is possible in equilibrium.

For all other parameter regions, let us first formulate necessary conditions for pairwise Nash equilibria. These conditions will differ for games of strategic complements ($\delta \geq \frac{1}{2}$) and for games of strategic substitutes ($\delta \leq \frac{1}{2}$).

Proposition 2 [Necessary conditions for a PNE]
Let $c < \max\{\delta + \lambda, 1 - \delta + \lambda\}$ and $s \in S^{PNE}$.

(i) If $\delta \geq \frac{1}{2}$, then for any $i, j \in N$ $x_i = x_j$ implies $p_{ij}p_{ji} = 1$.

(ii) If $\delta \leq \frac{1}{2}$, then for any $i, j \in N$ $x_i \neq x_j$ implies $p_{ij}p_{ji} = 1$.

Hence, if interactional incentives are such that players get higher utility from the links with matching actions than from the links with mismatching actions, then all players playing the same action must be linked in equilibium. If interactional incentives are the contrary, then all pairs of players playing different actions must be linked. This result, although intuitive, is not trivial, as the linking cost might still outweigh the benefits for many pairs of players (see Figure 2). On the other hand, it is a very important result, as together with Proposition 1 and the symmetry of the setting (ex ante, players differ only with respect to their action preferences) it already pins down six classes of equilibrium network structures described above as an exhaustive list.

Contrary to the region 5 with its whole variety of equilibrium action profiles, the regions 4, 9 and 10 demonstrate another extreme: a unique equilibrium action profile here is the fully satisfying one – such that coincides with the preference profile. Moreover, in each of these regions an induced equilibrium network is also unique, which results into a unique PNE.

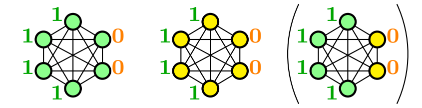
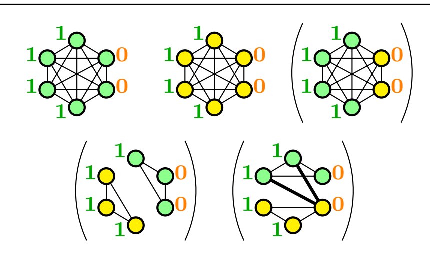
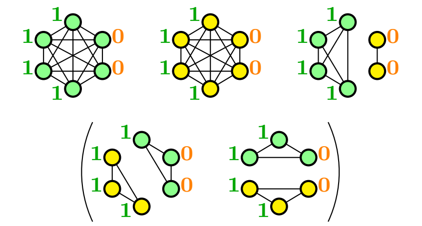
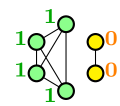
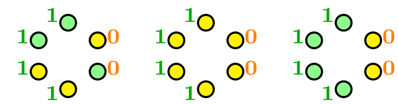
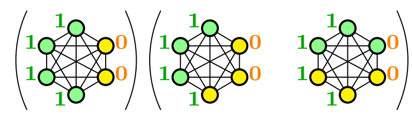
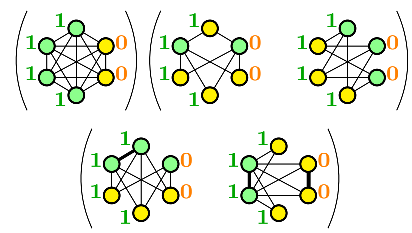
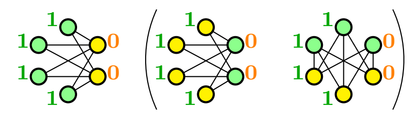
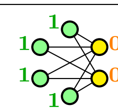
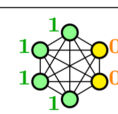
	Linking cost	*Game*	*PNE*	***Example:*** $\theta = (1,1,1,1,0,0)$
1	very low	coordination games	homogeneous action profile, complete network *if sufficient minority:* fully satisfying action profile, complete network	
2	low		homogeneous action profile, complete network *if sufficient minority:* fully satisfying action profile, complete network *if many players:* heterogeneous not fully satisfying action profile, two disconnected action cliques; *if many players and small minority:* heterogeneous not fully satisfying action profile, two partially connected action cliques	
3	medium		homogeneous action profile, complete network fully satisfying action profile, two disconnected action cliques *if many players:* heterogeneous not fully satisfying action profile, two disconnected action cliques	
4	high	games of strategic complements	fully satisfying action profile, two disconnected action cliques	
5	very high	all	any action profile, empty network	
6	very low	anti-coordination games	*if sufficient minority:* fully satisfying action profile, complete network *if small minority or few players*: heterogeneous not fully satisfying action profile, complete network	
7	low		*if sufficient minority:* fully satisfying action profile, complete network; heterogeneous not fully satisfying action profile, complete bipartite network (partition by action) *if many players and sufficient but not too large minority:* heterogeneous not fully satisfying action profile, union of a complete bipartite network and a clique	
8	medium		fully satisfying action profile, complete bipartite network (partition by action) *if many players:* heterogeneous not fully satisfying action profile, complete bipartite network (partition by action)	
9	high	games of strategic substitutes	fully satisfying action profile, complete bipartite network (partition by action)	
10	very low low medium	dominant action games	fully satisfying action profile, complete network	

Table 1: Pairwise Nash equilibria. Coloured numbers next to network nodes denote players' action preferences, colours of the nodes denote their actions (green – action 1, yellow – action 0).

Proposition 3 [Unique PNE: Fully satisfying action profile]

(i) Let $\max\{\delta, 1-\delta+\lambda\} < c < \delta+\lambda$. *A strategy profile* $s \in S^{PNE}$ *if and only if* $x_i = \theta_i \ \forall i \in N$ *and* $p_{ij} = 1 \Leftrightarrow \theta_i = \theta_j$.

(ii) Let $\max\{1-\delta, \delta+\lambda\} < c < 1-\delta+\lambda$. *A strategy profile* $s \in S^{PNE}$ *if and only if* $x_i = \theta_i \ \forall i \in N$ *and* $p_{ij} = 1 \Leftrightarrow \theta_i \neq \theta_j$.

(iii) Let $\lambda > |2\delta - 1|$ *and* $c < \min\{\delta+\lambda, 1-\delta+\lambda\}$. *A strategy profile* $s \in S^{PNE}$ *if and only if* $x_i = \theta_i \ \forall i \in N$ *and* $p_{ij} = 1 \ \forall i, j \in N$.

The intuition is the following. In the regions 4 and 9 (parts (i) and (ii) of the proposition respectively) the linking cost is not too high to exclude any possibility of a profitable link, but sufficiently high to prevent all except the most desirable types of links. Therefore, all players follow their action preferences and form links according to the interactional incentives (action matching in the region 4, or action mismatching in the region 9). In the region 10 (part (iii) of the proposition) the linking cost is lower, which permits links between players playing the same action as well as between those playing different actions. However, this region is characterized by strong action preferences ($\lambda > |2\delta - 1|$), which leads to a unique, fully satisfying equilibrium action profile.

In fact, these are the only regions in which a PNE is always unique. In particular, in the regions 1, 2 and 3 (coordination games with at most medium linking cost) there always exist at least two PNE – complete networks with a homogeneous action profile.

Proposition 4 [Homogeneous action profile]
Let $\delta > \frac{1+\lambda}{2}$ *and* $c < \delta$. *If* $x_i = x_j \ \forall i, j \in N$ *and* $p_{ij} = 1 \ \forall i, j \in N$, *then* $s \in S^{PNE}$.

In these regions links between players who play the same action are profitable regardless of whether these are their preferred actions or not. At the same time, coordination incentives secure homogeneous action profiles against unilateral action deviations.

The next three propositions concern other types of equilibria that can exist in these regions and provide sufficient conditions for their existence.

Proposition 5 [Fully satisfying action profile in coordination games]
Let $\delta > \frac{1+\lambda}{2}$. *If either of the conditions*

(i) $c < 1-\delta$ *and* $n^\pi \leq \frac{2\delta-1+\lambda}{2(2\delta-1)}(n-1) \ \forall \pi \in \{0, 1\}$,

(ii) $1-\delta < c < 1-\delta+\lambda$ *and* $n^\pi \leq \frac{\delta+\lambda-c}{3\delta-1-c}(n-1) \ \forall \pi \in \{0, 1\}$, *or*

(iii) $1 - \delta + \lambda < c < \delta$

holds, then there exists $s \in S^{PNE}$ *with* $x_i = \theta_i \; \forall i \in N$.

It can be shown (see the proof in the appendix) that for low and very low values of the linking cost a fully satisfying PNE inducing a complete network might exist (under additional conditions on the sizes of preferences groups), and for a medium linking cost there exists a fully satisfying PNE inducing two disconnected cliques corresponding to different actions. Table 1 illustrates these possibilities (see the third equilibrium for each respective region).

In the regions 1 and 2, a sufficient (and, in fact, also necessary) condition for existence of a fully satisfying equilibium is relative balancedness of the players' preference profile, that is, the preference majority must not be too large.[12] Note that as δ approaches $\frac{1+\lambda}{2}$, the conditions on relative sizes of preference groups become less stringent (the respective ratios tend to 1), due to the growing weight of idiosyncratic utility component relative to its interactional component.

The following proposition concerns another equilibrium network structure – a network consisting of two disjoint disconnected cliques.

Proposition 6 [Two disconnected action cliques]
Let $\delta > \frac{1+\lambda}{2}$. *If either of the conditions*

(i) $1 - \delta < c < 1 - \delta + \lambda$ *and* $n \geq 2\left\lceil \frac{3\delta-1-c}{2\delta-1-\lambda} + 1 \right\rceil$, *or*

(ii) $1 - \delta + \lambda < c < \delta$

holds, then there exists $s \in S^{PNE}$ *s.t.* $\forall i, j \in N \; p_{ij} = 1 \Leftrightarrow x_i = x_j$ *and* $\exists i, j \in N$ *with* $p_{ij} = 0$.

Due to coordination incentives of the game, these two disjoint cliques correspond to two different actions (see Proposition 2). Note that the two cliques are necessarily distinct, that is, not just a complete network with a homogeneous action profile as in Proposition 4. In the region 2 (part (i) of the proposition), an additional condition for existence of such a PNE is a sufficiently large number of players, as then the sizes of both action cliques can be sufficiently large to guarantee that pairwise deviations would be unprofitable.

Note that although this proposition concerns the regions 2 and 3, the same class of equilibiria exists in the region 4 (Proposition 3, part (i)), where it is a unique equilibrium.

[12]Note that $\frac{2\delta-1+\lambda}{2(2\delta-1)} \in [\frac{1}{2}, 1)$ (attaining the boundary value of $\frac{1}{2}$ when $\lambda = 0$) and $\frac{\delta+\lambda-c}{3\delta-1-c} \in (\frac{1}{2}, 1)$. In particular, when $\lambda = 0$, there is no equilibrium with a fully satisfying action profile in the region 1 (and the region 2 is empty).

Finally, Proposition 7 provides sufficient conditions for existence of the last possible class of equilibria for coordination games – a network consisting of two disjoint partially connected cliques.

