Multivariate Extremal Density Expansions and Residual Tail Dependence Structures

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Kurzzusammenfassung

In der multivariaten Extremwerttheorie kann man Abhängigkeitsstrukturen durch Pickands–Abhängigkeitsfunktionen modellieren. Insbesondere sind Extremwertverteilungsfunktionen (EVDs) mit standard invers exponentiellen Randverteilungen und die zugehörigen verallgemeinerten Pareto–Verteilungsfunktionen (GPDs) direkt mit Hilfe ihrer Pickands–Abhängigkeitsfunktion D darstellbar. Neben GPDs umfasst das untersuchte statistische Modell auch Verteilungsfunktionen in der Nachbarschaft von GPDs. Sie werden durch zwei Gruppen von Dichteentwicklungen charakterisiert, welche die Abhängigkeitsstruktur der zu Grunde liegenden Zufallsvektoren beschreiben und die Basis für das Testen auf Flankenabhängigkeit bilden.

Da Flankenunabhängigkeit in wichtigen Spezialfällen mit einer sehr geringen Rate angenommen wird, ist die residuale Abhängigkeitsstruktur bedeutsam. Um diese zu analysieren leiten wir Grenzverteilungen von Maxima unter Dreiecksschemata von Zufallsvektoren her. Solch ein Resultat wurde von Hüsler und Reiss bzw. Hashorva für normal– bzw. elliptisch verteilte Zufallsvektoren untersucht. Unser Ziel ist es, dieses Problem auf einem abstrakten Niveau zu behandeln. Dazu befassen wir uns mit technischen Bedingungen an die obigen Dichteentwicklungen und Verallgemeinerungen davon. Unsere Resultate erweitern wir zudem auf Modelle mit unterschiedlichen univariaten Randverteilungen.

Schließlich präsentieren wir mehrere Maße für bivariate und multivariate asymptotische Abhängigkeit. Analysen dieser Abhängigkeitsmaße innerhalb des statistischen Modells zeigen, dass sie in Beziehung stehen zu gewissen Dichteentwicklungen und insbesondere zur Pickands–Abhängigkeitsfunktion.

Abstract

In multivariate extreme value theory dependence structures can be modeled by using Pickands dependence functions. Extreme value distribution functions (EVDs) with standard reversely exponential margins and the pertaining generalized Pareto distribution functions (GPDs) can be directly represented in terms of their Pickands dependence function D. Besides GPDs our statistical model comprises multivariate distribution functions belonging to the neighborhood of GPDs. They are characterized by two groups of density expansions which describe the dependence structure of the underlying random vectors and are the basis for the establishment of a test on tail dependence.

Because in important cases tail independence is attained at a very slow rate, the residual dependence structure plays a significant role. To analyze the residual dependence structure we deduce limiting distributions of maxima under triangular schemes of random vectors. Such a result has been investigated by Hüsler and Reiss and Hashorva in the special cases of normally and elliptically distributed random vectors respectively. Our aim is to treat the problem on an abstract level. For this purpose we study technical conditions imposed on the above mentioned density expansions and generalizations of the same conditions. We also extend our results to models with different univariate margins.

Finally, we present various measures of asymptotic dependence in the bivariate and multivariate framework. Analyses of these dependence measures within our statistical model show that they are related to certain density expansions and, in particular, to the Pickands dependence function.

Contents

List of Special Symbols

c	radial component
D	Pickands dependence function
$\mathcal{D}(G)$	domain of attraction of the distribution function G
$E_d(\mu, \Sigma, g)$	set of elliptically distributed random vectors with parameters μ and Σ and density function g
f_D	Pickands density
G_D	extreme value distribution function (EVD) with Pickands dependence function D
G_i	univariate extreme value distribution function (i–th submodel), i–th marginal distribution function of G
H	distribution function; likewise we use the symbols F, G, W etc.
H^n	n–th power of the distribution function H with $H^n(x) := (H(x))^n$
$H_{\mathbf{z}}$	spectral distribution function belonging to H
$h_{\mathbf{z}}$	spectral density of H
I_d	d–dimensional unit matrix
$\mathbb{N}$	set of natural numbers
$\mathcal{N}(\mu, \Sigma)$	multivariate normal distribution with mean vector μ and covariance matrix Σ
$\mathcal{P}(H)$	spectral decomposition of the distribution function H
R	domain of a Pickands dependence function D
$\mathbb{R}$	set of real numbers
S	unit simplex
$T(\mathbf{X})$	Pickands transform of the random vector $\mathbf{X}$
W_D	generalized Pareto distribution function (GPD)
$\mathbf{z}$	angular component
$\alpha(H)$	left endpoint of the distribution function H
χ	tail dependence parameter
χ_z	angular tail dependence parameter

$\bar{\chi}$	residual dependence index
Φ	univariate standard normal (Gaussian) distribution function
φ_D	GPD–Pickands density with pertaining Pickands dependence function D
$\omega(H)$	right endpoint of the distribution function H
$X \stackrel{d}{=} Y$	equality in distribution of the random variables X and Y

1 Introduction

The modeling of asymptotic dependence structures has attracted attention more and more and gained in importance during the last years. Various models have been developed and applied in fields such as insurance, finance and hydrology. Recent developments in the financial market, for example, have shown once more that dependencies of extreme events in fact have to be taken seriously.

In models based on normal distributions, dependence structures are commonly described by the correlation coefficient or the covariance matrix. The latter is particularly used to estimate the Portfolio–VaR with the Variance–Covariance Method, cf. Reiss and Thomas [41], p. 387. Yet as soon as heavy tails occur, the assumption of normality has to be given up and the correlation is no longer an appropriate dependence measure. Schmidt [48], p. 6, also states that the dependence structure of extreme events should not be described by the covariance matrix.

To overcome these restrictions different dependence measures have been introduced in literature: the tail dependence parameter, cf., e.g., Falk et al. [10], p. 163, the coefficient of tail dependence, cf. Ledford and Tawn [33], the residual dependence index, cf. Hashorva [26], and Spearman's Rho, cf. Schmid and Schmidt [46], to name just a few.

Yet all these parameters give only one–dimensional information about the underlying asymptotic dependence structures. One possible approach for a more extensive modeling is given by the theory of copulas. Those multivariate distribution functions may be used to analyze the dependence structures of multivariate random vectors separately from the marginal distributions, cf., e.g., [41], p. 275. Copulas have enjoyed great popularity and have been applied for the modeling of dependencies in many contexts. In mathematical finance, for one field of application, they may be used to study dependence structures of asset returns, see [48], p. 6, with reference to several articles introducing copulas in mathematical finance.

Now in order to use copulas specifically for the modeling of dependencies in tail regions one defines so–called tail copulas or tail dependence copulas which provide a distributional description of tail dependence, cf. Juri and Wüthrich [31], Section 2.1, and [48], Section 2.2.4.

Nevertheless, in spite of successful applications of copulas, doubts about their usefulness have been expressed and discussed, cf. [41], p. 275, where the authors refer to Mikosch [37] and the attached discussions.

In this thesis we again choose a different way of modeling asymptotic dependence structures, which, however, is related to the above mentioned approaches. Our statistical model will enable us to look at these dependence structures from a distributional point of view but also includes the possibility to determine one–dimensional measures of asymptotic dependence.

The basis for the present work is given by the multivariate extreme value theory. Note that multivariate extreme value distribution functions (EVDs) with standard reversely exponential margins can be represented in terms of a Pickands dependence function D. The form of D determines the dependence structure: $D = 1$ stands for tail independence, $D \neq 1$ for tail dependence. Therefore we call random variables $X_1, \ldots, X_d$ tail independent if their joint distribution function belongs to the max–domain of attraction of an EVD with Pickands dependence function $D = 1$. Besides EVDs our model also comprises the pertaining multivariate generalized Pareto distribution functions (GPDs) and certain multivariate distribution functions which deviate from these GPDs and, thus, belong to their neighborhood. To characterize distribution functions in the neighborhood of a GPD we present two groups of extremal density expansions, cf. Frick [16], Chapter 3, Frick et al. [17], and Frick and Reiss [18]. The first one is made up of spectral expansions. The leading term is always a Pickands dependence function D followed by factorized terms containing regularly varying functions. If $D = 1$, the terms of lower order and in particular the exponents of variation determine the residual dependence structure. The second group of density expansions concerns Pickands densities, i.e. densities of the Pickands transform, with Pickands densities under GPDs as leading terms. The Pickands transform is the transformation of a random vector $(X_1, \ldots, X_d)$ onto its Pickands coordinates, i.e. the angular and the radial component, cf. Falk et al. [10], p. 150.

On the one hand, these density expansions are of interest on their own and shed light on the dependence structure of the underlying distribution function as indicated above. On the other hand, they are used to formulate technical conditions imposed on the upper tails of multivariate distributions. They thereby again serve to deduce information about asymptotic dependence structures in various ways. In the present text we especially focus on the deduction and analysis of asymptotic distributions of sample maxima under triangular schemes. Besides, we also show how a test on tail dependence can be established based on expansions of Pickands densities and we point out the relationship between spectral expansions and dependence measures. These topics briefly describe some of the main aims of this thesis.

In Chapter 2 we provide a basis for further understanding by introducing theoretical concepts from multivariate extreme value theory, namely EVD, max–domain of attraction, Pickands dependence function, GPD, Pickands coordinates, spectral decomposition, spectral distribution function and Pickands transform, tail independence and tail dependence.

The above mentioned multivariate density expansions are presented in Chapter 3. Spectral expansions are defined as expansions of densities of spectral distribution functions in the sense of Falk et al. [10]. Pickands densities as densities of the Pickands transform are first analyzed under a GP random vector. They are called GPD–Pickands densities in this case. Relationships between the multivariate and pairwise Pickands dependence functions are established which are of interest in their own right. Equivalences concerning the GPD–Pickands density and Pickands dependence functions are of particular interest for the testing problem of the subsequent chapter. In a second step we define expansions of Pickands densities with GPD–Pickands densities as leading terms. Afterwards we show how they can be deduced from the pertaining spectral expansions in the bivariate case. In the last part of Chapter 3 we provide examples of multivariate distributions for which one can calculate density expansions, e.g. the Crowder distribution, the bivariate standard normal distribution and Kotz type distributions which belong to the class of elliptically symmetric distributions.

Chapter 4 starts with a limit theorem for the radial component which is based on expansions of Pickands densities. The resulting limiting distribution functions enable a distinction to be made between tail dependence and marginal tail independence which leads to the formulation of the test on tail dependence generalizing a result in Falk et al. [10], Section 6.5, and Falk and Michel [11]. We present a uniformly most powerful test procedure and provide the pertaining power function and the p–value.

In Chapter 5 we consider limiting distributions and residual dependence structures of maxima under triangular schemes, i.e. in schemes of random vectors where the n–th line contains n random vectors. For triangular schemes of normally and elliptically distributed random vectors Hüsler and Reiss [29] and Hashorva [23], [22], [24], respectively, have already computed limiting distribution functions, which are called Hüsler–Reiss distribution functions in the Gaussian case.

Now the present work deals with this topic on an abstract level. Our aim is to deduce limiting distributions of maxima under triangular schemes while certain technical conditions are satisfied. For this purpose we extend our density expansions in such a way that they depend on the sample size n. In particular, we replace the exponent of variation β by a sequence $(\beta(n))_{n\in\mathbb{N}}$ which implies a varying dependence structure in the previous random vectors. We thereby receive sequences of density expansions on which we then impose convergence conditions. We compute limiting distribution functions based on these conditions imposed on both sequences of spectral expansions and sequences of Pickands density expansions. In the bivariate case it is shown that the resulting limiting distribution functions can be identified with each other. For further investigations we concentrate on spectral densities.

By establishing expansions for the distribution functions of maxima under triangular schemes we gain additional insight into the residual dependence structure which is essentially determined by the shape of the underlying spectral densities.

While presenting several examples in this context we also pay attention to the standard Gaussian case. It is stated that the spectral expansion pertaining to the bivariate normal distribution cannot be applied within the triangular scheme approach. Thus the asymptotic distribution of the sample maxima, i.e. the Hüsler–Reiss distribution, cannot be derived. However, this becomes possible in a more general framework.

We formulate a generalized condition to be imposed on sequences of spectral densities. Again, we compute and analyze limiting distribution functions of maxima under triangular schemes fulfilling this generalized technical condition. Besides the deduction of the Hüsler–Reiss distribution function an additional example is given by the bivariate Crowder distribution.

We finish Chapter 5 by showing how the test on tail dependence is affected by varying dependence structures in the underlying densities.

Chapter 6 deals with the extension of previous results to different univariate margins.

We consider precisely distribution functions with margins equal to or belonging to the max–domain of attraction of arbitrary univariate EVDs. Based on and as an extension of Chapter 2 we first present some important definitions and results, for example, we introduce modified Pickands coordinates and the modified Pickands transform. Using these modified concepts the spectral expansions and the expansions of Pickands densities coincide for different types of marginal distributions. This gives us the possibility to reformulate our results of Chapter 5, i.e. to deduce limiting distributions and residual dependence structures of maxima under triangular schemes whose univariate margins belong to the max–domain of attraction of any univariate EVD.

Chapter 7 is devoted to measures of asymptotic dependence. We first present some measures of bivariate dependence that can be found in literature, e.g. the above mentioned tail dependence parameter and dependence measures introduced by Coles et al. [6], Section 3.3. These parameters can also be computed within our statistical model comprising spectral expansions. They take a specific shape in this case and are related to the Pickands dependence function and the exponent of variation of the underlying density expansion.

Multivariate extensions are rare in relevant literature. Most considerations are restricted to the bivariate case. Yet there are still some multivariate approaches, cf. Falk et al. [10], Section 6.4, Schmidt [47], Definition 7.1, Schmid and Schmidt [46], and Weissman [52], Section 2. Taking into account these proposals we define different measures of multivariate asymptotic dependence which are extensions of the previously considered measures of bivariate asymptotic dependence. As before, we also analyze these measures within our statistical model. Additional modifications for the bivariate as well as for the multivariate case are enclosed at the end.

In Chapter 8 we conclude the thesis with some final remarks and an outlook concerning further research work.

2 Mathematical basics from multivariate extreme value theory

In this chapter some central terms as tail dependence, Pickands dependence function and Pickands coordinates are formally defined in the framework of the multivariate extreme value theory.

We start Section 2.1 with the characterization of certain max–stable multivariate distribution functions and introduce the notion of the Pickands dependence function. Multivariate extreme value distribution functions (EVDs) as well as multivariate generalized Pareto distribution functions (GPDs), which are also defined in this part, can be represented in terms of a Pickands dependence function D. These representations motivate the introduction of Pickands coordinates and of a spectral decomposition of multivariate distribution functions in Section 2.2. The transformation of a vector into its Pickands coordinates is also of importance in this context. Finally, in Section 2.3 we provide the formal definition of tail independence and tail dependence in this framework.

The outline of this chapter follows that of Falk et al. [10], Chapters 4 and 5, including Reiss and Thomas [41], Chapter 12.1, cf. also Frick [16], Chapter 2.

2.1 Extreme value and generalized Pareto distribution functions

In the univariate case, EVDs are defined as limiting distribution functions of maxima of random variables. Multivariate EVDs can be introduced in the same way. Let therefore $\mathbf{X}_i = (X_{i1}, \ldots, X_{id})$, $i \leq n$, be independent, identically distributed (iid) d–variate random vectors with common distribution function H, $d \geq 2$. The maximum of these random vectors is taken componentwise, namely

$$\max_{i \leq n} \mathbf{X}_i := \left(\max_{i \leq n} X_{i1}, \ldots, \max_{i \leq n} X_{id} \right),$$

and we have

$$P\left\{ \max_{i \leq n} \mathbf{X}_i \leq \mathbf{x} \right\} = H^n(\mathbf{x}). \tag{2.1}$$

The limiting distribution functions of such maxima of random vectors are again called EVDs.

Definition 2.1.1
If d–dimensional vectors $\mathbf{c}_n > 0$ *and* $\mathbf{d}_n$ *exist such that*

$$H^n(\mathbf{d}_n + \mathbf{c}_n\mathbf{x}) \to G(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{2.2}$$

as $n \to \infty$*, we call G a d–variate extreme value distribution function (EVD). In addition, we say H belongs to the max–domain of attraction of G, in short* $H \in \mathcal{D}(G)$*.*

As in the univariate case, multivariate EVDs can be characterized by the property of being max–stable.

A distribution function G is max–stable if we have

$$G^n(\mathbf{d}_n + \mathbf{c}_n\mathbf{x}) = G(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{2.3}$$

for every $n \in \mathbb{N}$ and for certain d–dimensional vectors $\mathbf{c}_n > 0$ and $\mathbf{d}_n$.

In what follows we primarily consider d–variate max–stable distribution functions with reversely exponential margins $G_{2,1}(x) = \exp(x)$, $x \leq 0$. The function $G_{2,1}$ is the standard Weibull distribution function with shape parameter $\alpha = -1$. Now, if a distribution function G is max–stable with margins G_j, $j = 1, \ldots, d$, it can be standardized by the following simple transformation:

$$G(G_1^{-1}(G_{2,1}(x_1)), \ldots, G_d^{-1}(G_{2,1}(x_d))), \quad \mathbf{x} < 0, \tag{2.4}$$

is a max–stable distribution function with reversely exponential margins, where G_j^{-1} denotes the quantile function of G_j.

The family of max–stable distribution functions with univariate margins $G_{2,1}$ can be characterized by the following theorem, cf. [10], Theorem 4.3.1.

Theorem 2.1.2
A d–variate function G is a max–stable distribution function with reversely exponential margins if, and only if, the equation

$$G(\mathbf{x}) = \exp\left(\int_S \min_{i \leq d}(u_i x_i)\, d\mu(\mathbf{u})\right), \quad \mathbf{x} < 0, \tag{2.5}$$

is valid, where μ *is a finite measure on the d–variate unit simplex*

$$S = \left\{\mathbf{u} : \sum_{i \leq d} u_i = 1,\, u_i \geq 0\right\} \tag{2.6}$$

with the property

$$\int_S u_i\, d\mu(\mathbf{u}) = 1, \quad i \leq d. \tag{2.7}$$

From this theorem we can deduce the representation of an EVD in terms of a Pickands dependence function. By further transformation of expression (2.5) we obtain

$$G(\mathbf{x}) = G_D(\mathbf{x}) := \exp\left(\left(\sum_{i \leq d} x_i\right) D\left(\frac{x_1}{\sum_{i \leq d} x_i}, \ldots, \frac{x_{d-1}}{\sum_{i \leq d} x_i}\right)\right), \tag{2.8}$$

for $\mathbf{x} = (x_1, \ldots, x_d) \in (-\infty, 0]^d$, $\mathbf{x} \neq \mathbf{0}$, where $D : R \to [0, \infty)$ is the Pickands dependence function

$$D(t_1, \ldots, t_{d-1}) := \int_S \max\left(u_1 t_1, \ldots, u_{d-1} t_{d-1}, u_d \left(1 - \sum_{i \le d-1} t_i\right)\right) d\mu(\mathbf{u}). \tag{2.9}$$

The domain R of D is given by

$$R := \left\{(t_1, \ldots, t_{d-1}) \in [0,1]^{d-1} : \sum_{i \le d-1} t_i \le 1\right\}. \tag{2.10}$$

We also call (2.8) the Pickands representation of G.

In [10], p. 162, it is shown that the measure μ on the simplex S in $\mathbb{R}^2$ can be replaced by a measure ν on $[0,1]$. Therewith one is able to reformulate Theorem 2.1.2 in the bivariate case, cf. [10], Lemma 6.1.1.

Lemma 2.1.3
A bivariate function G is a max–stable distribution function with univariate reversely exponential margins if, and only if, the representation

$$G(x,y) = \exp\left((x+y) D\left(\frac{x}{x+y}\right)\right), \quad x, y \le 0,\ (x,y) \neq (0,0),$$

is valid, where $D : [0,1] \to [0,1]$ is the Pickands dependence function

$$D(z) = \int_0^1 \max((1-u)z, u(1-z))\, d\nu(u)$$

and ν is an arbitrary measure on $[0,1]$ with the properties

$$\nu([0,1]) = 2 \quad \textit{and} \quad \int_0^1 u\, d\nu(u) = 1. \tag{2.11}$$

Now let $M(z) := \nu([0,z])$, $z \in [0,1]$, be the measure generating function corresponding to ν. Then one can verify the representation

$$D(z) = 1 - z + \int_0^z M(x)\, dx, \quad z \in [0,1], \tag{2.12}$$

which again implies that D is absolutely continuous with derivative

$$D'(z) := M(z) - 1, \quad z \in [0,1],$$

cf. [10], Lemma 6.2.1 with subsequent investigations.

In general the shape of the Pickands dependence function D determines the dependence structure of a d–dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)$ following the max–stable distribution function G_D represented by (2.8). In particular, the dependence functions $D(\mathbf{t}) = 1$ and $D(\mathbf{t}) = \max(t_1, \ldots, t_{d-1}, 1 - \sum_{i \le d-1} t_i)$, $\mathbf{t} \in R$, characterize the cases of independence and complete dependence of the random variables $X_1, \dots, X_d$. They can be regarded as the extremal points of the set of all dependence functions, cf. [10], p. 123.

Pickands dependence functions have some important properties listed below, cf. [10], pp. 122–123.

Lemma 2.1.4
Let $D : R \to [0, \infty)$ be some Pickands dependence function.

(i) The i–th unit vector $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ in $\mathbb{R}^{d-1}$ satisfies $D(\mathbf{e}_i) = 1$, $i \le d-1$. Moreover, we have $D(\mathbf{0}) = 1$.

(ii) D is continuous and convex.

(iii) For any $\mathbf{t} \in R$ we have

$$\frac{1}{d} \le \max\left(t_1, \dots, t_{d-1}, 1 - \sum_{i \le d-1} t_i\right) \le D(\mathbf{t}) \le 1.$$

(iv) If the function D satisfies the symmetry condition

$$D(t_1, \dots, t_{d-1}) = D(s_1, \dots, s_{d-1}) \tag{2.13}$$

for any subset $\{s_1, \dots, s_{d-1}\}$ of $\{t_1, \dots, t_d\}$ with $t_d := 1 - \sum_{i \le d-1} t_i$, it attains its minimum at $(1/d, \dots, 1/d) \in \mathbb{R}^{d-1}$, i.e.

$$D(1/d, \dots, 1/d) \le D(\mathbf{t}) \quad \text{for all } \mathbf{t} \in R.$$

Note that the symmetry condition (2.13) is satisfied if, and only if, the random variables $X_1, \dots, X_d$ are exchangeable, i.e. if $(X_{i_1}, \dots, X_{i_d})$ again follows the distribution function G_D for any permutation $(i_1, \dots, i_d)$ of $(1, \dots, d)$.

(v) The convex combination $D(\mathbf{t}) = (1-\lambda)D_1(\mathbf{t}) + \lambda D_2(\mathbf{t})$, $\lambda \in [0, 1]$, of two dependence functions D_1 and D_2 is also a dependence function.

For later purposes we mention a relationship between the d–variate and the pairwise Pickands dependence functions. Let $(X_1, \dots, X_d)$ be a d–variate random vector whose distribution function belongs to the max–domain of attraction of a d–variate EVD G_D with Pickands dependence function D. Then the bivariate marginal distribution function of the random vector (X_r, X_s) with $r, s \in \{1, \dots, d\}$, $r \neq s$, belongs to the max–domain of attraction of the bivariate EVD $G_{D_{rs}}$ with Pickands dependence function

$$D_{rs}(z) := D(z\mathbf{e_r} + (1-z)\mathbf{e_s}), \quad z \in [0, 1], \tag{2.14}$$

where $\mathbf{e_r}$ and $\mathbf{e_s}$ are the r–th and s–th unit vectors in $\mathbb{R}^{d-1}$ and $\mathbf{e_d} := \mathbf{0} \in \mathbb{R}^{d-1}$.

Now we introduce the so–called generalized Pareto (GP) function, pertaining to the max–stable distribution function G_D in $\mathbb{R}^d$, and write it in terms of the Pickands dependence function D:

$$W_D(\mathbf{x}) := 1 + \log\left(G_D(\mathbf{x})\right) \tag{2.15}$$
$$= 1 + \left(\sum_{i\le d} x_i\right) D\left(\frac{x_1}{\sum_{i\le d} x_i}, \dots, \frac{x_{d-1}}{\sum_{i\le d} x_i}\right), \quad \log(G_D(\mathbf{x})) \ge -1.$$

In the univariate and bivariate case, the GP function is a distribution function. This is not necessarily the case for $d \ge 3$. For a counter example see [10], p. 133.

In arbitrary dimensions $d \ge 2$, a distribution function is called a GPD — again denoted by W_D — if it has a representation as in (2.15) in a neighborhood of $\mathbf{0}$, cf. [10], pp. 132–136.

If we choose the function G_D as an EVD with reversely exponential margins in the bivariate case, then the margins of the pertaining bivariate GPD are uniform on $[-1, 0]$, thus following a specific univariate GPD.

2.2 Pickands coordinates, spectral decomposition, and Pickands transform

Looking at the functions G_D and W_D one recognizes that they both depend on the vector $\mathbf{x} = (x_1, \dots, x_d)$ and the Pickands dependence function D in a certain manner. In order to capture this structure we now introduce the so–called Pickands coordinates, cf. [10], p. 136.

Definition 2.2.1
In the unique representation

$$\mathbf{x} = \left(\sum_{i\le d} x_i\right)\left(\frac{x_1}{\sum_{i\le d} x_i}, \dots, \frac{x_{d-1}}{\sum_{i\le d} x_i}, 1 - \frac{\sum_{i\le d-1} x_i}{\sum_{i\le d} x_i}\right)$$
$$=: c\left(z_1, \dots, z_{d-1}, 1 - \sum_{i\le d-1} z_i\right)$$

of an arbitrary vector $\mathbf{x} = (x_1, \dots, x_d) \in (-\infty, 0]^d$ *with* $\mathbf{x} \ne \mathbf{0}$ *we call* $c < 0$ *and* $\mathbf{z} = (z_1, \dots, z_{d-1}) \in R$ *the Pickands coordinates of* $\mathbf{x}$*. In particular,* $\mathbf{z}$ *is said to be the angular component and* c *is called the radial component.*

This denomination indeed makes sense as $\mathbf{z}$ represents the angle and c the distance of the vector $\mathbf{x}$ from the origin. The radial component will gain a certain importance in what follows.

Now one can define spectral decompositions based on these Pickands coordinates. Let therefore H be an arbitrary distribution function with support in $(-\infty, 0]^d$. For

$\mathbf{z} \in R$ and $c \le 0$ put

$$H_{\mathbf{z}}(c) := H\left(c\left(z_1, \dots, z_{d-1}, 1 - \sum_{i \le d-1} z_i\right)\right). \tag{2.16}$$

This function $H_{\mathbf{z}}$ is a univariate distribution function on $(-\infty, 0]$ for any fixed $\mathbf{z}$. The distribution function H is uniquely determined by the family

$$\mathcal{P}(H) := \{H_{\mathbf{z}} : \mathbf{z} \in R\} \tag{2.17}$$

of the univariate spectral distribution functions $H_{\mathbf{z}}$. The family $\mathcal{P}(H)$ is called spectral decomposition of H, cf. [10], p. 137.

For a max–stable distribution function G_D with reversely exponential margins we have, e.g.,

$$G_{D,\mathbf{z}}(c) = \exp(cD(\mathbf{z})), \quad c \le 0, \, \mathbf{z} \in R, \tag{2.18}$$

and for a GPD $W_D = 1 + \log(G_D)$ we obtain

$$W_{D,\mathbf{z}}(c) = 1 + cD(\mathbf{z}), \quad c_0 \le c \le 0, \, \mathbf{z} \in R, \tag{2.19}$$

for a c_0 near 0.

It is also possible to represent any random vector $\mathbf{X} = (X_1, \dots, X_d)$ on $(-\infty, 0]^d$ in terms of Pickands coordinates. For this purpose we define the transformation

$$T : (-\infty, 0]^d \setminus \{\mathbf{0}\} \to R \times (-\infty, 0)$$

by

$$\begin{aligned} T(\mathbf{x}) &= (T_1(\mathbf{x}), T_2(\mathbf{x})) \\ &:= \left(\frac{x_1}{\sum_{i \le d} x_i}, \dots, \frac{x_{d-1}}{\sum_{i \le d} x_i}, \sum_{i \le d} x_i\right). \end{aligned} \tag{2.20}$$

This is the transformation of the vector $\mathbf{x} = (x_1, \dots, x_d)$ onto its Pickands coordinates $\mathbf{z} := T_1(\mathbf{x}) \in R$ and $c := T_2(\mathbf{x}) \in (-\infty, 0)$. It is one–to–one with the inverse function

$$T^{-1}(\mathbf{z}, c) = c\left(z_1, \dots, z_{d-1}, 1 - \sum_{i \le d-1} z_i\right). \tag{2.21}$$

The random vector

$$(\mathbf{Z}, C) := T(\mathbf{X})$$

is called the Pickands transform of the random vector $\mathbf{X}$ onto its Pickands coordinates, cf. [10], p. 150.

If the distribution function of the random vector $\mathbf{X} = (X_1, \dots, X_d)$ has continuous partial derivatives of the order d in a neighborhood of $\mathbf{0} \in \mathbb{R}^d$, there exists a density of the Pickands transform on $R \times (c_0, 0)$ for a $c_0 < 0$. The specific form of this density will play a central role in the following chapter.

2.3 Tail independence and tail dependence

As already mentioned, the dependence structure of a random vector $\mathbf{X} = (X_1, \dots, X_d)$ following an EVD G_D is determined by the form of the pertaining Pickands dependence function D. In particular, the random variables $X_1, \dots, X_d$ are independent if, and only if, we have $D = 1$, i.e., if $\mathbf{X}$ has the distribution function

$$G_D(\mathbf{x}) = \prod_{i \le d} \exp(x_i).$$

For random vectors whose distribution functions belong to the max–domain of attraction of the EVD G_D these statements are valid in an asymptotic way. This leads to the definition of tail independence and tail dependence.

Definition 2.3.1
Assume that the distribution function H of a d–variate random vector $\mathbf{X} = (X_1, \dots, X_d)$ belongs to the max–domain of attraction of an EVD G_D with Pickands dependence function D. In the case $D = 1$, the random variables $X_1, \dots, X_d$ are called tail independent. In the case $D \neq 1$, $X_1, \dots, X_d$ are said to be tail dependent.

To interpret the property of tail independence let $\mathbf{X}_i = (X_{i1}, \dots, X_{id})$, $i \le n$, again be independent, identically distributed d–variate random vectors with common distribution function H. Let H_j be the j–th marginal distribution function of H, $j \le d$. With (2.2) it follows that

$$P\left\{\max_{i \le n} X_{ij} \le d_{nj} + c_{nj}x_j\right\} = H_j^n(d_{nj} + c_{nj}x_j) \to G_{D,j}(x_j), \quad n \to \infty,$$

where $G_{D,j}$ is the j–th marginal distribution function of the EVD G_D. According to Definition 2.3.1, tail independence implies

$$P\left\{\max_{i \le n} \mathbf{X}_i \le \mathbf{d}_n + \mathbf{c}_n\mathbf{x}\right\} = H^n(\mathbf{d}_n + \mathbf{c}_n\mathbf{x}) \to \prod_{j \le d} G_{D,j}(x_j), \quad n \to \infty.$$

Thus we can interpret tail independence as a property of a multivariate distribution function meaning that the componentwise maxima are asymptotically independent.

3 Multivariate density expansions of finite length

We have seen that multivariate EVDs with standard reversely exponential margins and the pertaining multivariate GPDs can be parameterized in terms of their Pickands dependence function D with $D = 1$ representing tail independence. According to the definition of tail independence the Pickands dependence function also determines the asymptotic dependence structure of functions belonging to the max–domain of attraction of EVDs.

In the following text we will consider different multivariate density expansions related to Pickands dependence functions. They characterize a certain group of distribution functions which are in the max–domain of attraction of an EVD and deviate from the GPDs whereby EVDs serve as special cases. The outline of this chapter basically follows that of Frick [16], Chapter 3, cf. also Frick et al. [17], and Frick and Reiss [18].

Section 3.1 starts with the concept of the spectral δ–neighborhood of a GPD and strengthens it by introducing spectral expansions of finite length. These expansions contain regularly varying functions.

In Section 3.2 we introduce Pickands densities as densities of the Pickands transform, i.e. of the joint distribution of the angular and radial component. In the first part (Subsection 3.2.1) we study Pickands densities belonging to a GP random vector. They are denoted by φ_D. In this context we establish several relationships between the d–variate and pairwise Pickands dependence functions as well as between multivariate Pickands dependence functions D and the pertaining GPD–Pickands densities φ_D. These densities φ_D then provide the leading terms in the expansions of Pickands densities which are presented in Subsection 3.2.2. These density expansions also contain regularly varying functions. Subsection 3.2.3 shows how the expansion of a Pickands density can be deduced from the spectral expansion in the bivariate case.

Examples for spectral expansions and expansions of Pickands densities are given in Section 3.3. Several of them will be the basis for studies in the subsequent chapters. Because one of the presented density expansions belongs to an elliptical distribution, we give a short introduction to spherically and elliptically symmetric distributions.

3.1 Spectral expansions

Starting with the spectral decomposition of a distribution function H, i.e. the family $\mathcal{P}(H)$ of the univariate spectral distribution functions $H_{\mathbf{z}}$ in (2.17), we introduce so–called spectral expansions . First of all, consider the condition for a distribution function

$H(\mathbf{x})$, $\mathbf{x} \le \mathbf{0}$, to belong to the differentiable spectral neighborhood or the differentiable spectral δ–neighborhood of a GPD. It is provided in the following definition.

Definition 3.1.1
Let H be a distribution function on $(-\infty, 0]^d$ and assume that $H_{\mathbf{z}}(c)$ possesses a positive derivative

$$h_{\mathbf{z}}(c) := \frac{\partial}{\partial c} H_{\mathbf{z}}(c) \tag{3.1}$$

for $c < 0$ next to 0 and any $\mathbf{z} \in R$. If $h_{\mathbf{z}}$ satisfies the expansion

$$h_{\mathbf{z}}(c) = D(\mathbf{z})(1 + o(1)), \quad c \uparrow 0, \, \mathbf{z} \in R, \tag{3.2}$$

or

$$h_{\mathbf{z}}(c) = D(\mathbf{z}) \left(1 + O\left(|c|^{\delta}\right)\right), \quad c \uparrow 0, \, \mathbf{z} \in R, \tag{3.3}$$

respectively, for some $\delta \in (0, 1]$, we say that H belongs to the differentiable spectral neighborhood or the differentiable spectral δ–neighborhood, respectively, of the GPD W_D with Pickands dependence function D. Briefly, we call $h_{\mathbf{z}}(c)$ the spectral density of H.

An EVD G_D with reversely exponential margins, for example, belongs to the spectral δ–neighborhood of $W_D = 1 + \log(G_D)$ with $\delta = 1$, cf. [10], Section 5.3.

This conception of a spectral neighborhood of a GPD is due to the fact that GPDs satisfy expansions (3.2) and (3.3) precisely, i.e. without remainder term.

Lemma 3.1.2
For a GPD $W_D = 1 + \log(G_D)$ with Pickands dependence function D we get

$$w_{\mathbf{z}}(c) := \frac{\partial}{\partial c} W_{D,\mathbf{z}}(c) = D(\mathbf{z})$$

for $c < 0$ next to 0 and any $\mathbf{z} \in R$.

PROOF. The assertion follows directly from the spectral decomposition (2.19) of a GPD W_D. □

Therefore the densities in the expansions (3.2) and (3.3) coincide with $w_{\mathbf{z}}$ asymptotically, as $c \uparrow 0$. To strengthen these expansions we introduce modified representations with additional terms. Thus we obtain spectral expansions of length $k+1$ where $k \in \mathbb{N}$.

Definition 3.1.3
Let H be a distribution function on $(-\infty, 0]^d$ and assume that $H_{\mathbf{z}}(c)$ possesses a positive continuous derivative

$$h_{\mathbf{z}}(c) := \frac{\partial}{\partial c} H_{\mathbf{z}}(c)$$

for $c < 0$ next to 0 and any $\mathbf{z} \in R$. Assume that

$$h_{\mathbf{z}}(c) = D(\mathbf{z}) + \sum_{j=1}^{k} B_j(c) A_j(\mathbf{z}) + o(B_k(c)), \quad c \uparrow 0, \, k \in \mathbb{N}, \tag{3.4}$$

uniformly for $\mathbf{z} \in R$*, where* D *is a Pickands dependence function and the* $A_j : R \to \mathbb{R}$*,* $j = 1, \dots, k$*, are integrable functions. In addition, assume that the functions* $B_j : (-\infty, 0) \to (0, \infty)$*,* $j = 1, \dots, k$*, satisfy*

$$\lim_{c \uparrow 0} B_j(c) = 0 \tag{3.5}$$

and

$$\lim_{c \uparrow 0} \frac{B_j(ct)}{B_j(c)} = t^{\beta_j}, \quad t > 0,\ \beta_j > 0. \tag{3.6}$$

Without loss of generality, let $\beta_1 < \beta_2 < \cdots < \beta_k$*. Then we say that the distribution function* H *satisfies a spectral expansion of length* $k + 1$*.*

Property (3.6) means that the functions B_j are regularly varying in 0 with the parameters β_j, $j = 1, \dots, k$, being the exponents of variation. According to Resnick [42], p. 13, it is always possible to write a β–varying function as $|c|^\beta L(c)$, where L is slowly varying in 0, i.e., L has an exponent of variation equal to zero. As such functions play a decisive role in what follows, we list some important properties, cf. [42], pp. 12–25.

Remark 3.1.4
Let B*,* B_r *and* B_s *be regularly varying functions in* 0 *with positive exponents of variation* β*,* β_r *and* β_s*,* $\beta_r > \beta_s$*.*

(i) For B *we have*

$$\int_c^0 B(u)\, du \sim -\frac{1}{1+\beta} B(c)c, \quad c \uparrow 0,$$

cf. [42], p. 17.

(ii) Additionally, assume that the function B *is absolutely continuous and possesses a monotone derivative* b*. Then we have*

$$\lim_{c \uparrow 0} \frac{cb(c)}{B(c)} = \beta,$$

cf. [42], p. 21.

(iii) For B_r *and* B_s *we have*

$$\lim_{c \uparrow 0} \left| \frac{B_r(c)}{B_s(c)} \right| = 0,$$

i.e. $B_r(c) = o(B_s(c))$*,* $c \uparrow 0$*, cf. Frick and Reiss [18], Section 1.*

From part (iii) of the preceding remark we deduce that it is possible to write $h_\mathbf{z}$ as

$$h_\mathbf{z}(c) = D(\mathbf{z}) + \sum_{j=1}^{\kappa} B_j(c) A_j(\mathbf{z}) + o(B_\kappa(c)), \quad c \uparrow 0,$$

for any natural number κ between 1 and k.

Remark 3.1.5
The function $D(\mathbf{z})$ *in expansion* (3.4) *can be replaced by some function* $g(\mathbf{z})$ *which satisfies* $g(\mathbf{e}_i) = 1 = g(\mathbf{0})$, $i \le d-1$. *Because of* $h_{\mathbf{z}}(c) = g(\mathbf{z})(1+o(1))$ *we know that* H *is in the max–domain of attraction of some EVD with reversely exponential margins and* $g(\mathbf{z}) = D(\mathbf{z})$ *is the pertaining Pickands dependence function. We have*

$$H_{\mathbf{z}}^n\left(\frac{c}{n}\right) = H^n\left(\frac{c}{n}\left(z_1,\dots,z_{d-1},1-\sum_{i\le d-1} z_i\right)\right) \to \exp(cD(\mathbf{z})), \quad c \le 0, \tag{3.7}$$

as $n \to \infty$, *cf. [10], Theorem 5.3.2.*

Therefore, if the distribution function of some random vector $\mathbf{X} = (X_1,\dots,X_d)$ *satisfies a spectral expansion of length* $k+1$ *with some Pickands dependence function as the leading term, the random variables are tail independent if, and only if,* $D = 1$.

3.2 Expansions of Pickands densities

In the following text let T be the transform of a vector $\mathbf{x} = (x_1,\dots,x_d)$ onto its Pickands coordinates $\mathbf{z} = (z_1,\dots,z_{d-1})$ and c defined in (2.20) having the inverse

$$T^{-1}(\mathbf{z},c) = c\left(z_1,\dots,z_{d-1},1-\sum_{i\le d-1} z_i\right),$$

cf. (2.21). Consider an arbitrary random vector $\mathbf{X} = (X_1,\dots,X_d)$ which takes values in $(-\infty,0]^d$. Suppose that its distribution function H possesses continuous partial derivatives of the order d in a neighborhood of $\mathbf{0}$. Then

$$h(x_1,\dots,x_d) := \frac{\partial^d}{\partial x_1 \cdots \partial x_d} H(x_1,\dots,x_d) \tag{3.8}$$

is a density of H in a neighborhood of $\mathbf{0}$, cf. [41], p. 268. As a consequence we can specify the density of the Pickands transform.

Lemma 3.2.1
If the distribution function H *of* $\mathbf{X} = (X_1,\dots,X_d)$ *possesses continuous partial derivatives of the order* d *in a neighborhood of* $\mathbf{0}$, *there exists a* $c_0 < 0$ *such that the Pickands transform* $(\mathbf{Z},C) = T(\mathbf{X})$ *has the density*

$$f(\mathbf{z},c) = |c|^{d-1} h\left(T^{-1}(\mathbf{z},c)\right)$$

on $R \times (c_0, 0)$.

PROOF. See Falk and Reiss [13], Lemma 5.1. □

We call the density f in Lemma 3.2.1 the Pickands density.

3.2.1 The Pickands density of a GP random vector

The Pickands density of a GP random vector is of a particular form as noted in the subsequent lemma.

Lemma 3.2.2
Let $W_D = 1 + \log(G_D)$ be a GPD with pertaining Pickands dependence function D having continuous partial derivatives of the order d. Further, let $\mathbf{U} = (U_1, \ldots, U_d)$ be a random vector with distribution function W_D in a neighborhood of $\mathbf{0}$. Then the Pickands transform $(\mathbf{Z}, C) = T(\mathbf{U})$ has a density $f_D(\mathbf{z}, c)$ on $R \times (c_0, 0)$ for some $c_0 < 0$ near 0 independent of c, namely

$$\begin{aligned} f_D(\mathbf{z}, c) &= |c|^{d-1} \left(\frac{\partial^d}{\partial x_1 \ldots \partial x_d} W_D \right) \left(T^{-1}(\mathbf{z}, c) \right) \\ &= \varphi_D(\mathbf{z}), \quad \mathbf{z} \in R,\ c \in (c_0, 0). \end{aligned} \tag{3.9}$$

PROOF. See [13], Lemma 5.2. □

Remark 3.2.3
The constraint that the number c_0 has to be close to 0 is due to the fact that a GPD W_D coincides with a distribution function only in its upper tail and can be represented as $W_D = 1 + \log(G_D)$ with an EVD G_D there, cf. Lemma 5.1.3 and 5.4.1 in [10].

To distinguish φ_D from other Pickands densities one may call it the GPD–Pickands density, cf. Michel [36], Section 2.2.

Definition 3.2.4
For a differentiable GPD W_D with continuous partial derivatives of the order d we call the function

$$\varphi_D(\mathbf{z}) := |c|^{d-1} \left(\frac{\partial^d}{\partial x_1 \ldots \partial x_d} W_D \right) \left(T^{-1}(\mathbf{z}, c) \right)$$

from Lemma 3.2.2 the GPD–Pickands density.

