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Ambiguous Social Choice Functions

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Abstract: Call a mechanism that associates each profile of preferences over candidates to an ambiguous act an Ambiguous Social Choice Function (ASCF). This paper studies the strategy-proofness of ASCFs. We find that an ASCF is unanimous and strategyproof if and only if there exists a nonempty subset of voters, called the set of top voters, such that at each preference profile, the range of the selected act equals the set of top-ranked candidates of top voters. We provide a full characterization of the class of unanimous, strategyproof, and anonymous ASCFs, and provide a large subclass of ASCFs that satisfy the additional property of neutrality.

Keywords: Social Choice Function, Ambiguity Aversion, Ellsberg Urns, Strategy-proofness, Unanimity, Anonymity, Neutrality.

JEL classification: D71, D72, D81, D82

1 Introduction

A major weakness of voting mechanisms is their non-strategy-proofness: by misrepresenting their preferences, a voter can favor the selection of a candidate he or she prefers to the candidate that would otherwise be selected, see Gibbard (1973) and Satterthwaite (1975). Such an operation is generally possible because the voting mechanism is perfectly known and unambiguously associates each declared profile of preferences to the election winner. It

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is natural to wonder if strategyproof voting mechanisms can be constructed if the designer is allowed to leave some of their aspects uncertain. One can for instance easily conceive a voting situation where a menu of deterministic voting mechanisms is announced to the voters, from which one will be (ex-post) chosen, after they cast their ballots, to aggregate the preferences. Ex-post, the selection of the aggregating mechanism (within the given set) can for instance be conditioned on the outcome of a horse race or the realization of an imprecise probabilistic device, such as an Ellsberg urn. Think for instance of a peer-reviewing process, where the reviewers (voters in this case), prior to submitting their recommendations (preferences) for the paper under review, only have partial information about the aggregation process (or the type of the editor in charge): the editor is either pessimistic (aggregates the recommendations of reviewers using anti-plurality rule), neutral (aggregates the recommendations of reviewers using the Borda rule), or optimistic (aggregates the recommendations of reviewers using the plurality rule). In such a scenario, instead of pointing out a given candidate as the election winner, the proposed mechanism selects an act: a function that associates each possible state of nature to an election winner. Call such aggregating mechanism an *Ambiguous Social Choice Function* (ASCF).[2]

Call an ASCF unanimous if it selects the constant act which is equivalent to the top-ranked candidate of all players, whenever such candidate exists. This study provides a full characterization of the set of unanimous and strategyproof ASCFs, and a full characterization of ASCFs that are additionally anonymous. It is assumed that voters are ambiguity averse and aim to maximize their worst-case payoff: the preferences of voters over the set of candidates are extended to preferences over acts using the maxmin (Gilboa and Schmeidler (1989)) with lexicographic maxmin tie break rule (Pattanaik and Peleg (1984)). Our interest in this class of preferences is two-folds. In addition to capturing the aversion to ambiguity, it is complete, and indifference classes are smaller than those obtained under the maxmin extension (Gilboa and Schmeidler (1989)). See Barberà et al. (2004) for an overview of other possible extensions of preference relations and their axiomatizations.

We show that unanimous and strategyproof ASCFs are ex-post Pareto optimal: for each profile of preferences and each state of nature, the selected candidate is always the top-ranked candidate of at least one voter. The main result stipulates that an ASCF is unanimous and strategyproof if and only if it is a top selection: there exists a subgroup N_0 of voters, say the set of top voters, such that the range of the act selected at any profile of preferences

[2]Under an ASCF F, the election proceeds as follows. At time $t = -1$, the planner sets the ASCF F as well as the randomization device (think of an Ellsberg Urn) that will be used to determine the state of nature. At time $t = 0$, voters cast their ballots, and the profile of preferences, say R_N, is known. At time $t = 1$, the ASCF F is used to determine the selected act, say $F(R_N)$. At time $t = 2$, the state of nature, say ω, is drawn and candidate $F(R_N, \omega)$ is elected.

is always the set of top-ranked candidates of voters of the group N_0.[3] Preferences of other voters are taken into account only in the assignment of candidates to states of nature. In this sense, preferences of all voters can be taken into account by a unanimous and strategyproof ambiguous social choice function, even if the group N_0 contains only 2 voters, given that the state space is rich enough. In the case that the set N_0 is restricted to a singleton, the top selection is unique and is a dictatorship: at each profile of preferences, the selected act is constant and corresponds to the top-ranked candidate of the unique voter of the block N_0, the dictator. We show that an ASCF is unanimous, strategyproof, and anonymous if and only if it is a top selection with top voters $N_0 = N$, and each state of nature corresponds to a deterministic anonymous social choice function. We uncover a class of ASCFs that are unanimous, strategyproof, anonymous, and neutral. Each ASCF of the later class is constructed using a different deterministic anonymous and neutral aggregation process.[4]

Closest to our analysis is the recent paper by Bahel and Sprumont (2020). In that paper, the authors analyze unanimous and strategyproof ASCFs in presence of expected utility-maximizing voters. They show that unanimous and strategyproof ASCFs are ex-post Pareto optimal in the sense that for all preference profiles, the outcome of the selected act coincides with the top-ranked candidate of at least one voter in each state of nature. Our analysis confirms that such property remains true if voters have maxmin preferences with lexicographic maxmin tie break rule. Our findings are different from that of Bahel and Sprumont (2020) as the set of top voters (N_0) in our setting is constant, while in their setting the set of top voters might vary with the preference profile.

In this paper, we assume that the extended preferences of voters over the set of acts are of maxmin type. Voters, therefore, rank acts accordingly to their support: two acts with the same support are equivalent. In this sense, our analysis also belongs to the social choice correspondence literature. In particular, for the case of two voters, our main result (Theorem 1) is similar to the one of Barberà et al. (2001), even though voters are ambiguity averse in our model, while preferences of voters over sets of alternatives are conditionally expected utility consistent in their model. In the case that there are more than 2 voters, many other unanimous and strategyproof mechanisms emerge. We provide a full characterization of those mechanisms. In works as Pattanaik (1974), Pattanaik (1973), and Pattanaik (1976), Pattanaik studies the stability of social choice correspondences. In those papers, the extensions of preferences are of maxmin type, as the lexicographic maxmin extension used in our analysis. However, the research questions in those papers are different from that of

[3] The idea of top selection is closely related the omninomination rule defined in the social choice correspondence literature, see Brandt (2015), Example 6 in Benoit (2002) and Example 4 in Gärdenfors (1976).

[4] Think of a deterministic aggregation process as a mechanism that associates each profile of preferences over candidates to a social ranking.

the current paper. The maxmin extension with lexicographic maxmin rule generalises other support-based extensions as the Kelly-extension (Kelly (1977)), the Fishbrun-extension, and the Gardenfors-extension (Gärdenfors (1976)). In particular, our top selections are Kelly-strategyproof, Fishburn-strategyproof, and Gardenfors-strategyproof. But our main result, Theorem 1, does not necessarily hold under other support-based preference extensions. For instance, the Pareto rule, which associates each profile of preferences over candidates to an act whose support equals to the set of Pareto optimal candidates is Kelly-strategyproof.

Our analysis is also related to probabilistic social choice literature. Brandl et al. (2018) studies strategyproof social decision schemes. Those are social choice functions that assign each profile of preferences to a probability distribution over candidates. The authors show that there exists no unanimous and neutral social decision schemes that are efficient are strategyproof, assuming that preferences of voters over candidates are extended to lotteries over candidates in the stochastic dominance sense. Proposition 3 of this paper shows that the finding of Brandl et al. (2018) does not extend to the case where preferences of voters over the set of candidates are strict and are extended to preferences over lotteries in a lexicographic-maxmin fashion.

Gibbard (1977) studies the strategy-proofness of social decision schemes. When the unanimity condition is imposed, their main result says that a social decision scheme is strategyproof if and only if it is a convex combination of dictatorships. Our main result, Theorem 1, can be viewed as a robustness check of the findings of Gibbard (1977) when unanimity is required.[5] In particular, some top selection mechanisms can be viewed as sets of random dictatorships, and could alternatively be called ambiguous dictatorships. It is for instance the case if each state of nature corresponds (see Proposition 1) to a deterministic dictatorship. But in the general case, state space can be richer, and could for instance include (see Proposition 1) any deterministic social choice function that always selects a candidate top-ranked by at least one top voter.

The paper proceeds as follows. Section 2 presents the model and two real-life situations where the decision process can be modeled as an ambiguous social choice function, and Section 3 presents the main findings of the paper. Some properties and examples of ASCF are provided in Section 4, and Section 5 concludes the paper.

[5] Nandeibam (2013) provides another robustness check, assuming that voters have cardinal preferences over outcomes, and rank lotteries with respect to the expected utility.

2 The model

The set of voters is finite and denoted by $N = \{1, \cdots, n\}$. The set of alternatives or candidates or deterministic outcomes is also finite and denoted by $A = \{a_1, \cdots, a_m\}$. Subsets of candidates will be denoted by $\mathcal{A}, \mathcal{B}, \cdots$. We assume that there are at least two voters ($n \geq 2$) and three candidates ($m \geq 3$). Each voter has a strict preference (complete, transitive, and antisymmetric) over the set of candidates. Denote by L the set of strict preferences over A. For all candidates $x, y \in A$, and preference $R_i \in L$, xR_iy means that voter i stricly prefers candidate x to candidate y, or $x = y$. If $x \neq y$ and xR_iy, we simply write xP_iy. For all $i \in N$ and $R_i \in L$, $b(R_i) \in A$ denotes the candidate ranked first by voter i according to R_i. Denote by L_N the set of profiles of preferences. For all candidates $x, y \in A$ denote by $R_N^{x,y}$ the preference profile obtained from R_N by permuting the positions of candidates x and y. For all profile of preferences $R_N = (R_1, \cdots, R_n) \in L_N,\ i \in N$, and $Q_i \in L$, (Q_i, R_{-i}) denotes the profile obtained from R_N by replacing the $i-$th entry R_i by Q_i. For all $l \leq m,\ R_i \in L$ and $\mathcal{A} \subseteq A$, $\min_l(R_i, \mathcal{A})$ denotes, if it exists, the $l-$th worst candidate of $\mathcal{A}$ according to R_i.

A deterministic Social Choice Function (SCF) is represented with a map $f : L_N \to A$. It maps each profile of preferences over candidates R_N to the election winner $f(R_N) \in A$. Denote by $\mathcal{F} = \{f : L_N \to A\}$ the set of all deterministic SCF on the same set of candidates A, and the same set of voters N. We now introduce the concept of Ambiguous Social Choice Function (ASCF). Let Ω be a finite set of states of nature. A function $g : \Omega \to A$ is called *social act*. Denote by $\mathcal{G}$ the set of all social acts on the same set of states Ω and candidates A. For all candidates $x, y \in A$, denote by $g^{x,y}$ the social act obtained from g by permuting the occurrences of candidates x and y in the expression of g. For all social act $g \in \mathcal{G}$, $\mathcal{A}(g) = \{g(\omega) | \omega \in \Omega\}$ denotes the range of g. An ambiguous social choice function is represented with a map $F : L_N \to \mathcal{G}$. It maps each profile of preferences over candidates $R_N \in L_N$ to a social act $F(R_N) \in \mathcal{G}$. When election comes, each voter i casts his/her ballot R_i and the social act $F(R_N)$ is selected. If the state $\omega \in \Omega$ is realized, then the candidate $F(R_N)(\omega)$, simply denoted $F(R_N, \omega)$, is elected. Deterministic social choice functions are particular ASCFs where the set of states Ω is restricted to a single element.

