Testing the Standard Model with Precision Calculations of Semileptonic B-Decays

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Testen des Standardmodells mit Präzisionsrechnungen Semi-leptonischer B-Zerfälle

Zusammenfassung

Messungen im Flavour Sektor sind zum Test des Standardmodells sehr wichtig, da die meisten der freien Parameter in Beziehung zur Flavour Physik stehen. Wir diskutieren semileptonische B-Meson Zerfälle, aus denen ein wichtiger Parameter, $|V_{cb}|$, extrahiert wird.

Zunächst diskutieren wir nicht-perturbative Korrekturen höherer Ordnung in inklusiven semileptonischen Zerfällen von B Mesonen. Wir identifizieren die relevanten hadronischen Matrix Elemente bis zur Ordnung $1/m_b^5$ und schätzen diese mit einem Approximationsschema ab. Mit diesem Zugang wird der Effekt auf die integrierte Rate und kinematische Momente abgeschätzt. Ähnliche Abschätzungen werden für den $B \to X_s + \gamma$ Zerfall angegeben.

Weiterhin untersuchen wir in diesem Zerfall die Rolle von sogenannten „intrinsic-charm“ Operatoren, die ab der Ordnung $1/m_b^3$ in der schweren Quark Entwicklung auftreten. Durch explizite Rechnung zeigen wir, dass – bei Skalen $\mu \leq m_c$ – die Beiträge von „intrinsic-charm“ Effekten in die kurzreichweiten Koeffizienten Funktionen multipliziert z.B. mit dem Darwin Term, absorbiert werden können. Dann sind die einzigen Reste dieser Beiträge Logarithmen der Form $\ln(m_c^2/m_b^2)$, die mit Hilfe von Renormierungsgruppen Methoden resumiert werden können. Solange die Dynamik bei der Charm Quark Skala perturbativ ist, $\alpha_s(m_c) \ll 1$, impliziert dies, dass keine zusätzlichen nicht-perturbativen Matrix Elemente, neben dem Darwin und Spin-Orbit Term, zur Ordnung $1/m_b^3$ auftreten. Allerdings erzeugt „intrinsic charm“ zur nächsten Ordnung Terme proportional zu inversen Potenzen von der Charm Masse: $1/m_b^3 \times 1/m_c^2$. Parametrisch komplettieren diese die Abschätzung des Einflusses der $1/m_b^4$ Terme, die wir untersuchen werden. In diesem Kontext ziehen wir semiquantitative Schlussfolgerungen über die zu erwartende Größenordnung von „Weak Annihilation“ in semileptonischen B Zerfällen, sowohl für die Valenz als auch nicht-Valenz Komponenten.

Der letzte Teil ist der komplementären Messung von $|V_{cb}|$ durch exklusive $B \to D^{(*)}\ell\bar{\nu}_\ell$ Zerfälle gewidmet. Da diese Bestimmung eine kleine Inkonsistenz zu der inklusiven zeigt, untersuchen wir ob nicht Standardmodell Beiträge die Messung beeinflussen können.

Testing the Standard Model with Precision Calculations of Semileptonic B-Decays

Abstract

Measurements in the flavour sector are very important to test the Standard Model, since most of the free parameters are related to flavour physics. We are discussing semileptonic B-meson decays, from which an important parameter, $|V_{cb}|$, is extracted.

First we discuss higher-order non-perturbative corrections in inclusive semileptonic decays of B mesons. We identify the relevant hadronic matrix elements up to $1/m_b^5$ and estimate them using an approximation scheme. Within this approach the effects on the integrated rate and on kinematic moments are estimated. Similar estimates are presented for $B \to X_s + \gamma$ decays.

Furthermore we investigate the role of so-called "intrinsic-charm" operators in this decay, which appear first at order $1/m_b^3$ in the heavy-quark expansion. We show by explicit calculation that—at scales $\mu \leq m_c$—the contributions from "intrinsic-charm" effects can be absorbed into short-distance coefficient functions multiplying, for instance, the Darwin term. Then, the only remnant of "intrinsic charm" are logarithms of the form $\ln(m_c^2/m_b^2)$, which can be resummed by using renormalization-group techniques. As long as the dynamics at the charm quark scale is perturbative, $\alpha_s(m_c) \ll 1$, this implies that no additional non-perturbative matrix elements aside from the Darwin and the spin-orbit term have to be introduced at order $1/m_b^3$. However, "intrinsic charm" leads at the next order to terms with inverse powers of the charm mass: $1/m_b^3 \times 1/m_c^2$. Parametrically they complement the estimate of the potential impact of $1/m_b^4$ contributions, which we will explore. In this context, we draw semiquantitative conclusions for the expected scale of weak annihilation in semileptonic B decays, both for its valence and non-valence components.

The last part is dedicated to a complementary measurement of $|V_{cb}|$ from exclusive $B \to D^{(*)}\ell\bar{\nu}_\ell$. Since this determination shows a slight tension with respect to the inclusive one, we investigate wether a non standard model contribution may distort the extraction.

Contents

1. Introduction

At all times mankind has tried to explore nature just from observing their environment. In modern physics however, which starting is mostly linked with Galileo Galilei back in the 16th and 17th century, scientist realized that an active research using experiments widely improves their understanding. Nowadays the connection between theoretical work and experiments is inevitable for the understanding of elementary characteristics of the nature. Modern physics is basically nothing else than observing *interactions* of all different kinds depending on the field of research and then describing them in the best possible way. Interactions are the only way, how nature can be experienced and things in everyday life can be used. Even if you look at a beautiful picture, it is just the interaction of the photons with your eye. By forecasting different properties of nature, scientist try to verify or falsify their theories in experiments. Because nature has numerous phenomenon, their exist a lot of different physics research branches. Exploring the elementary phenomena in our world is doing fundamental research. Of course scientist pursue the goal to gain more and more knowledge about our environment where we all live in. It is one aspect where mankind should go for to advance in experience, but all times people have profited from this research. A lot of applications for our workaday life have only been possible due to knowledge gained from fundamental research. The major achievements of the last century were the development of the theory of *general relativity* by A. Einstein, which describes the gravitational force and interactions at large scales, e.g. the forces that stabilize our milky way and solar system. One other achievement was the establishment of the *Standard Model* (SM), which describes the fundamental interactions of the smallest particles that our world consists of. Unfortunately nobody has succeeded to unify these two theories, yet. To this end, more fundamental knowledge on both of these fields is necessary.

This thesis deals with an aspect of elementary particle physics. Particle physics is one important field of physics research, describing the fundamental *interaction* of the particles that our world consists of. It has already been widely explored in the 20th century. All begun with the theoretical development of quantum mechanics, connected with names like P.A.M. Dirac, W. Heisenberg, W. Pauli, E. Schrödinger, and even A. Einstein[1]. Then it was achieved to unify special relativity with quantum mechanics in quantum field theory. This was necessary to describe the observed properties. With the theoretical progress in quantum field theory and the improving accelerator technologies, it was possible to develop the so called *Standard Model*. Over the last approximately 40 years it has been widely tested and explored. Although it is checked to an incredible precision, most physicists believe that it can only be understood as an ef-

[1] Of course there were a lot more physicist who contributed to.

fective field theory, an approximation which we will explain in more detail later. To find deviations from the SM there are basically two ansatzes: One is to push calculations and also experimental efforts to an incredible precision in order to find inconsistencies in different determinations of the same parameters. The other is to directly apply theoretical models auf physics beyond the Standard Model (BSM) to experimental data.

1.1. Symmetries and Nature

There are a lot of different symmetries realized in our world. For example crystals grow in a specific way, and they look the same if one rotates them in a special way. Often it is very useful to find such symmetries in the system to be described. This simplifies the problem a lot: If the symmetry in the crystal is known there are less coordinates necessary to describe the whole crystal. Maybe also the beauty of such crystals like diamonds, guided the scientist to make use of symmetries.

In the theoretical description such symmetries manifest in mathematical descriptions which do not change, if one implies such symmetry. This reduces the degrees of freedom. In classical mechanics, for instance, the dynamics of a system is described by the Lagrange function. Since the equations of motions (e.o.m.) are derived from this function, all transformations that leave these e.o.m. invariant, are symmetries of the system. Of course in my position to be a student at the Emmy-Noether campus I have to mention the Noether theorem. This theorem states that for every symmetry of a system there is a conserved quantity. For example the homogeneity of space, which means a displacement does not change the system, leads to momentum conservation. Therefore the system can be described by less degrees of freedom.