In view of the analysis of the asymptotic dependence structure it will be of importance whether $\varphi_D = 0$ or, equivalently, $\int_R \varphi_D(\mathbf{z})\, d\mathbf{z} = 0$. In the bivariate case, if D is twice continuously differentiable, we have

$$\varphi_D(z) = D''(z) z (1 - z)$$

as well as

$$\int_R \varphi_D(z)\, dz = 2 \left(1 - \int_0^1 D(z)\, dz \right), \tag{3.10}$$

cf. Falk and Reiss [14], Section 2, which implies the equivalence

$$D = 1 \quad \Leftrightarrow \quad \int_0^1 \varphi_D(z)\, dz = 0. \tag{3.11}$$

In the multivariate case such an equivalence does not hold true in general. However, it can easily be deduced by differentiation that the d–variate GPD–Pickands density φ_D satisfies $\varphi_D(\mathbf{z}) = 0$, $\mathbf{z} \in R$, if $D = 1$, cf. [16], Lemma 3.2.8. The converse implication does not hold true in general. Yet by reducing the problem to the bivariate case one gets further insight in the relationship between a multivariate Pickands dependence function D and the pertaining GPD–Pickands density φ_D.

In [16], Section 3.5, an important relationship between D and the pairwise Pickands dependence functions D_{rs}, $r, s \in \{1, \dots, d\}$, $r \neq s$, in (2.14) is established, which is presented in the following lemma.

Lemma 3.2.5
For a Pickands dependence function $D : R \to [0, \infty)$ we have the equivalence

$$\begin{aligned} & D(\mathbf{z}) = 1,\ \mathbf{z} \in R, \\ \Leftrightarrow\quad & D_{rs}(z) = D(z\mathbf{e_r} + (1-z)\mathbf{e_s}) = 1,\ z \in [0,1],\ \text{for all } r,s \in \{1,\dots,d\},\ r \neq s. \end{aligned}$$

PROOF. See [18], Lemma 1.3, and [16], Corollary 3.5.2. □

With the help of this result it can further be shown that $\int_R \varphi_D(\mathbf{z})\, d\mathbf{z}$=0 is equivalent to tail independence in at least one bivariate marginal distribution.

Lemma 3.2.6
The Pickands density of a d–variate GPD W_D with pertaining Pickands dependence function D satisfies

$$\int_R \varphi_D(\mathbf{z})\, d\mathbf{z} = 0,$$

if, and only if, $D_{rs} = 1$ for at least one pair $r, s \in \{1, \dots, d\}$, $r \neq s$.

PROOF. See [18], Lemma 1.2. □

Note that in every case D must have continuous partial derivatives of the order d to secure the existence of a density w_D of W_D.

For a symmetric dependence function equivalence (3.11) still holds in the multivariate case.

Corollary 3.2.7
Let W_D be a d–dimensional GPD with pertaining Pickands dependence function D having continuous partial derivatives of the order d. If the function D is symmetric, it satisfies the equivalence

$$D(\mathbf{z}) = 1,\ \mathbf{z} \in R, \quad \Leftrightarrow \quad \int_R \varphi_D(\mathbf{z})\, d\mathbf{z} = 0.$$

To get an overview of the different relationships we list them up in a short form. We have

(a) $D = 1 \Leftrightarrow D_{rs} = 1$ for every pair r, s, cf. Lemma 3.2.5;

(b) $\int_R \varphi_D(\mathbf{z})\, d\mathbf{z} = 0 \Leftrightarrow D_{rs} = 1$ for at least one pair r, s , cf. Lemma 3.2.6, and, consequently,

(c) $D = 1 \Rightarrow \int_R \varphi_D(\mathbf{z})\, d\mathbf{z} = 0$,

(d) $D = 1 \Leftrightarrow \int_R \varphi_D(\mathbf{z})\, d\mathbf{z} = 0$ in the bivariate case and for a symmetric dependence function.

According to the equivalence in (b) the property of tail independence in at least one bivariate margin can be completely characterized by the integral $\int_R \varphi(\mathbf{z})\, d\mathbf{z}$. That the converse implication in (c) does not hold true in general can, for example, be seen by considering Example 1 in [18].

3.2.2 Expansions of finite length

Now we consider densities of Pickands transforms coinciding with $\varphi_D(\mathbf{z})$ asymptotically, as $c \uparrow 0$. A first general representation of such densities can be found in the framework of Pickands δ–neighborhoods of GPDs.

Definition 3.2.8
Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a random vector such that the pertaining Pickands transform has a density $f_D(\mathbf{z}, c)$ on $R \times (c_0, 0)$ for a $c_0 < 0$. If this density satisfies the expansion

$$f_D(\mathbf{z}, c) = \varphi_D(\mathbf{z}) + O\left(|c|^\delta\right), \quad c \uparrow 0, \tag{3.12}$$

uniformly for $z \in R$, for some $\delta > 0$, we say that the distribution function H of $\mathbf{X}$ belongs to the Pickands δ–neighborhood of the GPD W_D with dependence function D.

A max–stable distribution function G_D, for example, belongs to the Pickands δ–neighborhood of $W_D = 1 + \log(G_D)$ with $\delta = 1$, cf. [13], Section 5.

We again extend the representation of the density $f_D(\mathbf{z}, c)$ by adding further terms which contain regularly varying functions. Therewith we refine the first order condition (3.12) to a higher order condition by using an expansion of $f_D(\mathbf{z}, c)$ with a GPD–Pickands density $\varphi_D(\mathbf{z})$ as a leading term.

Definition 3.2.9
Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a random vector whose Pickands transform has a density $f_D(\mathbf{z}, c)$ on $R \times (c_0, 0)$ for $c_0 < 0$ close to 0. Assume that

$$f_D(\mathbf{z}, c) = \varphi_D(\mathbf{z}) + \sum_{j=1}^{k} B_j(c)\tilde{A}_j(\mathbf{z}) + o(B_k(c)), \quad c \uparrow 0,\ k \in \mathbb{N}, \tag{3.13}$$

uniformly for $\mathbf{z} \in R$, where φ_D is a GPD–Pickands density and the $\tilde{A}_j : R \to \mathbb{R}$, $j = 1, \ldots, k$, are integrable functions. In addition, assume that the functions $B_j : (-\infty, 0) \to (0, \infty)$, $j = 1, \ldots, k$, satisfy

$$\lim_{c \uparrow 0} B_j(c) = 0 \tag{3.14}$$

and

$$\lim_{c\uparrow 0} \frac{B_j(ct)}{B_j(c)} = t^{\beta_j}, \quad t>0,\ \beta_j>0. \tag{3.15}$$

Without loss of generality, let $\beta_1 < \beta_2 < \cdots < \beta_k$. Then we say that the density $f_D(\mathbf{z}, c)$ satisfies an expansion of length $k+1$.

Notice that this expansion can also be shortened by applying Remark 3.1.4 (iii). In particular, we have

$$f_D(\mathbf{z}, c) = \varphi_D(\mathbf{z}) + \sum_{j=1}^{\kappa} B_j(c)\tilde{A}_j(\mathbf{z}) + o(B_\kappa(c)), \quad c\uparrow 0, \tag{3.16}$$

for any natural number κ between 1 and k. We can also write

$$f_D(\mathbf{z}, c) = \varphi_D(\mathbf{z}) + o\left(|c|^\delta\right), \quad \text{as } c\uparrow 0, \tag{3.17}$$

for $0<\delta<\rho_1$ if the functions $\tilde{A}_j$ in (3.13) are bounded. This entails that the distribution function of $\mathbf{X}$ is in the Pickands δ–neighborhood of the GPD W_D with Pickands density φ_D.

There is another condition imposed on the functions $\tilde{A}_j$ which will be of importance later, namely the existence of a positive integer

$$\kappa := \min\left\{ j\in\{1,\dots,k\} : \int_R \tilde{A}_j(\mathbf{z})\, d\mathbf{z} \neq 0 \right\}. \tag{3.18}$$

Integrating with respect to $\mathbf{z}$ one gets an expansion of the marginal density in c.

Remark 3.2.10
If the Pickands density $f_D(\mathbf{z}, c)$ of a random vector $\mathbf{X}$ satisfies an expansion of length $k+1$ as given in (3.13)*, then the radial component $C = T_2(\mathbf{X})$ has the density $f_D(c)$ satisfying*

$$\begin{aligned} f_D(c) &= \int_R \varphi_D(\mathbf{z})\, d\mathbf{z} + \sum_{j=1}^{k} B_j(c) \int_R \tilde{A}_j(\mathbf{z})\, d\mathbf{z} + o(B_k(c)) \\ &= \int_R \varphi_D(\mathbf{z})\, d\mathbf{z} + B_\kappa(c) \int_R \tilde{A}_\kappa(\mathbf{z})\, d\mathbf{z} + o(B_\kappa(c)), \quad c\uparrow 0, \end{aligned} \tag{3.19}$$

on $(c_0, 0)$ with κ as in (3.18)*. The first expansion is immediate by integrating $f_D(\mathbf{z}, c)$ with respect to the variable $\mathbf{z}$.*

If, in addition, $\int_R \varphi_D(\mathbf{z})\, d\mathbf{z} = 0$, then we have, apparently,

$$f_D(c) = B_\kappa(c) \int_R \tilde{A}_\kappa(\mathbf{z})\, d\mathbf{z} + o(B_\kappa(c)), \quad c\uparrow 0,$$

on $(c_0, 0)$.

The expansion of $f_D(\mathbf{z}, c)$ can be reduced likewise as shown in the subsequent remark.

Remark 3.2.11
If $\int_R \varphi_D(\mathbf{z})\, d\mathbf{z} = 0$, then the expansion of $f_D(\mathbf{z}, c)$ in (3.13) *can be written in the reduced form*

$$f_D(\mathbf{z}, c) = B_\kappa(c)\tilde{A}_\kappa(\mathbf{z}) + o(B_\kappa(c)), \quad c \uparrow 0,$$

almost everywhere, cf. representation (33) in [18] and Theorem 3.2.10 in [16].

This implies in particular that an integer κ as in (3.18) *exists if $f_D(\mathbf{z}, c)$ satisfies an expansion with $\varphi_D(\mathbf{z}) = 0$.*

By analogy with (3.17) one obtains

$$f_D(\mathbf{z}, c) = o\left(|c|^\delta\right), \quad c \uparrow 0,$$

for $0 < \delta < \beta_\kappa$. Thus every distribution with the same pair (κ, β_κ) in its expansion of the pertaining Pickands density is in the same Pickands δ–neighborhood of the GPD W_D with $\varphi_D = 0$. If $\beta_\kappa \to \infty$, then $f(\mathbf{z}, c) \to 0$ and, hence, the Pickands density approaches the GPD–Pickands density.

Remark 3.2.12
According to [16], Lemma 3.3.3, the distribution function of the random vector $\mathbf{X}$ having the above Pickands density with a remainder term fulfilling some further assumptions belongs to the max–domain of attraction of an EVD G_D with reversely exponential margins and Pickands dependence function D.

3.2.3 The relationship to a bivariate spectral expansion

In the bivariate case it is possible to establish a relationship of the considered expansions of Pickands densities to the differentiable spectral expansions introduced in Frick et al. [17], Section 2.

By analogy with the multivariate case any distribution function H on $(-\infty, 0]^2$ may be written in the form $H(c(z, 1-z))$. Putting

$$H_z(c) = H(c(z, 1-z)), \quad z \in [0, 1],\ c \le 0,$$

one gets the spectral decomposition of H by means of the univariate distribution functions H_z, cf. (2.16) and [10], p. 137. We assume that H satisfies a spectral expansion of length $k+1$ according to Definition 3.1.3 (for $d = 2$)

$$h_z(c) = D(z) + \sum_{j=1}^{k} B_j(c)A_j(z) + o(B_k(c)), \quad c \uparrow 0,\ k \in \mathbb{N}, \tag{3.20}$$

uniformly for $z \in [0, 1]$.

By analogy with the multivariate case one can shorten this expansion to an expansion of length $\kappa + 1$ for any natural number κ between 1 and k. The function $D(z)$ in expansion (3.20) can again be replaced by some function $g(z)$ with $g(1) = 1 = g(0)$. Because $h_z(c) = g(z)(1 + o(1))$, $c \uparrow 0$, we know that H belongs to the max–domain of attraction of some EVD and $g(z) = D(z)$ is the pertaining Pickands dependence function, cf. [10], Theorem 5.3.2.

The following lemma shows that the existence of a spectral expansion of length $k+1$ for a distribution function H implies that the pertaining Pickands density also satisfies an expansion of finite length.

Lemma 3.2.13
Let H be the distribution function of a bivariate random vector $\mathbf{X} = (X_1, X_2)$ with values in $(-\infty, 0]^2$ satisfying the spectral expansion

$$h_z(c) = D(z) + \sum_{j=1}^{k} B_j(c)A_j(z) + o(B_k(c)), \quad c \uparrow 0, \tag{3.21}$$

uniformly for $z \in [0,1]$, where the Pickands dependence function D and the A_j, $j = 1, \dots, k$, are twice continuously differentiable.

(i) Putting

$$\tilde{A}_j(z) = -\beta_j A_j(z) - \frac{\beta_j}{1+\beta_j} A_j'(z)(1-2z) + \frac{1}{1+\beta_j} A_j''(z)z(1-z), \tag{3.22}$$

where β_j is the exponent of variation of the function B_j, one gets

$$\int_0^1 \tilde{A}_j(z)\, dz = -(2+\beta_j) \int_0^1 A_j(z)\, dz + A_j(0) + A_j(1)$$

for $j = 1, \dots, k$.

(ii) If the remainder term

$$R(z,c) := h_z(c) - D(z) - \sum_{j=1}^{k} B_j(c)A_j(z)$$

is positive and differentiable in c, then the density of the Pickands transform $(Z, C) = T(\mathbf{X})$ satisfies the expansion

$$f_D(z,c) = D''(z)z(1-z) + \sum_{j=1}^{k} B_j(c)\tilde{A}_j(z) + o(B_k(c)), \quad c \uparrow 0, \tag{3.23}$$

uniformly for $z \in [0,1]$ with $\tilde{A}_j$ as in (3.22). *The regularly varying functions B_j are the same as in expansion* (3.21).

(iii) The parameter κ in (3.18) *exists for the expansion* (3.23), *if and only if,*

$$(2+\beta_j) \int_0^1 A_j(z)\, dz - A_j(1) - A_j(0) \neq 0 \tag{3.24}$$

for some $j \in \{1, \dots, k\}$, that is, if condition (10) in [17] is fulfilled for some j.

PROOF. See Chapter 3.4 in [16] and Lemma 2.1 in [18]. □

Concerning a generalization to higher dimensions we refer to [16], Section 3.3.

3.3 Examples

In [17], Section 3 and Section 8, several bivariate distribution functions are presented which satisfy spectral expansions of finite length as pointed out in Section 3.2.3. For later purposes we present some of them again, namely certain mixtures of bivariate distribution functions as well as the lower tail of the bivariate Crowder distribution and the bivariate standard normal distribution. In the first cases we consider univariate beta margins whereas the margins of the latter two distributions are transformed in such a way that they follow the uniform distribution on the interval $[-1, 0]$, which belongs to the max–domain of attraction of the reversely exponential distribution.

Example 3.3.1
Let W_D be a bivariate GPD and let $W_{2,\alpha}(x) = 1 - (-x)^{-\alpha}$, $-1 \le x \le 0$, be a univariate beta distribution function with shape parameter $\alpha < -1$. Define a mixture distribution function H with weights p and $1-p$, where $p \in (0,1)$ also serves as a scale parameter, i.e.

$$H(x,y) := pW_D\left(\frac{x}{p}, \frac{y}{p}\right) + (1-p)W_{2,\alpha}\left(\frac{x}{p}\right)W_{2,\alpha}\left(\frac{y}{p}\right). \tag{3.25}$$

The distribution function H fulfills a spectral expansion of length 3

$$h_z(c) = D(z) + B_1(c)A_1(z) + B_2(c)A_2(z) \tag{3.26}$$

with

$$B_1(c) = |c|^{-\alpha-1}, \quad B_2(c) = |c|^{-2\alpha-1}$$

and

$$A_1(z) = -\alpha(1-p)p^{\alpha}(z^{-\alpha} + (1-z)^{-\alpha}), \quad A_2(z) = 2\alpha(1-p)p^{2\alpha}(z(1-z))^{-\alpha}.$$

From [17], Section 8, we know that (3.24) *is satisfied by the function A_2.*

Example 3.3.2
Let F be Joe's distribution function, cf. Heffernan [28] for its copula form, transformed to beta $W_{2,\alpha}$ margins, i.e.

$$F(x,y) = 1 - \{(-x)^{-\alpha\gamma} + (-y)^{-\alpha\gamma} - (-x)^{-\alpha\gamma}(-y)^{-\alpha\gamma}\}^{1/\gamma}$$

with parameters $\alpha > -1$ and $\gamma \ge 1$. By analogy with Example 3.3.1 let

$$H(x,y) = pW_D\left(\frac{x}{p}, \frac{y}{p}\right) + (1-p)F\left(\frac{x}{p}, \frac{y}{p}\right). \tag{3.27}$$

The partial derivatives of its spectral decomposition exist and are continuous. One obtains an expansion of length 2 with $B(c) = |c|^{-\alpha-1}$ and $A(z) = -\alpha(1-p)p^{\alpha}(z^{-\alpha\gamma} + (1-z)^{-\alpha\gamma})^{1/\gamma}$. Excluding the case $\gamma = 1$ which represents exact independence of the margins of Joe's distribution function it can be shown that the function A fulfills condition (3.24)*, cf. [17], Example 3.*

Example 3.3.3
Let H be the joint distribution function in the lower tail of the Crowder distribution, cf. [28] for its copula form, with $[-1,0]$*–uniform margins, i.e.*

$$\begin{aligned} H(x,y) &:= 1 + x + y \\ &\quad + \exp\left[-\left\{(\alpha - \log(-x))^\theta + (\alpha - \log(-y))^\theta - \alpha^\theta\right\}^{1/\theta} + \alpha\right] \end{aligned}$$

with $\alpha \geq 0$ *and* $\theta \geq 1$*. This distribution function satisfies the spectral expansion of length* 2

$$h_z(c) = 1 + B(c)A(z) + o(B(c)), \quad c \uparrow 0,$$

with

$$B(c) = |c|^{2^{1/\theta}-1} L(c)$$

where

$$L(c) = \exp\left(\alpha\left(1 - 2^{\frac{1}{\theta}}\right) + \frac{\alpha^\theta 2^{1/\theta - 1}}{\theta(\log|c|)^{\theta-1}}\right),$$

and

$$A(z) = 2^{1/\theta}(z(1-z))^{2^{1/\theta-1}}.$$

Because L is slowly varying, we know that B is regularly varying with the exponent of variation $\beta = 2^{1/\theta} - 1 \in (0,1]$ *for* $\theta \geq 1$*. In [17], Example 4, it is also shown that condition* (3.24) *is satisfied.*

Example 3.3.4
Consider the bivariate standard normal distribution with correlation $\rho \in (0,1)$*. Let H be the pertaining distribution function after transformations to* $[-1,0]$*–uniform margins, i.e.*

$$H(u,v) := \int_{-\infty}^{\Phi^{-1}(1+u)} \int_{-\infty}^{\Phi^{-1}(1+v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) dx\, dy$$

for u*,* $v \in [-1,0]$*. According to [17], Example 5, we can derive a spectral expansion of length* 2 *for this distribution function. It is given by*

$$h_z(c) = 1 + B(c)A(z) + o(B(c)), \quad c \uparrow 0,$$

with

$$B(c) = |c|^{\frac{2}{1+\rho}-1} L(c)$$

where

$$L(c) = (1+\rho)^{\frac{3}{2}}(1-\rho)^{-\frac{1}{2}}(4\pi)^{-\frac{\rho}{1+\rho}}(-\log|c|)^{-\frac{\rho}{1+\rho}},$$

and

$$A(z) = -\frac{2}{1+\rho}(z(1-z))^{\frac{1}{1+\rho}}.$$

The function L is slowly varying again. Therefore the function B is regularly varying with the exponent of variation $\beta = \frac{2}{1+\rho} - 1 \in (0,1)$ *for* $\rho \in (0,1)$*. Similarly as in Example 3.3.3 one can show that condition* (3.24) *is satisfied.*

According to Lemma 3.2.13 it is possible to deduce expansions of length 2 for the Pickands density in each of the examples in [17]. Several of them have been computed and then extended to the multivariate case in [16], Section 3.7, and [18], Section 2, where some further examples can also be found. In the sequel, we take the example of the multivariate standard normal distribution function up again and present an expansion for its Pickands density.

Example 3.3.5
The d–variate standard normal distribution $\mathcal{N}(\mathbf{0}, \Sigma)$ has the density

$$\varphi_\Sigma(\mathbf{y}) = \frac{1}{(2\pi)^{d/2}(\det \Sigma)^{1/2}} \exp\left(-\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y}\right).$$

Let the correlation matrix

$$\Sigma := \left(\rho_{ij}\right)_{i,j=1,\dots,d}$$

be positive definite and let $\rho_{ij} \in (0,1)$ for all $i, j \in \{1, \dots, d\}$, $i \neq j$.

After a transformation to reversely exponential margins we get the following density by applying the transformation theorem for densities:

$$h_\Sigma(\mathbf{x}) = \exp\left(\sum_{i\le d} x_i\right) \frac{1}{(\det \Sigma)^{1/2}} \exp\left(\frac{1}{2}F(\mathbf{x})^T\left(I_d - \Sigma^{-1}\right)F(\mathbf{x})\right)$$

where I_d is the d–dimensional unit matrix and

$$F(\mathbf{x}) := \left(\Phi^{-1}(\exp(x_1)), \dots, \Phi^{-1}(\exp(x_d))\right).$$

According to Lemma 3.2.1, the pertaining Pickands density on $R \times (c_0, 0)$ for $c_0 < 0$ near 0 is given by

$$f_\Sigma(\mathbf{z}, c) = \exp(c)|c|^{d-1}(\det \Sigma)^{-1/2} \exp\left(\frac{1}{2}F(c\mathbf{z})^T\left(I_d - \Sigma^{-1}\right)F(c\mathbf{z})\right)$$

with

$$c\mathbf{z} := \left(cz_1, \dots, cz_{d-1}, c\left(1 - \sum_{i\le d-1} z_i\right)\right).$$

Asymptotic considerations and computations show that it satisfies the following expansion of length 2:

$$f_\Sigma(\mathbf{z}, c) = B(c)\tilde{A}(\mathbf{z}) + o(B(c)), \quad c \uparrow 0,$$

with

$$B(c) = |c|^{\sum_{i,j=1}^d \sigma_{ij} - 1}L(c),$$
$$L(c) = (-\log|c|)^{\sum_{i,j=1}^d \sigma_{ij}/2 - d/2},$$

and

$$\tilde{A}(\mathbf{z}) = (\det \Sigma)^{-1/2}(4\pi)^{\sum_{i,j=1}^{d}\sigma_{ij}/2-d/2}\prod_{i,j=1}^{d}(z_i z_j)^{(\sigma_{ij}-\delta_{ij})/2},$$

where $I_d = (\delta_{ij})_{i,j=1,\dots,d}$ *and* $\Sigma^{-1} = (\sigma_{ij})_{i,j=1,\dots,d}$. *The function L is slowly varying in* 0*, hence the function B is regularly varying with the exponent of variation* $\beta = \sum_{i,j=1}^{d}\sigma_{ij} - 1 > 0$. *Obviously we have* $\int_R \tilde{A}(\mathbf{z}) > 0$ *because of* $\tilde{A}(\mathbf{z}) \geq 0$ *for all* $\mathbf{z} \in R$ *and* $\tilde{A}(\mathbf{z}) > 0$ *for* $\mathbf{z} \in \left\{(t_1, \dots, t_{d-1}) \in [0,1]^{d-1} : \sum_{i \leq d-1} t_i < 1\right\}$. *(The definition of R in* (2.10) *shows that the later set is the inner part if R which is no Lebesgue–null set.) This implies the existence of the parameter* κ *in* (3.18)*. Detailed calculations can be found in [16], Example 3.7.10.*

Multivariate normal distributions constitute a special case in the class of spherically and elliptically symmetric distributions. In the sequel we will shortly characterize these distributions by means of Fang et al. [15], Chapter 2, and Schmidt [47], Sections 2, 3 and 5, in order to present another example in the enlarged framework afterwards.

Definition 3.3.6
A d–dimensional random vector $\mathbf{Y}$ *is called spherically (symmetrically) distributed if the equality in distribution*

$$O\mathbf{Y} \stackrel{d}{=} \mathbf{Y}$$

is fulfilled for every orthogonal matrix $O \in \mathbb{R}^{d\times d}$.

Definition 3.3.7
A d–dimensional random vector $\mathbf{X}$ *is called elliptically (symmetrically) distributed with parameters* $\mu \in \mathbb{R}^d$ *and* $\Sigma \in \mathbb{R}^{d\times d}$ *if it satisfies the equality in distribution*

$$\mathbf{X} \stackrel{d}{=} \mu + A^T\mathbf{Y},$$

where $\mathbf{Y}$ *is an m–dimensional spherically distributed random vector and* $A \in \mathbb{R}^{m\times d}$ *is a matrix fulfilling* $A^T A = \Sigma$ *with* $rank(\Sigma) = d$.

A spherically or elliptically distributed random vector does not necessarily have a density. But in case a density exists it can be shown, cf. [15], Chapter 2, that it has to be of the form $g(x^T x)$.

Definition 3.3.8
Suppose a d–dimensional random vector $\mathbf{X}$ *which is elliptically distributed with parameters* μ *and* Σ *has a density function* $g(x^T x)$. *Then we call g the density generator of* $\mathbf{X}$. *It is also written* $\mathbf{X} \in E_d(\mu, \Sigma, g)$.

A special subclass of elliptically symmetric distributions is represented by the symmetric Kotz type distributions, which also include the multivariate normal distributions.

Definition 3.3.9
Let $\mathbf{X} \in E_d(\mu, \Sigma, g)$ and suppose that the density generator is of the form

$$g(u) = C_d u^{N-1} \exp(-ru^s), \quad r, s > 0,\, 2N + d > 2, \tag{3.28}$$

with a normalizing constant C_d. Then we say that $\mathbf{X}$ possesses a Kotz type distribution.

The density of a Kotz type distribution is given by

$$h(\mathbf{x}) = C_d (\det \Sigma)^{-1/2} \left[(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \right]^{N-1} \exp \left(-r \left[(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \right]^s \right)$$

and the normalizing constant C_d is

$$C_d = \frac{s\Gamma(d/2)}{\pi^{d/2}\Gamma((2N + d - 2)/(2s))} r^{(2N+d-2)/(2s)}, \tag{3.29}$$

cf. [15], p. 76. For $N = 1, s = 1$ and $r = 1/2$ we get the multivariate normal distribution.

In the subsequent example we provide an expansion for the Pickands density of this subclass of elliptical distributions.

Example 3.3.10
Suppose the random vector $\mathbf{X}$ possesses a Kotz type distribution with density generator g as in (3.28) *and correlation matrix Σ. We assume that*

$$a := \frac{2N - 1}{2s} - 1 < 0,$$

which includes the case of the multivariate normal distribution. (The cases where $a \geq 0$ can be dealt with similarly.) Then after a transformation to reversely exponential margins the pertaining Pickands density satisfies an expansion of length 2 *where the leading term φ_D is equal to zero, namely*

$$f_{\Sigma,r,s,N}(\mathbf{z}, c) = B(c)\tilde{A}(\mathbf{z}) + o(B(c)), \quad c \uparrow 0,$$

with

$$B(c) = |c|^{\left(\sum_{i,j=1}^d \sigma_{ij}\right)^s - 1} L(c),$$
$$L(c) = (-\log|c|)^{-\frac{d-1}{s}(N-1) - \left(\frac{2N-1}{2s} - 1\right)\left(\left(\sum_{i,j=1}^d \sigma_{ij}\right)^s - d\right)},$$

and

$$\tilde{A}(\mathbf{z}) = \frac{C_d}{C_1^{\left(\sum_{i,j=1}^d \sigma_{ij}\right)^s}} (\det \Sigma)^{-1/2} r^{-\frac{N-1}{s} - \frac{d}{2s} + \frac{2N-1}{2s}\left(\sum_{i,j=1}^d \sigma_{ij}\right)^s} (2s)^{\left(\sum_{i,j=1}^d \sigma_{ij}\right)^s - d} \left(\sum_{i,j=1}^d \sigma_{ij} \right)^{N-1}$$
$$\times \prod_{i,j=1}^d (z_i z_j)^{\left(\sum_{i,j=1}^d \sigma_{ij}\right)^{s-1} \sigma_{ij} - \delta_{ij}},$$

where $I_d = (\delta_{ij})_{i,j=1,\dots,d}$ *and* $\Sigma^{-1} = (\sigma_{ij})_{i,j=1,\dots,d}$. *The function L is slowly varying in* 0, *hence the function B is regularly varying with the exponent of variation* $\rho = \left(\sum_{i,j=1}^d \sigma_{ij}\right)^s - 1$. *By similar arguments as in Example 3.3.5 it follows that* $\int_R \tilde{A}(\mathbf{z})\, d\mathbf{z} > 0$ *which implies the existence of the parameter* κ *in* (3.18).

Before we prove the assertion of Example 3.3.10, we need an auxiliary result concerning the distribution function of **X** and its inverse.

Lemma 3.3.11
Suppose the random vector **X** *possesses a Kotz type distribution with density generator g as given in* (3.28), *where*

$$a = \frac{2N-1}{2s} - 1 < 0.$$

Then each marginal distribution function of **X** *is given by*

$$G(x) = 1 - \frac{C_1}{2s} r^{-(2N-1)/(2s)} \Gamma\left(\frac{2N-1}{2s}, rx^{2s}\right), \quad x \in \mathbb{R},$$

where

$$\Gamma(b,z) := \int_z^\infty t^{b-1} \exp(-t)\, dt \tag{3.30}$$

is the incomplete gamma function. The inverse of G can be approximated by

$$\begin{aligned} G^{-1}(u) \approx \Bigg[&\frac{1}{r}|a| \left(-\log|a| - \left(\log\left(\frac{2s}{C_1}(1-u) r^{(2N-1)/(2s)}\right) - 1\right) / |a|\right) \\ &\times \exp\left(\frac{\log\left(-\log|a| - \left(\log\left(\frac{2s}{C_1}(1-u) r^{(2N-1)/(2s)}\right) - 1\right)/|a|\right)}{\log|a| + \left(\log\left(\frac{2s}{C_1}(1-u) r^{(2N-1)/(2s)}\right) - 1\right)/|a|}\right) - \frac{1}{r}\Bigg]^{1/(2s)} \end{aligned}$$

if u is close to 1.

PROOF. According to [15], Section 2.2, each univariate margin of **X** possesses a one–dimensional Kotz type distribution with density generator

$$g_1(u) = C_1 u^{2N-2} \exp\left(-ru^{2s}\right),$$

where the normalizing constant C_1 is given by (3.29) for $d = 1$. Then we get the pertaining distribution function by integration, i.e.

$$\begin{aligned} G(x) &= \int_{-\infty}^x C_1 u^{2N-2} \exp\left(-ru^{2s}\right) du \\ &= 1 - \int_x^\infty C_1 u^{2N-2} \exp\left(-ru^{2s}\right) du \\ &= 1 - \int_{x^{2s}}^\infty \frac{C_1}{2s} u^{(2N-1)/(2s)-1} \exp(-ru)\, du \\ &= 1 - \frac{C_1}{2s} r^{-(2N-1)/(2s)} \Gamma\left(\frac{2N-1}{2s}, rx^{2s}\right), \end{aligned} \tag{3.31}$$

where Γ denotes the incomplete gamma function defined in (3.30). To derive the inverse function of G we need approximations of Γ and of its inverse function. According to Amore [1], formula (9), we have

$$\Gamma(b,z) \approx \exp(-z)(1+z)^{b-1}.$$

Thence we obtain

$$\begin{aligned}\Gamma\left(\frac{2N-1}{2s}, rx^{2s}\right) &\approx \exp\left(-rx^{2s}\right)\left(1+rx^{2s}\right)^{(2N-1)/(2s)-1} \\ &= \exp\left(-rx^{2s}\right)\left(1+rx^{2s}\right)^{a}.\end{aligned}$$

Now, if we put $rx^{2s} = z$ and solve

$$\exp(-z)(1+z)^a = q$$

for z, we obtain

$$z = \exp\left(-W\left(-\frac{1}{a}\exp\left(\frac{\log(q)-1}{a}\right)\right) + \frac{\log(q)-1}{a}\right) - 1 \tag{3.32}$$

where W is the principal branch of the Lambert W function, which can be approximated by

$$W(y) \approx \log(y) - \log(\log(y)) + \frac{\log(\log(y))}{\log(y)}$$

if y is sufficiently large, cf. de Bruijn [3], Section 2.4, and Corless et al. [7], Section 4. Inserting this expansion into equation (3.32) leads to

$$\begin{aligned} z &\approx \exp\Bigg(-\Bigg[\log\left(-\frac{1}{a}\right) + \frac{\log(q)-1}{a} - \log\left(\log\left(-\frac{1}{a}\right) + \frac{\log(q)-1}{a}\right) \\ &\qquad + \frac{\log\left(\log\left(-\frac{1}{a}\right) + \frac{\log(q)-1}{a}\right)}{\log\left(-\frac{1}{a}\right) + \frac{\log(q)-1}{a}}\Bigg] + \frac{\log(q)-1}{a}\Bigg) - 1 \\ &= \exp\left(\log|a| + \log\left(-\log|a| - \frac{\log(q)-1}{|a|}\right) + \frac{\log\left(-\log|a| - \frac{\log(q)-1}{|a|}\right)}{\log|a| + \frac{\log(q)-1}{|a|}}\right) - 1 \\ &=: h_a(q). \end{aligned}$$

Resubstituting z by rx^{2s} now gives us the possibility to solve equation (3.31) for x. We obtain

$$
\begin{aligned}
G^{-1}(u) &\approx \left[h_a\left(\frac{2s}{C_1}(1-u)r^{(2N-1)/(2s)}\right)\frac{1}{r}\right]^{1/(2s)} \\
&= \left[\frac{1}{r}\exp\left(\log|a| + \log\left(-\log|a| - \left(\log\left(\frac{2s}{C_1}(1-u)r^{(2N-1)/(2s)}\right)-1\right)/|a|\right)\right.\right. \\
&\qquad \left.\left. + \frac{\log\left(-\log|a| - \left(\log\left(\frac{2s}{C_1}(1-u)r^{(2N-1)/(2s)}\right)-1\right)/|a|\right)}{\log|a| + \left(\log\left(\frac{2s}{C_1}(1-u)r^{(2N-1)/(2s)}\right)-1\right)/|a|}\right) - \frac{1}{r}\right]^{1/(2s)} \\
&= \left[\frac{1}{r}|a|\left(-\log|a| - \left(\log\left(\frac{2s}{C_1}(1-u)r^{(2N-1)/(2s)}\right)-1\right)/|a|\right)\right. \\
&\qquad \left.\times\exp\left(\frac{\log\left(-\log|a| - \left(\log\left(\frac{2s}{C_1}(1-u)r^{(2N-1)/(2s)}\right)-1\right)/|a|\right)}{\log|a| + \left(\log\left(\frac{2s}{C_1}(1-u)r^{(2N-1)/(2s)}\right)-1\right)/|a|}\right) - \frac{1}{r}\right]^{1/(2s)}.
\end{aligned}
$$

□

We will now prove the assertion of Example 3.3.10.

PROOF. Let **X** be the random vector with density generator g as in (3.28). Then, according to [15], Section 3.2, it has the density

$$
\tilde{h}(\mathbf{x}) = C_d|\Sigma|^{-1/2} g\left(\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right),
$$

where Σ is the correlation matrix. After a transformation to reversely exponential margins we obtain the following density by applying the transformation theorem for densities:

$$
\begin{aligned}
h(\mathbf{x}) &= \tilde{h}\left(G^{-1}(\mathbf{x})\right)\left|\prod_{i\le d}\frac{g_1\left(G^{-1}(x_i)\right)}{\exp(x_i)}\right|^{-1} \\
&= C_d|\Sigma|^{-1/2}\prod_{i\le d}\exp(x_i) \\
&\quad\times\frac{g\left(G^{-1}(\exp(\mathbf{x}))^T\Sigma^{-1}G^{-1}(\exp(\mathbf{x}))\right)}{C_1^d\left(\prod_{i\le d}\left(G^{-1}(\exp(x_i))\right)^2\right)^{N-1}\exp\left(-r\sum_{i\le d}\left(G^{-1}(\exp(x_i))\right)^2\right)},
\end{aligned}
$$

where g_1 and G are the density and the distribution function, respectively, of any univariate margin and $G^{-1}(\mathbf{x})$ is defined by

$$
G^{-1}(\mathbf{x}) := \left(G^{-1}(x_1),\ldots,G^{-1}(x_d)\right).
$$

The pertaining Pickands density on $R \times (c_0, 0)$ for $c_0 < 0$ near 0 is given by

$$f_{\Sigma,r,s,N}(\mathbf{z}, c) = \exp(c)|c|^{d-1}\frac{C_d}{C_1^d}|\Sigma|^{-1/2}\frac{\left(\sum_{i,j=1}^d \sigma_{ij}G^{-1}(\exp(cz_i))G^{-1}(\exp(cz_j))\right)^{N-1}}{\left(\prod_{i\le d}\left(G^{-1}(\exp(cz_i))\right)^2\right)^{N-1}}$$
$$\times \exp\left(-r\left[\left(\sum_{i,j=1}^d \sigma_{ij}G^{-1}(\exp(cz_i))G^{-1}(\exp(cz_j))\right)^s - \sum_{i=1}^d \left(G^{-1}(\exp(cz_i))\right)^{2s}\right]\right). \tag{3.33}$$

