DISSERTATION

STOCHASTIC SINGULAR CONTROL: EXISTENCE, CHARACTERIZATION AND APPROXIMATION OF SOLUTIONS IN COST MINIMIZATION PROBLEMS AND GAMES

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Abstract

Stochastic singular control models (such as optimization problems, games and mean field games) refer to a class of problems in which some agents want to optimize a certain performance criterion by acting in a random environment which evolves in continuous time, and in which the effect of the agents' action on both the environment and on the performance is (linearly) proportional to the size of the action. Applications include investment and portfolio selection in finance, inventory management in operations research, control of queueing networks, dividend and equity issuance in insurance mathematics, spacecraft control in aerospace engineering, or rational harvesting in mathematical biology. Models involving stochastic singular controls raise many unsolved mathematical issues which represent a relevant limitation to their theoretical understanding.

In this thesis, we provide mathematical tools and structural conditions which allow to address problems of existence, characterization and approximation of solutions in optimization problems and games involving stochastic singular controls. For a class of optimal stochastic control problems with singular controls, we characterize the optimal control as the unique solution to a related Skorokhod reflection problem. We prove that the optimal control only acts when the underlying diffusion attempts to exit the so-called waiting region, and that the direction of this action is prescribed by the derivative of the value function. We next consider problems concerning existence and approximation of Nash equilibria in N-player stochastic games of multi-dimensional singular control with submodular costs. In a not necessarily Markovian setting, we establish the existence of Nash equilibria via Tarski's fixed point theorem, and we propose an algorithm to determine a Nash equilibrium. Moreover, we derive relations between weak (distributional) Nash equilibria of the game of singular control and the Nash equilibria of stochastic games with regular controls. Further, we study mean field games with regular and singular controls and costs that are submodular with respect to a suitable order relation on the state and measure space. The submodularity assumption allows us to prove existence of solutions via an application of Tarski's fixed point theorem, covering cases with discontinuous dependence on the measure variable. Also, it ensures that the set of solutions enjoys a lattice structure: in particular, there exist minimal and maximal solutions. Finally, it guarantees that those two solutions can be obtained through a simple learning procedure based on the iterations of the best-response-map. Our approach also allows to prove existence of a strong solution for a class of submodular mean field games with common noise, where the representative player at equilibrium interacts with the (conditional) mean of its state's distribution. Finally, we analyse

stationary mean field games with singular controls in which the representative player interacts with a long-time weighted average of the population through a discounted and an ergodic performance criterion. This class of games finds natural applications in the context of optimal productivity expansion in dynamic oligopolies. We prove existence and uniqueness of the mean field equilibria, which are completely characterized through nonlinear equations. Furthermore, we relate the mean field equilibria of the discounted and the ergodic games by showing the validity of an Abelian limit. The latter also allows to approximate Nash equilibria of symmetric N-player ergodic singular control games through the mean field equilibrium of the discounted game.

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Notation

General notation.
For $d \in \mathbb{N}$ with $d \geqslant 1$ and $x, y \in \mathbb{R}^d$, we denote by xy the scalar product in $\mathbb{R}^d$, as well as by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ we denote by $x^\top$ the transpose of x. The vector $e_i \in \mathbb{R}^d$ indicates the i-th element of the canonical basis of $\mathbb{R}^d$ and, for $x \in \mathbb{R}^d$ and $R > 0$, set $B_R(x) := \{y \in \mathbb{R}^d \,|\, |y - x| < R\}$.

For $x, y \in \mathbb{R}^d$ and $c \in \mathbb{R}$, we will write $x \leqslant y$ if $x^\ell \leqslant y^\ell$ for each $\ell = 1, ..., d$, as well as $x \leqslant c$ if $x^\ell \leqslant c$ for each $\ell = 1, ..., d$. Moreover, we set $x \wedge y := (x^1 \wedge y^1, ..., x^d \wedge y^d)$ and $x \vee y := (x^1 \vee y^1, ..., x^d \vee y^d)$, where $x^\ell \wedge y^\ell := \min\{x^\ell, y^\ell\}$ and $x^\ell \vee y^\ell := \max\{x^\ell, y^\ell\}$ for each $\ell = 1, ..., d$.

For d, $N \in \mathbb{N}$ with $d, N \geqslant 1$, and $a = (a^1, ..., a^N) \in \mathbb{R}^{Nd}$, for each $i = 1, ..., N$ set $a^{-i} := (a^1, ..., a^{i-1}, a^{i+1}, ..., a^N) \in \mathbb{R}^{(N-1)d}$ and, for $v \in \mathbb{R}^d$, set

$$(v, a^{-i}) := (a^1, ..., a^{i-1}, v, a^{i+1}, ..., a^N) \in \mathbb{R}^{Nd}.$$

Unless otherwise stated, C indicates a generic positive constant, which may change from line to line.

Functional spaces.
For $d \in \mathbb{N}$ with $d \geqslant 1$, an open set $B \subset \mathbb{R}^d$, $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ and a funciton $f : B \to \mathbb{R}$, we denote by $D^\alpha f := D_1^{\alpha_1} ... D_d^{\alpha_d} f$ the weak derivative of f, where $D_i f := f_{x_i} := \partial f / \partial x_i$, and we set $|\alpha| := \alpha_1 + ... + \alpha_d$.

For $\ell \in \mathbb{N}$, $q \in [1, \infty]$, and a measure space $(E, \mathcal{E}, m)$, we define:

- $\|f\|_{L^q(E)} := \int_E |f|^q dm$ if $q < \infty$, and $\|f\|_{L^\infty(E)} := \operatorname{ess\,sup}_E f$ for $q = \infty$
- $L^q(E) = L^q(E, m) := \{\text{measurable } f : E \to \mathbb{R} \text{ s.t. } \|f\|_{L^q(E)} < \infty\}$
- $C^\ell(B) := \{f : B \to \mathbb{R} \text{ with continuous } \ell\text{-order derivatives}\}$
- $C_c^\infty(B) := \{f : B \to \mathbb{R} \text{ with compact support, s.t. } f \in C^\ell(B) \text{ for each } \ell \in \mathbb{N}\}$
- $\|f\|_{C^0(B)} := \sup_{x \in B} |f(x)|$, $\|f\|_{\mathrm{Lip}\,(B)} := \sup_{x,y \in B} |f(y) - f(x)|/|y - x|$
- $\|f\|_{C^{\ell;1}(B)} := \sum_{|\alpha| \leqslant \ell} \|D^\alpha f\|_{C^0(B)} + \sum_{|\alpha| = \ell} \|D^\alpha f\|_{\mathrm{Lip}\,(B)}$
- $C^{\ell;1}(B) := \{f \in C^\ell(B) \text{ s.t. } \|f\|_{C^{\ell;1}(B)} < \infty\}$
- $\|f\|_{W^{\ell;q}(B)} := \sum_{|\alpha| \leqslant \ell} \|D^\alpha f\|_{L^q(B)}$
- $W^{\ell;q}(B) := \{f \in L^q(B) \text{ with } \|f\|_{W^{\ell;q}(B)} < \infty\}$
- $W^{\ell;q}_{loc}(B) := \{f \,|\, f \in W^{\ell;q}(D) \text{ for each bounded open set } D \subset B\}$
- $W^{\ell;q}_0(B)$ denotes the closure of $C_c^\infty(B)$ in the norm $\|\cdot\|_{W^{\ell;q}(B)}$

Probability.
Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $X \colon \Omega \to \mathbb{R}$, we use the notation $\mathbb{P} \circ X^{-1}$ for the law of X under $\mathbb{P}$, i.e., we set $(\mathbb{P} \circ X^{-1})[E] := \mathbb{P}[X \in E]$ for each Borel set E of $\mathbb{R}$. Finally, for a given $T \in (0, \infty)$ and a stochastic process $X = (X_t)_{t \in [0,T]}$, with a slight abuse of notation, we denote by $\mathbb{P} \circ X^{-1}$ the flow of measures associated to X; that is, we set $\mathbb{P} \circ X^{-1} := (\mathbb{P} \circ X_t^{-1})_{t \in [0,T]}$.

Chapter 1

Introduction

Stochastic optimal control theory (see, e.g., [45, 81, 143, 168]) deals with problems in which an agent wants to optimize a certain performance criterion by acting in a random environment which evolves in time. In a typical instance of the problem, the evolution of the environment is described by a stochastic process (the state process) which can be affected by another stochastic process (the control). The control is chosen by the agent in order to minimize a certain expected value of a random functional of the paths of the control and of the state process resulting from the chosen control (the controlled state process). When several agents want to optimize a performance criterion, depending also on the action of the other agents, a strategic interaction arises and we refer to the problem as to a stochastic game (see, e.g., [45]), to agents as to players, and to controls as to strategies. From this perspective, a control problem is a stochastic game with only one player. While many notions of solutions to the game can be introduced, the notion of Nash equilibrium seems to be the most appropriate for games in which players are assumed not to cooperate with each other. A Nash equilibrium is, roughly speaking, a vector of strategies (one for each specific player) such that no unilateral deviation is convenient for any of the players. The vector of outcomes (like a price, or a payoff) deriving from playing a Nash equilibrium is said to be a value of the game. Furthermore, taking inspiration from models of interacting particle systems, the recent theory of mean field games (MFGs) has been developed in order to study the (approximate) Nash equilibria in games with a large number of players under suitable symmetry assumptions (see, e.g., [45, 47]). The intuition is to replace the problem of a large number of identical players solving symmetric optimization problems, by the problem faced by a representative player, which plays against a limit distribution, describing the optimal state of the other players.

All these types of models have the challenging objective of describing and better understanding very complex phenomena emerging from (a simplified version of) many real world problems in which agents are competing in an environment. They are central in social sciences and find countless applications in economics, systems theory, engineering, operations research, biology, ecology, environmental sciences, among others. For example, players can be animals, computers, firms while the environment can be nature, a network or a market.

Stochastic control problems and games relate to other branches of mathematics and play a main role in the theory of partial differential equations (PDEs) as well as in probability theory. Indeed, due to their importance from an application point of view, since the early 50's these models captured the interests of many mathematicians, who introduced a wide range of tools in order to improve the theoretical understanding and to provide explicit solutions to real world problems (see [45, 143, 168]). One of these tools is the dynamic programming principle (DPP) developed by Bellman. In the case of stochastic control problems in continuous time with Markovian data, this principle connects the dynamic optimization problem to a PDE of second order, the so-called Hamilton-Jacobi-Bellman equation (HJB), allowing to characterize the value of the problem as the unique solution (in a proper sense) of such a PDE. This approach represents one of the main routes to find and characterize the solutions to a control problem. Indeed, one can hope to "solve" the HJB equation, which in turn allows (in many instances of the problem) to compute the optimal control. Another approach is the Pontryagin maximum principle (PMP), in which the optimization problem is related to a system of forward-backward stochastic differential equations (FBSDEs). Also in this case, solving the FBSDE system leads to a solution of the problem. Finally, one can employ techniques from the theory of weak convergence of probability measures in order to prove existence of optimal controls. This latter approach is known as compactification method. It allows to prove existence of optimal controls in great generality but it is not very useful from the perspective of providing explicit solutions to the problem, since the established controls often come with no further characterization.

Stochastic singular control problems (see, e.g., [81, 143]) represent a particular class of stochastic control problems, in which the effect of the agents' action on both the state of the system and on the performance is (linearly) proportional to the size of the action. This particular feature is very natural when modelling problems such as investment and portfolio selection in finance (see [14, 53, 63, 81], among many others), inventory management in operations research (see, e.g., [86, 94]), control of queueing networks (see, e.g., [114]), dividend and equity issuance in insurance mathematics (see [126], among others), spacecraft control in aerospace engineering (see, e.g., [135]), or rational harvesting in mathematical biology (see, e.g., [8]). Intuitively, while the (regular) control at time t represents the action taken by the agent at time t (e.g., the infinitesimal amount of money invested at that time), the singular control at time t represents the cumulative action taken up to time t (e.g., the total amount of money invested up to time t). The fact that the performance is proportional to the action brings as a consequence that the system can be divided in two states: one in which it is optimal not to act, and one in which an action is required. The term singular control arise indeed from the fact that, in many examples, it has been shown that the optimal control is a nondecreasing process which is singular with respect to the Lebesgue measure (see, e.g., [110]). Such a control keeps the underlying state process inside the no-action region by reflecting it at its boundary (often referred to as to the free boundary). For this reason, stochastic singular control theory is intimately related to the problem of existence of solutions to reflected stochastic differential equations (SDEs), also known as the Skorokhod problem for SDEs (see, e.g., [144]).

Classical tools from stochastic control theory (such as the aforementioned DPP approach, the PMP approach and various compactification methods) have been adapted to stochastic singular controls (see [37, 81, 96]). Nevertheless, the nature of a singular control problem brings substantial difficulties in employing such methods in order to provide or explicitly characterize the solutions. Indeed, having a solution to the HJB equation or to the related FBSDE system allows to find the optimal control only in a very limited number of examples, mostly in one-dimensional settings. Therefore, trying to employ these techniques in order to find equilibria in N-player games or in MFGs with singular controls appears even more challenging. Concluding, even though stochastic singular control models were introduced in the 60's, they still raise many unsolved mathematical issues which represent a relevant limitation to their theoretical understanding.

It is the aim of this thesis to provide mathematical tools and structural conditions which allow to address the following problems in the context of stochastic singular control:

1. Characterize the optimal control in terms of a related Skorokhod problem in multidimensional stochastic singular control problems;

2. Prove existence and approximation of Nash equilibria in N-player games with singular controls;

3. Prove existence and approximation of equilibria in MFGs with singular controls;

4. Illustrate some setups in which a MFG with singular controls can be (fairly) explicitly solved.

Problem 1 will be studied via a connection with a game of optimal stopping, which in turn will allow to prove some properties of the free boundary thanks to structural conditions on the data. These properties will then allow to construct a suitable probabilistic approximation of the optimal control, leading to its characterization as the solution to a related Skorokhod problem.

Problems 2 and 3 will be addressed enforcing the so-called submodularity condition. Submodular games were first introduced by Topkis in [157] (see also [10, 158, 160]) in the context of static non-cooperative N-player games. They are characterized by costs of the players that have decreasing differences with respect to a partial order induced by a lattice structure on the set of strategy vectors. Because the notion of submodularity is related to that of substitute goods in economics, submodular games have received large attention in the economic literature (see [9, 137], among many others). Also, this condition represents the situation in which players have an incentive to imitate each other, and, in the context of MFGs, it consists of a sort of antithetic version of the well known Larsy-Lions monotonicity condition (see [121]). The submodularity assumption, since the seminal work of [157], is known to enrich the game with some remarkable properties. Firstly, it allows us to prove existence of equilibria via an application of Tarski's fixed point theorem. Secondly, it ensures that the set of solutions enjoys a lattice structure: in particular, there exist minimal and maximal solutions. Thirdly, it guarantees that those two solutions can be obtained through a simple learning procedure based on the iterations of the best-response-map. Such a condition brings new existence and approximation results both in N-player games with singular control and in MFGs with regular or singular controls. Particularly relevant are new results in MFGs with common noise (an aggregate source of randomness), in which existence of strong solutions (i.e., adapted to the common noise) are proved.

From the methodological point of view, we also underline that, for Problems 2 and 3, a suitable approximation of the singular controls is often employed. In particular, exploiting the density (in a suitable topology) of the set of Lipschitz continuous functions in the set of càdlàg (i.e., right-continuous with left limits) nondecreasing functions, the N-player game with singular controls is related to games with regular controls. A similar technique is also employed to derive some new existence and approximation results for MFGs with singular controls, which we previously obtained for MFGs with regular controls.

Problem 4, instead, is addressed by formulating stationary MFGs with singular controls and discounted or ergodic payoff functional. By exploiting structural conditions on the data and characteristic properties of Itô-diffusion processes, the equilibria of these two MFGs are (fairly) explicitly provided by systems of equations. Relations among MFGs with discounted and ergodic payoffs are also established, as well as their connection with suitable N-player games with singular controls.

In the rest of this introduction we discuss the problems treated in this dissertation in more detail. In particular, for each problem we introduce the studied model, discuss the related literature and the motivations for our study, present the results and finally describe the solution approach.

1.1 Chapter 2: Singular control and Skorokhod problem

Chapter 2 considers the problem of characterizing optimal policies for stochastic singular control problems in multidimensional settings.[1] A stochastic singular control problem appears for the first time in [16], where the problem of controlling the motion of a spaceship has been addressed. Later on, examples of solvable stochastic singular control problems have been studied in [21]. Stochastic singular control problems of monotone-follower type have been introduced and studied in [106] and [110]. A monotone-follower problem is the problem of tracking a stochastic process by a nondecreasing process in order to optimize a certain performance criterion. Since then, this class of problems has found many applications in economics and finance (see [14, 53, 63], among many others), operations research (see, e.g., [86, 94]), queuing theory (see, e.g., [114]), mathematical biology (see, e.g., [8]), aerospace engineering (see, e.g., [135]), and insurance mathematics (see [126], among others). The literature on stochastic singular control problems experienced results on existence of minima (or maxima) (see [73] and [96], among others), characterization of the optimizers through first order conditions

[1] Parts of this introduction and of Chapter 2 are based on a joint work with Giorgio Ferrari, see [69].

(see, e.g., [12], [14] and [37]), as well as connections to optimal stopping problems (see, e.g., [110] or the more recent [28]) and to constrained backward stochastic differential equations [31].

We consider the problem of controlling, through a one-dimensional càdlàg (i.e., right-continuous with left limits) process v with locally bounded variation, the first component of a multidimensional diffusion with initial condition x. Namely, the controller can affect a state process $X^{x;v}$ which evolves according to the equation

$$dX_t^{x;v} = b(X_t^{x;v})dt + \sigma(X_t^{x;v})dW_t + e_1 dv_t,\ t \geqslant 0, \quad X_{0-}^{x;v} = x, \tag{1.1.1}$$

for a multidimensional Brownian motion W, a suitable convex Lipschitz function b, and a volatility matrix σ, which is either constant or linear in the state. The aim of the controller is to minimize the expected discounted cost

$$J(x;v) := \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X_t^{x;v})dt + \int_{[0,\infty)} e^{-\rho t} d|v|_t\bigg], \tag{1.1.2}$$

for a given convex function h and a suitable discount factor $\rho > 0$. Here, $|v|$ denotes the total variation of the process v. The value function V of the problem is defined, at any given initial condition x, as the minimum of $J(x;v)$ over the choice of controls v. Also, a control $\bar{v}$ is said to be optimal for x if $J(x;\bar{v}) = V(x)$. Existence of optimal controls can be proved in very general frameworks using different probabilistic compactification methods (see, e.g., [34, 58, 96, 123, 135]).

Natural questions that immediately arise are whether it is possible to characterize V, and how one should act on the system in order to obtain the minimal cost V. As a matter of fact, the Markovian nature of the problem together with mild regularity and growth conditions on b and h, allows to employ the dynamic programming approach. This leads to the characterization of the value function as a solution (in a suitable sense) to the Hamilton-Jacobi-Bellman equation

$$\max\{\rho V - bDV - \mathrm{tr}(\sigma\sigma^\top D^2 V)/2 - h, |V_{x_1}| - 1\} = 0. \tag{1.1.3}$$

This equation provides key insights on the way the controller should act on the system in order to minimize the cost of her actions. Indeed, when V is sufficiently regular, an application of Itô's formula suggests that the controller should make the state process not leaving the set $\mathcal{W} := \{|V_{x_1}| < 1\}$, usually referred to as the waiting region. In fact, in many examples (see, e.g., [66, 90, 113, 115, 130, 149, 167], among others) it is possible to construct the optimal control as the solution to a related Skorokhod reflection problem; that is, the optimal control can be characterized as that process $\bar{v}$, with minimal total variation, which is able to keep the process $X^{x;\bar{v}}$ inside the closure of the waiting region $\mathcal{W}$, by reflecting it in a direction prescribed by the gradient of the value function. However, in multidimensional settings, such a characterization often remains a conjecture (see the discussion in Chapter 6 in [152], Remark 5.2 in [28], and also [55, 56, 78, 77]), and many questions about the properties of optimal controls remain open, representing a strong limitation to the theory.

We now discuss more in detail the problem of the characterization of optimal rules. When the state process is one dimensional, optimal controls can be explicitly constructed as Skorokhod reflections in a general class of models (see [3, 19, 64, 79, 102, 103, 130, 162], among others). Also, in the (not necessarily Markovian) one dimensional case, a similar characterization of optimal controls has been achieved in [12, 13, 14], without relying on the dynamic programming approach. When the dimension of the problem becomes larger than one, the difficulty of characterizing optimal controls drastically increases. Indeed, classical results on the existence of solutions to the Skorokhod reflection problem in the multidimensional domain $\mathcal{W}$ require some regularity of the boundary of $\mathcal{W}$ and of the direction of reflection, which are, in most of the cases, unknown. When the value function V is convex, this difficulty is overcome in some specific settings. A celebrated example is presented in [149], where the problem of controlling a two-dimensional Brownian motion with a two-dimensional process of bounded variation is considered. There, the authors show that the boundary of the waiting region (the so-called free boundary) is of class C^2, and they are therefore able to construct the optimal policy as a solution to the associated Skorokhod problem. The problem of the characterization is also encountered in [56, 55, 78, 77], where the construction of the optimal control can be provided only by requiring additional properties on the boundary of the waiting region. Another example is exhibited in [64], in which the case of controlling a multidimensional Brownian motion with a multidimensional control is considered in the case of a radial running cost $h(x) = |x|^2$. We also refer to [113], where the construction of the optimal policy is provided in a two-dimensional context in which the drift is non-zero. To the best of our knowledge, in the case of a convex V, the most general multidimensional setting in which this characterization is shown is presented in [115], and in its finite time-horizon counterpart [30]. There, the problem of controlling a multidimensional Brownian motion with a multidimensional control is considered for a convex running cost. Remarkably, in [115] (and in [30]) the author presents an approach which allows to construct the unique optimal policy as a solution to the related Skorokhod problem bypassing the problems related to the regularity of the free boundary. In non-convex settings, the number of contributions are even rarer. The suitable regularity of the boundary of $\mathcal{W}$ is shown, in two-dimensional settings, in [90] and in [66], while a multidimensional case is considered in [167], via a connection with Dynkin games. We also mention that the construction of multidimensional reflected diffusions in polyhedral domains has been recently studied in [59, 89, 91], in the context of games with singular controls. To conclude, despite many decades of research in the field, the nature of optimal controls is, in general, far from being completely understood, and this motivates our study.

The main goal of Chapter 2 is to provide sufficient conditions for the characterization of the optimal policy of the singular control problem specified by (1.1.1) and (1.1.2) as the solution to the related Skorokhod reflection problem. Despite in our setting the control is one dimensional, the multidimensional nature of the problem arises from the fact that the components of the state process are interconnected; in particular, the action of the controller on the first component of the state process can affect all the other components. We will show the claimed characterization under two main classes of assumptions in which the volatility matrix is constant or linearly dependent on the state. In both cases additional monotonicity assumptions are enforced to the running cost h and to the drift b. These structural conditions are satisfied in a relevant class of linear-quadratic models, and in some specific settings considered in the literature for which the problem of constructing the optimal control remained partially open (see [56, 55, 78, 77]). The strategy of our proof is inspired by [115] and can be summarized in three main steps:

1. We first derive important monotonicity properties on V_{x_1}. This is done by identifying V_{x_1} as the value of a related Dynkin game, through a variational formulation in the spirit of [56].

2. We construct solutions $\bar{v}^\varepsilon$ to a family of Skorokhod problems in domains $\mathcal{W}_\varepsilon$ *approximating* $\mathcal{W}$. Here the monotonicity of V_{x_1} plays a crucial role in order to show the regularity of $\mathcal{W}_\varepsilon$. The controls $\bar{v}^\varepsilon$ are ε-optimal for (1.1.2); i.e. $J(x;\bar{v}^\varepsilon) \leqslant V(x)+\varepsilon$.

3. We find a control $\bar{v}$ such that $\bar{v}^\varepsilon \to \bar{v}$, as $\varepsilon \to 0$. This implies that $\bar{v}$ is optimal for x, and, thanks to the properties of $\bar{v}^\varepsilon$, that $\bar{v}$ solves the Skorokhod problem on the original domain $\mathcal{W}$. This then provides the desired characterization of the optimal policy $\bar{v}$.

As a consequence of our result, some works (in particular [56] and [167]) in the literature on singular control can be revisited, and the characterization of optimal controls can provided under slightly different assumptions. In addition, our approach allows to treat the singular control problems with degenerate diffusion matrix studied in [78, 77]. The results apply to problems with monotone controls, and to the case in which increasing the underlying diffusion has a different cost than decreasing it. The approach presented in this chapter seems to be suitable to treat also singular control problems in the finite time-horizon.

Clearly, our results relate to stochastic differential equations (SDEs, in short) with reflecting boundary conditions, also known as Skorokhod reflection problems for SDEs. In this field, existence and uniqueness of strong solutions to reflected SDEs in convex time-independent domains with normal reflection was first shown in the seminal [154]. These results were then generalized to non-convex smooth domains with smooth oblique reflection in [125], and subsequently refined in [146]. Existence of strong solutions in a class of non-smooth domains has been proved in [74], and therefore generalized to the time-dependent case in [128]. This list is, however, far from being exhaustive, and we therefore refer the interested reader to [35, 36, 60, 61, 141, 153] and to the references therein. From this point of view, our results provide existence and uniqueness of a (strong) solution to a Skorokhod problem in which the domain is given by the non-coincidence set $\mathcal{W}$ of a solution of the variational inequality with gradient constraint (1.1.3), and in which the reflection direction is prescribed by its gradient.

An essential tool for our analysis is the connection between optimal stopping and stochastic singular control theory. This connection is known since the seminal [16], where the authors observed that the derivative of the value function of a singular control problem identifies with the value of an optimal stopping problem. Since then, this connection has been elaborated through different approaches (see [24, 28, 110], among others), until the more recent interpretation given in [123]. When the control is assumed to be of locally bounded variation, and the system has dynamics with independent components, with one of them being controlled, the space derivative of the value function of the control problem coincides with the value of a zero-sum game of stopping; i.e., a Dynkin game (cf. [27, 56, 55, 90, 111]). This connection was described in a multi-dimensional setting with interconnected dynamics in [56] and [55] by employing a variational formulation of the problem. In Chapter 2, we employ essentially the formulation and the techniques elaborated in [56], however extending their arguments to fit our convex setting. An important aspect that needs to be underlined is that these types of connections are known only when the control is assumed to be of dimension one. This represents the main reason why our problem is formulated for one-dimensional bounded variation controls.

1.2 Chapter 3: Submodular N-player games with singular controls

In Chapter 3, we consider a class of stochastic N-player games over a finite time-horizon in which each player, indexed by $i = 1, ..., N$, faces a multi-dimensional stochastic singular control problem of monotone-follower type.[2] On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions, consider an adapted multi-dimensional càdlàg (i.e., right-continuous with left limits) process L and, for $i = 1, ..., N$, multi-dimensional continuous semimartingales f^i with nonnegative components. We call *monotone-follower game* the game in which each player i is allowed to choose a multi-dimensional control ξ^i in the set of *admissible strategies*

$$\mathcal{A} := \begin{Bmatrix} \mathbb{F}\text{-adapted processes with nondecreasing,} \\ \text{nonnegative and càdlàg components} \end{Bmatrix},$$

in order to minimize the cost functional

$$J^i(\xi^i, \xi^{-i}) := \mathbb{E}\bigg[\int_0^T h^i(L_t, \xi^i_t, \xi^{-i}_t)\, dt + g^i(L_T, \xi^i_T, \xi^{-i}_T) + \int_{[0,T]} f^i_t \, d\xi^i_t \bigg],$$

where $\xi^{-i} := (\xi^j)_{j \neq i}$. Here $T < \infty$ and h^i and g^i are suitable nonnegative convex cost functions. Furthermore, we introduce a sequence of approximating games with regular controls in the following way. For each $n \in \mathbb{N}$, define the *n-Lipschitz game* as the game in which players are restricted to pick a Lipschitz control in the set of *admissible n-Lipschitz strategies*

$$\mathcal{U}_n = \{\xi \in \mathcal{A} \,|\, \xi \text{ is Lipschitz with Lipschitz constant smaller that } n \text{ and } \xi_0 = 0\},$$

in order to minimize the cost functionals J^i.

The number of contributions on games of singular controls is still quite limited (see [65], [80], [89], [91], [119], [151], [161]), although these problems have received an increasing interest in the recent years. We briefly discuss here some of these works. In [151] it is determined a symmetric Nash equilibrium of a monotone-follower game with

[2] Parts of this introduction and of Chapter 3 are already published in a joint work with Giorgio Ferrari, see [68].

symmetric payoffs (i.e., the cost functional is the same for all players), and it is provided a characterization of any equilibria through a system of first order conditions. The same approach is followed in [80] for a game in which players are allowed to choose a regular control and a singular control. A general characterization of Nash equilibria through the Pontryagin maximum principle approach has been investigated in the recent [161] for regular-singular stochastic differential games. Connections between nonzero-sum games of singular control and games of optimal stopping have been tackled in [65]. It is also worth mentioning some recent works on mean field games with singular controls (see [83] and [87]) and their connection to symmetric N-player games (see [91]). A complete analysis of a Markovian N-player stochastic game in which players can control an underlying diffusive dynamic through a control of bounded-variation is provided in the recent [89]. There, the authors derive a Nash equilibrium by solving a system of *moving* free boundary problems. General existence results for stochastic games with multi-dimensional singular controls and non-Markovian costs were, however, missing in the literature, and this has motivated our study.

The main contributions presented in Chapter 3 are the following.

1. Under submodularity conditions on the functions h^i and g^i, we establish the existence of Nash equilibria for the monotone-follower and the n-Lipschitz games.

2. We show connections across these two classes of games. In particular:

   (a) Any sequence obtained by choosing, for each $n \in \mathbb{N}$, a Nash equilibrium of the n-Lipschitz game is relatively compact in the Meyer-Zheng topology, and any accumulation point of this sequence is the law of a *weak Nash equilibrium* of the monotone-follower game (see Definition 5 below). That is, any accumulation point is a Nash equilibrium on a suitable probability space on which are defined two processes $\bar{f}$ and $\bar{L}$ such that their joint law coincides with the joint law of f and L.

   (b) The N-dimensional vector whose components are the expected costs associated to any weak Nash equilibrium obtained through the previous approximation is a *Nash equilibrium value*. Moreover, for each $\varepsilon > 0$, there exist $n_\varepsilon \in \mathbb{N}$ large enough and a Nash equilibium of the n_ε-Lipschitz game which is an ε-Nash equilibrium of the monotone-follower game.

Furthermore, we provide applications of our results to deduce existence of Nash equilibria for a class of stochastic differential games with singular controls and non-Markovian random costs. Also, in the spirit of [157], we construct an algorithm to determine a Nash equilibrium of the monotone-follower game.

We now provide more details on our results by discussing the ideas and techniques of their proofs.

The existence results. Going back to the seminal ideas of J. Nash, a typical way to prove existence of Nash equilibria is to show existence of a fixed point for the best-reply map. In the spirit of [157], our strategy to prove existence of Nash equilibria in the monotone-follower game and in the n-Lipschitz game is to exploit the submodular structure of our games in order to apply a lattice-theoretical fixed point theorem:

the Tarski's fixed point theorem (see [155]). We proceed as follows. We first endow the spaces of admissible strategies $\mathcal{A}$ and $\mathcal{U}_n$ (defined above) with a lattice structure. While the lattice $\mathcal{U}_n$ is complete, the same does not hold true for $\mathcal{A}$. To overcome this problem, we show that, under suitable assumptions, each "reasonable" strategy lives in a bounded subset of $\mathcal{A}$, and we restrict our analysis to this subset. We then prove that the best-reply maps are nonempty. To accomplish this task in the n-Lipschitz game, we employ the so-called classical *direct method*. Indeed, since each strategy is forced to be n-Lipschitz, then the sequence of time-derivatives of any minimizing sequence is bounded in $\mathbb{L}^2$. Hence, Banach-Saks' theorem, together with the lower semi-continuity and the convexity of the costs, allows to conclude existence of the minima. On the other hand, for the monotone-follower game we use some more recent techniques already employed to prove existence of optimizers in stochastic singular control problems (see [14]). Assuming a uniform coercivity condition on the costs (which has to be, anyway, necessarily satisfied in any Nash equilibria; see Remark 3.1.6 below) we can use a theorem by Y.M. Kabanov (see Lemma 3.5 in [104]) which gives relative sequential compactness, in the Cesàro sense, of any minimizing sequence. Then, exploiting again the lower semi-continuity and the convexity of the cost functions, we conclude existence of the minima. Next, we show that the best-reply maps preserve the order in the spaces of admissible strategies, and for this the submodular condition is essential. The existence then follows by invoking Tarski's fixed point theorem.

Our result also generalizes to the infinite time-horizon case (see Remark 3.1.9) and to the monotone-follower game in which players are allowed to choose both a regular control and a singular control (see Remark 3.1.8). Moreover, some of our assumptions are not needed if we impose *finite-fuel constraints* (see Remark 3.1.7).

It is worth stressing that our proof strongly hinges on the submodularity assumption, which, however, is a typical requirement in many problems arising in applications (see, e.g., [137], [157], [159], or the books [158] and [160] and the references therein).

The approximation results. Singular control problems naturally arise to overcome the ill-posedness of standard stochastic control problems in which the control linearly affects the dynamics of the state variable, and the cost of control is proportional to the effort. Some kind of connection between regular control problems with the linear structure described above and singular control problems is then expected, and actually already discussed in the literature (see, e.g, the early [135] for an analytical approach, and [123] for a probabilistic approach). In Theorem 21 of [123], it is shown that any sequence obtained by choosing, for each $n \in \mathbb{N}$, a minimizer of the monotone-follower problem when the class of admissible controls is restricted to the set of n-Lipschitz controls, suitably approximates a (weak) optimal solution to the original monotone-follower problem.

We prove that any sequence of Nash equilibria of the n-Lipschitz game is weakly relatively compact, and that any accumulation point is a weak Nash equilibrium of the monotone-follower game. We first show that this sequence satisfies a tightness criterion for the Meyer-Zheng topology. Then, we prove that any Nash equilibrium of the n-Lipschitz game necessarily satisfies a system of stochastic equations. After changing the underlying probability space by a Skorokhod representation, we pass to the limit in these systems of equations and deduce that any accumulation point solves a new system of stochastic equations. These equations can be viewed as a version of the Pontryagin maximum principle, and they are sufficient to ensure that the limit point is a Nash equilibrium in the new probability space, hence a weak Nash equilibrium.

As a byproduct of this result, we are able to show that, for each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ large enough such that the Nash equilibrium of the n-Lipschitz game is an ε-Nash equilibrium of the monotone-follower game. This gives a clearer interpretation of the weak Nash equilibrium found through the approximation: the N-dimensional vector whose components are the expected costs associated to the weak Nash equilibrium is, in fact, a *Nash equilibrium value* (as defined in [33]) of the monotone-follower game.

Applications and examples. Our existence result applies to deduce existence of open-loop Nash equilibria in stochastic differential games with singular controls and non-Markovian random costs, whenever a certain structure is preserved by the dynamics. For the sake of illustration, we consider the case in which the dynamics of the state variable of each player are a linearly controlled geometric Brownian motion and a linearly controlled Ornstein-Uhlenbeck process.

Moreover, we consider the algorithm introduced by Topkis (see Algorithm II in [157]) for submodular games: given as initial point the constantly null profile strategy, this algorithm consists of an iteration of the best-reply map. We show that, also in our setting, this algorithm converges to a Nash equilibrium.

1.3 Chapter 4: Submodular mean field games with regular and singular controls

Chapter 4 is devoted to the study of mean field games with submodular costs, in which the representative agent's minimization problem is either a regular control problem or a singular control problem.[3] More precisely, on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, consider a standard Brownian motion W, and an $\mathcal{F}_0$ random variable x_0. The set of regular controls $\mathcal{U}$ is given by the set of square integrable progressively measurable processes, while the set of singular controls $\mathcal{A}$ is the set of nonnegative nondecreasing càdlàg adapted processes. For any flow of probability measures $\mu = (\mu_t)_{t\in[0,T]}$, we can consider two types of representative agent's minimization problem. In the case of regular controls, this is given by

$$\begin{cases} \inf_{u\in\mathcal{U}} \mathbb{E}\Big[\int_0^T (f(t,X_t,\mu_t) + l(t,X_t,u_t))dt + g(X_T,\mu_T)\Big], \\ \text{subject to } dX_t = b(t,X_t,u_t)dt + \sigma dW_t,\ t\in[0,T], \quad X_0 = x_0. \end{cases}$$

On the other hand, for the MFG with singular controls, the optimal control problem is

$$\begin{cases} \inf_{\xi\in\mathcal{A}} \mathbb{E}\Big[\int_0^T f(t,X_t,\mu_t)dt + g(X_T,\mu_T) + \int_{[0,T]} c_t d\xi_t\Big], \\ \text{subject to } dX_t = b(t,X_t)dt + \sigma dW_t + d\xi_t,\ t\in[0,T], \quad X_{0-} = x_0. \end{cases}$$

[3] Parts of this introduction and of Chapter 4 are already published in a joint work with Markus Fischer, Giorgio Ferrari and Max Nendel, see [70].

In both cases, a flow of measures μ is said to be a MFG equilibrium (or solution) if the fixed point condition holds; that is, if

$$\mu_t = \mathbb{P} \circ (X_t^\mu)^{-1}, \quad \text{for any } t \in [0, T],$$

where X^μ is one of the optimally controlled state process, when optimizing against μ. The diffusion coefficient, while independent of state and control, is possibly degenerate. Deterministic dynamics are thus included as a special case.

Mean field games, as introduced by Lasry and Lions [121] and, independently, by Huang, Malhamé and Caines [99], are limit models for non-cooperative symmetric N-player games with mean field interaction as the number of players N tends to infinity; see, for instance, [42], [45] and the recent two-volume work [47]. The related notion of oblivious equilibra for infinite models was also developed independently in [164]. The submodularity assumption has been applied to MFGs by Adlaka and Johari in [2] for a class of discrete time games with infinite horizon discounted costs, by Więcek in [165] for a class of finite state MFGs with total reward up to a time of first exit, and by Carmona, Delarue, and Lacker in [49] for mean field games of timing (optimal stopping), in order to study dynamic models of bank runs in a continuous time setting.

General existence of solutions to the MFG problem can be obtained through Banach's fixed point theorem if the time horizon is small (cf. [99]). For arbitrary time horizons, a version of the Brouwer-Schauder fixed point theorem, including generalizations to multi-valued maps, can be used; cf. [42] and [120] (see also [83] in the context of MFGs with singular controls). In the presence of a common noise (i.e., an aggregate source of randomness), the existence of a weak MFG solution (i.e., not adapted to the common noise) can be established for a general class of MFGs. On the other hand, the existence of a strong MFG solution (i.e., adapted to the common noise) seems to be addressed mainly under conditions which imply the uniqueness of equilibria. For example, in [48] an analogue of the famous result of Yamada and Watanabe is derived, and it is used to prove existence and uniqueness of a strong solution under the Lasry-Lions monotonicity conditions (see [121]). Under lack of uniqueness, existence of strong solutions remains mainly an open question.

Since uniqueness of equilibria in game theory is the exception rather than the rule, it is not surprising that existence of multiple solutions is quite common also in MFGs. This phenomenon has been investigated mainly on a case by case basis, and specific examples with multiple solutions have been presented in the recent [15, 51, 67, 156], among others. Interestingly, the submodularity assumption appears implicitly in a number of classical linear-quadratic models (see, e.g., [22]) and in [15, 38, 51, 67], even if this property is not exploited therein. Therefore, the increasing interest in the non uniqueness of solutions together with the perspective of characterizing many models through a unique *key property* motivates our study on submodular MFGs.

Finally, the problem of how to find solutions to a MFGs in a constructive way is of interest too. This problem has been addressed by Cardaliaguet and Hadikhanloo [44]. They analyze a learning procedure, similar to what is known as fictitious play (cf. [98] and the references therein), where the representative agent, starting from an arbitrary flow of measures, computes a new flow of measures by updating the average over past measure flows according to the best-response to that average. For potential mean field games, the authors establish convergence of this kind of fictitious play via PDE methods. Similar approaches have been further developed in some recent works (see [75, 142, 166], among others) with the help of machine learning techniques, providing a rich set of tools in order to address computational aspects in MFGs.

Driven by this questions, the aim of Chapter 4 is to investigate existence, structure and approximation of equilibria for MFGs with regular or singular controls in which the measure-dependent costs are assumed to be submodular with respect to the first order stochastic dominance on measures and the standard order relation on states (cf. Assumption 4.1.9 below). In the context of MFGs, the submodularity represents the situation in which players have an incentive to imitate each other, and it consists of a sort of antithetic version of the well known Larsy-Lions monotonicity condition.

The main results of Chapter 4 highlight the submodularity as relevant structural condition for MFG models, and can be summarized as follows.

1. The submodularity assumption yields an alternative way of establishing the existence of MFG solutions using Tarski's fixed point theorem [155]. This allows us to cover systems with coefficients that are possibly discontinuous in the measure variable. Furthermore, our lattice-theoretical approach allows to prove existence of strong solutions to a class of MFGs with common noise in which the representative agent faces a mean field interaction through the conditional mean of its state given the common noise.

2. The set of MFG solutions enjoys a lattice structure, so that there are a minimal solution and a maximal solution with respect to the order relation.

3. The learning procedure which consists of iterating the best-response-map (thus computing a new flow of measures as best-response to the previous measure flow) converges to the minimal (or the maximal) MFG solution, for appropriately chosen initial measure flows.

These results are proven first for a representative class of MFGs with regular controls, under the additional assumption that the representative agent's minimization problem admits a unique optimal control. Therefore, we extend them to MFGs with relaxed controls (see, e.g., [120]), in order to deal with control problems in which multiple optimal controls are allowed. The results obtained for regular and relaxed controls allow then to derive analogous conclusions for MFGs with singular controls, via an approximation of monotone càdlàg processes through Lipschitz processes, which in turn can be seen as regular controls.

Our lattice-theoretical approach works as long as some "good properties" are preserved by the costs and the dynamics, and some further setups are presented for the sake of illustration. In particular (yet relevant) cases we can also prove existence of MFG solutions when the dynamics of the state process depends on the measure (see Subsection 4.4.4). Finally, although our results strongly hinge on the one-dimensional nature of the setting, suitable multidimensional cases can also be considered. In particular, if the dependence on the measure is only through one of its one-dimensional marginals, existence and approximation of MFG solutions can still be obtained in some settings (cf. Subsection 4.4.1).

The approach that we follow in this thesis focuses exclusively on the representative agent minimization problem, without reformulating the problem in terms of a related forward-backward system or of the master equation (see, e.g., [20, 43, 46]). Whether those reformulations of the mean field game problem allow to obtain results of a similar fashion of ours is, to the best of our knowledge, an open question that we leave for future research.

1.4 Chapter 5: Stationary mean field games with singular control

In Chapter 5, we consider two stationary MFG models with singular controls, in which the representative agent optimization problem consists in the maximization of a discounted or ergodic profit functional.[4]

Our models are motivated by the aim of providing an explicit construction of equilibria in games of productivity expansion. Our analysis allows to treat the mean field version of a symmetric dynamic oligopoly model, which is described as follows. In the pre-limit, each company can instantaneously increase via costly investment its productivity, which is affected by idiosyncratic noise modeling, e.g., exogenous technological shocks. In the spirit of Chapter 11 in [71] or [25], each unit of investment gives rise to a proportional cost, and investments do not need to be necessarily performed at rates; also singularly continuous actions and gulps are allowed. The operating profit of each company increases with its productivity and decreases with the average long-term weighted mean productivity of other firsms. In the limit, the representative company is then expected to react to a weighted mean of the population's stationary productivity. In Chapter 5, we abstract from this concrete application and study a stationary mean field game where: (i) the state variable of the representative agent is a nonnegative singularly controlled Itô-diffusion; (ii) the interaction among players is through the reward functional, in which instantaneous profits depend on a suitable weighted average of the state process with respect to the stationary distribution; (iii) the representative agent maximizes either a discounted net profit functional or its ergodic version. The study of the ergodic mean field game is particularly relevant when one considers decisions in the context of sustainable development and management of public goods, in which it might be important to take care of the payoffs received by successive generations.

We now describe more in detail the models studied in Chapter 5. On a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, consider a standard Brownian motion W and the set $\mathcal{A}$ of nonnegative nondecreasing càdlàg adapted processes. For any $x > 0$ and $\xi \in \mathcal{A}$, denote by $X^{x;\xi}$ the solution to the singularly controlled Itô-diffusion

$$dX_t^{x;\xi} = b(X_t^{x;\xi})dt + \sigma(X_t^{x;\xi})dW_t + d\xi_t,\ t \geqslant 0, \quad X_{0-}^{x;\xi} = x.$$

The uncontrolled underlying state process is assumed to have 0 and $+\infty$ as natural boundaries. For any parameter $\theta > 0$, we consider discounted profit functionals J, and

[4] Parts of this introduction and of Chapter 5 are based on a joint work with Haoyang Cao and Giorgio Ferrari, see [40].

an ergodic profit G, respectively given by

$$J(x,\xi,\theta;r) := \mathbb{E}\bigg[\int_0^\infty e^{-rt}h(X_t^{x;\xi},\theta)dt - \int_{[0,T]} e^{-rt}d\xi_t\bigg], \quad \text{for } r>0,$$

and

$$G(x,\xi,\theta) := \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\bigg[\int_0^T h(X_t^{x;\xi},\theta)dt - \xi_T\bigg].$$

A parameter $\bar\theta > 0$ is a solution for the discounted (resp., ergodic) MFG if the optimally controlled process $\bar X^x$ for the profit functional $J(x,\cdot,\bar\theta;r)$ (resp., $G(x,\cdot,\bar\theta)$) admits a limiting distribution $\mathbb{P}_{\bar X^x_\infty}$ satisfying $\bar\theta = \theta(\mathbb{P}_{\bar X^x_\infty})$. Here, for two positive increasing functions f and F, we set

$$\theta(\mu) := F\bigg(\int_{\mathbb{R}_+} f(x)d\mu(x)\bigg),$$

for any probability measure μ such that $\int_{\mathbb{R}_+} f(x)d\mu(x) < \infty$.

The results in Chapter 5 contribute to the literature on mean field games with continuous time and continuous state-space. The closest work to ours is the recent paper [41], where the authors study a stationary discounted mean field game with two-sided singular controls, and its relation to the associated N-player game. However, differently to our general diffusive model, in [41] the dynamics of the state process is a geometric Brownian motion and the relation to the ergodic formulation of the mean field game is not addressed. In [38] and [91] mean field and N-player stochastic games for finite-fuel follower problems are studied, and the structure of equilibria is obtained. Finally, [83] provides a careful technical analysis of the existence of solutions to general mean field games involving singular controls.

The main contributions presented in Chapter 5 are the following. First of all, we are able to construct the unique mean field stationary equilibrium, both for a discounted and an ergodic reward functional. In both cases, the equilibrium control is of barrier-type: there exists an endogenously determined threshold $\bar x$ at which it is optimal to reflect the state process upward in a minimal way (i.e. according to a so-called Skorokhod reflection; see, e.g., Chapter 6 in [93]). The equilibrium stationary distribution is given by a truncated version of the speed-measure of the underlying Itô-diffusion; that is, it coincides with the speed measure on $[\bar x,\infty)$ and it is zero otherwise. The equilibrium triggers are characterized as the unique solutions to some nonlinear equations involving marginal profits, marginal cost of investment, and characteristic quantities of the Itô-diffusion. Those equations can be easily solved numerically. The approach leading to such a complete characterization of the discounted and ergodic mean field equilibria is as follows: we fix the stationary average θ of the population, and we solve one-dimensional stochastic singular control problem parametrized by θ. In line with [4, 5, 102, 103, 127], among others, we find that, for each given θ, it is optimal to reflect the state upward at some $\bar x(\theta)$. We then impose that the value of θ at equilibrium is the one which is computed through the stationary distribution of the state process reflected at $\bar x(\theta)$. We prove that the resulting fixed point problem admits a unique solution $\bar\theta$, which, in turn, leads to the equilibrium trigger $\bar x := \bar x(\bar\theta)$. It is worth observing that a byproduct of our analysis is the solution to a class of ergodic stochastic singular control problems, via exploiting a connection to optimal stopping in the spirit of [107].

Second, we can show that the so-called *Abelian limit* holds for our mean field games. This means that, when the representative agent discounts profits and costs at a rate r decreasing to zero, the expected reward associated to the mean field equilibrium of the r-discounted problem, multiplied by r, converges to a constant. The latter actually is the equilibrium value of the ergodic mean field game. Moreover, also the barrier triggering the equilibrium control of the discounted problem converges to that of the ergodic problem when $r \downarrow 0$. The proof of such a convergence requires a careful analysis of the dependency with respect to r of the equilibrium trigger and average arising in the discounted game. This is possible by analyzing the continuity with respect to r of the solution to the system of equations uniquely defining the equilibria.

A natural question is whether the determined mean field equilibria approximate the corresponding symmetric N-player games. Moreover, in light of the Abelian limit, one can wonder whether the mean field equilibrium for the discounted problem relates to ε-equilibria in the ergodic symmetric N-player game. The study of these two questions represents the third main contribution of Chapter 5. We introduce ergodic and discounted N-player symmetric games where each player reacts to the long-time average of an increasing function of a weighted mean of the opponents' states. We then show that the mean field equilibria of the ergodic and discounted problems realize an ε_N-Nash equilibrium for those N-player games, with ε_N converging to zero as N goes to infinity. Furthermore, when N is large and r is small, the validity of the Abelian limit allows to prove that the equilibrium of the discounted mean field game approximates a Nash equilibrium of the ergodic N-player singular control game. While N-player games with singular controls have already attracted some attention in the recent literature (see, among others, [65, 68, 80, 89, 91, 118, 119]), to our knowledge, singular control games with ergodic criterion have not yet been investigated. Thus, the previous approximation result sheds light on a novel class of dynamic stochastic games which naturally arise in applications.

1.5 Outline of the thesis

This thesis consists of 5 chapters and two appendices.

Chapter 2 addresses the problem of characterizing optimal singular controls, and it is organised as follows. In Section 2.1 we formulate the problem, we enforce some structural conditions, and we state the main result. The proof of the main result for a constant volatility is presented in Section 2.2, while the proof for a linear volatility is discussed in Section 2.3. Extensions and examples are provided in Section 2.4, while Section 2.5 and Section 2.6 are devoted to some auxiliary technical results.

In Chapter 3 we study N-player games with singular controls. In Section 3.1.1 we introduce the monotone-follower game. Sections 3.1.2 and 3.2 are devoted to the existence theorems of Nash equilibria for the submodular monotone-follower game and for the n-Lipschitz game, respectively. The approximation results are contained in Section 3.3. The application of our result to suitable stochastic differential games is provided in Section 3.4, together with the proof of the convergence to a Nash equilibrium of a certain algorithm.

Chapter 4 is devoted to the study of submodular MFGs. In Subsection 4.1.1, we introduce the controlled system dynamics and costs, together with our standing assumptions, and give the definition of a mean field game, where we take regular controls as admissible strategies. In Subsection 4.1.2, we define the order relation on probability measures which is crucial for the submodularity assumption on the cost coefficients of the game. That assumption is stated and discussed in Subsection 4.1.3, while Subsection 4.1.4 deals with properties of the best-response-map. Subsection 4.1.5 contains our main results for MFGs with regular controls. In Section 4.2, we extend the analysis of Section 4.1 to submodular mean field games defined over stochastic relaxed controls. This allows to re-obtain the existence and, especially, the convergence result under more general conditions. Finally, in Section 4.3 we derive similar results for MFGs with singular controls. Section 4.4 concludes with comments on the multidimensional setting, the linear-quadratic case, systems with multiplicative and mean field dependent dynamics, while some models with common noise are discussed in Section 4.5.

The stationary MFG models with singular controls are presented in Chapter 5. In Section 5.1 the probabilistic setting is introduced, while the mean field games are presented in Section 5.2. Section 5.3 collects the results of existence and uniqueness of mean field equilibria, and Section 5.4 derives the Abelian limit. The relation between the considered mean field games and their related symmetric N-player games is then discussed in Section 5.5.

In Appendix A we recall some results about the Meyer-Zheng topology, while some auxiliary results on first order stochastic dominance are collected in the Appendix B.

Chapter 2

Singular control and related Skorokhod problem

In this chapter, we characterize the optimal control for a class of stochastic singular control problems as the unique solution to a related Skorokhod reflection problem. The considered optimization problems concern the minimization of a discounted cost functional over an infinite time-horizon through a process of bounded variation affecting an Itô-diffusion. The setting is multidimensional, the dynamics of the state and the costs are convex, the volatility matrix can be constant or linear in the state. We prove that the optimal control acts only when the underlying diffusion attempts to exit the so-called waiting region, and that the direction of this action is prescribed by the derivative of the value function. Our approach is based on the study of a suitable monotonicity property of the derivative of the value function through its interpretation as the value of an optimal stopping game. Such a monotonicity allows to construct nearly optimal policies which reflect the underlying diffusion at the boundary of approximating waiting regions. The limit of this approximation scheme then provides the desired characterization. The main result of this chapter applies to a relevant class of linear-quadratic models, among others. Furthermore, it allows to construct the optimal control in degenerate and non degenerate settings considered in the literature, where this aspect was only partially addressed.

2.1 Problem formulation and main result

2.1.1 Singular control and Skorokhod problem

Fix $d \in \mathbb{N}$, $d \geqslant 2$, and a d-dimensional Brownian motion $W = (W^1, ..., W^d)$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions. For each $x = (x_1, ..., x_d) \in \mathbb{R}^d$, let the process $X^x = (X^{1,x}, ..., X^{d,x})$ denote the solution to the stochastic differential equation (SDE, in short)

$$\begin{cases} dX_t^{1,x} = (a_1 + b_1^1 X_t^{1,x})dt + \bar{\sigma}(X_t^{1,x})dW_t^1, & t \geqslant 0, \quad X_{0-}^{1,x} = x_1, \\ dX_t^{i,x} = b^i(X_t^{1,x}, X_t^{i,x})dt + \bar{\sigma}(X_t^{i,x})dW_t^i, & t \geqslant 0, \quad X_{0-}^{i,x} = x_i, \quad i = 2, ..., d. \end{cases} \tag{2.1.1}$$

Here a_1, b_1^1 are constants, while the coefficients $b^i \in C(\mathbb{R}^2)$ and $\bar{\sigma} \in C(\mathbb{R})$ are deterministic Lipschitz continuous functions. The drift $\bar{b}(x) := (a_1 + b_1^1 x_1, b^2(x_1, x_2), .., b^d(x_1, x_d))^\top$ and the function $\bar{\sigma}$ satisfy Assumption 2.1.1 below. Next, introduce the set of *admissible controls* as

$$\mathcal{V} := \{\mathbb{R}\text{-valued } \mathbb{F}\text{-adapted and càdlàg processes with locally bounded variation}\},$$

and, for each $v \in \mathcal{V}$ and $x \in \mathbb{R}^d$, let the process $X^{x;v} = (X^{1,x;v}, ..., X^{d,x;v})$ denote the unique strong solution to the controlled stochastic differential equation

$$\begin{cases} dX_t^{1,x;v} = (a_1 + b_1^1 X_t^{1,x;v})dt + \bar{\sigma}(X_t^{1,x;v})dW_t^1 + dv_t, & t \geqslant 0,\ X_{0-}^{1,x;v} = x_1, \\ dX_t^{i,x;v} = b^i(X_t^{1,x;v}, X_t^{i,x;v})dt + \bar{\sigma}(X_t^{i,x;v})dW_t^i, & t \geqslant 0,\ X_{0-}^{i,x;v} = x_i,\ i = 2, ..., d. \end{cases} \tag{2.1.2}$$

For any given initial condition $x \in \mathbb{R}^d$, consider the problem of minimizing the expected discounted cost

$$J(x; v) := \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X_t^{x;v})dt + \int_{[0,\infty)} e^{-\rho t} d|v|_t\bigg], \quad v \in \mathcal{V}, \tag{2.1.3}$$

where $|v|$ denotes the total variation of the process v, $h : \mathbb{R}^d \to \mathbb{R}$ is a continuous function, and $\rho > 0$ is a constant discount factor. We will say that the control $\bar{v} \in \mathcal{V}$ is optimal if

$$V(x) := \inf_{v \in \mathcal{V}} J(x; v) = J(x; \bar{v}), \tag{2.1.4}$$

and, in the following, we will refer to the function V as to the value function of the problem, and to $X^{x;\bar{v}}$ as to the optimal trajectory.

The second integral appearing in (2.1.3) has to be understood in the Lebesgue-Stieltjes sense, and it is defined as

$$\int_{[0,\infty)} e^{-\rho t} d|v|_t := |v|_0 + \int_0^\infty e^{-\rho t} d|v|_t, \tag{2.1.5}$$

in order to take into account possible jumps of the control at time zero. Moreover, for $v \in \mathcal{V}$ we will often write $dv = \gamma d|v|$ to denote the disintegration

$$v_t = \int_0^t \gamma_s d|v|_s, \quad \text{for each } t \geqslant 0,\ \mathbb{P}\text{-a.s.},$$

where $|v|$ denotes the total variation of the signed measure v, and the process γ is the Radon-Nikodym derivative of the signed measure v with respect to $|v|$. Also, for a control v, the nondecreasing càdlàg processes ξ^+, ξ^- will denote the minimal decomposition of the signed measure v; that is, $v = \xi^+ - \xi^-$, and $\xi^+ \leqslant \bar{\xi}^+$ and $\xi^- \leqslant \bar{\xi}^-$ for any other couple of nondecreasing càdlàg processes $\bar{\xi}^+$, $\bar{\xi}^-$ which satisfy $v = \bar{\xi}^+ - \bar{\xi}^-$.

Finally, recall from [115] the following notion of solution to the Skorokhod problem, which we adapt to our setting.

Definition 1. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$ with closure $\overline{\mathcal{O}}$, $x \in \overline{\mathcal{O}}$, and set $S := \partial\mathcal{O}$. Let $\bar{\nu}$ be a continuous vector field on S, with $\bar{\nu} = e_1\nu$ and $|\nu(y)| = 1$ for each $y \in S$.*

We say that the process $v \in \mathcal{V}$ is a solution to the modified Skorokhod problem for the SDE (2.1.2) in $\overline{\mathcal{O}}$ starting at x with reflection direction $\bar{\nu}$ if

1. $\mathbb{P}[X_t^{x;v} \in \overline{\mathcal{O}}, \forall t \geqslant 0] = 1$;
2. $\mathbb{P}$-a.s., for each $t \geqslant 0$ one has $dv = \gamma d|v|$, with
$$|v|_t = \int_0^t \mathbb{1}_{\{X_{s-}^{x;v} \in S,\, \nu(X_{s-}^{x;v}) = \gamma_s\}} d|v|_s;$$
3. $\mathbb{P}$-a.s., for each $t \geqslant 0$, a possible jump of the process $X^{x;v}$ at time t occurs on some interval $I \subset S$ parallel to the vector field $\bar{\nu}$; i.e., such that $\bar{\nu}(y)$ is parallel to I for each $y \in I$. If $X^{x;v}$ encounters such an interval I, it instantaneously jumps to its endpoint in the direction $\bar{\nu}$ on I.

Moreover, if v is continuous, then we say that v is a solution to the (classical) Skorokhod problem for the SDE (2.1.2) in $\overline{\mathcal{O}}$ starting at x with reflection direction $\bar{\nu}$.

2.1.2 Assumptions and main result

The main objective of this chapter is to characterize optimal control policies for Problem (2.1.4) as solutions to related Skorokhod problems.

We will prove our main result under the following structural conditions, which we enforce throughout the rest of this chapter. We postpone the discussion of some generalizations to Section 2.4.

Assumption 2.1.1. *For $p \geqslant 2$ we have:*

1. *The running cost h is $C^{2;1}(\mathbb{R}^d)$, convex, and, for suitable constants $K, \kappa_1, \kappa_2 > 0$, it satisfies, for each $x, y \in \mathbb{R}^d$ and for all $\lambda \in [0,1]$, the conditions*
$$\begin{aligned} \kappa_1|x_1|^p - \kappa_2 \leqslant h(x) &\leqslant K(1+|x|^p),\\ |h(y)-h(x)| &\leqslant K(1+|x|^{p-1}+|y|^{p-1})|y-x|,\\ \lambda h(x) + (1-\lambda)h(x) - h(\lambda x + (1-\lambda)y) &\leqslant K\lambda(1-\lambda)(1+|x|^{p-2}+|y|^{p-2})|x-y|^2,\\ 0 &< h_{x_1x_1}(x). \end{aligned}$$
2. *There exists a constant $\bar{L} \geqslant 0$ such that, for each $x, y \in \mathbb{R}^d$, we have*
$$\begin{aligned} |\bar{b}(x)| &\leqslant \bar{L}(1+|x|),\\ |\bar{b}(y)-\bar{b}(x)| &\leqslant \bar{L}|y-x|. \end{aligned}$$
The functions b^i are convex of class $C^{2;1}(\mathbb{R}^d)$. Furthermore, we assume that $h_{x_i} \geqslant 0$ and $b^i_{x_1}$, $b^i_{x_1x_i}$, $h_{x_1x_i} \leqslant 0$ for each $i = 2, ..., d$, and that $D\bar{b}$ is globally Lipschitz.
3. *For $\rho^* := p(2p-1)$ and a constant $\sigma > 0$, either of the two conditions below is satisfied:*
   (a) *$\bar{\sigma}(y) = \sigma$, $y \in \mathbb{R}$, and the discount factor satisfies the relation $\rho > 3\rho^*\bar{L}$.*
   (b) *$\bar{\sigma}(y) = \sigma y$, $y \in \mathbb{R}$, and the discount factor satisfies the relation $\rho > 2\rho^*(\bar{L} + \sigma^2(\rho^*-1))$. In this case, we also assume that there exists $x_1^* > 0$ such that $h_{x_1}(x) \leqslant \min\{0, -b^1_1\}$ for each x with $x_1 < 2x_1^*$, that $b^i(x_1, x_i) \geqslant 0$ for $x_1, x_i \geqslant 0$ for each $i = 2, ..., d$, and that $a_1 \geqslant 0$.*

Natural examples in which the conditions above are satisfied are given –after discussing generalizations of Assumption 2.1.1– in Section 2.4. These include a relevant class of *linear-quadratic* stochastic singular control problems (see Example 1 and Subsection 2.4.4 below). Notice that the nature of problem (2.1.4) is genuinely multidimensional, as the components of the dynamics (2.1.2) are interconnected.

Remark 2.1.2 (On the role of Assumption 2.1.1)**.** *We underline that the particular choice of $p \geqslant 2$ is motivated by quadratic running costs (cf. Example 1 in Section 2.4). From Condition 2 one can see that quite strong requirements are needed in order to treat models with a general b^i. However, when b^i has a simpler form, some conditions on the derivatives $b^i_{x_1}$, $b^i_{x_1x_i}$, h_{x_i}, $h_{x_1x_i}$ can be weakened (see Subsections 2.4.1 and 2.4.1). Also, the assumption on h_{x_1} in Condition 3b is to enforce that the optimal trajectories live in the set $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d |\, x_i > 0,\ i = 1, ..., d\}$, whenever the initial condition $x \in \mathbb{R}^d_+$ (cf. Lemma 2.3.1 below). This condition is a natural substitute, for minimization problems in dimension $d \geqslant 2$, of the classical Inada condition at 0 (see, e.g., equation (2.5) in [88]). The latter, is typically assumed in profit maximization problems, and it is satisfied by Cobb-Douglas production functions. Finally, the conditions on the discount factor ρ are in place in order to ensure a suitable "integrability" of the optimal trajectories, which allows to prove some semiconcavity estimates for the value function V (see steps 2 and 3 in the proof of Theorem 2.5.1 in Section 2.5).*

Observe that, when Condition 3a is in place, a generic controlled trajectory $X^{x;v}$, $v \in \mathcal{V}$, can reach the whole space with probability $\mathbb{P} > 0$. On the other hand, under Condition 3b, as mentioned in Remark 2.1.2, the natural domain for a controlled trajectory is $\mathbb{R}^d_+$. This suggest to define a domain D in the following way

$$D := \mathbb{R}^d \text{ if Condition 3a holds}, \quad D := \mathbb{R}^d_+ \text{ if Condition 3b holds}. \tag{2.1.6}$$

Indeed, it is possible to show that the value function V is finite and it is a convex solution in $W^{2;\infty}_{loc}(D)$ of the Hamilton-Jacobi-Bellman (HJB, in short) equation

$$\max\{\rho V - \mathcal{L}V - h, |V_{x_1}| - 1\} = 0, \quad \text{a.e. in } D, \tag{2.1.7}$$

where $\mathcal{L}V(x) := \bar{b}(x)DV(x) + \frac{1}{2}\sum_{i=1}^d \bar{\sigma}^2(x_i)V_{x_ix_i}(x),\ x \in D$, is the generator of the uncontrolled SDE (2.1.1). For completeness, a proof of this result is provided in Section 2.5 (see Theorem 2.5.1). During the proof of Theorem 2.5.1, the convergence of a certain penalization method is studied: This convergence will be a useful tool in many of the proofs in this chapter.

Define next the *waiting region* $\mathcal{W}$ as

$$\mathcal{W} := \{x \in D \,|\, |V_{x_1}(x)| < 1\}, \tag{2.1.8}$$

and notice that, by the $W^{2;\infty}_{loc}$-regularity of V, $\mathcal{W}$ is an open subset of D. Also, for each $z \in \mathbb{R}^{d-1}$, we define the sets

$$D_1(z) := \{y \in \mathbb{R} \,|\, (y, z) \in D\} \quad \text{and} \quad \mathcal{W}_1(z) := \{y \in \mathbb{R} \,|\, (y, z) \in \mathcal{W}\}.$$

In the sequel, the closure of $\mathcal{W}$ (resp. $\mathcal{W}_1(z)$) in D (resp. $D_1(z)$) will be denoted by $\overline{\mathcal{W}}$ (resp. $\overline{\mathcal{W}}_1(z)$). We state here a technical lemma, whose proof is given in Section 2.6.

Lemma 2.1.3. *For any $x = (x_1, z) \in D$, with $z \in \mathbb{R}^{d-1}$, the set $\mathcal{W}_1(z)$ is a nonempty open interval; in particular, $\mathcal{W}$ is nonempty.*

Remark 2.1.4 (Existence and uniqueness of optimal controls)**.** *For each $\bar{x} \in D$, it is possible to show that, under Assumption 2.1.1, there exists a unique optimal control $\bar{v} \in \mathcal{V}$. This is a classical result when the drift is affine. In the case of a convex drift, it essentially follows from the convexity of J w.r.t. (x, v). The latter in turn follows from the convexity of the drift, the monotonicity of h, and a comparison principle for SDEs. The argument can be recovered from the proof of Lemma 2.2.7 below, which works for any sequence of controls minimizing the cost functional J. Finally, the uniqueness of the optimal control is a consequence of the strict convexity of h in the variable x_1.*

The following is the main result of this chapter, characterizing the optimal policies in terms of the waiting region $\mathcal{W}$ and the derivative V_{x_1} in the sense of Definition 1.

Theorem 2.1.5. *Let $\bar{x} = (\bar{x}_1, \bar{z}) \in D$, with $\bar{z} \in \mathbb{R}^{d-1}$. The following statements hold true:*

1. *If $\bar{x} \in \overline{\mathcal{W}}$, then the optimal control $\bar{v}$ is the unique solution to the modified Skorokhod problem for the SDE (2.1.2) in $\overline{\mathcal{W}}$ starting at $\bar{x}$ with reflection direction $-V_{x_1}e_1$;*

2. *If $\bar{x} \notin \overline{\mathcal{W}}$, then the optimal control $\bar{v}$ can be written as $\bar{v} = \bar{y}_1 - \bar{x}_1 + \bar{w}$, where $\bar{y}_1$ is the metric projection of $\bar{x}_1$ into the set $\overline{\mathcal{W}}_1(\bar{z})$, and $\bar{w}$ is the unique solution to the modified Skorokhod problem for the SDE (2.1.2) in $\overline{\mathcal{W}}$ starting at $\bar{y} := (\bar{y}_1, \bar{z})$ with reflection direction $-V_{x_1}e_1$.*

In Section 2.2 we provide a proof of Theorem 2.1.5 under Condition 3a in Assumption 2.1.1. The strategy of the proof can be summarized in three main steps:

Step a. In Subsection 2.2.1 we study an important monotonicity property of V_{x_1}, through a connection with Dynkin games.

Step b. In Subsection 2.2.2, this property will allow us to construct ε-optimal policies as solutions to Skorokhod problems in domains $\overline{\mathcal{W}}_\varepsilon$ approximating $\overline{\mathcal{W}}$.

Step c. Finally, in Subsection 2.2.3 we prove that the ε-optimal policies approximate the optimal policy, and that the latter is a solution to the Skorokhod problem in the original domain $\overline{\mathcal{W}}$.

The proof of Theorem 2.1.5 under Condition 3b in Assumption 2.1.1 follows similar rationales, and it is discussed in Section 2.3. In particular, in Subsections 2.3.1 a preliminary lemma is proved, while in Subsection 2.3.2 we show how to use this lemma in order to repeat (with minor modifications) the arguments of Section 2.2.

2.2 Proof of Theorem 2.1.5 for constant volatility

In this section we assume that Condition 3a in Assumption 2.1.1 holds. To simplify the notation, the proof is given for $d = 2$, so that $D = \mathbb{R}^2$. The generalization to the case $d > 2$ is straightforward.

2.2.1 Step a: A connection to Dynkin games and the monotonicity property

In this subsection we adopt an approach based on the variational formulation of the problem in order to show, in the spirit of [56], a connection between the singular control problem (2.1.4) and a Dynkin game. This connection will enable us to prove a monotonicity property of V_{x_1}, which will be then fundamental in order to construct ε-optimal controls.

The related Dynkin game

We begin by characterizing V_{x_1} as a $W^{2;\infty}_{loc}$-solution to a *two-obstacle problem*. The proof of the next result borrows arguments from [56] (see in particular Theorem 3.9, Proposition 3.10, and Theorem 3.11 therein). However, since in our case b can be convex, the techniques used in [56] needs to be refined, and used along with suitable estimates (described more in detail in the proof of Theorem 2.5.1 in Section 2.5) on a penalization method. We provide a detailed proof for the sake of completeness.

Theorem 2.2.1. *The function V_{x_1} is a $W^{2;\infty}_{loc}(\mathbb{R}^2)$-solution to the equation*

$$max\{(\rho - b^1_1)V_{x_1} - \mathcal{L}V_{x_1} - \hat{h}, |V_{x_1}| - 1\} = 0, \quad a.e.\ in\ \mathbb{R}^2, \tag{2.2.1}$$

where $\hat{h} := h_{x_1} + b^2_{x_1} V_{x_2}$.

Proof. We organize the proof in two steps.

Step 1. In this step we show that the function V_{x_1} is a solution to a variational inequality with a local operator, and that $V_{x_1} \in W^{2;\infty}_{loc}(\mathbb{R}^2)$. Fix $B \subset \mathbb{R}^2$ open bounded and consider a nonnegative localizing function $\psi \in C^\infty_c(B)$. Define the sets

$$\mathcal{K} := \left\{ U \in W^{1;2}_{loc}(\mathbb{R}^2) \,|\, |U| \leqslant 1 \text{ a.e.} \right\} \quad \text{and} \quad \mathcal{K}_\psi := \{\psi U \,|\, U \in \mathcal{K}\}.$$

We show in the sequel that the function $W := V_{x_1}\psi$ is a solution in $\mathcal{K}_\psi$ to the variational inequality

$$A_B(W, U - W) \geqslant \langle \hat{H}, U - W \rangle_B, \quad \text{for each } U \in \mathcal{K}_\psi, \tag{2.2.2}$$

where $\hat{H} := \hat{h}\psi - V_{x_1}\mathcal{L}\psi - DV_{x_1}D\psi$, the operator $A_B : W^{1;2}(B) \times W^{1;2}(B) \to \mathbb{R}$ is given by

$$A_B(\bar{U}, U) := \frac{\sigma^2}{2}\sum_{i=1}^{2}\langle \bar{U}_{x_i}, U_{x_i}\rangle_B - \langle \bar{b}D\bar{U}, U\rangle_B + (\rho - b^1_1)\langle \bar{U}, U\rangle_B \quad \text{for each } \bar{U}, U \in W^{1;2}(B),$$

and $\langle \cdot, \cdot \rangle_B$ denotes the scalar product in $L^2(B)$.

Let us begin by introducing a family of penalized versions of the HJB equation (2.1.7). Let $\beta \in C^\infty(\mathbb{R})$ be a convex nondecreasing function with $\beta(r) = 0$ if $r \leqslant 0$ and $\beta(r) = 2r - 1$ if $r \geqslant 1$. For each $\varepsilon > 0$, let V^ε be defined as in (2.5.2). As in Step 1 in the proof of Theorem 2.5.1 in Section 2.5, V^ε is a C^2-solution to the partial differential equation

$$\rho V^\varepsilon - \mathcal{L}V^\varepsilon + \frac{1}{\varepsilon}\beta((V^\varepsilon_{x_1})^2 - 1) = h, \quad x \in \mathbb{R}^2. \tag{2.2.3}$$

It is possible to show (see Step 2 in the proof of Theorem 2.5.1 in Section 2.5) that, for each $R > 0$, there exists a constant C_R such that

$$\sup_{\varepsilon\in(0,1)} \|V^\varepsilon\|_{W^{2;\infty}(B_R)} \leqslant C_R. \tag{2.2.4}$$

Moreover (as in (2.5.18) in the proof of Theorem 2.5.1), as $\varepsilon \to 0$, on each subsequence we have:

$$\begin{aligned} &(V^\varepsilon, DV^\varepsilon) \text{ converges to } (V, DV) \text{ uniformly in } B_R; \\ &D^2V^\varepsilon \text{ converges to } D^2V \text{ weakly in } L^2(B_R). \end{aligned} \tag{2.2.5}$$

We now show that $V_{x_1} \in \mathcal{K}$. Since the $W^{1;2}_{loc}$-regularity of V_{x_1} is already known (cf. Theorem 2.5.1 in Section 2.5), we only need to show that $|V_{x_1}| \leqslant 1$ in $\mathbb{R}^2$. To this end, take $R > 0$ and observe that, by (2.2.4) and (2.2.3), we have

$$\sup_{\varepsilon\in(0,1)} \|\beta((V^\varepsilon_{x_1})^2 - 1)\|_{L^2(B_R)} \leqslant C_R\varepsilon, \tag{2.2.6}$$

where the constant $C_R > 0$ does not depend on ε. Moreover, unless to consider a larger C_R, by the estimate (2.2.4) and the definition of β, we also have the pointwise estimate

$$|\beta((V^\varepsilon_{x_1})^2 - 1)| \leqslant 2((V^\varepsilon_{x_1})^2 + 1) \leqslant C_R, \quad \text{on } B_R, \text{ for each } \varepsilon \in (0, 1). \tag{2.2.7}$$

Therefore, the limits in (2.2.5) and the estimates (2.2.7) allow to invoke the dominated convergence theorem to deduce, thanks to (2.2.6), that

$$\|\beta((V_{x_1})^2 - 1)\|_{L^2(B_R)} = \lim_{\varepsilon\to 0} \|\beta((V^\varepsilon_{x_1})^2 - 1)\|_{L^2(B_R)} = 0.$$

Since R is arbitrary, we conclude that $|V_{x_1}| \leqslant 1$ a.e. in $\mathbb{R}^2$, and therefore that $W \in \mathcal{K}_\psi$.