Proposition 7 [Two partially connected action cliques]
Let $\delta > \frac{1+\lambda}{2}$ and $1-\delta < c < 1-\delta+\lambda$. If $n^\pi < \min\{\frac{\delta+\lambda-c}{3\delta-1-c}\,(n-1)-3,\ n-4-\frac{\delta+\lambda-c}{2\delta-1-\lambda}\}$ for some $\pi \in \{0,1\}$, then there exists $s \in S^{PNE}$ s.t. $\forall i,j \in N\ x_i = x_j \Rightarrow p_{ij} = 1$, $\exists i,j \in N$ s.t. $x_i \neq x_j$ and $p_{ij} = 1$, and $\exists k,l \in N$ s.t. $x_k \neq x_l$ and $p_{kl} = 0$.

Again, the two cliques correspond to two different actions, but now they are connected by at least one link. However, they are not fully connected, that is, the network is not complete. Such an equilibrium network exists in the region 2 if the number of players is sufficiently large (so that $\min\{\frac{\delta+\lambda-c}{3\delta-1-c}\,(n-1)-3,\ n-4-\frac{\delta+\lambda-c}{2\delta-1-\lambda}\} > 2$, as $n^\pi \geq 2\ \forall\pi \in \{0,1\}$) and the preference minority is sufficiently small. This condition is sufficient but not necessary, however: in the example in Table 1 for parameter values $\lambda = 0.1$, $\delta = 0.7$ and $c = 0.35$ this condition is not satisfied, even though such an equilibrium exists (the last depicted equilibrium for the region 2, where thick lines indicate the links between two action cliques).

Let us now turn to the remaining regions 6, 7 and 8 – anti-coordination games with at most medium linking cost. The following three propositions describe classes of equilibria possible there and provide sufficient conditions for their existence.

Proposition 8 [Fully satisfying action profile in anti-coordination games]
Let $\delta < \frac{1-\lambda}{2}$. If either of the conditions

(i) $c < \delta$ *and* $n^\pi \geq \frac{1-2\delta-\lambda}{2(1-2\delta)}(n-1)\ \forall\pi \in \{0,1\}$,

(ii) $\delta < c < \delta+\lambda$ *and* $n^\pi \geq \frac{1-2\delta-\lambda}{2-3\delta-c}(n-1)\ \forall\pi \in \{0,1\}$, *or*

(iii) $\delta+\lambda < c < 1-\delta$,

holds, then there exists $s \in S^{PNE}$ with $x_i = \theta_i\ \forall i \in N$.

As the regions 6, 7 and 8 are symmetric to the regions 1, 2 and 3, respectively, fully satisfying equilibria there exist under similar conditions. For low or very low linking cost – the regions 6 and 7 – this condition is relative balancedness of the players' preference profile (the preference minority must not be too small).[13] If a fully satisfying PNE exists there,

[13] Note that $\frac{1-2\delta-\lambda}{2(1-2\delta)} \in (0, \frac{1}{2}]$ (attaining the boundary value of $\frac{1}{2}$ when $\lambda = 0$) and $\frac{1-2\delta-\lambda}{2-3\delta-c} \in (0, \frac{1}{2})$. Here, even when $\lambda = 0$, the existence of a fully satisfying equilibrium for a very low linking cost is not completely excluded (unlike coordination games – see footnote 12), since anti-coordination incentives favour action heterogeneity.

it induces a complete network (and hence, is unique). In the region 8, corresponding to the medium linking cost, there always exists a unique fully satisfying PNE, which induces a complete bipartite network with the bipartition $\{N^0, N^1\}$ (see Table 1).

The next proposition concerns more general complete bipartite networks as possible equilibrium network structures in anti-coordination games.

Proposition 9 [Complete bipartite network]
Let $\delta < \frac{1-\lambda}{2}$. If either of the conditions

(i) $\delta < c < \delta + \lambda$ and $n^\pi > \frac{1-\delta+\lambda-c}{1-2\delta-\lambda}$ $\forall \pi \in \{0, 1\}$, or

(ii) $\delta + \lambda < c < 1 - \delta$

holds, then there exists $s \in S^{PNE}$ s.t. $\forall i, j \in N$ $p_{ij} = 1 \Leftrightarrow x_i \neq x_j$.

Note that in these bipartite networks players are partitioned according to their actions. As we already know from Proposition 2, the links between players choosing different actions constitute the minimal set of links for all games with strategic substitutes. This proposition shows that for some of these games – namely, for the regions 7 and 8 (for the region 9 see part (ii) of Proposition 3) – this set of links can also be maximal.

Finally, the last proposition demonstrates the last possible class of equilibium network structures, that under some additional conditions on the sizes of preference groups is possible for games from the region 7.

Proposition 10 [Union of a complete bipartite network and a clique]
Let $\delta < \frac{1-\lambda}{2}$ and $\delta < c < \delta + \lambda$. If $\frac{1-\delta+\lambda-c}{1-2\delta-\lambda} < \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\, n^\pi \leq \min\{\frac{1-\delta+\lambda-c}{2-3\delta-c}\, n-2, n-n^\pi-2\}$ for some $\pi \in \{0, 1\}$, then there exists $s \in S^{PNE}$ s.t. $\forall i, j \in N$ $x_i \neq x_j \Rightarrow p_{ij} = 1$, $\exists i, j \in N$ s.t. $x_i = x_j$ and $p_{ij} = 1$, and $\exists k, l \in N$ s.t. $x_k = x_l$ and $p_{kl} = 0$.

Compared to a complete bipartite network from Proposition 9, this equilibrium network has additional links between some players from the same bipartition class (i.e. those playing the same action), but it is still less connected than a complete network. In particular, only players choosing their preferred actions can afford to have additional links (in Table 1, thick lines in the last two equilibium networks for the region 7 indicate these additional links). The symmetry of the setting implies that all such players will be linked to each other, which derives the union of the complete bipartite network with the clique of all players choosing their preferred actions.

A sufficient number of players and a sufficient but not too large preference minority guarantees existence of this type of an equilibrium. This condition is not a necessary one, however: in the example in Table 1 for parameter values $\lambda = 0.1$, $\delta = 0.3$ and $c = 0.35$, the last two equilibria for the region 7 exist, even though the condition of Proposition 10 is not satisfied.

4 Discussion and conclusions

4.1. The role of players' heterogeneity in action preferences

There are classes of pairwise Nash equilibria that exist only for $\lambda > 0$. These are, in particular, the two classes with heterogeneous but not fully satisfying action profiles and incomplete asymmetric network structures (either two partially connected cliques or a union of a complete bipartite network and a clique). If $\lambda = 0$ then the regions 2 and 7, which might give rise to such equilibria, are empty. Hence, the most irregular equilibrium structures that can be achieved are due to players' heterogeneous action preferences.

Another class of equilibria that, generically, exists only for $\lambda > 0$ is a fully satisfying action profile on a complete network. It might exist for up to medium values of the linking cost (in the regions 1, 2, 6, 7 or 10) and requires either a sufficiently balanced preference profile or relatively strong action preferences. If $\lambda = 0$, the equilibrium network there will still be complete, but the action profile will be either homogeneous (for coordination games) or heterogeneous but, generically, not fully satisfying (for anti-coordination games).[14]

4.2. Alternative equilibrium concepts

Obviously, pairwise deviations defined in this paper do not cover the whole range of deviations that a pair of players can implement. One natural refinement of the pairwise Nash equilibrium concept would be to consider equilibria proof against *all* possible bilateral deviations.[15]

Definition 2. A strategy profile s is a *bilateral equilibrium (BE)* of the above game if it is a Nash equilibrium and for any pair of players $i, j \in N$ and any strategy pair $s'_i \in S_i, s'_j \in S_j$,

$$u_i(s'_i, s'_j, s_{-i-j}) > u_i(s) \Rightarrow u_j(s'_i, s'_j, s_{-i-j}) < u_j(s).$$

Since $S^{BE} \subseteq S^{PNE}$, no new equilibria can be derived if we consider this alternative equilibrium concept. What would be interesting is to verify if all the classes of PNE that

[14]Equilibria in the case of anti-coordination games with homogeneous players are characterized in Bramoullé (2007): when the network is complete, the proportion of agents playing a strategy approximately equals the mixed equilibrium probability of this strategy.

[15]The following definition is adopted from Goyal and Vega-Redondo (2007), with the only difference that in this paper an action is also a part of a strategy.

we described can survive these more stringent equilibrium requirements. Although a complete characterization of bilateral equilibria lies outside the scope of this paper, let us make one important observation: irregular equilibrium structures are still possible under the BE equilibrium concept (the following figure provides an illustration).

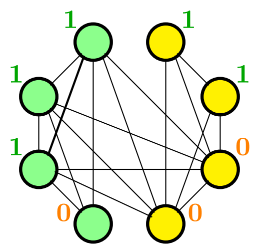


Figure 3: A bilateral equilibrium of the coordination game with $\delta = 0.7$, $\lambda = 0.1$ and $c = 0.35$. Coloured numbers denote players' action preferences, colours of the nodes – players' actions in this equilibrium (green corresponds to action 1, yellow – to action 0).

One could further refine the bilateral equilibrium set by considering deviations by coalitions consisting of more than two players. Dutta and Mutuswami (1997) and Jackson and van den Nouweland (2005) introduced the notion of strong stability of a network that refers to a situation where no coalition of players can rearrange their links to achieve a strong (or even weak – in the latter paper) improvement. These notions can be adapted to our framework with a simultaneous action choice. It would be interesting to verify if the conclusion of Jackson and van den Nouweland (2005) that strongly stable networks coincide with the set of efficient networks holds also in our framework.

4.3. Efficiency

A natural next step in our research is the efficiency analysis of possible outcomes. Figure 4 depicts the regions of parameter values corresponding to different sets of efficient strategy profiles. These regions partition the regions of different equilibrium sets in Figure 2.

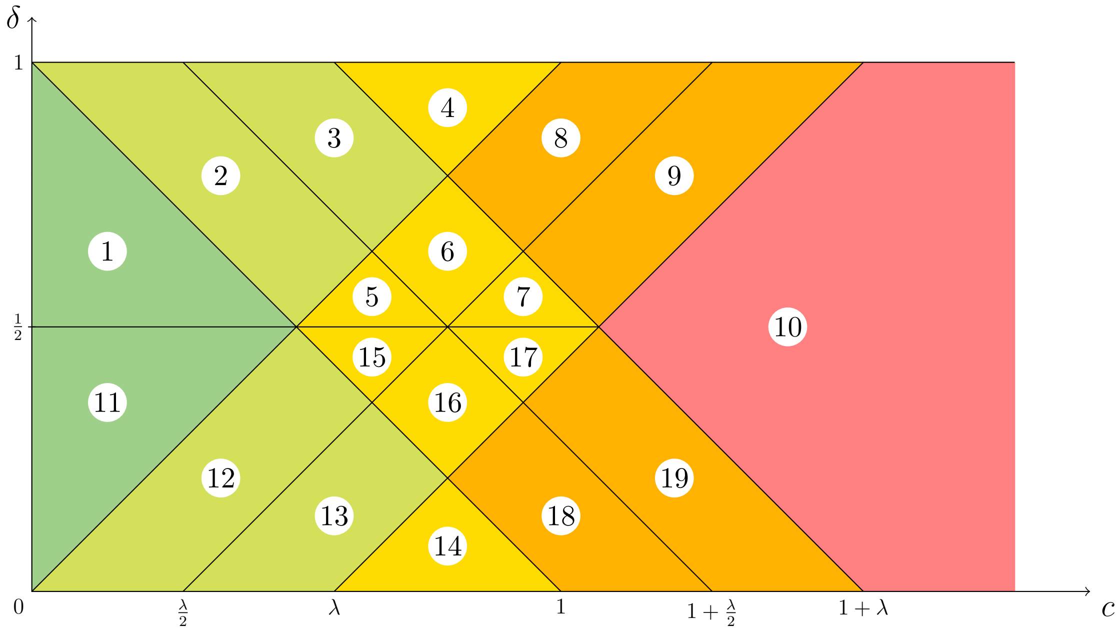


Figure 4: Regions of the linking cost values that correspond to different efficient networks. Each horizontal section defines a game (δ, λ).