Example 1 ***A real-life example of ambiguous social choice function***
Karin (a researcher) submitted a research article to a peer-reviewed journal. The editor in charge suitably chooses a set of 4 ($N = \{1, 2, 3, 4\}$) reviewers to evaluate the paper. In addition to their report, each reviewer has to make a recommendation. Each can recommend accepting the paper (AP), rejecting the paper (RP), or recommend revising and resubmitting

the paper (R&RS). After uploading their reports and recommendations, the decision is immediate if the reviewers have made the same recommendation. In the other case, the editor is asked if he or she is available to produce an additional report and make a decision (subjectively) within one week. (The editor is not allowed to make a decision that is worse or better for the researcher than each of the recommendations of reviewers: the editor can not reject (RP) or accept (AP) the paper if none of the reviewers recommended it.) If the editor is available, then his/her recommendation will be the decisive one, and Karin receives all the reports. Otherwise, the decision is given by the worst recommendation of the reviewers. In the latter case Karin receives all the negative reports, but a lower number of positive reports. This review process can be modeled as an ambiguous social choice function with voters $N = \{1, 2, 3, 4\}$*, candidates* $A = \{AP, RP, R\&RS\}$*, and state space* $\Omega = \{al, anl, na\}$*, where al means "the editor is available and likes the paper", anl means "the editor is available and does not like the paper", and na means that "the editor is unavailable". Each reviewer (voter) submits a ranking of preferences over the set A, even if the decision process uses only the best candidate (recommendation) of each voter. Given a profile of preferences of reviewers, the final decision is an act, as it depends on the availability of the editor and his/her preferences after he/she reads the paper, both unknown at the moment the reviewers submit their recommendations. Consider for instance a preference profile summarized by* $(R\&RS, R\&RS, AP, RP)$*, in which reviewers 1 and 2 recommend revising and resubmitting, reviewer 3 recommends accepting, and reviewer 4 recommends rejecting the paper. At this profile, the paper is automatically rejected if the editor is not (immediately) available. But if he/she is (immediately) available and likes the paper, then the paper might be accepted. In this example, the role of the editor is not to break the ties, as he/she can recommend a revise and resubmit even if no reviewer recommended it.*

Example 2 *A professorship position in a research center will be open in two years. To fulfill the position optimally (avoiding delay and reducing productivity uncertainty), the research center decided to immediately hire up to three 2-year-postdoctoral researchers and wishes to offer the professorship, after two years, to the most productive hired postdoctoral researcher. The productivity of a postdoctoral researcher will be measured with his number of publications and citations during those two years. This process of filling the professorship position can be modeled as an ambiguous social choice function. In this case, the set A of candidates corresponds to the set of applicants, and the set of states of nature equals the whole set of possible productivity levels of applicants. Obviously, the underlying hiring mechanism does not select a candidate as the election winner, but an act: at the moment the members of the committee cast their ballots, they do not know the true state of the nature.*[6]

[6] Similar hiring process has been used in Mannheim in order to fill a professorship position.

An ASCF is called *unanimous* if for all profile $R_N \in L_N$ such that $b(R_i) = b(R_j)$ for all $i, j \in N$, the social act $F(R_N)$ is constant, and $F(R_N, \omega) = b(R_1)$ for all $\omega \in \Omega$. We now extend the definition of strategy-proofness to ASCFs. At election time, voters neither know the realized state of nature nor the distribution it will be issued from. We assume that voters are ambiguity averse and wish to maximize their worst expected outcomes. If two acts have the same worst outcome, then voters prefer the act with the greatest second-worst outcome, and so on. We have the following definition.

Definition 1 (Extension of preferences) *Let $g, h \in \mathcal{G}, i \in N$, and $R_i \in L$. We say that voter i with preference R_i prefers act g over act h (and we write gR_ih) if $\mathcal{A}(g) \succ_{R_i} \mathcal{A}(h)$ where for all $\mathcal{A}, \mathcal{B} \subseteq A$, we have $\mathcal{A} \succeq_{R_i} \mathcal{B}$ if*

- $\min_1(R_i, \mathcal{A})\ P_i\ \min_1(R_i, \mathcal{B})$ *or*
- $\exists k \in \{1, \cdots, |\mathcal{B}|\} \mid \forall l = 1, \cdots, k-1,\ \min_l(R_i, \mathcal{A}) = \min_l(R_i, \mathcal{B})$ *and* $\min_k(R_i, \mathcal{A})\ P_i\ \min_k(R_i, \mathcal{B})$ *or*
- $\forall l = 1, \cdots, |\mathcal{B}|,\ \min_l(R_i, \mathcal{A}) = \min_l(R_i, \mathcal{B})$ *and* $|\mathcal{A}| \geq |\mathcal{B}|$.

Both the strict part of the preference relation $R_i \in L$ and its extension on the set $\mathcal{G}$ of acts are denoted P_i. The preference relation $\succeq_{R_i}$ is called *lexicographic maxmin extension* (lexmin) of the preference R_i. See Pattanaik and Peleg (1984) for an axiomatization of lexmin preferences.

Example 3 ***lexmin extension of preferences in presence of three candidates.*** *Assume that there are three candidates, x, y and z. For all subset $\mathcal{A} \subseteq A$ of candidates, let $g_{\mathcal{A}}$ be an arbitrary act whose range is $\mathcal{A}$. Let $R = xyz \in L$ be a preference such that x is the top-ranked candidate, y is the second-ranked candidate, and z is the worst candidate. Then the strict part P of the lexicographic maxmin extention of the preference R is such that* $g_{\{x\}}\ P\ g_{\{x,y\}}\ P\ g_{\{y\}}\ P\ g_{\{x,z\}}\ P\ g_{\{x,y,z\}}\ P\ g_{\{y,z\}}\ P\ g_{\{z\}}$.

An ASCF F is called *manipulable* if there exists $R_N \in L_N, i \in N$, and $Q_i \in L$ such that $F(Q_i, R_{-i}) P_i F(R_N)$, and called *strategyproof* if it is not manipulable.
An ASCF F is called *dictatorship* if there exists a voter $i \in N$ such that for all profile $R_N \in L_N,\ F(R_N)$ is the constant act which takes the value $b(R_i)$ in each state. Under the full preference domain condition (any ranking of acts is allowed), only the dictatorships are unanimous and strategyproof, see Gibbard (1973) and Satterthwaite (1975). We introduce a new class of unanimous and strategyproof mechanisms. Let F be an ASCF, and $N_0 \subseteq N$ a subset of voters. The ASCF F is called N_0-*top selection* if for all profile $R_N \in L_N$ of

preferences, the range of the act $F(R_N)$ is exactly $\{b(R_i), i \in N_0\}$. The ASCF F is called *top selection* if there exists a subset $N_1 \subseteq N$ such that F is a N_1-top selection. If the set N_0 is a singleton, then F is a dictatorship, and the unique voter $i \in N_0$ is the dictator.

Remark 1 *1) From the definition of the lexicographic maxmin extension of R_i over the set $\mathcal{G}$ of acts, voter i is indifferent between two acts if and only if they have the same range. Under the classic model of ambiguity aversion (Gilboa and Schmeidler (1989)), indifference classes are larger, and each contains all acts with the same worst outcome (candidate). 2) Pattanaik and Peleg (1984) provides an axiomatization of the lexicographic maxmin extension of a preference relation. Their characterization remains true in our setting, and the reader can check that their Strong Fishburn Monotonicity axiom is equivalent to the following independence axiom. Call two acts g and h independent if their respective ranges $\mathcal{A}(g)$ and $\mathcal{A}(h)$ are disjoint sets. Let $g, g' \in \mathcal{G}$. Then for any $\alpha \in (0,1)$ and $g'' \in \mathcal{G}$ independent of g and g', we have that*

$$g \ P_i \ g' \Leftrightarrow \alpha \cdot g + (1-\alpha) \cdot g'' \ P_i \ \alpha \cdot g' + (1-\alpha) \cdot g''.$$

To prove our main result, we use several properties of the lexicographic maxmin extension of a preference. Those properties are discussed in Appendix A.

3 Strategyproof ambiguous social choice functions

The following result characterises the ASCFs that are unanimous and strategyproof.

Theorem 1 *An Ambiguous Social Choice Function (ASCF) is unanimous and strategyproof if and only if it is a top selection.*

This theorem is a full characterization of the set of strategyproof and unanimous ambiguous social choice functions. It says that any ASCF with such properties has to be a top selection. As illustrated in Section 4 (see Example 5), a top selection does not have to be made of deterministic dictatorships.

Proof of Theorem 1. We proceed by induction. If there are only two voters ($n = 2$), a top selection is equivalent, according to the lexicographic maxmin extension of preferences, to a dictatorship or to a particular social choice correspondence Barberà et al. (2001) called a bi-dictatorship. In this case, Theorem 1 is similar to the characterization result (Theorem 3.3) obtained by Barberà et al. (2001). In Appendix B, we adopt their proof to our setting and obtain that with two voters, an ASCF is unanimous and strategyproof if and only if it is a top selection. Now let $n \geq 3$. Assume that the result is true in every scenario with less than n voters, a finite set of candidates, and with a finite set of states. Let $F : \Omega \to \mathcal{G}$ be a unanimous and strategyproof ASCF where the sets of state and candidates are both finite. We wish to show that F is a top selection.

Let $F_{1,2} : L_{N\setminus\{1\}} \to \mathcal{G}$ be the ASCF with voters $N\setminus\{1\}$ defined by $F_{1,2}(R_2, R_3, \cdots, R_n) = F(R_2, R_2, R_3, \cdots, R_n)$ for all $R_2, \cdots, R_n \in L$. The ASCF $F_{1,2}$ is unanimous and strategyproof. The former obviously follows from the fact that F is unanimous. We now prove that $F_{1,2}$ is strategyproof. Obviously, as F is strategyproof, no voter $j \notin \{1, 2\}$ can profitably manipulate the election. Let $R_2, R_3, \cdots, R_n, Q_2 \in L$. As F is strategyproof, we have that

$$F(R_2, R_2, R_3, \cdots, R_n)R_2F(R_2, Q_2, R_3, \cdots, R_n)R_2F(Q_2, Q_2, R_3, \cdots, R_n).$$

This implies that voter 2 can not manipulate the ASCF $F_{1,2}$. The ASCF $F_{1,2}$ is therefore strategyproof. From the induction hypothesis, $F_{1,2}$ is a top selection. We will show the following: if $F_{1,2}$ is a N_0-top selection where $1, 2 \notin N_0$, then F is N_0-top selection; if $F_{1,2}$ is a $\{2\}-$top selection, then there exists $N_1 \subseteq \{1, 2\}$ such that F is a N_1-top selection; if $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection where $1, 2 \notin N_0 \neq \emptyset$, then there exists a nonempty subset $N_1 \subseteq \{1, 2\}$ such that F is a $N_0 \cup N_1-$top selection.

1) Assume that $F_{1,2}$ is a N_0-top selection where $1, 2 \notin N_0$. We wish to show that F is a N_0-top selection. Let $R_N \in L_N$ be a profile of preferences. As $F_{1,2}$ is a N_0-top selection, acts $F(R_1, R_1, R_{-1,2})$ and $F(R_2, R_2, R_{-1,2})$ have the same range, which is $\{b(R_i), i \in N_0\}$. Furthermore, the strategy-proofness of F implies that

$$F(R_1, R_1, R_{-1,2})\ R_1\ F(R_1, R_2, R_{-1,2})\ R_1\ F(R_2, R_2, R_{-1,2}).$$

As acts $F(R_1, R_1, R_{-1,2})$ and $F(R_2, R_2, R_{-1,2})$ have the same range, any voter is indifferent between them, see Property P-1. Therefore, voter 1 (with preference R_1) is indifferent between acts $F(R_1, R_1, R_{-1,2})$ and $F(R_N)$. We conclude that the range of the act $F(R_N)$ is $\{b(R_i), i \in N_0\}$. That is F is a N_0-top selection.

2) Assume that $F_{1,2}$ is a $\{2\}-$top selection.

For all preference profile $R \in L_{N\setminus\{1,2\}}$ of voters $3, \cdots, n$, define the 2-voter ASCF $F^R_{1,2} : L \times L \to \mathcal{G}$ by $F^R_{1,2}(R_1, R_2) = F(R_1, R_2, R)$ for all $R_1, R_2 \in L$.

The ASCF $F^R_{1,2}$ is unanimous. Let $R_1, R_2 \in L$ such that $b(R_1) = b(R_2)$. As $F_{1,2}$ is a $\{2\}-$top selection, the act $F^R_{1,2}(R_2, R_2) = F(R_2, R_2, R)$ is constant and equal to $b(R_2)$.

This implies that $F(R_1, R_2, R)$ is also the constant act $b(R_2) = b(R_1)$. Voter 1 would otherwise manipulates the elections at the profile (R_1, R_2, R), favoring the selection of his/her best act, the constant act $b(R_1)$. The reader can also check that $F^R_{1,2}$ is strategyproof.

Now fix two profiles of preferences $R, Q \in L_{N\setminus\{1,2\}}$ of voters $3, \cdots, n$. Both ASCFs $F^R_{1,2}$ and $F^Q_{1,2}$ are unanimous and strategyproof. From our induction hypothesis, there exists two subsets N^R and N^Q of $\{1, 2\}$ such that $F^R_{1,2}$ is a N^R-top selection and $F^Q_{1,2}$ is a N^Q-top selection. We show now that $N^R = N^Q$. We proceed by contradiction. Assume that $N^R \neq N^Q$. Then $R \neq Q$. There is no loss to assume that there exists a unique $k \in \{3 \cdots, n\}$ such that $R_k \neq Q_k$. (Think of a sequence of one shot deviations from R to Q.) Let $(R_1, R_2) \in L \times L$ be a profile of preference of voters 1 and 2 such that $b(R_1) \neq b(R_2)$. The reader can check that if voter k with preference R_k strictly prefers $b(R_1)$ to $b(R_2)$, then, under F, he/she can profitably deviate either from (R_2, R_1, R) to (R_2, R_1, Q) or from (R_2, R_1, Q) to (R_2, R_1, R). In the other case, voter k can profitably deviate from (R_1, R_2, R) to (R_1, R_2, Q).