By using Lemma 3.3.11 we can deduce representations for $G^{-1}(\exp(cz_i))$, $i \le d$. We obtain

$$G^{-1}(\exp(cz))$$
$$\approx \left[\left(-\frac{|a|}{r}\log|a| - \frac{1}{r}\left(\log\left(\frac{2s}{C_1}\right) + \log(|c|z + o(|c|)) + \frac{2N-1}{2s}\log(r) - 1\right)\right)\right.$$
$$\times \exp\left(\frac{\log\left(-\log|a| - \left(\log\left(\frac{2s}{C_1}\right) + \log(|c|z+o(|c|)) + \frac{2N-1}{2s}\log(r) - 1\right)/|a|\right)}{\log|a| + \left(\log\left(\frac{2s}{C_1}\right) + \log(|c|z+o(|c|)) + \frac{2N-1}{2s}\log(r) - 1\right)/|a|}\right)$$
$$\left. -\frac{1}{r}\right]^{1/(2s)}$$
$$= \left[-\frac{|a|}{r}\log|a| - \frac{1}{r}\log\left(\frac{2s}{C_1}\right) - \frac{1}{r}\frac{2N-1}{2s}\log(r) + \frac{1}{r} - \frac{1}{r}\log(|c|z + o(|c|))\right.$$
$$\times\left(1 + \frac{\log\left(-\log|a| - \frac{1}{|a|}\log\left(\frac{2s}{C_1}\right) - \frac{1}{|a|}\frac{2N-1}{2s}\log(r) + \frac{1}{|a|} - \frac{1}{|a|}\log(|c|z+o(|c|))\right)}{\log|a| + \frac{1}{|a|}\log\left(\frac{2s}{C_1}\right) - \frac{1}{|a|}\frac{2N-1}{2s}\log(r) + \frac{1}{|a|} - \frac{1}{|a|}\log(|c|z+o(|c|))}\right.$$
$$+\frac{1}{2}\left(\frac{\log\left(-\log|a| - \frac{1}{|a|}\log\left(\frac{2s}{C_1}\right) - \frac{1}{|a|}\frac{2N-1}{2s}\log(r) + \frac{1}{|a|} - \frac{1}{|a|}\log(|c|z+o(|c|))\right)}{\log|a| + \frac{1}{|a|}\log\left(\frac{2s}{C_1}\right) - \frac{1}{|a|}\frac{2N-1}{2s}\log(r) + \frac{1}{|a|} - \frac{1}{|a|}\log(|c|z+o(|c|))}\right)^2$$
$$\left.\left. +o\left(\left(\frac{\log(-\log|c|)}{-\log|c|}\right)^2\right)\right) - \frac{1}{r}\right]^{1/(2s)}$$
$$= \left[-\frac{1}{r}\log(|c|z + o(|c|)) - \frac{|a|}{r}\log|a| - \frac{1}{r}\log\left(\frac{2s}{C_1}\right) - \frac{1}{r}\frac{2N-1}{2s}\log(r)\right.$$
$$-\frac{|a|}{r}\log\left(-\log|a| - \frac{1}{|a|}\log\left(\frac{2s}{C_1}\right) - \frac{1}{|a|}\frac{2N-1}{2s}\log(r) + \frac{1}{|a|} - \frac{1}{|a|}\log(|c|z+o(|c|))\right)$$
$$-\frac{|a|}{2r}\frac{\log\left(-\log|a| - \frac{1}{|a|}\log\left(\frac{2s}{C_1}\right) - \frac{1}{|a|}\frac{2N-1}{2s}\log(r) + \frac{1}{|a|} - \frac{1}{|a|}\log(|c|z+o(|c|))\right)}{-\log|a| - \frac{1}{|a|}\log\left(\frac{2s}{C_1}\right) - \frac{1}{|a|}\frac{2N-1}{2s}\log(r) + \frac{1}{|a|} - \frac{1}{|a|}\log(|c|z+o(|c|))}$$
$$\left. +o\left(\frac{(\log(-\log|c|))^2}{-\log|c|}\right)\right]^{1/(2s)}$$

$$= \Bigg[-\frac{1}{r}\log(|c|z+o(|c|)) + \frac{b}{r} - \frac{|a|}{r}\log\left(\frac{b}{|a|}+\frac{1}{|a|}-\frac{1}{|a|}\log(|c|z+o(|c|))\right)$$

$$-\frac{|a|}{2r}\frac{\left(\log\left(\frac{b}{|a|}+\frac{1}{|a|}-\frac{1}{|a|}\log(|c|z+o(|c|))\right)\right)^2}{\frac{b}{|a|}+\frac{1}{|a|}-\frac{1}{|a|}\log(|c|z+o(|c|))} + o\left(\frac{\log(-\log|c|)^2}{-\log|c|}\right)\Bigg]^{1/(2s)}$$

$$= r^{-1/(2s)}\Bigg[-\log(|c|z+o(|c|)) + b + |a|\log|a| - |a|\log(b+1-\log(|c|z+o(|c|)))$$

$$-\frac{|a|^2(-\log|a|+\log(b+1-\log(|c|z+o(|c|))))^2}{2(b+1-\log(|c|z+o(|c|)))} + o\left(\frac{\log(-\log|c|)^2}{-\log|c|}\right)\Bigg]^{1/(2s)}$$

$$= r^{-1/(2s)}\Bigg[-\log|c| - \log(z+o(1)) + b + |a|\log|a|$$

$$-|a|\log(b+1-\log|c|-\log(z+o(1)))$$

$$-\frac{|a|^2(-\log|a|+\log(b+1-\log|c|-\log(z+o(1))))^2}{2(b+1-\log|c|-\log(z+o(1)))}$$

$$+o\left(\frac{\log(-\log|c|)^2}{-\log|c|}\right)\Bigg]^{1/(2s)}, \quad c\uparrow 0,$$

where

$$b := -|a|\log|a| - \log\left(\frac{2s}{C_1}\right) - \frac{2N-1}{2s}\log(r).$$

By using the approximation

$$\begin{aligned}&\log(b+1-\log|c|-\log(z+o(1)))\\&= \log\left((-\log|c|)\left(1+\frac{b+1+\log(z+o(1))}{-\log|c|}\right)\right)\\&\approx \log(-\log|c|) + \frac{\log(z)-b-1}{\log|c|}, \quad \text{as } c\uparrow 0,\end{aligned}$$

we can continue the above chain of equations by

$$G^{-1}(\exp(cz))$$

$$\approx r^{-1/(2s)}(-\log|c|)^{1/(2s)}\Bigg[1+\frac{\log(z+o(1))}{\log|c|}+\frac{-|a|\log|a|-b}{\log|c|}$$

$$+|a|\frac{\log(b+1-\log|c|-\log(z+o(1)))}{\log|c|}$$

$$+\frac{|a|^2\left(\log(b+1-\log|c|-\log(z+o(1)))\right)^2}{2\log|c|(b+1-\log|c|-\log(z+o(1)))} + o\left(\frac{\log(-\log|c|)^2}{-\log|c|}\right)\Bigg]^{1/(2s)}$$

$$\approx r^{-1/(2s)}(-\log|c|)^{1/(2s)}\Bigg[1+\frac{\log(z)}{\log|c|}+\frac{-|a|\log|a|-b}{\log|c|}$$
$$+|a|\frac{\log(-\log|c|)}{\log|c|}+|a|\frac{\log(z)-b-1}{(\log|c|)^2}$$
$$+\frac{|a|^2\left(\log(-\log|c|)^2+2(\log(z-b-1))\frac{\log(-\log|c|)}{\log|c|}+\frac{(\log(z)-b-1)^2}{(\log|c|)^2}\right)}{-2(\log|c|)^2(1+(\log(z)-b-1)/\log|c|)}$$
$$+o\left(\left(\frac{\log(-\log|c|)}{-\log|c|}\right)^2\right)\Bigg]^{1/(2s)}$$
$$=r^{-1/(2s)}(-\log|c|)^{1/(2s)}\Bigg[1+|a|\frac{\log(-\log|c|)}{\log|c|}+\frac{\log(z)-b-|a|\log|a|}{\log|c|}$$
$$-|a|^2\frac{(\log(-\log|c|))^2}{2(\log|c|)^2}\frac{1}{1+(\log(z)-b-1)/\log|c|}+o\left(\left(\frac{\log(-\log|c|)}{-\log|c|}\right)^2\right)\Bigg]^{1/(2s)}$$
$$\approx r^{-1/(2s)}(-\log|c|)^{1/(2s)}\Bigg[1+\frac{|a|}{2s}\frac{\log(-\log|c|)}{\log|c|}+\frac{1}{\log|c|}\frac{1}{2s}(\log(z)-b-|a|\log|a|)$$
$$-\frac{|a|^2}{4s}\frac{(\log(-\log|c|))^2}{(\log|c|)^2}\frac{1}{1+(\log(z)-b-1)/\log|c|}+o\left(\left(\frac{\log(-\log|c|)}{-\log|c|}\right)^2\right)\Bigg],$$

as $c\uparrow 0$. Therewith we obtain

$$\frac{\left(\sum_{i,j=1}^d\sigma_{ij}G^{-1}(\exp(cz_i))G^{-1}(\exp(cz_j))\right)^{N-1}}{\left(\prod_{i\le d}(G^{-1}(\exp(cz_i)))^2\right)^{N-1}}$$
$$\approx\frac{\left(\sum_{i,j=1}^d\sigma_{ij}\right)^{N-1}\left(r^{-1/s}(-\log|c|)^{1/s}\right)^{N-1}\left(1+O\left(\frac{\log(-\log|c|)}{\log|c|}\right)\right)^{N-1}}{\left(r^{-d/s}(-\log|c|)^{d/s}\right)^{N-1}\left(1+O\left(\frac{\log(-\log|c|)}{\log|c|}\right)\right)^{N-1}}$$
$$=\left(\sum_{i,j=1}^d\sigma_{ij}r^{(d-1)/s}\right)^{N-1}(-\log|c|)^{(N-1)(1-d)/s}\left(1+O\left(\frac{\log(-\log|c|)}{\log|c|}\right)\right)^{N-1}$$
$$=\left(\sum_{i,j=1}^d\sigma_{ij}r^{(d-1)/s}\right)^{N-1}(-\log|c|)^{(N-1)(1-d)/s}+o\left((-\log|c|)^{(N-1)(1-d)/s}\right),\quad(3.34)$$

as $c\uparrow 0$. Furthermore, we have

$$\exp\left(-r\left[\left(\sum_{i,j=1}^d\sigma_{ij}G^{-1}(\exp(cz_i))G^{-1}(\exp(cz_j))\right)^s-\sum_{i=1}^d\left(G^{-1}(\exp(cz_i))\right)^{2s}\right]\right)$$

$$
\begin{aligned}
&\approx \exp\Bigg(\log|c| \Bigg[\Bigg(\sum_{i,j=1}^{d} \sigma_{ij}\Bigg[1+\frac{|a|}{2s}\frac{\log(-\log|c|)}{\log|c|}+\frac{1}{\log|c|}\frac{1}{2s}(\log(z_i)-b-|a|\log|a|)\\
&\quad -\frac{|a|^2}{4s}\left(\frac{\log(-\log|c|)}{\log|c|}\right)^2\frac{1}{1+(\log(z_i)-b-1)/\log|c|}+o\left(\left(\frac{\log(-\log|c|)}{\log|c|}\right)^2\right)\Bigg]\\
&\quad \times\Bigg[1+\frac{|a|}{2s}\frac{\log(-\log|c|)}{\log|c|}+\frac{1}{\log|c|}\frac{1}{2s}(\log(z_j)-b-|a|\log|a|)\\
&\quad -\frac{|a|^2}{4s}\left(\frac{\log(-\log|c|)}{\log|c|}\right)^2\frac{1}{1+(\log(z_j)-b-1)/\log|c|}\\
&\quad +o\left(\left(\frac{\log(-\log|c|)}{\log|c|}\right)^2\right)\Bigg]\Bigg)^s\\
&\quad -\Bigg(\sum_{i\le d}\Bigg[1+\frac{|a|}{2s}\frac{\log(-\log|c|)}{\log|c|}+\frac{1}{\log|c|}\frac{1}{2s}(\log(z_i)-b-|a|\log|a|)\\
&\quad -\frac{|a|^2}{4s}\left(\frac{\log(-\log|c|)}{\log|c|}\right)^2\frac{1}{1+(\log(z_i)-b-1)/\log|c|}\\
&\quad +o\left(\left(\frac{\log(-\log|c|)}{\log|c|}\right)^2\right)\Bigg]^{2s}\Bigg)\Bigg]\Bigg)\\
&= \exp\Bigg(\log|c|\Bigg[\Bigg(\sum_{i,j=1}^{d}\sigma_{ij}\Bigg[1+\frac{|a|}{s}\frac{\log(-\log|c|)}{\log|c|}+\frac{z_iz_j}{\log|c|2s}+\frac{-|a|\log|a|-b}{s\log|c|}\\
&\quad -\frac{|a|^2}{4s}\left(\frac{\log(-\log|c|)}{\log|c|}\right)^2\\
&\quad \times\left[\frac{1}{1+(\log(z_i)-b-1)/\log|c|}+\frac{1}{1+(\log(z_j)-b-1)/\log|c|}\right]\\
&\quad +\frac{|a|^2}{4s^2}\left(\frac{\log(-\log|c|)}{\log|c|}\right)^2+o\left(\left(\frac{\log(-\log|c|)}{\log|c|}\right)^2\right)\Bigg]\Bigg)^s\\
&\quad -\Bigg(\sum_{i\le d}\Bigg[1+\frac{|a|}{s}\frac{\log(-\log|c|)}{\log|c|}+\frac{\log(z_i^2)}{2s\log|c|}+\frac{-|a|\log|a|-b}{s\log|c|}\\
&\quad -\frac{|a|^2}{2s}\left(\frac{\log(-\log|c|)}{\log|c|}\right)^2\frac{1}{1+(\log(z_i)-b-1)/\log|c|}+\frac{|a|^2}{4s^2}\left(\frac{\log(-\log|c|)}{\log|c|}\right)^2\\
&\quad +o\left(\left(\frac{\log(-\log|c|)}{\log|c|}\right)^2\right)\Bigg]\Bigg)^s\Bigg]\Bigg)\\
&= \exp\left(\log|c|\left[\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^s-d\right]+|a|\log(-\log|c|)\left[\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^s-d\right]\right.
\end{aligned}
$$

$$+\frac{1}{2}\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s-1}\left(\sum_{i,j=1}^{d}\sigma_{ij}\log(z_iz_j)\right)-(|a|\log|a|+b)\left[\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s}-d\right]$$
$$-\frac{1}{2}\left(\sum_{i\le d}\log(z_i^2)\right)\Bigg)\left(1+O\left(\frac{\log(-\log|c|)^2}{\log|c|}\right)\right)$$
$$=|c|^{\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s}-d}(-\log|c|)^{|a|\left(\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s}-d\right)}\prod_{i,j=1}^{d}(z_iz_j)^{\frac{1}{2}\left(\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s-1}\sigma_{ij}-\delta_{ij}\right)}$$
$$\times\exp\left(\left[\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s}-d\right](-|a|\log|a|-b)\right)\left(1+O\left(\frac{\log(-\log|c|)^2}{\log|c|}\right)\right)$$
$$=|c|^{\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s}-d}(-\log|c|)^{-\left(\frac{2N-1}{2s}-1\right)\left(\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s}-d\right)}\prod_{i,j=1}^{d}(z_iz_j)^{\frac{1}{2}\left(\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s-1}\sigma_{ij}-\delta_{ij}\right)}$$
$$\left(\frac{2s}{C_1}r^{(2N-1)/(2s)}\right)^{\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s}-d}\left(1+O\left(\frac{\log(-\log|c|)^2}{\log|c|}\right)\right),\tag{3.35}$$

as $c\uparrow 0$. Inserting the representations (3.34) and (3.35) into (3.33) we deduce that the Pickands density satisfies (approximately)

$$f_{\Sigma,r,s,N}(\mathbf{z},c)=B(c)\tilde{A}(\mathbf{z})+o(B(c)),\quad c\uparrow 0,$$

with

$$B(c)=|c|^{\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s}-1}L(c),$$
$$L(c)=(-\log|c|)^{-\frac{d-1}{s}(N-1)-\left(\frac{2N-1}{2s}-1\right)\left(\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s}-d\right)},$$

and

$$\tilde{A}(\mathbf{z})=\frac{C_d}{C_1^{\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s}}}(\det\Sigma)^{-1/2}r^{-\frac{N-1}{s}-\frac{d}{2s}+\frac{2N-1}{2s}\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s}}(2s)^{\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s}-d}\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{N-1}$$
$$\times\prod_{i,j=1}^{d}(z_iz_j)^{\left(\sum_{i,j=1}^{d}\sigma_{ij}\right)^{s-1}\sigma_{ij}-\delta_{ij}},$$

where $I_d=(\delta_{ij})_{i,j=1,\dots,d}$ and $\Sigma^{-1}=(\sigma_{ij})_{i,j=1,\dots,d}$. This proves the assertion of Example 3.3.10. □

4 Testing the tail dependence based on Pickands densities

The aim of this chapter is to test the tail dependence against rates of tail independence based on the radial component.

In Section 4.1 we use expansions of Pickands densities to prove a limit theorem for the radial component. Within the statistical model of the limiting distribution functions we establish a uniformly most powerful test for testing the tail dependence, particularly, for testing the null hypothesis $\int_R \varphi_D(\mathbf{z})\,d\mathbf{z} > 0$ against the alternative $\int_R \varphi_D(\mathbf{z})\,d\mathbf{z} = 0$. The function φ_D is again the Pickands density of a GP random vector. The Neyman–Pearson test is provided in Section 4.2 as well as the pertaining power function and the p–value. The presentation runs along the lines of that in Frick and Reiss [18], Section 3, cf. also Frick [16], Section 4.1.

4.1 Limiting conditional distributions of the radial component

For a bivariate random vector (X, Y) it was proved in [17], Section 4, that under a spectral expansion of length 2 as in (3.20) the conditional limiting distribution function of $(X + Y)/c$, given $X + Y > c$, is $F_0(t) = t$ or $F_\beta(t) = t^{1+\beta}$ if $D \neq 1$ or $D = 1$ respectively. In the bivariate context these conditions are equivalent to $\int_0^1 \varphi_D(z)\,dz > 0$ and $\int_0^1 \varphi_D(z)\,dz = 0$ respectively. This result has been generalized to higher dimensions in [18], Section 3, where again limiting distribution functions of the radial component have been proved in a conditional set–up, namely that $\sum_{i\le d} X_i$ exceeds a threshold c. The special case of EVDs in higher dimensions has already been studied in [10], pp. 199–202, and Falk and Michel [11].

Theorem 4.1.1
Assume that the random vector $\mathbf{X} = (X_1, \ldots, X_d)$ which takes values in $(-\infty, 0]^d$ has a Pickands density satisfying conditions (3.13)–(3.15), *where φ_D is the GPD–Pickands density with pertaining Pickands dependence function D.*

(i) (Tail Dependence) If $\int_R \varphi_D(\mathbf{z})\,d\mathbf{z} > 0$, then

$$P\left(\sum_{i\le d} X_i > ct \,\middle|\, \sum_{i\le d} X_i > c\right) \to t, \quad c \uparrow 0,$$

uniformly for $t \in [0, 1]$.

(ii) *(Marginal Tail Independence) If* $\int_R \varphi_D(\mathbf{z})\,d\mathbf{z} = 0$ *and condition* (3.15) *holds with the inequalities* $0 < \beta_1 < \beta_2 < \cdots < \beta_k$*, then*

$$P\left(\sum_{i\le d} X_i > ct \,\middle|\, \sum_{i\le d} X_i > c\right) \to t^{1+\beta_\kappa}, \quad c \uparrow 0,$$

uniformly for $t \in [0,1]$*, with* κ *as in* (3.18)*, i.e.*

$$\kappa = \min\left\{ j \in \{1,\dots,k\} : \int_R \tilde{A}_j(\mathbf{z})\,d\mathbf{z} \neq 0 \right\}. \tag{4.1}$$

PROOF. See Theorem 3.1 in [17] or Theorem 3.6.1 in [16] respectively. □

Recall that the integer κ in (4.1) exists according to Remark 3.2.11.

The parameter β_κ may be regarded as a measure of tail dependence and of the degree of tail independence in at least one bivariate margin. If $\beta_\kappa \to 0$, then $F_{\beta_\kappa}(t) = t^{1+\beta_\kappa}$ converges to the distribution function $F_0(t) = t$ which represents tail dependence. In the bivariate case there is a relationship of β_κ to a dependence measure $\bar{\chi}$ introduced by Coles et al. [6], Section 3.2, and to the coefficient of tail dependence in Ledford and Tawn [33], Section 5, cf. [17], Section 8.2. In Chapter 7 we will investigate measures of asymptotic dependence more closely. In particular, we will present the mentioned relationship between the exponent of variation of a spectral expansion and the residual dependence index $\bar{\chi}$, cf. equation (7.9), and extend it to the multivariate case, see equation (7.21).

4.2 Testing the tail dependence

We want to detect — in other words, we want to prove — that there is a certain degree of tail independence in the bivariate marginal distribution functions of a random vector $\mathbf{X} = (X_1, \dots, X_d)$. Therefore tail dependence is tested against tail independence.

The first step consists in testing the pairwise tail dependence of the random variables $X_1, \dots, X_d$. We assume that the random vector $(X_1, \dots, X_d)$ has a Pickands density satisfying the conditions (3.13)–(3.15). (Recall that in the bivariate setup this is the case if its distribution function fulfills a spectral expansion of length 2, cf. Lemma 3.2.13.) Because of Theorem 4.1.1 this is possible by testing the functions $F_0(t) = t$, $t \in [0,1]$, and $F_\beta(t) = t^{1+\beta}$, $t \in [0,1]$, $\beta > 0$, against one another. A simple null hypothesis is thus tested against a compound alternative, i.e.

$$H_0 : F_0(t) = t \quad \text{against} \quad H_1 : F_\beta(t) = t^{1+\beta},\ \beta > 0. \tag{4.2}$$

The structure of this test problem is the same as presented in [17], Section 5, where tail dependence is tested against tail independence in the bivariate case. In the present context the alternative has to be interpreted slightly differently unless D is assumed to be symmetric. A rejection of the null hypothesis implies tail independence in at least one bivariate margin. Hence multivariate tail independence is included in the alternative. The parameter β in the alternative stands for the degree of tail independence. If the null hypothesis is accepted, we can assume multivariate tail dependence — even tail dependence in each bivariate margin.

As stated in [17], Section 5, it is possible to deduce a uniformly most powerful test for the test problem (4.2). In fact, the Neyman–Pearson test at the level α is given by

$$C_{m,\alpha} = \left\{ \sum_{i=1}^{m} \log(Y_i) > H_m^{-1}(1-\alpha) \right\},$$

where $Y_1, \ldots, Y_m$ are iid random variables with common distribution function F_0 and

$$H_m(t) = \exp(t) \sum_{i=0}^{m-1} \frac{(-t)^i}{i!}, \quad t \leq 0,$$

is the gamma distribution function on the negative half–line with parameter m.

The pertaining power function

$$\psi_{m,\alpha}(\beta) = 1 - H_m\Big((1+\beta)H_m^{-1}(1-\alpha)\Big), \quad \beta \geq 0,$$

can be approximated by

$$\psi_{m,\alpha}(\beta) \approx 1 - \Phi((1+\beta)\Phi^{-1}(1-\alpha) - \beta m^{1/2}), \quad \beta \geq 0, \tag{4.3}$$

with Φ denoting the standard normal distribution function. The p–value of the optimal test is given by

$$\begin{aligned} p(\mathbf{y}) &= 1 - \exp\left(\sum_{i=1}^{m} \log(y_i)\right) \sum_{j=0}^{m-1} \frac{\left(-\sum_{i=1}^{m} \log(y_i)\right)^j}{j!} \\ &\approx \Phi\left(-\frac{\sum_{i=1}^{m} \log(y_i) + m}{m^{1/2}}\right). \end{aligned}$$

Now suppose that the null hypothesis of the testing problem (4.2) is rejected, meaning that there is significance for tail independence in at least one bivariate marginal distribution of the random vector $(X_1, \ldots, X_d)$. The question arises whether this applies to each of the marginal distributions. It can be answered by using an intersection–union test. Tests of this type are treated by Casella and Berger [4], p. 357, and have been used, e.g., as goodness of fit tests, cf. Villaseñor et al. [51], Section 3. In the present context one can construct an intersection–union test by testing each bivariate marginal distribution on tail dependence by means of Neyman–Pearson tests. A detailed presentation of this test can be found in [16], Section 4.2.

5 Limiting distributions of maxima under triangular schemes

Until now we have only considered sequences of iid random vectors and have implicitly assumed that sample maxima are based on the first n random vectors. In the present chapter we intend to analyze triangular schemes of random vectors where the n–th line contains n random vectors. Therefore we give up the above assumption and consider limiting distributions of maxima under representations of spectral densities or Pickands densities that depend on the sample size n. This implies a varying dependence structure in the previous random vectors.

This procedure has been motivated particularly by Hüsler and Reiss [29] and by Hashorva [23]. They consider arrays of normally and elliptically distributed random vectors, respectively, whose marginal maxima are tail independent but can be forced to be tail dependent by letting certain parameters vary with the sample size.

If we take maxima of n independent bivariate normal random vectors with correlation coefficient $\rho < 1$, for example, we know according to Example 3.3.5 that the marginal maxima are tail independent, i.e., the limiting distribution of the maxima, as $n \to \infty$, is that of two independent random variables. Now Hüsler and Reiss [29], Section 2, show that the marginal maxima are no longer tail independent but tail dependent if $\rho \equiv \rho(n)$ varies with the sample size n and $(1 - \rho(n)) \log(n)$ converges to a positive constant, as $n \to \infty$. They also compute the limiting distribution function which is called the Hüsler–Reiss distribution function by Joe [30], Hashorva [22], [24] and other authors.

Hashorva [23], [22], [24] extends these considerations to certain bivariate and multivariate elliptically distributed random vectors.

Against this background our aim is to generalize those results by treating the problem on an abstract level. Strictly speaking, we will no longer start with various distributional assumptions — thereby considering confined classes of distributions — but with a technical condition.

In Section 5.1 this technical condition concerns density expansions, i.e. spectral expansions and expansions of Pickands densities. The exponents of variation in the expansions (3.4) and (3.13) are no longer assumed to be constant but we let them vary as the sample size increases. The Hüsler–Reiss example in connection with the previous Example 3.3.4 shows that a certain convergence condition has to be imposed on the exponent of variation in the pertaining spectral density because it directly depends on ρ. Such convergence conditions will also form part of our "technical condition". With this condition as our basis we consider limiting distributions and residual dependence structures of maxima under triangular schemes.

Section 5.2 generalizes the condition imposed on the spectral density. Limiting distribution functions of maxima under triangular schemes are computed under the established more general convergence conditions and the structure of the limiting distribution function is analyzed. As in Section 5.1 we explicitly provide the limit theorem for the bivariate case. We finish Section 5.2 by presenting previously considered examples within the more general context, thereby extending and unifying them.

In Section 5.3 we show in which way the power of the test on tail dependence is affected if the exponents of variation in the underlying density expansions fulfill the convergence conditions of Section 5.1.

5.1 Limiting distributions and residual dependence structures under density expansions

The first part of this section, i.e. Subsection 5.1.1, presents limiting distributions of maxima under triangular schemes fulfilling the convergence conditions imposed on sequences of spectral expansions. The same is done in Subsection 5.1.2 — this time based on expansions of Pickands densities. It is also shown that the limiting distribution functions can be identified with each other in the bivariate case. Subsection 5.1.3 extends the results of Section 5.1.1 by investigating the convergence rate for maxima in the multivariate framework. This also leads to representations of residual dependence structures and — in certain cases — to rates for the asymptotic independence of these maxima. Our considerations can be seen against the background of [41], pp. 293–294, and Reiss [40], Section 7.2. In Subsection 5.1.4 we consider the residual dependence structure of maxima of different bivariate random vectors.

5.1.1 Limiting distributions under spectral expansions

First we compute limiting distributions of maxima under triangular schemes requiring that the underlying distribution functions satisfy spectral expansions of finite length, where the exponents of variation depend on the sample size and certain convergence conditions are fulfilled. The leading term of the expansion $D(\mathbf{z})$ is supposed to equal 1 as we are dealing with the case of tail independence.

Theorem 5.1.1
Let $H_{\beta(n)}$, $\beta(n) = (\beta_1(n), \ldots, \beta_k(n))$, $n \in \mathbb{N}$, be d–dimensional distribution functions with support on $(-\infty, 0]^d$ and assume that the pertaining spectral densities satisfy expansions of length $k+1$

$$h_{\beta(n),\mathbf{z}}(c) = 1 + \sum_{j=1}^{k} B_{j,n}(c) A_{j,n}(\mathbf{z}) + R_n(\mathbf{z}, c), \quad k \in \mathbb{N}, \tag{5.1}$$

with $R_n(\mathbf{z}, c) = o\left(B_{k,n}(c)\right)$ uniformly for $\mathbf{z} \in R$, as $c \uparrow 0$, according to (3.4)*, such that*

$$R_n\left(\mathbf{z}, c/n\right) \to 0, \quad n \to \infty, \tag{5.2}$$

for every $c < 0$. Assume

$$\beta_j(n) \to 0, \quad n \to \infty, \tag{5.3}$$

$$B_{j,n}(c/n) \to \lambda_j, \quad n \to \infty, \tag{5.4}$$

for any $c < 0$, where $\lambda_j \in \mathbb{R}$, $j = 1, \dots, k$. Further, suppose

$$A_{j,n}(\mathbf{z}) \to A_j(\mathbf{z}), \quad n \to \infty, \tag{5.5}$$

uniformly for $\mathbf{z} \in R$, where the A_j, $j = 1, \dots, k$, are continuous, bounded and satisfy

$$\sum_{j=1}^{k} \lambda_j A_j(\mathbf{z}) \geq -1, \quad z \in R. \tag{5.6}$$

Then we have

$$H^n_{\beta(n)}\left(\frac{y_1}{n}, \dots, \frac{y_d}{n}\right) \to \exp\left(T_2(\mathbf{y})\left(1 + \sum_{j=1}^{k} \lambda_j A_j(T_1(\mathbf{y}))\right)\right) \tag{5.7}$$

$$=: G(y_1, \dots, y_d), \tag{5.8}$$

as $n \to \infty$, with T_1 and T_2 as in (2.20)*, and G is a distribution function.*

Proof. Starting with $H^n_{\beta(n)}\left(\frac{y_1}{n}, \dots, \frac{y_d}{n}\right)$, we can write

$$\begin{aligned} H^n_{\beta(n)}\left(\frac{y_1}{n}, \dots, \frac{y_d}{n}\right) &= \exp\left(n \log\left(H_{\beta(n)}\left(\frac{y_1}{n}, \dots, \frac{y_d}{n}\right)\right)\right) \\ &= \exp\left(-n\left(1 - H_{\beta(n)}\left(\frac{y_1}{n}, \dots, \frac{y_d}{n}\right)\right)\right) + o(1), \quad n \to \infty, \\ &= \exp\left(-n\left(1 - H_{\beta(n),T_1(\mathbf{y})}\left(\frac{T_2(\mathbf{y})}{n}\right)\right)\right) + o(1), \quad n \to \infty, \end{aligned} \tag{5.9}$$

using representation (2.16) as well as T_1 and T_2 defined in (2.20). Next we present the argument of (5.9) in terms of the partial derivative $h_{\beta(n),\mathbf{z}}(c)$ and its spectral expansion given in (5.1) which leads to

$$\begin{aligned} -n\left(1 - H_{\beta(n),T_1(\mathbf{y})}\left(\frac{T_2(\mathbf{y})}{n}\right)\right) &= -n \int_{\frac{T_2(\mathbf{y})}{n}}^{0} h_{\beta(n),T_1(\mathbf{y})}(c)\, dc \\ &= -n \int_{\frac{T_2(\mathbf{y})}{n}}^{0} \left(1 + \sum_{j=1}^{k} B_{j,n}(c) A_{j,n}(T_1(\mathbf{y})) + o(B_{k,n}(c))\right) dc, \quad c \uparrow 0, \\ &= T_2(\mathbf{y}) - \sum_{j=1}^{k} A_{j,n}(T_1(\mathbf{y})) n \int_{\frac{T_2(\mathbf{y})}{n}}^{0} B_{j,n}(c)\, dc - n \int_{\frac{T_2(\mathbf{y})}{n}}^{0} o(B_{k,n}(c))\, dc, \quad c \uparrow 0, \\ &= T_2(\mathbf{y}) + \sum_{j=1}^{k} A_{j,n}(T_1(\mathbf{y})) \frac{1}{1 + \beta_j(n)} T_2(\mathbf{y}) B_{j,n}\left(\frac{T_2(\mathbf{y})}{n}\right) \\ &\quad + o\left(T_2(\mathbf{y}) B_{k,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right), \quad T_2(\mathbf{y}) \uparrow 0, \end{aligned} \tag{5.10}$$

where the last step is due to Remark 3.1.4 (i) and an application of l'Hôpital's theorem to the remainder term, cf. the proof of Theorem 3.4.5 in [16]. From the assumptions (5.2)–(5.5) it follows that the expression (5.10) converges to

$$T_2(\mathbf{y})\left(1+\sum_{j=1}^{k}\lambda_j A_j(T_1(\mathbf{y}))\right),$$

as $n \to \infty$. From (5.9) and the continuity of the exponential function we can finally deduce the convergence (5.7). According to [10], Section 4.1, G is a distribution function. The continuity follows from the continuity of the exponential function, of T_1 and T_2 and of the functions A_j, $j = 1, \dots, k$. To see that G is normed notice that

$$y_{n,r}\left(1+\sum_{j=1}^{k}\lambda_j A_j(T_1(\mathbf{y}_n))\right) \to 0, \quad r = 1, \dots, d,$$

due to the boundedness of the A_j, which implies $G(\mathbf{y}_n) \to 1$, as $n \to \infty$, for any sequence $(\mathbf{y}_n)_{n\in\mathbb{N}}$ with $y_{n,r} \uparrow 0$, $r = 1, \dots, d$. If $y_{n,r} \downarrow -\infty$ for some $r \in \{1, \dots, d\}$, it follows from (5.6) that

$$y_{n,r}\left(1+\sum_{j=1}^{k}\lambda_j A_j(T_1(\mathbf{y}_n))\right) \to -\infty$$

and, thus, $G(\mathbf{y}_n) \to 0$, as $n \to \infty$. Note that we can assume strict inequality in (5.6) in this case. Otherwise G would be degenerate. The Δ–monotony holds because G is the pointwise limit of a sequence of distribution functions. □

Remark 5.1.2
The limiting distribution function G in (5.8) *is again max–stable according to* (2.3) *with normalizing vectors $\mathbf{c_n} = (1/n, \dots, 1/n)$ and $\mathbf{d_n} = \mathbf{0}$.*

There is a supplementary result concerning the univariate margins which we will need in the next section.

Lemma 5.1.3
Let $H_{\beta(n)}$, $n \in \mathbb{N}$, be d–variate distribution functions as in Theorem 5.1.1. If the limiting functions A_j, $j = 1, \dots, k$, in (5.5) *additionally satisfy*

$$\sum_{j=1}^{k}\lambda_j A_j(\mathbf{e}_i) = 0, \quad i = 1, \dots, d-1, \tag{5.11}$$

$$\sum_{j=1}^{k}\lambda_j A_j(\mathbf{0}) = 0, \tag{5.12}$$

where $\mathbf{e}_i$ is the i–th unit vector in $\mathbb{R}^{d-1}$, $i = 1, \dots, d-1$, then the margins of the limiting distribution with distribution function G in (5.8) *follow the reversely exponential distribution.*

If, conversely, the univariate marginal distribution functions of $H_{\beta(n)}$ belong to the max–domain of attraction of $\exp(x)$, $x \le 0$, *then the limiting functions A_j, $j = 1, \dots, k$, satisfy the properties* (5.11) *and* (5.12).

PROOF. The assertion can directly be deduced by setting $\mathbf{y} = y_i \mathbf{e}_i$, $i = 1, \dots, d$, in (5.7). □

Corollary 5.1.4
Let $H_{\beta(n)}$, $n \in \mathbb{N}$, again be d–variate distribution functions as in Theorem 5.1.1 and assume that the limiting functions A_j, $j = 1, \dots, k$, in (5.5) *satisfy the conditions* (5.11) *and* (5.12). *Then the representation* (5.7) *of the limiting distribution function G is a Pickands representation with Pickands dependence function*

$$D(\mathbf{z}) = 1 + \sum_{j=1}^{k} \lambda_j A_j(\mathbf{z}), \quad \mathbf{z} \in R. \tag{5.13}$$

The Pickands dependence function D in (5.13) describes the residual dependence structure of the limiting distribution in Theorem 5.1.1.

In the proof of Theorem 5.1.1 the Δ–monotony of G has been established by using the fact that G is the pointwise limit of a sequence of distribution functions. However, one may raise the question whether it is possible to identify functions of the form

$$\exp\left(T_2(\mathbf{y}) \left(1 + \sum_{j=1}^{k} \lambda_j A_j(T_1(\mathbf{y})) \right) \right)$$

as distribution functions by regarding them separately. This question is answered to some extent in the following lemma.

Lemma 5.1.5
Let

$$G(y_1, \dots, y_d) = \exp\left(T_2(\mathbf{y}) \left(1 + \sum_{j=1}^{k} \lambda_j A_j(T_1(\mathbf{y})) \right) \right)$$

be a function on $(-\infty, 0]^d$ with $\lambda_j \in \mathbb{R}$ and functions A_j defined on R, $j = 1, \dots, k$. If the A_j satisfy the conditions (5.11) *and* (5.12) *and if there exists a finite measure μ on the d–variate unit simplex S given in* (2.6) *such that*

$$1 + \sum_{j=1}^{k} \lambda_j A_j(z_1, \dots, z_{d-1}) = \int_S \max\left(u_1 z_1, \dots, u_{d-1} z_{d-1}, u_d \left(1 - \sum_{i \le d-1} z_i \right) \right) d\mu(\mathbf{u}) \tag{5.14}$$

for every $\mathbf{z} = (z_1, \dots, z_{d-1}) \in R$, then G is a max–stable distribution function with reversely exponential margins and Pickands dependence function

$$D(\mathbf{z}) = 1 + \sum_{j=1}^{k} \lambda_j A_j(\mathbf{z}), \quad \mathbf{z} \in R.$$

PROOF. The assertion follows directly from Theorem 2.1.2 together with the representation (2.9). The property (2.7) is fulfilled because

$$\int_S u_i\,d\mu(\mathbf{u}) = \begin{cases} 1+\sum_{j=1}^k \lambda_j A_j(\mathbf{e_i}) = 1, & i = 1,\dots,d-1, \\ 1+\sum_{j=1}^k \lambda_j A_j(\mathbf{0}) = 1, & i = d \end{cases}$$

according to (5.11) and (5.12). □

As we intend to investigate the bivariate case more closely, we provide the result of Theorem 5.1.1 for $d = 2$ in the following corollary. Without loss of generality we assume $k = 1$.

Corollary 5.1.6
Let $H_{\beta(n)}$, $n \in \mathbb{N}$, be bivariate distribution functions with support on $(-\infty, 0]^2$ and assume that the pertaining spectral densities satisfy expansions of length 2

$$h_{\beta(n),z}(c) = 1 + B_n(c)A_n(z) + R_n(z,c) \tag{5.15}$$

with $R_n(z,c) = o(B_n(c))$ uniformly for $z \in [0,1]$, as $c \uparrow 0$, according to (3.20), *such that*

$$R_n(z, c/n) \to 0, \quad n \to \infty, \tag{5.16}$$

for every $c < 0$. Assume

$$\beta(n) \to 0, \quad n \to \infty, \tag{5.17}$$

$$B_n(c/n) \to \lambda, \quad n \to \infty, \tag{5.18}$$

for any $c < 0$, where $\lambda \in \mathbb{R}$. Further, suppose

$$A_n(z) \to A(z), \quad n \to \infty, \tag{5.19}$$

uniformly for $z \in [0,1]$, where A is continuous, bounded and satisfies

$$\lambda A(z) \geq -1, \quad z \in [0,1].$$

Then we have

$$H^n_{\beta(n)}\left(\frac{x}{n}, \frac{y}{n}\right) \to \exp\left((x+y)\left(1+\lambda A\left(\frac{x}{x+y}\right)\right)\right), \quad n \to \infty, \tag{5.20}$$

and the limiting function is a bivariate distribution function.

For the special case of a limiting function as in (5.20) the condition (5.21) in Lemma 5.1.5 can be verified by simpler conditions given in the following lemma.

Lemma 5.1.7
Let

$$G(x,y) = \exp\left((x+y)\left(1+\lambda A\left(\frac{x}{x+y}\right)\right)\right)$$

be a function on $(-\infty, 0]^2$ *with* $\lambda \in \mathbb{R}$ *and a function* A *on the interval* $[0,1]$. *If* A *satisfies the condition* $A(1) = A(0) = 0$ *and if there exists a finite measure* ν *on* $[0,1]$ *such that*

$$1 + \lambda A(z) = \int_0^1 \max((1-u)z, u(1-z))\, d\nu(u) \tag{5.21}$$

for every $z \in [0,1]$, *then* G *is a max–stable distribution function with reversely exponential margins and Pickands dependence function*

$$D(z) = 1 + \lambda A(z), \quad z \in [0,1].$$

In particular, condition (5.21) *is fulfilled if* A *is differentiable and its derivative* $A'(z)$ *multiplied with* λ *is right–continuous, non–decreasing and satisfies* $\lambda A'(1) = 1$.

PROOF. The first part of the assertion follows directly from Lemma 2.1.3. The properties (2.7) are fulfilled because of the condition $A(1) = A(0) = 0$ implying

$$\int_0^1 u\, d\nu(u) = 1 + \lambda A(0) = 1$$

and

$$\begin{aligned}\nu([0,1]) &= \int_0^1 u\, d\nu(u) + \int_0^1 (1-u)\, d\nu(u) \\ &= 1 + \lambda A(0) + 1 + \lambda A(1) = 2.\end{aligned}$$

Now, if A is differentiable with $\lambda A'(z)$ being right–continuous and non–decreasing, then

$$M(z) = 1 + \lambda A'(z)$$

is a measure generating function on $[0,1]$. Using $A(0) = 0$ leads to the representation

$$1 + \lambda A(z) = 1 - z + \int_0^z M(x)\, dx, \quad z \in [0,1], \tag{5.22}$$

cf. representation (2.12). The measure ν generated by M is given by $\nu([0,z]) = M(z)$ and satisfies, cf. [10], p. 166,

$$\begin{aligned}\int_0^1 u\, d\nu(u) &= M(1) - \int_0^1 M(x)\, dx \\ &= 1 + \lambda A'(1) - \int_0^1 1 + \lambda A'(x)\, dx \\ &= 1 + \lambda A'(1) - (1 + \lambda A(1) - \lambda A(0)) \\ &= 1, \end{aligned} \tag{5.23}$$

where the last step is due to the assumption $\lambda A'(1) = 1$ and the condition $A(1) = A(0) = 0$. The equality $\lambda A'(1) = 1$ also implies $\nu([0,1]) = 2$. Finally, equation (5.23) is used to form representation (5.22) to

$$1 + \lambda A(z) = \int_0^1 \max((1-u)z, u(1-z))\, d\nu(u)$$

which is condition (5.21). □

5.1.2 Limiting distributions under expansions of Pickands densities

By analogy with Section 5.1.1 we now compute limiting distributions of maxima under triangular schemes requiring that the Pickands densities of the underlying distribution functions satisfy expansions of finite length which depend on the sample size and fulfill certain convergence conditions. For reasons of simplicity we restrict ourselves to the bivariate case. As we are still dealing with tail independence, the leading term of the expansion $\varphi_D(z)$ has to be equal to 0 (which is equivalent to $D(z) = 1$ in the bivariate framework according to (3.11)). Because of Remark 3.2.11 we can further restrict our considerations to expansions of length 2.

Lemma 5.1.8
For each $n \in \mathbb{N}$ let $H_{\beta(n)}$ be the distribution function of a bivariate random vector (X_n, Y_n) whose univariate margins belong to the max–domain of attraction of $G(x) = \exp(x)$, $x \le 0$, and which has the density $h_{\beta(n)}$. Further, assume that the Pickands density $f_{\beta(n)}$ of $H_{\beta(n)}$ satisfies a spectral expansion of length 2

$$f_{\beta(n)}(z, c) = B_n(c)\tilde{A}_n(z) + r_n(z, c), \tag{5.24}$$

with $r_n(z, c) = o(B_n(c))$ uniformly for $z \in [0, 1]$, as $c \uparrow 0$, such that

$$r_n(z, c/n) \to 0, \quad n \to \infty, \tag{5.25}$$

uniformly in z for every $c < 0$. Let

$$B_n(c) = |c|^{\beta(n)} L_n(c), \tag{5.26}$$

L_n being slowly varying, and assume

$$\beta(n) \to 0, \quad n \to \infty, \tag{5.27}$$

$$n^{-\beta(n)} L_n(c/n) \to \lambda, \quad n \to \infty, \tag{5.28}$$

for any $c < 0$, where $\lambda \in \mathbb{R}$. Further, suppose

$$\tilde{A}_n(z) \to \tilde{A}(z), \quad n \to \infty, \tag{5.29}$$

uniformly for $z \in [0, 1]$. Then we have

$$H^n_{\beta(n)}\left(\frac{x}{n}, \frac{y}{n}\right) \to \exp\left(x + y - x\lambda \int_{\frac{x}{x+y}}^1 \frac{1}{u}\tilde{A}(u)\, du - y\lambda \int_{\frac{y}{x+y}}^1 \frac{1}{u}\tilde{A}(1-u)\, du\right), \quad n \to \infty. \tag{5.30}$$

PROOF. Because the univariate margins belong to the max–domain of attraction of the reversely exponential distribution function, we can write

$$\begin{aligned} &H^n_{\beta(n)}\left(\frac{x}{n}, \frac{y}{n}\right) \\ &= \exp\left(-nP\left\{X_n > \frac{x}{n}\right\} - nP\left\{Y_n > \frac{y}{n}\right\} + nP\left\{X_n > \frac{x}{n}, Y_n > \frac{y}{n}\right\}\right) + o(1), \quad n \to \infty, \\ &= \exp\left(x + y + nP\left\{X_n > \frac{x}{n}, Y_n > \frac{y}{n}\right\}\right) + o(1), \quad n \to \infty. \end{aligned} \tag{5.31}$$

The probability $P\left\{X_n > \frac{x}{n}, Y_n > \frac{y}{n}\right\}$ can be written in terms of the density $h_{\beta(n)}$. From the transformation theorem for integrals and Lemma 3.2.1 it follows that

$$\begin{aligned}
&nP\left\{X_n > \frac{x}{n}, Y_n > \frac{y}{n}\right\} = n\int_{\frac{x}{n}}^{0}\int_{\frac{y}{n}}^{0} h_{\beta(n)}(u,v)\,dv\,du \\
&= n\int_{\frac{x+y}{n}}^{0}\int_{0}^{1} f_{\beta(n)}(z,c)\,dz\,dc - n\int_{\frac{x+y}{n}}^{\frac{y}{n}}\int_{0}^{1-\frac{y}{nc}} f_{\beta(n)}(z,c)\,dz\,dc - n\int_{\frac{x+y}{n}}^{\frac{x}{n}}\int_{\frac{x}{nc}}^{1} f_{\beta(n)}(z,c)\,dz\,dc.
\end{aligned}$$

Replacing $f_{\beta(n)}(z,c)$ by its expansion (5.24) and then using the substitution $u := \frac{y}{nc}$ and $u := \frac{x}{nc}$, respectively, we get

$$\begin{aligned}
&nP\left\{X_n > \frac{x}{n}, Y_n > \frac{y}{n}\right\} \\
&= n\int_{\frac{x+y}{n}}^{0} B_n(c)\left(\int_0^1 \tilde{A}_n(z)\,dz\right)dc - n\int_{\frac{x+y}{n}}^{\frac{y}{n}} B_n(c)\left(\int_0^{1-\frac{y}{nc}} \tilde{A}_n(z)\,dz\right)dc \\
&\quad - n\int_{\frac{x+y}{n}}^{\frac{x}{n}} B_n(c)\left(\int_{\frac{x}{nc}}^{1} \tilde{A}_n(z)\,dz\right)dc + S_n(z,c) \\
&= n\int_{\frac{x+y}{n}}^{0} B_n(c)\,dc\left(\int_0^1 \tilde{A}_n(z)\,dz\right) + y\int_{\frac{y}{x+y}}^{1} B_n\left(\frac{y}{n}\cdot\frac{1}{u}\right)\frac{1}{u^2}\left(\int_0^{1-u} \tilde{A}_n(z)\,dz\right)du \\
&\quad + x\int_{\frac{x}{x+y}}^{1} B_n\left(\frac{x}{n}\cdot\frac{1}{u}\right)\frac{1}{u^2}\left(\int_u^1 \tilde{A}_n(z)\,dz\right)du + S_n(z,c),
\end{aligned} \tag{5.32}$$

where the remainder term $S_n(z,c)$ is given by

$$\begin{aligned}
S_n(z,c) &= n\int_{\frac{x+y}{n}}^{0}\int_0^1 r_n(z,c)\,dz\,dc - n\int_{\frac{x+y}{n}}^{\frac{y}{n}}\int_0^{1-\frac{y}{nc}} r_n(z,c)\,dz\,dc \\
&\quad - n\int_{\frac{x+y}{n}}^{\frac{x}{n}}\int_{\frac{x}{nc}}^{1} r_n(z,c)\,dz\,dc \\
&= -(x+y)\int_0^1\int_0^1 r_n\left(z, \frac{x+y}{n}\cdot u\right)dz\,du + y\int_{\frac{y}{x+y}}^{1}\int_0^{1-u} r_n\left(z, \frac{y}{n}\cdot\frac{1}{u}\right)\frac{1}{u^2}\,dz\,du \\
&\quad + x\int_{\frac{x}{x+y}}^{1}\int_u^1 r_n\left(z, \frac{x}{n}\cdot\frac{1}{u}\right)\frac{1}{u^2}\,dz\,du
\end{aligned} \tag{5.33}$$

and converges to zero, as $n \to \infty$, according to the subsequent considerations. Because the convergences $r_n(z,c) \to 0$, as $c \uparrow 0$, and (5.25) are uniform in z, it follows that

$$R_n(a,b,c) := \int_a^b r_n(z,c)\,dz \to 0, \quad c \uparrow 0,$$

and

$$R_n\left(a,b,\frac{c}{n}\right) \to 0, \quad n \to \infty, \tag{5.34}$$

for any $a, b \in \mathbb{R}$ and $c < 0$. Now we will consider the three terms of (5.33) separately and show that each of them goes to zero, as $n \to \infty$.