We continue by proving (2.2.2). Since V^ε is a solution to (2.2.3), a standard bootstrapping argument (using Theorem 6.17 at p. 109 in [85]) allows to improve the regularity of V^ε and to prove that $V^\varepsilon \in C^4$. Therefore, we can differentiate equation (2.2.3) with respect to x_1 in order to get an equation for $V^\varepsilon_{x_1}$. That is,

$$[(\rho - b^1_1) - \mathcal{L}]V^\varepsilon_{x_1} + \frac{2}{\varepsilon}\beta'((V^\varepsilon_{x_1})^2 - 1)V^\varepsilon_{x_1}V^\varepsilon_{x_1x_1} = \hat{h}^\varepsilon, \quad x \in \mathbb{R}^2, \tag{2.2.8}$$

where we have defined $\hat{h}^\varepsilon := h_{x_1} + b^2_{x_1}V^\varepsilon_{x_2}$. Moreover, by (2.2.8), the localized function $V^\varepsilon_\psi := V^\varepsilon_{x_1}\psi$ is a solution to the equation

$$[(\rho - b^1_1) - \mathcal{L}]V^\varepsilon_\psi + \frac{2}{\varepsilon}\beta'((V^\varepsilon_{x_1})^2 - 1)V^\varepsilon_\psi V^\varepsilon_{x_1x_1} = \hat{H}^\varepsilon, \quad x \in \mathbb{R}^2, \tag{2.2.9}$$

where $\hat{H}^\varepsilon := \hat{h}^\varepsilon\psi - V^\varepsilon_{x_1}\mathcal{L}\psi - DV^\varepsilon_{x_1}D\psi$.

Let now $U \in \mathcal{K}_\psi$. Taking the scalar product of (2.2.9) with $U - V^\varepsilon_\psi$, an integration by parts gives

$$A_B(V^\varepsilon_\psi, U - V^\varepsilon_\psi) + \frac{2}{\varepsilon}\langle\beta'((V^\varepsilon_{x_1})^2 - 1)V^\varepsilon_\psi V^\varepsilon_{x_1x_1}, U - V^\varepsilon_\psi\rangle_B = \langle\hat{H}^\varepsilon, U - V^\varepsilon_\psi\rangle_B. \tag{2.2.10}$$

Moreover, since $\sigma > 0$, the operator $\left(\frac{\sigma^2}{2}\sum_{i=1}^2 \langle U_{x_i}, U_{x_i}\rangle_B\right)^{1/2}$, $U \in W^{1;2}(B)$, defines a norm on $W_0^{1;2}(B)$, and it is therefore lower semi-continuous with respect to the weak convergence in $W_0^{1;2}(B)$. By the limits in (2.2.5), this implies that

$$\liminf_{\varepsilon\to 0} \frac{\sigma^2}{2}\sum_{i=1}^2 \langle V^\varepsilon_{\psi x_i}, V^\varepsilon_{\psi x_i}\rangle_B \geqslant \frac{\sigma^2}{2}\sum_{i=1}^2 \langle W_{x_i}, W_{x_i}\rangle_B. \tag{2.2.11}$$

Therefore exploiting the convergences in (2.2.5) and (2.2.11), taking the liminf as $\varepsilon \to 0$ in (2.2.10), we obtain

$$A_B(W, U-W) + \liminf_{\varepsilon\to 0} \frac{2}{\varepsilon}\langle \beta'((V^\varepsilon_{x_1})^2 - 1)V^\varepsilon_\psi V^\varepsilon_{x_1x_1}, U - V^\varepsilon_\psi\rangle_B \geqslant \langle \hat{H}, U - W\rangle_B. \tag{2.2.12}$$

In order to prove (2.2.2), it thus only remains to show that the scalar product in (2.2.12) involving β' is nonpositive. Write U as $U = \psi\bar{U}$, with $\bar{U} \in \mathcal{K}$. If $x \in \mathbb{R}^2$ is such that $(V^\varepsilon_{x_1}(x))^2 \leqslant (\bar{U}(x))^2$, then $\beta'((V^\varepsilon_{x_1}(x))^2 - 1) = 0$ since $\bar{U} \in \mathcal{K}$. On the other hand, if $(V^\varepsilon_{x_1}(x))^2 > (\bar{U}(x))^2$ then we have $2V^\varepsilon_\psi(U - V^\varepsilon_\psi) \leqslant U^2 - (V^\varepsilon_\psi)^2 < 0$. Hence, since V^ε is convex and β' nonnegative, in both cases we deduce that

$$\frac{2}{\varepsilon}\beta'((V^\varepsilon_{x_1})^2 - 1)V^\varepsilon_\psi V^\varepsilon_{x_1x_1}(U - V^\varepsilon_\psi) \leqslant 0.$$

Therefore, we conclude that W is a solution to the variational inequality (2.2.2).

Finally, since $\sigma > 0$, the $W^{2;\infty}_{loc}$-regularity of V_{x_1} follows from Theorem 4.1 at p. 31 in [82], slightly modified in order to fit problem (2.2.2) (see Problem 1 at p. 44, combined with Problems 2 and 5 at p. 29 in [82]).

Step 2. We now prove that V_{x_1} is a pointwise solution to (2.2.1). For $B \subset \mathbb{R}^2$ open bounded and $\psi \in C_c^\infty(B)$, by Step 1 we have that $V_{x_1}\psi$ is a solution to the variational inequality (2.2.2). Moreover, thanks to the regularity of V_{x_1}, an integration by parts in (2.2.2) reveals that

$$\langle \hat{L}\psi, (U - V_{x_1})\psi\rangle_B \geqslant 0, \text{ for each } U \in \mathcal{K}, \tag{2.2.13}$$

where $\hat{L} := [(\rho - b_1^1) - \mathcal{L}]V_{x_1} - \hat{h}$. For every $\varepsilon > 0$, define the sets $\widehat{\mathcal{W}}_\varepsilon := \{|V_{x_1}| < 1 - \varepsilon\}$ and, for $N > 0$ and $0 < \eta < \varepsilon/N$, set $\hat{\psi} := -\eta\hat{L}\mathbb{1}_{\widehat{\mathcal{W}}_\varepsilon}\mathbb{1}_{\{\hat{L}<N\}}$. Define next $U := V_{x_1} + \hat{\psi}$, and observe that $U \in \mathcal{K}$. With this choice of U, the inequality (2.2.13) rewrites as

$$0 \leqslant \int_B \hat{L}(U - V_{x_1})\psi^2 dx = -\eta\int_{\mathbb{R}^2} \hat{L}^2\psi^2\mathbb{1}_{\widehat{\mathcal{W}}_\varepsilon}\mathbb{1}_{\{|\hat{L}|<N\}}dx,$$

which in turn implies that $\int_{\mathbb{R}^2}\hat{L}^2\psi^2\mathbb{1}_{\widehat{\mathcal{W}}_\varepsilon}\mathbb{1}_{\{|\hat{L}|<N\}}dx = 0$. Taking limits as $N \to \infty$ and $\varepsilon \to 0$, by monotone convergence theorem, we conclude that $\int_{\mathcal{W}}\hat{L}^2\psi^2 dx = 0$; that is, $\hat{L} = 0$ a.e. in $\mathcal{W}$.

Finally, defining the two regions

$$\mathcal{I}_- := \{x \in \mathbb{R}^2 \,|\, V_{x_1}(x) = -1\} \quad \text{and} \quad \mathcal{I}_+ := \{x \in \mathbb{R}^2 \,|\, V_{x_1}(x) = 1\}, \tag{2.2.14}$$

for $\hat{\psi} := -\eta\hat{L}^+\mathbb{1}_{\mathcal{I}_+}\mathbb{1}_{\{|\hat{L}|<N\}}$ and $\hat{\psi} := -\eta\hat{L}^-\mathbb{1}_{\mathcal{I}_-}\mathbb{1}_{\{|\hat{L}|<N\}}$, we can repeat the arguments above in order to deduce that $\hat{L} \leqslant 0$ a.e. in $\mathcal{I}_+ \cup \mathcal{I}_-$, and thus completing the proof of the theorem. ∎

Theorem 2.2.1 allows to provide a probabilistic representation of V_{x_1} in terms of a Dynkin game. Let $\mathcal{T}$ be the set of $\mathbb{F}$-stopping times, and, for $\tau_1, \tau_2 \in \mathcal{T}$, define the functional

$$G(x;\tau_1,\tau_2) := \mathbb{E}\bigg[\int_0^{\tau_1\wedge\tau_2} e^{-\hat{\rho}t}\hat{h}(X_t^x)dt - e^{-\hat{\rho}\tau_1}\mathbb{1}_{\{\tau_1\leqslant\tau_2,\,\tau_1<\infty\}} + e^{-\hat{\rho}\tau_2}\mathbb{1}_{\{\tau_2<\tau_1\}}\bigg],$$

where $\hat{h} = h_{x_1} + b^2_{x_1}V_{x_2}$ (cf. Theorem 2.2.1), the process X^x denotes the solution to the uncontrolled SDE (2.1.1), and $\hat{\rho} := \rho - b^1_1$. Consider the 2-player stochastic differential game of optimal stopping in which Player 1 (resp. Player 2) is allowed to choose a stopping time τ_1 (resp. τ_2) in order to maximize (resp. minimize) the functional G.

Recalling the definitions of $\mathcal{I}_-$ and $\mathcal{I}_+$ given in (2.2.14), from Theorem 2.2.1 we obtain the following verification theorem. Its proof is based on a generalized version of Itô's formula (see Theorem 1 at p. 122 in [116]) which can be applied to the process $(e^{-\hat{\rho}t}V_{x_1}(X_t^x))_{t\geqslant 0}$ because $V_{x_1} \in W^{2;\infty}_{loc}(\mathbb{R}^2)$ by Theorem 2.2.1. Since these arguments are standard, we omit the details in the interest of length.

Theorem 2.2.2. *For each $x \in \mathbb{R}^2$, the profile strategy $(\bar{\tau}_1, \bar{\tau}_2)$ given by the stopping times*

$$\bar{\tau}_1 := \inf\{t \geqslant 0 \,|\, X_t^x \in \mathcal{I}_-\} \quad \textit{and} \quad \bar{\tau}_2 := \inf\{t \geqslant 0 \,|\, X_t^x \in \mathcal{I}_+\}$$

is a saddle point of the Dynkin game, and its corresponding value equals $V_{x_1}(x)$; that is,

$$G(x;\tau_1,\bar{\tau}_2) \leqslant V_{x_1}(x) = G(x;\bar{\tau}_1,\bar{\tau}_2) \leqslant G(x;\bar{\tau}_1,\tau_2), \quad \textit{for each } \tau_1,\tau_2 \in \mathcal{T}.$$

Moreover, we have

$$V_{x_1}(x) = \sup_{\tau_1}\inf_{\tau_2} G(x;\tau_1,\tau_2) = \inf_{\tau_2}\sup_{\tau_1} G(x;\tau_1,\tau_2). \tag{2.2.15}$$

The monotonicity property

We now show how Condition 2 in Assumption 2.1.1 together with Theorems 2.2.1 and 2.2.2 lead to an important monotonicity of V_{x_1}.

Proposition 2.2.3. *We have $b^2_{x_1}V_{x_1x_2} \geqslant 0$ in $\mathbb{R}^2$.*

Proof. Since $b^2_{x_1} \leqslant 0$ by Condition 2 in Assumption 2.1.1, it is enough to show that $V_{x_1x_2} \leqslant 0$. Fix an initial condition $x \in \mathbb{R}^2$, take $r > 0$, and define a new initial condition $x^r \in \mathbb{R}^2$ by setting $x^r := x + re_2$. Let $X^{x^r} = (X^{1,x^r}, X^{2,x^r})$ be the solution to the uncontrolled dynamics (2.1.1), with initial condition x^r. By the structure we assumed on the drift, this perturbation of the initial condition will affect only the second component of X^{x^r}. Indeed, since $x_2^r \geqslant x_2$, a standard comparison principle for SDE (see [101]) gives $X_t^{2,x^r} - X_t^{2,x} \geqslant 0$ for each $t \geqslant 0$, $\mathbb{P}$-a.s., while $X^{1,x^r} = X^{1,x}$. Hence, since $h_{x_1x_2} \leqslant 0$, we have

$$h_{x_1}(X_t^{x^r}) \leqslant h_{x_1}(X_t^x), \quad \text{for each } t \geqslant 0, \ \mathbb{P}\text{-a.s.} \tag{2.2.16}$$

Moreover, since $b^2_{x_1} \leqslant 0$, we can exploit the convexity of V to obtain

$$\begin{aligned} &b^2_{x_1}(X^{x^r}_t)(V_{x_2}(X^{x^r}_t) - V_{x_2}(X^x_t)) \\ &\quad = b^2_{x_1}(X^{x^r}_t)(X^{2,x^r}_t - X^{2,x}_r)\int_0^1 V_{x_2x_2}(X^x_t + s(X^{x^r}_t - X^x_r))ds \\ &\quad \leqslant 0, \quad \text{for each } t \geqslant 0,\ \mathbb{P}\text{-a.s.} \end{aligned} \tag{2.2.17}$$

Let us now prove that $V_{x_2}(y) \geqslant 0$, for each $y \in \mathbb{R}^2$. Fix $y \in \mathbb{R}^2$ and let v be an optimal control for y. Observe that, for each $\delta > 0$ we can still employ a comparison principle to deduce that $X^{1,y;v}_t - X^{1,y-\delta e_2;v}_t = 0$, and $X^{2,y;v}_t - X^{2,y-\delta e_2;v}_t \geqslant 0$, for each $t \geqslant 0$, $\mathbb{P}$-a.s. This, since $h_{x_2} \geqslant 0$ and $V \in C^1(\mathbb{R}^2)$, in turn implies that

$$\begin{aligned} V_{x_2}(y) &= \lim_{\delta \to 0} \frac{V(y) - V(y - \delta e_2)}{\delta} \\ &\geqslant \lim_{\delta \to 0} \frac{J(y;v) - J(y - \delta e_2; v)}{\delta} \\ &\geqslant \lim_{\delta \to 0} \frac{1}{\delta}\mathbb{E}\left[\int_0^\infty e^{-\rho t}(h(X^{y;v}_t) - h(X^{y-\delta e_2;v}_t))dt\right] \geqslant 0, \end{aligned} \tag{2.2.18}$$

where we have used that the control v is suboptimal for the initial condition $y - \delta e_2$. Hence, since $b^2_{x_1x_2} \leqslant 0$, we obtain that

$$(b^2_{x_1}(X^{x^r}_t) - b^2_{x_1}(X^x_t))V_{x_2}(X^x_t) \leqslant 0, \quad \text{for each } t \geqslant 0,\ \mathbb{P}\text{-a.s.} \tag{2.2.19}$$

Summing now the inequalities (2.2.16), (2.2.17) and (2.2.19), we find, for each $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$h_{x_1}(X^{x^r}_t) + b^2_{x_1}(X^{x^r}_t)V_{x_2}(X^{x^r}_t) \leqslant h_{x_1}(X^x_t) + b^2_{x_1}(X^x_t)V_{x_2}(X^x_t); \tag{2.2.20}$$

that is, $\hat{h}(X^{x^r}) \leqslant \hat{h}(X^x)$. Therefore, for each stopping time τ_1, $\tau_2 \in \mathcal{T}$, we deduce that

$$G(x^r; \tau_1, \tau_2) \leqslant G(x; \tau_1, \tau_2).$$

Taking the supremum over $\tau_1 \in \mathcal{T}$ and the infimum over $\tau_2 \in \mathcal{T}$ in the latter inequality, we deduce, in light of (2.2.15) in Theorem 2.2.2, that $V_{x_1}(x^r) \leqslant V_{x_1}(x)$. Hence, we conclude that $V_{x_1x_2} \leqslant 0$ in $\mathbb{R}^2$, which completes the proof of the proposition. □

2.2.2 Step b: Construction of ε-optimal policies

For every $\varepsilon > 0$ define the sets

$$\mathcal{W}_\varepsilon := \{x \in \mathbb{R}^2 \,|\, V^2_{x_1}(x) < 1 - \varepsilon\}, \quad S_\varepsilon := \partial \mathcal{W}_\varepsilon.$$

The proof of the following lemma is obtained combining arguments from [115] together with the monotonicity property shown in Proposition 2.2.3.

Lemma 2.2.4. *For each $\varepsilon > 0$ such that $\bar{x} \in \mathcal{W}_\varepsilon$, there exists a solution $v^\varepsilon \in \mathcal{V}$ to the (classical) Skorokhod problem for the SDE (2.1.2) in $\overline{\mathcal{W}}_\varepsilon$ starting at $\bar{x}$ with reflection direction $-V_{x_1}/|V_{x_1}|e_1$.*

Proof. Fix $\varepsilon > 0$ such that $\bar{x} \in \mathcal{W}_\varepsilon$. In order to employ the results of [125] to construct v^ε as the solution of the Skorokhod problem with reflection along S_ε, we first show that S_ε is a C^3 hypersurface.

To this end, we begin the proof by showing that

$$V_{x_1x_1}(x) > 0, \quad \text{for each } x \in \mathcal{W}. \tag{2.2.21}$$

Take indeed $x \in \mathcal{W}$ and $\delta > 0$ such that $B_\delta(x) \subset \mathcal{W}$. Since V solves the linear equation $\rho V - \mathcal{L}V = h$ in $\mathcal{W}$, from Theorem 6.17 at p. 109 in [85] it follows that $V \in C^4(\mathcal{W})$. Therefore, we can differentiate two times with respect to x_1 the HJB equation (2.1.7), and obtain an equation for $V_{x_1x_1}$

$$(\rho - 2b_1^1)V_{x_1x_1} - \mathcal{L}V_{x_1x_1} = h_{x_1x_1} + 2b_{x_1}^2 V_{x_1x_2} + b_{x_1x_1}^2 V_{x_2}, \quad \text{in } B_\delta(x). \tag{2.2.22}$$

Since by assumption $h_{x_1x_1} > 0$, thanks to Proposition 2.2.3 we have that $h_{x_1x_1} + 2b_{x_1}^2 V_{x_1x_2} > 0$. By the inequality (2.2.18) in the proof of Proposition 2.2.3, and the fact that b^2 is convex, we deduce that $b_{x_1x_1}^2 V_{x_2} \geqslant 0$. Therefore, the right hand side of (2.2.22) is positive. Next, by the strong maximum principle (see Theorem 3.5 at p. 35 in [85]), $V_{x_1x_1}$ cannot achieve a nonpositive local minimum in $B_\delta(x)$, unless it is constant. If $V_{x_1x_1}$ is constant in $B_\delta(x)$, then by (2.2.22) we obtain $V_{x_1x_1} > 0$ as desired. If $V_{x_1x_1}$ attains its minimum at the boundary $\partial B_\delta(x)$, then by convexity of V we still have

$$V_{x_1x_1}(y) > \min_{\partial B_\delta(x)} V_{x_1x_1} \geqslant 0, \quad \text{for each } y \in B_\delta(x),$$

which also proves (2.2.21)

Next, define $\bar{\nu}(x) := V_{x_1}(x)/|V_{x_1}(x)|e_1$ for each $x \in S_\varepsilon$, and $w(y) := |V_{x_1}(y)|^2$ for each $y \in \mathcal{W}$. Notice that $\sqrt{w(y)} = |\partial_{\bar{\nu}} V(y)|$. For $R > 0$, by compactness of $\overline{\mathcal{W}}_{\varepsilon/2}^R := \overline{\mathcal{W}}_{\varepsilon/2} \cap \overline{B}_R$, in light of (2.2.21) we can find a constant $c_\varepsilon^R > 0$ such that

$$\inf_{x \in \overline{\mathcal{W}}_{\varepsilon/2}^R} V_{x_1x_1}(x) \geqslant c_\varepsilon^R > 0. \tag{2.2.23}$$

Therefore, for $x \in S_\varepsilon$ and R large enough, by (2.2.23), we have

$$\sqrt{w(x + \lambda\bar{\nu})} = \partial_{\bar{\nu}} V(x + \lambda\bar{\nu}) \geqslant \partial_{\bar{\nu}} V(x) + \lambda c_\varepsilon^R/2 = \sqrt{w(x)} + \lambda c_\varepsilon^R/2,$$

and hence

$$\partial_{\bar{\nu}} \sqrt{w(x)} \geqslant c_\varepsilon^R/2. \tag{2.2.24}$$

It thus follows that $\partial_{\bar{\nu}} w \neq 0$ on S_ε. This implies, by the implicit function theorem, that S_ε is a C^3-hypersurface.

Now, by (2.2.24), arguing as in Lemma 2.7 in [115] it is possible to show that the vector $-\bar{\nu}$ is not tangential to S_ε, and, by definition of $\mathcal{W}_\varepsilon$ and of $\bar{\nu}$, we observe that the vector $-\bar{\nu}$ points inside $\mathcal{W}_\varepsilon$. Therefore, we can employ a version of Theorem 4.4 in [125] for unbounded domains in order to find a solution $v^\varepsilon \in \mathcal{V}$ to the Skorokhod problem for the SDE (2.1.2) in $\overline{\mathcal{W}}_\varepsilon$ starting at $\bar{x}$, with reflection direction $-V_{x_1}/|V_{x_1}|e_1$. □

We conclude this section with the following lemma. We omit its proof since this can be established as in the proof of Lemma 2.8 in [115].

Lemma 2.2.5. *For each $\bar{x} \in \mathcal{W}$ and $\varepsilon > 0$ such that $\bar{x} \in \mathcal{W}_\varepsilon$, let the control v^ε be as in Lemma 2.2.4. Then $J(\bar{x}; v^\varepsilon) \to V(\bar{x})$ as $\varepsilon \to 0$.*

2.2.3 Step c: Characterization of the optimal control

Thanks to the results of Subsections 2.2.1 and 2.2.2 we can now prove Theorem 2.1.5. We provide a separate proof for each of the two claims.

Proof of Claim 1

We will first prove Claim 1 for $\bar{x} \in \mathcal{W}$, and then, at the end of this subsection, we will give a proof for a general $\bar{x} \in \overline{\mathcal{W}}$. Fix $\bar{x} \in \mathcal{W}$ and a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to zero. To simplify the notation, according to Lemma 2.2.4 we define the processes

$$X^n := X^{\bar{x};v^{\varepsilon_n}}, \ v^n := v^{\varepsilon_n}, \ \xi^n := |v^{\varepsilon_n}|, \quad \text{for each } n \in \mathbb{N}.$$

Bear in mind that the processes v^n, γ^n and ξ^n depend on the initial condition $\bar{x}$, and that, according to Lemma 2.2.5, the sequence of controls $\{v^n\}_{n\in\mathbb{N}}$ is a minimizing sequence; that is, $J(\bar{x}; v^n) \to V(\bar{x})$ as $n \to \infty$.

We begin with the following estimate.

Lemma 2.2.6. *Let $p' := (2p-1)/2$. We have*

$$\sup_n \int_0^\infty e^{-\rho t}(\mathbb{E}[|X_t^{1,n}|^p] + \mathbb{E}[|X_t^n|^{p'}])dt \leqslant C(1+|\bar{x}|^p). \tag{2.2.25}$$

Proof. Denoting by $X^{\bar{x}}$ the solution to (2.1.1), a standard use of Grönwall's inequality and of Burkhölder-Davis-Gundy's inequality leads to the classical estimate

$$\mathbb{E}[|X_t^{\bar{x}}|^p] \leqslant Ce^{p\bar{L}t}(1+|\bar{x}|^p) \quad \text{for each } t \geqslant 0,$$

where $\bar{L}$ is the Lipschitz constant of $\bar{b}$ and $C > 0$ is a generic constant. Therefore, since the control constantly equal to zero is not necessarily optimal for $\bar{x}$, from the latter estimate and the growth rate of h we obtain

$$\begin{aligned} V(\bar{x}) \leqslant \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X_t^{\bar{x}})dt\bigg] &\leqslant C\int_0^\infty e^{-\rho t}(1+\mathbb{E}[|X_t^{\bar{x}}|^p])dt \\ &\leqslant C\int_0^\infty e^{-(\rho-p\bar{L})t}(1+|\bar{x}|^p)dt \leqslant C(1+|\bar{x}|^p), \end{aligned}$$

where we have used that, by Condition 3a in Assumption 2.1.1, $\rho > p\bar{L}$. Therefore, since v^n is a minimizing sequence, for all n big enough we find the estimate

$$\kappa_1 \int_0^\infty e^{-\rho t}\mathbb{E}[|X_t^{1,n}|^p]dt - \kappa_2 \leqslant J(\bar{x}; v^n) \leqslant C(1+|\bar{x}|^p),$$

from which

$$\sup_n \int_0^\infty e^{-\rho t}\mathbb{E}[|X_t^{1,n}|^p]dt \leqslant C(1+|\bar{x}|^p). \tag{2.2.26}$$

Next, using again Grönwall's inequality and Burkhölder-Davis-Gundy's inequality, we find

$$\mathbb{E}[|X_t^{2,n}|^{p'}] \leqslant Ce^{p'\bar{L}t}\bigg(1+|\bar{x}|^{p'} + p_t + p_t\int_0^t \mathbb{E}[|X_s^{1,n}|^{p'}]ds\bigg), \quad \text{for each } t \geqslant 0,$$

where p_t is a suitable (deterministic) polynomial in t, not depending on n. Therefore

$$\begin{aligned}\int_0^\infty e^{-\rho t}\mathbb{E}[|X_t^{2,n}|^{p'}]dt \leqslant C\int_0^\infty e^{(p'\bar{L}-\rho)t}(1+|\bar{x}|^{p'}+p_t)dt \\ + C\int_0^\infty e^{[p'\bar{L}-\rho(1-p'/p)]t}p_t\int_0^t e^{-\rho(p'/p)s}\mathbb{E}[|X_s^{1,n}|^{p'}]dsdt \\ \leqslant C\int_0^\infty e^{(p'\bar{L}-\rho)t}(1+|\bar{x}|^{p'}+p_t)dt \\ + C\int_0^\infty e^{[p'\bar{L}-\rho(1-p'/p)]t}p_t\bigg(\int_0^\infty e^{-\rho s}\mathbb{E}[|X_s^{1,n}|^{p}]ds\bigg)^{\frac{p'}{p}}dt.\end{aligned} \tag{2.2.27}$$

After noticing that Condition 3a in Assumption 2.1.1 implies $p'\bar{L}-\rho<0$ and $p'\bar{L}-\rho(1-p'/p)<0$, using (2.2.26) in (2.2.27), we conclude that

$$\sup_n \int_0^\infty e^{-\rho t}\mathbb{E}[|X_t^{2,n}|^{p'}]dt \leqslant C(1+|\bar{x}|^p),$$

which, together with (2.2.26), completes the proof of the lemma. □

Lemma 2.2.7. *Let $\bar{v}\in\mathcal{V}$ be the unique optimal control for $\bar{x}$. We have that*

$$X_t^n \to X_t^{\bar{x};\bar{v}} \quad \text{and} \quad v^n\to\bar{v}, \quad \mathbb{P}\otimes dt\text{-a.e. in } \Omega\times[0,\infty), \quad \text{as } n\to\infty.$$

Proof. The proof employs arguments as those in the proof of Theorem 8 in [135], that however need to be suitably adapted in order to accommodate our more general convex setting.

We organize the proof in two steps.

Step 1. Arguing by contradiction, in this step we prove that the sequence X^n is Cauchy w.r.t. the convergence in the measure $\mathbb{P}\otimes e^{-\rho t}dt$; that is, for each $\delta>0$ we have

$$\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\mathbb{1}_{\{|X_t^n-X_t^m|>\delta\}}dt\bigg]\to 0, \quad \text{as } n,m\to\infty. \tag{2.2.28}$$

Indeed, suppose that for a subsequence (not relabelled), one has

$$\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\mathbb{1}_{\{|X_t^n-X_t^m|>\delta\}}dt\bigg]\geqslant \delta_0>0, \quad \text{for each } n,m \in\mathbb{N}, \tag{2.2.29}$$

for a certain constant $\delta_0>0$.

Fix $\lambda\in(0,1)$. We begin by defining the processes

$$Y^{n,m}:=X^{\bar{x};\lambda v^n+(1-\lambda)v^m} \quad \text{and} \quad Z^{n,m}:=\lambda X^n+(1-\lambda)X^m, \quad \text{for each } n,m \in\mathbb{N}.$$

We first need to show that

$$Y_t^{n,m}\leqslant Z_t^{n,m}, \quad \text{for each } t\geqslant 0, \ \mathbb{P}\text{-a.s.} \tag{2.2.30}$$

Since the drift $\bar{b}^1$ is affine, we have $Y^{1;n,m} = Z^{1;n,m}$. Moreover, since b^2 is convex, we find

$$\begin{aligned} Z_t^{2;n,m} &= \bar{x}_2 + \int_0^t (\lambda b^2(X_s^n) + (1-\lambda) b^2(X_s^m))dt + \sigma W_t^2 \\ &\geqslant \bar{x}_2 + \int_0^t b^2(Z_s^{1;n,m}, Z_s^{2;n,m})dt + \sigma W_t^2 \\ &= \bar{x}_2 + \int_0^t b^2(Y_s^{1;n,m}, Z_s^{2;n,m})dt + \sigma W_t^2, \end{aligned} \tag{2.2.31}$$

while $Y_t^{2;n,m} = \bar{x}_2 + \int_0^t b^2(Y_s^{1;n,m}, Y_s^{2;n,m})ds + \sigma W_t^2$. This, by the comparison principle for SDE (see [101]), implies that $Y_t^{2;n,m} \leqslant Z_t^{2;n,m}$, for each $t \geqslant 0$, $\mathbb{P}$-a.s., and (2.2.30) follows.

Next, in light of (2.2.30), by the monotonicity of h in x_2 we find

$$\begin{aligned} &\lambda J(\bar{x}; v^n) + (1-\lambda) J(\bar{x}; v^m) - J(\bar{x}; \lambda v^n + (1-\lambda) v^m) \\ &\quad = \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} (\lambda h(X_t^n) + (1-\lambda) h(X_t^m) - h(Y_t^{n,m}))dt \\ &\qquad\qquad + \int_{[0,\infty)} e^{-\rho t} (\lambda d\xi_t^n + (1-\lambda) d\xi_t^m - d|\lambda v^n + (1-\lambda) v^m|_t) \bigg] \\ &\quad \geqslant \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} (\lambda h(X_t^n) + (1-\lambda) h(X_t^m) - h(Z_t^{n,m}))dt \bigg], \end{aligned} \tag{2.2.32}$$

as we have that $|\lambda v^n + (1-\lambda) v^m|_t \leqslant \lambda \xi_t^n + (1-\lambda)\xi_t^m$, and that $e^{-\rho t}$ is positive and decreasing.

Then, using (2.2.29), for $M > 0$ we observe that

$$\begin{aligned} &\mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \mathbb{1}_{\{|X_t^n - X_t^m| > \delta\}} \mathbb{1}_{\{|X_t^n| \leqslant M, \, |X_t^m| \leqslant M\}} dt \bigg] \\ &\quad \geqslant \delta_0 - \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \mathbb{1}_{\{|X_t^n| > M\}} dt \bigg] - \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \mathbb{1}_{\{|X_t^m| > M\}} dt \bigg]. \end{aligned}$$

Moreover, the estimate in Lemma 2.2.6 and an application of Chebyshev's inequality yield

$$\mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \mathbb{1}_{\{|X_t^n| > M\}} dt \bigg] \leqslant \frac{C(1+|\bar{x}|^p)}{M^{p'}}, \quad \text{for each } n \in \mathbb{N},$$

so that we can find M big enough such that

$$\mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \mathbb{1}_{\{|X_t^n - X_t^m| > \delta\}} \mathbb{1}_{\{|X_t^n| \leqslant M, \, |X_t^m| \leqslant M\}} dt \bigg] \geqslant \frac{\delta_0}{2}, \quad \text{for each } n, m \in \mathbb{N}.$$

Combining the latter inequality with (2.2.32), we obtain

$$\begin{aligned} &\lambda J(\bar{x}; v^n) + (1-\lambda) J(\bar{x}; v^m) - J(\bar{x}; \lambda v^n + (1-\lambda) v^m) \\ &\quad \geqslant \delta_{(\delta_0, M)} \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \mathbb{1}_{\{|X_t^n - X_t^m| > \delta\}} \mathbb{1}_{\{|X_t^n| \leqslant M, \, |X_t^m| \leqslant M\}} dt \bigg] \\ &\quad \geqslant \delta_{(\delta_0, M)} \frac{\delta_0}{2}, \end{aligned} \tag{2.2.33}$$

where, by strict convexity of h in the variable x_1, we have $\delta_{(\delta_0,M)} > 0$, for

$$\delta_{(\delta_0,M)} := \inf\Big\{\lambda h(x) + (1-\lambda)h(y) - h(\lambda x + (1-\lambda)y)\Big|\, |x-y| > \delta_0, |x|, |y| \leqslant M\Big\}.$$

On the other hand, by Lemma 2.2.5, $J(\bar{x}; v^n)$ converges to $V(\bar{x})$ as $n \to \infty$. Therefore, from (2.2.33), we can find $\bar{n} \in \mathbb{N}$ such that

$$V(\bar{x}) \geqslant \delta_{(\delta_0,M)} \frac{\delta_0}{4} + J(\bar{x}; \lambda v^n + (1-\lambda)v^m), \quad \text{for each } n, m \geqslant \bar{n},$$

which contradicts the definition of V, completing the proof of (2.2.28).

Step 2. By the previous step, there exists a limit process $\hat{X}$ and, unless to consider a subsequence, we can assume that

$$X_t^n \to \hat{X}_t \quad \mathbb{P} \otimes dt\text{-a.e. in } \Omega \times [0,\infty), \quad \text{as } n \to \infty. \tag{2.2.34}$$

Next, defining the process $v_t := \hat{X}_t^1 - \bar{x}^1 - \int_0^t \bar{b}^1(\hat{X}_s^1)ds - \sigma W_t^1$, using the estimate from Lemma 2.2.6 and (2.2.34) we find

$$|v_t^n - v_t| \leqslant |X_t^{1,n} - \hat{X}_t^1| + \bar{L}\int_0^t |X_s^{1,n} - \hat{X}_s^1|ds \to 0 \quad \mathbb{P} \otimes dt\text{-a.e. in } \Omega \times [0,\infty),$$

which implies that

$$v_t^n \to v_t \quad \mathbb{P} \otimes dt\text{-a.e. in } \Omega \times [0,\infty), \text{ as } n \to \infty. \tag{2.2.35}$$

We also observe that, by using Lemma 3.5 in [104], we can assume the processes $\hat{X}^1$ and v to be càdlàg. Also, denoting with ξ the total variation of v, from (2.2.35) we easily find

$$\xi_t \leqslant \liminf_n \xi_t^n \quad \text{for each } t \geqslant 0. \tag{2.2.36}$$

Next, exploiting the limits in (2.2.34), the Lipschitz continuity of b^2 and the estimate from Lemma 2.2.6, we can see that the process $\hat{X}^2$ is continuous and it solves the SDE $d\hat{X}_t^2 = b^2(\hat{X}_t^1, \hat{X}_t^2)dt + \sigma dW_t^2,\ t \geqslant 0,\ \hat{X}_{0-}^2 = \bar{x}_2$. This, together with the definition of v, implies that

$$\hat{X} = X^{\bar{x};v}. \tag{2.2.37}$$

Finally, thanks to the limits in (2.2.34), (2.2.35) and (2.2.36), to the identity (2.2.37), and to the continuity of h, we invoke Fatou's lemma and, with an integration by parts (see, e.g., Corollary 2 at p. 68 in [145]), we find

$$\begin{aligned} J(\bar{x}; v) &= \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X_t^{\bar{x};v})dt + \rho \int_0^\infty e^{-\rho t}\xi_t dt\bigg] \\ &\leqslant \liminf_n \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X_t^n)dt + \rho\int_0^\infty e^{-\rho t}\xi_t^n dt\bigg] = \liminf_n J(\bar{x}; v^n) = V(\bar{x}), \end{aligned} \tag{2.2.38}$$

where we have used that the sequence $\{v^n\}_{n\in\mathbb{N}}$ is minimizing for $\bar{x}$, according to Lemma 2.2.5. Thus, the process v has locally bounded variation, and $v \in \mathcal{V}$. Also, from (2.2.38) we deduce that the control v is optimal for $\bar{x}$, and, by uniqueness of optimal controls (see Remark 2.1.4), we conclude that $v = \bar{v}$ and $\hat{X} = X^{\bar{x};\bar{v}}$, completing the proof of the lemma. □

The proofs of the next two propositions follow by employing arguments similar to those employed in Sections 2.3 and 2.4 in [115] (we provide details here in order to recall these arguments in the sequel).

Proposition 2.2.8. *We have that* $\mathbb{P}[X_t^{\bar{x};\bar{v}} \in \overline{\mathcal{W}},\ \forall t \geqslant 0] = 1$.

Proof. By Lemma 2.2.7, $X_t^n \to X_t^{\bar{x};\bar{v}}$, $\mathbb{P} \otimes dt$-a.e. in $\Omega \times [0,\infty)$, and, by Lemma 2.2.4, $\mathbb{P}[X_t^n \in \overline{\mathcal{W}}, t \geqslant 0] = 1$, as $\overline{\mathcal{W}}_{\varepsilon_n} \subset \overline{\mathcal{W}}$ for each $n \in \mathbb{N}$. Therefore, it is clear that $X_t^{\bar{x};\bar{v}} \in \overline{\mathcal{W}}$, $\mathbb{P} \otimes dt$-a.e. in $\Omega \times [0,\infty)$, which, by right-continuity, implies that $\mathbb{P}[X_t^{\bar{x};\bar{v}} \in \overline{\mathcal{W}}, t \geqslant 0] = 1$. □

Proposition 2.2.9. *We have* $d\bar{v} = \bar{\gamma} d|\bar{v}|$ *with*

$$|\bar{v}|_t = \int_0^t \mathbb{1}_{\{X_{s-}^{\bar{x};\bar{v}} \in S,\, -V_{x_1}(X_{s-}^{\bar{x};\bar{v}}) = \bar{\gamma}_s\}} d|\bar{v}|_s, \quad \text{for each } t \geqslant 0,\ \mathbb{P}\text{-a.s.}$$

Proof. Take $R > 0$ such that $\bar{x} \in B_R$ and define $\tau_R := \inf\{t \in [0,\infty) | X_s^{\bar{x};\bar{v}} \notin B_R\}$. For each $\varepsilon > 0$, let V^ε be as in (2.5.2). As in the Step 1 in the proof of Theorem 2.5.1 in Section 2.5, V^ε is a convex C^2-solution to (2.5.3). By Itô's formula for semimartingales (see, e.g., Theorem 33 at p. 81 in [145]), applied on the process $(e^{-\rho t} V^\varepsilon(X_t^{\bar{x};\bar{v}}))_{t \geqslant 0}$ on the time interval $[0, \tau_R]$, we find

$$\begin{aligned}
\mathbb{E}[e^{-\rho\tau_R} V^\varepsilon(X_{\tau_R}^{\bar{x};\bar{v}})] = V^\varepsilon(\bar{x}) \\
&+ \mathbb{E}\bigg[\int_0^{\tau_R} e^{-\rho t}(\mathcal{L}V^\varepsilon - \rho V^\varepsilon)(X_t^{\bar{x};\bar{v}}) dt + \int_{[0,\tau_R)} e^{-\rho t} V_{x_1}^\varepsilon(X_{t-}^{\bar{x};\bar{v}}) \bar{\gamma}_t d|\bar{v}|_t \\
&+ \sum_{0 \leqslant t \leqslant \tau_R} e^{-\rho t}(V^\varepsilon(X_t^{\bar{x};\bar{v}}) - V^\varepsilon(X_{t-}^{\bar{x};\bar{v}}) - V_{x_1}^\varepsilon(X_{t-}^{\bar{x};\bar{v}}) \bar{\gamma}_t(|\bar{v}|_t - |\bar{v}|_{t-}))\bigg].
\end{aligned}$$

By the convexity of V^ε, the last sum above is nonnegative. Also, since the function β in (2.5.3) in nonnegative, we have $\rho V^\varepsilon - \mathcal{L}V^\varepsilon \leqslant h$ a.e. in $\mathbb{R}^2$. Hence from the latter equality we deduce that

$$V^\varepsilon(\bar{x}) \leqslant \mathbb{E}\bigg[\int_0^{\tau_R} e^{-\rho t} h(X_t^{\bar{x};\bar{v}}) dt - \int_{[0,\tau_R)} e^{-\rho t} V_{x_1}^\varepsilon(X_{t-}^{\bar{x};\bar{v}}) \bar{\gamma}_t d|\bar{v}|_t\bigg]. \tag{2.2.39}$$

Therefore, taking first limits in (2.2.39) as $\varepsilon \to 0$ (using (2.5.18) and the dominated convergence theorem), and then letting $R \to \infty$ (using the monotone convergence theorem and the dominated convergence theorem), we obtain

$$V(\bar{x}) \leqslant \mathbb{E}\bigg[\int_0^{\infty} e^{-\rho t} h(X_t^{\bar{x};\bar{v}}) dt - \int_{[0,\infty)} e^{-\rho t} V_{x_1}(X_{t-}^{\bar{x};\bar{v}}) \bar{\gamma}_t d|\bar{v}|_t\bigg]. \tag{2.2.40}$$

Next, by the optimality of $\bar{v}$, we have that $V(\bar{x}) = J(\bar{x}; \bar{v})$, and, from (2.2.40), it follows that

$$\mathbb{E}\bigg[\int_{[0,\infty)} e^{-\rho t}(1 + V_{x_1}(X_{t-}^{\bar{x};\bar{v}}) \bar{\gamma}_t) d|\bar{v}|_t\bigg] \leqslant 0. \tag{2.2.41}$$

This in turn implies, using $0 \leqslant 1 - |V_{x_1}| \leqslant 1 + V_{x_1}\gamma$ for all $\gamma \in \mathbb{R}$ with $|\gamma| = 1$, that

$$0 \leqslant \mathbb{E}\bigg[\int_{[0,\infty)} e^{-\rho t}(1 - |V_{x_1}(X^{\bar{x};\bar{v}}_{t-})|)d|\bar{v}|_t\bigg] \leqslant \mathbb{E}\bigg[\int_{[0,\infty)} e^{-\rho t}(1 + V_{x_1}(X^{\bar{x};\bar{v}}_{t-})\bar{\gamma}_t)d|\bar{v}|_t\bigg] \leqslant 0.$$

From the latter chain of inequalities we deduce that the support of the random measure $d|\bar{v}|$ is $\mathbb{P}$-a.s. contained in the set $\{(\omega, t) \in \Omega \times [0,\infty) \,|\, X^{\bar{x};\bar{v}}_{t-}(\omega) \in \partial\mathcal{W}, \bar{\gamma}_t(\omega) = -V_{x_1}(X^{\bar{x};\bar{v}}_{t-}(\omega))\}$, which completes the proof of the proposition. □

The proof of the next proposition also follows by employing the arguments in [115]. Details are provided in Section 2.6 for the sake of completeness.

Proposition 2.2.10. *We have that, $\mathbb{P}$-a.s., a possible jump of the process $X^{\bar{x};\bar{v}}$ at time $t \geqslant 0$ occurs on some interval $I \subset \partial\mathcal{W}$ parallel to the vector field $-V_{x_1}e_1$, i.e., such that $-V_{x_1}(x)e_1$ is parallel to I for each $x \in I$. If $X^{\bar{x};\bar{v}}$ encounters such an interval I, it instantaneously jumps to its endpoint in the direction $-V_{x_1}e_1$ on I.*

Combining then the Propositions 2.2.8, 2.2.9 and 2.2.10, we see that, for $\bar{x} \in \mathcal{W}$, the optimal control $\bar{v} \in \mathcal{V}$ is a solution to the modified Skorokhod problem for the SDE (2.1.2) in $\overline{\mathcal{W}}$ starting at $\bar{x}$ with reflection direction $-V_{x_1}e_1$.

Take next $\bar{x} \in \overline{\mathcal{W}}$. By definition, there exists a sequence $\{x^k\}_{k\in\mathbb{N}} \subset \mathcal{W}$ such that $x^k \to \bar{x}$ as $k \to \infty$. For each k, let w^k be the optimal control for x^k, and consider the controls $x^k - \bar{x} + w^k$, which consist in following the policy w^k after an initial jump from $\bar{x}$ to x^k. Using the fact that $x^k \in \mathcal{W}$, from Proposition 2.2.8 we have that $\mathbb{P}[X^{x^k;w^k}_t \in \overline{\mathcal{W}}, t \geqslant 0] = 1$. Observe, moreover, that $X^{x^k;w^k} = X^{\bar{x};x^k-\bar{x}+w^k}$, and that $|J(\bar{x}; x^k - \bar{x} + w^k) - J(x^k; w^k)| = |\bar{x} - x^k|$. By the continuity of V, we now see that

$$V(\bar{x}) = \lim_k V(x^k) = \lim_k J(x^k; w^k) = \lim_k J(\bar{x}; x^k - \bar{x} + w^k).$$

Therefore, the sequence of controls $\{x^k - \bar{x} + w^k\}_{k\in\mathbb{N}}$ is a minimizing sequence for the initial condition $\bar{x}$. Repeating the proof of Lemma 2.2.7 with the sequence of controls $\{x^k - \bar{x} + w^k\}_{k\in\mathbb{N}}$, we see that $X^{x^k;w^k}_t \to X^{\bar{x};\bar{v}}_t$, $\mathbb{P} \otimes dt$-a.e. in $\Omega \times [0,\infty)$. This allows to repeat the arguments in the proofs of Propositions 2.2.8, 2.2.9 and 2.2.10 in order to conclude that, also for $\bar{x} \in \overline{\mathcal{W}}$, the optimal control $\bar{v} \in \mathcal{V}$ is a solution to the modified Skorokhod problem for the SDE (2.1.2) in $\overline{\mathcal{W}}$ starting at $\bar{x}$ with reflection direction $-V_{x_1}e_1$.

Finally, through a verification theorem (which can be proved by using Itô's formula as in the proof of Proposition 2.2.9, see also Theorem 4.1 at p. 300 in [81]), it is easy to show that any solution to the modified Skorokhod problem for the SDE (2.1.2) in $\overline{\mathcal{W}}$ starting at $\bar{x}$ with reflection direction $-V_{x_1}e_1$ is an optimal control. This, by uniqueness of the optimal control (see Remark 2.1.4) implies that such a solution is unique, completing the proof of Claim 1 of Theorem 2.1.5.

Proof of Claim 2

Fix $\bar{x} = (\bar{x}_1, \bar{z}) \notin \overline{\mathcal{W}}$ and denote again by $\bar{v}$ the optimal control for $\bar{x}$. Let $\bar{y}_1 \in \mathbb{R}$ be the metric projection of $\bar{x}_1$ into the set $\overline{\mathcal{W}}_1(\bar{z})$. The set $\overline{\mathcal{W}}_1(\bar{z})$ is a closed interval (cf. Lemma 2.1.3), hence the point $\bar{y}_1$ is uniquely determined. Set then $\bar{y} := (\bar{y}_1, \bar{z})$ and observe that $\bar{y} \in \partial\mathcal{W}$. Let $\bar{w}$ be the optimal control for $\bar{y}$. Notice that, since V_{x_1} is pointing outside $\overline{\mathcal{W}}_1(\bar{z})$, we have $V_{x_1}(\bar{y})(\bar{x}_1 - \bar{y}_1) = |\bar{x}_1 - \bar{y}_1|$. Therefore, since $(\bar{y}_1 + \lambda(\bar{x}_1 - \bar{y}_1), \bar{z}) \notin \mathcal{W}$ for each $\lambda \in (0,1)$, we get

$$V(\bar{x}) = V(\bar{y}_1, \bar{z}) + \int_0^1 V_{x_1}(\bar{y}_1 + \lambda(\bar{x}_1 - \bar{y}_1), \bar{z})(\bar{x}_1 - \bar{y}_1) d\lambda = V(\bar{y}) + |\bar{x}_1 - \bar{y}_1|.$$

This means that $V(\bar{x}) = J(\bar{y}; \bar{w}) + |\bar{x}_1 - \bar{y}_1| = J(\bar{x}; \bar{x}_1 - \bar{y}_1 + \bar{w})$, which, by uniqueness of the optimal control, implies that $\bar{v} = \bar{x}_1 - \bar{z}_1 + \bar{w}$. Moreover, since $\bar{y} \in \overline{\mathcal{W}}$ and $\bar{w}$ is optimal for $\bar{y}$, by Claim 1 we have that $\bar{w}$ is the unique solution to the modified Skorokhod problem for the SDE (2.1.2) in $\overline{\mathcal{W}}$ starting at $\bar{y}$ with reflection direction $-V_{x_1} e_1$. This completes the proof of Claim 2 and therefore also of Theorem 2.1.5.

2.3 On the proof of Theorem 2.1.5 for linear volatility

In this section we assume that Condition 3b in Assumption 2.1.1 holds. To simplify the notation, also this proof is given for $d = 2$, so that $D = \mathbb{R}^2_+ = \{x \in \mathbb{R}^2 \,|\, x_1, \, x_2 > 0\}$. The generalization to the case $d > 2$ is straightforward.

2.3.1 A preliminary lemma

Define the set

$$\mathcal{V}^x_+ := \{v \in \mathcal{V} \,|\, X^{1,x;v}_t, \, X^{2,x;v}_t > 0 \text{ for each } t \geqslant 0, \ \mathbb{P}\text{-a.s.}\}.$$

Lemma 2.3.1. *We have* $V(x) = \min_{v \in \mathcal{V}^x_+} J(x; v)$, *for each* $x \in \mathbb{R}^2_+$.

Proof. Let $v \in \mathcal{V}$ be an optimal control for $x \in \mathbb{R}^2_+$, and denote by (ξ^+, ξ^-) its minimal decomposition. In order to simplify the notation, set $X := X^{x;v}$. Assuming that $v_s = 0$ for each $s < 0$, define the family of random variables

$$\tau_k := \inf\{t \geqslant 0 \,|\, (X^1_t, \xi^-_{t+1/k} - \xi^-_{t-1/k}) \in (-\infty, x^*_1) \times (0, \infty)\}, \quad k \in \mathbb{N}.$$

Define the filtration $\mathbb{F}^k := (\mathcal{F}_{t+1/k})_{t \geqslant 0}$ and notice that, for each $k \geqslant 1$, τ_k is an $\mathbb{F}^k$-stopping time. Set $\tau := \sup_k \tau_k$, and observe that, for $k \leqslant \bar{k}$, we have $\tau_k \leqslant \tau_{\bar{k}}$. This implies that, for each $\bar{k} \geqslant 1$, $\tau = \sup_{k \geqslant \bar{k}} \tau_k$, so that τ is an $\mathbb{F}^{\bar{k}}$-stopping time, and, by right-continuity of the filtration $\mathbb{F}$, we deduce that τ is an $\mathbb{F}$-stopping time. Also, such a definition of τ is such that the negative part ξ^- of v acts at time τ; that is, τ is in the support of the measure ξ^-.

If $\mathbb{P}[\tau < \infty] = 0$, then the control ξ^- never acts when the state process X^1 lies in the region $(-\infty, x^*_1)$. Since $a_1 \geqslant 0$ and $b^2 \geqslant 0$, this is enough to ensure that $X^{1,x;v}_t$, $X^{2,x;v}_t > 0$ for each $t \geqslant 0$, $\mathbb{P}$-a.s., which in turn implies that $v \in \mathcal{V}^x_+$.

Arguing by contradiction, suppose that $\mathbb{P}[\tau < \infty] > 0$. Define the control $\tilde{v}_t := \mathbb{1}_{\{t<\tau\}} v_t + \mathbb{1}_{\{t \geqslant \tau\}} (\xi^+_t + \min\{x^*_1 - X^1_{\tau-}, 0\} \mathbb{1}_{\{\Delta \xi^-_\tau > 0\}})$, and the process $\tilde{X} := X^{x;\tilde{v}}$. Define next the stopping time $\bar{\tau} := \inf\{t \geqslant \tau \,|\, \tilde{X}_t^1 \geqslant 2x_1^*\}$, the control $\bar{v}_t := \mathbb{1}_{\{t<\bar{\tau}\}}\tilde{v} + \mathbb{1}_{\{t\geqslant\bar{\tau}\}}(X_{\bar{\tau}}^1 - \tilde{X}_{\bar{\tau}-}^1 + v_t - v_{\bar{\tau}})$ and the process $\bar{X} := X^{x;\bar{v}}$. Since at time τ only the negative part ξ^- of v acts, on $\{\tau < \infty\}$ we have $\tau < \bar{\tau}$. Also, by the definition of $\bar{v}$, for k such that $\tau + 1/k < \bar{\tau}$, on $\{\tau < \infty\}$ we have

$$v_{\tau+1/k} - \bar{v}_{\tau+1/k} \geqslant \xi^-_{\tau+1/k} \geqslant \xi^-_{\tau_k+1/k} > \xi^-_{\tau_k-1/k} \geqslant 0,$$

so that the processes v and $\bar{v}$ are not indistinguishable. Moreover, v and $\bar{v}$ are such that, on $\{\tau < \infty\}$, we have

$$\begin{cases} X_t^1 = \bar{X}_t^1 \text{ for } t \in [0,\tau) \cup [\bar{\tau},\infty), \\ X_t^1 \leqslant \bar{X}_t^1 \text{ for } t \in [\tau,\bar{\tau}). \end{cases} \tag{2.3.1}$$

After some manipulations, from (2.3.1) we deduce that

$$\begin{aligned} J(x;v) - J(x;\bar{v}) = \mathbb{E}\bigg[\mathbb{1}_{\{\tau<\infty\}}\bigg(\int_{(\tau,\bar{\tau})} e^{-\rho t} d\xi_t^- + \int_\tau^{\bar{\tau}} e^{-\rho t} Dh(\hat{X}_t)(X_t - \bar{X}_t)dt\bigg)\bigg] \\ + \mathbb{E}[\mathbb{1}_{\{\tau<\infty\}} e^{-\rho\tau}(|X_\tau^1 - X_{\tau-}^1| - |\bar{X}_\tau^1 - \bar{X}_{\tau-}^1|)] \\ + \mathbb{E}[\mathbb{1}_{\{\tau<\infty\}} e^{-\rho\bar{\tau}}(|X_{\bar{\tau}}^1 - X_{\bar{\tau}-}^1| - |\bar{X}_{\bar{\tau}}^1 - \bar{X}_{\bar{\tau}-}^1|)], \end{aligned} \tag{2.3.2}$$

for $\hat{X}_t = \lambda_t \bar{X}_t + (1-\lambda_t)X_t^{x;v} \in (-\infty, 2x_1^*) \times \mathbb{R}$, and suitable choice of $\lambda_t(\omega) \in [0,1]$. We point out that, the expectations in (2.3.2) are well defined also for $\bar{\tau} = \infty$. Indeed, since v is optimal, we have $\lim_{T\to\infty} \mathbb{E}[e^{-\rho T}|v|_T] = 0$, so that $e^{-\rho\bar{\tau}}(|X_{\bar{\tau}}^1 - X_{\bar{\tau}-}^1| - |\bar{X}_{\bar{\tau}}^1 - \bar{X}_{\bar{\tau}-}^1|) = 0$ $\mathbb{P}$-a.s. on $\{\bar{\tau} = \infty\}$. Since ξ^- acts at time τ, we have $X_\tau^1 - X_{\tau-}^1 \leqslant 0$. Also, at time τ the control $\bar{v}$ can only jump to the left, giving $\bar{X}_\tau^1 - \bar{X}_{\tau-}^1 \leqslant 0$. Hence, using $\bar{X}_\tau^1 - X_\tau^1 \geqslant 0$, we obtain

$$e^{-\rho\tau}(|X_\tau^1 - X_{\tau-}^1| - |\bar{X}_\tau^1 - \bar{X}_{\tau-}^1|) = e^{-\rho\tau}(\bar{X}_\tau^1 - X_\tau^1) \geqslant 0. \tag{2.3.3}$$

So that, from (2.3.2), we get

$$J(x;v) - J(x;\bar{v}) \geqslant \mathbb{E}[\mathbb{1}_{\{\tau<\infty\}}\Psi], \tag{2.3.4}$$

with

$$\begin{aligned} \Psi := \int_{(\tau,\bar{\tau})} e^{-\rho t} d\xi_t^- + \int_\tau^{\bar{\tau}} e^{-\rho t} Dh(\hat{X}_t)(X_t - \bar{X}_t)dt \\ + e^{-\rho\bar{\tau}}(|X_{\bar{\tau}}^1 - X_{\bar{\tau}-}^1| - |\bar{X}_{\bar{\tau}}^1 - \bar{X}_{\bar{\tau}-}^1|). \end{aligned} \tag{2.3.5}$$

Now, if $\bar{X}_{\bar{\tau}}^1 \geqslant \bar{X}_{\bar{\tau}-}^1$, then using (2.3.1) we find

$$|X_{\bar{\tau}}^1 - X_{\bar{\tau}-}^1| - |\bar{X}_{\bar{\tau}}^1 - \bar{X}_{\bar{\tau}-}^1| \geqslant \bar{X}_{\bar{\tau}-}^1 - X_{\bar{\tau}-}^1 \geqslant 0. \tag{2.3.6}$$

Therefore, plugging (2.3.6) into (2.3.5) and taking the expectation, we obtain the inequality

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{\{\tau<\infty,\,\bar{X}_{\bar{\tau}}^1\geqslant\bar{X}_{\bar{\tau}-}^1\}}\Psi] \geqslant \mathbb{E}\bigg[\mathbb{1}_{\{\tau<\infty,\,\bar{X}_{\bar{\tau}}^1\geqslant\bar{X}_{\bar{\tau}-}^1\}} \int_\tau^{\bar{\tau}} e^{-\rho t} h_{x_1}(\hat{X}_t)(X_t^1 - \bar{X}_t^1)dt\bigg] \\ + \mathbb{E}\bigg[\mathbb{1}_{\{\tau<\infty,\,\bar{X}_{\bar{\tau}}^1\geqslant\bar{X}_{\bar{\tau}-}^1\}} \int_\tau^{\bar{\tau}} e^{-\rho t} h_{x_2}(\hat{X}_t)(X_t^2 - \bar{X}_t^2)dt\bigg] \geqslant 0, \end{aligned} \tag{2.3.7}$$

where we have also used (2.3.1), Condition 3b in Assumption 2.1.1, and that, due to the monotonicity of b^2 in the variable x_1, via a comparison principle we have $X^2_t - \bar{X}^2_t \leqslant 0$ for $t \in (\tau, \bar{\tau})$. On the other hand, if $\bar{X}^1_{\bar{\tau}} \leqslant \bar{X}^1_{\bar{\tau}-}$, from (2.3.1) we obtain

$$\begin{aligned} |X^1_{\bar{\tau}} - X^1_{\bar{\tau}-}| - |\bar{X}^1_{\bar{\tau}} - \bar{X}^1_{\bar{\tau}-}| &\geqslant X^1_{\bar{\tau}-} - \bar{X}^1_{\bar{\tau}-} \\ &= X^1_\tau - \bar{X}^1_\tau + \int_\tau^{\bar{\tau}-} b^1_1(X^1_t - \bar{X}^1_t)dt + \int_\tau^{\bar{\tau}-} \sigma(X^1_t - \bar{X}^1_t)dW^1_t - \int_{(\tau,\bar{\tau})} d\xi^-_t. \end{aligned} \tag{2.3.8}$$

If $b^1_1 \leqslant 0$ substituting (2.3.8) into (2.3.5), as in (2.3.7) we obtain

$$\mathbb{E}[\mathbb{1}_{\{b^1_1 \leqslant 0,\, \tau<\infty,\, \bar{X}^1_{\bar{\tau}} \leqslant \bar{X}^1_{\bar{\tau}-}\}}\Psi] \geqslant 0. \tag{2.3.9}$$

Similarly, for $b^1_1 \geqslant 0$ we find

$$\begin{aligned} &\mathbb{E}[\mathbb{1}_{\{b^1_1 \geqslant 0,\, \tau<\infty,\, \bar{X}^1_{\bar{\tau}} \leqslant \bar{X}^1_{\bar{\tau}-}\}}\Psi] \\ &\quad \geqslant \mathbb{E}\left[\mathbb{1}_{\{b^1_1 \geqslant 0,\, \tau<\infty,\, \bar{X}^1_{\bar{\tau}} \leqslant \bar{X}^1_{\bar{\tau}-}\}} \int_\tau^{\bar{\tau}} (h_{x_1}(\hat{X}_t) - b^1_1)(X^1_t - \bar{X}^1_t)\right] \\ &\qquad + \mathbb{E}\left[\mathbb{1}_{\{b^1_1 \geqslant 0,\, \tau<\infty,\, \bar{X}^1_{\bar{\tau}} \leqslant \bar{X}^1_{\bar{\tau}-}\}} \int_\tau^{\bar{\tau}} h_{x_2}(\hat{X}_t)(X^2_t - \bar{X}^2_t)dt\right] \geqslant 0. \end{aligned} \tag{2.3.10}$$

Finally, adding the inequalities (2.3.7), (2.3.9) and (2.3.10) and using (2.3.4) we obtain

$$J(x; v) - J(x; \bar{v}) \geqslant 0,$$

which contradicts the uniqueness of the optimal control v, completing the proof of the lemma. □

2.3.2 Sketch of the proof of Theorem 2.1.5

Since we are interested in characterizing the optimal control for any given $\bar{x} \in \mathbb{R}^2_+$, thanks to Lemma 2.3.1 we can restrict the domain of the HJB equation to the set $\mathbb{R}^2_+$. We observe that, upon exploiting the ellipticity of the operator $\mathcal{L}$ in the domain $\mathbb{R}^2_+$ (and, in particular, the uniform ellipticity of $\mathcal{L}$ on each ball $B \subset \mathbb{R}^2_+$), all the results from Sections 2.2.1 and 2.2.2 can be recovered, with minimal adjustments of the arguments therein.

For $\bar{x} \in \mathcal{W}$ we can consider the processes $X^n := X^{\bar{x};v^n},\ v^n$ for $n \in \mathbb{N}$, with $\{v^n\}_{n\in\mathbb{N}}$ minimizing sequence of solutions to the Skorkokhod problems on domains $\overline{\mathcal{W}}_n$, according to Lemma 2.2.5 (here $\overline{\mathcal{W}}_n$ denotes the closure of $\mathcal{W}_n$ in $\mathbb{R}^2_+$).

Estimates as those of Lemma 2.2.6 can now be proved as follows. Denoting by $X^{\bar{x}}$ the solution to (2.1.1), by standard results (see, e.g., Theorem 4.1 at p. 59 in [133]) we have $\mathbb{E}[|X^{\bar{x}}_t|^p] \leqslant Ce^{p\,(2\bar{L}+\sigma^2(p-1))t}(1+|\bar{x}|^p)$ for each $t \geqslant 0$. Hence, arguing as in the proof of Lemma 2.2.6 and using the requirement on ρ from Condition 3b in Assumption 2.1.1, we find

$$\sup_n \int_0^\infty e^{-\rho t}\mathbb{E}[|X^{1,n}_t|^p]dt \leqslant C(1+|\bar{x}|^p). \tag{2.3.11}$$

Next, for $p' := (2p-1)/2$, we use (2.3.11) to estimate $|X^{2,n}|^{p'}$. We underline that, since $\overline{\mathcal{W}}_n \subset \mathcal{W}$, we have $X^n_t > 0$ $\mathbb{P}\otimes dt$-a.e. in $\Omega\times[0,\infty)$. For each $n\in\mathbb{N}$, define the process Λ^n as the solution to the SDE

$$d\Lambda^n_t = \bar{L}(1+|X^{1,n}_t|+\Lambda^n_t)dt + \sigma\Lambda^n_t dW^2_t,\ t\geqslant 0, \quad \Lambda^n_0 = \bar{x}_2.$$

Since $X^{2,n}_t \leqslant \bar{x}_2 + \int_0^t \bar{L}(1+|X^{1,n}_s|+|X^{2,n}_s|)ds + \sigma\int_0^t X^{2,n}_s dW^2_s$, by a comparison principle for SDE (see [101]) we obtain $X^{2,n}\leqslant\Lambda^n$. Therefore, using that

$$\Lambda^n_t = \hat{E}_t\bigg[\bar{x}_2 + \int_0^t \bar{L}(1+|X^{1,n}_s|)\hat{E}_s^{-1}ds\bigg],$$

with $\hat{E}_t := \exp[(\bar{L}-\sigma^2/2)t+\sigma W_t]$, we find

$$\begin{aligned}\int_0^\infty & e^{-\rho t}\mathbb{E}[|X^{2,n}_t|^{p'}]dt \\ &\leqslant \int_0^\infty e^{-\rho t}\mathbb{E}[|\Lambda^n_t|^{p'}]dt \\ &\leqslant C\int_0^\infty e^{-\rho t}\mathbb{E}\bigg[\hat{E}_t^{p'}\bar{x}_2^{p'} + p_t\int_0^t \hat{E}_t^{p'}\hat{E}_s^{-p'}ds\bigg] \\ &\quad + C\int_0^\infty\bigg(\int_0^t e^{-\rho s}\mathbb{E}[|X^{1,n}_s|^p]ds\bigg)^{\frac{1}{q}} e^{-\rho(1-\frac{1}{q})t}\bigg(\int_0^t \mathbb{E}[(\hat{E}_t/\hat{E}_s)^{p'q^*}]ds\bigg)^{\frac{1}{q^*}}dt,\end{aligned} \tag{2.3.12}$$

where we have also used Hölder's inequality with exponent $q=p/p'$, q^* denoting the conjugate of q. Exploiting the requirement on ρ made in Condition 3b in Assumption 2.1.1, after elementary computations one can see that

$$\int_0^\infty e^{-\rho(1-\frac{1}{q})t}\bigg(\int_0^t \mathbb{E}[(\hat{E}_t/\hat{E}_s)^{p'q^*}]ds\bigg)^{\frac{1}{q^*}}dt<\infty. \tag{2.3.13}$$

Finally, substituting (2.3.11) and (2.3.13) in (2.3.12), we conclude that

$$\sup_n\int_0^\infty e^{-\rho t}\mathbb{E}[|X^{2,n}_t|^{p'}]dt\leqslant C(1+|\bar{x}|^p),$$

which, combined with (2.3.11), gives

$$\sup_n\int_0^\infty e^{-\rho t}(\mathbb{E}[|X^{1,n}_t|^p]+\mathbb{E}[|X^n_t|^{p'}])dt\leqslant C(1+|\bar{x}|^p). \tag{2.3.14}$$

Thanks to the estimate (2.3.14), the arguments of Step 1 in the proof of Lemma 2.2.7 can be recovered, so that (up to a subsequence)

$$X^n_t\to\hat{X}_t\quad \mathbb{P}\otimes dt\text{-a.e. in }\Omega\times[0,\infty),\quad \text{as } n\to\infty, \tag{2.3.15}$$

for an adapted process $\hat{X}$. Using again (2.3.14) and the assumption $p\geqslant 2$, a standard use of Banach-Saks' theorem allows to find a subsequence of indexes $\{n_j\}_{j\in\mathbb{N}}$ such that the Cesàro means of $\{X^{1,n_j}\}_{j\in\mathbb{N}}$ converge in $\mathbb{L}^2$ to the process $\hat{X}^1$; that is, for each $T > 0$, we have

$$\bar{X}^{1,m} := \frac{1}{m}\sum_{j=1}^{m} X^{1,n_j} \to \hat{X}^1, \text{ as } m\to\infty, \quad \text{in } \mathbb{L}^2(\Omega\times[0,T];\mathbb{P}\otimes dt). \tag{2.3.16}$$

Next, defining the process $v_t := \hat{X}^1_t - \bar{x}_1 - \int_0^t \hat{X}^1_s ds - \int_0^t \hat{X}^1_s dW_s$, and exploiting the $\mathbb{L}^2$ convergence in (2.3.16) and the linearity of the dynamics for the first component, we deduce that

$$\bar{v}^m := \frac{1}{m}\sum_{j=1}^{m} v^{n_j} \to v, \text{ as } m\to\infty, \quad \text{in } \mathbb{L}^2(\Omega\times[0,T];\mathbb{P}\otimes dt), \text{ for each } T>0. \tag{2.3.17}$$

Again, by using Lemma 3.5 in [104], we can assume the processes $\hat{X}^1$ and v to be right-continuous. Next, observe that the processes $X^{2,n}$ can be expressed as

$$X^{2,n}_t = E_t\Big[\bar{x}_2 + \textstyle\int_0^t b^2(X^n_s)/E_s ds\Big], \quad \text{with} \quad E_t := \exp\Big(\sigma W^2_t - \frac{\sigma^2}{2}t\Big), \quad t \geqslant 0.$$

Hence, taking limits as $n\to\infty$ in the latter equality (exploiting (2.3.15) and the uniform integrability deriving from (2.3.14)), we deduce that

$$\hat{X}^2_t = E_t\Big[\bar{x}_2 + \textstyle\int_0^t b^2(\hat{X}_s)/E_s ds\Big], \quad t\geqslant 0,$$

so that, thanks also to the very definition of v, we have $\hat{X} = X^{\bar{x};v}$. Overall, from (2.3.15), (2.3.17) and the latter equality, we have

$$\bar{X}^m := \frac{1}{m}\sum_{j=1}^{m} X^{n_j} \to X^{\bar{x};v}, \text{ and } \bar{v}^m \to v, \quad \mathbb{P}\otimes dt\text{-a.e. in } \Omega\times[0,\infty), \tag{2.3.18}$$

as $m\to\infty$. It is however worth noticing that $\bar{X}^m$ is not the solution of the SDE controlled by $\bar{v}^m$, unless b^2 is affine. Similarly to (2.2.36), using the fact that the sequence of controls v^n is minimizing, and exploiting the limits in (2.3.18) and the convexity of h, we find

$$\begin{aligned} J(\bar{x};v) &= \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}h(X^{\bar{x};v}_t)dt + \rho\int_0^\infty e^{-\rho t}|v|_t dt\bigg] \\ &\leqslant \liminf_m \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}h(\bar{X}^m_t)dt + \rho\int_0^\infty e^{-\rho t}|\bar{v}^m|_t dt\bigg] \\ &\leqslant \liminf_m \frac{1}{m}\sum_{j=1}^{m}\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}h(X^{n_j}_t)dt + \rho\int_0^\infty e^{-\rho t}|v^{n_j}|_t dt\bigg] = V(\bar{x}), \end{aligned}$$

so that the control v has locally bounded variation and it is optimal. By uniqueness of the optimal control, we deduce that $\bar{v} = v$ and $\hat{X} = X^{\bar{x};\bar{v}}$.

Finally, thanks to the properties of (X^n, v^n), by repeating the arguments leading to Propositions 2.2.8, 2.2.9 and 2.2.10 (see Section 2.6), the optimal control $\bar{v}$ for $\bar{x}\in\mathcal{W}$ can be characterized as the unique solution to the modified Skorokhod problem for the

SDE (2.1.2) in $\overline{\mathcal{W}}$ starting at $\bar{x}$ with reflection direction $-V_{x_1}e_1$. On the other hand, for $\bar{x} \in \overline{\mathcal{W}}$, we can repeat the rationale at the end of Subsection 2.2.3, which yields that the optimal control can be characterized also for $\bar{x} \in \overline{\mathcal{W}}$, completing the proof of Claim 1 of Theorem 2.1.5.

When $\bar{x} \notin \overline{\mathcal{W}}$, following the arguments of Subsection 2.2.3, one can characterize the initial jump of $\bar{v}$. This completes the proof of Theorem 2.1.5 under Condition 3b in Assumption 2.1.1.

2.4 Comments, extensions and examples

2.4.1 Refinements of Assumption 2.1.1

Assumption 2.1.1 can be improved as follows.

Affine drift

If $\bar{\sigma}$ is constant, Theorem 2.1.5 holds also for a drift $\bar{b}(x) := a + bx$, for a vector $a \in \mathbb{R}^d$ and a matrix $b \in \mathbb{R}^{d\times d}$ such that the vector $\beta := (0, b_1^2, ..., b_1^d)^\top \in \mathbb{R}^d$ is an eigenvector of b and $h_{x_1\beta} \geqslant 0$. Here the vector $(0, b_1^2, ..., b_1^d)^\top$ is the first column of b, with b_1^1 replaced by 0, while $h_{x_1\beta}$ denotes the β-directional derivative of h_{x_1}. In this case, for $x \in \mathbb{R}^d$, $r > 0$ and $x^r := x + r\beta$, the solution X^{x^r} of (2.1.1) writes (see, e.g., p. 99 in [133]) as $X_t^{x^r} = e^{bt}x^r + P_t$, where P_t does not depend on x^r. Hence, since the vector β is by assumption an eigenvector of the matrix b with eigenvalue λ, we find $X_t^{x^r} - X_t^x = r\, e^{tb}\beta = re^{t\lambda}\beta$, for each $t \geqslant 0$, $\mathbb{P}$-a.s. This easily allows to repeat the arguments in the proof of Proposition 2.2.3, so that $V_{x_1\beta} \geqslant 0$, while all of the other results in this chapter still hold (often with less technical proofs). Also, in this case, for $p = 2$ it is sufficient to require that

$$\rho > 2\Lambda(b), \quad \Lambda(b) := \max\{\mathrm{Re}(\lambda) \,|\, \lambda \text{ eigenvalue of } b\}.$$

We refer to Lemma 2.2 and Theorem 2.3 in [56] for more details. Finally, all the results in this chapter apply for a constant volatility matrix $\bar{\sigma}$ such that $\bar{\sigma}\bar{\sigma}^\top$ is positive definite, $\bar{\sigma}^\top$ denoting the transpose of $\bar{\sigma}$.

