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Household Credit: Regulation, Constraints and Macroeconomic Implications

vorgelegt von:

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Household Credit: Regulation, Constraints and Macroeconomic Implications

a doctoral thesis written for the purpose

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Résumé

Le crédit aux ménages ne cesse de prendre de l'importance dans les systèmes financiers modernes, avec des effets réels immédiats. Cette thèse aborde les conséquences des mesures réglementaires sur les marchés du crédit aux ménages. Dans les chapitres 1 et 2, l'institution de faillite personnelle, telle qu'on peut la trouver aux États-Unis, est étudiée d'un point de vue théorique. Dans le premier chapitre, il est montré que la faillite personnelle implique l'efficacité de la dette publique en tant qu'outil de redistribution des ressources entre les générations. Dans le deuxième chapitre, il est démontré que la faillite personnelle ne provoque de l'instabilité macroéconomique à long terme que si elle est associée à des bulles d'actifs. Le chapitre 3 étudie la politique de restriction des taux d'intérêt d'un point de vue empirique, en utilisant les données administratives sur les prêts au logement en France. Il est démontré que cette réglementation entraîne un glissement de l'offre de crédit vers les prêts à court terme.

Mots-clés : faillite personnelle, redistribution intergénérationnelle, dette publique, cycle de vie, contrainte d'emprunt, bulles rationnelles, taux d'usure, durée des prêts.

Summary

Household credit is an ever-expanding part of modern financial systems with immediate real effects. This thesis addresses the consequences of regulatory measures on household credit markets. In Chapters 1 and 2, the personal bankruptcy institution of the type found in the United States is studied from a theoretical perspective. In the first chapter, personal bankruptcy is shown to imply efficiency of government debt as a tool for redistributing resources between generations. In the second chapter, personal bankruptcy is found to cause long-run macroeconomic instability only if it is associated with asset bubbles. Chapter 3 studies interest rate restriction policy from an empirical point of view, using administrative data on housing loans in France. This regulation is found to cause a shift in the supply of credit towards short-term loans.

Keywords : personal bankruptcy, intergenerational redistribution, government debt, life cycle, endogenous borrowing constraint, rational bubbles, interest rate regulation, loan duration.

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Finishing this thesis amid a global pandemic and social distancing, I realize more than ever I am only capable of intellectual effort when exchanging ideas and impressions with others. In this regard, I have been extremely lucky and privileged to be surrounded by wonderful minds and hearts during these years. Let me start with the heart side of life.

During the last three years of most intense work, Maria and Ierofey made me feel that I have a true home, and I'm not speaking about the small Parisian apartments where everyone is necessarily very close to each. Family has made it possible for me to remember that research is far from being the only thing that matters. I would barely make it this far without this perspective. Maria's humanist life outlook of a psychologist has also helped me remember that the economy, as well as economics, are nothing but a means of making people happier. With the help of my parents, Vadim and Maria, we have managed to visit Russia regularly, form where I have been coming back to Paris with a fresh mind and attitude to all kinds of issues. Together with my brothers Pavel and Nikita, we have managed to stay in touch on a regular basis, whereby we could exchange great ideas and feel fraternal support.

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During this PhD program, I have worked for at three university campuses and have participated in a dozen of conferences and workshops in around ten countries. Friendship with people from all these places is one of the things that have made the PhD years so worthwhile. In the old days at Maison des Sciences Economiques of Paris-1, I have shared office with many colleagues — Stefanija, Hamzeh, Baris, Yassine, Mathias and Diane — and it has been a fun and organic way to start the PhD path. These people, along with Anna, Anastasia, Bertrand, Lorenzo and others, have been my guides into the first year of work on the thesis, which has been an encouraging start. The stay in Bielefeld during the second year has brought a huge change of perspective and has left particularly dear memories. This is thanks to the company of Bulgan, Madhuri, Sevak, Vahe, Tigran, Martin, Quentin, Alessandro, Zhaojun, Gregor, and Elena. A special thanks goes to Elena for her help with housing and hints on administrative issues upon my arrival to Germany.

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My PhD degree is an orderly result of the efforts that so many people have put into my education. The first impetus has, of course, come from my parents, Maria and Vadim, who have emphasized the role of education in life since my youngest age. State school number 1944 in Moscow has then built the foundation for my subsequent academic path; it is this school that has given me the taste for maths and foreign languages. Studies in the Research University - Higher School of Economics has made it clear to me that research will play a decisive role in my life. In particular, Nikolay Arefiev has managed to show how fascinating macroeconomics can be (who knew!) to me and other students of the Economic Modelling and Economic Policy study group. It is worth noting that most of my fellow students from this group either pursue PhD degrees around the globe or work at the Central Bank of Russia. Finally, Irina Maltseva has been instrumental in organizing the smooth transition of french-speaking students of Higher School of Economics to the Paris-1 Master programmes.

Contents

Introduction (en français)

Le crédit aux ménages ne cesse de prendre de l'importance dans les systèmes financiers modernes, avec des effets réels immédiats. Au cours des dernières décennies, le crédit, garanti ou non, a augmenté dans les économies avancées comme dans les économies en voie de développement. Le rôle central de l'endettement des ménages dans la crise financière de 2007-2008 l'a soumis à un contrôle sans précédent des économistes et des décideurs politiques, entraînant une régulation accrue des prêts aux ménages comme du secteur bancaire en général. Dans cette thèse, j'étudie les effets de ces réglementations sur les contraintes de crédit, la redistribution optimale et la stabilité macroéconomique. En particulier, deux outils réglementaires sont analysés : la faillite personnelle et la restriction des taux d'intérêt.

Quelles sont les implications de ces réponses réglementaires à l'expansion du crédit des ménages ? Comment le crédit aux ménages interagit-il avec la politique macroéconomique et la transforme-t-elle en retour ? La réglementation empêche-t-elle effectivement l'endettement des ménages de provoquer de l'instabilité macroéconomique ou, au contraire, rend-elle l'endettement plus déstabilisant ? Cette thèse aborde ces grandes questions dans le contexte de deux outils réglementaires et de deux types de dettes. Dans le chapitre 1, j'étudie l'endettement lié à la consommation, avec possibilité de faillite personnelle — une institution que l'on retrouve surtout dans les économies avancées et qui est fréquemment utilisée aux États-Unis. Le chapitre se concentre sur l'interaction de ce type de réglementation avec la politique de redistribution à long terme. Le même marché et la même réglementation sont étudiés dans le chapitre 2, co-écrit avec Bertrand Wigniolle. La question abordée dans ce chapitre est celle de la stabilité macroéconomique : nous nous demandons si l'existence de la faillite personnelle crée une instabilité à long terme via des contraintes de crédit et/ou des bulles d'actifs. Le troisième chapitre est consacré à la restriction des taux d'intérêt et son influence sur la durée des prêts, dans le cas des prêts immobilier en France. J'étudie une réforme de la régulation, ce qui permet d'établir une relation causale entre la restriction du taux d'intérêt et l'offre de crédit. Les résultats de ces trois chapitres sont résumés ci-dessous, à la suite d'une brève description des faits stylisés de l'endettement des ménages dans le monde et de la réponse réglementaire qu'il a suscité.

L'endettement des ménages

Le crédit aux ménages occupe une part croissante de l'activité de prêt dans le monde entier, détournant progressivement l'attention que portaient les intermédiaires financiers aux prêts aux entreprises (Jordà et al., 2016). Le graphique 1 décrit l'évolution du crédit dans sept grandes économies qui présentent des différences significatives tant au niveau de leurs institutions que de leur géographie. Le premier graphique montre que la dette privée - la somme de la dette des ménages et des sociétés non financières - a augmenté régulièrement par rapport au PIB pour la plupart de ces pays. Dans le même temps, le deuxième graphique montre que la part du crédit des ménages a également eu tendance à augmenter sur la période.

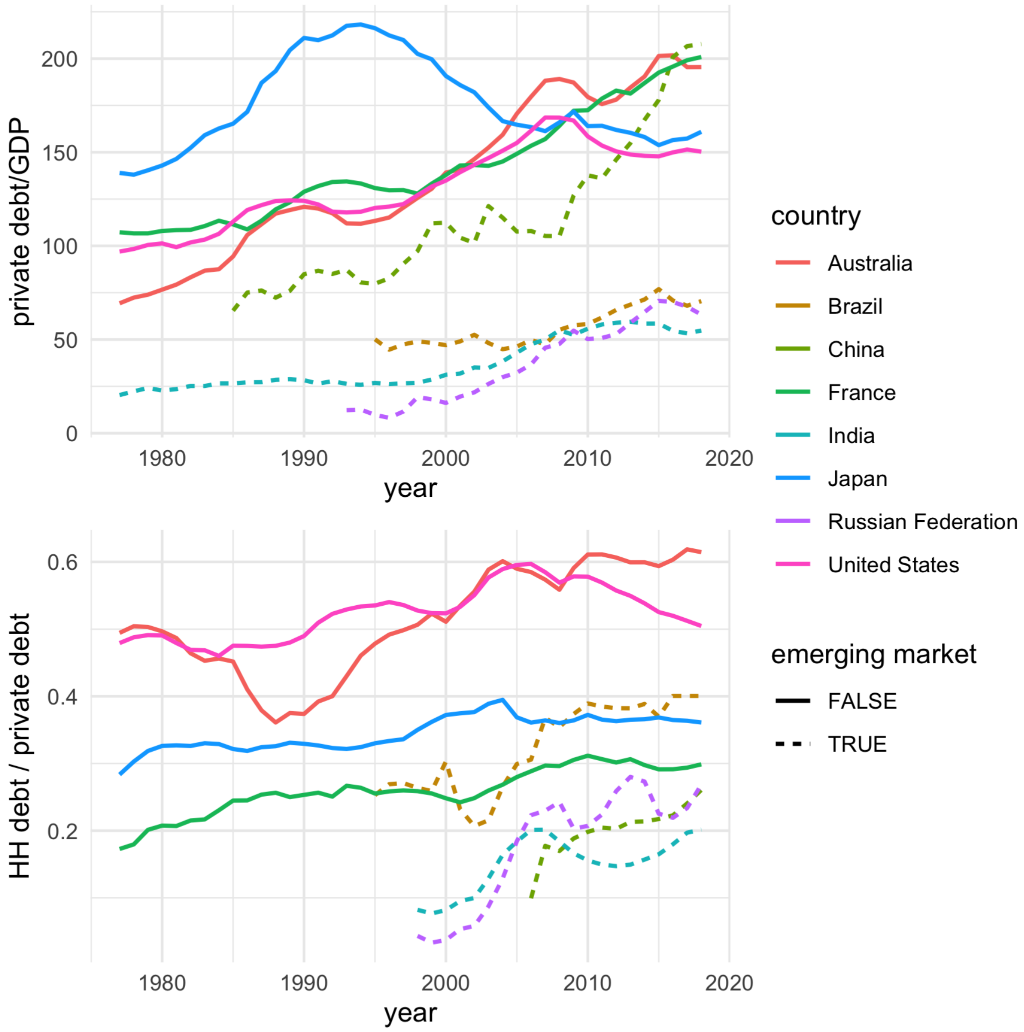


Figure 1: Dette privée par rapport au PIB (en haut) et part de la dette des ménages dans la dette privée (en bas) dans une sélection de pays, 1977-2018. La dette privée est la somme de la dette des ménages et de la dette des sociétés non financières. Les séries des économies émergentes sont en pointillié. Source : Global Debt Database du FMI.

Quelles sont les causes possibles d'une telle dynamique ? Tout d'abord, les taux d'intérêt réels ont considérablement baissé alors même que les primes de risque et de terme sont restés stables, ce qui désigne l'offre d'épargne excédentaire par rapport à la demande d'investissement comme force motrice principale de l'augmentation de la dette privée (Rachel and Summers, n.d.). Bernanke (2005) a notoirement expliqué l'accumulation de dettes aux États-Unis par une perspective internationale : un certain nombre de pays, d'Europe occidentale et d'Asie du Sud-Est notamment, ont connu une croissance de leur épargne telle qu'elle a entraîné des flux de capitaux vers le secteur bancaire américain. Coeurdacier et al. (2015) mettent en évidence le rôle des contraintes de crédit pour les ménages, qui ont été particulièrement fortes en Chine et faibles aux Etats-Unis. Pour autant, cette perspective n'explique pas pourquoi un certain nombre de grands pays, avec des niveaux de revenus différents, ont connu une croissance de leur ratio crédit/PIB largement tirée par le crédit aux ménages. La littérature récente souligne le rôle des excès d'épargne des menages aisés — en d'autres termes, la montée des inégalités de richesse. Kumhof et al. (2015) décomposent par groupe de revenu l'accumulation rapide de dettes qui a précédé la Grande Dépression et la Grande Récession aux États-Unis. Ils montrent que pour les deux périodes, les 5% supérieurs de la distribution des revenus ont connu une augmentation de leur épargne nette ainsi que de la part de leur revenu dans l'économie. Ce sont les 95% inférieurs qui ont contribué à l'augmentation des niveaux d'endettement aggrégé. En adoptant une perspective transversale, Mian et al. (2020a) établissent un lien entre les tendances de l'endettement et celles des inégalités de revenu entre États américains. Une tendance claire et cohérente se dégage : les ménages du 1% supérieur de la distribution des revenus ont considérablement augmenté leur épargne depuis les années 1980, tandis que les 90% inférieurs se sont fortement endettés sur la même période. De plus, l'épargne supplémentaire des plus riches a été principalement accumulée sous forme d'actifs financiers. Il est donc crucial de comprendre le rôle du secteur financier dans l'acheminement de cette épargne vers le crédit aux ménages plutôt que vers les prêts aux entreprises. Jordà et al. (2016) étudient les choix des intermédiaires financiers en matière d'actif dans le monde et constatent que les prêts hypothécaires dédiés à l'achat d'une maison ont largement contribué à l'expansion des bilans des banques tout au long du XXème siècle. Les auteurs établissent donc un lien entre cette évolution et les politiques d'accès à la propriété menées dans la plupart des économies avancées après la Seconde Guerre mondiale. Mian and Sufi (2015) détaillent par le menu les problèmes intrinsèques liés à ces contrats hypothécaires, qui impliquent en effet des externalités à cause des saisies de biens qui en découlent lors des crises comme celle de 2007-2008.

Bien que les hypothèques expliquent une grande partie de l'augmentation du solde de la dette des ménages au cours des dernières décennies, et qu'elles soient le coupable principal de la Grande Récession, il ne faut pas pour autant laisser de côté les autres types de dettes. Tout d'abord, dans certains pays, les contrats hypothécaires sont très peu utilisés, malgré un marché du crédit immobilier résidentiel développé. En France par exemple, les prêteurs s'appuient le plus souvent sur des prêts assurés qui n'utilisent pas le logement comme actif sous-jacent. Deuxièmement, la révolution informatique a rendu la gestion du risque de crédit plus flexible, ce qui a entraîné une expansion rapide des prêts non garantis depuis les années 1990 (Livshits et al., 2016). Enfin, étant donné qu'un même ménage peut accumuler à la fois des dettes garanties et non garanties, les deux sont susceptibles d'interagir et d'influer sur les stratégies financières et le comportement de l'emprunteur, en particulier lors d'épisodes de panique et de crise. Comme le souligne White (2007), les ménages américains en détresse peuvent se déclarer insolvable sur leurs dettes non garanties afin de retarder la saisie de leur maison pour prêts hypothécaires impayés. Motivé par ces considérations, j'étudie dans cette thèse le crédit non hypothécaire tel que les prêts à la consommation non garantis et les prêts au logement assurés. Leur principale différence avec les prêts hypothécaires est leur risque de défaut, qui a pour contrepartie soit un rationnement du crédit, soit des taux d'intérêt plus élevés.

Les ménages remboursent-ils leurs dettes de plus en plus importantes comme il se doit ? Pour la plupart des pays, les données accessibles sur le remboursement sont rares. Quelques données sont disponibles dans certaines enquêtes harmonisées sur les pays de l'UE : l'Enquête sur les Finances et la Consommation des Ménages menée par la BCE demande aux ménages s'ils ont été en retard dans le remboursement de l'un de leurs prêts au cours de l'année écoulée. On apprend que 5% des Français interrogés ont eu des retards de paiement en 2013, chiffre tombé à 4,2% en 2017. Aux États-Unis en revanche, les réponses à la même question, que l'on peut consulter dans l'Enquête sur les Finances des Consommateurs, brossent un tableau plus sombre. En effet, en 2013, la proportion de personnes interrogées ayant des retards de paiement s'élevait à 14,9% et n'a cessé de diminuer depuis, mais le chiffre était encore en 2019 au niveau alarmant de 12,3%.

Les importants effets sur le bien-être suscités par l'endettement des ménages ont conduit les gouvernements à réguler divers aspects des contrats de crédit. La plupart des mesures peuvent se ranger dans l'une des trois catégories suivantes : régulation de la divulgation d'informations, restriction des éléments tarifaires ou non des contrats de prêt, et intervention de l'État en cas d'insolvabilité. Dans cette thèse, je ne considèrerai pas les politiques de régulation de la divulgation d'informations [1]. Je me concentre plutôt sur les deux derniers types de mesures, qui sont les plus largement utilisées pour réguler les prêts non hypothécaires. Les dettes sans garantie posent deux défis singuliers aux prêteurs et aux régulateurs. En premier lieu, ces dettes peuvent être impayées, et en l'absence d'intervention réglementaire, le ménage peut se retrouver en situation de détresse permanente du fait de l'exigence de recouvrement de la dette. Deuxièmement, l'absence de garantie signifie que les prêteurs s'appuient sur les conditions du contrat, avec ou sans intérêt, pour évaluer le risque de défaut. Donc les régulateurs abordent généralement ces problèmes à l'aide de deux outils : d'une part les procédures comme celles de la faillite pour les particuliers, qui impliquent une libération de la dette, et d'autre part le contrôle des prix, qui limitent les intérêts et les coûts indépendants des intérêts des prêts. Pour simplifier, j'appelle le premier "faillite personnelle" et le second "restriction du taux d'intérêt".

[1] Voir Campbell et al. (2011) pour une discussion de plusieurs mesures qui ont été adoptées aux États-Unis.

D'un côté, les procédures de faillite personnelle permettent de pallier certaines asymétries d'information entre le prêteur et l'emprunteur en difficulté, permettant aux deux parties d'éviter les coûts d'un recouvrement excessif des dettes. De l'autre, les politiques de restriction des taux d'intérêt visent à prévenir les difficultés financières en ralentissant l'accumulation de dettes par l'effet des intérêts composés. Pour autant, ces deux outils ont un certain nombre d'effets indésirables, principalement liés aux contraintes de crédit. Ces effets indésirables sont examinés dans les sections suivantes et dans les trois principaux chapitres de la thèse. Les chapitres 1 et 2 étudient la faillite personnelle telle qu'on peut la trouver aux États-Unis, tandis que le chapitre 3 étudie la restriction des taux d'intérêt du type de celle appliquée en France.

Réponse réglementaire : surendettement, faillite personnelle

En théorie, le défaut de paiement est un élément nécessaire au bon fonctionnement des marchés financiers (Dubey et al., 1989; Zame, 1993). Pour résumer l'argument principal, si les prêteurs n'acceptaient des contrats tels que la dette devrait être systématiquement remboursée dans toutes les circonstances possibles, les prêts accordés seraient trop peu nombreux. Cependant, sur un marché concurrentiel, l'entente sur la possibilité de faire défaut donne naissance à une industrie du recouvrement des créances qui finit par se spécialiser dans la production de détresses pour les emprunteurs (Fedaseyeu and Hunt, 2018). La plupart des pays ont donc mis en place des systèmes de surveillance et de règlement judiciaire du surendettement des ménages. Toutefois, les systèmes varient grandement en ce qui concerne le degré d'apurement possible des dettes. Un exemple extrême des degrés que peuvent prendre ces libérations de dettes est la procédure de faillite personnelle *"fresh start"* aux États-Unis : la personne éligible à la procédure voit la quasi-totalité de ses dettes non garanties tout simplement effacées[2], et les créanciers de la personne n'ont pas le droit de saisir ses revenus du travail par la suite. Le principal facteur de discipline pour l'emprunteur est qu'une partie de ses biens peut être utilisée pour rembourser les créanciers pendant plusieurs années après la déclaration de faillite.

La France donne un exemple différent de l'intervention en cas de détresse de l'emprunteur. Une personne peut être déclarée surendettée et obtenir une libération partielle de ses dettes dans le cadre d'une procédure formelle administrée par la Banque de France. Cependant, seule une partie minime des actifs et des revenus futurs de l'emprunteur est protégée des créanciers. Par ailleurs, l'acquittement des dettes est conditionné par l'effort de l'emprunteur pour continuer à engranger des revenus. De manière attendue, moins d'emprunteurs subissent une telle procédure en France qu'aux Etats-Unis : environ 1% des ménages français y ont eu recours sur la période 2014-2019. De plus, le nombre de cas annuels a diminué d'environ 30% sur la période. Cela s'explique probablement par le

[2]Il est important de noter dans le contexte américain contemporain, que la dette étudiante n'est généralement pas annulée dans le cadre d'une faillite au titre de celle étudiée dans le chapitre 7. Cependant, elle peut être annulée grâce à une procédure spéciale visant à prouver les "difficultés financières indues" causées par la dette d'étude. Elle peut également être restructurée dans le cadre de la faillite telle qu'étudiée au chapitre 13.

durcissement significatif de la restriction des taux d'intérêt, principale forme de protection financière appliquée aux consommateurs français.

Réponse réglementaire : restriction du taux d'intérêt

Le crédit aux ménages avec des intérêts élevés a longtemps été dénoncé comme une pratique usurière — des profits excessifs pour les prêteurs les plus riches aux dépens des emprunteurs les plus pauvres. Aujourd'hui, ce récit est utilisé comme motif important des politiques de restriction du taux d'intérêt. En outre, les partisans de la mesure proposent des arguments plus subtils. Glaeser and Scheinkman (1998) font valoir qu'un transfert du surplus des prêteurs vers les emprunteurs dans les contrats de crédit améliore le partage global des risques. Si l'on ne sait pas si tel jeune finira par être plutôt un emprunteur ou un prêteur au long de sa vie, une répartition égale du surplus entre prêteurs et emprunteurs peut être un choix préférable pour tout le monde ex ante. Coco and De Meza (2009) montrent qu'un intérêt moindre atténue le problème d'aléa moral dans les cas de prêt aux entrepreneurs à responsabilité limitée, qui est également lié au crédit à la consommation non garanti avec possibilité de faillite personnelle.

Étant donné que la restriction du taux d'intérêt est un cas particulier des contrôles des prix, il n'est pas étonnant que la plupart des analyses dans la littérature lui trouvent des inconvénients significatifs. En plus de créer des pénuries, comme tout prix plafond est susceptible de le faire, les plafonds d'intérêt affectent la sélection des emprunteurs sur le marché, dans la mesure où des intérêts élevés peuvent être nécessaires pour les prêts présentant un risque de défaut élevé et/ou à long terme. Par exemple, une des justifications les plus répandues pour l'introduction de plafonds d'intérêt est la lutte contre les pratiques de prêts "prédatrices" sur le marché des prêts sur salaire dit "*payday lending*". Il a été établi qu'une telle politique a des conséquences néfastes importantes (Zinman, 2010; Morgan and Strain, 2008). En particulier, Zinman (2010) constate qu'après avoir été rationnés en raison des plafonds de taux d'intérêt, ceux qui étaient des emprunteurs sur salaire retardent leurs paiements et creusent leurs découverts. Cuesta and Sepulveda (2020) utilisent d'importantes bases de données administratives au Chili pour montrer que les emprunteurs ayant un faible score de crédit ont été rationnés après que les plafonds de taux d'intérêt pour les prêts à la consommation ont été revus à la baisse. Ils estiment également en terme de bien-être les gains résultant de la baisse des taux d'intérêt et les pertes dues au rationnement, en estimant un important modèle structurel du marché du crédit à la consommation. Les pertes dues au rationnement dépassent les gains des emprunteurs qui restent sur le marché, et le résultat serait ainsi en considérant un marché du crédit monopolistique.

Malgré ses inconvénients, ce type de régulation est omniprésent : Ferrari et al. (2018) constatent qu'au moins 76 pays ont une forme de restriction des taux d'intérêt pour les marchés du crédit. Comme dans le cas de la régulation du surendettement, la restriction des taux d'intérêt diffère grandement entre pays en ce qui concerne les niveaux des plafonds et la manière dont ils sont différenciés. Aux États-Unis, la plupart des États imposent des plafonds pour les petits prêts sur salaire et pour les prêts à tempérament plus importants. Cependant, le plus grand marché des crédits à la consommation - les cartes de crédit - a été déréglementé depuis les années 1970, de sorte que la restriction y est pratiquement inexistante. En France, au contraire, la réglementation est particulièrement stricte. Cela est dû à la règle d'actualisation des plafonds : pour plusieurs types de prêts, le régulateur calcule trimestriellement le taux d'intérêt moyen du marché et fixe ensuite le plafond comme une majoration par rapport à ce taux moyen. Les fortes baisses des taux directeurs des banques centrales sur la dernière décennie ont exercé une pression à la baisse sur les taux du marché du crédit, ce qui a entraîné des plafonds de taux d'intérêt particulièrement stricts. J'explore dans le Chapitre 3 les implications de cette régulation sur le marché français du crédit immobilier.

Résumé des résultats

Chapitre 1 : Redistribution intergénérationnelle avec contraintes endogènes sur la dette privée

Dans le chapitre 1 de cette thèse, j'étudie comment la possibilité de faillite personnelle et les contraintes de crédit qui en découlent façonnent le choix des instruments pour la redistribution à long terme. Je considère le modèle de Azariadis and Lambertini (2003) aux générations imbriquées de consommateurs qui vivent pendant trois périodes. Il y a une dotation exogène du bien de consommation à chaque période de la vie des agents, et l'État procède à des transferts forfaitaires, financés par des impôts forfaitaires et de la dette publique. Cette politique de redistribution est un moyen de lisser la consommation entre les périodes ; un autre canal pour ce faire est le marché des prêts à la consommation, où les agents peuvent emprunter lors de la première période, puis prêter dans la deuxième. Les agents peuvent se déclarer en faillite personnelle pour dettes à de la deuxième période, et ni leurs dotations ni leurs transferts ne peuvent être confisqués par leurs créanciers. La seule conséquence de la faillite personnelle dans le modèle est l'exclusion du système financier entre la deuxième et la troisième période de la vie des agents. La faillite augmente le niveau de consommation à la deuxième période, mais elle a pour conséquence l'incapacité de générer de l'épargne pour la vieillesse. L'offre de crédit dans ce cadre où il est possible de déclarer une faillite personnelle est modélisée comme dans Kehoe and Levine (1993) : les prêteurs limitent le montant du prêt de telle sorte que les emprunteurs soient indifférents entre la faillite et le remboursement de leur dette. On suppose alors que la faillite ne se produit jamais à l'équilibre.

La détermination de la contrainte d'emprunt endogène est illustrée dans le graphique 2, emprunté à Azariadis and Lambertini (2003). Ce diagramme montre le choix de consommation d'un individu dans la deuxième période — pour un adulte — qui peut soit déclarer faillite sur ses dettes de la première période, soit les rembourser. Après la faillite, tout ce qu'il peut consommer est le panier représenté par le cercle noir — les dotations et les transferts de l'agent dans les deux dernières périodes restantes. S'il rembourse ses dettes, sa consommation à l'âge adulte sera amoindrie, mais il pourra générer de l'épargne en vue de la triosième période – la vieillesse –, rémunérée au taux d'intérêt du marché. Le cercle gris correspond donc à la limite supérieure du remboursement des dettes issue de la première période autorisé à l'équilibre. Pour s'en rendre compte, il suffit de voir que le meilleur panier de consommation possible après remboursement de ce montant donne la même utilité que le panier de consommation post-faillite. L'agent déclare donc la faillite pour toute dette qui fait passer ses revenus à gauche du cercle gris. Les deux principaux déterminants de cette contrainte endogène sur l'emprunt sont les taux d'intérêt et la politique de transfert qui détermine la position du cercle noir.

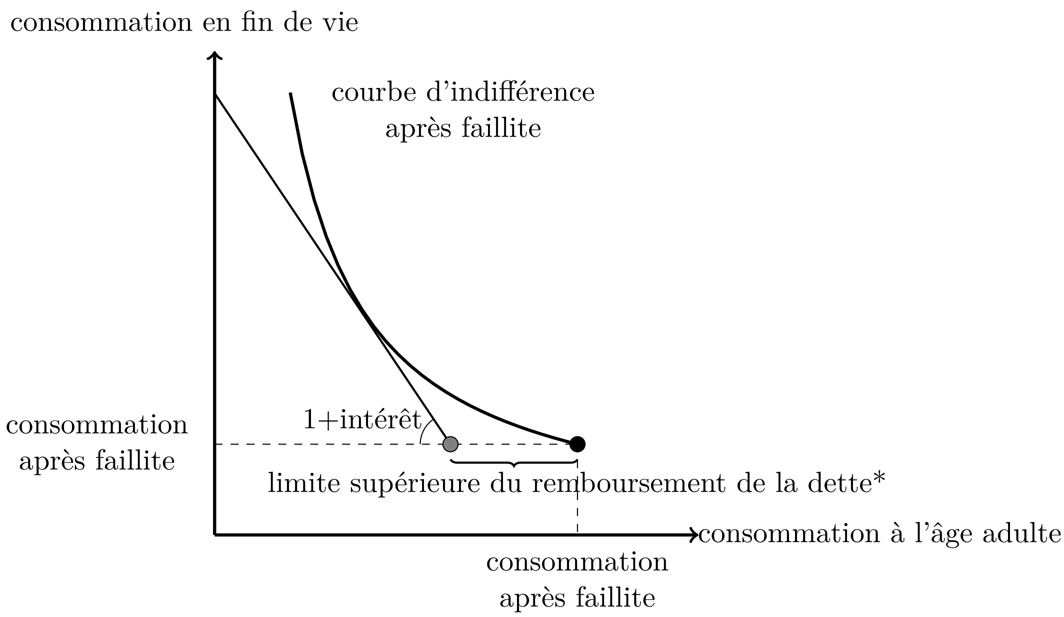


Figure 2: Illustration du mécanisme de contrainte d'emprunt endogène, chapitres 1 et 2. * : dans le chapitre 2, ce segment est le montant d'équilibre du remboursement de la dette et non la limite supérieure.

Pour comprendre intuitivement comment la politique de redistribution influence les contraintes d'emprunt, il suffit de considérer une économie avec un système de retraite par répartition où les cotisations et prestations sont relativement importantes. L'État s'occupe du niveau de consommation des ménages retraités, de sorte que l'accumulation d'actifs est relativement peu importante dans la logique du cycle de vie. Les ménages endettés ont alors une alternative relativement peu coûteuse en cas de faillite. En équilibre, les prêteurs reconnaissent cet aléa moral et diminuent en conséquence l'offre de crédit. Cela plaide donc contre l'utilisation de pensions par répartition en terme de redistribution entre les générations : si le niveau de consommation à la retraite est garanti par l'État, les jeunes emprunteurs se voient refuser leur crédit et ne peuvent pas consommer comme ils l'entendent. Je montre que le recours à la dette publique élargit l'ensemble des allocations optimales qui peuvent être décentralisées grâce à la politique de redistribution[3]. Dans ce modèle simple, la dette publique est un investissement parfait pour les ménages qui veulent financer leur consommation à la retraite. Si le gouvernement réduit le système par répartition et finance cette réforme avec une dette publique supplémentaire, il peut permettre une plus grande consommation de toutes les jeunes générations. Cela s'explique entièrement par les incitations à la faillite personnelle et la contrainte d'emprunt endogène qui en résulte, qui s'assouplie dans le nouvel équilibre. D'abord, un système de retraite par répartition plus petit rend l'accumulation d'actifs plus importante, de sorte que la faillite personnelle est plus coûteuse. Ensuite, l'introduction de la dette publique fait monter les taux d'intérêt, un résultat que l'on retrouve dans la majorité des modèles macroéconomiques. Cela crée des incitations supplémentaires à accumuler des actifs en vue de la retraite, ce qui rend encore une fois l'accès aux marchés financiers plus important avant la retraite. Ces deux canaux se traduisent par une contrainte d'emprunt qui devient moins stricte à l'équilibre : il y a moins d'incitations à faire faillite, de sorte que les jeunes ménages peuvent obtenir plus de crédit sans risque de repaiement supplémentaire.

Cet article fournit donc un nouvel argument contre les résultats bien connus sur l'équivalence de la dette publique et des impôts ou transferts forfaitaires. De nombreux auteurs, depuis la contribution fondamentale de Barro (1974), ont étudié les implications du niveau de la dette publique sur le bien-être dans les économies soumises à des contraintes de crédit (Woodford, 1990; Aiyagari and McGrattan, 1998; Eggertsson and Krugman, 2012). Contrairement à la plupart de cette littérature, j'analyse les contraintes de crédit endogènes, puisque je m'intéresse aux économies avec une possibilité de faillite personnelle. L'effet de la dette publique s'opère alors par l'offre de crédit et non par la demande de crédit.

En outre, je trouve une autre vertu de la dette publique dans mon modèle : elle réduit le problème des équilibres multiples, ou de l'instabilité du marché du crédit. Ce problème peut résulter de la possibilité d'une faillite personnelle, comme on le trouve couramment dans la littérature depuis Kehoe and Levine (1993). Le graphique 6 permet de décrire comment un équilibre sous-optimal avec un resserrement du crédit peut se produire. Si l'on s'attend à ce que le taux d'intérêt soit inférieur au taux marginal de substitution au niveau du panier de consommation post-faillite — le cercle noir — alors la limite d'emprunt est nulle. L'anticipatoin de taux d'intérêt aussi bas et d'une autarcie financière s'avère être auto-réalisatrice si et seulement si le gouvernement maintient son budget à l'équilibre. L'utilisation de la dette publique permet d'exclure des anticipations de taux d'intérêt aussi bas. En revanche, une politique d'équilibre budgétaire ne peut jamais exclure l'auto-réalisation des anticipations, de sorte que le gouvernement ne peut pas garantir que sa politique aboutisse à un équilibre optimal.

[3]Je compare donc deux politiques relativement à leur capacité à décentraliser les allocations optimales : l'une avec des transferts/impôts aux adultes et aux personnes âgées et sans dette publique ; l'autre avec des transferts/impôts aux personnes âgées et de la dette publique. Les allocations optimales sont définies comme des solutions aux problèmes d'optimisation d'un planificateur social bienveillant.

Chapitre 2 : Contraintes d'endettement endogène et bulles rationnelles dans un modèle de croissance à générations imbriquées

Dans le chapitre 2, Bertrand Wigniolle et moi étudions la faillite personnelle et les contraintes d'emprunt dans une économie de production avec du capital physique et des bulles d'actifs. Comme dans le chapitre précédent, le modèle s'appuie sur Azariadis and Lambertini (2003), mais nous supposons un secteur de production néoclassique standard au lieu de dotations exogènes. Cela nous permet d'étudier la dynamique des salaires comme un déterminant supplémentaire des incitations à la faillite personnelle. La structure du marché du crédit est telle que décrite ci-dessus. La détermination de la contrainte d'emprunt endogène est, là encore, comme dans le graphique 2, mais le revenu du travail à l'âge adulte et dans la vieillesse sont des variables d'équilibre déterminées par le niveau du capital dans l'économie. Contrairement à la plupart des modèles à contraintes d'emprunt endogènes, le nôtre permet d'étudier les équilibres qui passent d'un régime à emprunteurs sous contrainte de crédit à un régime où les contraintes ne sont pas saturées. Des formes fonctionnelles simples des fonctions de production et d'utilité nous permettent de caractériser la dynamique globale de l'équilibre.

L'économie a un comportement remarquablement stable en équilibre, malgré les rétroactions entre les contraintes de crédit, les salaires et le capital physique. L'état stationnaire est toujours unique et peut avoir des emprunteurs contraints ou non contraints. La transition vers l'état stationnaire est toujours monotone, avec un régime contraint ou non contraint pour toutes les périodes. À l'équilibre, la contrainte de crédit impose que les agents empruntent jusqu'à une fraction constante de leur revenu de vie, de sorte que l'économie ait un comportement similaire à l'économie avec des contraintes de crédit exogènes analysées par Jappelli and Pagano (1994a).

Nous explorons ensuite la possibilité de bulles financières liées à la faillite personnelle. L'hypothèse est que les ménages en faillite peuvent détenir un actif sans valeur intrinsèque — une bulle pure — à des fins de revente, même s'ils sont exclus du système financier formel. Pour définir les prix de l'actif de la bulle, nous considérons la possibilité d'une faillite en équilibre, en nous écartant du cadre de Kehoe and Levine (1993). En équilibre, les agents sont indifférents entre la faillite et le remboursement, donc une fraction des emprunteurs déclarent faillite. Cette fraction est une variable d'équilibre qui est toujours correctement prévue par les agents rationnels. Les prêteurs sont alors confrontés à un risque de faillite bien défini pour chaque prêt et fixent une marge sur les intérêts du prêt qui égalise le rendement attendu des prêts et du capital physique. La détermination conjointe de la fraction des emprunteurs en faillite, du taux d'intérêt sur les prêts et du rendement sur la bulle peut être décrite par le graphique 2. Le panier de consommation post-faillite ne dépend plus seulement du revenu du travail, mais aussi du rendement de la bulle, puisque cet actif peut être détenu après exclusion du crédit et du marché des capitaux. Le segment entre le cercle gris et le cercle noir correspond ainsi au montant du remboursement de la dette, y compris le taux d'intérêt avec la marge.

L'économie avec bulles est sujette à des équilibres multiples et à l'indétermination. Il existe trois configurations possibles des équilibres, selon les paramètres du modèle. Dans la première, il y a un état stationaire unique et il n'y a ni bulle ni faillite ; l'état stationnaire est déterminé localement. Dans la deuxième, il y a deux états stationnaires : un premier avec une bulle, une faillite permanente et une détermination locale ; un autre sans bulle ni faillite et une indétermination locale. Dans la troisième configuration, il y a également deux états stationnaires : un avec une bulle permanente, une faillite et une détermination locale ; un autre sans bulle, mais avec une faillite et une indétermination locale. L'analyse de la dynamique globale n'est faite que pour un cas particulier où les agents ne travaillent pas dans la dernière période. En résumé, l'économie avec bulles se comporte conformément aux conclusions fondamentales de Tirole (1985). Il existe cependant plusieurs différences importantes. Premièrement, les équilibres à bulles sont associés à des faillites et des pertes d'efficacité. Deuxièmement, l'existence de l'état stationnaire avec bulle permanente ne nécessite pas une suraccumulation de capital en l'absence de bulle. Ceci est dû à une autre propriété nouvelle de notre modèle : le capital physique, les prêts et les bulles ont nécessairement des taux de rendement différents en équilibre.

Chapitre 3 : La restriction des taux d'intérêt entraîne une réduction des prêts au logement : le cas de la France

Le chapitre 3 porte sur la restriction des taux d'intérêt pour les prêts immobiliers en France. Contrairement aux autres chapitres, l'analyse s'appuie sur le cadre institutionnel spécifique de la France et est réalisée avec une méthodologie de régression en forme réduite. J'étudie si la restriction des taux d'intérêt entraîne des distorsions dans les portefeuilles de prêts des établissements de crédit. Plus précisément, la question est de savoir si un plafond de taux d'intérêt uniforme pour différents types de prêts entraîne un raccourcissement de la durée moyenne des prêts. Le graphique 3 en explique la logique. Sur le côté gauche du graphique, une définition d'un taux d'intérêt effectif est donnée. C'est la mesure du prix du prêt qui est soumise à restriction en France. Sur la droite, ce prix est décomposé en une marge, diverses primes et divers coûts. Le but premier de la restriction du taux d'intérêt est de limiter la marge de prêt, ou les bénéfices. Cependant, une autre conséquence possible est un déplacement de l'offre de prêts vers des prêts qui nécessitent des primes plus faibles. La littérature s'est surtout concentrée sur la prime de risque de remboursement, et cherche à savoir si l'offre de prêts risqués se réduit lorsque les plafonds de prêts se resserrent (Cuesta and Sepulveda, 2020). Mon hypothèse porte sur la durée du prêt : sous un plafond de taux d'intérêt reserré, les prêteurs sont incités à proposer des prêts plus courts, qui nécessitent une prime de terme plus faible.

J'exploite une réforme de la restriction des taux d'intérêt en France qui a fixé des plafonds de taux d'intérêt distincts pour les prêts au logement à court et à long terme. Dans chaque catégorie de prêt, le plafond est déterminé en fonction du taux d'intérêt effectif moyen sur le marché, et ce taux a été plus élevé pour les prêts les plus longs —

la prime de terme a été positive. Les prêteurs pourraient alors accorder des prêts plus longs sans atteindre le plafond après la réforme. Je teste cette hypothèse en comparant la l'émission de prêts à court et long terme avant et après la réforme, qui a été annoncée en 2016 et mise en place en 2017.

Le graphique 4 représente la valeur totale des crédits nouveaux à long terme (plus de 20 ans) et à court terme trouvées dans M_CONTRAN, un large échantillon des crédits nouveaux administré par la Banque de France. Étant donné que la réforme en question a été annoncée en 2016 et mise en œuvre en 2017, ces données agrégées semblent soutenir l'hypothèse. Une régression de base des différences de différences sur ces données agrégées, ainsi que sur le nombre de prêts, confirme la significativité statistique des différences de différences. Cependant, les données agrégées peuvent avoir des interprétations qui ne sont pas liées à la réglementation des taux d'intérêt. Par exemple, le changement de la durée moyenne peut être du à des changements dans le pouvoir de marché relatif des établissements de crédit. Pour tenir compte de cela, je fournis des estimations analogues au niveau des établissements de crédit. Une autre explication alternative du changement de terme serait un choc de demande affectant de larges sous-populations d'emprunteurs. En effet, supposons que la population d'emprunteurs puisse être séparée en emprunteurs à long terme d'une part et emprunteurs à court terme d'autre part, en utilisant plusieurs caractéristiques socio-économiques — parmi les exemples de caractéristiques pertinentes, on peut citer le revenu, l'âge et le secteur d'emploi de l'emprunteur. Ces caractéristiques sont inégalement réparties dans les différents régions en France. Ensuite, si le changement des conditions de prêt s'explique par un choc de la demande spécifique à un groupe démographique, il est probable qu'il s'agisse d'un déplacement des prêts entre les différents lieux et non à l'intérieur de ceux-ci. J'étudie donc les glissements de termes à l'intérieur d'un même lieu en effectuant des régressions des différences de différences au niveau d'une agence bancaire. Au niveau des établissements de crédit et des succursales, j'obtiens des estimations qui soutiennent l'hypothèse principale et sont comparables en termes d'ampleur aux estimations globales.

Motivé par les estimations des différences de différences confirmant la pertinence de la réforme, je fournis des régressions qui contrôlent d'autres facteurs variables sur la durée du prêt et j'estime l'ampleur de l'effet attribuable à la réforme. Les courbes de rendement sur les différents marchés de la dette se sont considérablement aplaties au cours de la dernière décennie, ce qui a contribué à déplacer l'offre de prêts vers des durées plus longues, indépendamment des politiques de restriction des taux d'intérêt. J'utilise la dynamique des plafonds des prêts immobiliers à long et court terme comme variable de traitement dans une régression qui contrôle les taux de financement sur différents horizons. Les taux de financement sont mesurés par les prix des swaps sur les taux interbancaires avec une durée pertinente. Toutes les covariables temporelles, y compris les plafonds de taux d'intérêt, sont valables pour l'ensemble de l'économie, tandis que les résultats - les crédits nouveaux - sont utilisés au niveaux global, des prêteurs et des guichets. L'utilisation de données granulaires permet de traiter la dynamique des plafonds comme une dynamique exogène : bien que le plafond de chaque catégorie de prêt soit mis à jour en fonction de l'intérêt moyen du marché pour le type de prêt donné, un prêteur ou un guichet ne peut pas influencer la moyenne du marché de manière considérable. Je trouve que la différence de plafonds de taux d'intérêt a un impact statistiquement et économiquement significatif sur les prêts à long terme. La réforme a conduit à une augmentation nette de l'ensemble des prêts au logement estimée à environ 6% par trimestre, bien que les prêts à long terme aient évincé les prêts à court terme. Je calcule ensuite l'évolution des liquidités et du service de la dette d'un emprunteur type qui a contracté un prêt à long terme au lieu d'un prêt à court terme en raison de la réforme de la restriction des taux d'intérêt. Un tel emprunteur a vu ses paiements mensuels diminuer de 30%, mais le coût global du service de la dette a augmenté de 75% — effets qui s'expliquent principalement par la durée du prêt lui-même et non par le taux d'intérêt plus élevé qui est appliqué aux prêts plus longs.

Le raccourcissement des prêts en conséquence de la restriction des taux d'intérêt n'a pas été exploré dans la littérature auparavant. En même temps, il a été constaté que les prêts à long terme sont appréciés des consommateurs. Attanasio et al. (2008) constatent que la demande est plus sensible à la durée qu'au taux d'intérêt dans les prêts automobiles. Cela suggère que les consommateurs apprécient beaucoup la possibilité de prolonger leur dette et de diminuer les paiements réguliers. L'élasticité de la demande de crédit par rapport à la durée est particulièrement élevée pour les consommateurs à faibles revenus, ce que les auteurs interprètent comme une contrainte de liquidité atténuée par des prêts plus longs. Karlan and Zinman (2008) obtiennent le même résultat pour les microcrédits à la consommation. L'argument des contraintes de liquidité est probablement le plus pertinent pour les prêts immobiliers, en raison des montants importants empruntés. Dhillon et al. (1990) fournit des éléments d'enquête sur la préférence pour les prêts hypothécaires à 30 ans par rapport à ceux à 15 ans qui est plus forte pour les emprunteurs à faible revenu. Il est donc avantageux de définir les plafonds de taux d'intérêt de manière suffisamment souple pour que les prêteurs puissent fixer le prix des prêts en fonction de leur durée.

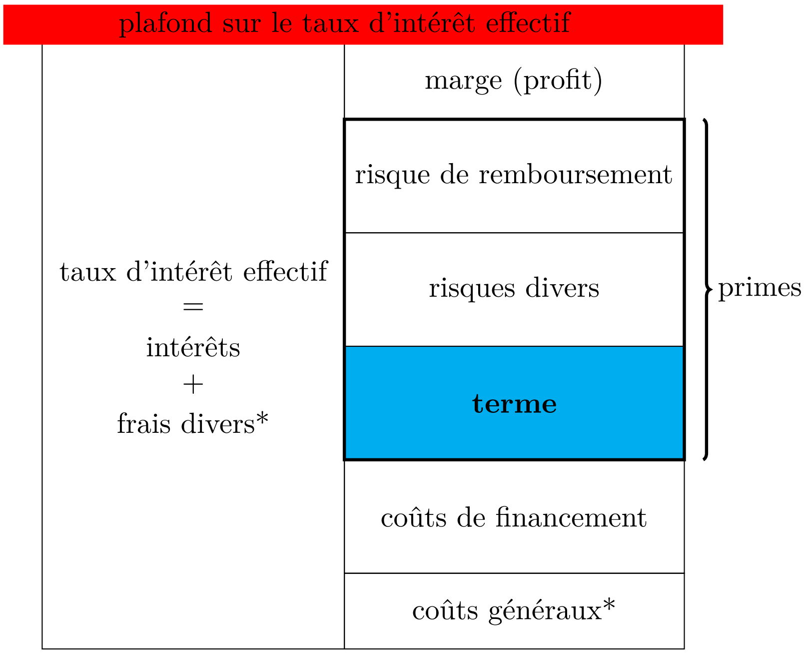


Figure 3: Le taux d'intérêt effectif d'un prêt — décomposé en marge, primes et coûts divers.
*: en pourcentage du montant du prêt, annualisé.

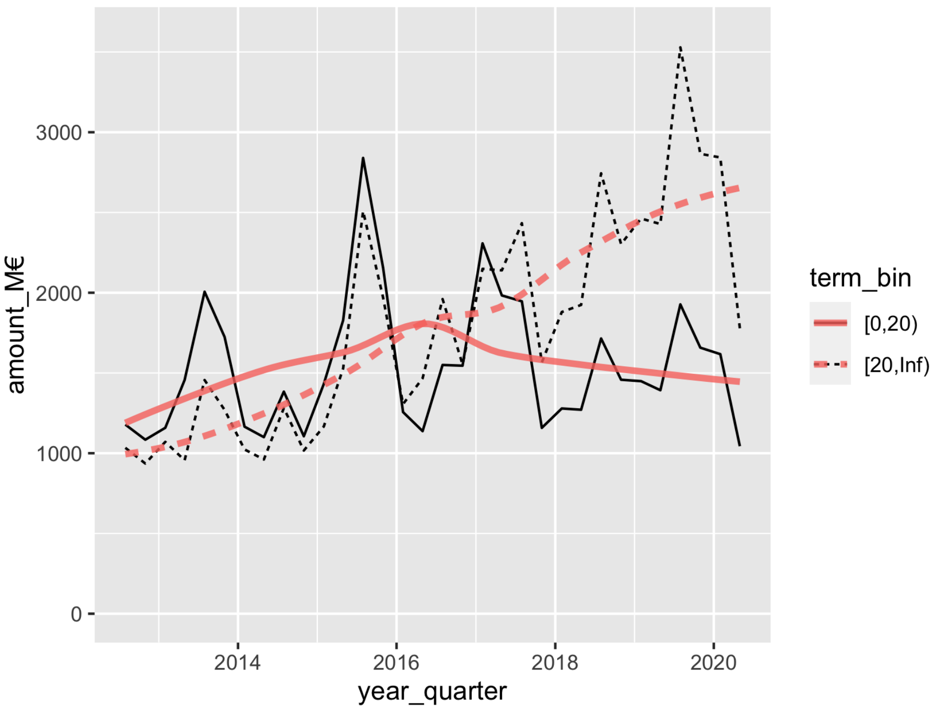


Figure 4: Montant total des prêts accordés, prêts à long terme (en pointillés) et court terme (traits pleins), en millions d'euros. Les lignes lisses sont les valeurs ajustées d'une régression LOESS estimée localement. Source : Jeux de données M_CONTRAN de la Banque de France, mes calculs.

Introduction (in English)

Household credit is an ever-expanding part of modern financial systems with immediate real effects. Both secured and unsecured credit has been on the rise in advanced and developing economies in the last decades. The central role of household debt to the Great Recession of 2007-2008 has brought it under unprecedented scrutiny of both economists and policy makers, resulting in additional regulation of loans to households and banking in general. In this thesis, I study implications of the regulatory measures in terms of credit constraints, optimal redistribution, and macroeconomic stability. In particular, two regulatory tools are studied: personal bankruptcy and interest rate restriction.

What are the implications of regulatory responses to household credit expansion? How does household credit interact with macroeconomic policy and transforms the latter? Does regulation prevent household debt from causing macroeconomic instability or on the contrary, makes debt more de-stabilizing? This thesis addresses these broad questions in the context of two regulatory tools and two types of debt. In Chapter 1, I study consumption debt with a possibility of personal bankruptcy — an institution mostly found in advanced economies and frequently used in the United States. The focus of the chapter is the interaction of this kind of regulation with long-run redistribution policy. The same market and regulation is studied in Chapter 2, co-written with Bertrand Wigniolle. The question addressed in this chapter is macroeconomic stability: we ponder whether the possibility of personal bankruptcy creates long-run instability via credit constraints and/or asset bubbles. The third chapter is devoted to interest rate restriction and its influence on the duration of loans, in the context of housing loans in France. I exploit on a reform that allows to establish a causal relationship between the interest rate restriction and credit supply. The findings of the three chapters are summarized below, after a brief description of stylized facts on household debt around the world and the regulatory response.

Facts on household debt

Household credit is occupies a growing share of lending activity around the globe, shifting the focus of financial intermediaries away from lending to firms (Jordà et al., 2016). Figure 5 describes the evolution of credit in seven large economies that differ significantly in both institutions and geography. The first plot shows that private debt — the sum of household and non-financial corporate debt — has been growing steadily with respect to GDP for most of these countries. At the same time, the second plot shows that the share of household credit has also tended to grow over the period.

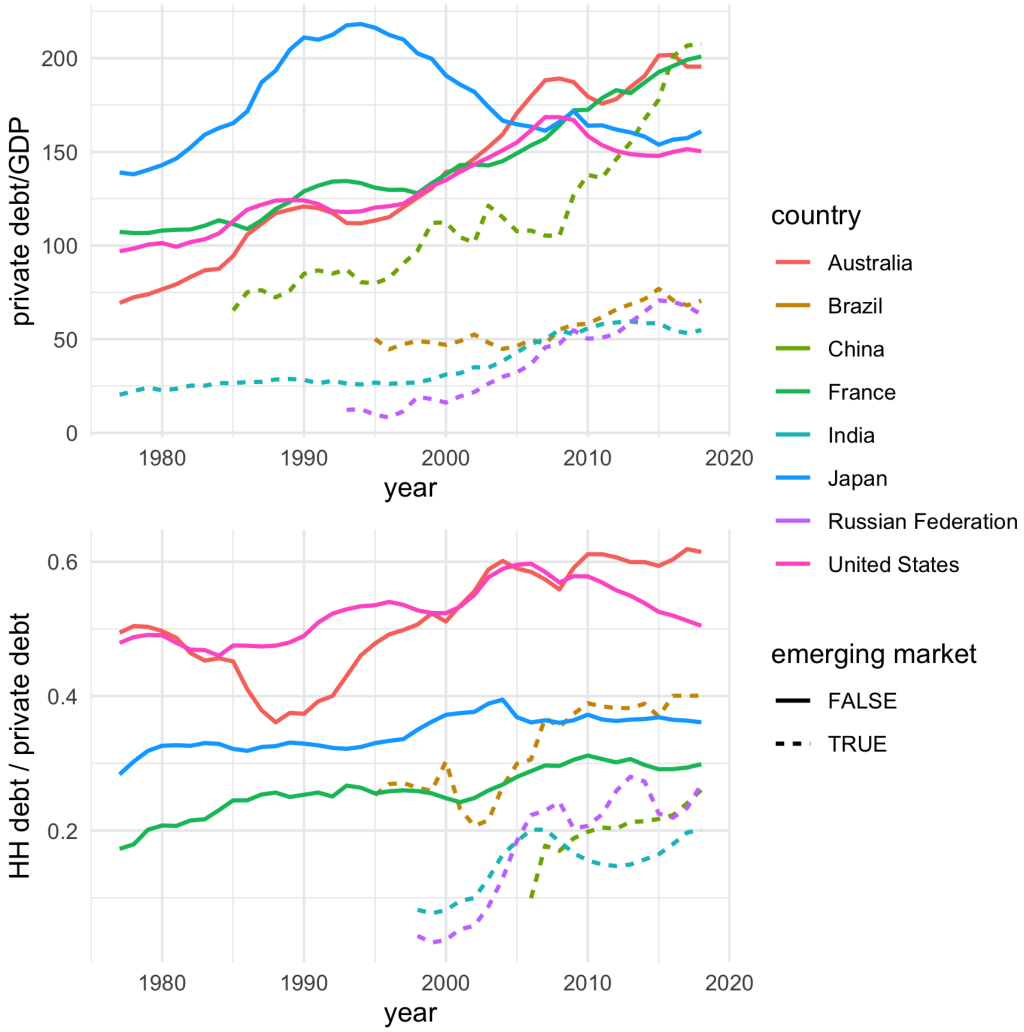


Figure 5: Private debt over GDP (top) and share of household debt in private debt (bottom) in selected countries, 1977-2018. Private debt is the sum of household debt and debt of non-financial corporations. Series of emerging markets in dashed lines. Source: IMF Global Debt Database.