Let $\epsilon > 0$. Because of (5.34) we can choose a number $N \in \mathbb{N}$ such that

$$\left| R_n \left(0, 1, \frac{x+y}{n} \cdot u \right) \right| < \epsilon \tag{5.35}$$

if $n > Nu$. By taking $n > N$ we achieve that (5.35) is true for every $u \in [0,1]$ which implies

$$\left| \int_0^1 \int_0^1 r_n \left(z, \frac{x+y}{n} \cdot u \right) dz\, du \right| \leq \int_0^1 \left| R_n \left(0, 1, \frac{x+y}{n} \cdot u \right) \right| du < \epsilon.$$

Therefore

$$\int_0^1 \int_0^1 r_n \left(z, \frac{x+y}{n} \cdot u \right) dz\, du \to 0, \quad n \to \infty. \tag{5.36}$$

For the second term of (5.33) we obtain

$$\left| \int_{\frac{y}{x+y}}^1 \int_0^{1-u} r_n \left(z, \frac{y}{n} \cdot \frac{1}{u} \right) \frac{1}{u^2}\, dz\, du \right| \leq \int_{\frac{y}{x+y}}^1 (1-u) \sup_{z \in [0,1]} \left| r_n \left(z, \frac{y}{n} \cdot \frac{1}{u} \right) \right| \cdot \frac{1}{u^2}\, du.$$

Now, the factor $(1-u)\frac{1}{u^2}$ is bounded on the interval $\left[\frac{y}{x+y}, 1\right]$ and the remaining integral $\int_{\frac{y}{x+y}}^1 \sup_{z\in[0,1]} \left| r_n \left(z, \frac{y}{n} \cdot \frac{1}{u} \right) \right| du$ converges to zero by similar arguments as above because the factor $1/u$ in the supremum is also bounded. This leads to

$$\int_{\frac{y}{x+y}}^1 \int_0^{1-u} r_n \left(z, \frac{y}{n} \cdot \frac{1}{u} \right) \frac{1}{u^2}\, dz\, du \to 0, \quad n \to \infty. \tag{5.37}$$

Likewise, we obtain

$$\int_{\frac{x}{x+y}}^1 \int_u^1 r_n \left(z, \frac{x}{n} \cdot \frac{1}{u} \right) \frac{1}{u^2}\, dz\, du \to 0, \quad n \to \infty. \tag{5.38}$$

From (5.36)–(5.38) it follows that the remainder term $S_n(z,c)$ in (5.32) converges to zero, as $n \to \infty$.

Now we consider the leading terms of (5.32). For the first term of (5.32) we compute

$$n \int_{\frac{x+y}{n}}^0 B_n(c)\, dc \int_0^1 \tilde{A}_n(z)\, dz \sim -(x+y) \frac{1}{1+\beta(n)} B_n \left(\frac{x+y}{n} \right) \int_0^1 \tilde{A}_n(z)\, dz,$$

as $n \to \infty$, due to Remark 3.1.4 (i). From the assumptions (5.26)–(5.29) it follows that

$$n \int_{\frac{x+y}{n}}^0 B_n(c)\, dc \int_0^1 \tilde{A}_n(z)\, dz \to -(x+y)\lambda \int_0^1 \tilde{A}(z)\, dz, \quad n \to \infty. \tag{5.39}$$

In particular, we have

$$\begin{aligned} \left| \int_0^1 \tilde{A}(z)\, dz - \int_0^1 \tilde{A}_n(z)\, dz \right| &= \left| \int_0^1 (\tilde{A}(z) - \tilde{A}_n(z))\, dz \right| \\ &\leq \int_0^1 |\tilde{A}(z) - \tilde{A}_n(z)|\, dz \\ &\leq \sup_{z\in[0,1]} |\tilde{A}(z) - \tilde{A}_n(z)| \to 0, \quad n \to \infty, \end{aligned}$$

because the convergence of $\tilde{A}_n$ to $\tilde{A}$ is uniform.

Writing the function B_n in the second term of (5.32) by means of representation (5.26) we get

$$\begin{aligned} B_n\left(\frac{y}{n}\cdot\frac{1}{u}\right) &= \left(\frac{y}{u}\right)^{\beta(n)} n^{-\beta(n)} L_n\left(\frac{y}{n}\cdot\frac{1}{u}\right) \\ &\approx \left(\frac{y}{u}\right)^{\beta(n)} n^{-\beta(n)} L_n(1/n) \\ &\to \lambda, \quad n\to\infty, \end{aligned} \tag{5.40}$$

according to (5.27) and (5.28). The approximation in (5.40) follows from the fact that L_n is slowly varying in 0. Hence we conclude

$$y\int_{\frac{y}{x+y}}^{1} B_n\left(\frac{y}{n}\cdot\frac{1}{u}\right)\frac{1}{u^2}\left(\int_0^{1-u}\tilde{A}_n(z)\,dz\right)du \to y\lambda\int_{\frac{y}{x+y}}^{1}\frac{1}{u^2}\left(\int_0^{1-u}\tilde{A}(z)\,dz\right)du, \tag{5.41}$$

as $n\to\infty$. Similarly,

$$x\int_{\frac{x}{x+y}}^{1} B_n\left(\frac{x}{n}\cdot\frac{1}{u}\right)\frac{1}{u^2}\left(\int_u^{1}\tilde{A}_n(z)\,dz\right)du \to x\lambda\int_{\frac{x}{x+y}}^{1}\frac{1}{u^2}\left(\int_u^{1}\tilde{A}_n(z)\,dz\right)du, \tag{5.42}$$

as $n\to\infty$. The integrals in the expressions (5.41) and (5.42) can be further transformed. We have

$$\begin{aligned} &\int_{\frac{y}{x+y}}^{1}\frac{1}{u^2}\left(\int_0^{1-u}\tilde{A}(z)\,dz\right)du \\ &= -\frac{1}{u}\int_0^{1-u}\tilde{A}(z)\,dz\Big|_{\frac{y}{x+y}}^{1} - \int_{\frac{y}{x+y}}^{1}\frac{1}{u}\tilde{A}(1-u)\,du \\ &= \frac{x+y}{y}\int_0^{\frac{x}{x+y}}\tilde{A}(z)\,dz - \int_{\frac{y}{x+y}}^{1}\frac{1}{u}\tilde{A}(1-u)\,du \end{aligned} \tag{5.43}$$

and

$$\begin{aligned} &\int_{\frac{x}{x+y}}^{1}\frac{1}{u^2}\int_u^{1}\tilde{A}(z)\,dz\,du \\ &= -\frac{1}{u}\int_u^{1}\tilde{A}(z)\,dz\Big|_{\frac{x}{x+y}}^{1} - \int_{\frac{x}{x+y}}^{1}\frac{1}{u}\tilde{A}(u)\,du \\ &= \frac{x+y}{x}\int_{\frac{x}{x+y}}^{1}\tilde{A}(z)\,dz - \int_{\frac{x}{x+y}}^{1}\frac{1}{u}\tilde{A}(u)\,du. \end{aligned} \tag{5.44}$$

Inserting (5.43) and (5.44) into (5.41) and (5.42), respectively, and combining (5.39), (5.41), and (5.42) we receive

$$nP\left\{X_n > \frac{x}{n}, Y_n > \frac{y}{n}\right\} \to -x\lambda\int_{\frac{x}{x+y}}^{1}\frac{1}{u}\tilde{A}(u)\,du - y\lambda\int_{\frac{y}{x+y}}^{1}\frac{1}{u}\tilde{A}(1-u)\,du.$$

Together with (5.31) this proves the assertion. □

Without any further information about the limiting function $\tilde{A}$ in (5.29) the integrals in (5.30) cannot be dissolved.

However, assuming the existence of both a spectral expansion and an expansion of the Pickands density pertaining to a bivariate distribution function as presented in Lemma 3.2.13, we can show that the limiting distributions in Corollary 5.1.6 and Lemma 5.1.8 are identical. In this case we can also dissolve the integrals in (5.30).

Lemma 5.1.9
Let $H_{\beta(n)}$, $n \in \mathbb{N}$, be bivariate distribution functions whose univariate marginal distribution functions belong to the max–domain of attraction of $G(x) = \exp(x)$, $x \leq 0$. Assume that they satisfy spectral expansions as given in Corollary 5.1.6, i.e.

$$h_{\beta(n),z}(c) = 1 + B_n(c)A_n(z) + o(B_n(c)), \quad c \uparrow 0. \tag{5.45}$$

Let A_n be twice continuously differentiable and assume that the convergence (5.19) *also holds for its derivatives. Moreover, assume that the remainder term*

$$R_n(z,c) = h_{\beta(n),z}(c) - 1 - B_n(c)A_n(z)$$

satisfies the convergence property (5.16) *uniformly in z, that it is positive and differentiable in c and its derivative is bounded in z. Then Lemma 5.1.8 can be applied to the Pickands density which exists according to Lemma 3.2.13 (ii) and the resulting limiting distribution function in* (5.30) *is the same as in* (5.20).

PROOF. According to Corollary 5.1.6 we have

$$H^n_{\beta(n)}\left(\frac{x}{n}, \frac{y}{n}\right) \to \exp\left((x+y)\left(1 + \lambda A\left(\frac{x}{x+y}\right)\right)\right), \quad n \to \infty.$$

Due to the conditions imposed on A_n and because the remainder term $R_n(z,c)$ is positive and differentiable, it follows from Lemma 3.2.13 that the Pickands density pertaining to $H_{\beta(n)}$ satisfies the expansion

$$f_{\beta(n)}(z,c) = B_n(c)\tilde{A}_n(z) + o(B_n(c)), \quad c \uparrow 0,$$

with

$$\tilde{A}_n(z) = -\beta(n)A_n(z) - \frac{\beta(n)}{1+\beta(n)}A_n'(z)(1-2z) + \frac{1}{1+\beta(n)}A_n''(z)z(1-z)$$

and the same function B_n as in (5.45). The remainder term

$$r_n(z,c) = f_{\beta(n)}(z,c) - B_n(c)\tilde{A}_n(z)$$

also satisfies the convergence property (5.25) because $R_n(z,c)$ fulfills (5.16) uniformly and its derivative is bounded, cf. again the proof of Theorem 3.4.5 in [16]. Thus the preconditions of Lemma 5.1.8 are fulfilled. In particular, because of the convergence (5.19) for A_n and its derivatives we know

$$\tilde{A}_n(z) \to A''(z)(1-z) =: \tilde{A}(z), \quad n \to \infty.$$

It follows that

$$H^{n}_{\beta(n)}\left(\frac{x}{n},\frac{y}{n}\right)$$
$$\to \exp\left(x+y-x\lambda\int_{\frac{x}{x+y}}^{1}\frac{1}{u}\tilde{A}(u)\,du-y\lambda\int_{\frac{y}{x+y}}^{1}\frac{1}{u}\tilde{A}(1-u)\,du\right),\quad n\to\infty.$$

It remains for us to demonstrate that

$$-x\lambda\int_{\frac{x}{x+y}}^{1}\frac{1}{u}\tilde{A}(u)\,du-y\int_{\frac{y}{x+y}}^{1}\frac{1}{u}\tilde{A}(1-u)\,du=(x+y)\lambda A\left(\frac{x}{x+y}\right). \tag{5.46}$$

For the integrals in (5.46) we compute

$$\begin{aligned}
&\int_{\frac{x}{x+y}}^{1}\frac{1}{u}\tilde{A}(u)\,du\\
&\quad=\int_{\frac{x}{x+y}}^{1}\frac{1}{u}A''(u)(1-u)\,du\\
&\quad=\int_{\frac{x}{x+y}}^{1}A''(u)\,du-\int_{\frac{x}{x+y}}^{1}uA''(u)\,du\\
&\quad=A'(1)-A'\left(\frac{x}{x+y}\right)-uA'(u)\Big|_{\frac{x}{x+y}}^{1}+\int_{\frac{x}{x+y}}^{1}A'(u)\,du\\
&\quad=-A'\left(\frac{x}{x+y}\right)+\frac{x}{x+y}A'\left(\frac{x}{x+y}\right)+A(1)-A\left(\frac{x}{x+y}\right)\\
&\quad=-\frac{y}{x+y}+A(1)-A\left(\frac{x}{x+y}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{y}{x+y}}^{1}\frac{1}{u}\tilde{A}(1-u)\,du\\
&\quad=\int_{\frac{y}{x+y}}^{1}A''(1-u)(1-u)\,du\\
&\quad=\int_{\frac{y}{x+y}}^{1}A''(1-u)\,du-\int_{\frac{y}{x+y}}^{1}uA''(1-u)\,du\\
&\quad=-A'(0)+A'\left(\frac{x}{x+y}\right)+uA'(1-u)\Big|_{\frac{y}{x+y}}^{1}-\int_{\frac{y}{x+y}}^{1}A'(1-u)\,du\\
&\quad=A'\left(\frac{x}{x+y}\right)-\frac{y}{x+y}A'\left(\frac{x}{x+y}\right)+A(0)-A\left(\frac{x}{x+y}\right)\\
&\quad=\frac{x}{x+y}A'\left(\frac{x}{x+y}\right)+A(0)-A\left(\frac{x}{x+y}\right).
\end{aligned}$$

Inserting these results into the left side of equation (5.46) we get

$$\begin{aligned}
&- x\lambda \int_{\frac{x}{x+y}}^{1} \frac{1}{u}\tilde{A}(u)\,du - y\lambda \int_{\frac{y}{x+y}}^{1} \frac{1}{u}\tilde{A}(1-u)\,du \\
&= \lambda \left(\frac{xy}{x+y} A'\left(\frac{x}{x+y}\right) - xA(1) + xA\left(\frac{x}{x+y}\right) \right. \\
&\quad \left. - \frac{xy}{x+y} A'\left(\frac{x}{x+y}\right) - yA(0) + yA\left(\frac{x}{x+y}\right) \right) \\
&= (x+y)\lambda A\left(\frac{x}{x+y}\right) - x\lambda A(1) - y\lambda A(0).
\end{aligned}$$

which is the right side of equation (5.46) because we have either $\lambda = 0$ or both $A(0) = 0$ and $A(1) = 0$ according to Lemma 5.1.3. Thus the assertion follows. □

5.1.3 Convergence rates under spectral expansions

We now consider convergence rates and residual dependence structures for multivariate maxima under triangular schemes. We no longer assume that $\beta(n)$ necessarily converges to zero, as $n \to \infty$, but we consider arbitrary sequences $(\beta(n))_{n\in\mathbb{N}}$ which may have any limit $\beta \geq 0$. To get an overview of the different assumptions imposed on the exponent of variation we list them up in the following lines — together with their area of application.

- $\beta \geq 0$ fixed: tail dependence structures
- β varying with $\beta(n) \to 0$, as $n \to \infty$: limiting distributions of maxima under triangular schemes
- β varying with $\beta(n) \to \beta \geq 0$, as $n \to \infty$: convergence rates and residual dependence structures

The main result within the framework of the third case is captured in the following theorem in which we establish an expansion for the distribution function $H^n_{\beta(n)}$ of the multivariate maxima.

Theorem 5.1.10
Let $H_{\beta(n)}$, $\beta(n) = (\beta_1(n), \ldots, \beta_k(n))$, $n \in \mathbb{N}$, be d–dimensional distribution functions with support on $(-\infty, 0]^d$ and assume that the pertaining spectral densities satisfy expansions of length $k+1$

$$h_{\beta(n),\mathbf{z}}(c) = 1 + \sum_{j=1}^{k} B_{j,n}(c) A_{j,n}(\mathbf{z}) + R_n(\mathbf{z}, c), \quad k \in \mathbb{N}, \tag{5.47}$$

with $R_n(\mathbf{z}, c) = o(B_{k,n}(c))$ uniformly for $\mathbf{z} \in R$, as $c \uparrow 0$, according to (3.4), *such that*

$$R_n(\mathbf{z}, c/n) \to 0, \quad n \to \infty,$$

for every $c < 0$. Let

$$B_{j,n}(c) = |c|^{\beta_j(n)} L_{j,n}(c), \quad j = 1, \ldots, k, \tag{5.48}$$

where the $L_{j,n}$ are slowly varying functions. Then we have

$$\begin{aligned} &H^n_{\beta(n)}\left(\frac{y_1}{n}, \ldots, \frac{y_d}{n}\right) \\ &= \exp(T_2(\mathbf{y})) \exp\left(-\sum_{j=1}^{k} |T_2(\mathbf{y})|^{1+\beta_j(n)} \frac{1}{1+\beta_j(n)} A_{j,n}(T_1(\mathbf{y})) n^{-\beta_j(n)} L_{j,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right) \\ &\quad \times \left(1 + o\left(n^{-\beta_k(n)} L_{k,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right)\right), \quad n \to \infty, \end{aligned} \tag{5.49}$$

with T_1 and T_2 defined in (2.20).

PROOF. From the proof of Theorem 5.1.1 and equation (5.48) we know that $H^n_{\beta(n)}$ can be represented by

$$\begin{aligned} &H^n_{\beta(n)}\left(\frac{y_1}{n}, \ldots, \frac{y_d}{n}\right) \\ &= \exp(T_2(\mathbf{y})) \exp\left(\sum_{j=1}^{k} T_2(\mathbf{y}) \frac{1}{1+\beta_j(n)} A_{j,n}(T_1(\mathbf{y})) B_{j,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right) \\ &\quad \times \left(1 + o\left(B_{k,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right)\right), \quad n \to \infty, \\ &= \exp(T_1(\mathbf{y})) \exp\left(-\sum_{j=1}^{k} |T_2(\mathbf{y})|^{1+\beta_j(n)} \frac{1}{1+\beta_j(n)} A_{j,n}(T_1(\mathbf{y})) n^{-\beta_j(n)} L_{j,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right) \\ &\quad \times \left(1 + o\left(n^{-\beta_j(n)} L_{k,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right)\right), \quad n \to \infty. \end{aligned}$$

□

Remark 5.1.11
Representation (5.47) *in Theorem 5.1.10 is meant to hold for a particular sequence $(\beta(n))_{n\in\mathbb{N}}$. This condition can be strengthened by assuming that H_β satisfies a spectral expansion for every sequence $\beta = (\beta(n))_{n\in\mathbb{N}}$. We will show later that there are distribution functions which fulfill condition* (5.47) *only for special sequences $(\beta(n))_{n\in\mathbb{N}}$, cf. Examples 5.1.17 and 5.1.18.*

Representation (5.49) shows that the convergence rate of the multivariate maxima is essentially determined by the convergences rates of the terms

$$n^{-\beta_j(n)} L_{j,n}\left(T_2(\mathbf{y})/n\right) \tag{5.50}$$

for $j = 1, \ldots, k$, cf. also [40], p. 234. As (5.50) constitutes a set of convergences, we speak of a field of convergence.

In the asymptotic expansion (5.49) the factor

$$\exp\left(-\sum_{j=1}^{k}|T_2(\mathbf{y})|^{1+\beta_j(n)}\frac{1}{1+\beta_j(n)}A_{j,n}(T_1(\mathbf{y}))n^{-\beta_j(n)}L_{j,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right)$$

represents the residual dependence structure of the distribution functions $H^n_{\beta(n)}$ of the maxima.

We have, obviously,

$$H^n_{\beta(n)}\left(\frac{y_1}{n},\dots,\frac{y_d}{n}\right)\to\exp\left(\sum_{i\le d}y_i\right)\quad n\to\infty,$$

if

$$n^{-\beta_j(n)}L_{j,n}\left(T_2(\mathbf{y})/n\right)\to 0,\quad n\to\infty,\ \text{for every } j\in\{1,\dots,k\}. \tag{5.51}$$

Therefore the marginal maxima are independent if condition (5.51) is fulfilled and the rate at which this independence is attained is essentially determined by the terms $n^{-\beta_j(n)}L_{j,n}\left(T_2(\mathbf{y})/n\right)$, cf. also [40], p. 236.

By dividing the field of convergence (5.50) into two subfields where the limits are either equal to zero or not we can modify expansion (5.49). Note that the convergence condition in (5.51) is particularly true for every j with $\beta_j(n)\equiv\beta_j$, i.e., $\beta_j(n)$ does not depend on n.

Corollary 5.1.12
Let $H_{\beta(n)}$, $n\in\mathbb{N}$, be d–variate distribution functions as given in Theorem 5.1.10 with

$$\beta_j(n)\to\beta_j\in\mathbb{R},\quad n\to\infty,$$
$$n^{-\beta_j(n)}L_{j,n}\left(c/n\right)\to\lambda_j\in\mathbb{R},\quad n\to\infty,$$

for $j=1,\dots,k$ and define

$$J:=\{j\in\{1,\dots,k\}:|\lambda_j|>0\}.$$

Further, suppose

$$A_{j,n}(\mathbf{z})\to A_j(\mathbf{z}),\quad n\to\infty,$$

uniformly for $\mathbf{z}\in R$. Then we have

$$H^n_{\beta(n)}\left(\frac{y_1}{n},\dots,\frac{y_d}{n}\right)=\exp(T_2(\mathbf{y}))\exp\left(-\sum_{j\in J}|T_2(\mathbf{y})|^{1+\beta_j}\frac{\lambda_j}{1+\beta_j}A_j(T_1(\mathbf{y}))\right)$$
$$\times\left(1+\sum_{j\in\{1,\dots,k\}\setminus J}O\left(n^{-\beta_j(n)}L_{j,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right)\right),\quad n\to\infty, \tag{5.52}$$

if $|J|<k$, and otherwise

$$H^n_{\beta(n)}\left(\frac{y_1}{n},\dots,\frac{y_d}{n}\right)=\exp(T_2(\mathbf{y}))\exp\left(-\sum_{j=1}^{k}|T_2(\mathbf{y})|^{1+\beta_j}\frac{\lambda_j}{1+\beta_j}A_j(T_1(\mathbf{y}))\right)$$
$$\times\left(1+o(1)\right),\quad n\to\infty.$$

If we impose an additional assumption on the slowly varying functions $L_{j,n}$, the expansion (5.52) can be further simplified.

Corollary 5.1.13
Let $H_{\beta(n)}$, $n \in \mathbb{N}$, be d–variate distribution functions as given in Theorem 5.1.10 with

$$\beta_j(n) \to \beta_j \in \mathbb{R}, \quad n \to \infty,$$
$$n^{-\beta_j(n)} L_{j,n}(c/n) \to \lambda_j \in \mathbb{R}, \quad n \to \infty,$$

for $j = 1, \dots, k$ and assume

$$|L_{r,n}(c)| \leq |L_{s,n}(c)| \quad \text{for all } n \in \mathbb{N}, c < 0, \tag{5.53}$$

if $r > s$. Then the set J in Corollary 5.1.12 is given by $J = \{1, \dots, \kappa\}$ with

$$\kappa := \max\{j \in \{1, \dots, k\} : |\lambda_j| > 0\}$$

if $J \neq \emptyset$. In case $J = \emptyset$ define $\kappa := 0$. Further, suppose

$$A_{j,n}(\mathbf{z}) \to A_j(\mathbf{z}), \quad n \to \infty,$$

uniformly for $\mathbf{z} \in R$. Then we have

$$H^n_{\beta(n)}\left(\frac{y_1}{n}, \dots, \frac{y_d}{n}\right) = \exp(T_2(\mathbf{y})) \exp\left(-\sum_{j=1}^{\kappa} |T_2(\mathbf{y})|^{1+\beta_j} \frac{\lambda_j}{1+\beta_j} A_j(T_1(\mathbf{y}))\right)$$
$$\times \left(1 + O\left(n^{-\beta_{\kappa+1}(n)} L_{\kappa+1,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right)\right), \quad n \to \infty, \tag{5.54}$$

if $\kappa < k$, and otherwise

$$H^n_{\beta(n)}\left(\frac{y_1}{n}, \dots, \frac{y_d}{n}\right) = \exp(T_2(\mathbf{y})) \exp\left(-\sum_{j=1}^{k} |T_2(\mathbf{y})|^{1+\beta_j} \frac{\lambda_j}{1+\beta_j} A_j(T_1(\mathbf{y}))\right)$$
$$\times (1 + o(1)), \quad n \to \infty.$$

PROOF. Recall that $\beta_r(n) > \beta_s(n)$, $n \in \mathbb{N}$, if $r > s$, according to Definition 3.1.3. Together with inequality (5.53) this implies

$$\left|n^{-\beta_r(n)} L_{r,n}(c/n)\right| \leq \left|n^{-\beta_s(n)} L_{s,n}(c/n)\right| \tag{5.55}$$

and, thus,

$$|\lambda_r| = \lim_{n\to\infty} \left|n^{-\beta_r(n)} L_{r,n}(c/n)\right| \leq \lim_{n\to\infty} \left|n^{-\beta_s(n)} L_{s,n}(c/n)\right| = |\lambda_s|.$$

From this inequality we can deduce the asserted structure of the set J and the representation

$$H^n_{\beta(n)}\left(\frac{y_1}{n},\dots,\frac{y_d}{n}\right) = \exp(T_2(\mathbf{y}))\exp\left(\sum_{j=1}^{\kappa} T_2(\mathbf{y})\lambda_j A_j(T_1(\mathbf{y}))\right) \times\left(1+\sum_{j=\kappa+1}^{k} O\left(n^{-\beta_j(n)}L_{j,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right)\right),\quad n\to\infty,$$

if $\kappa < k$. Inequality (5.55) further implies

$$O\left(n^{-\beta_j(n)}L_{j,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right) = O\left(n^{-\beta_{\kappa+1}(n)}L_{\kappa+1,n}\left(\frac{T_2(\mathbf{y})}{n}\right)\right),\quad j\geq \kappa+1,$$

which, finally, leads to expansion (5.54).

As the representation in the case $\kappa = k$ does not differ from (5.54), the proof is complete. □

Remark 5.1.14
If we consider a spectral expansion of length 2*, i.e.* $k = 1$ *in* (5.47)*, Corollary 5.1.12 and Corollary 5.1.13 are the same and we obtain*

$$H^n_{\beta(n)}\left(\frac{y_1}{n},\dots,\frac{y_d}{n}\right) = \exp(T_2(\mathbf{y}))\left(1+O\left(n^{-\beta(n)}L_n\left(\frac{T_2(\mathbf{y})}{n}\right)\right)\right),\quad n\to\infty,$$

if $\lambda = 0$*, and otherwise*

$$H^n_{\beta(n)}\left(\frac{y_1}{n},\dots,\frac{y_d}{n}\right) = \exp(T_2(\mathbf{y}))\exp\left(T_2(\mathbf{y})\lambda A\left(T_1(\mathbf{y})\right)\right)\left(1+o(1)\right),\quad n\to\infty.$$

5.1.4 Examples

In this section we will present different examples of bivariate distribution functions which satisfy spectral expansions as given in (5.47) and compute expansions for the distribution functions of the maxima, thereby considering their residual dependence structure.

First we present examples of distribution functions which do not belong to the class of elliptical distributions.

Example 5.1.15
Let H_α *be the mixture distribution function of Example 3.3.1 and let the parameter* α *of the univariate beta distribution functions vary in* n*, i.e.* $\alpha = \alpha(n) < -1$*. The spectral expansion* (3.26) *is fulfilled for every sequence* $(\alpha(n))_{n\in\mathbb{N}}$*. Thus we get*

$$h_{\alpha(n),z}(c) = 1 + B_{1,n}(c)A_{1,n}(z) + B_{2,n}(c)A_{2,n}(z)$$

with

$$B_{1,n}(c) = |c|^{-\alpha(n)-1},\quad B_{2,n}(c) = |c|^{-2\alpha(n)-1}$$

and

$$A_{1,n}(z) = -\alpha(n)(1-p)p^{\alpha(n)}\left(z^{-\alpha(n)} + (1-z)^{-\alpha(n)}\right),$$
$$A_{2,n}(z) = 2\alpha(n)(1-p)p^{2\alpha(n)}(z(1-z))^{-\alpha(n)}$$

for every member $\alpha(n)$ of any sequence $(\alpha(n))_{n\in\mathbb{N}}$. The exponents of variation are given by

$$\beta_1(n) = -\alpha(n) - 1$$

and

$$\beta_2(n) = -2\alpha(n) - 1.$$

From Theorem 5.1.10 it follows that

$$\begin{aligned} H^n_{\alpha(n)}\left(\frac{x}{n},\frac{y}{n}\right) &= \exp(x+y) \\ &\times \exp\left(-|x+y|^{-\alpha(n)}(1-p)p^{\alpha(n)}\left(\left(\frac{x}{x+y}\right)^{-\alpha(n)} + \left(\frac{y}{x+y}\right)^{-\alpha(n)}\right)n^{\alpha(n)+1}\right. \\ &\left. + |x+y|^{-2\alpha(n)}\frac{1}{2}(1-p)p^{2\alpha(n)}\left(\frac{xy}{(x+y)^2}\right)^{-\alpha(n)} n^{2\alpha(n)+1}\right). \end{aligned}$$

Now let $\alpha(n) \to -1$, $n \to \infty$, such that

$$(\alpha(n)+1)\log(n) \to \xi \in [-\infty, 0], \quad n \to \infty.$$

This implies $\beta_1(n) \to 0$ and $\beta_2(n) \to 1$ as well as

$$n^{\alpha(n)+1} \to \exp(\xi) =: \lambda \in [0,1]$$

and

$$n^{2\alpha(n)+1} \to 0,$$

as $n \to \infty$. Therewith we obtain the representation

$$\begin{aligned} H^n_{\alpha(n)}\left(\frac{x}{n},\frac{y}{n}\right) &= \exp(x+y) \\ &\times \exp\left(-|x+y|^{-\alpha(n)}(1-p)p^{\alpha(n)}\left(\left(\frac{x}{x+y}\right)^{-\alpha(n)} + \left(\frac{y}{x+y}\right)^{-\alpha(n)}\right)n^{\alpha(n)+1}\right) \\ &\times \left(1 + O\left(n^{2\alpha(n)+1}\right)\right), \quad n \to \infty, \end{aligned}$$

which, according to Remark 5.1.14, can be written as

$$H^n_{\alpha(n)}\left(\frac{x}{n},\frac{y}{n}\right) = \exp(x+y)\left(1 + O\left(n^{\alpha(n)+1}\right)\right), \quad n \to \infty,$$

if $\lambda = 0$, and otherwise

$$H^n_{\alpha(n)}\left(\frac{x}{n}, \frac{y}{n}\right) = \exp(x+y)\exp((x+y)(1-p)/p\lambda)(1+o(1)), \quad n \to \infty.$$

Obviously, the limiting distribution function G of $H^n_{\alpha(n)}(x/n, y/n)$, as $n \to \infty$, is given by the EVD

$$G(x,y) = \exp(x+y), \quad x, y \le 0,$$

with scale parameter 1 *if $\lambda = 0$ or with scale parameter*

$$\frac{p}{(1-p)\lambda}$$

if $\lambda \in (0,1]$. Thus in each case we have independence in the limit.

Example 5.1.16
Let H_α now be the mixture distribution function of Example 3.3.2 with varying parameter $\alpha = \alpha(n) < -1$. The asymptotic calculations leading to the spectral expansion of length 2 *given in Example 3.3.2 are valid for any sequence $(\alpha(n))_{n\in\mathbb{N}}$ and we obtain*

$$h_{\alpha(n),z}(c) = 1 + |c|^{-\alpha(n)-1}(-\alpha(n))(1-p)p^{\alpha(n)}\left(z^{-\alpha(n)\gamma} + (1-z)^{-\alpha(n)\gamma}\right)^{1/\gamma} + R_n(z,c)$$

with $R_n(z,c) = o\left(|c|^{-\alpha(n)-1}\right)$, as $c \uparrow 0$, and

$$R_n(z, c/n) \to 0, \quad n \to \infty,$$

for every $c < 0$ and $\gamma \ge 1$. The exponent of variation is given by

$$\beta(n) = -\alpha(n) - 1.$$

From Theorem 5.1.10 it follows that

$$\begin{aligned} H^n_{\alpha(n)}\left(\frac{x}{n}, \frac{y}{n}\right) &= \exp(x+y) \\ &\times \exp\left(-|x+y|^{-\alpha(n)}(1-p)p^{\alpha(n)}\left(\left(\frac{x}{x+y}\right)^{-\alpha(n)\gamma} + \left(\frac{y}{x+y}\right)^{-\alpha(n)\gamma}\right)^{1/\gamma} n^{\alpha(n)+1}\right) \\ &\times \left(1 + o\left(n^{\alpha(n)+1}\right)\right), \quad n \to \infty. \end{aligned}$$

Now let $\alpha(n) \to -1$, $n \to \infty$, such that

$$(\alpha(n)+1)\log(n) \to \xi \in [-\infty, 0], \quad n \to \infty.$$

This implies $\beta(n) \to 0$ and

$$n^{\alpha(n)+1} \to \exp(\xi) =: \lambda \in [0,1],$$

as $n \to \infty$. According to Remark 5.1.14 we can write

$$H^n_{\alpha(n)}\left(\frac{x}{n}, \frac{y}{n}\right) = \exp(x+y)\left(1 + O\left(n^{\alpha(n)+1}\right)\right),$$

as $n \to \infty$, if $\lambda = 0$, and otherwise

$$H^n_{\alpha(n)}\left(\frac{x}{n}, \frac{y}{n}\right)$$
$$= \exp(x+y)\exp\left((x+y)(1-p)/p\left(\left(\frac{x}{x+y}\right)^\gamma + \left(\frac{y}{x+y}\right)^\gamma\right)^{1/\gamma}\lambda\right)(1+o(1)),$$

as $n \to \infty$.

If $\lambda = 0$ or if $\lambda > 0$ and $\gamma = 1$, we have the same situation as in Example 5.1.15. Therefore assume $\lambda > 0$ and $\gamma > 1$. Then the limiting function is given by

$$G(x,y) = \exp\left((x+y)\left(1 + \lambda A\left(\frac{x}{x+y}\right)\right)\right)$$

with

$$A(z) = \frac{1-p}{p}\left(z^\gamma + (1-z)^\gamma\right)^{1/\gamma}, \quad z \in [0,1].$$

We have, obviously, $A(0) = A(1) = (1-p)/p$ which means that we cannot directly imply Lemma 5.1.7. Nevertheless, we are able to prove the assertion in a different way. Firstly we introduce a scale parameter

$$\sigma := \lambda\frac{1-p}{p} + 1$$

such that the limiting distribution function has standard reversely exponential margins. We have

$$H^n_{\alpha(n)}\left(\frac{x}{\sigma n}, \frac{y}{\sigma n}\right)$$
$$\to \exp\left((x+y)\frac{1}{\sigma}\left(1 + \lambda\frac{1-p}{p}\left(\left(\frac{x}{x+y}\right)^\gamma + \left(\frac{y}{x+y}\right)^\gamma\right)^{1/\gamma}\right)\right), \quad n \to \infty,$$
$$= \exp\left((x+y)D_\sigma\left(\frac{x}{x+y}\right)\right)$$
$$=: G_\sigma(x,y), \tag{5.56}$$

where D_σ is the convex combination of the Pickands dependence function $D_1 = 1$ and the Gumbel dependence function

$$D_2(z) = \left(z^\gamma + (1-z)^\gamma\right)^{1/\gamma},$$

cf. [10], p. 167, namely

$$D_\sigma = \frac{1}{\sigma}D_1 + \left(1 - \frac{1}{\sigma}\right)D_2.$$

Hence, according to Lemma 2.1.4 (v), D_σ is also a Pickands dependence function. Now equation (5.56) *shows that G_σ has a Pickands representation and it follows from Lemma 2.1.3 that G_σ is a max–stable distribution function with reversely exponential margins.*

The pertaining measure generating function M is given by

$$\begin{aligned} M(z) &= 1 + D'(z) \\ &= 1 + \left(1 - \frac{1}{\sigma}\right) \frac{z^{\gamma-1} - (1-z)^{\gamma-1}}{(z^\gamma + (1-z)^\gamma)^{1-1/\gamma}}, \quad z \in [0,1), \\ M(1) &= 2. \end{aligned}$$

The corresponding measure ν has mass $1/\sigma$ at each of the points 0 *and* 1 *and mass $2 - 2/\sigma$ on the interval $(0,1)$.*

Example 5.1.17

Consider again the joint distribution function H_θ in the lower tail of the Crowder distribution with $[-1,0]$–uniform margins as in Example 3.3.3. This time let the parameter θ vary in n, i.e. $\theta = \theta(n) \geq 1$. The parameter α remains fixed and non–negative. Thus we have to deal with sequences of spectral expansions

$$h_{\theta(n),z}(c) = 1 + B_n(c)A_n(z) + o(B_n(c)), \quad c \uparrow 0,$$

with

$$B_n(c) = |c|^{2^{1/\theta(n)}-1} L_n(c),$$

where

$$L_n(c) = \exp\left(\alpha\left(1 - 2^{1/\theta(n)}\right) + \frac{\alpha^{\theta(n)} 2^{1/\theta(n)-1}}{\theta(n)(\log|c|)^{\theta(n)-1}}\right)$$

and

$$A_n(z) = 2^{1/\theta(n)}(z(1-z))^{2^{1/\theta(n)-1}}.$$

Detailed calculations in the derivation of expansion (5.1.14) *in [17], Example 4, show that the remainder term of $h_{n,z}(c/n)$ converges to* 0*, as $n \to \infty$, only if $\theta(n) \equiv \theta$, i.e., $(\theta(n))_{n\in\mathbb{N}}$ is a constant sequence. Otherwise, one has to deal with a different representation of $h_{n,z}(c)$, cf. Example 5.2.18. Therefore we first restrict ourselves to the case that $(\theta(n))_{n\in\mathbb{N}}$ is constant. The exponent of variation $\beta(n)$ is given by*

$$\beta(n) = 2^{1/\theta(n)} - 1.$$

Because the term

$$\frac{\alpha^{\theta(n)} 2^{1/\theta(n)-1}}{\theta(n)(\log|c|)^{\theta(n)-1}}$$

converges to zero, we can write, according to Theorem 5.1.10,

$$\begin{aligned} &H^n_{\theta(n)}\left(\frac{x}{n},\frac{y}{n}\right) \\ &= \exp(x+y)\exp\left(-|x+y|^{2^{1/\theta(n)}}\left(\frac{xy}{(x+y)^2}\right)^{2^{1/\theta(n)-1}} n^{-2^{1/\theta(n)}+1}\exp\left(\alpha\left(1-2^{1/\theta(n)}\right)\right)\right) \\ &\quad\times\left(1+o\left(n^{-2^{1/\theta(n)}+1}\right)\right), \quad n\to\infty. \end{aligned}$$

Because of $\theta(n)\equiv\theta$ the factor $n^{-2^{1/\theta(n)}+1}$ goes to zero, as $n\to\infty$, and we get

$$H^n_{\theta(n)}\left(\frac{x}{n},\frac{y}{n}\right) = \exp(x+y)\left(1+O\left(n^{-2^{1/\theta(n)}+1}\right)\right), \quad n\to\infty.$$

The limiting function G of $H^n_{\theta(n)}(x/n,y/n)$, as $n\to\infty$, is given by

$$G(x,y)=\exp(x+y), \quad x,y\le 0,$$

which represents the case of independence.

The next example concerns the bivariate standard normal distribution with $[-1,0]$–uniform margins and correlation coefficient ρ which varies with the sample size. We will see that Theorem 5.1.10 is applicable only for special series $(\rho(n))_{n\in\mathbb{N}}$. In other cases, in particular, if

$$(1-\rho(n))\log(n)\to\lambda^2\in[0,\infty), \quad n\to\infty,$$

we will use a different approach in Section 5.2 that leads to the Hüsler–Reiss distribution function, cf. [29].

Example 5.1.18
Let H_ρ be the bivariate standard normal distribution function with $[-1,0]$–uniform margins as given in Example 3.3.4, but let the correlation coefficient vary in n, i.e. $\rho=\rho(n)\in(0,1)$. In this example, we only consider sequences $(\rho(n))_{n\in\mathbb{N}}$ which lead to spectral expansions

$$h_{\rho(n),z}(c)=1+B_n(c)A_n(z)+R_n(z,c) \tag{5.57}$$

with $R_n(z,c)=o(B_n(c))$ uniformly for $z\in[0,1]$, as $c\uparrow 0$, such that

$$R_n(z,c/n)\to 0, \quad n\to\infty, \tag{5.58}$$

for every $c<0$ and

$$B_n(c)=|c|^{2/(1+\rho(n))-1}L_n(c),$$

where

$$L_n(c)=(1+\rho(n))^{3/2}(1-\rho(n))^{-1/2}(4\pi)^{-\rho(n)/(1+\rho(n))}(-\log|c|)^{-\rho(n)/(1+\rho(n))}$$

and

$$A_n(z) = -\frac{2}{1+\rho(n)}(z(1-z))^{1/(1+\rho(n))}.$$

The exponent of variation is given by

$$\beta(n) = \frac{2}{1+\rho(n)} - 1 = \frac{1-\rho(n)}{1+\rho(n)}.$$

According to Theorem 5.1.10, we can write

$$\begin{aligned}
&H^n_{\rho(n)}\left(\frac{x}{n}, \frac{y}{n}\right) \\
&= \exp(x+y)\exp\Bigg(-|x+y|^{2/(1+\rho(n))}\left(\frac{xy}{(x+y)^2}\right)^{1/(1+\rho(n))} n^{-(1-\rho(n))/(1+\rho(n))} \\
&\quad \times (1+\rho(n))^{3/2}(1-\rho(n))^{-1/2}(4\pi)^{-\rho(n)/(1+\rho(n))}\left(\log(n) - \log|x+y|\right)^{-\rho(n)/(1+\rho(n))}\Bigg) \\
&\quad \times \left(1 + o\left(n^{-(1-\rho(n))/(1+\rho(n))}(\log(n))^{-\rho(n)/(1+\rho(n))}\right)\right), \quad n \to \infty.
\end{aligned}$$

Following [29], condition (2.8), let $\rho(n) \to 1$, as $n \to \infty$, such that

$$(1-\rho(n))\log(n) \to \lambda^2 \in [0,\infty], \quad n \to \infty. \tag{5.59}$$

We will show later that $h_{n,z}(c/n)$ satisfies the expansion (5.57) *with condition* (5.58) *if $\lambda = \infty$. This is particularly the case if $\rho(n) \equiv \rho$, i.e., $(\rho(n))_{n\in\mathbb{N}}$ is a constant sequence. It follows that the factor*

$$n^{-(1-\rho(n))/(1+\rho(n))}(\log(n))^{-\rho(n)/(1+\rho(n))}$$

goes to zero, as $n \to \infty$. Thus we obtain

$$H^n_{\rho(n)}\left(\frac{x}{n}, \frac{y}{n}\right) = \exp(x+y)\left(1 + O\left(n^{-(1-\rho(n))/(1+\rho(n))}(\log(n))^{-\rho(n)/(1+\rho(n))}\right)\right), \quad n \to \infty,$$

cf. [40], Example 7.2.7. As in the case of the Crowder distribution we get independence in the limit.

5.2 Limiting distributions under a generalized condition

In Example 5.1.18 we have seen that we cannot generally deduce a spectral expansion $h_{\rho(n),z}(c/n)$ for the bivariate standard normal distribution with varying correlation coefficients $\rho(n)$, $n \in \mathbb{N}$, as n goes to infinity. However, as we will see later, there is still some structure in the representation of the spectral density. This causes us to establish a more general representation of the spectral density and, thus, a generalized condition for the convergence of distribution functions of maxima under triangular schemes.