On Condition 2

A careful look into the proofs of Proposition 2.2.3 and of Lemma 2.2.7 reveals that the results in this chapter remain valid if the drift coefficients b^i in Condition 2 in Assumption 2.1.1 satisfy one of the following more general requirements.

1. Under Condition 3a, for $i = 2, ..., d$, either of the following is satisfied:
   (a) b^i is convex, $h_{x_i} \geqslant 0$, and either $b^i_{x_1}$, $b^i_{x_1x_i}$, $h_{x_1x_i} \leqslant 0$ or $b^i_{x_1}$, $b^i_{x_1x_i}$, $h_{x_1x_i} \geqslant 0$;
   (b) b^i is concave, $h_{x_i} \leqslant 0$, and either $b^i_{x_1}$, $-b^i_{x_1x_i}$, $h_{x_1x_i} \leqslant 0$ or $b^i_{x_1}$, $-b^i_{x_1x_i}$, $h_{x_1x_i} \geqslant 0$.
2. Under Condition 3b, for $i = 2, ..., d$, either of the following is satisfied:

(a) b^i is convex, $h_{x_i} \geqslant 0$, and $b^i_{x_1}$, $b^i_{x_1x_i}$, $h_{x_1x_i} \leqslant 0$;

(b) b^i is concave, $h_{x_i} \leqslant 0$, and $b^i_{x_1}$, $-b^i_{x_1x_i}$, $h_{x_1x_i} \leqslant 0$.

We point out that the conditions to deal with a linear volatility need to be compatible with the arguments in the proof of Lemma 2.3.1 and are, for this reason, more restrictive.

On the lower-growth of h

We underline that the lower-growth requirement on h in Condition 1 can be improved in some particular settings: If the drift is affine and the volatility is constant, for $p \leqslant 2$ it is sufficient to assume $h \geqslant -\kappa_2$. Indeed, in this case, the proof of the estimate (2.5.5) in Step 2 in the proof of Theorem 2.5.1 in Section 2.5 simplifies (in particular, in (2.5.6), $M_2 = 0$) and it can be provided without relying on Lemma 2.2.6. Also, for any $x \in \mathbb{R}^d$ and any sequence of minimizing controls $\{v^n\}_{n\in\mathbb{N}}$, we have the estimate

$$\sup_n \mathbb{E}\Big[\int_{[0,\infty)} e^{-\rho t} d|v^n|_t \Big] \leqslant C(1 + |x|^p),$$

which, combined with $\mathbb{E}[|X_t^{x;v^n}|] \leqslant C(1 + |x|^p + \mathbb{E}[|v^n|_t])e^{\bar{L}t}$, gives

$$\sup_n \mathbb{E}\Big[\int_{[0,\infty)} e^{-(\rho+\bar{L})t} |X_t^{x;v^n}| dt \Big] \leqslant \sup_n C\Big(1 + |x|^p + \mathbb{E}\Big[\int_0^\infty e^{-\rho t} |v^n|_t dt \Big]\Big) \leqslant C(1 + |x|^p).$$

Therefore, a limit process $\hat{X}$ such that $X_t^{x;v^n} \to \hat{X}_t$ $\mathbb{P} \otimes dt$-a.e. as $n \to \infty$ can be found, by adapting the reasoning in Step 1 in the proof of Lemma 2.2.7. Also, using Lemma 3.5 in [104], in the spirit of what has been done in Subsection 2.3.2, we can exploit the convexity of h and the fact that $\bar{b}$ is affine in order to prove that $\hat{X} = X^{x;v}$, with v optimal control for the given x. This allows to recover Lemma 2.2.7 and to characterize the optimal control v.

2.4.2 Some remarks

We provide here some extensions to the results contained in this chapter.

Remark 2.4.1 (Asymmetric costs of action)**.** *Unless to slightly modify some of the arguments in this chapter, Theorem 2.1.5 extends to the case in which increasing the first component of the state process has a different cost than decreasing it; that is, to the cost functional*

$$J_{\kappa_1,\kappa_2}(x; v) := \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} h(X_t^{x;v}) dt + \kappa_1 \int_{[0,\infty)} e^{-\rho t} d\xi_t^+ + \kappa_2 \int_{[0,\infty)} e^{-\rho t} d\xi_t^- \bigg], \quad \kappa_1,\, \kappa_2 > 0.$$

In this case, the value function V solves the HJB equation

$$\max\{\rho V - \mathcal{L}V - h, -V_{x_1} - \kappa_1, V_{x_1} - \kappa_2\} = 0, \quad a.e.\ in\ D.$$

This can be shown by employing arguments similar to those in the proof of Theorem 2.5.1 in Section 2.5, by replacing the penalizing term in (2.5.3) *with an "asymmetric" penalization $[\beta(-V_{x_1} - \kappa_1) + \beta(V_{x_1} - \kappa_2)]/\varepsilon$. Most of the arguments in this chapter remains essentially unchanged, and the optimal control can be characterized as the solution to a Skorokhod problem on the domain $\mathcal{W}_{\kappa_1,\kappa_2} := \{y \in \mathbb{R}^d \,|\, \kappa_1 < V_{x_1}(y) < \kappa_2\}$.*

Remark 2.4.2 (Monotone controls)**.** *Our approach allows also to characterize optimal controls for stochastic singular control problems where the minimization problem is formulated over the set of monotone controls; that is, when*

$$V(x) := \inf_{\xi \in \mathcal{A}} J(x;\xi) \quad \textit{with} \quad \mathcal{A} := \{\xi \in \mathcal{V},\, \xi \textit{ nondecreasing}\}.$$

In this case, V solves the HJB equation $\max\{\rho V - \mathcal{L}V - h, -V_{x_1} - 1\} = 0$, a.e. in D, and its derivative V_{x_1} is the value function of an optimal stopping problem (rather than a Dynkin game). The arguments in this chapter can be easily adapted, and the optimal control can be characterized as the solution to a Skorokhod problem on the domain $\mathcal{W}_+ := \{y \in \mathbb{R}^d \,|\, 1 < V_{x_1}(y)\}$. We stress that, in this case, the additional requirements on h and $\bar{b}$ in Condition 3b in Assumption 2.1.1 are not anymore needed (see Remark 2.1.2).

Remark 2.4.3 (Finite time horizon)**.** *A characterization result analogous to Theorem 2.1.5 could also be investigated for an optimal control problem over a finite time-horizon. For example, when $d = 2$ and b is affine, a connection with Dynkin games is known from [55]. Therefore, it seems possible to use this connection in order to investigate the monotonicity of the value of the game (as in Proposition 2.2.3), and to use this monotonicity in order to construct ε-optimal controls v^ε. In this case, building on the results in [30], one can try to study the limit as $\varepsilon \to 0$ of $\{v^\varepsilon\}_{\varepsilon>0}$, in order to provide a characterization of the optimal control.*

2.4.3 Examples

For the sake of illustration, we begin with the following:

Example 1. *For $d = 2$, ρ large enough, a convex nonincreasing function ϕ and a convex nondecreasing function f, in light of the discussion in Section 2.4.1 the optimal control can be characterized in the following settings:*

1. *$\bar{\sigma}$ as in Condition 3a and*
   - *(a) $b^2(x) = a^2 + b_1^2 x_1 + b_2^2 x_2$, $h(x) = |x|^2$, $h(x) = (x_1 - x_2)^2$ with $b_1^2 \leqslant 0$, $h(x) = (x_1 + x_2)^2$ with $b_1^2 \geqslant 0$;*
   - *(b) $b^2(x) = \phi(x_1) + b_2^2 x_2$, $h(x) = |x_1|^2 + f(x_2)$;*
2. *$\bar{\sigma}$ as in Condition 3b, $x_1^* > 0$ and*
   - *(a) $b^2(x) = a^2 + b_1^2 x_1 + b_2^2 x_2$, $h(x) = |x_1 - x_1^*|^2 + f(x_2)$;*
   - *(b) $b^2(x) = \phi(x_1) + b_2^2 x_2$, $h(x) = |x_1 - x_1^*|^2 + f(x_2)$, $h(x) = |x_1 - x_1^*|^2 + f(x_2 - x_1)$.*

In particular, Example 1a represents a relevant class of linear-quadratic *stochastic singular control problems, and it is the main example of Theorem 2.1.5.*

Example 2. *Here we discuss a model of pollution control. In the sequel, $x \in \mathbb{R}^2_+$ is the given and fixed initial condition of the state variable. Consider a company that can increase via an irreversible investment plan $\xi \in \mathcal{A}$ (cf. Remark 2.4.2) its production capacity $X^{1,x;\xi}$. The latter depreciates at constant rate $\delta > 0$ and is randomly fluctuating, e.g. because of technological uncertainty. Production leads to emissions of pollutants and thus impacts the level of a state process $X^{2,x;\xi}$ which summarizes one or more stocks of environmental pollutants (such as the average concentration of CO2 in the atmosphere). We assume that such an externality of production on the stock of pollutants is measured by a positive, convex, increasing, Lipschitz continuous function ϕ that has bounded second order derivative. Overall, the dynamics of $X^{x;\xi}$ is given by*

$$\begin{cases} dX_t^{1,x;\xi} = -\delta X_t^{1,x;\xi} dt + \sigma_1 X_t^{1,x;\xi} dW_t^1 + d\xi_t, \\ dX_t^{2,x;\xi} = (\phi(X_t^{1,x;\xi}) - X_2^{2,x;\xi})dt + \sigma_2 X_t^{2,x;\xi} dW_t^2. \end{cases}$$

The company aims at choosing a production plan that minimizes the sum of different costs: the cost of not meeting a given production level θ; the penalty of leading to a level of pollution that exceeds some environmental target ϑ; the proportional costs of investment. That is,

$$V(x) = \inf_{\xi \in \mathcal{A}} \mathbb{E}\bigg[\int_0^\infty e^{-\rho t} \Big((X_t^{1,x;\xi} - \theta)^2 + c(X_t^{2,x;\xi} - \vartheta) \Big) dt + \int_{[0,\infty)} e^{-\rho t} d\xi_t \bigg].$$

Here, $c \in C^{2;1}(\mathbb{R})$ is a nonnegative, nondecreasing, convex, Lipschitz continuous function such that $c(y) = 0$ for $y \leqslant 0$, and with bounded second order derivative. In light of the discussion in Subsections 2.4.1 and 2.4.2, the optimal control for V can be characterized as the solution to its related Skorokhod problem.

We next turn our focus to examples of bounded-variation problems treated in the literature and for which our results apply.

Example 3. *We discuss the model studied in [56]. For $d = 2$, consider the singular control problem with running cost $h(x_1, x_2) = \nu x_1^2 + x_2^2$, for $\nu > 0$, and drift $\bar{b}(x) = a + bx$, for a constant vector $a \in \mathbb{R}^2$ and a matrix*

$$b = \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix} \in \mathbb{R}^{2\times 2},$$

Observe that the requirements discussed in Subsection 2.4.1, are satisfied by assuming $b_2^1 = 0$ and $\rho > 2\Lambda(b)$. Therefore, Theorem 2.1.5 gives the optimal control as the solution of the related Skorokhod problem. The same result was obtained in [56] only under the additional assumption of a global Lipschitz-continuous free boundary.

Example 4. *Another example of setup similar to ours has been studied in [167], where a multidimesional singular control problem with $d \geqslant 2$ and constant drift and volatility is considered. There, the author shows the C^2-regularity of the value function, allowing for the characterization of the optimal policy as a solution to the related Skorokhod problem (even in the case of a state dependent cost of intervention). It is easy to see* *that, when the drift $\bar{b}$ is assumed to be constant, no monotonicity of the running cost h is required in order to obtain our Theorem 2.1.5. In comparison with [167], our main result (cf. Theorem 2.1.5) allows to characterize the optimal policy even in cases in which the dynamics are interconnected (at the cost of additional structural conditions on the running cost h).*

2.4.4 An example with degenerate dynamics

A more involved discussion is required to treat the degenerate singular control problem studied in [77] (see also [78]).

In this subsection, we take $d = 2$, h satisfying Condition 1 in Assumption 2.1.1, $\bar{b}(x) = a + bx = (\bar{b}^1(x), \bar{b}^2(x))^\top$, and

$$a = \begin{pmatrix} 0 \\ a^2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ b_1^2 & b_2^2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 \\ 0 & \eta \end{pmatrix}, \quad b_1^2,\, \eta,\, \rho > 0, \quad b_2^2 \leqslant 0, \quad h_{x_1 x_2} \geqslant 0. \tag{2.4.1}$$

In order to simplify the analysis of this example, assume $p = 2$ and $b_2^2 < 0$ and observe that, in this case, $\lambda(b) = 0$ (see the discussion in Subsection 2.4.1). The analysis of this subsection can be repeated also for $b_2^2 = 0$ and for a general $p \geqslant 1$.

Despite in this example the matrix $\sigma\sigma^\top$ is degenerate, the arguments in this chapter can be employed in order to characterize the optimal control. However, some extra care is needed in order to prove the regularity of the value function inside the waiting region, which in fact follows from the properties of the free boundary proved in [77] and [78].

We begin the discussion by observing that results analogous to the ones contained in Section 2.5 hold. In particular, Theorem 2.5.1 can be shown by using a suitable perturbation of the matrix σ (see Appendix A in [77], for more details). The connection with Dynkin games holds as well (see Theorem 3.1 in [77]), so that the arguments leading to Proposition 2.2.3 (which make no use of the non-degeneracy of $\sigma\sigma^\top$) can be recovered.

Regularity of V in $\mathcal{W}$

We enforce an additional hypothesis, which is satisfied by $h(x) = |x|^2$ or $h(x) = (x_1 + x_2)^2$.

Assumption 2.4.4.

1. $\lim_{x_2 \to \pm\infty} h_{x_2}(x_1, x_2) = \pm\infty$ *for any* $x_1 \in \mathbb{R}$*;*
2. *One of the following hold true:*
   - *(a)* $h_{x_1}(x_1, \cdot)$ *is strictly increasing for any* $x_1 \in \mathbb{R}$*;*
   - *(b)* $h_{x_1 x_2} = 0$ *and* $h(x_1, \cdot)$ *is strictly convex for any* $x_1 \in \mathbb{R}$*.*

As in Proposition 5.8 in [78] (see otherwise Proposition 4.25 at p. 92 in [147]), under the additional Assumption 2.4.4, there exist two nonincreasing locally Lipschitz continuous functions $g_1,\, g_2 : \mathbb{R} \to \mathbb{R}$ such that

$$\mathcal{I}_- = \{x \in \mathbb{R}^2 \,|\, x_2 \leqslant g_1(x_1)\} \quad \text{and} \quad \mathcal{I}_+ = \{x \in \mathbb{R}^2 \,|\, x_2 \geqslant g_2(x_1)\}. \tag{2.4.2}$$

For each $x \in \mathbb{R}^2$, recall the definition of $\bar{\tau}_1$, $\bar{\tau}_2$ given in Theorem 2.2.2 and define the stopping times

$$\bar{\tau}_1^\delta := \inf\{t \geqslant 0 \,|\, X_t^{x+\delta e_1} \in \mathcal{I}_-\}, \quad \bar{\tau}_2^\delta := \inf\{t \geqslant 0 \,|\, X_t^{x+\delta e_1} \in \mathcal{I}_+\}, \quad \delta \in \mathbb{R}. \tag{2.4.3}$$

The Lipschitz continuity of g_1 and of g_2 allows to prove the following lemma.

Lemma 2.4.5. *Under the additional Assumption 2.4.4, for $x \in \mathbb{R}^2$, we have*

$$\lim_{\delta\to 0} \bar{\tau}_1^\delta = \bar{\tau}_1, \quad \textit{and} \quad \lim_{\delta\to 0} \bar{\tau}_2^\delta = \bar{\tau}_2, \quad \mathbb{P}\textit{-a.s.}$$

Proof. We only prove the first of the two limits for $\delta \to 0^+$, since the same limit for $\delta \to 0^-$ follows by identical arguments, and the second limit can be proved in the same way. We first observe that, since g_1 is finite, we have $\mathbb{P}[\bar{\tau}_1 < \infty] = 1$. Also, when $\delta > 0$, we have, by convexity of V and by Proposition 2.2.3, that $V_{x_1}(x_1+\delta, X_t^{2,x+\delta e_1}) \geqslant V_{x_1}(x_1, X_t^{2,x+\delta e_1}) \geqslant V_{x_1}(x_1, X_t^{2,x})$, from which we deduce that

$$\bar{\tau}_1^\delta \geqslant \bar{\tau}_1, \quad \mathbb{P}\text{-a.s.} \tag{2.4.4}$$

We continue the proof arguing by contradiction. In light of (2.4.4), suppose that there exists $E \in \mathcal{F}$, with $\mathbb{P}[E] > 0$, such that for each $\omega \in E$ there exists $\varepsilon(\omega) > 0$ and a sequence $(\delta_j(\omega))_{j\in\mathbb{N}}$ with $\delta_j > 0$ and $\delta_j \to 0$ as $j \to \infty$, for which $\bar{\tau}_1^{\delta_j}(\omega) > \bar{\tau}_1(\omega) + \varepsilon(\omega)$ for each $j \in \mathbb{N}$. Using the representation in (2.4.2), (dropping the dependence on ω to simplify the notation) this is equivalent to

$$X_{\bar{\tau}_1}^{2,x} \leqslant g_1(x_1) \text{ and } X_{\bar{\tau}_1+s}^{2,x+\delta_j e_1} > g_1(x_1+\delta_j), \text{ for each } s \in [0,\varepsilon],\ j \in \mathbb{N}. \tag{2.4.5}$$

Notice that, due to the particular structure of the dynamics, we have

$$X_s^{2,x+\delta_j e_1} = X_s^{2,x} + \delta_j b_1^2\Big(e^{b_2^2 s} - 1\Big)/b_2^2, \quad s \geqslant 0,\ j \in \mathbb{N}, \tag{2.4.6}$$

from which we can write

$$\begin{aligned} X_{\bar{\tau}_1}^{2,x} &= (X_{\bar{\tau}_1}^{2,x} - X_{\bar{\tau}_1+s}^{2,x}) + X_{\bar{\tau}_1+s}^{2,x} \\ &= -\int_0^s (a_2 + b_1^2 x_1 + b_2^2 X_{\bar{\tau}_1+r}^{2,x})dr - \eta(W_{\bar{\tau}_1+s} - W_{\bar{\tau}_1}) \\ &\quad + X_{\bar{\tau}_1+s}^{2,x+\delta_j e_1} - \delta_j b_1^2\Big(e^{b_2^2(\bar{\tau}_1+s)} - 1\Big)/b_2^2, \end{aligned}$$

From the latter equality, using (2.4.5), by Lipschitz continuity of g_1, and pathwise boundedness of $X^{2,x}$ and of $\bar{\tau}_1$, we obtain

$$X_{\bar{\tau}_1}^{2,x} \geqslant -\delta_j C - sC + \eta(W_{\bar{\tau}_1+s} - W_{\bar{\tau}_1}) + g_1(x_1) - \delta_j, \text{ for each } s \in [0,\varepsilon],\ j \in \mathbb{N}, \tag{2.4.7}$$

where the constant C depends on $\sup_{r\in[0,\varepsilon]} X_{\bar{\tau}_1+r}^{2,x}$ and on $\bar{\tau}_1$ (which is finite, by assumption), but it is independent from s and j. Next, by the law of iterated logarithm (see, e.g., Theorem 9.23 at p. 112 in [109]) we find a sequence $(s_k)_{k\in\mathbb{N}}$ converging to zero and $\bar{k} \in \mathbb{N}$ (depending on ω) such that

$$(W_{\bar{\tau}_1+s_k} - W_{\bar{\tau}_1}) \geqslant \sqrt{s_k}\sqrt{\log\log(1/s_k)} \geqslant \sqrt{s_k}, \quad \text{for each } k \geqslant \bar{k}. \tag{2.4.8}$$

Finally, from (2.4.7) and (2.4.8), for suitable choice of δ_j and s_k, we conclude that

$$X^{2,x}_{\bar{\tau}_1} \geqslant -\delta_j(C+1) + \sqrt{s_k}(\eta - C\sqrt{s_k}) + g_1(x_1) > g_1(x_1),$$

which contradicts (2.4.5), and therefore completes the proof of the lemma. □

Lemma 2.4.6. *Under the additional Assumption 2.4.4, we have $V \in C^2(\mathcal{W})$.*

Proof. We split the proof in two steps.

Step 1. Take $z \in \mathcal{W}$ and $\varepsilon > 0$ such that $B^1_\varepsilon(z) \times B^2_\varepsilon(z) \subset \mathcal{W}$, where $B^1_\varepsilon(z) := \{x_1 \in \mathbb{R} \,|\, |z_1 - x_1| < \varepsilon\}$ and $B^2_\varepsilon(z) := \{x_2 \in \mathbb{R} \,|\, |z_2 - x_2| < \varepsilon\}$. We prove that $V_{x_2x_2}, V_{x_1x_2}$ are locally Lipschitz in $B^1_\varepsilon(z) \times B^2_\varepsilon(z)$ and that $V_{x_1x_1}(x_1, \cdot)$ is locally Lipschitz in $B^2_\varepsilon(z)$ for each $x_1 \in B^1_\varepsilon(z)$.

We begin by observing that, under (2.4.1), the HJB equation can be regarded a second order ordinary differential equation (ODE, in short) in the variable $x_2 \in \mathbb{R}$ depending on the parameter $x_1 \in \mathbb{R}$. In particular, V solves the equation

$$\rho V - \bar{b}^2 V_{x_2} - (\eta^2/2)V_{x_2x_2} = h, \quad \text{for a.a. } x_2 \in B^2_\varepsilon(z), \text{ for each fixed } x_1 \in B^1_\varepsilon(z). \tag{2.4.9}$$

Therefore we have $V(x_1, \cdot) \in C^{4;1}(B^2_\varepsilon(z))$, for each $x_1 \in B^1_\varepsilon(z)$. Next, for any y_1, $x_1 \in B^1_\varepsilon(z)$ we define the function $W(x_2) := V(y_1, x_2) - V(x_1, x_2),\ x_2 \in B^2_\varepsilon(z)$, which satisfies the ODE

$$\rho W - \bar{b}^2(y_1, \cdot)W_{x_2} - (\eta^2/2)W_{x_2x_2} = F, \quad x_2 \in B^2_\varepsilon(z),$$

where $F = h(y_1, \cdot) - h(x_1, \cdot) + b^2_1 V_{x_2}(x_1, \cdot)(y_1 - x_1)$. Therefore, by employing Schauder interior estimates (see Theorem 6.2 at p. 90 in [85]), we obtain

$$\|W\|_{C^{2;1}(B^2_{\varepsilon/2}(z))} \leqslant C(\|W\|_{C^0(B^2_\varepsilon(z))} + \|F\|_{C^{0;1}(B^2_\varepsilon(z))}).$$

Moreover, by the $W^{2;\infty}_{loc}$-regularity of V (cf. Theorem 2.5.1 in Section 2.5), the function F is Lipschitz in $B^1_\varepsilon(z) \times B^2_\varepsilon(z)$. Thus, the latter estimate implies that

$$\|V(y_1, \cdot) - V(x_1, \cdot)\|_{C^{2;1}(B^2_{\varepsilon/2}(z))} \leqslant C|y_1 - x_1|,$$

for a constant C which is independent from y_1 and x_1, as long as they are elements of $B^1_\varepsilon(z)$. Hence, the functions V, V_{x_2}, $V_{x_2x_2}$ are Lipschitz continuous in $B^1_\varepsilon(z) \times B^2_{\varepsilon/2}(z)$.

We can therefore compute the weak derivative of (2.4.9) with respect to x_1, obtaining, for each fixed $x_1 \in B^1_\varepsilon(z)$, the ODE

$$\rho V_{x_1} - b^2 V_{x_1x_2} - (\eta^2/2)V_{x_1x_2x_2} = h_{x_1} + b^2_1 V_{x_2}, \quad \text{for a.a. } x_2 \in B^2_{\varepsilon/2}(z). \tag{2.4.10}$$

Since $V_{x_2x_2}$ is Lipschitz, we have $V_{x_1}(x_1, \cdot) \in C^{3;1}(B^2_{\varepsilon/2}(z))$, for each $x_1 \in B^1_\varepsilon(z)$. Also, we can again define a function function $W^1(x_2) := V_{x_1}(y_1, x_2) - V_{x_1}(x_1, x_2),\ x_2 \in B^2_{\varepsilon/2}(z)$, which satisfies the elliptic equation

$$\rho W^1 - b^2(y_1, \cdot)W^1_{x_2} - (\eta^2/2)W^1_{x_2x_2} = F^1, \quad x_2 \in B^2_{\varepsilon/2}(z),$$

where $F^1 = h_{x_1}(y_1,\cdot) - h_{x_1}(x_1,\cdot) + b_1^2(V_{x_2}(y_1,\cdot) - V_{x_2}(x_1,\cdot)) + b_1^2 V_{x_1x_2}(x_1,\cdot)(y_1 - x_1)$. By employing again Schauder interior estimates, we obtain

$$\|W^1\|_{C^{2;1}(B^2_{\varepsilon/3}(z))} \leqslant C(\|W^1\|_{C^0(B^2_{\varepsilon/2}(z))} + \|F^1\|_{C^{0;1}(B^2_{\varepsilon/2}(z))}).$$

This, by the local Lipschitz continuity of V_{x_2} and $V_{x_1x_2}$ (since we have shown that $V_{x_1x_2x_2}$ exists bounded) in the variable x_2, implies that

$$\|V_{x_1}(y_1,\cdot) - V_{x_1}(x_1,\cdot)\|_{C^{2;1}(B^2_{\varepsilon/2}(z))} \leqslant C|y_1 - x_1|;$$

that is, the functions V_{x_1}, $V_{x_1x_2}$, $V_{x_1x_2x_2}$ are Lipschitz continuous in $B^1_\varepsilon \times B^2_{\varepsilon/3}(z)$.

This allows to compute once more the weak derivative w.r.t. x_1 in equation (2.4.10), obtaining for each fixed $x_1 \in B^1_\varepsilon(z)$, the ODE

$$\rho V_{x_1x_1} - b^2 V_{x_1x_1x_2} - (\eta^2/2) V_{x_1x_1x_2x_2} = h_{x_1x_1} + 2b_1^2 V_{x_1x_2}, \quad \text{for a.a. } x_2 \in B^2_{\varepsilon/3}(z). \tag{2.4.11}$$

Therefore, since we have shown that $V_{x_1x_2}$ is Lipschitz, after employing one more time Schauder interior estimates, we obtain

$$\|V_{x_1x_1}\|_{C^{2;1}(B^2_{\varepsilon/4}(z))} \leqslant C(\|V_{x_1x_1}\|_{C^0(B^2_{\varepsilon/3}(z))} + \|h_{x_1x_1} + 2b_1^2 V_{x_1x_2}\|_{C^{0;1}(B^2_{\varepsilon/3}(z))}) \leqslant C,$$

for $x_1 \in B^1_\varepsilon(z)$ and for C large enough, not depending on x_1. In particular we deduce that $V_{x_1x_1}(x_1,\cdot)$ is Lipschitz in $B^2_{\varepsilon/4}(z)$, with Lipschitz constant uniformly bounded for $x_1 \in B^1_\varepsilon(z)$.

Step 2. We now prove that $V_{x_1x_1}(\cdot,x_2)$ is continuous in $\mathcal{W}^1(x_2)$ (see Lemma 2.1.3), for each $x_2 \in \mathbb{R}$. This is done by employing a direct computation to find an expression for $V_{x_1x_1}$.

Fix $x \in \mathcal{W}$ and let $\hat{h}$ be as in Theorem 2.2.1. For $\delta > 0$, from (2.4.4) in the proof of Lemma 2.4.5, we have $\bar{\tau}_1^\delta \geqslant \bar{\tau}_1$. Then, from (2.4.6) and Theorem 2.2.2, we write

$$\begin{aligned}
&\frac{V_{x_1}(x+\delta e_1) - V_{x_1}(x)}{\delta} \leqslant \frac{G(x+\delta e_1;\bar{\tau}_1^\delta,\bar{\tau}_2) - G(x;\bar{\tau}_1^\delta,\bar{\tau}_2)}{\delta} \\
&= \mathbb{E}\bigg[\int_0^{\bar{\tau}_1^\delta\wedge\bar{\tau}_2} e^{-\rho t}\bigg(\frac{\hat{h}(X_t^{x+\delta e_1}) - \hat{h}(X_t^x)}{\delta}\bigg)dt\bigg] \\
&= \mathbb{E}\bigg[\int_0^{\bar{\tau}_1\wedge\bar{\tau}_2}\int_0^1 e^{-\rho t}\Big(\hat{h}_{x_1}(Z_t^{\delta,r}) + \hat{h}_{x_2}(Z_t^{\delta,r}) b_1^2 (e^{b_2^2 t} - 1)/b_2^2\Big) dr dt\bigg] \\
&\quad + \mathbb{E}\bigg[\int_{\bar{\tau}_1\wedge\bar{\tau}_2}^{\bar{\tau}_1^\delta\wedge\bar{\tau}_2}\int_0^1 e^{-\rho t}\Big(\hat{h}_{x_1}(Z_t^{\delta,r}) + \hat{h}_{x_2}(Z_t^{\delta,r}) b_1^2 (e^{b_2^2 t} - 1)/b_2^2\Big) dr dt\bigg] =: M_1^\delta + M_2^\delta,
\end{aligned} \tag{2.4.12}$$

where $Z_t^{\delta,r} := X_t^x + r(X_t^{x+\delta e_1} - X_t^x)$. Next, in order to study M_1^δ and M_2^δ, define

$$H(t,y) := \hat{h}_{x_1}(y) + \hat{h}_{x_2}(y) b_1^2 (e^{b_2^2 t} - 1)/b_2^2, \quad y \in \mathbb{R}^2. \tag{2.4.13}$$

Notice that, by (2.4.1), Proposition 2.2.3 (see the discussion in Subsection 2.4.1) and the convexity of V we have $h_{x_1x_1}$, $b_1^2 h_{x_1x_2}$, $b_1^2 V_{x_1x_2}$, $V_{x_2x_2} \geqslant 0$, and hence

$$H \geqslant 0. \tag{2.4.14}$$

Moreover, since $p = 2$, from Proposition 2.4 in [77], for each $\bar{y}$, $y \in \mathbb{R}^2$, and $\lambda \in [0,1]$, we have

$$\lambda V(\bar{y}) + (1-\lambda)V(y) - V(\lambda\bar{y} + (1-\lambda)y) \leqslant K\lambda(1-\lambda)|\bar{y} - y|^2, \tag{2.4.15}$$

for some $K > 0$. Hence, (2.4.14) and (2.4.15) together with Condition 1 in Assumption 2.1.1 give

$$0 \leqslant H(t,y) \leqslant C. \tag{2.4.16}$$

By Step 1, the function $H(t,\cdot)$ is continuous in $\mathcal{W}$. Moreover, since $Z^{\delta,r} \to X^x$ for $\mathbb{P}\otimes dt \otimes dr$-a.a. $(\omega,t,r) \in \Omega \times [0,\infty) \times (0,1)$, as $\delta \to 0$, we deduce that $H(t, Z_t^{\delta,r}) \to H(t, X_t^x)$, $\mathbb{P}\otimes dt\otimes dr$-a.e. as $\delta \to 0$. Therefore, thanks to (2.4.16), by the dominated convergence theorem we have

$$\lim_{\delta\to 0^+} M_1^\delta = \mathbb{E}\bigg[\int_0^{\bar{\tau}_1\wedge\bar{\tau}_2} e^{-\rho t}\Big(\hat{h}_{x_1}(X_t^x) + \hat{h}_{x_2}(X_t^x) b_1^2(e^{b_2^2 t} - 1)/b_2^2\Big)dt\bigg]. \tag{2.4.17}$$

Also, by Lemma 2.4.5 we have $\mathbb{1}_{(\bar{\tau}_1\wedge\bar{\tau}_2, \bar{\tau}_1^\delta\wedge\bar{\tau}_2)} \to 0$, $\mathbb{P}$-a.s. as $\delta \to 0$. Therefore we can again employ (2.4.16) and the dominated convergence theorem to conclude that

$$\lim_{\delta\to 0} M_2^\delta = 0. \tag{2.4.18}$$

Hence, since we already know that $V_{x_1x_1}$ exists a.e., (2.4.12), (2.4.17) and (2.4.18) implies that

$$V_{x_1x_1}(x) \leqslant \mathbb{E}\bigg[\int_0^{\bar{\tau}_1\wedge\bar{\tau}_2} e^{-\rho t}\Big(\hat{h}_{x_1}(X_t^x) + \hat{h}_{x_2}(X_t^x) b_1^2(e^{b_2^2 t} - 1)/b_2^2\Big)dt\bigg], \quad \text{a.e. in } \mathcal{W}. \tag{2.4.19}$$

Also, arguments similar to the one leading to (2.4.19), allow to estimate $V_{x_1x_1}$ from below, obtaining

$$V_{x_1x_1}(x) \geqslant \mathbb{E}\bigg[\int_0^{\bar{\tau}_1\wedge\bar{\tau}_2} e^{-\rho t}\Big(\hat{h}_{x_1}(X_t^x) + \hat{h}_{x_2}(X_t^x) b_1^2(e^{b_2^2 t} - 1)/b_2^2\Big)dt\bigg], \quad \text{a.e. in } \mathcal{W},$$

which, together with (2.4.19), implies that

$$V_{x_1x_1}(x) = \mathbb{E}\bigg[\int_0^{\bar{\tau}_1\wedge\bar{\tau}_2} e^{-\rho t}\Big(\hat{h}_{x_1}(X_t^x) + \hat{h}_{x_2}(X_t^x) b_1^2(e^{b_2^2 t} - 1)/b_2^2\Big)dt\bigg], \quad \text{a.e. in } \mathcal{W}. \tag{2.4.20}$$

We can finally study the continuity of $V_{x_1x_1}$ in the variable x_1. From (2.4.20) we have

$$\begin{aligned}|V_{x_1x_1}(x+\delta e_1) - V_{x_1x_1}(x)| &\leqslant \bigg|\mathbb{E}\bigg[\int_0^{\bar{\tau}_1\wedge\bar{\tau}_2} e^{-\rho t}(H(t, X_t^{x+\delta e_1}) - H(t, X_t^x))dt\bigg]\bigg| \\ &\quad + \bigg|\mathbb{E}\bigg[\int_{\bar{\tau}_1\wedge\bar{\tau}_2}^{\bar{\tau}_1^\delta\wedge\bar{\tau}_2^\delta} e^{-\rho t} H(t, X_t^{x+\delta e_1})dt\bigg]\bigg| =: N_1^\delta + N_2^\delta,\end{aligned} \tag{2.4.21}$$

with H defined in (2.4.13). Following arguments similar to the ones leading to (2.4.17) and (2.4.18), we can show that $\lim_{\delta\to 0} N_1^\delta = 0$ and that $\lim_{\delta\to 0} N_2^\delta = 0$. Therefore, taking limits as $\delta \to 0$ in (2.4.21), we deduce that $V_{x_1x_1}$ is a.e. equal to a function which is continuous the variable x_1.

By Step 1, the functions $V_{x_1x_1}(x_1,\cdot)$ are locally Lipschitz continuous, uniformly in x_1. Thus, by the continuity of $V_{x_1x_1}(\cdot, x_2)$, we conclude that the function $V_{x_1x_1}$ is jointly continuous in both variables in $\mathcal{W}$. This completes the proof of the lemma. □

Characterization of the optimal control

In light of Lemma 2.4.6, under the additional Assumption 2.4.4, we can construct the ε-optimal policies. Indeed, by employing the comparison principle to the second order ODE (2.4.11) (regarded as an equation in the variable x_2, depending on the parameter x_1), one still obtains that $V_{x_1x_1} > 0$ in $\mathcal{W}$. This, together with the fact that $V_{x_1} \in C^1(\mathcal{W})$ (by Lemma 2.4.6), allows to show that S_ε is a C^1 curve in $\mathbb{R}^2$ and that the vector field $-e_1 V_{x_1}/|V_{x_1}|$ is C^1 on S_ε, and nontangential to S_ε. All the assumptions in CASE 2 at p. 557 in [74] (up to the boundedness of $\mathcal{W}$) are then satisfied, and we can therefore employ (a suitable extension to unbounded domains of) Theorem 5.1 at p. 572 in [74] in order to find the ε-optimal controls as in Lemma 2.2.4. Finally, all the arguments in Section 2.2.3 can be repeated in the case in which $\sigma\sigma^\top$ is degenerate. Overall, we have proved the following result.

Theorem 2.4.7. *Consider the degenerate singular control problem described in* (2.4.1)*, with h satisfying Condition 1 in Assumption 2.1.1 and Assumption 2.4.4. Then, the thesis of Theorem 2.1.5 holds.*

Concluding, with respect to [77], we require in addition that $h_{x_1x_1} > 0$ and that Assumption 2.4.4 is satisfied. In this case, Theorem 2.4.7 applies, and the construction of the optimal control discussed in Section 7 in [77] can be provided. We underline that in [77] a construction of an optimal control is given in weak formulation, under a quite strong requirement on the running cost h. We refer to Proposition 7.3 in [77] for more details.

2.5 Auxiliary results: On the HJB equation

In this section we prove that V is a solution (in the a.e. sense) to the related HJB equation. The argument of the proof exploits the penalization method introduced in [76] for bounded domains (see also [100] and the references therein), which we extend to D thanks to suitable semiconcavity estimates, in the spirit of [32]. Although this result is somehow classical, we have not been able to find versions that exactly fit our setting, and we therefore provide its proofs in the following.

Theorem 2.5.1. *The value function V is a $W^{2;\infty}_{loc}(D)$-solution to the equation*

$$\max\{\rho V - \mathcal{L}V - h, |V_{x_1}| - 1\} = 0, \quad a.e. \text{ in } D. \tag{2.5.1}$$

Proof. We divide the proof in four steps.

Step 1. Let us start by introducing a family of penalized versions of the HJB equation (2.5.1). Let $\beta \in C^\infty(\mathbb{R})$ be a convex nondecreasing function with $\beta(r) = 0$ if $r \leqslant 0$ and $\beta(r) = 2r - 1$ if $r \geqslant 1$. For each $\varepsilon > 0$, let V^ε be the the value function of the penalized control problem

$$V^\varepsilon(x) := \inf_{\alpha \in \mathcal{U}_\varepsilon} J_\varepsilon(x;\alpha) := \inf_{\alpha \in \mathcal{U}_\varepsilon} \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}(h(X_t^{x;\alpha}) + |\alpha_t^1| + \alpha_t^2)dt\bigg], \quad x \in D, \tag{2.5.2}$$

where $\mathcal{U}_\varepsilon$ is the set of E_ε-valued $\mathbb{F}$-progressively measurable processes, wtih $E_\varepsilon := \{\alpha = (\alpha^1, \alpha^2) \in \mathbb{R} \times [0, \infty) \,|\, |\alpha^1| r - \frac{1}{\varepsilon}\beta(r(r+2)) \leqslant \alpha^2 \leqslant \frac{1}{\varepsilon},\ \forall r > 0\}$. Here, with a slight abuse of notation, $X^{x;\alpha}$ denotes the solution to $dX_t^{x;\alpha} = (b(X_t^{x;\alpha}) + e_1\alpha_t^1)dt + \sigma dW_t,\ t \geqslant 0,\ X_0^{x;\alpha} = x$. We point out that, under Condition 3b in Assumption 2.1.1, a result analogus to Lemma 2.3.1 holds. Arguing as in [100] (throught a localization argument), it is possible to show that V^ε is a $C^2(D)$ solution to the partial differential equation

$$\rho V^\varepsilon - \mathcal{L}V^\varepsilon + \frac{1}{\varepsilon}\beta((V^\varepsilon_{x_1})^2 - 1) = h, \quad \text{in } D. \tag{2.5.3}$$

Moreover, the family $(V^\varepsilon)_{\varepsilon \in (0,1)}$ provides an approximation of V; that is,

$$\lim_{\varepsilon \to 0} V^\varepsilon(x) = V(x), \quad \text{for each } x \in D. \tag{2.5.4}$$

Take indeed $x \in D$. Observe that, for each $\varepsilon > 0$, we have $V^\varepsilon(x) \geqslant V(x)$, as $\alpha^2 \geqslant 0$. Moreover, as in Theorem 2.2. in [57], one can show that for each $\delta > 0$ there exists a Lipschitz admissible process $w \in \mathcal{V}$ such that $J(x; w) \leqslant V(x) + \delta$. Since w is Lipschitz, we have $dw_t = \alpha_t^1 dt$, for some bounded progressively measurable process α^1. Then, defining $\alpha_t^2 = \rho\delta/2$, we can find $\bar{\varepsilon} > 0$ such that $\alpha := (\alpha^1, \alpha^2) \in \mathcal{U}_\varepsilon$ for each $\varepsilon \in (0, \bar{\varepsilon})$. Moreover, with this choice of α, we have that $J_\varepsilon(x; \alpha) \leqslant J(x; w) + \delta/2 \leqslant V(x) + \delta$, for each $\varepsilon \in (0, \bar{\varepsilon})$, completing the proof of (2.5.4).

Step 2. In this step we show that, under Condition 3a in Assumption 2.1.1, for each $R > 0$, there exists a constant C_R such that

$$0 \leqslant \lambda V^\varepsilon(\bar{x}) + (1-\lambda)V^\varepsilon(x) - V^\varepsilon(\lambda\bar{x} + (1-\lambda)x) \leqslant C_R \lambda(1-\lambda)|\bar{x} - x|^2, \tag{2.5.5}$$

for each $\lambda \in [0,1]$, $\bar{x}, x \in B_R$ and $\varepsilon > 0$. By the same arguments leading the convexity of V (cf. Remark 2.1.4), it is possible to show that, for each $\varepsilon > 0$, the function V^ε is convex. Therefore, we only need to prove the last inequality in (2.5.5). Take $\bar{x}$, $x \in B_R$, $\lambda \in [0,1]$ and set $x^\lambda := \lambda\bar{x} + (1-\lambda)x$. Fix $\varepsilon > 0$, an arbitrary $\delta > 0$, and let $\alpha \in \mathcal{U}_\varepsilon$ be a δ-optimal control for the problem (2.5.2) with initial condition x^λ; that is, $J_\varepsilon(x^\lambda; \alpha) \leqslant V^\varepsilon(x^\lambda) + \delta$. Since α is not necessarily optimal for x or $\bar{x}$, we have

$$\begin{aligned}
\lambda V^\varepsilon(\bar{x}) &+ (1-\lambda)V^\varepsilon(x) - V^\varepsilon(x^\lambda) - \delta \\
&\leqslant \lambda J_\varepsilon(\bar{x}; \alpha) + (1-\lambda)J_\varepsilon(x; \alpha) - J_\varepsilon(x^\lambda; \alpha) \\
&\leqslant \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\Big(\lambda h(X_t^{\bar{x};\alpha}) + (1-\lambda)h(X_t^{x;\alpha}) - h(X_t^{x^\lambda;\alpha})\Big)dt\bigg].
\end{aligned}$$

Setting $Z_t := \lambda X_t^{\bar{x};\alpha} + (1-\lambda)X_t^{x;\alpha}$, using Condition 1 in Assumption 2.1.1 , we continue the latter chain of estimates to find

$$\begin{aligned}
&\lambda V^{\varepsilon}(\bar{x}) + (1-\lambda)V^{\varepsilon}(x) - V^{\varepsilon}(x^{\lambda}) - \delta \\
&\quad\leqslant \mathbb{E}\bigg[\int_0^{\infty} e^{-\rho t}\Big(\lambda h(X_t^{\bar{x};\alpha}) + (1-\lambda)h(X_t^{x;\alpha}) - h(Z_t)\Big)dt\bigg] \\
&\qquad + \mathbb{E}\bigg[\int_0^{\infty} e^{-\rho t}\Big(h(Z_t) - h(X_t^{x^{\lambda};\alpha})\Big)dt\bigg] \\
&\quad\leqslant C\lambda(1-\lambda)\mathbb{E}\bigg[\int_0^{\infty} e^{-\rho t}\Big(1 + \Big|X_t^{x;\alpha}\Big|^{p-2} + \Big|X_t^{\bar{x};\alpha}\Big|^{p-2}\Big)\Big|X_t^{\bar{x};\alpha} - X_t^{x;\alpha}\Big|^2 dt\bigg] \\
&\qquad + C\mathbb{E}\bigg[\int_0^{\infty} e^{-\rho t}\Big(1 + \Big|Z_t\Big|^{p-1} + \Big|X_t^{x^{\lambda};\alpha}\Big|^{p-1}\Big)\Big|Z_t - X_t^{x^{\lambda};\alpha}\Big| dt\bigg] \\
&\quad =: M_1 + M_2.
\end{aligned} \tag{2.5.6}$$

We will now estimate M_1 and M_2 separately.

First of all, by a standard use of Grönwall's inequality, we find

$$\Big|X_t^{\bar{x};\alpha} - X_t^{x;\alpha}\Big| \leqslant Ce^{\bar{L}t}|\bar{x} - x|. \tag{2.5.7}$$

When $p = 2$, from (2.5.7) and our assumptions on ρ, we immediately deduce that

$$M_1 \leqslant C_R\lambda(1-\lambda)|\bar{x} - x|^2, \tag{2.5.8}$$

as desired. On the other hand, if $p > 2$, set $p' := (2p-1)/2$. Defining $q := p'/(p-2)$ and denoting by q^* its conjugate, we can employ Hölder's inequality and obtain

$$\begin{aligned}
M_1 &\leqslant C\lambda(1-\lambda)|\bar{x} - x|^2\bigg(\mathbb{E}\bigg[\int_0^{\infty} e^{(2\bar{L} - \rho(1-\frac{1}{q}))q^* t} dt\bigg]\bigg)^{\frac{1}{q^*}} \\
&\quad\times\bigg(\mathbb{E}\bigg[\int_0^{\infty} e^{-\rho t}\Big(\Big|X_t^{x;\alpha}\Big|^{p'} + \Big|X_t^{\bar{x};\alpha}\Big|^{p'}\Big)dt\bigg]\bigg)^{\frac{1}{q}} \\
&\leqslant C\lambda(1-\lambda)(1 + |x|^p + |\bar{x}|^p)^{\frac{1}{q}}|\bar{x} - x|^2 \\
&\leqslant C_R\lambda(1-\lambda)|\bar{x} - x|^2,
\end{aligned}$$

where we have used the requirements on ρ in Condition 3a in Assumption 2.1.1, and the estimate (2.2.26), which holds also for the penalized problem.

We next estimate M_2. Since the gradient Db is Lipschitz we have the estimate (see, e.g., Proposition 1.1.3 at p. 2 in [39])

$$|\lambda b(\bar{y}) + (1-\lambda)b(y) - b(\lambda\bar{y} + (1-\lambda)y)| \leqslant C\lambda(1-\lambda)|\bar{y} - y|^2, \quad \text{for each } \bar{y},\, y \in \mathbb{R}^2.$$

This, together with the Lipschitz proprerty of b, allows to obtain

$$\begin{aligned}
\Big|X_t^{x^{\lambda};\alpha} - Z_t\Big| &\leqslant \int_0^t \Big|b(X_s^{x^{\lambda};\alpha}) - \lambda b(X_s^{\bar{x};\alpha}) - (1-\lambda)b(X_s^{x;\alpha})\Big| ds \\
&\leqslant \bar{L}\int_0^t \Big(\Big|X_s^{x^{\lambda};\alpha} - Z_s\Big| + \lambda(1-\lambda)\Big|X_s^{\bar{x};\alpha} - X_s^{x;\alpha}\Big|^2\Big)ds, \\
&\leqslant \bar{L}\int_0^t \Big(\Big|X_s^{x^{\lambda};\alpha} - Z_s\Big| + \lambda(1-\lambda)|\bar{x} - x|^2 e^{2\bar{L}s}\Big)ds, \\
&\leqslant C\lambda(1-\lambda)|\bar{x} - x|^2 e^{2\bar{L}t} + \bar{L}\int_0^t \Big|X_s^{x^{\lambda};\alpha} - Z_s\Big| ds.
\end{aligned} \tag{2.5.9}$$

The latter estimate, after employing Grönwall's inequality, leads to

$$\left|X_t^{x^\lambda;\alpha} - Z_t\right| \leqslant C\lambda(1-\lambda)e^{3\bar{L}t}|\bar{x}-x|^2. \tag{2.5.10}$$

Defining $q := p'/(p-1)$ and denoting by q^* is conjugate, we can again employ Hölder's inequality and (2.5.10) in order to obtain

$$\begin{aligned}
M_2 &\leqslant C\lambda(1-\lambda)|\bar{x}-x|^2\mathbb{E}\bigg[\int_0^\infty e^{(3\bar{L}-\rho)t}\Big(1+\left|Z_t\right|^{p-1}+\left|X_t^{x^\lambda;\alpha}\right|^{p-1}\Big)dt\bigg] \\
&\leqslant C\lambda(1-\lambda)|\bar{x}-x|^2\bigg(\mathbb{E}\bigg[\int_0^\infty e^{(3\bar{L}-\rho(1-\frac{1}{q}))q^*t}dt\bigg]\bigg)^{\frac{1}{q^*}} \\
&\quad\times\bigg(\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\Big(\left|X_t^{x^\lambda;\alpha}\right|^{p'}+\left|Z_t\right|^{p'}\Big)dt\bigg]\bigg)^{\frac{1}{q}} \\
&\leqslant C\lambda(1-\lambda)(1+|x|^p+|\bar{x}|^p)^{\frac{1}{q}}|\bar{x}-x|^2 \\
&\leqslant C_R\lambda(1-\lambda)|\bar{x}-x|^2,
\end{aligned}$$

where we have used the estimate (2.2.26) and the requirements on ρ in Condition 3a in Assumption 2.1.1. This, together with (2.5.8) and (2.5.6), thanks again to the arbitrariness of δ, completes the proof of (2.5.5).

Step 3. We now prove the estimate (2.5.5) under Condition 3b in Assumption 2.1.1. To simplify the notation, we assume $d = 2$, the generalization to $d > 2$ being straightforward. We proceed from (2.5.6), and we estimate M_1 and M_2 from above. To this end, define the processes

$$E_t := \exp[(b_1^1 - \sigma^2/2)t + W_t^1] \quad \text{and} \quad \hat{E}_t := \exp[(\bar{L} - \sigma^2/2)t + \sigma W_t^2].$$

We first estimate M_1. Observe that

$$|X_t^{1,\bar{x};\alpha} - X_t^{1,x;\alpha}| = |\bar{x}_1 - x_1|E_t, \tag{2.5.11}$$

which we will use to estimate $|X_t^{2,\bar{x};\alpha} - X_t^{2,x;\alpha}|$. Define the process Δ as the solution to the SDE

$$d\Delta_t = \bar{L}(|X_t^{1,\bar{x};\alpha} - X_t^{1,x;\alpha}| + \Delta_t)dt + \sigma\Delta_t dW_t^2, \quad t \geqslant 0, \quad \Delta_0 = |\bar{x}_2 - x_2|.$$

Through a comparison principle, it is easy to check that $|X_t^{2,\bar{x};\alpha} - X_t^{2,x;\alpha}| \leqslant \Delta_t$, so that, using (2.5.11) and the explicit expression for Δ, we get

$$|X_t^{2,\bar{x};\alpha} - X_t^{2,x;\alpha}| \leqslant C|\bar{x}-x|\hat{E}_t\Big[1+\textstyle\int_0^t E_s/\hat{E}_s ds\Big] =: C|\bar{x}-x|P_t. \tag{2.5.12}$$

When $p = 2$, the estimate of M_1 can be easily deduced from (2.5.11) and (2.5.12). For $p > 2$, by employing Hölder's inequality with exponent $q = p'/(p-2)$, we find

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\Big(1+\left|X_t^{x;\alpha}\right|^{p-2}+\left|X_t^{\bar{x};\alpha}\right|^{p-2}\Big)(E_t^2+P_t^2)dt\bigg] \qquad (2.5.13)\\
&\leqslant C\bigg(\int_0^\infty e^{-\rho t}\mathbb{E}\Big[1+\left|X_t^{x;\alpha}\right|^{p'}+\left|X_t^{\bar{x};\alpha}\right|^{p'}\Big]dt\bigg)^{\frac{1}{q}}\bigg(\int_0^\infty e^{-\rho(1-\frac{1}{q})q^*t}\mathbb{E}\Big[E_t^{2q*}+P_t^{2q*}\Big]dt\bigg)^{\frac{1}{q^*}} \\
&\leqslant C(1+|x|^p)^{\frac{1}{q}}\bigg(\int_0^\infty e^{-\rho(1-\frac{1}{q})q^*t}\mathbb{E}\Big[E_t^{2q*}+P_t^{2q*}\Big]dt\bigg)^{\frac{1}{q^*}} \leqslant C_R < \infty.
\end{aligned}$$

Here, we have also used (2.3.14), while the finiteness of the latter integral follows, after some elementary computations, from the requirements on ρ in Condition 3b in Assumption 2.1.1. Finally, by (2.5.11), (2.5.12) and (2.5.13), we obtain

$$M_1 \leqslant C_R\lambda(1-\lambda)|\bar{x}-x|^2. \tag{2.5.14}$$

We next estimate M_2. Since $\bar{b}^1$ is affine, we have $Z^1 - X^{1,x^\lambda;\alpha}$. Similarly to (2.5.9), one has

$$\begin{aligned} Z_t^2 - X_t^{2,x^\lambda;\alpha} \leqslant \int_0^t \Big(C\lambda(1-\lambda)|X_s^{2,\bar{x};\alpha} - X_s^{2,x;\alpha}|^2 + \bar{L}\Big|X_s^{x^\lambda;\alpha} - Z_s\Big|\Big)ds \\ + \sigma\int_0^t (Z_s - X_s^{x^\lambda;\alpha})dW_s^2. \end{aligned}$$

Therefore, employing again a comparison principle and using (2.5.12), we see that

$$\begin{aligned} |Z_t^2 - X_t^{2,x^\lambda;\alpha}| &\leqslant C\lambda(1-\lambda)\hat{E}_t\int_0^t \frac{|X_s^{2,\bar{x};\alpha} - X_s^{2,x;\alpha}|^2}{\hat{E}_s}ds \\ &\leqslant C\lambda(1-\lambda)|\bar{x}-x|^2\int_0^t \frac{\hat{E}_t}{\hat{E}_s}P_s^2 ds. \end{aligned} \tag{2.5.15}$$

Also, Hölder's inequality with exponent $q = p'/(p-1)$ yields

$$\begin{aligned} &\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\Big(1 + \Big|Z_t\Big|^{p-1} + \Big|X_t^{x^\lambda;\alpha}\Big|^{p-1}\Big)\int_0^t \frac{\hat{E}_t}{\hat{E}_s}P_s^2 ds\, dt\bigg] \\ &\leqslant C\bigg(\int_0^\infty e^{-\rho t}\mathbb{E}\Big[1 + \Big|X_t^{x;\alpha}\Big|^{p'} + \Big|X_t^{\bar{x};\alpha}\Big|^{p'}\Big]dt\bigg)^{\frac{1}{q}} \\ &\quad \times\bigg(\mathbb{E}\bigg[\int_0^\infty e^{-\rho(1-\frac{1}{q})q^* t}\bigg(\int_0^t \frac{\hat{E}_t}{\hat{E}_s}P_s^2 ds\bigg)^{q*}dt\bigg]\bigg)^{\frac{1}{q^*}} \\ &\leqslant C(1+|x|^p)^{\frac{1}{q}}\bigg(\mathbb{E}\bigg[\int_0^\infty e^{-\rho(1-\frac{1}{q})q^* t}\bigg(\int_0^t \frac{\hat{E}_t}{\hat{E}_s}P_s^2 ds\bigg)^{q*}dt\bigg]\bigg)^{\frac{1}{q^*}} \leqslant C_R < \infty. \end{aligned} \tag{2.5.16}$$

Again, here we have also employed (2.3.14), while the finiteness of the latter integral follows, after some elementary computations, from the requirements on ρ in Condition 3b in Assumption 2.1.1. Finally, combining (2.5.15) and (2.5.16), we obtain $M_2 \leqslant C_R\lambda(1-\lambda)|\bar{x}-x|^2$, which, together with (2.5.14) and (2.5.6), implies (2.5.5).

Step 4. From (2.5.5) we deduce that, for each bounded open set $B \subset D$, there exists a constant $C_B > 0$ such that

$$\sup_{\varepsilon\in(0,1)} \|V^\varepsilon\|_{W^{2;\infty}(B)} \leqslant C_B. \tag{2.5.17}$$

This estimate allows, by mean of classical arguments (exploiting Sobolev compact embedding theorem of $W^{2;q}(B)$ into $C^1(B)$ for $q > 2+d$ and the weak compactness of the closed unit ball in $W^{2;2}(B)$) to improve the convergence in (2.5.4). Indeed (on each subsequence) we now have:

$$\begin{aligned} &(V^\varepsilon, DV^\varepsilon) \text{ converges to } (V, DV) \text{ uniformly in } B; \\ &D^2V^\varepsilon \text{ converges to } D^2V \text{ weakly in } L^2(B). \end{aligned} \tag{2.5.18}$$

Let us now prove that V solves the HJB equation (2.5.1). First of all observe that, from (2.5.3) and (2.5.17), (unless to take a larger C_B) we have

$$\frac{1}{\varepsilon}\beta((V^\varepsilon_{x_1})^2 - 1) \leqslant C_B, \quad \text{in } B. \tag{2.5.19}$$

Hence, taking pointwise limits in (2.5.3) and (2.5.19), we obtain

$$\rho V - \mathcal{L}V - h \leqslant 0, \quad \text{and} \quad |V_{x_1}| - 1 \leqslant 0 \quad \text{a.e. in } D.$$

Suppose now that the inequality $|V_{x_1}| - 1 \leqslant 0$ is strict in $\bar{x} \in D$. By continuity of V_{x_1}, there exist $\eta > 0$ and a neighborhod N of $\bar{x}$ such that $|V_{x_1}(x)| - 1 \leqslant -\eta$ for each $x \in N$. Therefore, by uniform convergence in N, for each ε small enough we have $|V^\varepsilon_{x_1}(x)| - 1 \leqslant -\eta/2$, and therefore, by (2.5.3), that $V^\varepsilon - \mathcal{L}V^\varepsilon - h = 0$ in N. Passing again to the limit, this in turn implies that $\rho V - \mathcal{L}V - h = 0$ in N, completing the proof of the theorem.

□

2.6 Auxiliary results: Proof of Lemma 2.1.3 and of Proposition 2.2.10

2.6.1 Proof of Lemma 2.1.3

We give a proof for $d = 2$, the case $d > 2$ is analogous. The set $\mathcal{W}_1(z)$ is an open interval, since, by convexity of V, the function $V_{x_1}(\cdot, z)$ is nondecreasing. We therefore show that the set $\mathcal{W}_1(z)$ is nonempty. Suppose that Condition 3a in Assumption 2.1.1 is in place. Arguing by contradiction, if $\mathcal{W}_1(z) = \emptyset$, then, by the continuity of V_{x_1}, we have $V_{x_1}(\cdot, z) = 1$ or $V_{x_1}(\cdot, z) = -1$. If $V_{x_1}(\cdot, z) = 1$, we have $V(x_1, z) + \kappa_2 \geqslant V(x_1, z) - V(y, z) = \int_y^{x_1} V_{x_1}(r, z)dr = x_1 - y \to \infty$ as $y \to -\infty$. Therefore $V(x_1, z) = \infty$, contradicting the finiteness of V (see Theorem 2.5.1 in Section 2.5). In the same way, we can not have that $V_{x_1}(\cdot, z) = -1$, which implies $\mathcal{W}_1(z) \neq \emptyset$.

On the other hand, suppose that Condition 3b in Assumption 2.1.1. Arguing by contradiction, we assume that $\mathcal{W}_1(z)$ is empty. From the continuity of V_{x_1}, we have $V_{x_1}(\cdot, z) = 1$ or $V_{x_1}(\cdot, z) = -1$. If $V_{x_1}(\cdot, z) = -1$, then we have $V(x_1, z) + \kappa_2 \geqslant V(x_1, z) - V(y, z) = -\int_{x_1}^{y} V_{x_1}(r, z)dr = y - x_1 \to \infty$ as $y \to \infty$. Therefore $V(x_1, z) = \infty$, contradicting the finiteness of V. We therefore assume that $V_{x_1}(\cdot, z) = 1$ and we show that this leads anyway to a contradiction.

For a generic $x_1 \in \mathbb{R}$ with $0 < x_1 < x_1^*$, let $v \in \mathcal{V}$ be optimal for the initial condition $x := (x_1, z)$, with $dv = \gamma d|v|$. By repeating the arguments leading to (2.2.41) in the proof of Proposition 2.2.9, an application of Itô's formula leads to

$$\mathbb{E}\bigg[\int_{[0,\infty)} e^{-\rho t}(1+V_{x_1}(X^{x;v}_{t-})\gamma_t)d|v|_t\bigg] \leqslant 0.$$

This in turn implies, using $0 \leqslant 1-|V_{x_1}| \leqslant 1+V_{x_1}u$ for all $u \in \mathbb{R}$ with $|u|=1$, that

$$\mathbb{E}[|v|_0(1+\gamma_0)] = \mathbb{E}[|v|_0(1+\gamma_0 V_{x_1}(X^{x;v}_{0-})] \leqslant \mathbb{E}\bigg[\int_{[0,\infty)} e^{-\rho t}(1+V_{x_1}(X^{x;v}_{t-})\gamma_t)d|v|_t\bigg] \leqslant 0,$$

where the first equality follows from the assumption $V_{x_1}(\cdot,z)=1$. Also, since $|\gamma_0|=1$, $\mathbb{E}[|v|_0(1+\gamma_0)] \geqslant 0$, which combined with the latter inequality gives $\mathbb{E}[|v|_0(1+\gamma_0)]=0$. In other words, a possible jump a time zero must be of negative size. Therefore, since $x_1 < x_1^*$, as in the proof of Lemma 2.3.1, we deduce that v has no jumps at time zero; that is,

$$\mathbb{P}[|v|_0 > 0] = 0. \tag{2.6.1}$$

Next, fix $0 < x_1 < y_1 < x_1^*$ and set $x=(x_1,z)$ and $y=(y_1,z)$. Since we are assuming that $V_{x_1}(\cdot,z)=1$, we have

$$V(y)-V(x) = \int_{x_1}^{y_1} V_{x_1}(r,z)dr = y_1 - x_1. \tag{2.6.2}$$

Next, denote by v and w the optimal control for the initial conditions x and y, respectively. By (2.6.1), neither v or w has a jump a time zero, so that, using (2.6.2), we find

$$J(y;v+x_1-y_1) = J(x;v)+|x_1-y_1| = V(x)+y_1-x_1 = V(y).$$

This, by uniqueness of the optimal control implies that $w = v+x_1-y_1$, so that, since $x_1 < y_1$, the control w has a negative jump at time zero, contradicting (2.6.1).

Therefore also the assumption $V_{x_1}(\cdot,z)=1$ leads to a contradiction, completing the proof of Lemma 2.1.3 under Condition 3b in Assumption 2.1.1.

2.6.2 Proof of Proposition 2.2.10

We split the proof in three steps.

Step 1. Let $x \in \partial\mathcal{W}$ be such that $x \in I$ for some interval $I \subset \mathbb{R}^2$, with $I \subset \partial\mathcal{W}$ and of the form

$$I = I_{a,c} := \{a + r\eta \,|\, r \in [0,c]\},$$

for some $a \in \mathbb{R}^2$, with $\eta = V_{x_1}(y)e_1$, for each $y \in I \setminus \{a\}$. Denote by $\mathcal{H}$ the set of all such x. Furthermore, assume that I in the above definition is maximal, in the sense that $a - r\eta \notin \partial\mathcal{W}$, for every $r>0$.

Observe that, since $\partial_\eta V(\cdot) = \eta DV = |V_{x_1}(\cdot)|^2 = 1$, then

$$V(a+r\eta) = V(a)+r, \quad \text{for each } r \in [0,c]. \tag{2.6.3}$$

We have that

$$\mathcal{H} = \bigcup_{i=1}^{\infty} \Big\{ y \in \partial\mathcal{W} \,|\, V(y) - V(y - V_{x_1}(y)/i) = 1/i \Big\}.$$

Suppose now that $\bar{x} \in \mathcal{H}$. Then there exists $a \in \mathbb{R}^2$ and $c > 0$ such that $x \in I_{a,c}$. Let $v^a \in \mathcal{V}$ be an optimal control for a. By (2.6.3), we find

$$J(\bar{x}; a - \bar{x} + v^a) = J(a; v^a) + |a - x| = V(a) + |a - x| = V(x),$$

which, by the uniqueness of the optimal control, implies that $\bar{v}_t = a - \bar{x} + v^a_t$, for any $t \geqslant 0$. This means exactly that the optimally controlled state starting from $\bar{x}$ jumps immediately to a.

Step 2. Let now $\bar{x} \in \overline{\mathcal{W}}$ be generic. We want to prove that $X^{\bar{x};\bar{v}}$ jumps only at those times t for which $X^{\bar{x};\bar{v}}_{t-} \in \mathcal{H}$. We argue by contradiction, and suppose that

$$\mathbb{P}[\,\omega \in \Omega \text{ s.t. there exists } t \geqslant 0 \text{ s.t. } X^{\bar{x};\bar{v}}_{t-}(\omega) \notin \mathcal{H} \text{ and } |X^{\bar{x};\bar{v}}_{t}(\omega) - X^{\bar{x};\bar{v}}_{t-}(\omega)| > 0] > 0.$$

For each $\varepsilon > 0$, let

$$\tau_\varepsilon := \inf\{t \geqslant 0 \,|\, X^{\bar{x};\bar{v}}_{t-} \notin \mathcal{H},\ |X^{\bar{x};\bar{v}}_{t} - X^{\bar{x};\bar{v}}_{t-}| \geqslant \varepsilon\}. \tag{2.6.4}$$

Take $\varepsilon > 0$ small enough such that $\mathbb{P}[\tau_\varepsilon < \infty] > 0$. Consider a sequence $(\bar{\tau}_k)_{k\in\mathbb{N}}$ of stopping times exhausting the jumps of $X^{\bar{x};\bar{v}}$ (see, e.g., Proposition 2.26 at p. 10 in [109], for a construction of such a sequence), so that

$$\tau_\varepsilon := \inf\{\bar{\tau}_k \,|\, k \in \mathbb{N},\ X^{\bar{x};\bar{v}}_{\bar{\tau}_k -} \notin \mathcal{H},\ |X^{\bar{x};\bar{v}}_{\bar{\tau}_k} - X^{\bar{x};\bar{v}}_{\bar{\tau}_k -}| \geqslant \varepsilon\}. \tag{2.6.5}$$

Since the jumps of $\bar{v}$ coincides with the jumps of $X^{\bar{x};\bar{v}}$, if $X^{\bar{x};\bar{v}}$ would have an infinite number of jumps of size grater than ε on some interval $[0,T]$ with $T \in (0,\infty)$, then $\bar{v}$ would not be of bounded variation on the interval $[0,T]$. Thus $X^{\bar{x};\bar{v}}$ has only a finite number of jumps of size grater than ε on each interval $[0,T]$. This reveals that τ_ε in (2.6.5) is actually the minimum of a finite number of stopping times, which implies that τ_ε is itself a stopping time.

Next, on $\{\tau_\varepsilon < \infty\}$, we find

$$\begin{aligned} V(X^{\bar{x};\bar{v}}_{\tau_\varepsilon}) - V(X^{\bar{x};\bar{v}}_{\tau_\varepsilon -}) &= \int_0^1 DV(\tau_\varepsilon, X^{\bar{x};\bar{v}}_{\tau_\varepsilon -} + \lambda(X^{\bar{x};\bar{v}}_{\tau_\varepsilon} - X^{\bar{x};\bar{v}}_{\tau_\varepsilon -}))(X^{\bar{x};\bar{v}}_{\tau_\varepsilon} - X^{\bar{x};\bar{v}}_{\tau_\varepsilon -})d\lambda \\ &= \int_0^1 V_{x_1}(\tau_\varepsilon, X^{\bar{x};\bar{v}}_{\tau_\varepsilon -} + \lambda(X^{\bar{x};\bar{v}}_{\tau_\varepsilon} - X^{\bar{x};\bar{v}}_{\tau_\varepsilon -}))\bar{\gamma}_{\tau_\varepsilon}(|\bar{v}|_{\tau_\varepsilon} - |\bar{v}|_{\tau_\varepsilon -})d\lambda \\ &> -|X^{\bar{x};\bar{v}}_{\tau_\varepsilon} - X^{\bar{x};\bar{v}}_{\tau_\varepsilon -}|, \end{aligned} \tag{2.6.6}$$

where the strict inequality follows from the fact that, by Proposition 2.2.8, $X^{\bar{x};\bar{v}}_{\tau_\varepsilon} \in \overline{\mathcal{W}}$ but τ_ε is such that $X^{\bar{x};\bar{v}}_{\tau_\varepsilon -} \notin \mathcal{H}$. Recalling that τ_ε is a stopping time, define the sequence of stopping times $\tau_k := (\tau_\varepsilon + \frac{1}{k}) \wedge T$. By the dynamic programming principle (see, e.g., [97]) we have, for each k

$$V(\bar{x}) = \mathbb{E}\bigg[\int_0^{\tau_k} e^{-\rho t} h(X^{\bar{x};\bar{v}}_t)dt + \int_{[0,\tau_k)} e^{-\rho t} d|\bar{v}|_t + e^{-\rho\tau_k} V(X^{\bar{x};\bar{v}}_{\tau_k -})\bigg]. \tag{2.6.7}$$

Therefore, taking limits as $k \to \infty$ in (2.6.7), using (2.6.6) we find

$$\begin{aligned} V(\bar{x}) &= \mathbb{E}\bigg[\int_0^{\tau_\varepsilon} e^{-\rho t} h(X_t^{\bar{x};\bar{v}}) dt + \int_{[0,\tau_\varepsilon]} e^{-\rho t} d|\bar{v}|_t + e^{-\rho \tau_\varepsilon} V(X_{\tau_\varepsilon}^{\bar{x};\bar{v}}) \bigg] \\ &= \mathbb{E}\bigg[\int_0^{\tau_\varepsilon} e^{-\rho t} h(X_t^{\bar{x};\bar{v}}) dt + \int_{[0,\tau_\varepsilon)} e^{-\rho t} d|\bar{v}|_t + e^{-\rho \tau_\varepsilon} |X_{\tau_\varepsilon}^{\bar{x};\bar{v}} - X_{\tau_\varepsilon -}^{\bar{x};\bar{v}}| + e^{-\rho \tau_\varepsilon} V(X_{\tau_\varepsilon -}^{\bar{x};\bar{v}}) \bigg] \\ &> \mathbb{E}\bigg[\int_0^{\tau_\varepsilon} e^{-\rho t} h(X_t^{\bar{x};\bar{v}}) dt + \int_{[0,\tau_\varepsilon)} e^{-\rho t} d|\bar{v}|_t + e^{-\rho \tau_\varepsilon} V(X_{\tau_\varepsilon -}^{\bar{x};\bar{v}}) \bigg] = V(\bar{x}), \end{aligned}$$

which is a contradiction, hence $X^{\bar{x};\bar{v}}$ jumps only at times t such that $X_{t-}^{\bar{x};\bar{v}} \in \mathcal{H}$.
Step 3. Suppose now that $X_{t-}^{\bar{x};\bar{v}} \in \mathcal{H}$ for some $t > 0$. It remains to prove that, also in this case, $\mathbb{P}$-a.s. the process $X^{\bar{x};\bar{v}}$ jumps at time t to the endpoint of the interval I. Now, for any $\mathbb{F}$-stopping time τ, for $\mathbb{P} \circ (X_\tau^{\bar{x};\bar{v}})^{-1}$-a.a. $x \in \mathbb{R}^2$, we have that the control

$$\bar{v}_t^\tau := \bar{v}_{\tau+t} - \bar{v}_{\tau-}, \quad t \geqslant 0, \tag{2.6.8}$$

is optimal for the initial condition $X_{\tau-}^{\bar{x};\bar{v}}$ (see Lemma 2.11 and the discussion at p. 1616 in [115]). Let now τ^1 be the first time at which the optimally controlled process $X^{\bar{x};\bar{v}}$ enters the set $\mathcal{H}$. Combining (2.6.8) together with Step 1, we obtain that $X^{\bar{x};\bar{v}}$ jumps to the endpoint of I. By constructing an increasing sequence τ_k of hitting times of the set $\mathcal{H}$, which exhausts the set in which $X^{\bar{x};\bar{v}} \in \mathcal{H}$, we conclude that $\mathbb{P}$-a.s. the process $X^{\bar{x};\bar{v}}$ jumps at time t to the endpoint of the interval I.

Chapter 3

Submodular N-player games with singular controls

We consider a class of N-player stochastic games of multi-dimensional singular control, in which each player faces a minimization problem of monotone-follower type with submodular costs. We call these games *monotone-follower games*. In a not necessarily Markovian setting, we establish the existence of Nash equilibria. Moreover, we introduce a sequence of approximating games by restricting, for each $n \in \mathbb{N}$, the players' admissible strategies to the set of Lipschitz processes with Lipschitz constant bounded by n. We prove that, for each $n \in \mathbb{N}$, there exists a Nash equilibrium of the approximating game and that the sequence of Nash equilibria converges, in the Meyer-Zheng sense, to a weak (distributional) Nash equilibrium of the original game of singular control. As a byproduct, such a convergence also provides approximation results of the equilibrium values across the two classes of games. We finally show how our results can be employed to prove existence of open-loop Nash equilibria in an N-player stochastic differential game with singular controls, and we propose an algorithm to determine a Nash equilibrium for the monotone-follower game.