What are the possible causes of such dynamics? First, real interest rates have declined significantly, while risk and term spreads have been much more stable, which points to excess supply of savings over demand for investment as the main driving force (Rachel and Summers, n.d.). Bernanke (2005) has famously explained the accumulation of debts in the US from an international perspective: a number of countries, of Western Europe and Southeast Asia in particular, have experienced a growth of savings that has resulted in capital flows into the U.S. banking sector. Coeurdacier et al. (2015) highlight the role of credit constraints for households that have been particularly strong in China and weak in the U.S. This perspective, however, does not explain why a number of large countries of different income levels have had a growth in the ratio of credit to GDP explained mostly by household credit. Recent literature highlights the role of internal saving gluts — in other words, rising wealth inequality. Kumhof et al. (2015) decompose by income group the fast debt accumulation that has preceded the Great Depression and the Great Recession in the U.S. They show that the top 5% of the income distribution has experienced an increase in net savings, as well as its share of income in the economy, in both periods. It is the bottom 95% that has accounted for the increase in aggregate debt levels. Taking a cross-sectional perspective, Mian et al. (2020a) relate the trends in debt accumulation to trends in income inequality across the U.S. states. A robust and consistent pattern emerges: households from the top 1% of income distribution have increased their savings substantially since the 80s, while the bottom 90% has accumulated debt. Furthermore, the additional savings of the rich have been done predominantly in financial assets. It is therefore crucial to understand the role of the financial sector in channelling these savings to household credit rather than business loans. Jordà et al. (2016) study the asset choice of financial intermediaries around the world and find that mortgages financing house purchase have driven the expansion of banks' balance sheets throughout the 20th century. The authors relate this transformation to the expansive housing policies that have promoted homeownership in most advanced economies after World War II. Mian and Sufi (2015) discuss at length the intrinsic problems of mortgage contracts, which involve externalities that arise due to foreclosure during crisis episodes such as the Great Recession of 2007-2008.

Although mortgages explain a bulk of the increase in household debt balances of the last decades and are the primary culprit for the Great Recession, other types of debt must not be ignored. First, in some countries, mortgage contracts are rarely used, despite a developed residential real estate credit market. For instance, lenders in France typically rely on insured loans that do not use the underlying housing as collateral. Second, the IT revolution has made credit risk management more scalable, resulting in a rapid expansion of unsecured lending since the 1990s (Livshits et al., 2016). Finally, given that the same household may accumulate both secured and unsecured debt, the two are likely to interact when affecting the borrower's financial behavior, especially in distress episodes. As pointed out by White (2007), U.S. households in distress may declare insolvency with respect to their unsecured debt in order to delay foreclosures on their homes due to unpaid mortgages. Motivated by these considerations, I study non-mortgage credit such as unsecured consumption loans and insured housing loans in this thesis. Their main differences from mortgages is default risk, resulting in either credit rationing or high interest rates.

Do the households repay duly their growing debt balances? For most countries, public data on repayment behavior is scant. Some harmonized survey data is available for the EU countries: the Household Finance and Consumption Survey of the ECB asks households if they have been late on any of their loan payments over the last year. 5% of French debt-holding respondents have had late payments in 2013, and the figure has decreased to 4.2% in 2017. In the U.S., the responses to the same question in Survey of Consumer Finances paint a starker picture. In 2013, the share of debt-holding respondents with late payments stood at 14.9% and has been decreasing ever since, but the figure was still at an alarming 12.3% in 2019.

The large welfare effects associated with household debt have lead governments to regulate various aspects of credit contracts. Most of the measures fit in the following three categories: information disclosure regulation, restriction of price and non-price elements of loan contracts, and state intervention in case of insolvency. I do not consider information disclosure policies in this thesis[4]. Instead, I focus on the latter two types of measures, as they are widely used to regulate non-mortgage loans. Debt without collateral poses two particular challenges to both lenders and the regulators. First, such debt can be defaulted on, and in absence of regulatory intervention the household can end up in permanent distress caused by debt collection. Second, the lack of collateral means that the lenders rely on interest or non-interest terms of the contract to price the repayment risk. Regulators commonly address these issues with two tools: bankruptcy-type procedures for individuals that involve debt discharge and price controls that restrict interest and non-interest cost of loans. For simplicity, I refer to the first one as personal bankruptcy and the second one as interest rate restriction. On the one hand, personal bankruptcy procedures allow to overcome some information asymmetries between the lender and the distressed borrower, letting both avoid the costs of excessive debt collection. On the other hand, interest rate restriction policies are meant to prevent financial distress by slowing down debt accumulation due to compound interest. However, both tools have a number of undesired effects, mostly related to credit constraints. This is explored in the following sections and the three main chapters of the thesis. Chapter 1 and 2 study personal bankruptcy of the type found in the U.S., while Chapter 3 studies on interest rate restriction of the type found in France.

Regulatory response: over-indebtedness, personal bankruptcy

In theory, debt default is a necessary part of well-functioning financial markets (Dubey et al., 1989; Zame, 1993). To summarize the main argument, if lenders only accepted such contracts where the debt is repaid any possible circumstance, too little loans would be made. However, a free-market arrangement of default gives rise to a debt collection industry that specializes at producing borrower distress (Fedaseyeu and Hunt, 2018). Most countries have therefore put in place some systems for supervision and legal settlement of household over-indebtedness. However, the systems vary greatly in the degree of possible debt discharge. A stark example of debt discharge is the "fresh start" personal bankruptcy procedure in the U.S. The person eligible for the procedure has almost all of her unsecured debts written off[5] and the creditors of the person have no right to garnish her labor income afterwards. The main disciplining factor for the borrower is that some of her assets can

[4]See Campbell et al. (2011) for a discussion of several measures that have been adopted in the U.S.

[5]Importantly for the recent U.S. context, student debt is typically not written off in Chapter 7 bankruptcy. However it can be done with a special procedure aimed at proving "undue financial hardship" caused by the student debt. It can be also be restructured under Chapter 13 bankruptcy.

be used to repay the creditors for several years after the bankruptcy filing.

Personal bankruptcy is a widespread phenomenon in the United States. According to the Survey of Consumer Finances, 2% of households have filed for bankruptcy at some point in 2014-2019, despite rather favorable macroeconomic conditions at the time. Fay et al. (2002) analyse survey data and find that bankruptcy filing has been mostly explained by rational, or strategic, considerations in the 1990s. They quantify these strategic incentives by simple accounting of net financial benefits of bankruptcy. Adverse unexpected events, on the contrary, account for a relatively small number of bankruptcies. Livshits et al. (2010) arrive at similar conclusions when fitting a large life-cycle model to survey data. In Chapters 1 and 2 of this thesis, I explore how public policy and macroeconomic conditions shape the incentives for personal bankruptcy of the "fresh start" type.

France offers an example of a different approach to intervention in case of borrower distress. A person can be declared over-indebted and get their debts partially discharged in formal procedure administered by Banque de France. However, only a minimal part assets and future income of the borrower is protected from the creditors. Furthermore, the discharge of debts is conditioned on the borrower's effort to keep earning. Predictably, fewer borrowers undergo such a procedure than in the U.S.: roughly 1% of French households have done it over 2014-2019. Moreover, the number of annual cases has declined by around 30% over the period. This is likely explained by the significant tightening of interest rate restriction, the main form of consumer financial protection in France.

Regulatory response: interest rate restriction

Credit to households for high interest has been long condemned as usury — excessive profits of richer lenders at the expense of poorer borrowers. Nowadays, this narrative is used to motivate interest rate restriction policies. In addition, proponents of the measure propose more subtle arguments. Glaeser and Scheinkman (1998) argue that a transfer of surplus from lenders to borrowers in credit contracts improves overall risk sharing. If it is not known whether a given young individual will end up being mainly a borrower or a lender throughout her life, an equal division of surplus between lenders and borrowers may be a preferred choice for all ex ante. Coco and De Meza (2009) show that smaller interest alleviates the moral hazard problem in a context of lending to entrepreneurs with limited liability, which is also related to unsecured consumer credit with possibility of personal bankruptcy.

Given that interest rate restriction is a particular case of price controls, it is no surprise that most analysis in the literature finds significant drawbacks of the measure. Apart from creating shortages, as any price ceiling may do, interest ceilings alter the selection of borrowers into the market, since high interest can be necessary for loans with high repayment risk and/or long term. For instance, a widespread motivation for introducing interest ceilings is to fight "predatory lending" practices of the payday loans market. Such a policy is found to have significant undesired consequences (Zinman, 2010; Morgan and

Strain, 2008). In particular, Zinman (2010) finds that former payday borrowers start delaying payments and incurring overdrafts after being rationed due to interest rate ceilings. Cuesta and Sepulveda (2020) use large administrative databases to show that borrowers with low credit scores have been rationed after interest rate ceilings for consumer loans have been revised downward in Chile. They also estimate welfare gains from smaller interest rates and losses due to rationing by estimating a large structural model of consumer credit market. Losses due to rationing outweigh the gains of borrowers that remain in the market, and it would be so even in a monopolistic credit market.

Despite the drawbacks, this kind of regulation is ubiquitous: Ferrari et al. (2018) find the at least 76 countries have some form of interest rate restriction for credit markets. As in the case of regulation of over-indebtedness, interest rate restriction differs greatly in the levels of interest ceilings and how differentiated they are. In the U.S., most states have ceilings for small payday loans and larger installment loans. However, the largest market for consumption debt — credit cards — has been deregulated since the 1970s such that the restriction is virtually nonexistent. In France, on the contrary, the regulation is particularly strict. This is a result of the updating rule for the ceilings: for several types of loans, the regulator calculates the average interest rate on the market quarterly and then sets the ceiling as a markup over the average rate. The sharp declines in central bank policy rates of last decade have put downward pressure on the credit market rates, resulting in particularly tight interest rate ceilings. I explore the implications of the policy for the French market of housing loans.

Results summary

Chapter 1: Intergenerational Redistribution with Endogenous Constraints to Private Debt

In Chapter 1 of this thesis, I study how the possibility of personal bankruptcy and the resulting credit constraints shape the choice of instruments for long-run redistribution. I consider the setup of Azariadis and Lambertini (2003) with overlapping generations of consumers that live for three periods. There is exogenous endowment of the consumption good specific to each period of life, and the government makes lump-sum transfers, financing them with lump-sum taxes and government debt. This redistribution policy is one channel of consumption smoothing between periods of life; another channel is the consumption loan market, where agents borrow in the first period of life, then lend in the second one. Agents can declare personal bankruptcy on debts in the second period of life, and neither their endowments nor transfers can be confiscated by their creditors. The only consequence of personal bankruptcy in the model is exclusion from the financial system between the second and the third period of life. Bankruptcy boosts the level of consumption in the second period of life, but comes at a cost of inability to make savings for old age. The credit supply under a possibility of personal bankruptcy is modeled as in Kehoe and Levine (1993): lenders constrain the amount of the loan such that the borrowers are indifferent between bankruptcy and repayment. It is then assumed that bankruptcy never takes place in equilibrium.

The determination of the endogenous borrowing constraint is illustrated in Figure 6, borrowed from Azariadis and Lambertini (2003). This diagram shows the consumption choice of an individual in the second period of life — an adult — that can either declare bankruptcy on her debts from first period or repay them. After bankruptcy, all she can consume is the bundle represented by the black circle — the endowments and transfers of the agent in the two remaining periods of life. If she repays her debts, her consumption when adult will be smaller, but she will will be able to make savings for old age, remunerated at the market interest rate. The gray circle then corresponds to the upper limit on the repayment of debts from the first period allowed in equilibrium. To see this, observe that the best possible consumption bundle after this amount has been repaid gives the same utility as the post-bankruptcy consumption bundle. The agent therefore declares bankruptcy under any debt that takes her income to the left of the gray circle. The two main determinants of this endogenous borrowing constraint are interest rates and the transfer policy that determines the position of the black circle.

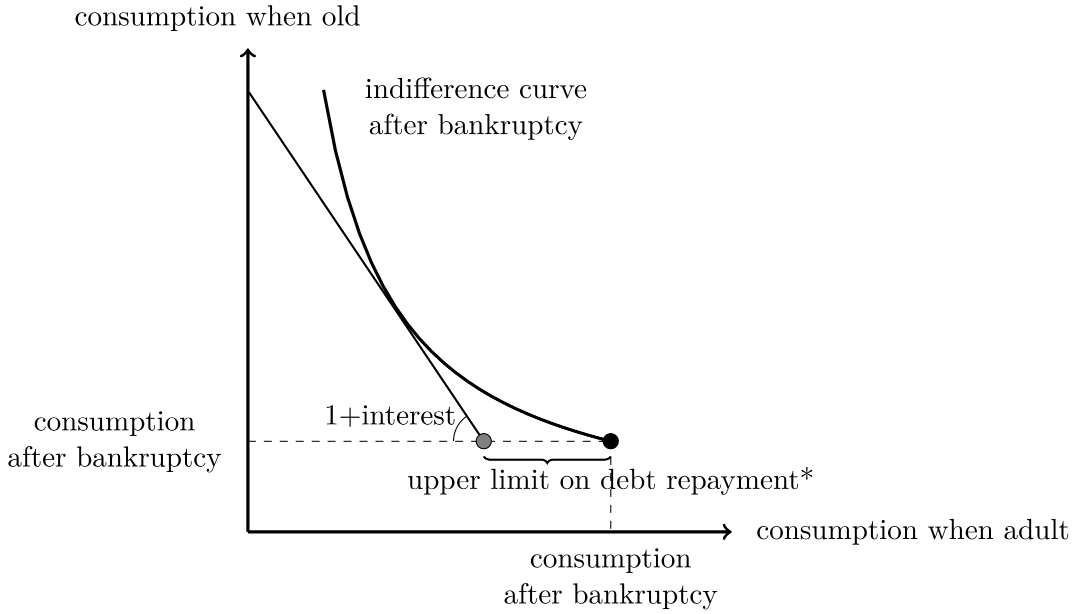


Figure 6: Illustration of the endogenous borrowing constraint mechanism, Chapters 1 and 2. *: in Chapter 2, this segment is the equilibrium amount of debt repayment and not the upper limit.

To understand intuitively how redistributive policy influences borrowing constraints, consider an economy with a pay-as-you-go pension system that has relatively large contributions and benefits. The state takes care of the consumption level of retired households, so that asset accumulation is relatively unimportant from the life-cycle perspective. Indebted households then have relatively small alternative cost of going bankrupt. In equilibrium, lenders recognize this moral hazard and decrease credit supply. This makes a case against the use of pay-as-you-go pensions for redistribution across generations: although consumption level at retirement is guaranteed by the state, the young borrowers are denied credit and cannot consume much. I show that the use of government debt expands the set of optimal allocations that can be decentralized with the redistribution policy[6]. In this simple model, government debt is a perfect investment vehicle for households that want to finance their retirement consumption. If the government reduces the pay-as-you-go system and finances this reform with additional government debt, it can allow for more consumption of all the young generations. This is explained entirely by the incentives for personal bankruptcy and the resulting endogenous borrowing constraint that is relaxed in the new equilibrium. First, a smaller pay-as-you-go pension system makes asset accumulation more important, so that personal bankruptcy is more costly. Second, introduction of government debt makes interest rates higher: a result found in a majority of macroeconomic models. This creates additional incentives of accumulating assets for retirement, which again makes access to financial markets more important prior to retirement. Both channels result in a borrowing constraint that becomes less tight in equilibrium: there are less incentives to go bankrupt, so that the young households can obtain more credit without additional repayment risk.

This article therefore provides a novel argument against the well known results on equivalence of government debt and lump sum taxes or transfers. Many authors since the seminal contribution of Barro (1974) have explored the welfare implications of the level of government debt in economies with credit constraints (Woodford, 1990; Aiyagari and McGrattan, 1998; Eggertsson and Krugman, 2012). In contrast with most of this literature, I consider endogenous credit constraints, since I am interested in economies with a possibility of personal bankruptcy. The effect of government debt then operates through credit supply and not credit demand.

In addition, I find another virtue of government debt in my framework: it alleviates the problem of multiple equilibria, or instability of the credit market. This problem can result from the possibility of personal bankruptcy, as it has been commonly found in the literature since Kehoe and Levine (1993). Figure 6 helps to describe how a suboptimal equilibrium with a credit crunch can occur. If the interest rate is expected to be lower than the marginal rate of substitution at the post-bankruptcy consumption bundle — the black circle — then the borrowing limit is null. Expectation of such low interest rates and a financial autarky turns out to be self-fulfilling if, and only if, government always balances its budget. The use of government debt can rule out the expectation of such low interest rates. In contrast, balanced-budget policies can never rule out the self-fulfilling expectations, so that the government cannot ensure that its policy results in an optimal equilibrium.

[6] I compare two policies with respect to their ability to decentralize optimal allocations: one with transfers/taxes to adults and to the old and no government debt; the second with transfers/taxes to the old and government debt. Optimal allocations are defined as solutions to optimization problems of a benevolent social planner.

Chapter 2: Endogenous Debt Constraints and Rational Bubbles in an OLG Growth Model

In Chapter 2, Bertrand Wigniolle and I study personal bankruptcy and borrowing constraints in a production economy with physical capital and asset bubbles. As in the previous chapter, the model builds on Azariadis and Lambertini (2003), but we assume a standard neoclassical production sector instead of exogenous endowments. This allows us to study wage dynamics as an additional determinant of the incentives for personal bankruptcy. The structure of the credit market is as described above. The determination of the endogenous borrowing constraint is, again, as in Figure 6, but labor income in adult and old age are equilibrium variables that are determined by capital intensity in the economy. Unlike most models with endogenous borrowing constraints, ours allows to study equilibria that switch between a regime with credit constrained borrowers and a regime where the constraints are not binding. Simple functional forms of the production and the utility functions allow us to characterize the global dynamics of the equilibrium.

The economy has a remarkably stable behavior in equilibrium, despite the feedbacks between credit constraints, wages and physical capital. The steady state is always unique and can have either constrained or unconstrained borrowers. The transition to the steady state is always monotonic, with either constrained or unconstrained regime in all periods. In equilibrium, the credit constraint is such that agents borrow up to a constant fraction of their lifetime income, so that the economy has similar behavior to economy with exogenous credit constraints analyzed by Jappelli and Pagano (1994a).

We then explore the possibility of financial bubbles that are associated with personal bankruptcy. The assumption is that bankrupt households can hold an asset without intrinsic value — a pure bubble — for resale purposes, even though they are excluded from the formal financial system. For the prices of the bubble asset to be defined, we consider the possibility of bankruptcy in equilibrium, departing from the framework of Kehoe and Levine (1993). In equilibrium, agents are indifferent between bankruptcy and repayment, and a fraction of borrowers declare bankruptcy. This fraction is an equilibrium variable that is always foreseen correctly by rational agents. Lenders then face a well defined risk of bankruptcy for each loan and set a markup on the loan interest that equalizes expected return on loans and on physical capital. The joint determination of the fraction of bankrupt borrowers, the interest rate on loans and the return on bubble can be described by Figure 6. The consumption bundle after bankruptcy now depends not only on the labor income, but also on the return on the bubble, since this asset can be held after exclusion from credit and capital market. The segment between the grey and the black circle now corresponds to the amount repayment of debt repayment, including the interest rate with the markup.

The economy with bubbles is prone to multiple equilibria and indeterminacy. There are three possible configurations of the equilibria, depending on the model parameters. In the first one, there is a unique steady state and it has no bubble nor bankruptcy; the steady state is locally determinate. In a second configuration, there are two steady states: one with a permanent bubble and bankruptcy and local determinacy; another with no bubble nor bankruptcy and local indeterminacy. In the third configuration, there are two steady states as well: one with permanent bubble and bankruptcy and local determinacy; another without bubble, but with bankruptcy and local indeterminacy. Global dynamics analysis is only done for a special case where agents do not work in the last period of life. In sum, the economy with bubbles behaves in line with the seminal findings of Tirole (1985). There are several important differences, though. First, the bubbly equilibria are associated with bankruptcies and losses of efficiency. Second, existence of the steady state with permanent bubble does not require overaccumulation of capital in absence of the bubble. This is due to another novel property of our framework: physical capital, loans and bubbles necessarily have different rates of return in equilibrium.

Chapter 3: Interest Rate Restriction Results in Shorter Housing Loans: Evidence from France

Chapter 3 focuses on interest rate restriction for housing loans in France. In contrast with the other chapters, the analysis relies on the specific institutional setting of France and is done with reduced-form regression methodology. I study whether interest rate restriction leads to distortions in loan portfolios of credit institutions. More specifically, the question is whether a uniform interest rate ceiling for different types of loans causes a shortening of the average loan term. The Figure 7 explains the logic. On the left-hand side of the Figure, a definition of an effective interest rate is given. This is the measure of loan price that is subject to restriction in France. On the right hand side, this price is decomposed into a margin, various premia and various costs. The primary aim of interest rate restriction is to constrain the lending margin, or profits. However, another possible consequence is a shift of the loan supply to loans that require smaller premia. The literature has mostly focused on the repayment risk premium and studied whether risky loan supply shrinks as loan ceilings become tighter (Cuesta and Sepulveda, 2020). My hypothesis is about the term of the loan: under a tight interest rate ceiling, lenders have have an incentive to supply shorter loans that require a smaller term premium.

I exploit a reform of interest rate restriction in France that has set separate interest rate ceilings for shorter- and longer-term housing loans. In each loan category, the ceiling is determined as a function of the average effective interest rate on the market, and this rate has been higher for longer loans — the term premium has been positive. The lenders could then make longer loans without hitting the ceiling after the reform. I test this hypothesis by comparing short-term and long-term loan originations before and after the reform, which has been announced in 2016 and enacted in 2017.

Figure 8 depicts the total value of long-term (more or equal to 20 years) and short-term credit originations found in M_CONTRAN, a large sample of credit originations administered by Banque de France. Given that the reform under study has been announced in

2016 and implemented in 2017, this aggregate evidence seems to support the hypothesis. A basic difference-in-differences regression on this aggregate data, as well as on the number of loans, confirms the statistical significance of the difference in differences. However, the aggregate evidence can have interpretations that are unrelated to interest rate regulation. For example, the change in the term structure might be due to changes in the relative market power of credit institutions. To account for this, I do analogous estimations on a credit institution level. Another alternative reason for the term shift is a demand shock affecting large sub-populations of borrowers. Indeed, assume that the population of borrowers can be separated into long-term borrowers and short-term borrowers using several socioeconomic characteristics. Examples of relevant characteristics include income, age and employment sector of the borrower. These characteristics are unevenly distributed in different locations in France. Then, if the shift in loan terms is explained by a demand shock specific to a demographic group, it is likely to happen as a shift in lending between, and not within, locations. I therefore study the term shifts within location by running the difference-in-differences regressions on the level of a bank branch. On both credit institution and branch level, I obtain estimates that support the main hypothesis and are comparable in magnitude to the aggregate ones.

Motivated by the difference-in-difference estimates confirming the relevance of the reform, I run regressions that control for other time-varying factors of loan duration and estimate the size of the effect attributable to the reform. Yield curves on various debt markets have flattened considerably over the last decade, contributing to the shift of loan supply towards longer terms, regardless of interest rate restriction policies. I use the dynamics of the ceilings on long-term and short-term housing loans as a treatment variable in a regression that controls for funding rates on different horizons. The funding rates are measured by prices of swaps on interbank rates with relevant duration. All the time-varying covariates, including the interest rate ceilings, are economy-wide, while the outcomes — the loan originations — are used on aggregate, on lender and on branch levels. The use of granular data allows to treat the ceiling dynamics as exogenous: although the ceiling for each loan category is updated depending on average market interest for the given loan type, one lender or branch cannot influence the market average considerably. I find that the difference in interest rate ceilings has statistically significant and economically meaningful impact on long-term originations. The reform has led to an estimated net increase in aggregate housing loan originations of around 6% quarterly, although long-term originations have crowded out shorter ones. I then calculate changes in liquidity and debt service of a typical borrower who has taken out a long-term loan instead of a short term due to the reform of interest rate restriction. Such a borrower has had monthly payment decreased by 30%, but the overall debt service cost increased by 75% — effects explained primarily by the loan term itself and not by the higher interest rate that is charged on longer loans.

The shortening of loans as consequence of interest rate restriction has not been explored in the literature before. At the same time, long-term loans have been found to be valued by consumers. Attanasio et al. (2008) find that higher sensitivity of demand to duration than to interest rate in car loans. This suggests that consumers value highly the opportunity to prolong their debt and decrease regular payments. The term elasticity is especially high for low income consumers, which the authors interpret as liquidity constraints being alleviated by longer loans. Karlan and Zinman (2008) obtain the same result for consumption microloans. The liquidity constraints argument is likely to to be most pertinent housing loans, because of the large amounts borrowed. Dhillon et al. (1990) provides survey evidence on the preference for 30-year mortgages against 15-year ones that is stronger for low-income borrowers. It is therefore beneficial to define interest rate ceilings flexibly enough so that lenders can price loans in accordance with their term.

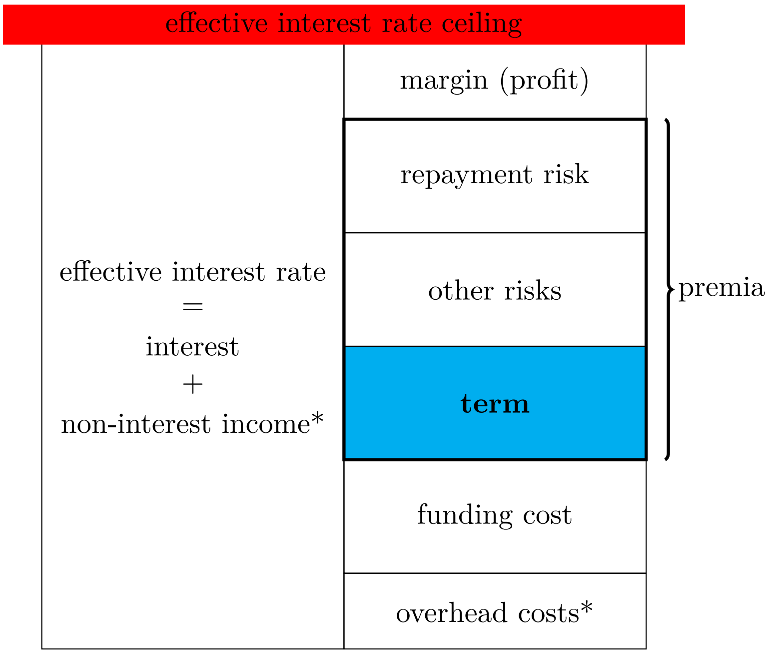


Figure 7: Effective interest rate of a loan — decomposed into margin, premia and costs. *: in percentage of loan amount, annualized.

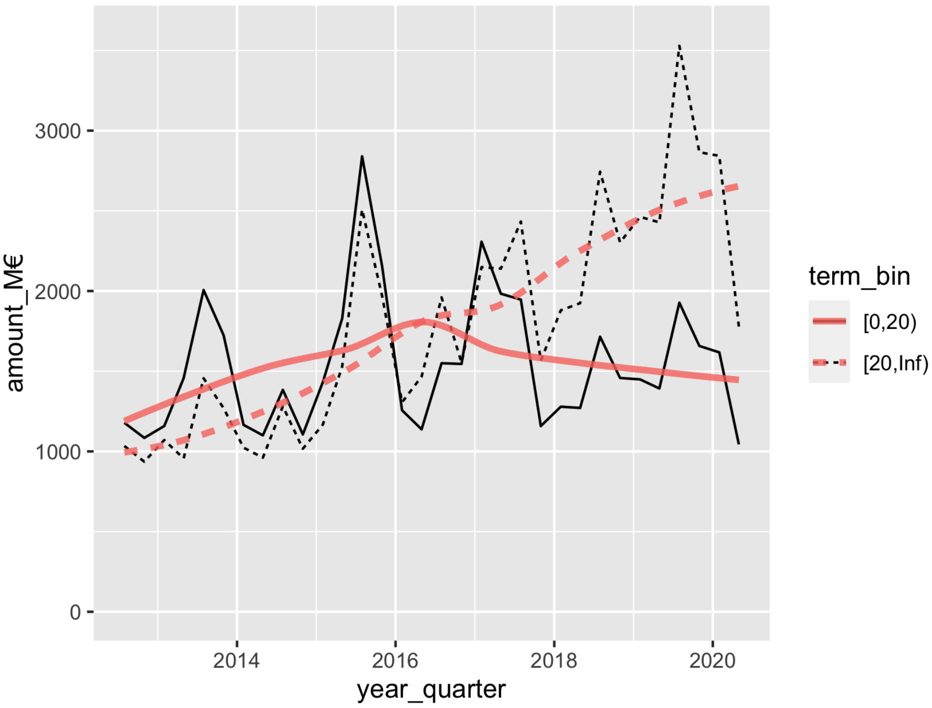


Figure 8: Aggregate amounts of loan originations, long-term (dashed) vs. short-term (solid) loans. Smooth lines are fitted values of a locally estimated scatterplot smoothing (LOESS) regression. Source: M_CONTRAN dataset of Banque de France, my calculations.

Chapter 1

Intergenerational Redistribution with Endogenous Constraints to Private Debt

1.1 Introduction

The financial system and government redistribution programmes both help to smooth household consumption, but the two systems can be in conflict. Consider an economy with a developed personal bankruptcy system, such as the United States. The personal bankruptcy institution may create moral hazard, which gives rise to borrowing constraints. Redistribution done by the government can makes the post-bankruptcy exclusion from the credit market less costly, making the moral hazard problem worse. The focus of this article is the interaction of personal bankruptcy decisions and the policies that redistribute resources between generations. I find that an optimal policy that takes into account households' incentives for bankruptcy should rely on government debt as a redistribution device.

To understand how redistribution policy can influence borrowing constraints, consider the trade-offs that an indebted person faces in presence of a personal bankruptcy system. I focus on bankruptcy rules as in Chapter 7 of the Bankruptcy Code of the United States[1]. According to this procedure, borrowers have their consumption loans written off and their subsequent labor earnings protected by law from creditors. On the other hand, making savings is not feasible for some time after the bankruptcy: if the bankrupt person manages to accumulate financial assets after the bankruptcy, these must be used to repay past creditors. Borrowers therefore have some incentives for debt repayment, but bankruptcy can still be an optimal choice: they have limited commitment to debt repayment. This

[1]The US personal bankruptcy law offers some of the highest levels of income and asset protection from creditors. White (2007) compares it to the systems in EU countries: for instance, in France a large part of incomes and assets must be used for repayment of creditors. In Germany, income protection is stronger, but most assets are not protected, too.

friction has been introduced in a general equilibrium setting by Kehoe and Levine (1993). In their framework, lenders impose constraints on the amount of credit just enough to make bankruptcy suboptimal for the borrower. Azariadis and Lambertini (2003) study this friction in an economy with overlapping generations, where individuals borrow early in life and save later on. If an agent decides not to repay her debts, she loses the opportunity to save for her old age.

I consider intergenerational redistribution in the model of Azariadis and Lambertini (2003). The main question is whether government debt and lump-sum taxation of adult agents are equivalent redistribution instruments. On the one hand, models with overlapping generations are known to produce excess of savings in equilibrium that the social planner can correct by either accumulating government debt (Diamond, 1965) or by a pay-as-you-go pension scheme (de la Croix and Michel (2002) survey the results). One the other hand, in the framework with limited commitment, the pay-as-you-go redistribution from adult to old agents affects adversely the endogenous borrowing constraint for the young. Indeed, taxing prime-age adults' income and paying out transfers to the old acts like doing savings on behalf of the former — savings that are not lost in case of personal bankruptcy[2]. This means that the inability to accumulate assets after personal bankruptcy is less costly under a large pay-as-you-go system, worsening the moral hazard and making the borrowing constraints tighter in equilibrium. The effort of the social planner to better allocate goods between agents then undermines the ability of credit markets to serve the same aim.

This article therefore provides a novel argument against the well-known results on equivalence of government debt and lump-sum taxes or transfers. These results date back to the intuition of David Ricardo, formalized in Barro (1974). Their famous result is as follows: in absence of financial frictions, the intertemporal budget constraint of the government implies that variations of government debt have no effect on private consumption. A related finding (Buiter and Kletzer, 1992) is that government debt and lump-sum taxes or transfers are equivalent when a social planner decentralizes optimal consumption allocations without financial frictions[3]. However, Barro (1974) also shows that decreasing taxes and increasing government debt can lead to a positive wealth effect if a reasonable financial friction is present — non-negativity of bequests. This friction has similarity to credit constraints, the focus of the present article. Non-negativity of bequests constrains the parents not to consume at the expense of the future income of their children — it is a borrowing constraint for the dynasty viewed as a single agent. By decreasing current taxes and increasing government debt, the government allows the current generation to consume more at the expense of the future generations, who will pay higher taxes. If the constraint was binding before the decrease in taxes, agents will consume more when smaller taxes are levied.

[2]US federal law protects main classes of retirement accounts from creditors of a bankrupt borrower.

[3]de la Croix and Michel (2002) provide several versions of this equivalence in Chapters 3 and 4. Gale (1990) finds some versions of the equivalence in economies with risky production and non-contingent transfers and government debt.

I show that limited commitment to debt repayment of young agents makes the use of government debt beneficial for a different reason. It relaxes the borrowing constraint on the young, but by affecting the credit supply and not the credit demand (demand for negative bequest) as in Barro (1974). In my setup, the government does not levy taxes nor give transfers directly to the young: there are three periods of life and the policy only redistributes incomes of the second and the third period. Therefore, taxes and transfers affect future income, but not the current income, of the young. However, future incomes have a strong effect on the endogenous credit constraint due to limited commitment: the policy shapes incentives for debt repayment in the future.

Formally, the superiority of government debt over lump-sum taxes and transfers for decentralization is stated as follows. I compare two policies that have the same number of policy instruments: one is making transfers between prime-age agents and old agents; another relies on government debt and gives transfers or levies taxes on the old agents. I call the first policy balanced-budget and the second a debt-based one. I compare the sets of optimal allocations of consumption that can be decentralized with the two policies. The main result is that the optimal allocations decentralized by the balanced-budget policy is a subset of those decentralized by the debt-based policy, under mild conditions.

A complementary finding is that the use of government debt eliminates a suboptimal equilibrium where private credit markets fail to operate due to limited commitment. It is a common property of models following Kehoe and Levine (1993) that equilibrium is not unique and at least one equilibrium is suboptimal in the sense of Pareto. In both Kehoe and Levine (1993) and Azariadis and Lambertini (2003) it is an equilibrium steady state with small interest rates and vanishing credit — a credit crunch. The intuition for the credit-crunch equilibrium is the same in both articles. If low interest rates are expected in future periods when a given agent has high income, this agent does not plan to make (large) savings at that time. Therefore, an exclusion from financial markets at that time is not costly and the agent can declare bankruptcy on her debts at present with a low opportunity cost. The lenders then constrain credit to such a borrower, so the latter enters the next period with small debts and will make large savings by the end of the period, leading to low interest rates. The expected decrease of interest rates is therefore self-fulfilling. In my setting, government debt prevents such an equilibrium because it pushes interest rates up by absorbing savings and crowding out the supply of credit.

The case for using government debt found in this article is based on the limited commitment along the life cycle and does not rely on standard arguments. Firstly, it is not about the use of government debt in addition to some set of transfers; instead, the main results are obtained by comparing two policies with an equal number of instruments. Second, the borrowing constraints are relaxed for the young because their incentives to declare bankruptcy are affected by the policy. This is an channel operating through the supply of credit, and not through the demand as in Barro (1974) or Woodford (1990). In addition, I show that policies with and without debt are equivalent if standard exogenous borrowing constraints are assumed. Finally, I do not rely on the logic of smoothened distortions form taxation (Barro, 1979), although this mechanism is in general complementary to my results.

The rest of the article is organized as follows. Section 1.2 relates this article to the literature. In Section 1.3 I introduce the setup, which is the one of Azariadis and Lambertini (2003) augmented with a government sector. Section 1.4 describes the equilibria of the model with two alternative institutional settings: with and without the possibility of personal bankruptcy. Section 1.5 then analyzes the problem of a social planner that decentralizes optimal allocations (optimality is defined in Section 1.5.1), with and without government debt. Finally, Section 1.5.6 highlights the role of endogenous borrowing constraints in the analysis by comparing it to common models with exogenous constraints. Section 1.6 concludes; the Appendix contains proofs of the article's propositions.

1.2 Related literature

An extant literature, starting with Woodford (1990), has explored whether public debt improves private consumption smoothing when markets are incomplete (Aiyagari and McGrattan, 1998; Floden, 2001; Eggertsson and Krugman, 2012; Guerrieri and Lorenzoni, 2017). The main intuition is that the increase of interest rates due an increase of public debt makes it easy to accumulate enough precautionary savings to avoid borrowing constraints in low-income periods. In another words, public debt is an additional savings device. Floden (2001) stresses the inequality aspect of the problem, showing that it is the wealthiest agents who benefit most from the additional savings device.

Closely related to the present article are Rohrs and Winter (2015), Antunes and Ercolani (2020), but both articles have results opposite to mine. These authors have studied the problem in the limited commitment setting of Kehoe and Levine (1993) and arrive to a conclusion that high government debt makes endogenous borrowing constraint more tight. Their mechanism works through interest rates that are increasing in the level of government debt. In these models, unlike in the present article, high interest rates are associated with tighter endogenous borrowing constraints, which deserves a discussion. The authors assume infinite life horizon and a Markov income process where a transition to very low levels of income can happen with positive probability in any state. This latter property is crucial, given that the authors use the definition of the endogenous borrowing limit in stochastic settings that follows Alvarez and Jermann (2000a). According to this definition, the credit limit must be such that the borrower doesn't go bankrupt in the following period in any state of the world. Therefore, the most relevant state of the world for setting the borrowing limit is the one with lowest borrower income, because marginal utility of consumption is then highest and so is the incentive to go bankrupt. Importantly, in this state of the world the borrower is also most likely to be a net borrower and not a net saver at the end of the period, if she doesn't go bankrupt. Going back to the effect of the government debt, this means that higher interest rates make the value of having access to financial market smaller in the worst state of the world, so that endogenous borrowing constraints become more tight. Note that this result is undone in my setting not only because of deterministic incomes, but also because the life-cycle income pattern. Retirement is an anticipated, large negative income shock for individuals. This makes it more likely that a person expects to be a net saver in any state of the world next period, so the endogenous borrowing limit increases in the expected interest rates.

The contribution of Antinolfi et al. (2007) is closely related to mine in terms of methodology. They study the problem of the social planner that decentralizes optimal allocations — both first-best and second-best — with lump-sum transfers and monetary policy, in an infinite horizon economy with limited commitment. Furthermore, money is equivalent to government debt in their framework. The two assumptions that make the model different with respect to the present article are infinite lives of the agents and possibility to renege on all the future taxes and transfers that make the agent worse off in sum. In this framework, government debt is found to crowd out private lending one for one — a property found in a more general setting by Hellwig and Lorenzoni (2009). Government debt, therefore, does not allow for additional opportunities to decentralize optimal allocations if lump-sum transfers are used, in contrast with my paper. Moreover, the authors' result on multiplicity of equilibria is opposite to mine — by using lump-sum transfers, the social planner can eliminate the autarkic steady state, while it is not the case if government debt is the only instrument[4].

Some of my findings parallel those that have been obtained recently in very different settings. Carapella and Williamson (2015) study the credit markets under limited commitment and the role of government debt in a Lagos and Wright (2005) type of framework. Their model has infinitely lived agents, asymmetric information and limited enforceability of taxes. Their main result is analogous to mine: the presence of government debt improves the endogenous borrowing limits, since it raises the opportunity cost of default. Mian et al. (2020b) study a model with infinitely-lived agents and credit that is backed with land-like assets and find that equilibrium multiplicity vanishes because of sufficiently high levels of government debt. As in my framework, government debt eliminates an equilibrium with low private debts.

The modelling of the production sector, crucial for the long-run analysis, is omitted from the present article. Yet there exist arguments in favor and against government debt in the literature on long-run growth. For instance, Greiner (2012) shows that government debt accumulation slows down growth in a basic endogenous growth setting via the incentives of households to work and to save. On the contrary, Andersen and Bhattacharya (2020) provide a case for optimal temporary government debt increases to finance growth-enhancing education spending. The optimal policy is found to rely on government debt. All in all, the normative status of government debt is not settled in literature and my results provide one additional argument in its favor.

[4]I study policies that use government debt and one transfer, so the analysis of Antinolfi et al. (2007) is of limited relevance to mine. However, in the special case when the social planner chooses optimal allocations according to the Golden Rule of Phelps (1961), government debt is rolled over forever without any tax nor transfer in my model, too. In this case, too, the autarkic equilibrium is eliminated.

1.3 Model

This article follows the framework of Azariadis and Lambertini (2003), with some changes of notation and an additional government sector. Time is discrete and starts at period $t = 0$. The economy is populated by generations of identical agents that live for three periods. I call the agents young, adult and old in their first, second and third period of life, respectively. Generation t is the set of agents that are born in $t-1$ and are adults in t. There are N_t identical agents in a generation t. Population grows at a constant rate n, so $N_{t+1} = (1+n)N_t$. There is one good in the economy and the agents can consume it when young, when adult and when old. Agents of each cohort are endowed with y_0, y_1 and y_2 units of the good in the three periods of life, with $(y_0, y_1, y_2) > 0$. The consumption good is not storable, so generations need to exchange claims on endowments if they need to smooth their consumption profile. The consumption in the first, second and third period of life of an agent of generation t is denoted c_{t-1}, d_t and e_{t+1}, respectively.

This agent's utility from the three periods' consumption is represented by the following function:

$$U(c_{t-1}, d_t, e_{t+1}) = u(c_{t-1}) + \beta u(d_t) + \beta^2 u(e_{t+1})$$

with $u(\cdot)$ of constant intertemporal elasticity of substitution (CIES) type: either $u(x) = \frac{x^{1-1/\sigma}}{1-1/\sigma}$ with $\sigma > 1$ or $u(x) = \log(x)$ $(\sigma \to 1)$. The assumption $\sigma \geq 1$ is used to simplify the equilibrium properties of the model: Kehoe and Levine (1990) have shown that $\sigma < 1$ generates equilibrium multiplicity in endowment OLG economies with three periods of life, even in absence of financial frictions.

The government can make positive and negative lump-sum transfers to different generations and finance them by issuing government debt. Without loss of generality, I abstract from government spending. Moreover, I abstract from transfers to the young generation — this is motivated in the Section 1.5. Two kinds of transfers are done in period t: a transfer to an adult and to an old agent, denoted τ_t^1 and τ_t^2, respectively. Any transfer can be negative, in which case I call it a tax. A positive sum $N_t\tau_t^1 + N_{t-1}\tau_t^2$ means there is a primary deficit of the government budget, which can be financed by government debt. The end-of-period government debt stock per adult of period t is denoted g_t and the gross interest paid on this stock of debt in the following period is R_{t+1}. The value of g_t can generally be negative, but the main results of the article will be obtained for non-negative government debt. The government budget constraint is the following:

$$N_t g_t = R_t N_{t-1} g_{t-1} + N_t \tau_t^1 + N_{t-1}\tau_t^2$$

or, using the constant population growth:

$$g_t = \frac{R_t}{1+n} g_{t-1} + \tau_t^1 + \frac{\tau_t^2}{1+n} \tag{1.1}$$

There is no uncertainty in the model and expectations are rational, so agents have perfect foresight. The policy, that is, the sequence of transfers and government debt $(\tau_t^1, \tau_t^2, g_t)_{t\geq 0}$, is announced by the government at the beginning of period 0 and is implemented as announced forever after. In the central part of the article on decentralization, the policy is endogenous : it depends on the allocation that the social planner aims to decentralize. However, some auxiliary results are formulated in terms of the model outcomes for different choices of policy, as if it was exogenous.

A private agent can borrow and lend by issuing and buying one-period assets. An agent of generation t then faces the following budget constraints:

$$\begin{cases} c_{t-1} + b_{t-1}^0 & \leq y_0 \\ d_t + b_t^1 & \leq y_1 + \tau_t^1 + R_t b_{t-1}^0 \\ e_{t+1} & \leq y_2 + \tau_{t+1}^2 + R_{t+1} b_t^1 \end{cases} \tag{1.2}$$

where b_{t-1}^0, b_t^1 are end-of-period asset positions of the agent when young and adult, respectively; τ_t^1, τ_{t+1}^2 are transfers or taxes paid when adult and when old; R_t is the interest factor on assets purchased or sold at $t-1$. Private and government debts are assumed to be perfect substitutes[5].

I will analyze two versions of institutional environment for asset markets in the model. They differ in their treatment of private debts, or, equivalently, negative asset positions b_t^0 and b_t^1. In the first environment, the agents are obliged to use their endowments and transfers to repay their debts. In this case, there are no borrowing constraints: agents can borrow up to the present value of their future endowments and transfers. I will call this environment full commitment to loan repayment, or full commitment in short. In the second environment, agents can declare bankruptcy when adult or when old and have a right to keep their endowments and transfers afterwards. Such agents do not participate in the asset markets for their remaining life. This environment will be labelled limited commitment. For sufficiently high debt levels, exclusion from asset markets does not discourage borrowers from bankruptcy, so lenders are assumed to limit lending as in Kehoe and Levine (1993), imposing a lower limit on b_t^0 and b_t^1. The following section analyzes equilibria in the two environments in turn, starting with the simpler case of full commitment.

1.4 Equilibrium

1.4.1 Equilibrium with full commitment

In this benchmark environment the agents face no credit constraints — they can borrow up to the present value of their incomes after transfers. First, note that the budget constraints in (1.2) are always binding when agents maximize utility, as the preferences are monotone.

[5] Although possibility of personal bankruptcy is introduced in Section 1.4.2, the bankruptcy is ruled out in equilibrium by the borrowing constraints, so the private assets are risk-free, as the government bonds.

Writing (1.2) with equality signs and eliminating b_{t-1}^0, b_t^1, one obtains the inter-temporal budget constraint:

$$c_{t-1} + \frac{d_t}{R_t} + \frac{e_{t+1}}{R_t R_{t+1}} = y_0 + \frac{y_1 + \tau_t^1}{R_t} + \frac{y_2 + \tau_{t+1}^2}{R_t R_{t+1}} \tag{1.3}$$

Then, the utility maximization problem of an agent of generation t can be written in a compact form:

$$\begin{aligned} \max_{(c_{t-1}, d_t, e_{t+1})>0} \quad & u(c_{t-1}) + \beta u(d_t) + \beta^2 u(e_{t+1}) \\ \text{s.t.} \quad & c_{t-1} + \frac{d_t}{R_t} + \frac{e_{t+1}}{R_t R_{t+1}} = y_0 + \frac{y_1 + \tau_t^1}{R_t} + \frac{y_2 + \tau_{t+1}^2}{R_t R_{t+1}} \end{aligned} \tag{1.4}$$

The fact that $\lim_{x \to 0+} u'(\cdot) = +\infty$, following from the CIES type of the function, guarantees that the solution is interior. The resulting asset positions can be obtained from $b_t^0 = y_0 - c_t$ and $b_t^1 = y_1 + R_t b_{t-1}^0 + \tau_t^1 - d_t$.

The asset market has the following structure. By adding up the aggregate asset positions of the young and the adults of a given period, one obtains the asset position of the private sector: $N_{t+1} b_t^0 + N_t b_t^1$. The economy is closed, so the government asset position, $-N_t g_t$, and private asset position must sum to zero:

$$N_{t+1} b_t^0 + N_t b_t^1 - N_t g_t = 0$$

$$\Leftrightarrow (1+n) b_t^0 + b_t^1 = g_t \tag{1.5}$$

A typical asset market equilibrium can be, but not limited to, the adults providing savings, while both the young and the government borrowing in each period.

When time begins, at $t = 0$, generations $-1, 0, 1$ are alive. The generation -1 is old and only consumes its endowment, the transfers and the assets with the interest income. For this generation, the assets and the interest, $R_0 b_{-1}^1$, are exogenous. Consumption of the generation -1 is then:

$$e_0 = y_2 + \tau_0^2 + R_0 b_{-1}^1 \tag{1.6}$$

The generation -1 therefore makes no decisions in the model and their consumption is determined by exogenous parameters and the transfer τ_0^2.

The generation 0 optimizes their adult and old age consumption, given that it has some exogenous assets and interest on them:

$$\begin{aligned} \max_{(d_0, e_1)>0, b_0^1} \quad & u(d_0) + \beta u(e_1) \\ \text{s.t.} \quad & \begin{cases} d_0 + b_0^1 & = y_1 + \tau_0^1 + R_0 b_{-1}^0 \\ e_1 & = y_2 + \tau_1^2 + R_1 b_0^1 \end{cases} \end{aligned} \tag{1.7}$$

Equilibrium with full commitment can now be defined:

Definition 1 (Equilibrium with full commitment)**.** *An equilibrium of an economy with a policy* $(\tau_t^1, \tau_t^2, g_t)_{t\geq 0}$ *and full commitment is a sequence of positive variables* $(c_t, d_t, e_t, R_t)_{t\geq 0}$ *such that:*

- $\forall t \geq 1, (c_{t-1}, d_t, e_{t+1})$ *solves* (1.4)
- $\forall t \geq 0,$ *the asset market clearing condition* (1.5) *holds*
- $\forall t \geq 0,$ *the government budget constraint* (1.1) *holds*
- $R_0, b_{-1}^0, b_{-1}^1, g_{-1}$ *are exogenous and* e_0 *is given by* (1.6)*;* d_1 *and* e_2 *solve* (1.7)*.*

The focus of the article is decentralization of given allocations as equilibrium of the economy, so I don't address the general question of existence and uniqueness of equilibria[6]. Furthermore, some results rely on stationarity of both policies and equilibria. A stationary policy is defined as follows:

$$\tau_t^1 = \tau^1,\ \tau_t^2 = \tau^2,\ g_t = \frac{1+n}{1+n-R}\left(\tau^1 + \frac{\tau^2}{1+n}\right)\ \forall t > 0 \tag{1.8}$$

where R is the endogenous, steady-state interest that results from the stationary transfers τ^1, τ^2. A particular class of stationary policies is balanced-budget stationary policies:

$$\tau_t^1 = \tau^1,\ \tau_t^2 = \tau^2 = -(1+n)\tau^1,\ g_t = 0\ \forall t > 0 \tag{1.9}$$

I order to get standard life-cycle patterns of borrowing and saving, the following assumption is made for the endowment profiles:

Assumption 1. *The life-cycle endowment profile is hump-shaped:* $y_1 > y_0;\ y_1 > y_2$*.*

Furthermore, an assumption is made about the transfers: I assume that the government never levies taxes that are larger or equal to the agent's endowment of the corresponding period:

Assumption 2. $\tau_t^1 > -y_1,\ \tau_t^2 > -y_2,\ \forall t \geq 0.$

This is introduced because negative after-tax endowments raise two issues. Firstly, it makes consumer behavior more complex in terms of demands. Secondly, it makes consumption negative in a situation where agents do not hold assets, whereas this situation must be studied in the environment of limited commitment, introduced below. The consequences of the Assumption are also twofold: it limits the scope of decentralization studied in Section 1.5 and it limits the domain of policies in some propositions that treat policies as exogenous.

[6] Sections 1.4.3, 1.5.5 discuss an important type of equilibrium steady state — an autarky — that can co-exist with steady-state equilibria with trade under limited commitment.

In what follows, I use the following notation for marginal rates of substitution:

$$MRS_{cd}(c_t, d_{t+1}) = \frac{u'(c_t)}{\beta u'(d_{t+1})}; \quad MRS_{de}(d_t, e_{t+1}) = \frac{u'(d_t)}{\beta u'(e_{t+1})} \tag{1.10}$$

One can now characterize steady state properties of economies with balanced-budget stationary policies.

Proposition 1. *Under balanced-budget stationary policies, given by* (1.9)*, a steady state with full commitment exists and is unique. The steady-state interest rate is decreasing in* τ^1*.*

Proof. See section 1.7.1 of the Appendix.

Note that economies with a balanced-budget stationary policy are closely related to the ones without government, studied by Azariadis and Lambertini (2003). Namely, one can re-define the endowments such that the transfers are included. All the results of Azariadis and Lambertini (2003) then hold for such economies, provided that the "post-transfer" endowments satisfy the Assumption 1.

The steady-state interest rate in equilibrium with full commitment and without policy is an important value for the subsequent analysis. I label it R^u as in Azariadis and Lambertini (2003), where u stands for "unconstrained", i.e. without the borrowing constraints that arise under limited commitment.

1.4.2 Equilibrium with limited commitment

Suppose now that indebted agents have a possibility to declare personal bankruptcy. For an adult of generation t with $b^0_{t-1} < 0$, this decision would be made at the beginning of t. Their debts are written off and neither their endowments nor transfers can be seized. The only consequence is that such an agent cannot borrow nor save after bankruptcy, as in Kehoe and Levine (1993)[7]. There is full information in the model, so lenders are aware of the incentives to declare bankruptcy and are assumed to limit lending. This implies two kinds of endogenous borrowing constraints for the model: one for adults and one for the young. The one for adults has a simple form, so I describe it first. If an adult borrows any amount, she has debts outstanding when old. However, she can declare bankruptcy at no cost, since she will neither save nor borrow in the last period of life in any case. There is no bankruptcy in equilibrium, so the utility from old-age consumption must never be lower than the utility from consuming the endowment and the transfer:

$$u(e_{t+1}) \geq u(y_2 + \tau^2_{t+1}) \tag{IR2}$$

[7] The impossibility to make savings after default has the following interpretation in the present model. After personal bankruptcy, endowments and transfers of individuals are protected form creditors, but asset income is not. Therefore, bankrupt adults cannot have any return on their savings and so choose not to save.

For an agent of generation t. The number 2 is in the label (IR2) because this is a second individual rationality constraint that an individual faces during her life; the first one is discussed below. This individual rationality and the budget constraint imply together that adults cannot borrow:

$$b_t^1 \geq 0 \tag{1.11}$$

This no-borrowing constraint for the adults is a consequence of the institutional environment of the model. However, it is not endogenous: the right hand side does not depend on other variables of the model. This the case for the borrowing constraint for the young, that I discuss next.

The individual rationality constraint for the young is defined in terms of their consumption when adult and when old:

$$u(d_t) + \beta u(e_{t+1}) \geq u(y_1 + \tau_t^1) + \beta u(y_2 + \tau_{t+1}^2) \tag{IR1}$$

Note that Assumption 2 on the policy makes sure that the right hand sides of (IR1), (IR2) are always defined. The constraint (IR1) is directly related to the saving decisions that the agent makes in her youth. Indeed, by they budget constraint, the constraint cannot hold if b_{t-1}^0 is too low. In order to obtain the endogenous borrowing limit for the young, consider the utility maximization problem of an adult individual of period t, taking the amount of b_{t-1}^0 as given:

$$\begin{aligned} \max_{(d_t, e_{t+1})>0, b_1^t} \quad & u(d_t) + \beta u(e_{t+1}) \\ \text{s.t.} \quad & \begin{cases} d_t + b_t^1 & = y_1 + \tau_t^1 + R_t b_{t-1}^0 \\ e_{t+1} & = y_2 + \tau_{t+1}^2 + R_{t+1} b_t^1 \\ b_t^1 & \geq 0 \end{cases} \end{aligned} \tag{1.12}$$

Denote the demands for consumption that result from this program $d(R_{t+1}, R_t b_{t-1}^0, \tau_t^1, \tau_{t+1}^2)$ and $e(R_{t+1}, R_t b_{t-1}^0, \tau_t^1, \tau_{t+1}^2)$ and the corresponding indirect utility $V(R_{t+1}, R_t b_{t-1}^0, \tau_t^1, \tau_{t+1}^2)$:

$$V(R_{t+1}, R_t b_{t-1}^0, \tau_t^1, \tau_{t+1}^2) \equiv u(d(R_{t+1}, R_t b_{t-1}^0, \tau_t^1, \tau_{t+1}^2)) + \beta u(e(R_{t+1}, R_t b_{t-1}^c, \tau_t^1, \tau_{t+1}^2)) \tag{1.13}$$

If this utility is lower than the utility after bankruptcy, the agent chooses bankruptcy. This leads to the individual rationality constraint (IR1), written for equilibrium consumption levels that maximize utility :

$$V(R_{t+1}, R_t b_{t-1}^0, \tau_t^1, \tau_{t+1}^2) \geq u(y_1 + \tau_t^1) + \beta u(y_2 + \tau_{t+1}^2) \tag{1.14}$$

The endogenous borrowing constraint for the young in equilibrium is such that the incentives to declare bankruptcy and to repay are equal, as in Kehoe and Levine (1993). If the young were allowed to borrow more than this amount, they would declare bankruptcy. If the constraints were more strict than this amount, lenders would forego opportunities of risk-free lending. Both situations are assumed to be out of equilibrium. Then the lower limit on assets in youth b_{t-1}^c is given in equilibrium by

$$V(R_{t+1}, R_t b_{t-1}^c, \tau_t^1, \tau_{t+1}^2) = u(y_1 + \tau_t^1) + \beta u(y_2 + \tau_{t+1}^2) \tag{1.15}$$

Thanks to the monotonicity of V in wealth, $R_t b_{t-1}^c$ can be expressed as a function of $R_{t+1}, \tau_t^1, \tau_{t+1}^2$.