Before we introduce the mentioned condition, we first set up some notations: For a vector $\mathbf{z} = (z_1, \dots, z_{d-1}) \in R$ with R as defined in (2.10) we set

$$\begin{aligned}
z_d &:= 1 - \sum_{i \le d-1} z_i, \\
\mathbf{z}^{(1)} &:= (z_1, z_2, \dots, z_{d-1}) \in R, \\
\mathbf{z}^{(i)} &:= (z_i, z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{d-1}) \in R, \quad i = 2, \dots, d-2, \\
\mathbf{z}^{(d-1)} &:= (z_{d-1}, z_1, \dots, z_{d-2}) \in R
\end{aligned}$$

and

$$\mathbf{z}^{(d)} := (z_d, z_1, \dots, z_{d-2}) \in R.$$

Another preliminary step shall be the deduction of an exact representation of a spectral density. Let therefore $(X_1, \dots, X_d)$ be a d–variate random vector with distribution function H and density h and let H_i be the univariate marginal distribution functions with densities h_i, $i = 1, \dots, d$. After a transformation to $[-1, 0]$–uniform margins we obtain the spectral distribution function

$$\begin{aligned}
H_{\mathbf{z}}(c) &= H\left(c\left(z_1, \dots, z_{d-1}, 1 - \sum_{i \le d-1} z_i\right)\right) \\
&= \int_{-\infty}^{H_1^{-1}(1+cz_1)} \cdots \int_{-\infty}^{H_d^{-1}(1+cz_d)} h(x_1, \dots, x_d)\, dx_d \cdots dx_1
\end{aligned}$$

whose derivative with respect to c, which is the spectral density, can be obtained by applying the multidimensional chain rule, cf. Theorem 41 in Erwe [8]. We have

$$\begin{aligned}
h_{\mathbf{z}}(c) &= z_1 \int_{-\infty}^{H_2^{-1}(1+cz_2)} \cdots \int_{-\infty}^{H_d^{-1}(1+cz_d)} h\left(H_1^{-1}(1+cz_1), x_2, \dots, x_d\right) dx_d \cdots dx_2 \\
&\quad \times \frac{1}{h_1\left(H_1^{-1}(1+cz_1)\right)} \\
&\quad + \cdots \\
&\quad + z_d \int_{-\infty}^{H_1^{-1}(1+cz_1)} \cdots \int_{-\infty}^{H_{d-1}^{-1}(1+cz_{d-1})} h\left(x_1, \dots, x_{d-1}, H_d^{-1}(1+cz_d)\right) dx_{d-1} \cdots dx_1 \\
&\quad \times \frac{1}{h_d\left(H_d^{-1}(1+cz_d)\right)} \\
&= z_1 \int_{-\infty}^{H_2^{-1}(1+cz_2)} \cdots \int_{-\infty}^{H_d^{-1}(1+cz_d)} h\left(x_2, \dots, x_d | H_1^{-1}(1+cz_1)\right) dx_d \cdots dx_2 \\
&\quad + \cdots \qquad\qquad (5.60) \\
&\quad + z_d \int_{-\infty}^{H_1^{-1}(1+cz_1)} \cdots \int_{-\infty}^{H_{d-1}^{-1}(1+cz_{d-1})} h\left(x_1, \dots, x_{d-1} | H_d^{-1}(1+cz_d)\right) dx_{d-1} \cdots dx_1
\end{aligned}$$

$$
\begin{aligned}
&= z_1 P\left(X_2 \le H_2^{-1}(1+cz_2),\dots,X_d \le H_d^{-1}(1+cz_d)|X_1 = H_1^{-1}(1+cz_1)\right)\\
&+\cdots\\
&+ z_d P\left(X_1 \le H_1^{-1}(1+cz_1),\dots,X_{d-1} \le H_{d-1}^{-1}(1+cz_{d-1})|X_d = H_d^{-1}(1+cz_d)\right). \quad (5.61)
\end{aligned}
$$

In what follows we assume that the random variables $X_1,\dots,X_d$ are exchangeable, cf. Lemma 2.1.4 (iv), which implies that all univariate margins and all conditional distribution functions are identical. Moreover, we assume that the $(d-1)$–variate conditional distribution functions are again unconditional (univariate) distribution functions, cf. [40], Example 7.2.7. In particular, we suppose that representation (5.61) can be rewritten in the following way.

Because the conditional distribution functions are identical we can write

$$
\begin{aligned}
h_{\mathbf{z}}(c) &= z_1 \tilde{F}\left(H_2^{-1}(1+cz_2),\dots,H_d^{-1}(1+cz_d),H_1^{-1}(1+cz_1)\right)\\
&+\cdots\\
&+ z_d \tilde{F}\left(H_1^{-1}(1+cz_1),\dots,H_{d-1}^{-1}(1+cz_{d-1}),H_d^{-1}(1+cz_d)\right),
\end{aligned}
\tag{5.62}
$$

where $\tilde{F}$ is a distribution function in the first $d-1$ components. Because we assume that $\tilde{F}$ is again an unconditional univariate distribution function, there exists a function $\tilde{g}:\mathbb{R}^d \to \mathbb{R}$ and a univariate distribution function F such that

$$
\tilde{F} = F \circ \tilde{g}. \tag{5.63}
$$

Inserting (5.63) into (5.62) we get

$$
\begin{aligned}
h_{\mathbf{z}}(c) &= z_1 F\left(\tilde{g}\left(H_2^{-1}(1+cz_2),\dots,H_d^{-1}(1+cz_d),H_1^{-1}(1+cz_1)\right)\right)\\
&+\cdots\\
&+ z_d F\left(\tilde{g}\left(H_1^{-1}(1+cz_1),\dots,H_{d-1}^{-1}(1+cz_{d-1}),H_d^{-1}(1+cz_d)\right)\right).
\end{aligned}
$$

Finally, we set

$$
g(z_1,\dots,z_{d-1},c) := \tilde{g}\left(H_2^{-1}(1+cz_2),\dots,H_d^{-1}(1+cz_d),H_1^{-1}(1+cz_1)\right) \tag{5.64}
$$

for a function $g:[0,1]^{d-1}\times(-\infty,0] \to \mathbb{R}$, where again $z_d = 1-\sum_{i\le d-1} z_i$. Thus we obtain the representation

$$
h_{\mathbf{z}}(c) = z_1 F(g(z_1,z_2,\dots,z_{d-1},c)) + \cdots + z_d F(g(z_d,z_1,\dots,z_{d-2},c))
$$

for the spectral density, which is picked up in the following condition.

Condition 5.2.1
Let H be a d–variate distribution function and assume that its spectral density satisfies

$$
h_{\mathbf{z}}(c) = z_1 F(g(z_1,z_2,\dots,z_{d-1},c)) + \cdots + z_d F(g(z_d,z_1,\dots,z_{d-2},c)), \tag{5.65}
$$

where F is a univariate distribution function and $g:[0,1]^{d-1}\times(-\infty,0)\to\mathbb{R}$ any measurable function.

In the bivariate case, representation (5.65) can be expressed as

$$h_z(c) = zF(g(z,c)) + (1-z)F(g(1-z,c)) \tag{5.66}$$

with g mapping from $[0,1] \times (-\infty, 0)$ to $\mathbb{R}$.

Under some additional assumptions for the functions F and g we again obtain an expansion for $h_{\mathbf{z}}(c)$ in (5.65). If F is the uniform distribution function on $[-1,0]$, i.e.

$$F(u) = 1 + u, \quad u \in [-1,0],$$

and g satisfies the expansion

$$g\left(\mathbf{z}^{(i)}, c\right) = \sum_{j=1}^{k} B_j(c) A_j\left(\mathbf{z}^{(i)}\right) + o(B_k(c)), \tag{5.67}$$

as $c \uparrow 0$, where the B_j are regularly varying functions fulfilling conditions (3.5) and (3.6), then (5.65) becomes a spectral expansion of length $k+1$ in the sense of Definition 3.1.3, i.e.

$$h_{\mathbf{z}}(c) = 1 + \sum_{j=1}^{k} B_j(c)\hat{A}_j(\mathbf{z}) + o(B_k(c)), \quad c \uparrow 0, \tag{5.68}$$

where $\hat{A}_j$, $j = 1, \ldots, k$, is defined by

$$\hat{A}_j(\mathbf{z}) = z_1 A_j\left(\mathbf{z}^{(1)}\right) + \cdots + z_d A_j\left(\mathbf{z}^{(d)}\right).$$

The leading term of the expansion, $D(\mathbf{z})$, is equal to 1 because we are dealing with the case of tail independence.

In the bivariate case, i.e. $d = 2$, and for $k = 1$ in (5.67) we get, in particular,

$$h_z(c) = 1 + B(c)\hat{A}(z) + o(B(c)), \quad c \uparrow 0,$$

with

$$\hat{A}(z) = zA(z) + (1-z)A(1-z), \quad z \in [0,1]. \tag{5.69}$$

We can now reformulate Theorem 5.1.1 on the basis of Condition 5.2.1 and, thus, generalize our previous result.

Theorem 5.2.2
Let H_n be a d–dimensional distribution function satisfying Condition 5.2.1, i.e., its spectral density can be represented by

$$h_{n,\mathbf{z}}(c) = z_1 F(g_n(z_1, z_2, \ldots, z_{d-1}, c)) + \cdots + z_d F(g_n(z_d, z_1, \ldots, z_{d-2}, c)), \tag{5.70}$$

where F is a continuous univariate distribution function and $g_n : [0,1]^{d-1} \times (-\infty, 0) \to \mathbb{R}$ is a measurable function for each $n \in \mathbb{N}$. Suppose that

$$g_n(\mathbf{z}, c/n) \to g(\mathbf{z}), \quad n \to \infty, \tag{5.71}$$

for every $\mathbf{z} \in R$ *with* $g : [0,1]^{d-1} \to \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ *being a continuous measurable function. Then we have*

$$H_n^n\left(\frac{y_1}{n}, \dots, \frac{y_d}{n}\right) \to \exp\left(y_1 F\left(g\left(T_1(\mathbf{y})^{(1)}\right)\right) + \dots + y_d F\left(g\left(T_1(\mathbf{y})^{(d)}\right)\right)\right) \tag{5.72}$$

$$=: G(y_1, \dots, y_d), \tag{5.73}$$

as $n \to \infty$*, with* $T_1(\mathbf{y})$ *as in* (2.20)*, and* G *is a distribution function.*

Proof. We again start as in the proof of Theorem 5.1.1 and get

$$\begin{aligned} H_n^n\left(\frac{y_1}{n}, \dots, \frac{y_d}{n}\right) &= \exp\left(-n \int_{\frac{T_2(\mathbf{y})}{n}}^0 h_{n,T_1(\mathbf{y})}(c)\, dc\right) \\ &= \exp\left(-\int_{T_2(\mathbf{y})}^0 h_{n,T_1(\mathbf{y})}(c/n)\, dc\right), \end{aligned} \tag{5.74}$$

where T_1 is given by $T_1(\mathbf{z}) = \left(z_1/\sum_{i\le d} z_i, \dots, z_{d-1}/\sum_{i\le d} z_i\right)$ as in (2.20). Inserting representation (5.70) for $h_{n,T_1(\mathbf{y})}$ and substituting c by $c \cdot T_2(\mathbf{y})$ leads to

$$\begin{aligned} -\int_{T_2(\mathbf{y})}^0 h_{n,T_1(\mathbf{y})}(c/n)\, dc = {} & y_1 \int_0^1 F\left(g_n\left(T_1(\mathbf{y})^{(1)}, \frac{cT_2(\mathbf{y})}{n}\right)\right) dc \\ & + \cdots \\ & + y_d \int_0^1 F\left(g_n\left(T_1(\mathbf{y})^{(d)}, \frac{cT_2(\mathbf{y})}{n}\right)\right) dc, \end{aligned}$$

which converges to

$$y_1 F\left(g\left(T_1(\mathbf{y})^{(1)}\right)\right) + \dots + y_d F\left(g\left(T_1(\mathbf{y})^{(d)}\right)\right),$$

as $n \to \infty$, because of (5.71). (Note that F is continuous and bounded.) From (5.74) and the continuity of the exponential function we can deduce the convergence (5.72). The assertion that G is a distribution function can be proved similarly to the proof of Theorem 5.1.1 by using the continuity of g and the boundedness of F. □

Remark 5.2.3
The limit of the spectral density $h_{n,\mathbf{z}}$ *of Theorem 5.2.2 with the argument* c/n *is given by*

$$\lim_{c \uparrow 0} h_{n,\mathbf{z}}(c/n) = z_1 F\left(g\left(\mathbf{z}^{(1)}\right)\right) + \dots + z_d F\left(g\left(\mathbf{z}^{(d)}\right)\right) \tag{5.75}$$

due to condition (5.71) *and the continuity of* F*.*

One can interpret the representation (5.75) *as a scalar product. The limit of the spectral density, as* $c \uparrow 0$*, is projected on the "direction vector"* $\mathbf{z}$ *— the angular component — and the resulting coefficients are given by* $F\left(g\left(\mathbf{z}^{(i)}\right)\right)$*,* $i = 1, \dots, d$*.*

Remark 5.2.4
Even under the generalized Condition 5.2.1 the limiting distribution function in (5.73) *is again max–stable according to* (2.3) *with normalizing vectors* $\mathbf{c_n} = (1/n, \dots, 1/n)$ *and* $\mathbf{d_n} = \mathbf{0}$*, cf. Remark 5.1.2.*

Concerning the univariate margins of G we present the following result which corresponds to Lemma 5.1.3.

Lemma 5.2.5
Let H_n *be a* d*–variate distribution function as in Theorem 5.2.2. If the limiting function* g *in* (5.72) *additionally satisfies*

$$g(\mathbf{e}_1) \geq \omega(F), \tag{5.76}$$

where $\omega(F)$ *is the right endpoint of* F *and* $\mathbf{e}_1$ *is the first unit vector in* $\mathbb{R}^{d-1}$*, then the margins of the limiting distribution with distribution function* G *in* (5.73) *follow the reversely exponential distribution.*

If, conversely, the univariate marginal distribution functions of H_n *belong to the max–domain of attraction of* $\exp(x)$*,* $x \leq 0$*, then the limiting function* g *satisfies the property* (5.76).

PROOF. The assertion can directly be deduced by setting $\mathbf{y} = y_i\mathbf{e}_i$, $i = 1, \dots, d$, in (5.72). For we have

$$\begin{aligned} T_1(\mathbf{e}_i)^{(i)} &= \mathbf{e}_1, \quad 1 \leq i \leq d-1, \\ T_1(\mathbf{0})^{(d)} &= \mathbf{e}_1. \end{aligned}$$

□

Corollary 5.2.6
Let H_n*,* $n \in \mathbb{N}$*, again be* d*–variate distribution functions as in Theorem 5.2.2 and assume that the limiting function* g *in* (5.72) *satisfies the condition* (5.76)*. Then the limiting distribution function* G *in* (5.73) *can be written as*

$$G(\mathbf{y}) = \exp\left(T_2(\mathbf{y})D(T_1(\mathbf{y}))\right)$$

with Pickands dependence function

$$D(\mathbf{z}) = z_1 F\left(g\left(\mathbf{z}^{(1)}\right)\right) + \dots + z_d F\left(g\left(\mathbf{z}^{(d)}\right)\right), \quad \mathbf{z} \in R. \tag{5.77}$$

Note that the Pickands dependence function D in (5.77) coincides with the limit of the spectral density $h_{n,\mathbf{z}}$ in (5.75). It again describes the residual dependence structure of the limiting distribution in Theorem 5.2.2.

By analogy with Lemma 5.1.5 we present an additional condition under which functions of the form

$$\exp\left(y_1 F\left(g\left(T_1(\mathbf{y})^{(1)}\right)\right) + \dots + y_d F\left(g\left(T_1(\mathbf{y})^{(d)}\right)\right)\right)$$

can be identified as distribution functions.

Lemma 5.2.7
Let

$$G(y_1,\dots,y_d) = \exp\left(y_1 F\left(g\left(T_1(\mathbf{y})^{(1)}\right)\right) + \dots + y_d F\left(g\left(T_1(\mathbf{y})^{(d)}\right)\right)\right)$$

be a function on $(-\infty,0]^d$ *with* F *being a continuous univariate distribution function and* g *being a continuous measurable function. If the condition* (5.76) *is satisfied by* g *and if there exists a finite measure* μ *on the* d*–variate unit simplex* S *given in* (2.6) *such that*

$$\begin{aligned} &z_1 F\left(g\left(\mathbf{z}^{(1)}\right)\right) + \dots + z_d F\left(g\left(\mathbf{z}^{(d)}\right)\right) \\ &= \int_S \max\left(u_1 z_1,\dots,u_{d-1}z_{d-1},u_d\left(1-\sum_{i\le d-1} z_i\right)\right)\,d\mu(\mathbf{u}) \end{aligned} \tag{5.78}$$

for every $\mathbf{z} = (z_1,\dots,z_{d-1}) \in R$ *with* $z_d = 1-\sum_{i\le d-1} z_i$*, then* G *is a max–stable distribution function with reversely exponential margins and Pickands dependence function*

$$D(\mathbf{z}) = z_1 F\left(g\left(\mathbf{z}^{(1)}\right)\right) + \dots + z_d F\left(g\left(\mathbf{z}^{(d)}\right)\right).$$

PROOF. The assertion follows directly from Theorem 2.1.2 together with representation (2.9). Property (2.7) is fulfilled because

$$\int_S u_i\,d\mu(\mathbf{u}) = F(g(\mathbf{e}_1)) = 1, \quad i=1,\dots,d,$$

according to (5.76). □

We provide the result of Theorem 5.2.2 for the bivariate case in the following corollary.

Corollary 5.2.8
Let H_n *be a bivariate distribution function satisfying Condition 5.2.1, i.e., its spectral density can be represented by*

$$h_{n,z}(c) = zF(g_n(z,c)) + (1-z)F(g_n(1-z,c)), \tag{5.79}$$

where F *is a continuous univariate distribution function and* $g_n : [0,1]\times(-\infty,0) \to \mathbb{R}$ *is a measurable function for each* $n \in \mathbb{N}$*. Suppose that*

$$g_n(z,c/n) \to g(z), \quad n\to\infty, \tag{5.80}$$

for every $z \in [0,1]$ *with* $g : [0,1] \to \bar{\mathbb{R}} := \mathbb{R}\cup\{-\infty,\infty\}$ *being a continuous measurable function. Then we have*

$$H_n^n\left(\frac{x}{n},\frac{y}{n}\right) \to \exp\left(xF\left(g\left(\frac{x}{x+y}\right)\right) + yF\left(g\left(\frac{y}{x+y}\right)\right)\right), \tag{5.81}$$

as $n\to\infty$*, and the limiting function is a bivariate distribution function.*

For the special case of a limiting function as in (5.81) the condition (5.78) in Lemma 5.2.7 can be verified by simpler conditions such as (5.84) in the following lemma.

Lemma 5.2.9
Let

$$G(x,y) = \exp\left(xF\left(g\left(\frac{x}{x+y}\right)\right) + yF\left(g\left(\frac{y}{x+y}\right)\right)\right)$$

be a function on $(-\infty, 0]^2$ *with* F *being a continuous univariate distribution function and* g *being a continuous measurable function. If* g *satisfies* $g(1) \geq \omega(F)$ *and if there exists a finite measure* ν *on* $[0,1]$ *such that*

$$zF(g(z)) + (1-z)F(g(1-z)) = \int_0^1 \max((1-u)z, u(1-z))\, d\nu(u) \tag{5.82}$$

for every $z \in [0,1]$*, then* G *is a max–stable distribution function with reversely exponential margins and Pickands dependence function*

$$D(z) = zF(g(z)) + (1-z)F(g(1-z)), \quad z \in [0,1]. \tag{5.83}$$

In particular, condition (5.82) *is fulfilled if* F *and* g *and, thus,* D *defined by* (5.83) *are differentiable, if* D' *is right–continuous, non–decreasing and the condition*

$$f(g(1))g'(1) = F(g(0)) \tag{5.84}$$

with $f = F'$ *is satisfied.*

Condition (5.84) can be identified with the previous condition in Lemma 5.1.7 if we choose $F(u) = 1 + u$, $u \in [-1, 0]$, and

$$g_n(z,c) = B_n(c)\hat{A}(z) + o(B_n(c)), \quad c \uparrow 0,$$

cf. (5.67), where

$$B_n(c/n) \to \lambda, \quad n \to \infty,$$

as in (5.18) and

$$\hat{A}(z) = zA(z) + (1-z)A(1-z), \quad z \in [0,1],$$

cf. (5.69). Then condition (5.84) becomes $\lambda A'(1) = 1$.

PROOF. The first part of the assertion of Lemma 5.2.9 follows directly from Lemma 2.1.3. The properties in (2.7) are fulfilled because of the condition $g(1) \geq \omega(F)$ implying

$$\int_0^1 u\, d\nu(u) = F(g(1)) = 1$$

and

$$\begin{aligned}\nu([0,1]) &= \int_0^1 u\, d\nu(u) + \int_0^1 (1-u)\, d\nu(u) \\ &= F(g(1)) + F(g(1)) = 2.\end{aligned}$$

Now, if F and g are differentiable, we can calculate the derivative D' of D

$$D'(z) = F(g(z)) + zf(g(z))g'(z) - F(g(1-z)) - (1-z)f(g(1-z))g'(1-z).$$

As D' is assumed to be right–continuous and non–decreasing,

$$M(z) = 1 + D'(z)$$

is a measure generating function on $[0,1]$. Using $F(g(1)) = 1$ leads to the representation

$$D(z) = 1 - z + \int_0^z M(x)\, dx, \quad z \in [0,1], \tag{5.85}$$

cf. representation (2.12). The measure ν generated by M is given by $\nu([0,z]) = M(z)$ and satisfies, cf. [10], p. 166,

$$\begin{aligned}
\int_0^1 u\, d\nu(u) &= M(1) - \int_0^1 M(x)\, dx \\
&= 1 + D'(1) - \int_0^1 1 + D'(x)\, dx \\
&= 1 + D'(1) - (1 + D(1) - D(0)) \\
&= 1 + F(g(1)) + f(g(1))g'(1) - F(g(0)) - 1 - F(g(1)) + F(g(1)) \\
&= 1,
\end{aligned} \tag{5.86}$$

where the last step is due to the assumption (5.84) and the condition $F(g(1)) = 1$. The equality (5.84) also implies $\nu([0,1]) = 2$. Finally, equation (5.86) is used to form representation (5.85) to

$$D(z) = \int_0^1 \max((1-u)z, u(1-z))\, d\nu(u)$$

which is condition (5.82). □

If the limiting function g in Theorem 5.2.2 and Corollary 5.2.8, respectively, is larger or equal to the right endpoint of the distribution function F for every $\mathbf{y} \in \mathbb{R}^d$, we again obtain the case of tail independence.

Remark 5.2.10
Let H_n, $n \in \mathbb{N}$, be multivariate distribution functions satisfying the assumptions of Theorem 5.2.2, where the function g in (5.71) *satisfies*

$$g(\mathbf{z}) \geq \omega(F) \quad \textit{for every } \mathbf{z} \in R, \tag{5.87}$$

with $\omega(F)$ being the right endpoint of the distribtion function F. Then $F(g(\mathbf{z})) = 1$ for every $\mathbf{z} \in R$ and we get

$$H_n^n\left(\frac{y_1}{n}, \ldots, \frac{y_d}{n}\right) \to \exp\left(\sum_{i \leq d} y_i\right), \quad n \to \infty.$$

In the bivariate case of Corollary 5.2.8 we get, analogously,

$$H_n^n\left(\frac{x}{n}, \frac{y}{n}\right) \to \exp(x+y), \quad n \to \infty,$$

if the function g in (5.80) *satisfies*

$$g(z) \geq \omega(F) \quad \textit{for every } z \in [0,1]. \tag{5.88}$$

If condition (5.87) or condition (5.88), respectively, is not fulfilled, then the limiting distribution functions in (5.72) and (5.81) are dominated by terms — composed of the distribution function F and the limiting function g — which determine the residual dependence structure.

Now, if the argument of F in the limiting distribution function in Theorem 5.2.2 is less or equal to the left endpoint of F for some $\mathbf{y} \in \mathbb{R}^d$ and/or $i \in \{1, \ldots, d\}$, we have the case of absolute dependence.

Remark 5.2.11

Let H_n, $n \in \mathbb{N}$, be multivariate distribution functions satisfying the assumptions of Theorem 5.2.2, where the function g in (5.71) *satisfies*

$$g(\mathbf{z}) \leq \alpha(F) \quad \textit{for some } \mathbf{z} \in R,$$

with $\alpha(F)$ being the left endpoint of the distribtion function F. Then

$$F\left(g\left(T_1(\mathbf{y})^{(i)}\right)\right) = 0$$

for some $\mathbf{y} \in \mathbb{R}^d$ and/or $i \in \{1, \ldots, d\}$. Thus the shape of the limiting distribution function depends on the structure of the function g.

An analogous assertion is true for the limiting distribution function in Corollary 5.2.8.

In the following lines we pick up the examples of the normal and the Crowder distribution again and put them in this more general context. Concerning the normal distribution we are now able to deduce limiting distribution functions of maxima of normal random vectors with varying correlation coefficients which satisfy condition (5.59) for any $\lambda^2 \in [0, \infty]$. In order to establish this result, we first need some additional statements. In the following lemma we provide a general representation of the density of the spectral distribution function in the standard normal case.

Lemma 5.2.12

Let H_ρ be the bivariate standard normal distribution function with $[-1,0]$–uniform margins and correlation coefficient $\rho \in (0,1)$ and let Φ be the univariate standard normal distribution function. Then the density of the spectral distribution function $H_{\rho,z}$ has the form

$$h_{\rho,z}(c) = z\Phi\left(\frac{\Phi^{-1}(1+c(1-z)) - \rho\Phi^{-1}(1+cz)}{\sqrt{1-\rho^2}}\right) \tag{5.89}$$

$$+ (1-z)\Phi\left(\frac{\Phi^{-1}(1+cz) - \rho\Phi^{-1}(1+c(1-z))}{\sqrt{1-\rho^2}}\right) \tag{5.90}$$

for $c < 0$ and any fixed $z \in [0,1]$.

PROOF. The spectral distribution function is given by

$$\begin{aligned} H_{\rho,z}(c) &= H_\rho(cz, c(1-z)) \\ &= \int_{-\infty}^{\Phi^{-1}(1+cz)} \int_{-\infty}^{\Phi^{-1}(1+c(1-z))} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right) dx\, dy \end{aligned}$$

for $c < 0$ and any fixed $z \in [0,1]$. Differentiation with respect to c leads to

$$h_{\rho,z}(c) = \frac{\partial}{\partial c} H_{\rho,z}(c)$$

$$= z \int_{-\infty}^{\Phi^{-1}(1+c(1-z))} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{\left(\Phi^{-1}(1+cz)\right)^2 - 2\rho\Phi^{-1}(1+cz)y + y^2}{2(1-\rho^2)}\right) dy \tag{5.91}$$

$$\times 1/\left(\varphi\left(\Phi^{-1}(1+cz)\right)\right) \tag{5.92}$$

$$+ (1-z) \int_{-\infty}^{\Phi^{-1}(1+cz)} \frac{1}{2\pi\sqrt{1-\rho^2}} \tag{5.93}$$

$$\times \exp\left(-\frac{x^2 - 2\rho x\Phi^{-1}(1+c(1-z)) + \left(\Phi^{-1}(1+c(1-z))\right)^2}{2(1-\rho^2)}\right) dx \tag{5.94}$$

$$\times 1/\left(\varphi\left(\Phi^{-1}(1+c(1-z))\right)\right), \tag{5.95}$$

where φ is the standard normal density. The term in (5.91) and (5.92) can be written as

$$\begin{aligned} & z \int_{-\infty}^{\Phi^{-1}(1+c(1-z))} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{\left(\Phi^{-1}(1+cz)\right)^2 - 2\rho\Phi^{-1}(1+cz)y + y^2}{2(1-\rho^2)}\right) dy \\ & \quad \times 1/\left(\varphi\left(\Phi^{-1}(1+cz)\right)\right) \\ &= z \int_{-\infty}^{\Phi^{-1}(1+c(1-z))} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \\ & \qquad \times \exp\left(-\frac{\rho^2\left(\Phi^{-1}(1+cz)\right)^2 - 2\rho\Phi^{-1}(1+cz)y + y^2}{2(1-\rho^2)}\right) dy \\ &= z \int_{-\infty}^{\Phi^{-1}(1+c(1-z))} \frac{1}{\sqrt{1-\rho^2}} \varphi\left(\frac{y-\rho\Phi^{-1}(1+cz)}{\sqrt{1-\rho^2}}\right) dy \\ &= z\Phi\left(\frac{\Phi^{-1}(1+c(1-z)) - \rho\Phi^{-1}(1+cz)}{\sqrt{1-\rho^2}}\right). \end{aligned}$$

An analogous result holds for the term in (5.93), (5.94) and (5.95). This proves the assertion. □

If we consider a correlation coefficient varying in n, i.e. $\rho(n) \in (0,1)$, $n \in \mathbb{N}$, and replace c by c/n, the representation in (5.89) and (5.90) becomes

$$h_{\rho(n),z}(c/n) = z\Phi\left(\frac{\Phi^{-1}(1+c(1-z)/n) - \rho(n)\Phi^{-1}(1+cz/n)}{\sqrt{1-\rho(n)^2}}\right) \tag{5.96}$$

$$+ (1-z)\Phi\left(\frac{\Phi^{-1}(1+cz/n) - \rho(n)\Phi^{-1}(1+c(1-z)/n)}{\sqrt{1-\rho(n)^2}}\right) \tag{5.97}$$

In the following lemma we deduce an expansion for the argument of Φ in (5.96).

Lemma 5.2.13
Let $\rho(n) \in (0,1)$, $n \in \mathbb{N}$, be correlation coefficients which vary with the sample size n and assume that the convergence (5.59) *holds, i.e.*

$$(1-\rho(n))\log(n) \to \lambda^2 \in [0,\infty], \quad n \to \infty.$$

If $\lambda = 0$, we assume that the convergence $\rho(n) \to 1$, $n \to \infty$, is not too fast such that we still have

$$(1-\rho(n))(\log(n))^2 \to \infty, \quad n \to \infty. \tag{5.98}$$

Then we obtain the expansion

$$\begin{aligned}
&\frac{\Phi^{-1}(1+c(1-z)/n) - \rho(n)\Phi^{-1}(1+cz/n)}{\sqrt{1-\rho(n)^2}} \\
&= \frac{\sqrt{2}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{1/2}}{\sqrt{1+\rho(n)}} - \frac{\sqrt{2}\sqrt{1-\rho(n)}\log(\log(n)-\log|c|)}{4\sqrt{1+\rho(n)}(\log(n)-\log|c|)^{1/2}} \\
&\quad + \frac{\log(z)-\log(1-z)}{\sqrt{2}\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{1/2}} + o\left(\frac{\log(\log(n))}{(\log(n))^{1/2}}\right),
\end{aligned}$$

as $n \to \infty$, for $c < 0$ and $z \in [0,1]$.

PROOF. Using the asymptotic expansion (A.2) in Ledford and Tawn [34], Appendix A, we obtain

$$\begin{aligned}
&\Phi^{-1}(1+c(1-z)/n) - \rho(n)\Phi^{-1}(1+cz/n) \\
&\sim (2\log(n) - 2\log|c| - \log(1-z))^{1/2} - \rho(n)(2\log(n) - 2\log|c| - 2\log(z))^{1/2} \\
&\quad - \frac{1}{2}(2\log(n) - 2\log|c| - 2\log(1-z))^{-1/2} \\
&\quad \times (\log(\log(n) - \log|c| - \log(1-z)) + \log(4\pi)) \\
&\quad + \frac{\rho(n)}{2}(2\log(n) - 2\log|c| - 2\log(z))^{-1/2} \\
&\quad \times (\log(\log(n) - \log|c| - \log(z)) + \log(4\pi)) \\
&= \sqrt{2}(\log(n) - \log|c|)^{1/2}\left(1 + \frac{\log(1-z)}{\log|c| - \log(n)}\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&- \rho(n)\sqrt{2}(\log(n) - \log|c|)^{1/2}\left(1 + \frac{\log(z)}{\log|c| - \log(n)}\right)^{1/2} \\
&- \frac{\sqrt{2}}{4}(\log(n) - \log|c|)^{-1/2}\left(1 + \frac{\log(1-z)}{\log|c| - \log(n)}\right)^{-1/2} \\
&\times (\log(\log(n) - \log|c| - \log(1-z)) + \log(4\pi)) \\
&+ \rho(n)\frac{\sqrt{2}}{4}(\log(n) - \log|c|)^{-1/2}\left(1 + \frac{\log(z)}{\log|c| - \log(n)}\right)^{-1/2} \\
&\times (\log(\log(n) - \log|c| - \log(z)) + \log(4\pi)) \\
\approx\ &\sqrt{2}(\log(n) - \log|c|)^{1/2}\left(1 + \frac{\log(1-z)}{\log|c| - \log(n)}\right)^{1/2} \\
&- \rho(n)\sqrt{2}(\log(n) - \log|c|)^{1/2}\left(1 + \frac{\log(z)}{\log|c| - \log(n)}\right)^{1/2} \\
&- \frac{\sqrt{2}}{4}(\log(n) - \log|c|)^{-1/2}\left(1 + \frac{\log(1-z)}{\log|c| - \log(n)}\right)^{-1/2} \\
&\times \left(\log(\log(n) - \log|c|) + \frac{\log(1-z)}{\log|c| - \log(n)} + \log(4\pi)\right) \\
&+ \rho(n)\frac{\sqrt{2}}{4}(\log(n) - \log|c|)^{-1/2}\left(1 + \frac{\log(z)}{\log|c| - \log(n)}\right)^{-1/2} \\
&\times \left(\log(\log(n) - \log|c|) + \frac{\log(z)}{\log|c| - \log(n)} + \log(4\pi)\right),
\end{aligned}
$$

as $n \to \infty$, cf. also [16], pp. 56–57. Inserting this expansion into the argument of Φ in (5.96) we can write

$$
\begin{aligned}
&\frac{\Phi^{-1}(1 + c(1-z)/n) - \rho(n)\Phi^{-1}(1 + cz/n)}{\sqrt{1 - \rho(n)^2}} \\
\approx\ &\frac{\sqrt{2}\sqrt{1-\rho(n)}(\log(n) - \log|c|)^{1/2}}{\sqrt{1+\rho(n)}} - \frac{\log(1-z)}{\sqrt{2}\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n) - \log|c|)^{1/2}} \\
&+ \frac{\rho(n)}{\sqrt{2}} \frac{\log(z)}{\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n) - \log|c|)^{1/2}} \\
&+ \frac{1}{4\sqrt{2}} \frac{\log(1-z)^2}{\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n) - \log|c|)^{3/2}} \\
&- \frac{\rho(n)}{4\sqrt{2}} \frac{\log(1-z)^2}{\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n) - \log|c|)^{3/2}} + o\left(\frac{1}{\sqrt{1-\rho(n)}(\log(n))^{3/2}}\right) \\
&- \frac{\sqrt{2}}{4} \frac{\log(\log(n) - \log|c|)}{\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n) - \log|c|)^{1/2}} \\
&- \frac{\sqrt{2}}{4} \frac{4\pi}{\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n) - \log|c|)^{1/2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\rho(n)\frac{\sqrt{2}}{4}\frac{\log(\log(n)-\log|c|)}{\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{1/2}}\\
&+\rho(n)\frac{\sqrt{2}}{4}\frac{4\pi}{\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{1/2}}\\
&-\frac{1}{4\sqrt{2}}\frac{\log(1-z)\log(\log(n)-\log|c|)}{\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{3/2}}\\
&+\frac{\rho(n)}{4\sqrt{2}}\frac{\log(z)\log(\log(n)-\log|c|)}{\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{3/2}}+o\left(\frac{\log(\log(n))}{\sqrt{1-\rho(n)}(\log(n))^{3/2}}\right)
\end{aligned} \tag{5.99}
$$

$$
\begin{aligned}
&=\frac{\sqrt{2}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{1/2}}{\sqrt{1+\rho(n)}}+\frac{\log(z)-\log(1-z)}{\sqrt{2}\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{1/2}}\\
&-\frac{\log(z)\sqrt{1-\rho(n)}}{\sqrt{2}\sqrt{1+\rho(n)}(\log(n)-\log|c|)^{1/2}}-\frac{\sqrt{2}\sqrt{1-\rho(n)}\log(\log(n)-\log|c|)}{4\sqrt{1+\rho(n)}(\log(n)-\log|c|)^{1/2}}\\
&-\frac{\sqrt{2}\sqrt{1-\rho(n)}\log(4\pi)}{4\sqrt{1+\rho(n)}(\log(n)-\log|c|)^{1/2}}+o\left(\frac{\log(\log(n))}{(\log(n))^{1/2}}\right)\\
&=\frac{\sqrt{2}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{1/2}}{\sqrt{1+\rho(n)}}-\frac{\sqrt{2}\sqrt{1-\rho(n)}\log(\log(n)-\log|c|)}{4\sqrt{1+\rho(n)}(\log(n)-\log|c|)^{1/2}}\\
&+\frac{\log(z)-\log(1-z)}{\sqrt{2}\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{1/2}}+o\left(\frac{\log(\log(n))}{(\log(n))^{1/2}}\right),
\end{aligned}
$$

as $n \to \infty$. Note that it is possible to shorten the expansion (5.99) because of the convergence (5.98). □

Of course, an analogous expansion is satisfied by the argument of Φ in (5.97).

Now we are able to present our result concerning the limiting distribution functions of normal random vectors under triangular schemes.

Example 5.2.14
Let $H_{\rho(n)}$ be the bivariate standard normal distribution function with with $[-1,0]$–uniform margins and correlation coefficient $\rho(n)$ which varies with the sample size n and assume that the convergences (5.59) *and* (5.98) *hold.*

From Lemma 5.2.12 and Lemma 5.2.13 we can deduce that the spectral density of $H_{\rho(n)}$ has a representation of the form (5.79) *with $F = \Phi$ and*

$$
\begin{aligned}
g_n(z,c/n)&=\frac{\sqrt{2}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{1/2}}{\sqrt{1+\rho(n)}}-\frac{\sqrt{2}\sqrt{1-\rho(n)}\log(\log(n)-\log|c|)}{4\sqrt{1+\rho(n)}(\log(n)-\log|c|)^{1/2}}\\
&+\frac{\log(z)-\log(1-z)}{\sqrt{2}\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{1/2}}+o\left(\frac{\log(\log(n))}{(\log(n))^{1/2}}\right),\quad n\to\infty,\\
&=\sum_{j=1}^{3}B_{j,n}(c/n)A_j(z)+o\left(B_{3,n}(c/n)\right),\quad n\to\infty,
\end{aligned}
$$

where

$$\begin{aligned}
B_{1,n}(c/n) &= \frac{\sqrt{2}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{1/2}}{\sqrt{1+\rho(n)}},\\
B_{2,n}(c/n) &= \frac{1}{\sqrt{2}\sqrt{1+\rho(n)}\sqrt{1-\rho(n)}(\log(n)-\log|c|)^{1/2}},\\
B_{3,n}(c/n) &= \frac{\sqrt{2}\sqrt{1-\rho(n)}\log(\log(n)-\log|c|)}{4\sqrt{1+\rho(n)}(\log(n)-\log|c|)^{1/2}}
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= A_3 \equiv 1,\\
A_2(z) &= \log(z)-\log(1-z).
\end{aligned}$$

Therefore we have some sort of expansion of length 3 *in the argument of the standard normal distribution function* Φ *in* (5.79)*. Obviously, the functions* $B_{j,n}$ *are slowly varying and we have*

$$\begin{aligned}
\lim_{n\to\infty} B_{1,n}(c/n) &= \lambda \in [0,\infty],\\
\lim_{n\to\infty} B_{2,n}(c/n) &= \frac{1}{2\lambda} \in [0,\infty]
\end{aligned}$$

for every $c<0$ *and*

$$\lim_{n\to\infty} B_{3,n}(c/n) = 0$$

for every sequence $(\rho(n))_{n\in\mathbb{N}}$ *and* $c<0$*. According to the convergence* (5.59) *we know that* $g_n(z,c/n)$ *converges to*

$$g(z) = \lambda + \frac{\log(z)-\log(1-z)}{2\lambda},$$

as $n\to\infty$*, if* $\lambda>0$*, which is a continuous measurable function. Now an application of Corollary 5.2.8 leads to*

$$\begin{aligned}
&H^n_{\rho(n)}\left(\frac{x}{n},\frac{y}{n}\right)\\
&\to \exp\left(x\Phi\left(\lambda+\frac{\log|x|-\log|y|}{2\lambda}\right)+y\Phi\left(\lambda+\frac{\log|y|-\log|x|}{2\lambda}\right)\right)\\
&=: H_\lambda(x,y),
\end{aligned} \tag{5.100}$$

as $n\to\infty$*, if* $\lambda>0$*. If* $\lambda=\infty$*, we obtain that*

$$g_n(z,c/n)\to g(z)\equiv\infty=\omega(\Phi),\quad n\to\infty,$$

and from Remark 5.2.10 it follows that

$$H^n_{\rho(n)}\left(\frac{x}{n},\frac{y}{n}\right)\to\exp(x+y)=:H_\infty(x,y),\quad n\to\infty.$$

Finally, Remark 5.2.11 implies that

$$H^n_{\rho(n)}\left(\frac{x}{n}, \frac{y}{n}\right) \to \exp\left(\min(x,y)\right) =: H_0(x,y), \quad n \to \infty,$$

if $\lambda = 0$. *In particular, we have*

$$\lim_{n\to\infty} \sum_{j=1}^{3} B_{j,n}\left((x+y)\frac{c}{n}\right) A_j\left(\frac{x}{x+y}\right) = 1$$

and

$$\lim_{n\to\infty} \sum_{j=1}^{3} B_{j,n}\left((x+y)\frac{c}{n}\right) A_j\left(\frac{y}{x+y}\right) = 0,$$

if $x < y$, *and vice versa, if* $y < x$.

Remark 5.2.15
In Example 5.2.14 we have, obviously,

$$H_0 = \lim_{\lambda\downarrow 0} H_\lambda \quad \text{and} \quad H_\infty = \lim_{\lambda\uparrow\infty} H_\lambda,$$

where $H_\lambda(x,y)$ *is the Hüsler–Reiss distribution function with reversely exponential margins, cf. [29], Section 2.*

Remark 5.2.16
In the bivariate case the general form (5.61) *of a spectral density becomes*

$$\begin{aligned} h_z(c) = {} & zP\left(Y \le H_2^{-1}(1+c(1-z)) | X = H_1^{-1}(1+cz)\right) \\ & + (1-z)P\left(X \le H_1^{-1}(1+cz) | Y = H_2^{-1}(1+c(1-z))\right), \end{aligned}$$

where (X,Y) *is a bivariate random vector with marginal distribution functions* H_1 *and* H_2. *Now let* $H_1 = H_2 = \Phi$ *and let* $h_z = h_{\rho,z}$ *be the spectral density pertaining to the bivariate standard normal distribution function with* $[-1,0]$*–uniform margins and correlation coefficient* $\rho \in (0,1)$. *Then X and Y are standard normal random variables and the conditional distribution of X given* $Y = y$ *is the normal distribution* $N_{(\rho y, 1-\rho^2)}$ *with mean value* ρy *and variance* $1-\rho^2$. *Thus we can identify the function* $\tilde{F}$ *in* (5.62) *by*

$$\tilde{F}(x,y) = \Phi\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)$$

and the functions F and $\tilde{g}$ *in* (5.63) *are given by*

$$F = \Phi, \quad \text{and} \quad \tilde{g}(x,y) = \frac{x-\rho y}{\sqrt{1-\rho^2}}.$$

Finally, the function g in (5.64) *can be defined as*

$$g(z,c) = \tilde{g}\left(\Phi^{-1}(1+c(1-z)), \Phi^{-1}(1+cz)\right) = \frac{\Phi^{-1}(1+c(1-z)) - \rho\Phi^{-1}(1+cz)}{\sqrt{1-\rho^2}}.$$

Therewith we obtain

$$h_{\rho,z}(c) = z\Phi\left(\frac{\Phi^{-1}(1+c(1-z)) - \rho\Phi^{-1}(1+cz)}{\sqrt{1-\rho^2}}\right) \tag{5.101}$$

$$+ (1-z)\Phi\left(\frac{\Phi^{-1}(1+cz) - \rho\Phi^{-1}(1+c(1-z))}{\sqrt{1-\rho^2}}\right), \tag{5.102}$$

which is again the result of Lemma 5.2.12.