One way to derive efficient strategy profiles is to maximize aggregate welfare over all created links (which is equivalent to maximizing aggregate welfare over all players). Altogether, six types of links are possible: three types that connect players from the same preference group and three types that connect players with different action preferences (see Table 2). Aggregate welfare is then the sum of corresponding link payoffs over all links.

	Link type	*Link payoff*	*Number of such links*
1	0 — 0 1 — 1	$2(\delta+\lambda-c)$	$\frac{m^0(m^0-1)}{2}+\frac{m^1(m^1-1)}{2}$
2	0 — 0 1 — 1	$2(1-\delta-c)+\lambda$	$m^0(n^0-m^0)+m^1(n^1-m^1)$
3	0 — 0 1 — 1	$2(\delta-c)$	$\frac{(n^0-m^0)(n^0-m^0-1)}{2}+\frac{(n^1-m^1)(n^1-m^1-1)}{2}$
4	0 — 1	$2(1-\delta+\lambda-c)$	m^0m^1
5	0 — 1 0 — 1	$2(\delta-c)+\lambda$	$m^0(n^1-m^1)+m^1(n^0-m^0)$
6	0 — 1	$2(1-\delta-c)$	$(n^0-m^0)(n^1-m^1)$

Table 2: Six possible types of links, their contributions to aggregate welfare and respective quantities. Here $m^\pi=|\{i\in N^\pi \mid x_i=\theta_i\}|$ *for* $\pi\in\{0,1\}$.

Each region in Figure 4 defines which types of links contribute to aggregate welfare (generate positive payoffs) and hence can be present in efficient profiles in this region. For instance, in the region 1 all links are profitable, which implies that efficient networks here are necessarily complete. Summing up link payoffs over all links and subtracting the constant part of aggregate welfare derives the following welfare maximization problem for this region:

$$\max_{\substack{m^0\in\{0,\dots,n^0\}\\ m^1\in\{0,\dots,n^1\}}} 2(2\delta-1)(m^0-m^1)^2+2(2\delta-1)(n^1-n^0)(m^0-m^1)+\lambda(n-1)(m^0+m^1),$$

which is equivalent to

$$\max_{\substack{x,y\in\mathbb{Z}_+\\ x+y\le 2n^0\\ x-y\le 2n^1}} 2(2\delta-1)y^2+2(2\delta-1)(n^1-n^0)y+\lambda(n-1)x.$$

For other regions, the maximization problems are derived in a similar way. After efficient strategy profiles are characterized, we can compare them with equilibrium profiles and find when achieving efficiency is guaranteed, when it is possible and when not. We leave these questions for future research.

A Appendix

Proof of Lemma 1

Proof. Take $s\in S^{NE}$ and suppose that $s\notin\bar{S}$. The latter implies that $\exists i,j\in N$ s.t. $p_{ij}=1$ and $p_{ji}=0$. Consider $s_i'=(x_i',p_i')$ with $x_i'=x_i$, $p_{ij}'=0$ and $p_{ik}'=p_{ik}$ $\forall k\neq j$. Then $u_i(s_i',s_{-i})=u_i(s)+\varepsilon>u_i(s)$, i.e. i's payoff is strictly higher with such a strategy deviation, and hence $s\notin S^{NE}$. By contradiction we proved that $s\in S^{NE}$ implies $s\in\bar{S}$. □

Proof of Lemma 2

Proof. Necessity follows trivially, so let us prove sufficiency. Let $s=(x,p)\in S$ be such that $u_i(s_i',s_{-i})\le u_i(s)$ $\forall i\in N$ $\forall s_i'\in\bar{S}_i(s_{-i})$. If $p_{ij}=1$ $\forall i,j\in N$, then $\bar{S}_i(s_{-i})=S_i$ $\forall i\in N$, and hence the proof is completed. Suppose $\exists i,j\in N$ s.t. $p_{ji}=0$, that is, $S_i\setminus\bar{S}_i(s_{-i})\neq\emptyset$. Take $s_i''=(x_i'',p_i'')\in S_i\setminus\bar{S}_i(s_{-i})$ and let us prove that $u_i(s_i'',s_{-i})\le u_i(s)$. Denote $J=\{j\in N\setminus\{i\}:p_{ij}''=1 \text{ and } p_{ji}=0\}$ and consider now $s_i'=(x_i',p_i')$ such that $x_i'=x_i''$, $p_{ij}'=0$ $\forall j\in J$ and $p_{ij}'=p_{ij}''$ $\forall j\notin J\cup\{i\}$. Then $u_i(s_i'',s_{-i})<u_i(s_i',s_{-i})$, which together with $s_i'\in\bar{S}(s_{-i})$ implies $u_i(s_i'',s_{-i})<u_i(s)$. Hence, s is a Nash equilibrium. □

Proof of Lemma 3

Proof. Let us first note that Assumption 1 implies that $u_{ij}(x_i, x_j) \neq 0 \ \forall i, j \in N$.

Necessity. Let $s \in S^{PNE}$ and pick $i, j \in N$ s.t. $p_{ij} = 1$. According to Lemma 1, $p_{ji} = p_{ij} = 1$. Without loss of generality, suppose $u_{ij}(x_i, x_j) < 0$. Consider now $s'_i = (x'_i, p'_i) \in S_i$ s.t. $x'_i = x_i$, $p'_{ij} = 0$ and $p'_{ik} = p_{ik} \ \forall k \neq j$. Then $u_i(s'_i, s_{-i}) > u_i(s)$, and hence $s \notin S^{PNE}$.

Sufficiency. Let $s \in S^{PNE}$ and $i, j \in N$ be s.t. both $u_{ij}(x_i, x_j) > 0$ and $u_{ji}(x_j, x_i) > 0$. Suppose, $p_{ij} = 0$. According to Lemma 1, $p_{ji} = p_{ij} = 0$. Consider a pairwise deviation s'_i, s'_j s.t. $x'_i = x_i$, $x'_j = x_j$, $p'_{ij} = p'_{ji} = 1$ and $p'_{lk} = p_{lk} \ \forall l \in \{i, j\} \ \forall k \notin \{i, j\}$. Then $u_i(s'_i, s'_j, s_{-i-j}) > u_i(s)$ and $u_j(s'_i, s'_j, s_{-i-j}) > u_j(s)$, and hence $s \notin S^{PNE}$. $\square$

Proof of Lemma 4

Proof. Necessity follows trivially, so let us prove sufficiency. Consider $s \in \bar{S}^+$ and let $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s) \ \forall i \in N$. We need to prove that $u_i(s'_i, s_{-i}) \leq u_i(s) \ \forall s'_i \in \bar{S}_i(s_{-i}) \ \forall i \in N$ (see Lemma 2). Fix any $i \in N$. If $s'_i = (x'_i, p'_i)$ is such that $x'_i = x_i$ and $p'_i \neq p_i$, then $s \in \bar{S}^+$ implies $u_i(s'_i, s_{-i}) < u_i(s)$. If s'_i is such that $x'_i \neq x_i$, then $x'_i = \tilde{x}_i$, and hence $u_i(s'_i, s_{-i}) \leq u_i(\tilde{s}_i, s_{-i}) = \sum\limits_{j:\ u_{ij}(\tilde{x}_i, x_j) > 0} p_{ij} p_{ji} u_{ij}(\tilde{x}_i, x_j) \leq u_i(s)$. Therefore, $u_i(s'_i, s_{-i}) \leq u_i(s) \ \forall s'_i \in \bar{S}_i(s_{-i}) \ \forall i \in N$. $\square$

Proof of Lemma 5

Proof. Necessity follows from Lemma 3 and the definition of a PNE. Let us prove sufficiency. Take $s \in \bar{S}^+$ and let conditions of the lemma hold for all $i \in N$ and for all $j \in N$ s.t. $p_{ij} p_{ji} = 0$. According to Lemma 4, condition (1) implies that $s \in S^{NE}$. We are left to prove that for all $i, j \in N$ s.t. $p_{ij} p_{ji} = 0$, s is proof against the pairwise deviation $\big((x_i, p'_i), (x_j, p'_j)\big)$.

Note that since $s \in \bar{S}^+$, $p_{ij} p_{ji} = 0$ together with Assumption 1 implies that either $u_{ij}(x_i, x_j) < 0$ or $u_{ji}(x_j, x_i) < 0$. Consequently, either $u_i((x_i, p'_i), (x_j, p'_j), s_{-i-j}) < u_i(s)$ or $u_j((x_i, p'_i), (x_j, p'_j), s_{-i-j}) < u_j(s)$ respectively. Hence, s is also proof against all possible pairwise deviations, i.e. $s \in S^{PNE}$.

$\square$

Proof of Lemma 6

Proof. Necessity. Let $s \in S^{PNE}$. Lemma 3 implies that $s \in \bar{S}^+$. It follows from the definition of a PNE that condition (1) holds for all $i \in N$. To prove the necessity of condition (2), consider arbitrary $i, j \in N$ with $p_{ij} p_{ji} = 0$. If $u_{ji}(\hat{x}_i, x_j) > 0$, then $u_j((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) = u_j(s) + u_{ji}(\hat{x}_i, x_j) > u_j(s)$, and condition (2) follows then from the definition of a PNE. Let now $u_{ji}(\hat{x}_i, x_j) < 0$ and assume that condition (2) does not hold, i.e. $u_i(s) \leq u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) = u_i((\hat{x}_i, p_i), s_{-i}) + u_{ij}(\hat{x}_i, x_j) \leq u_i(s) + u_{ij}(\hat{x}_i, x_j)$, which together with Assumption 1 implies $u_{ij}(\hat{x}_i, x_j) > 0$. The rest of the proof shows that in this case $s \notin S^{PNE}$.

Note that $u_{ji}(\hat{x}_i, x_j) < 0$ together with $u_{ij}(\hat{x}_i, x_j) > 0$ is only possible if $x_j \neq \theta_j$ and $\hat{x}_i = \theta_i$. Hence, $x_i \neq \theta_i$, and consequently, $u_{ij}(x_i, x_j) = u_{ji}(x_i, x_j) = \begin{cases} \delta & \text{if } \theta_i = \theta_j \\ 1-\delta & \text{if } \theta_i \neq \theta_j \end{cases}$. Since $s \in \bar{S}^+$ and $p_{ij}p_{ji} = 0$, it must be that $u_{ij}(x_i, x_j) = u_{ji}(x_i, x_j) < 0$. Noting additionally that $u_{ji}(\hat{x}_i, x_j) = 1 - u_{ji}(x_i, x_j) < 0$ derives $\max\{\delta, 1-\delta\} < c$. It must also be that $c < \max\{\delta + \lambda, 1 - \delta + \lambda\}$, since otherwise $u_{ij}(\hat{x}_i, x_j) > 0$ is violated. Finally, taking into account that $c \notin C^h_{\delta,\lambda}$, we derive $\max\{\delta, 1-\delta\} < c < \min\{\delta + \lambda, 1 - \delta + \lambda\}$. But then $u_{kl}(x_k, x_l) > 0$ if and only if $x_k = \theta_k$, and hence $\bar{S}^+ = \{s \in \bar{S} \mid p_{kl} = 1 \Leftrightarrow x_k = \theta_k \wedge x_l = \theta_l\}$. Above we derived that both $x_i \neq \theta_i$ and $x_j \neq \theta_j$, which implies $p_i = p_j = 0$, and thus $u_i(s) = u_j(s) = 0$. However, a pairwise deviation (s'_i, s'_j) with $x'_i = \theta_i$ and $x'_j = \theta_j$ would be Pareto improving for i and j: $u_i(s'_i, s'_j, s_{-i-j}) = u_{ij}(x'_i, x'_j) > 0 = u_i(s)$ and $u_j(s'_i, s'_j, s_{-i-j}) = u_{ji}(x'_i, x'_j) > 0 = u_j(s)$. This contradicts that $s \in S^{PNE}$. Thus, condition (2) is also necessary for $s \in S^{PNE}$.