Lemma 1. *Equation* (1.15) *defines $R_t b_{t-1}^c$ as a continuously differentiable function f of $(R_{t+1}, \tau_t^1, \tau_{t+1}^2)$ with a domain $R_{t+1} > 0,\ \tau_t^1 > -y_1,\ \tau_{t+1}^2 > -y_2$ and:*

$$\begin{aligned} f(R_{t+1}, \tau_t^1, \tau_{t+1}^2) < 0 \quad &\textit{iff } R_{t+1} > MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2), \\ f(R_{t+1}, \tau_t^1, \tau_{t+1}^2) = 0 \quad &\textit{iff } R_{t+1} \leq MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2) \end{aligned}$$

Proof. See section 1.7.2 of the Appendix.

According to the range of f, the value of $R_t b_{t-1}^c$, and hence of b_{t-1}^c, is always non-positive. It means that b_{t-1}^c is only a credit limit and never forces agents to make positive savings. The credit limit is null when interest rates are too low: this is because the agents have no incentive to make savings when adults: the exclusion from financial markets after bankruptcy has no cost for them. In this case the agents choose to declare bankruptcy on any debt, so borrowing when young is ruled out in equilibrium.

Focusing on stationary balanced-budget policies (1.9), I show the influence of the policy on the borrowing limit as a comparative statics exercise. The aim is to provide some intuition behind the main results on decentralization with limited commitment in Section 1.5.3.

Proposition 2. *In an economy with limited commitment and stationary balanced budget policy* (1.9)*, $R_t b_{t-1}^c$ is decreasing (has a null derivative) in τ^1 if $b_t^1 > 0$ $(= 0)$.*

Proof. See section 1.7.3 in the Appendix. □

The intuition of the proposition is as follows. A decrease in τ^1 with a corresponding increase in τ^2 crowds out savings. This means the opportunity to make savings for old age is valued less and the agents have more incentives to declare personal bankruptcy. The corresponding borrowing constraint is tighter in equilibrium — $R_t b_{t-1}^c$ increases.

The borrowing constraint for the young can be stated as follows:

$$b_{t-1}^0 \geq f(R_{t+1}, \tau_t^1, \tau_{t+1}^2)/R_t \tag{1.16}$$

The maximization problem of a young agent of generation $t \geq 1$ with limited commitment can be written:

$$\max_{(c_{t-1},d_t,e_{t+1})>0,b^0_{t-1},b^1_t} \quad u(c_{t-1})+\beta u(d_t)+\beta^2 u(e_{t+1})$$

$$\text{s.t.}\begin{cases} c_{t-1}+b^0_{t-1} & = y_0 \\ d_t+b^1_t & = y_1+\tau^1_t+R_t b^0_{t-1} \\ e_{t+1} & = y_2+\tau^2_{t+1}+R_{t+1}b^1_t \\ b^0_{t-1} & \geq f(R_{t+1},\tau^1_t,\tau^2_{t+1})/R_t \\ b^1_t & \geq 0 \end{cases} \tag{1.17}$$

The generations $t=-1,0$ need special treatment in the limited commitment context. Firstly, the generation -1 necessarily declares bankruptcy whenever their exogenous asset position is negative, $b^1_{-1}<0$. In what follows, I always assume an initial condition $b^1_{-1}\geq 0$. Secondly, The generation 0 might declare bankruptcy on their exogenous debts when $b^0_{-1}<0$, but this depends on the policy and interest rates. A definition of equilibrium must then include the individual rationality constraint (1.14) verified for the generation 0.

This leads to the following definition of an economy with limited commitment:

Definition 2 (Equilibrium with limited commitment.)**.** *An equilibrium of an economy with policy* $(\tau^1_t,\tau^2_t,g_t)_{t\geq 0}$ *and limited commitment is a sequence of positive variables* $(c_t,d_t,e_t,R_t)_{t\geq 0}$ *such that:*

- $\forall t\geq 0,(c_{t-1},d_t,e_{t+1})$ *solves* (1.17)
- $\forall t\geq 0,$ *asset market clearing condition* (1.5) *holds*
- $\forall t\geq 0,$ *government budget constraint* (1.1) *holds*
- $b^1_{-1}\geq 0,R_0b^0_{-1},g_{t-1}$ *are exogenous and* e_0 *is given by* (1.6)*;* d_1 *and* e_2 *are given by* (1.7)
- (1.14) *holds for period* 0*, so generation* 0 *does not go bankrupt.*

As shown by Azariadis and Lambertini (2003), an economy without transfers can have multiple equilibria with limited commitment. In the decentralization problems of Section 1.5, the question is whether a given allocation can be one of the equilibria with limited commitment if certain policy instruments are used. I address the question of multiple equilibria partially in the following section.

1.4.3 Equilibrium autarky

One type of equilibrium that is relevant for the analysis is autarky where no exchange of endowments takes place. I define autarky as an allocation of consumption where agents consume their endowments and transfers in each period of life, with null asset positions:

$$(c_t, d_t, e_t) = (y_0, y_1 + \tau_t^1, y_2 + \tau_t^2), \quad t \geq 0 \tag{1.18}$$

Note that the non-zero initial conditions for asset positions imply directly that generation -1 and/or 0 are not in autarky. It follows that the discussion of autarky is only relevant under the following initial conditions:

$$b_{-1}^0 = b_{-1}^1 = g_{-1} = 0 \tag{1.19}$$

Under full commitment, autarky is not a generic equilibrium, even if (1.19) is verified. First, autarky is impossible if policy included nonzero government debt, since government debt must have a private asset position as a counterpart. Second, it has been shown in Proposition 1 that economies without government debt and stationary transfers have only one steady state, which is generically not autarkic.

Under limited commitment, autarky is a prevalent equilibrium of the economy.

Proposition 3. *In an economy with limited commitment, null initial asset positions given by* (1.19) *and no government debt, any sequence of interest rates satisfying*

$$R_{t+1} \leq \min\{MRS_{cd}(y_0, y_1 + \tau_{t+1}^1),\ MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2)\}, \quad \forall t \geq 0 \tag{1.20}$$

results in equilibrium autarky with consumption levels given by (1.18).

Proof. See Section 1.7.4 in the Appendix. □

Note that autarky is generically not the only equilibrium of this economy. Section 1.5 shows that the social planner may be able to decentralize some optimal allocations equilibrium either with or without the use of government debt. Since autarky is suboptimal[8], then a decentralization without government debt produces equilibrium multiplicity with at least two equilibria: the optimal one, targeted by the social planner, and the autarky.

Finally, consider an economy that has nonzero public debt in at least one period. As discussed previously, this is incompatible with autarky. Intuitively, government debt can rule out autarky in two ways. First one, not specific to the limited commitment environment, is raising the interest rates sufficiently high for all agents to become net savers. The second one is based on limited commitment. Government debt may make interest rates such that only adult agents demand positive asset positions. This means that an exclusion from the financial markets in the event of bankruptcy is costly for the adults. By Lemma 1, this implies nonzero credit limits for the same agents when they are young. In this case, government debt can both incentivize savers to save and enable borrowers to borrow by ruling out low interest rates.

[8]The reason is discussed in the beginning of Section 1.5.3

1.5 Decentralization with transfers and government debt

The main question of this article is whether an allocation can be decentralized with a given set of policy instruments. In other words, whether the government can choose such a policy than a given allocation is an equilibrium, either with full or with limited commitment.

There are two frictions, or inefficiencies, that the social planner has to correct with the policy instruments. The first one is the overlapping generations structure: an equilibrium without policy intervention can have too much savings, as shown in the seminal papers on OLG (Samuelson, 1958; Diamond, 1965). This can be addressed by crowding out the private savings with either a pay-as-you-go transfer scheme from adults to the old or government debt used with at least one tax/transfer instrument in order to balance the government budget.

The second friction, central to this article, is the limited commitment of agents to repay debts and the resulting borrowing constraints. The choice of policy instruments used for decentralization has a number of indirect effects on the ability of agents to borrow. Firstly, the choice between transfers and government debt affects the set of equilibrium interest rates. This has been discussed in Section 1.4.3. The second effect is produced by the transfers entering directly the right hand side of individual rationality constraints (IR1), (IR2) or, equivalently, the borrowing constraints (1.11), (1.16). Proposition 2 has illustrated this in comparative statics, showing that smaller transfers (or larger taxes) to adults and corresponding larger transfers (lower taxes) to the old make the endogenous borrowing constraint more tight.

The goal of this part of the article is to study the indirect effects of policy on borrowing constraints, so I do not consider transfers to the young as a policy instrument, which alleviate the constraint directly. One can think of this analysis as the case where the government has done some transfers that are encapsulated in the endowment y_0 and has met some limit on such transfers. Instead, I address a question of whether the policy of transfers to the adults and to the old allows the young to finance their consumption with the help of the financial system. Under the constraints on transfers from Assumption 2, whether a given set of instruments is sufficient for decentralization is not trivial, even under full commitment. Indeed, even a full set of transfers would potentially fail to decentralize some allocations under such constraints. I address the problem in the following way: first I establish conditions for decentralization under full commitment in Section 1.5.2. For an economy with limited commitment, I then compare policies under the assumption that they are equally good for decentralization under full commitment, i.e. they satisfy Assumption 2. I abstract from the commitment problems of the social planner : the government debt is always repaid and transfers are always made as announced. Since there is full information in the model, and the government announces all the policy variables in period 0, private agents have perfect foresight on all the future transfers and government debt.

In what follows, I first describe the goal of the social planner: optimal allocations of consumption between generation. Optimality is defined formally by the maximization programs that are solved by such allocations. Then I study the benchmark decentralization problem: decentralization with full commitment. Finally, I address the same problem under limited commitment and contrast the results to the benchmark case and to standard models of exogenous borrowing constraints.

1.5.1 Optimal allocations

This section introduces optimal allocations of consumption between generations that the social planner needs to decentralize. I abstract from any heterogeneity of consumption profiles within generations. For the main results on decentralization, I also abstract from differences between generations, meaning that the social planner will aim at the same consumption profile for all generations. Below I present the social planner programs that result in such allocations.

All the allocations chosen by the social planner must be feasible, meaning that aggregate consumption of each given period must not exceed the aggregate endowment. This is also known as the resource constraint:

$$\begin{aligned} & N_{t+1}c_t + N_t d_t + N_{t-1}e_t \leq N_{t+1}y_0 + N_t y_1 + N_{t+1}y_2 \quad \forall t \geq 0 \\ \Leftrightarrow\ & c + \frac{d}{1+n} + \frac{e}{(1+n)^2} \leq y_0 + \frac{y_1}{1+n} + \frac{y_2}{(1+n)^2} \quad \forall t \geq 0 \end{aligned} \tag{1.21}$$

The baseline optimal allocation is such that all the generations born in $t \geq 0$ obtain the same level of lifetime utility and this utility is maximized. Such an allocation results from the Golden Rule of capital accumulation of Phelps (1961), so I label it the Golden Rule allocation. Agents of generations -1 and 0, for whom the social planner only chooses e_0 and (d_0, e_1) respectively, are constrained to have the same consumption levels as the subsequent generations $1, 2, \ldots$. To obtain the Golden Rule allocation, one has to maximize the utility of the representative generation, subject to the resource constraint.

$$\begin{aligned} & \max_{(c,d,e)} u(c) + \beta u(d) + \beta^2 u(e) \\ \text{s. t.} \quad & c + \frac{d}{1+n} + \frac{e}{(1+n)^2} = y_0 + \frac{y_1}{1+n} + \frac{y_2}{(1+n)^2} \end{aligned} \tag{1.22}$$

Here and in what follows I write the resource constraint as an equality since the objective function is always monotonic. The program has a concave objective function and a nonempty, convex set of feasible allocations, so a solution exists and is unique.

To obtain another class of optimal allocations, one can maximize a weighed sum of agents' utilities, such that all agents of a given generation have the same consumption profile. The following Proposition presents the program of the social planner that produces such allocations, together with some of their properties.

Proposition 4. *Any allocation maximizing a weighed sum of all generations' utility, with consumption homogeneous within generations, can be obtained a unique solution of a* *program:*

$$\max_{(c_t,d_t,e_t)_{t\geq 0}} \sum\nolimits_{t\geq 0} (\omega_{t+2}u(c_t) + \omega_{t+1}\beta u(d_t) + \omega_t\beta^2 u(e_t))$$
$$s.\ t.\ \forall t \geq 0, \quad c_t + \frac{d_t}{1+n} + \frac{e_t}{(1+n)^2} = y_0 + \frac{y_1}{1+n} + \frac{y_2}{(1+n)^2} \tag{1.23}$$

where $(\omega_t)_{t\geq 0}$ are such that their sum converges and ω_t is the weight of an agent of generation $t-1$ times the size of the generation, N_{t-1}.

Proof. See section 1.7.5 in the Appendix. □

I will call the above allocation utilitarian, by the name of the welfare criterion that sums the utilities of all agents. A utilitarian allocation is optimal in the sense of Pareto: there is no redistribution resources with a complete set of lump-sum transfers that makes no generation is worse off and at least one strictly better off. If it existed, it would increase the weighted sum of utilities, so the allocation would not be a solution to the above problem. On the other hand, not all the Pareto optimal allocations fall into the above definition: one can find others if heterogeneous consumption within a generation is allowed.

The notion of optimal allocation in this article includes utilitarian allocations and the Golden Rule. The main results on decentralization of optimal allocations will be formulated for stationary optimal allocations, either utilitarian or Golden Rule. The following Proposition formulates a program associated to this set of allocations:

Proposition 5. *A utilitarian allocation is stationary, i.e. it has $(\hat{c}_t, \hat{d}_t, \hat{e}_t) = (\hat{c}, \hat{d}, \hat{e})$, $\forall t \geq 0$ if, and only if, the associated program* (1.23) *has $\forall t \geq 0$ $\frac{\omega_{t+1}}{\omega_t} = \alpha \in]0,1[$. Such an allocation is also a solution to the following program:*

$$\max_{(c,d,e)} \alpha^2 u(c) + \alpha\beta u(d) + \beta^2 u(e)$$
$$s.\ t.\quad c + \frac{d}{1+n} + \frac{e}{(1+n)^2} = y_0 + \frac{y_1}{1+n} + \frac{y_2}{(1+n)^2} \tag{1.24}$$

Furthermore, the same program results in the Golden Rule allocation if $\alpha = 1$.

Proof. See section 1.7.6 of the Appendix.

The notation $(\hat{c}, \hat{d}, \hat{e})$ will be used in the article to denote all the possible stationary optimal allocations that solve program (1.24), which can be either be utilitarian of Golden Rule. Notation $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ is more general and includes non-stationary utilitarian allocations, too.

Lemma 2. *Any optimal allocation $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ has the following equality of marginal rates of substitution:*

$$\forall t \geq 0, \quad MRS_{cd}(\hat{c}_t, \hat{d}_{t+1}) = MRS_{de}(\hat{d}_t, \hat{e}_{t+1}) \tag{1.25}$$

Proof. See section 1.7.7 of the Appendix.

To shorten notation, I use the following shortcuts for any optimal allocation $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ and optimal stationary allocation $(\hat{c}, \hat{d}, \hat{e})$:

$$\hat{MRS}_t \equiv MRS_{cd}(\hat{c}_t, \hat{d}_{t+1}) = MRS_{de}(\hat{d}_t, \hat{e}_{t+1})$$
$$\hat{MRS} \equiv MRS_{cd}(\hat{c}, \hat{d}) = MRS_{de}(\hat{d}, \hat{e})$$

Lemma 3. *Any stationary optimal allocation* $(c_t, d_t, e_t) = (\hat{c}, \hat{d}, \hat{e})\ \ \forall t \geq 0$ *has* $\hat{MRS} = \frac{1+n}{\alpha}$ $\geq 1 + n$. *The last inequality is strict if the allocation is stationary utilitarian and is equality for the Golden Rule.*

Proof. $\hat{MRS} = \frac{1+n}{\alpha}$ follows directly from the first-order conditions of the program (1.24). Use the values of α from Proposition 5 to conclude. □

$\alpha^* \equiv \frac{1+n}{R^u}$ is an important threshold value for the social planner's parameter α of the program (1.24). As seen from the last Lemma, $\hat{MRS} = R^u$ when $\alpha = \alpha^*$, meaning that the equilibrium steady state with no policy results in an allocation that is optimal for the social planner. This will be used in the decentralization analysis below.

1.5.2 Decentralization with full commitment

I study three policy instruments with respect to their ability to decentralize optimal allocations. These are transfers to the adults $\{\tau_t^1\}_{t\geq 0}$, transfers to the old $\{\tau_t^2\}_{t\geq 0}$ and government debt $\{g_t\}_{t\geq 0}$. Policies that use the three instruments at once are the most flexible, so they are trivially superior to others and not studied. Instead, I focus on policies that use only two instruments at a time. The main question of this section is similar to the one of Barro (1974): can the social planner achieve more stationary allocations if transfers are substituted with government debt? I approach this by comparing two types of policy: a balanced-budget one, that uses two transfers and a debt-based one, which uses government debt and transfers to the old. These are the only two types of policy that do not use the three instruments at once and can decentralize non-trivial sets of optimal allocations. Firstly, a policy that uses only one type of instrument cannot satisfy the government budget constraint, (1.1). Secondly, a policy that does not use transfers to the old cannot decentralize any value of e_0 apart from $e_0 = y_2 + R_0 b_{-1}^1$. The notation for the two types of policy to be compared is:

1. Balanced-budget policies $(\bar{\tau}_t^1, \bar{\tau}_t^2)_{t\geq 0}$ with $g_t = 0\ \forall t \geq 0$.
   By government budget constraint, $\bar{\tau}_t^2 = -(1+n)\bar{\tau}_t^1,\ \forall t \geq 1$.

2. Debt-based policies $(\breve{g}_t, \breve{\tau}_t^2)_{t\geq 0}$ with $\tau_t^1 = 0\ \forall t \geq 0$.
   By government budget constraint, $\breve{\tau}_t^2 = (1+n)\breve{g}_t - R_t\breve{g}_{t-1},\ \forall t \geq 1$.

I will refer to the two policies as minimal policies since they use the smallest possible number of instruments. In line with the seminal result of Barro (1974), transfers or taxes on adults and government debt have similar role in decentralizing allocations in full commitment economies without borrowing constraints. The only difference that may arise between the two policies is whether the constraints on transfers from Assumption 2 are met.

Proposition 6. *An optimal allocation $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ can be decentralized with at most one balanced-budget policy $(\bar{\tau}_t^1, \bar{\tau}_t^2)_{t\geq 0}$, where:*

$$\bar{\tau}_0^1 = -\frac{R_0 g_{-1} + \bar{\tau}_0^2}{1+n} = \hat{d}_0 - y_1 - R_0 b_{-1}^0 + (1+n)(\hat{c}_0 - y_0) \tag{1.26}$$

$$\bar{\tau}_t^1 = -\frac{\bar{\tau}_t^2}{1+n} = \hat{d}_t - y_1 + \hat{MRS}_{t-1} \cdot (\hat{c}_{t-1} - y_0) + (1+n)(\hat{c}_t - y_0),\ \forall t > 0 \tag{1.27}$$

It can be decentralized with at most one debt-based policy $(\breve{g}_t, \breve{\tau}_t^2)_{t\geq 0}$, where:

$$\breve{g}_t = -\bar{\tau}_t^1 = -\hat{d}_t + y_1 - \hat{MRS}_{t-1} \cdot (\hat{c}_{t-1} - y_0) - (1+n)(\hat{c}_t - y_0),\ \forall t \geq 0 \tag{1.28}$$

$$\breve{\tau}_0^2 = (1+n)\breve{g}_0 - R_0 g_{-1} \tag{1.29}$$

$$\breve{\tau}_t^2 = (1+n)\breve{g}_t - \hat{MRS}_{t-1}\breve{g}_{t-1} \quad \forall t > 0 \tag{1.30}$$

For both policies, the decentralization is possible if and only if the transfers meet the Assumption 2, that is, $\forall t \geq 0,\ \bar{\tau}_t^1 > -y_1;\ \bar{\tau}_t^2 > -y_2$ and $\breve{\tau}_t^2 > -y_2$

Proof. See section 1.7.8 in the Appendix.

According to the Proposition, both minimal policies are equivalent for decentralization provided that Assumption 2 is respected. This is an equivalence of lump-sum transfers and government debt for an OLG economy without frictions, analogous to the result of Buiter and Kletzer (1992) for a Diamond economy with production and 2 periods of life, who do not impose constraints on transfers as in Assumption 2. In the present framework, the equivalence does not hold, as Assumption 2 might hold for one policy and not for the other. I explore this possibility below for the case of stationary optimal allocations.

Proposition 6 implies that transfers and government debt that decentralize stationary allocations are constant, except for period 0. For the balanced-budget case, I denote such stationary policies $(\bar{\tau}_0^1, \bar{\tau}_0^2, \bar{\tau}^1, \bar{\tau}^2)$, meaning that $\bar{\tau}_t^1 = \bar{\tau}^1; \bar{\tau}_t^2 = \bar{\tau}^2 \quad \forall t > 0$. For the debt-based case, the notation is $(\breve{g}_0, \breve{\tau}_0^2, \breve{g}, \breve{\tau}^2)$.

So far, no restriction on the sign of the government debt has been made, so the government can be a creditor of the private sector. The following Proposition provides the signs of the transfers and of government debt for decentralization of stationary optimal allocations. It relies on the threshold value $\alpha^* = \frac{1+n}{R^u}$ for the parameter α of the social planner's program (1.24).

Proposition 7. *Let $(\bar{\tau}_0^1, \bar{\tau}_0^2, \bar{\tau}^1, \bar{\tau}^2)$ and $(\breve{g}_0, \breve{\tau}_0^2, \breve{g}, \breve{\tau}^2)$ decentralize a stationary optimal allocation $(\hat{c}, \hat{d}, \hat{e})$ with full commitment. Then $\alpha < \alpha^* \Leftrightarrow \bar{\tau}^1 < 0 (\Leftrightarrow \breve{g} > 0)$, where α is the parameter from the social planner program (1.24) associated with $(\hat{c}, \hat{d}, \hat{e})$; $\alpha^* = \frac{1+n}{R^u}$ and R^u is the steady-state interest rate in the equilibrium with full commitment and no policy.*

Proof. If $\alpha = \alpha^*$, then $\hat{MRS} = R^u$ and no transfers are needed in the steady state to decentralize the optimal allocation: $\bar{\tau}^1 = 0$. By Proposition 1, the sign of $\bar{\tau}^1$ depends negatively on the steady-state interest rate, which is equal to $\hat{MRS}$ in the equilibrium where the optimal allocation is decentralized. This means $\hat{MRS} > R^u \Leftrightarrow \bar{\tau}^1 < 0 \Leftrightarrow \breve{g} > 0$, where the last equivalence is given by $\breve{g} = -\bar{\tau}^1$, from Proposition 6. Then $\alpha = (1+n)/\hat{MRS}$ (Lemma 3) can be used to conclude. □

The condition $\alpha < \alpha^*$ can be interpreted as sufficiently fast discounting in the social planner's objective function, with one caveat. α is initially defined as ω_{t+1}/ω_t, with ω_t the weight of generation t in the program (1.23). However, the simplified program (1.24) also produces the Golder Rule allocation if $\alpha = 1$, in which case α loses the interpretation of the discount rate technically, but is still related. Indeed, the Golden Rule means all generations $1, 2, \ldots$ are equally important for the social planner, which means there is no discounting.

The condition $\alpha < \alpha^*$ is always verified for economies with $\alpha^* > 1$ or, equivalently, $R^u < 1+n$. By Lemma 3, equilibrium steady-states with $R^u < 1+n$ are never optimal for the social planner. Therefore, the above Proposition leads to a well-known result on pay-as-you-go pensions and on government debt: if the steady state is suboptimal[9], either a balanced-budget pay-as-you-go pension scheme with $\tau_1 > 0, \tau_2 < 0$ or a policy with positive government debt decentralize optimal steady states.

A value of α sufficiently close to α^* guarantees that the Assumption 2 is met for both minimal policies. Indeed, little redistribution is necessary in this case: according to Proposition 7, $\bar{\tau}^1$ is close to zero, and then same holds for $\bar{\tau}^2, \breve{g}, \breve{\tau}^2$. Are there parametrizations of the economy where Assumption 2 holds for one minimal policy, but fails for the other? To begin answering this question, first note that one of the transfers is always non-negative in the balanced-budget policies. I will focus on the most empirically relevant case $\bar{\tau}^1 < 0, \bar{\tau}^2 > 0$, which is shown above to be equivalent to $\alpha < \alpha^*$. If the transfer $\breve{\tau}^2$, which is negative in this case, is at least al large as $\bar{\tau}^1$ in absolute value, then the two policies can be ranked. The reason is that $y_2 < y_1$ by Assumption 1, so when the social planner taxes the old to finance public debt service, it can well exceed their retirement income. I provide parametric conditions for this case below:

Proposition 8. *Let the economy have* $\frac{1+n}{2+n} < \alpha^*$ *and the social planner decentralize a stationary optimal allocation with* $\alpha \in [\frac{1+n}{2+n}, \alpha^*)$. *If the Assumption 2 holds for debt-based policy in* $t > 1$, *(i.e.* $\breve{\tau}^2 > -y_2$*) then it also holds for the balanced-budget policy (i.e.* $\bar{\tau}^1 > -y_1$, $\bar{\tau}^2 > -y_2$*), but the reverse is not true.*

Proof. From (1.30), $\breve{\tau}^2 = (\hat{MRS} - (1+n))\bar{\tau}^1$. Using $\hat{MRS} = \frac{1+n}{\alpha}$ from Lemma 3, one gets $\breve{\tau}^2 = (1/\alpha - 1)(1+n)\bar{\tau}^1$. On the other hand, $y_2 < y_1$ by Assumption 1, so $\bar{\tau}^1 > -y_1$ follows from $\breve{\tau}^2 > -y_2$ if $\breve{\tau}^2 \leq \bar{\tau}^1$. According to the expression of $\breve{\tau}^2$ above, the latter condition is equivalent to $(1/\alpha - 1)(1+n) \leq 1$, or $\alpha > \frac{1+n}{2+n}$. Finally, under $\alpha < \alpha^*$, $\bar{\tau}^2 > 0 > -y_2$. □

[9] Suboptimality is typically associated with dynamic inefficiency in production economies — the rate of return on capital being smaller than the growth rate of the aggregate labor income in the economy.

Note that the above Proposition does not take into account the initial period $t = 0$, but one can extend the result to this period by restricting the initial conditions of the economy.

The bottom line of the analysis of decentralization under full commitment is that the policies are generically not equivalent. Superiority of the balanced-budget policy is obtained under a number of parameter restrictions. In what follows, however, I will analyze limited commitment economies under an assumption that the two policies could decentralize the central planner's optimal allocation if full commitment was available.

1.5.3 Decentralization with limited commitment

The Definition 2 of equilibrium with limited commitment differs from its full commitment counterpart only by the presence of borrowing constraints (1.11), (1.16) and individual rationality (1.14) for adults of generation 0. However, borrowing constraints never bind when optimal allocations are decentralized in the present setting. This is implied by the programs (1.23), (1.24) that produce these allocations. Intuitively, borrowing constraints that bind lead to a static inefficiency of the allocation: the sum of utilities could be improved by transferring resources from savers to borrowers[10]. Such allocations cannot maximize a weighed sum of utilities, as do optimal allocations. It follows that an optimal allocation can be decentralized with limited commitment only if the borrowing constraints do not bind in the corresponding equilibrium. Equivalently, decentralization under limited commitment can happen only if decentralization of the same allocation can happen with full commitment. For the endogenous borrowing constraint on the young to be relevant, the optimal allocation should be such that agents are net borrowers: $\hat{c}_t \geq y_0,\ \forall t \geq 0$. If an optimal allocation does not satisfy this condition, it is easy to show that the no-borrowing constraint on the adults (1.11) makes the use of government debt necessary under limited commitment. Indeed, when the young are net savers, (1.11) implies that the asset position of the private sector is positive. By asset market clearing (1.5), government debt must be the counterpart of these aggregate savings. The rest of the section only focuses on optimal allocations with $\hat{c}_t \geq y_0,\ \forall t \geq 0$.

The following conditions will also be assumed for the analysis of this section:

1. the allocations can be decentralized with both minimal policies under full commitment (see Proposition 6);

2. the policies decentralizing the allocations with full commitment have non-negative government debt:
$$g_t \geq 0,\ \forall t \geq 0 \tag{1.31}$$

The first condition allows to focus on the constraints due to limited commitment, abstracting from constraints on taxes of Assumption 2 that are present both in the full and

[10] see the proof of Proposition 9 for a formal treatment of the case of allocations that have $c_t > y_0$, $\forall t \geq 0$ and are decentralized under full commitment with $g_t \geq 0,\ \forall t \geq 0$. The same argument is applicable to any optimal allocations.

the limited commitment cases. The second condition has can be motivated empirically: governments are usually net debtors and not net creditors of the economy. Note that all balanced-budget policies fall under the condition (1.31).

The following Proposition provides a necessary and sufficient condition for decentralization under limited commitment with $\hat{c}_t > y_0, \ \forall t \geq 0$ and corresponding initial conditions.

Proposition 9. *An optimal allocation* $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ *with* $\hat{c}_t > y_0, \ \forall t \geq 0$ *and initial conditions* $b^0_{-1} \leq 0, \ b^1_{-1} \geq 0$ *is decentralized under limited commitment by one of the two minimal policies with* $g_t \geq 0 \ \forall t \geq 0$ *if, and only if, two conditions hold:*

1. *the minimal policy decentralizes* $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ *with full commitment*
2. *individual rationality constraint* (IR1) *holds for each period*

Proof. See section 1.7.9 in the Appendix.

The Proposition allows to define a procedure of comparing balanced-budget and debt-based policies under limited commitment. If both policies can decentralize a given allocation with full commitment, it is sufficient to compare the individual rationality conditions implied by the two policies. As the left-hand side of (IR1) is determined by the allocation and is the same for any policy, only the right-hand sides need to be compared across policies. If one policy produces smaller right-hand side of (IR1) than another policy, it is more flexible for decentralization under limited commitment. This superiority can be stated as follows: a set of optimal allocations that one policy can decentralize is a subset of the set of optimal allocations decentralized by another policy.

The next result, central to this article, is on decentralization of stationary optimal allocations, with some restrictions on the parameters of the economy. It is useful to assume the social planner has $\alpha < \alpha^*$ in the program (1.24), for the following two reasons. First, by Proposition 7, the non-negativity of government debt (1.31) then holds when the allocation is decentralized with full commitment. Second, by the same Proposition, balanced-budget policies under $\alpha < \alpha^*$ have $\tau^1 < 0, \ \tau^2 > 0$, a pay-as-you-go pension system. The choice between a balanced-budget and a debt-based policy is then related to a comparison of pay-as-you-go and fully funded pension systems[11]. In order for the results on the steady-state generations $1, 2 \ldots$ to also hold for generations -1 and 0, I make additional assumptions for the initial conditions. Apart from $b^1_{-1} \geq 0$, assumed above for all limited commitment economies, I restrict the initial asset income (debt repayment) of adults of generation 0 to to be at least as small (at least as large) than that of subsequent generations: $R_0 b^0_{-1} \leq \hat{MRS} \cdot (y_0 - \hat{c})$. Intuitively, if this condition is verified as equality, generation 0 has the same budget constraint as all the other generations, so their incentive

[11]The debt-based policy taxes old-age income, which is in sharp contrast with pension systems. In this respect it is not a social security policy and is only studied as an extreme case. However, the benefits of decreasing taxes on the adults and increasing the government debt shown for this policy are also relevant for moderate policies that use the three instruments at once, allowing for positive transfers for the old.

compatibility must not be studied separately. If the condition holds as inequality, debt-based policies are preferable since the initial adults have a large debt and the incentives for savings are particularly important for incentive compatibility.

With all the above assumptions, one can rank the two minimal policies with respect to decentralization with limited commitment:

Proposition 10. *Assume the social planner has $\alpha < \alpha^*$ in the problem* (1.24)*, resulting in a stationary optimal allocation $(\hat{c}, \hat{d}, \hat{e})$ with $\hat{MRS} \cdot (y_0 - \hat{c}) \geq R_0 b^0_{-1}$ that can be decentralized with both a balanced budget and a debt-based policy under full commitment. If this allocation can be decentralized with a balanced-budget policy under limited commitment, it can also be done with a debt-based policy, but the reverse is not true.*

Proof. See section 1.7.10 in the Appendix.

To understand this result, recall the comparative statics exercise of Proposition 2. It says that in case of a balanced-budget stationary policy, larger redistribution from adults to the old makes endogenous borrowing constraints more tight. Moving from a balanced-budget policy to a debt-based one means going in the direction of larger τ^1 and smaller τ^2: the taxes on adults are null for the latter policy, and the old are being taxed instead of receiving transfers. Only the additional savings that the old agents hold in the form of government bonds allow them to maintain the optimal level of consumption. This additional reliance on savings instead of redistribution is what makes individual rationality constraints less tight and improves decentralization.

The following section provides a parametric example of a set of economies where Golden Rule allocations can be decentralized with debt-based policies but not with balanced-budget ones. The example also clarifies the roles of different assumptions in the analysis.

1.5.4 Decentralization with limited commitment: an example

Consider an economy with $n = 0, \beta = 1$ and:

$$0 < y_2 < y_0 < y_1 \tag{1.32}$$

which satisfies the Assumption 1. Suppose that the social planner aims to decentralize the Golden Rule allocation with limited commitment. In this case, the FOC of the program (1.22) lead to $u'(c) = u'(d) = u'(e)$, so the solution is:

$$\hat{c} = \hat{d} = \hat{e} = \frac{1}{3}(y_0 + y_1 + y_2) \tag{1.33}$$

For the borrowing constraints on the young to be relevant, I also assume that they are net borrowers, so:

$$\hat{c} > y_0 \;\Leftrightarrow \frac{1}{3}(y_0 + y_1 + y_2) \;\Leftrightarrow$$

$$\Leftrightarrow y_0 < (y_1 + y_2)/2 \tag{1.34}$$

Consider first the decentralization for generations $1, 2, 3, \ldots$; I will revisit initial generations at the end of the section. By Proposition 9, a policy that can carries out the decentralization under limited commitment necessarily does it under full commitment. By Proposition 6 the following two minimal policies can potentially do the decentralization, if they meet the constraints on the transfers of Assumption 2:

$$\bar{\tau}^1 = \hat{c} - y_1 - (M\hat{R}S + 1 + n)(y_0 - \hat{c}) = y_2 - y_0 < 0 \tag{1.35}$$

$$\bar{\tau}^2 = -(1+n)\tau^1 = y_0 - y_2 > 0 \tag{1.36}$$

$$\breve{g} = y_0 - y_2 > 0 \tag{1.37}$$

$$\breve{\tau}^2 = (1 + n - M\hat{R}S)\breve{g} = 0 \tag{1.38}$$

where Lemma 3 was used to obtain $M\hat{R}S = 1 + n = 1$. The constraints on the transfers of Assumption 2 are indeed met:

$$\bar{\tau}^1 = y_2 - y_0 > -y_1, \quad \bar{\tau}^2 = y_0 - y_2 > -y_2, \quad \breve{\tau}^2 = 0 > -y_2,$$

so both minimal policies decentralize the Golden Rule with full commitment.

The policies above also decentralize the Golden Rule with limited commitment for generations $1, 2, 3, \ldots$ if they result in the condition (IR1) being respected, as seen in Proposition 9. Furthermore, $\bar{\tau}^1 < 0 \Leftrightarrow \alpha < \alpha^*$ by Proposition 7, so the assumptions of Proposition 10 are verified and (IR1) is respected under a debt-based policy if they are so under the balanced-budget one, while the reverse is not true. As for (IR1), by substituting the values of transfers of the two policies, the constraint is:

$$u\left(\frac{1}{3}(y_0 + y_1 + y_2)\right) \geq \frac{1}{2}u(y_1 + y_2 - y_0) + \frac{1}{2}u(y_0) \qquad \text{for } \tau^1 = \bar{\tau}^1,\ \tau^2 = \bar{\tau}^2 \tag{1.39}$$

$$u\left(\frac{1}{3}(y_0 + y_1 + y_2)\right) \geq \frac{1}{2}u(y_1) + \frac{1}{2}u(y_2) \qquad \text{for } \tau^1 = 0,\ \tau^2 = \breve{\tau}^2 \tag{1.40}$$

According to Proposition 10, (1.39) is more strict than (1.40); Figure 1.1 illustrates it for one example of parameter values, while also showing that the result applies to any parameter values that respect (1.32) and (1.34).

When is government debt necessary for decentralization? One parametrization that makes decentralization possible with $(\breve{g}, \breve{\tau}^2)$ and impossible with $(\bar{\tau}^1, \bar{\tau}^2)$ is $y_0 = (y_1 + y_2)/2 - \varepsilon$, with ε small and positive. Indeed, in this case (IR1) for the two policies writes:

$$u\left(\frac{y_1 + y_2}{2} - \frac{\varepsilon}{3}\right) \geq \frac{1}{2}u\left(\frac{y_1 + y_2}{2} + \varepsilon\right) + \frac{1}{2}u\left(\frac{y_1 + y_2}{2} - \varepsilon\right) \text{ for } \tau^1 = \bar{\tau}^1,\ \tau^2 = \bar{\tau}^2 \tag{1.41}$$

$$u\left(\frac{y_1 + y_2}{2} - \frac{\varepsilon}{3}\right) \geq \frac{1}{2}u(y_1) + \frac{1}{2}u(y_2) \qquad \text{for } \tau^1 = 0,\ \tau^2 = \breve{\tau}^2 \tag{1.42}$$

In case $\varepsilon = 0$, (1.41) would be verified as equality. With a small increase of ε by $\Delta\varepsilon$, the change of the left hand side is $-\frac{1}{3}u'\left(\frac{y_1+y_2}{2}\right)\Delta\varepsilon < 0$ in first-order approximation. The same approximation for the right-hand side is null. Therefore, (1.41) does not hold. At

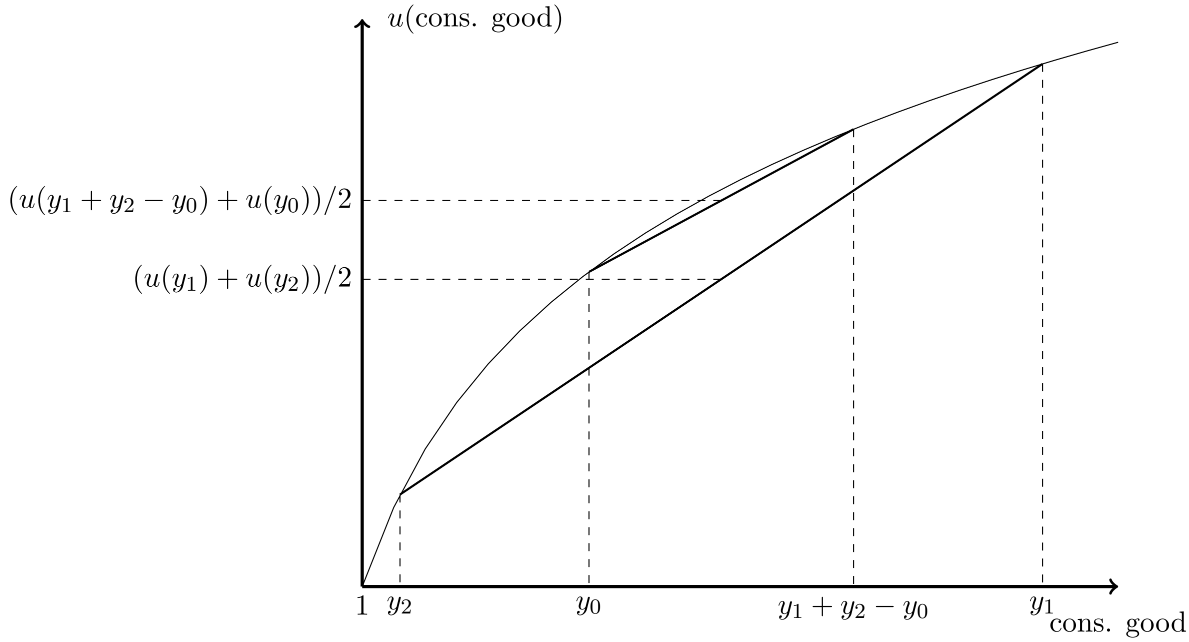


Figure 1.1: Example with $u(\cdot) = 3\ln(\cdot)$; $y_2 = 1.5$, $y_0 = 4$, $y_1 = 10$. The relationship $(u(y_1 + y_2 - y_0) + u(y_0))/2 > (u(y_1) + u(y_2))/2$ holds for any parameters satisfying the constraints assumed in the example: $0 < y_2 < y_0 < y_1$; $y_0 < (y_1 + y_2)/2$; $\sigma \geq 1$.

the same time, (1.42) does hold as inequality for $\varepsilon = 0$, by strict concavity of u. The difference of the right hand side and the left hand side is not marginal if IES is finite. Therefore, for a small ε, (1.42) is verified.

Finally, the above analysis is true for generations $-1, 0$ if $b^1_{-1} \geq 0$ and $R_0 b^0_{-1} \leq \hat{MRS} \cdot (y_0 - \hat{c}) = y_0 - \hat{c}$. The first inequality is necessary for any economy to have an equilibrium with limited commitment. The second one, used in Proposition 10, ensures that (IR1) holds for generation 0 if it holds for the subsequent ones. Indeed, from Proposition 6, $\bar{\tau}^1_0 = 2\hat{c} - y_1 - y_0 - R_0 b^0_{-1}$. If $R_0 b^0_{-1} \leq y_0 - \hat{c}$, then $\bar{\tau}^1_0 \geq \tau^1$, so the right hand side of (IR1) is more strict for this generation than for subsequent ones:

$$u(\hat{d}) + \beta u(\hat{e}) > u(y_1 + \bar{\tau}^1_0) + \beta u(y_2 + \bar{\tau}^2) > u(y_1 + \bar{\tau}^1) + \beta u(y_2 + \bar{\tau}^2)$$

The constraint (IR1) under the debt-based policy is the same for all generations as adults of $t = 0$ do not receive transfers. As a result, the constraint (IR1) in $t = 0$ is more strict under a balanced-budget policy that under a debt-based one.

1.5.5 Decentralization and autarky

The main result of the previous section is that, under mild conditions, it is easier for the social planner to make a given optimal allocation equilibrium with a debt-based minimal policy than with a balanced-budget one. However, Section 1.4.3 has shown that the equilibrium targeted by the social planner is not necessarily the only equilibrium of the economy under a given policy. In particular, autarky can be a second equilibrium steady state, which will prevail if all agents have corresponding expectations. Consider an economy that has $b^0_{-1} = 0$. The analysis of Section 1.4.3 implies that decentralization with a balanced-budget policy allows for autarky as an equilibrium, while a debt-based policy rules it out. Government debt rules out equilibrium autarky for two different reasons, depending on the optimal allocation that is decentralized. When the young are net borrowers, autarky is ruled out for the reasons discussed in Proposition 10. Namely, the use of government debt leads to individual rationality constraints — equivalently, the borrowing constraints — less strict than under balanced-budget policies. This is sufficient to rule out autarky in this setting, since the existence of an autarkic equilibrium relies on vanishing borrowing limits. In the case $\hat{c} < y_0$, considered in the beginning of Section 1.5.3, government debt ensures that interest rates are sufficiently high, so that both young and adult agents are willing to have net savings.

1.5.6 Comparison to simple borrowing constraints

In this section, I compare my results to simpler frameworks with constraints defined by an exogenous parameter. In particular, I study two version of the constraint commonly found in the literature. In the first version, the lower limit on the assets of the young is an exogenous constant. In the second version, it is a fraction of the present value of the agent's future income, as in Jappelli and Pagano (1994a). The results of the previous two sections do not hold in both cases: minimal policies are equivalent for decentralization under such constraints.

The first form of an exogenous borrowing constraint is the following:

$$b^0_{t-1} \geq \bar{b} \ \forall t \geq 0 \tag{1.43}$$

where $\bar{b} \leq 0$ is an exogenous parameter. Using the budget constraint, one obtains $y_0 - c_{t-1} \geq \bar{b}$, $\forall\ t \geq 0$. To examine the implications for decentralization of optimal allocations, note that $y_0 - \hat{c}_{t-1} \geq \bar{b}$, $\forall\ t \geq 0$ is only a constraint on the values of the optimal allocation and of exogenous parameters. It follows that the choice between two minimal policies has no influence on whether the constraint is verified.

Now assume the constraint is as in Jappelli and Pagano (1994a):

$$b^0_{t-1} \geq -\phi \cdot \left(\frac{y_1 + \tau^1_t}{R_t} + \frac{y_2 + \tau^2_{t+1}}{R_t R_{t+1}} \right) \ \forall t \geq 0 \tag{1.44}$$

where the exogenous constant $\phi \in [0, 1]$ can be interpreted as the fraction of lifetime after-transfer income that lenders can confiscate if the agent does not repay her debts. Although the transfers on the right hand side of (1.44) are different for a balanced-budget and a debt-based policy, the resulting present value of the lifetime income after transfers is the same.

Indeed, when a given optimal allocation is decentralized, the present value of the income after transfers is determined by the present value of the optimal allocation:

$$\hat{c}_{t-1} + \frac{\hat{d}_t}{M\hat{R}S_{t-1}} + \frac{\hat{e}_{t+1}}{M\hat{R}S_{t-1}M\hat{R}S_t} = y_0 + \frac{y_1 + \tau_t^1}{M\hat{R}S_{t-1}} + \frac{y_2 + \tau_{t+1}^2}{M\hat{R}S_{t-1}M\hat{R}S_t}$$
$$\Rightarrow -\phi \cdot \left(\frac{y_1 + \tau_t^1}{R_t} + \frac{y_2 + \tau_{t+1}^2}{R_t R_{t+1}} \right) = -\phi \cdot \left(\hat{c}_{t-1} + \frac{\hat{d}_t}{M\hat{R}S_{t-1}} + \frac{\hat{e}_{t+1}}{M\hat{R}S_{t-1}M\hat{R}S_t} - y_0 \right)$$

One concludes again that the borrowing constraint (1.44) is verified or not for a given optimal allocation, independently of the policy instruments used for decentralization. However, a less standard version of the latter constraint with only period t income entering the right-hand side of (1.44) would not result in the equivalence of the two minimal policies.

1.6 Conclusion

This paper provides a novel argument for the use of government debt for redistribution of resources between generations. Government debt accumulation, as opposed to taxation of adult workers, discourages consumers from personal bankruptcy. Endogenous borrowing constraints become less tight in response, so optimal allocations become feasible in equilibrium. I show this in a decentralization problem of a social planner that can use either a balanced-budget or debt-based policy with the same number of instruments. Under mild conditions, the set of allocations decentralized with a balanced-budget policy is a subset of those decentralized with a debt-based policy.

Furthermore, the use of public debt alleviates the problem of equilibrium multiplicity. If government debt is used, a suboptimal, autarkic equilibrium does not co-exist with the one targeted by the social planner, whereas such multiplicity is always present if government budgets are balanced in every period.

This article uses a simple structure for tractability: endowment economy with no uncertainty, identical agents within and across generations, three periods of life, no bequest motive, and so on. However, the mechanism identified in the model can generalize to larger, quantitative life-cycle models and produce results that differ from the recent models (Rohrs and Winter, 2015; Antunes and Ercolani, 2020) analyzing government debt and credit constraints in infinite-horizon economies.

1.7 Appendix

1.7.1 Proof of Proposition 1

Some additional notation is needed for the proof. The results of the utility maximisation problem of a generation t agent are the following functions of interest rates and transfers:

$$\begin{aligned}
c_{t-1} &= \sigma(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2) \\
d_t &= \delta(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2) \\
e_{t+1} &= \varepsilon(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2) \\
b_{t-1}^0 &= \beta^0(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2) = y_0 - \sigma(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2) \\
b_t^1 &= \beta^1(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2) = 1/R_{t+1}(\varepsilon(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2) - \tau_{t+1}^2 - y_2)
\end{aligned}$$

The following functions are the demands for consumption and saving of an agent facing a stationary interest rate R and stationary balanced-budget transfers $(\tau^1, -(1+n)\tau^1)$:

$$\begin{aligned}
c^s(R, \tau^1) &:= \sigma(R, R, \tau^1, -(1+n)\tau^1); \qquad & b^0(R, \tau^1) &:= \beta^0(R, R, \tau^1, -(1+n)\tau^1) \\
d^s(R, \tau^1) &:= \delta(R, R, \tau^1, -(1+n)\tau^1); \qquad & b^1(R, \tau^1) &:= \beta^1(R, R, \tau^1, -(1+n)\tau^1) \\
e^s(R, \tau^1) &:= \varepsilon(R, R, \tau^1, -(1+n)\tau^1)
\end{aligned}$$

Denote by W_t the present value in t of intertemporal wealth of an adult of period t: $W_t \equiv R_t y_0 + y_1 + \tau_t^1 + \frac{y_2 + \tau_{t+1}^2}{R_{t+1}}$. The following lemma gives a simplified description of demand functions:

Lemma 4. *The demands for c_{t-1}, d_t, e_{t+1} can be described by $R_t c_{t-1} = s^c(R_t, R_{t+1}) W_t$; $d_t = s^d(R_t, R_{t+1}) W_t$; $\frac{1}{R_{t+1}} e_{t+1} = s^e(R_t, R_{t+1}) W_t$, where $s^c(R_t, R_{t+1}) + s^d(R_t, R_{t+1}) + s^e(R_t, R_{t+1}) = 1$. For the case $u(\cdot) = \ln(\cdot)$, the expressions of $s^c(R_t, R_{t+1}), s^d(R_t, R_{t+1}), s^e(R_t, R_{t+1})$ hold true if one substitutes $\sigma = 1$ and the three functions are degenerate, i.e. constant with respect to their arguments.*

Proof. The FOC of the consumer with a CIES utility function and $\sigma > 1$ give:

$$d_t = (\beta R_t)^\sigma c_{t-1}; \; e_{t+2} = (R_{t+1} R_t)^\sigma \beta^{2\sigma} c_{t-1}$$

Substituting in the intertemporal budget constraint, one gets:

$$R_t c_{t-1} = R_t / \left(R_t + (\beta R_t)^\sigma + (R_t)^\sigma R_{t+1}^{\sigma-1} \beta^{2\sigma} \right) W_t$$

It follows that :

$$\begin{aligned}
s^c(R_t, R_{t+1}) &= R_t / \left(R_t + \beta^\sigma R_t^\sigma + \beta^{2\sigma} R_t^\sigma R_{t+1}^{\sigma-1} \right) \\
s^d(R_t, R_{t+1}) &= \beta^\sigma R_t^\sigma / \left(R_t + \beta^\sigma R_t^\sigma + \beta^{2\sigma} R_t^\sigma R_{t+1}^{\sigma-1} \right) \\
s^e(R_t, R_{t+1}) &= \beta^{2\sigma} R_t^\sigma R_{t+1}^{\sigma-1} / \left(R_t + \beta^\sigma R_t^\sigma + \beta^{2\sigma} R_t^\sigma R_{t+1}^{\sigma-1} \right)
\end{aligned}$$

and $s^c(R_t, R_{t+1}) + s^d(R_t, R_{t+1}) + s^e(R_t, R_{t+1}) = 1$ is verified. In the case $u(\cdot) = \ln(\cdot)$, one obtains $R_t c_{t-1} = \frac{1}{1+\beta+\beta^2} W_t$; $d_t = \frac{\beta}{1+\beta+\beta^2} W_t$; $e_{t+1}/R_{t+1} = \frac{\beta^2}{1+\beta+\beta^2} W_t$, which is consistent with substituting $\sigma = 1$ in the above formulas. □

By taking derivatives of the formulas of the Lemma, one can show that the consumption demands satisfy:

$$\partial\sigma(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2)/\partial R_t < 0,\ \ \partial\sigma(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2)/\partial R_{t+1} < 0$$
$$\partial\delta(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2)/\partial R_t > 0,\ \ \partial\delta(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2)/\partial R_{t+1} < 0$$
$$\partial\varepsilon(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2)/\partial R_t > 0,\ \ \partial\varepsilon(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2)/\partial R_{t+1} > 0$$

These derivatives have economic interpretation: consumption of each date is decreasing in its own relative price and increasing in the relative prices of consumption in all the other periods, if one sets the price of d_t to 1 the price of c_{t-1} to R_t and the price of e_{t+1} to $1/R_{t+1}$. I will call this property substitutability.

A steady state without government debt is given by

$$(1+n)b^0(R, \tau^1) + b^1(R, \tau^1) = 0 \tag{1.45}$$

The existence of at least one steady state can now be shown. First, the steady-state asset position of the private sector is a continuous function of R for $R > 0$, since the demands derived in Lemma 4 are continuous on this range. Then, consider $R \to 0+$. According to the formulas of Lemma 4, $c^s \to +\infty$ under such a stationary interest rate, so $b^0 < 0$. Moreover, $e^s \to 0$, so $b^1 < 0$ since $y_2 + \tau^2 > 0$ by Assumption 2. One gets $(1+n)b^0(R, \tau^1) + b^1(R, \tau^1) < 0$ for $R \to 0$. On the other hand, if $R \to +\infty$, then the opposite is true: $c^s \to 0$ and $e^s \to +\infty$, so $(1+n)b^0(R, \tau^1) + b^1(R, \tau^1) > 0$. One concludes that at least one steady state with $R > 0$ exists.

Showing uniqueness of the steady state is then equivalent to showing that the steady-state asset position of the private sector is increasing in the steady-state interest rate $R > 0$. To show this, use the substitutability features shown above: $\frac{\partial\sigma}{\partial R_t} < 0, \frac{\partial\sigma}{\partial R_{t+1}} < 0$ implies $\frac{\partial c^s}{\partial R} = \frac{\partial\sigma}{\partial R_t} + \frac{\partial\sigma}{\partial R_{t+1}} < 0$. Then, $\frac{\partial b^0}{R} = -\frac{\partial c^s}{\partial R} > 0$. To show the signs of derivatives of b^1, first write $b_t^1 = y_1 + \tau_t^1 + R_t b_{t-1}^0 - d_t$. It follows that $\frac{\partial\beta^1}{\partial R_{t+1}} = R_t \frac{\partial\beta^0}{\partial R_{t+1}} - \frac{\partial\delta}{\partial R_{t+1}} > 0$. Then write $b_t^1 = \frac{1}{R_{t+1}}(e_{t+1} + (1+n)\tau_{t+1}^1 - y_2)$. Then $\frac{\partial\beta^1}{\partial R_t} = \frac{1}{R_{t+1}}\frac{\partial\varepsilon}{\partial R_t} > 0$ as $\frac{\partial\varepsilon}{\partial R_t} > 0$ by substitutability. The signs of the two derivatives of β^1 imply $\frac{\partial b^1}{\partial R} > 0$. We finally get the sign of the derivative of the total asset position, $(1+n)\frac{\partial b^0}{\partial R} + \frac{\partial b^1}{\partial R} > 0$. This proves uniqueness of the steady state.