Today symmetry in physics is not only realized as a visual perception. As we have shown in the last paragraph, it is also seen as a symmetry in the mathematical sense. A nice illustrative example is electrodynamics. There a symmetry lies in the gauge transformations. The potentials that generate the electric and magnetic field cannot be observed. Nevertheless they can be calculated up to arbitrary gauge transformations. These transformations change the potentials but leave the physical observables invariant. The correct mathematical description for the transformation is group theory. Here the simplest group $U(1)$ corresponds to the class of gauge transformations. In fact we need to have a local symmetry, because it is not possible to form a Lorentz covariant vector with the properties of a photon: Helicity ± 1 and massless. One can only have a four-vector which transforms under Lorentz transformation as

$$U(\Lambda)A^\mu(x)U^{-1}(\Lambda) = (\Lambda^{-1})^\mu_\nu A^\nu(x) + \partial^\mu\theta(x)\,,$$

which is a gauge transformation. So we need a Lagrangian which is also invariant under this transformation. This will lead to an inner symmetry and the connected Noether current j^μ. This current in the end will describe the interaction of fermions with the photon. We will later see, that this principle inspired the physicists to construct the Standard Model in a similar manner.

1.2. Particle Physics and its Goals

Particle physics deals with the elementary particles out of which our world is build. These are the smallest particles we currently know and we believe that they are not made of even smaller constituents. The particles have to undergo certain interactions to build the matter we all know. To figure out the structure of the interactions is necessary for the understanding of our world. If all these different particles would not have exactly these properties, the world would look completely different. Besides we want to understand why there is only matter and not antimatter within the universe. By investigating these particles and establishing the correct theory, we can understand more and more about our own origin. And should it not be a quest for mankind, to find out their own dawn?

We have already seen, that electrodynamics is described by an $U(1)$ symmetry. The Standard Model bases upon a more complicated group structure. We will explain this on the next pages. But first it is time for a few comments. There are basically two different types of symmetries. One is to impose a global symmetry, so the gauge transformation is the same at all space-time points. This clearly does not induce any interesting physics, its just a phase which cannot be observed. But we have already seen, that we have to impose a local symmetry. Then we have different transformations at each space-time point. The symmetry principle, that the form of the corresponding Lagrangian density is invariant, demands to introduce auxiliary gauge fields. These fields turn into gauge forces. In other words imposing the requirement of a local gauge transformation leads inevitably to the introduction of gauge forces. These gauge forces are nothing else than the forces responsible for the interactions we observe. So we automatically introduce the forces by imposing a certain symmetry. This is nicely illustrated in fig. 1.1. The left picture shows the original system. For the picture in the middle a global gauge transformation has been imposed, which results in a total rotation of the system. For the picture on the right panel a local transformation has been performed, resulting in a deformation of the lines. In all cases the surface stays the same, which is the

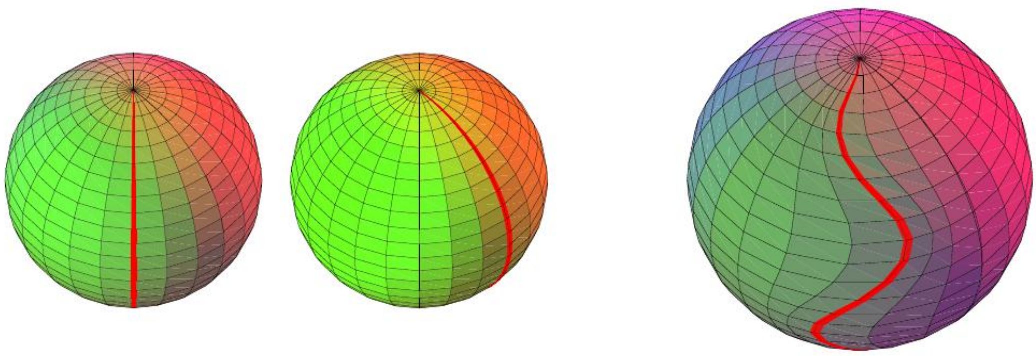


Figure 1.1.: The necessity of local force, picture taken from [1].

required symmetry. Nevertheless for the local symmetry a force is needed to account for the deformation. This fact is used in constructing the correct theory. Now the interplay between experiment and theory is a crucial point. The task of the theorists is to construct a consistent theory with a certain underlying symmetry[2]. Experiments now test this theory and determine the parameters which cannot be computed by first principles. If there are inconsistencies it is a hint, that the theory behind has to be modified. This has already been done, and so the Standard Model has been constructed in the last century. But nowadays this is seen as an approximate theory, valid only in some limits. In the next section we will explain the SM in more detail.

1.3. The Standard Model

The Standard Model has to describe the different interactions we know: The strong force, which binds the quarks inside the hadrons, the electromagnetic interaction as well as the decays of particles. In nature we observe the usual particles, which are fermions and the particles which mediate the forces. The latter ones follow the Bose statistic. A further complication are the masses of particles. On the one hand it is known that the particles are massive, on the other hand the range of the weak force is limited. Therefore the corresponding gauge bosons are also massive. But to implement the masses explicitly would break the symmetry of the Lagrangian. This leads to theoretical problems like the violation of gauge invariance. But the symmetry principle is exactly on what the theory bases. Furthermore gauge invariance requires, that the theory remains renormalizable. It turns out, that the so-called Higgs mechanism provides a very successful theoretical trick to achieve a consistent way of introducing masses for both particles and gauge bosons. It can be shown, that this procedure maintains the renormalizability of the theory.

1.3.1. Basic Ideas

To describe particles that obey different spin statistics, the corresponding Lagrangian densities were constructed. We have in general spin-1/2 fermions with the according Lagrangian density obeying the Dirac equation

$$\mathcal{L}_{\text{Fermion}} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi\,. \tag{1.1}$$

Additionally we need the Lagrangian density for a real scalar field, which describes a neutral spin-0 boson. This particles conform to the Klein-Gordan equation

$$\mathcal{L}_{\text{real KG}} = \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{1}{2}m^2\phi^2\,. \tag{1.2}$$

The corresponding equation for a complex scalar field describes charged spin-0 bosons

$$\mathcal{L}_{\text{complex KG}} = \left(\partial_\mu\phi\right)^\dagger\left(\partial^\mu\phi\right) - m^2\phi^\dagger\phi\,. \tag{1.3}$$

[2]At least if one does not have a brand-new concept. However this approach is too successful to be completely wrong.

We have massless spin-1 bosons, which fullfil the Maxwell equations for massless Bosons and the Proca equation with an additional term for massive ones

$$\mathcal{L}_{\text{Proca-Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \,, \tag{1.4}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right] \tag{1.5}$$

is the field-strength tensor.

It has been revealed, that the strong forces obey a $SU(3)_C$ symmetry, whereas the electro-weak force are combined into a $SU(2)_W \otimes U(1)_Y$ symmetry. We start to write down and explain the appropriate Lagrangian which satisfies these local symmetries in turn.