3.1 The monotone-follower game

3.1.1 Definition of the monotone-follower game

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions, a finite time horizon $T \in (0, \infty)$, an integer $N \geqslant 2$ and $k, d \in \mathbb{N}$. Consider an adapted càdlàg process $L : \Omega \times [0,T] \to \mathbb{R}^k$, and, for $i = 1, ..., N$, let $f^i : \Omega \times [0,T] \to \mathbb{R}^d_+$ be a continuous semimartingale, and set $f := (f^1, ..., f^N)$.

Define the space of *admissible strategies*

$$\mathcal{A} := \left\{ \xi : \Omega \times [0,T] \to \mathbb{R}^d \;\middle|\; \begin{array}{c} \xi \text{ is an } \mathbb{F}\text{-adapted càdlàg process, with} \\ \text{nondecreasing and nonnegative components} \end{array} \right\}, \quad (3.1.1)$$

and let $\mathcal{A}^N := \bigotimes_{i=1}^N \mathcal{A}$ denote the set of *admissible profile strategies*. In order to avoid confusion, in the following we will denote profile strategies in bold letters.

For each $i = 1, ..., N$, consider measurable functions $h^i, g^i : \mathbb{R}^k \times \mathbb{R}^{Nd} \to [0, \infty)$. We define the *monotone-follower game* as the game in which each player $i \in \{1, ..., N\}$ is allowed to choose an admissible strategy $\xi^i \in \mathcal{A}$ in order to minimize the cost functional

$$J^i(\xi^i, \xi^{-i}) := \mathbb{E}[C^i(f, L, \boldsymbol{\xi})] := \mathbb{E}\bigg[\int_0^T h^i(L_t, \boldsymbol{\xi}_t)\,dt + g^i(L_T, \boldsymbol{\xi}_T) + \int_{[0,T]} f_t^i\,d\xi_t^i\bigg],$$

where $\xi^{-i} := (\xi^j)_{j\neq i}$ and $\boldsymbol{\xi} := (\xi^i, \xi^{-i}) \in \mathcal{A}^N$. Here and in the sequel the integrals with respect to ξ^i are defined by

$$\int_{[0,T]} f_t^i\,d\xi_t^i := f_0^i \xi_0^i + \int_0^T f_t^i\,d\xi_t^i = \sum_{\ell=1}^d f_0^{\ell,i}\xi_0^{\ell,i} + \sum_{\ell=1}^d \int_0^T f_t^\ell\,d\xi_t^{\ell,i},$$

where the integrals on the right-hand side are intended in the standard Lebesgue-Stieltjes sense on the interval $(0, T]$.

We recall the notion of Nash equilibrium.

Definition 2. *An admissible profile strategy $\bar{\boldsymbol{\xi}} \in \mathcal{A}^N$ is a Nash equilibrium if, for every $i = 1, ..., N$, we have $J^i(\bar{\boldsymbol{\xi}}) < \infty$ and*

$$J^i(\bar{\xi}^i, \bar{\xi}^{-i}) \leqslant J^i(\zeta^i, \bar{\xi}^{-i}), \quad \textit{for every} \quad \zeta^i \in \mathcal{A}.$$

Letting $2^{\mathcal{A}}$ denote the set of all subsets of $\mathcal{A}$, for each $i = 1, ..., N$ define the *best-reply map* $R^i : \mathcal{A}^N \to 2^{\mathcal{A}}$ by

$$R^i(\boldsymbol{\xi}) := \arg\min_{\zeta^i \in \mathcal{A}} J^i(\zeta^i, \xi^{-i}). \tag{3.1.2}$$

Observe that the maps R^i are constant in the variable ξ^i. Moreover define the map

$$\mathbf{R} := (R^1, ..., R^N) : \mathcal{A}^N \to \bigotimes_{i=1}^N 2^{\mathcal{A}}, \tag{3.1.3}$$

and notice that the set of Nash equilibria coincides with the set of fixed points of the map $\mathbf{R}$ which have finite values; that is, the set of $\bar{\boldsymbol{\xi}} \in \mathcal{A}^N$ such that $\bar{\boldsymbol{\xi}} \in \mathbf{R}(\bar{\boldsymbol{\xi}})$ and $J^i(\bar{\boldsymbol{\xi}}) < \infty$ for every $i = 1, ..., N$.

Remark 3.1.1. *The notion of equilibrium introduced above is that of the so-called open-loop Nash equilibrium. While this equilibrium notion does not limit the ability of any player to optimize against given strategies of the others, it does limit the extent of dynamic interaction that can take place. Agents react to the evolving exogenous uncertainty, but take the actions of others as given and do not react to deviations from announced (equilibrium) play; in this sense, one might term such an equilibrium as one in precommitment strategies. In our general setting, allowing for more explicit feedback strategies – and therefore considering equilibria in closed-loop strategies – would make the analysis of our game much harder from a technical and conceptual point of view (see Section 2 of [11] or Chapter 3 of [150] for a discussion on open-loop and closed-loop equilibria in related irreversible investment games). A main issue is that, when looking for existence of Nash equilibria in closed-loop strategies, it is not clear whether our submodular condition on the cost functions (see Assumption 3.1.2 below) allows or not to* *prove the monotonicity of the best-reply-maps, a property of fundamental importance for our existence result to work (cf. Theorem 3.1.4 below). However, in specific Markovian settings, a construction of Nash equilibria with feedback strategies can be possible as it is shown in [89], [118], and [119], among others.*

We now specify the structural hypothesis on the costs.

Assumption 3.1.2. *For each $i = 1, ..., N$ and for $\phi^i \in \{h^i, g^i\}$ assume that:*

1. *for each $(l, a^{-i}) \in \mathbb{R}^k \times \mathbb{R}^{(N-1)d}$, the function $\phi^i(l, \cdot, a^{-i})$ is lower semi-continuous, and strictly convex;*

2. *for each $l \in \mathbb{R}^k$ the function $\phi^i(l, \cdot, \cdot)$ has decreasing differences in (a^i, a^{-i}), i.e.,*
$$\phi^i(l, \bar{a}^i, a^{-i}) - \phi^i(l, a^i, a^{-i}) \geqslant \phi^i(l, \bar{a}^i, \bar{a}^{-i}) - \phi^i(l, a^i, \bar{a}^{-i}),$$
for each $a, \bar{a} \in \mathbb{R}^{Nd}$ such that $\bar{a} \geqslant a$;

3. *for each $(l, a^{-i}) \in \mathbb{R}^k \times \mathbb{R}^{(N-1)d}$, the function $\phi^i(l, \cdot, a^{-i})$ is submodular, i.e.,*
$$\phi^i(l, \bar{a}^i, a^{-i}) + \phi^i(l, a^i, a^{-i}) \geqslant \phi^i(l, \bar{a}^i \wedge a^i, a^{-i}) + \phi^i(l, \bar{a}^i \vee a^i, a^{-i}),$$
for each $a, \bar{a} \in \mathbb{R}^{Nd}$.

In light of Conditions (2) and (3) of Assumption 3.1.2, in the following we refer to the game introduced above as to the *submodular monotone-follower game* (on submodular games see, e.g., [137], [157], [159], or the books [158] and [160] and the references therein). The submodular structure of our game will play a fundamental role in our subsequent analysis.

Remark 3.1.3. *Observe that, if $\phi^i \in \{h^i, g^i\}$ is twice-differentiable, then it fulfills Condition 2 of Assumption 3.1.2 if and only if*
$$\frac{\partial^2 \phi^i}{\partial a^{\ell,i} \partial a^{r,j}} \leqslant 0, \text{ for each } i,\ j = 1, ..., N \text{ with } i \neq j \text{ and } \ell,\ r = 1, ..., d.$$
Notice that Condition (3) in Assumption 3.1.2 is always satisfied in the case $d = 1$. If $d \geqslant 2$, it is verified if and only if, for each fixed $(l, a^{-i}) \in \mathbb{R}^k \times \mathbb{R}^{(N-1)d}$ and $\ell = 1, ..., d$, $h^i(l, \cdot, a^{-i})$ and $g^i(l, \cdot, a^{-i})$ have decreasing differences in $(a^{\ell,i}, a^{-\ell,i})$, where $a^{-\ell,i} = (a^{r,i})_{r \neq \ell}$ (see Theorem 2.6.1 and Corollary 2.6.1 at p. 44 in [158]). Hence, in the case of twice-differentiable cost functions $\phi^i \in \{h^i, g^i\}$, this condition corresponds to having
$$\frac{\partial^2 \phi^i}{\partial a^{\ell,i} \partial a^{r,i}} \leqslant 0 \text{ for each } i = 1, ..., N \text{ and } \ell,\ r = 1, ..., d, \text{ with } \ell \neq r.$$

Example 5. *For $N = 2$ and $d = 1$, Assumption 2.2 is satisfied in the following examples:*

1. *quadratic cost functions of type* $\phi(l, a^1, a^2) = F(l)(a^1 - a^2)^2$*, for any nonnegative* F*; more in general, for a cost function of the form* $\phi(l, a^1, a^2) = F(l, a^1 - a^2)$*, whenever* $F(l, \cdot)$ *is nonnegative, lower semi-continuous, and strictly convex for each* $l \in \mathbb{R}^k$*;*

2. *multiplicative cost functions of type* $\phi(l, a^1, a^2) = F^1(l, a^1)F^2(l, a^2)$*, whenever, for each* $l \in \mathbb{R}^k$*,* $F^1(l, \cdot)$ *is nonnegative, lower semi-continuous, decreasing, and strictly convex, and* $F^2(l, \cdot)$ *is nonnegative and increasing.*

Example 6. *In order to provide some intuition on Assumption 3.1.2, it is somehow easier to consider games in which players face maximization problems. To this end, we illustrate the following model.*

For $N \geqslant 2$ *and* $d = 1$*, consider* N *firms competing on a market. Firm* i *produces and sells a certain good* i*, by choosing an irreversible investment strategy* $\xi^i \in \mathcal{A}$ *in order to increase the production capacity of the good. Firm* i *aims at maximizing an expected net profit functional of type*

$$\Pi^i(\xi^i, \xi^{-i}) = \mathbb{E}\bigg[\int_0^T h^i(L_t, \xi^i_t, \xi^{-i}_t)\, dt - \int_{[0,T]} f^i_t \, d\xi^i_t\bigg],$$

where h^i *is a continuous running operating profit function. Notice that the instantaneous return of firm* i *depends on the investments of the other firms, thus creating a strategic interaction in the market. The processes* L *and* f^i *can be thought of as exogenous random factors affecting the prices of the goods and the costs of the investments, respectively.*

Since players maximize, we "reverse the signs" in Assumption 3.1.2 and we now take strictly concave h^i *such that, for each* $a, \bar{a} \in \mathbb{R}^N$ *with* $\bar{a} \geqslant a$*, we have*

$$h^i(l, \bar{a}^i, a^{-i}) - h^i(l, a^i, a^{-i}) \leqslant h^i(l, \bar{a}^i, \bar{a}^{-i}) - h^i(l, a^i, \bar{a}^{-i}). \tag{3.1.4}$$

This condition describes a situation in which whenever a firm increases its investment level, it creates an incentive for the other firms to increase their investment levels as well. This property is satisfied if firms produce so-called complementary goods (see [158], among others) like, e.g., different construction materials (bricks, cement, steel, etc.). The case $d > 1$ *would describe firms producing more than one good, again under the assumption that all goods are complementary.*

Condition (3.1.4) is satisfied, for example, by an operating profit function of power type

$$h^i(l, a^i, a^{-i}) = H^i(l)\bigg(a^i + \sum_{j \neq i} \lambda^j_i a^j\bigg)^{\gamma^i}, \quad \gamma^i \in (0,1), \quad \lambda^j_i \geqslant 0, \quad i, j = 1, ..., N,$$

for nonnegative H^i *such that* $H^i(L)$ *is sufficiently integrable. Requiring furthermore that the firms' total investment do not exceed a given and fixed amount* $w^i > 0$ *(that is, imposing the constraint* $\xi^i_T \leqslant w^i$*,* $\mathbb{P}$*-a.s.; see also [80]), (after the needed change of signs) Theorem 3.1.4 below, combined with Remark 3.1.7, ensures the existence of Nash equilibria for the model described in this example.*

3.1.2 Existence of Nash equilibria in the submodular monotone-follower game

Define the space of *extended admissible strategies*

$$\mathcal{A}_\infty := \left\{ \xi : \Omega \times [0,T] \to [0,\infty)^d \,\middle|\, \begin{array}{l} \xi \text{ is an } \mathbb{F}\text{-adapted càdlàg process,} \\ \text{with nondecreasing components} \end{array} \right\}, \tag{3.1.5}$$

and, on it, we define the order relation $\preccurlyeq$ such that, for ξ, $\zeta \in \mathcal{A}_\infty$, one has

$$\xi \preccurlyeq \zeta \quad \iff \quad \xi_t \leqslant \zeta_t \quad \forall\, t \in [0,T], \quad \mathbb{P}\text{-a.s.}$$

Moreover, we can endow the space $\mathcal{A}_\infty$ with a lattice structure, defining the processes $\xi \wedge \zeta$ and $\xi \vee \zeta$ as

$$(\xi \wedge \zeta)_t := \xi_t \wedge \zeta_t \quad \text{and} \quad (\xi \vee \zeta)_t := \xi_t \vee \zeta_t \quad \forall t \in [0,T], \quad \mathbb{P}\text{-a.s.}$$

In the same way, on the set of *extended profile strategies* $\mathcal{A}_\infty^N := \bigotimes_{i=1}^N \mathcal{A}_\infty$, define, for $\boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathcal{A}_\infty^N$, an order relation $\preccurlyeq^N$ by

$$\boldsymbol{\xi} \preccurlyeq^N \boldsymbol{\zeta} \quad \iff \quad \xi^i \preccurlyeq \zeta^i \quad \forall\, i \in \{1, ..., N\},$$

together with the lattice structure

$$\boldsymbol{\xi} \wedge \boldsymbol{\zeta} := (\xi^1 \wedge \zeta^1, ..., \xi^N \wedge \zeta^N) \quad \text{and} \quad \boldsymbol{\xi} \vee \boldsymbol{\zeta} := (\xi^1 \vee \zeta^1, ..., \xi^N \vee \zeta^N).$$

We now provide an existence result for the equilibria of the submodular monotone-follower game.

Theorem 3.1.4. *Let Assumption 3.1.2 hold and assume that the following uniform coercivity condition is satisfied: there exist two constants $K, \kappa > 0$ such that, for each $i = 1, ..., N$,*

$$J^i(\xi^i, \xi^{-i}) \geqslant \kappa\, \mathbb{E}[|\xi_T^i|] \quad \textit{for all} \quad \boldsymbol{\xi} \in \mathcal{A}^N \quad \textit{with} \quad \mathbb{E}[|\xi_T^i|] \geqslant K. \tag{3.1.6}$$

Suppose, moreover, that there exists a constant $M > 0$ such that, for each $i = 1, ..., N$,

$$\textit{for all} \quad \boldsymbol{\xi} \in \mathcal{A}^N \quad \textit{there exists} \quad r^i(\boldsymbol{\xi}) \in \mathcal{A} \quad \textit{such that} \quad J^i(r^i(\boldsymbol{\xi}), \xi^{-i}) \leqslant M. \tag{3.1.7}$$

Then the set of Nash equilibria $F \subset \mathcal{A}^N$ is nonempty, and the partially ordered set $(F, \preccurlyeq^N)$ is a complete lattice.

Proof. Our aim is to prove existence of a Nash equilibrium by applying Tarski's fixed point theorem (see Theorem 1 in [155]) to the map $\mathbf{R}$ (cf. (3.1.3)). For this, the assumption on the submodularity of h^i and g^i will play a crucial role.

First of all, recalling κ, K and M from (3.1.6) and (3.1.7), define the constant $w := \frac{2M}{\kappa} \vee K$, and introduce the set of restricted admissible strategies

$$\mathcal{A}(w) := \{ \xi \in \mathcal{A} \,|\, \mathbb{E}[\xi_T^l] \leqslant w,\, \forall\, l = 1, ..., d \}, \tag{3.1.8}$$

and the set of restricted profile strategies as $\mathcal{A}(w)^N := \bigotimes_{i=1}^N \mathcal{A}(w)$. In the following steps we will identify the proper framework allowing us to apply Tarski's fixed point theorem.

(Step 1) The best-reply maps $R^i : \mathcal{A}^N \to \mathcal{A}(w)$ are well defined.

Fix i and take $\boldsymbol{\xi} \in \mathcal{A}^N$. We have to prove that there exists a unique $\nu \in \mathcal{A}$ such that

$$J^i(\nu, \xi^{-i}) = \min_{\zeta \in \mathcal{A}} J^i(\zeta, \xi^{-i}),$$

and, moreover, that $\nu \in \mathcal{A}(w)$. Clearly, by (3.1.2), we have $\nu = (R^i(\boldsymbol{\xi})_t)_{t\in[0,T]}$.

Let $(\zeta^j)_{j\in\mathbb{N}} \subset \mathcal{A}$ be a minimizing sequence for the functional $J^i(\cdot, \xi^{-i})$. Thanks to the coercivity conditions (3.1.6) on the costs, we deduce that

$$\sup_{j\in\mathbb{N}} \mathbb{E}[|\zeta_T^j|] < \infty.$$

We can then use (a minimal adjustment of) Lemma 3.5 in [104], to find a càdlàg nondecreasing nonnegative $\mathbb{F}$-adapted process ν, and a subsequence of $\{\zeta^j\}_{j\in\mathbb{N}}$ (not relabeled) such that, $\mathbb{P}$-a.s.,

$$\lim_m \int_{[0,T]} \varphi_t \, d\nu_t^m = \int_{[0,T]} \varphi_t \, d\nu_t \quad \forall \varphi \in \mathcal{C}_b([0,T]; \mathbb{R}^d) \quad \text{and} \quad \lim_m \nu_T^m = \nu_T, \tag{3.1.9}$$

where we set, $\mathbb{P}$-a.s.

$$\nu_t^m := \frac{1}{m} \sum_{j=1}^m \zeta_t^j, \qquad \forall t \in [0,T]. \tag{3.1.10}$$

Moreover, from the limit in (3.1.9) we have that there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that, for each $\omega \in \Omega \setminus \mathcal{N}$ there exists a subset $\mathcal{I}(\omega) \subset [0,T)$ of null Lebesgue measure, such that

$$\lim_m \nu_t^m(\omega) = \nu_t(\omega) \quad \text{for each} \quad \omega \in \Omega \setminus \mathcal{N} \quad \text{and} \quad t \in [0,T] \setminus \mathcal{I}(\omega).$$

The latter convergence allows us to invoke Fatou's lemma which, together with the limit in (3.1.9) and thanks to the lower semi-continuity of the costs, allows us to conclude that

$$J^i(\nu, \xi^{-i}) \leqslant \liminf_m J^i(\nu^m, \xi^{-i}) \leqslant \liminf_m \frac{1}{m} \sum_{j=1}^m J^i(\zeta^j, \xi^{-i}) = \min_{\zeta\in\mathcal{A}} J^i(\zeta, \xi^{-i}),$$

where we have used the convexity of h^i and g^i and the minimizing property of ζ^j. Hence ν is a minimizer for $J^i(\cdot, \xi^{-i})$; in fact, ν is the unique minimizer of $J^i(\cdot, \xi^{-i})$ by strict convexity of the costs.

It remains to prove that $\nu \in \mathcal{A}(w)$, and to accomplish that we argue by contradiction. If there exists $l \in \{1, ..., d\}$ such that $\mathbb{E}[\nu_T^l] \geqslant w = \frac{2M}{\kappa} \vee K$, then we have $\mathbb{E}[|\nu_T|] \geqslant \frac{2M}{\kappa} \vee K$ and hence, by the coercivity condition (3.1.6) together with (3.1.7), we deduce that

$$J^i(\nu, \xi^{-i}) \geqslant \kappa \, \mathbb{E}[|\nu_T|] \geqslant 2M > J^i(r^i(\boldsymbol{\xi}), \xi^{-i}),$$

which contradicts the optimality of ν.

(Step 2) The best-reply maps R^i are increasing, i.e., if $\boldsymbol{\xi}, \bar{\boldsymbol{\xi}} \in \mathcal{A}^N$ are such that $\boldsymbol{\xi} \preccurlyeq^N \bar{\boldsymbol{\xi}}$, then $R^i(\boldsymbol{\xi}) \preccurlyeq R^i(\bar{\boldsymbol{\xi}})$.

First of all, observe that, by an integration by parts (see, e.g., Corollary 2 at p. 68 in [145]), the cost functional rewrites as

$$J^i(\xi^i, \xi^{-i}) = \mathbb{E}\bigg[\int_0^T h^i(L_t, \boldsymbol{\xi}_t)\,dt + g^i\,(L_T, \boldsymbol{\xi}_T) - \int_0^T \xi^i_{t-}\,df^i_t + f^i_T \xi^i_T\bigg], \tag{3.1.11}$$

where ξ^i_{t-} denotes the left-limit of ξ^i_t. Thanks to the optimality of $R^i(\boldsymbol{\xi})$ we have the inequality

$$J^i(R^i(\bar{\boldsymbol{\xi}}) \wedge R^i(\boldsymbol{\xi}), \xi^{-i}) - J^i(R^i(\boldsymbol{\xi}), \boldsymbol{\xi}^{-i}) \geqslant 0, \tag{3.1.12}$$

which by (3.1.11) and setting $R^i := R^i(\boldsymbol{\xi})$ and $\bar{R}^i := R^i(\bar{\boldsymbol{\xi}})$, can be rewritten as

$$\begin{aligned}\mathbb{E}\left[\int_0^T \Big(h^i(L_t, R^i_t \wedge \bar{R}^i_t, \xi^{-i}_t) - h^i(L_t, R^i_t, \xi^{-i}_t)\Big)\,dt\right] - \mathbb{E}\left[\int_0^T (R^i_{t-} \wedge \bar{R}^i_{t-} - R^i_{t-})\,df^i_t\right]\\ +\mathbb{E}\left[g^i(L_T, R^i_T \wedge \bar{R}^i_T, \xi^{-i}_T) - g^i(L_T, R^i_T, \xi^{-i}_T)\right] + \mathbb{E}\left[f^i_T(R^i_T \wedge \bar{R}^i_T - R^i_T)\right] \geqslant 0,\end{aligned}$$

By the submodularity Condition 3 in Assumption 3.1.2, we have

$$\begin{aligned}\mathbb{E}\left[\int_0^T \Big(h^i(L_t, R^i_t \wedge \bar{R}^i_t, \xi^{-i}_t) - h^i(L_t, R^i_t, \xi^{-i}_t)\Big)\,dt\right]\\ \leqslant \mathbb{E}\left[\int_0^T \Big(h^i(L_t, \bar{R}^i_t, \xi^{-i}_t) - h^i(L_t, R^i_t \vee \bar{R}^i_t, \xi^{-i}_t)\Big)\,dt\right],\end{aligned} \tag{3.1.13}$$

and

$$\begin{aligned}\mathbb{E}\left[g^i(L_T, R^i_T \wedge \bar{R}^i_T, \xi^{-i}_T) - g^i(L_T, R^i_T, \xi^{-i}_T)\right]\\ \leqslant \mathbb{E}\left[g^i(L_T, \bar{R}^i_T, \xi^{-i}_T) - g^i(L_T, R^i_T \vee \bar{R}^i_T, \xi^{-i}_T)\right].\end{aligned} \tag{3.1.14}$$

Moreover, one can easily verify that

$$\mathbb{E}\left[\int_0^T (R^i_{t-} \wedge \bar{R}^i_{t-} - R^i_{t-})\,df^i_t\right] = \mathbb{E}\left[\int_0^T (\bar{R}^i_{t-} - R^i_{t-} \vee \bar{R}^i_{t-})\,df^i_t\right] \tag{3.1.15}$$

and

$$\mathbb{E}\left[f^i_T(R^i_T \wedge \bar{R}^i_T - R^i_T)\right] = \mathbb{E}\left[f^i_T(\bar{R}^i_T - R^i_T \vee \bar{R}^i_T)\right]. \tag{3.1.16}$$

Using (3.1.13)-(3.1.16) we obtain

$$\begin{aligned}J^i(R^i(\bar{\boldsymbol{\xi}}) \wedge R^i(\boldsymbol{\xi}), \xi^{-i}) - J^i(R^i(\boldsymbol{\xi}), \xi^{-i})\\ \leqslant J^i(R^i(\bar{\boldsymbol{\xi}}), \xi^{-i}) - J^i(R^i(\boldsymbol{\xi}) \vee R^i(\bar{\boldsymbol{\xi}}), \xi^{-i}),\end{aligned}$$

so that, by (3.1.12), we deduce that

$$J^i(R^i(\bar{\boldsymbol{\xi}}), \xi^{-i}) - J^i(R^i(\boldsymbol{\xi}) \vee R^i(\bar{\boldsymbol{\xi}}), \xi^{-i}) \geqslant 0. \tag{3.1.17}$$

Now, by Condition 2 in Assumption 3.1.2, we have

$$J^i(R^i(\bar{\boldsymbol{\xi}}), \bar{\xi}^{-i}) - J^i(R^i(\boldsymbol{\xi}) \vee R^i(\bar{\boldsymbol{\xi}}), \bar{\xi}^{-i}) \geqslant J^i(R^i(\bar{\boldsymbol{\xi}}), \xi^{-i}) - J^i(R^i(\boldsymbol{\xi}) \vee R^i(\bar{\boldsymbol{\xi}}), \xi^{-i}),$$

and finally, by (3.1.17), we conclude that

$$J^i(R^i(\bar{\boldsymbol{\xi}}), \bar{\xi}^{-i}) - J^i(R^i(\boldsymbol{\xi}) \vee R^i(\bar{\boldsymbol{\xi}}), \bar{\xi}^{-i}) \geqslant 0.$$

Hence $R^i(\boldsymbol{\xi}) \vee R^i(\bar{\boldsymbol{\xi}})$ minimizes $J^i(\cdot, \bar{\xi}^{-i})$ as well as $R^i(\bar{\boldsymbol{\xi}})$ and, by uniqueness, it must be $R^i(\boldsymbol{\xi}) \vee R^i(\bar{\boldsymbol{\xi}}) = R^i(\bar{\boldsymbol{\xi}})$. That is $R^i(\bar{\boldsymbol{\xi}}) \preccurlyeq R^i(\boldsymbol{\xi})$, which shows the claimed monotonicity.

(Step 3) The lattices $(\mathcal{A}_\infty^N, \preccurlyeq^N)$ and $(\mathcal{A}_\infty, \preccurlyeq)$ are complete.

We prove the claim only for the lattice $(\mathcal{A}_\infty^N, \preccurlyeq^N)$, since an analogous rationale applies to show that the lattice $(\mathcal{A}_\infty, \preccurlyeq)$ is complete.

To prove that the lattice $(\mathcal{A}_\infty^N, \preccurlyeq^N)$ is complete we have to show that each subset of $\mathcal{A}_\infty^N$ has a least upper bound and a greatest lower bound. We now prove only the existence of a least upper bound, since the existence of a greatest lower bound follows by similar arguments.

Consider a subset $(\boldsymbol{\xi}^j)_{j \in \mathcal{I}}$ of $\mathcal{A}_\infty^N$, where $\mathcal{I}$ is a set of indexes. Define $Q := ([0,T] \cap \mathbb{Q}) \cup \{T\}$. For each $q \in Q$ we set

$$\tilde{\boldsymbol{\nu}}_q := \operatorname*{ess\,sup}_{j \in \mathcal{I}} \boldsymbol{\xi}_q^j, \tag{3.1.18}$$

and we recall that there exists a countable subset $\mathcal{I}_q$ of $\mathcal{I}$ such that

$$\tilde{\boldsymbol{\nu}}_q = \sup_{j \in \mathcal{I}_q} \boldsymbol{\xi}_q^j. \tag{3.1.19}$$

Define next the right-continuous process $\boldsymbol{\nu} : \Omega \times [0,T] \to [0,\infty]^{Nd}$ by

$$\boldsymbol{\nu}_T := \tilde{\boldsymbol{\nu}}_T, \quad \text{and} \quad \boldsymbol{\nu}_t := \inf\{\, \tilde{\boldsymbol{\nu}}_q \,|\, q > t, q \in Q\}, \quad \text{for} \quad t < T. \tag{3.1.20}$$

Observe that $\boldsymbol{\nu}$ is $\mathbb{F}$-adapted by right-continuity of the filtration. Hence, $\boldsymbol{\nu}$ lies in $\mathcal{A}_\infty^N$, and clearly $\boldsymbol{\xi}^j \preccurlyeq^N \boldsymbol{\nu}$ for each $j \in \mathcal{I}$.

Consider next an element $\boldsymbol{\zeta}$ of $\mathcal{A}_\infty^N$ such that $\boldsymbol{\xi}^j \preccurlyeq^N \boldsymbol{\zeta}$ for each $j \in \mathcal{I}$. For $q \in Q$ and $j \in \mathcal{I}_q$ there exists a $\mathbb{P}$-null set $\mathcal{M}_q^j$ such that $\boldsymbol{\xi}_q^j(\omega) \leqslant \boldsymbol{\zeta}_q(\omega)$ for all $\omega \in \Omega \setminus \mathcal{M}_q^j$. Defining then $\mathcal{M}_q := \bigcup_{j \in \mathcal{I}_q} \mathcal{M}_q^j$, we have $\boldsymbol{\xi}_q^j(\omega) \leqslant \boldsymbol{\zeta}_q(\omega)$ for all $\omega \in \Omega \setminus \mathcal{M}_q$ and $j \in \mathcal{I}_q$, which, by (3.1.19), implies that $\tilde{\boldsymbol{\nu}}_q(\omega) \leqslant \boldsymbol{\zeta}_q(\omega)$ for all $\omega \in \Omega \setminus \mathcal{M}_q$. Finally, introducing the $\mathbb{P}$-null set $\mathcal{M} := \bigcup_{q \in Q} \mathcal{M}_q$, we have $\tilde{\boldsymbol{\nu}}_q(\omega) \leqslant \boldsymbol{\zeta}_q(\omega)$ for all $\omega \in \Omega \setminus \mathcal{M}$ and $q \in Q$, and, by right-continuity, we deduce that $\boldsymbol{\nu} \preccurlyeq^N \boldsymbol{\zeta}$. Thus, $\boldsymbol{\nu}$ is the least upper bound of $(\boldsymbol{\xi}^j)_{j \in \mathcal{I}}$.

(Step 4) There exist increasing maps $\bar{R}^i : \mathcal{A}_\infty^N \to \mathcal{A}(w)$ such that $\bar{R}^i(\boldsymbol{\xi}) = R^i(\boldsymbol{\xi})$ for each $\boldsymbol{\xi} \in \mathcal{A}^N$.

For each $\boldsymbol{\xi} \in \mathcal{A}_\infty^N$, define $\bar{R}^i(\boldsymbol{\xi})$ as the least upper bound of the set $\{R^i(\boldsymbol{\eta}) \,|\, \boldsymbol{\eta} \in \mathcal{A}^N, \boldsymbol{\eta} \preccurlyeq^N \boldsymbol{\xi}\}$ in the complete lattice $(\mathcal{A}_\infty, \preccurlyeq)$. If $\boldsymbol{\xi} \in \mathcal{A}^N$, then $R^i(\boldsymbol{\xi}) \in \{R^i(\boldsymbol{\eta}) \,|\, \boldsymbol{\eta} \in \mathcal{A}^N, \boldsymbol{\eta} \preccurlyeq^N \boldsymbol{\xi}\}$ and, since R^i is increasing, $R^i(\boldsymbol{\eta}) \preccurlyeq R^i(\boldsymbol{\xi})$ for each $\boldsymbol{\eta} \in \mathcal{A}^N$ such that $\boldsymbol{\eta} \preccurlyeq^N \boldsymbol{\xi}$, which implies that $\bar{R}^i(\boldsymbol{\xi}) = R^i(\boldsymbol{\xi})$. Moreover, if $\boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathcal{A}_\infty^N$ are such that $\boldsymbol{\xi} \preccurlyeq^N \boldsymbol{\zeta}$, then we have $\{\boldsymbol{\eta} \in \mathcal{A}^N \,|\, \boldsymbol{\eta} \preccurlyeq^N \boldsymbol{\xi}\} \subset \{\boldsymbol{\eta} \in \mathcal{A}^N \,|\, \boldsymbol{\eta} \preccurlyeq^N \boldsymbol{\zeta}\}$ and hence that $\bar{R}^i(\boldsymbol{\xi}) \preccurlyeq \bar{R}^i(\boldsymbol{\zeta})$. We only remain to prove that $\bar{R}^i(\boldsymbol{\xi}) \in \mathcal{A}(w)$. In order to accomplish that, we observe that, for each $\boldsymbol{\eta}, \boldsymbol{\eta}' \in \mathcal{A}^N$ such that $\boldsymbol{\eta}, \boldsymbol{\eta}' \preccurlyeq^N \boldsymbol{\xi}$ we have that $\boldsymbol{\eta} \vee \boldsymbol{\eta}' \preccurlyeq^N \boldsymbol{\xi}$ and, since R^i is increasing, $R^i(\boldsymbol{\eta}) \vee R^i(\boldsymbol{\eta}') \preccurlyeq R^i(\boldsymbol{\eta} \vee \boldsymbol{\eta}')$. This implies that there exists a sequence $(\boldsymbol{\eta}^j)_{j\in\mathbb{N}} \subset \{\boldsymbol{\eta} \in \mathcal{A}^N \,|\, \boldsymbol{\eta} \preccurlyeq^N \boldsymbol{\xi}\}$ such that the sequence $(R^i(\boldsymbol{\eta}^j)_T)_{j\in\mathbb{N}}$ is increasing and, moreover,

$$\bar{R}^i(\boldsymbol{\xi})_T = \lim_j R^i(\boldsymbol{\eta}^j)_T, \quad \mathbb{P}\text{-a.s.}, \quad \text{and} \quad \mathbb{E}[\bar{R}^i(\boldsymbol{\xi})_T] = \lim_j \mathbb{E}[R^i(\boldsymbol{\eta}^j)_T], \tag{3.1.21}$$

where the latter equality follows from the monotone convergence theorem. Finally, by *Step 1* we have that $R^i(\boldsymbol{\eta}^j) \in \mathcal{A}(w)$ for each $j \in \mathbb{N}$, which by (3.1.21) implies that $\mathbb{E}[\bar{R}^i(\boldsymbol{\xi})_T] \leqslant w$ and hence that $\bar{R}^i(\boldsymbol{\xi})_T < \infty$ $\mathbb{P}$-a.s. By the completeness of the filtration – unless to consider an indistinguishable version of $\bar{R}^i(\boldsymbol{\xi})$ – with no loss of generality we can assume that $\bar{R}^i(\boldsymbol{\xi})(\omega)$ is finite for any $\omega \in \Omega$; that is, $\bar{R}^i(\boldsymbol{\xi}) \in \mathcal{A}(w)$. This completes the proof of Step 4.

(Step 5) Existence of Nash equilibria.

By the previous steps the lattice $(\mathcal{A}_\infty^N, \preccurlyeq^N)$ is complete and the map $\bar{\mathbf{R}} := (\bar{R}^1, ..., \bar{R}^N)$ from the set of extended profile strategies $\mathcal{A}_\infty^N$ into itself is monotone increasing. Then, by Tarski's fixed point theorem (see [155], Theorem 1), the set of fixed point of the map $\bar{\mathbf{R}}$ is a nonempty complete lattice. Now, by *Step 4*, the image of the map $\bar{\mathbf{R}}$ is contained in $\mathcal{A}(w)^N$, and the map $\bar{\mathbf{R}}$ coincides with the map $\mathbf{R}$ on $\mathcal{A}(w)^N$. This implies that the set of fixed points of $\mathbf{R}$ is equal to the set of fixed point of $\bar{\mathbf{R}}$, and since such a set coincides with the set of Nash equilibria, the proof is completed. □

3.1.3 Some remarks

In this subsection we collect some remarks concerning assumptions and extensions of the previous theorem.

Remark 3.1.5 (Comments on the Conditions of Theorem 3.1.4)**.** *A few comments are worth being done.*

1. *Condition (3.1.6) is satisfied if, for example, there exists a constant $c > 0$ such that*
$$\mathbb{P}\left[f_t^i \geqslant c, \,\forall i = 1, ..., N, \,\forall\, t \in [0, T]\right] = 1,$$
or if g^i are such that $g^i(l, a^i, a^{-i}) \geqslant \kappa\,|a^i|$.

2. *The role of Condition (3.1.7) is to force Nash equilibria, whenever they exist, to live in the bounded subset $\mathcal{A}^N(w)$ of $\mathcal{A}^N$. If there exist measurable functions H, G :*

$\mathbb{R}^k \to [0,\infty)$ such that, for each $i = 1, ..., N$ and for each $(l, a^{-i}) \in \mathbb{R}^k \times \mathbb{R}^{(N-1)d}$, we have $h^i(l, 0, a^{-i}) \leqslant H(l)$ and $g^i(l, 0, a^{-i}) \leqslant G(l)$, with

$$\mathbb{E}\left[\int_0^T H(L_s)\, ds + G(L_T)\right] < \infty,$$

then Condition (3.1.7) is satisfies with $r^i(\boldsymbol{\xi}) = 0$.

Remark 3.1.6. *Consider the case $N = 2$, $d = 1$. The costs relative to Player 1 are $f^1 = h^1 = 0$, $g^1(l, a^1, a^2) = e^{-a^1}(2 - e^{-a^2})$, while the costs of Player 2 can be generic functions satisfying our requirements. Then, all the assumptions of Theorem 3.1.4 are satisfied, with the exception of the coercivity condition (3.1.6), which is not satisfied by J^1. If now $(\hat{\xi}^1, \hat{\xi}^2)$ were a Nash equilibrium, then for the first player we could write*

$$0 < \mathbb{E}[e^{-\hat{\xi}^1_T}(2 - e^{-\hat{\xi}^2_T})] \leqslant \inf_{n\in\mathbb{N}} \mathbb{E}[e^{-n}(2 - e^{-\hat{\xi}^2_T})] = 0,$$

which is clearly a contradiction. This example shows that, at least in the Nash equilibria, the coercivity condition (3.1.6) is necessarily satisfied.

Remark 3.1.7 (Finite-Fuel Constraint)**.** *Many models in the literature on monotone-follower problems enjoy a so-called* finite fuel constraint *(see e.g. [108] for a seminal chapter, and the more recent [12] and [54]). This can be realized by requiring that the admissible control strategies stay bounded either $\mathbb{P}$-a.s. In our game, if we suppose that, for each $i = 1, ..., N$, the strategies of player i belongs to the set $\mathcal{A}(w^i) := \{\xi \in \mathcal{A} \,|\, \xi^\ell_T \leqslant w^i, \forall\, \ell = 1, ..., d\,\}$, a proof similar to that of Theorem 3.1.4 still shows existence of Nash equilibria without need of Conditions 3.1.6 and 3.1.7.*

Remark 3.1.8 (An Extension of Theorem 3.1.4 with Regular-Singular Controls)**.** *We here discuss how to extend Theorem 3.1.4 to a game in which players can choose both a regular and a singular control.*

Fix a square integrable random variable Θ and define the space of regular controls *$\mathcal{U}$ as the set of $\mathbb{R}^d$-valued $\mathbb{F}$-progressively measurable processes u such that $|u_t| \leqslant \Theta$ $\mathbb{P} \otimes dt - a.e.$ We consider the game of regular-singular controls, in which each player $i \in \{1, ..., N\}$ is allowed to choose an admissible strategy $Z^i = (u^i, \xi^i) \in \mathcal{U} \times \mathcal{A}$ in order to minimize the cost functional*

$$J^i(Z^i, Z^{-i}) := \mathbb{E}\left[\int_0^T h^i(L_t, Z^1_t, ..., Z^N_t)\, dt + g^i(L_T, \boldsymbol{\xi}_T) + \int_{[0,T]} f^i_t\, d\xi^i_t\right].$$

Define on $\mathcal{U}$ the order relation $\preccurlyeq$ by setting, for $u, v \in \mathcal{U}$, $u \preccurlyeq v$ if and only if $u_t \leqslant v_t$ $\mathbb{P}\otimes dt$-a.e. Next, consider on the lattice $(\mathcal{U}, \preccurlyeq)$ the topology $\mathcal{I}$ of intervals (see, e.g., p. 250 in [26]); that is, the topology for which the topology of closed sets is generated by the family of sets $\mathcal{I}_z := \{u \in \mathcal{U} \,:\, u \preccurlyeq z\}$ and $\mathcal{I}^z := \{u \in \mathcal{U} \,:\, z \preccurlyeq u\}$ for $z \in \mathcal{U}$. Since the topology $\mathcal{I}$ is included in the weak topology of $\mathbb{L}^2(\Omega \times [0,T]; \mathbb{R}^d)$ and $\mathcal{U}$ is bounded, then $\mathcal{U}$ is compact in the topology $\mathcal{I}$. Therefore, by a characterization of complete lattices (see Theorem 20 at p. 250 in [26]), it follows that the lattice $(\mathcal{U}, \preccurlyeq)$ is complete. Then, existence of Nash equilibria follows proceding as in the proof of Theorem 3.1.4.

Remark 3.1.9 (Infinite Time-Horizon Case: $T = \infty$)**.** *Theorem 3.1.4 can be proved also in the case $T = \infty$, which typically arises in applications. Indeed, we can consider the problem in which each player chooses a strategy in the set*

$$\mathcal{A}[0,\infty) = \left\{ \xi : [0,\infty) \times \Omega \to \mathbb{R}^d \;\middle|\; \begin{array}{c} \xi \text{ is an } \mathbb{F}\text{-adapted càdlàg process, with} \\ \text{nondecreasing and nonnegative components} \end{array} \right\},$$

in order to minimize the cost functional

$$J^i_\infty(\xi^i, \xi^{-i}) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} h^i(L_t, \boldsymbol{\xi}_t)\, dt + \int_{[0,\infty)} e^{-\rho t} f^i_t \, d\xi^i_t \right],$$

for a suitable intertemporal discount factor $\rho > 0$. The arguments developed in the proof of Theorem 3.1.4 carry on upon replacing ξ_T with $\xi_\infty := \sup_{t \in [0,\infty)} \xi_t$. Indeed, assuming the Condition (3.1.6) in terms of ξ_∞ (or simply working with finite-fuel constraints, cf. Remark 3.1.7), a suitable application of Komlós' theorem still allows to prove the well-posedness of the best-reply maps (cf. Step 1 of the proof of Theorem 3.1.4), while the completeness of the lattice $\mathcal{A}[0,\infty)$ and the submodularity of the costs still enable the application of Tarski's fixed point theorem.

Remark 3.1.10 (On the Uniqueness of Nash Equilibrium)**.** *In general, very little can be said about uniqueness of Nash equilibria. Although conditions ensuring uniqueness of equilibria for submodular games are available in the literature (see, e.g., [134] and references therein), it does not seem straightforward to us how to employ these techniques in our continuous-time stochastic setting. We leave this interesting question for future research.*

3.2 The n-Lipschitz game

In the notation of Section 3.1, for each $n \in \mathbb{N}$, define the space of *n-Lipschitz strategies*

$$\mathcal{U}_n = \{ \xi \in \mathcal{A} \,|\, \xi \text{ is Lipschitz with Lipschitz constant smaller than } n,\, \xi_0 = 0 \},$$

and the space of *n-Lipschitz profile strategies* as $\mathcal{U}_n^N := \bigotimes_{i=1}^N \mathcal{U}_n$. The set $\mathcal{U}_n$ (resp. $\mathcal{U}_n^N$) inherits from $\mathcal{A}$ (resp. $\mathcal{A}^N$) the order relation $\preccurlyeq$ (resp. $\preccurlyeq^N$) together with the associated lattice structure.

For each $n \in \mathbb{N}$, the set of n-Lipschitz profile strategies $\mathcal{U}_n^N$, together with the cost functionals J^i, define a game to which we will refer to as the *n-Lipschitz game*. We say that an n-Lipschitz profile strategy $\boldsymbol{\xi} \in \mathcal{U}_n^N$ is a Nash equilibrium of the n-Lipschitz game if, for each $i = 1, ..., N$, we have $J^i(\boldsymbol{\xi}) < \infty$ and

$$J^i(\xi^i, \xi^{-i}) \leqslant J^i(\zeta^i, \xi^{-i}), \quad \text{for every} \quad \zeta^i \in \mathcal{U}_n.$$

Theorem 3.2.1 (Existence of Nash Equilibria for the Submodular n-Lip. Game)**.** *Let Assumption 3.1.2 hold. Then, for each $n \in \mathbb{N}$, the set of Nash equilibria of the n-Lipschitz game $F \subset \mathcal{U}_n^N$ is nonempty, and the partially ordered set $(F, \preccurlyeq^N)$ is a complete lattice.*

Proof. As in the proof of Theorem 3.1.4, we identify the proper framework in order to apply Tarski's fixed point theorem. The completeness of the lattice $(\mathcal{U}_n^N, \preccurlyeq^N)$ follows by observing that the least upper bound (as well as the greatest lower bound) of any subset is still Lipschitz with Lipschitz constant bounded by n. Moreover, as in the proof of Proposition 26 at p. 109 in [123], we deduce that, for each $i = 1, ..., N$ and each $\boldsymbol{\xi} \in \mathcal{U}_n^N$, there exists a unique (by strict convexity of the costs) $R^i(\boldsymbol{\xi}) \in \mathcal{U}_n$ such that

$$J^i(R^i(\boldsymbol{\xi}), \xi^{-i}) = \min_{\zeta^i \in \mathcal{U}_n} J^i(\zeta^i, \xi^{-i}).$$

By employing arguments as those in the *Step 2* of the proof of Theorem 3.1.4 we conclude that the map $\mathbf{R} = (R^1, ..., R^N) : \mathcal{U}_n^N \to \mathcal{U}_n^N$ is monotone increasing in the complete lattice $(\mathcal{U}_n^N, \preccurlyeq^N)$. Then, the thesis of the theorem follows from Tarski's fixed point theorem. □

3.3 Existence and approximation of weak Nash equilibria in the submodular monotone-follower game

In this section we will investigate connections between the monotone-follower game and the n-Lipschitz games.

3.3.1 Weak formulation of the monotone-follower game.

For $T \in (0, \infty)$ and an arbitrary $m \in \mathbb{N}$, we introduce the following measurable spaces:

- $\mathcal{C}_+^m$ denotes the set of $\mathbb{R}^m$-valued continuous function on $[0, T]$ with nonnegative components, endowed with the Borel σ-algebra generated by the uniform convergence norm;
- $\mathcal{D}^m$ denotes the Skorokhod space of $\mathbb{R}^m$-valued càdlàg functions, defined on $[0, T]$, endowed with the Borel σ-algebra generated by the Skorokhod topology;
- $\mathcal{D}_\uparrow^m$ denotes the Skorokhod space of $\mathbb{R}^m$-valued nondecreasing, nonnegative càdlàg functions, defined on $[0, T]$, endowed with the Borel σ-algebra generated by the Skorokhod topology.

Also, let $\mathcal{P}(\mathcal{C}_+^m)$, $\mathcal{P}(\mathcal{D}^m)$ and $\mathcal{P}(\mathcal{D}_\uparrow^m)$ denote the set of probability measures on the Borel σ-algebras of $\mathcal{C}_+^m$, $\mathcal{D}^m$ and $\mathcal{D}_\uparrow^m$, respectively. Finally, denote by $\mathcal{P}(\mathcal{C}_+^m \times \mathcal{D}^m \times \mathcal{D}_\uparrow^m)$ the set of probability measures on the product σ-algebra.

Moreover, denote by $(\pi_f, \pi_L) : \mathcal{C}_+^{Nd} \times \mathcal{D}^k \times [0, T] \to \mathbb{R}^{Nd+k}$ the canonical projection, i.e., set $(\pi_f, \pi_L)_t(f, L) = (f_t, L_t)$ for each $(f, L) \in \mathcal{C}_+^{Nd} \times \mathcal{D}^k$ and $t \in [0, T]$. Also, for a probability measure $\mathbb{P} \in \mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k)$, denote by $\bar{\mathbb{F}}_+^{\pi_f, \pi_L}$ the right-continuous extension of the filtration on $\mathcal{C}_+^{Nd} \times \mathcal{D}^k$ generated by the canonical projections π_f and π_L, augmented by the $\mathbb{P}$-null sets.

We now give a weak formulation of the monotone-follower game. Assume given a distribution $\mathbb{P}_0 \in \mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k)$ such that the projection process $\pi_f : \mathcal{C}_+^{Nd} \times \mathcal{D}^k \times [0, T] \to \mathbb{R}^{Nd}$ is a semimartingale with respect to the filtration $\bar{\mathbb{F}}_+^{\pi_f, \pi_L}$.

Definition 3. *We call a basis a 5-tuple $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$ such that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, L is an $\mathbb{R}^k$-valued càdlàg process, $f = (f^1, ..., f^N)$ is an $\mathbb{R}^{Nd}$-valued continuous, nonnegative semimartingale with respect to the filtration $\bar{\mathbb{F}}_+^{f,L}$, and $\mathbb{P} \circ (f, L)^{-1} = \mathbb{P}_0$.*

For each basis β, we then give the relative notion of admissible strategy.

Definition 4. *Given a basis $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$, an* admissible strategy associated to *β is an $\mathbb{R}^d$-valued càdlàg, nondecreasing, nonnegative process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that, for each $t \in [0, T]$, the σ-algebras $\mathcal{F}_t^A$ and $\mathcal{F}_T^{(f,L)}$ are $\mathbb{P}$-independent, conditionally on $\mathcal{F}_{t+}^{f,L}$.*

We denote by $\mathcal{A}_\beta$ the set of admissible strategies associated to the basis β. Moreover, we define the space of admissible profile strategies associated to the basis β as $\mathcal{A}_\beta^N := \bigotimes_{i=1}^N \mathcal{A}_\beta$.

Given a basis $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$, for each $i \in \{1, ..., N\}$ and each admissible strategy $\xi^i \in \mathcal{A}_\beta$ we define the cost functionals

$$\begin{aligned} J_\beta^i(\xi^i, \xi^{-i}) &:= \mathbb{E}^{\mathbb{P}}[C^i(f, L, \boldsymbol{\xi})] \\ &= \mathbb{E}^{\mathbb{P}}\left[\int_0^T h^i(L_t, \boldsymbol{\xi}_t)\, dt + g^i(L_T, \boldsymbol{\xi}_T) + \int_{[0,T]} f_t^i \, d\xi_t^i\right], \end{aligned}$$

where $\xi^{-i} := (\xi^j)_{j \neq i}$, $\boldsymbol{\xi} := (\xi^i, \xi^{-i})$ and $\mathbb{E}^{\mathbb{P}}$ denotes the expectation under the probability measure $\mathbb{P}$.

We finally introduce a notion of equilibrium that we will refer to as *weak Nash equilibrium*.

Definition 5 (Weak Nash Equilibrium)**.** *We say that $(\bar{\beta}, \bar{\boldsymbol{\xi}})$ is a weak Nash equilibrium if $\bar{\beta}$ is a basis and $\bar{\boldsymbol{\xi}} \in \mathcal{A}_{\bar{\beta}}^N$ is an admissible profile strategy such that, for every $i = 1, ..., N$,*

$$J_{\bar{\beta}}^i(\bar{\xi}^i, \bar{\xi}^{-i}) \leqslant J_{\bar{\beta}}^i(\zeta^i, \bar{\xi}^{-i}), \quad \text{for every} \quad \zeta^i \in \mathcal{A}_{\bar{\beta}}.$$

3.3.2 Assumptions and a preliminary lemma

In this subsection we specify the main assumptions of this section, we introduce some notation, and we provide a preliminary lemma.

Assumption 3.3.1. *Let Assumption 3.1.2 hold and, for each $i = 1, ..., N$, assume that:*

1. *g^i and h^i are continuous and continuously differentiable in the variable $a^i \in \mathbb{R}^d$.*

2. *There exist $\gamma_1, \gamma_2 > 1$ such that the d-dimensional gradients $\nabla_i h^i$ and $\nabla_i g^i$ of the functions h^i and g^i with respect to the (d-dimensional) variable a^i satisfy*

$$|\nabla_i h^i(l, a)| + |\nabla_i g^i(l, a)| \leqslant C(1 + |l|^{\gamma_1} + |a|^{\gamma_2}), \tag{3.3.1}$$

for each $l \in \mathbb{R}^k$ *and* $a = (a^1, ..., a^N) \in \mathbb{R}^{Nd}$.
Moreover, there exist measurable functions $H^i, G^i : \mathbb{R}^k \to \mathbb{R}$ *such that* $h^i(l, 0, a^{-i}) \leqslant H^i(l)$ *and* $g^i(l, 0, a^{-i}) \leqslant G^i(l)$*, with*

$$\mathbb{E}^{\mathbb{P}_0}\left[\int_0^T |H^i((\pi_L)_s)|^q\, ds + |G^i((\pi_L)_T)|^q\right] < \infty \tag{3.3.2}$$

and

$$\mathbb{E}^{\mathbb{P}_0}\left[\sup_{s\in[0,T]} \left(|(\pi_L)_s|^{\alpha\gamma_1 p} + |(\pi_f)_s|^{\alpha p}\right)\right] < \infty, \tag{3.3.3}$$

where $q := \alpha \max\{\gamma_2\, p,\, p/(p-1)\}$ *for some* $p, \alpha > 1$.

3. *There exists a constant* $c > 0$ *such that*

$$\mathbb{P}_0\left[(\pi_f)_t^i \geqslant c,\ \forall\, t \in [0,T],\ \forall\, i = 1, ..., N\right] = 1, \tag{3.3.4}$$

and the total conditional variation (see definition (A.2) in Appendix A) of π_L *over the interval* $[0,T)$ *is finite; that is,* $V_T^{\mathbb{P}_0}(\pi_L) < \infty$ *.*

4. $\mathbb{P}_0$ *is Feller; i.e., for any* $t \in [0,T)$ *we have:*

   (a) *The* σ*-algebras* $\mathcal{F}_{t+}^{\pi_f,\pi_L}$ *and* $\mathcal{F}_t^{\pi_f,\pi_L}$ *on* $C_+^{Nd} \times \mathcal{D}^k$ *coincide* $\mathbb{P}_0$*-a.s.;*

   (b) *For each* $\psi \in C_c^\infty(\mathbb{R}^{Nd+k})$*, there exists* $\psi^* \in C_b(\mathbb{R}^{Nd+k})$ *such that*

$$\mathbb{E}^{\mathbb{P}_0}[\psi((\pi_f, \pi_L)_T)|\mathcal{F}_{t+}^{\pi_f,\pi_L}] = \psi^*((\pi_f,\pi_L)_t), \quad \mathbb{P}_0\text{-}a.s.$$

Moreover, the projection process (π_f, π_L) *is a quasi-martingale under* $\mathbb{P}_0$.

For a basis $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$, a profile strategy $\boldsymbol{\xi} = (\xi^1, ..., \xi^N) \in \mathcal{A}_\beta^N$ and an index $i \in \{1, ..., N\}$, we define the continuous (non adapted) subgradient process $\partial C^i(f, L, \boldsymbol{\xi}) : \Omega \times [0,T] \to \mathbb{R}^d$ by setting

$$\partial C^i(f, L, \boldsymbol{\xi})_t := \int_t^T \nabla_i h^i(L_t, \boldsymbol{\xi}_t)\, dt + \nabla_i g^i(L_T, \boldsymbol{\xi}_T) + f_t^i, \quad \forall\, t \in [0,T], \quad \mathbb{P}\text{-a.s.} \tag{3.3.5}$$

Furthermore, if $\boldsymbol{\xi}$ is such that $J_\beta^i(\boldsymbol{\xi}) < \infty$ for a certain $i \in \{1, ..., N\}$, then, exploiting the convexity of h^i and g^i and integrating by parts, we obtain the following subgradient inequality

$$J_\beta^i(\zeta^i, \xi^{-i}) - J_\beta^i(\xi^i, \xi^{-i}) \geqslant \mathbb{E}^{\mathbb{P}}\left[\int_{[0,T]} \partial C_t^i (d\zeta_t^i - dA_t^i)\right], \quad \text{for each} \quad \zeta^i \in \mathcal{A}_\beta. \tag{3.3.6}$$

Fix a basis $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$ and denote by $\bar{\mathbb{F}}_+^{f,L} = \{\bar{\mathcal{F}}_{t+}^{f,L}\}_{t\in[0,T]}$ the right-continuous extension of the filtration generated by f and L, augmented by the $\mathbb{P}$-null sets. For each $n \in \mathbb{N}$, consider a Nash equilibrium $\boldsymbol{\xi}^n = (\xi^{1,n}, ..., \xi^{N,n})$ of the n-Lipschitz game as in Theorem 3.2.1. The next lemma shows that any Nash equilibria of the n-Lipschtz game satisfy certain *first order conditions*. The proof of this claim follows arguments analogus to those used in the proof of Proposition 27 in [123].

Lemma 3.3.2. *For every* $n \in \mathbb{N}$ *and every* $i = 1, ..., N$*, set* $\partial C^{i,n} := \partial C^i(f, L, \boldsymbol{\xi}^n)$*. Then, under Assumption 3.3.1, defining* $\mathbf{1} := (1, ..., 1) \in \mathbb{R}^d$*, we have*

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \partial C_t^{i,n} d\xi_t^{i,n}\right] = -n\, \mathbb{E}^{\mathbb{P}}\left[\int_0^T (\partial C_t^{i,n})^-\, \mathbf{1}\, dt\right], \ \lim_n \mathbb{E}^{\mathbb{P}}\left[\int_0^T (\partial C_t^{i,n})^-\, dt\right] = 0. \tag{3.3.7}$$

3.3.3 Existence and approximation of weak Nash equilibria

We now state and prove the main result of this section, which can be thought of as a game-theoretic version of Theorem 21 in [123].

For an arbitrary $m \in \mathbb{N}$, consider on the space $\mathcal{C}_+^m$ the topology given by the convergence in the uniform norm. Furthermore, on the space $\mathcal{D}^m$ consider the *pseudopath topology* τ_{pp}^T; that is, the topology on $\mathcal{D}^m$ induced by the convergence in the measure $dt + \delta_T$ on the interval $[0, T]$, where dt denotes the Lebesgue measure, and δ_T denotes the Dirac measure at the terminal time T. The space $\mathcal{D}_\uparrow^m$ is a closed subset of the topological space $(\mathcal{D}^m, \tau_{pp}^T)$, and the Borel σ-algebra induced by the topology τ_{pp}^T, coincides with the σ-algebra induced by the Skorokhod topology (see also the appendix in [123]). Notice that the topological spaces $(\mathcal{D}^m, \tau_{pp}^T)$ and $(\mathcal{D}_\uparrow^m, \tau_{pp}^T)$ are separable, but not Polish (see, e.g., [136]). Finally, on the product space $\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd}$, consider the product topology, and on $\mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd})$ consider the topology of weak convergence of probability measures.

Fix a basis $\beta = (\Omega, \mathcal{F}, \mathbb{P}, f, L)$ and consider, for each $n \in \mathbb{N}$, a Nash equilibrium $\boldsymbol{\xi}^n = (\xi^{1,n}, ..., \xi^{N,n})$ of the n-Lipschitz game as in Theorem 3.2.1, with $\mathbb{F} = \bar{\mathbb{F}}_+^{f,L}$ (the right-continuous extension of the filtration generated by f and L, augmented by the $\mathbb{P}$-null sets). Observe that the processes $\boldsymbol{\xi}^n$ are $\bar{\mathbb{F}}_+^{f,L}$-adapted. Define, for $n \in \mathbb{N}$, the law $\mathbb{P}^n := \mathbb{P} \circ (f, L, \boldsymbol{\xi}^n)^{-1}$ in $\mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd})$; with a slight abuse of terminology, we will refer to the law $\mathbb{P}^n$ as the law of the Nash equilibrium $\boldsymbol{\xi}^n$. We then have the following theorem.

Theorem 3.3.3. *Under Assumption 3.3.1 the following statements hold.*

1. *The sequence $\{\mathbb{P}^n\}_{n \in \mathbb{N}}$ of the laws of the Nash equilibria of the n-Lipschitz games is weakly relatively compact in $\mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd})$.*
2. *Any accumulation point $\bar{\mathbb{P}}$ is the law of a weak Nash equilibrium of the monotone-follower game; that is, there exist a basis $\bar{\beta} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{Q}}, \bar{f}, \bar{L})$ and an admissible profile strategy $\bar{\boldsymbol{\xi}} \in \mathcal{A}_{\bar{\beta}}^N$, such that $(\bar{\beta}, \bar{\boldsymbol{\xi}})$ is a weak Nash equilibrium of the monotone-follower game and $\bar{\mathbb{P}} = \bar{\mathbb{Q}} \circ (\bar{f}, \bar{L}, \bar{\boldsymbol{\xi}})^{-1}$.*

Proof. We prove the two claims of the theorem separately.

Proof of Claim 1. By assumption we have $V_T^{\mathbb{P}}(L) < \infty$. Moreover, by employing arguments similar to those in the proof of Proposition 28 at p. 110 in [123], we find

$$\sup_n \mathbb{E}^{\mathbb{P}} [|\boldsymbol{\xi}_T^n|^q] < \infty, \tag{3.3.8}$$

where $q > 1$ is as in Assumption 3.3.1. Therefore, from Lemma A.1, we can deduce that the sequence $\{\boldsymbol{\xi}^n\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{D}_\uparrow^{Nd})$, and that L in tight in $\mathcal{P}(\mathcal{D}^k)$. Furthermore, since the space $\mathcal{C}_+^{Nd}$ is Polish, $\mathbb{P} \circ f^{-1}$ is regular, and hence f is tight in $\mathcal{P}(\mathcal{C}_+^{Nd})$ (see, e.g., Remark 13.27 at p. 260 in [112]). This implies that the sequence $\{(f, L, \boldsymbol{\xi}^n)\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd})$.

By Prokhorov's theorem (see, e.g., Theorem 13.29 at p. 261 in [112]), there exists a subsequence of indexes (still denoted by n) and a probability measure $\bar{\mathbb{P}} \in \mathcal{P}(\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd})$ such that the sequence $\mathbb{P}^n$ converges weakly to $\bar{\mathbb{P}}$. The first claim of the theorem is thus proved.

Proof of Claim 2. Thanks to an extension of Skorokhod's theorem for separable spaces (see Theorem 3 in [72]), there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{Q}})$, and, on it, a sequence $\{(\bar{f}^n, \bar{L}^n, \bar{\boldsymbol{\xi}}^n)\}_{n\in\mathbb{N}}$ of $\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd}$-valued random variables, and a $\mathcal{C}_+^{Nd} \times \mathcal{D}^k \times \mathcal{D}_\uparrow^{Nd}$-valued random variable $(\bar{f}, \bar{L}, \bar{\boldsymbol{\xi}})$, such that $\bar{\mathbb{Q}} \circ (\bar{f}^n, \bar{L}^n, \bar{\boldsymbol{\xi}}^n)^{-1} = \mathbb{P}^n$ and $\bar{\mathbb{Q}} \circ (\bar{f}, \bar{L}, \bar{\boldsymbol{\xi}})^{-1} = \bar{\mathbb{P}}$. Furthermore, this representation is such that, for almost all $\omega \in \bar{\Omega}$, we have

$$\bar{f}^n(\omega) \to \bar{f}(\omega) \quad \text{uniformly on the interval } [0,T], \tag{3.3.9}$$

as well as

$$(\bar{L}^n(\omega), \bar{\boldsymbol{\xi}}^n(\omega)) \to (\bar{L}(\omega), \bar{\boldsymbol{\xi}}(\omega)) \quad \text{in the measure } dt + \delta_T \text{ on } [0,T]. \tag{3.3.10}$$

Define then $\bar{\beta} := (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{Q}}, \bar{f}, \bar{L})$. Since $\mathbb{P} \circ (f, L)^{-1}$ is constantly $\mathbb{P}_0$, then the same holds for its limit; that is, $\bar{\mathbb{Q}} \circ (\bar{f}, \bar{L})^{-1} = \mathbb{P}_0$, and this implies that $\bar{\beta}$ is a basis.

Moreover, the fact that $\bar{\boldsymbol{\xi}} \in \mathcal{A}_{\bar{\beta}}^N$ can be proved following the same lines as in the proof of Proposition 29 in [123], using that the strategies $\boldsymbol{\xi}^n$ are $\bar{\mathbb{F}}_+^{f,L}$-adapted, and exploiting Condition 4 in Assumption 3.3.1.

Next, for every $i = 1, ..., N$ and $n \in \mathbb{N}$, recalling (3.3.5), we define on the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{Q}})$ the subgradient processes $\partial \bar{C}^{i,n} := \partial C^i(\bar{f}^n, \bar{L}^n, \bar{\boldsymbol{\xi}}^n)$ and $\partial \bar{C}^i := \partial C^i(\bar{f}, \bar{L}, \bar{\boldsymbol{\xi}})$. By the convergence at the terminal time (3.3.10) together with Fatou's lemma and the estimate (3.3.8) we have

$$\mathbb{E}^{\bar{\mathbb{Q}}}[|\bar{\boldsymbol{\xi}}_T|^q] \leqslant \sup_n \mathbb{E}^{\bar{\mathbb{Q}}}[|\bar{\boldsymbol{\xi}}_T^n|^q] = \sup_n \mathbb{E}^{\mathbb{P}}[|\boldsymbol{\xi}_T^n|^q] < \infty. \tag{3.3.11}$$

Let $Q := ([0,T) \cap \mathbb{Q}) \cup \{T\}$ and define the measurable function $\Phi : \mathcal{D}^k \to \mathbb{R}$ by

$$\Phi(X) := \sup_{t\in Q} |X_t|.$$

Being constantly equal to $\mathbb{P} \circ \Phi(L)^{-1}$, the sequence $\{\bar{\mathbb{Q}} \circ \Phi(\bar{L}^n)^{-1}\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\mathbb{R}^k)$. This allows to assume without loss of generality (modulo a further subsequence, a new Skorokhod representation of the sequence $\{(\bar{f}^n, \Phi(\bar{L}^n), \bar{L}^n, \bar{\boldsymbol{\xi}}^n)\}_{n\in\mathbb{N}}$, and exploiting the measurability of Φ), that $\Phi(\bar{L}^n)$ converges to $\Phi(\bar{L})$, $\bar{\mathbb{Q}}$-a.s. Furthermore, by (3.3.3) in Assumption 3.3.1, we have $\mathbb{E}^{\bar{\mathbb{Q}}}[\Phi(\bar{L})] = \mathbb{E}^{\mathbb{P}_0}[\Phi(\pi_L)] < \infty$. The latter, together with the $\bar{\mathbb{Q}}$-a.s. convergence of $\Phi(\bar{L}^n)$, the convergence in (3.3.10), and the integrability proved in (3.3.11), implies that, for $\bar{\mathbb{Q}}$-almost all $\omega \in \bar{\Omega}$, there exists a constant $M(\omega) < \infty$ such that

$$\sup_n \sup_{t\in[0,T]} (|\bar{L}_t^n(\omega)| + |\bar{\boldsymbol{\xi}}_t^n(\omega)| + |\bar{L}_t(\omega)| + |\bar{\boldsymbol{\xi}}_t(\omega)|) \leqslant M(\omega).$$

Thus, for $\bar{\mathbb{Q}}$-almost all $\omega \in \Omega$, we can find, by continuity of h^i, another constant $K(\omega) < \infty$ such that

$$\sup_n \sup_{t\in[0,T]} \Big[h^i(\bar{L}_t^n(\omega), \bar{\xi}_t^n(\omega), \bar{\xi}_t^{-i,n}(\omega)) + h^i(\bar{L}_t(\omega), \bar{\xi}_t(\omega), \bar{\xi}_t^{-i}(\omega)) \Big] \leqslant K(\omega).$$

Hence, for $\bar{\mathbb{Q}}$-almost all $\omega \in \Omega$, the bounded continuous function $\nabla_i h^i(l,a) \wedge K(\omega)$ coincides with the function $\nabla_i h^i(l,a)$ when evaluated along the sequence $(\bar{L}_s^n(\omega), \bar{\boldsymbol{\xi}}_s^n(\omega))$ and at the limit point $(\bar{L}_s(\omega), \bar{\boldsymbol{\xi}}_s(\omega))$.

Considering ω fixed and $\nabla_i h^i$ bounded by $K(\omega)$, this allows to use equation (A.1), together with standard arguments exploiting the compactness of $[0,T]$, in order to deduce that, $\bar{\mathbb{Q}}$-a.s.

$$\lim_n \sup_{t\in[0,T]} \left| \int_t^T \left(\nabla_i h^i(\bar{L}_s^n, \bar{\boldsymbol{\xi}}_s^n) - \nabla_i h^i(\bar{L}_s, \bar{\boldsymbol{\xi}}_s) \right) ds \right| = 0. \tag{3.3.12}$$

The latter, thanks to (3.3.9) and (3.3.10) and to the continuity of $\nabla_i g^i$, implies that,

$$\partial \bar{C}^{i,n} \to \partial \bar{C}^i \quad \text{uniformly on the interval } [0,T], \quad \text{for every } i = 1, ..., N, \quad \bar{\mathbb{Q}}\text{-a.s.} \tag{3.3.13}$$

The following claims summarize two key properties of the processes $\partial \bar{C}^i$ and $\bar{\boldsymbol{\xi}}$ that will guarantee that $(\bar{\beta}, \bar{\boldsymbol{\xi}})$ is a weak Nash equilibrium as in Definition 5.

For every $i = 1, ..., N$, we now prove that the following hold $\bar{\mathbb{Q}}$-a.s.:

(2.a) $\partial \bar{C}_t^i \geqslant 0$ for every $t \in [0,T]$;

(2.b) $\int_{[0,T]} \partial \bar{C}_t^i \, d\bar{\xi}_t^i = 0$.

Proof of 2.a. We begin by proving that $\partial \bar{C}^n \to \partial \bar{C}$ in $\mathbb{L}^1(\bar{\mathbb{Q}} \otimes dt)$. For $i = 1, ..., N$, from the convergence proved in (3.3.13) we have that $\bar{\mathbb{Q}} \otimes dt$-a.e. $\partial \bar{C}^{i,n}$ converges to $\partial \bar{C}^i$. Moreover, for $p > 1$ as in Assumption 3.3.1, by the growth condition (3.3.1) we easily find that

$$\mathbb{E}^{\bar{\mathbb{Q}}}\left[\sup_{t\in[0,T]} |\partial \bar{C}_t^{i,n}|^p \right] \leqslant \widetilde{C} \left(1 + \mathbb{E}^{\mathbb{P}}[|\boldsymbol{\xi}_T^n|^{\gamma_2 p}] + \mathbb{E}^{\mathbb{P}_0}\left[\sup_{t\in[0,T]} \left(|(\pi_L)_t|^{\gamma_1 p} + |(\pi_f)_t^i|^p \right) \right] \right), \tag{3.3.14}$$

for a suitable constant $\widetilde{C}$. Using then the integrability condition (3.3.3) in Assumption 3.3.1 and the estimates (3.3.8) (recall that by assumption $\gamma_2 p < q$), we have

$$\sup_n \mathbb{E}^{\bar{\mathbb{Q}}}\left[\sup_{t\in[0,T]} |\partial \bar{C}_t^{i,n}|^p \right] < \infty, \tag{3.3.15}$$

which implies that the sequence $\partial \bar{C}^{i,n}$ is uniformly integrable. From Theorem 6.25 at p. in [112], we deduce then that $\partial \bar{C}^n \to \partial \bar{C}$ in $\mathbb{L}^1(\bar{\mathbb{Q}} \otimes dt)$. Now, from the second equation in (3.3.7) in Lemma 3.3.2, we find

$$0 = \lim_n \mathbb{E}^{\mathbb{P}}\left[\int_0^T (\partial C_t^{i,n})^- \, dt \right] = \lim_n \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T (\partial \bar{C}_t^{i,n})^- \, dt \right] = \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T (\partial \bar{C}_t^i)^- \, dt \right],$$

and by continuity of $\partial \bar{C}^i$ we conclude that $\bar{\mathbb{Q}}$-a.s.

$$\partial \bar{C}_t^i \geqslant 0\,, \ \forall\, t \in [0,T], \ \forall\, i = 1, ..., N. \tag{3.3.16}$$

Proof of 2.b. Computations analogous to those employed in (3.3.14) yield

$$\mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\sup_{t\in[0,T]}|\partial\bar{C}^{i,n}_t|^{\alpha p}\bigg] \leqslant \widetilde{C}\bigg(1+\mathbb{E}^{\mathbb{P}}[|\boldsymbol{\xi}^n_T|^{\alpha\gamma_2 p}]+\mathbb{E}^{\mathbb{P}_0}\bigg[\sup_{t\in[0,T]}\Big(|(\pi_L)_t|^{\alpha\gamma_1 p}+|(\pi_f)^i_t|^{\alpha p}\Big)\bigg]\bigg), \tag{3.3.17}$$

as well as,

$$\mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\sup_{t\in[0,T]}|\partial\bar{C}^{i}_t|^{\alpha p}\bigg] \leqslant \widetilde{C}\bigg(1+\mathbb{E}^{\bar{\mathbb{Q}}}[|\bar{\boldsymbol{\xi}}_T|^{\alpha\gamma_2 p}]+\mathbb{E}^{\mathbb{P}_0}\bigg[\sup_{t\in[0,T]}\Big(|(\pi_L)_t|^{\alpha\gamma_1 p}+|(\pi_f)^i_t|^{\alpha p}\Big)\bigg]\bigg). \tag{3.3.18}$$

Now, the estimates (3.3.8), (3.3.11), (3.3.17) and (3.3.18) imply that

$$\sup_n \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\sup_{t\in[0,T]}|\partial\bar{C}^{i,n}_t|^{\alpha p}+\sup_{t\in[0,T]}|\partial\bar{C}^{i}_t|^{\alpha p}+|\bar{\boldsymbol{\xi}}^n_T|^{\frac{\alpha p}{p-1}}+|\bar{\boldsymbol{\xi}}_T|^{\frac{\alpha p}{p-1}}\bigg]<\infty,$$

which, together with the convergence established in (3.3.13), allows us to use Lemma A.2 in Appendix A in order to deduce that

$$\mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]}\partial\bar{C}^i_t\, d\bar{\xi}^i_t\bigg]=\lim_n \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]}\partial\bar{C}^{i,n}_t\, d\bar{\xi}^{i,n}_t\bigg]=\lim_n \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T\partial C^{i,n}_t\, d\xi^{i,n}_t\bigg]\leqslant 0, \tag{3.3.19}$$

where we have used the first equality of (3.3.7) in Lemma 3.3.2 and that, for each $n\in\mathbb{N}$, $\bar{\xi}^{i,n}_0=0$ $\bar{\mathbb{Q}}$-a.s. This implies, thanks to the non negativity of $\partial\bar{C}^i$ established in (3.3.16), that $\bar{\mathbb{Q}}$-a.s.

$$\int_{[0,T]}\partial\bar{C}^i_t d\bar{\xi}^i_t=0;$$

i.e. $(2.b)$ is proved.