To show the relationship between τ^1 and R, note that the steady-state asset position of the private sector is continuous in τ^1. The steady state interest can then be given by an implicit function of τ^1:

$$R^s: \ ]-y_1, y_2/(1+n)[\rightarrow \mathbb{R};\ (1+n)b^0(R^s(\tau^1),\tau^1)+b^1(R^s(\tau^1),\tau^1)=0 \tag{1.46}$$

The claim that the steady state interest rate decreases in τ^1 can then be summarized as $\frac{dR^s}{d\tau^1}<0$. By the implicit function theorem, this derivative exists and is given by

$$\frac{dR^s}{d\tau^1}=\frac{(1+n)\frac{\partial b^0}{\partial \tau^1}+\frac{\partial b^1}{\partial \tau^1}}{-(1+n)\frac{\partial b^0}{\partial R}-\frac{\partial b^1}{\partial R}} \tag{1.47}$$

The remaining proof relies on the signs and the magnitudes of $\frac{\partial b^0}{\partial R}; \frac{\partial b^0}{\partial \tau^1}; \frac{\partial b^1}{\partial R}; \frac{\partial b^1}{\partial \tau^1}$.

As shown above, substitutability leads to $\frac{\partial b^0}{\partial R}>0$, $\frac{\partial b^1}{\partial R}>0$. One obtains the negative sign of $-(1+n)\frac{\partial b^0}{\partial R}-\frac{\partial b^1}{\partial R}$, the denominator of $\frac{dR^s}{d\tau^1}$.

To get the sign of $(1+n)\frac{\partial b^0}{\partial \tau^1}+\frac{\partial b^1}{\partial \tau^1}$, use definitions of $\beta^0, \beta^1, b^0, b^1$ the budget constraints and Lemma 4 to get:

$$\begin{aligned}
\frac{\partial b^0}{\partial \tau^1}&=-\frac{\partial c^s}{\partial \tau^1}; & \frac{\partial b^1}{\partial \tau^1}&=\frac{1}{R}\left(\frac{\partial e^s}{\partial \tau^1}+1+n\right)\\
\frac{\partial c^s}{\partial \tau^1}&=\frac{s^c(R,R)}{R}\left(1-\frac{1+n}{R}\right); & \frac{\partial e^s}{\partial \tau^1}&=s^e(R,R)(R-(1+n))
\end{aligned}$$

To see the sign of $(1+n)\frac{\partial b^0}{\partial \tau^1}+\frac{\partial b^1}{\partial \tau^1}$, two different cases must be studied depending on the sign of $R-(1+n)$:

<u>Case $R\leq 1+n$</u>: $\frac{\partial b^0}{\partial \tau^1}=-\frac{\partial c^s}{\partial \tau^1}=-s^c(R,R)\frac{R-(1+n)}{R^2}\geq 0$ and $\frac{\partial b^1}{\partial \tau^1}>0$ (the latter is true in both cases), so $(1+n)\frac{\partial b^0}{\partial \tau^1}+\frac{\partial b^1}{\partial \tau^1}>0$.

<u>Case $R>1+n$</u>: $(1+n)\frac{\partial b^0}{\partial \tau^1}+\frac{\partial b^1}{\partial \tau^1}=(1+n)\left(\frac{1}{R}-\frac{\partial c^s}{\partial \tau^1}\right)+\frac{1}{R}\frac{\partial e^s}{\partial \tau^1}$ and since $s^c(R,R)<1$, one gets $\frac{\partial c}{\partial \tau^1}<\frac{R-(1+n)}{R^2}$. This leads to

$$\begin{aligned}
(1+n)\frac{\partial b^0}{\partial \tau^1}+\frac{\partial b^1}{\partial \tau^1}&>(1+n)\left(\frac{1}{R}-\frac{R-(1+n)}{R^2}\right)+\frac{1}{R}\frac{\partial e^s}{\partial \tau^1}\\
&=\frac{1}{R}\left((1+n)\frac{1+n}{R}+\frac{\partial e^s}{\partial \tau^1}\right)>0
\end{aligned}$$

where the last inequality is due to $\frac{\partial e^s}{\partial \tau^1}=s^e(R,R)(R-(1+n))>0$. One obtains $(1+n)\frac{\partial b^0}{\partial \tau^1}+\frac{\partial b^1}{\partial \tau^1}>0$. The numerator of $\frac{dR^s}{d\tau^1}$ is shown to be positive and the denominator is shown to be negative. As a result, $\frac{dR^s}{d\tau^1}<0$.

1.7.2 Proof of Lemma 1

The function f exists since V is continuous (by the Maximum theorem), monotonic in the second argument and has a sufficiently wide range. Indeed, it is monotonic since a increase of b^0_{t-1} expands the budget set; this makes utility larger because the utility function is monotonic. Then, note that $V(R_{t+1},0,\tau^1_t,\tau^2_{t+1})\geq u(y_1+\tau^1_t)+\beta u(y_2+\tau^2_{t+1})$ since $(y_1+\tau^1_t, y_2+\tau^2_{t+1})$ is in the budget set of the corresponding utility maximization problem (1.12) with $b^0_{t-1} = 0$. At the same time, denoting $\zeta \equiv -y_1 - \tau^1_t - (y_2 + \tau^2_{t+1})/R_{t+1}$, one obtains $\lim_{R_t b^0_{t-1} \to \zeta+} V < u(y_1 + \tau^1_t) + \beta u(y_2 + \tau^2_{t+1})$ since the budget set collapses in this limit. This proves the existence of f. Continuous differentiability of the function follows from continuous differentiability of the utility function and the implicit function theorem. Furthermore, since $\zeta < 0$ by Assumption 2, the proof of existence also implies that f never takes on positive values, i. e., it only defines a borrowing limit and not a positive lower bound on savings in youth.

To show the conditions under which $f(R_{t+1}, \tau^1_t, \tau^2_{t+1}) = 0$, first note that if the adults could borrow (as in the full commitment environment), $b^0_{t-1} \leq 0$ and $R_{t+1} \leq MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1})$ would imply $b^1_t \leq 0$. Indeed, $R_{t+1} = MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1})$ is the condition for an adult with $b^0_{t-1} = 0$ to demand $b^1_t = 0$, so if $b^0_{t-1} < 0$, the demand for e_{t+1} decreases by normality of old-age consumption, leading to demand for $b^1_t < 0$. When the no-borrowing constraint (1.11) is taken into account, $b^1_t = 0$ for $R_{t+1} \leq MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1})$. It then follows that the value of f for such interest rates is 0. To verify this, substitute $b^0_{t-1} = 0$ in the adult agent's problem to obtain the demands for adult and old age consumption under such interest rates equal to $d_t = y_1 + \tau^1_t$; $e_{t+1} = y_2 + \tau^2_{t+1}$. This implies $V(R_{t+1}, 0, \tau^1_t, \tau^2_{t+1}) = u(y_1 + \tau^1_t) + \beta u(y_2 + \tau^2_{t+1})$ under such interest rates, or equivalently $f(R_{t+1}, \tau^1_t, \tau^2_{t+1}) = 0$.

Finally, $f(R_{t+1}, \tau^1_t, \tau^2_{t+1}) < 0$ for $R_{t+1} > MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1})$ since f is decreasing in the first argument. This is true since the corresponding partial derivative is $\frac{\partial f}{\partial R_{t+1}} = 0$ for $R_{t+1} = MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1})$ and $\frac{\partial f}{\partial R_{t+1}} < 0$ for $R_{t+1} > MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1})$. Indeed, $\frac{\partial f}{\partial R_{t+1}} = -\frac{\partial V}{\partial R_{t+1}} / \frac{\partial V}{\partial (R_t b^0_{t-1})}$ by implicit function theorem. For interest higher than (equal to) $MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1})$, the agent is a net saver (has null savings) when adult, so $\frac{\partial V}{\partial R_{t+1}}$ is positive (null). Finally, $\frac{\partial V}{\partial (R_t b^0_{t-1})} > 0$ is proved above. One gets $f(R_{t+1}, \tau^1_t, \tau^2_{t+1}) = 0$ for $R_{t+1} \leq MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1})$ and is decreasing in the interest when the interest exceeds these values. This completes the proof.

1.7.3 Proof of Proposition 2

Define a function for the optimal asset position of an adult of generation t that solves the program (1.12) as:

$$z(R_{t+1}, R_t b^0_{t-1}, \tau^1, -(1+n)\tau^1) = y_1 + \tau^1 + R_t b^0_{t-1} - d(R_{t+1}, R_t b^0_{t-1}, \tau^1, -(1+n)\tau^1)$$

The equation (1.15) that defines $R_t b^c_{t-1}$ can then be written:

$$\begin{aligned} &u(y_1 + \tau^1 + R_t b^0_{t-1} - z(R_{t+1}, R_t b^c_{t-1}, \tau^1, -(1+n)\tau^1)) \\ &\quad + \beta u(y_2 - (1+n)\tau^1 + R_{t+1} z(R_{t+1}, R_t b^c_{t-1}, \tau^1, -(1+n)\tau^1)) \\ &\quad = u(y_1 + \tau^1) + \beta u(y_2 - (1+n)\tau^1) \end{aligned} \tag{1.48}$$

Then, define a function $F(\tau^1, R_t b^c_{t-1})$ as a difference of the left-hand side and the right-hand side of the equation (1.48). The expression $F(\tau^1, R_t b^c_{t-1}) = 0$ then defines $R_t b^c_{t-1}$ as an implicit function of τ^1 by the same manner as the more general function f has been defined in Lemma 1. F increases in the second argument by the wealth effect discussed in the proof of Lemma 1, so the proposition can be concluded by showing that $\frac{\partial F}{\partial \tau^1} \geq 0$. This derivative is simplified by the envelope theorem: since $z(R_{t+1}, R_t b^c_{t-1}, \tau^1, -(1+n)\tau^1)$ maximizes the utility of the adult agent, all the terms of $\frac{\partial F}{\partial \tau^1}$ involving derivatives of z sum to zero. One then obtains:

$$\begin{aligned}\frac{\partial F}{\partial \tau^1} =& u'(d(R_{t+1}, R_t b^c_{t-1}, \tau^1, -(1+n)\tau^1)) - \beta(1+n)u'(e(R_{t+1}, R_t b^c_{t-1}, \tau^1, -(1+n)\tau^1)) \\ &- \Big(u'(y_1 + \tau^1) - \beta(1+n)u'(y_2 - (1+n)\tau^1)\Big)\end{aligned}$$

where the two lines correspond to derivatives of left- and right-hand sides of (1.48). Rearranging terms, one obtains:

$$\begin{aligned}\frac{\partial F}{\partial \tau^1} =& \Big(u'(d(R_{t+1}, R_t b^c_{t-1}, \tau^1, -(1+n)\tau^1)) - u'(y_1 + \tau^1)\Big) + \\ & \beta(1+n)\Big(u'(y_2 - (1+n)\tau^1) - u'(e(R_{t+1}, R_t b^c_{t-1}, \tau^1, -(1+n)\tau^1))\Big)\end{aligned}$$

Both differences in brackets are positive when the asset position of the adult agent is positive and null when the asset position is null; this follows from $u'' < 0$, $b^c_{t-1} \leq 0$ (by Lemma 1) and:

$$d(R_{t+1}, R_t b^c_{t-1}, \tau^1, (1+n)\tau^1) = y_1 + \tau^1 + R_t b^c_{t-1} - z(R_{t+1}, R_t b^c_{t-1}, \tau^1, -(1+n)\tau^1);$$

$$e(R_{t+1}, R_t b^c_{t-1}, \tau^1, -(1+n)\tau^1) = y_2 - (1+n)\tau^1 + R_{t+1} z(R_{t+1}, R_t b^c_{t-1}, \tau^1, -(1+n)\tau^1)$$

One obtains $\frac{\partial F}{\partial \tau^1} > 0$ (=0) and $R_t b^c_{t-1}$ decreasing (having a null derivative) in τ^1 if $b^1_t = z(R_{t+1}, R_t b^0_{t-1}, \tau^1, -(1+n)\tau^1) > 0$ $(= 0)$.

1.7.4 Proof of Proposition 3

As discussed in the proof of Lemma 1, $R_t \leq MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1})$ implies $b^1_t = 0$ if $b^0_{t-1} \leq 0$. At the same time, the demanded level of b^0_t is $b^0_{t-1} < 0$ for $b^1_t = 0$ and $R_t \leq MRS_{cd}(y_0, y_1 + \tau^1_t)$, and the borrowing limit is null according to Lemma 1. Then the young and the adults are constrained to have $b^0_t = 0, b^1_t = 0$, $\forall t \geq 0$ under such interest rates, and the asset market clears since $g_t = 0$, $\forall t \geq 0$ by assumption. Autarky then satisfies all the properties of equilibrium from Definition 2.

1.7.5 Proof of Proposition 4

The following program maximizes a weighed sum of utilities of all the generations while constraining all agents within a generation to have the same consumption:

$$\max_{(c_t,d_t,e_t)_{t\geq 0}} \{\theta_{-1}N_{-1}\beta^2 u(e_0) + \theta_0 N_0 \left(\beta u(d_0) + \beta^2 u(e_1)\right) + \sum\nolimits_{t\geq 0} \theta_{t+1}N_{t+1}\left(u(c_t) + \beta u(d_{t+1}) + \beta^2 u(e_{t+2})\right)\}$$
$$\text{s. t.} \quad c_t + \frac{d_t}{1+n} + \frac{e_t}{(1+n)^2} = y_0 + \frac{y_1}{1+n} + \frac{y_2}{(1+n)^2}, \ \forall t \geq 0$$

where is θ_t is the weight of one agent of generation t. The solution does not exist only if the series under the objective function does not converge. According to the resource constraint, aggregate consumption in any period is finite, so the sum of agents' instantaneous utilities is finite, too. The objective function is then defined if, and only if, the sum of generations' weights converges. To obtain a simpler formulation of the same program, define $\omega_t \equiv \theta_{t-1}N_{t-1}$. The objective function can then be rewritten:

$$\omega_0\beta^2 u(e_0) + \omega_1\left(\beta u(d_0) + \beta^2 u(e_1)\right) + \sum\nolimits_{t\geq 0}\omega_{t+2}\left(u(c_t) + \beta u(d_{t+1}) + \beta^2 u(e_{t+2})\right)$$
$$= \sum\nolimits_{t\geq 0}(\omega_{t+2}u(c_t) + \omega_{t+1}\beta u(d_t) + \omega_t\beta^2 u(e_t))$$

This results in the sought program (1.23). If the sum of $(\omega_t)_{t\geq 0}$ converges, the solution exists. It is unique, since the objective function is strictly concave and the set of feasible allocations is convex.

1.7.6 Proof of Proposition 5

The FOC of (1.23) can be written as:

$$\begin{cases} \dfrac{u'(c_t)}{\beta u'(d_t)} = \dfrac{\omega_{t+1}}{\omega_{t+2}}(1+n) & (1.49) \\ \dfrac{\beta u'(d_t)}{\beta^2 u'(e_t)} = \dfrac{\omega_t}{\omega_{t+1}}(1+n) & (1.50) \\ c_t + \dfrac{d_t}{1+n} + \dfrac{e_t}{(1+n)^2} = y_0 + \dfrac{y_1}{1+n} + \dfrac{y_2}{(1+n)^2} & (1.51) \end{cases}$$

The first order conditions are static, so the solution is stationary if, and only if, the weights are such that (1.49), (1.50) give same relation of c_t, d_t and e_t every period. The latter condition means $\frac{\omega_{t+1}}{\omega_{t+2}}$ is constant and $\frac{\omega_t}{\omega_{t+1}}$ is constant, the two statements being equivalent. As the sum of $(\omega_t)_{t\geq 0}$ must converge, we get $\omega_{t+1} = \alpha\omega_t$ and $\alpha < 1$.

The utilitarian objective function with a constant discount factor α writes:

$$\mathcal{W}^\alpha = \sum\nolimits_{t\geq 0}\alpha^t(\alpha^2 u(c_t) + \alpha\beta u(d_t) + \beta^2 u(e_t))$$

Knowing that c_t, d_t, e_t are constant, one can solve an equivalent constrained problem with $c_t = c, d_t = d, e_t = e$. Substituting the constant consumption levels in the sum above, a geometric series is obtained:

$$\begin{aligned}\mathcal{W}^{\alpha} &= \sum\nolimits_{t\geq 0} \alpha^t(\alpha^2 u(c) + \alpha\beta u(d) + \beta^2 u(e)) \\ &= \frac{1}{1-\alpha}(\alpha^2 u(c) + \alpha\beta u(d) + \beta^2 u(e))\end{aligned}$$

Finally, (1.24) has $\mathcal{W}^S = (1-\alpha)\mathcal{W}^{\alpha}$, so they have the same maximum point.

1.7.7 Proof of Lemma 2

The first order conditions of (1.23) for variables of period t are:

$$\begin{cases} \alpha^{t+2}\omega_{t+2}u'(c_t) = \lambda_t & (1.52) \\ \alpha^{t+1}\omega_{t+1}\beta u'(d_t) = \dfrac{\lambda_t}{1+n} & (1.53) \\ \omega_t \beta^2 u'(e_t) = \dfrac{\lambda_t}{(1+n)^2} & (1.54) \\ c_t + \dfrac{d_t}{1+n} + \dfrac{e_t}{(1+n)^2} = y_0 + \dfrac{y_1}{1+n} + \dfrac{y_2}{(1+n)^2} & (1.55) \end{cases}$$

where λ_t is a Lagrange multiplier associated to a period t resource constraint. Dividing the terms of (1.53) by the terms of (1.54) written for period t+1, one gets $MRS(d_t, e_{t+1}) = \frac{\lambda_t(1+n)}{\lambda_{t+1}}$. At the same time, dividing the terms of (1.52) by the terms of (1.53) written for period $t+1$, one gets the equality of $MRS(c_t, d_{t+1})$ to the same expression.

1.7.8 Proof of Proposition 6

The FOC of agents' maximisation problem with full commitment gives

$$\forall t \geq 0, \quad M\hat{R}S_t = R_{t+1}$$

It is then sufficient to show that a given sequence of debt and transfers satisfies the budget constraints of all agents and the market asset clearing conditions with interest rates replaced by the MRS.

A general policy $(\tau_1^1, \tau_t^2, g_t)_{t\geq 0}$ decentralizing $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ must then solve the following system:

$$\begin{cases} \hat{c}_{t-1} + b_{t-1}^0 = y_0 & (1.56) \\ \hat{d}_t + b_t^1 = y_1 + \tau_t^1 + \hat{M}RS_{t-1}b_{t-1}^0 & (1.57) \\ \hat{e}_{t+1} = y_2 + \tau_{t+1}^2 + \hat{M}RS_t b_t^1 & (1.58) \\ (1+n)b_t^0 + b_t^1 = g_t & (1.59) \\ g_t = \dfrac{R_t}{1+n}g_{t-1} + \tau_t^1 + \dfrac{\tau_t^2}{1+n} & (1.60) \end{cases}$$

The two minimal sets of policy instruments $(\bar{\tau}^1_t, \bar{\tau}^2_t)_{t\geq 0}$ and $(\breve{g}_t, \breve{\tau}^2_t)_{t\geq 0}$ are particular solutions of the above system that have $g_t = 0$ and $\tau^1_t = 0$, correspondingly.

Balanced-budget policy : First, consider periods $t > 0$. To get the expression for $\bar{\tau}^1_t, t \geq 1$, substitute expressions of asset positions from (1.56),(1.57) in (1.59), then solve the latter for τ^1_t. To check that (1.58) also holds, substitute $-(1+n)\tau^1_{t+1}$ for τ^2_{t+1} and use the expression for $\bar{\tau}^1_t$ obtained above. This gives:

$$\hat{e}_{t+1} = y_2 - (1+n)(\hat{c}_t - y_0)\hat{M}RS_t - (1+n)\hat{d}_{t+1} + (1+n)y_1 + (1+n)^2(y_0 - \hat{c}_{t+1}) + \hat{M}RS_t b^1_t$$

while from (1.59) one has $b^1_t = -(1+n)b^0_t = -(1+n)(y_0 - \hat{c}_t)$, so:

$$\begin{aligned}\hat{e}_{t+1} &= y_2 - (1+n)(\hat{c}_t - y_0)\hat{M}RS_t - (1+n)\hat{d}_{t+1} + (1+n)y_1 + (1+n)^2(y_0 - \hat{c}_{t+1}) + \hat{M}RS_t b^1_t \\ &\Leftrightarrow \hat{e}_{t+1} = y_2 + (1+n)(y_1 - \hat{d}_{t+1}) + (1+n)^2(y_0 - \hat{c}_{t+1}) \\ &\Leftrightarrow \hat{c}_{t+1} + \frac{\hat{d}_{t+1}}{1+n} + \frac{\hat{e}_{t+1}}{(1+n)^2} = y_0 + \frac{y_1}{1+n} + \frac{y_2}{(1+n)^2}\end{aligned}$$

The last equation is the resource constraint for period $t+1$, always verified for an optimal allocation.

The initial transfers $\bar{\tau}^1_0, \bar{\tau}^2_0$ are obtained in analogous manner, using R_0 instead of $\hat{M}RS_{-1}$ and $R_0, b^0_{-1}, g_{-1}, b^1_{-1}$ being exogenous.

Debt-based policy: Denote the asset positions of private agents $(\breve{b}^0_t, \breve{b}^1_t)_{t\geq 0}$ under debt-based policy and $(\bar{b}^0_t, \bar{b}^1_t)_{t\geq 0}$ under the balanced-budget one. Since transfers to the young are absent in both cases, we have $\bar{b}^0_t = \breve{b}^0_t$. Moreover, since transfer to adults are absent in the debt-based policy, one obtains $\breve{b}^1_t = \bar{b}^1_t - \bar{\tau}^1_t$. Asset market equilibrium then leads to $\breve{g}_t = -\bar{\tau}^1_t$. Indeed,

$$\begin{cases}(1+n)\breve{b}^0_t + \breve{b}^1_t & = \breve{g}_t \\ (1+n)\bar{b}^0_t + \bar{b}^1_t & = 0\end{cases} \Leftrightarrow \begin{cases}(1+n)\bar{b}^0_t + \bar{b}^1_t - \bar{\tau}^1_t & = \breve{g}_t \\ (1+n)\bar{b}^0_t + \bar{b}^1_t & = 0\end{cases}$$

so $\breve{g}_t = -\bar{\tau}^1_t$. The last equation that should be verified is (1.58). Using $\breve{\tau}^2_t = (1+n)\breve{g}_t - \hat{M}RS_{t-1}\breve{g}_{t-1}$ and $\breve{g}_t = -\bar{\tau}^1_t$ in (1.58), one gets:

$$\begin{aligned}\hat{e}_{t+1} &= y_2 + \hat{M}RS_t\breve{b}^1_t + \breve{\tau}^2_{t+1} = y_2 + \hat{M}RS_t(\bar{b}^1_t - \bar{\tau}^1_t) - (1+n)\bar{\tau}^1_{t+1} + \hat{M}RS_t\bar{\tau}^1_t \\ &= y_2 + \hat{M}RS_t\bar{b}^1_t - (1+n)\bar{\tau}^1_{t+1} = y_2 + \hat{M}RS_t\bar{b}^1_t + \bar{\tau}^2_{t+1},\end{aligned}$$

where the last expression for $\hat{e}_{t+1}$ is true since (1.58) is respected under the balanced-budget policy.

1.7.9 Proof of Proposition 9

First, $b^1_{-1} \geq 0$, $b^0_{-1} \leq 0$ and $\hat{c}_t > y_0$, $\forall t \geq 0$ implies that (IR2) holds for $t \geq 0$. Indeed, $b^1_t = -(1+n)b^0_t + g_t > 0$, so (IR2) holds since it is equivalent to $b^1_t > 0$. The following lemma is used for the analysis of the constraint (IR1).

Lemma 5. *If an optimal allocation* $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ *with* $\hat{c}_t > y_0$, $\forall t \geq 0$ *is decentralized under limited commitment with* $g_t \geq 0$*, the corresponding equilibrium has* $b^0_{t-1} \geq f(R_{t+1}, \tau^1_t, \tau^2_{t+2})/R_t$ $\forall t > 0$*, so the young agents are not constrained in equilibrium.*

Proof. Rewrite the utility maximization program with limited commitment (1.17) as:

$$\max_{(c_{t-1}, d_t, e_{t+1})>0} \quad u(c_{t-1}) + \beta u(d_t) + \beta^2 u(e_{t+1})$$
$$\text{s.t.} \begin{cases} R_t c_{t-1} + d_t + \frac{e_{t+1}}{R_{t+1}} & = R_t y_0 + y_1 + \tau^1_t + \frac{y_2 + \tau^2_{t+1}}{R_{t+1}} \\ y_0 - c_{t-1} & \geq f(R_{t+1}, \tau^1_t, \tau^2_{t+1})/R_t \\ y_1 - d_t + R_t(y_0 - c_{t-1}) + \tau^1_t & \geq 0 \end{cases} \tag{1.61}$$

This form is obtained by eliminating b^0_{t-1}, b^1_t from all the constraints of (1.17). The last constraint can be omitted for the current context as it is proved above that the constraint is not binding in the decentralized equilibria in question. Denote λ_t the Lagrange multiplier associated with the first constraint and χ_t with the second. Then, the FOC with respect to c_{t-1}, d_t are $u'(c_{t-1}) = (\lambda_t + \chi_t)R_t$ and $\beta u'(d_t) = \lambda_t$. Dividing the FOC by each other, one gets

$$MRS_{cd}(c_{t-1}, d_t) = (1 + \chi_t/\lambda_t)R_t$$

Solving the same problem for an agent of the generation $t-1$, one gets FOC with respect to d_{t-1} and e_t are $\beta u'(d_{t-1}) = \lambda_{t-1}$ and $\beta^2 u'(e_t) = \lambda_{t-1}/R_t$. Dividing the FOC by each other, one gets

$$MRS_{de}(d_{t-1}, e_t) = R_t$$

From Lemma 2,$MRS_{cd}(\hat{c}_{t-1}, \hat{d}_t) = MRS_{de}(\hat{d}_{t-1}, \hat{e}_t)$ for $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$. This allocation is a solution to the FOC when it is decentralized, so one obtains

$$MRS_{cd}(\hat{c}_{t-1}, \hat{d}_t) = (1 + \chi_t/\lambda_t)MRS_{de}(\hat{d}_{t-1}, \hat{e}_t) = (1 + \chi_t/\lambda_t)MRS_{cd}(\hat{c}_{t-1}, \hat{d}_t),$$

so $\chi_t = 0$ and the second constraint is not binding. This is true for any generation $t \geq 1$. □

The main proposition is an "⇔" statement, where the part "⇐" follows from the equivalence of borrowing constraints and individual rationality constraints. Indeed, if a minimal policy decentralizes an allocation under full commitment and both (IR1) and (IR2) hold in $t \geq 0$, then all the conditions for decentralization under limited commitment are verified. To prove the part "⇒", first note that Lemma 5 implies the allocation is decentralized under limited commitment with borrowing constraints not binding for the young. (IR1) is then satisfied in each period for the allocation and the minimal policy in question. A solution to (1.17) with slack constraints is also a solution to (1.4), so $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ fulfills all the conditions for a full commitment equilibrium with the minimal policy in question.

Equivalently, this policy decentralizes the allocation with full commitment.

1.7.10 Proof of Proposition 10

As discussed in Section 1.5.3, the current proposition is trivially verified for allocations with $\hat{c} < y_0$: balanced-budget policies never decentralize such allocations with limited commitment. The remaining proof therefore assumes $\hat{c} > y_0$.

Let a balanced-budget policy decentralize the allocation under limited commitment. Then, given the Proposition 9, the current proposition is equivalent to (IR1), being more strict under under the balanced-budget policy than under the debt-based one. For a given optimal allocation, the left hand side of (IR1) is the same for any policy, so the proof can be concluded by comparing the right hand sides of (IR1) for different policies. I first do it for periods $t > 0$ and then come back to $t = 0$. The constraint (IR1) in periods $t > 0$ under the two policies is:

$$u(\hat{d}) + \beta u(\hat{e}) \geq u(y_1 + \bar{\tau}^1) + \beta u(y_2 + \bar{\tau}^2) \equiv \bar{L} \quad \text{for balanced-budget}$$
$$u(\hat{d}) + \beta u(\hat{e}) \geq u(y_1) + \beta u(y_2 + \check{\tau}^2) \equiv \check{L} \quad \text{for debt-based}$$

where the label $\bar{L}$ is used for for lower limit on adults' utility under the first policy and the label $\check{L}$ is used for the same limit under the second policy. The current proposition then holds if $\bar{L} > \check{L}$. The values of transfers $\bar{\tau}^1, \bar{\tau}^2, \check{\tau}^2$ are given by Proposition 6 since they decentralize the allocation under full commitment, according to Proposition 9. One gets:

$$\bar{\tau}^1 = \hat{d} - y_1 + (\hat{MRS} + 1 + n)(\hat{c} - y_0) < 0 \text{ (by } \hat{MRS} > R^u\text{, Proposition 7)}$$
$$\bar{\tau}^2 = -(1+n)\bar{\tau}^1 > 0$$
$$\check{\tau}^2 = (\hat{MRS} - 1 - n)\bar{\tau}^1 < 0 \text{ (since } \hat{MRS} > 1 + n \text{ by Lemma 3)}$$

To show that $\bar{L} > \check{L}$, define a function L as follows:

$$L : \left]\frac{-y_2 - \bar{\tau}^2}{\hat{MRS}}, y_1 - \frac{\bar{\tau}^2}{1+n}\right[\to \mathbb{R}$$
$$L(x) = u\left(y_1 - \frac{\bar{\tau}^2}{1+n} - x\right) + \beta u\left(y_2 + \bar{\tau}^2 + \hat{MRS} \cdot x\right) \tag{1.62}$$

As u is twice continuously differentiable, so is L. The function describes the utility from adult and old age consumption of a fictional adult agent that has $b^0 = 0$ and faces transfers $(-\bar{\tau}^2/(1+n), \bar{\tau}^2)$ and an interest rate $\hat{MRS}$. This agent chooses consumption on a budget line

$$\mathcal{B}^f = \{(d, e) \in \mathbb{R}^2_{++} : d + e/\hat{MRS} = y_1 - \bar{\tau}^2/(1+n) + (y_2 + \bar{\tau}^2)/\hat{MRS}\}$$

by doing savings of size x. The superscript f in $\mathcal{B}^f$ stands for ““fictional”. The rest of the proof uses a utility maximisation argument for the fictional agent to show that $\bar{L}$ is a utility level that is higher than $\check{L}$. First, according to (1.62), $\bar{L} = L(0)$ and $\check{L} = L(-\bar{\tau}^2/(1+n))$, since $\check{\tau}^2 = (1 - \frac{M\hat{R}S}{1+n})\bar{\tau}^2$. As $-\bar{\tau}^2/(1+n) < 0$ the proof for $t > 0$ can be concluded by showing that $L(\cdot)$ is increasing on $]-\bar{\tau}^2/(1+n), 0[$.

Since $(\bar{\tau}^1, \bar{\tau}^2)$ decentralize $(\hat{c}, \hat{d}, \hat{e})$ under full commitment in $t > 0$, the consumption levels $(\hat{d}, \hat{e})$ maximize the utility of an adult that has $Rb^0 = M\hat{R}S \cdot (y_0 - \hat{c}) < 0$, transfers $(\bar{\tau}^1, \bar{\tau}^2)$ and interest rate $M\hat{R}S$. I will call this adult ""real" to distinguish from the fictional one mentioned above. The budget line of the real agent is

$$\mathcal{B}^r = \{(d, e) \in \mathbb{R}^2_{++} : d + e/M\hat{R}S = y_1 + M\hat{R}S \cdot (y_0 - \hat{c}) - \bar{\tau}^2/(1+n) + (y_2 + \bar{\tau}^2)/M\hat{R}S\}$$

Since $(\hat{c}, \hat{d}, \hat{e})$ can be decentralized by a balanced-budget policy under limited commitment, it follows that the real adult has non-negative optimal savings in equilibrium. On the other hand, the only difference between the budget lines $\mathcal{B}^f$ and $\mathcal{B}^r$ is initial assets: 0 in the first case and $(y_0 - \hat{c}) < 0$ in the second. By normality of old-age consumption, we get that the fictional agent maximizes utility with strictly positive savings. It follows that $\arg\max L > 0$. At the same time, L is concave by concavity of u: $L''(x) = u''(y_1 - \bar{\tau}^2/(1+n) - x) + \beta M\hat{R}S^2 u''(y_2 + \bar{\tau}^2 + M\hat{R}S \cdot x) < 0$. We get that $L'(x) = 0$ in only one point, namely $x = \arg\max L$, and $L'(x) > 0$ for all $x < \arg\max L$. It follows that L is indeed increasing on $]-\bar{\tau}^2/(1+n), 0[$ as $\arg\max L > 0$.

In the initial period $t = 0$, generation -1 is old and generation 0 is adult. Although (IR2) is verified for generations $t \geq 1$ by Proposition 9, it is not always the case for the two initial generations. The constraint (IR2) on generation -1 depends only on the initial condition and is verified under any policy iff $b^1_{-1} > 0$. For generation 0, (IR2) is more strict under the balanced-budget than under the debt-based policy, since $\bar{\tau}^2_1 = \bar{\tau}^2 > \check{\tau}^2_1 = \check{\tau}^2$.

The constraint (IR1) for generation 0 is:

$$\begin{aligned} u(\hat{d}) + \beta u(\hat{e}) &\geq u(y_1 + \bar{\tau}^1_0) + \beta u(y_2 + \bar{\tau}^2) \equiv \bar{L}_0 && \text{for balanced-budget} \\ u(\hat{d}) + \beta u(\hat{e}) &\geq u(y_1) + \beta u(y_2 + \check{\tau}^2) \equiv \check{L}_0 && \text{for debt-based} \end{aligned}$$

which uses the fact that transfers are at their stationary values beginning with $t = 2$. The initial transfer for the balanced-budget policy is $\bar{\tau}^1_0 = \hat{d} - y_1 - R_0 b^0_{-1} + (1+n)(\hat{c} - y_0)$. One can write $\bar{\tau}^1_0 = \bar{\tau}^1 - R_0 b^0_{-1} + M\hat{R}S \cdot (y_0 - \hat{c})$. Then, $\bar{L}_0 \geq \bar{L} \Leftrightarrow R_0 b^0_{-1} \leq M\hat{R}S \cdot (y_0 - \hat{c})$. This is assumed for this Proposition, so one obtains $\bar{L}_0 \geq \bar{L} > \check{L} = \check{L}_0$. This means the constraint (IR1) on the generation 0 is more strict under the balanced-budget policy than under the debt-based one. This completes the proof.

Chapter 2

Endogenous Debt Constraints and Rational Bubbles in an OLG Growth Model (with Bertrand Wigniolle)

2.1 Introduction

General equilibrium models with imperfect capital markets have received a renewed interest, following the financial crisis of 2007. In particular, the recent literature stresses the implications of imperfect markets for financial bubbles: market imperfections may play a role with respect to the existence conditions for bubbles, on their impact on the real economy, or on their crowding out or liquidity effect.

Our article is a contribution to this literature. We consider a general equilibrium model with imperfect capital markets, more precisely with a borrowing constraint that is defined endogenously, following Kehoe and Levine (1993), Kehoe and Levine (2001), Kocherlakota (1996), Alvarez and Jermann (2000b), and Azariadis and Lambertini (2003). This type of constraint naturally exists if borrowers cannot credibly commit to repay their loans. In case of default, defaulters are excluded from the credit market. Lenders impose a limit to the size of loans in such a way that agents have no incentive to default at equilibrium. Azariadis and Lambertini (2003) have studied this type of constraint in an OLG model with exogenous endowments. They show that such a framework can generate a multiplicity of stationary states and indeterminacy of equilibrium.

This article makes two contributions to the literature. Firstly, we introduce the above-mentioned friction in a simple OLG model *à la* Diamond (1965) with three periods of life; in other words we make agents' endowments in Azariadis and Lambertini (2003) endogenous by introducing a production sector that uses capital and labor as inputs. Agents have a sequence of incomes with a hump-shaped profile: they borrow when young and save when middle aged. Using simple functional forms, the model allows for a complete characterization of the global dynamics with regime changes. We adopt a general definition of the intertemporal equilibrium that allows to take into account possible regime switches from an unconstrained to a constrained equilibrium or vice versa. A constrained equilibrium corresponds to the case where households cannot borrow when young the optimal amount due to the endogenous borrowing constraint. The introduction of endogenous endowments tends to stabilize the dynamics with respect to Azariadis and Lambertini's results: we prove that the intertemporal equilibrium always exists and is determined. Moreover, there is no regime changes along the dynamics and the economy remains at all periods with either constrained or unconstrained borrowers.

How to understand the difference in our results with respect to Azariadis and Lambertini (2003)? In Azariadis and Lambertini (2003), when high expected interest rates make saving more desirable, opportunity cost of default rises and endogenous borrowing constraints become less tight. Consumption in youth and retirement becomes complements, and this allows for the existence of multiple equilibria and indeterminacy. In our framework with endogenous endowments, an increase in savings tends to decrease the interest rate. The complementarity between consumption in youth and retirement is lost and there exists only one equilibrium without regime switches.

The second contribution of this article is related to financial bubbles. If we assume that defaulters who are excluded from the formal market for loans have access to another market for savings that is based on a bubbly asset, a new type of equilibrium is possible with equilibrium endogenous default. Lenders react to the risk of default by increasing the interest rate on loans to compensate the loss on defaulters, instead of rationing the quantity that is borrowed. They cannot select between defaulters and non defaulters that are identical ex-ante and that borrow the same amount when young. The fraction of defaulters and the price of the bubbly asset are determined at equilibrium in such a way that agents are indifferent between making default or not. In this framework, a bubbly equilibrium with interesting features may exist. There exist three interest rates in such an equilibrium: the interest rate on loans, which is higher that the interest rate for lenders, which exceeds the one on the bubbly asset. The return on the bubbly asset is equal to the growth rate of the economy as in Tirole (1985). This implies that the rental rate of capital is higher than the growth rate: bubbles existence is associated with underaccumulation.

The intertemporal equilibrium can be reduced to a dynamic system of order 1 associated with two forward looking variables. Three types of stationary equilibria may exist in the economy with a bubbly asset: the bubbly steady state where the bubble has a positive value and is held by a positive fraction of agents and two steady states for which the aggregate value of the bubbly asset is zero. These two types of bubbleless equilibria are obtained either because the bubble is held by a null fraction of agents, or because the value of the asset is zero. We show that at most two of these three steady states may exist together. When the bubbly steady state exists, it is unstable (a source) so the equilibrium is locally determinate since all the variables are forward-looking. When one of the two bubbleless steady state exists together with the bubbly steady state, it is a saddle point, which means local indeterminacy since all the variables are forward-looking . When a bubbleless steady state is the only stationary equilibrium, it is unstable (a source), meaning it is locally determinate.

The introduction of an endogenous borrowing constraint in a standard OLG model with capital accumulation seems a relevant way to represent the possibility of personal bankruptcy in modern economies. Personal bankruptcy is gaining importance as a legal institution for market-based economies. As in Kehoe and Levine (1993), we model personal bankruptcy as a stripped-down version of the "fresh start " procedure described by Chapter 7 of the Bankruptcy Code in the USA. Under this type of bankruptcy, almost all of unsecured debt is discharged, and future wage earnings are protected from debt collection.

Another real world element of the bankruptcy-repayment trade-off motivates the bubble part of our model. There are more and more bubble-like assets that a bankrupt person can hide from creditors, such as cryptocurrencies. Still largely unregulated, these assets have attracted considerable investment over the last decade. The intrinsic value of cryptocurrencies is out of scope of this paper, but as long as they can be accumulated for resale purposes, they alter the incentives for personal bankruptcy.

This paper is related to different strands of macroeconomic literature. Firstly, we contribute to the growth and finance literature by incorporating the limited commitment assumption in a tractable growth model. Using a very similar overlapping generations framework, Jappelli and Pagano (1994b) study the impact of exogenous constraints on consumption debt on growth and welfare. The borrowing limits are modelled as a fixed share of the present value of future income. They find that tightening of the borrowing limit unambiguously reduces steady-state capital level and even the balanced growth rate in the endogenous growth version of the model. We depart from this comparative statics approach and study joint dynamics of borrowing constraints and capital accumulation. For equilibria without bubbles, we find that debt limit can be expressed as a constant share of present value income in equilibrium, but this share is determined by the primitives of the model.

Our work is also related to all the literature that has associated financial frictions and rational bubbles. As in Martin and Ventura (2012) and Farhi and Tirole (2012), imperfect capital markets allow to disconnect the rate of return on capital from the rate of return on the bubbly asset. It is possible to obtain a bubbly equilibrium associated with a rate of return of capital that is higher than the growth rate. The seminal paper of Tirole (1985) lays out the basic tension between growth and bubbles: investment in a rational bubble crowds out capital accumulation, but this can be efficient in an OLG environment. In the present paper, crowding out is also at play. However, bubbles are less likely to be Pareto improving as they make bankruptcy more attractive and, as a result, distort the credit market. Recent research has focused mostly on firms' liquidity constraints that can be alleviated by issuing or holding rational bubbles: Martin and Ventura (2012), Farhi and Tirole (2012), Miao and Wang (2018), Raurich and Seegmuller (2019). In our setting, bubbles also have some productive effect as they make constraints on consumption debt more tight. Indeed, the opportunity to invest in bubbles after bankruptcy makes limited commitment problem worse, and the loan supply shrinks. Savings are channelled away from the "unproductive" consumption credit market towards the physical capital market, as in Jappelli and Pagano (1994b). However, in our setting this effect does not compensate the basic crowding out effect of Tirole (1985), so bubbly steady states always have less capital that the bubbleless ones.

Section 2.2 analyses the bubbleless model. Section 2.3 considers the bubbly economy. Section 2.4 concludes. The Appendix gathers different proofs.

2.2 Endogenous constraint and global dynamics

This section focuses on an OLG model with production and endogenous borrowing constraints. In the first part, we present the model. In the second part, we characterize the global dynamics with possible regime switching and some properties of the equilibrium. Jappelli and Pagano (1994b).

2.2.1 The model

Agents and endogenous borrowing constraint

We consider an overlapping generations model in which agents are alive during three periods. We label the three generations the young, adults and the old. Agents are homogenous within a generation and we define generation t as the N_t agents that are adults in period t. A generation t agent has no income in $t-1$ during youth and must borrow to consume c_{t-1}. When adult, she inelastically supplies one unit of labor. When old, she works during a fraction h of the period and is retired during the remaining time $1-h$. d_t is her consumption level in t (when adult) and e_{t+1} in $t+1$ (when old). w_t is her income in period t and hw_{t+1} the $t+1$ income, with w_t the real wage for one unit of labor.

Each agent is endowed with an inter-temporal utility function:

$$U\left(c_{t-1}, d_t, e_{t+1}\right) = \gamma_1 \ln c_{t-1} + \gamma_2 \ln d_t + \gamma_3 \ln e_{t+1}$$

with $\gamma_1 + \gamma_2 + \gamma_3 = 1$.

Agents have access to an imperfect capital market on which borrowing may be constrained. In $t-1$, c_{t-1} must be borrowed.

The budget constraints in t and $t+1$ are:

$$\begin{aligned} d_t + s_t &= w_t - R_t c_{t-1} \\ e_{t+1} &= R_{t+1} s_t + hw_{t+1} \end{aligned}$$

R_t is the interest factor between periods $t-1$ and t, s_t the amount of savings of the agent when adult.

Agents are subject to a borrowing constraint in their youth that limits the level of their first period consumption: $c_{t-1} \leq \bar{c}_{t-1}$, with $\bar{c}_{t-1}$ the borrowing limit. Therefore, the program of an agent is defined as:

$$\begin{cases} \max\limits_{(c_{t-1},\, d_t,\, e_{t+1},\, s_t)} \gamma_1 \ln c_{t-1} + \gamma_2 \ln d_t + \gamma_3 \ln e_{t+1} \\ \qquad\qquad \text{s.t.} \quad d_t + s_t = w_t - R_t c_{t-1} \\ \qquad\qquad\qquad\quad e_{t+1} = R_{t+1} s_t + h w_{t+1} \\ \qquad\qquad\qquad\quad c_{t-1} \leq \bar{c}_{t-1} \end{cases} \tag{P}$$

The borrowing limit $\bar{c}_{t-1}$ is endogenous and defined as in Azariadis and Lambertini (2003). It is the value that makes an agent indifferent between reimbursing her first period debt and making default. In the case of default, the agent is excluded from the capital and lending markets and cannot save between her adult and old ages.

Let us define V_t^D the indirect utility from period t of a defaulter, and V_t^{ND} the indirect utility from period t of a non defaulter. By definition, V_t^D is a function of the incomes at periods t and $t+1$ of the agent: $V_t^D(w_t,\, hw_{t+1})$. V_t^{ND} also depends on the interest factor between t and $t+1$ and on the debt $R_t c_{t-1}$ that is repaid in t for the first period consumption: $V_t^{ND}(w_t,\, hw_{t+1}, R_{t+1}, R_t c_{t-1})$.

By definition of the endogenous borrowing constraint, $\bar{c}_{t-1}$ is defined as:

$$V_t^D(w_t, hw_{t+1}) = V_t^{ND}(w_t, hw_{t+1}, R_{t+1}, R_t \bar{c}_{t-1})$$

Lemma 6. *The borrowing limit is*

$$\bar{c}_{t-1} = \frac{1}{R_t}\left[w_t + \frac{hw_{t+1}}{R_{t+1}} - \frac{w_t^{1-\eta}}{\lambda}\left(\frac{hw_{t+1}}{R_{t+1}}\right)^{\eta}\right]$$

with $\eta \equiv \gamma_3/(\gamma_2+\gamma_3)$ *and* $\lambda \equiv \eta^{\eta}(1-\eta)^{1-\eta}$.

Proof. See Section 2.5.1 in the Appendix. □

Following this result, the first period consumption of agents is constrained:

$$R_t c_{t-1} \leq w_t + \frac{hw_{t+1}}{R_{t+1}} - \frac{w_t^{1-\eta}}{\lambda}\left(\frac{hw_{t+1}}{R_{t+1}}\right)^{\eta} \tag{2.1}$$

The program (P) may have two types of results, depending on the borrowing constraint that may be binding or not. When the constraint binds in $t-1$, the agent is said to be constrained. When the constraint does not bind in $t-1$, the agent is unconstrained.

Proposition 11. *If*

$$\frac{w_t^{1-\eta}}{\lambda}\left(\frac{hw_{t+1}}{R_{t+1}}\right)^{\eta} \leq (1-\gamma_1)\left(w_t + \frac{hw_{t+1}}{R_{t+1}}\right) \tag{2.2}$$

the agent is unconstrained in $t-1$, and her intertemporal choices are given by

$$R_t c_{t-1} = \gamma_1\left(w_t + \frac{hw_{t+1}}{R_{t+1}}\right) \tag{2.3}$$

$$d_t = \gamma_2\left(w_t + \frac{hw_{t+1}}{R_{t+1}}\right) \tag{2.4}$$

$$e_{t+1} = \gamma_3 R_{t+1}\left(w_t + \frac{hw_{t+1}}{R_{t+1}}\right) \tag{2.5}$$

$$s_t = \gamma_3 w_t - (\gamma_1+\gamma_2)\frac{hw_{t+1}}{R_{t+1}} \tag{2.6}$$

If

$$\frac{w_t^{1-\eta}}{\lambda}\left(\frac{hw_{t+1}}{R_{t+1}}\right)^{\eta} > (1-\gamma_1)\left(w_t + \frac{hw_{t+1}}{R_{t+1}}\right) \tag{2.7}$$

the agent is constrained in $t-1$, and her intertemporal choices are given by

$$R_t c_{t-1} = w_t + \frac{hw_{t+1}}{R_{t+1}} - \frac{w_t^{1-\eta}}{\lambda}\left(\frac{hw_{t+1}}{R_{t+1}}\right)^{\eta} \tag{2.8}$$

$$d_t = (1-\eta)\frac{w_t^{1-\eta}}{\lambda}\left(\frac{hw_{t+1}}{R_{t+1}}\right)^{\eta} \tag{2.9}$$

$$e_{t+1} = R_{t+1}\eta\frac{w_t^{1-\eta}}{\lambda}\left(\frac{hw_{t+1}}{R_{t+1}}\right)^{\eta} \tag{2.10}$$

$$s_t = \eta\frac{w_t^{1-\eta}}{\lambda}\left(\frac{hw_{t+1}}{R_{t+1}}\right)^{\eta} - \frac{hw_{t+1}}{R_{t+1}} \tag{2.11}$$

Proof. See appendix. □

Production

The economy is endowed with a Cobb-Douglas production function:

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}$$

Capital is held by old agents and rented by firms. It depreciates fully in one period. w_t is the cost of labor and R_t the capital cost. The firm has a competitive behavior and maximizes its profit:

$$K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - R_t K_t$$

This leads to the standard conditions:

$$w_t = (1-\alpha) K_t^{\alpha} L_t^{-\alpha}$$
$$R_t = \alpha K_t^{\alpha-1} L_t^{1-\alpha}$$

Intertemporal Equilibrium

Considering the labor market, total labor supply in t is the sum of the contributions of adults and old agents and is equal to labor demand: $L_t = N_t + hN_{t-1}$.

From this condition, the equilibrium prices are obtained through:

$$w_t = (1-\alpha)k_t^{\alpha} \tag{2.12}$$
$$R_t = \alpha k_t^{\alpha-1} \tag{2.13}$$
$$\text{with } k_t = K_t/(N_t + hN_{t-1}) \tag{2.14}$$

Considering the capital market equilibrium, savings of adult agents must finance the capital stock for the next period plus the amount of loans that finance young agents consumption:

$$K_{t+1} = N_t s_t - N_{t+1} c_t$$

or, dividing by $N_{t+1} + hN_t$

$$k_{t+1} = \frac{s_t}{1+n+h} - \frac{1+n}{1+n+h} c_t \tag{2.15}$$

The particular case of initial period 0 must be treated separately.

In period 0, generations $-1, 0$ and 1 are alive. The capital stock K_0 and the consumption level of generation 0 in period -1, c_{-1}, are given. The value of K_0 determines k_0 : $k_0 = K_0/(N_0 + hN_{-1})$. $R_0 c_{-1}$ is the value of debt that must be repaid in period 0, and its amount is given exogenously in the beginning of period 0.

It is assumed that generation 0 has no interest to default on its debt and repays $R_0 c_{-1}$.[1] Otherwise generation 0 would make no saving and there would be no production nor consumption from period 1 onwards. From (2.1), this assumption is true if:

$$R_0 c_{-1} \leq w_0 + \frac{hw_1}{R_1} - \frac{w_0^{1-\eta}}{\lambda}\left(\frac{hw_1}{R_1}\right)^{\eta} \tag{2.16}$$

[1] This hypothesis is consistent with the assumption of a borrowing constraint that is taken from period 0 : at each period, it is impossible to borrow such an amount that makes default advantageous.

The choices of consumption and saving for a generation 0 agent are:

$$d_0 = (1-\eta)\left(w_0 + \frac{hw_1}{R_1} - R_0 c_{-1}\right) \tag{2.17}$$

$$e_1 = \eta R_1 \left(w_0 + \frac{hw_1}{R_1} - R_0 c_{-1}\right) \tag{2.18}$$

$$s_0 = \eta\left(w_0 - R_0 c_{-1}\right) - (1-\eta)\frac{hw_1}{R_1} \tag{2.19}$$

The value of saving in -1 must be consistent with the given values of K_0 and c_{-1}, which leads to:

$$K_0 = N_0 s_{-1} - N_0 c_{-1}$$

or

$$k_0 = \frac{s_{-1}}{1+n+h} - \frac{1+n}{1+n+h}c_{-1} \tag{2.20}$$

Finally, for generation -1, consumption is determined by their budget constraint:

$$e_0 = R_0 s_{-1} + hw_0 \tag{2.21}$$

The capital market equilibrium in period 0 is also dependent on the initial conditions:

$$k_1 = \frac{s_0}{1+n+h} - \frac{1+n}{1+n+h}c_0 \tag{2.22}$$

Indeed, s_0 is given by (2.19) and depends on the value of c_{-1} inherited from the past.

Equation (2.15) shows that the dynamics of k_t will be driven by the expressions of s_t and c_t. The value of s_t depends on the state of generation t in $t-1$, constrained or unconstrained. The value of c_t depends on the state of generation $t+1$ in t, constrained or unconstrained. Therefore, the dynamics of the economy may take 4 different expressions depending on the state that are experienced in $t-1$ and t.

It is useful to introduce $\Sigma = \{U, C\}$ as the set of the two possible states at each period t : unconstrained or constrained. The state in period t is denoted by $S_t \in \Sigma$. From the previous analysis, if (2.2) is satisfied, $S_{t-1} = U$, if (2.7) is satisfied, $S_{t-1} = C$.

To simplify, we also adopt a new notation for variables: $c(t-1, S_{t-1})$, $d(t, S_{t-1})$, $e(t+1, S_{t-1})$ and $s(t, S_{t-1})$ are respectively the expressions for a generation t agent of her consumptions when young, adult, old and her saving given by (2.3), (2.4), (2.5), and (2.6) if $S_{t-1} = U$, and by (2.8), (2.9), (2.10), and (2.11) if $S_{t-1} = C$.

Definition 3. *An intertemporal equilibrium is a sequence of non negative variables* $(c_t,\ d_t,\ e_t,\ s_t,\ k_{t+1})_{t\geq 0}$ *and a sequence of states* $(S_t)_{t\geq 0}$ *with* $S_t \in \Sigma$*, such that:*

- $\forall t \geq 1$*, if (2.2) is satisfied,* $S_{t-1} = U$*, if (2.7) is satisfied,* $S_{t-1} = C$*;*
- $\forall t \geq 1$*, agents make optimal choices* $c_{t-1} = c(t-1, S_{t-1})$*,* $d_t = d(t, S_{t-1})$*,* $e_{t+1} = e(t+1, S_{t-1})$*,* $s_t = s(t, S_{t-1})$*;*

- $\forall t \geq 0$, *the optimal behavior of firms at equilibrium determine the wage and gross interest rates by (2.12), (2.13).*

- $\forall t \geq 1$, *there is an equilibrium on the capital market (2.15).*

- *For* $t = 0$, $k_0 = K_0/(N_0 + hN_{-1})$, c_{-1} *and* s_{-1} *are exogenously given such that (2.22) and (2.16) hold, and* d_0, e_1, s_0, e_0 *are respectively given by (2.17), (2.18), (2.19), and (2.21).*

Characterization of the dynamics of the economy

From this definition, it is possible to express the dynamics of the economy with respect to one variable, k_t. Using equation (2.15), we know that this dynamics may take 4 different expressions depending on the states that are experienced in $t-1$ and t.

An example is given below. Assume that the economy is unconstrained both in t and $t-1$. Then, c_t and s_t are given respectively by (2.3) and (2.6). Replacing in (2.15), it is obtained:

$$(1+n+h)k_{t+1} = \gamma_3 w_t - (\gamma_1+\gamma_2)\frac{hw_{t+1}}{R_{t+1}} - (1+n)\gamma_1\left(\frac{w_{t+1}}{R_{t+1}} + \frac{hw_{t+2}}{R_{t+1}R_{t+2}}\right)$$

From (2.12), (2.13), w_t and R_t only depends on k_t. We get:

$$\begin{aligned}(1+n+h)k_{t+1} = {}& \gamma_3(1-\alpha)k_t^\alpha - (\gamma_1+\gamma_2)\frac{h(1-\alpha)k_{t+1}}{\alpha} \\ & - (1+n)\gamma_1\left(\frac{(1-\alpha)k_{t+1}}{\alpha} + \frac{h(1-\alpha)k_{t+2}}{\alpha^2 k_{t+1}^{\alpha-1}}\right)\end{aligned}$$

Finally, we simplify by introducing a new variable:

$$x_t \equiv k_{t+1}/k_t^\alpha$$

Dividing the previous expression by k_{t+1}, we get first order dynamics in x_t:

$$\begin{aligned}(1+n+h) = {}& \frac{\gamma_3(1-\alpha)}{x_t} - (\gamma_1+\gamma_2)h\frac{1-\alpha}{\alpha} \\ & - (1+n)\gamma_1\frac{1-\alpha}{\alpha} - (1+n)\gamma_1\frac{1-\alpha}{\alpha}\frac{h}{\alpha}x_{t+1}\end{aligned}$$

The same calculations are done for the three other possible cases, using the homogeneity properties of the model.