3) It is left to show that if $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection where $1, 2 \notin N_0 \neq \emptyset$, then there exists $N_1 \subseteq \{1, 2\}$ such that F is a $N_0 \cup N_1-$top selection. This final step requires some specific properties of the lexmin extension of preferences, see Proposition 4 in Section A, and few intermediate results, see Appendix C. Lemma 6 concludes the proof.

■

The next result provides a characterization of the set of ambiguous social choice functions that are unanimous, strategyproof, and anonymous. We consider the following definition.

Definition 2 *An ambiguous social choice function $F : L_N \to \mathcal{G}$ is Anonymous if for all preference profiles $R_N, Q_N \in L_N$, and voters $i, j \in N$, we have that*

$$\textit{if } (R_i, R_j) = (Q_j, Q_i) \textit{ and } R_k = Q_k \textit{ for all } k \neq i, j, \textit{ then } F(R_N) = F(Q_N) \tag{3.1}$$

Definition 2 says that an ambiguous social choice function F is anonymous if it treats the voters equally: the act selected at a given preference profile remains the same if the preferences of two arbitrary voters are permuted.

Theorem 2 *Let F be an Ambiguous Social Choice Function (ASCF). If F is unanimous, strategyproof, and anonymous, then F is a top selection with top voters N.*

Theorem 2 provides a necessary condition for an ASCF to be unanimous, strategyproof, and anonymous. The condition says that the ASCF must be a top selection, and each voter must be a top voter. In section 4, we uncover a class of finitely many unanimous strategyproof and anonymous ASCFs, see Example 6, and we provide a complete characterization of ASCFs that have such properties, see Proposition 2.

Proof of Theorem 2. Let F be an ASCF. Assume that F is unanimous, strategyproof, and anonymous. We show that F is a top selection with top voters N. As F is strategyproof and unanimous, it follows from Theorem 1 that F is a top selection. Now assume that there exists a voter $i_0 \in N$ such that i_0 is not a top voter. Let $x, y \in A$ be two different candidates, and let $R_N \in L_N$ be a profile of preferences such that $b(R_{i_0}) = x$ and $b(R_i) = y$ for all $i \neq i_0$: the top-ranked candidate of voter i_0 is x, and the top-ranked candidate of any other voter is y. As all top voters have the same top-ranked candidate according to R_N, the range of the act $F(R_N)$ is y. That is the act $F(R_N)$ is the constant act y. Now let $i_1 \in N$ be a top voter. Denote by $Q_N \in L_N$ the preference profile obtained from R_N by permuting the preferences of voters i_0 and i_1. As i_1 is a top voter, the range of the act selected at the profile Q_N includes candidate x. It follows that the acts $F(R_N)$ and $F(Q_N)$ are not the same. This is a violation of the anonymity of F. We conclude that the set of top voters is N. ■

4 Simple ambiguous social choice functions

This section is devoted to the study of simple ASCFs. Those are special cases of ASCFs where the state space is a subset of the set $\mathcal{F} = \{f : L_N \to A\}$ of deterministic SCFs. Those mechanisms are interesting because they are intuitive, and can be easily implemented. Think for instance of a peer-reviewing process, where the reviewers (voters in this case), before submitting their recommendations for the paper under review, only have partial information about the aggregation process (or the type of the editor in charge). The editor is either pessimistic (aggregates the recommendations of reviewers using anti-plurality rule), neutral (aggregates the recommendations of reviewers using the Borda rule), or optimistic (aggregates the recommendations of reviewers using the plurality rule). The timing is the same as in the general model: the planner announces, prior to the election, a set $\Omega \subseteq \mathcal{F}$ of deterministic voting mechanisms from which one will be selected to aggregate the preferences. After the participants have cast their ballots, an imprecise probabilistic device (think of an Ellsberg urn) is used to determine the deterministic mechanism $f \in \Omega$ that will be used to aggregate the preferences. The election winner is $F(R_N, f) = f(R_N)$. We have the following definition.

Definition 3 *An ambiguous social choice function $F : L_N \to \mathcal{G}$ is called **simple** if $\Omega \subseteq \mathcal{F}$. In this case, if the realized state is $f \in \Omega$, then $F(R_N, f) = f(R_N)$ is the election winner.*

From Definition 3, a simple mechanism is completely defined by the set of states Ω. We will denote such mechanism by F_Ω.

Proposition 1 *Any ambiguous social choice function admits a simple representation.*

Proposition 1 says that any ASCF admits a simple representation. This result follows from the fact that for all ASCF $F : L_N \to \mathcal{G}$ and for all state $\omega \in \Omega$, the map $F_\omega : L_N \to A$ defined by $F_\omega(R_N) = F(R_N, \omega)$ is a deterministic social choice function. A simple representation of $F : L_N \to \mathcal{G}$ is obtained with the new states space $\Omega' = \{F_\omega \mid \omega \in \Omega\}$. We now study some examples of simple mechanisms.

Example 4 *Ambiguous social choice function made of Borda, the plurality, and the anti-plurality voting mechanism.*
Let $\Omega_1 = \{Bord, Pl, Apl\}$ where Bord, Pl, and Apl are respectively the deterministic Borda voting mechanism, the plurality mechanism, and the anti-plurality mechanism, each with the lexicographic tie-break rule (ties are broken according to the lexicographic ranking of candidates). In this case, the election winner will be selected by one of the three previous rules, but none of the voters knows which one at the moment he/she casts his/her ballot. It is well known that all of the three above mechanisms while having some nice properties, are unfortunately not strategyproof. It is therefore interesting to know if the ASCF F_{Ω_1} is strategyproof. The answer is negative. As a simple illustration, consider a three-candidate-three-voter election with candidates x, y, z and voters $1, 2$ and 3. Consider the profile $R_N = (R_1, R_2, R_3)$ where $R_1 = xyz$, $R_2 = yzx$ and $R_3 = zxy$. At the profile R_N, the lexicographic advantage of candidate x makes him the election winner under each deterministic mechanism in Ω_1. That is the range $\{Bord(R_N), Pl(R_N), Apl(R_N)\}$ of the act $F_{\Omega_1}(R_N)$ is reduced to the singleton $\{x\}$, which is the worst act of voter 2, see Example 3. Now consider a strategic preference of voter $Q_2 = zyx$ of voter 2. At the profile (Q_2, R_{-2}), candidate z is elected under the plurality and the Borda mechanisms, while candidate x is elected under the anti-plurality mechanism. That is the range $\{Bord(Q_2, R_{-2}), Pl(Q_2, R_{-2}), Apl(Q_2, R_{-2})\}$ of the act $F_{\Omega_1}(Q_2, R_{-2})$ is $\{x, z\}$. As voter 2 strictly prefers any act with range $\{x, z\}$ to the constant act x, see Example 3, the deviation Q_2 of voter 2 is profitable. This implies that the ASCF F_{Ω_1} is not strategyproof. As the set of voters whose top-ranked candidate belongs to the range of the selected acts differs from the profile R_N to the profile (Q_2, R_{-2}), we also conclude that F_{Ω_1} is not a top selection.

Remark 2 *a) Deterministic SCFs are particular cases of simple ASCFs where the set of state Ω has only one element. In particular, deterministic dictatorships are unanimous and strategyproof simple ASCFs.*
b) If $\Omega_2 = \mathcal{F}$, then the ASCF F_{Ω_2} is strategyproof as the range of the act $F_{\Omega_2}(R_N)$ coincides with the whole set A of candidates for all $R_N \in L_N$. Unfortunately F_{Ω_2} is not unanimous: the selected act still has full range even if the preferences of voters are the same. One can check that a simple ASCF F_Ω is unanimous if and only if each $f \in \Omega$ is unanimous. If the set of states is reduced to the set of deterministic and unanimous SCF, then the resulting ASCF is not strategyproof. To see this, consider an election with candidates x, y, and z. Let R^N be a preference profile in which the best candidate of any voter $i \neq 1$ is candidate x, and the best candidate of voter 1 is y while his/her second-best candidate is x. Such profile is manipulable under F_Ω by voter 1. In fact, the deterministic social choice function f_z that elects a candidate a if he is the top-ranked candidate of each voter and elects z otherwise is unanimous. Therefore the range of the act F_Ω includes candidate z, and voter 1 will find it profitable to strategically rank candidate x first and to favor the selection of the constant act x. On the other hand, F_Ω can be unanimous and strategyproof while no $f \in \Omega$ is strategyproof. Example 5 provides an illustration. It follows that the strategy-proofness of a simple ASCF F_Ω might be lost if we add one element to the state space, or delete one existing one.

Example 5 *In this example, we provide a simple strategyproof ASCF F_{Ω_3} which has the property that no $f \in \Omega_3$ is strategyproof. There are three candidates (x, y, and z) and 2 voters (1 and 2). Now consider the deterministic SCF f_1 and f_2 defined as follows:*

$$f_1(R_1, R_2) = \begin{cases} b(R_1) & \text{if } R_1 \in \{xyz, xzy, yxz\} \\ b(R_2) & \text{if } R_1 \in \{yzx, zxy, zyx\} \end{cases} \qquad \text{and}$$

$$f_2(R_1, R_2) = \begin{cases} b(R_1) & \text{if } R_1 \in \{yzx, zxy, zyx\} \\ b(R_2) & \text{if } R_1 \in \{xyz, xzy, yxz\} \end{cases} .$$

Let $\Omega_3 = \{f_1, f_2\}$. At each state $f \in \Omega_3$, the elected candidate is the best candidate of at least one voter. This implies that F_{Ω_3} is unanimous. Neither f_1 nor f_2 is strategyproof. For instance voter 1 can manipulate f_1 at the profile $R_N = (zxy, yxz)$ by chosing $Q_1 = xyz$; voter 1 can manipulate f_2 at the profile $R_N = (xzy, yzx)$ by choosing $Q_1 = zxy$. Observe that the range of the act $F_{\Omega_3}(R_1, R_2)$ equals $\{b(R_1), b(R_2)\}$ for all profile of preferences $(R_1, R_2) \in L_N$. This means that F_{Ω_3} is a top selection, and is therefore strategyproof.

Example 6 *From Theorem 2, any unanimous, strategyproof and anonymous ASCF has to be a top selection with top voters $N_0 = N$. In this example we present a class of finitely many*

ASCFs that satisfy those three requirements. Denote by $\Omega_1 \subseteq \mathcal{F}$ an arbitrary subset of $\mathcal{F}$ that contains deterministic unanimous and anonymous SCFs. The set Ω_1 could for instance be empty or contain the plurality or the Borda count with lexicographic tie break rule. Now let $\Omega = \{f_1, \cdots, f_n\} \cup \Omega_1$ where for all $k \in \{1, \cdots, n\}$ and profile $R_N \in L_N$,

$$f_k(R_N) = \begin{cases} min_k(R_0, \{b(R_i), i \in N\}) & \textit{if } k < |\{b(R_i), i \in N\}| \\ b(R_0, \{b(R_i), i \in N\}) & \textit{if not,} \end{cases}$$

where $b(R_0, \{b(R_i), i \in N\})$ is the top-ranked candidate of the set $\{b(R_i), i \in N\}$ according to the lexicographic preference R_0. The simple ASCF F_Ω is a top selection and therefore unanimous and strategyproof, see Theorem 1. As each $f \in \Omega$ is anonymous, we conclude that F_Ω is anonymous as well.

Proposition 2 *A simple Ambiguous Social Choice Function F_Ω is unanimous, strategyproof and anonymous if and only if F_Ω is a top selection with top voters N, and each $f \in \Omega$ is anonymous.*

Proof of Proposition 2. Let F_Ω be a simple unanimous, strategyproof and anonymous ASCF. From Theorem 2, F_Ω is a top selection with top voters N. From Definition 2, it follows directly that each F_Ω is anonymous if and only if each $f \in \Omega$ is anonymous. Finally recall that top selections are unanimous and strategyproof. ■

Definition 4 *An ambiguous social choice function $F : L_N \to \mathcal{G}$ is neutral if for all preference profile $R_N \in L_N$, act $g \in \mathcal{G}$, and candidates $x, y \in A$,*

$$\textit{if } F(R_N) = g, \ \textit{then } F(R_N^{x,y}) = g^{x,y} \tag{4.1}$$

Definition 4 says that an ASCF F is neutral if it treats candidates equally: if a profile of preferences $R_N^{x,y}$ is obtained from a given one R_N by permuting the positions of candidates x and y, then the act $F(R_N^{x,y})$ selected under F at the profile $R_N^{x,y}$ is obtained from the act $F(R_N)$ by permuting the positions of x and y.