This introduces directly all kind of interactions, by the principle of local gauge interactions. The interaction terms are higher than bilinear terms with more than two fields in the Lagrangian density. To compensate the additional term due to the local symmetry in the kinetic term, we have to introduce a gauge field with the correct (transformation) properties. This will be combined in the covariant derivative to render the total Lagrangian gauge-invariant

$$i\partial_\mu \rightarrow iD_\mu = i(\partial_\mu - igA_\mu^a T^a) \,,$$

where $A_\mu^a T^a$ is the gauge field transforming under the adjoint transformation, thus T^a are the adjoint generators for the corresponding symmetry with $a = 1, \ldots, N^2 - 1$. A generator acting on a field is an eigenvalue equation with the quantum number of the field as its eigenvalue. Using the covariant derivative we can write the field strength as

$$F_{\mu\nu} = F_{\mu\nu}^a T^a = -\frac{i}{g}\left[iD_\mu, iD_\nu\right] \,. \tag{1.6}$$

The fields transform as

$$A'_\mu = U(x) A_\mu(x) U^\dagger(x) + \frac{i}{g} U(x)(\partial_\mu U^\dagger(x)) \tag{1.7}$$

$$\psi' = U(x)\psi(x) \,, \tag{1.8}$$

where $U \in SU(N)$ with $U(x) = \exp\left[i\theta^a(x)T^a\right]$ is a finite symmetry transformation. Additionally we now have to put in the kinetic terms for the gauge bosons. From eq. (1.5) we know the transformation properties for the field strength tensor

$$F'_{\mu\nu}(x) = U(x) F_{\mu\nu} U^\dagger(x) \,. \tag{1.9}$$

From this we know, that the field strength transforms in the adjoint representation, and so it is composed of the generators in the adjoint representation. Therefore the simplest gauge invariant form for the kinetic term is

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\text{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] = -\frac{1}{2}F_{\mu\nu}^a F^{b,\mu\nu}\text{Tr}\left[T^a T^b\right] = -\frac{1}{4}F_{\mu\nu}^a F^{a,\mu\nu} \,, \tag{1.10}$$

where the prefactor ensures the correct outcome of the Maxwell equation of motion. In total this leads to the following Lagrangian density for a fermionic field with an imposed local symmetry

$$\mathcal{L} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] .$$

It can be directly seen that this Lagrangian is invariant under the symmetry transformation and that the additional term through the derivative of the transformed field is canceled by the transformation properties of this gauge field. For a $U(1)$ symmetry this is the Lagrangian density of QED.

1.3.2. The Electroweak Interaction

In experiment it was observed, that the interactions in quantum electrodynamics are mediated through a vector current. For the charged current interaction it was noticed, that this involves a vector minus axial vector coupling, only. Whereas for the neutral current interaction there is a mixture between vector and axial vector.

The only way to combine this consistently is to impose a $SU(2)_W \otimes U(1)_y$ symmetry, where W is the weak isospin and Y is the hypercharge quantum number.

Now we have to introduce the masses for obvious reasons, but the masses for the particles would destroy the $SU(2)$ symmetry explicitly. Additionally we know from the small range of the weak forces, that the mediating particles have to be massive. But an explicit mass term would as it was mentioned destroy gauge invariance.

Higgs Mechanism

Therefore we make use of the so-called Higgs mechanism. The idea is, that we introduce a spin-0 auxiliary field ϕ with the shortcut notation $\phi\phi^\dagger = \rho$. By a certain choice of a potential we give this field a non-vanishing vacuum expectation value (VEV). The Lagrangian density is then given by

$$\mathcal{L}_{\text{Higgs}} = \left(\partial_\mu\phi\right)^\dagger\left(\partial^\mu\phi\right) - V(\phi\phi^\dagger)\,. \tag{1.11}$$

The interaction term of a field with this Higgs boson then leads to a mass term, if the VEV is acquired. This mechanism will be explained in the next few steps in more details. There are two alternatives for the ground states, which means the states with lowest energy.

1. There exist only one unique ground state. That means it is invariant under **all** symmetry transformations. This corresponds to the trivial case.

2. There can be a degenerate ground state which transforms under a subgroup of the symmetry transformations, where all the different ground states are physically equivalent. But we can choose one of them to define a specific direction in representation space. This choice of the ground state is then called spontaneous breaking of the symmetry.

In fact a degenerate ground state is typical for systems with infinitely many degrees of freedom, only [2]. This is always the case for a quantum field theory, since we there have a set of fields for every of the infinitely many space-time points.

In our case it means, the related potential is adjusted in such a way, that the energetic minimum is displaced from the point, where the field vanishes. We will demonstrate this with the so-called Goldstone model. Take the following potential as given

$$\begin{aligned} V(\phi\phi^\dagger) &= \mu^2\rho + \lambda\rho^2 \\ &= \lambda\left(\rho + \frac{\mu^2}{2\lambda}\right)^2 - \frac{(\mu^2)^2}{4\lambda} \,. \end{aligned}$$

The parameter λ has to be positive, because we need a lower bound for the energy. For the parameter μ^2, we have two possibilities[3]. Since $\rho \geq 0$ we have a minimum at $\rho \equiv 0$ for $\mu^2 > 0$. This is the trivial case. For the other case $\mu^2 < 0$ we have a minimum at $\rho_0 = v^2 = -\frac{\mu^2}{2\lambda}$. This means the fields acquires its minimum at a non-zero point for the field. So due to the specific choice of the potential the field gets a constant VEV v, and we will make use of this to create the masses for the gauge bosons. The form of this potential for $\mu^2 < 0$ is known as the Mexican-hat potential, and shown in fig. 1.2.

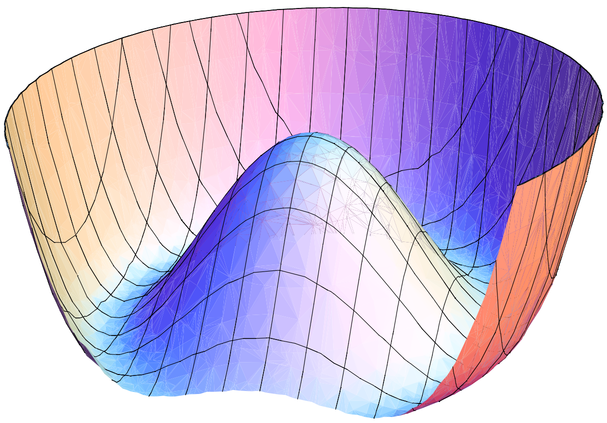


Figure 1.2.: 2D model of a Higgs potential with non-vanishing VEV.

Now we parameterize the complex field around the VEV with two real fields η and ξ

$$\phi = \left(v + \frac{1}{\sqrt{2}}\eta(x)\right) \exp\left[i\left(\theta + \frac{\xi(x)}{\sqrt{2}v}\right)\right] \,. \tag{1.12}$$

θ is some arbitrary phase, which can be rotated away by the symmetry. Choosing a particular phase leads to *spontaneous symmetry breaking*. Plugging in the reparametrized field in (1.11), we see that the field η becomes massive with $m_\eta = -2\mu^2$ whereas ξ stays massless. The latter field corresponds to the excitation along the minimal potential line in fig 1.2, more precisely it is associated to the normal mode with zero

[3]Note, that we do not want to have absorption and thus require the potential to be purely real.

eigenfrequency. The η field is the radial excitation and becomes massive due to the non-vanishing curvature of the potential.

We have seen, that by a specific choice of the potential the formally massless field becomes massive. The next step is to extend this trick to give masses to the gauge bosons. By imposing a local symmetry, the kinetic term has to be modified such, that we replace the derivatives with a covariant derivate. In this manner we automatically introduce the interaction of the gauge fields with the Higgs boson just by the requirement of gauge invariance. We achieve the mass terms for the gauge bosons due to their coupling to the constant VEV. Nevertheless the imposed symmetry is broken, but through a dynamical process and therefore it is spontaneously broken. In this toy model we assume one gauge field A_μ with the coupling constant g, to illustrate the general principle. The ξ field can be gauged away by a unitary gauge, and therefore turns into the longitudinal polarization mode of the massive gauge boson

$$B_\mu = A_\mu - \frac{1}{gv}\partial_\mu \eta \tag{1.13}$$

$$\phi = v + \frac{\eta}{\sqrt{2}}\,, \tag{1.14}$$

which simplifies the Lagrangian density a lot

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2}\left(\partial_\mu \eta\right)\left(\partial^\mu \eta\right) + \frac{1}{2} g^2 \left(v + \eta\right)^2 B_\mu B^\mu - \frac{1}{2}\text{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] - V\left((v + \frac{\eta}{\sqrt{2}})^2\right)\,. \tag{1.15}$$

The mass and interaction terms can now be directly read off, as we have already seen $m_\eta = -2\mu^2$, and $m_B = v\,g$.