The condition to be constrained in $t-1$, (2.7), can also be written using the variable x_t:

$$\left(\frac{h}{\alpha}\right)^\eta \frac{x_t^\eta}{\lambda} - (1-\gamma_1)\left(1+\frac{h}{\alpha}x_t\right) > 0 \tag{2.23}$$

This inequality determines the regime of the economy in $t-1$ with respect to x_t. It is possible to obtain a more precise condition by using some non-negativity constraints. From equation (2.15), s_t must be non negative as c_t and k_{t+1} are non negative. Then, the expressions of s_t in both regimes (2.6) and (2.11) impose two additional constraints on x_t. Finally, we get the following results:

Lemma 7. *Along an intertemporal equilibrium, generation t agents are unconstrained in $t-1$ iff $x_t \in (0, \underline{x}]$, with $\underline{x}$ the smallest solution of the equation*

$$\left(\frac{h}{\alpha}\right)^{\eta} \frac{x_t^{\eta}}{\lambda} - (1-\gamma_1)\left(1 + \frac{h}{\alpha}x_t\right) = 0;$$

generation t agents are constrained in $t-1$ iff $x_t \in (\underline{x}, \hat{x})$, with

$$\hat{x} = \frac{\alpha}{h}\frac{\gamma_3}{\gamma_2}$$

Proof. See Section 2.5.3 in the Appendix. □

From this lemma, the dynamics of x_t can be restricted to the interval $(0, \hat{x})$. If $x_t \in (0, \underline{x}]$, generation t agents are unconstrained in $t-1$ whereas if $x_t \in (\underline{x}, \hat{x})$, they are constrained in $t-1$.

The following proposition shows that the dynamics of the economy can be characterized in a simple way. Let us define:

$$\begin{aligned}
G_U(x) &= -\frac{\gamma_3(1-\alpha)}{x} + (1+n+h) + (\gamma_1+\gamma_2)\,h\frac{1-\alpha}{\alpha} + (1+n)\frac{1-\alpha}{\alpha} \\
G_C(x) &= -\eta\frac{1-\alpha}{\lambda}x^{\eta-1}\left(\frac{h}{\alpha}\right)^{\eta} + \frac{(1+n+h)}{\alpha} \\
F_U(x) &= -(1+n)\gamma_1\frac{1-\alpha}{\alpha}\frac{h}{\alpha}x + (1+n)(1-\gamma_1)\frac{1-\alpha}{\alpha} \\
F_C(x) &= -(1+n)\frac{1-\alpha}{\alpha}\left[\frac{h}{\alpha}x - \left(\frac{h}{\alpha}\right)^{\eta}\frac{x^{\eta}}{\lambda}\right] \\
G_I\left(x, \frac{c_{-1}}{k_0}\right) &= \frac{1}{\alpha}(1+n+h(1-\eta(1-\alpha))) - \eta\left[(1-\alpha) - \alpha\frac{c_{-1}}{k_0}\right]\frac{1}{x}
\end{aligned}$$

Proposition 12. *Starting from given initial conditions $k_0 = K_0/(N_0 + hN_{-1})$ and c_{-1}, the equilibrium dynamics can be characterized as a sequence $(k_t, x_t, S_t)_{t\geq 0} \in \mathbb{R}_+^{*2} \times \Sigma$ such that:*

- $\forall t \geq 1 : G_{S_{t-1}}(x_t) = F_{S_t}(x_{t+1})$;
- $k_{t+1} = x_t k_t^{\alpha}$;
- $\forall t \geq 1,\ S_{t-1} = C \Leftrightarrow x_t \in (\underline{x}, \hat{x})$, *and* $S_{t-1} = U \Leftrightarrow x_t \in (0, \underline{x}]$;
- *For* $t = 0$, $x_0 \in (0, \hat{x}]$ *such that* $G_I\left(x_0, \frac{c_{-1}}{k_0}\right) = F_{S_0}(x_1)$ *and* $\frac{c_{-1}}{k_0}$ *satisfies the constraint* $\frac{c_{-1}}{k_0} \leq \frac{1-\alpha}{\alpha} - \frac{F_C(x_0)}{1+n}$.

Proof. See Section 2.5.4 in the Appendix. □

The equation $G_{S_{t-1}}(x_t) = F_{S_t}(x_{t+1})$ that governs the dynamics of x_t has a simple interpretation. (2.15) can be written, using the previous notations,

$$k_{t+1} = \frac{s(t, S_{t-1})}{1+n+h} - \frac{1+n}{1+n+h} c(t, S_t)$$

or,

$$1+n+h - \frac{s(t, S_{t-1})}{k_{t+1}} + (1+n)\frac{1-\alpha}{\alpha} = (1+n)\frac{1-\alpha}{\alpha} - (1+n)\frac{c(t, S_t)}{k_{t+1}}$$

It appears in the proof of Proposition (12) that in all cases, $\frac{s(t,S_{t-1})}{k_{t+1}}$ only depends on x_t and $\frac{c(t,S_t)}{k_{t+1}}$ only depends on x_{t+1}. The functions $G_{S_{t-1}}(x_t)$ and $F_{S_t}(x_{t+1})$ are defined as:

$$G_{S_{t-1}}(x_t) = 1+n+h - \frac{s(t, S_{t-1})}{k_{t+1}} + (1+n)\frac{1-\alpha}{\alpha}$$
$$F_{S_t}(x_{t+1}) = (1+n)\frac{1-\alpha}{\alpha} - (1+n)\frac{c(t, S_t)}{k_{t+1}}$$

Note that x_t is a forward looking variable with no initial condition. We can think of the dynamics as first order dynamics in x_t with possible jumps between unconstrained and constrained regime. If the sequence $(x_t)_{t\geq 0}$ is known, the sequence $(k_t)_{t\geq 0}$ is also known and all variables of the economy can be calculated.

2.2.2 The equilibrium dynamics

Steady states.

There may exist two types of steady states: in the unconstrained or in the constrained regime. The main result of the section is that the two types of steady states never co-exist. An economy has only one stationary equilibrium, which is either in constrained or unconstrained regime, depending on the parameter values. For the uniqueness result, a non-restrictive assumption on the productive function is sufficient, namely, that $\alpha > 0.14115$. In any case, the steady state is unstable.

Proposition 13. *Assume that $\alpha > \bar{\alpha}$, with $\bar{\alpha}$ a threshold approximately equal to* 0.14115.

1. *The dynamical equation $F_U(x_{t+1}) = G_U(x_t)$ has a unique stationary solution $x^* > 0$, that is unstable. If $x^* \in (0, \underline{x}]$, x^* is a steady state of the unconstrained regime*
2. *The dynamical equation $F_C(x_{t+1}) = G_C(x_t)$ has a unique stationary solution $\tilde{x} > 0$, that is unstable. If $\tilde{x} \in (\underline{x}, \hat{x})$, $\tilde{x}$ is a steady state of the constrained regime.*
3. *There exists a unique steady state for the dynamics of the economy. If $x^* \leq \underline{x}$, this steady state is in the unconstrained regime. If $x^* > \underline{x}$, then $\tilde{x} > \underline{x}$ and the steady state is in the constrained regime.*

Proof. See Section 2.5.5 in the Appendix. □

Global dynamics

The dynamics of the economy has two phases: the initial period with a specific equation, and all periods after $t \geq 1$. As x_t is a forward looking variable, we first solve the global dynamics from period 1. Then, we consider the specific case of period 0.

The following proposition proves that, from period $t = 1$, the economy has a constant value of x_t and is either constrained or unconstrained forever.

Proposition 14. *If $x^* \leq \underline{x}$, for any $t \geq 1$, $x_t = x^*$ and $S_{t-1} = U$. If $x^* > \underline{x}$, for any $t \geq 1$, $x_t = \tilde{x}$ and $S_{t-1} = C$.*

Proof. See Section 2.5.6 in the Appendix. □

We call this a steady state in the sense of the main dynamic variable x_t being constant and no regime switches. Note that $\{k_t\}$ is generically not constant but rather increasing or decreasing monotonically. Indeed, by definition of x, $k_{t+1} = x^{ss} k_t^{\alpha}$ for $t \geq 0$, in a steady state, where x^{ss} is either x^* or $\tilde{x}$. It is then straightforward to proof that $\{k_t\}$ converges monotonically to its steady state value $k^{ss} = (x^{ss})^{1/(1-\alpha)}$.

The proof of Proposition 14 shows that any path of x_t that is different from the steady state is impossible as an intertemporal equilibrium. When $x^* \leq \underline{x}$, x_t jumps from period 1 to x^* and the economy experiences the unconstrained regime for all periods $t \geq 0$. When $x^* > \underline{x}$, x_t jumps from period 1 to $\tilde{x}$ and the economy experiences the constrained regime for all periods $t \geq 0$.

It is possible to have an intuition of the proof by looking at the Figure 2.1, where the two plots are obtained with numerical parameter values corresponding to two cases $x^* < \underline{x}$ and $x^* > \underline{x}$.

Corollary 1. *Along an intertemporal equilibrium, x_0 is given by:*

$$G_I\left(x_0, \frac{c_{-1}}{k_0}\right) = F_U(x^*), \quad \textit{if } x^* \leq \underline{x} \tag{2.24}$$

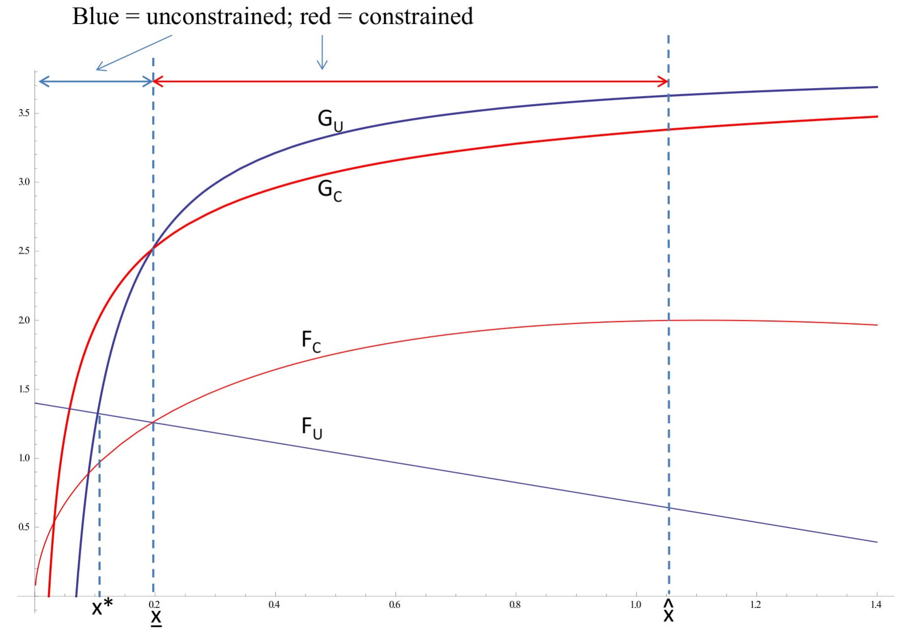
$$G_I\left(x_0, \frac{c_{-1}}{k_0}\right) = F_C(\tilde{x}), \quad \textit{if } x^* > \underline{x} \tag{2.25}$$

Proof. The Corollary is simply derived from the fourth point of Proposition (13), using the results obtained in Proposition (14). □

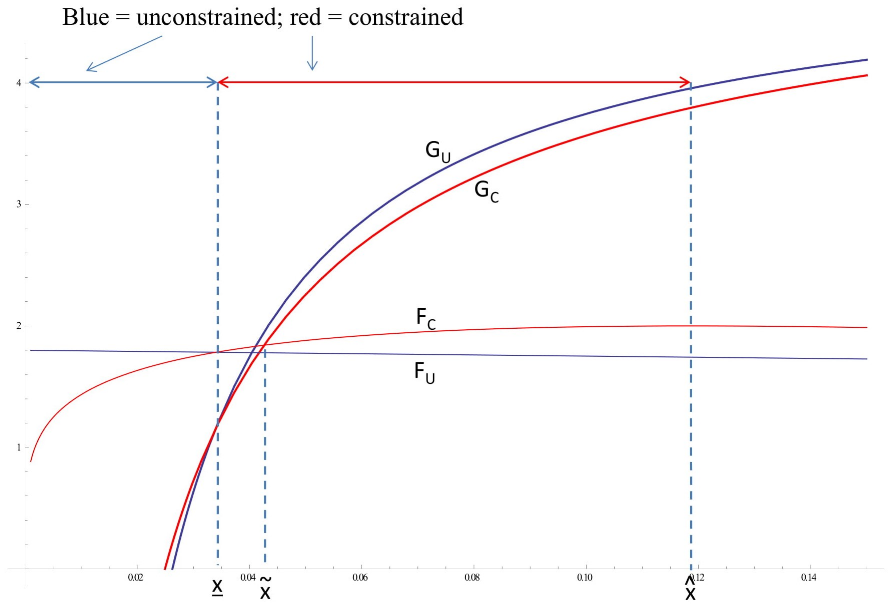
For the case $x^* \leq \underline{x}$, we define x_l and l as follows: $x_l < x^*$ is such that $G_C(x_l) = F_U(x^*)$; and

$$l = \frac{1-\alpha}{\alpha} - \frac{F_C(x_l)}{1+n}$$

To understand these definitions, we know from Proposition (12) that an equilibrium must satisfy $\frac{c_{-1}}{k_0} \leq \frac{1-\alpha}{\alpha} - \frac{F_C(x_0)}{1+n}$ to avoid default in period 0. We also know from Corollary (1) that along the equilibrium, $G_I\left(x_0, \frac{c_{-1}}{k_0}\right) = F_U(x^*)$. Finally, it is easy to check that $G_I\left(x_0, \frac{1-\alpha}{\alpha} - \frac{F_C(x_0)}{1+n}\right) = G_C(x_0)$. This property is intuitive: when $\left(\frac{c_{-1}}{k_0}\right)$ takes its limit value $\frac{1-\alpha}{\alpha} - \frac{F_C(x_0)}{1+n}$, the one that would correspond to the constrained consumption level,



(a) case $x^* < \underline{x}$, with $\gamma_1 = 0.3$, $\gamma_2 = 0.3$, $\gamma_3 = 0.4$, $h = 0.4$, $n = 0$ and $\alpha = 1/3$.



(b) case $x^* > \underline{x}$, with $\gamma_1 = 0.1$, $\gamma_2 = 0.7$, $\gamma_3 = 0.2$, $h = 0.8$, $n = 0$ and $\alpha = 1/3$.

Figure 2.1: Equilibrium characterisation: examples

$G_I\left(x_0, \frac{c_{-1}}{k_0}\right)$ must correspond to the function that defines the constrained dynamics. From these properties, l is the maximum admissible level of initial consumption per capital intensity $\left(\frac{c_{-1}}{k_0}\right)$. For this threshold level, we get $x_0 = x_l$ as $G_I\left(x_l, \frac{1-\alpha}{\alpha} - \frac{F_C(x_l)}{1+n}\right) = G_C(x_l) = F_U(x^*)$. If $x_0 = x_l$, then, the threshold level that allows avoiding default is $\frac{1-\alpha}{\alpha} - \frac{F_C(x_l)}{1+n} = l$.

In the same way, for the case $x^* > \underline{x}$ we define $\tilde{l}$ as

$$\tilde{l} = \frac{1-\alpha}{\alpha} - \frac{F_C(\tilde{x})}{1+n}$$

$\tilde{l}$ is the maximum admissible level of initial consumption per capital intensity $\left(\frac{c_{-1}}{k_0}\right)$. For this threshold level, we get $x_0 = \tilde{x}$. Indeed, we know that $G_I\left(\tilde{x}, \frac{1-\alpha}{\alpha} - \frac{F_C(\tilde{x})}{1+n}\right) = G_C(\tilde{x}) = F_C(\tilde{x})$.

Proposition 15. *For any initial conditions $k_0 = K_0/(N_0 + hN_{-1})$ and c_{-1}, there exists at most one equilibrium.*

1. *Consider the first case $x^* \leq \underline{x}$.For any $\left(\frac{c_{-1}}{k_0}\right) \leq l$, there exists a unique intertemporal equilibrium with $x_0 \in [x_l, \hat{x})$ and $\forall t \geq 1$, $x_t = x^*$ and $S_{t-1} = U$. Moreover, x_0 is a decreasing function of $\left(\frac{c_{-1}}{k_0}\right)$ with $x_0 = x_l$ for $\left(\frac{c_{-1}}{k_0}\right) = l$. For $\left(\frac{c_{-1}}{k_0}\right) > l$, generation* 0 *makes default on its debt and no equilibrium exists.*

2. *Consider the second case $x^* > \underline{x}$. For any $\left(\frac{c_{-1}}{k_0}\right) \leq \tilde{l}$, there exists a unique intertemporal equilibrium with $x_0 \in [\tilde{x}, \hat{x}]$ and $\forall t \geq 1$, $x_t = \tilde{x}$ and $S_{t-1} = C$. Moreover, x_0 is a decreasing function of $\left(\frac{c_{-1}}{k_0}\right)$ with $x_0 = \tilde{x}$ when $\left(\frac{c_{-1}}{k_0}\right) = \tilde{l}$. For $\left(\frac{c_{-1}}{k_0}\right) > \tilde{l}$, generation* 0 *makes default on its debt and no equilibrium exists.*

Proposition 14 provided the characterization of the global dynamics from $t = 1$. Proposition 15 gives the result obtained for $t = 0$. Then, the dynamics if fully characterized.

Our results show that the intertemporal equilibrium always exists and is determined. Moreover, there is no regime changes along the dynamics and the economy remains at all periods with either constrained or unconstrained borrowers. These results differ sharply from Azariadis and Lambertini (2003), who find the possible existence of multiple equilibria and indeterminacy. In their framework with exogenous endowment, when agents expect a high future interest rate which makes savings more desirable, opportunity cost of default rises and endogenous borrowing constraints become less tight. Consumption in youth and retirement becomes complements, and this allows the existence of indeterminacy and multiple equilibria. In our framework with endogenous endowments, an increase in savings tends to decrease the interest rate. The complementarity between consumption in youth and retirement is lost and there exists only one equilibrium without regime switches.

Endogenous borrowing constraint tightness

Having characterized the equilibrium dynamics, we now compare the credit constraint prevailing in equilibrium to a standard exogenous credit constraint for the OLG literature — the one used in Jappelli and Pagano (1994b). In their paper, the borrowing limit is the present discounted value of the agent's lifetime income, multiplied by an exogenous constant — the tightness of the borrowing constraint. In our framework, we can define endogenous tightness of the credit constraint. Tightness for a generation t agent, ϕ_t, is defined as the borrowing limit, given by the right hand side of (2.1) (divided by R_t), divided by the present value income of the same agent:

$$\phi_t = \frac{\frac{w_t}{R_t} + \frac{hw_{t+1}}{R_t R_{t+1}} - \frac{w_t^{1-\eta}}{\lambda R_t}\left(\frac{hw_{t+1}}{R_{t+1}}\right)^\eta}{\frac{w_t}{R_t} + \frac{hw_{t+1}}{R_t R_{t+1}}}$$

Using $x_t = k_{t+1}/k_t{}^\alpha$ and the equilibrium values of wage and interest:

$$\phi_t = 1 - \frac{\left(\frac{h}{\alpha}\right)^\eta x_t^\eta}{\lambda\left(1 + \frac{h}{\alpha}x_t\right)} \tag{2.26}$$

The previous section has shown that x_t is constant for all periods $t \geq 1$. It follows that the tightness of the credit constraint is constant for all except the very first generation. Note that given model primitives, the relationship between capital accumulation and credit in the constrained regime is same as in Jappelli and Pagano (1994b). Namely, capital accumulation is slowed down by consumption credit, with a constant fraction of the growing lifetime incomes consumed by the young and not invested in physical capital. However, in our setting any change in the primitives of the economy affects the level of tightness, both directly in the equation (2.26) and through the steady-state value of x.

2.3 Bubbles and equilibrium default

In this section, we show that another type of equilibrium is possible when a bubbly asset is introduced in the economy. The first subsection presents the model. The second studies the global dynamics in the particular case $h = 0$. The last subsection considers the case $h > 0$ and characterizes the existence of steady states and their local dynamical properties.

2.3.1 The model with bubbles and default

We now include an additional asset, without fundamental value, that is available to agents after default. We also consider a different equilibrium concept where default may happen at equilibrium. For this new type of equilibrium, lenders react to the risk of default by increasing the interest rate on loans to compensate the loss on defaulters, instead of rationing the quantity that is borrowed. They cannot select between defaulters and non defaulters that are identical ex-ante and that borrow the same amount when young. The fraction of defaulters and the price of the bubbly asset are determined at equilibrium in such a way that agents are indifferent between making default and repaying the debts.

When young in $t-1$, all generation t agents borrow the same amount and are identical. At adult age, in period t, only a share $1-\pi_t$ of agents repay the debt and the other part makes default. The non defaulters have access to the capital market to save for their old age. The fraction π_t of defaulters are excluded from the financial market, but have access to the market of the bubbly asset. The return on this market is lower than the one on the formal market. The probability of default and the return of the bubbly asset are determined endogenously in such a way that a defaulter has the same indirect utility as a non defaulter. Default is a random choice and there is no ex ante difference between defaulters and non defaulters. The instantaneous gain of defaulters that do not reimburse the loan is compensated in the indirect utility by the lower return on savings. Finally, lenders take into account the probability of default and charge a higher interest rate to compensate.

The equilibrium concept is then different from to the one used in the previous part. The bubbly asset lessens the punishment associated with default as agents have access to another asset for saving. It is valuated at some period only if agents expect that defaults will occur in the next period, in such a way that it will be possible to resell the asset. Therefore, the possibility of default at equilibrium is essentially related to the existence of the bubbly asset.

Lenders no longer ration credit to prevent default, but charge higher interest rates to compensate for the default risk, which is now well defined and given by π_t. The interest factor on consumption loans is ω_t. For lenders, a consumption loan of 1 unit in $t-1$ has a return of $(1-\pi_t)\,\omega_t$ in t. If 1 unit is rented to firms as capital in $t-1$, the return is R_t. In equilibrium, expected returns are equalized:

$$(1-\pi_t)\,\omega_t = R_t$$

For a non defaulter, the budget constraints now are:

$$\begin{aligned} d_t^{nd} + s_t &= w_t - \omega_t c_{t-1} \\ e_{t+1}^{nd} &= R_{t+1} s_t + h w_{t+1} \end{aligned}$$

where the letters "*nd*' are added to the variables d_t^{nd} and e_{t+1}^{nd} that are specific to the non defaulters. The program of an agent is defined as:

$$\begin{cases} \max\limits_{(c_{t-1},\, d_t,\, e_{t+1},\, s_t)} \gamma_1 \ln c_{t-1} + \gamma_2 \ln d_t^{nd} + \gamma_3 \ln e_{t+1}^{nd} \\ \text{s.t.} \quad d_t^{nd} + s_t = w_t - \omega_t c_{t-1} \\ \qquad e_{t+1}^{nd} = R_{t+1} s_t + h w_{t+1} \end{cases} \tag{2.27}$$

The solution is similar to the one of a non constrained agent of the previous section:

$$\omega_t c_{t-1} = \gamma_1 \left(w_t + \frac{hw_{t+1}}{R_{t+1}} \right) \Leftrightarrow R_t c_{t-1} = (1 - \pi_t) \gamma_1 \left(w_t + \frac{hw_{t+1}}{R_{t+1}} \right) \tag{2.28}$$

$$d_t^{nd} = \gamma_2 \left(w_t + \frac{hw_{t+1}}{R_{t+1}} \right) \tag{2.29}$$

$$e_{t+1}^{nd} = \gamma_3 R_{t+1} \left(w_t + \frac{hw_{t+1}}{R_{t+1}} \right) \tag{2.30}$$

and

$$s_t = \gamma_3 w_t - (\gamma_1 + \gamma_2) \frac{hw_{t+1}}{R_{t+1}} \tag{2.31}$$

A generation t defaulter consumes the same amount c_{t-1} in period $t-1$ as a non defaulter. She makes default in period t and has no more access to formal financial markets, but she can buy the bubbly asset in a quantity λ_t in order to save, which has a price of p_t in units of the good. The total quantity of asset supply is normalized to 1.

The budget constraints of a defaulter from period t are:

$$\begin{aligned} d_t^d + p_t \lambda_t &= w_t \\ e_{t+1}^d &= p_{t+1} \lambda_t + hw_{t+1} \end{aligned}$$

where the letter "d' is added to the variables d_t^d and e_{t+1}^d that are specific to the defaulters.

The gross return on the bubble is $\rho_{t+1} = p_{t+1}/p_t$.

The program from period t of the agent is defined as:

$$\begin{cases} \max\limits_{(d_t,\, e_{t+1},\, \lambda_t)} \gamma_2 \ln d_t^d + \gamma_3 \ln e_{t+1}^d \\ \text{s.t.} \quad d_t^d + p_t \lambda_t = w_t \\ \qquad\qquad e_{t+1}^d = p_{t+1} \lambda_t + hw_{t+1} \end{cases} \tag{2.32}$$

and the optimal choices are:

$$d_t^d = (1 - \eta) \left(w_t + \frac{hw_{t+1}}{\rho_{t+1}} \right) \tag{2.33}$$

$$\frac{e_{t+1}^d}{\rho_{t+1}} = \eta \left(w_t + \frac{hw_{t+1}}{\rho_{t+1}} \right) \tag{2.34}$$

$$p_t \lambda_t = \eta w_t - (1 - \eta) \frac{hw_{t+1}}{\rho_{t+1}} \tag{2.35}$$

At equilibrium, prices and the share of defaulters are such that indirect utilities of defaulters and non defaulters are equal:

$$\gamma_1 \ln c_{t-1} + \gamma_2 \ln d_t^{nd} + \gamma_3 \ln e_{t+1}^{nd} = \gamma_1 \ln c_{t-1} + \gamma_2 \ln d_t^d + \gamma_3 \ln e_{t+1}^d$$

or

$$\gamma_2 \ln \gamma_2 + \gamma_3 \ln \gamma_3 + (\gamma_2+\gamma_3)\ln\left(w_t + \frac{hw_{t+1}}{R_{t+1}}\right) + \gamma_3 \ln R_{t+1}$$
$$= \gamma_2 \ln \frac{\gamma_2}{\gamma_2+\gamma_3} + \gamma_3 \ln \frac{\gamma_3}{\gamma_2+\gamma_3} + (\gamma_2+\gamma_3)\ln\left(w_t + \frac{hw_{t+1}}{\rho_{t+1}}\right) + \gamma_3 \ln \rho_{t+1}$$

After simple calculations, we get

$$(1-\gamma_1)\left(w_t + \frac{hw_{t+1}}{R_{t+1}}\right) R^{\eta}_{t+1} = \left(w_t + \frac{hw_{t+1}}{\rho_{t+1}}\right)\rho^{\eta}_{t+1}$$

or, using $x_t = k_{t+1}/k_t^{\alpha}$:

$$(1-\gamma_1)\left(1+\frac{h}{\alpha}x_t\right)\left(\frac{R_{t+1}}{\rho_{t+1}}\right)^{\eta} = \left(1+\frac{h}{\alpha}x_t\frac{R_{t+1}}{\rho_{t+1}}\right) \tag{2.36}$$

Note that this equation implies $\rho_{t+1} < R_{t+1}$. This is intuitive: if the return on the formal market was not larger than the return on bubble, default would always be preferred to repayment. We impose a short-selling constraint on bubbles for the non-defaulting agents; in absence of such a constraint the return differential could have been used for arbitrage.

On the capital market, there is now only a fraction $(1-\pi_t)$ of adult agents that save. The equilibrium of the market is obtained with:

$$K_{t+1} = (1-\pi_t)N_t s_t - N_{t+1}c_t$$

or

$$k_{t+1} = \frac{(1-\pi_t)s_t}{1+n+h} - \frac{1+n}{1+n+h}c_t \tag{2.37}$$

Replacing s_t and c_t by (2.31) and (2.28):

$$k_{t+1}\left[1+\frac{(1-\pi_t)}{1+n+h}(\gamma_1+\gamma_2)h\frac{1-\alpha}{\alpha} + \gamma_1\frac{(1-\pi_{t+1})(1+n)}{1+n+h}\frac{1-\alpha}{\alpha}\right] +$$
$$\gamma_1\frac{(1-\pi_{t+1})(1+n)}{1+n+h}\frac{h}{\alpha}\frac{1-\alpha}{\alpha}k_{t+2}k_{t+1}^{1-\alpha}$$
$$= \frac{(1-\pi_t)\gamma_3(1-\alpha)}{1+n+h}k_t^{\alpha}$$

Finally, using $x_t = k_{t+1}/k_t^{\alpha}$:

$$x_t\left[1+n+h+(1-\pi_t)(\gamma_1+\gamma_2)h\frac{1-\alpha}{\alpha} + \gamma_1(1-\pi_{t+1})(1+n)\frac{1-\alpha}{\alpha}\right] + \tag{2.38}$$
$$\gamma_1(1-\pi_{t+1})(1+n)\frac{h}{\alpha}\frac{1-\alpha}{\alpha}x_{t+1}x_t = (1-\pi_t)\gamma_3(1-\alpha)$$

From the capital market equilibrium condition (2.37), s_t must be positive. Using

(2.31), we get:

$$x_t < \frac{\alpha}{h}\frac{\gamma_3}{\gamma_1+\gamma_2} = \frac{\alpha}{h}\frac{\gamma_3}{1-\gamma_3} \tag{2.39}$$

For the bubble market, the market clearing condition implies $\pi_t N_t \lambda_t = 1$. Using optimal bubble holding (2.35), we obtain:

$$\frac{p_t}{\pi_t N_t} = \eta w_t - (1-\eta)\frac{h w_{t+1}}{\rho_{t+1}}$$

Combining the above relations for t and $t+1$ and using $\rho_{t+1} = p_{t+1}/p_t$:

$$\pi_t \frac{\rho_{t+1}}{1+n}\left[\eta w_t - (1-\eta)\frac{h w_{t+1}}{\rho_{t+1}}\right] = \pi_{t+1}\left[\eta w_{t+1} - (1-\eta)\frac{h w_{t+2}}{\rho_{t+2}}\right]$$

We introduce a new variable:

$$\delta_t \equiv R_{t+1}/\rho_{t+1}$$

The last equation becomes:

$$\frac{\pi_t R_{t+1}}{\delta_t(1+n)}\left[\eta w_t - (1-\eta)\frac{h w_{t+1}\delta_t}{R_{t+1}}\right] = \pi_{t+1}\left[\eta w_{t+1} - (1-\eta)\frac{h w_{t+2}\delta_{t+1}}{R_{t+2}}\right]$$

and using $x_t = k_{t+1}/k_t^{\alpha}$,

$$\frac{\pi_t \alpha}{(1+n)}\left[\eta - (1-\eta)\frac{h x_t \delta_t}{\alpha}\right] = \pi_{t+1} x_t \delta_t \left[\eta - (1-\eta)\frac{h x_{t+1}\delta_{t+1}}{\alpha}\right] \tag{2.40}$$

It is worth noting that p_t is non negative, which implies at each period t,

$$\eta - (1-\eta)\frac{h x_t \delta_t}{\alpha} \geq 0$$

or

$$x_t \delta_t \leq \frac{\alpha}{h}\frac{\eta}{1-\eta} = \frac{\alpha}{h}\frac{\gamma_3}{\gamma_2} \tag{2.41}$$

Using the variable δ_t, the condition (2.36), which translates the equality of the indirect utilities for defaulters and non defaulters, becomes:

$$(1-\gamma_1)\left(1+\frac{h}{\alpha}x_t\right)(\delta_t)^{\eta} = 1 + \frac{h}{\alpha}x_t\delta_t \tag{2.42}$$

Finally we have obtained a simple characterization of the intertemporal equilibrium with a bubbly asset. The equilibrium dynamics can be characterized with two forward looking dynamic variables (x_t, π_t) and one static variable δ_t, solutions of the three equations (2.38), (2.40) and (2.42), and satisfying the two constraints (2.39) and (2.41).

2.3.2 Equilibrium dynamics in the case $h = 0$

Before analyzing the general case, it is useful to study first the case $h = 0$ that allows an explicit resolution of the dynamics.

Equations (2.38), (2.40) and (2.42) become:

$$x_t(1+n)\left[1+\gamma_1(1-\pi_{t+1})\frac{1-\alpha}{\alpha}\right] = (1-\pi_t)\gamma_3(1-\alpha) \tag{2.43}$$

$$\frac{\pi_t\alpha}{(1+n)} = \pi_{t+1}x_t\delta_t \tag{2.44}$$

$$\delta_t = (1-\gamma_1)^{-1/\eta} \tag{2.45}$$

A new variable is introduced: $\chi_t \equiv 1/\pi_t$. Combining (2.43) and (2.44) to eliminate x_t, we get:

$$\left(1+\gamma_1\frac{1-\alpha}{\alpha}\right)\chi_{t+1} = (1-\gamma_1)^{-1/\eta}\gamma_3\frac{1-\alpha}{\alpha}\chi_t + \left(\gamma_1 - (1-\gamma_1)^{-1/\eta}\gamma_3\right)\frac{1-\alpha}{\alpha}$$

This equation has a constant solution:

$$\chi = \breve{\chi} = \frac{(1-\gamma_1)^{-1/\eta}\gamma_3 - \gamma_1}{(1-\gamma_1)^{-1/\eta}\gamma_3 - \gamma_1 - \frac{\alpha}{1-\alpha}}$$

The general solution is:

$$\chi_t = \left[\frac{(1-\gamma_1)^{-1/\eta}\gamma_3}{\gamma_1 + \frac{\alpha}{1-\alpha}}\right]^t (\chi_0 - \breve{\chi}) + \breve{\chi}$$

As $\chi_t = 1/\pi_t$, the bubbly equilibrium can exist only if at all periods, $0 < 1/\chi_t < 1$. This leads to the following result:

Proposition 16. *For an economy with $h = 0$, assume that*

$$(1-\gamma_1)^{-1/\eta}\gamma_3 > \gamma_1 + \frac{\alpha}{1-\alpha} \tag{2.46}$$

1. *The economy has two steady states*

   - *a bubbleless steady state (x^*, π^*) such that*

$$x^* = \frac{\gamma_3(1-\alpha)}{(1+n)\left(1+\gamma_1\frac{1-\alpha}{\alpha}\right)}, \ \pi^* = 0$$

- *A bubbly steady state* $(\breve{x}, \breve{\pi}, \breve{\delta})$ *such that*

$$\breve{x} = \frac{(1-\gamma_1)^{1/\eta}\,\alpha}{(1+n)}$$
$$\breve{\pi} = \frac{(1-\gamma_1)^{-1/\eta}\,\gamma_3 - \gamma_1 - \frac{\alpha}{1-\alpha}}{(1-\gamma_1)^{-1/\eta}\,\gamma_3 - \gamma_1}$$
$$\breve{\delta} = (1-\gamma_1)^{-1/\eta}$$

2. *An intertemporal equilibrium exists iff* $\pi_0 \leq \hat{\pi}$.

   - *If* $\pi_0 = \breve{\pi}$, *the economy converges to the bubbly steady state.*
   - *If* $\pi_0 < \breve{\pi}$, *the economy converges to the bubbleless steady state.*

For $(1-\gamma_1)^{-1/\eta}\,\gamma_3 < \gamma_1 + \frac{\alpha}{1-\alpha}$, *no bubbly equilibrium exists. The variable* x_t *reaches the stationary unconstrained solution*

$$x^* = \frac{\gamma_3(1-\alpha)}{(1+n)\left(1+\gamma_1\frac{1-\alpha}{\alpha}\right)}$$

This proposition gives results that are close to the standard model of Tirole (1985). Under (2.46), multiple equilibria exist. There is one initial value for the bubble that leads to the bubbly steady state (when $\pi_0 = \breve{\pi}$). There also exists a multitude of paths that converge to the bubbleless steady state. This implies equilibrium indeterminacy since π is not predetermined and only depends on the expectations. Note that the bubbleless steady state is an unconstrained one, as the endogenous debt constraint disappears when $h = 0$.

In the bubbly steady state, the gross return on the bubble is given by:

$$\breve{\rho} = \frac{\breve{R}}{\breve{\delta}} = \frac{\alpha \breve{k}^{\alpha-1}}{\breve{\delta}} = \frac{\alpha}{\breve{x}\breve{\delta}} = 1+n$$

where we have used the fact that $x = k^{1-\alpha}$ in any steady state. We get the standard property that the rate of return is equal to the population growth rate for the bubble asset. For non defaulters that have access to the formal credit market, the interest factor for lenders is given by $\breve{R} = \alpha/\breve{x} = (1+n)\,\breve{\delta} = (1+n)\,(1-\gamma_1)^{-1/\eta} > 1+n$. The interest rate is greater on the formal credit market than on the bubble market, and greater than the GDP growth rate. Finally, the growth factor for borrowers is given by $\breve{\omega} = \breve{R}/(1-\breve{\pi}) > \breve{R}$. To sum up, three interest rates coexist in the equilibrium with bubbles: the lowest one is obtained by investing in the bubble on the informal market; the intermediate value is the interest rate for savers on the formal market; the highest value is the interest rate for borrowers on the formal market.

In this particular case it is also easy to check that $\frac{\partial \breve{\pi}}{\partial \alpha} < 0$. The more capital intensive the production is, the more beneficial it is to invest in capital for the agents, so they have less incentives to default.

2.3.3 The case $h > 0$.

In the general case, we are not able to characterize the global dynamics. We analyze the existence of different possible steady states and study the local dynamics around them.

Simplification of the dynamical system

For $h > 0$, equations (2.38), (2.40) and (2.42) allow to define a dynamical system associated with two variables $(x_t,\ \pi_t)$, as (2.42) is a static equation allowing to determine δ_t as a function of x_t. Note that all the dynamic variables are forward-looking, jumping variables. It means that there is equilibrium determinacy only if the system has one possible value for each variable given initial conditions. If this value is a steady state, it must be locally unstable — a source — for determinacy to occur.

The following lemma shows that an intertemporal equilibrium may exist only under the condition $x_t \in [0, \underline{x}]$ that allows to define δ_t as a function of x_t, $\delta_t = \Delta(x_t)$.

Lemma 8. *Let us consider $\underline{\delta}$ defined as $\underline{\delta} = \frac{1}{\underline{x}}\frac{\alpha}{h}\frac{\gamma_3}{\gamma_2}$. $\underline{\delta}$ is such that $\underline{\delta} > (1-\gamma_1)^{-1/\eta}$.*

- *For any $x_t \in [0, \underline{x}]$, $\exists! \delta_t \in \left[(1-\gamma_1)^{-1/\eta}, \underline{\delta}\right]$ such that (2.42) is satisfied. This value of δ_t is denoted by $\Delta(x_t)$. Moreover, $\Delta(x_t)$ is an increasing function with $\Delta(0) = (1-\gamma_1)^{-1/\eta}$ and $\Delta(\underline{x}) = \underline{\delta}$. When $x_t < \underline{x}$, $x_t\Delta(x_t) < \frac{\alpha}{h}\frac{\gamma_3}{\gamma_2}$.*
- *It is impossible to have $x_t > \underline{x}$ along an equilibrium.*

Proof. See Section 2.5.8 in the Appendix. $\square$

Following this result, it is convenient to define $\Gamma(x) = x\Delta(x)$. The intertemporal equilibrium can be characterized by a system of two dynamical equations:

$$1 + n + h + (1-\pi_t)(\gamma_1 + \gamma_2)\, h\frac{1-\alpha}{\alpha} + \gamma_1(1-\pi_{t+1})(1+n)\frac{1-\alpha}{\alpha} + \gamma_1(1-\pi_{t+1})(1+n)\frac{h}{\alpha}\frac{1-\alpha}{\alpha}x_{t+1} = \frac{(1-\pi_t)\gamma_3(1-\alpha)}{x_t} \tag{2.47}$$

$$\frac{\pi_t\alpha}{(1+n)}\left[\frac{\gamma_3}{\Gamma(x_t)} - \gamma_2\frac{h}{\alpha}\right] = \pi_{t+1}\left[\gamma_3 - \gamma_2\frac{h}{\alpha}\Gamma(x_{t+1})\right] \tag{2.48}$$

First the existence of different types of steady states will be characterized. Then, the local dynamics around these steady states will be analyzed.

Steady state analysis

A steady state is a solution (x, π, δ) of the following system:

$$x\left[1+n+h+(1-\pi)(\gamma_1+\gamma_2)h\frac{1-\alpha}{\alpha}+\gamma_1(1-\pi)(1+n)\frac{1-\alpha}{\alpha}\right]+ \\ \gamma_1(1-\pi)(1+n)\frac{h}{\alpha}\frac{1-\alpha}{\alpha}x^2=(1-\pi)\gamma_3(1-\alpha) \tag{2.49}$$

$$\frac{\pi\alpha}{(1+n)}\left[\eta-(1-\eta)\frac{hx\delta}{\alpha}\right]=\pi x\delta\left[\eta-(1-\eta)\frac{hx\delta}{\alpha}\right] \tag{2.50}$$

$$(1-\gamma_1)\left(1+\frac{h}{\alpha}x\right)\delta^{\eta}=1+\frac{h}{\alpha}x\delta \tag{2.51}$$

This system may admit three types of solutions.

First consider the solution obtained for $\pi = \pi^* \equiv 0$. Then, (2.50) is satisfied. (2.49) has a unique solution x^*, which corresponds to the unconstrained steady state of the economy without bubbles. We know that it exists if $x^* \leq \underline{x}$. Finally, $\delta = \delta^*$ is defined by (2.51). For this steady state, the economy corresponds to the unconstrained steady state of the economy without bubbles as nobody holds the bubble ($\pi = 0$).

Secondly, we are looking for steady states such that $\pi \neq 0$. From (2.50), there are two possibilities:

- either $\eta - (1-\eta)\frac{hx\delta}{\alpha} > 0$ and $x\delta = \frac{\alpha}{1+n}$,
- or $\eta - (1-\eta)\frac{hx\delta}{\alpha} = 0$.

The first case leads to a steady state denoted by $(\breve{x}, \breve{\pi}, \breve{\delta})$. This is a bubbly steady state where a fraction $\breve{\pi}$ of agents make default and invest in the bubble. Some conditions must be fulfilled for the existence of such a steady state: the value of the bubble must be positive, or $\eta - (1-\eta)\frac{h\breve{x}\breve{\delta}}{\alpha} > 0$; $\breve{\pi}$ solution of (2.49) must satisfy $0 < \breve{\pi} < 1$. As $\breve{x}\breve{\delta} = \frac{\alpha}{1+n}$, the first condition is satisfied if $\frac{\gamma_3}{\gamma_2} > \frac{h}{1+n}$. The second condition holds only if the value of $\breve{x}$, solution of $\breve{x}\Delta(\breve{x}) = \frac{\alpha}{1+n}$, is such that $\breve{x} < x^*$.

The second case leads to a steady state denoted by $(\underline{x}, \underline{\pi}, \underline{\delta})$. The price of the bubble is 0. From lemma 8, the condition $\eta - (1-\eta)\frac{hx\delta}{\alpha} = 0$ leads to the limit values $x = \underline{x}$ and $\delta = \underline{\delta}$. Finally, (2.49) with $x = \underline{x}$ gives a value $\underline{\pi}$ for π, that must be such that $0 < \underline{\pi} < 1$. This is the case only if $\underline{x} < x^*$. $\underline{\pi}$ may well be positive even though the steady state value of the bubble is zero: there is some positive return on the asset price as it crashes, so default and repayment can still give the same utility in this steady-state.

From this analysis, a necessary condition to obtain the stationary bubbly equilibrium $(\breve{x}, \breve{\pi}, \breve{\delta})$ is $\frac{\gamma_3}{\gamma_2} > \frac{h}{1+n}$. We then limit our analysis to this case, which allows to get the following proposition:

Proposition 17. *Assume that $\frac{\gamma_3}{\gamma_2} > \frac{h}{1+n}$. Then there are at most two steady states in the economy.*

- *If $\breve{x} < \underline{x} < x^*$, two steady states exist, one bubbly $(\breve{x}, \breve{\pi}, \breve{\delta})$ and one bubbleless $(\underline{x}, \underline{\pi}, \underline{\delta})$;*

- *If $\breve{x} < x^* \leq \underline{x}$, two steady states exist, one bubbly $(\breve{x}, \breve{\pi}, \breve{\delta})$ and one bubbleless (x^*, π^*, δ^*);*

- *If $x^* < \breve{x} \leq \underline{x}$, only one bubbleless steady state exists (x^*, π^*, δ^*).*

Proof. See Section 2.5.9 in the Appendix. □

Local dynamics around the steady states

The dynamical system can be expressed with respect to the two variables (x_t, π_t). (2.38) is written under the form:

$$1 + n + h + (1 - \pi_t)(\gamma_1 + \gamma_2)\, h\frac{1-\alpha}{\alpha} + \gamma_1(1 - \pi_{t+1})(1+n)\frac{1-\alpha}{\alpha} + \gamma_1(1 - \pi_{t+1})(1+n)\frac{h}{\alpha}\frac{1-\alpha}{\alpha}x_{t+1} = \frac{(1-\pi_t)\gamma_3(1-\alpha)}{x_t} \tag{2.52}$$

Using Lemma 8, (2.40) can be written:

$$\frac{\pi_t\alpha}{(1+n)}\left[\frac{\eta}{\Gamma(x_t)} - (1-\eta)\frac{h}{\alpha}\right] = \pi_{t+1}\left[\eta - (1-\eta)\frac{h\Gamma(x_{t+1})}{\alpha}\right] \tag{2.53}$$

with $\Gamma(x) \equiv x\Delta(x)$.

(x_t, π_t) are forward looking variables. Therefore, a steady state is locally determinate if it is unstable.

Proposition 18. *Assume that $\frac{\gamma_3}{\gamma_2} > \frac{h}{1+n}$.*

- *The bubbleless steady state (x^*, π^*, δ^*) is a saddle point when it exists together with the bubbly steady state $(\breve{x}, \breve{\pi}, \breve{\delta})$ (case $\breve{x} < x^* \leq \underline{x}$), and is unstable when it exists alone (case $x^* < \breve{x} \leq \underline{x}$).*

- *The bubbleless steady state $(\underline{x}, \underline{\pi}, \underline{\delta})$ is a saddle point when it exists together with the bubbly steady state $(\breve{x}, \breve{\pi}, \breve{\delta})$ (case $\breve{x} < \underline{x} < x^*$).*

- *If γ_2 is small enough, the bubbly steady state $(\breve{x}, \breve{\pi}, \breve{\delta})$ is locally unstable.*

Proof. See Section 2.5.10 in the Appendix. □

As the dynamical system has a complex form, it was only possible to characterize the local dynamical properties of the bubbly steady state $(\breve{x}, \breve{\pi}, \breve{\delta})$ in the particular case of a small value of γ_2. The results are close to Tirole (1985): when a bubbleless steady state exists together with the bubbly one, it is indeterminate; it becomes determinate if it exists alone.

For the bubbly steady state, we get the same properties as the one obtained in the case $h = 0$: three interest rates coexist in the equilibrium with bubbles. The gross return on the bubble is $\breve{\rho} = 1 + n$, whereas the return on the savings of non defaulters is higher $\breve{R} = (1+n)\,\breve{\delta} > 1 + n$. Finally, for young borrowers, the interest rate is greater $\breve{\omega} = \breve{R}/(1 - \breve{\pi}) > \breve{R}$. The existence of the bubble is compatible with an interest rate on the formal market that is higher than the growth rate.

Bubble existence and overaccumulation

As the return on the bubble is not equal to the return on capital, bubble existence does not rely on overaccumulation of capital in the bubbleless equilibrium, in contrast with Tirole (1985). In a bubbly steady state, the return on the bubble is $1 + n$ and the return on capital is higher than that, so there is capital underaccumulation. If the bubble is sufficiently small, the capital intensity is close to the one that the same economy has in in the bubbleless steady state, so capital might also be underaccumulated in that state. For tractability, we prove this possibility in the particular case of $h = 0$.

Proposition 19. *Assume $h = 0$, and $(1 - \gamma_1)^{-1/\eta}\gamma_3 - \gamma_1 - \frac{\alpha}{1-\alpha}$ is positive and sufficiently close to 0. Then a bubbly and a bubbleless steady states exist and there is capital underaccumulation in the bubbleless steady state.*

Proof. See Section 2.5.11 in the Appendix. □

2.4 Conclusion

In this paper, we have studied the dynamics of an economy with physical capital, and a consumption credit market where borrowers have limited commitment to repayment. Two types of equilibrium may emerge in the same economy: one with bubbles and no default; another with bubbles and default. The bubbleless equilibrium is always unique and determined, unlike in the endowment economy of Azariadis and Lambertini (2003). Capital intensity converges monotonically to its steady-state value and the endogenous credit limit is a constant fraction of the borrower's net present income. This is a global dynamics result that holds for whatever initial conditions under which the initial generation does not default.

The equilibrium with bubbles and default has similar dynamic properties to the seminal model of Tirole (1985). A steady state with a permanent bubble may exist, and the equilibrium converging to it is determined. In the other steady-state, the bubble is vanishing, but agents still default and invest in the bubble on the transition. Such equilibrium is indeterminate. In any bubbly equilibria, three rates of return co-exist in the economy. For this reason, existence of the bubbly steady-state does not rely on the overaccumulation of capital without the bubble — a necessary condition found in Tirole (1985).

2.5 Appendix

2.5.1 Proof of Lemma 6

Consider an agent who repays the debt when adult in period t. She has the following intertemporal budget constraint:

$$d_t + \frac{e_{t+1}}{R_{t+1}} = w_t + \frac{hw_{t+1}}{R_{t+1}} - R_t c_{t-1} \equiv W_t$$

Optimal consumption and saving from period t are:

$$d_t = (1-\eta)W_t \tag{2.54}$$

$$e_{t+1} = \eta R_{t+1} W_t \tag{2.55}$$

$$s_t = \eta\left(w_t - R_t c_{t-1}\right) - (1-\eta)\frac{hw_{t+1}}{R_{t+1}} \tag{2.56}$$

with $\eta = \gamma_3/(\gamma_2+\gamma_3)$.

These choices yield the following indirect utility from t:

$$\gamma_2 \ln \frac{\gamma_2}{\gamma_2+\gamma_3} + \gamma_3 \ln \frac{\gamma_3}{\gamma_2+\gamma_3} + (\gamma_2+\gamma_3)\ln W_t + \gamma_3 \ln R_{t+1}$$

Now consider an agent defaulting in period t on the debt. She consumes in t and $t+1$ her income, as she has no access to financial markets:

$$d_t = w_t$$
$$e_{t+1} = hw_{t+1}$$

The indirect utility from t is:

$$\gamma_2 \ln w_t + \gamma_3 \ln\left(hw_{t+1}\right)$$

The constraint on debt is defined as the limit value of c_{t-1} such that:

$$\gamma_2 \ln \frac{\gamma_2}{\gamma_2+\gamma_3} + \gamma_3 \ln \frac{\gamma_3}{\gamma_2+\gamma_3} + (\gamma_2+\gamma_3)\ln W_t + \gamma_3 \ln R_{t+1} = \gamma_2 \ln w_t + \gamma_3 \ln\left(hw_{t+1}\right)$$

or

$$R_t c_{t-1} \leq w_t + \frac{hw_{t+1}}{R_{t+1}} - \frac{w_t^{1-\eta}}{\lambda}\left(\frac{hw_{t+1}}{R_{t+1}}\right)^{\eta} \tag{2.57}$$

with $\lambda = \eta^{\eta}(1-\eta)^{1-\eta}$.

2.5.2 Proof of Proposition 11

An agent is unconstrained if the optimal consumption level c_{t-1} obtained by solving program (P) without the constraint $c_{t-1} \leq \bar{c}_{t-1}$ satisfies the constraint. The optimal intertemporal choices without the borrowing constraint are given by

$$R_t c_{t-1} = \gamma_1 \left(w_t + \frac{h w_{t+1}}{R_{t+1}} \right)$$
$$d_t = \gamma_2 \left(w_t + \frac{h w_{t+1}}{R_{t+1}} \right)$$
$$e_{t+1} = \gamma_3 R_{t+1} \left(w_t + \frac{h w_{t+1}}{R_{t+1}} \right)$$

$$s_t = \gamma_3 w_t - (\gamma_1 + \gamma_2) \frac{h w_{t+1}}{R_{t+1}}$$

The condition $c_{t-1} \leq \bar{c}_{t-1}$ with Lemma (6) leads to:

$$\frac{w_t^{1-\eta}}{\lambda} \left(\frac{h w_{t+1}}{R_{t+1}} \right)^{\eta} \leq (1-\gamma_1) \left(w_t + \frac{h w_{t+1}}{R_{t+1}} \right)$$

If this inequality is not satisfied, the agent is constrained in $t-1$. Consumption level in $t-1$ is determined by $\bar{c}_{t-1}$. d_t, e_{t+1} and s_t are optimal choices calculated from period t :

$$R_t c_{t-1} = w_t + \frac{h w_{t+1}}{R_{t+1}} - \frac{w_t^{1-\eta}}{\lambda} \left(\frac{h w_{t+1}}{R_{t+1}} \right)^{\eta}$$
$$d_t = (1-\eta) \frac{w_t^{1-\eta}}{\lambda} \left(\frac{h w_{t+1}}{R_{t+1}} \right)^{\eta}$$
$$e_{t+1} = R_{t+1} \eta \frac{w_t^{1-\eta}}{\lambda} \left(\frac{h w_{t+1}}{R_{t+1}} \right)^{\eta}$$

$$s_t = \eta \frac{w_t^{1-\eta}}{\lambda} \left(\frac{h w_{t+1}}{R_{t+1}} \right)^{\eta} - \frac{h w_{t+1}}{R_{t+1}}$$

2.5.3 Proof of Lemma 7

We consider the equation

$$\left(\frac{h}{\alpha} \right)^{\eta} \frac{x^{\eta}}{\lambda} - (1-\gamma_1) \frac{h}{\alpha} x = 1 - \gamma_1$$

This equation has 2 positive solutions $\underline{x}$ and $\bar{x}$, with $\underline{\mathrm{x}} < \bar{x}$, if the function

$$h(x) \equiv \left(\frac{h}{\alpha} \right)^{\eta} \frac{x^{\eta}}{\lambda} - (1-\gamma_1) \frac{h}{\alpha} x$$

has a maximum higher than $1-\gamma_1$. The maximum is obtained in $\acute{x}$ such that

$$\left(\frac{h}{\alpha} \right)^{\eta} \frac{\eta \acute{x}^{\eta-1}}{\lambda} = (1-\gamma_1) \frac{h}{\alpha}$$

or

$$\acute{x} = \frac{\alpha}{h} \left[\frac{\eta}{\lambda(1-\gamma_1)} \right]^{\frac{1}{1-\eta}}$$

and

$$h(\acute{x}) = \left(\frac{\eta}{\lambda}\right)^{\frac{1}{1-\eta}} (1-\gamma_1)^{\frac{-\eta}{1-\eta}} \frac{1-\eta}{\eta}$$

Therefore, the condition $h(\acute{x}) > 1-\gamma_1$ gives:

$$\left(\frac{\eta}{\lambda}\right)^{\frac{1}{1-\eta}} (1-\gamma_1)^{\frac{-1}{1-\eta}} \frac{1-\eta}{\eta} > 1$$

or

$$\left(\frac{\eta}{\lambda}\right)^{\frac{1-\eta}{\eta}} \left(\frac{1-\eta}{\eta}\right)^{1-\eta} > 1-\gamma_1$$

As $\lambda = \eta^{\eta}(1-\eta)^{1-\eta}$, we get: $1 > 1-\gamma_1$, which is true.