Let $\mathcal{F}^*$ be the set of simple ASCFs defined as follows. For all neutral and anonymous deterministic aggregation process $f : L_N \to L$, let the family $\Omega(f) = \{f_1, \cdots, f_n\}$ be the familly of deterministic SCFs such that for all $k \in \{1, \cdots, n\}$ and all profile $R_N \in L_N$,

$$f_k(R_N) = \begin{cases} min_k(f(R_N), \{b(R_i), i \in N\}) & \text{if } k < |\{b(R_i), i \in N\}| \\ b(f(R_N), \{b(R_i), i \in N\}) & \text{if not,} \end{cases}$$

where $b(f(R_N), \{b(R_i), i \in N\})$ is the top-ranked candidate of the set $\{b(R_i), i \in N\}$ according to the preference $f(R_N)$. [7] A simple ASCF F belongs to $\mathcal{F}^*$ if and only if there exists a deterministic neutral aggregation process f such that $F = F_{\Omega(f)}$.

Proposition 3 *Any simple ambiguous social choice function $F \in \mathcal{F}^*$ is unanimous, anonymous, neutral and strategyproof.*

Proof of Proposition 3. Let $F_{\Omega(f)} \in \mathcal{F}^*$. By construction, the range of the act $F_{\Omega(f)}(R_N)$ selected at a given profile $R_N \in L_N$ is exactly the set of top ranked candidates $\{b(R_i), i \in N\}$. That is $F_{\Omega(f)}$ is a top selection. From Theorem 1, it follows that F is unanimous and strategyproof. Furthermore, as f is anonymous, each $f_k, k \in \{1, \cdots, n\}$ is anonymous. It follows from Proposition 2 that $F_{\Omega(f)}$ is anonymous. The neutrality of $F_{\Omega(f)}$ follows from that of f. ■

5 Conclusion and discussion

This paper analyzes unanimous and strategyproof ambiguous social choice functions. The main result, Theorem 1, tells that an ambiguous social choice function is unanimous and strategyproof if and only if it is a top selection. That is, there exists a subset N_0 of voters, called the set of top voters, such that at each preference profile, the range of the selected act coincides with the set of top-ranked candidates of top voters. We show that an ambiguous social choice function is unanimous, strategyproof, and anonymous if and only if it is a top selection with top voters $N_0 = N$, and each state of nature corresponds to a deterministic anonymous social choice function. See Theorem 2 and Proposition 2. We also uncover a large class of ambiguous social choice functions that additionally satisfy the neutrality, see Proposition 3.

In the analysis of this paper, we assume that all voters are ambiguity averse. In the case that Subjective Expected Utility (SEU) maximizers and ambiguity-averse voters co-exist, new unanimous and strategyproof mechanisms arise. As an illustration, consider an election with set of candidates $A = \{a, b, c\}$, and voters $N = \{1, 2, 3\}$. Assuming that voters 1 and 2 are ambiguity averse and voter 3 is a SEU maximizer, the ambiguous social choice function F that selects the most preferred act $F(R_N)$ of voter 3 within the set of acts with range $\{b(R_1), b(R_2)\}$ is unanimous and strategyproof. A full characterization of the set of

[7] Given a profile of preferences $R_N \in L_N$, the ranking $f(R_N)$ could for instance be obtained using the Borda rule where ties are broken according to the preference R_1 of voter 1: candidates are ranked according to their Borda score, and if two or more candidates have the same score, they are ranked according to the preferences of voter 1. Such aggregation process is obviously anonymous and neutral.

unanimous and strategyproof ambiguous social choice functions in such cases remains to be found. As argued in the introduction, Bahel and Sprumont (2020) together with our findings show that in presence of SEU maximizers or ambiguity averse voters, any unanimous and strategyproof ambiguous social choice function is ex-post Pareto optimal: for all profile of preferences, the outcome of the selected act coincides with the best candidate of at least one voter in each state of nature. One might wonder if such property holds for other classes of preferences as the $\alpha-$maxmin (Ghirardato et al. (2004)), or the smooth preference (Klibanoff et al. (2005)).

The idea of ASCFs easily extends to Arrowian aggregation setup, see Arrow (2012). The natural extension of the *Pareto efficiency* and the *Independence of Irrelevant Alternatives* (IIA) properties to *Ambiguous Aggregation Procedure* (AAP) is such that an AAP F that transforms any arbitrary profile of preferences $R_N = (R_1, \cdots, R_n)$ into an act $F(R_N)$ whose outcomes $F_\omega(R_N)$ in each state $\omega \in \Omega$ of nature is a ranking over the set of candidates, satisfies Pareto efficiency and the IIA properties if and only if each single aggregation process $R_N \mapsto F_\omega(R_N)$ satisfies them, and additionally the *unrestricted domain* property. In the latter case, each aggregation process F_ω must be a dictatorship in the sense that there exists a player $i(\omega)$ such that for all profile of preferences R_N, the outcome $F_\omega(R_N)$ of the act $F(R_N)$ in the state ω always coincides with the preference R_i of voter $i(w)$, see Arrow (2012). This means that an AAP F satisfies the Pareto efficiency, and the IIA properties if and only if there exists a subset N_0 of voters, which we refer to as top voters, such that for all profile of preferences R_N, the range of the act R_N is exactly the set of preferences of top voters.

Our method can be used to analyse the group-strategy-proofness of ASCFs.[8] In fact, our characterized mechanisms, the top selections, give no room for compromises. Voters, therefore, need to vote strategically if they have conflicting preferences, and wish to favor the selection of a compromise act. In particular, if there are at least two top voters (the set N_0 has more than one element), the coalition N_0 can jointly manipulate the election. A simple illustration is an election with three candidates (a, b, and c), where top voters have the same second-ranked candidate, candidate b, but do not have the same top-ranked candidate. In this case, sincere voting under the given top selection mechanism leads to the selection of an act with range $\{a, c\}$, while a strategic voting, for instance favoring the selection of the constant act b, might be profitable for each top voter. In the case that there

[8] See Barbera (1979), Green and Laffont (1979), and Bennett and Conn (1977) for examples of study of group-strategyproof of aggregation mechanisms.

is only one top voter, the top selection is a dictatorship and is not vulnerable to coalitional deviations. That is dictatorships are the only unanimous and group strategyproof ASCFs. Sufficient conditions provided by Barberà et al. (2010), which guarantee the equivalence between group strategy-proofness and individual strategy-proofness, therefore do not hold in our setting.

References

Kenneth J Arrow. Social choice and individual values. Yale University Press, 2012.

Eric Bahel and Yves Sprumont. Strategyproof choice of social acts. American Economic Review, 110(2): 596–627, 2020.

Salvador Barbera. A note on group strategy-proof decision schemes. Econometrica, pages 637–640, 1979.

Salvador Barberà, Bhaskar Dutta, and Arunava Sen. Strategy-proof social choice correspondences. Journal of Economic Theory, 101(2): 374–394, 2001.

Salvador Barberà, Walter Bossert, and Prasanta K Pattanaik. Ranking sets of objects. In Handbook of utility theory, pages 893–977. Springer, 2004.

Salvador Barberà, Dolors Berga, and Bernardo Moreno. Individual versus group strategy-proofness: When do they coincide? Journal of Economic Theory, 145(5): 1648–1674, 2010.

Elaine Bennett and David Conn. The group incentive properties of mechanisms for the provision of public goods. Public Choice, 29(2): 95–102, 1977.

Jean-Pierre Benoit. Strategic manipulation in voting games when lotteries and ties are permitted. Journal of Economic Theory, 102(2): 421–436, 2002.

Florian Brandl, Felix Brandt, Manuel Eberl, and Christian Geist. Proving the incompatibility of efficiency and strategyproofness via smt solving. Journal of the ACM (JACM), 65(2): 1–28, 2018.

Felix Brandt. Set-monotonicity implies kelly-strategyproofness. Social Choice and Welfare, 45(4): 793–804, 2015.

Peter Gärdenfors. Manipulation of social choice functions. Journal of Economic Theory, 13 (2): 217–228, 1976.

Paolo Ghirardato, Fabio Maccheroni, and Massimo Marinacci. Differentiating ambiguity and ambiguity attitude. Journal of Economic Theory, 118(2): 133–173, 2004.

Allan Gibbard. Manipulation of voting schemes: a general result. Econometrica, pages 587–601, 1973.

Allan Gibbard. Manipulation of schemes that mix voting with chance. Econometrica, pages 665–681, 1977.

Itzhak Gilboa and David Schmeidler. Maxmin expected utility with a non-unique prior. Journal of Mathematical Economics, 18: 141–153, 1989.

Jerry Green and Jean-Jacques Laffont. On coalition incentive compatibility. The Review of Economic Studies, 46(2): 243–254, 1979.

Jerry S Kelly. Strategy-proofness and social choice functions without singlevaluedness. Econometrica, pages 439–446, 1977.

Peter Klibanoff, Massimo Marinacci, and Sujoy Mukerji. A smooth model of decision making under ambiguity. Econometrica, 73(6): 1849–1892, 2005.

Shasikanta Nandeibam. The structure of decision schemes with cardinal preferences. Review of Economic Design, 17(3):205–238, 2013.

Prasanta K Pattanaik. On the stability of sincere voting situations. Journal of Economic Theory, 6(6):558–574, 1973.

Prasanta K Pattanaik. Stability of sincere voting under some classes of non-binary group decision procedures. Journal of Economic Theory, 8(2):206–224, 1974.

Prasanta K Pattanaik. Collective rationality and strategy-proofness of group decision rules. Theory and Decision, 7(3):191, 1976.

Prasanta K Pattanaik and Bezalel Peleg. An axiomatic characterization of the lexicographic maximin extension of an ordering over a set to the power set. Social Choice and Welfare, 1(2): 113–122, 1984.

Mark Allen Satterthwaite. Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. Journal of economic theory, 10(2): 187–217, 1975.

Appendix

A Properties of the lexmin extension of preferences

In this section, we discuss some properties of the lexmin extension of preferences. We will use them later to prove Theorem 1.

Proposition 4 *The lexicographic maxmin extension of a preference relation $R_i \in L$ over the set $\mathcal{G}$ of acts satisfies the following properties.*

P-1) Voter i (with preference R_i) is indifferent between two acts f and g if and only those acts have the same range.

P-2) If the least preferred candidate $b_m(R_i)$ of voter i (with preference R_i) does not belong to the range of the act f, then voter i strictly prefers the act f to any act other act g whose range includes the candidate $b_m(R_i)$.

P-3) Let $\mathcal{A} \subseteq A$, $x, y \in A$ such that $x, y \notin \mathcal{A}$. Then voter i (with preference R_i) strictly prefers an act with range $\mathcal{A} \cup \{x\}$ to an act with range $\mathcal{A} \cup \{y\}$ if and only if he/she strictly prefers candidate x over candidate y.

P-4) If voter i (with preference R_i) strictly prefers an act f to a constant act x, then voter i weakly prefers any candidate y that belongs to the range of the act f to the candidate x.

P-5) If $x \neq b(R_i)$, then voter i (with preference R_i) weakly prefers any act f with range $\{x, b(R_i)\}$ to any other act g whose range contains x. Furthermore, if the range of the act g contains more than 2 candidates, then voter i strictly prefers the act f to the act g.

P-6) Voter i (with preference R_i) weakly prefers any act with range $\mathcal{A} \cup \{b(R_i)\}$ to any other act whose range contains $\mathcal{A}$. The preference is strict if $b(R_i) \notin \mathcal{A}$.

P-7) Let $\mathcal{A} \subseteq A$, $x, y \in A$ such that $x \neq y$. Let $R_i \in L$ such that $b(R_i) = x$ and $b_2(R_i) = y$. Then an act f whose range includes $\mathcal{A}$ is weakly preferred to a given act with range $\mathcal{A} \cup \{y\}$ if and only if the range of the act f is either $\mathcal{A} \cup \{y\}$ or $\mathcal{A} \cup \{x\}$ or $\mathcal{A} \cup \{x, y\}$.

P-8) Let $\mathcal{A} \subseteq A$, $\mathcal{B} \subseteq \mathcal{A}$ and $x \in A$. If voter i (with preference R_i) strictly prefers any candidate $y \in \mathcal{A}$ to candidate x, then he/she strictly prefers any act with range $\mathcal{B}$ to any act with range $\mathcal{A} \cup \{x\}$.

Properties stated in Proposition 4 follow directly from the definition of the lexicographic maxmin extension of a preference relation. We therefore omit the proof.

B Proof of the main result: case with 2 voters

When there are only two voters, top selections are equivalent, according to the lexmin extension, to the class of dictatorships or bi-dictatorships social choice correspondences. In this case, Theorem 1 is similar to the characterization result (Theorem 3.3) obtained by Barberà et al. (2001). In their paper, the authors provide a characterization of unanimous and strategyproof social choice correspondences, assuming that preferences of voters over sets of alternatives are conditionally expected utility consistent. In this section, we adapt their method to the setting of this paper and obtain a proof of Theorem 1 for the case $n = 2$. In the following, an element $x \in A$ will also refer to the constant act that has range $\{x\}$, and the set $\mathcal{G} \cap A$ will denote the set of all constant acts. We will need the following definition.