It does remain to conclude that the couple $(\bar{\beta},\bar{\boldsymbol{\xi}})$ is a weak Nash equilibrium of the game. Fix $i\in\{1,...,N\}$, and consider an admissible strategy $\zeta^i\in\mathcal{A}_{\bar{\beta}}$. By (3.3.6) and Claims $(2.a)$ and $(2.b)$ we have

$$J^i_{\bar{\beta}}(\zeta^i,\bar{\xi}^{-i})-J^i_{\bar{\beta}}(\bar{\xi}^i,\bar{\xi}^{-i})\geqslant \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]}\partial\bar{C}^i_t(d\zeta^i_t-d\bar{\xi}^i_t)\bigg]=\mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]}\partial\bar{C}^i_t d\zeta^i_t\bigg]\geqslant 0,$$

which in fact completes the proof. □

3.3.4 On Lipschitz ε-Nash equilibria for the monotone-follower game

In this subsection we prove another connection between the Lipschitz games and the monotone-follower game by showing that ε-Nash equilibria of the monotone-follower game can be realized as Nash equilibria of the n-Lipschitz game, for n sufficiently large. The proof of this result exploits Theorem 3.3.3, combined with a contradiction scheme.

As in Subsection 3.3.3, in the following we consider fixed a basis $\beta=(\Omega,\mathcal{F},\mathbb{P},f,L)$, and, for each $n\in\mathbb{N}$, let $\boldsymbol{\xi}^n=(\xi^{1,n},...,\xi^{N,n})$ be a Nash equilibrium of the n-Lipschitz game as in Theorem 3.2.1, with $\mathbb{F}=\bar{\mathbb{F}}^{f,L}_+$. Observe that the processes $\boldsymbol{\xi}^n$ are $\bar{\mathbb{F}}^{f,L}_+$-adapted.

Theorem 3.3.4. *Suppose that Assumption 3.3.1 holds and that there exists a constant $C>0$ such that*

$$|h^i(l,a)| + |g^i(l,a)| \leqslant C(1+|l|^{\gamma_1} + |a^{-i}|^{\gamma_2}), \tag{3.3.20}$$

for each $l \in \mathbb{R}^k$ and $a=(a^1,...,a^N) \in \mathbb{R}^{Nd}$.

Then, for each $\varepsilon > 0$, there exists n_ε such that the Nash equilibrium $\boldsymbol{\xi}^{n_\varepsilon}$ of the n_ε-Lipschitz game is an ε-Nash equilibrium of the monotone-follower game; that is, for each $i=1,...,N$

$$J^i_\beta(\xi^{i,n_\varepsilon}, \xi^{-i,n_\varepsilon}) \leqslant J^i_\beta(\zeta^i, \xi^{-i,n_\varepsilon}) + \varepsilon \quad \textit{for each} \quad \zeta^i \in \mathcal{A}_\beta.$$

Proof. We argue by contraddiction and we suppose that the thesis is false. Then, there exists $\varepsilon>0$ such that, for each $n \in \mathbb{N}$, there exist $i_n \in \{1,...,N\}$ and an admissible strategy $\zeta^n \in \mathcal{A}$ (i.e., adapted to $\bar{\mathbb{F}}^{f,L}_+$) with

$$J^{i_n}_\beta(\boldsymbol{\xi}^n) > J^{i_n}_\beta(\zeta^n, \xi^{-i_n,n}) + \varepsilon.$$

Since the number of indexes of the players is finite, we can suppose that there exists $i \in \{1,...,N\}$ such that, for each $n \in \mathbb{N}$,

$$J^i_\beta(\boldsymbol{\xi}^n) > J^i_\beta(\zeta^n, \xi^{-i,n}) + \varepsilon. \tag{3.3.21}$$

Recall now that, for each $n \in \mathbb{N}$, $\boldsymbol{\xi}^n$ is a Nash equilibrium for the n-Lipschitz game and notice that the process constantly equal to zero is admissible. Hence, from (3.3.21), and using the coercivity condition (3.3.4) and the integrability condition (3.3.2) in Assumption 3.3.1, we find

$$\begin{aligned} c\,\mathbb{E}^{\mathbb{P}}[|\zeta^n_T|] \leqslant J^i_\beta(\zeta^n, \xi^{-i,n}) < J^i_\beta(\boldsymbol{\xi}^n) - \varepsilon \leqslant J^i_\beta(0, \xi^{-i,n}) \\ \leqslant \mathbb{E}^{\mathbb{P}_0}\left[\int_0^T H^i((\pi_L)_t)\,dt + G^i((\pi_L)_T)\right] < \infty, \end{aligned}$$

which implies that

$$\sup_n \mathbb{E}^{\mathbb{P}}[|\zeta^n_T|] < \infty. \tag{3.3.22}$$

With arguments analogous to those employed in the proof of *Claim 1* of Theorem 3.3.3, from the tightness condition (3.3.22) we deduce that there exists a subsequence of indexes (still denoted by n) and a probability measure $\tilde{\mathbb{P}} \in \mathcal{P}(\mathcal{C}^{Nd}_+ \times \mathcal{D}^k \times \mathcal{D}^{(1+N)d}_\uparrow)$ such that the sequence $\mathbb{P} \circ (f, L, \zeta^n, \boldsymbol{\xi}^n)^{-1}$ converges weakly to $\tilde{\mathbb{P}}$.

Then, thanks to an extension of Skorokhod's theorem (see Theorem 3 in [72]), there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{Q}})$, and, on it, a sequence $\{(\bar{f}^n, \bar{L}^n, \bar{\zeta}^n, \bar{\boldsymbol{\xi}}^n)\}_{n\in\mathbb{N}}$ of $\mathcal{C}^{Nd}_+ \times \mathcal{D}^k \times \mathcal{D}^{(1+N)d}_\uparrow$-valued random variables, and a $\mathcal{C}^{Nd}_+ \times \mathcal{D}^k \times \mathcal{D}^{(1+N)d}_\uparrow$-valued random variable $(\bar{f}, \bar{L}, \bar{\zeta}, \bar{\boldsymbol{\xi}})$, such that

$$\tilde{\mathbb{P}}^n = \bar{\mathbb{Q}} \circ (\bar{f}^n, \bar{L}^n, \bar{\zeta}^n, \bar{\boldsymbol{\xi}}^n)^{-1} \text{ and } \tilde{\mathbb{P}} = \bar{\mathbb{Q}} \circ (\bar{f}, \bar{L}, \bar{\zeta}, \bar{\boldsymbol{\xi}})^{-1}.$$

Furthermore, this representation is such that, for $\bar{\mathbb{Q}}$-almost all $\omega \in \bar{\Omega}$, we have

$$\bar{f}^n(\omega) \to \bar{f}(\omega) \quad \text{uniformly on the interval } [0,T], \tag{3.3.23}$$

as well as

$$(\bar{L}^n(\omega), \bar{\zeta}^n(\omega), \bar{\boldsymbol{\xi}}^n(\omega)) \to (\bar{L}(\omega), \bar{\zeta}(\omega), \bar{\boldsymbol{\xi}}(\omega)) \quad \text{in the measure } dt + \delta_T \text{ on } [0,T]. \tag{3.3.24}$$

A rationale similar to that yelding (3.3.12) can be employed to show that, $\bar{\mathbb{Q}}$-a.s.,

$$\begin{aligned} \lim_n \int_0^T h^i(\bar{L}_t^n, \bar{\zeta}_t^n, \bar{\xi}_t^{-i,n})\, dt + g^i(\bar{L}_T^n, \bar{\zeta}_T^n, \bar{\xi}_T^{-i,n}) \\ = \int_0^T h^i(\bar{L}_t, \bar{\zeta}_t, , \bar{\xi}_t^{-i})\, dt + g^i(\bar{L}_T, \bar{\zeta}_T, \bar{\xi}_T^{-i}), \end{aligned} \tag{3.3.25}$$

where we have also used that h^i and g^i are continuous. Furthermore, thanks to the growth condition (3.3.20), for $p > 1$ as in Assumption 3.3.1, we can find a suitable constant $\tilde{C} > 0$ such that

$$\begin{aligned} \sup_n \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\bigg| \int_0^T h^i(\bar{L}_t^n, \bar{\zeta}_t^n, \bar{\xi}_t^{-i,n})\, dt + g^i(\bar{L}_T^n, \bar{\zeta}_T^n, \bar{\xi}_T^{-i,n}) \bigg|^p\bigg] \\ \leqslant \tilde{C} \sup_n \left(1 + \mathbb{E}^{\mathbb{P}_0}\left[\sup_{t\in[0,T]} |(\pi_L)_t|^{\gamma_1 p}\right] + \mathbb{E}^{\mathbb{P}}\left[|\boldsymbol{\xi}_T^n|^{\gamma_2 p}\right]\right) < \infty, \end{aligned} \tag{3.3.26}$$

where the integrability of the right-hand side follows from Condition (3.3.3) and the estimate (3.3.8). Finally, the limit in (3.3.25), together with the uniform integrability in (3.3.26), allows us to conclude that

$$\begin{aligned} \lim_n \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T h^i(\bar{L}_t^n, \bar{\zeta}_t^n, \bar{\xi}_t^{-i,n})\, dt + g^i(\bar{L}_T^n, \bar{\zeta}_T^n, \bar{\xi}_T^{-i,n})\right] \\ = \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T h^i(\bar{L}_t, \bar{\zeta}_t, \bar{\xi}_t^{-i})\, dt + g^i(\bar{L}_T, \bar{\zeta}_T, \bar{\xi}_T^{-i})\right]. \end{aligned} \tag{3.3.27}$$

With a similar reasoning we also find

$$\begin{aligned} \lim_n \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T h^i(\bar{L}_t^n, \bar{\xi}_t^{i,n}, \bar{\xi}_t^{-i,n})\, dt + g^i(\bar{L}_T^n, \bar{\xi}_T^{i,n}, \bar{\xi}_T^{-i,n})\right] \\ = \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T h^i(\bar{L}_t, \bar{\xi}_t^i, \bar{\xi}_t^{-i})\, dt + g^i(\bar{L}_T, \bar{\xi}_T^i, \bar{\xi}_T^{-i})\right]. \end{aligned} \tag{3.3.28}$$

Moreover, Condition (3.3.3) yields

$$\sup_n \mathbb{E}^{\bar{\mathbb{Q}}}\left[\sup_{t\in[0,T]} |\bar{f}_t^n|^{\alpha p} + \sup_{t\in[0,T]} |\bar{f}_t|^{\alpha p}\right] = 2\, \mathbb{E}^{\mathbb{P}_0}\left[\sup_{t\in[0,T]} |(\pi_f)_t|^{\alpha p}\right] < \infty. \tag{3.3.29}$$

The latter, together with (3.3.8) and (3.3.11), allows to use Lemma A.2 in Appendix A in order to deduce that

$$\lim_n \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_0^T \bar{f}_t^{i,n}\, d\bar{\xi}_t^{i,n}\right] = \mathbb{E}^{\bar{\mathbb{Q}}}\left[\int_{[0,T]} \bar{f}_t^i\, d\bar{\xi}_t^i\right],$$

which, together with (3.3.28), gives

$$\lim_n J^i_\beta(\boldsymbol{\xi}^n) = J^i_{\bar\beta}(\bar\xi^i, \bar\xi^{-i}). \tag{3.3.30}$$

Fix now $M \in \mathbb{N}$ and define the sequence of processes $\{\bar\zeta^{n,M}\}_{n\in\mathbb{N}}$ by $\bar\zeta^{n,M}_t := \bar\zeta^n_t \wedge M$ as well as the process $\bar\zeta^M_t := \bar\zeta_t \wedge M$. Observe that, for each $n \in \mathbb{N}$, from (3.3.21) and the definition of $\bar B^{n,M}$ we have

$$\begin{aligned} J^i_{\bar\beta}(\bar{\boldsymbol{\xi}}^n) > \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_0^T h^i(\bar L^n_t, \bar\zeta^n_t, \bar\xi^{-i,n}_t)\,dt + g^i(\bar L^n_T, \bar\zeta^n_T, \bar\xi^{-i,n}_T)\bigg] \\ + \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]} \bar f^{i,n}_t \, d\bar\zeta^{n,M}_t\bigg] + \varepsilon. \end{aligned} \tag{3.3.31}$$

Moreover, notice that the convergence established in (3.3.24) implies that, $\bar{\mathbb{Q}}$-a.s., the sequence $\{\bar\zeta^{n,M}\}_{n\in\mathbb{N}}$ converges to $\bar\zeta^M$ in the measure $dt + \delta_T$ on $[0,T]$.

Now, since the sequence $\{\bar\zeta^{n,M}\}_{n\in\mathbb{N}}$ is bounded by the constant M, we can use again Lemma A.2 in Appendix A to deduce that

$$\lim_n \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]} \bar f^{i,n}_t \, d\bar\zeta^{n,M}_t\bigg] = \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]} \bar f^{i}_t \, d\bar\zeta^{M}_t\bigg]. \tag{3.3.32}$$

Hence, thanks to (3.3.30), (3.3.27) and (3.3.32), for each fixed M we can pass to the limit in the inequality (3.3.31), in order to obtain that

$$J^i_{\bar\beta}(\bar{\boldsymbol{\xi}}) \geqslant \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_0^T h^i(\bar L_t, \bar\zeta_t, \bar\xi^{-i}_t)\,dt + g^i(\bar L_T, \bar\zeta_T, \bar\xi^{-i}_T)\bigg] + \mathbb{E}^{\bar{\mathbb{Q}}}\bigg[\int_{[0,T]} \bar f^{i}_t \, d\bar\zeta^{M}_t\bigg] + \varepsilon.$$

Finally, by the monotone convergence theorem, we can take the limit as $M \to \infty$ in the latter inequality to deduce that

$$J^i_{\bar\beta}(\bar\xi^i, \bar\xi^{-i}) \geqslant J^i_{\bar\beta}(\bar\zeta, \bar\xi^{-i}) + \varepsilon. \tag{3.3.33}$$

On the other hand, the probability measure $\bar{\mathbb{Q}} \circ (\bar f, \bar L, \bar{\boldsymbol{\xi}})^{-1}$ is an accumulation point of the sequence $\mathbb{P} \circ (f, L, \boldsymbol{\xi}^n)^{-1}$, and hence, by Theorem 3.3.3, the couple $(\bar\beta, \bar{\boldsymbol{\xi}})$ is a weak Nash equilibrium of the monotone-follower game, with $\bar\beta := (\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbb{Q}}, \bar f, \bar L)$. Moreover, $\bar\zeta$ is an admissible strategy associated to the basis $\bar\beta$ (this can be proved following the same lines as in the proof of Proposition 29 in [123], using that the strategies $\bar\zeta^n$ are $\bar{\mathbb{F}}^{f,L}_+$-adapted, and exploiting Condition 4 in Assumption 3.3.1). Therefore, we also have

$$J^i_{\bar\beta}(\bar\xi^i, \bar\xi^{-i}) \leqslant J^i_{\bar\beta}(\bar\zeta, \bar\xi^{-i}),$$

which, together with (3.3.33), leads to a contradiction, and thus completes the proof. □

3.3.5 Some remarks

In this subsection we collect some remarks concerning extensions and comments of the previous theorems.

Remark 3.3.5. *Theorem 3.3.3 and Theorem 3.3.4 can be thought of as the game-theoretic counterpart of Theorem 21 and Corollary 24 in [123], respectively. In comparison to Theorem 21 in [123], the existence result for the n-Lipschitz game contained in Theorem 3.2.1 – on which Theorem 3.3.3 is based – requires the study of a fixed point problem (cf. Sections 3.1 and 3.2). The convergence result of Theorem 3.3.3 is only slightly more involved than that in [123], even if some extra care is needed when performing the necessary limits. On the other hand, while Corollary 24 in [123] directly follows from Theorem 21 in [123], the proof of Theorem 3.3.4 needs a new argument, which relies on the estimates and limits previously obtained in the proof of Theorem 3.3.3. Finally, differently to [123], we can also allow for a stochastic cost of control f.*

Remark 3.3.6 (Infinite Time-Horizon Case: $T = \infty$)**.** *The techniques and the methodologies used in the proof of Theorems 3.3.3 and 3.3.4 can be extended so to handle also the infinite time-horizon case (cf. the setup discussed in Remark 3.1.9). With respect to the case $T < \infty$, the main technical difference is that one now needs to introduce a different pseudopath topology in order to be able to work on the Skorokhod space of $\mathbb{R}^{k+Nd}$-valued càdlàg functions defined on the half line $[0, \infty)$. To this end, one can use the pseudopath topology τ^ρ_{pp} induced by the convergence in the measure λ^ρ on the half line $[0, \infty)$, where λ^ρ is defined by $d\lambda^\rho := e^{-\rho t}dt$ and $\rho > 0$ denotes a suitable intertemporal discount factor. The tightness of the sequence of Nash equilibria of the n-Lipschitz games can then be deduced whenever the standard Meyer-Zheng tightness criteria are satisfied (see Theorem 4 in [136]). These conditions are automatically fulfilled in the finite-fuel case (see Remark 3.1.7). Moreover, with such a choice of the topology, the convergence of the subgradient processes (cf. the limit obtained in (3.3.13)) can be proved by exploiting the characterization of the convergence in τ^ρ_{pp} given in the proof of Lemma 1 in [136].*

Remark 3.3.7. *Theorem 3.3.4 can also be understood in a different way. Fix a weak Nash equilibrium $(\bar\beta, \bar{\boldsymbol\xi})$, which is an accumulation point of a sequence of Nash equilibria of the n-Lipschitz game on a fixed basis β, and define*

$$\mathbf{V} = (V^1, ..., V^N) := (J^1_{\bar\beta}(\bar{\boldsymbol\xi}), ..., J^N_{\bar\beta}(\bar{\boldsymbol\xi})).$$

Then, $\mathbf{V}$ is a Nash equilibrium value *of the monotone-follower game (see, e.g., Definition 2.7 in [33], or [124]), in the sense that, for each $\varepsilon > 0$, there exists $\boldsymbol\xi^\varepsilon \in \mathcal{A}^N_\beta$ such that, for each $i = 1, ..., N$, we have:*

1. $J^i_\beta(\xi^{i,\varepsilon}, \xi^{-i,\varepsilon}) \leqslant J^i_\beta(\zeta^i, \xi^{-i,\varepsilon}) + \varepsilon$, *for each* $\zeta^i \in \mathcal{A}_\beta$;
2. $|J^i_\beta(\boldsymbol\xi^\varepsilon) - V^i| \leqslant \varepsilon$.

Moreover, Theorem 3.3.4 shows that the Nash equilibrium value $\mathbf{V}$ is such that, for each $\varepsilon > 0$, the profile strategy $\boldsymbol\xi^\varepsilon$, which satisfies the conditions of the definition above, can be chosen as a Nash equilibrium of the n-Lipschitz game, for n large enough.

Remark 3.3.8. *Notice that the submodularity conditions (2) and (3) in Assumption 3.1.2 are not necessarily needed in the proof of Theorems 3.3.3 and 3.3.4. Indeed,* *only the requirement that, for each $n \in \mathbb{N}$, there exists a Nash equilibrium for the n-Lipschitz game is needed. The latter games can be seen as stochastic differential games, where the set of strategies is the set of progressively measurable stochastic processes $u^i : \Omega \times [0,T] \to [0,n]^d$, with degenerate dynamics $\xi^i_t = \int_0^t u^i_s ds$. This fact suggests that, whenever the submodularity requirement does not hold, one might exploit, on a case by case basis, existence results on equilibria for stochastic differential games (see [92, 131, 132] for related results on stochastic differential games).*

3.4 Applications and examples

3.4.1 Existence of equilibria in a class of stochastic differential games

This subsection is devoted to show that Theorem 3.1.4 applies to deduce existence of open-loop Nash equilibria in stochastic differential games with singular controls, whenever a certain structure is preserved by the dynamics. For the sake of illustration, we propose the following model.

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions and consider on it N standard $\mathbb{F}$-Brownian motions W^i. Suppose given, for $i = 1, ..., N$, measurable functions $g^i, h^i : \mathbb{R}^k \times \mathbb{R}^N \to \mathbb{R}$, as well as constants $\mu^i, \sigma^i \in \mathbb{R}$ and continuous $\mathbb{F}$-adapted stochastic processes $f^i : \Omega \times [0,T] \to [0,\infty)$. Moreover, assume given an $\mathbb{F}$-adapted process $L : [0,T] \times \Omega \to \mathbb{R}^k$ with càdlàg components. The set of admissible strategies $\mathcal{A}$ is defined as the set of nondecreasing, nonnegative, càdlàg, $\mathbb{F}$-adapted, $\mathbb{R}$-valued stochastic processes, whereas $\mathcal{A}^N := \bigotimes_{i=1}^N \mathcal{A}$ denotes the set of asmissible profile strategies.

We consider the N-player stochastic differential game of singular controls in which, for $i = 1, ..., N$, player i chooses an admissible strategy $\xi^i \in \mathcal{A}$ to control her private state, which evolves according to the stochastic differential equation

$$dX^i_t = \mu^i X^i_t \, dt + \sigma^i X^i_t \, dW^i_t + d\xi^i_t, \quad t \in [0,T], \quad X^i_{0-} = x^i_0 > 0, \tag{3.4.1}$$

in order to minimize her expected cost

$$J^i(\xi^i, \xi^{-i}) := \mathbb{E}\bigg[\int_0^T h^i(L_t, X^i_t, X^{-i}_t) dt + g^i(L_T, X^i_T, X^{-i}_T) + \int_{[0,T]} f^i_t d\xi^i_t \bigg].$$

Observe that, for $i = 1, ..., N$, the solution to equation (3.4.1) is given by

$$X^i_t = E^i_t \bigg[x^i_0 + \int_{[0,t]} \frac{1}{E^i_s} d\xi^i_s \bigg] = E^i_t \left[x^i_0 + \bar{\xi}^i_t \right], \tag{3.4.2}$$

where the processes $(E^i_t)_{t \in [0,T]}$ and $(\bar{\xi}^i_t)_{t \in [0,T]}$ are defined by

$$E^i_t := \exp\bigg[\bigg(\mu^i - \frac{(\sigma^i)^2}{2} \bigg) t + \sigma^i W^i_t \bigg] \quad \text{and} \quad \bar{\xi}^i_t := \int_{[0,t]} \frac{1}{E^i_s} d\xi^i_s. \tag{3.4.3}$$

Assumption 3.4.1. *Let h^i and g^i satisfy Assumption 3.1.2. Suppose moreover that:*

1. *for each* $i = 1, ..., N$, *there exist functions* $\widetilde{H}^i, \widetilde{G}^i : \mathbb{R}^k \times \mathbb{R} \to [0, \infty)$ *such that* $h^i(l, x^i, x^{-i}) \leqslant \widetilde{H}^i(l, x^i)$ *and* $g^i(l, x^i, x^{-i}) \leqslant \widetilde{G}^i(l, x^i)$, *for each* $(l, x) \in \mathbb{R}^k \times \mathbb{R}^N$, *with*
$$\mathbb{E}\left[\int_0^T \widetilde{H}^i(L_t, x_0 E_t^i)\, dt + \widetilde{G}^i(L_T, x_0 E_T^i)\right] < \infty;$$

2. *there exists a constant* k_1 *such that, for each* $i = 1, ..., N$, *we have* $g^i(l, x) \geqslant k_1 x^i$ *for each* $(l, x) \in \mathbb{R}^k \times \mathbb{R}^N$.

Theorem 3.4.2. *Under Assumption 3.4.1, there exists an open-loop Nash equilibrium of the previously introduced stochastic differential game.*

Proof. Thanks to (3.4.2), the cost functional of player i can be rewritten in terms of $\bar{\xi}^i$ (cf. (3.4.3)), that is

$$J^i(\xi^i, \xi^{-i}) = \mathbb{E}\bigg[\int_0^T h^i\left(L_t, E_t^i\left[x_0^i + \bar{\xi}_t^i\right], \left\{E_t^j\left[x_0^j + \bar{\xi}_t^j\right]\right\}_{j \neq i}\right) dt + g^i\left(L_T, E_T^i\left[x_0^i + \bar{\xi}_T^i\right], \left\{E_T^j\left[x_0^j + \bar{\xi}_T^j\right]\right\}_{j \neq i}\right) + \int_{[0,T]} f_t^i E_t^i\, d\bar{\xi}_t^i\bigg]. \tag{3.4.4}$$

This leads to define the new functions $\bar{h}^i, \bar{g}^i : \mathbb{R}^k \times (0, \infty)^N \times \mathbb{R}^N \to [0, \infty)$ by

$$\begin{aligned} \bar{h}^i(l, e, z^i, z^{-i}) &:= h^i(l, e^i[x_0^i + z^i], \{e^j[x_0^j + z^j]\}_{j \neq i}) \\ \bar{g}^i(l, e, z^i, z^{-i}) &:= g^i(l, e^i[x_0^i + z^i], \{e^j[x_0^j + z^j]\}_{j \neq i}), \end{aligned}$$

as well as the continuous processes $\bar{f}^i : \Omega \times [0, T] \to \mathbb{R}$ by $\bar{f}_t^i := f_t^i\, E_t^i$. These definitions allows us to introduce new cost functionals in terms of new profile strategies $\boldsymbol{\zeta} = (\zeta^1, ..., \zeta^N) \in \mathcal{A}^N$ setting

$$\bar{J}^i(\zeta^i, \zeta^{-i}) := \mathbb{E}\bigg[\int_0^T \bar{h}^i(L_t, E_t, \zeta_t^i, \zeta_t^{-i}) dt + \bar{g}^i(L_T, E_T, \zeta_T^i, \zeta_T^{-i}) + \int_{[0,T]} \bar{f}_t^i d\zeta_t^i\bigg].$$

Notice that, by (3.4.4) and the definition of $\bar{\xi}^i$ in (3.4.3) as a function of ξ^i, we have that

$$\bar{J}^i(\bar{\xi}^i, \bar{\xi}^{-i}) = J^i(\xi^i, \xi^{-i}), \quad \forall\, \boldsymbol{\xi} \in \mathcal{A}^N, \quad \forall\, i \in \{1, ..., N\}.$$

Furthermore, for each $\boldsymbol{\zeta} \in \mathcal{A}^N$ there exists a unique $\boldsymbol{\xi} \in \mathcal{A}^N$ such that $\zeta^i = \bar{\xi}^i$ for each $i \in \{1, ..., N\}$. This means that solving the stochastic differential game in the class of profile strategies $\xi \in \mathcal{A}$ and with cost functionals J^i is equivalent to solve the monotone-follower game for $\zeta \in \mathcal{A}$ and cost functionals $\bar{J}^i$. The rest of the proof is then mainly devoted to show that the costs $\bar{h}^i$ and $\bar{g}^i$, together with the processes $\bar{f}^i$, satisfy the conditions of Theorem 3.1.4.

Since the functions h^i and g^i satisfy Assumption 3.1.2, for each $(l, e, z^{-i}) \in \mathbb{R}^k \times (0, \infty)^N \times \mathbb{R}^{N-1}$ the functions $\bar{h}^i(l, e, \cdot, z^{-i})$ and $\bar{g}^i(l, e, \cdot, z^{-i})$ are clearly continuous and strictly convex. Moreover, for $(l, e) \in \mathbb{R}^k \times (0, \infty)^N$ and $z, \bar{z} \in \mathbb{R}^N$ such that $z \leqslant \bar{z}$, we have $e^j[x_0^j + z^j] \leqslant e^j[x_0^j + \bar{z}^j]$ for each $j = 1, ..., N$, since the components of e are positive. Therefore, because h^i has decreasing differences, we deduce that

$$\begin{aligned}
&\bar{h}^i(l, e, \bar{z}^i, z^{-i}) - \bar{h}^i(l, e, z^i, z^{-i}) \\
&\qquad = h^i(l, e^i[x_0^i + \bar{z}^i], \{e^j[x_0^j + z^j]\}_{j\neq i}) - h^i l, (e^i[x_0^i + z^i], \{e^j[x_0^j + z^j]\}_{j\neq i}) \\
&\qquad \geqslant h^i(l, e^i[x_0^i + \bar{z}^i], \{e^j[x_0^j + \bar{z}^j]\}_{j\neq i}) - h^i(l, e^i[x_0^i + z^i], \{e^j[x_0^j + \bar{z}^j]\}_{j\neq i}) \\
&\qquad = \bar{h}^i(l, e, \bar{z}^i, \bar{z}^{-i}) - \bar{h}^i(l, e, z^i, \bar{z}^{-i}),
\end{aligned}$$

which means that $\bar{h}^i$ has decreasing difference as well. In the same way it is possible to show that $\bar{g}^i$ has decreasing differences, and this allows to conclude that the functions $\bar{h}^i$ and $\bar{g}^i$ satisfy Assumption 3.1.2. Moreover, thanks to (1) in Assumption 3.4.1, Condition 3.1.7 is clearly satisfied with $r^i(\boldsymbol{\zeta}) = 0$ for each $\boldsymbol{\zeta} \in \mathcal{A}^N$.

We prove now that the functionals $\bar{J}^i$ satisfy a slightly different version of Condition 3.1.6. The superlinear condition (2) in Assumption 3.4.1 implies that

$$\begin{aligned}
\bar{J}^i(\zeta^i, \zeta^{-i}) \geqslant & \mathbb{E}\left[\bar{g}^i(L_T, \zeta_T^i, \zeta_T^{-i})\right] = \mathbb{E}\left[g^i\left(L_T, E_T^i\left[x_0^i + \zeta_T^i\right], \left\{E_T^j\left[x_0^j + \zeta_T^j\right]\right\}_{j\neq i}\right)\right] \\
& \geqslant k_1 \mathbb{E}\left[E_T^i\left[x_0^i + \zeta_T^i\right]\right] \geqslant k_1 \mathbb{E}\left[E_T^i\, \zeta_T^i\right] = k_1\, \mathbb{E}[E_T^i]\, \mathbb{E}^{\tilde{\mathbb{P}}^i}\left[\zeta_T^i\right],
\end{aligned}$$

where $\tilde{\mathbb{P}}^i$ is the probability measure on $(\Omega, \mathcal{F})$ given by

$$d\tilde{\mathbb{P}}^i := \frac{E_T^i}{\mathbb{E}[E_T^i]}\, d\mathbb{P},$$

and equivalent to $\mathbb{P}$.

We can therefore apply Theorem 3.1.4 (in fact a slightly different version of it, in which the expectation in Condition 3.1.6 is replaced by the expectation under an equivalent probability measure) to deduce existence of a Nash equilibrium $\hat{\boldsymbol{\zeta}} = (\hat{\zeta}^1, ..., \hat{\zeta}^N)$ of the monotone-follower game with cost functionals $\bar{J}^i$. Hence the process $\hat{\boldsymbol{\xi}} = (\hat{\xi}^1, ..., \hat{\xi}^N)$ defined by

$$\hat{\xi}_t^i := \int_{[0,t]} E_s^i\, d\hat{\zeta}_s^i$$

is an open-loop Nash equilibrium of the stochastic differential game. □

Remark 3.4.3. *The same arguments employed in the proof of Theorem 3.4.2 apply if we replace the dynamics of the controlled geometric Brownian motion in (3.4.1) by the dynamics of a controlled Ornstein–Uhlenbeck process*

$$dX_t^i = \theta^i(\mu^i - X_t^i)\, dt + \sigma^i\, dW_t^i + d\xi_t^i, \quad t \in [0, T], \quad X_{0-}^i = x_0^i > 0, \tag{3.4.5}$$

for some parameters $\theta^i, \sigma^i > 0$ and $\mu^i \in \mathbb{R}$. Mean-reverting dynamics (as the Ornstein–Uhlenbeck one) find important application in the energy and commodity markets (see, e.g., [23] or Chapter 2 in [129]).

3.4.2 An algorithm to approximate the least Nash equilibrium

In this subsection we prove that, also in our setting, the algorithm introduced by Topkis (see Algorithm II in [157]) for submodular games converges to the least Nash equilibrium of the game.

According to the notation of Section 2, define the sequence of processes $\{\boldsymbol{\xi}^n\}_{n\in\mathbb{N}} \subset \mathcal{A}^N$ in the following way:

- $\boldsymbol{\xi}^0 = 0 \in \mathcal{A}^N$;
- for each $n \geqslant 1$, set $\boldsymbol{\xi}^{n+1} := \mathbf{R}(\boldsymbol{\xi}^n)$.

Theorem 3.4.4. *Suppose that the assumptions of Theorem 3.1.4 hold. Assume, moreover, that there exists a constant $C > 0$ such that, for each $i = 1, ..., N$,*

$$h^i(l,a) + g^i(l,a) \leqslant C(1+|a|),\ \forall\,(l,a) \in \mathbb{R}^k \times \mathbb{R}^{Nd} \text{ and } |f_t^i| \leqslant C,\ \forall\, t \in [0,T],\ \mathbb{P}\text{-a.s.} \tag{3.4.6}$$

Then the sequence $(\boldsymbol{\xi}^n)_{n\in\mathbb{N}}$ is monotone increasing in the lattice $(\mathcal{A}^N, \preccurlyeq^N)$ and it converges to the least Nash equilibrium of the game.

Proof. Since the map $\mathbf{R} : \mathcal{A}^N \to \mathcal{A}^N$ is increasing (cf. *Step 2* in the proof of Theorem 3.1.4), the sequence $(\boldsymbol{\xi}^n)_{n\in\mathbb{N}}$ is clearly monotone increasing with respect to the order relation in $\mathcal{A}^N$.

Define now the process $\boldsymbol{\nu} := (\nu^1, ..., \nu^N) \in \mathcal{A}^N_\infty$ as the least upper bound of the sequence $(\boldsymbol{\xi}^n)_{n\in\mathbb{N}}$ in the lattice $(\mathcal{A}^N_\infty, \preccurlyeq^N)$. Recall the construction of $\boldsymbol{\nu}$ and $\tilde{\boldsymbol{\nu}}$ (cf. (3.1.19) and (3.1.20) in *Step 3* in the proof of Theorem 3.1.4). Notice that, since the sequence $(\boldsymbol{\xi}^n)_{n\in\mathbb{N}}$ is increasing in the lattice $(\mathcal{A}^N, \preccurlyeq^N)$, there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that

$$\tilde{\boldsymbol{\nu}}_q(\omega) = \lim_n \boldsymbol{\xi}^n_q(\omega) = \sup_n \boldsymbol{\xi}^n_q(\omega), \quad \forall\, q \in Q := ([0,T] \cap \mathbb{Q}) \cup \{T\}, \quad \forall\, \omega \in \Omega \setminus \mathcal{N}.$$

Take now $\bar{t} \in (0,T)$ and $\omega \in \Omega \setminus \mathcal{N}$. If $\boldsymbol{\xi}^n_{\bar{t}}(\omega)$ does not converge to $\boldsymbol{\nu}_{\bar{t}}(\omega)$, then we find $\varepsilon > 0$ such that,

$$\tilde{\boldsymbol{\nu}}_q(\omega) + \varepsilon = \sup_n \boldsymbol{\xi}^n_q(\omega) + \varepsilon \leqslant \sup_n \boldsymbol{\xi}^n_{\bar{t}}(\omega) + \varepsilon \leqslant \boldsymbol{\nu}_{\bar{t}}(\omega).$$

for each $q \in Q$ such that $q < \bar{t}$. This implies that $\boldsymbol{\nu}_{\bar{t}-}(\omega) + \varepsilon \leqslant \boldsymbol{\nu}_{\bar{t}}(\omega)$, which means that $\bar{t}$ is in the set $\mathcal{I}(\omega)$ of discontinuity points of $\boldsymbol{\nu}(\omega)$. Thus, we conclude that there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that,

$$\boldsymbol{\nu}_t(\omega) = \lim_n \boldsymbol{\xi}^n_t(\omega) \quad \forall\, t \in [0,T] \setminus \mathcal{I}(\omega), \quad \forall\, \omega \in \Omega \setminus \mathcal{N}, \tag{3.4.7}$$

since, for each $\omega \in \Omega \setminus \mathcal{N}$, the latter convergence is verified in T by the definition of $\boldsymbol{\nu}_T$.

We next show that the limit point $\boldsymbol{\nu}$ is a Nash equilibrium. By *Step 1* in the proof of Theorem 3.1.4, we know that there exists a suitable constant $\widetilde{C}$ such that, for each $n \in \mathbb{N}$, $\mathbb{E}[|\boldsymbol{\xi}^n_T|] \leqslant \widetilde{C}$. Hence, by the monotone convergence theorem, we deduce that

$$\mathbb{E}[|\boldsymbol{\nu}_T|] \leqslant \widetilde{C}, \tag{3.4.8}$$

which in turn implies that $\boldsymbol{\nu} \in \mathcal{A}^N$. Fix then $i \in \{1, ..., N\}$ and $\zeta^i \in \mathcal{A}$. If $\mathbb{E}[|\zeta^i_T|] = \infty$, then, by the coercivity condition (3.1.6), we would automatically have $J^i(\nu^i, \nu^{-i}) \leqslant J^i(\zeta^i, \nu^{-i}) = \infty$. Hence, without loss of generality, we can assume that

$$\mathbb{E}[|\zeta^i_T|] < \infty. \tag{3.4.9}$$

Now, since $\xi^{i,n+1}$ minimizes $J^i(\cdot, \xi^{-i,n})$, for each $n \in \mathbb{N}$ we can write

$$\begin{aligned} &\mathbb{E}\bigg[\int_0^T h^i(L_t, \xi^{i,n+1}_t, \xi^{-i,n}_t)\, dt + g^i(L_T, \xi^{i,n+1}_T, \xi^{-i,n}_T) + \int_{[0,T]} f^i_t \, d\xi^{i,n+1}_t \bigg] \\ &\leqslant \mathbb{E}\bigg[\int_0^T h^i(L_t, \zeta^i_t, \xi^{-i,n}_t)\, dt + g^i(L_T, \zeta^i_T, \xi^{-i,n}_T) + \int_{[0,T]} f^i_t \, d\zeta^i_t \bigg]. \end{aligned}$$

Moreover, the limit in (3.4.7), together with conditions (3.4.6) and the estimates (3.4.8) and (3.4.9), allows us to invoke the dominated convergence theorem and to take the limit as n goes to infinity in the last inequality in order to deduce that $J^i(\nu^i, \nu^{-i}) \leqslant J^i(\zeta^i, \nu^{-i})$. Hence $\boldsymbol{\nu}$ is a Nash equilibrium.

Finally, we prove that $\boldsymbol{\nu}$ is the least Nash equilibrium. Suppose that $\bar{\boldsymbol{\nu}}$ is another Nash equilibrium. By definition we have $\boldsymbol{\xi}^0 = 0 \preccurlyeq^N \bar{\boldsymbol{\nu}}$. If, for an arbitrary $n \in \mathbb{N}$, we have $\boldsymbol{\xi}^n \preccurlyeq^N \bar{\boldsymbol{\nu}}$, then, since the map $\boldsymbol{\xi}$ is increasing and $\bar{\boldsymbol{\nu}}$ is a fixed point of $\mathbf{R}$, we have $\boldsymbol{\xi}^{n+1} = \mathbf{R}(\boldsymbol{\xi}^n) \preccurlyeq^N \mathbf{R}(\bar{\boldsymbol{\nu}}) = \bar{\boldsymbol{\nu}}$. Hence, by induction, we deduce that $\boldsymbol{\xi}^n \preccurlyeq^N \bar{\boldsymbol{\nu}}$ for each $n \in \mathbb{N}$, which in turn implies that $\boldsymbol{\nu} \preccurlyeq^N \bar{\boldsymbol{\nu}}$, since $\boldsymbol{\nu}$ is the least upper bound of the sequence $\{\boldsymbol{\xi}^n\}_{n \in \mathbb{N}}$. □

Chapter 4

Submodular mean field games with regular and singular controls

We study mean field games with scalar Itô-type dynamics and costs that are *submodular* with respect to a suitable order relation on the state and measure space. The submodularity assumption has a number of remarkable consequences. Firstly, it allows us to prove existence of solutions via an application of Tarski's fixed point theorem, covering cases with discontinuous dependence on the measure variable. Secondly, it ensures that the set of solutions enjoys a lattice structure: in particular, there exist minimal and maximal solutions. Thirdly, it guarantees that those two solutions can be obtained through a simple learning procedure based on the iterations of the best-response-map. The mean field game is first defined over ordinary stochastic controls, then extended to relaxed controls, and finally to singular controls. Our approach also allows to prove existence of a strong solution for a class of submodular mean field games with common noise, where the representative player at equilibrium interacts with the (conditional) mean of its state's distribution.

4.1 The submodular mean field game

In this section we develop our setup for submodular mean field games. This setup allows us to prove existence of MFG solutions without using a weak formulation or the notion of relaxed controls. Instead, we combine probabilistic arguments together with a lattice-theoretical approach in order to prove existence and approximation of MFG solutions.

4.1.1 The mean field game problem

Let $T > 0$ be a fixed time horizon and W be a Brownian Motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let $x_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ and σ be a nonnegative progressively measurable square integrable stochastic process. Notice that we allow the volatility process to be zero on a progressively measurable set $E \subset [0,T] \times \Omega$ with positive measure, thus leading to a degenerate dynamics. For a closed and convex set

$U \subset \mathbb{R}$, define the the set of admissible controls $\mathcal{U}$ as the set of all square integrable progressively measurable processes $u\colon \Omega\times[0,T]\to U$. For a measurable function $b\colon \Omega\times[0,T]\times\mathbb{R}\times U\to\mathbb{R}$ and an admissible process u, we consider the controlled SDE (SDE(u), in short)

$$dX_t = b(t, X_t, u_t)dt + \sigma_t dW_t, \qquad t\in[0,T], \quad X_0 = x_0. \tag{4.1.1}$$

With no further reference, throughout this chapter we will assume that for each $(x,u)\in\mathbb{R}\times U$ the process $b(\cdot,\cdot,x,u)$ is progressively measurable and that the usual Lipschitz continuity and growth conditions are satisfied; that is, there exists a constant $C_1>0$ such that for each $(\omega,t,u)\in\Omega\times[0,T]\times U$ we have

$$\begin{aligned} &|b(\omega,t,x,u)-b(\omega,t,y,u)|\leqslant C_1|x-y|, \quad \forall x,y\in\mathbb{R},\\ &|b(\omega,t,x,u)|\leqslant C_1(1+|x|+|u|), \quad \forall x\in\mathbb{R}. \end{aligned} \tag{4.1.2}$$

Under the standing assumption, by standard SDE theory, for each $u\in\mathcal{U}$ there exists a unique strong solution $X^u := (X_t^u)_{t\in[0,T]}$ to the controlled SDE(u) (4.1.1).

Let $\mathcal{P}(\mathbb{R})$ denote the space of all probability measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$, endowed with the classical (C_b-)weak topology, i.e. the topology induced by the weak convergence of probability measures. The costs of the problem are given by three measurable functions

$$\begin{aligned} f &: \Omega\times[0,T]\times\mathbb{R}\times\mathcal{P}(\mathbb{R})\to\mathbb{R},\\ l &: \Omega\times[0,T]\times\mathbb{R}\times U\to\mathbb{R},\\ g &: \Omega\times\mathbb{R}\times\mathcal{P}(\mathbb{R})\to\mathbb{R}, \end{aligned} \tag{4.1.3}$$

such that, for each $(x,\mu,u)\in\mathbb{R}\times\mathcal{P}(\mathbb{R})\times U$, the processes $f(\cdot,\cdot,x,\mu)$, $l(\cdot,\cdot,x,u)$ are progressively measurable and the random variable $g(\cdot,x,\mu)$ is $\mathcal{F}_T$-measurable. We underline that the cost processes f, l and g are not necessarily Markovian.

For any given and fixed measurable flow $\mu=(\mu_t)_{t\in[0,T]}$ of probability measures on $\mathcal{B}(\mathbb{R})$, we introduce the cost functional

$$J(u,\mu) := \mathbb{E}\left[\int_0^T \Big[f(t,X_t^u,\mu_t)+l(t,X_t^u,u_t)\Big]dt + g(X_T^u,\mu_T)\right], \quad u\in\mathcal{U}, \tag{4.1.4}$$

and consider the optimal control problem $\inf_{u\in\mathcal{U}} J(u,\mu)$.

We say that (X^μ,u^μ) is an *optimal pair* for the flow μ if $-\infty < J(u^\mu,\mu)\leqslant J(u,\mu)$ for each admissible $u\in\mathcal{U}$ and $X^\mu = X^{u^\mu}$.

Remark 4.1.1. *The subsequent results of this section remain valid if we consider a geometric dynamics for X (cf. Subsection 4.4.3 below). Moreover, for suitable choices of the costs, we can also allow for geometric or mean-reverting state processes with dependence on the measure in the dynamics (see Subsection 4.4.4 for more details).*

We make the following standing assumption.

Assumption 4.1.2.

1. *For each measurable flow μ of probability measures on $\mathcal{B}(\mathbb{R})$, there exists a unique (up to indistinguishability) optimal pair (X^μ,u^μ).*

2. *There exists a continuous and strictly increasing function* $\psi\colon [0,\infty) \to [0,\infty)$ *with* $\lim_{s\to\infty} \psi(s) = \infty$ *and a constant* $M > \psi(0)$ *such that*

$$\mathbb{E}\Big[\psi\Big(|X_t^\mu|\Big)\Big] \leqslant M \quad \text{for all measurable flows of probabilities } \mu \text{ and } t \in [0,T]. \tag{4.1.5}$$

Remark 4.1.3. *To underline the flexibility of our setup, Condition (1) in Assumption 4.1.2 is stated at an informal level. Condition (1) holds, for example, in the case of a linear-convex setting in which* $b(t,x,u) = c_t + p_t x + q_t u$*, for suitable processes* c_t*,* p_t*,* q_t*,* $l(t,\cdot,\cdot)$ *is strictly convex and lower semi-continuous,* $f(t,\cdot,\mu)$ *and* $g(\cdot,\mu)$ *are lower semi-continuous, and* U *is convex and compact (see e.g. Theorem 5.2 at p. 68 in [168]). More general conditions ensuring existence and uniqueness of an optimal pair in the strong formulation of the control problem can be found in [84] and in Chapter II of [45], among others.*

Remark 4.1.4. *Notice that Condition (2) in Assumption 4.1.2 is equivalent to the tightness of the family of laws* $\Big\{\mathbb{P}\circ(X_t^\mu)^{-1} \,|\, \mu \text{ is a measurable flow}, \ t\in[0,T]\Big\}$ *(cf. [52], [122] or [139]). The latter is satisfied, for example, if* U *is compact or if* b *is bounded in* u*. Alternatively, one can assume that* U *is closed and convex and that there exist exponents* $p' > p \geqslant 1$ *and constants* κ*,* $K > 0$ *such that* $\mathbb{E}\Big[|x_0|^{p'} + \Big(\int_0^T |\sigma_t|^2 dt\Big)^{p'/2}\Big] < \infty$ *and*

$$\begin{aligned} &|f(t,x,\mu)| + |g(x,\mu)| \leqslant K(1+|x|^p), \\ &\kappa|u|^{p'} - K(1+|x|^p) \leqslant l(t,x,u) \leqslant K(1+|x|^p+|u|^{p'}), \end{aligned} \tag{4.1.6}$$

for all $(t,x,\mu,u) \in [0,T]\times\mathbb{R}\times\mathcal{P}(\mathbb{R})\times U$*. Indeed, following the proof of Lemma 5.1 in [120], these conditions allow to have an a priori bound on the* p'*-moments of the minimizers independent of the measure* μ*; that is, there exists a constant* $M > 0$ *such that*

$$\mathbb{E}[|X_t^\mu|^{p'}] \leqslant M \quad \text{for all measurable flows of probabilities } \mu \text{ and } t\in[0,T]. \tag{4.1.7}$$

Remark 4.1.5 (On the topology on $\mathbb{R}\times\mathcal{P}(\mathbb{R})$ and the non-continuity of the costs)**.** *We point out that the space* $\mathbb{R}$ *is endowed with the usual Euclidean distance, while the set* $\mathcal{P}(\mathbb{R})$ *is endowed with the classical* (C_b-)*weak topology, i.e. the topology induced by the weak convergence of probability measures. Also, we say that sequence of probability measures converges weakly if it converges in the* (C_b-)*weak topology. Unless otherwise stated, the set* $\mathbb{R}\times\mathcal{P}(\mathbb{R})$ *will always be endowed with the product topology, and the continuity of* f, g *will mean continuity with respect to this topology.*

Alternatively, for $p \geqslant 1$*, one could work on the space*

$$\mathcal{P}_p(\mathbb{R}) := \{\mu\in\mathcal{P}(\mathbb{R}) \,|\, \int_{\mathbb{R}} |y|^p d\mu(y) < \infty\},$$

endowed with the p*-Wasserstein distance*

$$W_p(\mu,\nu) := \Bigg(\inf_{\gamma\in\Gamma(\mu,\nu)} \int_{\mathbb{R}^2} |x-y|^p d\gamma(x,y)\Bigg)^{1/p}, \quad \mu,\nu\in\mathcal{P}_p(\mathbb{R}),$$

where $\Gamma(\mu, \nu)$ denotes the set of probability measures γ on the Borel sets of $\mathbb{R}^2$, such that $\gamma(E \times \mathbb{R}) = \mu(E)$ and $\gamma(\mathbb{R} \times E) = \nu(E)$ for each $E \in \mathcal{B}(\mathbb{R})$. The latter distance is usually used in the literature to address the continuity of the costs (see e.g. [120]).

Differently from the standard conditions in the literature on mean field games, our existence result (Theorem 4.1.14) does not require any continuity of the costs f and g with respect to the measure μ. In fact, f and g can be discontinuous with respect to the weak topology or with respect to any Wasserstein distance.

For each measurable flow μ of probability measures on $\mathcal{B}(\mathbb{R})$, we now define the *best-response* by $R(\mu) := \mathbb{P} \circ (X^\mu)^{-1}$. The map $\mu \mapsto R(\mu)$ is called the *best-response-map.*

Definition 6 (MFG Solution)**.** *A measurable flow μ^* of probability measures on $\mathcal{B}(\mathbb{R})$ is a mean field game solution if it is a fixed point of the best-response-map R; that is, if $R(\mu^*) = \mu^*$.*

4.1.2 The lattice structure

In this section, we endow the space of measurable flows with a suitable lattice structure, which is fundamental for the subsequent analysis. We start by identifying the set of probability measures $\mathcal{P}(\mathbb{R})$ with the set of distribution functions on $\mathbb{R}$, setting $\mu(s) := \mu(-\infty, s]$ for each $s \in \mathbb{R}$ and $\mu \in \mathcal{P}(\mathbb{R})$. On $\mathcal{P}(\mathbb{R})$ we then consider the order relation $\leqslant^{\text{st}}$ given by the *first order stochastic dominance*, i.e. we write

$$\mu \leqslant^{\text{st}} \nu \text{ for } \mu, \nu \in \mathcal{P}(\mathbb{R}) \text{ if and only if } \mu(s) \geqslant \nu(s) \text{ for each } s \in \mathbb{R}. \tag{4.1.8}$$

The partially ordered set $(\mathcal{P}(\mathbb{R}), \leqslant^{\text{st}})$ is then endowed with a lattice structure by defining

$$(\mu \wedge^{\text{st}} \nu)(s) := \mu(s) \vee \nu(s) \quad \text{and} \quad (\mu \vee^{\text{st}} \nu)(s) := \mu(s) \wedge \nu(s) \quad \text{for each } s \in \mathbb{R}. \tag{4.1.9}$$

Observe that (see e.g. [148]), for $\mu, \nu \in \mathcal{P}(\mathbb{R})$, we have

$$\mu \leqslant^{\text{st}} \nu \text{ if and only if } \langle \varphi, \mu \rangle \leqslant \langle \varphi, \nu \rangle \tag{4.1.10}$$

for any increasing function $\varphi \colon \mathbb{R} \to \mathbb{R}$ such that $\langle \varphi, \mu \rangle$ and $\langle \varphi, \nu \rangle$ are finite, where $\langle \varphi, \mu \rangle := \int_{\mathbb{R}} \varphi(y) d\mu(y)$.

Recall that by (4.1.5),

$$\mathbb{E}\Big[\psi\Big(|X_t^\mu|\Big)\Big] \leqslant M \quad \text{for all measurable flows } \mu \text{ and } t \in [0, T].$$

Then, following the arguments in the proof of Lemma B.2, we can define $\mu^{\text{Min}}, \mu^{\text{Max}} \in \mathcal{P}(\mathbb{R})$ with

$$\mu^{\text{Min}} \leqslant^{\text{st}} \mathbb{P} \circ (X_t^\mu)^{-1} \leqslant^{\text{st}} \mu^{\text{Max}} \quad \text{for all measurable flows } \mu \text{ and } t \in [0, T],$$

where, extending ψ to $(-\infty, 0)$ by $\psi(s) := \psi(0)$ for $s < 0$, μ^{Min} and μ^{Max} are given by

$$\mu^{\text{Min}}(s) := \frac{M}{\psi(-s)} \wedge 1 \quad \text{and} \quad \mu^{\text{Max}}(s) := \left(1 - \frac{M}{\psi(s)}\right) \vee 0, \quad \text{for all } s \in \mathbb{R}. \tag{4.1.11}$$

This observation suggests to consider the interval

$$[\mu^{\text{Min}}, \mu^{\text{Max}}] = \left\{ \mu \in \mathcal{P}(\mathbb{R}) \,\middle|\, \mu^{\text{Min}} \leqslant^{\text{st}} \mu \leqslant^{\text{st}} \mu^{\text{Max}} \right\}$$

endowed with the Borel σ-algebra induced by the weak topology, i.e. the topology related to the weak convergence of probability measures. We consider the finite measure $\pi := \delta_0 + dt + \delta_T$ on the Borel σ-algebra $\mathcal{B}([0,T])$ of the interval $[0,T]$, where δ_t denotes the Dirac measure at time $t \in [0,T]$. Notice that we include δ_0 into the definition of the measure π in order to prescribe the initial law $\mathbb{P} \circ \xi^{-1}$. We then define the set L of feasible flows of measures as the set of all equivalence classes (w.r.t. π) of measurable flows $(\mu_t)_{t\in[0,T]}$ with $\mu_t \in [\mu^{\text{Min}}, \mu^{\text{Max}}]$ for π-almost all $t \in (0,T]$ and $\mu_0 = \mathbb{P} \circ \xi^{-1}$. On L we consider the order relation $\leqslant^L$ given by $\mu \leqslant^L \nu$ if and only if $\mu_t \leqslant^{\text{st}} \nu_t$ for π-a.a. $t \in [0,T]$. This order relation implies that L can be endowed with the lattice structure given by

$$(\mu \wedge^L \nu)_t := \mu_t \wedge^{\text{st}} \nu_t \quad \text{and} \quad (\mu \vee^L \nu)_t := \mu_t \vee^{\text{st}} \nu_t \quad \text{for } \pi\text{-a.a. } t \in [0,T].$$

Notice that $\mathbb{P} \circ (X^\mu)^{-1} \in L$ for every $\mu \in L$. In particular, the best-response-map $R\colon L \to L$ is well defined.

Remark 4.1.6. *We point out that if $\psi(x) = x^2$, then each element of $[\mu^{\text{Min}}, \mu^{\text{Max}}]$ has finite first-order moment, i.e. $\int_{\mathbb{R}} |y| d\mu(y) < \infty$ for each $[\mu^{\text{Min}}, \mu^{\text{Max}}]$. This follows directly from Lemma B.3. Notice also that a higher integrability requirement in (4.1.5) implies the existence and uniform boundedness of higher moments for the elements of $[\mu^{\text{Min}}, \mu^{\text{Max}}]$. More precisely, if $\psi(x) = x^{p'}$ for some $p' \in (1,\infty)$, then*

$$\sup_{\mu \in [\mu^{\text{Min}}, \mu^{\text{Max}}]} \int_{\mathbb{R}} |y|^p d\mu(y) < \infty \quad \textit{for all } p \in (1, p').$$

We now turn our focus on the main result of this subsection, which is the following lemma. Its proof follows from the more general Proposition B.4, which is relegated to the Appendix B.

Lemma 4.1.7. *The lattice $(L, \leqslant^L)$ is complete. That is, each subset of L has a least upper bound and a greatest lower bound.*

Remark 4.1.8. *We underline that, in general,* $\inf L$ *and* $\sup L$ *are given by*

$$(\inf L)_t := \mathbb{1}_{\{0\}}(t)\mathbb{P} \circ \xi^{-1} + \mathbb{1}_{(0,T]}(t)\mu^{\text{Min}}, \quad (\sup L)_t := \mathbb{1}_{\{0\}}(t)\mathbb{P} \circ \xi^{-1} + \mathbb{1}_{(0,T]}(t)\mu^{\text{Max}},$$

with μ^{Min}, μ^{Max} defined in (4.1.11) in terms of ψ and M. In particular, according to Remark 4.1.4, if U is compact, if b is bounded or if Condition (4.1.6) is satisfied, then Condition (2) in Assumption 4.1.2 is satisfied with $\psi(s) = s^p$ for $s \geqslant 0$ and some $p \geqslant 1$. In this case, $\inf L$ *and* $\sup L$ *are explicitly given by*

$$(\inf L)_t(s) := \mathbb{1}_{\{0\}}(t)\mathbb{P} \circ \xi^{-1}(s) + \mathbb{1}_{(0,T]}(t)\left[\mathbb{1}_{\{s<0\}}\left(\frac{M}{(-s)^p} \wedge 1\right) + \mathbb{1}_{\{s\geqslant 0\}}\right], \tag{4.1.12}$$

for a.a. $t \in [0,T]$ and for each $s \in \mathbb{R}$, and

$$(\sup L)_t(s) := \mathbb{1}_{\{0\}}(t)\mathbb{P} \circ \xi^{-1}(s) + \mathbb{1}_{(0,T]}(t)\left[\mathbb{1}_{\{s\leqslant 0\}} + \mathbb{1}_{\{s>0\}}\left(1 - \frac{M}{(s)^p}\right) \vee 0\right], \tag{4.1.13}$$

for a.a. $t \in [0,T]$ and for each $s \in \mathbb{R}$.

4.1.3 The submodularity condition

Our subsequent results rely on the following key assumption.

Assumption 4.1.9 (Submodularity condition)**.** *For $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0, T]$, the functions $f(t, \cdot, \cdot)$ and g have decreasing differences in (x, μ); that is, for $\phi \in \{f(t, \cdot, \cdot), g\}$,*

$$\phi(\bar{x}, \bar{\mu}) - \phi(x, \bar{\mu}) \leqslant \phi(\bar{x}, \mu) - \phi(x, \mu),$$

for all $\bar{x}, x \in \mathbb{R}$ and $\bar{\mu}, \mu \in \mathcal{P}(\mathbb{R})$ s.t. $\bar{x} \geqslant x$ and $\bar{\mu} \geqslant^{st} \mu$.

We list here two examples in which Assumption 4.1.9 is satisfied.

Example 7 (mean field interaction of scalar type)**.** *Consider a mean field interaction of scalar type; that is, $\phi(x, \mu) = \gamma(x, \langle \varphi, \mu \rangle)$ for given measurable maps $\gamma : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$. If the map φ is increasing and the map $\gamma : \mathbb{R}^2 \to \mathbb{R}$ has decreasing differences in $(x, y) \in \mathbb{R}^2$, then Assumption 4.1.9 is satisfied. Observe that a function $\gamma \in \mathcal{C}^2(\mathbb{R}^2)$ has decreasing differences in (x, y) if and only if*

$$\frac{\partial^2 \gamma}{\partial x \partial y}(x, y) \leqslant 0 \quad \textit{for each} \quad (x, y) \in \mathbb{R}^2.$$

Example 8 (mean field interactions of order-1)**.** *Another example is provided by the interactions of order-1, i.e. when ϕ is of the form*

$$\phi(x, \mu) = \int_{\mathbb{R}} \gamma(x, y) d\mu(y).$$

It is easy to check that, thanks to (4.1.10), Assumption 4.1.9 holds when γ has decreasing differences in (x, y).

A natural and relevant question related to Assumption 4.1.9 concerns its link to the so-called *Lasry-Lions monotonicity condition*, i.e. the condition

$$\int_{\mathbb{R}} (\phi(x, \bar{\mu}) - \phi(x, \mu)) d(\bar{\mu} - \mu)(x) \geqslant 0, \quad \forall\, \bar{\mu}, \mu \in \mathcal{P}(\mathbb{R}). \tag{4.1.14}$$

In general, there is no relation between the submodularity condition and (4.1.14). However, since Assumption 4.1.9 is equivalent to the fact that the map $\phi(\cdot, \bar{\mu}) - \phi(\cdot, \mu)$ is decreasing for $\mu, \bar{\mu} \in \mathbb{P}(\mathbb{R})$ with $\bar{\mu} \geqslant^{\text{st}} \mu$, Assumption 4.1.9 and (4.1.10) imply that

$$\int_{\mathbb{R}} (\phi(x, \bar{\mu}) - \phi(x, \mu)) d(\bar{\mu} - \mu)(x) \leqslant 0, \quad \forall\, \bar{\mu}, \mu \in \mathcal{P}(\mathbb{R}) \text{ with } \bar{\mu} \geqslant^{\text{st}} \mu;$$

the latter, roughly speaking, being sort of an opposite version of the Lasry-Lions monotonicity condition (4.1.14).

Remark 4.1.10. *Specific cost functions satisfying Assumption 4.1.9 are, for example,*

$$f(t, x, \mu) \equiv 0, \quad l(t, x, u) = \frac{u^2}{2}, \quad g(x, \mu) = \Big(x - \mathbb{1}_{[0, \infty)}(\langle id, \mu \rangle)\Big)^2,$$

where $id(y) = y$. Notice that the function $\mu \mapsto g(x, \mu)$ is discontinuous, in contrast to the typical continuity requirement assumed in the literature (see, e.g., [120]).

4.1.4 The best-response-map

In the following lemma, we show that the set of admissible trajectories is a lattice.

Lemma 4.1.11. *If u and $\bar{u}$ are admissible controls, then there exists an admissible control $u^\vee$ such that $X^u \vee X^{\bar{u}} = X^{u^\vee}$. Moreover, there exists an admissible control $u^\wedge$ such that $X^u \wedge X^{\bar{u}} = X^{u^\wedge}$.*

Proof. Let u and $\bar{u}$ be admissible controls and define the process $u^\vee$ by

$$u_s^\vee := \begin{cases} u_s & \text{on} \quad \{X_s^u > X_s^{\bar{u}}\} \cup \{X_s^u = X_s^{\bar{u}},\ b(s, X_s^u, u_s) \geqslant b(s, X_s^{\bar{u}}, \bar{u}_s)\}, \\ \bar{u}_s & \text{on} \quad \{X_s^u < X_s^{\bar{u}}\} \cup \{X_s^u = X_s^{\bar{u}},\ b(s, X_s^u, u_s) < b(s, X_s^{\bar{u}}, \bar{u}_s)\}. \end{cases}$$

The process $u^\vee$ is clearly progressively measurable and square integrable, hence admissible.

We want to show that $X^u \vee X^{\bar{u}} = X^{u^\vee}$; that is,

$$X_t^u \vee X_t^{\bar{u}} = x_0 + \int_0^t b(s, X_s^u \vee X_s^{\bar{u}}, u_s^\vee) ds + \int_0^t \sigma_s dW_s, \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.} \tag{4.1.15}$$

In order to do so, observe that the process $X^u \vee X^{\bar{u}}$ satisfies, $\mathbb{P}$-a.s. for each $t \in [0, T]$, the following integral equation

$$X_t^u \vee X_t^{\bar{u}} = x_0 + \int_0^t \sigma_s dW_s + \left(\int_0^t b(s, X_s^u, u_s) ds \right) \vee \left(\int_0^t b(s, X_s^{\bar{u}}, \bar{u}_s) ds \right). \tag{4.1.16}$$

Furthermore, defining the two processes A and $\bar{A}$ by

$$A_t := \int_0^t b(s, X_s^u, u_s) ds \quad \text{and} \quad \bar{A}_t := \int_0^t b(s, X_s^{\bar{u}}, \bar{u}_s) ds,$$

we see that the process S, defined by $S_t := A_t \vee \bar{A}_t$, is $\mathbb{P}$-a.s. absolutely continuous. Hence the time derivative of S exists a.e. in $[0, T]$ and, in view of (4.1.16), in order to prove (4.1.15) it suffices to show that $dS_t/dt = b(t, X_t^u \vee X_t^{\bar{u}}, u_t^\vee)$ for $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0, T]$.

Since the processes A, $\bar{A}$ and S are $\mathbb{P}$-a.s. absolutely continuous, for each ω in a set of full probability, the paths $A(\omega)$, $\bar{A}(\omega)$ and $S(\omega)$ admit time derivatives in a subset $E(\omega) \subset [0, T]$ with full Lebesgue measure. We now use a pathwise argument, without stressing the dependence on $\omega \in \Omega$. Take $t \in E$ such that $X_t^u > X_t^{\bar{u}}$. By continuity, there exists a (random) neighborhood I_t of t in $\mathbb{R}$ such that $X_s^u > X_s^{\bar{u}}$ for each $s \in I_t \cap [0, T]$, which, by (4.1.16), is true if and only if $A_s > \bar{A}_s$ for each $s \in I_t \cap [0, T]$. Hence, by definition of S, we have

$$\frac{dS_s}{ds} = \frac{dA_s}{ds} = b(s, X_s^u, u_s), \quad \forall\, s \in I_t \cap [0, T],$$

and, in particular, $dS_s/ds = b(s, X_s^u \vee X_s^{\bar{u}}, u_s^\vee)$ for each $s \in I_t \cap [0, T]$.

Take now $t \in E$ such that $X_t^u = X_t^{\bar{u}}$ and $b(t, X_t^u, u_t) \geqslant b(t, X_t^{\bar{u}}, \bar{u}_t)$. From (4.1.16) it follows that $A_t = \bar{A}_t$, which in turn implies that

$$\frac{dS_t}{dt} = \lim_{h \to 0} \frac{A_{t+h} \vee \bar{A}_{t+h} - A_t \vee \bar{A}_t}{h} \geqslant \frac{dA_t}{dt} \vee \frac{d\bar{A}_t}{dt}.$$

By construction,

$$\frac{dA_t}{dt} = b(t, X_t^u, u_t) \geqslant b(t, X_t^{\bar{u}}, \bar{u}_t) = \frac{d\bar{A}_t}{dt}. \tag{4.1.17}$$

If there exists a sequence $\{h^j\}_{j\in\mathbb{N}}$ converging to 0 such that $A_{t+h^j} \geqslant \bar{A}_{t+h^j}$ for each $j \in \mathbb{N}$, then clearly $dS_t/dt = dA_t/dt = b(t, X_t^u, u_t) = b(t, X_t^u \vee X_t^{\bar{u}}, u_t^\vee)$, as desired. On the other hand, if such a sequence does not exist, then there exists some $\delta > 0$ such that $A_{t+h} \leqslant \bar{A}_{t+h}$ for each $h \in (-\delta, \delta)$. Recalling (4.1.17), this implies that $dA_t/dt \leqslant dS_t/dt = d\bar{A}_t/dt \leqslant dA_t/dt$, hence we obtain again that $dS_t/dt = dA_t/dt$.

Altogether, we have proved that for a.a. $t \in [0,T]$ with $X_t^u > X_t^{\bar{u}}$ or $X_t^u = X_t^{\bar{u}}$ and $b(t, X_t^u, u_t) \geqslant b(t, X_t^{\bar{u}}, \bar{u}_t)$, we have $dS_t/dt = b(t, X_t^u, u_t) = b(t, X_t^u \vee X_t^{\bar{u}}, u_t^\vee)$. Analogously, one can prove that $dS_t/dt = b(t, X_t^{\bar{u}}, \bar{u}_t) = b(t, X_t^u \vee X_t^{\bar{u}}, u_t^\vee)$ for a.a. $t \in [0,T]$ with $X_t^u < X_t^{\bar{u}}$ or $X_t^u = X_t^{\bar{u}}$ and $b(t, X_t^u, u_t) < b(t, X_t^{\bar{u}}, \bar{u}_t)$. Therefore $dS_t/dt = b(t, X_t^u \vee X_t^{\bar{u}}, u_t^\vee)$ for $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0,T]$, which proves (4.1.15).

The arguments employed above allow to prove that the process $X^u \wedge X^{\bar{u}}$ satisfies the SDE controlled by $u^\wedge$; i.e.

$$X_t^u \wedge X_t^{\bar{u}} = x_0 + \int_0^t b(s, X_s^u \wedge X_s^{\bar{u}}, u_s^\wedge) ds + \int_0^t \sigma_s dW_s, \quad \forall t \in [0,T], \quad \mathbb{P}\text{-a.s.},$$

where $u^\wedge$ is defined by

$$u_s^\wedge := \begin{cases} \bar{u}_s & \text{on} \quad \{X_s^u > X_s^{\bar{u}}\} \cup \{X_s^u = X_s^{\bar{u}},\ b(s, X_s^u, u_s) \geqslant b(s, X_s^{\bar{u}}, \bar{u}_s)\}, \\ u_s & \text{on} \quad \{X_s^u < X_s^{\bar{u}}\} \cup \{X_s^u = X_s^{\bar{u}},\ b(s, X_s^u, u_s) < b(s, X_s^{\bar{u}}, \bar{u}_s)\}. \end{cases}$$

The proof of the lemma is therefore completed. □

We now prove the fundamental property of the best-response-map.

Lemma 4.1.12. *The best-response-map R is increasing in $(L, \leqslant^L)$.*

Proof. Take $\bar{\mu}, \mu \in L$ such that $\mu \leqslant^L \bar{\mu}$ and let $(X^{\bar{\mu}}, u^{\bar{\mu}})$ and (X^μ, u^μ) be the optimal pairs related to $\bar{\mu}$ and μ, respectively. For $t \in [0,T]$, we define the event

$$B_t := \{X_t^\mu > X_t^{\bar{\mu}}\} \cup \{X_t^\mu = X_t^{\bar{\mu}},\ b(t, X_t^\mu, u_t^\mu) \geqslant b(t, X_t^{\bar{\mu}}, u_t^{\bar{\mu}})\}. \tag{4.1.18}$$

As it is shown in Lemma 4.1.11, the process $X^\mu \vee X^{\bar{\mu}}$ is the solution to the dynamics (4.1.1) controlled by $u_t^\vee := u_t^\mu \mathbb{1}_{B_t} + u_t^{\bar{\mu}} \mathbb{1}_{B_t^c}$, and the process $X^\mu \wedge X^{\bar{\mu}}$ is the solution to the dynamics controlled by $u_t^\wedge := u_t^\mu \mathbb{1}_{B_t^c} + u_t^{\bar{\mu}} \mathbb{1}_{B_t}$.