If the unconstrained regime exists in $t-1$, the condition $s_t \geq 0$ in (2.6) leads to

$$\gamma_3 w_t \geq (\gamma_1+\gamma_2) \frac{h w_{t+1}}{R_{t+1}}$$

Using (2.12) and (2.13), this condition becomes:

$$\gamma_3 (1-\alpha) k_t^{\alpha} \geq (\gamma_1+\gamma_2) \frac{h(1-\alpha) k_{t+1}}{\alpha}$$

or

$$x_t \leq \frac{\gamma_3}{\gamma_1+\gamma_2} \frac{\alpha}{h}$$

Now, if the constrained regime exists in $t-1$, the condition $s_t \geq 0$ in (2.11) leads to

$$\eta \frac{w_t^{1-\eta}}{\lambda} \left(\frac{h w_{t+1}}{R_{t+1}}\right)^{\eta} \geq \frac{h w_{t+1}}{R_{t+1}}$$

Using (2.12) and (2.13),

$$\eta (1-\alpha) \frac{k_t^{(1-\eta)\alpha}}{\lambda} \left(\frac{h}{\alpha}\right)^{\eta} k_{t+1}^{\eta} \geq \frac{h(1-\alpha) k_{t+1}}{\alpha}$$

or

$$x_t \leq \frac{\alpha}{h} \frac{\eta}{1-\eta} = \frac{\gamma_3}{\gamma_2} \frac{\alpha}{h} \equiv \hat{x}$$

It is obvious that

$$\frac{\alpha}{h} \frac{\gamma_3}{\gamma_1+\gamma_2} < \frac{\alpha}{h} \frac{\gamma_3}{\gamma_2} < \acute{x} = \frac{\alpha}{h} \frac{\gamma_3}{\gamma_2} \left(\frac{1}{1-\gamma_1}\right)^{\frac{1}{1-\eta}}$$

with $\acute{x}$ the maximum of the function h introduced above. From these inequalities, we deduce that

$$\frac{\alpha}{h} \frac{\gamma_3}{\gamma_1+\gamma_2} < \frac{\alpha}{h} \frac{\gamma_3}{\gamma_2} < \acute{x} = \frac{\alpha}{h} \frac{\gamma_3}{\gamma_2} \left(\frac{1}{1-\gamma_1}\right)^{\frac{1}{1-\eta}} < \bar{x}$$

Moreover, it is easy to prove that

$$\underline{x} < \frac{\alpha}{h}\frac{\gamma_3}{\gamma_1+\gamma_2} < \frac{\alpha}{h}\frac{\gamma_3}{\gamma_2}$$

Indeed, we need to prove that $h\left(\frac{\alpha}{h}\frac{\gamma_3}{\gamma_1+\gamma_2}\right) > 1-\gamma_1$. This condition gives:

$$\left(\frac{h}{\alpha}\right)^{\eta}\frac{\left(\frac{\alpha}{h}\frac{\gamma_3}{\gamma_1+\gamma_2}\right)^{\eta}}{\lambda} > (1-\gamma_1)\left(\frac{\gamma_3}{\gamma_1+\gamma_2}+1\right)$$

with

$$\lambda = \eta^{\eta}(1-\eta)^{1-\eta} = \frac{\gamma_3^{\eta}\gamma_2^{1-\eta}}{1-\gamma_1}$$

After some simplifications, we get:

$$\left(\frac{\gamma_1+\gamma_2}{\gamma_2}\right)^{\eta} > \gamma_2$$

which is true.

2.5.4 Proof of Proposition 12

The case of an economy that is unconstrained in t and $t-1$ has been studied in the text. We have shown that it leads to the following evolution of x_t:

$$\begin{aligned}(1+n+h) =& \frac{\gamma_3(1-\alpha)}{x_t} - (\gamma_1+\gamma_2)\,h\frac{1-\alpha}{\alpha}\\ & -(1+n)\gamma_1\frac{1-\alpha}{\alpha} - (1+n)\gamma_1\frac{1-\alpha}{\alpha}\frac{h}{\alpha}x_{t+1}\end{aligned}$$

This can be written

$$\begin{aligned}&-(1+n)\gamma_1\frac{1-\alpha}{\alpha}\frac{h}{\alpha}x_{t+1} + (1+n)(1-\gamma_1)\frac{1-\alpha}{\alpha}\\ &= -\frac{\gamma_3(1-\alpha)}{x_t} + (1+n+h) + (\gamma_1+\gamma_2)\,h\frac{1-\alpha}{\alpha} + (1+n)\frac{1-\alpha}{\alpha}\end{aligned}$$

or $G_U(x_t) = F_U(x_{t+1})$.

Assume now that borrowing is unconstrained in t and constrained in $t-1$. Then, c_t and s_t are given respectively by (2.3) and (2.11). Replacing in (2.15), we obtain:

$$(1+n+h)k_{t+1} = \eta\frac{w_t^{1-\eta}}{\lambda}\left(\frac{hw_{t+1}}{R_{t+1}}\right)^{\eta} - \frac{hw_{t+1}}{R_{t+1}} - (1+n)\gamma_1\left(\frac{w_{t+1}}{R_{t+1}} + \frac{hw_{t+2}}{R_{t+1}R_{t+2}}\right)$$

Replacing w_t and R_t with respect to k_t, we get:

$$(1+n+h)k_{t+1} = \eta\,(1-\alpha)\,\frac{k_t^{(1-\eta)\alpha}}{\lambda}\left(\frac{h}{\alpha}\right)^{\eta} k_{t+1}^{\eta} - \frac{h\,(1-\alpha)\,k_{t+1}}{\alpha}$$
$$-(1+n)\gamma_1\left(\frac{(1-\alpha)\,k_{t+1}}{\alpha} + \frac{h\,(1-\alpha)\,k_{t+2}}{\alpha^2 k_{t+1}^{\alpha-1}}\right)$$

Dividing by k_{t+1} and rearranging, we obtain:

$$-(1+n)\gamma_1\frac{1-\alpha}{\alpha}\frac{h}{\alpha}x_{t+1} + (1+n)(1-\gamma_1)\frac{1-\alpha}{\alpha} = -\eta\frac{1-\alpha}{\lambda}x_t^{\eta-1}\left(\frac{h}{\alpha}\right)^{\eta} + \frac{(1+n+h)}{\alpha}$$

or $G_C(x_t) = F_U(x_{t+1})$.

Assume now that borrowing is constrained in t and unconstrained in $t-1$. Then, c_t and s_t are given respectively by (2.8) and (2.6). Replacing in (2.15), we obtain:

$$(1+n+h)k_{t+1} = \gamma_3 w_t - (\gamma_1+\gamma_2)\,\frac{hw_{t+1}}{R_{t+1}} - \frac{(1+n)}{R_{t+1}}\left[w_{t+1} + \frac{hw_{t+2}}{R_{t+2}} - \frac{w_{t+1}^{1-\eta}}{\lambda}\left(\frac{hw_{t+2}}{R_{t+2}}\right)^{\eta}\right]$$

Replacing w_t and R_t with respect to k_t, we get:

$$(1+n+h)k_{t+1} = \gamma_3\,(1-\alpha)\,k_t^{\alpha} - (\gamma_1+\gamma_2)\,\frac{h\,(1-\alpha)\,k_{t+1}}{\alpha}$$
$$-(1+n)\left[\frac{(1-\alpha)\,k_{t+1}}{\alpha} + \frac{h\,(1-\alpha)\,k_{t+2}}{\alpha^2 k_{t+1}^{\alpha-1}} - \frac{(1-\alpha)}{\alpha}\frac{k_{t+1}^{1-\alpha\eta}}{\lambda}\left(\frac{h}{\alpha}\right)^{\eta} k_{t+2}^{\eta}\right]$$

Dividing by k_{t+1} and rearranging, we obtain:

$$-(1+n)\frac{1-\alpha}{\alpha}\left[\frac{h}{\alpha}x_{t+1} - \left(\frac{h}{\alpha}\right)^{\eta}\frac{x_{t+1}^{\eta}}{\lambda}\right]$$
$$= -\frac{\gamma_3\,(1-\alpha)}{x_t} + (1+n+h) + (\gamma_1+\gamma_2)\,h\frac{1-\alpha}{\alpha} + (1+n)\frac{1-\alpha}{\alpha}$$

or $G_U(x_t) = F_C(x_{t+1})$.

Assume now that borrowing is constrained in t and constrained in $t-1$. Then, c_t and s_t are given respectively by (2.8) and (2.11). Replacing in (2.15), we obtain:

$$(1+n+h)k_{t+1} = \eta\frac{w_t^{1-\eta}}{\lambda}\left(\frac{hw_{t+1}}{R_{t+1}}\right)^{\eta} - \frac{hw_{t+1}}{R_{t+1}} - \frac{(1+n)}{R_{t+1}}\left[w_{t+1} + \frac{hw_{t+2}}{R_{t+2}} - \frac{w_{t+1}^{1-\eta}}{\lambda}\left(\frac{hw_{t+2}}{R_{t+2}}\right)^{\eta}\right]$$

Replacing w_t and R_t with respect to k_t, we get:

$$(1+n+h)k_{t+1} = \eta\,(1-\alpha)\,\frac{k_t^{(1-\eta)\alpha}}{\lambda}\left(\frac{h}{\alpha}\right)^{\eta} k_{t+1}^{\eta} - \frac{h\,(1-\alpha)\,k_{t+1}}{\alpha}$$
$$-\,(1+n)\left[\frac{(1-\alpha)\,k_{t+1}}{\alpha} + \frac{h\,(1-\alpha)\,k_{t+2}}{\alpha^2 k_{t+1}^{\alpha-1}} - \frac{(1-\alpha)}{\alpha}\frac{k_{t+1}^{1-\alpha\eta}}{\lambda}\left(\frac{h}{\alpha}\right)^{\eta} k_{t+2}^{\eta}\right]$$

Dividing by k_{t+1} and rearranging, we obtain :

$$-(1+n)\frac{1-\alpha}{\alpha}\left[\frac{h}{\alpha}x_{t+1} - \left(\frac{h}{\alpha}\right)^{\eta}\frac{x_{t+1}^{\eta}}{\lambda}\right] = -\eta\frac{1-\alpha}{\lambda}x_t^{\eta-1}\left(\frac{h}{\alpha}\right)^{\eta} + \frac{(1+n+h)}{\alpha}$$

or $G_C(x_t) = F_C(x_{t+1})$.

At the initial period $t=0$, capital market equilibrium (2.22) and saving (2.19) gives:

$$(1+n+h)k_1 = \eta\,(w_0 - R_0 c_{-1}) - (1-\eta)\frac{hw_1}{R_1} - (1+n)c_0$$

or

$$\frac{1}{\alpha}(1+n+h(1-\eta(1-\alpha))) - \eta\left[(1-\alpha) - \alpha\frac{c_{-1}}{k_0}\right]\frac{1}{x_0} = F_{S_0}(x_1)$$

from which the left hand side will be treated as a function analogous to G_U and G_C, and parametrized by initial condition $\frac{c_{-1}}{k_0}$:

$$G_I\left(x, \frac{c_{-1}}{k_0}\right) = \frac{1}{\alpha}(1+n+h(1-\eta(1-\alpha))) - \eta\left[(1-\alpha) - \alpha\frac{c_{-1}}{k_0}\right]\frac{1}{x}$$

Note that saving s_0 must be non negative, or

$$\eta\,(w_0 - R_0 c_{-1}) \geq (1-\eta)\frac{hw_1}{R_1}$$

Dividing by k_0^{α}, this leads to

$$\eta\left[(1-\alpha) - \alpha\frac{c_{-1}}{k_0}\right] \geq h(1-\eta)\frac{1-\alpha}{\alpha}x_0$$

as $\frac{c_{-1}}{k_0} \geq 0$, this inequality implies

$$\eta(1-\alpha) \geq h(1-\eta)\frac{1-\alpha}{\alpha}x_0$$

or $x_0 \leq \frac{\gamma_3}{\gamma_2}\frac{\alpha}{h} = \hat{x}$.

Finally, the condition (2.16) must be satisfied:

$$R_0 c_{-1} \leq w_0 + \frac{hw_1}{R_1} - \frac{w_0^{1-\eta}}{\lambda}\left(\frac{hw_1}{R_1}\right)^{\eta}$$

Dividing by $R_0 k_0$, this condition can be written

$$\frac{c_{-1}}{k_0} \leq \frac{1-\alpha}{\alpha} + \frac{1-\alpha}{\alpha}\frac{h}{\alpha}x_0 - \frac{1-\alpha}{\alpha}\left(\frac{h}{\alpha}\right)^{\eta}\frac{x_0^{\eta}}{\lambda}$$

or

$$\frac{c_{-1}}{k_0} \leq \frac{1-\alpha}{\alpha} - \frac{F_C(x_0)}{1+n}.$$

2.5.5 Proof of Proposition 13

Proof of part 1.

We study the dynamical equation $F_U(x_{t+1}) = G_U(x_t)$. The equation $F_U(x) = G_U(x)$ has a unique solution x^*. It is unique as $F_U(x)$ is decreasing and $G_U(x)$ is increasing. Existence comes from the fact that $G_U(x)$ increases from $-\infty$ to $(1+n+h)+(\gamma_1+\gamma_2)\,h\frac{1-\alpha}{\alpha}+(1+n)\frac{1-\alpha}{\alpha} > 0$, while $F_U(x)$ decreases from $(1+n)(1-\gamma_1)\frac{1-\alpha}{\alpha} > 0$ to $-\infty$. Moreover, x^* is unstable. Indeed,

$$\left.\frac{dx_{t+1}}{dx_t}\right|_{x_t=x^*} = \frac{G'_U(x^*)}{F'_U(x^*)} = -\frac{\gamma_3}{(1+n)\gamma_1\frac{1}{\alpha}\frac{h}{\alpha}\,(x^*)^2}$$

Therefore, instability is obtained if

$$(x^*)^2 < \frac{\gamma_3}{(1+n)\gamma_1\frac{1}{\alpha}\frac{h}{\alpha}} \Leftrightarrow x^* < \alpha\sqrt{\frac{\gamma_3}{(1+n)\gamma_1 h}} \equiv x^l$$

which is true, as

$$G_U\left(\alpha\sqrt{\frac{\gamma_3}{(1+n)\gamma_1 h}}\right) > F_U\left(\alpha\sqrt{\frac{\gamma_3}{(1+n)\gamma_1 h}}\right)$$

Indeed,

$$\begin{aligned} G_U\left(x^l\right) - F_U\left(x^l\right) = & -\frac{\sqrt{\gamma_3(1+n)\gamma_1 h}\,(1-\alpha)}{\alpha} + (1+n+h) + (\gamma_1+\gamma_2)\,h\frac{1-\alpha}{\alpha} + \\ & (1+n)\frac{1-\alpha}{\alpha} + \sqrt{\gamma_3(1+n)\gamma_1 h}\frac{1-\alpha}{\alpha} - (1+n)(1-\gamma_1)\frac{1-\alpha}{\alpha} \end{aligned}$$

or

$$G_U\left(x^l\right) - F_U\left(x^l\right) = (1+n+h) + (\gamma_1+\gamma_2)\,h\frac{1-\alpha}{\alpha} + (1+n)\gamma_1\frac{1-\alpha}{\alpha} > 0.$$

Proof of part 2.

We study the dynamical equation $F_C(x_{t+1}) = G_C(x_t)$. We consider the solutions to the equation $F_C(x) = G_C(x)$. $F_C(x)$ is a concave inverse U-shape function with a maximum reached for $x = \hat{x} = \frac{\alpha}{h}\frac{\eta}{1-\eta} = \frac{\alpha}{h}\frac{\gamma_3}{\gamma_2}$. Moreover, $F_C(0) = 0$.

$G_C(x)$ is a concave increasing function with $G_C(0+) = -\infty$. It is possible to show that $G_C(\hat{x}) > F_C(\hat{x})$:

$$G_C(\hat{x}) = \frac{1+n+h/\alpha}{\alpha} > F_C(\hat{x}) = \frac{(1+n)(1-\alpha)}{\alpha}$$

Therefore, there exists at least one solution to $F_C(x) = G_C(x)$, and possible solutions belong to the interval $(0, \hat{x})$. In fact, as $G_C(\hat{x}) > F_C(\hat{x})$, a solution must belong to the interval $(0, \check{x})$ with $\check{x}$ defined as $G_C(\check{x}) = F_C(\hat{x})$. This equation gives:

$$\check{x} = \frac{\eta}{1-\eta} \left(\frac{h}{\alpha}\right)^{\frac{\eta}{1-\eta}} \frac{(1-\alpha)^{\frac{1}{1-\eta}}}{(1+n+h/\alpha)^{\frac{1}{1-\eta}}}$$

We want to prove that only one solution is possible on the interval $(0, \check{x})$ when $\alpha > \bar{\alpha}$. If it is not true, there must exist a least three solutions on $(0, \check{x})$ such that $G_C(x) - F_C(x) = 0$. This implies that there must exist at least two solutions on $(0, \check{x})$ of the equation $G'_C(x) - F'_C(x) = 0$. This implies that there must exist at least one value of x on $(0, \check{x})$ such that $G''_C(x) - F''_C(x) = 0$. This last condition gives:

$$\begin{aligned} G''_C(x) - F''_C(x) = & -\eta(1-\eta)(2-\eta)\frac{1-\alpha}{\lambda}\left(\frac{h}{\alpha}\right)^{\eta} x^{\eta-3} + \\ & \eta(1-\eta)(1+n)\frac{1-\alpha}{\alpha\lambda}\left(\frac{h}{\alpha}\right)^{\eta} x^{\eta-2} = 0 \end{aligned}$$

or

$$x = \frac{(2-\eta)\,\alpha}{1+n}$$

Finally, if $\frac{(2-\eta)\alpha}{1+n} > \check{x}$, it is impossible to have more than one stationary solution in the constrained regime. This last condition can be written

$$\frac{\alpha}{1-\alpha}\frac{1+n+h/\alpha}{(1+n)^{1-\eta}h^{\eta}}\left[\frac{(2-\eta)\,(1-\eta)}{\eta}\right]^{1-\eta} > 1 \tag{2.58}$$

We consider the term $(1+n+h/\alpha)\,h^{-\eta}$ as a function of h. This is a U-shape function, with a minimum for

$$h = \frac{\eta\alpha(1+n)}{1-\eta}$$

Therefore, a stronger condition than (2.58) can be obtained for this value of h :

$$\frac{\alpha^{1-\eta}}{1-\alpha}\frac{(2-\eta)^{1-\eta}}{\eta} > 1$$

This last condition can be written:

$$X(\eta) = (1-\eta)\ln\alpha - \ln(1-\alpha) + (1-\eta)\ln(2-\eta) - \ln\eta > 0 \tag{2.59}$$

We have:

$$\begin{aligned} X'(\eta) &= -\ln\alpha - \ln(2-\eta) - \frac{(1-\eta)}{(2-\eta)} - \frac{1}{\eta} \\ X''(\eta) &= \frac{1}{(2-\eta)} + \frac{1}{(2-\eta)^2} + \frac{1}{\eta^2} > 0 \end{aligned}$$

From these results, two cases must be studied. If $X'(1) < 0$, which means $\alpha > 1/e$, X is decreasing in η and a stronger condition than (2.59) is $X(1) > 0$ or $-\ln(1-\alpha) > 0$, which is satisfied.

If $\alpha < 1/e$, $X(\eta)$ reaches a minimum at some value $\grave{\eta}$ defined by $X'(\grave{\eta}) = 0$. Then, (2.59) is satisfied if α is greater than some threshold $\bar{\alpha}$ defined by the solution α of the system of two equations:

$$
\begin{aligned}
X(\eta) &= (1-\eta)\ln\alpha - \ln(1-\alpha) + (1-\eta)\ln(2-\eta) - \ln\eta = 0 \\
X'(\eta) &= -\ln\alpha - \ln(2-\eta) - \frac{(1-\eta)}{(2-\eta)} - \frac{1}{\eta} = 0
\end{aligned}
$$

This system allows to calculate $\bar{\alpha}$ numerically as $\bar{\alpha} \simeq 0.14115$.

For $\alpha > \bar{\alpha}$, we denote by $\tilde{x}$ the unique steady state in the constrained regime. This steady state is unstable as

$$\left.\frac{dx_{t+1}}{dx_t}\right|_{x_t=\tilde{x}} = \frac{G'_C(\tilde{x})}{F'_C(\tilde{x})}$$

and $G'_C(\tilde{x}) > F'_C(\tilde{x})$ as $G_C(x) < F_C(x)$ for $x < \tilde{x}$.

Proof of part 3.

One technical result is needed to prove this part:

Lemma 9. *The functions F_C, F_U, G_C and G_U have the following properties on* $(0, \hat{x})$:

- $F_C(x) > F_U(x) \Leftrightarrow x \in (\underline{x}, \hat{x})$ *and* $G_C(x) < G_U(x) \Leftrightarrow x \in (\underline{x}, \hat{x})$.
- $F_C(x) = F_U(x) \Leftrightarrow x = \underline{x}$ *and* $G_C(x) = G_U(x) \Leftrightarrow x = \underline{x}$.
- $F_C(x) < F_U(x) \Leftrightarrow x \in (0, \underline{x})$ *and* $G_C(x) < G_U(x) \Leftrightarrow x \in (0, \underline{x})$.

Proof. The lemma results from the following properties:

$$
\begin{aligned}
F_C(x) - F_U(x) &= (1+n)\frac{1-\alpha}{\alpha}\left[\left(\frac{h}{\alpha}\right)^{\eta}\frac{x^{\eta}}{\lambda} - (1-\gamma_1)\left(\frac{h}{\alpha}x + 1\right)\right] \\
G_U(x) - G_C(x) &= \frac{\gamma_3(1-\alpha)}{(1-\gamma_1)x}\left[\left(\frac{h}{\alpha}\right)^{\eta}\frac{x^{\eta}}{\lambda} - (1-\gamma_1)\left(\frac{h}{\alpha}x + 1\right)\right]
\end{aligned}
$$

Then, lemma 7 can be used to conclude. □

Therefore, $\underline{x} < \hat{x} < \bar{x}$.

Moreover, from the proof of part 2, we know that $\tilde{x} < \hat{x}$, and then $\tilde{x} < \hat{x} < \bar{x}$.

Assume first that $x^* < \underline{x}$. Consequently, there exists a steady state in the unconstrained regime. From lemma 9, as $x^* < \underline{x}$, we have $G_C(x^*) > G_U(x^*) = F_U(x^*) > F_C(x^*)$. As $G_C(x^*) > F_C(x^*)$, this implies $\tilde{x} < x^*$and therefore $\tilde{x} < \underline{\mathrm{x}}$: the unconstrained steady state does not exist.

The second possible case is when $x^* > \underline{\mathrm{x}}$. In this case, there does not exist a steady state in the unconstrained regime. In the interval $(\underline{\mathrm{x}}, \hat{x})$, we know from Lemma 9 that

$G_C(x) < G_U(x)$ and $F_C(x) > F_U(x)$. Therefore we get: $F_C(x^*) > F_U(x^*) = G_U(x^*) > G_C(x^*)$. The inequality $F_C(x^*) > G_C(x^*)$ implies that $x^* < \tilde{x}$. As we have proved that $\tilde{x} < \hat{x}$ is always fulfilled, we get finally $\underline{x} < x^* < \tilde{x} < \bar{x}$. There exists a steady state in the constrained regime.

2.5.6 Proof of Proposition 14

From proposition 13, two cases must be studied: $\tilde{x} < x^* < \underline{x}$ (unconstrained steady state), and $\underline{x} < x^* < \tilde{x}$ (constrained steady state).

Proof in case 1: $\tilde{x} < x^* < \underline{x}$

The proof results from different lemma. Assume that $(x_t)_{t\geq 0}$ is an equilibrium dynamics.

Lemma 10. *Assume that $x_\tau \in (\underline{x}, \hat{x})$ for some date $\tau \in \mathbb{N}$ (constrained regime in period $\tau - 1$). Then, the dynamics cannot stay in the constrained regime at all future periods.*

Proof. Assume that $x_t \in (\underline{x}, \hat{x})$ from period τ. Two cases may arise.

1st case: $G_U(\underline{\mathrm{x}}) = G_C(\underline{\mathrm{x}}) > F_C(\hat{x})$. As G_C is increasing, and F_C is an inverse U-shape function that reaches its maximum for $x = \hat{x}$, for any $x_\tau \in (\underline{x}, \hat{x})$, there does not exist $x_{\tau+1}$ such that $G_C(x_\tau) = F_C(x_{\tau+1})$.

2nd case: $G_U(\underline{\mathrm{x}}) = G_C(\underline{\mathrm{x}}) < F_C(\hat{x})$. From the proof of Proposition 13 part 2, it is known that $G_C(\hat{x}) > F_C(\hat{x})$. Then it is possible to define x^s such that $G_C(x^s) = F_C(\hat{x})$. If $x_\tau \in [x^s, \hat{x}]$, there does not exist $x_{\tau+1}$ such that $G_C(x_\tau) = F_C(x_{\tau+1})$. If $x_\tau \in (\underline{x}, x^s)$, the dynamics defined by $G_C(x_t) = F_C(x_{t+1})$ is increasing as both functions are increasing and $G_C(x) > F_C(x)$. In a finite number of periods i, $x_{\tau+i}$ is higher than x^s and the dynamics is no more defined in the constrained regime. □

Lemma 11. *For $x \in [0, \underline{x}]$, the function $F_U^{-1} \circ G_U(x)$ is decreasing. The dynamics $x_{t+1} = F_U^{-1} \circ G_U(x_t)$ admits a steady state x^* and no limit cycle of period 2. If $x_\tau \neq x^*$, in a finite number of period, there does not exist $x_{t+1} \in [0, \underline{x}]$ such that $x_{t+1} = F_U^{-1} \circ G_U(x_t)$.*

Proof. $F_U^{-1} \circ G_U(x)$ is decreasing as F_U is decreasing and G_U is increasing. From proposition 13, $x_{t+1} = F_U^{-1} \circ G_U(x_t)$ admits a steady state x^*. Consider the second iterate:

$$
\begin{aligned}
&F_U^{-1} \circ G_U \circ F_U^{-1} \circ G_U(x_t)\\
&= \frac{-\gamma_3(1-\alpha)x_t}{-\gamma_3(1-\alpha) + \left(1 + n + h + (\gamma_1+\gamma_2)h^{\frac{1-\alpha}{\alpha}} + \gamma_1(1+n)^{\frac{1-\alpha}{\alpha}}\right)x_t}\\
&\quad - \left(\frac{1}{(1+n)\gamma_1^{\frac{1-\alpha}{\alpha}}\frac{h}{\alpha}}\right)\left(1 + n + h + (\gamma_1+\gamma_2)h\frac{1-\alpha}{\alpha} + \gamma_1(1+n)\frac{1-\alpha}{\alpha}\right)
\end{aligned}
$$

This function is increasing, it is an hyperbola with a positive vertical asymptote for $x_t = x^a$ with

$$
x^a = \frac{\gamma_3(1-\alpha)}{1 + n + h + (\gamma_1+\gamma_2)h^{\frac{1-\alpha}{\alpha}} + \gamma_1(1+n)^{\frac{1-\alpha}{\alpha}}}
$$

and $F_U^{-1} \circ G_U \circ F_U^{-1} \circ G_U(0) < 0$. Therefore, the function as a unique positive steady state value $x^* < x^a$. As no other steady state exists, $F_U^{-1} \circ G_U(x_t)$ does not admit a limit cycle of period 2.

The dynamics $x_{t+1} = F_U^{-1} \circ G_U(x_t)$ is oscillating and divergent. If $x_\tau \neq x^*$, in a finite number of period, there does not exist $x_{t+1} \in [0, \underline{x}]$ such that $x_{t+1} = F_U^{-1} \circ G_U(x_t)$: either $G_U(x_t) < F_U(\underline{\mathrm{x}})$ or $G_U(x_t) > F_U(0)$. □

Corollary 2. *For $\tilde{x} < x^* < \underline{x}, F_U^{-1} \circ G_U(\underline{x}) < G_U^{-1} \circ F_U(\underline{x})$.*

Proof. From the preceding lemma, it is known that x^* is the only value of $x \in [0, \underline{x}]$ such that: $F_U^{-1} \circ G_U(x^*) = G_U^{-1} \circ F_U(x^*)$. Indeed, such value x must be solution of $F_U^{-1} \circ G_U \circ F_U^{-1} \circ G_U(x) = x$. Moreover, for $x = 0$, $F_U^{-1} \circ G_U(0) = +\infty > G_U^{-1} \circ F_U(0)$. Then, $F_U^{-1} \circ G_U(x) > G_U^{-1} \circ F_U(x)$ for $x < x^*$ and $F_U^{-1} \circ G_U(x) < G_U^{-1} \circ F_U(x)$ for $x > x^*$ by continuity of $F_U^{-1} \circ G_U$ and $G_U^{-1} \circ F_U$. As $\underline{x} > x^*$, $F_U^{-1} \circ G_U(\underline{x}) < G_U^{-1} \circ F_U(\underline{x})$. □

Lemma 12. *Assume that $x_\tau \in (\underline{x}, \hat{x})$ for some date $\tau \in \mathbb{N}$ (constrained regime in period $\tau - 1$). Then, either a transition to the unconstrained regime is impossible, or, if it is possible, the dynamics is no more defined one period later.*

Proof. Assume that $x_t \in (\underline{x}, \hat{x})$ from period τ. Two cases may arise.

1st case: $F_U(0) < G_U(\underline{x}) = G_C(\underline{x})$. As F_U is a decreasing function and G_C is increasing, for any $x_\tau \in (\underline{x}, \hat{x})$, there does not exist $x_{\tau+1}$ such that $G_C(x_\tau) = F_U(x_{\tau+1})$. In this case, a transition to the unconstrained regime is impossible.

2nd case: $F_U(0) > G_U(\underline{x}) = G_C(\underline{x})$. In this case, for some value $x_\tau \in (\underline{x}, \hat{x})$, there may exist $x_{\tau+1}$ such that $G_C(x_\tau) = F_U(x_{\tau+1})$. But it is possible to prove that $x_{\tau+2}$ cannot exist.

Two sub-cases must be studied. First, it is impossible to find $x_{\tau+2}$ in the constrained regime. Indeed, as $x_\tau > \underline{x}$ and G_C is increasing, $G_C(x_\tau) > G_C(\underline{x})$. As F_U is decreasing, $F_U^{-1} \circ G_C(x_\tau) < F_U^{-1} \circ G_C(\underline{x})$. From the previous corollary, we get $x_{\tau+1} = F_U^{-1} \circ G_C(x_\tau) < G_U^{-1} \circ F_U(\underline{x})$. Therefore, we get: $G_U(x_{\tau+1}) < F_U(\underline{x}) = F_C(\underline{x})$. This last inequality implies that there does not exist $x_{\tau+2}$ in the constrained regime such that $F_C(x_{\tau+2}) = G_U(x_{\tau+1})$ as $F_C(x) > G_U(x_{\tau+1})$ for all $x \in (\underline{x}, \hat{x})$.

Secondly, it remains to prove that it is impossible to find $x_{\tau+2}$ in the unconstrained regime. We know that $G_U(x_{\tau+1}) < F_U(\underline{\mathrm{x}})$. But $F_U(\underline{\mathrm{x}})$ is the lowest value of F_U on $[0, \underline{x}]$ as F_U is decreasing. Therefore, there does not exist $x_{\tau+2}$ in the unconstrained regime such that $F_U(x_{\tau+2}) = G_U(x_{\tau+1})$ as $F_U(x) > G_U(x_{\tau+1})$ for all $x \in [0, \underline{x}]$. □

Proof in case 2: $\underline{x} < x^* < \tilde{x}$

The proof is achieved in two steps. First, we show that no period can be in the unconstrained regime.

Lemma 13. *Assume that $x_\tau \in (0, \underline{x})$ for some date $\tau \in \mathbb{N}$ (unconstrained regime in period $\tau - 1$). Then, it is impossible to find $x_{\tau+1}$ and S_τ such that $G_U(x_\tau) = F_{S_\tau}(x_{\tau+1})$.*

Proof. Assume that $0 < x_\tau < \underline{x}$ (unconstrained regime in $\tau - 1$). Then one can prove equilibrium is not defined for $\tau + 1$. Firstly, we prove that there is no $x_{\tau+1} < \underline{x}$ such that $F_U(x_{\tau+1}) = G_U(x_\tau)$. As $\underline{\mathrm{x}} < x^*$ and $F_U(x) - G_U(x)$ is a decreasing function of x, we get $F_U(\underline{x}) > G_U(\underline{x})$. As F_U is decreasing on $(0, \underline{x})$ and G_U is increasing, there is no $x_{\tau+1} < \underline{x}$ such that $F_U(x_{\tau+1}) = G_U(x_\tau)$. It means there is no unconstrained equilibrium for period τ. Secondly, we prove that there is no $x_{\tau+1} > \underline{x}$ such that $F_C(x_{\tau+1}) = G_U(x_\tau)$. We know that $F_C(\underline{x}) = F_U(\underline{x}) > G_U(\underline{x}) > G_U(x_\tau)$ and that F_C is increasing for $x \in [\underline{x}, \hat{x})$. This implies there is no $x_{\tau+1} \in [\underline{x}, \hat{x})$ such that $F_C(x_{\tau+1}) = G_U(x_\tau)$. It means there is no constrained equilibrium in period τ. □

Secondly, we show that it is impossible to stay in the constrained regime if x_t is not at the stationary value $\tilde{x}$.

Lemma 14. *Assume that $x_\tau \in (\underline{x}, \tilde{x}) \cup (\tilde{x}, \hat{x})$ for some date $\tau \in \mathbb{N}$ (constrained regime in period $\tau - 1$). Then, it is impossible to stay at all future periods in the constrained regime: there does not exist a sequence $(x_t)_{t \geq \tau}$ with $G_C(x_t) = F_C(x_{t+1})$ and $x_t \in (\underline{x}, \hat{x})$.*

Proof. As G_C and F_C are both increasing functions, the dynamics in the constrained regime $G_C(x_t) = F_C(x_{t+1})$ is monotonic. If $x_\tau > \tilde{x}$, x_t will become greater than $\hat{x}$ in a finite number of periods, which is impossible. If $x_\tau < \tilde{x}$, x_t will become smaller than $\underline{\mathrm{x}}$ in a finite number of periods, which is impossible. □

2.5.7 Proof of Proposition 15

Proof in the case $x^* \leq \tilde{x}$.

From Corollary (1), we have to study the existence of a value x_0 such that $G_I\left(x_0, \frac{c-1}{k_0}\right) = F_U(x^*)$.

Lemma 15. *For $\frac{c-1}{k_0} < \frac{1-\alpha}{\alpha}$, the equation $G_I\left(x_0, \frac{c-1}{k_0}\right) = F_U(x^*)$ has a unique solution $x_0 \in (0, \hat{x})$. Moreover, the solution is a decreasing function of $\frac{c-1}{k_0}$.*

Proof. For $\frac{c-1}{k_0} < \frac{1-\alpha}{\alpha}$, $G_I\left(x, \frac{c-1}{k_0}\right)$ is an increasing function with respect to x and $\frac{c-1}{k_0}$. When x tends to 0, $G_I\left(x, \frac{c-1}{k_0}\right)$ tends to $-\infty$. Then $G_I\left(x, \frac{c-1}{k_0}\right) < F_U(x^*)$ for x close to 0. For $x = \hat{x}$, we get $G_I\left(\hat{x}, \frac{c-1}{k_0}\right) > G_I(\hat{x}, 0) = \frac{1+n}{\alpha} + h$. We have also $F_U(0) = (1+n)(1-\gamma_1)\frac{1-\alpha}{\alpha} > F_U(x^*)$ as F_U is decreasing. As $G_I(\hat{x}, 0) - F_U(0) > 0$, we conclude that $G_I\left(\hat{x}, \frac{c-1}{k_0}\right) > F_U(x^*)$. Finally, there exists a unique solution $x_0 \in (0, \hat{x})$ to the equation $G_I\left(x_0, \frac{c-1}{k_0}\right) = F_U(x^*)$. From the properties of G_I, we deduce that this solution is a decreasing function of $\frac{c-1}{k_0}$. □

From this Lemma, it is easy to prove the first part of Proposition 15.

If $\frac{c-1}{k_0} = l$, $x_0 = x_l$ is the unique solution as $G_I(x_l, l) = G_C(x_l) = F_U(x^*)$. x_l is smaller than x^* as $G_C(x) > G_U(x)$ for $x < \underline{\mathrm{x}}$ and $G_U(x^*) = F_U(x^*)$.

If $\frac{c-1}{k_0} < l$, There exists a unique solution $x_0 = x_l$ to the equation $G_I\left(x_0, \frac{c-1}{k_0}\right) = F_U(x^*)$ with $x_0 > x_l$. Moreover, x_0 is a decreasing function of $\frac{c-1}{k_0}$.

For $\left(\frac{c_{-1}}{k_0}\right) > l$, the equation $G_I\left(x_0, \frac{c_{-1}}{k_0}\right) = F_U(x^*)$ has a solution x_0 that is smaller than x_l. Assume that this solution satisfies the non default constraint $\frac{c_{-1}}{k_0} \leq \frac{1-\alpha}{\alpha} - \frac{F_C(x_0)}{1+n}$. Then, we get:

$$F_U(x^*) = G_I\left(x_0, \frac{c_{-1}}{k_0}\right) \leq G_I\left(x_0, \frac{1-\alpha}{\alpha} - \frac{F_C(x_0)}{1+n}\right)$$

as G_I increases with $\frac{c_{-1}}{k_0}$. But $G_I\left(x_0, \frac{1-\alpha}{\alpha} - \frac{F_C(x_0)}{1+n}\right) = G_C(x_0)$ and $F_U(x^*) = G_C(x_l)$ by definition, so we get: $G_C(x_l) \leq G_C(x_0)$. As G_C is an increasing function, we get $x_l \leq x_0$, which contradicts the previous result $x_0 < x_l$. Then the non default constraint cannot be satisfied for $\left(\frac{c_{-1}}{k_0}\right) > l$.

Proof in the case $x^* > \tilde{x}$.

From Corollary (1), we have to study the existence of a value x_0 such that $G_I\left(x_0, \frac{c_{-1}}{k_0}\right) = F_C(\tilde{x})$.

Lemma 16. *For $\frac{c_{-1}}{k_0} < \frac{1-\alpha}{\alpha}$, the equation $G_I\left(x_0, \frac{c_{-1}}{k_0}\right) = F_C(\tilde{x})$ has a unique solution $x_0 \in (0, \hat{x})$. Moreover, the solution is a decreasing function of $\frac{c_{-1}}{k_0}$.*

Proof. For $\frac{c_{-1}}{k_0} < \frac{1-\alpha}{\alpha}$, $G_I\left(x, \frac{c_{-1}}{k_0}\right)$ is an increasing function with respect to x and $\frac{c_{-1}}{k_0}$. When x tends to 0, $G_I\left(x, \frac{c_{-1}}{k_0}\right)$ tends to $-\infty$. Then $G_I\left(x, \frac{c_{-1}}{k_0}\right) < F_C(\tilde{x})$ for x close to 0. For $x = \hat{x}$, we get $G_I\left(\hat{x}, \frac{c_{-1}}{k_0}\right) > G_I(\hat{x}, 0) = \frac{1+n}{\alpha} + h$. We have also $F_C(\tilde{x}) = (1+n)\frac{1-\alpha}{\alpha} < \frac{1+n}{\alpha} + h$. Then, $G_I\left(\hat{x}, \frac{c_{-1}}{k_0}\right) > F_U(x^*)$. Finally, there exists a unique solution $x_0 \in (0, \hat{x})$ to the equation $G_I\left(x_0, \frac{c_{-1}}{k_0}\right) = F_C(\tilde{x})$. From the properties of G_I, we deduce that this solution is a decreasing function of $\frac{c_{-1}}{k_0}$. □

From this Lemma, it is easy to prove the second part of Proposition 15.

If $\frac{c_{-1}}{k_0} = \tilde{l}$, $x_0 = \tilde{x}$ is the unique solution as

$$G_I(\tilde{x}, \tilde{l}) = G_I\left[\tilde{x}, \frac{1-\alpha}{\alpha} - \frac{F_C(\tilde{x})}{1+n}\right] = G_C(\tilde{x}) = F_C(\tilde{x}).$$

For $\frac{c_{-1}}{k_0} < \tilde{l}$, there exists a unique solution x_0 with $x_0 > \tilde{x}$. Moreover, x_0 is a decreasing function of $\frac{c_{-1}}{k_0}$.

For $\left(\frac{c_{-1}}{k_0}\right) > \tilde{l}$, the equation $G_I\left(x_0, \frac{c_{-1}}{k_0}\right) = F_C(\tilde{x})$ has a solution x_0 that is smaller than $\tilde{x}$. Assume that this solution satisfies the non default constraint $\frac{c_{-1}}{k_0} \leq \frac{1-\alpha}{\alpha} - \frac{F_C(x_0)}{1+n}$. Then, we get:

$$F_C(\tilde{x}) = G_I\left(x_0, \frac{c_{-1}}{k_0}\right) \leq G_I\left(x_0, \frac{1-\alpha}{\alpha} - \frac{F_C(x_0)}{1+n}\right) = G_C(x_0)$$

By definition, $F_C(\tilde{x}) = G_C(\tilde{x})$. Finally, we get: $G_C(\tilde{x}) \leq G_C(x_0)$. As G_C is an increasing function, $\tilde{x} \leq x_0$ which contradicts the previous result $x_0 < \tilde{x}$. Then the non default constraint cannot be satisfied for $\left(\frac{c_{-1}}{k_0}\right) > \tilde{l}$.

2.5.8 Proof of Lemma 8

For a given value of x_t, we want to find a solution δ_t to the equation (2.42). Let us define

$$H(x,\delta) = \left(1+\frac{h}{\alpha}x\delta\right) - (1-\gamma_1)\left(1+\frac{h}{\alpha}x\right)(\delta)^{\eta}$$

For $x=0$, $H(x,\delta)=0$ has the solution $\delta = (1-\gamma_1)^{-1/\eta}$.

For $x>0$, we consider the derivative $H'_\delta(x,\delta)$:

$$H'_\delta(x,\delta) = \frac{h}{\alpha}x - (1-\gamma_1)\left(1+\frac{h}{\alpha}x\right)\eta\,(\delta)^{\eta-1}$$

We get $H'_\delta(x,\delta)=0$ for

$$\delta = \delta_x \equiv \left[\frac{(1-\gamma_1)\,\eta\left(1+\frac{h}{\alpha}x\right)}{\frac{h}{\alpha}x}\right]^{\frac{1}{1-\eta}}$$

$H(x,\delta)$ is a U-shaped function that reaches a minimum in δ_x. $H(x,\delta)=0$ has a solution δ only if $H(x,\delta_x)\leq 0$. When $H(x,\delta_x)<0$, the equation $H(x,\delta)=0$ has 2 solutions.

We get:

$$H(x,\delta_x) = 1 - \left[\frac{(1-\gamma_1)\,\eta\left(1+\frac{h}{\alpha}x\right)}{\frac{h}{\alpha}x}\right]^{\frac{1}{1-\eta}} x\frac{h}{\alpha}\frac{1-\eta}{\eta}$$

The condition $H(x,\delta_x)\leq 0$ is equivalent to:

$$(1-\gamma_1)\left(1+\frac{h}{\alpha}x\right) \geq \left(\frac{h}{\alpha}\right)^{\eta}\frac{x^{\eta}}{\lambda}$$

From Lemma (7), we know that this condition is satisfied for $x\in[0,\underline{x}]\cup[\bar{x},+\infty)$, with an equality for $x=\underline{x}$ or $x=\bar{x}$.

The case $x\in[\bar{x},+\infty)$ is impossible. Indeed, we know that an equilibrium must satisfy $x_t < \frac{\alpha}{h}\frac{\gamma_3}{1-\gamma_3}$. It is easy to check that $\underline{x} < \frac{\alpha}{h}\frac{\gamma_3}{1-\gamma_3} < \bar{x}$ because

$$(1-\gamma_1)\left(1+\frac{h}{\alpha}\frac{\alpha}{h}\frac{\gamma_3}{1-\gamma_3}\right) < \left(\frac{h}{\alpha}\right)^{\eta}\frac{\left(\frac{\alpha}{h}\frac{\gamma_3}{1-\gamma_3}\right)^{\eta}}{\lambda}$$

as this is equivalent to $\gamma_2 < 1-\gamma_3$.

Therefore, the only possible case is $x\in[0,\underline{x}]$.

For $x\in(0,\underline{x})$, the equation $H(x,\delta)=0$ has two solutions in δ, one on $(0,\delta_x)$ for which $H(x,\delta)$ is decreasing in δ, one on $(\delta_x,+\infty)$ for which $H(x,\delta)$ is increasing in δ. The second solution is not possible along an equilibrium, from condition (2.41). Indeed, (2.41) implies that $\delta\leq\frac{1}{x}\frac{\alpha}{h}\frac{\gamma_3}{\gamma_2}$. We calculate:

$$H\left(x,\frac{1}{x}\frac{\alpha}{h}\frac{\gamma_3}{\gamma_2}\right) = \left(\frac{1}{x}\frac{\alpha}{h}\frac{\gamma_3}{\gamma_2}\right)^{\eta}\left[\left(\frac{h}{\alpha}\right)^{\eta}\frac{x^{\eta}}{\lambda} - (1-\gamma_1)\left(1+\frac{h}{\alpha}x\right)\right] \tag{2.60}$$

Then, $H\left(x, \frac{1}{x}\frac{\alpha}{h}\frac{\gamma_3}{\gamma_2}\right) < 0$ for $x \in (0, \underline{x})$. As δ must be smaller than $\frac{1}{x}\frac{\alpha}{h}\frac{\gamma_3}{\gamma_2}$, the only possible solution is on $(0, \delta_x)$ for which $H(x,\delta)$ is decreasing in δ.

Finally, the equation $H(x,\delta) = 0$ has a unique solution $\delta = \Delta(x)$ for $x \in (0, \underline{x})$, with $H'_\delta(x, \Delta(x)) < 0$. Moreover, $H'_x(x,\delta) > 0$ as

$$H'_x(x,\delta) = \frac{h}{\alpha}\delta - (1-\gamma_1)\,\frac{h}{\alpha}\,(\delta)^\eta$$

which is positive for $\delta > (1-\gamma_1)^{\frac{1}{1-\eta}}$. This condition is true as $\delta > 1$ and $(1-\gamma_1)^{\frac{1}{1-\eta}} < 1$.

Finally we get:

$$\Delta'(x) = -\frac{H'_x(x,\Delta(x))}{H'_\delta(x,\Delta(x))} > 0.$$

In the case $x = \underline{x}$, from (2.60), we get that $\underline{\delta} = \frac{1}{\underline{x}}\frac{\alpha}{h}\frac{\gamma_3}{\gamma_2}$ is the solution.

2.5.9 Proof of Proposition 17

Under the assumption $\frac{\gamma_3}{\gamma_2} > \frac{h}{1+n}$, we know that $\breve{x}, \breve{\delta}$ solutions of equations (2.50) and (2.51) exist with $\breve{x} < \underline{x}$. Then, the steady state $(\breve{x}, \breve{\pi}, \breve{\delta})$ exists if equation (2.49) leads to a value of $\breve{\pi}$ such that $0 < \breve{\pi} < 1$. In the same way, $(\underline{x}, \underline{\pi}, \underline{\delta})$ will exist if equation (2.49) leads to a value of $\underline{\pi}$ such that $0 < \underline{\pi} < 1$.

Equation (2.49) can be written:

$$(1-\pi) = \frac{1+n+h}{\frac{\gamma_3(1-\alpha)}{x} - (\gamma_1+\gamma_2)\,h\frac{1-\alpha}{\alpha} - \gamma_1(1+n)\frac{1-\alpha}{\alpha} - \gamma_1(1+n)\frac{h}{\alpha}\frac{1-\alpha}{\alpha}x}$$

The condition $0 < \pi < 1$ is satisfied iff

$$\begin{aligned} 0 &< \frac{\gamma_3(1-\alpha)}{x} - (\gamma_1+\gamma_2)\,h\frac{1-\alpha}{\alpha} - \gamma_1(1+n)\frac{1-\alpha}{\alpha} - \gamma_1(1+n)\frac{h}{\alpha}\frac{1-\alpha}{\alpha}x \\ 1+n+h &< \frac{\gamma_3(1-\alpha)}{x} - (\gamma_1+\gamma_2)\,h\frac{1-\alpha}{\alpha} - \gamma_1(1+n)\frac{1-\alpha}{\alpha} - \gamma_1(1+n)\frac{h}{\alpha}\frac{1-\alpha}{\alpha}x \end{aligned}$$

The second condition is stronger and is the only condition to be satisfied. We define a function $\xi(x)$ as

$$\xi(x) = \frac{\gamma_3(1-\alpha)}{x} - (\gamma_1+\gamma_2)\,h\frac{1-\alpha}{\alpha} - \gamma_1(1+n)\frac{1-\alpha}{\alpha} - \gamma_1(1+n)\frac{h}{\alpha}\frac{1-\alpha}{\alpha}x - (1+n+h)$$

and the previous condition then becomes $\xi(x) > 0$. $\xi(x)$ is a decreasing function of x with $\xi(x^*) = 0$ by definition of x^*. Therefore, the condition $\xi(x) > 0$ is equivalent to $x < x^*$. Following this result, it is easy to conclude on the three cases of the proposition.

If $\breve{x} < \underline{x} < x^*$, as $x^* > \underline{x}$, the steady state (x^*, π^*, δ^*) cannot exist. Both steady states $(\breve{x}, \breve{\pi}, \breve{\delta})$ and $(\underline{x}, \underline{\pi}, \underline{\delta})$ exist as $\breve{x} < x^*$ and $\underline{x} < x^*$.

If $\breve{x} < x^* \leq \underline{x}$, the steady state $(\underline{x}, \underline{\pi}, \underline{\delta})$ cannot exist. Both steady states $(\breve{x}, \breve{\pi}, \breve{\delta})$ and (x^*, π^*, δ^*) exist as $\breve{x} < x^*$ and $x^* < \underline{x}$.

If $x^* < \breve{x} \leq \underline{x}$, the steady state (x^*, π^*, δ^*) is the only one. Both steady states $(\breve{x}, \breve{\pi}, \breve{\delta})$

and $(\underline{x}, \overline{\pi}, \underline{\delta})$ do not exist as $\breve{x} > x^*$ and $\underline{x} > x^*$.

2.5.10 Proof of Proposition 18

Consider the dynamical system given by (2.47) and (2.48). Consider a steady state solution (x, π) of this system. The local dynamics is governed by the Jacobean matrix at that steady state. The partial derivatives are solutions of the system:

$$
\begin{aligned}
\gamma_1(1-\pi)(1+n)\frac{1-\alpha}{\alpha}\frac{h}{\alpha}\frac{\partial x_{t+1}}{\partial x_t} - \gamma_1(1+n)\frac{1-\alpha}{\alpha}\left(1+\frac{h}{\alpha}x\right)\frac{\partial \pi_{t+1}}{\partial x_t} &= -\frac{(1-\pi)\gamma_3(1-\alpha)}{x^2} \\
\gamma_1(1-\pi)(1+n)\frac{1-\alpha}{\alpha}\frac{h}{\alpha}\frac{\partial x_{t+1}}{\partial \pi_t} - \gamma_1(1+n)\frac{1-\alpha}{\alpha}\left(1+\frac{h}{\alpha}x\right)\frac{\partial \pi_{t+1}}{\partial \pi_t} &= (\gamma_1+\gamma_2)\, h\frac{1-\alpha}{\alpha} \\
&\quad - \frac{\gamma_3(1-\alpha)}{x} \\
-\pi\gamma_2\frac{h}{\alpha}\Gamma'(x)\frac{\partial x_{t+1}}{\partial x_t} + \frac{\partial \pi_{t+1}}{\partial x_t}\left[\gamma_3 - \gamma_2\frac{h}{\alpha}\Gamma(x)\right] &= -\frac{\pi\alpha}{(1+n)}\frac{\gamma_3\Gamma'(x)}{\Gamma(x)^2} \\
-\pi\gamma_2\frac{h}{\alpha}\Gamma'(x)\frac{\partial x_{t+1}}{\partial \pi_t} + \frac{\partial \pi_{t+1}}{\partial \pi_t}\left[\gamma_3 - \gamma_2\frac{h}{\alpha}\Gamma(x)\right] &= \frac{\alpha}{(1+n)}\left[\frac{\gamma_3}{\Gamma(x)} - \gamma_2\frac{h}{\alpha}\right]
\end{aligned}
$$

Steady state (x^*, π^*, δ^*): for this state, $\pi^* = 0$, and we get:

$$
\begin{aligned}
\frac{\partial \pi_{t+1}}{\partial x_t} &= 0 \\
\frac{\partial \pi_{t+1}}{\partial \pi_t} &= \frac{\alpha}{(1+n)\,\Gamma(x^*)} \\
\frac{\partial x_{t+1}}{\partial x_t} &= -\frac{\gamma_3}{\gamma_1\frac{(1+n)}{\alpha}\frac{h}{\alpha}x^{*2}}
\end{aligned}
$$

As the Jacobean matrix is triangular, the eigen values are $\frac{\alpha}{(1+n)\Gamma(x^*)}$ and $-\frac{\gamma_3}{\gamma_1\frac{(1+n)}{\alpha}\frac{h}{\alpha}x^{*2}}$.

This steady state exists if either $\breve{x} < x^* \leq \underline{x}$, or $x^* < \breve{x} \leq \underline{x}$. By definition of $\breve{x}$, $\Gamma(\breve{x}) = \frac{\alpha}{1+n}$. Therefore, when $x^* < \breve{x}$, $\frac{\alpha}{(1+n)\Gamma(x^*)} > 1$, and when $x^* > \breve{x}$, $\frac{\alpha}{(1+n)\Gamma(x^*)} < 1$.

For the second eigen value, it has been proved for Proposition 13 that $\frac{\gamma_3}{\gamma_1\frac{(1+n)}{\alpha}\frac{h}{\alpha}x^{*2}} > 1$.

Finally, in the case $\breve{x} < x^* \leq \underline{x}$, where there exists both the bubbly steady state $\breve{x}$ and the bubbleless x^*, x^* is a saddlepoint (one stable and one unstable eigenvalue). In the case $x^* < \breve{x} \leq \underline{x}$, where there exists only the bubbleless steady state x^*, x^* is unstable (two unstable eigenvalues).