Definition 5 *Let $R_N \in L_N$ be a profile of preferences, and let $g \in \mathcal{G}$ be an act. We say that the act g is achievable by voter i given R_N (or simply R_{-i}) if there exists a preference $Q_i \in L$ of voter i such that $F(Q_i, R_{-i}) = g$.*

Lemma 1 *Suppose that there are two voters (1 and 2). Let F be a unanimous and strategyproof ASCF. Then the set $\mathcal{O}_2(R_1) = \{g \in \mathcal{G} \cap A \mid \exists R_2 \in L \text{ such that } F(R_1, R_2) = g\}$ of constant acts achievable by voter 2 given R_1 depends only on the best candidate $b(R_1)$ of voter 1.*

From the unanimity condition, the set of constant acts $\mathcal{O}_2(R_1)$ achievable by voter 2 given the preference R_1 of voter 1 is always nonempty, as it contains the constant act which is equal to the candidate ranked first by voter 1. Lemma 1 says that $\mathcal{O}_2(R_1)$ depends only on the candidate $b(R_1)$ ranked first by voter 1.

Proof of Lemma 1. Let $R_1, R_1' \in L$ such that $b(R_1) = b(R_1')$. Let $a_j = b(R_1)$. Assume that there exists a constant act a_k such that $a_k \in \mathcal{O}_2(R_1) \setminus \mathcal{O}_2(R_1')$. Let $R_2 \in L$ be a preference of voter 2 such that $b(R_2) = a_k$ and $b_2(R_2) = a_j$. Notice that the lexicographic maxmin extension of the preferences relations R_1, R_1', R_2 to the set $\mathcal{G}$ satisfy

- $a_j \; P_1 \; g \; P_1 \; a_k$ for all acts $g \in \mathcal{G}$ such that the range of g is $\{a_j, a_k\}$;
- $a_j \; P_1' \; g \; P_1' \; a_k$ for all acts $g \in \mathcal{G}$ such that the range of g is $\{a_j, a_k\}$.

- $a_k \ P_2 \ g \ P_2 \ a_j \ P_2 \ h$ for all acts $g, h \in \mathcal{G}$ such that the range of g is $\{a_j, a_k\}$ and the range of h is neither $\{a_j\}$ nor $\{a_k\}$ nor $\{a_j, a_k\}$.

The constant act a_k has to be selected by F at the profile (R_1, R_2). If not voter 2 will profitably deviate from (R_1, R_2), as $a_k \in \mathcal{O}_2(R_1)$. As F satisfies the unanimity condition, we have $a_j \in \mathcal{O}_2(R'_1)$. Furthermore, as F is strategyproof, the range of the act $F(R'_1, R_2)$ must be either $\{a_j, a_k\}$ or $\{a_j\}$. (Recall that $a_k \notin \mathcal{O}_2(R'_1)$.) In both cases, voter 1 manipulates at (R_1, R_2) by choosing R'_1. We conclude that such a_k can not exist. That is $\mathcal{O}_2(R_1) = \mathcal{O}_2(R'_1)$. ■

Lemma 2 *Suppose that there are two voters (1 and 2). Let F be a unanimous and strategyproof ASCF, and $R_1 \in L$. Then $|\mathcal{O}_2(R_1)| \in \{1, m\}$.*

This lemma says that given any preference $R_1 \in L$ of voter 1, either all constant acts are achievable by voter 2, or only one constant act is achievable by voter 2. Notice that the constant act $m(R_1)$ is always achievable by voter 2 at R_1. This follows from the unanimity condition.

Proof of Lemma 2. We proceed by contradiction. Let $R_1 \in L$ such that $1 < |\mathcal{O}_2(R_1)| < m$. Let $a_j = b(R_1)$ and $a_k, a_l \in A \backslash \{a_j\}$ such that $a_k \in \mathcal{O}_2(R_1)$ and $a_l \notin \mathcal{O}_2(R_1)$. If $a_k P_1 a_l$, then exchange the position of a_k and a_l in R_1. This operation will not change the fact that $a_k \in \mathcal{O}_2(R_1)$ and $a_l \notin \mathcal{O}_2(R_1)$ as from Lemma 1, $\mathcal{O}_2(R_1)$ depends only on $b(R_1)$. Let $R_2 \in L$ such that $b(R_2) = a_l$ and $b_2(R_2) = a_k$. The lexicographic maxmin extension of the preferences relations R_1, R_2 to the set $\mathcal{G}$ satisfy

- $a_l \ P_1 \ g \ P_1 \ a_k$ for all acts $g \in \mathcal{G}$ such that the range of g is $\{a_k, a_l\}$.
- $a_l \ P_2 \ g \ P_2 \ a_k \ P_2 \ h$ for all acts $g, h \in \mathcal{G}$ such that the range of g is $\{a_l, a_k\}$ and the range of h is neither $\{a_l\}$ nor $\{a_k\}$ nor $\{a_l, a_k\}$

As $a_k \in \mathcal{O}_2(R_1)$, the strategy-proofness of F implies that the range of $F(R_1, R_2)$ must be either $\{a_l, a_k\}$ or $\{a_k\}$. (Recall that $a_l \notin \mathcal{O}_2(R_1)$.) In both cases, voter 1 can manipulate the elections at (R_1, R_2) by choosing any preference $Q_1 \in L$ such that $b(Q_1) = a_l$. ■

Lemma 3 *Suppose that there are two voters (1 and 2). Let F be a unanimous and strategyproof ASCF. If there exists $R_1 \in L$ such that $|\mathcal{O}_2(R_1)| = m$, then $|\mathcal{O}_2(Q_1)| = m$ for all $Q_1 \in L$.*

This lemma says that the number of constant acts achievable by voter 2 is the same for all preferences of voter 1. Either all constant acts are achievable by voter 2 at any given preference of voter 1, or only one constant act is achievable by voter 2 at any given preference of voter 1.

Proof of Lemma 3. Assume that there exists $R_1, R_1' \in L$ such that $|\mathcal{O}_2(R_1')| = 1$ and $|\mathcal{O}_2(R_1)| = m$. Then $\mathcal{O}_2(R_1') = \{b(R_1')\}$. Let $a_j = b(R_1)$ and $a_k = b(R_1')$. From Lemma 1, we have that $a_j \neq a_k$. Let $a_l \in A\backslash\{a_j, a_k\}$. If $a_l P_1 a_k$, then exchange the positions of a_l and a_k in the ranking R_1. This operation does not change the number of elements of $\mathcal{O}_2(R_1)$, see Lemma 1. Now let $R_2 \in L$ such that $b(R_2) = a_l$ and $b_2(R_2) = a_k$. The lexicographic maxmin extension of the preferences relations R_1, R_2 to the set $\mathcal{G}$ satisfy

- $a_k\ P_1\ g\ P_1\ a_l$ for all acts $g \in \mathcal{G}$ such that the range of g is $\{a_k, a_l\}$.
- $a_l\ P_2\ g\ P_2\ a_k\ P_2\ h$ for all acts $g, h \in \mathcal{G}$ such that the range of g is $\{a_l, a_k\}$ and the range of h is neither $\{a_l\}$ nor $\{a_k\}$ nor $\{a_l, a_k\}$.

As $|\mathcal{O}_2(R_1)| = m$, we have that the best act of voter 2 (with preference R_2), a_l, is achievable by voter 2 at (R_1, R_2). Therefore $F(R_1, R_2) = a_l$. Furthermore, as the constant act $b_2(R_2) = a_k$ belongs to $\mathcal{O}_2(R_1')$, the range of the act selected at the profile (R_1', R_2) must be either $\{a_k, a_l\}$ or $\{a_k\}$. Recall that $a_l \notin \mathcal{O}_2(R_1')$. In each case, voter 1 can manipulate at (R_1, R_2) by choosing R_1'. This contradicts the fact that F is strategyproof. ■

Lemma 4 *Suppose that there are two voters (1 and 2). Let F be a unanimous and strategyproof ASCF. If there exists a preference profile $R_N \in L_N$ such that $b(R_1) \neq b(R_2)$ and that the constant act $b(R_1)$ is selected at the profile R_N, then voter 1 is a dictator.*

Proof of Lemma 4. Assume that $b(R_1) \neq b(R_2)$ and that the constant act $b(R_1)$ is selected at the profile R_N. Let $Q_N \in L_N$. We show that the constant act $b(Q_1)$ is selected at the profile Q_N. As the constant act $b(R_1)$ is selected at the profile R_N, both constant acts $b(R_1)$ and $b(R_2)$ are achievable by voter 1 if voter 2 chooses R_2. From Lemma 2, all constant acts are achievable by voter 1 given that voter 2 chooses R_2. And from Lemma 3, all constant acts are achievable by voter 1 given that voter 2 chooses Q_2. The strategy-proofness of F implies that the best act of voter 1, $b(Q_1)$, is selected at Q_N. We conclude that voter 1 is a dictator. That is F is a $\{1\}-$top selection. ■

Lemma 5 *Suppose that there are two voters (1 and 2). Let F be a unanimous and strategyproof ASCF, and $R_N = (R_1, R_2) \in L_N$ be a profile of preferences. Then the range of the act selected at R_N is included in $\{b(R_1), b(R_2)\}$.*

Proof of Lemma 5. Without loss of generality, we assume in this proof that F is not a dictatorship. Let $R_N \in L_N$. We wish to show that the range of the act $F(R_N)$ is included in the set $\{b(R_1), b(R_2)\}$. As F is strategyproof, we have that $F(R_1, R_1)\ R_1\ F(R_1, R_2)\ R_1 F(R_2, R_2)$. Therefore, if $b(R_1) = b(R_2)$, then the act $F(R_N)$ is constant, and equals $b(R_1)$. This follows from Unanimity. In that case, the range of the act $F(R_N)$ is included in the set $\{b(R_1), b(R_2)\}$. From now we assume that $b(R_1) \neq b(R_2)$.

- Let Q_N be another arbitrary profile such that $b(Q_1) \neq b(Q_2)$. If $F(Q_N) = b(Q_1)$ or $F(Q_N) = b(Q_2)$, then F is a dictatorship, see Lemma 4. This is a contradiction. Assume that there exists a candidate $x \in A$ such that $F(Q_N) = x$ with $x \notin \{b(Q_1), b(Q_2)\}$. Then from Lemma 2, all constant acts are achievable by voter 1 at the profile Q_N. The strategy-proofness of F therefore implies that $F(Q_N) = b(Q_1)$. This contradicts the fact that $x \notin \{b(Q_1), b(Q_2)\}$. From now on, we assume that

(H 1) whenever the two voters do not have the same top-ranked candidate, the range of the selected act has at least two elements.

- Let $x, y \in A$ such that $x \neq b(R_2)$ and $y \neq b(R_1)$. Consider two preferences $Q_1, Q_2 \in L$ such that

$$\begin{cases} b(Q_1) = x \text{ and } b_2(Q_1) = b(R_2); \\ b(Q_2) = y \text{ and } b_2(Q_2) = b(R_1). \end{cases}$$

  As $b_1(R_2) = b_2(Q_1) \in \mathcal{O}_1(R_2)$, the range of the act $F(Q_1, R_2)$ must be either $\{x\}$ or $\{b(R_2)\}$ or $\{x, b(R_2)\}$. From (H 1), the latter range is neither $\{x\}$ nor $\{b(R_2)\}$. Therefore, the range of the act $F(Q_1, R_2)$ is $\{x, b(R_2)\}$. Similarly, the range of the act $F(R_1, Q_2)$ is $\{b(R_1), y\}$. This implies the following.

(H 2) For all candidate $x \in A$, at least one act with range $\{b(R_1), x\}$ is achievable by voter 2 at (R_1, R_2), and at least one act with range $\{b(R_2), x\}$ is achievable by voter 1 at (R_1, R_2).

- If the range of the act $F(R_N)$ is $\{b(R_1), x\}$ with $x \neq b(R_2)$, then voter 2 will manipulate the election, as at least one act with the range $\{b(R_1), b(R_2)\}$ is achievable by voter 2 at R_N, and any act with range $\{b(R_1), b(R_2)\}$ is strictly preferred by voter 2 (with preference R_2) to any act with range $\{b(R_1), x\}$. Similarly, the range of the act $F(R_N)$ can not be $\{x, b(R_2)\}$ with $x \neq b(R_2)$.

- Now we show that the range of the act $F(R_N)$ can not be $\{x, y\}$ with $x, y \in A \backslash \{b(R_1), b(R_2)\}$. We proceed by contradiction. Suppose that the range of the act $F(R_N)$ is $\{x, y\}$ with $x, y \in A \backslash \{b(R_1), b(R_2)\}$. Without loss of generality, suppose that $x\ P_1\ y$. Consider the preference $Q_2 \in L$ such that $b(Q_2) = x$ and $b_2(Q_2) = y$. Then the range of the act $F(R_1, Q_2)$ is $\{x, y\}$. In fact, if the constant act x is achievable by voter 2 at (R_1, Q_2), then all constant acts would be achievable to voter 2 at R_N, see Lemma 2, and therefore the act $F(R_N)$ would be the constant act $b(R_2)$. This contradicts the fact that the range of the act $F(R_N)$ is $\{x, y\}$. As at least one act with range $\{x, y\}$ is achievable by voter 2 at R_N, the range of the act $F(R_1, Q_2)$ has to be $\{x, y\}$. But then voter 1 can profitably deviate (manipulate) at (R_1, Q_2), by ranking candidate x first. We therefore have the following.