Standard Model Gauge Part

In the special case of the Standard Model, where a non-Abelian symmetry with in total four generators is at hand, we have to generalize the latter toy example. In general we have to introduce for N generators in total N real scalar Higgs fields. In the SM we make the therefore the special choice of a doublett with two complex scalar Higgs fields

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} \tag{1.16}$$

to account for the 4 generators A_μ of $U(1)_Y$ with coupling g' and B^i_μ with $i = 1$,2,3 and coupling g for $SU(2)_W$. The Higgs field has to transform under the adjoint representation with hypercharge $Y = -1$ and we write it in an exponential parameterization

$$\Phi = e^{\frac{i}{2\rho_0}\tau^a \xi^a(x)} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(\eta(x) + v) \end{pmatrix}\,. \tag{1.17}$$

Now everything follows the same line as in the toy example, but now we have different generators which connect the fields in the equations. Choosing the vacuum expectation value as

$$\Phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{1.18}$$

we find three broken generators, and one remains unbroken, which is identified with the charge,

$$Q\Phi_0 \equiv (\tau_3 + Y)\Phi_0 = 0 \,. \tag{1.19}$$

Plugging everything into the Higgs part of the Lagrangian density, we compute a similar result as in the toy model case

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} =& \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - V\left(\frac{1}{2}(\eta + v)^2\right) \\ &+ \frac{1}{8} g^2 (\eta + v)^2 \left(B_\mu^1 - iB_\mu^2\right)\left(B^{\mu\,1} - iB^{\mu\,2}\right) \\ &+ \frac{1}{8}(\eta + v)^2 \left(g' A_\mu - g B_\mu^3\right)\left(g' A^\mu - g B^{\mu\,3}\right) \,. \end{aligned} \tag{1.20}$$

Thus we end up with three massive gauge bosons

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(B_\mu^1 \mp B_\mu^2\right) \qquad \text{with} \qquad M_{W^\pm} = \frac{1}{2} g^2 v^2 \tag{1.21a}$$

$$Z_\mu^0 = B_\mu^3 \cos\theta_W - A_\mu \sin\theta_W \qquad \text{with} \qquad M_{Z^0} = \frac{1}{2} v^2 (g^2 + g'^2) \,, \tag{1.21b}$$

where we have introduced the shortcut $\sin\theta_W = g'/\sqrt{g^2 + g'^2}$, and θ_W is the Cabbibo Angle. The Higgs mass is obviously the same as in the toy model $M_H = -2\mu^2$. The orthogonal field combination to the neutral current

$$\mathcal{A}_\mu = B_\mu^3 \cos\theta_W + A_\mu \sin\theta_W \tag{1.22}$$

is identified with the massless photon. The breaking is therefore organized in such a way, that the theory is broken as $SU(2)_W \otimes U(1)_y \to U(1)_{\text{em}}$. In the end we have for the unbroken $U(1)$ generator one massless gauge boson, which is identified with the massless photon of the electromagnetic interaction.

Fermionic Part

The same arguments as for the gauge bosons hold true for the fermionic matter fields: We cannot implement mass terms explicitly, since they destroy gauge invariance. Therefore masses are created dynamically. For the particle masses we have to introduce Yukawa couplings by hand in the following way.

First of all experimentally it was noticed, that the weak force violates parity. Thus we treat left- and right-handed fields differently. The projector $P_{R/L} = 1/2(1 \pm \gamma_5)$ projects onto the right(left)-handed component. We put the left-handed fermions in a left-handed doublett under $SU(2)_W$, whereas the right-handed fields are singletts. For the leptons we have

$$L_A = \begin{pmatrix} \nu_{A_L} \\ A_L \end{pmatrix} , \qquad R_A = (A) , \qquad \text{with} \qquad A = e, \mu, \tau , \tag{1.23}$$

where we have no right handed neutrinos, for reasons we will see in a moment[4]. The left-handed handed leptons have a hypercharge of $Y = -1$, for the right-handed singletts we have $Y = -2$. For quarks we have for both "up-type" quarks u

$$L_u = \begin{pmatrix} u_L \\ d_L \end{pmatrix} , \qquad L_c = \begin{pmatrix} c_L \\ s_L \end{pmatrix} , \qquad L_t = \begin{pmatrix} t_L \\ b_L \end{pmatrix} , \tag{1.24}$$

as well as "down-type" quarks d a right-handed part

$$R^{(u)} = (u_R, c_R, t_R) , \qquad R^{(d)} = (d_R, s_R, b_R) . \tag{1.25}$$

The corresponding hypercharges are given by $YL_u = 1/3L_u$, $YR^{(u)} = 4/3R^{(u)}$ and $YR^{(d)} = -2/3R^{(d)}$.

First of all, the coupling between gauge bosons and fermionic fields are constructed in the very same way as we have already seen. We have to introduce the covariant derivatives which then couple the gauge bosons to the matter fields.

We begin with the generation of the mass terms for the leptons. Therefore we introduce Yukawa couplings of the form

$$\mathcal{L}_{\text{Yuk}} = \lambda \, \bar{\psi}(x)\Phi(x)\psi(x) \tag{1.26}$$

with our scalar Higgs field $\Phi(x)$. The VEV leads to terms, which have the form of mass terms for fermionic particles

$$\mathcal{L}_{\text{Mass}} = \lambda \, \langle \Phi \rangle \bar{\psi}(x)\psi(x) . \tag{1.27}$$

This kind of interaction term couples the right-handed fields to the left-handed ones. Note, that these terms are gauge invariant and for sake of the notation we have grouped the right-handed fields in the same doublett as the left-handed fields with a vanishing upper-component,

$$\mathcal{L}_{\text{Yuk}} = -\sum_{A,B} G_{AB} L_A \Phi R_B + \text{h.c.} \tag{1.28}$$

Thereby we sum over all leptons $A,B = e,\mu,\tau$ and (G_{AB}) is in general a complex 3×3 matrix. In the next step we compute the diagonal mass matrix, which is always feasible by a bi-unitary transformation according to $M_{\text{diag}} = UGV^\dagger$, where we use the notation

$$\Lambda = \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix} \qquad \hat{\Lambda} = \begin{pmatrix} \nu_{e_L} \\ \nu_{\mu_L} \\ \nu_{\tau_L} \end{pmatrix} \qquad \mathcal{R} = \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} . \tag{1.29}$$

[4] We keep the neutrinos massless, although we know they are massive. However the SM was originally constructed without neutrino masses. For their masses, different scenarios are discussed, which are beyond the scope of this thesis.

Then, in unitary gauge, the Yukawa term takes the following form

$$\begin{aligned}\mathcal{L}_{\text{Yuk}} &= -\frac{1}{\sqrt{2}}(\eta+v)\bar{\Lambda}\underbrace{U^\dagger U}_{\mathbf{1}} G \underbrace{V^\dagger V}_{\mathbf{1}} \mathcal{R} + \text{ h.c.} \\ &= -\frac{1}{\sqrt{2}}(\eta+v)\bar{\Lambda}' M_{\text{diag}} \mathcal{R}' + \text{ h.c.} \end{aligned} \tag{1.30}$$

For obtaining the diagonal mass matrix we pay the price of a mismatch between mass and weak eigenstates of the fermionic fields, which is defined as $\Lambda' = U\Lambda$ and $\mathcal{R}' = V\mathcal{R}$. In experiment, we detect the mass eigenstates, only. In case of the leptons, we have a special situation. Since neutrinos are massless - there are no right-handed neutrinos within the Standard Model - we can choose the same transformation rule for the neutrinos as well as for the charged leptons $\hat{\Lambda}' = U\hat{\Lambda}$. The kinetic terms are not changed, because only fields with the same handedness are coupled and thus the transformation cancels out. One direct consequence of the massless neutrinos is the separate conservation of the lepton flavours n_e, n_μ, n_τ.