Sufficiency. Take $s \in \bar{S}^+$ and let conditions of the lemma hold for all $i \in N$ and for all $j \in N$ s.t. $p_{ij}p_{ji} = 0$. According to Lemma 4, condition (1) implies that $s \in S^{NE}$. We are left to prove that for all $i, j \in N$ s.t. $p_{ij}p_{ji} = 0$, s is proof against two pairwise deviations: $((x_i, \hat{p}_i), (x_j, \hat{p}_j))$ and $((\hat{x}_i, \hat{p}_i), (\hat{x}_j, \hat{p}_j))$ with $\hat{x}_i \neq x_i$ and $\hat{x}_j \neq x_j$. For the first one, see the analogous proof of Lemma 5. Consider now the deviation $((\hat{x}_i, \hat{p}_i), (\hat{x}_j, \hat{p}_j))$.

Case 1: $c > \max\{\delta + \lambda, 1 - \delta + \lambda\}$. Then $u_{kl}(x_k, x_l) < 0\ \forall k, l \in N$, and hence $\bar{S}^+ = \{s \in \bar{S} \mid p_{kl} = 0\ \forall k, l \in N\}$, which implies $u_i(s) = u_j(s) = 0$. Since then $u_i((\hat{x}_i, \hat{p}_i), (\hat{x}_j, \hat{p}_j), s_{-i-j}) < 0 = u_i(s)$, s is proof against the deviation $((\hat{x}_i, \hat{p}_i), (\hat{x}_j, \hat{p}_j))$.

Case 2: $c < \max\{\delta + \lambda, 1 - \delta + \lambda\} = \delta + \lambda$. Then $\delta \geq \frac{1}{2}$. Taking into account that $c \notin C^h_{\delta,\lambda}$, it must be that $c < \max\{\delta, 1 - \delta + \lambda\}$. First, let $x_i \neq x_j$. Then $1 - \delta \leq \delta$ implies $u_{ij}(\hat{x}_i, \hat{x}_j) \leq u_{ij}(\hat{x}_i, x_j)$, and hence, due to condition (2) of the lemma, $u_i((\hat{x}_i, \hat{p}_i), (\hat{x}_j, \hat{p}_j), s_{-i-j}) = u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) - u_{ij}(\hat{x}_i, x_j) + u_{ij}(\hat{x}_i, \hat{x}_j) < u_i(s)$, i.e. s is proof against this deviation. Second, let $x_i = x_j$. We show that this leads to a contradiction. Note that it must be that $c > \delta$, as otherwise both $u_{ij}(x_i, x_j) > 0$ and $u_{ji}(x_i, x_j) > 0$, which together with $p_{ij}p_{ji} = 0$ contradicts $s \in \bar{S}^+$. Then $c < \max\{\delta, 1 - \delta + \lambda\}$ implies $c < 1 - \delta + \lambda$. Note also that either $x_i \neq \theta_i$ or $x_j \neq \theta_j$ (or both), as otherwise $u_{ij}(x_i, x_j) = u_{ji}(x_i, x_j) = \delta + \lambda - c > 0$. Without loss of generality, let $x_i \neq \theta_i$. Since $s \in \bar{S}^+$ and $c > \delta \geq 1 - \delta$, it follows that $p_i = 0$. But then $u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) = u_{ij}(\hat{x}_i, x_j) = 1 - \delta + \lambda - c > 0 = u_i(s)$, which contradicts condition (2) of the lemma.

Case 3: $c < \max\{\delta + \lambda, 1 - \delta + \lambda\} = 1 - \delta + \lambda$. Then $\delta \leq \frac{1}{2}$. Taking into account that

$c \notin C^h_{\delta,\lambda}$, it must be that $c < \max\{\delta + \lambda, 1 - \delta\}$. The rest of the proof is symmetric to the previous case. First, let $x_i = x_j$. Then $\delta \leq 1 - \delta$ implies $u_{ij}(\hat{x}_i, \hat{x}_j) \leq u_{ij}(\hat{x}_i, x_j)$, and hence, due to condition (2) of the lemma, $u_i((\hat{x}_i, \hat{p}_i), (\hat{x}_j, \hat{p}_j), s_{-i-j}) = u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) - u_{ij}(\hat{x}_i, x_j) + u_{ij}(\hat{x}_i, \hat{x}_j) < u_i(s)$, i.e. s is proof against this deviation. Second, we show that $x_i \neq x_j$ is impossible, as it leads to a contradiction. Let $x_i \neq x_j$. Note that it must be that $c > 1 - \delta$, as otherwise both $u_{ij}(x_i, x_j) > 0$ and $u_{ji}(x_i, x_j) > 0$. Hence, $c < \delta + \lambda$. Note also that, like in the previous case, either $x_i \neq \theta_i$ or $x_j \neq \theta_j$ (or both). Without loss of generality, let $x_i \neq \theta_i$, and consequently, $p_i = 0$. But then $u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) = u_{ij}(\hat{x}_i, x_j) = \delta + \lambda - c > 0 = u_i(s)$, which contradicts condition (2) of the lemma.

In all cases s is proof against the deviation $((\hat{x}_i, \hat{p}_i), (\hat{x}_j, \hat{p}_j))$, and thus $s \in S^{PNE}$. $\square$

Proof of Proposition 1

Proof. Let $c > \max\{\delta + \lambda, 1 - \delta + \lambda\}$. Since $\lambda \geq 0$, it means that $u_{ij}(x_i, x_j) < 0 \ \forall x_i, x_j \in \{0, 1\}$. Hence, $\bar{S}^+ = \{s \in S \mid p_{ij} = 0 \ \forall i, j \in N\}$. According to Lemma 3, $S^{PNE} \subseteq \bar{S}^+$, which proves necessity of $p_{ij} = 0 \ \forall i, j \in N$ for every $s \in S^{PNE}$. To prove its sufficiency, we can use Lemma 5. First, observe that $\forall i \in N \ \forall s \in \bar{S}^+ \ u_i(s) = 0$, and hence $u_i(\tilde{s}_i, s_{-i}) = u_i(s)$. Second, $\forall i, j \in N \ \forall s \in \bar{S}^+ \ u_i((x'_i, p'_i), (x'_j, p'_j), s_{-i-j}) < 0 = u_i(s)$ for all admissible x'_i and x'_j. Hence, every $s \in \bar{S}^+$ is proof against all admissible deviations, which implies $s \in S^{PNE}$. $\square$

Proof of Proposition 2

Proof. Let $s \in S^{PNE}$ and $c < \max\{\delta + \lambda, 1 - \delta + \lambda\}$. Lemma 1 implies $p_{ij} = p_{ji} \ \forall i, j \in N$.

(i) Let $\delta \geq \frac{1}{2}$, which implies $c < \delta + \lambda$. Suppose $\exists i, j \in N$ s.t. $x_i = x_j$ but $p_{ij} = p_{ji} = 0$. We show that in each of the following cases (that cover all possibilities) there exists a profitable pairwise deviation, which contradicts $s \in S^{PNE}$.

Case 1: $c < \delta$. Consider a pairwise deviation (s'_i, s'_j) with $x'_i = x_i$ and $x'_j = x_j$. $u_i(s'_i, s'_j, s_{-i-j}) = u_i(s) + u_{ij}(x_i, x_j) > u_i(s)$, as $u_{ij}(x_i, x_j) \geq \delta - c > 0$, and similarly, $u_j(s'_i, s'_j, s_{-i-j}) > u_j(s)$.

Case 2: $x_i = \theta_i$ and $x_j = \theta_j$. Again, a pairwise deviation (s'_i, s'_j) with $x'_i = x_i$ and $x'_j = x_j$ is profitable for i and j: $u_i(s'_i, s'_j, s_{-i-j}) > u_i(s)$ and $u_j(s'_i, s'_j, s_{-i-j}) > u_j(s)$, as $u_{ij}(x_i, x_j) = u_{ji}(x_i, x_j) = \delta + \lambda - c > 0$.

Case 3: $\delta < c < 1 - \delta + \lambda$ and either $x_i \neq \theta_i$ or $x_j \neq \theta_j$ (or both). Without loss of generality, let $x_i \neq \theta_i$. Since $c > \max\{\delta, 1 - \delta\}$, Lemma 3 implies $p_i = 0$. Consider a pairwise deviation (s'_i, s'_j) with $x'_i = \theta_i$ and $x'_j = \theta_j$: $u_i(s'_i, s'_j, s_{-i-j}) = u_{ij}(x'_i, x'_j) > 0 = u_i(s)$, as $c < \min\{\delta + \lambda, 1 - \delta + \lambda\}$, and similarly, $u_j(s'_i, s'_j, s_{-i-j}) = u_j(s) + u_{ji}(x'_i, x'_j) >$

$u_j(s)$ (note that the last equality holds both for $x_j = \theta_j$ and for $x_j \neq \theta_j$, as in the latter case $p_j = 0$).

Case 4: $c > \max\{\delta, 1 - \delta + \lambda\}$ and either $x_i \neq \theta_i$ or $x_j \neq \theta_j$ (or both). Without loss of generality, let $x_i \neq \theta_i$. As in the previous case, Lemma 3 implies $p_i = 0$. There are two possibilities: either $\theta_i = \theta_j$ or $\theta_i \neq \theta_j$. Consider the first possibility. Then a pairwise deviation (s'_i, s'_j) with $x'_i = \theta_i$ and $x'_j = \theta_j$ is profitable for i and j: $u_i(s'_i, s'_j, s_{-i-j}) = u_{ij}(x'_i, x'_j) = \delta + \lambda - c > 0 = u_i(s)$ and similarly, $u_j(s'_i, s'_j, s_{-i-j}) = u_j(s) + u_{ji}(x'_i, x'_j) > u_j(s)$ (note that the last equality holds both for $x_j = \theta_j$ and for $x_j \neq \theta_j$, as in the latter case $p_j = 0$). Consider the second possibility, $\theta_i \neq \theta_j$. Take a third player k with $\theta_k = \theta_i$. As $p_i = 0$, a pairwise deviation (s'_i, s'_k) with $x'_i = \theta_i$ and $x'_k = \theta_k$ is feasible and, moreover, profitable for i and k (see the above reasoning for i and j).

As in each case there exists a profitable pairwise deviation, our assumption contradicts $s \in S^{PNE}$. Hence, it must be that $\forall i, j \in N$ $x_i = x_j$ implies $p_{ij} = p_{ji} = 1$.