Steady state $(\underline{x}, \underline{\pi}, \underline{\delta})$: for this state, $\gamma_3 - \gamma_2 \frac{h}{\alpha}\Gamma(\underline{x}) = 0$, and we get:

$$\frac{\partial x_{t+1}}{\partial \pi_t} = 0$$
$$\frac{\partial x_{t+1}}{\partial x_t} = \frac{\alpha^2}{(1+n)\,h} \frac{\gamma_3}{\gamma_2 \Gamma(\underline{x})^2} = \frac{\gamma_2}{\gamma_3} \frac{h}{1+n}$$
$$\frac{\partial \pi_{t+1}}{\partial \pi_t} = \frac{\frac{\gamma_3(1-\alpha)}{\underline{x}} - (\gamma_1 + \gamma_2)\,h \frac{1-\alpha}{\alpha}}{\gamma_1 (1+n) \frac{1-\alpha}{\alpha} \left(1 + \frac{h}{\alpha}\underline{x}\right)}$$

This steady state exists only if $\breve{x} < \underline{x} < x^*$. As the Jacobean matrix is triangular, the eigen values are $\frac{\partial x_{t+1}}{\partial x_t}$ and $\frac{\partial \pi_{t+1}}{\partial \pi_t}$. By assumption, $\frac{\gamma_2}{\gamma_3} \frac{h}{1+n} < 1$ which implies $\frac{\partial x_{t+1}}{\partial x_t} < 1$. Then, it is possible to prove that $\frac{\partial \pi_{t+1}}{\partial \pi_t} > 1$. Indeed, the condition $\frac{\partial \pi_{t+1}}{\partial \pi_t} > 1$ is equivalent to:

$$\frac{\gamma_3(1-\alpha)}{\underline{x}} - (\gamma_1 + \gamma_2)\,h \frac{1-\alpha}{\alpha} - \gamma_1 (1+n) \frac{1-\alpha}{\alpha} \left(1 + \frac{h}{\alpha}\underline{x}\right) > 0$$

Using previous notations, this condition can be written: $F_U(\underline{x}) - G_U(\underline{x}) + 1 + n + h > 0$. Moreover, we now that the function $F_U(x) - G_U(x)$ is decreasing and that $F_U(x^*) - G_U(x^*) = 0$. As the steady state $\underline{x}$ only exists when $\underline{x} < x^*$, we deduce that $F_U(\underline{x}) - G_U(\underline{x}) + 1 + n + h > 1 + n + h > 0$. Then $\frac{\partial \pi_{t+1}}{\partial \pi_t} > 1$ and the steady state is a saddle point.

Steady state $(\breve{x}, \breve{\pi}, \breve{\delta})$ in the case γ_2 close to 0.

Consider the case $\gamma_2 = 0$. The function Γ has an explicit form:

$$\Gamma(x) = \frac{x}{1 - \gamma_1 - \gamma_1 \frac{h}{\alpha} x}$$

From this equation,

$$\frac{1}{\Gamma(x)} = \frac{1-\gamma_1}{x} - \gamma_1 \frac{h}{\alpha}$$

and

$$\frac{\Gamma'(x)}{\Gamma(x)^2} = \frac{1-\gamma_1}{x^2}$$

Moreover, $\breve{x}$ is defined by the condition $\Gamma(\breve{x}) = \frac{\alpha}{1+n}$.

The partial derivatives are solutions of a simplified system:

$$\gamma_1(1-\breve{\pi})(1+n)\frac{1-\alpha}{\alpha}\frac{h}{\alpha}\frac{\partial x_{t+1}}{\partial x_t} - \gamma_1(1+n)\frac{1-\alpha}{\alpha}\left(1 + \frac{h}{\alpha}\breve{x}\right)\frac{\partial \pi_{t+1}}{\partial x_t} = -\frac{(1-\breve{\pi})\gamma_3(1-\alpha)}{\breve{x}^2}$$
$$\gamma_1(1-\breve{\pi})(1+n)\frac{1-\alpha}{\alpha}\frac{h}{\alpha}\frac{\partial x_{t+1}}{\partial \pi_t} - \gamma_1(1+n)\frac{1-\alpha}{\alpha}\left(1 + \frac{h}{\alpha}\breve{x}\right)\frac{\partial \pi_{t+1}}{\partial \pi_t} = (\gamma_1 + \gamma_2)\,h\frac{1-\alpha}{\alpha} - \frac{\gamma_3(1-\alpha)}{\breve{x}}$$
$$\frac{\partial \pi_{t+1}}{\partial x_t} = -\frac{\breve{\pi}\alpha}{(1+n)}\frac{\Gamma'(\breve{x})}{\Gamma(\breve{x})^2} = -\frac{\breve{\pi}\alpha}{(1+n)}\frac{1-\gamma_1}{\breve{x}^2}$$
$$\frac{\partial \pi_{t+1}}{\partial \pi_t} = \frac{\alpha}{(1+n)}\frac{1}{\Gamma(\breve{x})} = 1$$

Therefore, after some calculations and using $\gamma_1 = 1 - \gamma_3$, we get:

$$\frac{\partial x_{t+1}}{\partial x_t} = -\frac{\gamma_3 \alpha^2}{\gamma_1 \breve{x}^2 h(1+n)} \frac{1 - \breve{\pi} + \breve{\pi}\gamma_1 \left(1 + \frac{h}{\alpha}\breve{\pi}\breve{x}\right)}{1-\breve{\pi}} < 0$$

$$\frac{\partial x_{t+1}}{\partial \pi_t} = -\frac{\left[\frac{\gamma_3(1-\alpha)}{\breve{x}} - (\gamma_1+\gamma_2)\, h\frac{1-\alpha}{\alpha} - \gamma_1(1+n)\frac{1-\alpha}{\alpha}\left(1+\frac{h}{\alpha}\breve{x}\right)\right]}{\gamma_1(1-\breve{\pi})(1+n)\frac{1-\alpha}{\alpha}\frac{h}{\alpha}} < 0$$

$$\frac{\partial \pi_{t+1}}{\partial x_t} = -\frac{\breve{\pi}\alpha}{(1+n)}\frac{1-\gamma_1}{\breve{x}^2} < 0$$

$$\frac{\partial \pi_{t+1}}{\partial \pi_t} = 1$$

From these results, the determinant D and the trace T of the Jacobian matrix are both negative. Indeed, for $T < 0$, it is easy to check that

$$\frac{\gamma_3 \alpha^2}{\gamma_1 \breve{x}^2 h(1+n)} > 1$$

as we have both $\breve{x} < \frac{\alpha\gamma_3}{h\gamma_1}$ and $\breve{x} < \frac{\alpha}{1+n}$. For D and T negative, the steady state is unstable iff $T - D - 1 > 0$.

Calculating $T - D - 1$, it is obtained:

$$\begin{aligned}
&- \frac{\gamma_3 \alpha^2}{\gamma_1 \breve{x}^2 h(1+n)} \frac{1 - \breve{\pi} + \breve{\pi}\gamma_1 \left(1 + \frac{h}{\alpha}\breve{\pi}\breve{x}\right)}{1-\breve{\pi}} + 1 + \frac{\gamma_3 \alpha^2}{\gamma_1 \breve{x}^2 h(1+n)} \frac{1 - \breve{\pi} + \breve{\pi}\gamma_1 \left(1 + \frac{h}{\alpha}\breve{\pi}\breve{x}\right)}{1-\breve{\pi}} \\
&+ \frac{\left[\frac{\gamma_3(1-\alpha)}{\breve{x}} - (\gamma_1+\gamma_2)\, h\frac{1-\alpha}{\alpha} - \gamma_1(1+n)\frac{1-\alpha}{\alpha}\left(1+\frac{h}{\alpha}\breve{x}\right)\right]}{\gamma_1(1-\breve{\pi})(1+n)\frac{1-\alpha}{\alpha}\frac{h}{\alpha}} \frac{\breve{\pi}\alpha}{(1+n)}\frac{1-\gamma_1}{\breve{x}^2} - 1 \\
&= \frac{\left[\frac{\gamma_3(1-\alpha)}{\breve{x}} - (\gamma_1+\gamma_2)\, h\frac{1-\alpha}{\alpha} - \gamma_1(1+n)\frac{1-\alpha}{\alpha}\left(1+\frac{h}{\alpha}\breve{x}\right)\right]}{\gamma_1(1-\breve{\pi})(1+n)\frac{1-\alpha}{\alpha}\frac{h}{\alpha}} \frac{\breve{\pi}\alpha}{(1+n)}\frac{1-\gamma_1}{\breve{x}^2}
\end{aligned}$$

Moreover, by definition of the steady state,

$$\frac{\gamma_3(1-\alpha)}{\breve{x}} - (\gamma_1+\gamma_2)\, h\frac{1-\alpha}{\alpha} - \gamma_1(1+n)\frac{1-\alpha}{\alpha}\left(1+\frac{h}{\alpha}\breve{x}\right) = \frac{1+n+h}{1-\breve{\pi}} > 0$$

Therefore, the steady state is unstable.

2.5.11 Proof of Proposition 19

In the bubbly steady state, the return on the bubble is $\breve{\rho} = 1+n$ and the return on capital is $\breve{R} = \delta(1+n) = (1-\gamma_1)^{-1/\eta}(1+n) > 1+n$, so there is underaccumulation of capital. If one chooses α such that $(1-\gamma)^{-1/\eta}\gamma_3 - \gamma_1 - \frac{\alpha}{1-\alpha} \to +0$, then the bubbly steady state exists, but

$\breve{\pi} \to 0$, and $\breve{x} \to x^*$, so $\breve{R} \to R^*$. At the same time, $\breve{R} - (1+n) = (1+n)((1-\gamma_1)^{-1/\eta} - 1)$ is positive and not affected by the choice of α, so there exists such α that $R^* > 1 + n$.

Chapter 3

Interest Rate Restriction Results in Shorter Housing Loans: Evidence from France

3.1 Introduction

Interest rate restriction is a common consumer protection tool that can distort household credit supply in at least two ways. It is commonly argued that tight interest rate ceilings lead to reduced credit access for borrowers with higher repayment risks (Cuesta and Sepulveda, 2020; Zinman, 2010). The argument is intuitive: for such loans, high interest is necessary to price repayment risk, even under perfect competition of lenders. However, high interest rates can also reflect another loan characteristic: its duration, or term. In this paper, I show that tight interest rate ceilings decrease the supply of long-term loans, a necessary part of banks' balance sheets (Drechsler et al., 2020) and a preferred product for liquidity-constrained households (Attanasio et al., 2008).

I focus on residential real estate loans in France and exploit changes in interest rate regulation on this market. The French market for housing loans is a suitable laboratory for studying the relation between the term of the loans and interest rate restriction. First, a majority of loans is not collateralized and instead relies on insurance and guarantees. All these costs are included in the effective interest rate — the comprehensive measure of consumer cost of credit that is restricted by interest rate regulation. Second, the term of loans on the market varies widely, ranging from less than one to 30 years. Third, the interest rate ceilings, set as a markup over average market rates and updated quarterly, have decreased significantly amid low policy rates of the last decade.

I study a reform that has introduced separate interest rate ceilings for housing loans of different terms in 2017. Using administrative loan-level data, I first document a divergence in the aggregate trends of long-term loan originations of 20 years and more and shorter ones, around the time of the reform. These two categories of loan term are those that have had diverging interest rate ceilings after the reform. I exploit this in a series of difference-in-differences regressions on aggregate, lender and branch levels. The use of panel data on different levels allows to isolate alternative factors of the aggregate shift of loan originations toward longer loans, such as market structure and demand shocks. In the aggregate as well as panel estimations, long-term loan originations are found to increase on impact relative to short-term ones after the reform announcement. As the interest rate ceilings on long-term loans of 20 years has diverged away from the ceiling on shorter loans, the difference in originations has increased accordingly.

With difference-in-difference estimates confirming the role of the reform, I run regressions that control for other time-varying factors of loan duration and estimate the size of the effect attributable to the reform. Yield curves on various debt markets have flattened considerably over the last decade, contributing to the shift of loan supply towards longer duration, regardless of interest rate regulation policies. I use the dynamics of the ceilings on long-term and short-term housing loans as a treatment variable in a regression that controls for funding rates on different horizons. The funding rates are measured by prices of swaps on interbank rates with relevant duration. All the time-varying covariates, including the interest rate ceilings, are economy-wide, while the outcomes — the loan originations — are used on aggregate, on lender and on branch levels. The use of granular data allows to treat the ceiling dynamics as exogenous: although the ceiling for each loan category is updated depending on average market interest for the loans in question, one lender or branch cannot influence the market average considerably. I find that the difference in interest rate ceilings has statistically significant and economically meaningful impact on long-term originations. I find that the reform has led to a net increase in aggregate housing loan originations of around 6% quarterly, although long-term originations have crowded out shorter ones. I then calculate changes in liquidity and debt service of a typical borrower who has taken out a long-term loan instead of a short term due to the reform of interest rate restriction. Such a borrower has had monthly payment decreased by 30%, but the overall debt service cost increased by 75% — effects explained primarily by the loan term itself and not by the higher interest rate that is charged on longer loans.

In hindsight, the reform has been a timely effort to protect the profits of French credit institutions from deterioration. French banks have been financing housing purchase of households at zero net interest margins since the end of 2016[1]. Further easing of monetary policy was not a likely event for the observers of 2017, but the unexpected shock of the COVID-19 pandemic has made this scenario come true[2]. Holding loans with pre-crisis interest rates for a longer horizon is a natural way to increase the stability of banks' interest income. This can support banks' equity positions and their ability to sustain lending, both in expansion and in crisis periods. The lending horizon on the French housing market is particularly long, however: a typical contract in 2019 had a 25-year term. Since an significant increase of interest rates is not ruled out on such a long horizon,

[1]see: Assessment of risks in the residential real estate sector report of the Higher Council of Financial Stability of France in 2019

[2]the ECB has announced, on 4 June 2020, the Pandemic Emergency Purchase Programme, a €1350b extension of the Bank's Asset Purchase Programme.

the measures discussed above have resulted in some additional interest rate risk exposure of banks in the long term. This explains more recent recommendations of the French regulator to constrain the term of loan originations.

Maturity mismatch in the assets and liabilities is a conventional element of banking business. Recent literature has found it to also mediate the transmission of monetary policy, and more so in the low-interest environment (Claessens et al., 2018; Paul, 2020). The choice of term, or maturity, of banks' assets has been explored in the literature on the negative interest rate policy (Arce et al., 2018; Bottero et al., 2019). When highly liquid short-term assets such as interbank loans start bearing negative rates, banks typically rebalance their portfolios towards loans of longer term. The term choice that I study in this paper is between different time horizons: a typical "short term" (with respect to the interest rate regulation) housing loan in France is for around 15 years, while a "long term" one is between 20 and 25 years. To my knowledge, the composition of housing loan portfolio with respect to term is not yet studied in the empirical banking literature, especially in the low interest rates context.

The motivation for interest rate restriction on the household credit market is consumer protection. Does additional flexibility of ceilings with respect to loan term come at the expense of consumer protection? Attanasio et al. (2008) find that higher sensitivity of demand to duration than to interest rate in car loans. This suggests that consumers value highly the opportunity to prolong their debt and decrease regular payments. The term elasticity is especially high for low income consumers, which the authors interpret as liquidity constraints being alleviated by longer loans. Karlan and Zinman (2008) obtain the same result for consumption microloans. The liquidity constraints argument is likely to to be most pertinent housing loans, because of the large amounts borrowed. Dhillon et al. (1990) provides survey evidence on the preference for 30-year mortgages against 15-year ones that is stronger for low-income borrowers.

The rest of the paper is organized as follows. Section 3.2 provides background on the housing credit and interest rate regulation in France. Section 3.3 describes the data and Section 3.4 provides some descriptive evidence. Section 3.5 then formulates the hypotheses and Section 3.6 describes the empirical strategy. Section 3.7 describes the main results, with some additional ones left for Section 3.8. Section 3.9 concludes.

3.2 Institutional framework

3.2.1 Housing credit market in France

This section discusses the characteristics of the French market of residential real estate credit, labelling it housing credit throughout the article. The market in France is large and has been growing fast over the last decades. By the end of 2019, households' housing loans outstanding have reached €5900b, of which €265b has originated in 2019, which amounts to roughly 11% of GDP of the same year. For comparison, this ratio has been at around 8% in Germany, 9% in the UK and 10 % in the US in 2019. Over the last decade, the French housing loan market has grown as fast as 4.6% per year, well above the average GDP growth rate over the period — around 1%.

The housing loans market is highly concentrated. In 2019, 5 banking groups were holding around 97% of the total housing debt outstanding[3]. These are first and foremost the cooperative banking groups, or *banques mutualistes*, Credit Agricole (35%), BPCE (26%), Credit Mutuel (17%). The remaining two large players are banking groups that do not have the cooperative bank structure: BNP Paribas (9.4%) and Societe Generale (9.2%). The distinctive feature of cooperative banks is their bottom-up structure: they consist of local entities owned by local customers and a set of governing bodies — regional and central. A set of local entities and a regional governing body constitute a regional bank that can be autonomous with respect to the central governing bodies; each cooperative banking group counts tens of such regional banks. In addition, all the above mentioned banking groups include non-bank specialized credit institutions that operate on the housing credit market. The main difference of specialized institutions from banks is that they are not deposit takers.

A sizeable portion of new contracts on the housing loans market are not brand new loans. Firstly, a contract can represent renegotiation of an existing loan. Another case is loan refinancing, where an existing loan is transferred to another credit institution that has offered the borrower better conditions. A survey-based measure of the share of refinancing operations in new contracts is depicted in Figure A.13. This share has peaked at 40% in 2015.

The majority of housing loans in France are not secured by the purchased property, in contrast with countries like the UK and the US. Mortgages, or loans secured with land or the house being purchased, account for only 28% of housing debt outstanding in 2019[4]. Instead, French lenders rely on two credit risk management principles: detailed information on the borrowers' income stream and obligatory insurance, or third-party guarantee, of loans. Both the insurance and the guarantee can be provided by the government.

Since insurance and guarantee fees are ubiquitous in housing loan contract, they must be taken into account when the cost of credit is analysed. In this paper, I focus on the effective interest rate[5], a rate that includes various fees and additional loan-related costs, as opposed to narrowly defined interest rates. For short, I call them "interest rates". This price measure takes into account such transaction costs as insurance costs, contract creation fee, contract maintenance fee and the cost of evaluation of the acquired good. The interest rate restriction studied in this article applies to the effective interest rate and not only the conventional interest rate, in line with the aim of consumer protection.

[3]Source: ACPR (Banque de France) 2019 report on housing finance. The banking groups include both banks and specialized credit institutions.

[4]Ibid. The share includes another variety of secured contract used in France — lender's lien.

[5]the exact term in France is *Taux Effectif Global* or *TEG*; the definition is given in the chapter R. 314-11 of the Consumption Code and is summarized in Section A.1 of the Appendix.

3.2.2 Interest rate restriction

The interest rate regulation in France consists of a relatively large number of interest rates ceilings specific to a loan type[6], but a ceiling for a given loan type is the same for any credit institution in France. I will only focus on loan categories within the housing loan market. Housing loans have separate ceilings for fixed-rate and adjustable-rate loans, and a reform in 2016-2017 has introduced separate ceilings for loans of different term, or duration. I exploit this reform as the exogenous variation in interest rate ceilings. All the housing loans categories subject to dedicated ceilings are listed in the Table A.1, separately for the pre-reform and post-reform periods.

All loan categories have the same formula for updating the interest rate ceiling. The formula relies on a lagged market average of the interest rate for a given loan category. Indexing with k one of the categories of the Table A.1, the ceiling updating rule can be written as follows:

$$ceiling_{k,t} = \frac{4}{3}(average\ rate)_{k,t-1} \tag{3.1}$$

where the averages are weighted by loan amount. The measurement of the average rates is carried out by Banque de France, and in the Section 3.3 I introduce the data that the regulators use for this purpose. The evolution of ceilings on different categories of housing loans is depicted in Figure A.1[7]. On 12 July 2016, the Financial Stability Council of France has adopted a new definition of regulatory categories, making ceilings specific to loan term. To my best knowledge, the motivation for this reform has not been communicated to the public. As reported in the Table A.1, the fixed interest loans category has been decomposed into loans of up to 10 year term, loans from 10 up to 20 years term, and loans of 20 and more years term.

At the beginning of 2017, the average rates of the previous quarter have been such that the two new categories of fixed rate loans have had the same interest rate ceiling. In three quarters, however, the averages of the two categories have diverged[8], leading to a less tight ceiling for the longest loans. At the same time, the two categories of shorter loans have converged in average rates and have had minor difference ever since[9]. For this reason, I abstract from the difference between the two shorter categories of loans throughout the article. The loan term will almost exclusively be studied as a binary category: whether the loan is less than, or more/equal to, 20 years.

[6]For comparison to other EU countries, see the following report: https://op.europa.eu/en/publication-detail/-/publication/46a336d0-18a0-4b46-8262-74f0e0f47eb3

[7]the level of the unique ceiling on all fixed-rate loans prior to 2017 is currently not provided on the Banque de France web page and was obtained from espacecredit.com. The level of the pre-2016 ceiling is not used in regression analysis; only the fact that the ceiling was unique for all terms is taken into account.

[8]the reasons for such market dynamics are currently not explored in this paper. Collusion of large players with the aim of pushing up the average interest is not ruled out.

[9]the difference of the two ceilings has reached a maximum of 14 bps in 2017:Q2 and has not been more than 7 bps afterwards.

3.3 Data

The main data source is the M_CONTRAN dataset of Banque de France. The dataset was initiated in 2012 as part of a reform of interest rate regulation of 2010-2012. Banque de France uses the dataset to measure the average interest for each loan category in the formula (3.1) as well as to evaluate the effect of the 2010-2012 reform, that was primarily affecting consumption loans. The dataset covers loans of various types to all sectors of the economy, but I focus on the housing loans to households. The observation period is 2012:Q3 – 2020:Q2. For a detailed description of the sampling, the unit of observation and the variables, see Appendix A.1.

The dataset is a quarterly register of credit transactions of a large sample of French credit institutions. A credit institution is identified by a code called *CIB*, attributed by the French Prudential Supervision and Resolution Authority (*ACPR*) together with the licence for lending activities. The CIB are anonymized: I am not able to identify banks belonging to the same group or regional entities of cooperative banks. Contracts are reported for either a sample of branches or of all the branches of a given credit institution; within a given branch, the set of contracts is exhaustive. Branch identifiers are observed for banks, but not for specialized institutions — the latter are marked by a dummy variable. There is a total of 166 credit institutions in the dataset, of which 111 are banks. Banks have a total of 5064 branches.

A number of loan contract characteristics are observed. These include the type of loan product, amount and term of loan, effective interest rate, narrowly defined interest rate, interest fixation regime, and some others. For a full list of observable characteristics, see Appendix A.1. A dummy variable for loan refinancing operations is only available from 2016 on, which is a potential issue for the empirical strategy presented below. My sample therefore includes loan originations that are part of refinancing operations. I explore to which extent these observations drive my results in Section 3.8.2. On the other hand, I exclude from loan modifications, or renegotiations, from the analysis. Furthermore, for the main estimations I also abstract from variable-rate loans. These loans have not been subject to the reform of interest rate regulation and their share in the sample is small — around 3% in the total value of the loans. Tables A.2,A.3 present main summary statistics for the resulting sample of contracts. The sub-sample of bank loans is largely comparable, in terms of the main contract characteristics, to the full sample including specialized credit institutions.

Figure A.4 presents the total housing loans value of the part of M_CONTRAN used in this paper versus the aggregate amount of housing loan originations in France. The two series do not have the same scope: I exclude loan renegotiations and variable-rate loans from M_CONTRAN, while the aggregate series includes those categories. It is a wave of loan renegotiations in 2017 that explains the peak of loan contracts on the aggregate level in that year, a peak absent in my sub-sample of M_CONTRAN. This discrepancy notwithstanding, the share of loan originations covered by the sample used in this paper is on average 6.5%, with a slight decreasing trend.

Figure A.5 presents total of loans over the sample period for each credit institutions in the sample. For banks, it also plots the number of branches. The concentration of the market appears much smaller than presented in the section 3.2. This has two reasons. First, the banking groups are split into member banks and specialized institutions. Second, two of the three main cooperative banks appear in the sample as a set of regional banks. One outlier bank is apparent from the Figure. It represents a bulk of the total loan value, but at the same time has only one branch identifier, unlike all the other large banks in the sample. Given the extreme volume of credit attributed to this branch identifier, I interpret it as miscoding of multiple branches with a single identifier. I therefore exclude these miscoded observations from the branch-level estimations, but not from the aggregate and institution-level estimations.

The dataset is an unbalanced panel in terms of participating credit institutions. Table A.4 indicates that only 112 out of 166 credit institution identifiers are found in the sample in both 2012 and 2020[10]. More problematic for the study is that a significant part of institutions exits the sample before the reform — in 2013-2016. This motivates the estimation strategies presented in Section 3.6.

3.4 Stylized facts

Ceilings and interest rate distribution. How important is the interest rate regulation for the housing loans market? An intuitive way to answer this question is by studying the distribution of interest rates on the loans that have been contracted under relatively loose and relatively tight ceilings. Figures A.6 and A.7 present the distribution of effective interest rates of long and short loans, together with the corresponding ceilings. Before the reform, the ceilings do not seem to bind until 2016[11]. By 2016, the distribution of interest rates becomes visibly truncated at the ceiling, whereas bunching at the ceiling level is not apparent. After the reform of 2017, two ceilings emerge. As shown on the Figure A.1, the difference between the two is not apparent since the beginning of 2017 and in particular, the ceiling on the shorter loans even under-shoots the one on longer loans in 2017:Q2. Starting from 2017:Q3, the longer term loans ceiling becomes progressively less tight than the one on shorter loans. Importantly, a non-negligible part of the longer term contracts has the interest rate lying between the two ceilings; lenders are able to charge effective interest this high only because the term of these loans in in the longer regulatory category.

[10]two aspect of unbalancedness are missing from the table. First, it only tracks the year and not quarter of entry and exit, so entities that exit on 2020:Q1, one quarter before the end of the sample, still have the longest spell possible in terms of years. Secondly, the consistency of the spells is not verified: whether entities disappear for some quarters to appear again. This last pattern does exist in the data: in 2015:Q1, the number of bank identifiers drops to around 20 to get back to normal numbers in 2015:Q2. At the same time, aggregate lending does not drop on that quarter. This signals problems with the credit institution id variable.

[11]interest rates occasionally set above the ceilings could be explained by a lag between the date of the contract itself and its reporting to the regulator.

From 2018 on, both ceilings become low enough to cause considerable bunching in both loan term categories. To sum up the histogram analysis, interest rate ceilings do seem to constrain the lending on the housing loan market so lenders do exploit the difference between short and long term loan ceilings to set higher effective interest rates.

Short and long loans aggregates. Figure A.2 displays time series of aggregate amounts lent under short and long contracts. The long-term loans seem to follow a linear trend that is similar before and after the reform. At the same time, shorter loans change the trend from ascending to descending just around 2017. The same pattern holds for numbers of loans in the two bins: number of long loans increases steadily, while the number of short-term loans starts to decrease around 2017. This aggregate evidence can be interpreted as substitution of long-term loans for short-term ones that starts with the reform announcement and continues until at least early 2020.

Loan terms distribution. As a third piece of evidence, I report the terms of the loans in a less aggregated way. Figure A.10 depicts the kernel density functions of the term rates in 2012-2016 and in 2017-2020. The 25-year term was slightly less frequent than the 20-year one before the reform, but became by far more important after. This increase of frequency is at the expense of the three other largest modes: 10, 15 and 20 years. Therefore, the shift of terms into the "$\geq$ 20 years" bin is driven by the 25-year loans. While consistent with the logic of binding interest rate ceilings, this evidence suggests other factors for increasing loan term are also at play. If the interest rate restriction reform was the only reason to supply longer loans, one would observe bunching of terms at the left bracket of the regulatory bin: 20 years. This motivates an empirical strategy that takes into account other factors of loan term choice than interest rate regulation.

3.5 Hypotheses

Uniform interest rate ceilings across loans of different terms restrict portfolio choice of credit institutions with respect to duration of their assets. The separate interest rate ceilings for shorter and longer housing loans, announced in 2016 and enacted in 2017 are expected to loosen a binding constraint on the credit institutions: the constraint on the effective interest rate spread between shorter and longer loans. The main testable prediction is then an larger growth in loan originations of 20 years and more relative to originations of up to 20 years — a positive difference in differences. This must also hold after controlling for other time-varying variables that affect loan supply at different horizons. In particular, the changes in the funding rates have affected the choice of loan term around the time of the reform: decrease in the long-term funding rates has made long-term loan originations increase.

To sum up, the two main hypotheses are formulated as follows:

- H1 Additional flexibility of interest rate ceilings for loans with a term of 20 years and more has caused additional loan originations in this category with respect to originations of shorter term.

H2 Term structure of funding rates also explains the changes in housing loan originations of different terms.

Both hypotheses are operationalized in the following section.

3.6 Empirical strategy

I use difference-in-differences regressions and regressions with time-varying covariates such as in interest rate ceilings and funding rates. In a first set of regressions, I estimate yearly differences in short and long term loan outcomes — total value and number of loans. The main identification assumption is that no events around 2017 have affected housing loans market significantly and differentially for loans of less and more than 20 years' term — this assumption is discussed below. In a second set of regressions, I use the difference of interest rate ceilings as a treatment variable and study its differential impact on loans of shorter and longer term. This second set of regressions also includes time-varying controls that allow to isolate a number of alternative causes of a shift in loan terms. All estimations are done in pooled and panel versions.

3.6.1 Differences in term and in time

The baseline specification compares the difference of outcomes in the two term bins before and after the reform. One problem to be taken into account is unequal length of pre- and post-reform periods — 2012:Q3–2016:Q4 against 2017:Q1–2020:Q2, or 18 quarters against 14. This makes the interpretation of time difference coefficients unclear, but also can bias difference-in-differences coefficients. To counter the problem, I use the following specification with quarterly outcomes and time fixed effects:

$$y_{term,t} = \beta D_{term \geq 20years} + \delta D_{after2016} D_{term \geq 20years} + \varphi_t + \varepsilon_{term,t} \tag{3.2}$$

$y_{term,t}$ is either the aggregate value or the number of housing loans of a given term bin at a period (quarter) t, under logarithm transformation in most specifications. The term bins are "$<$ 20 years" and "$\geq$ 20 years", with the dummy $D_{term \geq 20year}$ equal to 1 for the second bin. The time dummy $D_{after2016}$ is equal to 1 for periods $t \in$ [2017Q1, 2020Q2] and 0 for $t \in [2012Q3, 2016Q4]$. The coefficient of interest is δ, which is the expectation of the quarterly difference in the outcome between the two term categories in the post-reform period minus the same difference in the pre-reform period. φ_t are period dummies, or time fixed effects. Time fixed effects absorb any time variation that is common to the two term bins.

In order to better identify the timing of the shift in the term structure, I then do a version of regression (3.3) with yearly difference-in-differences coefficients[12]:

$$y_{term,t} = \beta D_{term \geq 20years} + \sum_{year(t)} \delta_{year(t)} D_{term \geq 20years} + \varphi_t + \varepsilon_{term,t} \tag{3.3}$$

The difference-in-differences coefficients of interest are $(\delta_{2013}, \ldots \delta_{2020})$, with 2012 taken as the base year. The coefficients measures difference of the outcome in the two term categories specific to a given year, relative to the difference in 2012. Again, time fixed effects absorb the time variation common for the two term bins — both across years and within each year.

The coefficients of interest $(\delta_{2013}, \ldots \delta_{2020})$ are expected to be stable and not statistically significant for the years 2013 to 2015. This is an assumption on parallel pre-trends for the loans in the two term bins. The coefficients are expected to be positive and significant starting from either 2016 or 2017. The first case would suggest that the announcement of the reform modifies lending on impact, while the second would mean that the effect appears only after a de-facto divergence of the interest rate ceilings. Finally, the largest coefficients are expected in 2018-2019, with a decrease in 2020, when the two ceilings converge again.

The aggregate level estimates can have interpretations that are unrelated to interest rate regulation. For example, the change in the term structure might be due to changes in the relative market power of credit institutions. To account for this, I do estimations analogous to the regressions (3.2),(3.3) on a credit institution level. Another identification issue is a demand shock that can potentially explain the shift in loan terms. Indeed, assume that the population of borrowers can be separated into long-term borrowers and short-term borrowers using several socioeconomic characteristics. Examples of relevant characteristics include income, age and employment sector of the borrower. These characteristics are unevenly distributed in space. Then, if the shift in loan terms is explained by a demand shock specific to a demographic group, it is more likely to happen as a shift in lending between, and not within, locations. I therefore study the term shifts within location by running the regressions (3.2),(3.3) on the level of a bank branch[13].

The panel regressions on both levels are done by aggregating the outcome variable on the level and adding corresponding fixed effects:

$$y_{term,i,t} = \beta D_{term \geq 20years} + \delta D_{after2016} D_{term \geq 20years} + \varphi_t + \gamma_i + \varepsilon_{term,i,t} \tag{3.4}$$

[12] to be consistent with the notation in the regression (3.2), one would have to write the second term of the equation as $\sum_k \delta_k D_{year(t)=k} D_{term \geq 20years}$. The notation omitting the year dummy is more conventional.

[13] for branch-level regressions, the specialized institutions are dropped from the sample since their branch identifiers are not available. Section 3.8.3 reports split-sample regressions comparing banks to specialized institutions.

$$y_{term,i,t} = \beta D_{term \geq 20years} + \sum_{year(t)} \delta_{year(t)} D_{term \geq 20years} + \varphi_t + \gamma_i + \varepsilon_{term,i,t} \tag{3.5}$$

for the two-period and the yearly specification, respectively.

$y_{term,i,t}$ is the (logarithm of) value or number of housing loans of a given term bin in an entity i at a period t, where an entity is either a credit institution or a branch. γ_i is an entity fixed effect. The regressions have a "within " interpretation: it estimates the effect of the regressors on deviations of the outcome variables from the entity average.

All the panel regressions are weighted by the total value (from 2012:Q3 to 2020:Q2) of loans of the corresponding entity. For credit institutions, these values are found in Figure A.5. Such weighting penalizes the entities that have been in the sample for less then the whole observation period, which is a significant part of the sample as seen in Tables A.4, A.5. In regression (3.5), the errors $\varepsilon_{term,i,t}$ are clustered on entity level i to make the inference robust to entity level autocorrelation of unobservables, as suggested by Bertrand et al. (2004).

3.6.2 Difference in term and ceilings

The regression specification is closely related to the previous ones, with the difference of rate ceilings and time-varying controls used instead of the time dummies:

$$\begin{aligned} y_{term,i,t} = \alpha + \beta D_{term \geq 20years} + \beta_c \Delta_{ceilings} + \delta_c D_{term \geq 20years} \times \Delta_{ceilings} \\ + \beta_r r_{<20y,t} + \delta_r D_{term \geq 20years} \times r_{<20y,t} \\ + \beta_s \Delta r_t + \delta_s D_{term \geq 20years} \times \Delta r_t + \gamma_i + \varepsilon_{term,i,t} \end{aligned} \tag{3.6}$$

The variable of interest $\Delta_{ceilings}$ can also be written as $ceiling_{term \geq 20y,t} - ceiling_{term<20y,t}$, where $ceiling_{term<20y,t}$ is the maximum of the ceilings of regulatory bins $[0, 10)$ years and $[10, 20)$ years.

The variable $r_{<20y,t}$ is a measure of the credit institutions' funding rate that corresponds to housing loans of the shorter term bin, while Δr_t is a difference of such measures for the longer and the shorter bins: $\Delta r_t = r_{\geq 20y,t} - r_{<20y,t}$. I follow the literature on mortgage lending[14] to approximate the funding rate corresponding to different terms with rates of interest rate swaps on interbank market rates. One also has to aggregate the rates of swaps of different terms to have a measure of a funding rate corresponding to a term bin. The variables $r_{\geq 20y,t}$, $r_{<20y,t}$ are averages of interest rate swap rates, of different terms, on the 1-month EURIBOR. I use the following rule of thumb to measure funding rates for the two term bins. First, observe form Figure A.10 that the bin "< 20 years" has the two largest modes at 10 and 15 years and the bin "≥ 20 years" has modes of 20

[14] see Basten and Ongena (2019), Basten (2019)

and 25 years. I then use a simple average of the two modes in each bin to measure the corresponding funding rate:

$$\begin{aligned} r_{<20y,t} =& 0.5(\text{10-year swap, 1-month EURIBOR})_t + 0.5(\text{15-year swap, 1-month EURIBOR})_t \\ r_{\geq 20y,t} =& 0.5(\text{20-year swap, 1-month EURIBOR})_t + 0.5(\text{25-year swap, 1-month EURIBOR})_t \end{aligned} \tag{3.7}$$

The series on the right hand side are obtained from Factset and were only available from 2013 on, which means 2012:Q1-Q2 are dropped from the regression.

The main coefficients of interest are β_c, δ_c. The hypothesis H1 allows for several cases for the signs of the coefficients. β_c is expected to be non-positive and δ_c non-negative, but both coefficients cannot be null at once. If $\beta_c = 0, \delta_c > 0$, then the interest rate restriction reform has lead to an increase in long-term originations without crowding-out of the short-term ones. If $\beta_c < 0, \delta_c = 0$, the reform only results in a decrease of short-term originations. Finally, the case $\beta_c < 0, \delta_c > 0$ means there is a substitution of long-term loans for short-term ones, and the net effect depends on the sign of $\beta_c + \delta_c$.

According to H2, one expects $\beta_r < 0$: short-term originations are decreasing in the funding rate on the corresponding horizon. I further assume a substitution effect between short-term and long-term lending: ceteris paribus, the long-run loan supply is increasing in the short-run cost. Given that $\Delta r_t = r_{\geq 20y,t} - r_{<20y,t}$, the effect of $r_{\geq 20}$ is given by $\beta_r + \beta_s$. I then expect $\beta_r + \beta_s > 0$, which requires $\beta_s > 0$. Correspondingly, the effects of the rates on long-term originations are expected to be the opposite: $\delta_r > 0$ and $\delta_r + \delta_s < 0$, so $\delta_s < 0$.

Figure A.9 plots the three time-varying covariates together. The difference of the funding rates and the rate corresponding to the short term appear highly correlated (correlation 0.8); both variables decrease towards the end of the sample, contributing to a shift of originations to the long-term bin. This makes the funding rates necessary controls when an impact of the interest rate restriction reform is studied.

The regression is done on the aggregate, on the level of credit institutions and on the level of branches. The estimation within credit institution weights observations by total loan value of the credit institution throughout 2013:Q1 – 2020:Q2. Finally, the standard errors are clustered on entity level i for panel regressions to avoid the under-estimation of variance raised by Moulton (1986).

3.6.3 Discussion

Loan refinancing

Loan refinancing is prevalent in the French housing loan market, but more so for the years 2015-2017 that other years in the sample period. Figure A.13 shows that the share of refinancing peaked in 2015 and early 2017, at least in the institutions participating in the ACPR survey of housing loan production. Loan refinancing operations may drive the change in average term of loan originations: according to the same survey, the contracts that refinance existing loans have around 2 years shorter term than brand-new loans. The difference-in-differences coefficients of the regressions (3.2)-(3.5) are then lower for periods that have the largest share of refinancing contracts. In the case of two-period estimations, the influence on the difference-in-differences coefficient is most likely to be positive, since the pre-reform period has had the largest shares of refinancing operations. For the yearly estimates, it is the years 2015-2017 that are more likely to have positively influenced coefficients than other years, with the largest influence for the year 2015.

Relying on the information of the Figure A.13, I do the following robustness check to minimize the influence of refinancing operations on two-period estimates. As plotted on the Figure, periods 2012:Q3–2014:Q4 and 2017:Q1–2020Q2 have the same average share of refinancing operations (0.14 for the first period and 0.142 for the second), although it has been more volatile for the first period. By excluding years 2015 and 2016 for the 2-period estimations, I make the influence of refinancing equal for the average term structure in the pre-reform and the post-reform periods. Section 3.8.2 reports results for a sub-sample that excludes the years 2015 and 2016. For the yearly coefficients, the results are discussed taking into consideration the potential upward bias for the years 2016, 2017 and especially the year 2015.

Regressions with ceiling difference and other time-varying covariates takes into account loan refinancing in a different way. Section 3.8.2 reports results of regressions with ceiling differences where the share of refinancing operations of the ACPR survey is one of the covariates. The potential issue with this robustness check is measurement error. The survey only includes 12 institutions and the share is a median share among these responders, not reflecting the aggregate average share.

Finally, loan refinancing is likely to influence estimations using loan value less than those using loan quantities. Loan refinancing is done on smaller (those that remain to be repaid) values than brand-new loans, so their share in loan quantities is likely to be larger than their share in loan value that is presented in Figure A.13.

Confounding events

The difference-in-differences estimation relies on the timing of the reform alone, which requires an absence of confounding events in 2016-2017 leading to a shift of credit supply towards long-term loans. The main candidates for such events are discussed below.

ECB Asset Purchase Programme. The Asset Purchase Programme, labelled QE in its most active phase, has influenced the yield curves on assets across the EU, with the government bonds affected more than any other assets. However, there are two reasons to think that it is not a major confounder for the analysis. First, the programme has mostly influenced 5 to 10 years spreads, with this spread dropping on impact for French government bonds in 2015, as seen in Figure A.12. In 2017, when the programme has phased out, the spread has rebounded. As it is evident from the Figure A.10, the 5-10 years part of the yield curve is not immediately relevant for the dynamics of housing loan terms. The more relevant spreads, 25 to 20 years and 20 to 15 years, have been relatively stable over the period. An increase of the 20 to 15 years spread around 2017 reverts in 2018.

Government bond rates do not reflect directly the incentives of private credit institutions to lend at different horizons. Interbank and financial corporates' bond rates could have been influenced by the APP on the relevant parts of the yield curve. Time-varying measures of bank funding rates at different horizons are included in the regression specification of Section 3.6.2.

Fiscal reforms. In 2017, the newly elected president and government have enacted a reform of wealth and capital income taxation. A tax that has been levied on all property of the most wealthy part of population has been replaced with a property tax with same parameters. Furthermore, capital income, previously taxed at the same progressive scale as labor income, has become subject to a flat tax with a relatively low rate. The response of the housing credit could have been different depending on which type of borrowers are considered. For those who have been subject to the wealth tax prior to the reform, the incentive to hold non-housing wealth has increased, decreasing the demand for loans. If the most wealthy are more likely to borrow at shorter terms, such an effect can confound the analysis. However, the wealth tax has been applicable to less than 350 thousand households in France, and the number of housing loans contracted in the M_CONTRAN sample is around 130 thousand yearly, so the behavior of the most wealthy is not likely to have a significant influence on the estimations of number of loans[15]. For the aggregate values and not numbers of loans, the influence of the wealth tax contraction remains a possible confounder.

The transition to the flat capital income tax increases incentives for investment in relatively expensive real estate. ACPR, the prudential authority within Banque de France, administers a survey credit institutions survey on housing loans that includes questions on aims of the new loans, their size and term[16]. This survey has shown a rise of the share of loans financing investment in real estate in the total value of credit institutions' housing loans originations. However, the term of such loans is around 2 years shorter than the average, which means the share of such loans rising would imply a decrease in the share of longest term loans. This goes in the opposite direction to the effect of the interest restriction reform.

Variable-rate loans regulation. In 2014, the European Commission has adopted a directive on housing lending practices that obliges residential housing loan providers to increase transparency of the loan contract. As stated in the summary[17] of the directive, one of the aims is to inform consumers on "worst-case scenarios regarding variable interest and

[15]assume the households in question take a housing loan once every 10 years. Then they would account for around 35 thousand contracts yearly. Since M_CONTRAN is a representative sample of around 6.5% of all contracts, the affluent households in question then account for roughly 2300 M_CONTRAN contracts yearly — 2% of the yearly number of contracts.

[16]https://acpr.banque-france.fr/publications/etudes-et-recherche/statistiques/suivi-mensuel-de-la-production-de-credits-lhabitat

[17]https://eur-lex.europa.eu/legal-content/EN/LSU/?uri=celex:32014L0017

foreign currency loans so as to alert consumers of potential interest rate variations". The directive seems to have influenced housing credit market in France in at least one relevant dimension: the share of variable-rate contracts. The share of such contracts in the total value of housing loans has dropped from around 8% in 2015 to 3% in 2015, converging to around 1.5% afterwards. An alternative explanation to this fact is the QE that might have made the rise of interest rates in the next two decades less likely. Regardless of which cause is the right one, the change in incentives to offer variable-rate contracts might have influenced the term structure of housing loans, since the interest rate risk of a fixed-rate contract increases with the contract term. However, this means that disincentive to offer variable-rate contracts could cause a shift to shorter, not longer, loans.

The variable-rate loans category in itself can be thought of as a control group for the estimation of the effect of the interest rate restriction reform. Indeed, this category has had another ceiling, which has had a decreasing trend, but has not been differentiated for loans of different terms. However, the aggregate value of these loans is too small for comparing them to fixed-rate loans in a single regression. Instead, I report in Section 3.8.1 the results of the main estimations carried out on this sample. This can be thought of as placebo regressions.

3.7 Results

3.7.1 Difference in terms and in time

Aggregate M_CONTRAN. Table A.6 presents results of regressions (3.2), (3.3) for quarterly aggregate value and quantity of housing loans in M_CONTRAN. This kind of estimation is closely related to the visualization of the series in Figures A.2,A.3. Prior to the reform, there are less loan originations in the long-term category — measured both in total amount (-11%) and in total number (-58%)[18]. The difference-in-differences estimate for the two-period case is the growth rate of long-term loan originations being 50 percentage points larger than the growth rate of shorter-term loans. The yearly estimates show a relatively insignificant term shift in 2016, upon reform announcement, and a further shift that accelerates until 2019, with a small reversion in 2020. The significant increase of the difference in differences from 2018 to 2019 cannot be explained by the ceiling dynamics alone: Figure A.9 shows that the difference in the long- and short-term loan ceilings has peaked in 2018. One possible explanation is that the adjustment of lenders' loan menu has some inertia: the investment in the marketing of longer loan products may take more than a year.

The number of loans shows a similar pattern, consistent with the Figure A.3. First, the difference in the number of contracts in the two term bin is apparent: before the reform, there is, on average, 58% less long-term than short-term loan originations per quarter.

[18] I interpret the difference in logarithms as a relative variation for clarity of presentation, although large differences in logs are imprecise approximations of large relative variations.

The two-period difference-in-differences estimate is same as in the loan value estimates: 51%. Yearly estimation shows larger difference-in-differences in 2016-2017, compared to the value estimates: the initial increase is therefore stronger in term on number of loans than in value.

Overall, the aggregate estimates are in line with the hypotheses stated above. Changes in sample composition reflected in Tables A.4, A.5, as well changes in the credit industry organisation and aggregate demand shocks, could drive the estimates. This is addressed by estimations within credit institutions and within branches.

Within credit institutions. The estimates within credit institutions, reported in Table A.7, repeat the patterns the aggregate estimates. The difference-in-difference estimates in the two-period case are strikingly robust to the introduction of credit institution fixed effects. The yearly estimates are also largely in line with the aggregate ones. The estimates of the number of loans are not significantly different from those of the total value. The yearly difference-in-differences coefficients for total loan value are plotted in the Figure A.14. In the number of loans yearly regression, no effect is detectable prior to 2018. Technically, the pre-trends in both yearly regressions are now not parallel: the year 2013 has a smaller difference of total long and short loan values and quantities than 2012, the base year.

Within bank branches. The difference-in-difference estimates on branch level are more mixed in the branch-level estimates. First, they are significantly smaller in the two-period case (38% and 35% for loan value and quantity, respectively). Second, the parallel pre-trends assumption seems violated. It is worth studying the main effect of the longer term bin first — the coefficient β in the regression equations (3.4), (3.5). It is significantly larger in all the specifications than both in the aggregate and the institution-level case. This makes all the difference-in-difference smaller, ceteris paribus. Correspondingly, the yearly estimates have a similar qualitative pattern to the one seen before: relatively small coefficients for 2013-2015, larger ones for 2016-2017 and yet larger for 2018-2020. The estimates in 2013-2015 are not monotonic, however, which cannot be explained by the interest rate ceilings alone. One reason for the estimates to be different from those found on other levels is the exclusion of the largest bank from the sample, due to the lack of branch identifiers. The yearly difference-in-difference estimates for total loan value are plotted in Figure A.15. The loan originations of the largest bank, excluded from branch-level estimations, are described in Section 3.8.4.

3.7.2 Difference in terms and in ceilings

Table A.9 reports the main estimates of a regression of total loan values[19] on the difference in ceilings, a measure of funding cost corresponding to the short bin and the difference of funding costs corresponding to the two bins. According to hypothesis H1, one expects

[19]number of loans as outcome variable is omitted in this section, since it has been found previously to have a similar behaviour to the loan value variable. The estimates reported in this section are therefore similar for regressions of number of loans.

a negative and statistically significant coefficient on the ceilings difference Δceiling, a positive and statistically significant sign on the interaction of this variable and the long-term loan dummy, or both. According to H2, one expects a negative sign for the funding cost corresponding to short-term loans, $r_{<20y}$, a positive sign for the difference in the funding costs in the long-term and short-term bins, Δr, and the inverse signs for the interactions of these variables with the long-term bin.

On the aggregate, all the coefficients have the expected signs, except for the interaction of the funding cost of short-term loans and the long-term dummy. The only coefficient that has statistical significance is the interaction term of the long-term bin and the difference in ceilings. The positive effect for the long-term loans being statistically significant and the negative effect for short-term loans being statistically insignificant could suggest that the effect of the ceiling divergence is a net increase in originations and not a substitution of long-term loans for short-term ones. However, the magnitude of the difference in ceilings coefficient is non-negligible and there is a clear lack of power in the aggregate sample: the regression has only 52 degrees of freedom.

In the panel regressions, the signs of all the coefficients are the same as on aggregate. In the institution-level regression, the negative effect of the difference in ceilings on short-term loans is smaller in magnitude and, again, statistically insignificant. The interaction term of the variable with the long-term loan dummy is the same on aggregate and within credit institutions. Moreover, all the effects of funding rates are larger in magnitude in the latter regression. The branch-level estimates show a different picture of the shift in loan terms due to the reform. The negative main effect of the difference in ceilings is larger in magnitude than found in the previous regressions ($-.84$ vs. $-.45$ on institution level and $-.6$ on aggregate) and is statistically significant. The effect of interaction with the long-term dummy is, on the contrary, somewhat smaller (1.36 vs. 1.53). On the branch level then the shift towards longer loans explained by the reform is, to a larger extent, a substitution of long-term loans for short-term ones. Finally, the effects of funding rates are smaller than found in the previous regression, except for the effect of the finding rates difference on the long-term loans.

3.7.3 Magnitude of the reform effects

Regression (3.6) allows for an estimation of the effect of difference in ceilings by loan term on the value of loan originations. Specification without the log transformation of the results, reported in Table A.10, is most suitable for obtaining the estimates of absolute changes in loan originations. Since the coefficients in the aggregate-, institution- and branch-level regressions are shown to have comparable magnitudes in Table A.9 with log specifications, one can use the regression on the aggregate level to estimate the effect.

The coefficient of interest, δ_c, measures the change in the long-term loan originations corresponding to a 1 percentage point increase in the difference between the long- and the short-term ceiling. Furthermore, the coefficient β_c measures the simultaneous change in the short-term loan originations, so that the estimated change in aggregate loan originations corresponding to divergence of ceilings by 1 p.p. is $\hat{\delta}_c + \hat{\beta}_c = 1742$ (M€). This estimate should be scaled up by a factor of 15.9 to have the effect for the total loan originations in France, of which M_CONTRAN represents around 6.3% (see Figure A.4). Finally, the average post-2016 difference in the ceilings has been at 0.146 p.p., so the quarterly net increase in loan originations due to the reform is $1742 \times 15.9 \times 0.146 = 254$ (M€), or 6.4% of the average quarterly loan originations in France after 2016.

The estimate for the change in short-term loans due to the reform is negative, so the net increase in loan originations can be interpreted as partial substitution of short-term loans for longer ones. Consider a borrower that has been offered a longer loan instead of a shorter loan of the same size because of the reform. This has effects on two important dimensions of the borrower's financial position: the liquidity, depending on the size of regular payments, and the overall debt service cost. All other characteristics held constant, a larger loan term makes regular payments smaller and overall debt service cost higher. In addition, longer loans have larger interest rates, which make both regular payments and service cost larger. I do a back-of-the-envelope calculation to see how substitution of a short loan for a longer one influences the financial position of a typical borrower.

Suppose a borrower takes out a loan of average size in the sample according to Table A.2, 117.600€. Assume then that in absence of the reform, the borrower would get the typical loan of regulatory category of shorter loans, which is a 15-year loan. Assume further that because of the reform this loan is substituted with one of the same size and a typical term in the longer regulatory category, which is 25 years. I take the average interest rates of the two regulatory categories after 2016 to be the interest rates for these hypothetical loans: 1.76 p.p. for the short-term one and 1.88 p.p. for the long-term one.

If a loan is paid back in regular, constant monthly payments, the following formula calculates the size of the payment[20]:

$$monthly\ payment = amount \times \frac{\left((1+r)^{1/12} - 1\right)(1+r)^{term}}{(1+r)^{term} - 1} \tag{3.8}$$

Then, the sum of all the payments minus the size of the loan gives the total debt service cost:

$$debt\ service\ cost = term \times 12 \times monthly\ payment - amount \tag{3.9}$$

The short-term and the long-term loan would have a monthly payment of 743€ and 524€, correspondingly. The decrease in monthly payments due to longer term is then 30%, and given that amount enters (3.8) multiplicatively, this relative decrease is the same for any amount assumed for the two loans. This means that the direct effect of term on the size of the payment compensates by large the effect of the interest rate, which makes

[20]the derivation of the formula relies on finding a monthly amount such that the present discounted value of the stream of payments is equal to the loan amount. For exposition, see, e.g. https://www.boe.ca.gov/info/tvm/lesson7.html

the payment on the longer loan larger. Such significant improvement in the liquidity of the new borrowers can potentially improve borrowers' consumption levels, as found by Agarwal et al. (2015), Agarwal et al. (2017) for additional liquidity due to government programs for mortgage modification and refinancing in the U.S. .

The debt service cost is 27094€ for the long loan and 16155€ for the short one. This is largely due to the longer term and not to the higher interest rate: if the long loan had an interest rate of the short one, the debt service cost would still be at 25272€. In the life-cycle perspective, this 75% increase in debt service is a drag on the borrowers' ability to accumulate assets and sustain consumption later in life. Overall, an economy with a larger fraction of incomes devoted to debt service can experience lower long-run level of GDP, as argued in Mian et al. (2020c).

3.8 Robustness and additional results

3.8.1 Placebo: loans unaffected by reform

As discussed in Section 3.2.2, the interest rate restriction has been imposed not only on fixed-rate, but also on variable rate loans. The variable-rate loans category has not been reformed throughout 2012-2020 and has had the same interest ceiling for all loan terms. This makes it suitable for a placebo difference-in-differences estimation. Figure A.18 plots the difference-in-differences coefficients of the regression (3.5) on sums of lenders' variable rate loans of the two term bins[21]. If anything, the term structure of variable-rate loans has shifted to shorter, not longer loans. However, the main feature of the Figure A.18 is wide confidence intervals that point to insignificant, rather than negative, estimates.

The shift in term structure is also not apparent for consumption loans of pre-defined term, or personal loans, that is the second largest category of household credit in France. This is consistent with the interest rate regulation being the same for personal loans of all terms. Figure A.11 plots the empirical density for the periods 2012-2016 and 2017-2020, in the same way as it has been constructed for housing loans before. Loans from 1- to 7-year term are common on the market — the distribution has seven corresponding modes. Loans of 1- to 5-year term have become less frequent, but the mode that has increased most after 2016 is 6 years, not 7 years. The less frequent modes at 10 and 12 years have also gained in frequency somewhat. Overall, the term on average has increased, but the shift of term modes has been much less apparent than in the case of housing loans.

3.8.2 Influence of loan refinancing

Table A.12 reports results of two-period panel estimations that exclude the years 2015 and 2016. This leaves the pre- and post-reform periods with the same average share of refinancing operations, as depicted in Figure A.13. Even though high shares of refinancing op-

[21] For technical reasons, the regression is reported without the logarithmic transformation of the dependent variable. The result is qualitatively null with the transformation, too.

erations in 2015 and 2016 suggest a priori positive influence on the difference-in-differences coefficients, the removal of these two years from the sample makes the coefficients even larger, both for value and number of loans, on credit institution and branch level (see Tables A.7, A.8 for comparison to full sample). The yearly estimates on the full sample, plotted in Figures A.14, A.15 clarify this issue. In the pre-reform period, the year 2016 has a higher coefficient than other years. At the same time, the coefficient for 2017 is lower than the ones for the other post-reform years. Exclusion of 2016 and 2017 then lowers the average term shift pre-reform and raises it post-reform. This robustness check suggests that the positive and large difference-in-differences estimates found in the two-period regressions of the full sample are not driven by waves for loan refinancing.