In the following remark we show that the spectral density in (5.96)–(5.97) fulfills the spectral expansion presented in Example 5.1.18 if the correlation coefficient satisfies the condition $(1-\rho(n))\log(n) \to \infty$, as $n \to \infty$. Thereby we relate the previous results to earlier findings.

Remark 5.2.17
From Mill's ratio, cf. Ruben [44], expansion (2.6), it follows that

$$\Phi(y) \sim 1 - \frac{1}{y}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y^2\right), \quad y \to \infty. \tag{5.103}$$

This asymptotic equivalence leads to the spectral expansion of the standard normal distribution function if the arguments of Φ *in* (5.96) *and* (5.97) *converge to* ∞*, as* $n \to \infty$*. According to Lemma 5.2.13, this is the case if* $(1-\rho(n))\log(n) \to \infty$*, as* $n \to \infty$*, which is, of course, always true if* ρ *is constant.*

Assume now that the before named condition is fulfilled. Then we can write the spectral density given in (5.96) *and* (5.97) *in the form*

$$\begin{aligned} h_{\rho(n),z}(c/n) = 1 - z\frac{1}{\sqrt{2\pi}}&\frac{\sqrt{1-\rho(n)^2}}{\Phi^{-1}(1+c(1-z)/n) - \rho(n)\Phi^{-1}(1+cz/n)} \\ \times \exp&\left(-\frac{1}{2}\frac{\left(\Phi^{-1}(1+c(1-z)/n)\right)^2}{1-\rho(n)^2} + \frac{\rho(n)\Phi^{-1}(1+c(1-z)/n)\Phi^{-1}(1+cz/n)}{1-\rho(n)^2}\right. \\ &\left.-\frac{1}{2}\frac{\rho(n)^2}{1-\rho(n)^2}\left(\Phi^{-1}(1+cz/n)\right)^2\right) \\ - (1-z)\frac{1}{\sqrt{2\pi}}&\frac{\sqrt{1-\rho(n)^2}}{\Phi^{-1}(1+cz/n) - \rho(n)\Phi^{-1}(1+c(1-z)/n)} \\ \times \exp&\left(-\frac{1}{2}\frac{\left(\Phi^{-1}(1+cz/n)\right)^2}{1-\rho(n)^2} + \frac{\rho(n)\Phi^{-1}(1+c(1-z)/n)\Phi^{-1}(1+cz/n)}{1-\rho(n)^2}\right. \\ &\left.-\frac{1}{2}\frac{\rho(n)^2}{1-\rho(n)^2}\left(\Phi^{-1}(1+c(1-z)/n)\right)^2\right), \quad n \to \infty. \end{aligned} \tag{5.104}$$

Using an asymptotic result from [16], p. 58, we get

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\frac{\left(\Phi^{-1}(1+c(1-z)/n)\right)^2}{1-\rho(n)^2}\right)\\
&\quad= (|c|/n)^{1/(1-\rho(n)^2)}(\log(n)-\log|c|)^{1/(2(1-\rho(n)^2))}(4\pi)^{1/(2(1-\rho(n)^2))}(1-z)^{1/(1-\rho(n)^2)}\\
&\qquad\times\left(1-\frac{1}{16}\frac{1}{1-\rho(n)^2}\frac{(\log(\log(n)-\log|c|))^2}{\log(n)-\log|c|}+o\left(\frac{(\log(\log(n)))^2}{\log(n)}\right)\right),\quad n\to\infty,\\
&\exp\left(\frac{\rho(n)\Phi^{-1}(1+c(1-z)/n)\Phi^{-1}(1+cz/n)}{1-\rho(n)^2}\right)\\
&\quad= (|c|/n)^{2\rho(n)/(1-\rho(n)^2)}(\log(n)-\log|c|)^{-\rho(n)/(1-\rho(n)^2)}(4\pi)^{-\rho(n)/(1-\rho(n)^2)}\\
&\qquad\times(z(1-z))^{-\rho(n)/(1-\rho(n)^2)}\\
&\qquad\times\left(1-\frac{1}{8}\frac{\rho(n)}{1-\rho(n)^2}\frac{\log(\log(n)-\log|c|)^2}{\log(n)-\log|c|}+o\left(\frac{(\log(\log(n)))^2}{\log(n)}\right)\right),\quad n\to\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\frac{\left(\Phi^{-1}(1+cz/n)\right)^2}{1-\rho(n)^2}\right)\\
&\quad= (|c|/n)^{1/(1-\rho(n)^2)}(\log(n)-\log|c|)^{1/(2(1-\rho(n)^2))}(4\pi)^{1/(2(1-\rho(n)^2))}z^{1/(1-\rho(n)^2)}\\
&\qquad\times\left(1-\frac{1}{16}\frac{1}{1-\rho(n)^2}\frac{(\log(\log(n)-\log|c|))^2}{\log(n)-\log|c|}+o\left(\frac{(\log(\log(n)))^2}{\log(n)}\right)\right),\quad n\to\infty,
\end{aligned}$$

which implies

$$\begin{aligned}
&\exp\Bigg(-\frac{1}{2}\frac{\left(\Phi^{-1}(1+c(1-z)/n)\right)^2}{1-\rho(n)^2}+\frac{\rho(n)\Phi^{-1}(1+c(1-z)/n)\Phi^{-1}(1+cz/n)}{1-\rho(n)^2}\\
&\qquad\qquad-\frac{1}{2}\frac{\rho(n)^2}{1-\rho(n)^2}\left(\Phi^{-1}(1+cz/n)\right)^2\Bigg)\\
&= (|c|/n)^{(1-\rho(n))/(1+\rho(n))}(\log(n)-\log|c|)^{(1-\rho(n))/(2(1+\rho(n)))}(4\pi)^{(1-\rho(n))/(2(1+\rho(n)))}\\
&\quad\times(1-z)^{1/(1+\rho(n))}z^{-\rho(n)/(1+\rho(n))}\\
&\quad+o\left(n^{-(1-\rho(n))/(1+\rho(n))}(\log(n))^{(1-\rho(n))/(2(1+\rho(n)))}\right),\quad n\to\infty.
\end{aligned}$$

Together with Lemma 5.2.13 this leads to

$$\begin{aligned}
&z\frac{1}{\sqrt{2\pi}}\frac{\sqrt{1-\rho(n)^2}}{\Phi^{-1}(1+c(1-z)/n)-\rho(n)\Phi^{-1}(1+cz/n)}\\
&\quad\times\exp\Bigg(-\frac{1}{2}\frac{\left(\Phi^{-1}(1+c(1-z)/n)\right)^2}{1-\rho(n)^2}+\frac{\rho(n)\Phi^{-1}(1+c(1-z)/n)\Phi^{-1}(1+cz/n)}{1-\rho(n)^2}
\end{aligned}$$

$$-\frac{1}{2}\frac{\rho(n)^2}{1-\rho(n)^2}\left(\Phi^{-1}(1+cz/n)\right)^2\Bigg)$$
$$= (|c|/n)^{(1-\rho(n))/(1+\rho(n))}(\log(n)-\log|c|)^{-\rho(n)/(1+\rho(n))}(4\pi)^{-\rho(n)/(1+\rho(n))}$$
$$\times (1-\rho(n))^{-1/2}(1+\rho(n))^{3/2}\frac{1}{(1+\rho(n))}(z(1-z))^{1/(1+\rho(n))}$$
$$+ o\left(n^{-(1-\rho(n))/(1+\rho(n))}(\log(n))^{-\rho(n)/(1+\rho(n))}\right), \quad n\to\infty.$$

Analogous considerations for the third term of representation (5.104) *finally give us the spectral expansion*

$$h_{\rho(n),z}(c) = 1 + B_n(c)A_n(z) + R_n(z,c)$$

with $R_n(z,c) = o(B_n(c))$ *uniformly for* $z \in [0,1]$, *as* $c \uparrow 0$, *such that*

$$R_n(z, c/n) \to 0, \quad n \to \infty,$$

for every $c < 0$ *and*

$$B_n(c) = |c|^{2/(1+\rho(n))-1}L_n(c),$$

where

$$L_n(c) = (1+\rho(n))^{3/2}(1-\rho(n))^{-1/2}(4\pi)^{-\rho(n)/(1+\rho(n))}(-\log|c|)^{-\rho(n)/(1+\rho(n))}$$

and

$$A_n(z) = -\frac{2}{1+\rho(n)}(z(1-z))^{1/(1+\rho(n))}.$$

Let us now come back to the Crowder distribution. In Example 5.1.17 it has been said that one has to modify the representation of the spectral density of $H_{\theta(n)}$, the joint distribution function of the lower tail of the Crowder distribution, in order to deduce the limiting distribution of the maxima for a non–constant sequence $(\theta(n))_{n\in\mathbb{N}}$. Such a modified representation will be computed in the subsequent lines, thereby giving another example of Corollary 5.2.8.

Example 5.2.18
Let $H_{\theta(n)}$ *be the joint distribution function of the lower tail of the Crowder distribution with* $[-1,0]$*–uniform margins and parameters* $\alpha \geq 0$ *and* $\theta(n) \geq 1$, $n \in \mathbb{N}$, *cf. Example 3.3.3. Let us now compute its spectral density for each* $n \in \mathbb{N}$ *by differentiating the spectral distribution*

function $H_{\theta(n),z}$ with respect to c. We have

$$
\begin{aligned}
&h_{n,z}(c) = \frac{\partial}{\partial c} H_{\theta(n),z}(c) \\
&= \frac{\partial}{\partial c}\Bigg(1 + c \\
&\quad + \exp\left[-\left\{(\alpha - \log(-cz))^{\theta(n)} + (\alpha - \log(-c(1-z)))^{\theta} - \alpha^{\theta(n)}\right\}^{1/\theta(n)} + \alpha\right]\Bigg) \\
&= 1 + \exp\left[-\left\{(\alpha - \log(-cz))^{\theta(n)} + (\alpha - \log(-c(1-z)))^{\theta(n)} - \alpha^{\theta(n)}\right\}^{1/\theta(n)} + \alpha\right] \\
&\quad \times \left(\frac{1}{\theta(n)}\right)\left\{(\alpha - \log(-cz))^{\theta(n)} + (\alpha - \log(-c(1-z)))^{\theta(n)} - \alpha^{\theta(n)}\right\}^{1/\theta(n)-1} \\
&\quad \times \left(\theta(n)(\alpha - \log(-cz))^{\theta(n)-1}\frac{(-z)}{cz} + \theta(n)(\alpha - \log(-c(1-z)))^{\theta(n)-1}\frac{(-(1-z))}{c(1-z)}\right) \\
&= z\Bigg(1 + \exp\left[-\left\{(\alpha - \log(-cz))^{\theta(n)} + (\alpha - \log(-c(1-z)))^{\theta(n)} - \alpha^{\theta(n)}\right\}^{1/\theta(n)} + \alpha\right] \\
&\quad \times \left\{(\alpha - \log(-cz))^{\theta(n)} + (\alpha - \log(-c(1-z)))^{\theta(n)} - \alpha^{\theta(n)}\right\}^{1/\theta(n)-1} \\
&\quad \times \frac{(\alpha - \log(-cz))^{\theta(n)-1}}{cz}\Bigg) \\
&+ (1-z) \\
&\quad \times \Bigg(1 + \exp\left[-\left\{(\alpha - \log(-cz))^{\theta(n)} + (\alpha - \log(-c(1-z)))^{\theta(n)} - \alpha^{\theta(n)}\right\}^{1/\theta(n)} + \alpha\right] \\
&\quad \times \left\{(\alpha - \log(-cz))^{\theta(n)} + (\alpha - \log(-c(1-z)))^{\theta(n)} - \alpha^{\theta(n)}\right\}^{1/\theta(n)-1} \\
&\quad \times \frac{(\alpha - \log(-c(1-z)))^{\theta(n)-1}}{c(1-z)}\Bigg) \\
&= z\Bigg(1 + \exp\Bigg[2^{1/\theta(n)}\log|c|\Bigg\{\frac{1}{2}\left(1 + \frac{\log(z)}{\log|c|} - \frac{\alpha}{\log|c|}\right)^{\theta(n)} \\
&\quad + \frac{1}{2}\left(1 + \frac{\log(1-z)}{\log|c|} - \frac{\alpha}{\log|c|}\right)^{\theta(n)} + \frac{\alpha^{\theta(n)}}{2(-\log|c|)^{\theta(n)}}\Bigg\}^{1/\theta(n)} + \alpha\Bigg] \\
&\quad \times 2^{1/\theta(n)}\Bigg\{\frac{1}{2}\left(1 + \frac{\log(z)}{\log|c|} - \frac{\alpha}{\log|c|}\right)^{\theta(n)} + \frac{1}{2}\left(1 + \frac{\log(1-z)}{\log|c|} - \frac{\alpha}{\log|c|}\right)^{\theta(n)} \\
&\quad + \frac{\alpha^{\theta(n)}}{2(-\log|c|)^{\theta(n)}}\Bigg\}^{1/\theta(n)-1}\frac{1}{2}\left(1 + \frac{\log(z)}{\log|c|} - \frac{\alpha}{\log|c|}\right)^{\theta(n)-1}\frac{1}{cz}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+(1-z)\left(1+\exp\left[2^{1/\theta(n)}\log|c|\left\{\frac{1}{2}\left(1+\frac{\log(z)}{\log|c|}-\frac{\alpha}{\log|c|}\right)^{\theta(n)}\right.\right.\right.\\
&\quad\left.\left.+\frac{1}{2}\left(1+\frac{\log(1-z)}{\log|c|}-\frac{\alpha}{\log|c|}\right)^{\theta(n)}+\frac{\alpha^{\theta(n)}}{2(-\log|c|)^{\theta(n)}}\right\}^{1/\theta(n)}+\alpha\right]\\
&\quad\times 2^{1/\theta(n)}\left\{\frac{1}{2}\left(1+\frac{\log(z)}{\log|c|}-\frac{\alpha}{\log|c|}\right)^{\theta(n)}+\frac{1}{2}\left(1+\frac{\log(1-z)}{\log|c|}-\frac{\alpha}{\log|c|}\right)^{\theta(n)}\right.\\
&\quad\left.\left.+\frac{\alpha^{\theta(n)}}{2(-\log|c|)^{\theta(n)}}\right\}^{1/\theta(n)-1}\frac{1}{2}\left(1+\frac{\log(1-z)}{\log|c|}-\frac{\alpha}{\log|c|}\right)^{\theta(n)-1}\frac{1}{c(1-z)}\right)\\
&=z\left(1+\exp\left[2^{1/\theta(n)}\log|c|\left\{\frac{1}{2}\left(1+\frac{\log(z)}{\log|c|}-\frac{\alpha}{\log|c|}\right)^{\theta(n)}\right.\right.\right.\\
&\quad\left.\left.+\frac{1}{2}\left(1+\frac{\log(1-z)}{\log|c|}-\frac{\alpha}{\log|c|}\right)^{\theta(n)}+\frac{\alpha^{\theta(n)}}{2(-\log|c|)^{\theta(n)}}\right\}^{1/\theta(n)}+\alpha\right]\\
&\quad\left.\times\frac{1}{cz}\left\{1+\left(\frac{1+\frac{\log(1-z)}{\log|c|}-\frac{\alpha}{\log|c|}}{1+\frac{\log(z)}{\log|c|}-\frac{\alpha}{\log|c|}}\right)^{\theta(n)}+\frac{\alpha^{\theta(n)}}{(-\log|c|-\log(z)-\alpha)^{\theta(n)}}\right\}^{1/\theta(n)-1}\right)\\
&+(1-z)\left(1+\exp\left[2^{1/\theta(n)}\log|c|\left\{\frac{1}{2}\left(1+\frac{\log(z)}{\log|c|}-\frac{\alpha}{\log|c|}\right)^{\theta(n)}\right.\right.\right.\\
&\quad\left.\left.+\frac{1}{2}\left(1+\frac{\log(1-z)}{\log|c|}-\frac{\alpha}{\log|c|}\right)^{\theta(n)}+\frac{\alpha^{\theta(n)}}{2(-\log|c|)^{\theta(n)}}\right\}^{1/\theta(n)}+\alpha\right]\\
&\quad\times\frac{1}{c(1-z)}\left\{1+\left(\frac{1+\frac{\log(z)}{\log|c|}-\frac{\alpha}{\log|c|}}{1+\frac{\log(1-z)}{\log|c|}-\frac{\alpha}{\log|c|}}\right)^{\theta(n)}\right.\\
&\qquad\qquad\left.\left.+\frac{\alpha^{\theta(n)}}{(-\log|c|-\log(1-z)-\alpha)^{\theta(n)}}\right\}^{1/\theta(n)-1}\right).
\end{aligned}
$$

Hence the spectral density $h_{n,z}(c)$ *possesses a representation of the form* (5.79) *with*

$$F(u)=1-\exp(-u),\quad u\in[0,\infty),$$

which is the distribution function of the exponential distribution on $[0,\infty)$*, and*

$$\begin{aligned} g_n(z,c) &= 2^{1/\theta(n)}(-\log|c|)\left\{\frac{1}{2}\left(1+\frac{\log(z)}{\log|c|}-\frac{\alpha}{\log|c|}\right)^{\theta(n)}\right.\\ &\left.+\frac{1}{2}\left(1+\frac{\log(1-z)}{\log|c|}-\frac{\alpha}{\log|c|}\right)^{\theta(n)}+\frac{\alpha^{\theta(n)}}{2(-\log|c|)^{\theta(n)}}\right\}^{1/\theta(n)}-\alpha+\log(|c|z)\\ &+\left(1-\frac{1}{\theta(n)}\right)\log\left\{1+\left(\frac{1+\frac{\log(1-z)}{\log|c|}-\frac{\alpha}{\log|c|}}{1+\frac{\log(z)}{\log|c|}-\frac{\alpha}{\log|c|}}\right)^{\theta(n)}+\frac{\alpha^{\theta(n)}}{(-\log|c|-\log(z)-\alpha)^{\theta(n)}}\right\}. \end{aligned}$$

To compute the limiting function g of $g_n(z,c/n)$, as $n\to\infty$, cf. (5.80)*, we have to do some preliminary considerations.*

Let us assume that the sequence $(\theta(n))_{n\in\mathbb{N}}$ satisfies the convergences

$$\theta(n)\to\infty$$

and

$$\left(2^{1/\theta(n)}-1\right)\log(n)\to\xi\in(0,\infty),$$

as $n\to\infty$.

First of all, it follows that

$$\frac{\alpha^{\theta(n)}}{(\log(n))^{\theta(n)}}=\left(\frac{\alpha}{\log(n)}\right)^{\theta(n)}\to 0,\quad n\to\infty,$$

which implies

$$\frac{\alpha^{\theta(n)}}{2(\log(n)-\log|c|)^{\theta(n)}}\to 0 \tag{5.105}$$

and

$$\frac{\alpha^{\theta(n)}}{2(\log(n)-\log|c|-\log(z)-\alpha)^{\theta(n)}}\to 0, \tag{5.106}$$

as $n\to\infty$.

Moreover, we have

$$\begin{aligned} \theta(n)\left(2^{1/\theta(n)}-1\right) &= \theta(n)\left(2^{1/\theta(n)}-1\right)\\ &= -\theta(n)\left(1-\exp\left(\frac{1}{\theta(n)}\log(2)\right)\right)\\ &= \log(2)+\frac{(\log(2))^2}{2}\frac{1}{\theta(n)}+o\left(\frac{1}{\theta(n)}\right)\\ &\to\log(2),\quad n\to\infty, \end{aligned}$$

which implies

$$\frac{\theta(n)}{\log(n)}\to\frac{\log(2)}{\xi},\quad n\to\infty. \tag{5.107}$$

From (5.107) *we can again deduce*

$$\left(1+\frac{x}{\log|c|-\log(n)}\right)^{\theta(n)} \to 2^{x/\xi}, \quad n\to\infty,$$

for any $x \in \mathbb{R}$, *because of*

$$\begin{aligned}\theta(n)\log\left(1+\frac{x}{\log|c|-\log(n)}\right) &= \theta(n)\left(\frac{x}{\log|c|-\log(n)}+o\left(\frac{1}{\log(n)}\right)\right)\\ &= \theta(n)\frac{x}{\log|c|-\log(n)}+o(1)\\ &= x\frac{\log(n)}{\log|c|-\log(n)}\frac{\theta(n)}{\log(n)}+o(1)\\ &\to x\frac{\log(2)}{\xi},\end{aligned}$$

as $n \to \infty$. *This leads to*

$$\left(\frac{1+\frac{\log(1-z)}{\log|c|-\log(n)}-\frac{\alpha}{\log|c|-\log(n)}}{1+\frac{\log(z)}{\log|c|-\log(n)}-\frac{\alpha}{\log|c|-\log(n)}}\right)^{\theta(n)} \to 2^{(\log(1-z)-\log(z))/\xi}, \quad n\to\infty, \tag{5.108}$$

and

$$\begin{aligned}\psi(n) := &\frac{1}{2}\left(1+\frac{\log(z)}{\log|c|-\log(n)}-\frac{\alpha}{\log|c|-\log(n)}\right)^{\theta(n)}\\ &+\frac{1}{2}\left(1+\frac{\log(1-z)}{\log|c|-\log(n)}-\frac{\alpha}{\log|c|-\log(n)}\right)^{\theta(n)}+\frac{\alpha^{\theta(n)}}{2(\log(n)-\log|c|)^{\theta(n)}}\\ &\to 2^{\log(z)-\alpha-1}+2^{\log(1-z)-\alpha-1}, \quad n\to\infty,\end{aligned}$$

where the latter convergence is also due to (5.105). *Next we have*

$$\begin{aligned}&2^{1/\theta(n)}(\log(n)-\log|c|)(\psi(n))^{1/\theta(n)}-\log(n)+\log(|c|z)\\ &= -2^{1/\theta(n)}\log|c|(\psi(n))^{1/\theta(n)}+\log(n)\left((2\psi(n))^{1/\theta(n)}-1\right)+\log|c|+\log(z)\\ &\to \xi+\log(z), \quad n\to\infty,\end{aligned} \tag{5.109}$$

because of

$$\begin{aligned}&\log(n)\left((2\psi(n))^{1/\theta(n)}-1\right)\\ &= \log(n)\left(\exp\left(\frac{1}{\theta(n)}\log(2\psi(n))\right)-1\right)\\ &= \log(n)\left(\frac{1}{\theta(n)}\log(2)+\frac{1}{\theta(n)}\log(\psi(n))+o\left(\frac{1}{\theta(n)}\right)\right)\\ &\to \xi, \quad n\to\infty.\end{aligned}$$

The convergences (5.106), (5.108) *and* (5.109) *finally lead to*

$$g_n(z, c/n) \to \log(z) + \xi - \alpha + \log\left(1 + 2^{(\log(1-z)-\log(z))/\xi}\right) =: g(z),$$

as $n \to \infty$. Obviously, the function g is continuous and measurable. Applying Corollary 5.2.8 we obtain

$$\begin{aligned}
&H^n_{\theta(n)}\left(\frac{x}{n}, \frac{y}{n}\right)\\
&\to \exp\left(x\left(1-\exp\left(-\log\left(\frac{x}{x+y}\right) - \xi + \alpha - \log\left(1+2^{\log(y)-\log(x)}\right)\right)\right)\right.\\
&\qquad \left. + y\left(1-\exp\left(-\log\left(\frac{y}{x+y}\right) - \xi + \alpha - \log\left(1+2^{\log(x)-\log(y)}\right)\right)\right)\right)\\
&= \exp\Bigg(x+y\\
&\qquad -(x+y)\exp(\alpha-\xi)\left(\left\{1+2^{\log(y)-\log(x)}\right\}^{-1} + \left\{1+2^{\log(x)-\log(y)}\right\}^{-1}\right)\Bigg)\\
&= \exp((x+y)(1-\exp(\alpha-\xi))),
\end{aligned}$$

as $n \to \infty$. Obviously, $\xi = \alpha$ is a degenerate case. For $\xi \neq \alpha$ we have independence in the limit again because the limiting function is the product distribution function — this time with scale parameter

$$\frac{1}{1-\exp(\alpha-\xi)}.$$

5.3 Effects on the power of the test on tail dependence

As spectral expansions or expansions of Pickands densities, respectively, are the basis of the test on tail dependence, our next aim is to analyze what effect it has on the power of the test if the exponent of variation in the underlying expansion varies with the sample size.

Consider again the approximate power function

$$\psi_{m,\alpha}(\beta) \approx 1 - \Phi\left((1+\beta)\Phi^{-1}(1-\alpha) - \beta m^{1/2}\right), \quad \beta \geq 0,$$

in (4.3) where m is the sample size. If β is fixed, i.e. under condition (3.13) or (3.20) in the bivariate case, we obtain

$$\psi_{m,\alpha}(\beta) \to \psi_\alpha(\beta) := \begin{cases} \alpha, & \beta = 0 \\ 1, & \beta > 0 \end{cases} \tag{5.110}$$

as $m \to \infty$, meaning that the type I error rate converges to α whereas the type II error rate converges to 0.

Now let the parameter β vary as the sample size increases, i.e. $\beta = \beta(m) \geq 0$, and assume

$$\beta(m) \to 0, \quad m \to \infty,$$

cf. the conditions (5.24) and (5.27) or (5.15) and (5.17) in the bivariate case. Obviously,

$$(1+\beta(m))\Phi^{-1}(1-\alpha) \to \Phi^{-1}(1-\alpha), \quad m \to \infty,$$

so we have to deal only with the convergence of the term $\beta(m)m^{1/2}$. Because $\beta(m) \geq 0$, we get

$$\beta(m)m^{1/2} \to \mu \in [0, \infty], \quad m \to \infty.$$

Let us consider three cases:

$$\mu = 0 \quad \Rightarrow \quad \psi_{m,\alpha}(\beta(m)) \to \alpha, \quad m \to \infty, \tag{5.111}$$

$$\mu \in (0, \infty) \quad \Rightarrow \quad \psi_{m,\alpha}(\beta(m)) \to 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \mu\right) \in (\alpha, 1), \quad m \to \infty, \tag{5.112}$$

$$\mu = \infty \quad \Rightarrow \quad \psi_{m,\alpha}(\beta(m)) \to 1, \quad m \to \infty. \tag{5.113}$$

The first and the third case contain the convergence in (5.110) if the sequence $(\beta(m))_{m\in\mathbb{N}}$ is constant, i.e. $\beta(m) \equiv 0$ or $\beta(m) \equiv \beta > 0$ respectively. In the sequel we assume that $\beta(m) > 0$ for every $m \in \mathbb{N}$ so that we are considering the alternative of the test on tail dependence. In the case (5.111) the type II error rate converges to $1-\alpha$, meaning that the power of the test becomes as bad as possible as the sample size tends to infinity. In case (5.112) the type II error rate tends to a value between 0 and $1-\alpha$ in the limit and the limiting power of the test depends on μ. Finally, the best possible performance of the test is achieved in case (5.112) where the type II error rate converges to 0.

An example for the last case is given by the bivariate standard normal distribution with a sequence of correlation coefficients $(\rho(m))_{m\in\mathbb{N}}$ satisfying

$$(1-\rho(m))\log(m) \to \lambda^2 \in (0, \infty], \quad m \to \infty, \tag{5.114}$$

cf. condition (5.59) in Example 5.1.18. Because $\beta(m)$ is given by

$$\beta(m) = \frac{1-\rho(m)}{1+\rho(m)},$$

the convergence (5.114) implies

$$\beta(m)m^{1/2} \to \infty, \quad m \to \infty.$$

6 Results for other univariate marginal distributions

Till now we have mainly considered d–variate max–stable distribution functions with reversely exponential margins as stated in Chapter 2. Distribution functions with other margins have been standardized by the transformation given in (2.4) or have at least been transformed to margins belonging to the max–domain of attraction of the reversely exponential distribution function. In this chapter we deal with distribution functions having margins that belong to the max–domain of attraction of arbitrary univariate EVDs. Section 6.1 gives an overview of basic definitions from Chapter 2 transferred to this more general situation. The modified Pickands transform for general EVDs constitutes an important part thereof. We also add some basic results which are taken from Falk et al. [10], pp. 156–157. Most of the content and notation in this section follows [10], p. 144, pp. 156–159, and pp. 201–202. In Section 6.2 we show that spectral densities and Pickands densities and, thus, their expansions are the same for different types of marginal distributions provided that the modified Pickands transform is used. Therefore, in Section 6.3 we are able to reformulate results of Chapter 5 for univariate margins belonging to the max–domain of attraction of any univariate EVD.

6.1 Basic definitions and results

In the following part we consider d–dimensional EVDs G with margins following an arbitrary univariate EVD. Let the i–th marginal distribution function of G be given by

$$G_i(x) = \exp(\psi_{\alpha_i}(x)), \quad 1 \le i \le d, \tag{6.1}$$

where

$$\psi_{\alpha_i}(x) := \begin{cases} -(-x)^{-\alpha_i}, & x \le 0, \quad \text{if } \alpha_i < 0 \\ -x^{-\alpha_i}, & x > 0, \quad \text{if } \alpha_i > 0 \\ -\exp(-x), & x \in \mathbb{R}, \quad \text{if } \alpha_i = 0. \end{cases} \tag{6.2}$$

We thereby get the family of (reverse) Weibull, Fréchet and Gumbel distribution functions. In each case ψ_{α_i} is a strictly monotone and continuous function whose range is given by $(-\infty, 0)$. Note that G_i with $\alpha_i = -1$ is again the reversely exponential marginal distribution function.

In the sequel we denote a d–dimensional max–stable distribution function with univariate marginal distribution functions G_i, $i \le d$, by $G_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$. The distribution function G with reversely exponential margins is now denoted by $G_{(-1,\ldots,-1)}$. The subsequent lemma shows that the max–stability of $G_{\boldsymbol{\alpha}}$ is preserved if the margins are transformed to follow the reversely exponential distribution function, cf. Lemma 5.4.7 in [10].

Lemma 6.1.1
Suppose that the random vector $\mathbf{X} = (X_1, \ldots, X_d)$ *has the distribution function* $G_{\boldsymbol{\alpha}}$*. Let*

$$U_i := \psi_{\alpha_i}(X_i), \quad 1 \le i \le d.$$

Then the random vector $\mathbf{U} = (U_1, \ldots, U_d)$ *has the distribution function* $G_{(-1,\ldots,-1)}$.

PROOF. See [10], Lemma 5.4.7. □

According to [10], p. 157, we can reformulate Lemma 6.1.1 as

$$G_{\boldsymbol{\alpha}}\left(\psi_{\alpha_1}^{-1}(x_1), \ldots, \psi_{\alpha_d}^{-1}(x_d)\right) = G_{(-1,\ldots,-1)}(x_1, \ldots, x_d), \quad \mathbf{x} < 0. \tag{6.3}$$

This result together with representation (2.8) of $G_{(-1,\ldots,-1)} = G_D$ leads to the representation

$$\begin{aligned} &G_{\boldsymbol{\alpha}}(x_1, \ldots, x_d) \\ &= G_{(-1,\ldots,-1)}\left(\psi_{\alpha_1}(x_1), \ldots, \psi_{\alpha_d}(x_d)\right) \\ &= \exp\left(\left(\sum_{i \le d} \psi_{\alpha_i}(x_i)\right) D\left(\frac{\psi_{\alpha_1}(x_1)}{\sum_{i\le d}\psi_{\alpha_i}(x_i)}, \ldots, \frac{\psi_{\alpha_{d-1}}(x_{d-1})}{\sum_{i\le d}\psi_{\alpha_i}(x_i)}\right)\right) \end{aligned} \tag{6.4}$$

of an arbitrary max–stable distribution function $G_{\boldsymbol{\alpha}}$, where D is a Pickands dependence function as defined in (2.9).

From Chapter 2 we know that the random variables $U_i = \psi_{\alpha_i}(X_i)$, $1 \le i \le d$, from Lemma 6.1.1 with joint distribution function $G_{(-1,\ldots,-1)}$ are independent if, and only if, $D = 1$. According to representation (6.4) this is the case if, and only if, the random variables $X_1, \ldots, X_d$ are independent. The same is true asymptotically for distribution functions belonging to the max–domain of attraction of an EVD. To see this we first repeat the assertion of Lemma 6.1.1 for arbitrary distribution functions and apply a result from [10], p. 144, concerning max–domains of attractions: Consider an arbitrary d–dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)$ with distribution function $H_{\boldsymbol{\alpha}}$ and let $U_i = \psi_{\alpha_i}(X_i)$, $1 \le i \le d$. Then $\mathbf{U} = (U_1, \ldots, U_d)$ has the distribution function $H_{(-1,\ldots,-1)}$. In other words, we have

$$\begin{aligned} H_{\boldsymbol{\alpha}}\left(\psi_{\alpha_1}^{-1}(x_1), \ldots, \psi_{\alpha_d}^{-1}(x_d)\right) &= H_{(-1,\ldots,-1)}(x_1, \ldots, x_d) \\ &=: H(x_1, \ldots, x_d), \quad \mathbf{x} < 0. \end{aligned} \tag{6.5}$$

Then, according to [10], p. 144, it follows that

$$H \in \mathcal{D}\left(G_{(-1,\ldots,-1)}\right) \Leftrightarrow H_{\boldsymbol{\alpha}} \in \mathcal{D}(G_{\boldsymbol{\alpha}}), \tag{6.6}$$

cf. Definition 2.1.1.

Now we can formulate our result concerning tail independence.

Lemma 6.1.2
Let $H_{\boldsymbol{\alpha}}$ be the distribution function of a d–variate random vector $\mathbf{X} = (X_1, \dots, X_d)$ and assume that $H_{\boldsymbol{\alpha}}$ belongs to the max–domain of attraction of an EVD $G_{\boldsymbol{\alpha}}$. Then the random variables $X_1, \dots, X_d$ are tail independent if, and only if, the random variables $U_i = \psi_{\alpha_i}(X_i)$, $1 \le i \le d$, are tail independent.

PROOF. The assertion follows directly from Definition 2.3.1, equivalence (6.6), and representation (6.4). □

From Lemma 6.1.2 we deduce that it is possible to test the tail dependence of the random variables $X_1, \dots, X_d$, whose joint distribution function coincides in its upper tail with $G_{\boldsymbol{\alpha}}$, by testing the tail dependence of the random variables $\psi_{\alpha_1}(X_1), \dots, \psi_{\alpha_d}(X_d)$, see also [10], p. 202.

The GPD corresponding to an EVD $G_{\boldsymbol{\alpha}}$ is again given by any distribution function $W_{\boldsymbol{\alpha}}$ that has the representation

$$W_{\boldsymbol{\alpha}}(\mathbf{x}) = 1 + \log(G_{\boldsymbol{\alpha}}(\mathbf{x})), \quad \log(G_{\boldsymbol{\alpha}}(\mathbf{x})) \ge -1,$$

in a neighborhood of $\mathbf{0}$. The upper tails of the univariate margins W_i of $W_{\boldsymbol{\alpha}}$ coincide with those of univariate GPDs, i.e.

$$\begin{array}{lll} W_{2,\alpha_i}(x) = 1 - (-x)^{-\alpha_i}, & -1 \le x \le 0, & \text{if } \alpha_i < 0, \\ W_{1,\alpha_i}(x) = 1 - x^{-\alpha_i}, & x \ge 1, & \text{if } \alpha_i > 0, \\ W_0(x) = 1 - \exp(-x), & x \ge 0, & \text{if } \alpha_i = 0. \end{array} \tag{6.7}$$

The distribution functions in (6.7) constitute the family of beta, Pareto and exponential distribution functions, cf. [41], p. 24. The GPD pertaining to the EVD $G_{(-1,\dots,-1)}$ with reversely exponential margins is now denoted by $W_{(-1,\dots,-1)}$.

For GPDs we obtain a result that is analogue to Lemma 6.1.1 for EVDs, cf. Corollary 5.4.8 in [10].

Corollary 6.1.3
Suppose that the random vector $\mathbf{X} = (X_1, \dots, X_d)$ has the distribution function $W_{\boldsymbol{\alpha}}$. Let

$$U_i := \psi_{\alpha_i}(X_i), \quad 1 \le i \le d.$$

Then the random vector $\mathbf{U} = (U_1, \dots, U_d)$ has the distribution function $W_{(-1,\dots,-1)}$.

PROOF. See [10], Corollary 5.4.8. □

According to [10], pp. 157–158, we can again reformulate Corollary 6.1.3 as

$$W_{\boldsymbol{\alpha}}\left(\psi_{\alpha_1}^{-1}(x_1), \dots, \psi_{\alpha_d}^{-1}(x_d)\right) = W_{(-1,\dots,-1)}(x_1, \dots, x_d), \quad c_0 < x_i < 0, 1 \le i \le d,$$

for c_0 next to 0. If $\sum_{i \le d} \psi_{\alpha_i}(x_i)$ is close enough to 0, we have

$$\begin{aligned} W_{\boldsymbol{\alpha}}(x_1, \dots, x_d) &= W_{(-1,\dots,-1)}\left(\psi_{\alpha_1}(x_1), \dots, \psi_{\alpha_d}(x_d)\right) \\ &= 1 + \left(\sum_{i \le d} \psi_{\alpha_i}(x_i)\right) D\left(\frac{\psi_{\alpha_1}(x_1)}{\sum_{i \le d} \psi_{\alpha_i}(x_i)}, \dots, \frac{\psi_{\alpha_{d-1}}(x_{d-1})}{\sum_{i \le d} \psi_{\alpha_i}(x_i)}\right). \end{aligned} \tag{6.8}$$

Motivated by the representations (6.4) and (6.8) we introduce the modified Pickands coordinates

$$c_{\boldsymbol{\alpha}} := \sum_{i \le d} \psi_{\alpha_i}(x_i) \le 0 \tag{6.9}$$

and

$$\mathbf{z}_{\boldsymbol{\alpha}} := \left(\frac{\psi_{\alpha_1}(x_1)}{c_{\boldsymbol{\alpha}}}, \dots, \frac{\psi_{\alpha_{d-1}}(x_{d-1})}{c_{\boldsymbol{\alpha}}} \right) \in R \tag{6.10}$$

of an arbitrary vector $\mathbf{x} = (x_1, \dots, x_d)$ with x_i in the domain of ψ_{α_i}, $1 \le i \le d$, cf. (6.2). For $\alpha_i = -1, 1 \le i \le d$, these modified Pickands coordinates coincide with the Pickands coordinates given in Definition 2.2.1. As in Chapter 2 we can now define spectral decompositions based on these modified Pickands coordinates. For a d–dimensional distribution function $H_{\boldsymbol{\alpha}}$ whose i–th margin has a support belonging to the domain of ψ_{α_i}, $1 \le i \le d$, put

$$(H_{\boldsymbol{\alpha}})_{\mathbf{z}}(c) := H_{\boldsymbol{\alpha}} \left(\psi_{\alpha_1}^{-1}(cz_1), \dots, \psi_{\alpha_{d-1}}^{-1}(cz_{d-1}), \psi_{\alpha_d}^{-1} \left(c \left(1 - \sum_{i \le d-1} z_i \right) \right) \right) \tag{6.11}$$

for $c \le 0$ and $\mathbf{z} \in R$. The function $(H_{\boldsymbol{\alpha}})_{\mathbf{z}}$ is again a univariate distribution function on $(-\infty, 0]$ for any fixed $\mathbf{z}$.

Obviously, we obtain

$$\begin{aligned}
(H_{\boldsymbol{\alpha}})_{\mathbf{z}}(c) &= H_{\boldsymbol{\alpha}} \left(\psi_{\alpha_1}^{-1}(cz_1), \dots, \psi_{\alpha_{d-1}}^{-1}(cz_{d-1}), \psi_{\alpha_d}^{-1} \left(c \left(1 - \sum_{i \le d-1} z_i \right) \right) \right) \\
&= H \left(cz_1, \dots, cz_{d-1}, c \left(1 - \sum_{i \le d-1} z_i \right) \right) && (6.12) \\
&= H_{\mathbf{z}}(c). && (6.13)
\end{aligned}$$

For an EVD $G_{\boldsymbol{\alpha}}$ and the corresponding GPD $W_{\boldsymbol{\alpha}}$ we put

$$G_{\boldsymbol{\alpha}}^{\boldsymbol{\psi}}(x_1, \dots, x_d) := G_{\boldsymbol{\alpha}} \left(\psi_{\alpha_1}^{-1}(x_1), \dots, \psi_{\alpha_d}^{-1}(x_d) \right), \quad \mathbf{x} < 0,$$

and

$$W_{\boldsymbol{\alpha}}^{\boldsymbol{\psi}}(x_1, \dots, x_d) := W_{\boldsymbol{\alpha}} \left(\psi_{\alpha_1}^{-1}(x_1), \dots, \psi_{\alpha_d}^{-1}(x_d) \right), \quad \mathbf{x} < 0.$$

Thus, from Lemma 6.1.1 and Corollary 6.1.3 together with (2.18) and (2.19), we obtain

$$\left(G_{\boldsymbol{\alpha}}^{\boldsymbol{\psi}} \right)_{\mathbf{z}}(c) = \exp(cD(\mathbf{z})), \quad c \le 0, \mathbf{z} \in R,$$

and

$$\left(W_{\boldsymbol{\alpha}}^{\boldsymbol{\psi}} \right)_{\mathbf{z}}(c) = 1 + cD(\mathbf{z}), \quad c_0 \le c \le 0, \mathbf{z} \in R,$$

cf. [10], p. 158.

Let us now have a look at one special example of a bivariate EVD, namely the Hüsler–Reiss EVD, cf. Example 5.4.9 in [10].

Example 6.1.4
Usually the bivariate Hüsler–Reiss EVD is given with Gumbel margins, i.e.

$$H_\lambda(x,y) = \exp\left(-\Phi\left(\lambda + \frac{y-x}{2\lambda}\right)\exp(-x) - \Phi\left(\lambda + \frac{x-y}{2\lambda}\right)\exp(-y)\right), \quad x, y \in \mathbb{R},$$

where Φ is the standard normal distribution function, see also representation (2.7) in [29]. Thus, in the notation of this section, we have $H_\lambda = G_{(0,0)}$. According to (6.3) *we obtain*

$$\begin{aligned} G_{(-1,-1)} &= H_\lambda\left(\psi_0^{-1}(x), \psi_0^{-1}(y)\right) \\ &= \exp\left(x\Phi\left(\lambda + \frac{\log|x| - \log|y|}{2\lambda}\right) + y\Phi\left(\lambda + \frac{\log|y| - \log|x|}{2\lambda}\right)\right), \quad x, y \le 0, \end{aligned}$$

which is again the Hüsler–Reiss distribution function with reversely exponential margins as in (5.100).