(H 3) The range of the act $F(R_N)$ can not be $\{x, y\}$ with $x, y \in A \backslash \{b(R_1), b(R_2)\}$.

- Let us show that the range $\mathcal{A}$ of the act $F(R_N)$ can not have more than two elements. We proceed by contradiction. Without loss of generality, we suppose that $|\mathcal{A}|$ is minimal. This means that there exists no profile $Q_N \in L_N$ such that the range of the act $F(Q_N)$ has more than two elements, but fewer elements than the set $\mathcal{A}$.

  Observe that $b(R_1) \notin \mathcal{A}$. In fact, as at least one act with range $\{b(R_1), b(R_2)\}$ is achievable by voter 2 at R_N, see (H 2), voter 2 will profitably (see Property P-5) deviate from R_N favoring the selection of an act with range $\{b(R_1), b(R_2)\}$, if $b(R_1) \in \mathcal{A}$. This contradicts the fact that F is strategyproof.

  Write $\mathcal{A} = \{x_1, \cdots, x_k\}$ where $x_1 \ P_1 \ x_2 \cdots \ x_{k-1} \ P_1 \ x_k$ and let $Q_2 \in L$ be such that $b_l(Q_2) = x_l$ for all $l = 1, \cdots, k$ and $b_m(Q_2) = b(R_1)$. The range of the act $F(R_1, Q_2)$ must be $\mathcal{A}$. The latter holds for the following reasons:

  - The only constant act achievable by voter 2 at the profile (R_1, Q_2) is the constant act $b(R_1)$, which can not be selected at (R_1, Q_2) as at least one act with range $\mathcal{A}$ is achievable by voter 2, and such act is strictly preferred to $b(R_1)$ according to Q_2, see Property P-2.
  - As seen above, see (H 2) and (H 3), an act whose range has two elements is achievable by voter 2 at the profile (R_1, Q_2) if and only if the later range is of the type $\{b(R_1), x\}$ with $x \in A$. Such act can not be selected at (R_1, Q_2) since $\mathcal{A}$ is strictly preferred to each $\{b(R_1), x\}$, see Property P-2.
  - Obviously, any act with range $\mathcal{A}$ is strictly preferred by voter 2 (according to Q_2) to any other act whose range $\mathcal{B} \neq \mathcal{A}$ has at least as many elements as $\mathcal{A}$. Therefore, the range of the act $F(R_1, Q_2)$ must be $\mathcal{A}$.

  But now voter 1 can manipulate the elections by strategically ranking candidate x_1 first. This is a contradiction.

- The only possibility left is that the range of the act $F(R_N)$ is $\{b(R_1), b(R_1)\}$.

■

Proof of Theorem 1 with 2 voters. The reader can check that top selections are unanimous and strategyproof. Now let F be a unanimous and strategyproof ASCF. From Lemma 5, the range of the act selected at an arbitrary profile R_N has to be included in $\{b(R_1), b(R_2)\}$. Let $R_N \in L_N$ such that $b(R_1) \neq b(R_2)$. We distinguish 3 cases. Case 1: if the act $F(R_N)$ is constant and equals $b(R_1)$, then voter 1 is a dictator, see Lemma 4. Case 2: if the act $F(R_N)$ is constant and equals $b(R_2)$, then voter 2 is a dictator, see Lemma 4. Case 3: if the range of the act $F(R_N)$ is exactly $\{b(R_1), b(R_2)\}$, then F is a $N-$top selection. If not, there exists a profile $Q_N \in L_N$ with $b(Q_1) \neq b(Q_2)$, such that the act $F(Q_N)$ is constant, see Lemma 5. Once more Lemma 4 implies that F is a dictatorship. This contradicts the fact that the range of the act $F(R_N)$ is $\{b(R_1), b(R_2)\}$. This concludes the proof of Theorem 1 when there are only 2 voters. ■

C Intermediate results with many voters

In this section, we assume that there are at least 3 voters. The set N_0 denotes a nonempty subset of $\{3, \cdots, n\}$, and $F_{1,2}$ is the ASCF defined on the set of voters $\{2, \cdots, n\}$ by $F_{1,2}(R_2, R_3, \cdots, R_n) = F(R_2, R_2, R_3, \cdots, R_n)$ for all $R_2, \cdots, R_n \in L$. Our aim is to prove the following result.

Lemma 6 *Let F be a unanimous and strategyproof ASCF, such that $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection. Then there exists $N_1 \subseteq \{1, 2\}$ such that F is a $N_0 \cup N_1-$top selection.*

We split the proof of Lemma 6 in small steps. In particular, Lemma 8 shows that the range of the act $F(R_N)$ always includes the set $\{b(R_i), i \in N_0\}$ of candidates top-ranked by voters of the block N_0. Lemma 12 shows that the range of the act $F(R^N)$ is included in the set $\{b(R_i), i \in N_0 \cup \{1, 2\}\}$. The existence and unicity of N_1 are obtained from Lemma 11 and Lemma 14.

Lemma 7 *Let F be a unanimous and strategyproof ASCF, such that $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection. Let $R_N = (R_1, \cdots, R_n) \in L_N$ be a profile of preferences such that $b(R_1) = b(R_2)$. Then acts $F(R_N)$, $F(R_1, R_1, R_3, \cdots, R_n)$ and $F(R_2, R_2, R_3, \cdots, R_n)$ have the same range, which is $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$.*

Proof of Lemma 7. As F is strategyproof, we have that

$$F(R_1, R_1, R_3, \cdots, R_n) \ R_1 \ F(R_1, R_2, R_3, \cdots, R_n) \ R_1 \ F(R_2, R_2, R_3, \cdots, R_n).$$

As $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection and $b(R_1) = b(R_2)$, acts $F(R_1, R_1, R_3, \cdots, R_n)$ and $F(R_2, R_2, R_3, \cdots, R_n)$ have the same range. From Property P-1, voter 1 is indifferent between acts $F(R_1, R_1, R_3, \cdots, R_n)$ and $F(R_2, R_2, R_3, \cdots, R_n)$. Voter 1 is therefore indifferent between acts $F(R_N)$, $F(R_1, R_1, R_3, \cdots, R_n)$ and $F(R_2, R_2, R_3, \cdots, R_n)$. It follows from Property P-1 that the latter acts have the same range, which is $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$. ■

Lemma 8 *Let F be a unanimous and strategyproof ASCF, such that $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection. Then for all profile $R_N \in L_N$ and voter $i \in N_0$, the candidate $b(R_i)$ belongs to the range of the act $F(R_N)$.*

Proof of Lemma 8. If $b(R_1) = b(R_2)$, then the range of the act $F(R_N)$ is exactly $\{b(R_i), i \in N_0 \cup \{2\}\}$, see Lemma 7. Now assume that $b(R_1) \neq b(R_2)$. Let $x \in \{b(R_i), i \in N_0\}$. Then either $x \in \{b(R_i), i \in N_0\} \backslash \{b(R_1)\}$ or $x \in \{b(R_i), i \in N_0\} \backslash \{b(R_2)\}$.

Case 1 *If $x \in \{b(R_i), i \in N_0\} \backslash \{b(R_2)\}$. Write $\{b(R_i), i \in N_0\} \backslash \{b(R_2)\} = \{x_s, \cdots, x_m\}$, where $x_m = x$. Consider a preference $Q_1 \in L$ such that $b(Q_1) = b(R_2)$ and $m_r(Q_1) = x_r$ for all $r = s, \cdots, m$. From Lemma 7, acts $F(Q_1, R_2, R_3, \cdots, R_n)$ and $F(R_2, R_2, R_3, \cdots, R_n)$ have the same range, which is $\{b(R_i), i \in N_0 \cup \{2\}\}$. This implies that $x = x_m$ belongs to the range of the act $F(Q_1, R_2, R_3, \cdots, R_n)$. As $x = x_m$ is the worst candidate of voter 1 (with preference Q_1), x must belong to the range of any act achievable by voter 1 at the profile $(Q_1, R_2, R_3, \cdots, R_n)$, see Property P-2. The strategy-proofness of F would otherwise be violated. In particular x belongs to the range of the act $F(R_N)$.*

Case 2 *If $x \in \{b(R_i), i \in N_0\} \backslash \{b(R_1)\}$. This case is similar to Case 1. Write $\{b(R_i), i \in N_0\} \backslash \{b(R_1)\} = \{x_s, \cdots, x_m\}$, where $x_m = x$. Consider a preference $Q_2 \in L$ such that $b(Q_2) = b(R_1)$ and $m_r(Q_2) = x_r$ for all $r = s, \cdots, m$. As above, the worst candidate of voter 2 (with preference Q_2), x, belongs to the range of the act $F(R_1, Q_2, R_3, \cdots, R_n)$ and therefore to the range of any act achievable by voter 2 at the profile $(R_1, Q_2, R_3, \cdots, R_n)$, see Property P-2. In particular, x belongs to the range of the act $F(R_N)$.*

■

Proposition 5 *Let F be a unanimous and strategyproof ASCF such that $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection. Let $R_N \in L_N$, and $Q_1 \in L$ be a preference of voter 1 such that $b(Q_1) = b(R_1)$ and $b_2(Q_1) = b(R_2)$. Then the range of the act $F(Q_1, R_{-1})$ is either $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$.*

Proof. As F is strategyproof, voter 1 (with preference Q_1) weakly prefers the act $F(Q_1, R_{-1})$ to the act $F(R_2, R_2, R_{-1,2})$. Lemma 7 tells that the range of the act $F(R_2, R_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$, and Lemma 8 tells that the range of the act $F(Q_1, R_{-1})$ includes the set $\{b(R_i), i \in N_0\}$. It therefore follows from Property P-7 that the range of the act $F(Q_1, R_{-1})$ is either $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$. ∎

In Lemmata 9, 10 and 11 below, voter 1 and voter 2 play symmetric roles. We leave this verification to the reader.

Lemma 9 *Let F be a unanimous and strategyproof ASCF such that $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection. Let $x \in A$, $R_N \in L_N$ and $R'_1 \in L$ such that $b(R_1) = b(R'_1)$. If at least one act with range $\{b(R_i), i \in N_0\} \cup \{x\}$ is achievable by voter 2 at R_N (under F), then at least one act with range $\{b(R_i), i \in N_0\} \cup \{x\}$ is achievable by voter 2 at the profile (R'_1, R_{-1}).*

Proof of Lemma 9. We proceed by contradiction. Assume that at least one act with range $\{b(R_i), i \in N_0\} \cup \{x\}$ is achievable by voter 2 at R_N, and that no act with range $\{b(R_i), i \in N_0\} \cup \{x\}$ is achievable by voter 2 at the profile (R'_1, R_{-1}). Let $a_j = b(R_1) = b(R'_1)$. Let $Q_2 \in L$ be a preference of voter 2 such that $b(Q_2) = x$ and $b_2(Q_2) = a_j$. The range of the act $F(Q_2, R_{-2})$ must be $\{b(R_i), i \in N_0\} \cup \{x\}$. In fact Lemma 8 implies that the range of the act $F(Q_2, R_{-2})$ includes the set $\{b(R_i), i \in N_0\}$, and by hypothesis at least one act with range $\{b(R_i), i \in N_0\} \cup \{x\}$ is achievable by voter 2 at R_N. The strategy-proofness of F and Property P-6 therefore imply that the range of the act $F(Q_2, R_{-2})$ must be $\{b(R_i), i \in N_0\} \cup \{x\}$.
From Lemma 7, at least one act with range $\{b(R_i), i \in N_0\} \cup \{a_j\}$ is achievable by voter 2 at (R'_1, R_{-1}). As F is strategyproof and no act with range $\{b(R_i), i \in N_0\} \cup \{x\}$ is achievable by voter 2 at the profile (R'_1, R_{-1}), the range of the act selected at $(R'_1, Q_2, R_{-1,2})$ must either be $\{b(R_i), i \in N_0\} \cup \{a_j, x\}$ or $\{b(R_i), i \in N_0\} \cup \{a_j\}$. In both cases, voter 1 manipulates at (Q_2, R_{-2}) by choosing R'_1. ∎

Lemma 10 *Let F be a unanimous and strategyproof ASCF such that $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection. Let $R_N \in L_N$. Assume that there exits $x \in A$ such that no act with range $\{b(R_i), i \in N_0\} \cup \{x\}$ is achievable by voter 2 at R_N. Then for all $y \notin \{b(R_i), i \in N_0\} \cup \{b(R_1)\}$, no act with range $\{b(R_i), i \in N_0\} \cup \{y\}$ is achievable by voter 2 at R_N.*

Proof of Lemma 10. We proceed by contradiction. Assume that there exists a candidate $y \notin \{b(R_i), i \in N_0\} \cup \{b(R_1)\}$, a preference $Q_2 \in L$ such that the range of the act $F(Q_2, R_{-2})$ is $\{b(R_i), i \in N_0\} \cup \{y\}$. Without loss of generality, assume that xP_1y. In the case that yP_1x, exchange the position of x and y in R_1. This operation will not change our hypothesis, as

$x \neq b(R_1)$ and $y \neq b(R_1)$, see Lemma 9.