By the admissibility of $u^\vee$ and the optimality of $u^{\bar{\mu}}$ we can write

$$\begin{aligned} 0 \leqslant J(u^\vee, \bar{\mu}) - J(u^{\bar{\mu}}, \bar{\mu}) = \mathbb{E}\left[\int_0^T \left[f(t, X_t^\mu \vee X_t^{\bar{\mu}}, \bar{\mu}_t) - f(t, X_t^{\bar{\mu}}, \bar{\mu}_t) \right] dt \right] \\ + \mathbb{E}\left[\int_0^T \left[l(t, X_t^\mu \vee X_t^{\bar{\mu}}, u_t^\vee) - l(t, X_t^{\bar{\mu}}, u_t^{\bar{\mu}}) \right] dt \right] \\ + \mathbb{E}\left[g(X_T^\mu \vee X_T^{\bar{\mu}}, \bar{\mu}_T) - g(X_T^{\bar{\mu}}, \bar{\mu}_T) \right]. \end{aligned} \tag{4.1.19}$$

Next, from the definition of B_t in (4.1.18) and the trivial identity $1 = \mathbb{1}_{B_t} + \mathbb{1}_{B_t^c}$, we find

$$\begin{aligned}\mathbb{E}\bigg[\int_0^T \Big[f(t, X_t^{\mu} \vee X_t^{\bar\mu}, \bar\mu_t) - f(t, X_t^{\bar\mu}, \bar\mu_t)\Big] dt\bigg] \\ &= \mathbb{E}\bigg[\int_0^T \mathbb{1}_{B_t}\Big[f(t, X_t^{\mu}, \bar\mu_t) - f(t, X_t^{\bar\mu}, \bar\mu_t)\Big] dt\bigg] \\ &= \mathbb{E}\bigg[\int_0^T \Big[f(t, X_t^{\mu}, \bar\mu_t) - f(t, X_t^{\mu} \wedge X_t^{\bar\mu}, \bar\mu_t)\Big] dt\bigg],\end{aligned}$$

as well as

$$\mathbb{E}\left[g(X_T^{\mu} \vee X_T^{\bar\mu}, \bar\mu_T) - g(X_T^{\bar\mu}, \bar\mu_T)\right] = \mathbb{E}\left[g(X_T^{\mu}, \bar\mu_T) - g(X_T^{\mu} \wedge X_T^{\bar\mu}, \bar\mu_T)\right].$$

In the same way, by the definition of $u^\vee$ and $u^\wedge$, we see that

$$\begin{aligned}\mathbb{E}\bigg[\int_0^T \Big[l(t, X_t^{\mu} \vee X_t^{\bar\mu}, u_t^\vee) - l(t, X_t^{\bar\mu}, u_t^{\bar\mu})\Big] dt\bigg] \\ &= \mathbb{E}\bigg[\int_0^T \mathbb{1}_{B_t}\Big[l(t, X_t^{\mu}, u_t^{\mu}) - l(t, X_t^{\bar\mu}, u_t^\wedge)\Big] dt\bigg] \\ &= \mathbb{E}\bigg[\int_0^T \Big[l(t, X_t^{\mu}, u_t^{\mu}) - l(t, X_t^{\mu} \wedge X_t^{\bar\mu}, u_t^\wedge)\Big] dt\bigg].\end{aligned}$$

Now, the latter three equalities allow to rewrite (4.1.19) as

$$\begin{aligned}0 \leqslant J(u^\vee, \bar\mu) - J(u^{\bar\mu}, \bar\mu) = &\ \mathbb{E}\bigg[\int_0^T \Big[f(t, X_t^{\mu}, \bar\mu_t) - f(t, X_t^{\mu} \wedge X_t^{\bar\mu}, \bar\mu_t)\Big] dt\bigg] \\ &+ \mathbb{E}\bigg[\int_0^T \Big[l(t, X_t^{\mu}, u_t^{\mu}) - l(t, X_t^{\mu} \wedge X_t^{\bar\mu}, u_t^\wedge)\Big] dt\bigg] \\ &+ \mathbb{E}\left[g(X_T^{\mu}, \bar\mu_T) - g(X_T^{\mu} \wedge X_T^{\bar\mu}, \bar\mu_T)\right],\end{aligned} \tag{4.1.20}$$

which reads as

$$J(u^\vee, \bar\mu) - J(u^{\bar\mu}, \bar\mu) = J(u^{\mu}, \bar\mu) - J(u^\wedge, \bar\mu) \tag{4.1.21}$$

Finally, exploiting Assumption 4.1.9 in the expectations in (4.1.20), we deduce that

$$\begin{aligned}0 \leqslant J(u^\vee, \bar\mu) - J(u^{\bar\mu}, \bar\mu) \leqslant &\ \mathbb{E}\bigg[\int_0^T \Big[f(t, X_t^{\mu}, \mu_t) - f(t, X_t^{\mu} \wedge X_t^{\bar\mu}, \mu_t)\Big] dt\bigg] \\ &+ \mathbb{E}\bigg[\int_0^T \Big[l(t, X_t^{\mu}, u_t^{\mu}) - l(t, X_t^{\mu} \wedge X_t^{\bar\mu}, u_t^\wedge)\Big] dt\bigg] \\ &+ \mathbb{E}\left[g(X_T^{\mu}, \mu_T) - g(X_T^{\mu} \wedge X_T^{\bar\mu}, \mu_T)\right] \\ = &\ J(u^{\mu}, \mu) - J(u^\wedge, \mu).\end{aligned} \tag{4.1.22}$$

Hence the control $u^\wedge$ is a minimizer for $J(\cdot, \mu)$, and, by uniqueness of the minimizer, we conclude that $X^{\mu} \wedge X^{\bar\mu} = X^{\mu}$; that is, $X_t^{\mu} \leqslant X_t^{\bar\mu}$ for each $t \in [0, T]$ $\mathbb{P}$-a.s., which implies that $R(\mu) \leqslant^L R(\bar\mu)$. □

Remark 4.1.13. *For later use, we point out that we have actually proved that for $\bar\mu, \mu \in L$ such that $\mu \leqslant^L \bar\mu$ we have that $X_t^{\mu} \leqslant X_t^{\bar\mu}$ for each $t \in [0, T]$, $\mathbb{P}$-a.s.*

4.1.5 Existence and approximation of MFG solutions

We finally obtain an existence result for the mean field game solutions.

Theorem 4.1.14. *Under the assumptions 4.1.2 and 4.1.9, the set of MFG solutions* $(\mathcal{M}, \leqslant^L)$ *is a nonempty complete lattice: in particular there exist a minimal and a maximal MFG solution.*

Proof. Combining Lemma 4.1.7 together with Lemma 4.1.12, we have that the best-response-map R is an increasing map from the complete lattice $(L, \leqslant^L)$ into itself. The statement then follows from Tarski's fixed point theorem (see Theorem 1 in [155]). □

Following [157], we introduce *learning procedures* $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$, $\{\overline{\mu}^n\}_{n\in\mathbb{N}} \subset L$ for the mean field game problem as follows:

- $\underline{\mu}^0 := \inf L$, $\overline{\mu}^0 := \sup L$;
- $\underline{\mu}^{n+1} = R(\underline{\mu}^n)$, $\overline{\mu}^{n+1} = R(\overline{\mu}^n)$ for each $n \geqslant 1$.

For simplicity, we make the following assumption. A discussion on the role of these conditions is postponed to Remark 4.1.19 below.

Assumption 4.1.15.

1. *The control set* $U \subset \mathbb{R}$ *is compact and there exists some* $p > 1$ *such that* $\mathbb{E}[|x_0|^p] < \infty$.
2. *The dynamics of the system given by* $b(t, x, u) = c_t + p_t x + q_t u$, *where* c_t, p_t *and* q_t *are deterministic and continuous in* t. *The volatility* σ *is constant.*
3. *For* $\mathbb{P} \otimes dt$*-a.a.* (ω, t) *in* $\Omega \times [0, T]$, *the cost functions* $f(t, \cdot, \cdot), g$ *are continuous in* (x, μ), *and the cost function* $l(t, \cdot, \cdot)$ *is convex and lower semi-continuous in* (x, u).
4. f, l *and* g *have subpolynomial growth; that is, there exists a constant* $C > 0$ *such that for all* $(\omega, t, x, u, \mu) \in \Omega \times [0, T] \times \mathbb{R} \times U \times [\mu^{\mathrm{Min}}, \mu^{\mathrm{Max}}]$,

$$|f(t, x, \mu)| + |l(t, x, u)| + |g(x, \mu)| \leqslant C(1 + |x|^p).$$

Remark 4.1.16. *Under Assumption 4.1.15 it can be easily verified that for each admissible control* u *the map* $t \mapsto \mathbb{P} \circ (X_t^u)^{-1}$ *is continuous in the weak topology.*

We then have the following convergence result.

Theorem 4.1.17. *Under Assumptions 4.1.2, 4.1.9 and 4.1.15 we have:*

(i) The sequence $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$ *is increasing in* $(L, \leqslant^L)$ *and it converges weakly to the minimal MFG solution,* π*-a.e.*

(ii) The sequence $\{\overline{\mu}^n\}_{n\in\mathbb{N}}$ *is decreasing in* $(L, \leqslant^L)$ *and it converges weakly to the maximal MFG solution,* π*-a.e.*

Proof. We only prove the first claim, since the second follows by analogous arguments.

By Lemma 4.1.12 the sequence $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$ is clearly increasing. Moreover, the completeness of the lattice L allows to define $\underline{\mu}^*$ as the least upper bound in the lattice $(L, \leqslant^L)$ of $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$, and, by Remark B.5 in Appendix B, the sequence $\underline{\mu}^n$ converges weakly to $\underline{\mu}^*$ π-a.e.

Define now, for each $n \geqslant 1$, the optimal pairs $(X^n, u^n) := (X^{\underline{\mu}^{n-1}}, u^{\underline{\mu}^{n-1}})$. Since the controls u^n take values in the compact set U, the processes X^n are pathwise equicontinuous and equibounded. Moreover, by Remark 4.1.13, the sequence $\{X^n\}_{n\in\mathbb{N}}$ is increasing. Therefore, by Arzelà-Ascoli's theorem, we can find an adapted process X such that X^n converges uniformly on $[0,T]$ to X, $\mathbb{P}$-a.s.

We now prove that $\underline{\mu}^*$ is a MFG solution. Since $\underline{\mu}^n_t = R(\underline{\mu}^{n-1})_t = \mathbb{P}\circ(X_t^{\underline{\mu}^{n-1}})^{-1} = \mathbb{P}\circ(X^n_t)^{-1}$ and since X^n converges uniformly to X $\mathbb{P}$-a.s. and $\underline{\mu}^n_t$ converges weakly to $\underline{\mu}^*_t$ for π-a.a. $t\in[0,T]$, we deduce that $\underline{\mu}^*_t = \mathbb{P}\circ X_t^{-1}$ for π-a.a $t\in[0,T]$. Hence, by the continuity of the map $t\mapsto \mathbb{P}\circ X_t^{-1}$ in the weak topology (see Remark 4.1.16), we can take $\mathbb{P}\circ X^{-1}$ as a continuous version of μ^*; that is, $\underline{\mu}^*_t = \mathbb{P}\circ X_t^{-1}$ for each $t\in[0,T]$. It remains to find an admissible control u such that $X = X^u$ and (X,u) is the optimal pair for $\underline{\mu}^*$.

In order to do so, thanks to the compactness of U, we invoke the Banach-Saks theorem to find a subsequence of indexes $\{n_j\}_{j\in\mathbb{N}}$ such that the Cesàro means of $\{u^{n_j}\}_{j\in\mathbb{N}}$ converge in L^2 to a process u. Up to a subsequence, we can assume that the convergence of the Cesàro means to the process u is pointwise; that is,

$$\beta^m_t := \frac{1}{m}\sum_{j=1}^m u^{n_j}_t \to u_t, \text{ as } m\to\infty, \ \mathbb{P}\otimes dt\text{-a.e.} \tag{4.1.23}$$

Moreover, observe that, by Assumption 4.1.15-(2), we have $X^{\beta^m} = \frac{1}{m}\sum_{j=1}^m X^{n_j}$. Hence, because we already know that X^{n_j} converges to X uniformly in $[0,T]$, $\mathbb{P}$-a.s. as $n_j\to\infty$, we deduce that X^{β^m} converges uniformly to X $\mathbb{P}$-a.s. as $m\to\infty$, and that

$$X_t = x_0 + \int_0^t (c_s + p_s X_s + q_s u_s)ds + \sigma W_t, \quad \forall\, t\in[0,T], \ \mathbb{P}\text{-a.s.};$$

that is, the process X is the solution to the dynamics controlled by u. Furthermore, by the subpolynomial growth of the costs, we have $-\infty < J(u, \underline{\mu}^*)$.

We now prove that the pair (X,u) is optimal for the flow $\underline{\mu}^*$. Observe that, for each admissible w and each $n_j\geqslant 1$, by the optimality of the pair (X^{n_j}, u^{n_j}) for the flow $\underline{\mu}^{n_j-1}$, we have

$$J(u^{n_j}, \underline{\mu}^{n_j-1}) \leqslant J(w, \underline{\mu}^{n_j-1}).$$

Summing over $j\leqslant m$, we write

$$\frac{1}{m}\sum_{j=1}^m \mathbb{E}\left[\int_0^T \left[f(t, X_t^{n_j}, \underline{\mu}_t^{n_j-1}) + l(t, X_t^{n_j}, u_t^{n_j})\right]dt + g(X_T^{n_j}, \underline{\mu}_T^{n_j-1})\right] \leqslant \frac{1}{m}\sum_{j=1}^m J(w, \underline{\mu}^{n_j-1}),$$

which, by convexity of l, in turn implies that

$$\mathbb{E}\bigg[\int_0^T l(t, X_t^{\beta^m}, \beta_t^m)dt\bigg] + \frac{1}{m}\sum_{j=1}^m \mathbb{E}\bigg[\int_0^T f(t, X_t^{n_j}, \underline{\mu}_t^{n_j-1})dt + g(X_T^{n_j}, \underline{\mu}_T^{n_j-1})\bigg] \leqslant \frac{1}{m}\sum_{j=1}^m J(w, \underline{\mu}^{n_j-1}). \tag{4.1.24}$$

By the compactness of U and the subpolynomial growth of l, the sequence $l(t, X_t^{\beta^m}, \beta_t^m)$ is clearly uniformly integrable with respect to the measure $\mathbb{P}\otimes dt$. Moreover, by the convergence of X^{β^m} and β^m, thanks to the lower semi-continuity of l, we obtain the pointwise limit

$$l(t, X_t, u_t) \leqslant \liminf_m l(t, X_t^{\beta^m}, \beta_t^m), \quad \mathbb{P}\otimes dt\text{-a.e.}$$

Therefore, we can take limits as $m \to \infty$ in the first expectation in (4.1.24) to find that

$$\mathbb{E}\bigg[\int_0^T l(t, X_t, u_t)dt\bigg] \leqslant \liminf_m \mathbb{E}\bigg[\int_0^T l(t, X_t^{\beta^m}, \beta_t^m)dt\bigg]. \tag{4.1.25}$$

Furthermore, by the convergence of X^n and of $\underline{\mu}^n$ and the continuity of the costs f and g, we can use the subpolynomial growth of f and g and the boundedness of the sequence μ^n (cf. Remark 4.1.6) to deduce that

$$\mathbb{E}\bigg[\int_0^T f(t, X_t, \underline{\mu}_t^*)dt + g(X_T, \underline{\mu}_T^*)\bigg] = \lim_m \frac{1}{m}\sum_{j=1}^m \mathbb{E}\bigg[\int_0^T f(t, X_t^{n_j}, \underline{\mu}_t^{n_j-1})dt + g(X_T^{n_j}, \underline{\mu}_T^{n_j-1})\bigg] \tag{4.1.26}$$

and that

$$J(w, \underline{\mu}^*) = \lim_m \frac{1}{m}\sum_{j=1}^m J(w, \underline{\mu}^{n_j-1}). \tag{4.1.27}$$

Finally, using (4.1.25), (4.1.26) and (4.1.27) in (4.1.24), we conclude that $J(u, \underline{\mu}^*) \leqslant J(w, \underline{\mu}^*)$, which, in turn, proves the optimality of (X, u) for $\underline{\mu}^*$, by arbitrariness of w. Hence, $\underline{\mu}^*$ is a MFG solution.

It only remains to prove the minimality of $\underline{\mu}^*$. Suppose that $\nu^* \in L$ is another MFG solution. By definition, $\inf L = \underline{\mu}^0 \leqslant^L \nu^*$. Since R is increasing, we have $\underline{\mu}^1 = R(\underline{\mu}^0) \leqslant^L R(\nu^*) = \nu^*$ and, by induction, we conclude that $\underline{\mu}^n \leqslant^L \nu^*$ for each $n \in \mathbb{N}$. This implies that the same inequality holds for the least upper bound of $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$; that is, $\underline{\mu}^* \leqslant^L \nu^*$, which completes the proof of the claim. □

4.1.6 Remarks and examples

In this subsection we collect some remarks and some examples concerning the previous theorems.

Remark 4.1.18. *In light of Theorem 4.1.17, a natural question is whether the minimal (resp. maximal) MFG solution is associated to the minimal expected cost. In fact, this relation does not hold in general (see Example 9 below). Nevertheless, it is easy to see that whenever $f(t,x,\cdot)$ and $g(x,\cdot)$ are increasing (resp. decreasing) in μ for each $(t,x) \in [0,T] \times \mathbb{R}$, the minimal (resp. maximal) solution leads to the minimal expected cost and can be approximated via the learning procedure above.*

We also mention that the study of possible relations between our minimal and maximal solutions and the equilibria obtained via the "zero-noise limit" and the "N-player game limit" approaches (see, e.g., [17, 18, 50, 67]) represents a challenging open question (see also Example 9 below).

Remark 4.1.19 (On Assumption 4.1.15)**.** *We point out that the linear-convex structure required in conditions (2) and (3) of Assumption 4.1.15 is crucial for our proof of Theorem 4.1.17. Indeed, the linear-convex structure is employed, together with a Banach-Sacks compactification argument, in order to characterize the limit points of the learning procedure as MFG solutions. In the next section, we extend Theorem 4.1.17 to a non-convex setting, by employing a weak formulation of the problem (see also Remark 4.2.7). Clearly, also the continuity of the costs f and g in the measure μ plays an essential role in the proof of Theorem 4.1.17. Alternatively, one could require the continuity of f and g with respect to a Wasserstein distance (see Remark 4.1.5).*

On the other hand, conditions (1) and (4) can be replaced by the growth condition (4.1.6) *(when $p' \geqslant 2$), unless to slightly extend some of the arguments. Also, if the a priori estimate* (4.1.7) *is satisfied, one can see that the continuity of f and g in the weak topology can be replaced by the continuity in the p-Wasserstein distance, where $p' > p \geqslant 1$ are as in Remark 4.1.4.*

Remark 4.1.20 (On the initialization of the learning procedure)**.** *Theorem 4.1.17 assumes a more concrete meaning observing that, according to Remark 4.1.8, the initial conditions of the learning procedure can be written in terms of the data of the problem. In particular, if U is compact, if b is bounded or if the growth condition* (4.1.6) *is satisfied (see also Remark 4.1.19),* (4.1.12) *and* (4.1.13) *provides an explicit expression for $\inf L$ and $\sup L$, respectively.*

Moreover, let μ be a generic flow of probabilities, which is not necessarily an element of L. Define the sequence $\mu^0 := \mu$ and $\mu^{n+1} := R(\mu^n)$ for $n \in \mathbb{N}$. Following the proof of Theorem 4.1.17 we see that, if $\mu^0 \leqslant^L R(\mu^0) = \mu^1$ (resp. $\mu^0 \geqslant^L R(\mu^0) = \mu^1$), then the sequence $\{\mu^n\}_{n\in\mathbb{N}}$ is increasing (resp. decreasing) in $(L, \leqslant^L)$ and it converges to a MFG equilibrium. In other words, if the learning procedure of Theorem 4.1.17 starts from an arbitrary element, then it converges to a MFG equilibrium whenever the first and the second element of the sequence are comparable. In particular, in order to approximate the minimal (resp. the maximal) MFG equilibrium, it is sufficient to start the learning procedure from a generic flow of measures μ^0 such that $\mu^0 \leqslant^L \inf L$ (resp. $\geqslant^L \sup L$).

Example 9. *We discuss here the setting studied in [67] in order to draw a connection between the solutions selected therein and our maximal and minimal solutions. Consider the case $U = \mathbb{R}$, $x_0 = 0$, $b(t,x,u) = cx + u$, $c \in \mathbb{R}$, σ constant, $f(t,x,\mu) = 0$,*

$l(t,x,u) = (x^2+u^2)/2$ and $g(x,\mu) = (x+\varphi(\langle id, \mu\rangle)^2/2$. Here φ is defined as

$$\varphi(y) := -\frac{y}{r_\delta}\mathbb{1}_{\{|y|\leqslant r_\delta\}} - sign(y)\mathbb{1}_{\{|y|>r_\delta\}},\ y\in\mathbb{R},\quad \delta\in(0,T),\quad r_\delta := \int_\delta^T w_s^{-2}ds,$$

with $w_t := \exp\Big[\int_t^T(-c+\eta_s)ds\Big]$, η solution to the Riccati equation $\frac{d\eta_t}{dt} = \eta_t^2 - 2c\eta_t - 1$, $\eta_T = 1$.

By the monotonicity of φ (see also Example 7), we can easily verify that g satisfies the Submodularity Assumption 4.1.9, while existence and uniqueness of optimal pairs is a consequence of the strict convexity of the costs, and of the linearity of b (we refer to [67] for more datails). Moreover, by the boundedness of φ, we have that $g(x,\mu)\leqslant x^2+1$. Hence, for any flow of measures μ we see that the optimal control u^μ must satisfy

$$\mathbb{E}\bigg[\int_0^T\frac{(u_t^\mu)^2}{2}dt\bigg]\leqslant J(u^\mu,\mu)\leqslant J(0,\mu)\leqslant 1+\mathbb{E}\bigg[\int_0^T\frac{(X_t^0)^2}{2}dt+(X_T^0)^2\bigg]<\infty,$$

where 0 denotes the control constantly equal to zero. From the latter estimate, and a standard use of Grönwall's inequality, we deduce that (4.1.7) is satisfied with $p'=2$. All the requirements of Theorem 4.1.14 are then fulfilled. Moreover, the proof of Theorem 4.1.17 can be easily modified to fit the example under consideration (see also Remark 4.1.19). Therefore, the set of MFG solution is a nonempty complete lattice, and the minimal and maximal MFG solutions can be selected by the learning procedure introduced in the previous subsection.

It is shown in [67] that the set of MFG solutions $\mathcal{M}$ has exactly three elements, namely $\mathcal{M} = \{\mu^{-1},\mu^0,\mu^1\}$, satisfying

$$\langle id, \mu_t^A\rangle := A\,w_t\int_0^t w_s^{-2}ds,\quad \textit{for each } t\in[0,T],\quad A\in\{-1,0,1\}. \tag{4.1.28}$$

Since $w>0$, we immediately see that $\langle id,\mu_t^{-1}\rangle < \langle id,\mu_t^0\rangle < \langle id,\mu_t^1\rangle$ for each $t\in[0,T]$, which can happen only if $\mu^{-1}\leqslant^L\mu^0\leqslant^L\mu^1$. We finally draw a connection between the solutions selected in [67] and our maximal and minimal solutions, recalling from [67] the following facts:

- *The equilibrium with minimal cost is μ^0.*
- *The "zero-noise limit" and the "N-player game limit" select a randomized equilibrium, given by a combination of the maximal and the minimal MFG solution, both with probability $1/2$; that is, with law $\frac{1}{2}\delta_{\mu^{-1}}+\frac{1}{2}\delta_{\mu^1}$.*

4.2 Relaxed submodular mean field games

In this section we aim at allowing for multiple solutions of the individual optimization problem, and at overcoming the linear-convex setting in the convergence result. This comes with the price of pushing the analysis to a more technical level, by working with a *weak formulation* of the problem and with the so-called *relaxed controls*. The use of relaxed controls in mean field games (and in control problems) allows to gain compactness on the set of admissible controls and, for this reason, this technique has been widely used in this field (see [83, 120], among many others).

4.2.1 The relaxed mean field game

Let b, σ, f, l, g, U be given as in Section 4.1 (see (4.1.2) and (4.1.3)), with the additional assumption that b, f, l, g are deterministic and, for simplicity, that σ is constant. Let $\mathcal{C}$ denote the set of continuous functions on $[0,T]$. In view of a weak formulation of the problem, the initial value of the dynamics will be described through an initial fixed probability distribution $\nu_0 \in \mathcal{P}(\mathbb{R})$.

Let Λ denote the set of *deterministic relaxed controls* on $[0,T] \times U$; that is, the set of positive measures λ on $[0,T] \times U$ such that $\lambda([s,t] \times U) = t - s$ for all $s, t \in [0,T]$ with $s < t$.

Definition 7. *A tuple $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \lambda)$ is said to be an admissible relaxed control if*

1. *$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions;*
2. *x_0 is an $\mathcal{F}_0$-measurable $\mathbb{R}$-valued random variable (r.v.) such that $\mathbb{P} \circ x_0^{-1} = \nu_0$;*
3. *W is a standard $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$-Brownian motion;*
4. *λ is a Λ-valued r.v. defined on Ω such that $\sigma\{\lambda([0,t] \times E) \,|\, E \in \mathcal{B}(U)\} \subset \mathcal{F}_t$, $\forall t \in [0,T]$.*

We denote by $\widetilde{\mathcal{U}}$ the set of admissible relaxed controls.

The set of admissible ordinary controls is naturally included in the set of relaxed controls via the map $u \mapsto \lambda^u(dt, du) := \delta_{u_t}(du)dt$. Any admissible relaxed control $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \lambda) \in \widetilde{\mathcal{U}}$, on the other hand, can be factorized in that one can find an adapted process $\lambda \colon \Omega \times [0,T] \to \mathcal{P}(U)$ such that $\lambda(dt, du) = \lambda_t(du)dt$ $\mathbb{P}$-almost surely.

Furthermore, since b is assumed to satisfy the usual Lipschitz continuity and growth conditions, there exists a unique process $X^\rho : \Omega \times [0,T] \to \mathbb{R}$, solving the system's dynamics equation that now reads as

$$X_t^\rho = x_0 + \int_0^t \int_U b(t, X_t^\rho, u)\lambda_t(du)dt + \sigma W_t, \;\; t \in [0,T]. \tag{4.2.1}$$

Then, for a measurable flow of probability measures μ, we define the cost functional

$$J(\rho, \mu) := \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T \int_U \Big[f(t, X_t^\rho, \mu_t) + l(t, X_t^\rho, u) \Big] \lambda_t(du)dt + g(X_T^\rho, \mu_T) \bigg], \quad \rho \in \widetilde{\mathcal{U}},$$

and we say that $\rho \in \widetilde{\mathcal{U}}$ is an *optimal relaxed control* for the flow of measures μ if it solves the optimal control problem related to μ; that is, if $-\infty < J(\rho, \mu) = \inf J(\cdot, \mu)$.

We now make the following assumptions, which will be employed in the existence result of Theorem 4.2.6.

Assumption 4.2.1.

1. *The control space U is compact.*

2. *The costs* $f(t,\cdot,\mu)$*,* $l(t,\cdot,\cdot)$ *and* $g(\cdot,\mu)$ *are lower semi-continuous in* (x,u) *for each* $(t,\mu)\in[0,T]\times\mathcal{P}(\mathbb{R})$*.*

3. *There exist exponents* $p'>p\geqslant 1$ *and a constant* $K>0$ *such that*

$$|\nu_0|^{p'} := \int_{\mathbb{R}} |y|^{p'} d\nu_0(y) < \infty$$

and such that, for all $(t,x,\mu,u)\in[0,T]\times\mathbb{R}\times\mathcal{P}(\mathbb{R})\times U$*,*

$$\begin{aligned}&|g(x,\mu)| \leqslant K(1+|x|^p+|\mu|^p),\\&|f(t,x,\mu)|+|l(t,x,u)| \leqslant K(1+|x|^p+|\mu|^p),\end{aligned}$$

where $|\mu|^p=\int_{\mathbb{R}}|y|^p d\mu(y)$*.*

4. f *and* g *satisfy the Submodularity Assumption 4.1.9.*

Remark 4.2.2. *Alternatively, as discussed also in Remark 4.1.4, we can replace (1) in Assumption 4.2.1 by requiring* U *to be closed and the growth condition* (4.1.6) *to be satisfied.*

Remark 4.2.3. *Under Assumption 4.2.1, it is well-known that for each measurable flow* μ*,* $\arg\min J(\cdot,\mu)$ *is nonempty. This can be proved using the so-called "compactification-method" (see e.g. [140] and [95], among others). For later use, we now sketch the main argument. Let* $\{\rho_n\}_{n\in\mathbb{N}}$ *be a minimizing sequence for* $J(\cdot,\mu)$*, with*

$$\rho_n=(\Omega^n,\mathcal{F}^n,\mathbb{F}^n,\mathbb{P}^n,x_0^n,W^n,\lambda^n).$$

Then, since U *is compact, thanks to the growth conditions on* b*, the sequence* $\mathbb{P}^n\circ(x_0^n,W^n,\lambda^n,X^{\rho_n})^{-1}$ *is tight in* $\mathcal{P}(\mathbb{R}\times\mathcal{C}\times\Lambda\times\mathcal{C})$*, so that, up to a subsequence,* $\mathbb{P}^n\circ(x_0^n,W^n,\lambda^n,X^{\rho_n})^{-1}$ *converges weakly to a probability measure* $\bar{\mathbb{P}}\in\mathcal{P}(\mathbb{R}\times\mathcal{C}\times\Lambda\times\mathcal{C})$*. Moreover, through a Skorokhod representation argument, we can find an admissible relaxed control*

$$\rho_*=(\Omega_*,\mathcal{F}_*,\mathbb{F}_*,\mathbb{P}_*,x_0^*,W_*,\lambda_*)$$

such that $\bar{\mathbb{P}}=\mathbb{P}_*\circ(x_0^*,W_*,\lambda_*,X^{\rho_*})^{-1}$*. Finally, the continuity assumptions on the costs together with their polynomial growth, allows to conclude that*

$$J(\rho_*,\mu)\leqslant\liminf_n J(\rho_n,\mu)=\inf J(\cdot,\mu);$$

i.e., $\rho_*\in\arg\min J(\cdot,\mu)$*. In particular, this argument shows that for any sequence* $\{\rho_n\}_{n\in\mathbb{N}}\subset\arg\min J(\cdot,\mu)$ *we can find an admissible relaxed control*

$$\rho_*=(\Omega_*,\mathcal{F}_*,\mathbb{F}_*,\mathbb{P}_*,x_0^*,W_*,\lambda_*)\in\arg\min J(\cdot,\mu)$$

such that, up to a subsequence, $\mathbb{P}^n\circ(X^{\rho_n})^{-1}$ *converges weakly to* $\mathbb{P}_*\circ(X^{\rho_*})^{-1}$ *in* $\mathcal{P}(\mathcal{C})$*.*

The compactness of U and (4.1.2) immediately imply that there exists a constant $M > 0$ such that

$$\mathbb{E}^{\mathbb{P}}[|X_t^\rho|^{p'}] \leqslant M, \quad \text{for any } t \in [0,T], \text{ and } \rho \in \widetilde{\mathcal{U}}.$$

Hence, Lemma B.2 in the Appendix B allows to find μ^{Min}, $\mu^{\text{Max}} \in \mathcal{P}(\mathbb{R})$ with

$$\mu^{\text{Min}} \leqslant^{\text{st}} \mathbb{P} \circ (X_t^\rho)^{-1} \leqslant^{\text{st}} \mu^{\text{Max}}, \quad \text{for any } t \in [0,T], \text{ and } \rho \in \widetilde{\mathcal{U}}.$$

Moreover, as it is shown in Remark 4.1.6, we have uniform boundedness of the moments

$$\sup_{\mu \in [\mu^{\text{Min}}, \mu^{\text{Max}}]} |\mu|^q < \infty, \quad q < p'. \tag{4.2.2}$$

Next, define the set of feasible flows of measures L as the set of all equivalence classes (w.r.t. $\pi := \delta_0 + dt + \delta_T$) of measurable flows $(\mu_t)_{t\in[0,T]}$ with $\mu_t \in [\mu^{\text{Min}}, \mu^{\text{Max}}]$ for π-almost all $t \in (0,T]$ and $\mu_0 = \nu_0$. Let 2^L be the set of all subsets of L, and define the best-response-correspondence $\mathcal{R} : L \to 2^L$ by

$$\mathcal{R}(\mu) := \left\{ \mathbb{P} \circ (X^\rho)^{-1} \,\middle|\, \rho \in \arg\min J(\cdot, \mu) \right\} \subset L, \quad \mu \in L. \tag{4.2.3}$$

We can then give the following definition.

Definition 8. *The flow of measures μ^* is a relaxed mean field game solution if $\mu^* \in \mathcal{R}(\mu^*)$.*

4.2.2 Existence and approximation of relaxed MFG solutions

We now move on to proving the existence and approximation of relaxed mean field game solutions. In order to keep a self-contained but concise analysis, the proofs of the subsequent results will be only sketched whenever their arguments follow along the same lines of those employed in the proofs of Section 4.1.

We begin by proving some monotonicity properties for the sets of best-responses.

Lemma 4.2.4. *Under Assumption 4.2.1, the best-response-correspondence satisfies the following:*

(i) For all $\mu \in L$, we have that $\inf \mathcal{R}(\mu)$, $\sup \mathcal{R}(\mu) \in \mathcal{R}(\mu)$.

(ii) $\inf \mathcal{R}(\mu) \leqslant^L \inf \mathcal{R}(\overline{\mu})$ *and* $\sup \mathcal{R}(\mu) \leqslant^L \sup \mathcal{R}(\overline{\mu})$ *for all $\mu, \overline{\mu} \in L$ with $\mu \leqslant^L \overline{\mu}$.*

Proof. We prove the two claims separately.

Proof of (i). Take $\mu \in L$. In order to show that $\inf \mathcal{R}(\mu) \in \mathcal{R}(\mu)$, we recall that, as it is shown in the proof of Lemma B.4 in the Appendix B, we can select a sequence of relaxed controls $\{\rho_n\}_{n\in\mathbb{N}} \subset \arg\min J(\cdot,\mu)$ such that $\inf\{\mathbb{P}^n \circ X^{\rho_n} | \, n \in \mathbb{N}\} = \inf \mathcal{R}(\mu)$.

Without loss of generality, we can assume that the relaxed controls ρ_n are defined on the same stochastic basis; that is, $\rho^n = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \lambda^n)$ for each $n \in \mathbb{N}$. Indeed, we can choose

$$\Omega := \mathbb{R} \times \mathcal{C} \times \Lambda^{\mathbb{N}}$$

as sample space and take x_0, W, λ^n, $n \in \mathbb{N}$, as the canonical projections. Let $\hat{\mathcal{F}}$ be the Borel σ-algebra on Ω (w.r.t. the product topology), and let $\hat{\mathbb{F}}$ be the natural filtration induced by x_0, W, λ^n, $n \in \mathbb{N}$; that is, $\hat{\mathbb{F}}_t := \sigma(x_0, W(s), \lambda^n(C) : s \in [0,t], C \in \mathcal{B}([0,t] \times U), n \in \mathbb{N})$, $t \in [0,T]$. Thus, W corresponds to a continuous real-valued $\hat{\mathbb{F}}$-adapted process, while λ^n can be identified with a $\mathcal{P}(U)$-valued $\hat{\mathbb{F}}$-predictable process (see, for instance, Lemma 3.2 in [120]). Recall that ν_0 denotes the common initial distribution. Let γ denote standard Wiener measure on $\mathcal{B}(\mathcal{C})$. If $\bar{\rho}^n = (\Omega_n, \mathcal{F}_n, \mathbb{F}^n, \mathbb{P}_n, x_0^n, W^n, \bar{\lambda}^n)$, $n \in \mathbb{N}$, are stochastic relaxed controls with $\mathbb{P}_n \circ (x_0^n)^{-1} = \nu_0$, hence $\mathbb{P}_n \circ (x_0^n, W^n)^{-1} = \nu_0 \otimes \gamma$, then let Q_n denote the Markov kernel from $\mathbb{R} \times \mathcal{C}$ to Λ that corresponds to (a version of) the regular conditional distribution of $\bar{\lambda}^n$ given (x_0^n, W^n). Let $\mathbb{P}$ be the probability measure on $\hat{\mathcal{F}}$ determined by

$$\mathbb{P}\left(\{x_0 \in B_0\} \cap \{W \in B\} \cap \bigcap_{i \in I} \{\lambda^i \in C_i\}\right) := \int_{B_0 \times B} \left(\prod_{i \in I} Q_i(x, w; C_i)\right) \nu_0 \otimes \gamma(dx, dw)$$

for any choice of $B_0 \in \mathcal{B}(\mathbb{R})$, $B \in \mathcal{B}(\mathcal{C})$, $I \subset \mathbb{N}$ a finite subset, and $C_i \in \mathcal{B}(\Lambda)$, $i \in I$. Then $\mathbb{P} \circ (x_0, W, \lambda^n)^{-1} = \mathbb{P}_n \circ (x_0^n, W^n, \bar{\lambda}^n)^{-1}$ for all $n \in \mathbb{N}$. As last step, define $\mathcal{F}$ to be the $\mathbb{P}$-completion of $\hat{\mathcal{F}}$, and let $\mathbb{F}$ be the right-continuous $\mathbb{P}$-augmentation of $\hat{\mathbb{F}}$.

We will now employ an inductive scheme. Let ρ^1, ρ^2 be the first two elements of the sequence $\{\rho_n\}_{n \in \mathbb{N}}$. As in Lemma 4.1.11, we can define two Λ-valued r.v.'s $\lambda^\vee$ and $\lambda^\wedge$ and two admissible relaxed controls $\rho^\vee = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \lambda^\vee)$ and $\rho^\wedge = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \lambda^\wedge)$ such that $X^{\rho_1} \vee X^{\rho_2} = X^{\rho^\vee}$ and $X^{\rho_1} \wedge X^{\rho_2} = X^{\rho^\wedge}$. In fact, defining the (random) intervals

$$\begin{cases} B_s^1 := \{X_s^{\rho_1} < X_s^{\rho_2}\} \cup \{X_s^{\rho_1} = X_s^{\rho_2},\ \int_U b(s, X_s^{\rho_1}, u)\lambda_s^1(du) < \int_U b(s, X_s^{\rho_2}, u)\lambda_s^2(du)\}, \\ B_s^2 := \{X_s^{\rho_1} > X_s^{\rho_2}\} \cup \{X_s^{\rho_1} = X_s^{\rho_2},\ \int_U b(s, X_s^{\rho_1}, u)\lambda_s^1(du) \geqslant \int_U b(s, X_s^{\rho_2}, u)\lambda_s^2(du)\}, \end{cases}$$

we have

$$\lambda_s^\wedge := \begin{cases} \lambda_s^1 & \text{on } B_s^1, \\ \lambda_s^2 & \text{on } B_s^2, \end{cases}$$

where $\lambda^1(ds, du) = \lambda_s^1(du)ds$, $\lambda^2(ds, du) = \lambda_s^2(du)ds$, and $\lambda^\wedge(ds, du) := \lambda_s^\wedge(du)ds$. The definition of $\lambda^\vee$ is analogous. Repeating the same arguments which lead to (4.1.21) in the proof of Lemma 4.1.12, we see that

$$0 \leqslant J(\rho^\vee, \mu) - J(\rho_1, \mu) = J(\rho_2, \mu) - J(\rho^\wedge, \mu) = 0,$$

which implies that $\mathbb{P} \circ (X^{\rho^\wedge})^{-1} = \mathbb{P} \circ (X^{\rho_1} \wedge X^{\rho_2})^{-1} \in \mathcal{R}(\mu)$. Moreover, since $X^{\rho_1} \wedge X^{\rho_2} = X^{\rho^\wedge}$, we obviously have $\mathbb{P} \circ (X^{\rho^\wedge})^{-1} \leqslant^L \mathbb{P} \circ (X^{\rho^1})^{-1} \wedge^L \mathbb{P} \circ (X^{\rho^2})^{-1}$. Repeating this construction inductively, for each $n \in \mathbb{N}$ we find an admissible relaxed control $\rho^{\wedge n} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \lambda^{\wedge n})$ such that $\mathbb{P} \circ (X^{\rho^{\wedge n}})^{-1} \in \mathcal{R}(\mu)$ and $\mathbb{P} \circ (X^{\rho^{\wedge n}})^{-1} \leqslant^L \mathbb{P} \circ (X^{\rho^1} \wedge^L ... \wedge^L \mathbb{P} \circ X^{\rho^n})^{-1}$. Furthermore, the sequence $\mathbb{P} \circ (X^{\rho^{\wedge n}})^{-1}$ is decreasing in L, since for each n we have $X_t^{\rho^{\wedge n}} = X_t^1 \wedge ... \wedge X_t^n \leqslant X_t^1 \wedge ... \wedge X_t^{n-1}$ for each $t \in [0,T]$ $\mathbb{P}$-a.s. Hence,

$$\inf \mathcal{R}(\mu) = \inf \left\{ \mathbb{P} \circ (X^{\rho_n})^{-1} \,\middle|\, n \in \mathbb{N} \right\} = \inf \left\{ \mathbb{P} \circ (X^{\rho^{\wedge n}})^{-1} \,\middle|\, n \in \mathbb{N} \right\},$$

which implies that the sequence $\mathbb{P}\circ(X^{\rho^{\wedge n}})^{-1}$ converges weakly to $\inf\mathcal{R}(\mu)$, π-a.e. Since $\left\{\mathbb{P}\circ(X^{\rho^{\wedge n}})^{-1}\right\}_{n\in\mathbb{N}}\subset\mathcal{R}(\mu)$, by the closure property of $\mathcal{R}(\mu)$ (see Remark 4.2.3), we conclude that $\inf\mathcal{R}(\mu)\in\mathcal{R}(\mu)$.

Analogously, it can be shown that $\sup\mathcal{R}(\mu)\in\mathcal{R}(\mu)$.

Proof of (ii). Let $\mu,\bar{\mu}\in L$ with $\mu\leqslant^L\bar{\mu}$ and $\rho,\bar{\rho}\in\widetilde{\mathcal{U}}$ with $\rho\in\arg\min J(\cdot,\mu)$ and $\bar{\rho}\in\arg\min J(\cdot,\bar{\mu})$. As in the proof of claim (i), we may assume that ρ and $\bar{\rho}$ are defined on the same stochastic basis; that is, $\rho=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},x_0,W,\lambda)$ and $\rho=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},x_0,W,\bar{\lambda})$. As above, we can then define two Λ-valued r.v.'s $\lambda^\vee$ and $\lambda^\wedge$ and two admissible relaxed controls $\rho^\vee=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},x_0,W,\lambda^\vee)$ and $\rho^\wedge=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},x_0,W,\lambda^\wedge)$ such that $X^\rho\vee X^{\bar{\rho}}=X^{\rho^\vee}$ and $X^\rho\wedge X^{\bar{\rho}}=X^{\rho^\wedge}$.

Repeating the arguments which lead to (4.1.22) in the proof of Lemma 4.1.12, we exploit the submodularity of the costs and the definitions of $\lambda^\vee$ and $\lambda^\wedge$ to find

$$0\leqslant J(\rho^\vee,\bar{\mu})-J(\bar{\rho},\bar{\mu})\leqslant J(\rho^\vee,\mu)-J(\bar{\rho},\mu)=J(\rho,\mu)-J(\rho^\wedge,\mu)\leqslant 0, \tag{4.2.4}$$

where the first and the last inequality hold because of the optimality of ρ and $\bar{\rho}$.

By claim (i), we have that $\sup\mathcal{R}(\mu)\in\mathcal{R}(\mu)$ and $\sup\mathcal{R}(\bar{\mu})\in\mathcal{R}(\bar{\mu})$, therefore, we can choose ρ and $\bar{\rho}$ such that $\mathbb{P}\circ(X^{\bar{\rho}})^{-1}=\sup\mathcal{R}(\bar{\mu})$ and $\mathbb{P}\circ(X^{\rho})^{-1}=\sup\mathcal{R}(\mu)$. From (4.2.4) we see that $\rho^\vee\in\arg\min J(\cdot,\bar{\mu})$, which implies that $\mathbb{P}\circ(X^{\rho^\vee})^{-1}\leqslant^L\sup\mathcal{R}(\bar{\mu})$. This, by construction of $\rho^\vee$, in turn implies that

$$\sup\mathcal{R}(\mu)=\mathbb{P}\circ(X^\rho)^{-1}\leqslant^L\mathbb{P}\circ(X^\rho)^{-1}\vee^L\mathbb{P}\circ(X^\rho)^{-1}\leqslant^L\mathbb{P}\circ(X^{\rho^\vee})^{-1}\leqslant^L\sup\mathcal{R}(\bar{\mu});$$

that is, $\sup\mathcal{R}(\mu)\leqslant^L\sup\mathcal{R}(\bar{\mu})$. In the same way, choosing ρ and $\bar{\rho}$ such that $\mathbb{P}\circ(X^{\bar{\rho}})^{-1}=\inf\mathcal{R}(\bar{\mu})$ and $\mathbb{P}\circ(X^\rho)^{-1}=\inf\mathcal{R}(\mu)$ we conclude that $\inf\mathcal{R}(\mu)\leqslant^L\inf\mathcal{R}(\bar{\mu})$. □

Remark 4.2.5. *For later use, we point out that that the crucial point in the proof the previous lemma is in showing the following statement: For*

$$\rho_i=(\Omega^i,\mathcal{F}^i,\mathbb{F}^i,\mathbb{P}^i,x_0^i,W^i,\lambda^i)\in\widetilde{\mathcal{U}},\quad i=1,2,$$

there exist admissible relaxed controls $\rho^\wedge=(\Omega^\wedge,\mathcal{F}^\wedge,\mathbb{F}^\wedge,\mathbb{P}^\wedge,x_0^\wedge,W^\wedge,\lambda^\wedge)$ *and* $\rho^\vee=(\Omega^\vee,\mathcal{F}^\vee,\mathbb{F}^\vee,\mathbb{P}^\vee,x_0^\vee,W^\vee,\lambda^\vee)$ *such that:*

1. $\mathbb{P}^\wedge\circ(X^{\rho^\wedge})^{-1}\leqslant^L\mathbb{P}^1\circ(X^{\rho_1})^{-1}\wedge^L\mathbb{P}^2\circ(X^{\rho_2})^{-1}$;
2. $\mathbb{P}^1\circ(X^{\rho_1})^{-1}\vee^L\mathbb{P}^2\circ(X^{\rho_2})^{-1}\leqslant^L\mathbb{P}^\vee\circ(X^{\rho^\vee})^{-1}$;
3. $J(\rho^\wedge,\mu)+J(\rho^\vee,\mu)=J(\rho_1,\mu)+J(\rho_2,\mu)$, *for any* $\mu\in L$.

This, together with the submodularity of the costs, in turn implies that, for μ^1, $\mu^2\in L$ *such that* $\mu^1\leqslant^L\mu^2$, *one has*

$$J(\rho^\vee,\mu^2)-J(\rho_2,\mu^2)\leqslant J(\rho_1,\mu^1)-J(\rho^\wedge,\mu^1),$$

from which the monotonicity of $\inf\mathcal{R}$ *and* $\sup\mathcal{R}$ *can be derived.*

We can finally state the main results of this section.

Theorem 4.2.6. *Under Assumption 4.2.1, we have that*

(i) The set of mean field game solutions $\mathcal{M}$ is nonempty and admits a minimal and a maximal element.

Assume moreover that the costs $f(t,\cdot,\cdot)$ and $g(\cdot,\cdot)$ are continuous in (x,μ). Then

(ii) For $\underline{\mu}^0 := \inf L$ and $\underline{\mu}^n := \inf \mathcal{R}(\underline{\mu}^{n-1})$ for $n \in \mathbb{N}$, we have that the learning procedure $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$ is increasing and it converges weakly to $\inf \mathcal{M}$, π-a.e.

(iii) For $\overline{\mu}^0 := \sup L$ and $\overline{\mu}^n := \sup \mathcal{R}(\overline{\mu}^{n-1})$ for $n \in \mathbb{N}$, we have that the learning procedure $\{\overline{\mu}^n\}_{n\in\mathbb{N}}$ is decreasing and it converges weakly to $\sup \mathcal{M}$, π-a.e.

Proof. Claim (i) follows from Lemma 4.2.4 and Theorem 4.1 in [159].

We only prove (ii), since the proof of (iii) is similar. By Lemma 4.2.4 the sequence $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$ is increasing, hence it converges weakly to its least upper bound $\underline{\mu}_*$, π-a.e. For each $n \in \mathbb{N}$, let $\rho^n = (\Omega^n, \mathcal{F}^n, \mathbb{F}^n, \mathbb{P}^n, x_0^n, W^n, \lambda^n)$ be an admissible relaxed control such that $\mathbb{P}^n \circ (X^{\rho^n})^{-1} = \inf \mathcal{R}(\underline{\mu}^{n-1})$. As in Remark 4.2.3, the sequence $\{\mathbb{P} \circ (x_0^n, W^n, \lambda^n, X^{\rho^n})^{-1}\}_{n\in\mathbb{N}}$ is tight, so that, up to a subsequence, we can assume that the sequence $\mathbb{P}^n \circ (x_0, W, \lambda^n, X^{\rho^n})^{-1}$ converges weakly to a probability measure $\bar{\mathbb{P}} \in \mathcal{P}(\mathbb{R} \times \mathcal{C} \times \Lambda \times \mathcal{C})$. Moreover, we can find an admissible relaxed control $\rho_* = (\Omega_*, \mathcal{F}_*, \mathbb{F}_*, \mathbb{P}_*, x_0^*, W_*, \lambda_*)$ such that $\bar{\mathbb{P}} = \mathbb{P}_* \circ (x_0^*, W_*, \lambda_*, X^{\rho_*})^{-1}$, and this implies that $\mu_* = \mathbb{P}_* \circ (X^{\rho_*})^{-1}$.

By the optimality of ρ^n for the flow of measures $\underline{\mu}^{n-1}$, we have

$$J(\rho^n, \underline{\mu}^{n-1}) \leqslant J(\rho, \underline{\mu}^{n-1}), \ \forall \rho \in \widetilde{\mathcal{U}}. \tag{4.2.5}$$

Now, the continuity of the costs f, l and g, together with their polynomial growth and the uniform integrability condition (4.2.2), allow to show the continuity of the functional J along the sequences $\{\rho^n, \underline{\mu}^{n-1}\}_{n\in\mathbb{N}}$ and $\{\rho, \underline{\mu}^{n-1}\}_{n\in\mathbb{N}}$. This in turn enables us to take limits as $n \to \infty$ in (4.2.5) and to deduce that $J(\rho_*, \underline{\mu}_*) \leqslant J(\rho, \underline{\mu}_*)$ for each $\rho \in \widetilde{\mathcal{U}}$. Hence, X^{ρ_*} is an optimal trajectory for the flow $\underline{\mu}_*$ and, since $\underline{\mu}_* = \mathbb{P}_* \circ (X^{\rho_*})^{-1}$, we have $\underline{\mu}_* \in \mathcal{R}(\underline{\mu}_*)$; that is, $\underline{\mu}_*$ is a mean field game solution.

It remains to show that $\underline{\mu}_* = \inf \mathcal{M}$. Let $\nu \in \mathcal{M}$. By definition, we have $\underline{\mu}^0 = \inf L \leqslant^L \nu$. Since $\inf \mathcal{R}$ is increasing by (ii) in Lemma 4.2.4, $\underline{\mu}^1 = \inf \mathcal{R}(\underline{\mu}^0) \leqslant^L \inf \mathcal{R}(\nu) \leqslant^L \nu$, where the last inequality follows from $\nu \in \mathcal{R}(\nu)$. By induction, we deduce that $\underline{\mu}^n \leqslant^L \nu$ for each $n \in \mathbb{N}$. Recalling that $\underline{\mu}_* = \sup\{\underline{\mu}^n | n \in \mathbb{N}\}$, we conclude that $\underline{\mu}_* \leqslant^L \nu$, which completes the proof. □

Remark 4.2.7. *Notice that the role of the compactification through the problem's weak formulation and the use of relaxed controls is twofold. On the one hand, it ensures that the sets of best-responses $\mathcal{R}(\cdot)$ admit minimal and maximal elements, which is essential for our arguments in the case in which $\mathcal{R}(\cdot)$ are not singletons. On the other hand, regarding the convergence of the learning procedure, it replaces the compactification via Banach-Saks' theorem used in the proof of Theorem 4.1.17, for which the additional linear-convex structure (enforced in Assumption 4.1.15) is necessary (see also Remark 4.1.19).*

4.3 Submodular mean field games with singular controls

We now present some results on existence and approximation of solutions for mean field games with singular controls. While many of the techniques developed in the Sections 4.1 and 4.2 work unchanged also in this new setting, some of the arguments need to be adapted exploiting the density of Lipschitz processes in the set of singular controls.

The results in this section will be presented under the weak formulation of the problem (in the spirit of Section 4.2), similar statements will be discussed for the strong formulation (i.e., in the spirit of Section 4.1) in Remark 4.3.6 and in Subsection 4.4.2.

4.3.1 Model formulation

Let b, σ, f, g be given as in Section 4.1 (see (4.1.2) and (4.1.3)), with b independent from u. Assume, in addition, that b, f, g are deterministic and, for simplicity, that σ is constant. Consider, moreover, a deterministic measurable function $c : [0,T] \to \mathbb{R}$. In view of a weak formulation of the problem, the initial value of the dynamics will be described through an initial fixed probability distribution $\nu_0 \in \mathcal{P}(\mathbb{R})$.

Definition 9. *A tuple $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi)$ is said to be an admissible singular control if*

1. *$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions;*
2. *x_0 is an $\mathcal{F}_0$-measurable $\mathbb{R}$-valued random variable such that $\mathbb{P} \circ x_0^{-1} = \nu_0$;*
3. *W is a standard $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$-Brownian motion;*
4. *$\xi : \Omega \times [0,T] \to [0,\infty)$ is an $\mathbb{F}$-adapted nondecreasing càdlàg process.*

We denote by $\mathcal{A}$ the set of admissible singular controls.

Again, since b is assumed to satisfy the usual Lipschitz continuity and growth conditions, for any $\rho \in \mathcal{A}$ there exists a unique process $X^\rho : \Omega \times [0,T] \to \mathbb{R}$, solving the system's dynamics equation, that now reads as

$$X_t^\rho = x_0 + \int_0^t b(t, X_t^\rho) dt + \sigma W_t + \xi_t, \quad t \in [0,T]. \tag{4.3.1}$$

Then, for a measurable flow of probability measures μ, we define the cost functional

$$J(\rho, \mu) := \mathbb{E}^{\mathbb{P}} \bigg[\int_0^T f(t, X_t^\rho, \mu_t) dt + g(X_T^\rho, \mu_T) + \int_{[0,T]} c_t d\xi_t \bigg], \quad \rho \in \mathcal{A},$$

and we say that $\rho \in \mathcal{A}$ is an *optimal singular control* for the flow of measures μ if it solves the optimal control problem related to μ; that is, if $-\infty < J(\rho, \mu) = \inf_{\mathcal{A}} J(\cdot, \mu)$.

The data of the problem are subject to the following requirements.

Assumption 4.3.1.

1. *c is nonincreasing and continuously differentiable, with $c \geqslant 0$.*
2. *The costs $f(t,\cdot,\mu)$ and $g(\cdot,\mu)$ are lower semi-continuous in x for each $(t,\mu) \in [0,T] \times \mathcal{P}(\mathbb{R})$.*
3. *There exist an exponent $p > 1$ and constants $K, \kappa > 0$ such that*
$$|\nu_0|^p := \int_{\mathbb{R}} |y|^p d\nu_0(y) < \infty$$
and such that, for all $(t,x,\mu) \in [0,T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R})$,
$$\kappa(|x|^p - 1) \leqslant g(x,\mu) \leqslant K(1 + |x|^p),$$
$$\kappa(|x|^p - 1) \leqslant f(t,x,\mu) \leqslant K(1 + |x|^p).$$
4. *f and g satisfy the Submodularity Assumption 4.1.9.*

Remark 4.3.2 (A priori estimates)**.** *Let $\bar{\rho}_0 = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{x}_0, \bar{W}, \bar{\xi})$ be an admissible singular control such that $\bar{\xi}_t = 0$ for any $t \in [0,T]$, $\bar{\mathbb{P}}$-a.s. Under Assumption 4.3.1, it can be shown that there exists a constant $M > 0$ such that, for any measurable flow of probabilities μ and each singular control $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi) \in \mathcal{A}$ such that $J(\rho,\mu) \leqslant J(\bar{\rho}_0,\mu)$, one has*

$$\mathbb{E}^{\mathbb{P}}[|X_t^\rho|^p] < M, \text{ for each } t \in [0,T]. \tag{4.3.2}$$

Indeed, using the growth conditions of f and g in Assumption 4.3.1 and arguing as in the proof of Lemma 2.2.6 in Chapter 2, we find a constant $M_1 > 0$ such that, if $\rho \in \mathcal{A}$ is such that $J(\rho,\mu) \leqslant J(\bar{\rho}_0,\mu)$, then

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T |X_t^\rho|^p dt + |X_T^\rho|^p\right] < M_1.$$

Therefore, using the SDE (4.3.1), we obtain $\mathbb{E}^{\mathbb{P}}[|\xi_T|^p] < M_2$, for a suitable constant $M_2 > 0$ which does not depend on μ. Finally, using Grönwall's inequality, we obtain the estimate as in (4.3.2), for a constant $M > 0$ not depending on μ.

Remark 4.3.3 (Existence of optimal controls)**.** *Under the standing assumptions, it is shown in [96] that, for each measurable flow of probabilities μ, the set $\arg\min_{\mathcal{A}} J(\cdot,\mu) \subset \mathcal{A}$ is nonempty. Also, as in Remark 4.2.3, one can show that, for each sequence of singular controls $\{\rho_n\}_{n\in\mathbb{N}} \subset \arg\min_{\mathcal{A}} J(\cdot,\mu)$ we can find an admissible singular control $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi) \in \arg\min_{\mathcal{A}} J(\cdot,\mu)$ such that, up to a subsequence, $\mathbb{P}^n \circ (X^{\rho_n})^{-1}$ converges to $\mathbb{P} \circ (X^\rho)^{-1}$ in the Meyer-Zheng topology (see again [96], or Appendix A for further details on this topology).*

Combining the previous remarks, we deduce that there exists a constant $M > 0$ such that, for any flow of measures μ, one has

$$\mathbb{E}^{\mathbb{P}}[|X_t^\rho|^p] \leqslant M, \quad \text{for any } t \in [0,T],\ \rho \in \arg\min J(\cdot,\mu).$$

Hence, Lemma B.2 in the Appendix B allows to find μ^{Min}, $\mu^{\text{Max}} \in \mathcal{P}(\mathbb{R})$ such that, for any flow of measures μ, one has

$$\mu^{\text{Min}} \leqslant^{\text{st}} \mathbb{P} \circ (X_t^\rho)^{-1} \leqslant^{\text{st}} \mu^{\text{Max}}, \quad \text{for any } t \in [0,T],\ \rho \in \arg\min J(\cdot, \mu). \tag{4.3.3}$$

Next, we define the set L^{ext} as the set of all equivalence classes (w.r.t. the measure $dt + \delta_T$ on the interval $(-\infty, T]$) of measurable flows of probabilities $\mu : (-\infty, T] \to \mathcal{P}(\mathbb{R})$, which are constant for negative times. To simplify the subsequent notation (avoiding to re-define flows of probability measures for negative times), we say that two elements of L^{ext} coincide $\delta_{0-} + dt + \delta_T$-a.e. in $[0,T]$ if they coincide $dt + \delta_T$-a.e. on the interval $(-\infty, T]$. In this spirit, define the set of feasible flows of measures L as the set of all elements of L^{ext} with $\mu_t \in [\mu^{\text{Min}}, \mu^{\text{Max}}]$ for π-almost all $t \in [0,T]$ and $\mu_{0-} = \nu_0$. On L we consider the order relation $\leqslant^L$ given by $\mu \leqslant^L \nu$ if and only if $\mu_t \leqslant^{\text{st}} \nu_t$ for π-a.a. $t \in [0,T]$, with the lattice structure given by

$$(\mu \wedge^L \nu)_t := \mu_t \wedge^{\text{st}} \nu_t \quad \text{and} \quad (\mu \vee^L \nu)_t := \mu_t \vee^{\text{st}} \nu_t \quad \text{for } \pi\text{-a.a. } t \in [0,T].$$

As in Lemma 4.1.7, the lattice L is complete. Finally, letting 2^L be the set of all subsets of L, thanks to (4.3.3) the best-response-correspondence $\mathcal{R} : L \to 2^L$ given by

$$\mathcal{R}(\mu) := \Big\{ \mathbb{P} \circ (X^\rho)^{-1} \,\Big|\, \rho \in \arg\min J(\cdot, \mu) \Big\}, \quad \mu \in L, \tag{4.3.4}$$

is well defined.

We can then give the following definition.

Definition 10. *The flow of measures $\mu^* \in L$ is a solution to the mean field game with singular controls if $\mu^* \in \mathcal{R}(\mu^*)$.*

4.3.2 Existence and approximation of solutions

In order to recover the results of the previous sections, we prove the following technical lemma.

Lemma 4.3.4. *Given $\rho_i = (\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i, x_0^i, W^i, \xi^i) \in \mathcal{A}$, $i = 1, 2$, such that*

$$\mathbb{E}^{\mathbb{P}^i}[|\xi_T^i|^p] < \infty, \quad i = 1, 2,$$

there exist admissible singular controls

$$\rho^\wedge = (\Omega^\wedge, \mathcal{F}^\wedge, \mathbb{F}^\wedge, \mathbb{P}^\wedge, x_0^\wedge, W^\wedge, \xi^\wedge) \quad \text{and} \quad \rho^\vee = (\Omega^\vee, \mathcal{F}^\vee, \mathbb{F}^\vee, \mathbb{P}^\vee, x_0^\vee, W^\vee, \xi^\vee)$$

such that:

1. *$\mathbb{P}^\wedge \circ (X^{\rho^\wedge})^{-1} \leqslant^L \mathbb{P}^1 \circ (X^{\rho_1})^{-1} \wedge^L \mathbb{P}^2 \circ (X^{\rho_2})^{-1}$;*
2. *$\mathbb{P}^1 \circ (X^{\rho_1})^{-1} \vee^L \mathbb{P}^2 \circ (X^{\rho_2})^{-1} \leqslant^L \mathbb{P}^\vee \circ (X^{\rho^\vee})^{-1}$;*
3. *$J(\rho^\wedge, \mu) + J(\rho^\vee, \mu) = J(\rho_1, \mu) + J(\rho_2, \mu)$, for any $\mu \in L$.*

Proof. The argument exploits an approximation scheme of the singular controls through regular controls and the results from the Subsection 4.1.4. We divide the proof in four steps.

Step 1. Take $\rho_i = (\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i, x_0^i, W^i, \xi^i) \in \mathcal{A}$, $i = 1, 2$. Without loss of generality, we can assume that the controls ρ_1, ρ_2 are defined on a same stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W)$; that is, $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i, x_0^i, W^i) = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W)$, $i = 1, 2$. Indeed, this can be deduced employing the same rationales as in the proof of Lemma 4.2.4 and by observing that the singular controls ξ^1, ξ^2 can be see as r.v.'s in a Polish space (the space of right-continuous, nondecreasing, nonnegative functions on $[0,T]$, together with the strong Skorokhod M_1 topology, we refer to [83] for more details).

Introduce a Wong-Zakai-type approximation of ξ^i by defining the sequences of processes $\{\xi^{i,n}\}_{n\in\mathbb{N}}$ setting, for each $n \in \mathbb{N}$,

$$\xi_t^{i,n} := \begin{cases} n \int_{t-1/n}^t \xi_s^i ds & t \in [0,T), \\ \xi_T^i & t = T. \end{cases} \tag{4.3.5}$$

Recall that processes are always (implicitly) assumed to be equal to 0 for negative times. Notice also that, since $\mathbb{E}^{\mathbb{P}}[|\xi_T^i|^p] < \infty$ by assumption, the processes $\xi^{i,n}$ are Lipschitz continuous on the time interval $[0,T)$. However, they may have discontinuities at time T. Moreover, for each $i = 1, 2,\ n \in \mathbb{N}$, denote by $X^{i,n}$ the solution to the controlled SDE

$$X_t^{i,n} = x_0 + \int_0^t b(s, X_s^{i,n}) ds + \sigma W_t + \xi_t^{i,n}, \quad t \in [0,T].$$

Next, since the processes $\xi^{i,n}$ have Lipschitz paths and are nondecreasing, we can find $\mathbb{F}$-adapted processes $u^{i,n} : \Omega \times [0,T] \to [0,\infty)$ such that

$$\xi_t^{i,n} = \int_0^t u_s^{i,n} ds, \quad t \in [0,T).$$

Observing that the processes $u^{i,n}$ can be regarded as regular controls (see Section 4.1), we wish to employ the results from Section 4.1 in order to construct $\rho^\wedge$, $\rho^\vee$. However, we need to take care of possible discontinuities at time T.

As in Lemma 4.1.11, for each $n \in \mathbb{N}$ we find two $\mathbb{F}$-adapted $[0,\infty)$-valued processes $u^{\wedge,n}, u^{\vee,n}$ such that, defining

$$\xi_t^{\wedge,n} := \int_0^t u_s^{\wedge,n} ds \quad \text{and} \quad \xi_t^{\vee,n} := \int_0^t u_s^{\vee,n} ds, \quad \text{for each } t \in [0,T),\ \mathbb{P}\text{-a.s.}, \tag{4.3.6}$$

we have, for each $t \in [0,T)$, $\mathbb{P}$-a.s.,

$$\begin{aligned} X_t^{1,n} \wedge X_t^{2,n} &= x_0 + \int_0^t b(s, X_s^{1,n} \wedge X_s^{2,n}) ds + \sigma W_t + \xi_t^{\wedge,n}, \\ X_t^{1,n} \vee X_t^{2,n} &= x_0 + \int_0^t b(s, X_s^{1,n} \vee X_s^{2,n}) ds + \sigma W_t + \xi_t^{\vee,n}. \end{aligned} \tag{4.3.7}$$

This suggest to define the processes $\xi^{\wedge,n}$ and $\xi^{\vee,n}$ at time T by setting, $\mathbb{P}$-a.s.,

$$\begin{aligned} \xi_T^{\wedge,n} &:= X_T^{1,n} \wedge X_T^{2,n} - x_0 - \int_0^T b(s, X_s^{1,n} \wedge X_s^{2,n}) ds - \sigma W_T, \\ \xi_T^{\vee,n} &:= X_T^{1,n} \vee X_T^{2,n} - x_0 - \int_0^T b(s, X_s^{1,n} \vee X_s^{2,n}) ds - \sigma W_T. \end{aligned}$$

In this way, by setting

$$
\begin{aligned}
\rho^{\wedge,n} &:= (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi^{\wedge,n}),\\
\rho^{\vee,n} &:= (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi^{\vee,n}),\\
\rho^{i,n} &:= (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi^{i,n}),\ i = 1, 2,
\end{aligned}
$$

by (4.3.7) and the definition of $\xi_T^{\wedge,n}$ and $\xi_T^{\vee,n}$, we have, $\mathbb{P}$-a.s.,

$$
X_t^{1,n} \wedge X_t^{2,n} = X_t^{\rho^{\wedge,n}}, \quad \text{and} \quad X_t^{1,n} \vee X_t^{2,n} = X_t^{\rho^{\vee,n}}, \quad \text{for any } t \in [0,T]. \tag{4.3.8}
$$

Moreover, we observe that the processes $\xi^{\wedge,n}$ and $\xi^{\vee,n}$ are nondecreasing.
Step 2. In this step we prove that

$$
J(\rho^{\wedge,n}, \mu) + J(\rho^{\vee,n}, \mu) = J(\rho^{1,n}, \mu) + J(\rho^{2,n}, \mu). \tag{4.3.9}
$$

This is done again by adapting arguments from Subsection 4.1.4, taking care of possible discontinuities of the processes $\xi^{i,n},\ \xi^{\wedge,n},\ \xi^{\vee,n}$ at time T.

For a generic admissible control $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi) \in \mathcal{A}$, using integration by parts and the controlled SDE (4.3.1), we rewrite the cost functional as

$$
\begin{aligned}
J(\xi, \mu) &= \mathbb{E}^{\mathbb{P}}\left[\int_0^T f(t, X_t^\rho, \mu_t)dt + g(X_T^\rho, \mu_T) + c_T \xi_T - \int_0^T \xi_t c_t' dt\right] \qquad (4.3.10)\\
&= \mathbb{E}^{\mathbb{P}}\left[\int_0^T \Big(f(t, X_t^\rho, \mu_t) - c_T b(t, X_t^\rho) - \xi_t c_t'\Big)dt + g(X_T^\rho, \mu_T) + c_T X_T^\rho\right] - c_T \mathbb{E}^{\mathbb{P}}[x_0]\\
&= G^1(\rho, \mu) - G^2(\rho, \mu) + H(\rho, \mu) - c_T \mathbb{E}^{\mathbb{P}}[x_0],
\end{aligned}
$$

where we have set

$$
\begin{aligned}
G^1(\rho, \mu) &:= \mathbb{E}^{\mathbb{P}}\left[\int_0^T \Big(f(t, X_t^\rho, \mu_t) - c_T b(t, X_t^\rho)\Big)dt\right],\\
G^2(\rho, \mu) &:= \mathbb{E}^{\mathbb{P}}\left[\int_0^T \xi_t c_t' dt\right],\\
H(\rho, \mu) &:= \mathbb{E}^{\mathbb{P}}[g(X_T^\rho, \mu_T) + c_T X_T^\rho].
\end{aligned}
$$

Observing that the functional G^1 depends on the control only on the interval $[0,T)$, thanks to the construction of $u^{\wedge,n}$, $u^{\vee,n}$ provided in the Step 1, we can repeat the arguments in Lemma 4.1.12 obtaining,

$$
G^1(\rho^{\wedge,n}, \mu) + G^1(\rho^{\vee,n}, \mu) = G^1(\rho^{1,n}, \mu) + G^1(\rho^{2,n}, \mu). \tag{4.3.11}
$$

Moreover, from the definition of $u^{\wedge,n},\ u^{\vee,n}$ in Step 1, as in the proof of Lemma 4.1.12, we see that

$$
\xi_t^{\wedge,n} + \xi_t^{\vee,n} = \int_0^t (u_s^{\wedge,n} + u_s^{\vee,n})ds = \int_0^t (u_s^{1,n} + u_s^{2,n})ds = \xi_t^{1,n} + \xi_t^{2,n}, \quad \text{for each } t \in [0,T),
$$

so that

$$
G^2(\rho^{\wedge,n}, \mu) + G^2(\rho^{\vee,n}, \mu) = G^2(\rho^{1,n}, \mu) + G^2(\rho^{2,n}, \mu). \tag{4.3.12}
$$

Finally, we easily find

$$H(X_T^{\rho^{\wedge,n}},\mu)+H(X_T^{\rho^{\vee,n}},\mu)=H(X_T^{\rho^{1,n}},\mu)+H(X_T^{\rho^{2,n}},\mu). \tag{4.3.13}$$

Therefore, adding (4.3.11), (4.3.12) and (4.3.13), and using the representation in (4.3.10), we obtain (4.3.9).

Step 3. Set $X^i := X^{\rho_i}$, $i = 1,2$, and define the right-continuous processes $\xi^\wedge$, $\xi^\vee$ by setting

$$\begin{aligned}\xi_t^\wedge &:= X_t^1 \wedge X_t^2 - x_0 - \int_0^t b(s, X_s^1 \wedge X_s^2)ds - \sigma W_t, \qquad (4.3.14)\\ \xi_t^\vee &:= X_t^1 \vee X_t^2 - x_0 - \int_0^t b(s, X_s^1 \vee X_s^2)ds - \sigma W_t.\end{aligned}$$

In this step, we want to prove that the controls $\rho^\wedge := (\Omega,\mathcal{F},\mathbb{F},\mathbb{P},x_0,W,\xi^\wedge)$ and $\rho^\vee := (\Omega,\mathcal{F},\mathbb{F},\mathbb{P},x_0,W,\xi^\vee)$ are admissible, and satisfy Claims 1, 2.

From (4.3.5), we immediately see that, $\mathbb{P}$-a.s.,

$$\begin{cases}\xi_t^{i,n} \to \xi_t^i \text{ as } n\to\infty \text{ for all continuity points } t\in[0,T) \text{ of } \xi^i,\\ \xi_T^{i,n}\to\xi_T^i \text{ as } n\to\infty.\end{cases} \tag{4.3.15}$$

Therefore, using (4.3.15) and Grönwall's inequality, we deduce that, $\mathbb{P}$-a.s.,

$$\begin{cases}X_t^{i,n} \to X_t^i \text{ as } n\to\infty \text{ for all continuity points } t\in[0,T) \text{ of } X^i,\\ X_T^{i,n}\to X_T^i \text{ as } n\to\infty.\end{cases} \tag{4.3.16}$$

This allows to take limits in (4.3.8) in order to conclude that, $\mathbb{P}$-a.s., for $dt+\delta_T$-a.a. $t\in[0,T]$ we have

$$X_t^{1,n}\wedge X_t^{2,n}\to X_t^1\wedge X_t^2,\quad X_t^{1,n}\vee X_t^{2,n}\to X_t^1\vee X_t^2,\quad \xi_t^{\wedge,n}\to\xi_t^\wedge,\quad \xi_t^{\vee,n}\to\xi_t^\vee. \tag{4.3.17}$$

Since the processes $\xi^{\wedge,n}$ and $\xi^{\vee,n}$ are nonnegative and nondecreasing, from the latter limit we deduce that the processes $\xi^\wedge$ and $\xi^\vee$ are nonnegative and nondecreasing, hence $\rho^\wedge$ and $\rho^\vee$ are admissible. Moreover, by definition of $\xi^\wedge$ and $\xi^\vee$, we have

$$X_t^{\rho^\wedge} = X_t^1\wedge X_t^2 \leqslant X_t^1\vee X_t^2 \leqslant X_t^{\rho^\vee}, \quad \text{for each } t\in[0,T],\ \mathbb{P}\text{-a.s.},$$

which proves Claims 1 and 2.

Step 4. We finally prove Claim 3. We begin by observing that, for a generic constant $C>0$, by Grönwall's inequality we have

$$|X_t^{i,n}|^p \leqslant C\Big(1+|x_0|^p+\sigma^p\sup_{s\in[0,T]}|W_s|^p+|\xi_T^{i,n}|^p\Big),$$

so that, by definition of $\xi^{i,n}$, we obtain

$$\sup_n\sup_{t\in[0,T]}|X_t^{i,n}|^p \leqslant C\Big(1+|x_0|^p+\sigma^p\sup_{s\in[0,T]}|W_s|^p+|\xi_T^{i}|^p\Big)\in\mathbb{L}^1(\Omega;\mathbb{P}), \tag{4.3.18}$$

where the integrability condition of the right hand side follows from our assumptions. Therefore, thanks to the limits in (4.3.15) and (4.3.16) and the estimate (4.3.18), the growth conditions on f and g allows employ the dominated convergence theorem in order to obtain

$$\begin{aligned} J(\rho_i,\mu) &= \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T f(t,X_t^{\rho_i},\mu_t)dt + g(X_T^{\rho_i},\mu_T) + \int_{[0,T]} c_t d\xi_t^i\bigg] \\ &= \lim_n \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T f(t,X_t^{\rho^{i,n}},\mu_t)dt + g(X_T^{\rho^{i,n}},\mu_T) + \int_{[0,T]} c_t d\xi_t^{i,n}\bigg] = \lim_n J(\rho^{i,n},\mu). \end{aligned} \tag{4.3.19}$$

Next, with an integration by parts, we can use the limits in (4.3.17) and Fatou's lemma obtaining

$$\begin{aligned} J(\rho^\wedge,\mu) &= \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T f(t,X_t^{\rho^\wedge},\mu_t)dt + g(X_T^{\rho^\wedge},\mu_T) + c_T\xi_T^\wedge - \int_0^T \xi_t^\wedge c_t' dt\bigg] \\ &\leqslant \liminf_n \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T f(t,X_t^{\rho^{\wedge,n}},\mu_t)dt + g(X_T^{\rho^{\wedge,n}},\mu_T) + c_T\xi_T^{\wedge,n} - \int_0^T \xi_t^{\wedge,n} c_t' dt\bigg] \\ &= \liminf_n J(\rho^{\wedge,n},\mu). \end{aligned} \tag{4.3.20}$$

Similarly, we obtain

$$J(\rho^\vee,\mu) \leqslant \liminf_n J(\rho^{\vee,n},\mu). \tag{4.3.21}$$

Finally, exploiting (4.3.19), (4.3.20) and (4.3.21), we can take limits in (4.3.9) in order to obtain Claim 3. This completes the proof of the lemma. □

As pointed out in Remark 4.2.5, Lemma 4.3.4 together with Remark 4.3.3 allows to adapt the the proof of Lemma 4.2.4 in order to deduce analogous monotonicity properties of the best-response-correspondence for MFGs with singular controls. This allows, together with the completeness of the lattice L, to repeat the proof of Theorem 4.2.6 in order to obtain the following result.