Now turning to the quark system, we will point out the difference. Here both up-types and down-types are charged and massive. Again we construct the mass terms along the same line using the Higgs doublett Φ

$$\mathcal{L}^{(d)}_{\text{Yuk}} = -\sum_{A,B} G'_{AB}\bar{L}_A \Phi R^{(d)}_B + \text{ h.c.} \tag{1.31a}$$

$$\mathcal{L}^{(u)}_{\text{Yuk}} = -\sum_{A,B} G''_{AB}\bar{L}_A \tilde{\Phi} R^{(u)}_B + \text{ h.c.}\,, \tag{1.31b}$$

where we have to introduce in principle two different Higgs fields Φ and $\tilde{\Phi}$, one for the up-type right-handed fields and one for the down-type ones. Note that we have again grouped the right-handed singletts into an appropriate doublett with a vanishing component, in order to introduce a compact notation. Now we can make use of a special phenomenon in $SU(2)$, where the complex conjugate representation is equivalent to the fundamental one. We notice that to render both terms invariant, the field $\tilde{\Phi}$ has to have again the hypercharge $Y = -1$. Then by using a $SU(2)$ transformation, we can represent the field $\tilde{\Phi}$ through the Higgs field Φ itself, and write again in unitary gauge

$$\tilde{\Phi} = \frac{1}{\sqrt{2}}\begin{pmatrix}\eta+v \\ 0\end{pmatrix}. \tag{1.32}$$

Following the same line as in the case of leptons, we can determine the mass matrix for the up-type quarks. But now we do not have the freedom, to choose the same transformation for both up and down-type quarks, since now right-handed quarks exist for both types. In general the matrices G' and G'' are not commutative $[G',G''] \neq 0$, and therefore we cannot diagonalize both matrices at the same time.

Therefore we resort the quarks in the following way

$$D_L := \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} \qquad D_R := \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} \tag{1.33a}$$

$$U_L := \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} \qquad U_R := \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} , \tag{1.33b}$$

such that now the Yukawa Lagrangian density is given by

$$\mathcal{L}_{\text{Yuk}} = -\frac{1}{\sqrt{2}}(\rho + \rho_0) \left[\bar{D}_L G' D_R + \bar{U}_L G'' U_R\right] + \text{ h.c.} \tag{1.34}$$

We can again diagonalize both matrices, but we pay the price to introduce two different transformations

$$V^{(d)} G' W^{(d)\,\dagger} := G'_{\text{diag}} = \frac{\sqrt{2}}{v} \text{Diag}\,(m_d, m_s, m_b) \tag{1.35a}$$

$$V^{(u)} G'' W^{(u)\,\dagger} := G''_{\text{diag}} = \frac{\sqrt{2}}{v} \text{Diag}\,(m_u, m_c, m_t) \; . \tag{1.35b}$$

In the end we obtain the physical quark fields with the diagonal mass matrices. The physical quark fields are thus defined as

$$D'_L = V^{(d)} D_L \qquad D'_R = W^{(d)} D_R \tag{1.36a}$$

$$U'_L = V^{(u)} U_L \qquad U'_R = W^{(u)} U_R \tag{1.36b}$$

The difference to the lepton case is now, that the kinetic term changes. As we have seen before, the neutral current does not change. But in the case of charged currents there are some implications, which allow for quark flavours transition similar to the $\ell \leftrightarrow \nu_\ell$ process. The analogous process in the quark sector is the up-type $\leftrightarrow$ down-type quark transition. Compared to the leptonic case, where both neutrinos and charged leptons obey the same transformation, we have here a missmatch as explained above. This leads to the Cabbibo Kobayashi Maskawa (CKM) matrix, which we will discuss in more detail at the end of this section.

After having introduced the general principle of the theory construction, we will take a closer look on the different forces in turn.

1.3.3. Quantum Chromo Dynamics

First we will give a few important facts on how Quantum Chromo Dynamics (QCD) has been founded. Then we will give some more mathematical details.

QCD describes the strong interaction of quarks inside a hadron. The force is mediated via the gluons. The gluons couple to quarks and have a self-coupling responsible for the special properties. Historically in the late 60s it was suggested that hadrons are built out of fundamental partons, namely the quarks [3][5]. A further step was made by

[5] A more detailed historical introduction can be found in [4], we just give a brief overview here.

the discovery of asymptotic freedom in non-Abelian gauge theories [5,6], for which the self coupling of the gauge bosons are important. This allows the perturbative treatment at short-distances, as it was observed [7], and at the same time provides the strong binding to hadrons at long-distances. In experiments it then turns out that an additional quantum number for the description of quarks was necessary: The color degree of freedom. Furthermore no states with a non-vanishing colour quantum number were observed. For that reason, the additional colour symmetry has to be exact and it was proposed that this symmetry of the non-Abelian gauge theory has to be identified with the color symmetry [8]. In experiment it was established, that there should be three different colours. With the additional requirement, that anti-quarks should behave as a complex conjugate representation from the quarks, the only simple non-Abelian symmetry group left is $SU(3)$. This group additionally contains the singlett pieces, which describe the colour neutral hadrons. Therefore the symmetry group for QCD is $SU(3)_C$, which is also established experimentally very well. In agreement with experimental data, hadrons can then be classified as mesons containing a quark-antiquark pair, as well as baryons made up of three quarks. Both these states can be described as singlett pieces of $SU(3)_C$ as required. The feature of $SU(3)$ in contrast to $U(3)$ is that no singlett gluons appear. Thus there is no color neutral gluon, and the range is limited to the size of the hadron[6]. The massless photons have infinite range, and the range of the weak bosons is limited due to their mass.

The quark fields are put into a triplet under the colour symmetry

$$\psi_q^1 = \psi_q(x)\begin{pmatrix}1\\0\\0\end{pmatrix}\,, \qquad \psi_q^2 = \psi_q(x)\begin{pmatrix}0\\1\\0\end{pmatrix}\,, \qquad \psi_q^3 = \psi_q(x)\begin{pmatrix}0\\0\\1\end{pmatrix}\,, \tag{1.37}$$

where $\{1{,}2{,}3\} \mathrel{\hat{=}}$ "red, green, blue" corresponds to the color quantum number and q is the flavour of the quark field. The fields transforms as

$$\psi_q^{\prime a} = e^{-ig_s\theta^b(x)T^b}\psi_q^a\,, \tag{1.38}$$

where $T^b = \lambda^b/2$ are the generators of $SU(3)_C$ and λ^b are the Gell-Mann matrices. The generators fulfill the Lie algebra

$$\left[T^a{,}T^b\right] = if^{abc}T^c\,, \tag{1.39}$$

where f^{abc} are the $SU(3)$ structure constants.

As in the sections before, we construct a Lagrangian that fulfills the requirement of the underlying gauge symmetry. The QCD Lagrangian is then given by

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,,a} + \sum_{q=1}^{N_f}\sum_{a,b=1}^{3}\bar{\psi}_q^a\left(i\not{D}_{ab} - m_q\delta_{ab}\right)\psi_q^b\,, \tag{1.40}$$

[6]In fact due to polarization effects protons and neutrons are bound inside the atomic core.

which has to be added to the Lagrangian from the electroweak sector to form the complete description of the SM. However this part is completely independent from the electroweak sector.

Nevertheless it is time for a few comments. The covariant derivative and the field-strength tensor are defined as

$$D^{\mu}_{ab} = \partial_\mu \delta_{ab} - ig_s\, A^c_\mu T^c_{ab} \tag{1.41a}$$

$$F^a_{\mu\nu} T^a = \frac{i}{g_s}\left[D_\mu, D_\nu\right] = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\mu\right) T^a\,, \tag{1.41b}$$

from which the self interaction terms, three and four gluon vertices, can be read of directly due to the kinetic gauge boson term. The gauge invariance of this term can be directly seen, by using

$$\sum_a F^a_{\mu\nu} F^{\mu\nu\,,a} = \sum_{a,b} F^a_{\mu\nu} F^{\mu\nu\,,b}\delta_{ab} = 2\sum_{a,b} F^a_{\mu\nu} F^{\mu\nu\,,b}\mathrm{Tr}\left[T^a T^b\right] \equiv 2\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] \tag{1.42}$$

Here the sum over q goes over all $N_f = 6$ quark flavours The sum over a is the sum over all three colors $a = 1{,}2{,}3$ The strong coupling constant is denoted by g_s.

1.3.4. Running Coupling and Renormalization

In general a quantum field theory cannot be solved exactly in four dimensions. But it can be expanded in powers of the coupling constant $\alpha = \frac{g^2}{4\pi}$. As long as $\alpha \ll 1$ is given, the expansion is valid and applicable. This allows for an order by order computation of the considered interaction. To have a consistent description therefore these quantum corrections[7] have to be computed. The first order is the tree-level and describes in principle "classical" processes, this is also known as the Born level. Higher order corrections are purely quantum effects; they are organized in a series of $\frac{\alpha}{4\pi}$, where α is the coupling constant of the according QFT. Higher orders are called "loop-level", because here internal interactions occur, where the momentum is not fixed and we therefore have to integrated over this loop-momenta.