Let $\overline{Q}_2 \in L$ be such that $b(\overline{Q}_2) = x$ and $b_2(\overline{Q}_2) = y$. We show that the range of the act $F(\overline{Q}_2, R_{-2})$ is either $\{b(R_i), i \in N_0\} \cup \{y\}$ or $\{b(R_i), i \in N_0\} \cup \{x, y\}$. As the act $F(Q_2, R_{-2})$ is achievable by voter 2 at the profile $(\overline{Q}_2, R_{-2})$, voter 2 (with preference $\overline{Q}_2$) must weakly prefer the act $F(\overline{Q}_2, R_{-2})$ to the act $F(Q_2, R_{-2})$. The strategy-proofness of F would otherwise be violated. Property P-7 therefore implies that the range of the act $F(\overline{Q}_2, R_{-2})$ is either $\{b(R_i), i \in N_0\} \cup \{y\}$ or $\{b(R_i), i \in N_0\} \cup \{x\}$ or $\{b(R_i), i \in N_0\} \cup \{x, y\}$. As no act with range $\{b(R_i), i \in N_0\} \cup \{x\}$ is achievable by voter 2 at R_N, no act with the later range is achievable by voter 2 at $(\overline{Q}_2, R_{-2})$. The conclusion follows.

In both cases, voter 1 can manipulate the elections at $(\overline{Q}_2, R_{-2})$ by choosing any strategic preference $Q_1 \in L$ such that $b(Q_1) = x$, see Lemma 7. ■

Lemma 11 *Let F be a unanimous and strategyproof ASCF such that $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection. Let $R_{-1,2} \in L_{N\setminus\{1,2\}}$, $R_1, Q_2 \in L$ such that $b(R_1) \notin \{b(R_i), i \in N_0\} \cup \{b(Q_2)\}$. Assume that the range of the act $F(R_1, Q_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$. Then for all $R_2 \in L$, the range of the act $F(R_1, R_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$.*

Proof of Lemma 11. We proceed by contradiction. Suppose that there exists $R_2 \in L$ such that the range of the act $F(R_1, R_2, R_{-1,2})$ is not $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$. The strategy-proofness of F and Property P-6 imply that

(H 4) no act with range $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$ is achievable by voter 1 at the profile $(R_1, R_2, R_{-1,2})$.

We also have that $b(R_2) \neq b(Q_2)$, $b(R_2) \neq b(Q_2)$, and $b(R_1) \neq b(R_2)$. The first difference follows from Lemma 7, and the second one follows from Lemma 9. If $b(R_1) = b(R_2)$, then range of the act $F(R_1, R_2, R_{-1,2})$ would be $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$, see Lemma 7. This contradicts (H 4).

There is no loss of generality to assume that voter 2 with preference Q_2 strictly prefers $b(R_2)$ to $b(R_1)$. If the opposite holds, then exchange the position of $b(R_1)$ and $b(R_2)$ in the ranking Q_2. This will neither change our hypothesis that $b(R_1) \notin \{b(R_i), i \in N_0\} \cup \{b(Q_2)\}$ nor the hypothesis (H 5) that

(H 5) the range of the act $F(R_1, Q_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$.

In fact Lemma 9 implies that at least one act with range $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$ remains achievable by voter 1 at the profile $(R_1, Q_2, R_{-1,2})$. Similarly, it remains true that the range of the act $F(R_1, R_2, R_{-1,2})$ is not $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$.

Let $Q_1 \in L$ be such that $b(Q_1) = b(R_1)$ and $b_2(Q_1) = b(R_2)$. From (H 5), at least one act with range $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$ is achievable by voter 1 at $(Q_1, Q_2, R_{-1,2})$. Lemma 8 and strategy-proofness of F imply that the range of the act $F(Q_1, Q_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1) = b(Q_1)\}$. From Proposition 5 and (H 4), the range of the act $F(Q_1, R_2, R_{-1,2})$ must either be $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$. In both cases, voter 2 manipulates the election from $(Q_1, Q_2, R_{-1,2})$ to $(Q_1, R_2, R_{-1,2})$. This is a contradiction. ■

Lemma 12 *Let F be a unanimous and strategyproof ASCF such that $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection. Let $R_N \in L_N$ be a profile of preferences. Then the range of the act $F(R_N)$ is included in the set $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$.*

Proof of Lemma 12. We proceed by contradiction. Assume that

(H 6) the range of the act $F(R_N)$ is not included in $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$.

Obviously, $b(R_1) \neq b(R_2)$. Otherwise the range act $F(R_N)$ would be $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$, see Lemma 7.

Case 3 *Assume that there exists a candidate $x \in A$ such that the range of the act $F(R_N)$ is $\{b(R_i), i \in N_0\} \cup \{x\}$ with $x \notin \{b(R_i), i \in N_0\} \cup \{b(R_1)\}$. Then from Lemma 10, at least one act with range $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$ is achievable by voter 2 at R_N. As $\{b(R_i), i \in N_0\}$ is included in the range of the act $F(R_N)$, see Lemma 8, the Property P-6 and the strategy-proofness of F imply that $x = b(R_2)$: the range of the act $F(R_N)$ must be $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$. This contradicts (H 6). Similarly, the range of the act $F(R_N)$ can not be $\{b(R_i), i \in N_0\} \cup \{x\}$ with $x \notin \{b(R_i), i \in N_0\} \cup \{b(R_2)\}$. We conclude that the range of the act $F(R_N)$ can not take the form $\{b(R_i), i \in N_0\} \cup \{x\}$ with $x \notin \{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$.*

Case 4 *Assume that the range of the act $F(R_N)$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1), x\}$ for some $x \notin \{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$.*
a) If $b(R_1) \in \{b(R_i), i \in N_0\}$, then the range of the act $F(R_N)$ takes the form $\{b(R_i), i \in N_0\} \cup \{x\}$ with $x \notin \{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$. This is not possible, see the case 3.
b) If $b(R_1) \notin \{b(R_i), i \in N_0\}$, consider the preference $Q_2 \in L$ such that $b(Q_2) = b(R_2)$ and $b_2(Q_2) = b(R_1)$. From Proposition 5, the range of the act $F(R_1, Q_2, R_{-1,2})$ must either be $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$, or $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$. If the range of the act $F(R_1, Q_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$, or $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$, then voter 2 can profitably manipulate the election at the profile $(R_1, R_2, R_{-1,2})$ by choosing Q_2. If the range of the act $F(R_1, Q_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$, then

the range of the act $F(R_1, R_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$, see Lemma 11. This contradicts the fact the range of the act $F(R_1, R_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1), x\}$ with $x \notin \{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$. Similarly, the range of the act $F(R_N)$ can not take the form $\{b(R_i), i \in N_0\} \cup \{y, b(R_2)\}$ with $y \notin \{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$.

Case 5 *Now we show that the range of the act $F(R_N)$ can not be $\{b(R_i), i \in N_0\} \cup \{x, y\}$ with $x, y \notin \{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$. We proceed by contradiction. Suppose that the range of the act $F(R_N)$ is $\{b(R_i), i \in N_0\} \cup \{x, y\}$ for some $x, y \notin \{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$. Without loss of generality, suppose that $x\ P_1\ y$. Consider a preference $Q_2 \in L$ such that $b(Q_2) = x$ and $b_2(Q_2) = y$. If an act with range $\{b(R_i), i \in N_0\} \cup \{x\}$ is achievable by voter 2 at R_N (and therefore at $(R_1, Q_2, R_{-1,2})$), then at least one act with range $\{b(R_i), i \in N_0\} \cup \{z\}$ would be achievable by voter 2 at the profile R_N, for all $z \in A$, see Lemma 10. In that case, the range of the act $F(R_N)$ is $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$. This is a contradiction. In the case that no act with range $\{b(R_i), i \in N_0\} \cup \{x\}$ is achievable by voter 2 at R_N, the range of the act $F(R_1, Q_2, R_{-1,2})$ must be $\{b(R_i), i \in N_0\} \cup \{x, y\}$, as at least one act with range $\{b(R_i), i \in N_0\} \cup \{x, y\}$ is achievable by voter 2 at $(R_1, Q_2, R_{-1,2})$. But then voter 1 can profitably deviate (manipulate), by ranking candidate x first. This contradicts the strategy-proofness of F.*

Case 6 *Now we show that the range of the act $F(R_N)$ can not take the form $\{b(R_i), i \in N_0\} \cup \mathcal{A}$ where $\mathcal{A} \cap \{b(R_i), i \in N_0\} = \emptyset$ and $\emptyset \neq \mathcal{A} \not\subseteq \{b(R_1), b(R_2)\}$. We proceed by contradiction. Without loss of generality, we suppose that $|\mathcal{A}|$ is minimal in the sense that there exists no profile $R'_N \in L_N$ such that the range of the act $F(R'_N)$ contains more elements of $A \backslash \{b(R_i), i \in N_0\}$ than $\mathcal{A}$.*
First observe that $b(R_1) \notin \mathcal{A}$. If $b(R_1) \in \mathcal{A}$, then $b(R_1) \notin \{b(R_i), i \in N_0\}$. Consider $Q_2 \in L$ such that $b(Q_2) = b(R_2)$ and $b_2(Q_2) = b(R_1)$. From Proposition 5, the range of the act $F(R_1, Q_2, R_{-1,2})$ must either be $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$. If the range of the act $F(R_1, Q_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$, then voter 2 can profitably deviate from R_N by choosing Q_2, as $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$ is included in the range of the act $F(R_N)$. If the range of the act $F(R_1, Q_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$, then the range of the act $F(R_1, R_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$, see Lemma 11. This contradicts the fact that $\mathcal{A} \not\subseteq \{b(R_1), b(R_2)\}$.

Write $\mathcal{A} = \{x_1, \cdots, x_k\}$ where $x_1\ P_1\ x_2 \cdots\ x_{k-1}\ P_1\ x_k$ and let $Q_2 \in L$ be such that $b_l(Q_2) = x_l$ for all $l \in \{1, \cdots, k\}$ and $b_m(Q_2) = b(R_1)$. The preference Q_2 is such that $x\ Q_2\ y$ for all $x \in \mathcal{A}$ and $y \in \{b(R_i), i \in N_0\}$. It therefore follows that

(H 7) *for all non empty $\mathcal{B} \subsetneq \mathcal{A}$, voter 2 (with preference Q_2) prefers any act with range $\{b(R_i), i \in N_0\} \cup \mathcal{B}$ to any other act with range $\{b(R_i), i \in N_0\}$.*

The range of the act $F(Q_2, R_{-2})$ must be $\{b(R_i), i \in N_0\} \cup \mathcal{A}$. The latter holds for the following reasons.

- *If the range of the act $F(Q_2, R_{-2})$ is $\{b(R_i), i \in N_0\}$, then voter 2 will manipulate the election at the profile (Q_2, R_{-2}) by strategically announcing R_2, see (H 7).*
- *If the range of the act $F(Q_2, R_{-2})$ is $\{b(R_i), i \in N_0\} \cup \{x\}$ with $x \notin \{b(R_i), i \in N_0\} \cup \{b(R_1)\}$, then from Lemma 10, at least one act with range $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$ is achievable by voter 2 at (Q_2, R_{-2}). This implies that the range of the act $F(R_N)$ is $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$. This contradicts the fact that $\mathcal{A} \not\subseteq \{b(R_1), b(R_2)\}$.*
- *Similarly as above, see Case 4 and Case 5, the range of the act $F(Q_2, R_{-2})$ takes the form $\{b(R_i), i \in N_0\} \cup \{x, y\}$ with $x, y \notin \{b(R_i), i \in N_0\}$ only if $\{x, y\} = \{b(R_1), b(Q_2)\}$. This in turn is not possible as the act $F(R_N)$ would be strictly preferred by voter 2 (with preference Q_2) to the act $F(Q_2, R_{-2})$, see Property P-2, contradicting the strategy-proofness of F.*
- *Obviously, any act with range $\{b(R_i), i \in N_0\} \cup \mathcal{A}$ is strictly preferred by voter 2 (with preference Q_2) to any other act with range $\{b(R_i), i \in N_0\} \cup \mathcal{B}$, where $\mathcal{B} \neq \mathcal{A}$, $\mathcal{B} \cap \{b(R_i), i \in N_0\} = \emptyset$ and Y has at least as many elements as $\mathcal{A}$. Therefore, the range of the act $F(Q_2, R_{-2})$ must be $\{b(R_i), i \in N_0\} \cup \mathcal{A}$.*

But now voter 1 can manipulate the elections by strategically ranking candidate x_1 first, see Lemma 7. This is a contradiction. We conclude that the range of the act $F(R_N)$ is included in $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$.