(ii) Let $\delta \leq \frac{1}{2}$, which implies $c < 1 - \delta + \lambda$. Suppose $\exists i, j \in N$ s.t. $x_i \neq x_j$ but $p_{ij} = p_{ji} = 0$. We show that in each of the following cases (that cover all possibilities) there exists a profitable pairwise deviation, which contradicts $s \in S^{PNE}$. The cases are symmetric to those in part (i), and hence the rest of the proof is analogous to the proof of (i).

Case 1: $c < 1 - \delta$. A pairwise deviation (s'_i, s'_j) with $x'_i = x_i$ and $x'_j = x_j$ is profitable for i and j, as $u_{ij}(x_i, x_j) \geq 1 - \delta - c > 0$ and $u_{ji}(x_i, x_j) \geq 1 - \delta - c > 0$.

Case 2: $x_i = \theta_i$ and $x_j = \theta_j$. Again, a pairwise deviation (s'_i, s'_j) with $x'_i = x_i$ and $x'_j = x_j$ is profitable for i and j, as $u_{ij}(x_i, x_j) = u_{ji}(x_i, x_j) = 1 - \delta + \lambda - c > 0$.

Case 3: $1 - \delta < c < \delta + \lambda$ and either $x_i \neq \theta_i$ or $x_j \neq \theta_j$ (or both). Without loss of generality, let $x_i \neq \theta_i$, which implies $p_i = 0$. A pairwise deviation (s'_i, s'_j) with $x'_i = \theta_i$ and $x'_j = \theta_j$ is profitable for i and j: $u_i(s'_i, s'_j, s_{-i-j}) = u_{ij}(x'_i, x'_j) > 0 = u_i(s)$ and $u_j(s'_i, s'_j, s_{-i-j}) = u_j(s) + u_{ji}(x'_i, x'_j) > u_j(s)$ (the last equality holds both for $x_j = \theta_j$ and for $x_j \neq \theta_j$).

Case 4: $c > \max\{\delta + \lambda, 1 - \delta\}$ and either $x_i \neq \theta_i$ or $x_j \neq \theta_j$ (or both). Without loss of generality, let $x_i \neq \theta_i$, which implies $p_i = 0$. If $\theta_i \neq \theta_j$, then a pairwise deviation (s'_i, s'_j) with $x'_i = \theta_i$ and $x'_j = \theta_j$ is profitable for i and j, as $u_{ij}(x'_i, x'_j) = u_{ji}(x'_i, x'_j) = 1 - \delta + \lambda - c > 0$. If $\theta_i = \theta_j$, then there must be a player k with $\theta_k \neq \theta_i$, and a pairwise deviation (s'_i, s'_k) with $x'_i = \theta_i$ and $x'_k = \theta_k$ is profitable for i and k, as $u_{ik}(x'_i, x'_k) = u_{ki}(x'_i, x'_k) = 1 - \delta + \lambda - c > 0$.

In each case our assumption contradicts $s \in S^{PNE}$. Thus, it must be that $\forall i, j \in N$ $x_i \neq x_j$ implies $p_{ij} = p_{ji} = 1$.

$\square$

Proof of Proposition 3

Proof. (i) Let $\max\{\delta, 1-\delta+\lambda\} < c < \delta + \lambda$. Since $\lambda \geq 0$, it means that $u_{ij}(x_i, x_j) > 0$ if and only if $\theta_i = x_i = x_j$, otherwise $u_{ij}(x_i, x_j) < 0$. Hence, $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow \theta_i = x_i = x_j = \theta_j\}$.

Necessity. Consider a strategy profile $s \in S^{PNE}$, and suppose that $x_i \neq \theta_i$ for some $i \in N$. Note that $S^{PNE} \subseteq \bar{S}^+$ (see Lemma 3). Then it must be that $p_{ij} = 0$ $\forall j \in N$. Take another player j with $\theta_j = \theta_i$ (such a player must exist, as $n^\pi \geq 2$ $\forall \pi \in \{0,1\}$). A pairwise deviation (s'_i, s'_j) with $x'_i = \theta_i$ and $x'_j = \theta_j$ is Pareto improving for i and j: $u_i(s'_i, s'_j, s_{-i-j}) = u_{ij}(x'_i, x'_j) > 0 = u_i(s)$ and $u_j(s'_i, s'_j, s_{-i-j}) = u_j(s) + u_{ji}(x'_i, x'_j) > u_j(s)$ (note that the last equality holds both for $x_j = \theta_j$ and for $x_j \neq \theta_j$, as in the latter case $p_j = 0$). Hence, $s \notin S^{PNE}$.

Now let $s \in S^{PNE}$ and $x_i = \theta_i$ $\forall i \in N$. Since $S^{PNE} \subseteq \bar{S}^+$, it must be that $p_{ij} = 1 \Leftrightarrow \theta_i = \theta_j$, which completes this part of the proof.

Sufficiency. Consider a strategy profile with $x_i = \theta_i$ $\forall i \in N$ and $p_{ij} = 1 \Leftrightarrow \theta_i = \theta_j$. Then $s \in \bar{S}^+$. It suffices to verify that conditions of Lemma 5 hold. First, fix a player i. Since $u_{ij}(\tilde{x}_i, x_j) < 0$ $\forall x_j \in \{0,1\}$, it must be that $\tilde{p}_{ij} = 0$ $\forall j \in N$, and hence $u_i(\tilde{s}_i, s_{-i}) = 0 \leq u_i(s)$. Second, fix a pair of players i and j s.t. $p_{ij}p_{ji} = 0$. Consider a pairwise deviation (s'_i, s'_j) with either $x'_i \neq x_i$ or $x'_j \neq x_j$. Without loss of generality, let $x'_i \neq x_i$. Since $x'_i \neq \theta_i$ implies $u_{ik}(x'_i, x_k) < 0$ $\forall x_k \in \{0,1\}$, it follows that $u_i(s'_i, s'_j, s_{-i-j}) = u_i((x'_i, p_i), s_{-i}) + u_{ij}(x'_i, x'_j) \leq u_i(s) + u_{ij}(x'_i, x'_j) < u_i(s)$. Hence, s is proof against all admissible deviations, which implies $s \in S^{PNE}$.

(ii) Let $\max\{1-\delta, \delta+\lambda\} < c < 1-\delta+\lambda$. Then $u_{ij}(x_i, x_j) > 0$ if and only if $\theta_i = x_i \neq x_j$, and hence, $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow \theta_i = x_i \neq x_j = \theta_j\}$. The rest of the proof is identical to that of part (i) with the only difference: the proof of necessity of $x_i = \theta_i$ $\forall i \in N$ for $s \in S^{PNE}$ uses a pairwise deviation of i and j s.t. $\theta_j \neq \theta_i$.

(iii) Let $\lambda > |2\delta - 1|$ and $c < \min\{\delta + \lambda, 1 - \delta + \lambda\}$. The first inequality is equivalent to $\frac{1-\lambda}{2} < \delta < \frac{1+\lambda}{2}$, which implies $\max\{\delta, 1-\delta\} < \min\{\delta + \lambda, 1 - \delta + \lambda\}$. Several cases have to be considered separately:

Case 1: $c < \min\{\delta, 1-\delta\}$. Then $u_{ij}(x_i, x_j) > 0\ \forall x_i, x_j \in \{0,1\}$, and hence $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1\ \forall i, j \in N\}$.

Case 2: $1-\delta < c < \delta$. Then $u_{ij}(x_i, x_j) > 0$ if and only if either $x_i = \theta_i$ or $x_i = x_j$, and hence $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow ((x_i = \theta_i \wedge x_j = \theta_j) \vee x_i = x_j)\}$.

Case 3: $\delta < c < 1-\delta$. Then $u_{ij}(x_i, x_j) > 0$ if and only if either $x_i = \theta_i$ or $x_i \neq x_j$, and hence $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow ((x_i = \theta_i \wedge x_j = \theta_j) \vee x_i \neq x_j)\}$.

Case 4: $c > \max\{\delta, 1-\delta\}$. Then $u_{ij}(x_i, x_j) > 0$ if and only if $x_i = \theta_i$, and hence $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow (x_i = \theta_i \wedge x_j = \theta_j)\}$.

Necessity. Consider a strategy profile $s \in S^{PNE}$, and suppose that $x_i \neq \theta_i$ for some $i \in N$. Lemma 3 implies that $s \in \bar{S}^+$. First, consider cases 1, 2 and 3. Let us prove that in each of these cases $\exists j \in N$ s.t. $p_{ij}p_{ji} = 1$. Suppose not, that is $p_{ij}p_{ji} = 0\ \forall j \in N$. Getting a contradiction in case 1 is trivial. In cases 2 and 3 any pairwise deviation (s'_i, s'_j) s.t. $x'_i = \theta_i$ and $x'_j = \theta_j$ is Pareto improving for i and j: $u_i(s'_i, s'_j, s_{-i-j}) = u_{ij}(x'_i, x'_j) > 0 = u_i(s)$ and $u_j(s'_i, s'_j, s_{-i-j}) \geq u_j(s) + u_{ji}(x'_i, x'_j) > u_j(s)$. Note that the penultimate inequality holds as equality for $x_j = \theta_j$, and if $x_j \neq \theta_j$ then for all $k \in N$ s.t. $p_{jk}p_{kj} = 1$ (if they exist) it holds that either $u_{jk}(x_j, x_k) = \delta - c < 1 - \delta + \lambda - c = u_{jk}(x'_j, x_k)$ (case 2) or $u_{jk}(x_j, x_k) = 1 - \delta - c < \delta + \lambda - c = u_{jk}(x'_j, x_k)$ (case 3), and hence $u_j(s) = \sum_{\{k \in N:\, p_{jk}p_{kj}=1\}} u_{jk}(x_j, x_k) \leq u_j((x'_j, p_j), s_{-j})$ (with equality if $\nexists k \in N$ s.t. $p_{jk}p_{kj} = 1$). Thus, we have proved that $\exists j \in N$ s.t. $p_{ij}p_{ji} = 1$.

Now consider the unilateral deviation $\tilde{s}_i$. Since $\tilde{x}_i = \theta_i$, $u_{ij}(\tilde{x}_i, x_j) > 0\ \forall j \in N$, and hence $\tilde{p}_{ij} = p_{ij}$. Note also that for all $j \in N$ s.t. $p_{ij}p_{ji} = 1$ (above we have shown that at least one such j exists) $u_{ij}(\tilde{x}_i, x_j) \geq \min\{\delta + \lambda - c, 1 - \delta + \lambda - c\} > \max\{\delta - c, 1 - \delta - c\} \geq u_{ij}(x_i, x_j)$. Then $u_i(\tilde{s}_i, s_{-i}) = u_i((\tilde{x}_i, p_i), s_{-i}) = \sum_{\{j \in N:\, p_{ij}p_{ji}=1\}} u_{ij}(\tilde{x}_i, x_j) > \sum_{\{j \in N:\, p_{ij}p_{ji}=1\}} u_{ij}(x_i, x_j) = u_i(s)$, i.e. $\tilde{s}_i$ is a payoff-improving deviation. Hence, $s \notin S^{PNE}$. By contradiction, we have proved that $s \in S^{PNE}$ implies $x_i = \theta_i\ \forall i \in N$ in cases 1, 2 and 3.