In contrast to the electroweak sector, the coupling constant of the strong interaction becomes stronger for lower energies. This results in the binding of quarks to hadrons. In this regime the coupling constant is of order one, and therefore perturbative techniques cannot be applied. To describe hadrons several non-perturbative methods have been developed. We will use and explain Heavy Quark Effective Theory (HQET) in more detail. This theory uses the fact, that the heavy b-quark mass is much larger than the typical momentum transfer in weak decay processes.

In computing the loop diagrams we notice that divergences can appear. The integrals are of the form

$$\int \mathrm{d}^4 k\, \frac{k^i}{\left[k^2 - m^2\right]^j}\,, \tag{1.43}$$

We can immediately see, that we can have

[7] In principle this is also an expansion in $\hbar$ and therefore quantum effects play a role.

- *Infrared divergences*: If $m \to 0$ and $4 + i \leq j$ we have a divergence in the vicinity of small k.
- *Ultraviolet divergences*: If $4 + i \geq j$ we have a divergence for large k.

In total we have to recognize that we have to deal with this divergences if we compute quantum corrections. The reason for this is, that we treat the theory valid for all momenta, which might not be correct. See for example the four-fermion effective interaction [10]: Here we make use of the fact, that the momentum transfer is much smaller than the mass of the mediating gauge boson. Nevertheless we integrate over all momenta and therefore have to deal with these divergences. How this is done, and how this is interpreted we will describe below.

Regularization

The basic idea behind regularization is to express the physical observables through physical quantities and hide the divergences in unmeasurable parameters. This means such unphysical parameters drop out at the end in observables. A variety of different regularization schemes have been invented. Among these are Pauli-Villars and Cut-off schemes. Nowadays mostly dimensional regularization is used. There the physical dimension of Minkowskian space is analytically continued to $D = 4 - 2\epsilon$ dimensions, and after performing the integration the singularity in $\epsilon \to 0$ can be identified. Each scheme has its advantages. A cut-off scheme deals with the problem, that it violates Lorentz invariance. Dimensional regularization however is rather "unnatural". UV and IR singularities are regulated along the same line, also stemming from different divergent regions of the complex dimension D.

Nevertheless the physical result in the end has to be independent of the used scheme. Also its common to all schemes, that an additional scale, the renormalization scale, is introduced in the theory.

Renormalization

It may happen, that the regulator scales do not drop out of the physical observable. In such cases, we have to reinterpret the actual physical observable. The quantity in the Lagrangian is called *bare*-quantity. Additionally we introduce the physical *renormalized* quantity. In this way we include the divergence in a renormalization factor. Commonly a multiplicative renormalization prescription is used, as e.g. for the QCD fields and parameters

$$\psi_0^q = \sqrt{Z_q}\psi^q \qquad (m_q)_0 = Z_m m_q \tag{1.44a}$$

$$(A_\mu^a)_0 = \sqrt{Z_A} A_\mu^a \qquad (g_s)_0 = \mu^\epsilon Z_g g_s \tag{1.44b}$$

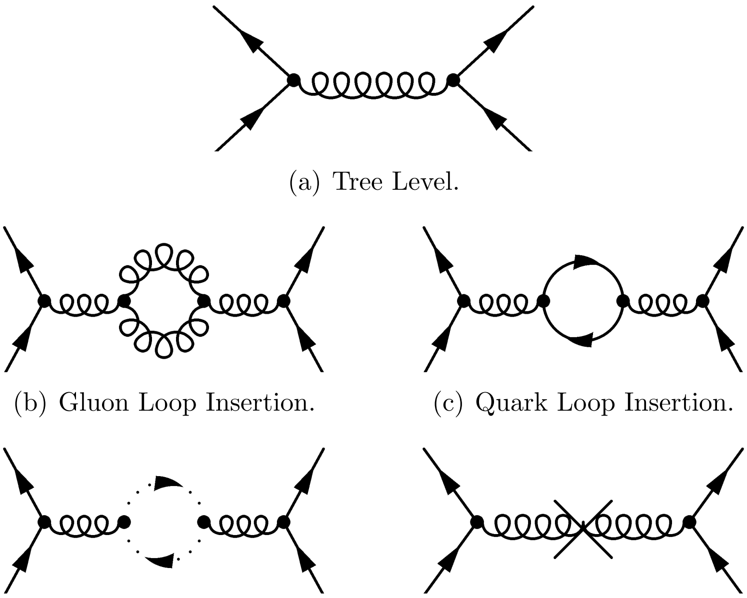
Here the bare quantities are labeled by a subscript 0. We had to introduce the scale μ in order to keep the coupling constant g_s dimensionless. Now putting this physical fields in the Lagrangian the usual choice is to express everything through the physical fields and keep the known Feynman rules

$$\mathcal{L} = \bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 = \bar{\psi}(i\partial\!\!\!/ - m)\psi + (Z_q - 1)\bar{\psi}i\partial\!\!\!/\psi - (Z_2 Z_m - 1)\bar{\psi}m\psi\,. \tag{1.45}$$

Since we evaluate the loop integrals order by order, the renormalization constants are defined order by order as well.

In physical observables, by summing up to all orders, the renormalization scale should drop out. However, since the expansion is only performed to a finite number of terms, there is always a remnant dependence. The dependence is usually of the form $\log \mu^2/Q^2$, where Q is the momentum transfer. Therefore the typical choice is to set the renormalization scale around Q^2 to keep these logarithms small. However one should keep in mind to check the dependence on this scale, and that higher order computations should fix the scale better.

Renormalization Group and Asymptotic Freedom



(a) Tree Level.

(b) Gluon Loop Insertion. (c) Quark Loop Insertion.

(d) Ghost Loop Insertion. (e) Counter Term.

Figure 1.3.: All necessary diagrams for computing the QCD β-function at the one loop level in background field gauge.

The convergence of the higher-order calculations strongly depends on the magnitude of the coupling constant. At the usual scales we observe, we have $\alpha_{\text{em}} = 1/137$ and $\alpha_s \approx 0.2$. Therefore considering QCD corrections is very important for the phenomenological analysis of weak decays, whereas in most applications QED and weak corrections are negligible. We explicitly have argued, that the bare quantities do not depend on μ.

Thus we can compute the behavior of the μ-dependence from eq.(1.44). This leads to the renormalization group equations (RGE)[8]

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} g_s(\mu) = -\epsilon g_s(\mu) \underbrace{-g_s Z_g^{-1} \frac{\mathrm{d}Z_g}{\mathrm{d}\ln\mu}}_{\beta(g_s)}$$

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\ln\mu} \alpha_s(\mu) = 2\beta(\mu) = -2\beta_0 \frac{\alpha_s^2}{4\pi} - 2\beta_1 \frac{\alpha_s^3}{(4\pi)^2} \tag{1.46}$$

This is a special RGE, called the beta-function. β_i are the coefficients to loop order $i+1$, which can be computed from the coupling constant renormalization constant. They have already been calculated and are given by

$$\beta_0 = \frac{11N_C - 2N_f}{3}\,, \qquad \beta_1 = \frac{34}{3}N_C^2 - \frac{10}{3}N_C N_f - 2C_F N_f \qquad C_F = \frac{N_C^2 - 1}{2N_C}\,,$$

where N_f denotes the number of quark flavours[9], and N_c is the number of colours. For calculating this you need to consider the diagrams shown in Figure 1.3, and then extract the corresponding renormalization constant Z_g. We have only shown the diagrams for the one loop contribution. The two gluon-ghost vertex is necessary to extract the correct physical polarization of non-Abelian gauge fields in loop diagrams.

For a complete review on this topic, please refer to references [2, 4, 9–11].

1.3.5. The Standard Model Particles

Now we are in the position to give a complete summary over the particle content of the SM. First we summarize the gauge bosons, mediating the four known forces of the Standard Model. We also list their experimentally determined properties. Then we will focus on the particles, leptons and quarks, which build the matter.

The Standard Model itself, as it has been constructed along the line which has been described above, has in total 19 different parameters. These parameters arise from the proposed structure and from the symmetry breaking sector.

3 Coupling constants: $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$

1 Higgs Mass

1 Higgs vacuum expectation value (VEV)

6 Quark masses

3 Lepton masses

[8] Its called a group, because these differential equations link the different sets of quantities at different values of μ.