■

Lemma 13 *Let F be a unanimous and strategyproof ASCF such that $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection. Let $R_N \in L_N$ be a profile of preferences. Then the range of the act $F(R_N)$ contains at least one element of the set $\{b(R_1), b(R_2)\}$.*

Proof of Lemma 13. In this proof, the range of the act $F(R_N)$ selected at a given profile R_N will be denoted $\mathcal{A}_F(R_N)$. Let $R_N \in L_N$.
a) If $b(R_1) = b(R_2)$, then the range $\mathcal{A}_F(R_N)$ of the act $F(R_N)$ is exactly $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$, see Lemma 7. We therefore have that $\mathcal{A}_F(R_N) \cap \{b(R_1), b(R_2)\} \neq \emptyset$.
b) If $b(R_1) \in \{b(R_i), i \in N_0\}$ or $b(R_2) \in \{b(R_i), i \in N_0\}$, then $\mathcal{A}_F(R_N) \cap \{b(R_1), b(R_2)\} \neq \emptyset$,

see Lemma 8.

c) If voter 2 (with preference R_2) strictly prefers $b(R_1)$ to any candidate $x \in \{b(R_i), i \in N_0\}$, then, as at least one act with range $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$ is achievable by voter 2 at R_N, see Lemma 7, the range of the act $F(R_N)$ is either $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$, see Lemma 12. In each case we have $\mathcal{A}_F(R_N) \cap \{b(R_1), b(R_2)\} \neq \emptyset$.

d) Now assume that $b(R_1) \neq b(R_2)$, $b(R_1), b(R_2) \notin \{b(R_i), i \in N_0\}$, and that there exists $x \in \{b(R_i), i \in N_0\}$ such that voter 2 with preference R_2 strictly prefers x to $b(R_1)$. In this case,

(H 8) voter 2 with preference R_2 strictly prefers any act with range $\{b(R_i), i \in N_0\}$ to any act with range $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$.

From here we proceed by contradiction. Assume that

(H 9) the range $\mathcal{A}_F(R_N)$ of the act $F(R_N)$ is $\{b(R_i), i \in N_0\}$ and that $b(R_1), b(R_2) \notin \mathcal{A}_F(R_N)$.

The later assumption is without loss of generality, see Lemma 8 and Lemma 12. Let $Q_1 \in L$ such that $b(Q_1) = b(R_1)$ and $b_2(Q_1) = b(R_2)$.

d-1) The range of the act $F(Q_1, R_{-1})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$. From Proposition 5, the range of the act $F(Q_1, R_{-1})$ is either $\{b(R_i), i \in N_0\} \cup \{b(R_1), b(R_2)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$ or $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$. If the range of the act $F(Q_1, R_{-1})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$, then the range of the act $F(R_N)$ is $\{b(R_i), i \in N_0\} \cup \{b(R_2)\}$, see Lemma 11. This contradicts (H 9). If the range of the act $F(Q_1, R_{-1})$ is $\{b(R_i), i \in N_0\} \cup \{b(R_1)\}$, then voter 1 will manipulate the election at R_N, favoring the selection of the act $F(Q_1, R_{-1})$, see Property P-6.

d-2) Let $Q_2 \in L$ be such that $b(Q_2) \in \{b(R_i), i \in N_0\}$. Then the range of the act $F(R_1, Q_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\}$, as at least one act with range $\{b(R_i), i \in N_0\}$ is achievable by voter 2 at $(R_1, Q_2, R_{-1,2})$, see (H 9), and $\{b(R_i), i \in N_0\} \subseteq \mathcal{A}_F(R_1, Q_2, R_{-1,2})$, see Lemma 8. From Lemma 12, the range of the act $F(Q_1, Q_2, R_{-1,2})$ is included in $\{b(R_i), i \in N_0\} \cup \{b(Q_1)\}$, and must contain $\{b(R_i), i \in N_0\}$, see Lemma 8. The latter range can not be $\{b(R_i), i \in N_0\} \cup \{b(Q_1)\}$, as voter 1 would profitably deviate from $(R_1, Q_2, R_{-1,2})$ to $(Q_1, Q_2, R_{-1,2})$, see Property P-6. Therefore, the range of the act $F(Q_1, Q_2, R_{-1,2})$ is $\{b(R_i), i \in N_0\}$. But now voter 2 can profitably deviate from $(Q_1, R_{-1}) = (Q_1, R_2, R_{-1,2})$ to $(Q_1, Q_2, R_{-1,2})$, see (H 8). This contradicts the strategy-proofness of F. ∎

From Lemmata 11, 12, and 13, it follows that for all profile of preferences $R = (R_3, \cdots, R_n) \in L_{N\setminus\{1,2\}}$ of voters $\{3, \cdots, n\}$, there exists a nonempty subset $N^R \subseteq \{1, 2\}$ such that the range of the act $F(R_1, R_2, R)$ is $\{b(R_i), i \in N_0 \cup N^R\}$ for all $R_1, R_2 \in L$. We have the following lemma.

Lemma 14 *Let F be a unanimous and strategyproof ASCF such that $F_{1,2}$ is a $N_0 \cup \{2\}-$top selection, where $|N_0| \geq 2$. Let $i, j \in N_0$, $R \in L_{N\backslash\{1,2\}}$ be a profile of preferences of voters of the block $N\backslash\{1,2\}$ such that $\{b(R_k), k \in N_0\} \neq A$. Let $Q_i \in L$ be a preference of voter i such that $Q_i = R_j$ and let $Q = (Q_i, R_{-i})$. Then $N^R = N^Q$.*

In Lemma 14, the profile Q is obtained from the profile R be replacing the entry R_i of voter i by R_j. The lemma says that the set N^R remains unchanged after such an operation.

Proof of Lemma 14. We proceed by contradiction. Assume that $N^R \neq N^Q$. We introduce an additional notation: for all $\mathcal{A} \subseteq A$ and preference $R_i \in L$, $w(\mathcal{A}, R_i)$ denotes the least preferred candidate $a \in \mathcal{A}$ with respect to R_i. We consider 3 cases.

Case 7 *$N^R = \{1\}$ and $N^Q = \{2\}$. Let $R_1, R_2 \in L$ such that $b(R_1) = w(A\backslash\{b(R_i), i \in N_0\backslash\{i\}\}, R_i)$, and $R_2 = R_i$. As the range of the acts $F(R_1, R_2, R)$ and $F(R_1, R_2, Q)$ are respectively $\{b(R_k), k \in N_0\} \cup \{b(R_1)\}$ and $\{b(R_k), k \in N_0\}$, voter i can profitably manipulate the election from (R_1, R_2, R) to (R_1, R_2, Q), see Property P-8.*

Case 8 *$N^R = \{1\}$ and $N^Q = \{1,2\}$. Let $R_1, R_2 \in L$ such that $R_1 = R_i$ and $b(R_2) = w(A\backslash\{b(R_k), k \in N_0\}, Q_i)$. As the ranges of the acts $F(R_1, R_2, R)$ and $F(R_1, R_2, Q)$ are respectively $\{b(R_k), k \in N_0\}$ and $\{b(R_k), k \in N_0\} \cup \{b(R_2)\}$, voter i can profitably manipulate the election from (R_1, R_2, Q) to (R_1, R_2, R), see Property P-8.*

Case 9 *$N^R = \{1,2\}$ and $N^Q = \{1\}$. Similarly as above, let $R_1, R_2 \in L$ such that $R_1 = R_i$ and $b(R_2) = w(A\backslash\{b(R_k), k \in N_0\}, R_i)$. As the ranges of the acts $F(R_1, R_2, R)$ and $F(R_1, R_2, Q)$ are respectively $\{b(R_k), k \in N_0\} \cup \{b(R_2)\}$ and $\{b(R_k), k \in N_0\}$, voter i can profitably manipulate the election from (R_1, R_2, R) to (R_1, R_2, Q), see Property P-8.*

Any other possible scenario is equivalent to one of the three previous cases. ■

Proof of Lemma 6. We show that N^R is independent from R whenever $\{b(R_i), i \in N_0\} \neq A$.

a) Let $R = (R_3, \cdots, R_n) \in L_{N\backslash\{1,2\}}$ such that $\{b(R_i), i \in N_0\} \neq A$. Let $j \in \{3, \cdots, n\}\backslash N_0$, and $Q_j \in L$. Let $Q = (Q_j, R_{-j})$. The reader can check that if $N^R \neq N^Q$, then voter j can profitably manipulate the elections. This implies that the set N^R is independent of the preference of any voter $j \in \{3, \cdots, n\}\backslash N_0$.

b) Assume that $|N_0| = 1$, and write $N_0 = \{i_0\}$. Let $R \in L_{N\setminus\{1,2\}}$ such that $\{b(R_i), i \in N_0\} \neq A$. Let $Q_{i_0} \in L$. Let $Q = (Q_{i_0}, R_{-i_0})$. We show by contradiction that $N^R = N^Q$. Assume that $N^R \neq N^Q$. There is no loss to further assume that Q_{i_0} is obtained from R_{i_0} by permuting two consecutive candidates. This implies that $b(R_{i_0}) \neq b_m(Q_{i_0})$ and $b(Q_{i_0}) \neq b_m(R_{i_0})$.

- If $N^R = \{1\}$ and $N^Q = \{2\}$. Let $R_1, R_2 \in L$ such that $b(R_1) = b_m(R_{i_0})$ and $R_2 = R_{i_0}$. As the ranges of the acts $F(R_1, R_2, R)$ and $F(R_1, R_2, Q)$ are respectively $\{b_m(R_{i_0}), b(R_{i_0})\}$ and $\{b(R_{i_0}), b(Q_{i_0})\}$, voter i_0 can manipulate the election from (R_1, R_2, R) to (R_1, R_2, Q), see Property P-2.

- If $N^R = \{1\}$ and $N^Q = \{1, 2\}$. Let $R_1, R_2 \in L$ such that $R_1 = R_{i_0}$ and $b(R_2) = b_m(Q_{i_0})$. As the ranges of the acts $F(R_1, R_2, R)$ and $F(R_1, R_2, Q)$ are respectively $\{b(R_{i_0})\}$ and $\{b(R_{i_0}), b_m(Q_{i_0}), b(Q_{i_0})\}$, voter i_0 can manipulate the election from (R_1, R_2, Q) to (R_1, R_2, R), see Property P-8.

- If $N^R = \{1, 2\}$ and $N^Q = \{1\}$. Similarly as above, let $R_1, R_2 \in L$ such that $R_1 = R_{i_0}$ and $b(R_2) = b_m(R_{i_0})$. As the ranges of the acts $F(R_1, R_2, R)$ and $F(R_1, R_2, Q)$ are respectively $\{b(R_{i_0}), b_m(R_{i_0})\}$ and $\{b(R_{i_0}), b(Q_{i_0})\}$, voter i_0 can manipulate the election from (R_1, R_2, R) to (R_1, R_2, Q), see Property P-2.

- Any other case is similar to one of the three cases above.

c) Let $R, Q \in L_{N\setminus\{1,2\}}$ be two profiles such that $\{b(R_i), i \in N_0\} \neq A$ and $\{b(Q_i), i \in N_0\} \neq A$. We show that $N^R = N^Q$. Consider the profile $\overline{Q}$ obtained from the profile R by replacing each entry R_i by Q_i for all $i \in \{3, \cdots, n\}\setminus N_0$. From a) $N^R = N^{\overline{Q}}$. From b) $N^{\overline{Q}} = N^Q$ if $|N_0| = 1$. In the case that $|N_0| > 1$, we have $N^{\overline{R}} = N^{(Q_{i_0}, \overline{R}_{-i_0})} = N^Q$, see Lemma 14. Therefore, N^R is independent of R whenever $\{b(R_i), i \in N_0\} \neq A$.

d) Let $x, y, z \in A$ be 3 distinct candidates. Let $R \in L_{N\setminus\{1,2\}}$ be such that $b(R_i) = z$ for all $i \in \{3, \cdots, n\}$. Let $R_1, R_2 \in L$ such that $b(R_1) = x$ and $b(R_2) = y$. From Lemmata 8, 12, and 13, the range of the act $F(R_1, R_2, R_{-1,2})$ must be either $\{x, z\}$, $\{y, z\}$, or $\{x, y, z\}$. From c), we have the following:

- if the range of the act $F(R_1, R_2, R_{-1,2})$ is $\{x, z\}$, then F is a $N_0 \cup \{1\}-$top selection;

- if the range of the act $F(R_1, R_2, R_{-1,2})$ is $\{y, z\}$, then F is a $N_0 \cup \{2\}-$top selection;

- if the range of the act $F(R_1, R_2, R_{-1,2})$ is $\{x, y, z\}$, then F is a $N_0 \cup \{1, 2\}-$top selection.

This concludes the proof of Lemma 6. ∎