Consider case 4. Since $s \in \bar{S}^+$ and $x_i \neq \theta_i$, it must be that $p_{ij} = 0\ \forall j \in N$. But then a pairwise deviation (s'_i, s'_j) s.t. $x'_i = \theta_i$ and $x'_j = \theta_j$ is Pareto improving for i and j: $u_i(s'_i, s'_j, s_{-i-j}) = u_{ij}(x'_i, x'_j) > 0 = u_i(s)$ and $u_j(s'_i, s'_j, s_{-i-j}) = u_j(s) + u_{ji}(x'_i, x'_j) > u_j(s)$. Note that the last equality holds both for $x_j = \theta_j$ and for $x_j \neq \theta_j$, as in the latter case $u_j(s) = 0$. Hence, we have proved that $s \in S^{PNE}$ must have $x_i = \theta_i\ \forall i \in N$ also in case 4.

Now let $s \in S^{PNE}$ and $x_i = \theta_i\ \forall i \in N$. Since $s \in \bar{S}^+$, it must be that $p_{ij} = 1\ \forall i, j \in N$ (in each of the four cases), which completes this part of the proof.

Sufficiency. Consider a strategy profile with $x_i = \theta_i\ \forall i \in N$ and $p_{ij} = 1\ \forall i, j \in N$. Then $s \in \bar{S}^+$ (in each of the four cases). As there are no feasible pairwise deviations, it suffices to verify that the first condition of Lemma 5 holds. Fix a player i. Note that for all $j \in N$ it holds that $u_{ij}(x_i, x_j) \geq \min\{\delta+\lambda-c, 1-\delta+\lambda-c\} > \max\{\delta-c, 1-\delta-c\} \geq u_{ij}(\tilde{x}_i, x_j)$. Then $u_i(\tilde{s}_i, s_{-i}) = \sum_{\{j\in N:\, \tilde{p}_{ij}p_{ji}=1\}} u_{ij}(\tilde{x}_i, x_j) = \sum_{j\in N} \max\{u_{ij}(\tilde{x}_i, x_j), 0\} < \sum_{j\in N} u_{ij}(x_i, x_j) = u_i(s)$, and hence, according to Lemma 5, $s \in S^{PNE}$.

□

Proof of Proposition 4

Proof. Let $\delta > \frac{1+\lambda}{2}$ and $c < \delta$ and consider a strategy profile with $x_i = x_j\ \forall i, j \in N$ and $p_{ij} = 1\ \forall i, j \in N$. For all $i, j \in N$ either $u_{ij}(x_i, x_j) = \delta - c > 0$ or $u_{ij}(x_i, x_j) = \delta + \lambda - c > 0$, hence $s \in \bar{S}^+$. We can now apply Lemma 5.

First, take a player with $x_i = \theta_i$. Then $u_i(s) = (\delta+\lambda-c)(n-1)$, while with the unilateral deviation $\tilde{s}_i$ she gets $u_i((\tilde{x}_i, \tilde{p}_i), s_{-i}) = \sum_{\{j\in N:\, j\neq i\}} \max\{1-\delta-c, 0\} \leq (1-\delta-c)(n-1) < (\delta+\lambda-c)(n-1) = u_i(s)$ (the last inequality follows from $\delta > \frac{1+\lambda}{2}$, which is equivalent to $1-\delta+\lambda < \delta$).

Now, take a player with $x_i \neq \theta_i$. Then $u_i(s) = (\delta-c)(n-1)$, while with the unilateral deviation $\tilde{s}_i$ she gets $u_i((\tilde{x}_i, \tilde{p}_i), s_{-i}) = \sum_{\{j\in N:\, j\neq i\}} \max\{1-\delta+\lambda-c, 0\} \leq (1-\delta+\lambda-c)(n-1) < (\delta-c)(n-1) = u_i(s)$.

In either case, such a unilateral deviation is unprofitable for i. And since there are no feasible pairwise deviations, Lemma 5 implies that $s \in S^{PNE}$. □

Proof of Proposition 5

Proof. Note that if any of the conditions (i), (ii) or (iii) holds, then $c \notin C^h_{\delta,\lambda}$, and hence Lemma 6 is applicable here.

(i) Let $\delta > \frac{1+\lambda}{2}$, $c < 1-\delta$ and $n^\pi \leq \frac{2\delta-1+\lambda}{2(2\delta-1)}(n-1)\ \forall \pi \in \{0,1\}$. We will prove that the strategy profile with $x_i = \theta_i\ \forall i \in N$ and $p_{ij} = 1\ \forall i, j \in N$ is a PNE.

Note that $u_{ij}(x_i, x_j) > 0\ \forall x_i, x_j \in \{0,1\}$, and hence $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1\ \forall i, j \in N\}$. Obviously, $s \in \bar{S}^+$. There are no possible pairwise deviations from s, hence it suffices to show that $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)\ \forall i \in N$ (see Lemma 6). Without loss of generality, take a player with $\theta_i = 0$. Then $u_i(s) = (\delta+\lambda-c)(n^0-1) + (1-\delta+\lambda-c)n^1$, and $u_i((\tilde{x}_i, \tilde{p}_i), s_{-i}) = u_i((\tilde{x}_i, p_i), s_{-i}) = (1-\delta-c)(n^0-1) + (\delta-c)n^1$. Consequently, $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $(1-\delta)(n^0-1) + \delta n^1 \leq (\delta+\lambda)(n^0-1) + (1-\delta+\lambda)n^1$. Rearranging terms and substituting n^0 for $n-n^1$, we can get an equivalent inequality: $n^1 \leq \frac{2\delta-1+\lambda}{2(2\delta-1)}(n-1)$. Similarly, for a player with $\theta_i = 1$, $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $n^0 \leq \frac{2\delta-1+\lambda}{2(2\delta-1)}(n-1)$. In either case, such a unilateral deviation is unprofitable for i, and hence $s \in S^{PNE}$.

(ii) Let $\delta > \frac{1+\lambda}{2}$, $1-\delta < c < 1-\delta+\lambda$ and $n^\pi \leq \frac{\delta+\lambda-c}{3\delta-1-c}(n-1)$ $\forall \pi \in \{0,1\}$. We will prove that the strategy profile with $x_i = \theta_i$ $\forall i \in N$ and $p_{ij} = 1$ $\forall i, j \in N$ is a PNE.

Note that $u_{ij}(x_i, x_j) > 0$ if and only if either $x_i = \theta_i$ or $x_i = x_j$, and hence $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow ((x_i = \theta_i \wedge x_j = \theta_j) \vee x_i = x_j)\}$. Then $s \in \bar{S}^+$. The rest of the proof is similar to the proof of part (i): it suffices to show that $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ $\forall i \in N$. Take a player with $\theta_i = 0$. Then $u_i(s) = (\delta+\lambda-c)(n^0-1) + (1-\delta+\lambda-c)n^1$, and $u_i((\tilde{x}_i, \tilde{p}_i), s_{-i}) = \sum_{\{j \in N:\, x_j = \tilde{x}_i\}} u_{ij}(\tilde{x}_i, x_j) = (\delta-c)n^1$. Hence, $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $\delta n^1 \leq (\delta+\lambda-c)(n^0-1) + (1-\delta+\lambda)n^1$, or equivalently, $n^1 \leq \frac{\delta+\lambda-c}{3\delta-1-c}(n-1)$. Similarly, for a player with $\theta_i = 1$, $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $n^0 \leq \frac{\delta+\lambda-c}{3\delta-1-c}(n-1)$. In either case, such a unilateral deviation is unprofitable for i, and hence $s \in S^{PNE}$.

(iii) Let $\delta > \frac{1+\lambda}{2}$ and $1-\delta+\lambda < c < \delta$. We will prove that the strategy profile with $x_i = \theta_i$ $\forall i \in N$ and $p_{ij} = 1 \Leftrightarrow \theta_i = \theta_j$ is a PNE.

Here $u_{ij}(x_i, x_j) > 0$ if and only if $x_i = x_j$, and hence $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow x_i = x_j\}$. Note that $s \in \bar{S}^+$. As in the previous two parts of the proof, we can apply Lemma 6. First, take a player with action preference θ_i and consider the unilateral deviation $\tilde{s}_i$. Then $u_i(s) = (\delta+\lambda-c)(n^{\theta_i}-1) \geq 0 = u_i(\tilde{s}_i, s_{-i})$, which implies that $\tilde{s}_i$ is unprofitable. Second, take two unlinked players i and j (with respective action preferences $\theta_i \neq \theta_j$) and consider their pairwise deviation $((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j))$. Since $n^\pi \geq 2$ $\forall \pi \in \{0,1\}$, we can derive: $u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) = (1-\delta-c)(n^{\theta_i}-1) + (\delta-c) < \delta - c < (\delta+\lambda-c)(n^{\theta_i}-1) = u_i(s)$. Hence, according to Lemma 6, $s \in S^{PNE}$.

□

Proof of Proposition 6

Proof. (i) Let $1-\delta < c < 1-\delta+\lambda < \delta$ (the last inequality is equivalent to $\delta > \frac{1+\lambda}{2}$). Then $u_{ij}(x_i, x_j) > 0$ if and only if either $x_i = \theta_i$ or $x_i = x_j$. Hence, $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow ((x_i = \theta_i \wedge x_j = \theta_j) \vee x_i = x_j)\}$.

Let $n \geq 2\left\lceil \frac{3\delta-1-c}{2\delta-1-\lambda} + 1 \right\rceil$. Then it must be that $n^\pi \geq \left\lceil \frac{3\delta-1-c}{2\delta-1-\lambda} + 1 \right\rceil$ for some $\pi \in \{0,1\}$. Without loss of generality, let $n^0 \geq \left\lceil \frac{3\delta-1-c}{2\delta-1-\lambda} + 1 \right\rceil$ and consider such a strategy profile s that $|\{i \in N \mid x_i = 1\}| = |\{i \in N^0 \mid x_i = 1\}| = \left\lceil \frac{3\delta-1-c}{2\delta-1-\lambda} + 1 \right\rceil$ and $\forall i, j \in N$ $p_{ij} = 1 \Leftrightarrow x_i = x_j$. Then $a^\pi := |\{i \in N \mid x_i = \pi\}| \geq \left\lceil \frac{3\delta-1-c}{2\delta-1-\lambda} + 1 \right\rceil$ $\forall \pi \in \{0,1\}$. Note that $s \in \bar{S}^+$ (as $x_i = \theta_i$ and $x_j = \theta_j$ implies $x_i = x_j = 0$), and let us apply Lemma 6.

First, fix a player i. If $\tilde{x}_i = \theta_i$, then $u_{ij}(\tilde{x}_i, x_j) = 1-\delta+\lambda-c > 0 \; \forall j$ s.t. $p_{ij} = 1$, hence $\tilde{p}_i = p_i$ and $u_i(\tilde{s}_i, s_{-i}) = \sum_{\{j \in N:\, p_{ij}=1\}}(1-\delta+\lambda-c) < \sum_{\{j \in N:\, p_{ij}=1\}}(\delta - c) = u_i(s)$. If $\tilde{x}_i \neq \theta_i$, then $u_{ij}(\tilde{x}_i, x_j) = 1-\delta-c < 0 \; \forall j$ s.t. $p_{ij} = 1$, hence $\tilde{p}_i = 0$ and $u_i(\tilde{s}_i, s_{-i}) = 0 \leq u_i(s)$.