For the spectral decomposition we obtain

$$\begin{aligned} \left(H_{\lambda,(0,0)}\right)_z(c) &= H_\lambda\left(\psi_0^{-1}(cz), \psi_0^{-1}(c(1-z))\right) \\ &= \exp\left(cD_\lambda(z)\right), \end{aligned}$$

where

$$D_\lambda(z) := z\Phi\left(\lambda + \frac{\log(z/(1-z))}{2\lambda}\right) + (1-z)\Phi\left(\lambda + \frac{\log((1-z)/z)}{2\lambda}\right), \quad z \in [0,1].$$

By analogy with Chapter 2 the introduction of the modified Pickands coordinates (6.9) and (6.10) motivates the definition of the corresponding modified Pickands transform of a random vector $\mathbf{X} = (X_1, \dots, X_d)$ with distribution function $H_{\boldsymbol{\alpha}}$. Let Ψ_i be the domain of ψ_i, $1 \le i \le d$, according to (6.2), i.e.

$$\Psi_i := \begin{cases} (-\infty, 0], & \text{if } \alpha_i < 0 \\ (0, \infty), & \text{if } \alpha_i > 0 \\ \mathbb{R}, & \text{if } \alpha_i = 0. \end{cases} \tag{6.14}$$

Then we define the transformation

$$T_{\boldsymbol{\alpha}} : \mathop{\times}_{i \le d} \Psi_i \to R \times (-\infty, 0)$$

by

$$\begin{aligned} T_{\boldsymbol{\alpha}}(\mathbf{x}) &= (T_{\boldsymbol{\alpha},1}(\mathbf{x}), T_{\boldsymbol{\alpha},2}(\mathbf{x})) \\ &:= \left(\frac{\psi_{\alpha_1}(x_1)}{\sum_{i \le d}\psi_{\alpha_i}(x_i)}, \dots, \frac{\psi_{\alpha_{d-1}}(x_{d-1})}{\sum_{i \le d}\psi_{\alpha_i}(x_i)}, \sum_{i \le d}\psi_{\alpha_i}(x_i)\right). \end{aligned} \tag{6.15}$$

This is the transformation of $\mathbf{x} = (x_1, \dots, x_d)$ onto its modified Pickands coordinates $\mathbf{z}_{\boldsymbol{\alpha}} := T_{\boldsymbol{\alpha},1}(\mathbf{x}) \in R$ and $c_{\boldsymbol{\alpha}} := T_{\boldsymbol{\alpha},2}(\mathbf{x}) \in (-\infty, 0)$. It is one–to–one with the inverse function

$$T_{\boldsymbol{\alpha}}^{-1}(\mathbf{z}, c) = \left(\psi_{\alpha_1}^{-1}(cz_1), \dots, \psi_{\alpha_{d-1}}^{-1}(cz_{d-1}), \psi_{\alpha_d}^{-1}\left(c \left(1 - \sum_{i \le d-1} z_i \right) \right) \right). \tag{6.16}$$

We call the random vector

$$(\mathbf{Z}_{\boldsymbol{\alpha}}, C_{\boldsymbol{\alpha}}) := T_{\boldsymbol{\alpha}}(\mathbf{X})$$

the modified Pickands transform of the random vector $\mathbf{X}$ onto its modified Pickands coordinates, cf. [10], p. 158.

6.2 Spectral densities and Pickands densities

After the definition of spectral decompositions for distribution functions $H_{\boldsymbol{\alpha}}$ and of the modified Pickands transform we now analyze the pertaining spectral densities and Pickands densities.

Let $H_{\boldsymbol{\alpha}}$ be a distribution function on $\times_{i \le d} \Psi_i$ as defined in (6.14) and assume that its spectral decomposition $(H_{\boldsymbol{\alpha}})_{\mathbf{z}}$ possesses a positive derivative with respect to c. Then we can define the spectral density of $H_{\boldsymbol{\alpha}}$ in the same way as in Definition 3.1.1, namely by

$$\left(h_{\boldsymbol{\alpha}}\right)_{\mathbf{z}}(c) := \frac{\partial}{\partial c}(H_{\boldsymbol{\alpha}})_{\mathbf{z}}(c)$$

for c near 0 and $\mathbf{z} \in R$. As a consequence of (6.12) we get

$$\left(h_{\boldsymbol{\alpha}}\right)_{\mathbf{z}}(c) = h_{\mathbf{z}}(c), \tag{6.17}$$

where $h_{\mathbf{z}}$ is the spectral density of $H = H_{(-1,\dots,-1)}$. These results lead to the following lemma about the spectral densities of $H_{\boldsymbol{\alpha}}$ and H.

Lemma 6.2.1
Let H be a distribution function on $(-\infty, 0]^d$ and let $H_{\boldsymbol{\alpha}}$ be the distribution function given by

$$H_{\boldsymbol{\alpha}}(x_1, \dots, x_d) = H(\psi_{\alpha_1}(x_1), \dots, \psi_{\alpha_d}(x_d)).$$

Then H satisfies a spectral expansion of length $k+1$ in the sense of Definition 3.1.3 if, and only if, $H_{\boldsymbol{\alpha}}$ satisfies a spectral expansion of length $k+1$. In this case the two of them coincide.

PROOF. The assertion is due to equation (6.17). □

Next we will deduce a similar result for the Pickands density pertaining to $H_{\boldsymbol{\alpha}}$, i.e. the density of the modified Pickands transform $(\mathbf{Z}_{\boldsymbol{\alpha}}, C_{\boldsymbol{\alpha}}) = T_{\boldsymbol{\alpha}}(\mathbf{X})$ of a random vector $\mathbf{X}$ with distribution function $H_{\boldsymbol{\alpha}}$. Therefore we first calculate the determinant of the functional matrix of $T_{\boldsymbol{\alpha}}$.

Lemma 6.2.2
Let $J_{\boldsymbol{\alpha}}$ be the functional matrix of the transformation $T_{\boldsymbol{\alpha}}$ defined in (6.15). *Then we have*

$$\det(J_{\boldsymbol{\alpha}}(\mathbf{x})) = \frac{\prod_{i\le d}\psi'_{\alpha_i}(x_i)}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^{d-1}}. \tag{6.18}$$

PROOF. The functional matrix $J_{\boldsymbol{\alpha}}$ of $T_{\boldsymbol{\alpha}}$ is given by

$$J_{\boldsymbol{\alpha}}(\mathbf{x}) = M_{\boldsymbol{\alpha}}(\mathbf{x})N_{\boldsymbol{\alpha}}(\mathbf{x}),$$

where $M_{\boldsymbol{\alpha}}$ and $N_{\boldsymbol{\alpha}}$ are defined by

$$M_{\boldsymbol{\alpha}}(\mathbf{x}) := \begin{pmatrix} \psi'_{\alpha_1}(x_1) & & & 0 \\ & \psi'_{\alpha_2}(x_2) & & \\ & & \ddots & \\ 0 & & & \psi'_{\alpha_d}(x_d) \end{pmatrix}$$

and

$$N_{\boldsymbol{\alpha}}(\mathbf{x}) := \begin{pmatrix} \frac{\sum_{i\le d}\psi_{\alpha_i}(x_i)-\psi_{\alpha_1}(x_1)}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & \frac{-\psi_{\alpha_2}(x_2)}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & \cdots & \frac{-\psi_{\alpha_{d-1}}(x_{d-1})}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & 1 \\ \frac{-\psi_{\alpha_1}(x_1)}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & \frac{\sum_{i\le d}\psi_{\alpha_i}(x_i)-\psi_{\alpha_2}(x_2)}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & \cdots & \frac{-\psi_{\alpha_{d-1}}(x_{d-1})}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & 1 \\ \vdots & & \ddots & & \vdots \\ \frac{-\psi_{\alpha_1}(x_1)}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & \frac{-\psi_{\alpha_2}(x_2)}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & \cdots & \frac{\sum_{i\le d}\psi_{\alpha_i}(x_i)-\psi_{\alpha_{d-1}}(x_{d-1})}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & 1 \\ \frac{-\psi_{\alpha_1}(x_1)}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & \frac{-\psi_{\alpha_2}(x_2)}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & \cdots & \frac{-\psi_{\alpha_{d-1}}(x_{d-1})}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & 1 \end{pmatrix}$$

respectively. By elementary matrix manipulation we obtain

$$EN_{\boldsymbol{\alpha}}(\mathbf{x}) = \widetilde{N_{\boldsymbol{\alpha}}}(\mathbf{x})$$

with the elementary matrix

$$E := \begin{pmatrix} 1 & & & 0 & -1 \\ & 1 & & & -1 \\ & & \ddots & & \\ & & & 1 & -1 \\ 0 & & & & 1 \end{pmatrix}$$

and

$$\widetilde{N_{\boldsymbol{\alpha}}}(\mathbf{x}) := \begin{pmatrix} \frac{1}{\left|\sum_{i\le d}\psi_{\alpha_i}(x_i)\right|} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\left|\sum_{i\le d}\psi_{\alpha_i}(x_i)\right|} & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & \frac{1}{\left|\sum_{i\le d}\psi_{\alpha_i}(x_i)\right|} & 0 \\ \frac{-\psi_{\alpha_1}(x_1)}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & \frac{-\psi_{\alpha_2}(x_2)}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & \cdots & \frac{-\psi_{\alpha_{d-1}}(x_{d-1})}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^2} & 1 \end{pmatrix}.$$

The determinants of the diagonal or triangular matrices $M_{\boldsymbol{\alpha}}(\mathbf{x})$, E, and $\widetilde{N_{\boldsymbol{\alpha}}}(\mathbf{x})$ can easily be computed, they are simply the product of the particular diagonal elements. Thus, applying the Cauchy–Binet theorem for the calculation of the determinant of a product of matrices, cf. Theorem 6.1 in Marcus and Minc [35], Section 2.6, we get

$$\begin{aligned}\det(J_{\boldsymbol{\alpha}}(\mathbf{x})) &= \det(M_{\boldsymbol{\alpha}}(\mathbf{x}))(\det(E))^{-1}\det\left(\widetilde{N_{\boldsymbol{\alpha}}}(\mathbf{x})\right)\\ &= \left(\prod_{i\le d}\psi'_{\alpha_i}(x_i)\right)\cdot 1\cdot\frac{1}{\left(\sum_{i\le d}\psi_{\alpha_i}(x_i)\right)^{d-1}},\end{aligned}$$

which proves the assertion. □

As another auxiliary result we establish a relationship between the Pickands density and the density of $H_{\boldsymbol{\alpha}}$. Provided that $H_{\boldsymbol{\alpha}}$ possesses continuous partial derivatives of the order d next to its upper endpoint, a density of $H_{\boldsymbol{\alpha}}$ is given by

$$h_{\boldsymbol{\alpha}}(x_1,\dots,x_d) := \frac{\partial^d}{\partial x_1\cdots\partial x_d}H_{\boldsymbol{\alpha}}(x_1,\dots,x_d)$$

for $\mathbf{x} = (x_1,\dots,x_d)$ next to the upper endpoint of $H_{\boldsymbol{\alpha}}$.

Lemma 6.2.3
If the distribution function $H_{\boldsymbol{\alpha}}$ of $\mathbf{X} = (X_1,\dots,X_d)$ possesses a density $h_{\boldsymbol{\alpha}}(x_1,\dots,x_d)$ next to its upper endpoint, there exists a $c_0 < 0$ such that the modified Pickands transform $(\mathbf{Z}_{\boldsymbol{\alpha}}, C_{\boldsymbol{\alpha}}) = T_{\boldsymbol{\alpha}}(\mathbf{X})$ has the density

$$f_{\boldsymbol{\alpha}}(\mathbf{z},c) = \frac{|c|^{d-1}}{\left(\prod_{i\le d-1}\psi'_{\alpha_i}\left(\psi^{-1}_{\alpha_i}(cz_i)\right)\right)\cdot\psi'_{\alpha_d}\left(\psi^{-1}_{\alpha_d}\left(c\left(1-\sum_{i\le d-1}z_i\right)\right)\right)}h_{\boldsymbol{\alpha}}\left(T^{-1}_{\boldsymbol{\alpha}}(\mathbf{z},c)\right) \tag{6.19}$$

on $R\times(c_0,0)$.

PROOF. The transformation theorem for densities implies that $(\mathbf{Z}_{\boldsymbol{\alpha}}, C_{\boldsymbol{\alpha}})$ has the density

$$f_{\boldsymbol{\alpha}}(\mathbf{z},c) = \left|\det\left(J_{\boldsymbol{\alpha}}\left(T^{-1}_{\boldsymbol{\alpha}}(\mathbf{z},c)\right)\right)\right|^{-1} h_{\boldsymbol{\alpha}}\left(T^{-1}_{\boldsymbol{\alpha}}(\mathbf{z},c)\right), \tag{6.20}$$

where $J_{\boldsymbol{\alpha}}$ is the functional matrix of $T_{\boldsymbol{\alpha}}$. Substituting $\mathbf{x}$ in equation (6.18) of Lemma 6.2.2 by $T^{-1}_{\boldsymbol{\alpha}}(\mathbf{z},c)$ as given in (6.16) leads to

$$\begin{aligned}&\left|\det\left(J_{\boldsymbol{\alpha}}\left(T^{-1}_{\boldsymbol{\alpha}}(\mathbf{z},c)\right)\right)\right|^{-1}\\ &= \frac{\left|\sum_{i\le d-1}\psi_{\alpha_i}\left(\psi^{-1}_{\alpha_i}(cz_i)\right)+\psi_{\alpha_d}\left(\psi^{-1}_{\alpha_d}\left(c\left(1-\sum_{i\le d-1}z_i\right)\right)\right)\right|^{d-1}}{\left(\prod_{i\le d-1}\left|\psi'_{\alpha_i}\left(\psi^{-1}_{\alpha_i}(cz_i)\right)\right|\right)\cdot\left|\psi'_{\alpha_d}\left(\psi^{-1}_{\alpha_d}\left(c\left(1-\sum_{i\le d-1}z_i\right)\right)\right)\right|}\\ &= \frac{|c|^{d-1}}{\left(\prod_{i\le d-1}\psi'_{\alpha_i}\left(\psi^{-1}_{\alpha_i}(cz_i)\right)\right)\cdot\psi'_{\alpha_d}\left(\psi^{-1}_{\alpha_d}\left(c\left(1-\sum_{i\le d-1}z_i\right)\right)\right)}.\end{aligned} \tag{6.21}$$

Note that the derivatives ψ'_{α_i} of ψ_{α_i}, $1\le i\le d$, are always positive. Inserting (6.21) into (6.20) proves the assertion. □

Our next auxiliary result concerns the relationship between the densities $h_{\boldsymbol{\alpha}}$ and h, where the latter is given by

$$h(x_1, \dots, x_d) := \frac{\partial^d}{\partial x_1 \cdots \partial x_d} H(x_1, \dots, x_d)$$

for $\mathbf{x}$ in a neighborhood of $\mathbf{0}$, cf. (3.8).

Lemma 6.2.4
Consider the distribution functions $H_{\boldsymbol{\alpha}}$ and H, where H is defined on $(-\infty, 0]^d$ and $H_{\boldsymbol{\alpha}}$ is given by

$$H_{\boldsymbol{\alpha}}(x_1, \dots, x_d) = H\left(\psi_{\alpha_1}(x_1), \dots, \psi_{\alpha_d}(x_d)\right). \tag{6.22}$$

Assume that H possesses the density h in a neighborhood of $\mathbf{0}$. Then $H_{\boldsymbol{\alpha}}$ has a density $h_{\boldsymbol{\alpha}}$ next to its upper endpoint, too, which satisfies

$$h_{\boldsymbol{\alpha}}(x_1, \dots, x_d) = \left(\prod_{i \le d} \psi'_{\alpha_i}(x_i)\right) h\left(\psi_{\alpha_1}(x_1), \dots, \psi_{\alpha_d}(x_d)\right). \tag{6.23}$$

PROOF. The assertion is immediate from (6.22) and the application of the chain rule. □

Now we are able to formulate our result concerning the Pickands densities belonging to $H_{\boldsymbol{\alpha}}$ and H.

Lemma 6.2.5
Let $\mathbf{X} = (X_1, \dots, X_d)$ be a random vector with distribution function $H_{\boldsymbol{\alpha}}$. Assume that the distribution function H of the random vector $\mathbf{Y} := (\psi_{\alpha_1}(X_1), \dots, \psi_{\alpha_d}(X_d))$ possesses a density h such that the Pickands transform $(\mathbf{Z}, C) = T(\mathbf{Y})$ has the density

$$f(\mathbf{z}, c) = |c|^{d-1} h\left(T^{-1}(\mathbf{z}, c)\right)$$

on $R \times (c_0, 0)$ for a $c_0 < 0$. Then the modified Pickands transform $(\mathbf{Z}_{\boldsymbol{\alpha}}, C_{\boldsymbol{\alpha}}) = T_{\boldsymbol{\alpha}}(\mathbf{X})$ has the same density, i.e.

$$f_{\boldsymbol{\alpha}}(\mathbf{z}, c) = f(\mathbf{z}, c).$$

PROOF. Because the distribution function H of $\mathbf{Y}$ has a density h, the distribution function $H_{\boldsymbol{\alpha}}$ possesses a density $h_{\boldsymbol{\alpha}}$ satisfying equation (6.23), according to Lemma 6.2.4. Now from Lemma 6.2.3 it follows that the modified Pickands transform $(\mathbf{Z}_{\boldsymbol{\alpha}}, C_{\boldsymbol{\alpha}}) = T_{\boldsymbol{\alpha}}(\mathbf{X})$ has the density $f_{\boldsymbol{\alpha}}$ given in (6.19). Thus, using equation (6.16) for $T_{\boldsymbol{\alpha}}^{-1}$, we get

$$\begin{aligned}
f_{\boldsymbol{\alpha}}(\mathbf{z}, c) &= \frac{|c|^{d-1}}{\left(\prod_{i \le d-1} \psi'_{\alpha_i}\left(\psi_{\alpha_i}^{-1}(cz_i)\right)\right) \cdot \psi'_{\alpha_d}\left(\psi_{\alpha_d}^{-1}\left(c\left(1 - \sum_{i \le d-1} z_i\right)\right)\right)} h_{\boldsymbol{\alpha}}\left(T_{\boldsymbol{\alpha}}^{-1}(\mathbf{z}, c)\right) \\
&= |c|^{d-1} h\left(cz_1, \dots, cz_{d-1}, c\left(1 - \sum_{i \le d-1} z_i\right)\right) \\
&= |c|^{d-1} h\left(T^{-1}(\mathbf{z}, c)\right) \\
&= f(\mathbf{z}, c),
\end{aligned}$$

which completes the proof. □

6.3 Modified limiting distributions of maxima under triangular schemes

Against the background of Section 6.2 we are able to deduce limiting distributions of multivariate maxima under triangular schemes whose univariate margins belong to the max–domain of attraction of any univariate EVD.

Let therefore $\mathbf{X}_i = (X_{i1}, \ldots, X_{id})$, $i \leq n$, be independent identically distributed d–variate random vectors with common distribution function $H_{\boldsymbol{\alpha}}$. From Chapter 2 we know that the distribution function of the componentwise taken maximum of these random vectors is given by $H_{\boldsymbol{\alpha}}^n$, cf. (2.1). Our aim is to find vectors $\mathbf{d}_n$ and $\mathbf{c}_n$ and a suitable function $G_{\boldsymbol{\alpha}}$ so that

$$H_{\boldsymbol{\alpha}}^n(\mathbf{d}_n + \mathbf{c}_n\mathbf{x}) \to G_{\boldsymbol{\alpha}}(\mathbf{x}), \quad n \to \infty,$$

provided that $H_{\boldsymbol{\alpha}}$ fulfills certain preliminarities. We will present our results by reformulating some of the most important theorems of Chapter 5.

We choose the vectors $\mathbf{d}_n = (d_{n,1}, \ldots, d_{n,d})$ and $\mathbf{c}_n = (c_{n,1}, \ldots, c_{n,d})$ by using the normalizing constants $d_{n,i}$ and $c_{n,i}$ of the univariate marginal maxima. Assume that the i–th margin of $H_{\boldsymbol{\alpha}}$ belongs to the max–domain of attraction of the univariate EVD $\exp(\psi_{\alpha_i}(x))$ with $\psi_{\alpha_i}(x)$ as defined in (6.2). This means that we have

$$H_{\boldsymbol{\alpha},i}^n(d_{n,i} + c_{n,i}x_i) \to \exp(\psi_{\alpha_i}(x_i)), \quad n \to \infty,$$

for $1 \leq i \leq d$. We only consider univariate margins for which the normalizing constants $d_{n,i}$ and $c_{n,i}$ are given by

$$\begin{array}{lll} d_{n,i} = 0, & c_{n,i} = n^{1/\alpha_i}, & \text{if } \alpha_i < 0, \\ d_{n,i} = 0, & c_{n,i} = n^{1/\alpha_i}, & \text{if } \alpha_i > 0, \\ d_{n,i} = \log(n), & c_{n,i} = 1, & \text{if } \alpha_i = 0, \end{array} \tag{6.24}$$

according to [41], p. 18. In other cases the univariate margins have to be transformed appropriately first.

Because $H_{\boldsymbol{\alpha}}(\mathbf{d}_n + \mathbf{c}_n\mathbf{x})$ can be expressed in terms of H, i.e.

$$H_{\boldsymbol{\alpha}}(\mathbf{d}_n + \mathbf{c}_n\mathbf{x}) = H(\psi_{\alpha_1}(d_{n,1} + c_{n,1}x_1), \ldots, \psi_{\alpha_d}(d_{n,d} + c_{n,d}x_d)),$$

cf. (6.5), we will now have a closer look at the terms $\psi_{\alpha_i}(d_{n,i} + c_{n,i}x_i)$, $1 \leq i \leq d$, by using the constants given in (6.24). For $\alpha_i < 0$ we have

$$\begin{aligned} \psi_{\alpha_i}(d_{n,i} + c_{n,i}x_i) &= -\left(-n^{1/\alpha_i}x_i\right)^{-\alpha_i} \\ &= \frac{1}{n}\left(-(-x_i)^{-\alpha_i}\right) \\ &= \frac{1}{n}\psi_{\alpha_i}(x_i). \end{aligned}$$

If $\alpha_i > 0$, we obtain

$$\begin{aligned}\psi_{\alpha_i}(d_{n,i} + c_{n,i}x_i) &= -\left(n^{1/\alpha_i}x_i\right)^{-\alpha_i} \\ &= \frac{1}{n}\left(-x_i^{-\alpha_i}\right) \\ &= \frac{1}{n}\psi_{\alpha_i}(x_i).\end{aligned}$$

Finally, $\alpha_i = 0$ implies

$$\begin{aligned}\psi_{\alpha_i}(d_{n,i} + c_{n,i}x_i) &= -\exp(-(\log(n) + x_i)) \\ &= \exp(\log(1/n))(-\exp(-x_i)) \\ &= \frac{1}{n}\psi_{\alpha_i}(x_i).\end{aligned}$$

Thus, in each case, we have

$$\psi_{\alpha_i}(d_{n,i} + c_{n,i}x_i) = \frac{1}{n}\psi_{\alpha_i}(x_i), \quad 1 \le i \le d, \tag{6.25}$$

which implies

$$H_{\boldsymbol{\alpha}}(\mathbf{d}_n + \mathbf{c}_n\mathbf{x}) = H\left(\frac{\psi_{\alpha_1}(x_1)}{n}, \dots, \frac{\psi_{\alpha_d}(x_d)}{n}\right). \tag{6.26}$$

We can now reformulate our results of Chapter 5. The following theorem corresponds to Theorem 5.1.1.

Theorem 6.3.1
Let $H_{\boldsymbol{\alpha},\beta(n)}$, $\beta(n) = (\beta_1(n), \dots, \beta_k(n))$, $n \in \mathbb{N}$, be d–dimensional distribution functions and assume that the pertaining spectral densities satisfy expansions of length $k+1$

$$\left(h_{\boldsymbol{\alpha},\beta(n)}\right)_{\mathbf{z}}(c) = 1 + \sum_{j=1}^{k} B_{j,n}(c)A_{j,n}(\mathbf{z}) + R_n(\mathbf{z}, c), \quad k \in \mathbb{N}, \tag{6.27}$$

with $R_n(\mathbf{z}, c) = o(B_{k,n}(c))$ uniformly for $\mathbf{z} \in R$, as $c \uparrow 0$, according to (3.4)*, such that the conditions* (5.2) – (5.6) *of Theorem 5.1.1 are fulfilled. Then we have*

$$H^n_{\boldsymbol{\alpha},\beta(n)}(d_{n,1} + c_{n,1}y_1, \dots, d_{n,d} + c_{n,d}y_d) \to \exp\left(T_{\boldsymbol{\alpha},2}(\mathbf{y})\left(1 + \sum_{j=1}^{k}\lambda_j A_j(T_{\boldsymbol{\alpha},1}(\mathbf{y}))\right)\right) \tag{6.28}$$

$$=: G_{\boldsymbol{\alpha}}(y_1, \dots, y_d), \tag{6.29}$$

as $n \to \infty$, with the normalizing constants $d_{n,i}$ and $c_{n,i}$, $1 \le i \le d$, as given in (6.24) *and $T_{\boldsymbol{\alpha},1}$ and $T_{\boldsymbol{\alpha},2}$ as defined in* (6.15)*. Moreover, the limiting function $G_{\boldsymbol{\alpha}}$ is a distribution function.*

PROOF. In the sequel the convergence (6.28) is proved by referring the present situation back to that given in Theorem 5.1.1.

Due to Lemma 6.2.1 the distribution function $H_{\beta(n)}$ corresponding to $H_{\boldsymbol{\alpha},\beta(n)}$ satisfies the spectral expansion (6.27) as well. Therefore we can apply Theorem 5.1.1 to $H_{\beta(n)}$. The normalizing constants are given by $d_{n,i} = 0$ and $c_{n,i} = 1/n$, $1 \le i \le d$, in this case. Now, for $H_{\boldsymbol{\alpha},\beta(n)}$, we use the normalizing constants $d_{n,i}$ and $c_{n,i}$, $1 \le i \le d$, as given in (6.28). Then equation (6.26) and Theorem 5.1.1 directly imply the convergence (6.28).

The properties that characterize $G_{\boldsymbol{\alpha}}$ as a distribution function can again be proved similarly to the proof of Theorem 5.1.1. The continuity of $G_{\boldsymbol{\alpha}}$ follows from the continuity of the exponential function, of $T_{\boldsymbol{\alpha},1}$ and $T_{\boldsymbol{\alpha},2}$ and of the functions A_j, $j = 1, \dots, k$. Now let $\alpha(\psi_i)$ and $\omega(\psi_i)$ be the left and right endpoint of Ψ_i, respectively, i.e.

$$\alpha(\psi_i) := \begin{cases} -\infty, & \text{if } \alpha_i \le 0 \\ 0, & \text{if } \alpha_i > 0 \end{cases}$$

and

$$\omega(\psi_i) := \begin{cases} 0, & \text{if } \alpha_i < 0 \\ \infty, & \text{if } \alpha_i \ge 0, \end{cases}$$

cf. (6.14). For any sequence $(\mathbf{y}_n)_{n\in\mathbb{N}}$ with $y_{n,r} \uparrow \omega(\psi_r)$, $r = 1, \dots, d$, it follows that $\psi_r(y_{n,r}) \to 0$, $r = 1, \dots, d$, which implies

$$\psi_r(y_{n,r}) \left(1 + \sum_{j=1}^{k} \lambda_j A_j(T_{\boldsymbol{\alpha},1}(\mathbf{y}_n)) \right) \to 0, \quad r = 1, \dots, d,$$

as $n \to \infty$, due to the boundedness of the A_j. Therefore we have $G_{\boldsymbol{\alpha}}(\mathbf{y}_n) \to 1$, as $n \to \infty$, which means that $G_{\boldsymbol{\alpha}}$ is normed. Similarly, if $y_{n,r} \downarrow \alpha(\psi_r)$ for some $r \in \{1, \dots, d\}$, it follows that $\psi_r(y_{n,r}) \to -\infty$ and together with property (5.6) we obtain that

$$\psi_r(y_{n,r}) \left(1 + \sum_{j=1}^{k} \lambda_j A_j(T_{\boldsymbol{\alpha},1}(\mathbf{y}_n)) \right) \to -\infty$$

and, thus, $G_{\boldsymbol{\alpha}}(\mathbf{y}_n) \to 0$, as $n \to \infty$. We can again assume strict inequality in (5.6). Otherwise $G_{\boldsymbol{\alpha}}$ would be degenerate. The Δ–monotony holds because $G_{\boldsymbol{\alpha}}$ is the pointwise limit of a sequence of distribution functions. Thus, according to [10], Section 4.1, $G_{\boldsymbol{\alpha}}$ is a distribution function. □

We can again provide an additional result concerning the univariate margins of $G_{\boldsymbol{\alpha}}$.

Lemma 6.3.2
Let $H_{\boldsymbol{\alpha},\beta(n)}$, $n \in \mathbb{N}$, be d–variate distribution functions as in Theorem 6.3.1. If the limiting functions A_j, $j = 1, \dots, k$, additionally satisfy

$$\sum_{j=1}^{k} \lambda_j A_j(\mathbf{e_i}) = 0, \quad i = 1, \dots, d-1, \tag{6.30}$$

$$\sum_{j=1}^{k} \lambda_j A_j(\mathbf{0}) = 0, \tag{6.31}$$

where $\mathbf{e_i}$ *is the i–th unit vector in* $\mathbb{R}^{d-1}$, $i = 1, \ldots, d-1$, *then the i–th marginal distribution function belonging to* $G_{\boldsymbol{\alpha}}$ *is given by*

$$G_i(x) = \exp(\psi_{\alpha_i}(x)),$$

cf. (6.1).

If, conversely, the i–th univariate marginal distribution function of $H_{\boldsymbol{\alpha},\beta(n)}$ *belongs to the max–domain of attraction of* G_i, $i = 1, \ldots, d$, *then the limiting functions* A_j, $j = 1, \ldots, k$ *satisfy the properties* (6.30) *and* (6.31).

PROOF. For $i = 1, \ldots, d$ consider $\mathbf{y} = (y_1, \ldots, y_d)$ and let $y_r \to \omega(\psi_r)$, $r \neq i$. From the proof of Theorem 6.3.1 we know that this leads to $\psi_r(y_r) \to 0$, $r \neq i$. It follows that $T_{\alpha,1}(\mathbf{y}) \to \mathbf{e_i}$ if $i \in \{1, \ldots, d-1\}$, and $T_{\alpha,1}(\mathbf{y}) \to 0$ if $i = d$. Together with the continuity of the A_j this implies the assertion. □

The max–stability of the limiting distribution function still holds in the present context.

Lemma 6.3.3
The limiting distribution function $G_{\boldsymbol{\alpha}}$ *in* (6.29) *is max–stable according to* (2.3) *with the same normalizing constants as used in* (6.28).

PROOF. We have

$$\begin{aligned}
&G^n_{\boldsymbol{\alpha}}(d_{n,1} + c_{n,1}y_1, \ldots, d_{n,d} + c_{n,d}y_d)\\
&= \exp\left(n\left(\sum_{i\le d}\psi_{\alpha_i}(d_{n,i} + c_{n,i}y_i)\right)\right.\\
&\qquad\left.\times\left(1 + \sum_{j=1}^k \lambda_j A_j\left(\frac{\psi_{\alpha_1}(d_{n,1} + c_{n,1}y_1)}{\sum_{i\le d}\psi_{\alpha_i}(d_{n,i} + c_{n,i}y_i)}, \ldots, \frac{\psi_{\alpha_{d-1}}(d_{n,d-1} + c_{n,d-1}y_{d-1})}{\sum_{i\le d}\psi_{\alpha_i}(d_{n,i} + c_{n,i}y_i)}\right)\right)\right)\\
&= \exp\left(n\left(\sum_{i\le d}\frac{1}{n}\psi_{\alpha_i}(y_i)\right)\right.\\
&\qquad\left.\times\left(1 + \sum_{j=1}^k \lambda_j A_j\left(\frac{\frac{1}{n}\psi_{\alpha_1}(y_1)}{\sum_{i\le d}\frac{1}{n}\psi_{\alpha_i}(y_i)}, \ldots, \frac{\frac{1}{n}\psi_{\alpha_{d-1}}(y_{d-1})}{\sum_{i\le d}\frac{1}{n}\psi_{\alpha_i}(y_i)}\right)\right)\right)\\
&= G_{\boldsymbol{\alpha}}(y_1, \ldots, y_d),
\end{aligned}$$

where the second equality follows from (6.25). □

We now reformulate Theorem 5.1.10, which establishes an expansion for the distribution function of the multivariate maximum.

Theorem 6.3.4
Let $H_{\boldsymbol{\alpha},\beta(n)}$, $\beta(n) = (\beta_1(n), \ldots, \beta_k(n))$, $n \in \mathbb{N}$, *be d–dimensional distribution functions and assume that the pertaining spectral densities satisfy expansions of length* $k+1$

$$\left(h_{\boldsymbol{\alpha},\beta(n)}\right)_{\mathbf{z}}(c) = 1 + \sum_{j=1}^k B_{j,n}(c)A_{j,n}(\mathbf{z}) + R_n(\mathbf{z}, c), \quad k \in \mathbb{N},$$

with $R_n(\mathbf{z}, c) = o(B_{k,n}(c))$ *uniformly for* $\mathbf{z} \in R$*, as* $c \uparrow 0$*, according to* (3.4)*, such that*

$$R_n(\mathbf{z}, c/n) \to 0, \quad n \to \infty,$$

for every $c < 0$*. Let*

$$B_{j,n}(c) = |c|^{\beta_j(n)} L_{j,n}(c), \quad j = 1, \dots, k,$$

where the $L_{j,n}$ *are slowly varying functions. Then we have*

$$\begin{aligned}
&H^n_{\boldsymbol{\alpha},\beta(n)}\left(d_{n,1} + c_{n,1}y_1, \dots, d_{n,d} + c_{n,d}y_d\right)\\
&= \exp(T_{\boldsymbol{\alpha},2}(\mathbf{y}))\\
&\quad \times \exp\left(-\sum_{j=1}^{k} |T_{\boldsymbol{\alpha},2}(\mathbf{y})|^{1+\beta_j(n)} \frac{1}{1+\beta_j(n)} A_{j,n}(T_{\boldsymbol{\alpha},1}(\mathbf{y})) n^{-\beta_j(n)} L_{j,n}\left(\frac{T_{\boldsymbol{\alpha},2}(\mathbf{y})}{n}\right)\right)\\
&\quad \times \left(1 + o\left(n^{-\beta_k(n)} L_{k,n}\left(\frac{T_{\boldsymbol{\alpha},2}(\mathbf{y})}{n}\right)\right)\right), \quad n \to \infty,
\end{aligned}$$

with the normalizing constants $d_{n,i}$ *and* $c_{n,i}$*,* $1 \le i \le d$*, as given in* (6.24) *and* $T_{\boldsymbol{\alpha},1}$ *and* $T_{\boldsymbol{\alpha},2}$ *as defined in* (6.15).

PROOF. The proof runs along the lines of that belonging to Theorem 6.3.1. □

The pertaining Corollaries 5.1.12 and 5.1.13 can be modified in like manner.

Next we present a modified version of Theorem 5.2.2, which gives us a limiting distribution function based on a generalized condition for convergence.

Theorem 6.3.5
Let $H_{\boldsymbol{\alpha},n}$ *be a d–dimensional distribution function satisfying Condition 5.2.1, i.e., its spectral density can be represented by*

$$\left(h_{\boldsymbol{\alpha},n}\right)_{\mathbf{z}}(c) = z_1 F(g_n(z_1, z_2, \dots, z_{d-1}, c)) + \dots + z_d F(g_n(z_d, z_1, \dots, z_{d-2}, c))$$

with $z_d := 1 - \sum_{i \le d-1} z_i$*, where F is a continuous univariate distribution function and the* $g_n : [0,1]^{d-1} \times (-\infty, 0) \to \mathbb{R}$ *are any measurable functions for each* $n \in \mathbb{N}$*. Suppose that*

$$g_n(\mathbf{z}, c/n) \to g(\mathbf{z}), \quad n \to \infty,$$

for every $\mathbf{z} \in R$ *with* $g : [0,1]^d \to \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ *being a continuous measurable function. Then we have*

$$\begin{aligned}
&H^n_{\boldsymbol{\alpha},n}\left(d_{n,1} + c_{n,1}y_1, \dots, d_{n,d} + c_{n,d}y_d\right) && (6.32)\\
&\to \exp\left(\psi_{\alpha_1}(y_1) F\left(g\left(T_{\boldsymbol{\alpha},1}(\mathbf{y})^{(1)}\right)\right) + \dots + \psi_{\alpha_d}(y_d) F\left(g\left(T_{\boldsymbol{\alpha},1}(\mathbf{y})^{(d)}\right)\right)\right) && (6.33)\\
&=: G_{\boldsymbol{\alpha}}(y_1, \dots, y_d), && (6.34)
\end{aligned}$$

as $n \to \infty$*, with the normalizing constants* $d_{n,i}$ *and* $c_{n,i}$*,* $1 \le i \le d$*, as given in* (6.24) *and* $T_{\boldsymbol{\alpha},1}$ *and* $T_{\boldsymbol{\alpha},2}$ *as defined in* (6.15)*, and* $G_{\boldsymbol{\alpha}}$ *is a distribution function.*

PROOF. The proof is again analogue to that of Theorem 6.3.1. □

In conformity with Section 5.2 we present the following result concerning the univariate margins of $G_{\boldsymbol{\alpha}}$ which corresponds to Lemma 6.3.2.

Lemma 6.3.6
Let $H_{\boldsymbol{\alpha},n}$ be a d–variate distribution function as in Theorem 6.3.5. If the limiting function g in (6.33) *additionally satisfies*

$$g(\mathbf{e}_1) \geq \omega(F), \tag{6.35}$$

where $\omega(F)$ is the right endpoint of F and $\mathbf{e_1}$ is the first unit vector in $\mathbb{R}^{d-1}$, then the i–th marginal distribution function belonging to G_α is given by

$$G_i(x) = \exp(\psi_{\alpha_i}(x)), \quad i = 1, \ldots, d,$$

cf. (6.1).

If, conversely, the i–th univariate marginal distribution function of $H_{\boldsymbol{\alpha},n}$ belongs to the max–domain of attraction of G_i, $i = 1, \ldots, d$, then the limiting functions A_j, $j = 1, \ldots, k$ satisfy the property (6.35).

PROOF. The assertion can be proved with the same argumentation as in the proof of Lemma 6.3.2. □

We close this section with a remark about the max–stability of the limiting distribution function under the generalized condition for convergence.

Remark 6.3.7
Under the generalized Condition 5.2.1 the limiting distribution function $G_{\boldsymbol{\alpha}}$ in (6.34) *is again max–stable according to* (2.3) *with the normalizing vectors used in* (6.32), *cf. Lemma 6.3.3.*

7 Measures of asymptotic dependence

In the preceding chapters we have modeled asymptotic dependence structures of multivariate random vectors by expansions of densities and limiting distribution functions. However, in some contexts one wishes to obtain information about the asymptotic dependence structure by just one parameter — as in Chapter 4 where the exponent of variation is used to distinguish between tail dependence and tail independence. In this context we have already mentioned that there exists a relationship to certain dependence measures in literature.

Our present aim is to show which measures of asymptotic dependence are available and to extend them to more general cases. We also point out how they are related to each other and how our model comprising spectral expansions can be embedded into this framework.

We start Section 7.1 with the classical definition of tail independence in terms of a tail dependence parameter and show that it coincides with our previously given definition. Following [6], Section 3.3, we present two more dependence measures in the bivariate case — the tail dependence parameter and the residual dependence index — and use them to characterize asymptotic dependence structures. Under a spectral expansion of length 2 they are related to the pertaining Pickands dependence function and the exponent of variation. Taking into account several proposals in literature we extend these dependence parameters to the multivariate case in Section 7.2. Again, we analyze their structure under spectral expansions and establish relationships to the Pickands dependence function and the exponent of variation. In Section 7.3 we present an additional modification of one of the tail dependence measures, thereby defining the angular tail dependence parameter. Section 7.4 again focusses on the residual dependence. We compute the tail dependence parameter and the residual dependence parameter under sequences of spectral expansions fulfilling the convergence conditions from Chapter 5. It transpires that in this context the tail dependence parameter no longer measures the tail dependence but the residual dependence.

7.1 Measures of bivariate dependence

The most common definition of tail dependence and tail independence, respectively, of a bivariate random vector (X, Y) with distribution function H and continuous marginal distribution functions H_X and H_Y uses the so–called tail dependence parameter χ defined by

$$\chi = \lim_{q \to 1} P\left(Y > H_Y^{-1}(q) \,\middle|\, X > H_X^{-1}(q)\right), \tag{7.1}$$

where H_X^{-1} and H_Y^{-1} are the quantile functions of H_X and H_Y, cf., e.g., [10], p. 163, [41], p. 75, Geffroy [20], [21] and Sibuya [49]. If $\chi = 0$, then X and Y are called tail independent; otherwise, X and Y are tail dependent.

Tail independence may as well be expressed within the framework of copulas. With $U = H_X(X)$ and $V = H_Y(Y)$ we have

$$\chi = \lim_{u \to 1} P(V > u | U > u),$$

where U and V are uniformly distributed on the interval $[0, 1]$, cf. [17], Section 1.

For random variables X and Y with support in $(-\infty, 0]$ the tail dependence parameter is given by

$$\chi = \lim_{c \uparrow 0} P(Y > c | X > c). \tag{7.2}$$

It can easily be shown that the relationship

$$\chi = 2(1 - D(1/2)) \tag{7.3}$$

holds if the joint distribution function of X and Y satisfies a spectral expansion with Pickands dependence function D according to Definition 3.1.3. From the convexity of D it follows that

$$\chi = 0 \quad \Leftrightarrow \quad D(z) = 1, \quad \text{for every } z \in [0, 1]. \tag{7.4}$$

Therefore X and Y are tail independent if, and only if, $D(z) = 1$, $z \in [0, 1]$, which coincides with the Definition 2.3.1 of tail independence.

Coles et al. [6], Section 3.3.1, call χ a dependence measure. They show that χ can be received as the limit of another, asymptotically identical, function. For U and V in the copula framework we have

$$P(V > u | U > u) \sim 2 - \frac{\log P\{U < u, V < u\}}{\log P\{U < u\}},$$

as $u \to 1$. Therefore they define

$$\chi(u) := 2 - \frac{\log P\{U < u, V < u\}}{\log P\{U < u\}}, \quad 0 \le u \le 1,$$

so that

$$\chi = \lim_{u \to 1} \chi(u).$$

According to [41], p. 75 the dependence parameters χ and $\chi(u)$ possess the following properties:

- $\chi(u)$ and χ are symmetric in U and V;
- $\chi(u)$ and χ range between 0 and 1;

- if U and V are stochastically independent, then $\chi = 0$; hence independence implies tail independence;
- if $U = V$, then $\chi = 1$.

On the negative quadrant we define $\chi(c)$ analogously by

$$\chi(c) := 2 - \frac{\log P\{X < c, Y < c\}}{\log P\{X < c\}}, \quad c < 0, \tag{7.5}$$

which leads to

$$\chi = \lim_{c \uparrow 0} \chi(c). \tag{7.6}$$

Coles et al. [6], Section 3.3.2, consider an additional dependence measure at level u

$$\bar{\chi}(u) := \frac{2 \log P\{U > u\}}{\log P\{V > u, U > u\}} - 1,$$

where U and V are $[0,1]$–uniformly distributed, and the pertaining limit

$$\bar{\chi} := \lim_{u \to 1} \bar{\chi}(u).$$

According to [6], Section 3.3.2, and [41], p. 322, this second dependence parameter is introduced to be able to distinguish between pairs of random variables that are both tail independent, i.e., $\chi = 0$, but have different degrees of residual dependence at an asymptotic level of higher order. Therefore we call $\bar{\chi}$ the residual dependence index.

Again, we list some properties of $\bar{\chi}$ and $\bar{\chi}(u)$, cf. [41], p. 323.

- $\bar{\chi}(u)$ and $\bar{\chi}$ are symmetric in U and V;
- $\bar{\chi}(u)$ and $\bar{\chi}$ range between -1 and 1;
- if $U = V$, then $\bar{\chi} = 1$.

For normal copula random vectors $(U, V) = (\Phi(X), \Phi(Y))$, where (X, Y) follows a bivariate standard normal distribution with correlation coefficient ρ and Φ is the univariate standard normal distribution function, we have

$$\bar{\chi} = \rho,$$

cf. [41], equation (13.21).