Table A.11 reports the regression on ceiling difference and funding rates with the aggregate share of refinancing operations as another time-varying covariate. This additional control is significant and makes the main estimates smaller in magnitude, as expected. On the aggregate level, the difference in ceilings does not explain the difference in long- and short-term loan value any more. However, within credit institutions and within branches, the main estimate remains statistically significant.

3.8.3 Banks vs. specialized institutions

Since the branch level analysis excludes specialized institutions, the difference of those results from aggregate and institution-level ones might be explained by heterogeneous effects of the reform on banks and on specialized institutions. The contribution of this difference is potentially sizeable, since specialized institutions account for roughly 40% of the aggregate value and the number of housing credit contracts in France.

Table A.13 reports the yearly panel regression 3.5 on credit institution level, with the sample split into banks and specialized institutions. For both loan values and quantities, the patterns are similar for the two types of institutions. In the specialized institutions, the share of long-term loans is smaller in 2012-2013, and it rises faster until 2019. The difference-in-differnces estimate of 2019 is particularly large for specialized institutions, which cannot be explained by interest rate restriction. In general, the estimates for specialized institutions lack statistical significance, which is explained by more than a fourfold difference in sample size with respect to banks.

3.8.4 Large bank sub-sample

The largest bank, excluded from the estimations within branch for technical reasons, has had the same pattern of loan term structure as found in main estimations. This pattern is shown in Figure A.17, which is an analogous plot to Figure A.2 on the M_CONTRAN aggregates. However, it has some distinctions from the aggregate and within credit institutions results. Firstly, the declining trend in short-term loans has started by the end of 2015, around a year before the reform has been announced. Secondly, the pre-reform difference in the total value of short- and long-term loans is roughly as large as the post-reform one, but has opposite sign. This means the substitution of short loans for long ones has been most dramatic in this bank. The total value of the loans in the bank is also considerable, meaning that the aggregate and institution-level estimates are, to a large extent, driven by this influential bank.

3.8.5 Quarterly differences in term and in time

Figures A.16, provide estimates of a quarterly version of regression (3.5), on a credit institution level. This allows to see which quarters contribute most to the yearly estimates discussed in Section 3.7, while taking into account the loan refinancing bias that is most relevant to several quarters, such as 2015:Q3 and 2017:Q1. The effect is already found in 2016:Q2, which is even before the announcement of the reform by the High Council of Financial Stability in July. On the one hand, this might invalidate the attribution of the effect to the reform. On the other, it may reflect the information that the industry participants might have had before the official announcement has been made to the general public. In 2017, the term shift starts to take effect since 2017:Q3, which is the first quarter were interest rate ceilings have diverged.

3.9 Conclusion

I find that additional flexibility in the interest rate restriction policy has a significant impact on the term structure of housing loan originations in France. This evidence suggests that uniform interest rate ceilings for short-term and long-term loans crowd out the supply of the latter. Given that the ceilings are often set as a function of average market interest rates, the recent macroeconomic conditions make such constraints particularly binding. While some borrowers gain from smaller interest rate, others are left out, since short-term loans might imply regular payments too high for their incomes. However, households that are able to obtain short-term, low-interest loans because of the interest rate restriction have a significant benefit in the long run because of the overall debt service cost.

Overall, refining the interest rate restriction with respect to loan term might be desirable for a number of economies, and the marginal cost of such additional rules is small compared to that of the other micro-prudential measures introduced over the last decade.

Appendix A

Appendix for Chapter 3

A.1 Dataset description

M_CONTRAN is a quarterly register of loan originations of credit institutions that have a licence of the French Prudential Supervision and Resolution Authority, ACPR. Only loans in euro are included. The dataset includes loans to all sectors of the economy; I focus on loans to households. Among different types of loans to households available, I focus on residential real estate (housing) loans.

Sample In terms of participating credit institutions, M_CONTRAN is an unbalanced panel; in the observation window 2012:Q3-2020:Q2 the number of participating number of institutions varies between 150 and 200. A list of institutions has been publicly available by Banque de France only once, before a revision of the sample in 2015[1]. The three main cooperative banking groups, *Crédit Agricole*, *Crédit Mutuel* and *Banque Populaire – Caisse d'Epargne* are all found in the 2015 list, along with their largest non-cooperative competitors, *Société Générale* and *BNP Paribas*. *Crédit Agricole*, and *Banque Populaire – Caisse d'Epargne* are represented by their regional entities — the *Caisse*'s. The two cooperative banks have at least as many regional entities in the list as their total number in 2020, which allows to say that a vast majority, if not all, of their regional entities are represented in the sample. At the same time, *Crédit Mutuel* is only represented by its central body, although it has the same decentralized organizational structure as the two banks mentioned above. This particularity of the data is in line with the presence of an abnormally large lender in the sample, reported in Figure A.5, for which the branch identifiers are not available.

The sample is stratified with an aim of unbiased calculation of the average interest rates. The share in the total amount of originations of each lender is representative. The sampling strategy within a credit institution depends on its size. For small institutions, the universe of originations is reported. For large institutions, only a part of branches is included in the sample, but those branches report the universe of their originations.

[1] see https://www.banque-france.fr/sites/default/files/media/2016/11/18/liste_des_assujetis_mensuels_taux_2015.pdf

Unit of observation An observation represents credit origination, done either under a new credit contract or under an existing contract involving multiple originations (examples of contracts with multiple originations include credit lines, revolving loans or account contracts allowing for overdrafts). The housing loans sub-sample, studied in this paper, does not include contracts with multiple originations. Apart from brand new housing loans, it includes two other types of originations:

1. Modification of some terms of an existing contract. Such observations are marked with a renegotiation (*renégociation*) dummy. No reference to the previous contract is observed. Modifications that take place in accordance with the initial contract terms (e.g. evolution of interest rate on a variable-rate loan) are not in the sample.

2. Transfer of a loan from one lender to another. Since 2016, such observations are marked with a refinancing (*rachat*) dummy. No reference to the previous contract is observed.

Variables The following variables are observed for the housing loans sub-sample:

- Amount in euros.
- Term in months.
- Effective interest rate (*TEG*) — an annualized measure of consumer price of credit, percentage of the loan amount. Includes contract administration fees, remuneration of intermediaries, cost of obligatory insurance and guarantees, cost of real estate evaluation. Interest rate restriction applies to this value.
- Interest rate, narrowly defined (*TESE*).
- Adjustment of interest rate: whether the rate is fixed or variable.
- For variable-rate loans: reference rate (e.g. 3-month EURIBOR); period of interest rate fixation (*PFIT*), maximum rate.
- Purpose of the housing loan: primary residence purchase, secondary residence purchase, bridging loan (*prêt relais*), etc. Observed for 2 quarters out of 32.
- Modification of loan terms (*renégociation*).
- Loan refinancing (*rachat*). Variable observed only since 2016.
- Credit institution identifier (*CIB*) of lender, anonymized.
- "Specialized credit institution" status of lender.
- Bank branch identifier. Not available for specialized credit institutions.
- Annual labor income of borrower. Variable not used in this paper due to data quality considerations.

A.2 Tables

A.2.1 Institutional framework

Period	Interest rate regulation loan categories
2012 – 2016	Fixed rate, variable rate*, *prets relais**
2017 –	Fixed rate $[0, 10)$ years term, fixed rate $[10, 20)$ years term, fixed rate ≥ 20 years term, variable rate*, *prets relais**

Table A.1: Regulatory categories of housing loans subject to interest rate restriction before and after the reform in 2017. Pre-2012 out of scope of this paper.
*: categories omitted from main estimations because of small aggregate values. Section 3.8.1 presents additional estimations for variable-rate loans.

A.2.2 Descriptive statistics

Statistic	N	Mean	St. Dev.	Min	Max	Pctl(25)	Median	Pctl(75)
amount	1,125,420	116.266	100.538	0.001	9,000.000	51.808	95.432	153.260
term	1,125,420	16.233	6.758	0.083	41.667	10.917	15.167	20.500
eff_interest	1,125,420	2.932	1.000	0.276	15.398	2.176	2.711	3.543

Table A.2: Contract descriptive statistics — full sample.

Statistic	N	Mean	St. Dev.	Min	Max	Pctl(25)	Median	Pctl(75)
amount	709,299	117.614	99.293	0.001	9,000.000	55.000	96.076	152.631
term	709,299	16.626	6.445	0.083	41.333	12.000	16.167	21.333
eff_interest	709,299	2.959	0.964	0.446	15.398	2.193	2.760	3.648

Table A.3: Contract descriptive statistics – banks only.

start \end	2013	2014	2015	2016	2017	2018	2019	2020	total
2012	2	10	5	9	5	2	2	112	147
2013	1	0	0	0	0	0	0	1	2
2014	0	0	1	0	0	0	0	2	3
2015	0	0	4	0	0	0	1	2	7
2016	0	0	0	0	0	0	0	1	1
2017	0	0	0	0	1	1	0	1	3
2018	0	0	0	0	0	0	0	0	0
2019	0	0	0	0	0	0	1	1	2
2020	0	0	0	0	0	0	0	1	1
total	3	10	10	9	6	3	4	121	166

Table A.4: Panel balance: number of credit institutions entering and exiting sample on given years

start \end	2012	2013	2014	2015	2016	2017	2018	2019	2020	total
2012	19	54	1822	10	47	9	8	56	512	2537
2013	0	47	207	5	5	2	4	22	53	345
2014	0	0	43	1	2	1	2	11	16	76
2015	0	0	0	20	22	34	46	103	1398	1623
2016	0	0	0	0	17	6	10	3	54	90
2017	0	0	0	0	0	36	6	47	280	369
2018	0	0	0	0	0	0	4	1	6	11
2019	0	0	0	0	0	0	0	7	5	12
2020	0	0	0	0	0	0	0	0	1	1
total	19	101	2072	36	93	88	80	250	2325	5064

Table A.5: Panel balance: number of bank branches entering and exiting sample on given years

A.2.3 Difference in terms and time estimates

	aggregate M CONTRAN			
	ln(loans €)	ln(loans n)	ln(loans €)	ln(loans n)
	(1)	(2)	(3)	(4)
term $\geq$ 20	-0.11^{**}	-0.58^{***}	-0.14^{*}	-0.70^{***}
	(0.04)	(0.04)	(0.07)	(0.07)
term$\geq$20$\times$after 2016	0.50^{***}	0.51^{***}		
	(0.07)	(0.07)		
term$\geq$20$\times$year 2013			-0.14	-0.03
			(0.08)	(0.08)
term$\geq$20$\times$year 2014			0.03	0.05
			(0.08)	(0.08)
term$\geq$20$\times$year 2015			-0.01	0.13
			(0.08)	(0.08)
term$\geq$20$\times$year 2016			0.27^{***}	0.41^{***}
			(0.08)	(0.08)
term$\geq$20$\times$year 2017			0.27^{***}	0.40^{***}
			(0.08)	(0.08)
term$\geq$20$\times$year 2018			0.57^{***}	0.66^{***}
			(0.08)	(0.08)
term$\geq$20$\times$year 2019			0.70^{***}	0.76^{***}
			(0.08)	(0.08)
term$\geq$20$\times$year 2020			0.69^{***}	0.79^{***}
			(0.10)	(0.10)
Year-quarter FE	Y	Y	Y	Y
Observations	64	64	64	64
R^2	0.92	0.92	0.98	0.98
Adjusted R^2	0.84	0.83	0.96	0.95
Residual Std. Error	0.13	0.13	0.07	0.07
Note:				*p<0.1; **p<0.05; ***p<0.01

Table A.6: Aggregate-level regressions. 2-period specification of equation (3.2) in columns 1 and 2; yearly specification of equation (3.3) in columns 3 and 4.

	credit institution level			
	ln(loans €)	ln(loans n)	ln(loans €)	ln(loans n)
	(1)	(2)	(3)	(4)
term $\geq$ 20	-0.09^{***}	-0.49^{***}	-0.14	-0.55^{***}
	(0.02)	(0.02)	(0.11)	(0.11)
term$\geq$20$\times$after 2016	0.52^{***}	0.52^{***}		
	(0.04)	(0.04)		
term$\geq$20$\times$year 2013			-0.12^{***}	-0.12^{***}
			(0.05)	(0.04)
term$\geq$20$\times$year 2014			0.05	0.01
			(0.06)	(0.05)
term$\geq$20$\times$year 2015			0.02	0.04
			(0.09)	(0.09)
term$\geq$20$\times$year 2016			0.30^{**}	0.32^{***}
			(0.11)	(0.12)
term$\geq$20$\times$year 2017			0.27^{***}	0.28^{***}
			(0.09)	(0.09)
term$\geq$20$\times$year 2018			0.58^{***}	0.58^{***}
			(0.10)	(0.10)
term$\geq$20$\times$year 2019			0.79^{***}	0.79^{***}
			(0.10)	(0.09)
term$\geq$20$\times$year 2020			0.71^{***}	0.69^{***}
			(0.11)	(0.10)
Year-quarter FE	Y	Y	Y	Y
Cred. institution FE	Y	Y	Y	Y
Observations	8,052	8,052	8,052	8,052
R^2	0.78	0.77	0.78	0.78
Adjusted R^2	0.77	0.77	0.78	0.77
Residual Std. Error	23.26	23.41	23.13	23.29
Note:			*p<0.1; **p<0.05; ***p<0.01	

Table A.7: Lender-level regressions. 2-period specification of equation (3.4) in columns 1 and 2; yearly specification of equation (3.5) in columns 3 and 4.

	branch level			
	ln(loans €)	ln(loans n)	ln(loans €)	ln(loans n)
	(1)	(2)	(3)	(4)
term $\geq$ 20	0.21***	−0.25***	0.33***	−0.21***
	(0.01)	(0.01)	(0.04)	(0.03)
term≥20×after 2016	0.38***	0.35***		
	(0.01)	(0.01)		
term≥20×year 2013			−0.26***	−0.18***
			(0.04)	(0.03)
term≥20×year 2014			−0.12***	−0.08***
			(0.04)	(0.03)
term≥20×year 2015			−0.21***	−0.09***
			(0.04)	(0.03)
term≥20×year 2016			−0.002	0.10***
			(0.04)	(0.03)
term≥20×year 2017			−0.01	0.09***
			(0.04)	(0.03)
term≥20×year 2018			0.27***	0.33***
			(0.04)	(0.03)
term≥20×year 2019			0.43***	0.45***
			(0.04)	(0.04)
term≥20×year 2020			0.44***	0.45***
			(0.04)	(0.04)
Year-quarter FE	Y	Y	Y	Y
Branch FE	Y	Y	Y	Y
Observations	112,847	112,847	112,847	112,847
R^2	0.43	0.49	0.44	0.50
Adjusted R^2	0.41	0.47	0.41	0.47
Residual Std. Error	104.45	82.58	103.98	82.16
Note:				*p<0.1; **p<0.05; ***p<0.01

Table A.8: Branch-level regressions. 2-period specification of equation (3.4) in columns 1 and 2; yearly specification of equation (3.5) in columns 3 and 4. Largest bank excluded from estimation since all its observations are miscoded as originations of a single branch.

A.2.4 Difference in term and ceilings estimates

	Dependent variable:		
	ln(loans €)		
	aggregate	cred. inst.	branch
	(1)	(2)	(3)
term $\geq$ 20	0.22	0.34***	0.81***
	(0.37)	(0.12)	(0.04)
Δ ceiling	−0.60	−0.45	−0.84***
	(0.49)	(0.40)	(0.06)
$r_{<20y}$	−0.19	−0.54**	−0.18***
	(0.20)	(0.21)	(0.05)
Δr	0.25	0.53**	0.26***
	(0.25)	(0.22)	(0.04)
term$\geq$20 $\times\Delta$ ceiling	1.54**	1.53***	1.36***
	(0.69)	(0.21)	(0.07)
term$\geq$20 $\times$ $r_{<20y}$	−0.43	−0.40***	−0.11***
	(0.29)	(0.10)	(0.04)
term$\geq$20 $\times\Delta r$	−0.08	−0.17**	−0.43***
	(0.36)	(0.07)	(0.04)
Constant	7.12***		
	(0.26)		
Year-quarter FE	N	N	N
Cred. institution FE	N	Y	N
Branch FE	N	N	Y
Observations	60	7,513	106,183
R^2	0.41	0.75	0.40
Adjusted R^2	0.33	0.74	0.37
Residual Std. Error	0.26 (df = 52)	24.30 (df = 7340)	108.18 (df = 101132)
Note:			*p<0.1; **p<0.05; ***p<0.01

Table A.9: Regressions with time-varying covariates for total loan amounts, specification of equation (3.6). Δ ceiling is the interest rate ceiling of the "$\geq$ 20 years" category minus the ceiling on the "$<$ 20 years" category. $r_{<20y}$ is the swap-based measure of funding rate corresponding to the short loan category, described in Section 3.6.2. Δr is the funding rate measure for the long loan category minus that of the short loan category.

	Dependent variable:		
	loans_EUR_M		loans_EURk
	aggregate	cred. inst.	branch
	(1)	(2)	(3)
term $\geq$ 20	628.51	8.97*	682.35***
	(656.37)	(4.55)	(148.24)
Δ ceiling	−1,103.61	−114.70	−485.22***
	(872.73)	(81.83)	(42.84)
$r_{<20y}$	−342.91	19.25	−130.73
	(364.79)	(20.70)	(108.13)
Δr	445.66	7.13	153.48***
	(452.79)	(4.73)	(50.23)
term$\geq$20 $\times\Delta$ ceiling	2,845.61**	146.95*	1,241.55***
	(1,234.22)	(86.67)	(281.22)
term$\geq$20 $\times$ $r_{<20y}$	−618.96	−40.30*	−149.16***
	(515.89)	(24.12)	(46.30)
term$\geq$20 $\times\Delta r$	−343.24	−4.98	−383.05***
	(640.35)	(3.07)	(93.93)
Constant	1,206.57**		
	(464.12)		
Year-quarter FE	N	N	N
Cred. institution FE	N	Y	N
Branch FE	N	N	Y
Observations	60	7,513	106,183
R^2	0.41	0.84	0.47
Adjusted R^2	0.33	0.83	0.45
Residual Std. Error	467.70 (df = 52)	1,307.50 (df = 7340)	114,443.50 (df = 101132)
Note:			*p<0.1; **p<0.05; ***p<0.01

Table A.10: Regressions with time-varying covariates for total loan amount, specification of equation (3.6) without log transformation of outcome variable. Δ ceiling is the interest rate ceiling of the "$\geq$ 20 years" category minus the ceiling on the "< 20 years" category. $r_{<20y}$ is the swap-based measure of funding rate corresponding to the short loan category, described in Section 3.6.2. Δr is the funding rate measure for the long loan category minus that of the short loan category.

A.2.5 Additional results, robustness

	Dependent variable:		
	ln(loans €)		
	aggregate	cred. inst.	branch
	(1)	(2)	(3)
term ≥ 20	0.46	0.62***	0.95***
	(0.38)	(0.14)	(0.05)
Δ ceiling	0.70	1.61***	0.65***
	(0.59)	(0.60)	(0.09)
$r_{<20y}$	−0.14	−0.46**	−0.08*
	(0.18)	(0.20)	(0.04)
Δr	0.22	0.49**	0.02
	(0.23)	(0.21)	(0.04)
refinancing	2.09***	3.33***	2.26***
	(0.62)	(0.53)	(0.10)
term$\geq$20 $\times\Delta$ ceiling	0.85	0.73***	0.56***
	(0.83)	(0.23)	(0.09)
term$\geq$20 $\times$ $r_{<20y}$	−0.45*	−0.42***	−0.20***
	(0.26)	(0.10)	(0.04)
term$\geq$20 $\times\Delta r$	−0.07	−0.15**	−0.28***
	(0.32)	(0.07)	(0.04)
term$\geq$20 $\times$ refinancing	−1.11	−1.29***	−1.23***
	(0.88)	(0.23)	(0.11)
Constant	6.67***		
	(0.27)		
Year-quarter FE	N	N	N
Cred. institution FE	N	Y	N
Branch FE	N	N	Y
Observations	60	7,513	106,183
R^2	0.54	0.76	0.41
Adjusted R^2	0.45	0.76	0.38
Residual Std. Error	0.24 (df = 50)	23.66 (df = 7338)	107.27 (df = 101130)
Note:			*p<0.1; **p<0.05; ***p<0.01

Table A.11: Regressions with time-varying covariates for total loan amounts, specification of equation (3.6) with aggregate share of loan refinancing as an additional covariate. Δ ceiling is the interest rate ceiling of the "$\geq$ 20 years" category minus the ceiling on the "< 20 years" category. $r_{<20y}$ is the swap-based measure of funding rate corresponding to the short loan category, described in Section 3.6.2. Δr is the funding rate measure for the long loan category minus that of the short loan category. Refinancing is the median share of refinancing operations in housing loan originations among 12 credit institutions surveyed in the ACPR monthly survey of credit production. The latter covariate is a measure of the economy-wide share of refinancing operations.

	Dependent variable:			
	ln(loans €)	ln(loans n)	ln(loans €)	ln(loans n)
	cred. inst.		branch	
	(1)	(2)	(3)	(4)
term $\geq$ 20	-0.17^{***}	-0.59^{***}	0.17^{***}	-0.31^{***}
	(0.03)	(0.03)	(0.01)	(0.01)
term$\geq$20$\times$after 2016	0.60^{***}	0.61^{***}	0.41^{***}	0.42^{***}
	(0.04)	(0.04)	(0.01)	(0.01)
Year-quarter FE	Y	Y	Y	Y
Cred. institution FE	Y	Y	N	N
Branch FE	N	N	Y	Y
Observations	6,065	6,065	88,637	88,637
R^2	0.77	0.76	0.45	0.51
Adjusted R^2	0.76	0.75	0.41	0.48
Residual Std. Error	24.60	24.78	102.91	80.43
Note:			*p<0.1; **p<0.05; ***p<0.01	

Table A.12: 2-period panel regressions, specification of equation (3.4); sample excludes years 2015 and 2016 for a comparable aggregate share of refinancing operations pre- and post- reform — see Figure A.13. Largest bank excluded from branch-level estimation since all its observations are miscoded as originations of a single branch.

	credit institution level: split sample			
	ln(loans €)	ln(loans n)	ln(loans €)	ln(loans n)
	banks		specialized	
	(1)	(2)	(3)	(4)
term $\geq$ 20	−0.11	−0.51***	−0.29	−0.69
	(0.11)	(0.10)	(0.43)	(0.49)
term$\geq$20$\times$year 2013	−0.12**	−0.12***	−0.16	−0.11
	(0.05)	(0.04)	(0.14)	(0.12)
term$\geq$20$\times$year 2014	0.04	−0.002	0.08	0.06
	(0.06)	(0.06)	(0.14)	(0.08)
term$\geq$20$\times$year 2015	0.01	0.02	0.09	0.12
	(0.10)	(0.09)	(0.21)	(0.26)
term$\geq$20$\times$year 2016	0.27**	0.30**	0.39	0.40
	(0.13)	(0.12)	(0.28)	(0.34)
term$\geq$20$\times$year 2017	0.25**	0.26***	0.35	0.38
	(0.10)	(0.09)	(0.30)	(0.36)
term$\geq$20$\times$year 2018	0.58***	0.56***	0.57*	0.64*
	(0.11)	(0.09)	(0.30)	(0.36)
term$\geq$20$\times$year 2019	0.73***	0.71***	1.02***	1.08***
	(0.11)	(0.10)	(0.19)	(0.24)
term$\geq$20$\times$year 2020	0.71***	0.69***	0.66*	0.63
	(0.10)	(0.09)	(0.39)	(0.43)
Year-quarter FE	Y	Y	Y	Y
Cred. institution FE	Y	Y	Y	Y
Observations	6,397	6,397	1,655	1,655
R^2	0.84	0.83	0.57	0.58
Adjusted R^2	0.83	0.83	0.54	0.55
Residual Std. Error	20.84	20.86	26.94	27.72
Note:			*p<0.1; **p<0.05; ***p<0.01	

Table A.13: Lender-level split-sample regressions, specification of equation (3.5).

A.3 Graphs

Data used for plots is M_CONTRAN and my calculations, unless otherwise stated.

A.3.1 Descriptive graphs

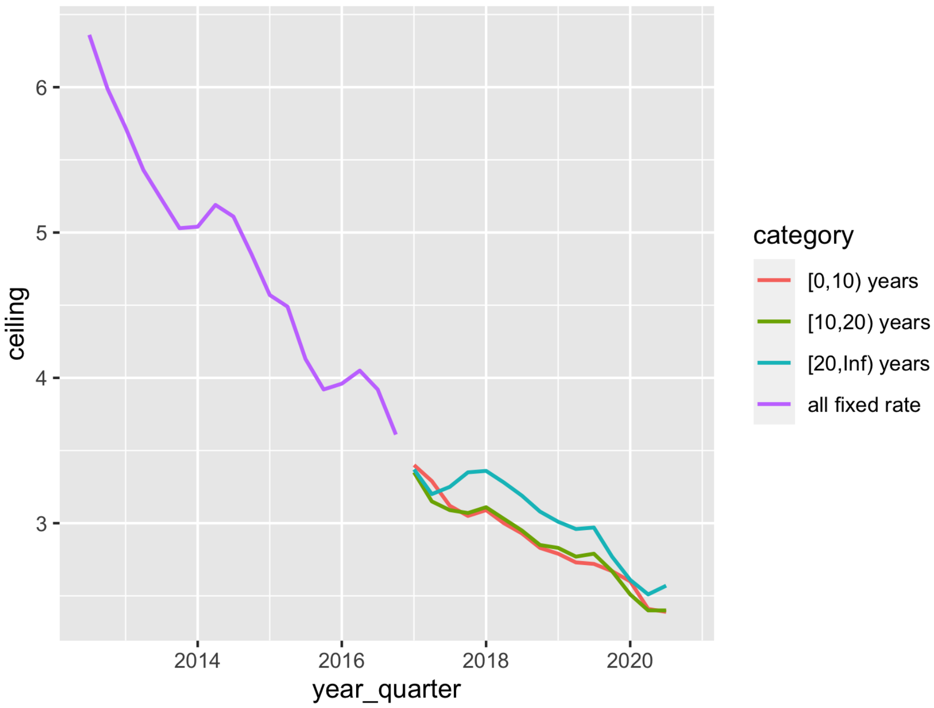


Figure A.1: Interest rate ceilings for housing loans. Source: Banque de France for 2017-2020 ceilings; espacecredit.com for pre-2017 ceiling.

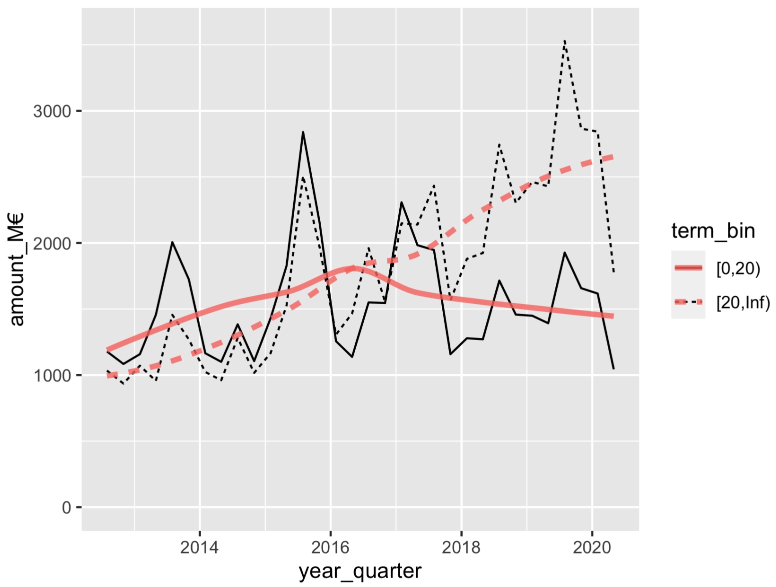


Figure A.2: Total M_CONTRAN value of loans by term bin, €millions. Smooth lines are LOESS fit on the corresponding series.

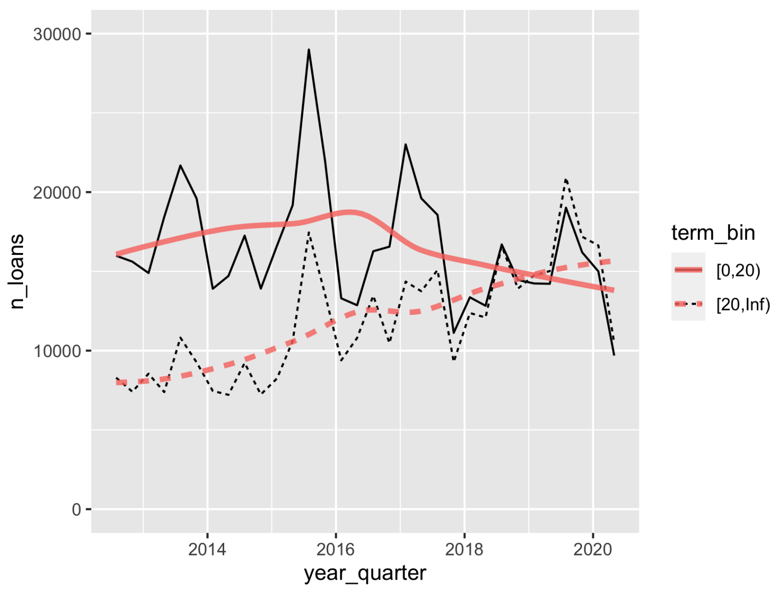


Figure A.3: Total M_CONTRAN number of loans by term bin. Smooth lines are LOESS fit on the corresponding series.

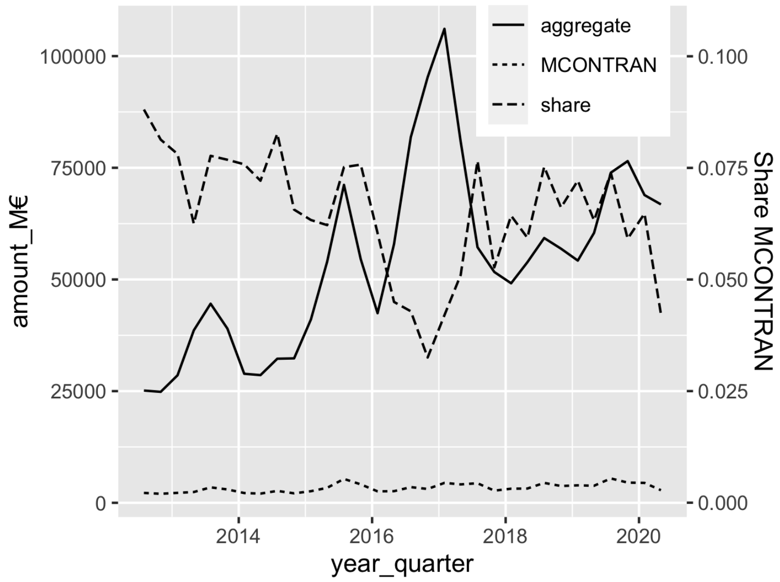


Figure A.4: Total quarterly value of loans originations in M_CONTRAN (excluding renegotiations and variable-rate loans) and on aggregate (no exclusion). Right axis for the share of M_CONTRAN in aggregate. Aggregate series source: Webstat of Banque de France.

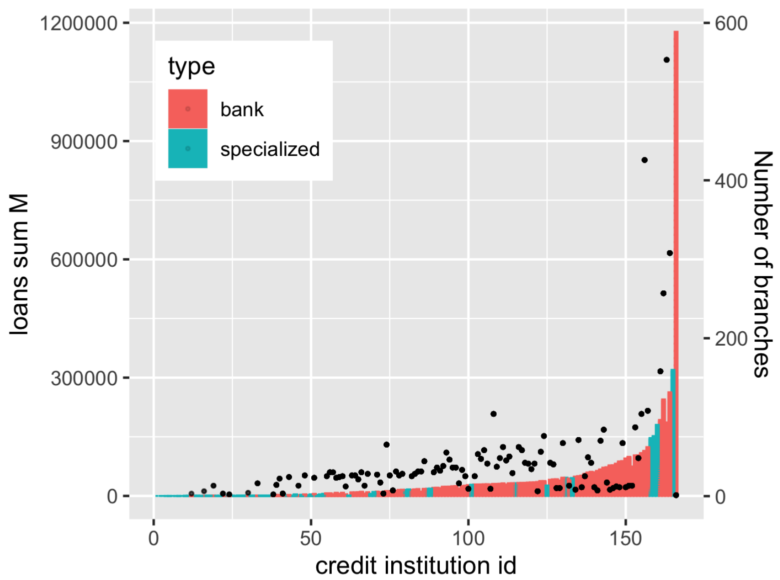


Figure A.5: Total value of loans (bar) and number of branches (dots, only for banks) per credit institution throughout 2012:Q3-2020:Q2. The top bank accounts for 17% of the loan value of M_CONTRAN, while having only one branch identifier.

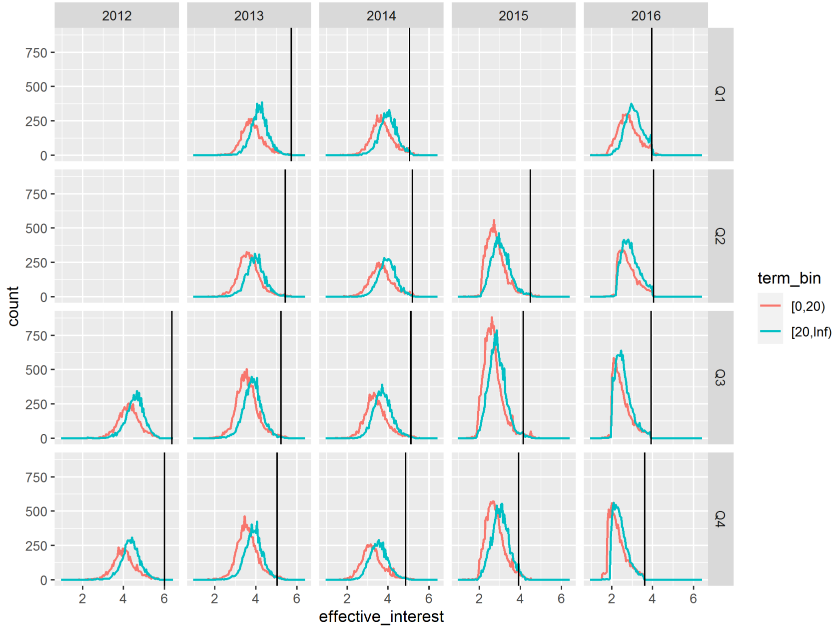


Figure A.6: Quarterly histograms of effective interest rates on short and long term loans, 2012-2016. Vertical line on the level of interest rate ceiling. Histogram bin width 5 basis points.

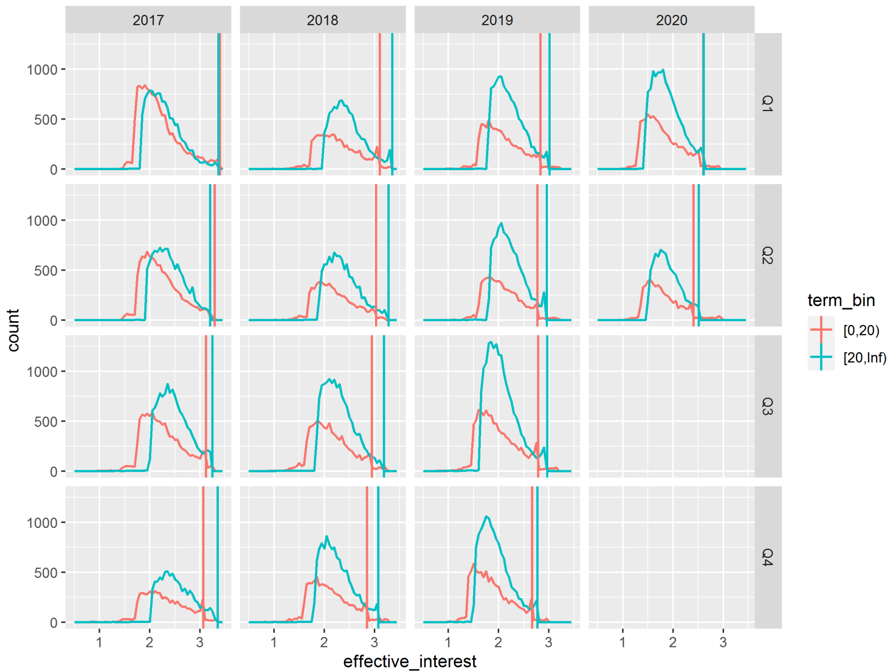


Figure A.7: Quarterly histograms of effective interest rates on short and long term loans, 2017-2020. Vertical lines on the levels of interest rate ceilings. The ceiling for $[0, 20)$ years is the maximum of the ceilings for the $[0, 10)$ and $[10, 20)$ years regulatory bins. Histogram bin width 5 basis points.

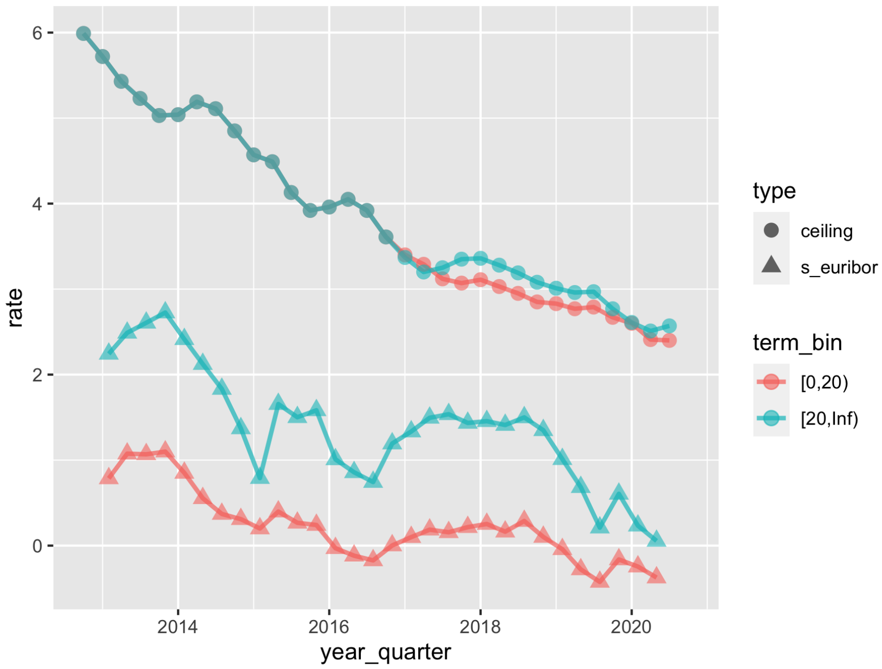


Figure A.8: Interest rate ceilings for two term bins and measures of corresponding EURIBOR swap rates. Unique ceiling for all term bins prior to 2017. Ceiling for $[0, 20)$ years is the maximum of the ceilings for the $[0, 10)$ and $[10, 20)$ years regulatory bins. For EURIBOR swap calculation by term bin, see Section 3.4. Source: Banque de France, espacecredit.com, Factset.

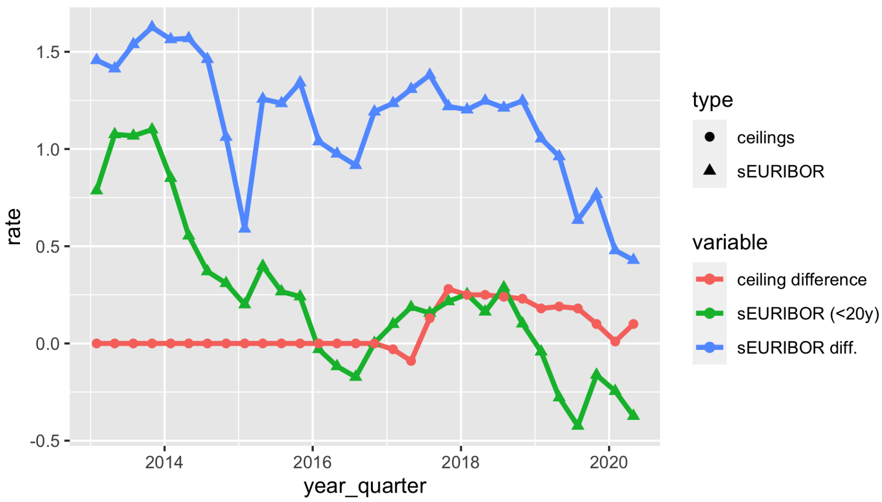


Figure A.9: Time-varying covariates of regression (3.6): the difference of ceilings between long and short loan bins ; swap EURIBOR rate of the short bin, difference of swap EURIBOR of the two bins. Source: Banque de France, Factset.

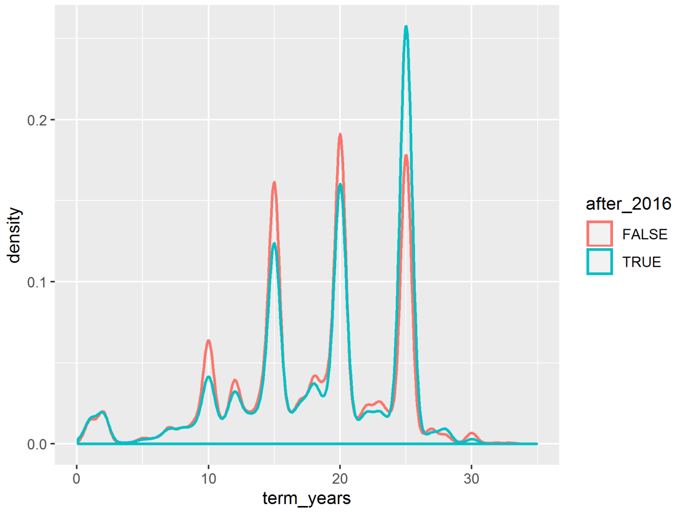


Figure A.10: Empirical density functions for housing loan terms, 2012-2016 vs. 2017-2020.

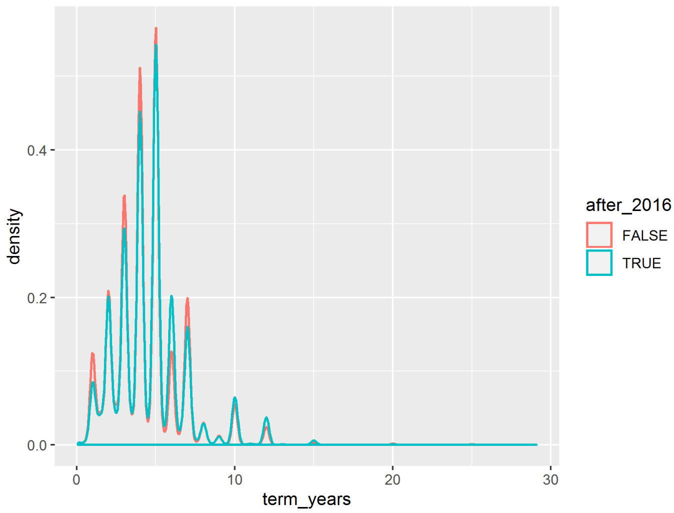


Figure A.11: Empirical density functions for personal loan terms, 2012-2016 vs. 2017-2020.

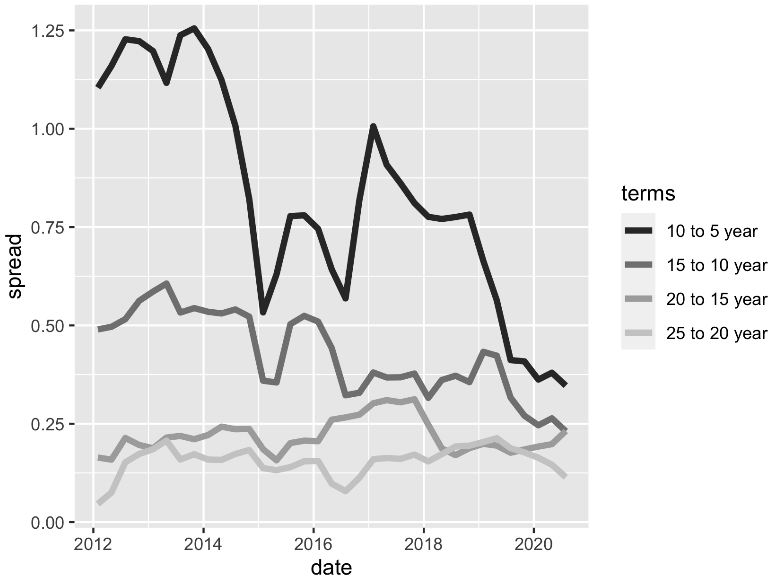


Figure A.12: Term spreads on French Treasury bill constant maturity swaps (*TEC*). Source: Webstat of Banque de France.

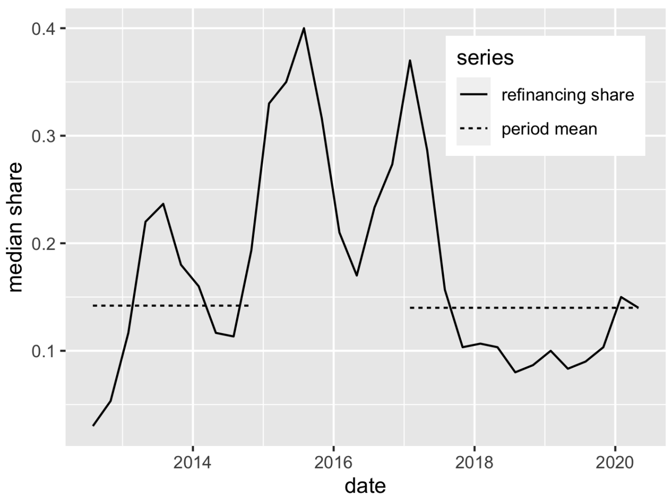


Figure A.13: Share of housing loan refinancing in total loan originations — median among 12 credit institutions participating in survey. Source: ACPR (Banque de France) monthly survey of housing credit production.

A.3.2 Difference-in-difference estimates

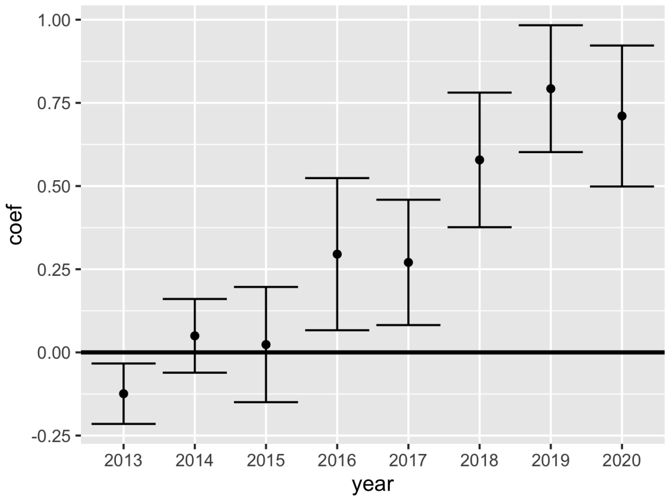


Figure A.14: Interaction coefficient estimates of Table A.7 lender-level panel regression of loan value, with 95% confidence intervals.

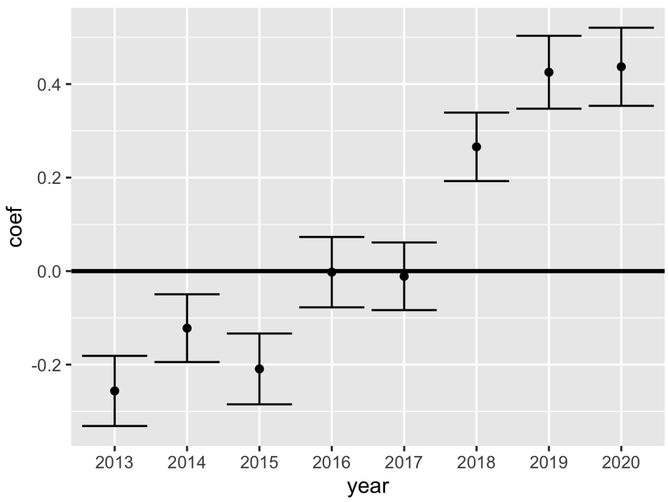


Figure A.15: Interaction coefficient estimates of Table A.8 branch-level panel regression of loan value, with 95% confidence intervals.

A.3.3 Additional results, robustness

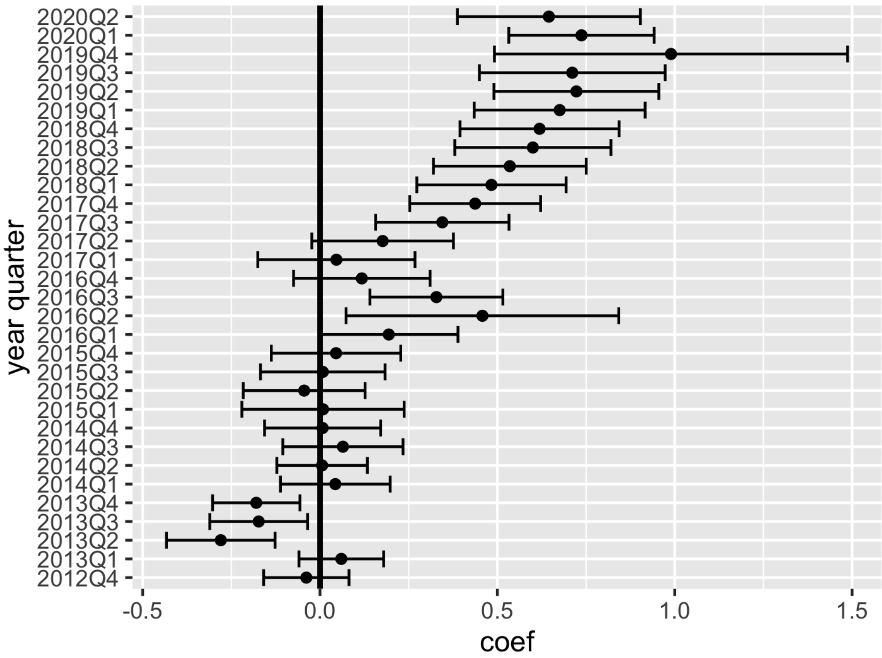


Figure A.16: Interaction coefficient estimates of a quarterly version of a lender-level panel regression (3.5) of loan value, with 95% confidence intervals.

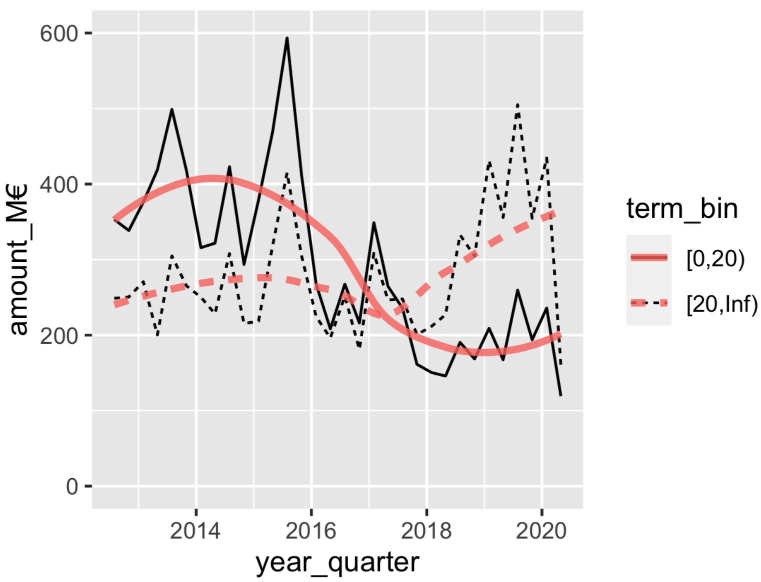


Figure A.17: Total value of loans, by term bin, of the largest bank in the sample, €millions. Smooth lines are LOESS fit on the corresponding series.

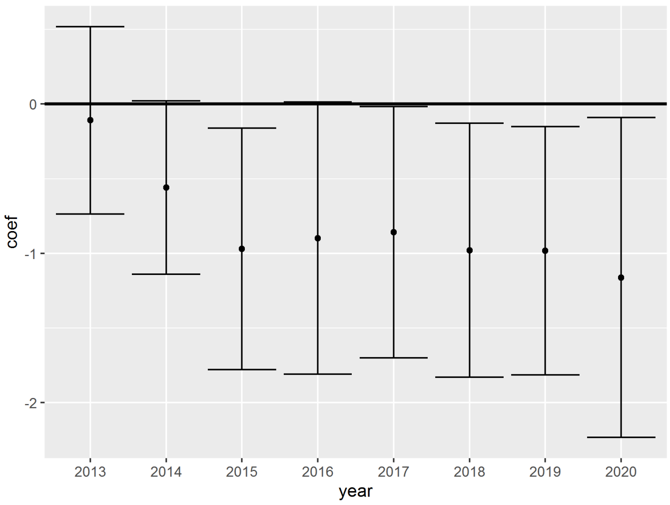


Figure A.18: Placebo test. Interaction coefficient estimates of regression 3.5 lender-level panel regression of value of *variable-rate* loans, with 95% confidence intervals.

Conclusion

In this thesis, I have analysed various consequences of regulatory measures on the household credit market. In a theoretical analysis of personal bankruptcy, I show implications for optimal redistribution policy, as well as for macroeconomic stability. First, the use of government debt for redistribution of resources between generations alleviates the credit constraints that are due to the possibility of personal bankruptcy. This is an additional argument in favor of deficit finance in the lasting debate on fiscal policy in the United States (Blanchard, 2019), which is especially topical today, amid an unprecedented recession. Second, I do not find de-stabilizing effects of the personal bankruptcy institution for long-term macroeconomic dynamics, unless bankrupt individuals have access to a large market of informal assets that can be hidden from creditors. The latter concern is valid today, since advances in information technology give rise to new kinds of assets and markets, such as the booming cryptocurrency market. Finally, I analyze interest rate restriction policies on the housing loan market in France and show that this regulation results in a shift of credit supply towards shorter loans. This is important for allocation of credit, since it can create credit constraints for young and income-poor households.

All the findings of this thesis suggest further investigation. The theoretical analysis of the redistribution policy under the personal bankruptcy friction of Chapter 1 is done in a stylized model that allows to isolate the life-cycle motive for borrowing from other reasons for debt accumulation. However, the results of this analysis are at odds with some recent contributions, where government debt is found to make credit constraints tighter in an environment with personal bankruptcy (Rohrs and Winter, 2015; Antunes and Ercolani, 2020). The reason is as follows: these papers, as most of the literature, focus on negative income shocks as the main reason to borrow. It is my focus on the life-cycle motive and several simplifying assumptions that help to highlight the novel relationship between government debt and credit constraint. It is therefore important to reproduce the analysis in a larger model that combines the life-cycle motive for credit with the negative income shock motive.

In the analysis of macroeconomic instability of Chapter 2, the simple equilibrium properties of the model might be due to the choice of functional forms of production and utility functions. A number of macroeconomic models displays complex equilibrium behaviour under specific functional forms and parameters (Nourry et al., 2013; Dufourt et al., 2015); the model explored in Chapter 2 may be among them. The robustness of the analysis of bubbles must also be addressed. For instance, the asset used after bankruptcy may have both a fundamental value and a bubble, i.e., a land-like asset may be used. Furthermore, the welfare properties of equilibria with and without bubbles remain to be explored in our setting.

In the empirical analysis of Chapter 3, the implications of the shift in duration of new housing loans must be further addressed. The claim about the credit constraints for young and income-poor borrowers remains hypothetical at this stage, but can be tested using the loan-level data merged with local borrower characteristics, following the various studies surveyed in Mian and Sufi (2015). The interaction with local housing price dynamics is another important dimension.

All in all, given the importance of household debts for today's economies, the topic deserves further investigation. These efforts are necessary to make credit markets both inclusive and sound at the same time.

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