[9] Please note, that the number of quark flavours depends on the scale at which we are computing the diagrams: we have to take into account all quark flavours with $m_q \lesssim \mu$, the others are integrated out of the theory and do not appear as dynamical degrees of freedom.

3 Quark mixing angles

1 CP-violating phase

1 Strong CP violating phase (often assumed to be zero)

The Strong CP violating phase arises due to a total derivative term in the Lagrangian of the form

$$\mathcal{L}_\theta = -\frac{\theta}{2}\epsilon_{\mu\nu\rho\sigma}F_a^{\mu\nu}F_a^{\rho\sigma} = -\theta\partial^\mu k_\mu\,, \qquad k_\mu = 4\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}\big[A_\nu\partial_\rho A_\sigma - \frac{2ig}{3}A_\nu A_\rho A_\sigma\big]\,. \quad (1.47)$$

Due to the non-Abelian structure of QCD such a term can contribute, although in classical mechanics such terms vanish in the equation of motion. Nevertheless θ is measured to be extremely small, compatible with zero.

Furthermore the non-vanishing neutrino masses were established. Now incorporating them into the Standard Model, which is just an extension, leads to nine additional parameters

3 Neutrino masses

3 Neutrino mixing angles

1 CP violating phase

2 Majorana phases.

Thus depending on the point of view, we have at least 18, at maximum 28 parameters in the Standard Model which have to be extracted from experiment. Please note, that almost all of these parameters are related to flavour physics.

The Gauge Bosons

Altogether we have twelve different gauge bosons stemming from the Standard Model gauge group

$$SU(3)_C \otimes SU(2)_W \otimes U(1)_Y$$

with different properties, where eight different gluons exist. For completeness their properties are listed in the table 1.1. The photon and gluon are massless due to the exact remaining symmetry

$$SU(3)_C \otimes U(1)_{em}$$

after the electroweak breaking. The massive weak bosons mediate the charged and neutral currents, which range is limited due to their mass. As it was mentioned their exist eight gluons, corresponding to the eight possible color charges: It transforms under the adjoint representation

$$3 \otimes \bar{3} = 1 \oplus 8\,. \qquad (1.48)$$

The gluon singlett, which corresponds to the colorless symmetric admixture does not exist, because we have $SU(3)$ to conserve unitarity: It removes the $U(1)$ singlett piece.

Boson	Mass	Coupling	Theory
γ	0	e	QED
$W^{\pm}$	80.403 ± 0.029 GeV	G_F	GWS
Z^0	91.1876 ± 0.0021 GeV	G_F	GWS
g	0	g_s	QCD

Table 1.1.: Fundamental Gauge Bosons of the Standard Model.

The Matter Particles and their Properties

The particles are divided into leptons, quarks and gauge bosons. Quarks are the only particles that undergo strong interaction. For some reason nature duplicated the leptons and quarks into three families, also called generations. Each family is divided into a doublett, in each case an "up-type" quark with the electric charge $+2/3e$ and a "down-type" quark with the electric charge $-1/3e$. Here e denotes the elementary charge of the electron, defined through $-e$. Additionally there is a charged lepton with charge $-e$ corresponding to the up-type quark, and a neutral lepton (neutrino) corresponding to the "down-type" quark. The families are sorted according to their experimentally determined mass; an overview is given in table 1.2, taken from [13].

Our usual matter consist of the first generation particles, only. The ones from the second and third generation are produced in high energetic collisions. They obey the usual principle that all systems tend to go to the most favourable energetic state, and hence decay in a chain into the first generation. As we have seen this decays are mediated through the weak bosons and this fact is most important for this thesis. Thus we will discuss this in much more detail later.

	Family			Electric Charge
	I	II	III	(e)
Leptons	$\begin{pmatrix} e \\ \nu_e \end{pmatrix}$	$\begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}$	$\begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}$	-1 0
Quarks	$\begin{pmatrix} u \\ d \end{pmatrix}$	$\begin{pmatrix} c \\ s \end{pmatrix}$	$\begin{pmatrix} t \\ b \end{pmatrix}$	$\frac{2}{3}$ $-\frac{1}{3}$

Table 1.2.: Elementary particles sorted after families.

We have already learned that there exist no free quarks[10], so only bound hadronic states, baryons and mesons, can be experimentally measured.

[10] The top quark is an exception, because it is heavy enough to radiate a real W-boson, responsible for the decay. Therefore it decays faster than the time it takes to form a bound hadronic state.

Hence the quark masses are only parameters in the theory, but not physical observables. They have to be defined in certain schemes, which can be translated into each other. Besides the masses also the coupling constants are such scheme dependent parameters. All these parameters depend on a scale parameter, usually called μ. In the end physical observables have to be independent of this parameter. Therefore this scale just shifts contributions from one piece to the other, depending on how one looks at the process. It is known that below approximately $\mu_H \approx 1$ GeV the strong force becomes too strong to be treated perturbatively. This sets a reference point. All quarks that are lighter than than μ_H are light quarks, the other heavy quarks. Typically the masses of the quarks are given in the $\overline{MS}$ scheme of the "running" mass. The light quarks with masses below 1 GeV are up down and strange. They are quoted at the reference scale $\mu = 2$ GeV. The heavy quarks charm and bottom are listed at the scale of their own mass, whereas the heavy top quark is defined in the pole scheme.

Sometimes people quote the so called $1s$ mass of 4,63−4,77 GeV for the b-quark. This is approximately half of the physical $\Upsilon(1S)$ resonance mass. $\Upsilon(1S)$ is a bound state of $b\bar{b}$, and due to the properties of the b-quark, this meson can be treated perturbatively very well.

The determination of the physical lepton masses can be done pretty well, with very small experimental errors.

	I	II	III
Leptons	$\begin{pmatrix} 511\text{keV} \\ < 2\text{ eV} \end{pmatrix}$	$\begin{pmatrix} 106\text{MeV} \\ < 0.19\text{ MeV} \end{pmatrix}$	$\begin{pmatrix} 1777\text{ MeV} \\ < 18.2\text{ MeV} \end{pmatrix}$
Quarks	$\begin{pmatrix} 1.5 - 3.3\text{MeV} \\ 3.5 - 6.0\text{ MeV} \end{pmatrix}$	$\begin{pmatrix} 1.16 - 1.34\text{ GeV} \\ 70 - 130\text{ MeV} \end{pmatrix}$	$\begin{pmatrix} 169.7 - 172.9\text{ GeV} \\ 4.13 - 4.37\text{ GeV} \end{pmatrix}$

Table 1.3.: Masses of the Elementary Particles

It is a challenge to measure the neutrino masses. These particles can only be seen indirectly, since the neutrinos just interact weakly. Therefore at the moment it is only possible to give upper bounds.

1.3.6. Cabbibo-Kobayashi-Maskawa Matrix

As we have seen before, we diagonalize the Yukawa coupling terms in order to identify the mass terms in the Standard Model Lagrangian. This can always be done by a bi-unitary transformation, with one rotation matrix for up-type and one for down-type quarks. Plugging this transformation into the charged current, we are left with terms that couple up-type quarks and down-type quarks. With the rotated fields the Lagrangian density of the charged current reads

$$\mathcal{L}_{\text{cc}} = -\frac{1}{\sqrt{2}} g_2 \left[\bar{U}'_L V^{(u)} \not{W}^+ W^{(d)\dagger} D'_L + \bar{D}'_L W^{(d)} \not{W}^- V^{(u)\dagger} U'_L \right] . \tag{1.49}$$

From this we define the Cabbibo-Kobayashi-Maskawa (CKM) matrix as

$$V_{\text{CKM}} = V^{(u)} W^{(d)\,\dagger} = \begin{pmatrix} V_{\text{ud}} & V_{\text{us}} & V_{\text{ub}} \\ V_{\text{cd}} & V_{\text{cs}} & V_{\text{cb}} \\ V_{\text{td}} & V_{\text{ts}} & V_{\text{tb}} \end{pmatrix}\,, \tag{1.50}$$

which is a 3×3 unitary matrix. Hence in principle it has nine parameters, but as we will explain below, due to constraints we are left with 3 mixing angles of the three families and one complex phase, responsible for CP violation.