Theorem 4.3.5. *Under Assumption 4.3.1, we have that*

(i) The set of solutions $\mathcal{M}$ to the mean field game with singular controls is nonempty and admits a minimal and a maximal element.

Assume moreover that the costs $f(t,\cdot,\cdot)$ and $g(\cdot,\cdot)$ are continuous in (x,μ). Then

(ii) For $\underline{\mu}^0 := \inf L$ and $\underline{\mu}^n := \inf \mathcal{R}(\underline{\mu}^{n-1})$ for $n \in \mathbb{N}$, we have that the learning procedure $\{\underline{\mu}^n\}_{n\in\mathbb{N}}$ is increasing and it converges weakly to $\inf \mathcal{M}$, π-a.e.

(iii) For $\overline{\mu}^0 := \sup L$ and $\overline{\mu}^n := \sup \mathcal{R}(\overline{\mu}^{n-1})$ for $n \in \mathbb{N}$, we have that the learning procedure $\{\overline{\mu}^n\}_{n\in\mathbb{N}}$ is decreasing and it converges weakly to $\sup \mathcal{M}$, π-a.e.

4.3.3 Remarks and extensions

We discuss here how the previous arguments can be adapted in order to fit some settings which are typical in the literature on stochastic singular control.

Remark 4.3.6 (On the strong formulation of the problem). *The techniques presented in this section can also be used to treat MFGs with singular controls under the strong formulation. In this case, one can fix a stochastic basis* $\beta = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{x}_0, \bar{W})$ *satisfying the requirements of Definition 9, and define the set of admissible controls* $\mathcal{A}_\beta$ *as set of singular controls* $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi) \in \mathcal{A}$ *with* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W) = \beta$*. For any measurable flow of probabilities* μ*, the representative agent's minimization problem is then given by*

$$\inf_{\rho \in \mathcal{A}_\beta} J(\rho, \mu).$$

In the strong formulation, existence and uniqueness of optimal controls can be shown in linear-convex settings, in which the drift b *is affine in the state, and in which the costs* f*,* g *are strictly convex in the state. Alternatively, existence of optimal controls can be investigated through the DPP approach, by trying to adapt the results in [64, 79, 102, 130, 162] to running costs* $f(t, x, \mu_t)$ *which are discontinuous in time.*

Remark 4.3.7 (Bounded-variation control problem). *We point point out that MFGs with controls of bounded variation (see, e.g., Chapter 2) can also be treated with the approach presented in this section. In this case, the set of admissible controls is given by the set* $\mathcal{V}$ *of tuples* $\rho = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W, \xi)$ *in which the stochastic basis* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, x_0, W)$ *satisfies the requirements in Definition 9 and in which* v *is an* $\mathbb{R}$*-valued* $\mathbb{F}$*-adapted càdlàg processes of bounded variation. For a flow of probabilities* μ*, the representative agent's minimization problem is given by*

$$\begin{cases} \inf_{\rho \in \mathcal{V}} \mathbb{E}\Big[\int_0^T f(t, X_t^\rho, \mu_t) dt + \int_{[0,T]} c_t d|v|_t \Big] \\ dX_t^\rho = b(t, X_t^\rho) dt + \sigma dW_t + + dv_t, \ t \geqslant 0, \quad X_{0-}^\rho = x_0. \end{cases}$$

Here, as in Chapter 2, the process $|v|$ *denotes the total variation of* v*.*

4.4 Concluding remarks and further extensions

In the following, we provide some comments on our assumptions and further extensions of the techniques elaborated in the previous sections. For simplicity, we discuss only the case of mean field games with regular controls, although analogous conclusions can be derived also for mean field games with singular controls.

4.4.1 On the multidimensional case

Our approach can be extended only to some particular multidimensional cases. Indeed, although the first order stochastic dominance induces a lattice structure on $\mathcal{P}(\mathbb{R})$, it does not induce a lattice order on $\mathcal{P}(\mathbb{R}^d)$ for $d > 1$ (cf. [105] and [138]). Also, Lemma 4.1.11 does not hold, in general, for multidimensional settings, as the following counterexample shows.

Example 10. *Consider a 2-dimensional Brownian motion $W = (W^1, W^2)$. For any $\mathbb{R}$-valued integrable progressively measurable process u, let $X^u = (X^{1,u}, X^{2,u})$ be the solution to*

$$X_t^{1,u} = \int_0^t u_s ds + W_t^1, \quad X_t^{2,u} = -\int_0^t u_s ds + W_t^2.$$

Taking a positive u, we see that $X^{1,u} \vee X^{1,-u} = X^{1,u}$, while $X^{2,u} \vee X^{2,-u} = X^{2,-u}$. This means that the first component of $X^u \vee X^{-u}$ should be controlled by u, while the second component should be controlled by $-u$. Therefore, $X^\beta \neq X^u \vee X^{-u}$ for any control β.

Nevertheless, the results in this chapter can be extended to suitable multidimensional settings where the actual dependence on the measure is only through one of its one-dimensional marginals, and Lemma 4.1.11 and Proposition 4.1.12 hold.

For example, take $d > 1$ and a d-dimensional Brownian motion $W = (W^1, \dots, W^d)$, on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Consider closed sets $U^i \subset \mathbb{R}$, $i = 1, \dots, d$. Admissible controls are d-dimensional square integrable progressively measurable processes $u = (u^1, \dots, u^d)$ taking values in $U^1 \times ... \times U^d$. Take measurable functions

$$\begin{aligned} b^i, l^i &: \Omega \times [0,T] \times \mathbb{R} \times U^i \to \mathbb{R}, \; i = 1, ..., d, \\ f &: \Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}, \quad g : \Omega \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}, \end{aligned}$$

and a d-dimensional $\mathcal{F}_0$-measurable square integrable random variable $x_0 = (x_0^1, ..., x_0^d)$. For each admissible control u, let the process $X^u = (X^{1,u}, ..., X^{d,u})$ denote the solution to the system

$$dX_t^{i,u} = b^i(t, X_t^{i,u}, u_t^i)dt + dW_t^i, \qquad t \in [0,T], \quad X_0^{i,u} = x_0^i, \quad i = 1, ..., d.$$

Next, for any given measurable flow $\mu = (\mu_t)_{t \in [0,T]}$ of probability measures on $\mathcal{B}(\mathbb{R})$, we consider the cost functional

$$J(u, \mu) := \mathbb{E}\bigg[\int_0^T \Big[f(t, X_t^u, \mu_t) + \sum_{i=1}^d l^i(t, X_t^{i,u}, u_t^i) \Big] dt + g(X_T^u, \mu_T) \bigg].$$

We enforce an analogous of Assumption 4.1.2; that is, we assume that for each flow μ the exists a unique optimal pair (X^μ, u^μ) with X^μ satisfying some tightness condition uniformly in μ.

Notice that we assume that the minimization problem depends on a measure on $\mathbb{R}$, not on $\mathbb{R}^d$. For example, the problem can depend only on one fixed marginal, say the first. In this spirit, a MFG solution is a measurable flow $\mu^* = (\mu_t^*)_{t \in [0,T]}$ of probabilities such that

$$\mu_t^* = \mathbb{P} \circ (X_t^{1,\mu^*})^{-1} \quad \text{for each } t \in [0,T].$$

Now, since the components of X^u are decoupled, we easily see that Lemma 4.1.11 can be recovered. However, in order to deal with the multidimensional setting, we need to enforce a stronger version of Assumption 4.1.9.

Assumption 4.4.1. *For $\mathbb{P}\otimes dt$ a.a. $(\omega,t)\in\Omega\times[0,T]$, for $\phi\in\{f(t,\cdot,\cdot),g\}$, we have*

$$\phi(\bar{x}\vee x,\mu)-\phi(\bar{x},\mu)\leqslant\phi(x,\mu)-\phi(\bar{x}\wedge x,\mu),$$

for all $\bar{x},x\in\mathbb{R}^d$ and $\mu\in\mathcal{P}(\mathbb{R})$, and

$$\phi(\bar{x},\bar{\mu})-\phi(x,\bar{\mu})\leqslant\phi(\bar{x},\mu)-\phi(x,\mu),$$

for all $\bar{x},x\in\mathbb{R}^d$ and $\bar{\mu},\mu\in\mathcal{P}(\mathbb{R})$ s.t. $\bar{x}\geqslant x$ and $\bar{\mu}\geqslant^{st}\mu$.

By the additive structure of the running cost involving the controls, using Assumption 4.4.1 we can adapt the proof of Proposition 4.1.12 to prove that the best-reply-map is increasing. Therefore, for this particular setup, the arguments of Section 4.1 can be recovered, and Theorems 4.1.14 and (by making an analogous of Assumption 4.1.15) 4.1.17 can be extended.

4.4.2 On linear-quadratic MFG

Assumption 4.1.9 is fulfilled in the linear-quadratic case

$$\begin{aligned}
b(t,x,u)&=c_t+p_tx+q_tu,\\
f(t,x,\mu)+l(t,x,u)&=\frac{1}{2}n_tu^2+\frac{1}{2}(m_tx+\widehat{m}_t\langle\mathrm{id},\mu\rangle)^2,\\
g(x,\mu)&=\frac{1}{2}(h_tx+\widehat{h}_t\langle\mathrm{id},\mu\rangle)^2,
\end{aligned}$$

where $\mathrm{id}(y)=y$, and for deterministic continuous functions c_t, p_t, q_t, n_t, m_t, $\widehat{m}_t$, h_t, and $\widehat{h}_t$ such that $\inf_{t\in[0,T]}q_t>0$, $\inf_{t\in[0,T]}n_t>0$, $n_t\widehat{m}_t\leqslant 0$ and $h_t\widehat{h}_t\leqslant 0$ for each $t\in[0,T]$.

However, the tightness condition (2) in Assumption 4.1.2 is not satisfied unless we consider a compact control set U. In fact, when U is not compact, there is a counterexample in Section 7 of [120], which shows that a mean field game solution may not exist.

Nevertheless, our approach allows to treat non-standard linear-quadratic mean field games, as for example the one considered in Subsection 2.2 in [67].

4.4.3 On a geometric dynamics

Our results still hold true if we replace (4.1.1) with a dynamics of the geometric form

$$dX_t=b(t,X_t,u_t)X_tdt+\sigma_tX_tdW_t,\qquad t\in[0,T],\quad X_0=x_0,\tag{4.4.1}$$

for some square-integrable positive r.v. x_0, a bounded drift b and a bounded stochastic process σ. Indeed, for each square-integrable process u there exists a unique strong solution X^u to the latter SDE, and classical estimates show that there exists a constant $M>0$ such that

$$\sup_{t\in[0,T]}\mathbb{E}[|X_t^u|^2]\leqslant M;$$

hence, the tightness condition in Assumption 4.1.2 is satisfied. Moreover, the solution to (4.4.1) can be represented as

$$X_t^u = x_0 \exp\Big(\int_0^t \Big(b(s, X_s^u, u_s) - \frac{1}{2}\sigma_s^2\Big) ds + \int_0^t \sigma_s dW_s \Big), \quad t \in [0,T],$$

and the mapping $x \mapsto \exp(x)$ is monotone. Hence, since x_0 is positive, for any couple of admissible controls u, $\bar{u}$, we have that for each $t \in [0,T]$ $\mathbb{P}$-a.s.

$$X_t^{\bar{u}} \geqslant X_t^u \qquad \text{if and only if} \qquad \int_0^t b(s, X_s^{\bar{u}}, \bar{u}_s) ds \geqslant \int_0^t b(s, X_s^u, u_s) ds.$$

The latter property allows to repeat all the arguments employed in the proof of Lemma 4.1.11 and (mutatis mutandis) to carry on the analysis that lead to the existence results of Theorems 4.1.14 and 4.2.6.

4.4.4 On mean field dependent dynamics

For a suitable choice of the costs f, g and l, Theorem 4.1.14 still holds if we have a "sufficiently simple" mean field dependence in the dynamics of the system. For the sake of illustration, we discuss here two examples.

Let U be a compact subset of $\mathbb{R}$. For any admissible process u and any measurable flow of probability measures μ, consider a state process given by

$$dX_t = X_t(u_t + m(\mu_t))dt + \sigma X_t dW_t, \quad t \in [0,T], \quad X_0 = x_0, \tag{4.4.2}$$

where x_0 is a positive square-integrable r.v. and $m\colon \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ is a bounded function which is measurable with respect to the Borel σ-algebra associated to the topology of weak convergence of probability measures. Assume moreover that m is increasing with respect to the first order stochastic dominance.

Notice that, for each measurable flow μ and for each admissible u, the SDE (4.4.2) admits the explicit solution

$$X_t^{u,\mu} = E_t(u) M_t(\mu), \tag{4.4.3}$$

where

$$E_t(u) := x_0 \exp\bigg(\int_0^t \Big(u_s - \frac{\sigma^2}{2}\Big) ds + \sigma W_t \bigg) \quad \text{and} \quad M_t(\mu) := \exp\bigg(\int_0^t m(\mu_s) ds \bigg).$$

Since U is compact and m is bounded, we can find a constant $K > 0$ which is independent of μ, such that

$$\sup_{t \in [0,T]} \mathbb{E}[|X_t^{u,\mu}|^2] \leqslant K.$$

The latter implies the tightness condition in Assumption 4.1.2. As in Subsection 4.1.2, this allows us to define a set L of feasible flows of measures, and to show that $(L, \leqslant^L)$ is a complete lattice.

Given $\mu \in L$ and two admissible controls u and $\bar{u}$, as in Lemma 4.1.11 we can construct $u^\vee$ and $u^\wedge$ such that $X_t^{u,\mu} \vee X_t^{\bar{u},\mu} = X_t^{u^\vee,\mu}$ and $X_t^{u,\mu} \wedge X_t^{\bar{u},\mu} = X_t^{u^\wedge,\mu}$. Moreover, due to the particular structure of (4.4.2), the construction of $u^\vee$ and $u^\wedge$ does not depend on μ.

Consider now cost functions $l(t,x,u) = u^2/2$ and $f(t,x,\mu) = x\psi(\mu)$, for a measurable function $\psi\colon \mathcal{P}(\mathbb{R}) \to \mathbb{R}_-$ which is decreasing w.r.t. the first order stochastic dominance. With such a choice of the costs, the functional J is strictly convex w.r.t. u. Hence, for each $\mu \in L$, there exists a unique minimizer u of $J(\cdot,\mu)$ (see, e.g., Theorem 5.2 in [168]). We then have the following result.

Lemma 4.4.2. *The best-response-map $R\colon L \to L$ is increasing.*

Proof. Take μ, $\bar{\mu} \in L$ with $\mu \leqslant^L \bar{\mu}$. Let $u \in \arg\min J(\cdot,\mu)$ and $\bar{u} \in \arg\min J(\cdot,\bar{\mu})$.

Similarly to Lemma 4.1.12, we first see that

$$0 \geqslant J(u,\mu) - J(u^\wedge,\mu) = J(u^\vee,\mu) - J(\bar{u},\mu). \tag{4.4.4}$$

We also observe that, exploiting (4.4.3), the monotonicity of m and the fact that ψ is negative and decreasing, one has

$$\begin{aligned}(X_t^{u^\vee,\mu} - X_t^{\bar{u},\mu})\psi(\mu_t) &= (E_t(u^\vee) - E_t(\bar{u}))M_t(\mu)\psi(\mu_t) \\ &\geqslant (E_t(u^\vee) - E_t(\bar{u}))M_t(\bar{\mu})\psi(\bar{\mu}_t) = (X_t^{u^\vee,\bar{\mu}} - X_t^{\bar{u},\bar{\mu}})\psi(\bar{\mu}_t).\end{aligned} \tag{4.4.5}$$

Thus, combining (4.4.4) and (4.4.5), we obtain

$$\begin{aligned}0 \geqslant J(u^\vee,\mu) - J(\bar{u},\mu) &= \mathbb{E}\bigg[\int_0^T \bigg(\frac{(u_t^\vee)^2}{2} - \frac{\bar{u}_t^2}{2} + (X_t^{u^\vee,\mu} - X_t^{\bar{u},\mu})\psi(\mu_t)\bigg)dt\bigg] \\ &\geqslant \mathbb{E}\bigg[\int_0^T \bigg(\frac{(u_t^\vee)^2}{2} - \frac{\bar{u}_t^2}{2} + (X_t^{u^\vee,\bar{\mu}} - X_t^{\bar{u},\bar{\mu}})\psi(\bar{\mu}_t)\bigg)dt\bigg] = J(u^\vee,\bar{\mu}) - J(\bar{u},\bar{\mu}).\end{aligned}$$

Hence $u^\vee \in \arg\min J(\cdot,\bar{\mu})$, which, by uniqueness, implies that $u^\vee = \bar{u}$. This in turn implies that $E_t(u^\vee) = E_t(u) \vee E_t(\bar{u}) = E_t(\bar{u})$. Hence, $E_t(u) \leqslant E_t(\bar{u})$ and, by monotonicity of m, we find $X_t^{u,\mu} = E_t(u)M_t(\mu) \leqslant E_t(\bar{u})M_t(\bar{\mu}) = X_t^{\bar{u},\bar{\mu}}$ and $R(\mu) = \mathbb{P}\circ(X_t^{u,\mu})^{-1} \leqslant^L \mathbb{P}\circ(X_t^{\bar{u},\bar{\mu}})^{-1} = R(\bar{\mu})$, which completes the proof. □

Thanks to Lemma 4.4.2, we can invoke Tarski's fixed point theorem in order to deduce that the set of mean field game equilibria is a nonempty and complete lattice.

Remark 4.4.3. *Statements analogous to the previous ones still hold if we consider a controlled Ornstein-Uhlenbeck process with mean field term in the dynamics; that is, if the state process is given by*

$$dX_t = \Big(\kappa X_t + u_t + m(\mu_t)\Big)dt + \sigma dW_t,\ t \in [0,T], \quad X_0 = x_0, \quad \kappa \in \mathbb{R}, \sigma \geqslant 0, \tag{4.4.6}$$

for a measurable bounded increasing function $m : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$.

4.5 On mean field games with common noise

Our approach allows also to treat submodular mean field games with common noise. We refer to the recent works [48], [67] and [156] for a related setup. In the following we discuss two examples of mean field games with regular or singular controls, and in which the representative player interacts with the population through the conditional mean of its state given the common noise.

We fix the probabilistic setup for this section. Let W and B be two independent Brownian motions on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Here, the Brownian motion B stands for the common noise, while W represents the idiosyncratic noises affecting the state processes in the pre-limit N-player game. Let $\mathbb{F}^B$ be the natural filtration generated by B augmented by all $\mathbb{P}$-null sets, and define M^B to be the set of all real-valued $\mathbb{F}^B$-progressively measurable processes. Finally, let $x_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$, $\sigma \geqslant 0$, and $\sigma_0 > 0$.

4.5.1 Regular controls and common noise

For each $u \in \mathcal{U}$ (see the beginning of Subsection 4.1.1), consider a dynamics of the system given by

$$dX_t = b(t, X_t, u_t)dt + \sigma dW_t + \sigma_0 dB_t, \quad t \in [0, T], \quad X_0 = x_0, \tag{4.5.1}$$

for some measurable function b satisfying the requirements in (4.1.2).

For any given process $m \in M^B$, consider the optimization problem $\inf J(\cdot, m)$, with J defined by

$$J(u, m) := \mathbb{E}\left[\int_0^T \Big[f(t, X_t^u, m_t) + l(t, X_t^u, u_t)\Big]dt + g(X_T^u, m_T)\right], \quad u \in \mathcal{U},$$

for appropriately measurable functions $f\colon \Omega \times [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, $l\colon \Omega \times [0,T] \times \mathbb{R} \times U \to \mathbb{R}$ and $g\colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$. Notice that f and g are now functions of the process m, which represents the conditional mean of the population given the common noise B.

We enforce the following conditions.

Assumption 4.5.1.

1. *The control space U is compact.*

2. *For each process $m \in L^B$, there exists a unique optimal pair (X^m, u^m).*

3. *For $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0, T]$, the functions $f(t, \cdot, \cdot)$ and g have decreasing differences in (x, y); that is, for $\phi \in \{f(t, \cdot, \cdot), g\}$,*

$$\phi(\bar{x}, \bar{y}) - \phi(x, \bar{y}) \leqslant \phi(\bar{x}, y) - \phi(x, y),$$

for all $\bar{x}, x, \bar{y}, y \in \mathbb{R}$ s.t. $\bar{x} \geqslant x$ and $\bar{y} \geqslant y$.

Notice that, since we assume the control set U to be compact, then, by a standard use of Grönwall inequality, we can find a constant $C > 0$ such that the solution X^u to the SDE (4.5.1) satisfies ($\mathbb{P}$-a.s.) the estimate

$$|X_t^u| \leqslant C\left(1 + |x_0| + \sigma \sup_{s\in[0,t]} |W_s| + \sigma_0 \sup_{s\in[0,t]} |B_s|\right) =: Y_t \quad \text{for all } t \in [0,T] \text{ and } u \in \mathcal{U}.$$

Moreover, notice that the process $Y := (Y_t)_{t\in[0,T]}$ belongs to $L^2(\Omega \times [0,T])$, and that the process Z defined by

$$Z_t := \mathbb{E}[Y_t|\mathcal{F}_T^B], \quad \mathbb{P}\text{-a.s.}, \ t \in [0,T],$$

is $\mathbb{F}^B$-progressively measurable.

Therefore, define L^B to be the set of all real-valued $\mathbb{F}^B$-progressively measurable processes $m = (m_t)_{t\in[0,T]}$ such that $|m_t| \leqslant Y_t$ $\mathbb{P}$-a.s., for each $t \in [0,T]$. Next, introduce the map $R\colon L^B \to L^B$ defined by

$$R(m)_t := \mathbb{E}[X_t^m|\mathcal{F}_T^B] \quad \mathbb{P}\text{-a.s.}, \ t \in [0,T].$$

Notice that $R(m)$ is $\mathbb{F}^B$-adapted (see Remark 1 in [156]) and continuous in t, and therefore $\mathbb{F}^B$-progressively measurable.

Definition 11. *A process $m^* \in L^B$ is a strong MFG solution to the MFG with common noise if*

$$m_t^* = \mathbb{E}[X_t^{m^*}|\mathcal{F}_T^B], \quad \mathbb{P}\text{-a.s.}, \ t \in [0,T].$$

Consider on L^B the order relation given by $m \leqslant \overline{m}$ if and only if $m_t \leqslant \overline{m}_t$ $\mathbb{P}\otimes dt$-a.e. Since L^B is a bounded subset of the Dedekind complete lattice $L^2(\Omega \times [0,T])$, it is a complete lattice. Moreover, as in Remark 4.1.13, for $\overline{m}, m \in L^B$ with $m \leqslant \overline{m}$ we have that $X_t^m \leqslant X_t^{\overline{m}}$ for each $t \in [0,T]$, $\mathbb{P}$-a.s., and hence

$$R(m)_t = \mathbb{E}[X_t^m|\mathcal{F}_T^B] \leqslant \mathbb{E}[X_t^{\overline{m}}|\mathcal{F}_T^B] = R(\overline{m})_t, \quad \mathbb{P}\text{-a.s., for every } t \in [0,T],$$

which implies that $R\colon L^B \to L^B$ is increasing. Once more, using Tarski's fixed point theorem, we have proved the following result.

Theorem 4.5.2. *Under Assumption 4.5.1, the set of strong solutions of the MFG with common noise is a nonempty complete lattice.*

Remark 4.5.3. *We point out that Theorem 4.5.6 guarantees existence of a strong solution to the MFG; that is, a solution which is adapted to the common noise. As a matter of fact, results on the existence of strong solutions are still relatively limited in the literature on MFGs with common noise, and they are usually proved through uniqueness results (see e.g. Section 6 in [48]), in the spirit of the Yamada-Watanabe theory for weak and strong solutions to standard SDEs.*

Remark 4.5.4. *Notice that the crucial step in order to obtain Theorem 4.5.6 is the inequality $X_t^m \leqslant X_t^{\overline{m}}$, for each $t \in [0,T]$, whenever $m \leqslant \overline{m}$. Following the arguments developed in Subsection 4.4.4 for MFG without common noise, a similar relation can be established also in the case of mean field dependent dynamics as in (4.4.2) or (4.4.6) with an additional common noise term $\sigma_0 dB_t$. Note that the latter mean-reverting dynamics is exactly the one considered in [67] and [156].*

4.5.2 Singular controls and common noise

For simplicity, we will work under the finite fuel assumption. Indeed, for a constant $\Psi > 0$, define the set of admissible singular controls as the set $\mathcal{A}(\Psi)$ of all $\mathbb{F}$-adapted càdlàg, nondecreasing and nonnegative processes ξ satisfying $\xi_T \leqslant \Psi$, $\mathbb{P}$-a.s. For each $\xi \in \mathcal{A}(\Psi)$, let X^ξ denote a linearly controlled Ornstein–Uhlenbeck process; that is, the solution of the SDE

$$dX_t^\xi = \theta(\lambda - X_t^\xi)dt + \sigma dW_t + \sigma_0 dB_t + d\xi_t,\ t \in [0,T], \quad X_{0-}^\xi = x_0. \tag{4.5.2}$$

For any given process $m \in M^B$, consider the optimization problem $\inf J(\cdot, m)$, with J defined by

$$J(\xi, m) := \mathbb{E}\bigg[\int_0^T f(t, X_t^\xi, m_t)dt + g(X_T^\xi, m_T) + \int_{[0,T]} c_t d\xi_t\bigg], \quad \xi \in \mathcal{A}(\Psi),$$

for appropriately measurable functions $f\colon \Omega \times [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, $c\colon \Omega \times [0,T] \to [0,\infty)$ and $g\colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$.

We enforce the following requirements.

Assumption 4.5.5.

1. *$\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0,T]$, the functions $f(t,\cdot,y)$ and $g(\cdot,y)$ are strictly convex and lower semi-continuous, satisfying, for all $(\omega,t,x,y) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}$ and some $K > 0$, the growth conditions*

$$|f(t,x,y)| + |g(x,y)| \leqslant K(1 + |x|^2);$$

2. *For $\mathbb{P} \otimes dt$ a.a. $(\omega, t) \in \Omega \times [0,T]$, the functions $f(t,\cdot,\cdot)$ and g have decreasing differences in (x,y); that is, for $\phi \in \{f(t,\cdot,\cdot), g\}$,*

$$\phi(\bar{x}, \bar{y}) - \phi(x, \bar{y}) \leqslant \phi(\bar{x}, y) - \phi(x, y),$$

for all $\bar{x}, x, \bar{y}, y \in \mathbb{R}$ s.t. $\bar{x} \geqslant x$ and $\bar{y} \geqslant y$.

3. *The function c is continuous.*

Notice that the controlled state processes can be explicitly written as

$$X_t^\xi = e^{-\theta t}\bigg(x_0 + \lambda(e^{\theta t} - 1) + \int_0^t e^{\theta s}(\sigma dW_s + \sigma_0 dB_s) + \int_{[0,t]} e^{\theta s} d\xi_s\bigg). \tag{4.5.3}$$

Also, by the finite fuel assumption, we have

$$0 \leqslant \int_{[0,t]} e^{\theta s} d\xi_s \leqslant 2\Psi e^{|\theta| T}, \quad \text{for all } t \in [0,T],\ \mathbb{P}\text{-a.s.}$$

Hence, defining the square integrable stochastic process Y^Ψ by

$$Y_t^\Psi := e^{-\theta t}\bigg(|x_0 + \lambda(e^{\theta t} - 1)| + \bigg|\sigma \int_0^t e^{\theta s}\sigma dW_s\bigg| + \sigma_0 \bigg|\int_0^t e^{\theta s} dB_s\bigg| + 2\Psi e^{|\theta| T}\bigg),$$

we have that the solution X^ξ to the SDE (4.5.2) satisfies ($\mathbb{P}$-a.s.) the estimate

$$|X_t^\xi| \leqslant Y_t^\Psi \quad \text{for all } t \in [0,T] \text{ and } \xi \in \mathcal{A}(\Psi). \tag{4.5.4}$$

Therefore, defining the $\mathbb{F}^B$-progressively measurable process Z^Ψ by $Z_t^\Psi := \mathbb{E}[Y_t^\Psi|\mathcal{F}_T^B]$, introduce the set L^B as the set of all real-valued $\mathbb{F}^B$-progressively measurable processes m such that $|m_t| \leqslant Z_t^\Psi$ $\mathbb{P}$-a.s., for each $t \in [0,T]$. As in Subsection 4.5.1, L^B is a complete lattice.

Under Assumption 4.5.5, for any process m there exists a unique optimal pair (X^m, ξ^m). This can be shown adapting arguments from the proof of Lemma 2.2.7 in Chapter 2, or following the proof of Theorem 8 in [135]. Therefore, we can introduce the map $R \colon L^B \to L^B$ defined by

$$R(m)_t := \mathbb{E}[X_t^m|\mathcal{F}_T^B], \quad \mathbb{P}\text{-a.s., } t \in [0,T].$$

Via an approximation of singular controls through Lipschitz controls (see the Step 1 in the proof of Lemma 4.3.4), using Remark 1 in [156] one can show that $R(m)$ is $\mathbb{F}^B$-progressively measurable. Therefore, we can define the notion of strong solution to the MFG with singular controls and common noise analogously to Definition 11.

Next, thanks to the explicit expression (4.5.3), one can easily check that, for ξ, $\bar\xi \in \mathcal{A}(\Psi)$, we have $X^\xi \wedge X^{\bar\xi} = X^{\xi^\wedge}$ and $X^\xi \vee X^{\bar\xi} = X^{\xi^\vee}$ by setting

$$\xi_t^\wedge := \int_{[0,t]} e^{-\theta s} d(\zeta \wedge \bar\zeta)_s,\ \xi_t^\vee := \int_{[0,t]} e^{-\theta s} d(\zeta \vee \bar\zeta)_s,\ \zeta := \int_{[0,t]} e^{\theta s} d\xi_s,\ \bar\zeta := \int_{[0,t]} e^{\theta s} d\bar\xi_s.$$

Furthermore, we have $\xi^\wedge$, $\xi^\vee \in \mathcal{A}(\Psi)$ and the construction of $\xi^\wedge$, $\xi^\vee$ allows to repeat the arguments of Lemma 3.1.2 in order to deduce that the best-reply-map R is increasing. This allows to prove the following result.

Theorem 4.5.6. *Under Assumption 4.5.5, the set of strong solutions of the MFG with singular controls and common noise is a nonempty complete lattice.*

Chapter 5

Stationary mean field games with singular controls

We study stationary mean field games with singular controls in which the representative player interacts with a long-time weighted average of the population through a discounted and an ergodic performance criterion. This class of games finds natural applications in the context of optimal productivity expansion in dynamic oligopolies. We prove existence and uniqueness of the mean field equilibria, which are completely characterized through nonlinear equations. Furthermore, we relate the mean field equilibria for the discounted and the ergodic games by showing the validity of an Abelian limit. The latter allows also to approximate Nash equilibria of symmetric N-player ergodic singular control games through the mean field equilibrium of the discounted game.

5.1 The probabilistic setting

We introduce here the probabilistic setting for our model. On a given complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions, consider an $\mathbb{F}$-Brownian motion W. Set $\mathbb{R}_+ := (0, \infty)$, and for any $x \in \mathbb{R}_+$, let the process X^x denote the unique strong solution to the uncontrolled SDE

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dW_t, \ t \geqslant 0, \quad X_0^x = x. \tag{5.1.1}$$

Existence and uniqueness of such a solution is ensured by the following assumption (cf. Theorem 7 in Chapter V of [145]).

Assumption 5.1.1. *The coefficients $b : \mathbb{R}_+ \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuously differentiable. Furthermore, b and σ are (globally) Lipschitz continuous, and $\sigma\sigma'$ is locally Lipschitz.*

The locally Lipschitz property of $\sigma\sigma'$, as well as the Lipschitz continuity of b, will be needed in our subsequent analysis (cf. (5.1.3) and Proof of Lemma 5.4.1, respectively).

Furthermore, under Assumption 5.1.1, the process X^x is nondegenerate, and for any $x_o \in \mathbb{R}_+$ there exists $\varepsilon > 0$ (depending on x_o) such that

$$\int_{x_o-\varepsilon}^{x_o+\varepsilon} \frac{1 + |b(z)|}{\sigma^2(z)} \, dz < +\infty. \tag{5.1.2}$$

The latter guarantees that X^x is a regular diffusion. That is, starting from $x \in \mathbb{R}_+$, X^x reaches any other $y \in \mathbb{R}_+$ in finite time with positive probability.

In our subsequent analysis, an important role will be also played by the one-dimensional Itô-diffusion $\widehat{X}^x$ evolving as

$$d\widehat{X}^x_t = \Big[b(\widehat{X}^x_t) + (\sigma\sigma')(\widehat{X}^x_t)\Big]dt + \sigma(\widehat{X}^x_t)d\widehat{W}_t, \quad \widehat{X}^x_0 = x \in \mathbb{R}_+, \tag{5.1.3}$$

for some one-dimensional $\mathbb{F}$-Brownian motion $\widehat{W}$.

Notice that, under Assumption 5.1.1, there exists a unique strong solution to (5.1.3), up to a possible explosion time. Moreover, one has that for any $x_o \in \mathbb{R}_+$ there exists $\varepsilon > 0$ such that

$$\int_{x_o-\varepsilon}^{x_o+\varepsilon} \frac{1 + |b(z)| + |\sigma\sigma'(z)|}{\sigma^2(z)} dz < +\infty, \tag{5.1.4}$$

ensuring that $\widehat{X}^x$ is a regular diffusion as well.

5.1.1 Characteristics and requirements on the diffusion process

In this subsection we recall useful basic characteristics of the diffusion processes X^x and $\widehat{X}^x$. We refer to Chapter II in [29] for further details.

The infinitesimal generator related to the uncontrolled SDE (5.1.1) is denoted by $\mathcal{L}_X$ and is defined as the second-order differential operator

$$\left(\mathcal{L}_X f\right)(x) := \frac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x), \quad f \in C^2(\mathbb{R}_+), \quad x \in \mathbb{R}_+. \tag{5.1.5}$$

On the other hand, the infinitesimal generator of (5.1.3) is denoted by $\mathcal{L}_{\widehat{X}}$ and is such that

$$\left(\mathcal{L}_{\widehat{X}} f\right)(x) := \frac{1}{2}\sigma^2(x)f''(x) + (b(x) + \sigma(x)\sigma'(x))f'(x), \quad f \in C^2(\mathbb{R}_+), \quad x \in \mathbb{R}_+. \tag{5.1.6}$$

For $r > 0$, we introduce ψ_r and ϕ_r as the fundamental solutions to the ordinary differential equation (ODE),

$$\mathcal{L}_X u(x) - ru(x) = 0, \qquad x \in \mathbb{R}_+, \tag{5.1.7}$$

and we recall that they are strictly increasing and decreasing, respectively. For an arbitrary $x_o \in \mathbb{R}_+$ we also denote by

$$S'(x) := \exp\left(-\int_{x_o}^{x} \frac{2b(z)}{\sigma^2(z)}\, dz\right), \qquad x \in \mathbb{R}_+,$$

the derivative of the scale function of X, and we observe that the derivative of the speed measure of X is given by

$$m'(x) := \frac{2}{\sigma^2(x)\, S'(x)}.$$

Together with the killing measure, scale function and speed measure represent the basic characteristics of any diffusion process. In particular, S is related to the drift of the diffusion and, more specifically, to the probability of the diffusion leaving an interval either from its left or right endpoint. On the other hand, it can be shown that the transition probability of a regular diffusion is absolutely continuous with respect to the speed measure.

Throughout this chapter we assume that

$$\int_a^\infty m'(y)dy < \infty, \quad \text{for any} \quad a > 0.$$

Moreover, when $r - b'(x) \geqslant r_o > 0$ for $x \in \mathbb{R}_+$, any solution to the ODE

$$\mathcal{L}_{\widehat{X}} u(x) - (r - b'(x))u(x) = 0, \qquad x \in \mathbb{R}_+, \tag{5.1.8}$$

can be written as a linear combination of the fundamental solutions $\widehat{\psi}_r$ and $\widehat{\phi}_r$, which are strictly increasing and decreasing, respectively. Finally, letting $x_o \in \mathbb{R}_+$ to be arbitrary, we denote by

$$\widehat{S}'(x) := \exp\left(-\int_{x_o}^x \frac{2b(z) + 2\sigma(z)\sigma'(z)}{\sigma^2(z)}\, dz\right), \qquad x \in \mathbb{R}_+,$$

the derivative of the scale function of $\widehat{X}$, and by

$$\widehat{m}'(x) := \frac{2}{\sigma^2(x)\,\widehat{S}'(x)}$$

the density of its speed measure. One can easily check that the scale functions and speed measures of X and $\widehat{X}$ are related through $\widehat{S}'(x) = S'(x)/\sigma^2(x)$ and $\widehat{m}'(x) = 2/S'(x)$, for $x \in \mathbb{R}_+$.

Concerning the boundary behavior of the real-valued Itô-diffusions X and $\widehat{X}$, in the rest of this chapter we assume that 0 and $+\infty$ are natural boundaries for those two processes. In particular, this means that 0 and ∞ are unattainable in finite time and that, for each $r > 0$, we have

$$\lim_{x\downarrow 0} \psi_r(x) = 0, \quad \lim_{x\downarrow 0} \phi_r(x) = +\infty, \quad \lim_{x\uparrow\infty} \psi_r(x) = +\infty, \quad \lim_{x\uparrow\infty} \phi_r(x) = 0, \tag{5.1.9}$$

$$\lim_{x\downarrow 0} \frac{\psi_r'(x)}{S'(x)} = 0, \quad \lim_{x\downarrow 0} \frac{\phi_r'(x)}{S'(x)} = -\infty, \quad \lim_{x\uparrow\infty} \frac{\psi_r'(x)}{S'(x)} = +\infty, \quad \lim_{x\uparrow\infty} \frac{\phi_r'(x)}{S'(x)} = 0. \tag{5.1.10}$$

Also, when $r - b'(x) \geqslant r_o > 0$ for each $x \in \mathbb{R}_+$, we have

$$\lim_{x\downarrow 0} \widehat{\psi}_r(x) = 0, \quad \lim_{x\downarrow 0} \widehat{\phi}_r(x) = +\infty, \quad \lim_{x\uparrow\infty} \widehat{\psi}_r(x) = +\infty, \quad \lim_{x\uparrow\infty} \widehat{\phi}_r(x) = 0, \tag{5.1.11}$$

$$\lim_{x\downarrow 0} \frac{\widehat{\psi}_r'(x)}{\widehat{S}'(x)} = 0, \quad \lim_{x\downarrow 0} \frac{\widehat{\phi}_r'(x)}{\widehat{S}'(x)} = -\infty, \quad \lim_{x\uparrow\infty} \frac{\widehat{\psi}_r'(x)}{\widehat{S}'(x)} = +\infty, \quad \lim_{x\uparrow\infty} \frac{\widehat{\phi}_r'(x)}{\widehat{S}'(x)} = 0. \tag{5.1.12}$$

Furthermore, we require that

$$\lim_{x\downarrow 0} \phi_r'(x) = -\infty \quad \text{and} \quad \lim_{x\uparrow\infty} \psi_r'(x) = \infty.$$

Then, by arguing as in the second part of the proof of Lemma 4.3 in [7], one can show that, under our conditions on X and $\widehat{X}$, one has $\widehat{\phi}_r = -\phi_r'$ and $\widehat{\psi}_r = \psi_r'$.

Finally, the following useful equations hold for any $0 < a < b < \infty$:

$$\begin{cases} \dfrac{\widehat{\psi}_r'(b)}{\widehat{S}'(b)} - \dfrac{\widehat{\psi}_r'(a)}{\widehat{S}'(a)} = \displaystyle\int_a^b \widehat{\psi}_r(y)(r - b'(y))\widehat{m}'(y)dy, \\ \dfrac{\widehat{\phi}_r'(b)}{\widehat{S}'(b)} - \dfrac{\widehat{\phi}_r'(a)}{\widehat{S}'(a)} = \displaystyle\int_a^b \widehat{\phi}_r(y)(r - b'(y))\widehat{m}'(y)dy. \end{cases} \tag{5.1.13}$$

We summarize the requirements made in this subsection in the following assumptions.

Assumption 5.1.2.

1. *$\int_a^\infty m'(y)dy < \infty$, for any $a > 0$;*

2. *The points 0 and $+\infty$ are natural boundaries for the processes X and $\widehat{X}$;*

3. *$\lim_{x\downarrow 0} \phi_r'(x) = -\infty$ and $\lim_{x\uparrow\infty} \psi_r'(x) = \infty$.*

We conclude this discussion by noticing that all the requirements on X (and, consequently, on $\widehat{X}$) assumed so far are satisfied, for example, by the relevant cases in which X is a geometric Brownian motion with drift $b(x) = -\delta x$, $\delta > 0$, or an affine mean-reverting dynamics with drift $b(x) = \kappa(\lambda - x)$ and volatility $\sigma(x) = \sigma x$, for positive κ, λ, σ.

5.2 The stationary mean field games

In this section we introduce the stationary mean field games (MFGs) that will be the object of our study.

Introduce the set of singular controls as

$$\mathcal{A} := \{\text{nonnegative nondecreasing } \mathbb{F}\text{-adapted càdlàg processes}\},$$

and, for $x \in \mathbb{R}_+$ and $\xi \in \mathcal{A}$, let $X^{x;\xi}$ denote the unique strong solution to the controlled SDE

$$dX_t^{x;\xi} = b(X_t^{x;\xi})dt + \sigma(X_t^{x;\xi})dW_t + d\xi_t, \quad X_{0-}^{x;\xi} = x \in \mathbb{R}_+. \tag{5.2.1}$$

Next, for any $\xi \in \mathcal{A}$, $x \in \mathbb{R}_+$, and $\theta \in \mathbb{R}_+$ we consider the *discounted* expected profit

$$J(x, \xi, \theta; r) := \mathbb{E}\bigg[\int_0^\infty e^{-rs} h(X_s^{x;\xi}, \theta)ds - \int_{[0,\infty)} e^{-rs} d\xi_s\bigg], \quad r > 0, \tag{5.2.2}$$

as well as the *ergodic* expected profit

$$G(x, \xi, \theta) := \limsup_{T\uparrow\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T h(X_s^{x;\xi}, \theta)ds - \xi_T\bigg]. \tag{5.2.3}$$

In (5.2.2) and (5.2.3), $h : \mathbb{R}^2 \to [0, \infty)$ is an instantaneous profit function and the control processes are picked from the following two classes of admissible controls

$$\mathcal{A}_d := \Big\{ \xi \in \mathcal{A} \Big| \mathbb{E}\Big[\int_{[0,\infty)} e^{-rs} d\xi_s \Big] < \infty \Big\}, \quad (5.2.4)$$
$$\mathcal{A}_e := \Big\{ \xi \in \mathcal{A} \Big| \mathbb{E}\Big[\xi_T \Big] < \infty, \text{ for all } T > 0 \Big\},$$

respectively. In order to simplify notation, in the sequel we shall omit the dependency on r of the set $\mathcal{A}_d$, which in fact will be clear from the context. Furthermore, as in (2.1.5) in Chapter 2, the integral $\int_{[0,\infty)} e^{-rs} d\xi_s$ in (5.2.2) is intended in the Lebesgue-Stieltjes sense and it includes the cost of a possible initial jump of ξ of amplitude ξ_0.

For a probability measure μ on $\mathbb{R}_+$ such that $\int_{\mathbb{R}_+} f(x)\mu(dx) < \infty$, we define

$$\theta(\mu) := F\bigg(\int_{\mathbb{R}_+} f(x)\mu(dx) \bigg), \quad (5.2.5)$$

where F and f are strictly increasing nonnegative functions. In the mean field games defined through the next Definitions 12 and 13, the term $\theta = \theta(\mu)$ appearing in (5.2.2) and (5.2.3) describes a suitable mean with respect to the stationary distribution $\mu = \mathbb{P}_{X_\infty^{x;\xi}}$ of the optimally controlled state process $X^{x;\xi}$ (provided that one exists). For example, if $X^{x;\xi}$ describes the productivity of the representative company, then μ provides the distribution of the asymptotic productivity, and its weighted average – with weight function f – defines a price index through the function F (cf. Remark 5.2.2 below).

In the sequel, we focus on the following definition of MFG equilibria.

Definition 12 (Equilibrium of the discounted MFG)**.** *For $r > 0$ and $x \in \mathbb{R}_+$, a couple $(\bar{\xi}^r, \bar{\theta}_r) \in \mathcal{A}_d \times \mathbb{R}_+$ is said to be an equilibrium of the discounted MFG for the initial condition x if*

1. *$J(x, \bar{\xi}^r, \bar{\theta}_r; r) \geqslant J(x, \xi, \bar{\theta}_r; r)$, for any $\xi \in \mathcal{A}_d$;*
2. *The optimally controlled process $\bar{X}^x := \bar{X}^{x;\bar{\xi}^r}$ admits a limiting distribution $\mathbb{P}_{\bar{X}_\infty^x}$ satisfying $\bar{\theta}_r = \theta(\mathbb{P}_{\bar{X}_\infty^x})$.*

Definition 13 (Equilibrium of the ergodic MFG)**.** *For $x \in \mathbb{R}_+$, a couple $(\bar{\xi}^e, \bar{\theta}_e) \in \mathcal{A}_e \times \mathbb{R}_+$ is said to be an equilibrium of the ergodic MFG for the initial condition x if*

1. *$G(x, \bar{\xi}^e, \bar{\theta}_e) \geqslant G(x, \xi, \bar{\theta}_e)$, for any $\xi \in \mathcal{A}_e$;*
2. *The optimally controlled process $\bar{X}^x := \bar{X}^{x;\bar{\xi}^e}$ admits a limiting distribution $\mathbb{P}_{\bar{X}_\infty^x}$ satisfying $\bar{\theta}_e = \theta(\mathbb{P}_{\bar{X}_\infty^x})$.*

We enforce the following structural conditions on the running profit and weight function.

Assumption 5.2.1.

1. *The running profit $h : \mathbb{R}_+^2 \to [0, +\infty)$ belongs to $C^2(\mathbb{R}_+^2)$. Furthermore,*

(a) $h(\cdot,\theta)$ *is concave and nondecreasing for any* $\theta \in \mathbb{R}_+$;

(b) h *has strictly decreasing differences; that is,* $h_{x\theta}(x,\theta) < 0$ *for any* $(x,\theta) \in \mathbb{R}^2_+$;

(c) *For any* $x \in \mathbb{R}_+$,

$$\lim_{\theta \downarrow 0} h_x(x,\theta) = +\infty \quad \textit{and} \quad \lim_{\theta \uparrow \infty} h_x(x,\theta) = 0;$$

(d) *For any* $0 < a < b < \infty$ *there exists a function* $h^{a,b} : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|h_{x\theta}(x,\theta)|\, \widehat{m}'(x) \leqslant h^{a,b}(x), \quad \textit{for any } x \in \mathbb{R}_+,\ \theta \in (a,b),$$

with $h^{a,b} \in \mathbb{L}^1(\kappa,\infty)$ *for any* $\kappa \in \mathbb{R}_+$.

2. *The weight functions* F *and* f *appearing in* (5.2.5) *satisfies:*

(a) *The functions* $F, f : [0,\infty) \to [0,\infty)$ *are continuously differentiable with* $F', f' > 0$ *and, for* $\beta \in (0,1)$ *and a constant* $C > 0$, *they satisfy the growth conditions:*

$$\begin{aligned} f(x) &\leqslant C(1+|x|^{\beta}), \\ F(x) &\leqslant C(1+|x|^{\frac{1}{\beta}}), \\ |F(y)-F(x)| &\leqslant C(1+|x|+|y|)^{\frac{1}{\beta}-1}|y-x|, \end{aligned}$$

for any $y, x \in \mathbb{R}_+$;

(b) $\lim_{y\uparrow\infty} F(y) = +\infty$ *and* $\lim_{y\uparrow\infty} f(y) = +\infty$.

Remark 5.2.2. *With regard to the formulation of a game of productivity expansion as a stationary mean field game, a benchmark example of running profit function and average satisfying Assumption 5.2.1 are*

$$h(x,\theta) := x^{\beta}\,\theta^{-(1+\beta)}, \quad \theta := \theta(\mu) = \left(\int_{\mathbb{R}_+} x^{\beta}\mu(dx)\right)^{\frac{1}{\beta}}, \quad \beta \in (0,1).$$

Such a form of interaction can be obtained from the so-called isoelastic demand obtained from Spence-Dixit-Stiglitz preferences (see, e.g., footnote 5 in [1] for such a derivation).

5.3 Existence, uniqueness, and characterization of the mean field equilibria

In this section, the discounted MFG problem and the ergodic MFG problem are solved. In particular, existence and uniqueness of equilibria is shown by charaterizing the equilibria in terms of the unique solution to systems of nonlinear equations.

5.3.1 On the discounted stationary MFG

In order to deal with the discounted MFG problem for a fixed discount factor $r > 0$, we make the following additional requirement (see also [102], [118], among others).

Assumption 5.3.1.

1. *For each $x \in \mathbb{R}_+$ we have $r - b'(x) \geqslant 2c > 0$, for a constant $c > 0$;*
2. *For any $\theta \in \mathbb{R}_+$, there exists $\widehat{x}_r(\theta) \in \mathbb{R}_+$ such that*

$$h_x(x,\theta) - r + b'(x) \begin{cases} < 0, & x > \widehat{x}_r(\theta), \\ = 0, & x = \widehat{x}_r(\theta), \\ > 0, & x < \widehat{x}_r(\theta). \end{cases}$$

Condition 1 above guarantees that the discount rate is (uniformly) larger than the marginal growth rate of the diffusion X. It is automatically satisfied in the particular cases in which X is a geometric Brownian motion with drift $b(x) = -\delta x$, $\delta > 0$, or it is an affine mean-reverting process with drift $b(x) = \kappa(\lambda - x)$, $\kappa, \lambda > 0$ (and volatility $\sigma(x) = \sigma x$, $\sigma > 0$). Moreover, bearing in mind the mean field game of productivity expansion discussed in the introduction, Condition 2 in Assumption 5.3.1 ensures the following: The marginal running profit h_x, net of the "user cost of capital" $r - b'$, changes sign at most once. Such a requirement guarantees that the mean field equilibrium is of threshold type, as in fact, for any given θ, it should not be profitable to increase productivity via costly investment when $h_x(x,\theta) - r + b'(x) < 0$. This is formalized in the following theorem.

Theorem 5.3.2. *Let $r > 0$, and let Assumptions 5.1.1, 5.1.2, 5.2.1, and 5.3.1 hold. For any $x \in \mathbb{R}_+$, there exists a unique equilibrium $(\bar{\xi}^r, \bar{\theta}_r)$ of the discounted MFG.*

Moreover, $\bar{\xi}^r$ makes the state process reflected upward at the barrier $\bar{x}_r < \widehat{x}_r(\bar{\theta}_r)$, and the couple $(\bar{x}_r, \bar{\theta}_r)$ is determined as the unique solution to the system

$$\int_{\bar{x}_r}^{\infty} \widehat{\phi}_r(y)\Big(h_x(y,\bar{\theta}_r) - r + b'(y)\Big)\widehat{m}'(y)dy = 0 \quad \text{and} \quad \int_{\bar{x}_r}^{\infty} (f(y) - F^{-1}(\bar{\theta}_r))m'(y)dy = 0. \tag{5.3.1}$$

Proof. The proof is organized in two steps.

Step 1. For any fixed $\theta \in \mathbb{R}_+$, here we solve the problem

$$V(x,\theta;r) := \sup_{\xi \in \mathcal{A}_d} \mathbb{E}\bigg[\int_0^{\infty} e^{-rs} h(X_s^{x;\xi},\theta)ds - \int_{[0,\infty)} e^{-rs} d\xi_s\bigg]. \tag{5.3.2}$$

We shall see that an optimal control for (5.3.2) is such that to keep (with minimal effort) the state process above a trigger $\bar{x}(\theta)$. Although the arguments of this step are somehow classical (see, e.g., [102]) we sketch here their main ideas for the sake of completeness. In the following, in order to simplify the exposition, we do not explicitly stress the dependency on r, unless strictly necessary.

Motivated by the intuition that a costly investment should be made only when the productivity is sufficiently low, for any $x \in \mathbb{R}_+$ we define the candidate value

$$v(x,\theta) := \begin{cases} A\phi_r(x) + \overline{v}(x,\theta), & x > \bar{x}(\theta), \\ (x - \bar{x}(\theta)) + v(\bar{x}(\theta),\theta), & x \leqslant \bar{x}(\theta), \end{cases} \tag{5.3.3}$$

for constants A and $\bar{x}(\theta)$ to be found, and with

$$\overline{v}(x,\theta) := \mathbb{E}\bigg[\int_0^\infty e^{-rs} h(X_s^x,\theta)ds\bigg],$$

which is finite due to Conditions 1 and 2 in Assumption 5.3.1.

In order to determine A and $\bar{x}(\theta)$ we impose that $v(\cdot,\theta)$ belongs to $C^2(\mathbb{R}_+)$, from which we obtain that

$$A = -\frac{\overline{v}_{xx}(\bar{x}(\theta),\theta)}{\phi_r''(\bar{x}(\theta))} \tag{5.3.4}$$

and

$$\overline{v}_x(\bar{x}(\theta),\theta)\phi_r''(\bar{x}(\theta)) - \overline{v}_{xx}(\bar{x}(\theta),\theta)\phi_r'(\bar{x}(\theta)) = \phi_r''(\bar{x}(\theta)).$$

Now, using that $\phi_r'(x) = -\widehat{\phi}_r(x)$ and dividing both members of the latter by $\widehat{S}'(\bar{x}(\theta))$ we obtain

$$\frac{\overline{v}_{xx}(\bar{x}(\theta),\theta)\widehat{\phi}_r(\bar{x}(\theta)) - \overline{v}_x(\bar{x}(\theta),\theta)\widehat{\phi}_r'(\bar{x}(\theta))}{\widehat{S}'(\bar{x}(\theta))} = -\frac{\widehat{\phi}_r'(\bar{x}(\theta))}{\widehat{S}'(\bar{x}(\theta))}. \tag{5.3.5}$$

Notice now that for any function $w \in C^2(\mathbb{R}_+)$, standard differentiation, and the fact that $\mathcal{L}_{\widehat{X}}\widehat{S} = 0$ and $(\mathcal{L}_{\widehat{X}} - (r - b'))g = 0$ for $g \in \{\widehat{\psi}, \widehat{\phi}\}$, yield

$$\frac{d}{dx}\left[\frac{w'(x)}{\widehat{S}'(x)}\widehat{\phi}_r(x) - \frac{\widehat{\phi}_r'(x)}{\widehat{S}'(x)}w(x)\right] = \widehat{\phi}_r(x)\widehat{m}'(x)\Big(\mathcal{L}_{\widehat{X}} - (r - b'(x))\Big)w(x). \tag{5.3.6}$$

This last relation applied to the left-hand side of (5.3.5) with $w = \overline{v}_x$, and to the right-hand side of (5.3.5) with $w = 1$ gives

$$-\int_{\bar{x}(\theta)}^\infty \widehat{\phi}_r(y)\widehat{m}'(y)(\mathcal{L}_{\widehat{X}} - (r - b'(y)))\overline{v}_x(y,\theta)dy = \int_{\bar{x}(\theta)}^\infty \widehat{\phi}_r(y)(r - b'(y))\widehat{m}'(y)dy. \tag{5.3.7}$$

Using now that $(\mathcal{L}_{\widehat{X}} - (r - b'(y)))\overline{v}_x(y,\theta) = -h_x(y,\theta)$ we obtain from (5.3.7) a nonlinear equation for $\bar{x}(\theta)$:

$$K(\bar{x}(\theta),\theta) = 0, \quad \text{where} \quad K(x,\theta) := \int_x^\infty \widehat{\phi}_r(y)\Big(h_x(y,\theta) - r + b'(y)\Big)\widehat{m}'(y)dy. \tag{5.3.8}$$

Due to Assumption 5.2.1 it is easy to see that $K(\widehat{x}_r(\theta),\theta) < 0$. Moreover,

$$K_x(x,\theta) = -\widehat{\phi}_r(x)\Big(h_x(x,\theta) - r + b'(x)\Big)\widehat{m}'(x)\begin{cases} \geqslant 0, & x \geqslant \widehat{x}_r(\theta) \\ < 0, & x < \widehat{x}_r(\theta). \end{cases} \tag{5.3.9}$$

Also, for any $x < \widehat{x}_r(\theta) - \varepsilon := \widehat{x}_\varepsilon(\theta)$, for suitable $\varepsilon > 0$, and for $z \in (x, \widehat{x}_\varepsilon(\theta))$, by the integral mean-value theorem we find

$$\begin{aligned} K(x,\theta) &= \int_x^{\widehat{x}_\varepsilon(\theta)} \widehat{\phi}_r(y)\Big(h_x(y,\theta) - r + b'(y)\Big)\widehat{m}'(y)dy + K(\widehat{x}_\varepsilon(\theta),\theta) \\ &= \frac{h_x(z,\theta) - r + b'(z)}{r - b'(z)} \left(\frac{\widehat{\phi}_r'(\widehat{x}_\varepsilon(\theta))}{\widehat{S}'(\widehat{x}_\varepsilon(\theta))} - \frac{\widehat{\phi}_r'(x)}{\widehat{S}'(x)}\right) + K(\widehat{x}_\varepsilon(\theta),\theta), \end{aligned}$$

where (5.1.13) have been used in the last step. Using now that $h_x(z,\theta) - r + b'(z) > 0$ by Assumption 5.2.1, that $r - b'(z) \geqslant 2c > 0$, and the equations in (5.1.12) we see that that $\lim_{x\downarrow 0} K(x,\theta) = \infty$. The previous considerations thus lead to the existence of a unique $\bar{x}(\theta) \in (0, \widehat{x}_r(\theta))$ solving (5.3.8). For later use, we stress that

$$K_x(\bar{x}(\theta),\theta) < 0. \tag{5.3.10}$$

It can then be checked that $v(x,\theta)$ as in (5.3.3) is a C^2-solution to the HJB equation

$$\min\Big\{(\mathcal{L}_X - r)u(x,\theta) + h(x,\theta), 1 - u_x(x,\theta)\Big\} = 0. \tag{5.3.11}$$

In turn, this allows to show, via a classical verification theorem (see, e.g., Theorem 4.1 at p. 300 in [81]), that $v(x,\theta) = V(x,\theta)$ and that the control $\bar{\xi}(\theta)$ such that

$$X_t^{x;\bar{\xi}(\theta)} \geqslant \bar{x}(\theta) \quad \text{and} \quad \bar{\xi}_t(\theta) = \int_0^t \mathbb{1}_{\{X_s^{x;\bar{\xi}(\theta)} \leqslant \bar{x}(\theta)\}} d\bar{\xi}_s(\theta), \quad \forall t \geqslant 0 \ \mathbb{P}\text{-a.s.}, \tag{5.3.12}$$

belongs to $\mathcal{A}_d$ and is optimal. As a matter of fact, since the free boundary $\bar{x}(\theta)$ is a constant, the latter control rule exists by classical results on the Skorokhod reflection problem (cf. Chapter 6 in [93] and Chapter 3.6 in [109]).

Step 2. Since the control $\bar{\xi}(\theta)$ reflects upward the process $X^{x;\bar{\xi}(\theta)}$ at $\bar{x}(\theta)$, thanks to Assumption 5.1.2 the optimally controlled process $X^{x;\bar{\xi}(\theta)}$ is positively recurrent and its stationary distribution is such that (cf. Section 12 of Chapter II in [29])

$$\mathbb{P}_{X_\infty^{x;\bar{\xi}(\theta)}}(dx) = \frac{m'(x)\mathbb{1}_{[\bar{x}(\theta),\infty)}(x)}{\int_{\bar{x}(\theta)}^\infty m'(y)dy} dx.$$

It thus follows that the consistency equation (i.e., (2) in Definition 12) reads

$$\theta = F\left(\int_{\bar{x}(\theta)}^\infty f(y)\mathbb{P}_{X_\infty^{x;\bar{\xi}(\theta)}}(dy)\right) = F\left(\frac{\int_{\bar{x}(\theta)}^\infty f(y)m'(y)dy}{\int_{\bar{x}(\theta)}^\infty m'(y)dy}\right);$$

that is,

$$Q(\theta) := \int_{\bar{x}(\theta)}^\infty (f(y) - F^{-1}(\theta))m'(y)dy = 0. \tag{5.3.13}$$

We now show that (5.3.13) admits a unique solution $\bar{\theta}$ so that $(\bar{\xi},\bar{\theta}) := (\bar{\xi}(\bar{\theta}),\bar{\theta})$ is the mean field equilibrium for the discounted stationary MFG.

Recall the definition of $\bar{x}(\theta)$ and K in (5.3.8). Since $K \in C^1(\mathbb{R}^2_+)$ due to Condition 1 in Assumption 5.2.1, by the implicit function theorem we find that $\theta \mapsto \bar{x}(\theta)$ is continuously differentiable and has derivative

$$\frac{d}{d\theta}\bar{x}(\theta) = -\frac{K_\theta(\bar{x}(\theta),\theta)}{K_x(\bar{x}(\theta),\theta)} < 0, \tag{5.3.14}$$

where the last inequality follows from (5.3.10) and from the fact that $\theta \mapsto h_x(x,\theta)$ is strictly decreasing, by Assumption 5.2.1.

We thus have that Q as in (5.3.13) is continuously differentiable with derivative

$$\frac{d}{d\theta}Q(\theta) = -(f(\bar{x}(\theta)) - F^{-1}(\theta))m'(\bar{x}(\theta))\frac{d}{d\theta}\bar{x}(\theta) - \frac{1}{F'(F^{-1}(\theta))}\int_{\bar{x}(\theta)}^{\infty} m'(y)dy. \tag{5.3.15}$$

Let now $\widehat{\theta}$ be the unique solution to $f(\bar{x}(\theta)) - F^{-1}(\theta) = 0$. Such a value indeed exists. To see this notice that $f \circ \bar{x}$ is strictly decreasing and continuous. Moreover, by using Condition 1c in Assumption 5.2.1, it can be shown that $\bar{x}(\theta) \to +\infty$ as $\theta \downarrow 0$ and $\bar{x}(\theta) \to 0$ as $\theta \uparrow \infty$, which, Condition 1b in Assumption 5.2.1, in turn gives

$$\lim_{\theta\downarrow 0} f(\bar{x}(\theta)) - F^{-1}(\theta) = \infty \quad \text{and} \quad \lim_{\theta\uparrow\infty} f(\bar{x}(\theta)) - F^{-1}(\theta) = -\infty.$$

Then $Q(\widehat{\theta}) > 0$ and $\frac{d}{d\theta}Q(\theta) < 0$ for any $\theta \geqslant \widehat{\theta}$. Moreover, for any $\theta < \widehat{\theta}$,

$$Q(\theta) \geqslant (f(\bar{x}(\theta)) - F^{-1}(\theta))\int_{\bar{x}(\theta)}^{\infty} m'(y)dy > (f(\bar{x}(\widehat{\theta})) - F^{-1}(\widehat{\theta}))\int_{\bar{x}(\widehat{\theta})}^{\infty} m'(y)dy = 0,$$

where the strictly decreasing property of $f \circ \bar{x}$ has been used. Finally, for $\theta > \theta_o > \widehat{\theta}$ we see that

$$\frac{d}{d\theta}Q(\theta) < -\frac{1}{F'(F^{-1}(\theta))}\int_{\bar{x}(\theta_o)}^{\infty} m'(y)dy,$$

where we have used that $\bar{x}(\cdot)$ is decreasing. Hence,

$$\begin{aligned} Q(\theta) - Q(\theta_o) &< -\left(\int_{\theta_o}^{\theta} \frac{1}{F'(F^{-1}(z))}dz\right)\int_{\bar{x}(\theta_o)}^{\infty} m'(y)dy \\ &= -(F^{-1}(\theta) - F^{-1}(\theta_0))\int_{\bar{x}(\theta_o)}^{\infty} m'(y)dy, \end{aligned}$$

and, taking limits as $\theta \uparrow \infty$ in the latter, and using that $F^{-1}(\theta) \to \infty$, we obtain $Q(\theta) \to -\infty$.

All the previous properties of Q imply that there exists a unique $\bar{\theta} > \widehat{\theta}$ solving the consistency equation (5.3.13). Therefore, stressing now the dependency of the involved quantities with respect to r, and setting $(\bar{x}_r, \bar{\theta}_r) := (\bar{x}(\bar{\theta}), \bar{\theta})$ and $\bar{\xi}^r := \bar{\xi}(\bar{\theta}_r)$, we conclude that $(\bar{\xi}^r, \bar{\theta}_r)$ is the unique equilibrium of the discounted stationary MFG, and that it is characterized by the couple $(\bar{x}_r, \bar{\theta}_r)$ solving the system of equations (5.3.1). This completes the proof of the theorem. □

5.3.2 On the ergodic stationary MFG

Our analysis of the ergodic MFG problem is subject to the following requirements, which are consistent to those in Assumption 5.3.1 when $r = 0$.

Assumption 5.3.3.

1. *For each $x \in \mathbb{R}_+$ we have $b'(x) < -2c < 0$, for a constant $c > 0$;*

2. *For any $\theta \in \mathbb{R}_+$, there exists $\widehat{x}_0(\theta) \in \mathbb{R}_+$ such that*

$$h_x(x,\theta) + b'(x) \begin{cases} < 0, & x > \widehat{x}_0(\theta), \\ = 0, & x = \widehat{x}_0(\theta), \\ > 0, & x < \widehat{x}_0(\theta). \end{cases}$$

Notice that the condition $b'(x) < -2c < 0$ is easily seen to be verified in the relevant cases of X being a geometric Brownian motion and a mean-reverting affine process with drift $b(x) = \kappa(\lambda - x)$, $\kappa, \lambda > 0$, (and volatility $\sigma(x) = \sigma x$, $\sigma > 0$).

Recall now the mean field game problem with ergodic net profit given by (5.2.3), together with its notion of solution given in Definition 13. For each $\theta > 0$, set

$$\lambda(\theta) := \sup_{\xi \in \mathcal{A}_e} \limsup_{T \uparrow \infty} \frac{1}{T} \mathbb{E}\bigg[\int_0^T h(X_s^{x;\xi}, \theta) ds - \xi_T \bigg] \tag{5.3.16}$$

The next result provides a complete characterization of the ergodic mean field equilibrium.

Theorem 5.3.4. *Let Assumptions 5.1.1, 5.1.2, 5.2.1, and 5.3.3 hold. For any $x \in \mathbb{R}_+$, there exists a unique equilibrium $(\bar{\xi}^e, \bar{\theta}^e)$ of the ergodic MFG.*

Moreover, the process $\bar{\xi}^e$ reflects the state process at the barrier $\bar{x}_e < \widehat{x}_0(\bar{\theta}_e)$, and the couple $(\bar{x}_e, \bar{\theta}_e)$ is determined as the unique solution to the system

$$\int_{\bar{x}_e}^{\infty} \widehat{\phi}_0(y)\Big(h_x(y, \bar{\theta}_e) + b'(y)\Big)\widehat{m}'(y) dy = 0 \quad \text{and} \quad \int_{\bar{x}_e}^{\infty} (f(y) - F^{-1}(\bar{\theta}_e)) m'(y) dy = 0. \tag{5.3.17}$$

Finally, the value of the ergodic MFG at equilibrium is given by

$$\lambda(\bar{\theta}_e) = b(\bar{x}_e) + h(\bar{x}_e, \bar{\theta}_e). \tag{5.3.18}$$

Proof. We divide the proof into two steps.

Step 1. We fix $\theta > 0$ and we solve the control problem with ergodic profit (5.3.16). To this aim, define $\mathcal{T}$ as the set of $\mathbb{F}$-stopping times, and, recalling that $b'(x) < -2c$ by Condition 1 in Assumption 5.3.3, consider the auxiliary optimal stopping problem

$$u(x,\theta) := \inf_{\tau \in \mathcal{T}} \mathbb{E}_x \bigg[\int_0^\tau e^{\int_0^t b'(\widehat{X}_s) ds} h_x(\widehat{X}_t, \theta) dt + e^{\int_0^\tau b'(\widehat{X}_s) ds} \bigg]. \tag{5.3.19}$$

By employing methods as in [4] (see in particular Theorem 5 therein), one can prove that the value function $u(\cdot,\theta)$ is $C^1(\mathbb{R}_+)$ with $u_{xx}(\cdot,\theta) \in L^\infty_{\text{loc}}(\mathbb{R}_+)$, and that the optimal stopping time is given by $\bar{\tau}(x,\theta) := \inf\{t \geqslant 0 \,|\, \widehat{X}^x_t \leqslant \bar{x}(\theta)\}$, where $\bar{x}(\theta)$ uniquely solves

$$\int_{\bar{x}(\theta)}^{\infty} \widehat{\phi}_0(y)\Big(h_x(y,\theta) + b'(y)\Big)\widehat{m}'(y)dy = 0. \tag{5.3.20}$$

Existence of a unique solution $\bar{x}(\theta)$ to the equation (5.3.20) can be deduced from Assumption 5.3.3 as in Step 1 of the proof of Theorem 5.3.2. Also, we have $\bar{x}(\theta) < \widehat{x}_0(\theta)$ (cf. Assumption 5.3.3). Moreover, it can be shown that

$$\begin{cases} \mathcal{L}_{\widehat{X}} u(x,\theta) + b'(x)u(x,\theta) + h_x(x,\theta) = 0, & x > \bar{x}(\theta), \\ u(x,\theta) = 1, & x \leqslant \bar{x}(\theta), \end{cases} \tag{5.3.21}$$

as well as

$$\begin{cases} \mathcal{L}_{\widehat{X}} u(x,\theta) + b'(x)u(x,\theta) + h_x(x,\theta) \geqslant 0, & x < \bar{x}(\theta), \\ u(x,\theta) \leqslant 1, & x \geqslant \bar{x}(\theta). \end{cases} \tag{5.3.22}$$

Next, define the function $U(\cdot,\theta)$ such that $U_x(x,\theta) = u(z,\theta)$. By the regularity of $u(\cdot,\theta)$, the function $U(\cdot,\theta)$ is $C^2(\mathbb{R}_+)$ and we observe that $U_x(x,\theta) = u(x,\theta) \leqslant 1$ for each $x \in \mathbb{R}_+$. Furthermore, setting $\Lambda := b(\bar{x}(\theta)) + h(\bar{x}(\theta),\theta)$, we find

$$\begin{aligned} &\frac{\sigma^2(x)}{2}U_{xx}(x,\theta) + b(x)U_x(x,\theta) + h(x,\theta) \qquad (5.3.23) \\ &\qquad = \frac{\sigma^2(x)}{2}u_x(x,\theta) + b(x)u(x,\theta) + h(x,\theta) \\ &\qquad = \int_{\bar{x}(\theta)}^{x} \left(\frac{\sigma^2(z)}{2}u_x(z,\theta) + b(z)u(z,\theta) + h(z,\theta)\right)_z dz \\ &\qquad\quad + \frac{\sigma^2(\bar{x}(\theta))}{2}u_x(\bar{x}(\theta),\theta) + b(\bar{x}(\theta))u(\bar{x}(\theta),\theta) + h(x,\theta) \\ &\qquad = \int_{\bar{x}(\theta)}^{x} \Big(\mathcal{L}_{\widehat{X}} u(z,\theta) + b'(z)u(z,\theta) + h_x(z,\theta)\Big)dz + \Lambda, \end{aligned}$$

where we have used (5.3.21) and (5.3.22) in the last equality. Now, if $x < \bar{x}(\theta)$, the integral in the right-hand side of (5.3.23) is nonpositive, so that

$$\frac{\sigma^2(x)}{2}U_{xx}(x,\theta) + b(x)U_x(x,\theta) + h(x,\theta) \leqslant \Lambda.$$

On the other hand, if $x > \bar{x}(\theta)$, from (5.3.23) and (5.3.21) we deduce that

$$\frac{\sigma^2(x)}{2}U_{xx}(x,\theta) + b(x)U_x(x,\theta) + h(x,\theta) = \Lambda.$$

Overall, we have shown that $U(\cdot,\theta)$ is a $C^2(\mathbb{R}_+)$ function satisfying

$$\mathcal{L}U(x,\theta) + h(x,\theta) \leqslant \Lambda \quad \text{and} \quad U_x(x,\theta) \leqslant 1.$$

Let now $\xi(\bar{x}(\theta)) \in \mathcal{A}_e$ be the control that keeps the state process above the threshold $\bar{x}(\theta)$. Since U is bounded from below as $U_x(x,\theta) \geqslant 0$ on $\mathbb{R}_+$, and because $\xi(\bar{x}(\theta))$ increases only when $X^{x;\xi(\bar{x}(\theta))} \geqslant \bar{x}(\theta)$, a verification theorem (similar to Theorem 4.1 at p. 300 in [81]) shows that $\lambda(\theta) = \Lambda = b(\bar{x}(\theta)) + h\big(\bar{x}(\theta), \theta\big)$, and that the process $\xi(\bar{x}(\theta))$ is optimal.

Step 2. Given $\bar{x}(\theta)$ as in Step 1, we impose the consistency condition on θ; that is, we look for $\bar{\theta}$ such that

$$\int_{\bar{x}(\bar{\theta})}^{\infty} (f(y) - F^{-1}(\bar{\theta}))m'(y)dy = 0.$$

As in Step 2 in the proof of Theorem 5.3.2, we can show that such a $\bar{\theta}$ exists and it is in fact unique. Therefore, setting $(\bar{x}_e, \bar{\theta}_e) := (\bar{x}(\bar{\theta}), \bar{\theta})$ and $\bar{\xi}^e := \xi(\bar{x}_e)$ we conclude that $(\bar{\xi}^e, \bar{\theta}_e)$ is the unique equilibrium of the ergodic MFG problem. Moreover, such equilibrium is characterized by the couple $(\bar{x}_e, \bar{\theta}_e)$ uniquely solving (5.3.17), and the value at equilibrium is given by $b(\bar{x}_e) + h(\bar{x}_e, \bar{\theta}_e)$. This completes the proof of the theorem. □

5.4 Connecting discounted and ergodic MFGs: The Abelian limit

A natural question is whether the mean field equilibrium and the relative equilibrium value of the discounted game can be related to those of the ergodic game in the limit $r \downarrow 0$. In this section we provide a positive answer to the previous question by showing the validity of the so-called Abelian limit for the equilibrium value of the discounted game. Moreover, we also prove convergence of the equilibrium boundary of the discounted game towards that of the ergodic game. Although similar results are known in the literature on stochastic singular control problems (cf. [6], [107], [163]), to our knowledge they appear here for the first time within this literature in the mean field context.