Second, fix a pair of players i and j s.t. $p_{ij} = 0$ (i.e. $x_i \neq x_j$) and consider a pairwise deviation $((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j))$. If $\hat{x}_i = \theta_i$, then $u_i(s) = \sum_{\{j \in N:\, p_{ij}=1\}}(\delta - c) = (a^{1-\theta_i} - 1)(\delta - c)$, while $u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) = (a^{1-\theta_i} - 1)(1-\delta+\lambda-c) + (\delta+\lambda-c)$. Thus, $u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) < u_i(s)$ if and only if $(a^{1-\theta_i} - 1)(1-\delta+\lambda-c) + (\delta+\lambda-c) < (a^{1-\theta_i} - 1)(\delta - c)$, which is equivalent to $a^{1-\theta_i} > \frac{3\delta-1-c}{2\delta-1-\lambda}$. This last inequality holds true due to $a^\pi \geq \lceil \frac{3\delta-1-c}{2\delta-1-\lambda} + 1 \rceil \; \forall \pi \in \{0,1\}$, and hence condition (2) of Lemma 6 is satisfied. Similarly, if $\hat{x}_i \neq \theta_i$, then $u_i(s) = (a^{1-\theta_i} - 1)(\delta+\lambda-c)$, while $u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) = (a^{1-\theta_i} - 1)(1-\delta-c) + (\delta - c)$. In this case, $u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) < u_i(s)$ if and only if $(a^{1-\theta_i} - 1)(1-\delta-c) + (\delta - c) < (a^{1-\theta_i} - 1)(\delta+\lambda-c)$, which is equivalent to $a^{1-\theta_i} > \frac{3\delta-1+\lambda-c}{2\delta-1+\lambda}$. However, this last inequality is weaker than the respective one in the previous case (using $c < \delta$, one can show that $\frac{3\delta-1+\lambda-c}{2\delta-1+\lambda} < \frac{3\delta-1-c}{2\delta-1-\lambda}$). Hence, condition (2) of Lemma 6 is satisfied also in this case, and we can conclude that $s \in S^{PNE}$.

(ii) Let $1-\delta+\lambda < c < \delta$. Then the strategy profile with $x_i = \theta_i \; \forall i \in N$ and $p_{ij} = 1 \Leftrightarrow x_i = x_j$ is a PNE (see the proof of part (iii) of Proposition 5).

□

Proof of Proposition 7

Proof. Let $1-\delta < c < 1-\delta+\lambda < \delta$ (the last inequality is equivalent to $\delta > \frac{1+\lambda}{2}$). Then $u_{ij}(x_i, x_j) > 0$ if and only if either $x_i = \theta_i$ or $x_i = x_j$, and $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow ((x_i = \theta_i \wedge x_j = \theta_j) \vee x_i = x_j)\}$. Without loss of generality, let $n^0 < \min\{\frac{\delta+\lambda-c}{3\delta-1-c}\,(n-1) - 3,\; n - 4 - \frac{\delta+\lambda-c}{2\delta-1-\lambda}\}$.

Let us introduce some additional notation, $a^\pi := |\{i \in N \mid x_i = \pi\}|$ for $\pi \in \{0,1\}$, and consider such a strategy profile s that $x_i = 0 \; \forall i \in N^0$, $|\{i \in N^1 \mid x_i = 1\}| = \lceil \frac{2\delta-1-\lambda}{\delta+\lambda-c}\, n^0 \rceil + 2$ and $p_{ij} = 1 \Leftrightarrow ((x_i = \theta_i \wedge x_j = \theta_j) \vee x_i = x_j)$. Note that here $a^1 = |\{i \in N^1 \mid x_i = 1\}|$. First of all, we show that $0 < a^1 < n^1$, and hence that s indeed induces two partially connected action cliques: $\exists i, j \in N$ s.t. $\theta_i = 0 = x_i \neq x_j = 1 = \theta_j$ and $p_{ij} = 1$, and $\exists k, l \in N$ s.t. $\theta_k \neq 0 = x_k \neq x_l = 1 = \theta_l$ and $p_{kl} = 0$.

Thus, we need to prove that $0 < \lceil \frac{2\delta-1-\lambda}{\delta+\lambda-c}\, n^0 \rceil + 2 < n^1$. As $\frac{2\delta-1-\lambda}{\delta+\lambda-c} \in (0,1)$, the first inequality is obvious, while the second one can be derived from $n^0 < \frac{\delta+\lambda-c}{3\delta-1-c}\,(n-1) - 3$ in several steps: using that $\frac{\delta+\lambda-c}{3\delta-1-c} \in (\frac{1}{2}, 1)$, we derive $n^0 < \frac{\delta+\lambda-c}{3\delta-1-c}\,(n-3)$, or equivalently, $\frac{3\delta-1-c}{\delta+\lambda-c}\, n^0 + 1 < n-2$, which implies $\lceil \frac{3\delta-1-c}{\delta+\lambda-c}\, n^0 \rceil < n-2$, or equivalently, $\lceil (1 + \frac{2\delta-1-\lambda}{\delta+\lambda-c})\, n^0 \rceil <$

$n-2$, and hence $\lceil \frac{2\delta-1-\lambda}{\delta+\lambda-c}\, n^0 \rceil < n - n^0 - 2 = n^1 - 2$. Hence, the strategy profile s indeed induces two connected action cliques. The rest of the proof shows that $s \in S^{PNE}$.

We can apply Lemma 6. Note that $s \in \bar{S}^+$ and fix a player i. First, let $\theta_i = 0$. Then $u_i(s) = (\delta+\lambda-c)(a^0-1) + (1-\delta+\lambda-c)\, a^1$ and $u_i(\tilde{s}_i, s_{-i}) = (\delta-c)\, a^1$. It follows that $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $a^1 \leq \frac{\delta+\lambda-c}{3\delta-1-c}\,(n-1)$. Substituting $a^1 = \lceil \frac{2\delta-1-\lambda}{\delta+\lambda-c}\, n^0 \rceil + 2$, we get an equivalent expression: $\lceil \frac{2\delta-1-\lambda}{\delta+\lambda-c}\, n^0 \rceil \leq \frac{\delta+\lambda-c}{3\delta-1-c}\,(n-1) - 2$. This, however, is always true, as $n^0 < \frac{\delta+\lambda-c}{3\delta-1-c}\,(n-1) - 3$ and $\frac{2\delta-1-\lambda}{\delta+\lambda-c} \in (0,1)$. Second, let $\theta_i = x_i = 1$. Then $u_i(s) = (\delta+\lambda-c)(a^1-1) + (1-\delta+\lambda-c)\, n^0$ and $u_i(\tilde{s}_i, s_{-i}) = (\delta-c)\, n^0$. Consequently, $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $a^1 \geq \frac{2\delta-1-\lambda}{\delta+\lambda-c}\, n^0 + 1$, i.e. $\lceil \frac{2\delta-1-\lambda}{\delta+\lambda-c}\, n^0 \rceil + 2 \geq \frac{2\delta-1-\lambda}{\delta+\lambda-c}\, n^0 + 1$, which obviously holds. Finally, let $\theta_i = 1 \neq x_i$. Then $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $(1-\delta+\lambda-c)(a^0-1) \leq (\delta-c)(a^0-1)$, which holds true due to $\delta > \frac{1+\lambda}{2}$.

Now, fix a pair of players i and j s.t. $p_{ij} = 0$. It must be that $x_i \neq x_j$ and, without loss of generality, $x_i \neq \theta_i$ and $x_j = \theta_j$ (if also $x_j \neq \theta_j$, it would contradict $x_i \neq x_j$). It is left to check that condition (2) of Lemma 6 holds for both i and j. For i, $u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) < u_i(s)$ if and only if $(1-\delta+\lambda-c)(a^0-1) + (\delta+\lambda-c) < (\delta-c)(a^0-1)$, which is equivalent to $a^1 < n-1-\frac{\delta+\lambda-c}{2\delta-1-\lambda}$, or $\lceil \frac{2\delta-1-\lambda}{\delta+\lambda-c}\, n^0 \rceil + 2 < n-1-\frac{\delta+\lambda-c}{2\delta-1-\lambda}$. However, our assumption $n^0 < n-4-\frac{\delta+\lambda-c}{2\delta-1-\lambda}$ together with $\frac{2\delta-1-\lambda}{\delta+\lambda-c} \in (0,1)$ makes it hold true. For j, $u_j((x_i, \hat{p}_i), (\hat{x}_j, \hat{p}_j), s_{-i-j}) < u_j(s)$ if and only if $(1-\delta-c)(a^1-1) + (\delta-c)\, n^0 + (\delta-c) < (\delta+\lambda-c)(a^1-1) + (1-\delta+\lambda-c)\, n^0$, or equivalently, $a^1 > \frac{2\delta-1-\lambda}{2\delta-1+\lambda}\, n^0 + \frac{\delta-c}{2\delta-1+\lambda} + 1$, or $\lceil \frac{2\delta-1-\lambda}{\delta+\lambda-c}\, n^0 \rceil + 2 > \frac{2\delta-1-\lambda}{2\delta-1+\lambda}\, n^0 + \frac{\delta-c}{2\delta-1+\lambda} + 1$. Note, however, that $\frac{\delta-c}{2\delta-1+\lambda} \in (0,1)$ and $\frac{2\delta-1-\lambda}{\delta+\lambda-c} > \frac{2\delta-1-\lambda}{2\delta-1+\lambda}$, hence condition (2) holds also for j and, according to Lemma 6, $s \in S^{PNE}$. □

Proof of Proposition 8

Proof. The proof is analogous to the proof of Proposition 5. Note that if any of the conditions (i), (ii) or (iii) holds, then $c \notin C^h_{\delta,\lambda}$, and hence Lemma 6 is applicable here.

(i) Let $\delta < \frac{1-\lambda}{2}$, $c < \delta$ and $n^\pi \geq \frac{1-2\delta-\lambda}{2(1-2\delta)}(n-1)\ \forall \pi \in \{0,1\}$. We will prove that the strategy profile with $x_i = \theta_i\ \forall i \in N$ and $p_{ij} = 1\ \forall i,j \in N$ is a PNE.

   Note that $u_{ij}(x_i, x_j) > 0\ \forall x_i, x_j \in \{0,1\}$, and hence $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1\ \forall i,j \in N\}$. Obviously, $s \in \bar{S}^+$. There are no possible pairwise deviations from s, hence it suffices to show that $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)\ \forall i \in N$ (Lemma 6). Without loss of generality, take a player with $\theta_i = 0$. Then, as in Proposition 5, $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $(1-\delta)(n^0-1) + \delta n^1 \leq (\delta+\lambda)(n^0-1) + (1-\delta+\lambda)n^1$. Rearranging terms and substituting n^0 for $n - n^1$, we get an equivalent inequality: $n^1 \geq \frac{1-2\delta-\lambda}{2(1-2\delta)}(n-1)$. Similarly, for a player with $\theta_i = 1$, $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $n^0 \geq \frac{1-2\delta-\lambda}{2(1-2\delta)}(n-1)$. In either case, such a unilateral deviation is unprofitable for i, and hence $s \in S^{PNE}$.

(ii) Let $\delta < \frac{1-\lambda}{2}$, $\delta < c < \delta + \lambda$ and $n^\pi \geq \frac{1-2\delta-\lambda}{2-3\delta-c}(n-1)$ $\forall \pi \in \{0,1\}$. We will prove that the strategy profile with $x_i = \theta_i$ $\forall i \in N$ and $p_{ij} = 1$ $\forall i, j \in N$ is a PNE.