If X and Y are again random variables with support in $(-\infty, 0]$, we define

$$\bar{\chi}(c) := \frac{2 \log P\{X > c\}}{\log P\{Y > c, X > c\}} - 1, \quad c < 0, \tag{7.7}$$

and

$$\bar{\chi} := \lim_{c \uparrow 0} \bar{\chi}(c).$$

Now, if the distribution function H of (X, Y) satisfies a spectral expansion of length 2 with Pickands dependence function D and the exponent of variation β, we obtain

$$\bar{\chi} = 1 \tag{7.8}$$

if $D \neq 1$, and

$$\bar{\chi} = \frac{1-\beta}{1+\beta} \tag{7.9}$$

if $D = 1$, cf. [17], Section 8.2.

Combining (7.8) and (7.9) with equivalence (7.4) we get

$$D \neq 1 \quad \Leftrightarrow \quad \chi > 0, \bar{\chi} = 1$$

and

$$D = 1 \quad \Leftrightarrow \quad \chi = 0, \bar{\chi} < 1.$$

Therefore, following Coles et al. [6], Section 3.3.2, we can also use the pair $(\chi, \bar{\chi})$ instead of the dependence function D to characterize the asymptotic dependence structure of two random variables:

- $\chi > 0, \bar{\chi} = 1$: tail dependence with χ determining the degree of dependence
- $\chi = 0, \bar{\chi} < 1$: tail independence with $\bar{\chi}$ determining the residual dependence,

cf. also [41], p. 323.

7.2 Multivariate extensions

In literature, definitions and detailed considerations of dependence measures and tail dependence parameters have usually been restricted to the bivariate case. However, there are still a few multivariate approaches some of which we will present in the subsequent lines.

Recall that tail independence in the d–variate context may be characterized by bivariate considerations, cf. Galambos [19], p. 301, [42], Proposition 5.27, and [40], Theorem 7.2.5. Hence a first intuitive extension of the tail dependence parameter (7.2) can be obtained by simultaneous pairwise bivariate considerations. Therefore we define

$$\begin{aligned}\chi_{ij} &:= \lim_{c \uparrow 0} P(X_j > c | X_i > c) \\ &= 2\left(1 - D\left(\frac{1}{2}\mathbf{e_i} + \frac{1}{2}\mathbf{e_j}\right)\right) \\ &= 2(1 - D_{ij}(1/2))\end{aligned}$$

for $i, j = 1, \dots, d$, $d \geq 2$, with D_{ij} as defined in (2.14). To obtain a multivariate dependence measure we set

$$\chi := \frac{2}{d(d-1)} \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \chi_{ij}.$$

It follows that $0 \leq \chi \leq 1$ and

$$\begin{aligned} & \chi = 0 \\ \Leftrightarrow \quad & \chi_{ij} = 0 \text{ for every pair } i, j \\ \Leftrightarrow \quad & D_{ij} = 1 \text{ for every pair } i, j \\ \Leftrightarrow \quad & D = 1, \end{aligned}$$

where the last equivalence is due to Lemma 3.2.5.

Another suggestion of a measure of multivariate tail dependence can be found in [47], Definition 7.1. After a transformation to reversely exponential margins it takes the following form.

Definition 7.2.1
Let $\mathbf{X}$ be a d–dimensional random vector with reversely exponential margins. Then $\mathbf{X}$ is said to be multivariate tail dependent if for some sets $I \cup J = \{1, \dots, d\}$ and $I \cap J = \emptyset$ the limit

$$\lambda := \lim_{c \uparrow 0} P(X_i > c, \forall i \in I | X_j > c, \forall j \in J)$$

exists and is larger than zero. If $\lambda = 0$, $\mathbf{X}$ is said to be tail independent. The parameter λ is called the (upper) multivariate tail dependence coefficient.

In an article by Schmid and Schmidt [46], a new measure of multivariate tail dependence is introduced and its relationship to the upper and lower multivariate tail dependence coefficients is shown.

Weissman [52], Section 2, defines a coefficient of tail dependence in terms of a Pickands dependence function. Translating his definition into our context we obtain the parameter

$$\chi_d := \frac{d\left(1 - D\left(\frac{1}{d}, \dots, \frac{1}{d}\right)\right)}{d-1}, \tag{7.10}$$

where D is a Pickands dependence function. We call χ_d the multivariate tail dependence parameter. According to the properties of D described in Chapter 2 we have

$$\chi_d = 0 \quad \Leftrightarrow \quad D(\mathbf{z}) = 1 \quad \text{for every } \mathbf{z} \in R, \tag{7.11}$$

if D is symmetric, cf. Lemma 2.1.4 (iv). The case $\chi_d = 1$ represents total dependence whereas $\chi_d = 0$ stands for tail independence. For $d = 2$ we again obtain the dependence parameter given in (7.3).

Both in the bivariate and in the multivariate case, the tail dependence parameter χ_d can be generalized to arbitrary $D(\mathbf{z})$. Falk et al. [10] define the so–called canonical dependence function or tail dependence function by

$$\vartheta(\mathbf{z}) := \frac{1 - D(\mathbf{z})}{1 - \max\left(z_1, \dots, z_{d-1}, 1 - \sum_{i \leq d-1} z_i\right)}, \quad \mathbf{z} \in R,$$

where D is a Pickands dependence function, cf. [10], Section 6.4. We set

$$\vartheta(\mathbf{0}) := \lim_{\mathbf{z}\downarrow\mathbf{0}} \vartheta(\mathbf{z}) \quad \text{and} \quad \vartheta(\mathbf{e_i}) := \lim_{\mathbf{z}\to\mathbf{e_i}} \vartheta(\mathbf{z}),$$

where $\mathbf{e_i}$ is the i–th unit vector in $\mathbb{R}^{d-1}$, $i = 1, \dots, d-1$. From the properties of the function D it follows that $0 \le \vartheta(\mathbf{z}) \le 1$. The dependence functions $D(\mathbf{z}) = 1$ and $D(\mathbf{z}) = \max\left(z_1, \dots, z_{d-1}, 1 - \sum_{i\le d-1} z_i\right)$, which characterize the cases of independence and complete dependence, are mapped onto $\vartheta_1(\mathbf{z}) = 0$ and $\vartheta_2(\mathbf{z}) = 1$ respectively. We then have, obviously,

$$\vartheta\left(\frac{1}{d}, \dots, \frac{1}{d}\right) = \frac{d\left(1 - D\left(\frac{1}{d}, \dots, \frac{1}{d}\right)\right)}{d-1} = \chi_d,$$

which is also called the canonical parameter.

As in the bivariate case we now define a multivariate tail dependence parameter at level c and show that under a spectral expansion the limit, as $c \uparrow 0$, will again be the multivariate tail dependence parameter χ_d.

Definition 7.2.2
Let $(X_1, \dots, X_d)$ be a random vector on $(-\infty, 0]^d$. Then the parameter

$$\chi_d(c) := \left(d - \frac{\log P\{X_1 < c, \dots, X_d < c\}}{\log P\{X_1 < c\}}\right) / (d-1) \tag{7.12}$$

is called the multivariate tail dependence parameter at level c.

Comparing the definition (7.12) to (7.5) we see that it actually is a multivariate extension of the bivariate case.

Let us now compute the limit of $\chi_d(c)$.

Lemma 7.2.3
Assume that the distribution function H of the random vector $(X_1, \dots, X_d)$ on $(-\infty, 0]^d$ satisfies a spectral expansion of length 2 according to Definition 3.1.3. Then we have

$$\chi_d = \lim_{c\uparrow 0} \chi_d(c).$$

PROOF. Using the spectral decomposition of H we obtain

$$P\{X_1 < c, \dots, X_d < c\} = H(c, \dots, c) = H_{(1/d, \dots, 1/d)}(c).$$

Thus, together with the asymptotic representation

$$\log(1-u) \sim -u,$$

as $u \to 0$, we obtain

$$\begin{aligned}
&\lim_{c\uparrow 0} \frac{\log P\{X_1 < c, \dots, X_d < c\}}{\log P\{X_1 < c\}} \\
&= \lim_{c\uparrow 0} \frac{1 - H_{(1/d,\dots,1/d)}(dc)}{1 - H_{(1,0,\dots,0)}(c)} \\
&= \lim_{c\uparrow 0} \frac{dh_{(1/d,\dots,1/d)}(dc)}{h_{(1,0,\dots,0)}(c)} \\
&= \lim_{c\uparrow 0} \frac{d\left(D(1/d,\dots,1/d) + B(dc)A(1/d,\dots,1/d) + o(B(dc))\right)}{1 + B(c)A(1,0,\dots,0) + o(B(c))} \\
&= dD(1/d,\dots,1/d),
\end{aligned}$$

which justifies the assertion. □

After having extended the tail dependence parameter χ to the multivariate case the next step consists in extending the dependence measure $\bar{\chi}(c)$ at level c in (7.7) and its limit $\bar{\chi}$ as well.

Definition 7.2.4
Let $(X_1, \dots, X_d)$ be a random vector on $(-\infty, 0]^d$. Then the parameter

$$\bar{\chi}_d(c) := \frac{d \log P\{X_1 > c\}}{\log P\{X_1 > c, \dots, X_d > c\}} - (d-1) \tag{7.13}$$

is called the multivariate residual dependence index at level c. If the limit

$$\bar{\chi}_d := \lim_{c\uparrow 0} \bar{\chi}_d(c) \tag{7.14}$$

exists, we call it the multivariate residual dependence index.

Our next aim is to show which structure $\bar{\chi}$ takes under a spectral expansion. Therefore we first need an auxiliary result concerning the survivor functions of a d–dimensional distribution function based on its spectral decomposition and its spectral expansion.

Lemma 7.2.5
Let H_β be the distribution function of the d–dimensional random vector $(X_1, \dots, X_d)$ and assume that it possesses a spectral density $h_{\beta,\mathbf{z}}$. Then the pertaining survivor function is given by

$$\begin{aligned}
&P\{X_1 > c, \dots, X_d > c\} \\
&= 1 + \sum_{m \in \{0,1\}^d \setminus (1,\dots,1)} (-1)^{d - \sum_{i\le d} m_i} H_{\beta, \left(d - \sum_{i\le d} m_i\right)^{-1} z_m} \left(\left(d - \sum_{i \le d} m_i \right) c \right),
\end{aligned} \tag{7.15}$$

where

$$z_m := (1-m_1,\dots,1-m_{d-1}) \in \{0,1\}^{d-1}. \tag{7.16}$$

Its derivative with respect to c is

$$\frac{\partial}{\partial c} P\{X_1 > c,\dots,X_d > c\}$$
$$= \sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i} \left(d - \sum_{i\le d} m_i\right) h_{\beta,\left(d-\sum_{i\le d} m_i\right)^{-1} z_m} \left(\left(d - \sum_{i\le d} m_i\right) c\right). \tag{7.17}$$

PROOF. As a multivariate distribution function H_β is Δ–monotone. This implies, cf. [10], pp. 107–108,

$$P\{X_1 > c,\dots,X_d > c\}$$
$$= P\left\{(X_1,\dots,X_d) \in (c,0]^d\right\}$$
$$= \sum_{m\in\{0,1\}^d} (-1)^{d-\sum_{i\le d} m_i} H_\beta\left(0^{m_1}c^{1-m_1},\dots,0^{m_d}c^{1-m_d}\right) \tag{7.18}$$
$$= H_\beta(0,\dots,0) + \sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i} H_\beta\left(0^{m_1}c^{1-m_1},\dots,0^{m_d}c^{1-m_d}\right) \tag{7.19}$$
$$= 1 + \sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i} H_{\beta,\left(d-\sum_{i\le d} m_i\right)^{-1} z_m} \left(\left(d - \sum_{i\le d} m_i\right) c\right)$$

with z_m as defined in (7.16). In (7.18) and (7.19) we set $0^0 := 1$. Thus we have deduced representation (7.15). The representation of the derivative (7.17) follows directly from the definition (3.1) of a spectral density. □

Lemma 7.2.6
Assume that the distribution function H_β of the d–dimensional random vector $(X_1,\dots,X_d)$ satisfies a spectral expansion of length 2

$$h_{\beta,\mathbf{z}}(c) = D(\mathbf{z}) + B(c)A(\mathbf{z}) + o(B(c)), \quad c \uparrow 0, \tag{7.20}$$

where B is a regularly varying function with the exponent of variation β. Additionally, assume that B is absolutely continuous and possesses a monotone derivative b. Then we have

$$\bar{\chi}_d = 1,$$

if

$$\sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i} \left(d - \sum_{i\le d} m_i\right) D\left(\left(d - \sum_{i\le d} m_i\right)^{-1} z_m\right) \ne 0$$

and

$$\bar{\chi}_d = \frac{1+(1-d)\beta}{1+\beta}, \tag{7.21}$$

if

$$\sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i} \left(d-\sum_{i\le d} m_i\right) D\left(\left(d-\sum_{i\le d} m_i\right)^{-1} z_m\right) = 0 \tag{7.22}$$

with z_m as defined in (7.16).

PROOF. Regarding the definitions (7.13) and (7.14) we first compute the limit

$$\eta := \lim_{c\uparrow 0} \frac{\log P\{X_1 > c\}}{\log P\{X_1 > c,\dots,X_d > c\}}. \tag{7.23}$$

Using the spectral decomposition of H_β we get

$$P\{X_1 > c\} = 1 - H_{\beta,(1,0,\dots,0)}(c)$$

and the derivative of $H_{\beta,(1,0,\dots,0)}(c)$ with respect to c is given by $h_{\beta,(1,0,\dots,0)}(c)$. Therefore, by applying l'Hôpital's theorem and Lemma 7.2.5, we obtain

$$\eta = \lim_{c\uparrow 0}\left(\frac{P\{X_1 > c,\dots,X_d > c\}}{1-H_{\beta,(1,0,\dots,0)}(c)} \cdot \frac{-h_{\beta,(1,0,\dots,0)}(c)}{g(c)}\right), \tag{7.24}$$

where

$$\begin{aligned} g(c) := &\sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i}\left(d-\sum_{i\le d} m_i\right) h_{\beta,\left(d-\sum_{i\le d} m_i\right)^{-1} z_m}\left(\left(d-\sum_{i\le d} m_i\right)c\right)\\ = &\sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i}\left(d-\sum_{i\le d} m_i\right)\left[D\left(\left(d-\sum_{i\le d} m_i\right)^{-1} z_m\right)\right.\\ &\left. + B\left(\left(d-\sum_{i\le d} m_i\right)c\right) A\left(\left(d-\sum_{i\le d} m_i\right)^{-1} z_m\right)\right] + o(B(c)), \end{aligned}$$

as $c\uparrow 0$, with z_m as defined in (7.16). We then have, obviously,

$$\begin{aligned} &\lim_{c\uparrow 0} g(c) \neq 0\\ \Leftrightarrow &\sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i}\left(d-\sum_{i\le d} m_i\right) D\left(\left(d-\sum_{i\le d} m_i\right)^{-1} z_m\right) \neq 0. \end{aligned}$$

In this case it follows from l'Hôpital's theorem and

$$\begin{aligned} h_{\beta,(1,0,\dots,0)}(c) &= 1 + B(c)A(1,0,\dots,0) + o(B(c)), \quad c\uparrow 0,\\ &\to 1, \quad c\uparrow 0, \end{aligned} \tag{7.25}$$

that
$$\lim_{c\uparrow 0} \frac{P\{X_1 > c, \dots, X_d > c\}}{P\{X_1 > c\}} = \lim_{c\uparrow 0} \frac{g(c)}{-h_{\beta,(1,0,\dots,0)}} \neq 0$$
and, thus,
$$\eta = 1 \Rightarrow \bar{\chi}_d = 1.$$
Let us now consider the case when
$$\sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i} \left(d - \sum_{i\le d} m_i\right) D\left(\left(d - \sum_{i\le d} m_i\right)^{-1} z_m\right) = 0$$
which implies
$$\begin{aligned} g(c) = &\sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i} \left(d - \sum_{i\le d} m_i\right) \\ &\times \left[B\left(\left(d - \sum_{i\le d} m_i\right) c\right) A\left(\left(d - \sum_{i\le d} m_i\right)^{-1} z_m\right)\right] + o(B(c)), \quad c\uparrow 0, \end{aligned} \tag{7.26}$$
and
$$\begin{aligned} g'(c) = &\sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i} \left(d - \sum_{i\le d} m_i\right)^2 \\ &\times \left[b\left(\left(d - \sum_{i\le d} m_i\right) c\right) A\left(\left(d - \sum_{i\le d} m_i\right)^{-1} z_m\right)\right] + o(b(c)), \quad c\uparrow 0. \end{aligned} \tag{7.27}$$
Now, starting with equation (7.24), we obtain
$$\begin{aligned} \eta &= \lim_{c\uparrow 0} \left(\frac{P\{X_1 > c, \dots, X_d > c\}}{1 - H_{\beta,(1,0,\dots,0)}(c)} \cdot \frac{-h_{\beta,(1,0,\dots,0)}(c)}{g(c)} \right) \\ &= \lim_{c\uparrow 0} \frac{P\{X_1 > c, \dots, X_d > c\}}{(1 - H_{\beta,(1,0,\dots,0)}(c))(-g(c))}, \end{aligned}$$
where the last step is due to (7.25). By applying l'Hôpital's theorem and Lemma 7.2.6 again we obtain
$$\begin{aligned} \eta &= \lim_{c\uparrow 0} \frac{g(c)}{h_{\beta,(1,0,\dots,0)}(c)g(c) - (1 - H_{\beta,(1,0,\dots,0)}(c))g'(c)} \\ &= \lim_{c\uparrow 0} \frac{1}{h_{\beta,(1,0,\dots,0)}(c) + \frac{(1-H_{\beta,(1,0,\dots,0)}(c))g'(c)}{-g(c)}} \\ &= \lim_{c\uparrow 0} \frac{1}{h_{\beta,(1,0,\dots,0)}(c) + \frac{1-H_{\beta,(1,0,\dots,0)}(c)}{-cg(c)/B(c)} \cdot cg'(c)/B(c)}. \end{aligned} \tag{7.28}$$

Let us now compute the limits of $cg'(c)/B(c)$ and of $\frac{1-H_{\beta,(1,0,\dots,0)}(c)}{-cg(c)/B(c)}$. First, using representation (7.27) and applying condition (3.6) and Remark 3.1.4 (ii) to the function B in the spectral expansion (7.20), we deduce

$$
\begin{aligned}
\frac{cg'(c)}{B(c)} &= \sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i}\left(d-\sum_{i\le d} m_i\right)\frac{B\left(\left(d-\sum_{i\le d} m_i\right)c\right)}{B(c)} \\
&\quad\times\left[\frac{\left(d-\sum_{i\le d} m_i\right)cb\left(\left(d-\sum_{i\le d} m_i\right)c\right)}{B\left(\left(d-\sum_{i\le d} m_i\right)c\right)}A\left(\left(d-\sum_{i\le d} m_i\right)^{-1} z_m\right)\right] \\
&\quad + o\left(\frac{cb(c)}{B(c)}\right), \quad c\uparrow 0, \\
&\to \beta\sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i}\left(d-\sum_{i\le d} m_i\right)^{1+\beta} A\left(\left(d-\sum_{i\le d} m_i\right)^{-1} z_m\right) \\
&=: \beta M,
\end{aligned}
\tag{7.29}
$$

as $c\uparrow 0$. Similarly, the second limit can be computed by using representation (7.26) and by applying l'Hôpital's theorem and condition (3.6).

$$
\lim_{c\uparrow 0}\frac{1-H_{\beta,(1,0,\dots,0)}(c)}{-cg(c)/B(c)} = \lim_{c\uparrow 0}\frac{h_{\beta,(1,0,\dots,0)}(c)}{N(c)},
$$

where

$$
\begin{aligned}
N(c) &:= -\sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i}\left(d-\sum_{i\le d} m_i\right) \\
&\quad\times c\frac{B\left(\left(d-\sum_{i\le d} m_i\right)c\right)}{B(c)}A\left(\left(d-\sum_{i\le d} m_i\right)^{-1} z_m\right) + o(c), \quad c\uparrow 0, \\
&= \sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i}\left(d-\sum_{i\le d} m_i\right)\frac{B\left(\left(d-\sum_{i\le d} m_i\right)c\right)}{B(c)} \\
&\quad\times A\left(\left(d-\sum_{i\le d} m_i\right)^{-1} z_m\right) + \sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i}\left(d-\sum_{i\le d} m_i\right) \\
&\quad\times\frac{\left(d-\sum_{i\le d} m_i\right)cb\left(\left(d-\sum_{i\le d} m_i\right)c\right)B(c) - cB\left(\left(d-\sum_{i\le d} m_i\right)c\right)b(c)}{B(c)^2} \\
&\quad\times A\left(\left(d-\sum_{i\le d} m_i\right)^{-1} z_m\right) + o(1), \quad c\uparrow 0, \\
&\to \sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i}\left(d-\sum_{i\le d} m_i\right)^{1+\beta} A\left(\left(d-\sum_{i\le d} m_i\right)^{-1} z_m\right) = M,
\end{aligned}
$$

as $c \uparrow 0$, because of

$$\begin{aligned}
&\frac{\left(d-\sum_{i\le d} m_i\right) cb\left(\left(d-\sum_{i\le d} m_i\right)c\right)B(c) - cB\left(\left(d-\sum_{i\le d} m_i\right)c\right)b(c)}{B(c)^2}\\
&= \frac{\left(d-\sum_{i\le d} m_i\right) cb\left(\left(d-\sum_{i\le d} m_i\right)c\right)}{B\left(\left(d-\sum_{i\le d} m_i\right)c\right)} \cdot \frac{B\left(\left(d-\sum_{i\le d} m_i\right)c\right)}{B(c)}\\
&\quad - \frac{B\left(\left(d-\sum_{i\le d} m_i\right)c\right)}{B(c)} \cdot \frac{cb(c)}{B(c)}\\
&\to \beta\left(d-\sum_{i\le d} m_i\right)^{\beta} - \left(d-\sum_{i\le d} m_i\right)^{\beta}\beta = 0,
\end{aligned}$$

as $c \uparrow 0$. From (7.25) we can deduce that

$$\lim_{c\uparrow 0} \frac{1-H_{\beta,(1,0,\ldots,0)}(c)}{-cg(c)/B(c)} = \frac{1}{M}.$$

Inserting this result as well as the limits (7.25) and (7.29) into (7.28) leads to

$$\eta = \frac{1}{1+\beta}.$$

Thus, remembering the definition (7.23) of η, we get

$$\begin{aligned}
\bar{\chi}_d &= \lim_{c\uparrow 0} \bar{\chi}_d(c)\\
&= \lim_{c\uparrow 0} \frac{d\log P\{X_1>c\}}{\log P\{X_1>c,\ldots,X_d>c\}} - (d-1)\\
&= d\eta - (d-1)\\
&= d\left(\frac{1}{1+\beta}\right) - (d-1)\\
&= \frac{d-(d-1)(1+\beta)}{1+\beta}\\
&= \frac{1+(1-d)\beta}{1+\beta}.
\end{aligned}$$

□

Obviously, we again obtain $\bar{\chi}$ with representation (7.9) if we put $d = 2$ in Lemma 7.2.6. In the bivariate case we have

$$\bar{\chi} \ne 1 \quad \text{if, and only if,} \quad D = 1, \tag{7.30}$$

cf. [17], Section 8.2. In order to deduce a relationship between the Pickands dependence function D and the form of the residual dependence index $\bar{\chi}_d$, one has to consider condition (7.22). We investigate only the if–part of the equivalence (7.30) in the multivariate case, which is straightforward.

Lemma 7.2.7
Let D be a Pickands dependence function on the $(d-1)$–dimensional space R. If $D = 1$, it follows that

$$\sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\leq d} m_i} \left(d - \sum_{i\leq d} m_i\right) D\left(\left(d-\sum_{i\leq d} m_i\right)^{-1} z_m\right) = 0$$

with z_m as defined in (7.16).

PROOF. For $D = 1$ we get

$$\begin{aligned}
&\sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\leq d} m_i} \left(d - \sum_{i\leq d} m_i\right) D\left(\left(d-\sum_{i\leq d} m_i\right)^{-1} z_m\right)\\
&= \sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\leq d} m_i} \left(d - \sum_{i\leq d} m_i\right)\\
&= \sum_{r=1}^{d} (-1)^{r-1} r \binom{d}{r}\\
&= d\sum_{r=1}^{d} (-1)^{r-1} \binom{d-1}{r-1}\\
&= d\sum_{r=0}^{d-1} \binom{d-1}{r} (-1)^{r}\\
&= d(1-1)^{d-1}\\
&= 0
\end{aligned}$$

by using the calculation rules for binomial coefficients and the binomial theorem, cf. Königsberger [32], p. 4. □

7.3 The angular tail dependence parameter

The definition of the tail dependence parameter χ in (7.1) and (7.6), respectively, may be modified in such a way that it depends on the angular component z.

In order to obtain an appropriate definition of the angular tail dependence parameter let us first consider the conditional probability $P(Y > c(1-z)|X > cz)$ for $[-1,0]$–uniformly distributed random variables X and Y. In this copula framework on the negative quadrant we obtain

$$\begin{aligned}
&P(Y > c(1-z)|X > cz) \\
&= \frac{P\{X > cz, Y > c(1-z)\}}{1 - P\{X \le cz\}} \\
&= \frac{1 - P\{X \le cz\} - P\{Y \le c(1-z)\} + P\{X \le cz, Y \le c(1-z)\}}{1 - P\{X \le cz\}} \\
&= 1 + \frac{1 - P\{Y \le c(1-z)\}}{1 - P\{X \le cz\}} - \frac{1 - P\{X \le cz, Y \le c(1-z)\}}{1 - P\{X \le cz\}} \\
&\sim 1 + \frac{1-z}{z} - \frac{\log P\{X \le cz, Y \le c(1-z)\}}{\log P\{X \le cz\}},
\end{aligned}$$

as $c \uparrow 0$. Therefore we choose the following definition.

Definition 7.3.1
Let (X, Y) be a bivariate random vector on $(-\infty, 0]^2$. Then the parameter

$$\chi_z(c) := 1 + \frac{1-z}{z} - \frac{\log P\{X \le cz, Y \le c(1-z)\}}{\log P\{X \le cz\}}, \quad z \in [0,1],$$

is called the angular tail dependence parameter at level c. The limit

$$\chi_z := \lim_{c \uparrow 0} \chi_z(c), \quad z \in [0,1],$$

is called the angular tail dependence parameter.

It can be easily shown that the relationship

$$\chi_z = \frac{1 - D(z)}{z} \tag{7.31}$$

holds for every $z \in [0,1]$, if the joint distribution function of X and Y satisfies a spectral expansion with Pickands dependence function D according to Definition 3.1.3. From the convexity of D it follows that

$$\chi_z = 0 \quad \Leftrightarrow \quad D(z) = 1$$

for every $z \in [0,1]$. Comparing χ_z in (7.31) with χ in (7.3) one obviously has $\chi = \chi_{1/2}$.

Combining the previous considerations we also obtain the convergence

$$zP(Y > c(1-z)|X > cz) \to 1 - D(z),$$

as $c \uparrow 0$, for every $z \in [0,1]$.

Again, it is possible to extend the angular tail dependence parameter to the multivariate case. In fact, we define several multivariate versions of $\chi_z(c)$ and compute the limits, as $c \uparrow 0$, afterwards.

Definition 7.3.2
Let $(X_1, \dots, X_d)$ be a random vector on $(-\infty, 0]^d$. Then the parameter

$$\chi_{d,\mathbf{z},i}(c) := \left(\frac{1}{z_i} - \frac{\log P\{X_1 < cz_1, \dots, X_d < cz_d\}}{\log P\{X_i < cz_i\}}\right) / (d-1), \quad \mathbf{z} \in R,$$

where $z_d := 1 - \sum_{i \le d-1} z_i$, is called the i–th multivariate angular tail dependence parameter at level c.

Similar to the proof of Lemma 7.2.3 it can be shown that the convergence

$$\chi_{d,\mathbf{z},i}(c) \to \frac{1 - D(\mathbf{z})}{z_i(d-1)} =: \chi_{d,\mathbf{z},i}, \quad c \uparrow 0, \tag{7.32}$$

holds for every $i \in \{1, \dots, d\}$ and $\mathbf{z} \in R$. We call $\chi_{d,\mathbf{z},i}$ the i–th multivariate angular tail dependence parameter. Setting $\mathbf{z} = (1/d, \dots, 1/d)$ makes it clear that $\chi_{d,\mathbf{z},i}$, $i = 1, \dots, d$, is a special case of the multivariate dependence parameter χ_d defined in (7.10) because one obtains

$$\chi_d = \chi_{d,(1/d,\dots,1/d),i}, \quad \text{for every } i \in \{1, \dots, d\}.$$

Another interesting limiting behavior can be deduced by putting the previous dependence parameters together, namely by considering the sum $\sum_{i \le d} z_i \chi_{d,\mathbf{z},i}(c)$. From the convergence (7.32) we deduce

$$\sum_{i \le d} z_i \chi_{d,\mathbf{z},i}(c) \to \frac{d(1 - D(\mathbf{z}))}{d-1} =: \chi_{d,\mathbf{z}}, \quad \mathbf{z} \in R,$$

as $c \uparrow 0$. The limit $\chi_{d,\mathbf{z}}$ may be called the multivariate angular tail dependence parameter. As in the bivariate case we get the equivalence

$$\chi_{d,\mathbf{z}} = 0 \quad \Leftrightarrow \quad D(\mathbf{z}) = 1$$

for every $\mathbf{z} \in R$. In contrast to the relationship (7.11), this equivalence is true for every Pickands dependence function D.

7.4 Measures of residual dependence under sequences of spectral expansions

As we have seen in the preceding Sections 7.1 and 7.2 the residual dependence index $\bar{\chi}_d$ can be used to quantify the degree of residual dependence in case two or more random variables are tail independent, i.e., the tail dependence parameter satisfies $\chi_d = 0$. If the joint distribution function of these random variables satisfies a spectral expansion of length 2, the value of $\bar{\chi}_d$ is determined by the exponent of variation β, cf. (7.21).

Instead of using this dependence measure it is also possible to investigate the residual dependence structure by considering maxima of random vectors under triangular schemes as in Chapter 5.

In this section we combine both approaches. In particular, we deduce the above dependence measures under sequences of spectral expansions fulfilling the convergence conditions from Section 5.1.3. Therefore we first define the multivariate dependence parameter at level c depending on n, i.e.

$$\chi_{d,n}(c) := \left(d - \frac{\log P\{X_{1,n} < c, \dots, X_{d,n} < c\}}{\log P\{X_{1,n} < c\}} \right) / (d-1), \tag{7.33}$$

where $(X_{1,n}, \dots, X_{d,n})$ is a random vector on $(-\infty, 0]^d$ for each $n \in \mathbb{N}$.

Let us now compute the limit of $\chi_{d,n}(c)$ under sequences of spectral expansions. We set $c := -1/n$, thereby letting the level and the dependence structure vary simultaneously. One may interpret this as a triangular scheme of tail dependence parameters.

Lemma 7.4.1
For each $n \in \mathbb{N}$ let $H_{\beta(n)}$ be the distribution function of a d–dimensional random vector $(X_{1,n}, \dots, X_{d,n})$ on $(-\infty, 0]^d$ and assume that the pertaining spectral density $h_{\beta(n),\mathbf{z}}$ fulfills the conditions of Theorem 5.1.1 and Lemma 5.1.3. Then we have

$$\lim_{n\to\infty} \chi_{d,n}(-1/n) = \frac{d\left(1 - D\left(\frac{1}{d}, \dots, \frac{1}{d}\right)\right)}{d-1},$$

where D is the Pickands dependence function of the limiting distribution function G of $H^n_{\beta(n)}$, as $n \to \infty$.

PROOF. Combining the arguments of the proofs of Theorem 5.1.1 and Lemma 7.2.3 we obtain

$$\begin{aligned}
&\lim_{c\uparrow 0} \frac{\log P\{X_{1,n} < -1/n, \dots, X_{d,n} < -1/n\}}{\log P\{X_{1,n} < -1/n\}} \\
&= \lim_{c\uparrow 0} \frac{1 - H_{\beta(n),(1/d,\dots,1/d)}(-d/n)}{1 - H_{\beta(n),(1,0,\dots,0)}(-1/n)} \\
&= \lim_{c\uparrow 0} \frac{-n\left(1 - H_{\beta(n),(1/d,\dots,1/d)}(-d/n)\right)}{-n\left(1 - H_{\beta(n),(1,0,\dots,0)}(-1/n)\right)} \\
&\to \frac{-d\left(1 + \sum_{j=1}^k \lambda_j A_j\left(\frac{1}{d}, \dots, \frac{1}{d}\right)\right)}{-1\left(1 + \sum_{j=1}^k \lambda_j A_j(\mathbf{e_1})\right)}, \quad n \to \infty, \\
&= d\left(1 + \sum_{j=1}^k \lambda_j A_j\left(\frac{1}{d}, \dots, \frac{1}{d}\right)\right),
\end{aligned}$$

where the last step is due to condition (5.11) in Lemma 5.1.3. According to Corollary 5.1.4 the Pickands dependence function of G is given by

$$D(\mathbf{z}) = 1 + \sum_{j=1}^k \lambda_j A_j(\mathbf{z}), \quad \mathbf{z} \in R.$$

Inserting these results into (7.33) leads to

$$\lim_{n\to\infty} \chi_{d,n}(-1/n) = \frac{d - dD\left(\frac{1}{d},\ldots,\frac{1}{d}\right)}{d-1} = \frac{d\left(1 - D\left(\frac{1}{d},\ldots,\frac{1}{d}\right)\right)}{d-1},$$

which completes the proof. □

For each fixed n the tail dependence parameter

$$\chi_{d,n} := \lim_{c\uparrow 0} \chi_{d,n}(c)$$

is equal to zero because the leading term of the spectral expansions is always given by the constant Pickands dependence function 1, which stands for tail independence. However, the limit in Lemma 7.4.1 no longer contains the Pickands dependence function pertaining to the underlying spectral expansions. Instead it contains the Pickands dependence function of the limiting distribution function G of the maxima under triangular schemes. Because the latter describes the residual dependence structure, the tail dependence parameter becomes a measure for the residual dependence if it is computed under triangular schemes.

Similarly, we consider the residual dependence index $\bar{\chi}_d$ under sequences of spectral expansions. Let $(X_{1,n},\ldots,X_{d,n})$ again be a random vector on $(-\infty,0]^d$ for each $n \in \mathbb{N}$ and define

$$\bar{\chi}_{d,n}(c) := \frac{d \log P\{X_{1,n} > c\}}{\log P\{X_{1,n} > c,\ldots,X_{d,n} > c\}} - (d-1).$$

We call $\bar{\chi}_{d,n}(c)$ the residual dependence index at level c depending on n.

As above, we compute the limit of $\bar{\chi}_{d,n}(c)$ under sequences of spectral expansions by setting $c := -1/n$.

Lemma 7.4.2

For each $n \in \mathbb{N}$ let $H_{\beta(n)}$ be the distribution function of a d–dimensional random vector $(X_{1,n},\ldots,X_{d,n})$ on $(-\infty,0]^d$ and assume that the pertaining spectral density $h_{\beta(n),\mathbf{z}}$ fulfills the conditions of Theorem 5.1.1 and Lemma 5.1.3. Then we have

$$\lim_{n\to\infty} \bar{\chi}_{d,n}(-1/n) = 1$$

provided that

$$\sum_{m\in\{0,1\}^d\setminus(1,\ldots,1)} (-1)^{d-\sum_{i\le d} m_i} \left(d - \sum_{i\le d} m_i\right) \sum_{j=1}^{k} \lambda_j A_j \left(\left(d - \sum_{i\le d} m_i\right)^{-1} z_m\right) \neq 0 \qquad (7.34)$$

with z_m as defined in (7.16).

PROOF. Using the spectral decomposition of $H_{\beta(n)}$ and the result of Lemma 7.2.5 we obtain

$$\frac{\log P\{X_{1,n} > -1/n\}}{\log P\{X_{1,n} > -1/n, \dots, X_{d,n} > -1/n\}}$$
$$= \left(\log\left(1 - H_{\beta(n),(1,0,\dots,0)}(-1/n)\right)\right) /$$
$$\left(\log\left(\sum_{m\in\{0,1\}^d,\, d-\sum_{i\le d} m_i \text{ even}} - \left(1 - H_{\beta(n),\left(d-\sum_{i\le d}\right)^{-1} z_m}\left(-\left(d - \sum_{i\le d}\right)/n\right)\right)\right.\right.$$
$$\left.\left. + \sum_{m\in\{0,1\}^d,\, d-\sum_{i\le d} m_i \text{ odd}} \left(1 - H_{\beta(n),\left(d-\sum_{i\le d}\right)^{-1} z_m}\left(-\left(d - \sum_{i\le d}\right)/n\right)\right)\right)\right)$$
$$= \left(\log\left(n\left(1 - H_{\beta(n),(1,0,\dots,0)}(-1/n)\right)\right) - \log(n)\right) / \tag{7.35}$$
$$\left(\log\left(\sum_{m\in\{0,1\}^d,\, d-\sum_{i\le d} m_i \text{ even}} -n\left(1 - H_{\beta(n),\left(d-\sum_{i\le d}\right)^{-1} z_m}\left(-\left(d - \sum_{i\le d}\right)/n\right)\right)\right.\right. \tag{7.36}$$
$$\left.\left. + \sum_{m\in\{0,1\}^d,\, d-\sum_{i\le d} m_i \text{ odd}} n\left(1 - H_{\beta(n),\left(d-\sum_{i\le d}\right)^{-1} z_m}\left(-\left(d - \sum_{i\le d}\right)/n\right)\right)\right) - \log(n)\right). \tag{7.37}$$

As in the proof of Theorem 5.1.1 the argument of the first logarithmic expression in (7.35) converges to $1 + \sum_{j=1}^{k} \lambda_j A_j(\mathbf{e}_1)$ which is equal to 1 according to condition (5.11) in Lemma 5.1.3. Similarly, the argument of the logarithm in (7.36) and (7.37) converges to

$$M := \sum_{m\in\{0,1\}^d\setminus(1,\dots,1)} (-1)^{d-\sum_{i\le d} m_i}\left(d - \sum_{i\le d} m_i\right)\sum_{j=1}^{k}\lambda_j A_j\left(\left(d - \sum_{i\le d} m_i\right)^{-1} z_m\right),$$

which is supposed to be different from zero, cf. (7.34). Therefore we have

$$\lim_{c\uparrow 0} \frac{\log P\{X_{1,n} > -1/n\}}{\log P\{X_{1,n} > -1/n, \dots, X_{d,n} > -1/n\}}$$
$$= \lim_{c\uparrow 0} \frac{-\log(n)}{\log(M) - \log(n)} = 1,$$

which justifies the assertion. □

The result of Lemma 7.4.2 suits the preceding considerations very well. Because the Pickands dependence function in the underlying spectral expansions is equal to 1, one obtains the residual dependence index

$$\bar{\chi}_{d,n} := \lim_{c\uparrow 0} \bar{\chi}_{d,n}(c) = \frac{1 + (1-d)\beta(n)}{1 + \beta(n)} < 1 \tag{7.38}$$

for each fixed n according to Lemma 7.2.6 and Lemma 7.2.7. However, the limit in Lemma 7.4.2 equals 1. Thus we again capture the dependence structure of the limiting distribution function G. Nevertheless, as $\beta(n) \to 0$, which forms part of the conditions imposed on the spectral densities, cf. (5.3), the residual dependence index $\bar{\chi}_{d,n}$ in (7.38) converges to 1. This convergence shows that the result of Lemma 7.4.2 is indeed meaningful.

8 Conclusion and outlook

In the preceding text we investigated the modeling of residual tail dependence structures within multivariate extreme value theory. Spectral densities and Pickands densities played a central role. In particular, technical conditions imposed on these densities led to limiting distribution functions of maxima under triangular schemes. The shape of the limiting distribution functions then gave information about the residual dependence structure of the underlying random variables. Pickands densities were used to deduce a test on tail dependence. Moreover, various measures of asymptotic dependence were established on the basis of spectral density expansions.

For future investigations this work offers several starting points. On the one hand, there is still room for advanced research within the presented model. On the other hand, one can think of possible modifications and extensions to gain further insight and to derive additional advantage from the previous results.

Some open questions concern the limiting distribution functions of the maxima under triangular schemes. From Theorem 5.1.1 and Theorem 5.2.2 we know that the limiting function G is actually a distribution function. Additional conditions imposed on the functions A_j, F and g , which determine G, imply that these functions G are distribution functions even without being the pointwise limit of a sequence of distribution functions, cf. condition (5.14) in Lemma 5.1.5 and condition (5.78) in Lemma 5.2.7. It is worthwhile to investigate these conditions more closely. This may also shed further light on the limiting distribution functions themselves. In addition, one may ask for necessary properties of the functions $A_{j,n}$ which are contained in the spectral expansion of $H^n_{\beta(n)}$ and converge to A_j, as $n \to \infty$. Properties of the $A_{j,n}$ are of particular interest in view of the expansions (5.49). Since these expansions can be regarded as signed measures represented in form of a measure generating function, one might look for suitable penultimate distributions, cf. [40], pp. 172–173.

Another point that could be studied more closely concerns the limit of the spectral density $h_{n,\mathbf{z}}$. In Remark 5.2.3 we have only given a hint that there is a geometric interpretation of this limiting function, which is simultaneously the Pickands dependence function of the limiting distribution function G, cf. (5.77) in Corollary 5.2.6. Therefore future research work may comprise further investigations of this geometric aspect.

In the framework of maxima under triangular schemes and especially for the computation of their limiting distribution functions, spectral densities have turned out to be more adequate and easier to handle than Pickands densities. We have indeed established limiting distributions under expansions of Pickands densities in Section 5.1.2, where we have also seen that they coincide with the limiting distributions under spectral expansions provided that some conditions are satisfied. However, we have restricted ourselves to the bivariate case. Therefore, an extension of Lemma 5.1.8 and

Lemma 5.1.9 to higher dimensions is still a task as yet untackled. Although for our purpose of investigating residual dependence structures spectral densities have been sufficient, additional results under Pickands densities might lead to a deeper insight into this matter.

A separate chapter of this work has been dedicated to the derivation of results for univariate margins belonging to the max–domain of any univariate EVD, not only to the reversely exponential distribution function. Yet we have not investigated whether the density expansion of a given multivariate distribution function and, thus, the limiting distribution function of the maxima changes due to a transformation of the univariate margins. One step further would consist in analyzing the effect of a transformation to univariate margins which belong to the max–domain of attraction of an EVD.

The problem of marginal transformations is also of interest if one intends to compute spectral densities or Pickands densities of specific distributions. In Section 3.3 we chose transformations to either reversely exponential or $[-1, 0]$–uniform margins. One of the examples in this section presented an expansion of the Pickands density of an elliptical random vector following the Kotz type distribution. However, it is still an untackled problem to compute spectral expansions for elliptical distributions and to verify a condition which leads to limiting distribution functions G of maxima under triangular schemes. Against the background of the work by Hashorva, cf. [22] and [23], we expect G to be the Hüsler–Reiss distribution function if the distribution function F of the radius of the elliptical random vector is in the max–domain of attraction of the unit Gumbel distribution. If F belongs to the max–domain of attraction of the Weibull distribution, G is supposed to be the max–infinitely divisible (max–id) distribution function $H_{\alpha,\lambda}$ given in Theorem 2.1 of [23]. One may even extend Theorem 5.2.2 and Theorem 6.3.5, respectively, in such a way that the limiting distribution functions are no longer max–stable in general but max–id.

In addition to elliptical distributions it might also be interesting to investigate other classes of distributions, e.g. sum–stable distributions.

Throughout this thesis we have considered distributions and limiting distributions of maxima. However, an alternative formulation of the preceding results in the framework of multivariate generalized Pareto distributions as limiting distributions of exceedances over high thresholds instead of maxima should be possible.

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