The main idea of the subsequent analysis is to show suitable regularity, with respect to the discount factor r in a neighborhood of 0, of the solutions to the systems of equations provided in Theorems 5.3.2 and 5.3.4, which in fact completely characterize the MFG equilibria.

Throughout this section, we let Assumptions 5.1.1, 5.1.2, 5.2.1, and 5.3.3 hold, and we require that Assumption 5.3.1 is also satisfied for any $r > 0$.

Let then $c > 0$ be as in Assumption 5.3.3, and define the functions

$$\begin{aligned}
\Pi(y,\theta;r) &:= \widehat{m}'(y)\Big(h_x(y,\theta) - (r - b'(y))\Big), && x,\ \theta > 0,\ r \in [-c,1], \\
K(x,\theta;r) &:= \int_x^{\infty} \widehat{\phi}_r(y)\Pi(y,\theta,r)dy, && x,\ \theta > 0,\ r \in [-c,1], \\
\widehat{K}(x,\theta;r) &:= K(x,\theta;r)/\widehat{\phi}_r(x), && x,\ \theta > 0,\ r \in [-c,1], \\
\widehat{G}(x,\theta) &:= \int_x^{\infty} (f(y) - F^{-1}(\theta))m'(y)dy, && x,\ \theta > 0.
\end{aligned}$$

Define next $\Phi : \mathbb{R}_+^2 \times (-c,1) \to \mathbb{R}^2$ by setting

$$\Phi(x,\theta;r) := (\widehat{K}(x,\theta;r), \widehat{G}(x,\theta)). \tag{5.4.1}$$

The function Φ describes the system of equations determining the MFG equilibria of the discounted problem and of the ergodic problem. Indeed, for each $r > 0$, since $\widehat{\phi}_r > 0$, we have $K(x, \theta; r) = 0$ if and only if $\widehat{K}(x, \theta; r) = 0$; hence, according to Theorem 5.3.2, for each $r > 0$, there exists a unique $(\bar{x}_r, \bar{\theta}_r)$ such that $\Phi(\bar{x}_r, \bar{\theta}_r; r) = 0$. Analogously, according to Theorem 5.3.4, there exists a unique $(\bar{x}_e, \bar{\theta}_e)$ such that $\Phi(\bar{x}_e, \bar{\theta}_e; 0) = 0$.

Clearly, continuity of Φ is a necessary ingredient for the previously discussed convergence of the equilibrium of the discounted MFG towards that of the ergodic MFG.

Lemma 5.4.1. *The function $\Phi : \mathbb{R}^2_+ \times (-c, 1) \to \mathbb{R}^2$ is continuous.*

Proof. We prove only the continuity of $\widehat{K}$, the continuity of $\widehat{G}$ being obvious. Fix $(x, \theta, r) \in \mathbb{R}^2_+ \times (-c, 1)$, and a sequence $\{x^n, \theta^n, r^n\}_{n\in\mathbb{N}}$ converging to (x, θ, r). Without loss of generality, we can assume that $a := x/2 < x^n < 2x$ and that $\theta/2 < \theta^n < 2\theta$ for each $n \in \mathbb{N}$. Also, since the functions $\widehat{\phi}_r$ are defined up to a positive multiplicative factor, we can assume that $\widehat{\phi}_r(a) = 1$ for each $r \in (-c, 1)$. Hence, for $0 < a < y$, defining $\tau_a^y := \inf\{t \geqslant 0 \,|\, \widehat{X}_t^y \leqslant a\}$, we have (cf. Chapter II in [29])

$$\widehat{\phi}_r(y) = \frac{\widehat{\phi}_r(y)}{\widehat{\phi}_r(a)} = \mathbb{E}\bigg[\exp\bigg(\int_0^{\tau_a^y} (b'(\widehat{X}_s^y) - r)ds\bigg)\bigg]. \tag{5.4.2}$$

Therefore, for each $0 < a < y$ and $-c \leqslant r < \bar{r} \leqslant 1$, one has

$$\widehat{\phi}_{\bar{r}}(y) = \mathbb{E}\bigg[\exp\bigg(\int_0^{\tau_a^y} (b'(\widehat{X}_s^y) - \bar{r})ds\bigg)\bigg] \leqslant \mathbb{E}\bigg[\exp\bigg(\int_0^{\tau_a^y} (b'(\widehat{X}_s^y) - r)ds\bigg)\bigg] = \widehat{\phi}_r(y),$$

so that

$$\widehat{\phi}_1(y) \leqslant \widehat{\phi}_r(y) \leqslant \widehat{\phi}_{-c}(y), \quad y > a, \quad r \in (-c, 1). \tag{5.4.3}$$

We next prove that $\widehat{\phi}_{r^n}(x^n) \to \widehat{\phi}_r(x)$ as $n \to \infty$. In order to do so, set

$$\alpha^n := \int_0^{\tau_a^{x^n}} (b'(\widehat{X}_s^{x^n}) - r^n)ds \quad \text{and} \quad \alpha := \int_0^{\tau_a^x} (b'(\widehat{X}_s^x) - r)ds,$$

and observe that $X_s^{x^n} \to X_s^x$ $\mathbb{P} \otimes ds$-a.e. and that $\tau_a^{x^n} \to \tau_a^x$ $\mathbb{P}$-a.s., as $n \to \infty$. Hence,

$$\mathbb{1}_{(0,\tau_a^{x^n})}(s)(b'(\widehat{X}_s^{x^n}) - r^n) \to \mathbb{1}_{(0,\tau_a^x)}(s)(b'(\widehat{X}_s^x) - r), \quad \mathbb{P} \otimes ds\text{-a.s., as } n \to \infty.$$

This, thanks to the Liptschitz continuity of b, allows to invoke the dominated convergence theorem in order to deduce that

$$\alpha^n \to \alpha, \ \mathbb{P}\text{-a.s., as } n \to \infty. \tag{5.4.4}$$

From (5.4.4) and (5.4.2), using that $b'(X_s^{x^n}) - r^n < -c$ for each $n \in \mathbb{N}$, we can employ the dominated convergence theorem once more in order to conclude that

$$\widehat{\phi}_{r^n}(x^n) = \mathbb{E}[\exp(\alpha^n)] \to \widehat{\phi}_r(x) = \mathbb{E}[\exp(\alpha)], \ \text{as } n \to \infty. \tag{5.4.5}$$

In the same way, we can prove that

$$\widehat{\phi}_{r^n}(y) \to \widehat{\phi}_r(y), \ \text{for each } y > a, \text{ as } n \to \infty. \tag{5.4.6}$$

Next, from (5.4.5) and (5.4.6), we have, for a.a. $y > a$,

$$\mathbb{1}_{(x^n,\infty)}(y)\frac{\widehat{\phi}_{r^n}(y)}{\widehat{\phi}_{r^n}(x^n)}\Pi(y,\theta^n,r^n) \to \mathbb{1}_{(x,\infty)}(y)\frac{\widehat{\phi}_r(y)}{\widehat{\phi}_r(x)}\Pi(y,\theta,r), \text{ as } n\to\infty. \tag{5.4.7}$$

Moreover, thanks to (5.4.3), we have the estimate

$$\left|\mathbb{1}_{(x^n,\infty)}(y)\frac{\widehat{\phi}_{r^n}(y)}{\widehat{\phi}_{r^n}(x^n)}\Pi(y,\theta^n,r^n)\right| \leqslant \mathbb{1}_{(a,\infty)}(y)\frac{\widehat{\phi}_{-c}(y)}{\widehat{\phi}_1(2x)}\Pi(y,\theta/2,-c) \in \mathbb{L}^1(\mathbb{R}), \qquad n\in\mathbb{N}.$$

This, together with (5.4.7), allows to invoke the dominated convergence theorem and obtain that

$$\widehat{K}(x^n,\theta^n;r^n) = \int_{x^n}^{\infty}\frac{\widehat{\phi}_{r^n}(y)}{\widehat{\phi}_{r^n}(x^n)}\Pi(y,\theta^n,r^n)dy \to \widehat{K}(x,\theta;r) = \int_x^{\infty}\frac{\widehat{\phi}_r(y)}{\widehat{\phi}_r(x)}\Pi(y,\theta,r)dy,$$

as $n\to\infty$, thus providing the claimed continuity of $\widehat{K}$.

□

For each $r \in (0,1]$, denote now by $V(x,r;\bar{\theta}_r)$ the equilibrium value of the MFG with discount factor r. We are then in the condition of stating the main result of this section.

Theorem 5.4.2. *For any $x \in \mathbb{R}_+$, one has*

$$\lim_{r\downarrow 0}(\bar{x}_r,\bar{\theta}_r) = (\bar{x}_e,\bar{\theta}_e) \quad \textit{and} \quad \lim_{r\downarrow 0} rV(x,\bar{\theta}_r;r) = \lambda(\bar{\theta}_e).$$

Proof. We divide the proof in two steps.

Step 1. In this step we prove the first of the two claimed limits. This is done via a suitable application of the implicit function theorem on the function Φ (cf. (5.4.1)), that defines the system of equations characterizing the MFG equilibria.

For convenience of notation, set $(\bar{x}_0,\bar{\theta}_0) := (\bar{x}_e,\bar{\theta}_e)$. Thanks to Lemma 5.4.1, the map Φ is continuous and, by Theorem 5.3.4, we have $\Phi(\bar{x}_0,\bar{\theta}_0;0) = 0$. By invoking Theorem 1.1 in [117], the function $r \mapsto (\bar{x}_r,\bar{\theta}_r)$ is continuous in a neighborhood $(-\delta,\delta)$ of 0 if and only if there exists neighborhoods $(-\varepsilon,\varepsilon) \subset (-c,1)$ and $B \subset \mathbb{R}^2_+$ of 0 and of $(\bar{x}_0,\bar{\theta}_0)$ respectively, such that the map $\Phi(\cdot,\cdot;r) : B \to \mathbb{R}^2$ is locally injective for each $r \in (-\varepsilon,\varepsilon)$. Therefore, we only need to prove local injectivity of the map $\Phi(\cdot,\cdot;r)$, and, in order to accomplish that, we will employ the local inversion theorem. In particular, by observing that, for each $r \in (-c,1)$, we have $\Phi(\cdot,\cdot;r) \in C^1(\mathbb{R}^2_+)$, it is enough to show that $\det \mathcal{J}\Phi(\bar{x}_0,\bar{\theta}_0;0) \neq 0$ and that $\det \mathcal{J}\Phi$ is continuous in a neighborhood of $(\bar{x}_0,\bar{\theta}_0;0)$, where $\det \mathcal{J}\Phi$ denotes the determinant of the Jacobian matrix of Φ in the variable (x,θ).

We begin by computing the partial derivatives of $\widehat{K}$:

$$\begin{aligned}\partial_x\widehat{K}(x,\theta;r) &= -\Pi(x,\theta;r) - \widehat{K}(x,\theta;r)\frac{\widehat{\phi}_r'(x)}{\widehat{\phi}_r(x)} \\ &= -\Pi(x,\theta;r) + \widehat{K}(x,\theta;r)\widehat{S}'(x)\int_x^{\infty}\frac{\widehat{\phi}_r(y)}{\widehat{\phi}_r(x)}(r-b'(y))\widehat{m}'(y)dy,\end{aligned}$$

where, in the second equality, we have used (5.1.13). In particular, by repeating arguments similar to those in the proof of Lemma 5.4.1, one can show that $\partial_x \widehat{K}(x,\theta;r)$ is continuous in (x,θ,r). Also, by Theorem 5.3.4, we have $K(\bar{x}_0,\bar{\theta}_0;0)=0$, so that

$$\partial_x \widehat{K}(\bar{x}_0,\bar{\theta}_0;0) = -\Pi(\bar{x}_0,\bar{\theta}_0;0) < 0, \tag{5.4.8}$$

where the latter inequality follows from the fact that, arguing as in the proof of Theorem 5.3.2, one has

$$h_x(\bar{x}_0,\bar{\theta}_0) + b'(\bar{x}_0) > 0 \quad \text{and} \quad f(\bar{x}_0) - F^{-1}(\bar{\theta}_0) < 0. \tag{5.4.9}$$

Next, thanks to Condition 1d in Assumption 5.2.1, we find

$$\partial_\theta \widehat{K}(x,\theta;r) = \int_x^\infty \frac{\widehat{\phi}_r(y)}{\widehat{\phi}_r(x)}\Big(h_{x\theta}(y,\theta) - \big(r - b'(y)\big)\widehat{m}'(y)dy,$$

which, through arguments similar to those in the proof of Lemma 5.4.1, can be shown to be continuous in (x,θ,r). Moreover, since $h_{x\theta} \leqslant 0$ and $b' - r < 0$, we have

$$\partial_\theta \widehat{K}(\bar{x}_0,\bar{\theta}_0;0) < 0. \tag{5.4.10}$$

Finally, the function $\widehat{G}$ clearly belongs to $C^1(\mathbb{R}^2_+)$. Moreover,

$$\partial_\theta \widehat{G}(\bar{x}_0,\bar{\theta}_0) = -\frac{1}{F'(F^{-1}(\bar{\theta}_0))}\int_{\bar{x}_0}^\infty m'(y)dy < 0, \tag{5.4.11}$$

and by (5.4.9) we have

$$\partial_x \widehat{G}(\bar{x}_0,\bar{\theta}_0) = -(f(\bar{x}_0) - F^{-1}(\bar{\theta}_0))m'(\bar{x}_0) > 0. \tag{5.4.12}$$

Therefore, by employing (5.4.8), (5.4.10), (5.4.11) and (5.4.12), and using the continuity of $\det \mathcal{J}\Phi$, we find neighborhoods $(-\varepsilon,\varepsilon)$ and B of 0 and $(\bar{x}_0,\bar{\theta}_0)$ such that

$$\det \mathcal{J}\Phi(x,\theta;r) = \Big[\partial_x \widehat{K}\,\partial_\theta \widehat{G} - \partial_\theta \widehat{K}\,\partial_x \widehat{G}\Big](x,\theta;r) > 0, \qquad (x,\theta) \in B,\ r \in (-\varepsilon,\varepsilon).$$

By the latter inequality we can then invoke the local inversion theorem in order to deduce that, for each $r \in (-\varepsilon,\varepsilon)$, the function $\Phi(\cdot,\cdot;r): B \to \mathbb{R}^2$ is locally invertible. Therefore, by Theorem 1.1 in [117], the map $r \to (\bar{x}_r,\bar{\theta}_r)$ is continuous in $(-\varepsilon,\varepsilon)$.

Step 2. With regard to Theorem 5.3.2 and its proof, we have that

$$rV(x,\bar{\theta}_r;r) = rV(\bar{x}_r,\bar{\theta}_r;r) + r(x-\bar{x}_r), \quad x \leqslant \bar{x}_r.$$

Since $V(\cdot,\theta;r) \in C^2(\mathbb{R}_+)$, by using the fact that $(\mathcal{L}_X - r)V(x,\bar{\theta}_r;r) + h(x,\bar{\theta}_r) = 0$ for $x \geqslant \bar{x}_r$, we find that

$$rV(\bar{x}_r,\bar{\theta}_r;r) = b(\bar{x}_r) + h(\bar{x}_r,\bar{\theta}_r).$$

Therefore, for $x \in \mathbb{R}_+$ we can write

$$\begin{aligned} rV(x,\bar{\theta}_r;r) &= r\int_{\bar{x}_r}^x V_x(z,\bar{\theta}_r;r)dz + rV(\bar{x}_r,\bar{\theta}_r;r) \\ &= r\int_{\bar{x}_r}^x V_x(z,\bar{\theta}_r;r)dz + b(\bar{x}_r) + h(\bar{x}_r,\bar{\theta}_r). \end{aligned} \tag{5.4.13}$$

Moreover, we have $V_x(\cdot,\bar\theta_r;r) \geqslant 0$ for each $r>0$. Indeed, for $z \leqslant \bar z$ and any control $\xi \in \mathcal{A}_d$, by a comparison theorem (see, e.g., Theorem 54 at p. 324 in [145]) we have $X_t^{z;\xi} \leqslant X_t^{\bar z;\xi}$ for each $t \geqslant 0$, $\mathbb{P}$-a.s. This, together with the monotonicity of $h(\cdot,\bar\theta_r)$, implies that

$$V(z,\bar\theta_r;r) = \sup_{\xi\in\mathcal{A}_d} J(z,\xi,\bar\theta_r;r) \leqslant \sup_{\xi\in\mathcal{A}_d} J(\bar z,\xi,\bar\theta_r;r) = V(\bar z,\bar\theta_r;r),$$

so that $V_x(\cdot,\bar\theta_r;r) \geqslant 0$ for each $r>0$. Also, since $V(\cdot,\bar\theta_r;r)$ solves the equation (5.3.11) in the proof of Theorem 5.3.2, we have $V_x(\cdot,\bar\theta_r;r) \leqslant 1$ for each $r>0$, which allows to conclude that

$$0 \leqslant V_x(\cdot,\bar\theta_r;r) \leqslant 1, \qquad \text{for each } r>0.$$

The latter, together with the limits proved in Step 1, allows to use the dominated convergence theorem to take limits as $r \downarrow 0$ in (5.4.13), and to conclude that

$$\lim_{r\downarrow 0} rV(x,\bar\theta_r;r) = \lim_{r\downarrow 0}(b(\bar x_r) + h(\bar x_r,\bar\theta_r)) = b(\bar x_e) + h(\bar x_e,\bar\theta_e) = \lambda(\bar\theta_e),$$

where the last equality follows from Theorem 5.3.4. This completes the proof of the theorem.

□

5.5 MFG vs. *N*-player games: approximation results

In the previous sections, we have established existence and uniqueness of the solutions to both the discounted and ergodic MFGs with singular controls. Here we provide the connection of these mean field solutions to symmetric N-player games. In particular, we show that each mean field game solution approximates the Nash equilibrium of a suitable N-player game. Furthermore, by exploiting the Abelian limit, we find that the mean field equilibrium of the discounted game realizes an ε-Nash equilibrium for the N-player ergodic game, when N is large and r is small. These results have the two following implications: on the one hand, they shed light on the "closeness" of N-player discounted games with the N-player ergodic games, when N is large and r is small; on the other hand, they provide an operative way of constructing approximate equilibria.

Throughout this section, we let Assumptions 5.1.1, 5.1.2, 5.2.1, and 5.3.3 hold, and we require that Assumption 5.3.1 is also satisfied for any $r>0$.

5.5.1 *N*-player games and the MFGs with random initial conditions

The N-player games are described as follows. Let the filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ support a standard Brownian motion W, and a sequence $(W^i)_{i\in\mathbb{N}}$ of independent $\mathbb{F}$-Brownian motions, independent from W. Suppose also that the filtered probability space is rich enough to allow for a sequence $(z_0^i)_{i\in\mathbb{N}}$ of i.i.d. square-integrable

$\mathbb{R}_+$-valued $\mathcal{F}_0$-random variables, independent from W and $(W^i)_{i\in\mathbb{N}}$, and with distribution μ_0. For each $i \in \{1,\dots,N\}$, player i chooses an (open-loop) strategy $\xi^i \in \mathcal{A}$ in order to control its state process X^{i,ξ^i}, which evolves according to

$$dX_t^{i,\xi^i} = b(X_t^{i,\xi^i})dt + \sigma(X_t^{i,\xi^i})dW_t^i + d\xi_t^i, \quad X_{0-}^{i,\xi^i} = z_0^i. \tag{5.5.1}$$

We point out that, in this section, we will not stress anymore the dependence of the processes X^{i,ξ^i} on the initial conditions z_0^i.

For strategies $\xi^i \in \mathcal{A}$, we denote by $\boldsymbol{\xi}^{-i} = (\xi^1,\dots,\xi^{i-1},\xi^{i+1},\dots,\xi^N)$ the vector of strategies picked by player i's opponents, and we define profile strategies by $(\xi^i,\boldsymbol{\xi}^{-i}) := (\xi^1,\dots,\xi^N)$. We set

$$\theta^N_{\boldsymbol{\xi}^{-i}} := \lim_{t\to\infty} \frac{1}{t}\int_0^t F\Big(\frac{1}{N-1}\sum_{j\neq i} f(X_s^{j,\xi^j})\Big)ds, \tag{5.5.2}$$

and, for $q \in \{d,e\}$ and $\mathcal{A}_q$ as in (5.2.4), introduce the sets:

$$\widehat{\mathcal{A}}_q^{N-1} := \Big\{\boldsymbol{\xi}^{-i} \in \mathcal{A}_q^{N-1} : \theta^N_{\boldsymbol{\xi}^{-i}} \text{ exists finite a.s.}\Big\}.$$

Then, for any $\boldsymbol{\xi}^{-i} \in \widehat{\mathcal{A}}_e^{N-1}$ or $\boldsymbol{\xi}^{-i} \in \widehat{\mathcal{A}}_d^{N-1}$, the ergodic and the discounted payoffs of player i, reacting to her opponent's strategies $\boldsymbol{\xi}^{-i}$, are respectively given by

$$G^i(\xi^i,\boldsymbol{\xi}^{-i}) := \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\int_0^T h\Big(X_t^{i,\xi^i},\theta^N_{\boldsymbol{\xi}^{-i}}\Big)dt - \xi_T^i\Big], \quad \xi^i \in \mathcal{A}_e, \tag{5.5.3}$$

and

$$J^i(\xi^i,\boldsymbol{\xi}^{-i};r) := \mathbb{E}\Big[\int_0^\infty e^{-rt}h\Big(X_t^{i,\xi^i},\theta^N_{\boldsymbol{\xi}^{-i}}\Big)dt - \int_{[0,\infty)} e^{-rt}d\xi_t^i\Big], \quad \xi^i \in \mathcal{A}_d. \tag{5.5.4}$$

Definition 14 (ε-Nash Equilibrium)**.** *For $\varepsilon > 0$,*

1. *$\bar{\boldsymbol{\xi}} = (\bar\xi^1,\dots,\bar\xi^N) \in \mathcal{A}_e^N$ is called ε-Nash equilibrium (ε-NE) of the ergodic N-player game if for any $i = 1,\dots,N$ we have $\bar{\boldsymbol{\xi}}^{-i} \in \widehat{\mathcal{A}}_e^{N-1}$ and*

$$G^i(\bar\xi^i,\bar{\boldsymbol{\xi}}^{-i}) \geqslant G^i(\xi^i;\bar{\boldsymbol{\xi}}^{-i}) - \varepsilon, \quad \xi^i \in \mathcal{A}_e;$$

2. *$\bar{\boldsymbol{\xi}} = (\bar\xi^1,\dots,\bar\xi^N) \in \mathcal{A}_d^N$ is called ε-Nash equilibrium (ε-NE) of the discounted N-player game if for any $i = 1,\dots,N$ we have $\bar{\boldsymbol{\xi}}^{-i} \in \widehat{\mathcal{A}}_d^{N-1}$ and*

$$J^i(\bar\xi^i,\bar{\boldsymbol{\xi}}^{-i};r) \geqslant J^i(\xi^i;\bar{\boldsymbol{\xi}}^{-i};r) - \varepsilon, \quad \xi^i \in \mathcal{A}_d.$$

In order to approximate Nash equilibria, for any $\theta > 0$ we define (with slight abuse of notation) the profit functionals for the mean field game problems when the initial conditions for the SDEs (5.5.1) are random variables:

$$G(\xi,\theta) := \int_{\mathbb{R}_+} G(x,\xi,\theta)\mu_0(dx), \quad \xi \in \mathcal{A}_e, \tag{5.5.5}$$
$$J(\xi,\theta;r) := \int_{\mathbb{R}_+} J(x,\xi,\theta;r)\mu_0(dx) \quad \xi \in \mathcal{A}_d,\ r > 0,$$

where $G(x,\xi,\theta)$ and $J(x,\xi,\theta;r)$ are defined in (5.2.3) and (5.2.2), respectively.

Remark 5.5.1 (On the initial distribution)**.** *We point out that all the results in the previous sections hold true also for profit functionals as in (5.5.5); that is, if the deterministic initial condition $X_{0-} = x \in \mathbb{R}_+$ is replaced by $X_{0-} = z_0$, for a positive square-integrable $\mathcal{F}_0$-random variable z_0, with distribution μ_0. In particular, by the Markov property of the solution to the reflected Skorokhod problem (cf. Theorem 1.2.2 and Exercise 1.2.2 in [144]), the MFG equilibria $(\bar{\xi}^r, \bar{\theta}_r)$ and $(\bar{\xi}^e, \bar{\theta}_e)$ are still characterized by couples $(\bar{x}_r, \bar{\theta}_r)$ and $(\bar{x}_e, \bar{\theta}_e)$ solving the systems of equations provided in Theorems 5.3.2 and 5.3.4, respectively.*

5.5.2 Strategies and preliminary estimates

For any $i = 1, \dots, N$, consider the policy $\bar{\xi}^{i,e} \in \mathcal{A}_e$ according to which the state is reflected upward at the boundary $\bar{x}_e$. Similarly, for $r > 0$, the policy $\bar{\xi}^{i,r} \in \mathcal{A}_d$ makes the state upward reflected at $\bar{x}_r$. We observe that, for $i = 1, ..., N$ and $q \in \{d, e\}$, the profile strategies $(\bar{\xi}^{1,q}, \dots, \bar{\xi}^{i-1,q}, \bar{\xi}^{i+1,q}, \dots, \bar{\xi}^{N,q}) \in \widehat{\mathcal{A}}_q^{N-1}$. Then, define accordingly:

$$\bar{\boldsymbol{\xi}}^e := (\bar{\xi}^{1,e}, \dots, \bar{\xi}^{N,e}), \quad \bar{\boldsymbol{\xi}}^r := (\bar{\xi}^{1,r}, \dots, \bar{\xi}^{N,r}), \quad \theta_e^{i,N} := \theta^N_{\bar{\boldsymbol{\xi}}^{-i,e}}, \quad \theta_r^{i,N} := \theta^N_{\bar{\boldsymbol{\xi}}^{-i,r}}. \tag{5.5.6}$$

To facilitate our discussion, we enforce some additional requirements on the dynamics of the state processes and on the profit function.

Assumption 5.5.2.

1. *There exists $x_{b,\sigma} > 0$ such that $2\,x\,b(x) + \sigma^2(x) \leqslant 0$ for any $x \geqslant x_{b,\sigma}$;*
2. *For any $a > 0$, there exists a constant $C > 0$ such that*

$$|h(x, \theta_1) - h(x, \theta_2)| \leqslant C(1 + |x|)|\theta_1 - \theta_2|, \quad \forall \theta_1, \theta_2 \geqslant a,$$

for all $x \in \mathbb{R}$.

Notice that the previous conditions are satisfied by the benchmark cases in which $b(x) = -\delta x$ or $b(x) = \delta(\lambda - x)$ and $\sigma(x) = \sigma x$ (that is, geometric or affine dynamics) when $2\delta \geqslant \sigma^2$, and for a profit function $h(x, \theta) = x^\beta\,\theta^{-(1+\beta)}$, for some elasticity $\beta \in (0, 1)$.

For $\theta > 0$ and $r \geqslant 0$, let $\widehat{x}_r(\theta)$ be as in Assumption 5.3.1 and 5.3.3. It is easy to show that the function $\widehat{x}_r(\theta)$ is continuous in (θ, r) so that, by the convergence in Theorem 5.4.2, we can set

$$\widehat{B} := 2 \max\left\{ \sup_{r \in (0,1]} \widehat{x}_r(F(f(\bar{x}_r))),\, \widehat{x}_0(F(f(\bar{x}_e))),\, \sup_{r \in (0,1]} \bar{x}_r,\, \bar{x}_e \right\} < \infty.$$

Next, for any $i = 1, ..., N$, by definition of $\bar{\xi}^{i,e}$, we have $X_t^{i,\bar{\xi}^{i,e}} \geqslant \bar{x}_e$, $\mathbb{P}$-a.s., for any $t > 0$. This fact, for $\theta_e^{i,N}$ as in (5.5.6), by monotonicity of f and F implies that $\theta_e^{i,N} \geqslant F(f(\bar{x}_e))$, $\mathbb{P}$-a.s. In the same way, $\theta_r^{i,N} \geqslant F(f(\bar{x}_r))$, $\mathbb{P}$-a.s. for each $i = 1, ..., N$ and $r > 0$. Therefore, since for $r \geqslant 0$ the functions $\widehat{x}_r$ are nonincreasing in θ, by definition of $\widehat{B}$ we have

$$\widehat{x}_0(\theta_e^{i,N}) \leqslant \widehat{x}_0(F(f(\bar{x}_e))) \leqslant \widehat{B}, \quad \widehat{x}_r(\theta_r^{i,N}) \leqslant \widehat{x}_r(F(f(\bar{x}_r))) \leqslant \widehat{B},\ r > 0. \tag{5.5.7}$$

Next, define the sets

$$\mathcal{A}_q(\widehat{B}) := \{\xi \in \mathcal{A}_q \mid \operatorname{supp}(d\xi) \cap \{X^{i,\xi} \geqslant \widehat{B}\} = \emptyset, \mathbb{P}\text{-a.s.}\}, \quad \text{for } q \in \{d, e\}.$$

Notice that, since $\bar{x}_r, \bar{x}_e \leqslant \widehat{B}/2$, we have $\bar{\xi}^{i,r} \in \mathcal{A}_d(\widehat{B})$ and $\bar{\xi}^{i,e} \in \mathcal{A}_e(\widehat{B})$. Moreover, we have the following a priori estimates.

Lemma 5.5.3. *We have*

$$\sup_{\xi \in \mathcal{A}_e(\widehat{B})} \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\bigg[\int_0^T |X_t^{i,\xi}|^2 dt\bigg] < \infty$$

and

$$\sup_{\xi \in \mathcal{A}_d(\widehat{B})} \mathbb{E}\bigg[\int_0^T e^{-rt} |X_t^{i,\xi}|^2 dt\bigg] < \infty, \quad \textit{for } r > 0.$$

Proof. We prove only the first estimate, the proof of the second being analogous. Let $i \in \{1, \dots, N\}$ be given and fixed. Recall the definition of $x_{b,\sigma}$, set $\widehat{L} := \max\{\widehat{B}, x_{b,\sigma}\}$, and let $\xi \in \mathcal{A}_e(\widehat{B})$. Let then $(\tau_k^i, \bar{\tau}_k^i)_{k \geqslant 1}$ be a sequence of stopping times such that $0 \leqslant \tau_1^i \leqslant \bar{\tau}_1^i \leqslant \tau_2^i \leqslant \bar{\tau}_2^i \leqslant \dots$, $\mathbb{P}$-a.s., and such that $\{X^{i,\xi} \geqslant \widehat{L}\} = \bigcup_{k \geqslant 1} [\tau_k^i, \bar{\tau}_k^i]$.

By employing Itô's rule on the process $\{|X_t^{i,\xi}|^2\}_{t \in [\tau_k^i, \bar{\tau}_k^i]}$, we obtain

$$\begin{aligned}
|X_t^{i,\xi}|^2 &= |X_{\tau_k^i}^{i,\xi}|^2 + \int_{\tau_k^i}^t \Big(2X_s^{i,\xi} b(X_s^{i,\xi}) + \sigma^2(X_s^{i,\xi})\Big) ds + \int_{\tau_k^i}^t 2X_s^{i,\xi} \sigma(X_s^{i,\xi}) dW_s^i \\
&\leqslant |X_{\tau_k^i}^{i,\xi}|^2 + \int_{\tau_k^i}^t 2X_s^{i,\xi} \sigma(X_s^{i,\xi}) dW_s^i.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^T \mathbb{E}[|X_t^{i,\xi}|^2] dt &= \int_0^T \mathbb{E}\Big[|X_t^{i,\xi}|^2 \mathbb{1}_{\{X_t^{i,\xi} \leqslant \widehat{L}\}} + |X_t^{i,\xi}|^2 \mathbb{1}_{\{X_t^{i,\xi} \geqslant \widehat{L}\}}\Big] dt \\
&\leqslant \widehat{L}^2 T + \sum_{k \geqslant 1} \int_0^T \mathbb{E}\bigg[\mathbb{1}_{(\tau_k^i, \bar{\tau}_k^i)}(t) \bigg(|X_{\tau_k^i}^{i,\xi}|^2 + \int_{\tau_k^i}^t 2X_s^{i,\xi} \sigma(X_s^{i,\xi}) dW_s^i\bigg)\bigg] dt \\
&\leqslant (2\widehat{L}^2 + \mathbb{E}[|z_0^i|^2]) T. \qquad (5.5.8)
\end{aligned}$$

In the last inequality above we have used that the expectation of the stochastic integral vanishes and that, because $\xi \in \mathcal{A}_e(\widehat{B})$, one has either $X_{\tau_k^i}^{i,\xi} = z_0^i$ if $\tau_k^i = 0$ or $X_{\tau_k^i}^{i,\xi} = \widehat{L}$ if $\tau_k^i > 0$.

Since the right-hand side of (5.5.8) does not depend on the choice of $\xi \in \mathcal{A}_e(\widehat{B})$, the claim is then easily obtained.

□

Lemma 5.5.4. *We have*

$$\sup_{\xi \in \mathcal{A}_e} G^i(\xi, \bar{\boldsymbol{\xi}}^{-i,e}) = \sup_{\xi \in \mathcal{A}_e(\widehat{B})} G^i(\xi, \bar{\boldsymbol{\xi}}^{-i,e}),$$

and

$$\sup_{\xi \in \mathcal{A}_d} J^i(\xi, \bar{\boldsymbol{\xi}}^{-i,r}; r) = \sup_{\xi \in \mathcal{A}_d(\widehat{B})} J^i(\xi, \bar{\boldsymbol{\xi}}^{-i,r}; r), \quad \textit{for } r > 0.$$

Proof. For $z > 0$, set

$$m_z'(x) := \mathbb{1}_{[z,\infty)}(x)\frac{m'(x)}{\int_z^\infty m'(y)dy}.$$

Exploiting the estimates from Lemma 5.5.3 and using results from the ergodic theory (see, e.g., p. 37 in [29]), for $q \in \{d, e\}$ we find, $\mathbb{P}$-a.s.,

$$\begin{aligned}\theta_q^{i,N} &= \lim_{t\to\infty}\frac{1}{t}\int_0^t F\Big(\frac{1}{N-1}\sum_{j\neq i} f(X_s^{j,\bar{\xi}^{j,q}})\Big)ds \\ &= \int_{\mathbb{R}^{N-1}} F\Big(\frac{1}{N-1}\sum_{j\neq i} f(x^j)\Big)\prod_{j\neq i} m'_{\bar{x}_q}(x^j)dx^j,\end{aligned} \tag{5.5.9}$$

so that $\theta_q^{i,N}$ is in fact deterministic. It is shown in the proofs of Theorems 5.3.2 and 5.3.4 that, for any $\theta > 0$, the optimal control never acts when the optimally controlled state process lies in the set $\{y \,|\, h_x(y,\theta) - (r - b'(y)) < 0\} = \{y \,|\, y > \widehat{x}_r(\theta)\}$. This, by the definition (5.5.7), completes the proof of the lemma. □

5.5.3 Approximation of Nash equilibria

We are finally ready to state the main result of this section. It states that mean field game equilibria realize approximate Nash equilibria in the related symmetric N-player games defined in Definition 14, when N is large and/or r is small.

Theorem 5.5.5. *The following approximations hold true:*

1. $\bar{\boldsymbol{\xi}}^e$ *is an* ε_N*-NE for the ergodic* N*-player game with* $\varepsilon_N \to 0$ *as* $N \to \infty$*;*
2. $\bar{\boldsymbol{\xi}}^r$ *is an* $\varepsilon_{N,r}$*-NE for the ergodic* N*-player game with* $\varepsilon_{N,r} \to 0$ *as* $N \to \infty$ *and* $r \to 0$*;*
3. $\bar{\boldsymbol{\xi}}^r$ *is an* ε_N*-NE for the discounted* N*-player game with* $\varepsilon_N \to 0$ *as* $N \to \infty$*;*
4. $\bar{\boldsymbol{\xi}}^e$ *is an* $\varepsilon_{N,r}$*-NE for the discounted* N*-player game with* $\varepsilon_{N,r} \to 0$ *as* $N \to \infty$ *and* $r \to 0$.

Proof. We will prove only Claims 1 and 2, as the proof of Claims 3 and 4 follows similar arguments.

Proof of Claim 1. For $\xi \in \mathcal{A}_e(\widehat{B})$, set

$$R^N(\xi) := G^i(\xi, \bar{\boldsymbol{\xi}}^{-i,e}) - G(\xi, \bar{\theta}_e).$$

By Theorem 5.3.4, the control policy $\bar{\xi}^{i,e}$ is optimal for the MFG problem with ergodic cost. Hence

$$G^i(\bar{\xi}^{i,e}, \bar{\boldsymbol{\xi}}^{-i,e}) \geqslant G^i(\xi, \bar{\boldsymbol{\xi}}^{-i,e}) + R^N(\bar{\xi}^{i,e}) - R^N(\xi), \quad \xi \in \mathcal{A}_e(\widehat{B}). \tag{5.5.10}$$

Therefore, since $\bar{\xi}^{i,e} \in \mathcal{A}_e(\widehat{B})$ by definition of $\widehat{B}$, we only need to show that $|R^N(\xi)| \to 0$ as $N \to \infty$, uniformly for $\xi \in \mathcal{A}_e(\widehat{B})$. In order to do so, we first observe that

$$|R^N(\xi)| \leqslant \limsup_{T\to\infty}\left|\frac{1}{T}\mathbb{E}\left[\int_0^T h(X_t^{i,\xi}, \theta_e^{i,N})dt\right] - \frac{1}{T}\mathbb{E}\left[\int_0^T h(X_t^{i,\xi}, \bar{\theta}_e)dt\right]\right|. \tag{5.5.11}$$

Next, using that $\theta_e^{i,N}$ are deterministic (see (5.5.9)) and that $\theta_e^{i,N} \geqslant F(f(\bar{x}_e))$, by Assumption 5.5.2 we have

$$\begin{aligned}
&\sup_{\xi \in \mathcal{A}_e(\widehat{B})} \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T \frac{1}{T}\left|h(X_t^{i,\xi},\theta_e^{i,N}) - h(X_t^{i,\xi},\bar{\theta}_e)\right| dt\right] \\
&\qquad \leqslant \sup_{\xi \in \mathcal{A}_e(\widehat{B})} \limsup_{T\to\infty} \mathbb{E}\left[\int_0^T \frac{1}{T} C(1+|X_t^{i,\xi}|)\left|\theta_e^{i,N} - \bar{\theta}_e\right| dt\right] \\
&\qquad \leqslant \left|\theta_e^{i,N} - \bar{\theta}_e\right| \sup_{\xi \in \mathcal{A}_e(\widehat{B})} \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T C(1+|X_t^{i,\xi}|) dt\right] \\
&\qquad \leqslant \bar{C}\left|\theta_e^{i,N} - \bar{\theta}_e\right|,
\end{aligned} \tag{5.5.12}$$

for a constant $\bar{C} < \infty$ (depending on the initial conditions, but not on $\xi \in \mathcal{A}_e(\widehat{B})$), and where the last inequality follows from Lemma 5.5.3.

For $z > 0$, set

$$m_z'(x) := \frac{m'(x)\mathbb{1}_{[z,\infty)}(x)}{\int_z^\infty m'(y)dy}. \tag{5.5.13}$$

Exploiting the estimates from Lemma 5.5.3 and using results from the ergodic theory (see, e.g., p. 37 in [29]), we find

$$\begin{aligned}
\left|\theta_e^{i,N} - \bar{\theta}_e\right| &= \left|\lim_{t\to\infty} \frac{1}{t}\int_0^t F\Big(\frac{1}{N-1}\sum_{j\neq i} f(X_s^{j,\bar{\xi}^{j,e}})\Big)ds - \bar{\theta}_e\right| \\
&= \left|\int_{\mathbb{R}^{N-1}} F\Big(\frac{1}{N-1}\sum_{j\neq i} f(x^j)\Big)\prod_{j\neq i} m_{\bar{x}_e}'(x^j)dx^j - \bar{\theta}_e\right|.
\end{aligned}$$

Thanks to the assumption of local Lipschitz continuity of F, the growth conditions on f and F (see Condition 2 in Assumption 5.2.1), and the estimates from Lemma 5.5.3, we can then employ a suitable version of Hewitt and Savage's theorem (see Corollary 5.13 in [42]), obtaining

$$\lim_{N\to\infty}\left|\theta_e^{i,N} - \bar{\theta}_e\right| = \left|F\Big(\int_{\mathbb{R}_+} f(z)m_{\bar{x}_e}'(z)dz\Big) - \bar{\theta}_e\right| = 0.$$

The latter, together with (5.5.12) and (5.5.11), gives $|R^N(\xi)| \to 0$ as $N \to \infty$, uniformly over $\xi \in \mathcal{A}_e(\widehat{B})$. Hence, from (5.5.10) and Lemma 5.5.4, we conclude the proof of Claim 1.

Proof of Claim 2. Following an argument similar to the one adopted in the previous step, we use Theorem 5.4.2 and the optimality of $\bar{\xi}^{i,e}$ for the MFG problem with ergodic cost in order to obtain, for any $\xi \in \mathcal{A}_d(\widehat{B})$, the inequality

$$\begin{aligned}
G^i(\bar{\xi}^{i,r},\bar{\boldsymbol{\xi}}^{-i,r}) \geqslant\ & G^i(\xi,\bar{\boldsymbol{\xi}}^{-i,r}) \\
&+ G(\xi,\bar{\theta}_e) - G^i(\xi,\bar{\boldsymbol{\xi}}^{-i,r}) + G^i(\bar{\xi}^{i,r},\bar{\boldsymbol{\xi}}^{-i,r}) - G(\bar{\xi}^{i,r},\bar{\theta}_r) \\
&+ G(\bar{\xi}^{i,r},\bar{\theta}_r) - G(\bar{\xi}^{i,e},\bar{\theta}_r) + G(\bar{\xi}^{i,e},\bar{\theta}_r) - G(\bar{\xi}^{i,e},\bar{\theta}_e) \\
=\ & G^i(\xi,\bar{\boldsymbol{\xi}}^{-i,r}) + G(\bar{\xi}^{i,r},\bar{\theta}_r) - G(\bar{\xi}^{i,e},\bar{\theta}_r) + \varepsilon_{N,r},
\end{aligned} \tag{5.5.14}$$

with $\varepsilon_{N,r}$ vanishing as $N \to \infty$ and $r \to 0$. Hence, it only remains to show that $G(\bar{\xi}^{i,r}, \bar{\theta}_r) - G(\bar{\xi}^{i,e}, \bar{\theta}_r) \to 0$ as $r \to 0$. Let now m'_z be as in (5.5.13). By the ergodic theory (see, e.g., p. 37 in [29]) and estimates from Lemma 5.5.3 , for $q \in \{d, e\}$, we have

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T h(X_t^{i,\bar{\xi}^{i,q}}, \bar{\theta}_r)dt\right] = \int_{\bar{x}_q}^{\infty} h\left(x, \bar{\theta}_r\right) m'_{\bar{x}_q}(x)dx. \tag{5.5.15}$$

Also, as in the proof of Lemma 5.5.3, one can prove that for any $T > 0$ one has $\mathbb{E}[X_T^{i,\bar{\xi}^{i,q}}] \leqslant 2\widehat{L} + \mathbb{E}[z_0^i]$. This allows to deduce that

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[\bar{\xi}_T^{i,q}] &= \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\bigg[X_T^{i,\bar{\xi}^{i,q}} - z_0^i - \int_0^T b(X_s^{i,\bar{\xi}^{i,q}})ds\bigg] \\ &= -\int_{\bar{x}_q}^{\infty} b(x)m'_{\bar{x}_q}(x)dx.\end{aligned} \tag{5.5.16}$$

By the convergence in Theorem 5.4.2, using (5.5.15) and (5.5.16), we conclude that $G(\bar{\xi}^{i,r}, \bar{\theta}_r) - G(\bar{\xi}^{i,e}, \bar{\theta}_r) \to 0$ as $r \to 0$, thus completing the proof of Claim 2. □

Remark 5.5.6. *We point out that results analogous to Claims 1 and 2 in Theorem 5.5.5 can be obtained even if the mean field interaction term* (5.5.2) *in the N-player game is replaced by a time-dependent interaction. As a matter of fact, one can consider*

$$\Theta^N_{\boldsymbol{\xi}^{-i}}(t) := F\bigg(\frac{1}{N-1}\sum_{j\neq i} f(X_t^{j,\xi^j})\bigg),$$

and define accordingly, for $\boldsymbol{\xi}^{-i} \in \mathcal{A}_e^{N-1}$, player i's ergodic profit functional

$$G^i(\xi^i, \boldsymbol{\xi}^{-i}) := \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\bigg[\int_0^T h\left(X_t^{i,\xi^i}, \Theta^N_{\boldsymbol{\xi}^{-i}}(t)\right)dt - \xi_T^i\bigg], \quad \xi^i \in \mathcal{A}_e.$$

Appendices

Appendix A

On the Meyer-Zheng convergence

In this appendix we recall some fact about the so-called Meyer-Zheng topology (see [136]) and we provide some results concerning the tightness of càdlàg processes in such a topology.

Pseudopath topology. Recall that we have defined (cf. Subsection 3.3.3) the pseudopath topology τ_{pp}^{T} on the space $\mathcal{D}^m$ as the topology induced by the convergence in the measure $dt + \delta_T$ on the interval $[0, T]$, where dt denotes the Lebesgue measure and δ_T denotes the Dirac measure at the terminal point T. Notice that we introduce the pseudo-path topology through its characterization proved in Lemma 1 in [136]. Observe that the topology τ_{pp}^{T} is metrizable. If $\{x^n\}_{n\in\mathbb{N}}$ is a sequence of functions in $\mathcal{D}^m$ converging to a function $x \in \mathcal{D}^m$ in the pseudopath topology τ_{pp}^{T}, then we have that (see, e.g., Appendix A.3. at p. 116 in [123])

$$\lim_n \int_0^T \phi(s, x_s^n)\, ds = \int_0^T \phi(s, x_s)\, ds, \quad \text{and} \quad \lim_n x_T^n = x_T, \tag{A.1}$$

for each bounded continuous function $\phi : [0, T] \times \mathbb{R}^m \to \mathbb{R}$.

Meyer-Zheng topology and tightness criteria. The *Meyer-Zheng topology* on $\mathcal{P}(\mathcal{D}^m)$ is the topology of weak convergence of probability measures on the topological space $(\mathcal{D}^m, \tau_{pp}^{T})$.

For a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ consider a càdlàg process $X : \Omega \times [0, T] \to \mathbb{R}^m$, and consider the *conditional variation* of X over the interval $[0, T]$, defined as

$$V_T^{\mathbb{P}}(X) := \sup \sum_{i=1}^{n} \mathbb{E}\left[\left|\mathbb{E}[X_{t_i} - X_{t_{i-1}} | \mathcal{F}_{t_{i-1}}]\right|\right] + \mathbb{E}[|X_{t_n}|], \tag{A.2}$$

where the supremum is taken over all the partitions $0 = t_0 < ... < t_n \leqslant T$, $n \in \mathbb{N}$.

We finally prove, for the sake of completeness, a slightly different version of the classical Meyer-Zheng tightness criterion (see Theorem 4 at p. 360 in [136]), that is useful in many occasions during our study. Notice that, differently to Theorem 34 at p. 116 in [123], the next lemma allows us to handle a stochastic cost of control f.

Lemma A.1. *The following tightness criteria hold true.*

1. *Let* $\{X^n\}_{n\in\mathbb{N}}$ *be a sequence of* $\mathbb{R}^m$*-valued càdlàg processes defined on* $[0,T]$ *such that*
$$\sup_n V_T^{\mathbb{P}}(X^n) < \infty.$$
Then $\{\mathbb{P}\circ X^n\}_{n\in\mathbb{N}}$ *is tight in* $\mathcal{P}(\mathcal{D}^m)$*.*

2. *Let* $\{X^n\}_{n\in\mathbb{N}}$ *be a sequence of nondecreasing, nonnegative,* $\mathbb{R}^m$*-valued càdlàg processes defined on* $[0,T]$ *such that*
$$\sup_n \mathbb{E}[|X_T^n|] < \infty.$$
Then $\{\mathbb{P}\circ X^n\}_{n\in\mathbb{N}}$ *is tight in* $\mathcal{P}(\mathcal{D}^m_{\uparrow})$*.*

Proof. We will prove only the claim (1), since the proof of claim (2) follows by an analogous rationale.

Let $\mathcal{D}^m[0,\infty)$ be the space of $\mathbb{R}^m$-valued càdlàg functions on $[0,\infty)$, with the Borel σ-algebra generated by the Skorokhod topology. On the half line $[0,\infty)$, consider the measure λ given by $d\lambda := e^{-t}dt$. On $\mathcal{D}^m[0,\infty)$ consider the *pseudopath topology* τ_{pp}; that is, the topology induced by the convergence in the measure λ on the interval $[0,\infty)$. Define, moreover, the space $\widetilde{\mathcal{D}}^m[0,\infty)$ as the set of elements of $\mathcal{D}^m[0,\infty)$ which are constant on $[T,\infty)$, and notice that $\widetilde{\mathcal{D}}^m[0,\infty)$ is a closed subset of $\mathcal{D}^m[0,\infty)$. Also, observe that the *extension map* $\Psi : \mathcal{D}^m \to \widetilde{\mathcal{D}}^m[0,\infty)$, defined by
$$\Psi(x)_t := \begin{cases} x_t & \text{if} \quad t\in[0,T] \\ x_T & \text{if} \quad t\in(T,\infty), \end{cases} \tag{A.3}$$
is an omeomorphism between the topological spaces $(\mathcal{D}^m, \tau_{pp}^T)$ and $(\widetilde{\mathcal{D}}^m[0,\infty), \tau_{pp})$.

Now, using the uniform boundedness of $V_T^{\mathbb{P}}(X^n)$, we notice that the sequence $\Psi(X^n)$ satisfies the requirement of Theorem 4 in [136], and, as shown in its proof, it follows that the sequence $\{\mathbb{P}\circ\Psi(X^n)\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{D}^m[0,\infty))$. Furthermore, since $\widetilde{\mathcal{D}}^m[0,\infty)$ is a closed subset of $\mathcal{D}^m[0,\infty)$, we have that $\{\mathbb{P}\circ\Psi(X^n)\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\widetilde{\mathcal{D}}^m[0,\infty))$. Finally, since the map Ψ is an omeomorphism, we conclude that the sequence $\{\mathbb{P}\circ X^n\}_{n\in\mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{D}^m)$ in the Meyer-Zheng topology. □

We finally summarize in a lemma a result on the convergence of stochastic integrals.

Lemma A.2. *Let* $\{F^n\}_{n\in\mathbb{N}}$ *be a sequence of* $\mathbb{R}^m$*-valued continuous processes which converges* $\mathbb{P}$*-a.s. to an* $\mathbb{R}^m$*-valued continuous process* F *uniformly on* $[0,T]$*. Let* $\{X^n\}_{n\in\mathbb{N}}$ *be a sequence of nondecreasing, nonnegative,* $\mathbb{R}^m$*-valued càdlàg processes defined on* $[0,T]$*, which converges* $\mathbb{P}$*-a.s. to nondecreasing, nonnegative,* $\mathbb{R}^m$*-valued cadlag process* X *in the pseudopath topology* τ_{pp}^T*. Suppose, moreover, that there exists two constant* $\alpha, p > 1$ *such that*
$$\sup_n \mathbb{E}\left[\sup_{t\in[0,T]}(|F_t^n|^{\alpha p} + |F_t|^{\alpha p}) + |X_T^n|^{\frac{\alpha p}{p-1}} + |X_T|^{\frac{\alpha p}{p-1}}\right] < \infty. \tag{A.4}$$
Then
$$\lim_n \mathbb{E}\left[\int_{[0,T]} F_t^n \, dX_t^n\right] = \mathbb{E}\left[\int_{[0,T]} F_t \, dX_t\right]. \tag{A.5}$$

Proof. We will prove that for each subsequence of indexes there exists a further subsequence for which the limit in (A.5) holds true.

Consider then a subsequence of indexes (not relabeled). From Condition (A.4), Hölder's inequality with p as in the assumptions easily reveals that

$$\sup_n \mathbb{E}\left[\left|\int_{[0,T]} F_t^n dX_t^n\right|^\alpha\right] + \sup_n \mathbb{E}\left[\left|\int_{[0,T]} F_t dX_t^n\right|^\alpha\right] < \infty. \tag{A.6}$$

Since $\alpha > 1$, by the reflexivity of $\mathbb{L}^\alpha(\mathbb{P})$, there exists a subsequence of indexes n_j and a random variable $Z \in \mathbb{L}^\alpha(\mathbb{P})$, for which

$$\lim_j \mathbb{E}\left[\int_{[0,T]} F_t^{n_j} dX_t^{n_j}\right] = \lim_j \mathbb{E}\left[\int_{[0,T]} F_t \, dX_t^{n_j}\right] = \mathbb{E}[Z], \tag{A.7}$$

where the equality of the two limits follows from the $\mathbb{P}$-a.s. uniform convergence of F^n to F and from the integrability condition (A.4).

Next, since by Condition (A.4) the sequence $\{X_T^{n_j}\}_{j\in\mathbb{N}}$ is bounded in $\mathbb{L}^1(\mathbb{P})$, by Lemma 3.5 in [104] there exist a nondecreasing, nonnegative, $\mathbb{R}^m$-valued càdlàg process B defined on $[0,T]$ and a subsequence (not relabeled) of $\{X^{n_j}\}_{j\in\mathbb{N}}$ such that, $\mathbb{P}$-a.s.,

$$\lim_m \int_{[0,T]} \varphi_t dB_t^m = \int_{[0,T]} \varphi_t dB_t \quad \forall\, \varphi \in \mathcal{C}_b([0,T];\mathbb{R}^d) \quad \text{and} \quad \lim_m B_T^m = B_T, \tag{A.8}$$

where we have set, $\mathbb{P}$-a.s.

$$B_t^m := \frac{1}{m}\sum_{j=1}^m X_t^{n_j}, \qquad \forall t \in [0,T]. \tag{A.9}$$

Moreover, for $\varphi \in \mathcal{C}_c^\infty([0,T);\mathbb{R}^d)$, the limit in (A.8) and an integration by parts, together with the limit in (A.1) (observing that the sequence $\{|X_T^n|\}_{n\in\mathbb{N}}$ is $\mathbb{P}$-a.s. bounded), imply that, $\mathbb{P}$-a.s.,

$$\int_{[0,T]} \varphi_t dB_t = \lim_m \frac{1}{m}\sum_{j=1}^m \int_{[0,T]} \varphi_t dX_t^{n_j} = -\lim_m \frac{1}{m}\sum_{j=1}^m \int_0^T X_t^{n_j}\varphi_t' dt = \int_{[0,T]} \varphi_t dX_t.$$

Therefore, by the fundamental lemma of the Calculus of Variation (see Theorem 1.24 at p. 26 in [62]), the right-continuity of X and B, and the convergence of $X_T^{n_j}$ to X_T, we have $B_t = X_t$ for all $t \in [0,T]$, $\mathbb{P}$-a.s. This identification allows to conclude, using (A.7) and uniform integrability estimates as in (A.6), that

$$\mathbb{E}[Z] = \lim_m \frac{1}{m}\sum_{j=1}^m \mathbb{E}\left[\int_{[0,T]} F_t \, dX_t^{n_j}\right] = \lim_m \mathbb{E}\left[\int_{[0,T]} F_t \, dB_t^m\right] = \mathbb{E}\left[\int_{[0,T]} F_t \, dX_t\right].$$

The latter, combined with (A.7), completes the proof of the lemma. □

Appendix B

Results on lattices of measures

In this section, we derive some technical results concerning the first order stochastic dominance introduced in Subsection 4.1.2. As in Subsection 4.1.2, we identify the set of probability measures $\mathcal{P}(\mathbb{R})$ with the set of distribution functions on $\mathbb{R}$, setting $\mu(s) := \mu(-\infty, s]$ for each $s \in \mathbb{R}$ and $\mu \in \mathcal{P}(\mathbb{R})$. On $\mathcal{P}(\mathbb{R})$ we then consider the lattice ordering of first order stochastic dominance given by (4.1.8) and (4.1.9). In the following remark, we collect some fundamental observations that are crucial for the analysis in this section.

Remark B.1.

a) *Notice that by identifying μ by its distribution function, $\mathcal{P}(\mathbb{R})$ coincides with the set of all nondecreasing right-continuous functions $F\colon \mathbb{R} \to [0,1]$ with*

$$\lim_{s\to-\infty} F(s) = 0 \quad \text{and} \quad \lim_{s\to\infty} F(s) = 1.$$

Moreover, we would like to recall that the weak topology is metrizable and that the weak convergence coincides with the pointwise convergence of distribution functions at every continuity point, i.e. $\mu_n \to \mu$ if and only if

$$\mu_n(s) \to \mu(s) \quad \text{as } n \to \infty \quad \text{for every continuity point } s \in \mathbb{R} \text{ of } \mu.$$

Therefore, the weak convergence behaves well with the pointwise lattice operations $\vee^{\mathrm{st}}$ and $\wedge^{\mathrm{st}}$. In particular, the maps $(\mu,\nu) \mapsto \mu \vee^{\mathrm{st}} \nu$ and $(\mu,\nu) \mapsto \mu \wedge^{\mathrm{st}} \nu$ are continuous $\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$.

b) *Recall that a nondecreasing function $\mathbb{R} \to \mathbb{R}$ is right-continuous if and only if it is upper semi-continuous (usc). Hence, for a sequence $\{\mu^n\}_{n\in\mathbb{N}} \in \mathcal{P}(\mathbb{R})$ which is bounded above, the supremum $\sup_{n\in\mathbb{N}} \mu^n$ is exactly the pointwise infimum of the distribution functions $\{\mu^n\}_{n\in\mathbb{N}}$.*

c) *For a nondecreasing function $F\colon \mathbb{R} \to \mathbb{R}$, we define its usc-envelope $F^*\colon \mathbb{R} \to \mathbb{R}$ by*

$$F^*(s) := \inf_{\delta>0} F(s+\delta) \quad \text{for all } s \in \mathbb{R}.$$

Notice that

$$F(s) \leqslant F^*(s) \leqslant F(s+\varepsilon) \quad \text{for all } s \in \mathbb{R} \text{ and } \varepsilon > 0. \tag{B.1}$$

Intuitively speaking, F^* is the right-continuous version of F. That is, F^* differs from F only at discontinuity points of F. For a sequence $\{\mu^n\}_{n\in\mathbb{N}} \in \mathcal{P}(\mathbb{R})$ which is bounded below, the infimum $\inf_{n\in\mathbb{N}} \mu^n$ is then given by the usc-envelope of the pointwise supremum of the distribution functions $\{\mu^n\}_{n\in\mathbb{N}}$. That is, one has to modify the pointwise supremum at all its discontinuity points in order to be right-continuous. In fact, let $\underline{\mu} = F^*$ denote the usc-envelope of the pointwise supremum F of $\{\mu^n\}_{n\in\mathbb{N}}$. By Equation (B.1), $\underline{\mu}(s) \leqslant F(s+\varepsilon) \leqslant \underline{\mu}(s+\varepsilon)$ for all $s \in \mathbb{R}$ and $\varepsilon > 0$, i.e., $\underline{\mu}$ is nondecreasing and $\underline{\mu} \leqslant^{\text{st}} \mu^n$ for all $n \in \mathbb{N}$. Moreover, by definition, μ is usc, and thus right-continuous. Since $\underline{\mu} \leqslant^{\text{st}} \mu^1$, $\underline{\mu}(s) \geqslant \mu^1(s) \to 1$ as $s \to \infty$. Let ν be a lower bound of $\{\mu^n\}_{n\in\mathbb{N}}$. Then, $\underline{\mu}(s) \leqslant F(s+\varepsilon) \leqslant \nu(s+\varepsilon)$ for all $s \in \mathbb{R}$ and $\varepsilon > 0$. Taking the limit $\varepsilon \to 0$, we may conclude that $\underline{\mu}(s) \leqslant \nu(s)$ for all $s \in \mathbb{R}$. In particular, $\underline{\mu}(s) \leqslant \nu(s) \to 0$ as $s \to -\infty$. Altogether, we have shown that $\underline{\mu}$ is a distribution function with $\nu \leqslant^{\text{st}} \underline{\mu} \leqslant^{\text{st}} \mu^n$ for all $n \in \mathbb{N}$ and every lower bound ν of $\{\mu^n\}_{n\in\mathbb{N}}$.

d) *Combining the previous remarks, leads to the following insight: If $\{\mu^n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R})$ is a bounded and nondecreasing or non-increasing sequence, then $\{\mu^n\}_{n\in\mathbb{N}}$ converges weakly to its supremum or infimum, respectively. In fact, we have seen that the supremum $\bar{\mu}$ of $\{\mu^n\}_{n\in\mathbb{N}}$ exists, and that its distribution function is given by the pointwise supremum of the sequence of distribution functions of $\{\mu^n\}_{n\in\mathbb{N}}$. In particular, $\mu^n(s) \to \bar{\mu}(s)$ as $n \to \infty$ for all $s \in \mathbb{R}$. Moreover, it is shown that infimum $\underline{\mu}$ of $\{\mu^n\}_{n\in\mathbb{N}}$ exists, and its distribution function is given by the usc-envelope of the pointwise supremum of the sequence of distribution functions of $\{\mu^n\}_{n\in\mathbb{N}}$. Therefore, the distribution function of $\underline{\mu}$ coincides with the pointwise supremum of the sequence of distribution functions of $\{\mu^n\}_{n\in\mathbb{N}}$ at every continuity point of the distribution function of $\underline{\mu}$. In particular, $\mu^n(s) \to \underline{\mu}(s)$ as $n \to \infty$ for every continuity point $s \in \mathbb{R}$ of the distribution function of $\underline{\mu}$. Since the weak convergence of probability measures is equivalent to the pointwise convergence of the distribution functions at every continuity point of the distribution function of the limit, we obtain that $\mu^n \to \bar{\mu}$ and $\mu^n \to \underline{\mu}$ weakly as $n \to \infty$.*

Lemma B.2. *Let $K \subset \mathcal{P}(\mathbb{R})$ and $\psi\colon [0,\infty) \to [0,\infty)$ be continuous and strictly increasing with $\psi(s) \to \infty$ as $s \to \infty$ and*

$$\sup_{\mu\in M} \int_{\mathbb{R}} \psi(|x|)d\mu(x) < \infty.$$

Then, there exist $\mu^{\text{Min}}, \mu^{\text{Max}} \in \mathcal{P}(\mathbb{R})$ with $\mu^{\text{Min}} \leqslant^{\text{st}} \mu \leqslant^{\text{st}} \mu^{\text{Max}}$ for all $\mu \in K$.

Proof. We extend ψ to $(-\infty,0)$ by $\psi(s) := \psi(0)$ for $s < 0$. Moreover, let $C \geqslant \psi(0)$ with

$$\sup_{\mu\in K} \int_{\mathbb{R}} \psi(|x|)d\mu(x) \leqslant C.$$

Then, we define $\mu^{\text{Min}}, \mu^{\text{Max}}\colon \mathbb{R} \to [0,1]$ by

$$\mu^{\text{Min}}(s) := \frac{C}{\psi(-s)} \wedge 1 \quad \text{and} \quad \mu^{\text{Max}}(s) := \left(1 - \frac{C}{\psi(s)}\right) \vee 0 \tag{B.2}$$

for all $s \in \mathbb{R}$. Since ψ is strictly increasing with $\psi(s) \to \infty$ as $s \to \infty$, $\mu^{\text{Min}}(s) = 1$ for $s \geqslant -\psi^{-1}(C)$ and $\mu^{\text{Max}} = 0$ for $s \leqslant \psi^{-1}(C)$. In particular, $\lim_{s\to-\infty} \mu^{\text{Min}}(s) = 0$ and $\lim_{s\to\infty} \mu^{\text{Max}}(s) = 1$. Moreover, μ^{Min} and μ^{Min} are nondecreasing and (right) continuous, which shows that $\mu^{\text{Min}}, \mu^{\text{Max}} \in \mathcal{P}(\mathbb{R})$. Now, let $\mu \in K$. Then, recalling that ψ is nondecreasing, one has

$$1 - \mu(s) \leqslant \frac{1}{\psi(s)} \int_s^\infty \psi(|x|) d\mu(x) \leqslant \frac{1}{\psi(s)} \int_{\mathbb{R}} \psi(|x|) d\mu(x) \leqslant \frac{C}{\psi(s)} = 1 - \mu^{\text{Max}}(s)$$

for all $s \in \mathbb{R}$ with $\psi(s) > C$. Since $\mu^{\text{Max}}(s) = 0$ for all $s \in \mathbb{R}$ with $\psi(s) \leqslant C$, it follows that $\mu \leqslant \mu^{\text{Max}}$. On the other hand,

$$\mu(s) \leqslant \frac{1}{\psi(-s)} \int_{-\infty}^s \psi(|x|) d\mu(x) \leqslant \frac{1}{\psi(-s)} \int_{\mathbb{R}} \psi(|x|) d\mu(x) \leqslant \frac{C}{\psi(-s)} = \mu^{\text{Min}}(s)$$

for all $s \in \mathbb{R}$ with $\psi(-s) > C$. Since $\mu^{\text{Min}}(s) = 1$ for all $s \in \mathbb{R}$ with $\psi(-s) \leqslant C$, it follows that $\mu \geqslant \mu^{\text{Min}}$. □

Lemma B.3. *Let $K \subset \mathcal{P}(\mathbb{R})$ and $\psi\colon [0,\infty) \to [0,\infty)$ be continuous and strictly increasing with $\psi(s) \to \infty$ as $s \to \infty$ and*

$$\sup_{\mu \in M} \int_{\mathbb{R}} \psi(|x|) d\mu(x) < \infty.$$

Further, let μ^{Min} and μ^{Max} be given by (B.2) and $0 \leqslant \alpha < 1$. Then, the map $x \mapsto \psi(|x|)^\alpha$ is u.i for $[\mu^{\text{Min}}, \mu^{\text{Max}}]$, i.e.

$$\sup_{\mu \in [\mu^{\text{Min}}, \mu^{\text{Max}}]} \int_{\mathbb{R}} 1_{(M,\infty)}(|x|) \cdot \psi(|x|)^\alpha d\mu(x) \to 0 \quad \textit{as } M \to \infty.$$

Proof. Let $\beta \in (\alpha, 1)$. Then, by (B.2),

$$\begin{aligned} \psi(s) &= \frac{C}{1 - \mu^{\text{Max}}(s)} \quad \text{for } s \geqslant \psi^{-1}(C), \\ \psi(-s) &= \frac{C}{\mu^{\text{Min}}(s)} \quad \text{for } s \leqslant -\psi^{-1}(C). \end{aligned} \tag{B.3}$$

Recall $\psi^{-1}(C) = \max\left\{s \in \mathbb{R} \,\middle|\, \left(\mu^{\text{Max}}\right)(s) = 0\right\}$ and $-\psi^{-1}(C) = \min\left\{s \in \mathbb{R} \,\middle|\, \left(\mu^{\text{Min}}\right)(s) = 1\right\}$. This together with (B.3) implies that

$$\int_0^\infty \psi(s)^\beta d\mu^{\text{Max}}(s) = \int_{\psi^{-1}(C)}^\infty \left(\frac{C}{1 - \mu^{\text{Max}}(s)}\right)^\beta d\mu^{\text{Max}}(s) = \int_0^1 \left(\frac{C}{1-u}\right)^\beta du < \infty$$

and

$$\int_{-\infty}^0 \psi(-s)^\beta d\mu^{\text{Min}}(s) = \int_{-\infty}^{-\psi^{-1}(C)} \left(\frac{C}{\mu^{\text{Min}}(s)}\right)^\beta d\mu^{\text{Min}}(s) = \int_0^1 \left(\frac{C}{u}\right)^\beta du < \infty,$$

where, in both equalities, we used the transformation lemma. It follows that

$$\sup_{\mu\in[\mu^{\mathrm{Min}},\mu^{\mathrm{Max}}]}\int_{\mathbb{R}}\psi(|x|)^\beta d\mu(x)\leqslant\int_0^\infty\psi(s)^\beta d\mu^{\mathrm{Max}}(s)+\int_{-\infty}^0\psi(-s)^\beta d\mu^{\mathrm{Min}}(s).$$

By the De La Vallée-Poussin Lemma, it follows that $|x|\mapsto\psi(|x|)^\alpha$ is u.i. for $[\mu^{\mathrm{Min}},\mu^{\mathrm{Max}}]$. In particular, if $\psi(s)\geqslant s^p$ for some $p\in(0,\infty)$, then, $x\mapsto|x|^q$ is u.i. for $[\mu^{\mathrm{Min}},\mu^{\mathrm{Max}}]$ for all $q\in(0,p)$.

□

We now turn our focus on measurable flows of probability measures. The following proposition is the starting point in order to apply Tarski's fixed point theorem in the proof of the existence of mean field game solutions. We start by building up the setup. Let $\underline{\mu},\overline{\mu}\in\mathcal{P}(\mathbb{R})$ with $\underline{\mu}\leqslant^{\mathrm{st}}\overline{\mu}$ and $(S,\mathcal{S},\pi)$ be a finite measure space. We denote by $\mathcal{B}$ the Borel σ-algebra on $\mathcal{P}(\mathbb{R})$ generated by the weak topology. We denote the lattice of all equivalence classes of $\mathcal{S}$-$\mathcal{B}$-measurable functions $S\to[\underline{\mu},\overline{\mu}]$ by $L=L^0(S,\pi;[\underline{\mu},\overline{\mu}])$. An arbitrary element μ of L will be denoted in the form $\mu=(\mu_t)_{t\in S}$. On L we consider the order relation $\leqslant^L$ given by $\mu\leqslant^L\nu$ if and only if $\mu_t\leqslant^{\mathrm{st}}\nu_t$ for π-a.a. $t\in S$. The following proposition can be found in a more general form in [139]. However, for the sake of a self-contained exposition, we provide a short proof below.

Proposition B.4. *The lattice L is complete.*

Proof. Let $M\subset L$ be a nonempty subset of L. Then, for every countable set $\Psi\subset M$, we denote by $\mu^\Psi:=\sup_{\mu\in\Psi}\mu$. Let Ψ be a countable subset of M, and $\{\Psi^n\}_{n\in\mathbb{N}}$ be a sequence of finite subsets of Ψ with $\Psi^n\subset\Psi^{n+1}$ for all $n\in\mathbb{N}$ and $\bigcup_{n\in\mathbb{N}}\Psi^n=\Psi$. As Ψ^n is finite, by Remark B.1 b), $\mu^{\Psi^n}\in L$ with $\mu^{\Psi^n}\leqslant^L\mu^{\Psi^{n+1}}$ for all $n\in\mathbb{N}$. By Remark B.1 d), if follows that $\{\mu^{\Psi_n}\}_{n\in\mathbb{N}}$ converges weakly π-a.e. to μ^Ψ. As a consequence, $\mu^\Psi\in L$ for every countable set $\Psi\subset M$. Let

$$c:=\sup\left\{\int_S\int_{\mathbb{R}}\arctan(x)d\mu_t^\Psi(x)d\pi(t)\,\middle|\,\Psi\subset M\text{ countable}\right\}.$$

Notice that the map $t\mapsto\int_{\mathbb{R}}\arctan(x)\,d\mu_t$ is measurable for every $\mu\in L$ since $\arctan\in C_b(\mathbb{R})$ induces a continuous (w.r.t. to the weak topology) linear functional $\mathcal{P}(\mathbb{R})\to\mathbb{R}$. By definition of the constant c, there exists a sequence $\{\Psi^n\}_{n\in\mathbb{N}}$ of countable subsets of M with

$$\int_S\int_{\mathbb{R}}\arctan(x)d\mu_t^{\Psi^n}(x)d\pi(t)\to c\quad\text{as }n\to\infty.$$

Let $\Psi^*:=\bigcup_{n\in\mathbb{N}}\Psi^n$ and $\mu^*:=\mu^{\Psi^*}$. We now show that $\mu\leqslant^L\mu^*$ π-a.s. for all $\mu\in M$. In order to see this, fix some $\mu\in M$ and let $\Psi':=\Psi^*\cup\{\mu\}$. Then, it follows that

$$c=\int_S\int_{\mathbb{R}}\arctan(x)d\mu_t^*(x)d\pi(t)\leqslant\int_S\int_{\mathbb{R}}\arctan(x)d\mu_t^{\Psi'}(x)d\pi(t)\leqslant c.$$

Since arctan is strictly increasing it follows that $\mu^{\Psi'}=\mu^*$, i.e. $\mu\leqslant^L\mu^*$. Moreover, for any upper bound $\mu\in L$ of M it is easily seen that $\mu^*\leqslant^L\mu$. Altogether, we have shown that $\mu^*=\sup M$. In a similar way, one shows that M has an infimum. □

Remark B.5. *Let $M \subset L$ be nonempty. Then, we say that M is directed upwards or directed downwards if for all $\mu, \nu \in M$ there exists some $\eta \in M$ with $\mu \vee \nu \leqslant^L \eta$ or $\eta \leqslant^L \mu \wedge \nu$, respectively.*

a) The proof of the previous theorem shows that if M is directed upwards, then there exists a nondecreasing sequence $\{\mu^n\}_{n\in\mathbb{N}} \subset M$ with $\mu^n \to \sup M$ weakly π-a.e. as $n \to \infty$. The analogous statement holds for the infimum if M is directed downwards. In particular, if $\{\mu^n\}_{n\in\mathbb{N}}$ is a nondecreasing or non-increasing sequence in L, then it converges weakly π-a.e. to its least upper bound or greatest lower bound, respectively.

b) Assume that S is a singleton with $\pi(S) > 0$. Then, the previous remark implies the following: For any nonempty set $K \subset \mathcal{P}(\mathbb{R})$ that is bounded above and directed upwards, its supremum exists and can be weakly approximated by a monotone sequence. An analogous statement holds for the infimum if the set K is bounded below and directed downwards.

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