Note that $u_{ij}(x_i, x_j) > 0$ if and only if either $x_i = \theta_i$ or $x_i \neq x_j$, and hence $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow ((x_i = \theta_i \wedge x_j = \theta_j) \vee x_i \neq x_j)\}$. Then $s \in \bar{S}^+$. According to Lemma 6, it suffices to show that $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ $\forall i \in N$. Take a player with $\theta_i = 0$. Then $u_i(s) = (\delta + \lambda - c)(n^0 - 1) + (1 - \delta + \lambda - c)n^1$, and $u_i((\tilde{x}_i, \tilde{p}_i), s_{-i}) = \sum_{\{j \in N: x_j \neq \tilde{x}_i\}} u_{ij}(\tilde{x}_i, x_j) = (1 - \delta - c)(n^0 - 1)$. Hence, $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $(1-\delta)(n^0-1) \leq (\delta+\lambda)(n^0-1) + (1-\delta+\lambda-c)n^1$, or equivalently, $n^1 \geq \frac{1-2\delta-\lambda}{2-3\delta-c}(n-1)$. Similarly, for a player with $\theta_i = 1$, $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $n^0 \geq \frac{1-2\delta-\lambda}{2-3\delta-c}(n-1)$. In either case, such a unilateral deviation is unprofitable for i, and hence $s \in S^{PNE}$.

(iii) Let $\delta < \frac{1-\lambda}{2}$ and $\delta + \lambda < c < 1 - \delta$. We will prove that the strategy profile with $x_i = \theta_i$ $\forall i \in N$ and $p_{ij} = 1 \Leftrightarrow \theta_i \neq \theta_j$ is a PNE.

Here $u_{ij}(x_i, x_j) > 0$ if and only if $x_i \neq x_j$, and hence $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow x_i \neq x_j\}$. Note that $s \in \bar{S}^+$ and let us check the other conditions of Lemma 6. First, take a player with action preference θ_i and consider the unilateral deviation $\tilde{s}_i$. Then $u_i(s) = (1 - \delta + \lambda - c)n^{1-\theta_i} \geq 0 = u_i(\tilde{s}_i, s_{-i})$, which implies that $\tilde{s}_i$ is unprofitable. Second, take two unlinked players i and j (with $\theta_i = \theta_j$) and consider their pairwise deviation $((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j))$. Since $n^\pi \geq 2$ $\forall \pi \in \{0,1\}$, we can derive: $u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) = (\delta - c)n^{1-\theta_i} + (1 - \delta - c) < 1 - \delta - c < (1 - \delta + \lambda - c)n^{1-\theta_i} = u_i(s)$. Hence, according to Lemma 6, $s \in S^{PNE}$.

□

Proof of Proposition 9

Proof. (i) Let $\delta < c < \delta + \lambda < 1 - \delta$ (the last inequality is equivalent to $\delta < \frac{1-\lambda}{2}$). Then $u_{ij}(x_i, x_j) > 0$ if and only if either $x_i = \theta_i$ or $x_i \neq x_j$. Hence, $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow ((x_i = \theta_i \wedge x_j = \theta_j) \vee x_i \neq x_j)\}$.

Let $n^\pi > \frac{1-\delta+\lambda-c}{1-2\delta-\lambda}$ $\forall \pi \in \{0,1\}$, and consider such a strategy profile s that $x_i \neq \theta_i$ $\forall i \in N$ and $p_{ij} = 1 \Leftrightarrow x_i \neq x_j$ $\forall i, j \in N$. Note that $s \in \bar{S}^+$ and let us apply Lemma 6. First, fix a player i with action preference θ_i. Then $u_i(\tilde{s}_i, s_{-i}) = (\delta + \lambda - c)\, n^{1-\theta_i} < (1 - \delta - c)\, n^{1-\theta_i} = u_i(s)$. Second, fix a pair of players i and j s.t. $p_{ij} = 0$ (i.e. $x_i = x_j$). Then $u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) < u_i(s)$ if and only if $(\delta + \lambda - c)\, n^{1-\theta_i} + (1 - \delta + \lambda - c) < (1 - \delta - c)\, n^{1-\theta_i}$, or equivalently, $n^{1-\theta_i} > \frac{1-\delta+\lambda-c}{1-2\delta-\lambda}$. Similarly, $u_j((x_i, \hat{p}_i), (\hat{x}_j, \hat{p}_j), s_{-i-j}) < u_j(s)$ if and only if $n^{\theta_i} > \frac{1-\delta+\lambda-c}{1-2\delta-\lambda}$. As all conditions of Lemma 6 are satisfied, we conclude that $s \in S^{PNE}$.

(ii) Let $\delta + \lambda < c < 1 - \delta$. Then the strategy profile with $x_i = \theta_i \; \forall i \in N$ and $p_{ij} = 1 \Leftrightarrow x_i \neq x_j$ is a PNE (see the proof of part (iii) of Proposition 8).

$\square$

Proof of Proposition 10

Proof. Let $\delta < c < \delta + \lambda < 1 - \delta$ (the last inequality is equivalent to $\delta < \frac{1-\lambda}{2}$). Then $u_{ij}(x_i, x_j) > 0$ if and only if either $x_i = \theta_i$ or $x_i \neq x_j$, and $\bar{S}^+ = \{s \in \bar{S} \mid p_{ij} = 1 \Leftrightarrow ((x_i = \theta_i \wedge x_j = \theta_j) \vee x_i \neq x_j)\}$. Without loss of generality, let $\frac{1-\delta+\lambda-c}{1-2\delta-\lambda} < \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\, n^0 \leq \min\{\frac{1-\delta+\lambda-c}{2-3\delta-c}\, n - 2, n - n^0 - 2\}$.

Let us introduce some additional notation, $a^\pi := |\{i \in N \mid x_i = \pi\}|$ for $\pi \in \{0,1\}$, and consider such a strategy profile s that $x_i = 0 \; \forall i \in N^0$, $|\{i \in N^1 \mid x_i = 1\}| = \lceil \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\,(n^0 - 1) \rceil + 1$ and $p_{ij} = 1 \Leftrightarrow ((x_i = \theta_i \wedge x_j = \theta_j) \vee x_i \neq x_j)$. Note that here $a^1 = |\{i \in N^1 \mid x_i = 1\}|$. As $a^0 \geq n^0 \geq 2$, there exist $i, j \in N$ s.t. $\theta_i = x_i = x_j = \theta_j = 0$ and $p_{ij} = 1$. Let us show that $a^1 < n^1$, and hence $a^0 > n^0$, implying that there exist also $k, l \in N$ s.t. $0 = \theta_k = x_k = x_l \neq \theta_l = 1$ and $p_{kl} = 0$. Noting that $\frac{1-2\delta-\lambda}{1-\delta+\lambda-c} \in (0,1)$, we can derive $\lceil \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\,(n^0 - 1) \rceil - 1 < \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\,(n^0 - 1) < \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\, n^0 \leq n - n^0 - 2 = n^1 - 2$, which is equivalent to $\lceil \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\,(n^0 - 1) \rceil + 1 < n^1$ and is exactly what we wanted to show. The rest of the proof shows that $s \in S^{PNE}$.

We can apply Lemma 6. Note that $s \in \bar{S}^+$ and fix a player i. First, let $\theta_i = 0$. Then $u_i(s) = (1 - \delta + \lambda - c)\, a^1 + (\delta + \lambda - c)(n^0 - 1)$ and $u_i(\tilde{s}_i, s_{-i}) = (1 - \delta - c)\,(n^0 - 1)$. It follows that $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $a^1 \geq \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\,(n^0 - 1)$, which is always true, as $a^1 = \lceil \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\,(n^0 - 1) \rceil + 1$. Second, let $\theta_i = x_i = 1$. Then $u_i(s) = (1 - \delta + \lambda - c)\, a^0 + (\delta + \lambda - c)(a^1 - 1)$ and $u_i(\tilde{s}_i, s_{-i}) = (1 - \delta - c)(a^1 - 1)$. Consequently, $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $a^1 \leq \frac{1-\delta+\lambda-c}{2-3\delta-c}\, n + \frac{1-2\delta-\lambda}{2-3\delta-c}$, which also holds, as $a^1 = \lceil \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\,(n^0 - 1) \rceil + 1 < \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\,(n^0 - 1) + 2 \leq (\frac{1-\delta+\lambda-c}{2-3\delta-c}\, n - 2) - \frac{1-2\delta-\lambda}{1-\delta+\lambda-c} + 2 < \frac{1-\delta+\lambda-c}{2-3\delta-c}\, n + \frac{1-2\delta-\lambda}{2-3\delta-c}$. Finally, let $\theta_i = 1 \neq x_i$. Then $u_i(\tilde{s}_i, s_{-i}) \leq u_i(s)$ if and only if $(\delta + \lambda - c)\, a^1 \leq (1 - \delta - c)\, a^1$, which holds true due to $\delta < \frac{1-\lambda}{2}$.

Now, fix a pair of players i and j s.t. $p_{ij} = 0$. It must be that $x_i = x_j$ and, without loss of generality, $x_i \neq \theta_i$ (if both $x_i = \theta_i$ and $x_j = \theta_j$, it would contradict $p_{ij} = 0$). It is left to check that condition (2) of Lemma 6 holds for both i and j. For i, $u_i((\hat{x}_i, \hat{p}_i), (x_j, \hat{p}_j), s_{-i-j}) < u_i(s)$ if and only if $(\delta + \lambda - c)a^1 + (1 - \delta + \lambda - c) < (1 - \delta - c)a^1$, which is equivalent to $a^1 > \frac{1-\delta+\lambda-c}{1-2\delta-\lambda}$. It holds true, as $a^1 = \lceil \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\,(n^0 - 1) \rceil + 1 \geq \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\,(n^0 - 1) + 1 > \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\, n^0 > \frac{1-\delta+\lambda-c}{1-2\delta-\lambda}$. If $x_j \neq \theta_j$, then condition (2) for j coincides with the above one for i. If $x_j = \theta_j$, then $u_j((x_i, \hat{p}_i), (\hat{x}_j, \hat{p}_j), s_{-i-j}) < u_j(s)$ if and only if $(\delta - c)\, a^1 + (1 - \delta - c)(n^0 - 1) + (1 - \delta - c) < (1 - \delta + \lambda - c)\, a^1 + (\delta + \lambda - c)(n^0 - 1)$, or equivalently, $a^1 > \frac{1-2\delta-\lambda}{1-2\delta+\lambda}\,(n^0 - 1) + \frac{1-\delta-c}{1-2\delta+\lambda}$. Note, however, that $\frac{1-\delta-c}{1-2\delta+\lambda} \in (0,1)$ and $\frac{1-2\delta-\lambda}{1-\delta+\lambda-c} > \frac{1-2\delta-\lambda}{1-2\delta+\lambda}$ implies $a^1 = \lceil \frac{1-2\delta-\lambda}{1-\delta+\lambda-c}\,(n^0 - 1) \rceil + 1 >$

$\frac{1-2\delta-\lambda}{1-2\delta+\lambda}(n^0-1)+\frac{1-\delta-c}{1-2\delta+\lambda}$, and hence condition (2) holds also for j. According to Lemma 6, $s \in S^{PNE}$. $\square$

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