In case of the neutral current, there are no transitions between different quark flavours at tree level, because there are no transitions between up- and down-type quarks. Therefore the rotation matrices cancel out.

This concept is different to the lepton sector. There we have only one rotation due to the massless neutrinos, because we have argued that in this case the same rotation for both charged and neutral leptons can be chosen. Therefore the charged current mediates only transitions between charged leptons and neutrinos of the same family, because in the analogous equation (1.49) of the lepton sector, $W^{(d)}$ can be chosen as $V^{(u)}$ and therefore (1.50) becomes unity.

Parameterization of the CKM Matrix

In general we have for n generations a $n \times n$ complex matrix with a priori

$$N = 2n^2$$

parameters, from which n^2 can be eliminated by the unitarity condition. Furthermore have $N_{\text{unob.}} = 2n - 1$ relative unobservable phases. Thus in total we have

$$N_{\text{Phases}} = \frac{(n-1)(n-2)}{2} \qquad N_{\text{Angles}} = \frac{n(n-1)}{2}\,, \tag{1.51}$$

where the number of rotation angles is known from the n-dimensional rotation in real Euclidean space. This makes in total

$$N_{\text{Parameter}} = N_{\text{Phases}} + N_{\text{Angles}} = (n-1)^2 \tag{1.52}$$

physical parameters. Therefore in case of four quarks, corresponding to $n = 2$ generations, we are left with one physical parameter: The Cabbibo angle. This was also assumed phenomenological in the beginning, and used to forecast the charm quark [21].

In the real physical situation, we have three families[11] and hence three rotation angles and one CP violating complex phase. The most common parameterization of the CKM matrix is given by the Particle Data Group (PDG) [13] by the means of Euler rotations among the three families plus a CP violating phase put to the 1−3 transition

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & e^{-i\delta}s_{13} \\ -c_{23}s_{12} - e^{i\delta}c_{12}s_{13}s_{23} & c_{12}c_{23} - e^{i\delta}s_{12}s_{13}s_{23} & c_{13}s_{23} \\ s_{12}s_{23} - e^{i\delta}c_{12}c_{23}s_{13} & -e^{i\delta}c_{23}s_{12}s_{13} - c_{12}s_{23} & c_{13}c_{23} \end{pmatrix}.$$

[11] We do not assume a fourth generation here, also this is discussed in the literature.

Here $s_{ij} \equiv \sin\theta_{ij}$ and $c_{ij} \equiv \cos\theta_{ij}$. To visualize the strength, often the so-called Wolfenstein parameterization is adopted with

$$s_{12} = \lambda\,, \qquad s_{23} = A\lambda^2\,, \qquad s_{13}e^{i\delta} = A\lambda^3(\rho + i\eta)\,,$$

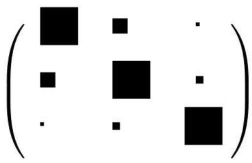
where $\lambda \approx 0.22$. This allows to expand the CKM matrix in powers of λ

$$V_{\text{ckm}} \approx \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \tag{1.53}$$

Hierarchical Structure of the CKM Matrix

Plugging in the numbers, we can immediately see the hierarchical structure of the CKM matrix, where we restrict ourselves to the absolute values

$$|V_{\text{ckm}}| \approx \begin{pmatrix} 0.97425 \pm 0.00022 & 0.2252 \pm 0.0009 & (3.89 \pm 0.44)\cdot 10^{-3} \\ 0.230 \pm 0.011 & 1.023 \pm 0.036 & (41.5 \pm 0.7)\cdot 10^{-3} \\ (8.4 \pm 0.6)\cdot 10^{-3} & (38.7 \pm 2.1)\cdot 10^{-3} & 0.88 \pm 0.07 \end{pmatrix}$$

$\approx$  .

In fact the more you go away from the main diagonal, the smaller the values are. This basically means, that transitions within the same family are enhanced and transitions from the third family to the other ones are suppressed, where the transition from the third to the first family has the smallest value.

The values are taken from [13] in the one sigma interval. But this values have to be taken with some care, because in all determinations subtleties are hidden. Especially the determination of $|V_{cb}|$, where most parts of this thesis deals with, has some discrepancy for two different methods, today. The values are taken from the determination from inclusive measurements. The exclusive experiment shows slight deviations, namely

$$|V_{cb}|^{\text{incl.}} = (41.5 \pm 0.7)\cdot 10^{-3}$$
$$|V_{cb}|^{\text{excl.}} = (38.7 \pm 1.1)\cdot 10^{-3}\,.$$

Measurements and the Unitary Triangle

The CKM mechanism is a very important part of the Standard Model. It is therefore natural to investigate this sector carefully in order to test the SM. All measurements are combined in the so-called unitary triangle. For this we use the unitarity condition of the CKM matrix and keep in mind, that a sum of three complex numbers equal to zero, corresponds to a closed triangle in the complex plane. It can be shown, that all six unitary triangles (and their complex conjugate ones) are equivalent. The area A of the unitary triangles, proportional to the CP violation, are given by the Jarlskog invariant $J = 2A$, defined by

$$J \sum_{m,n} \epsilon_{ikm}\epsilon_{jln} = \mathrm{Im}\left[V_{ij}V_{kl}V_{il}^{*}V_{kj}^{*}\right] . \tag{1.54}$$

Therefore we choose one particular triangle, which has the property of all sides of the same order of magnitude in length

$$V_{CKM}^{\dagger}V_{CKM} = \begin{pmatrix} V_{ud}^{*} & V_{cd}^{*} & V_{td}^{*} \\ V_{us}^{*} & V_{cs}^{*} & V_{ts}^{*} \\ \boxed{V_{ub}^{*} \quad V_{cb}^{*} \quad V_{tb}^{*}} \end{pmatrix} \begin{pmatrix} \boxed{V_{ud}} & V_{us} & V_{ub} \\ \boxed{V_{cd}} & V_{cs} & V_{cb} \\ \boxed{V_{td}} & V_{ts} & V_{tb} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boxed{0} & 0 & 1 \end{pmatrix} .$$

This triangle is then visualized in the ρ–η plane, see Figure 1.4, where we have used the slightly better converging definition of $\bar{\rho}$ and $\bar{\eta}$ with

$$s_{12} = \lambda = \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}} , \qquad s_{23} = A\lambda^2 = \lambda \left|\frac{V_{cb}}{V_{us}}\right| ,$$
$$s_{13}e^{i\delta} = V_{ub}^{*} = A\lambda^3(\rho + i\eta) = A\lambda^3(\bar{\rho} + i\bar{\eta})\frac{\sqrt{1 - A^2\lambda^4}}{\sqrt{1-\lambda^2}\left[1 - A^2\lambda^4(\bar{\rho} + i\bar{\eta})\right]} . \tag{1.55}$$

In this convention the triangle condition takes the form

$$1 - (\bar{\rho} + i\bar{\eta}) - (1 - \bar{\rho} - i\bar{\eta}) = 0 . \tag{1.56}$$

Additionally this convention ensures, that the tip of the triangle

$$\bar{\rho} + i\bar{\eta} = -\frac{V_{ud}V_{ub}^{*}}{V_{cd}V_{cb}^{*}} \tag{1.57}$$

is independent of the phase-convention, because all possible transitions between the families are accounted for. Now different measurements lead to different constraints

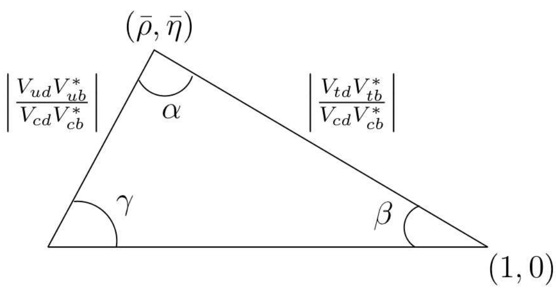


Figure 1.4.: Sketch of the Unitary Triangle.

for the triangle. The sides are measured from CP-conserving quantities, whereas the angles are extracted from CP-violating processes. All measurements are then combined in a fit, where the latest result from the CKM-Fitter group [14] is displayed in fig. 1.5. We see, that all measurements agree very well within their errors until now. There is not much space for deviations. The values as presented in [13] are given by

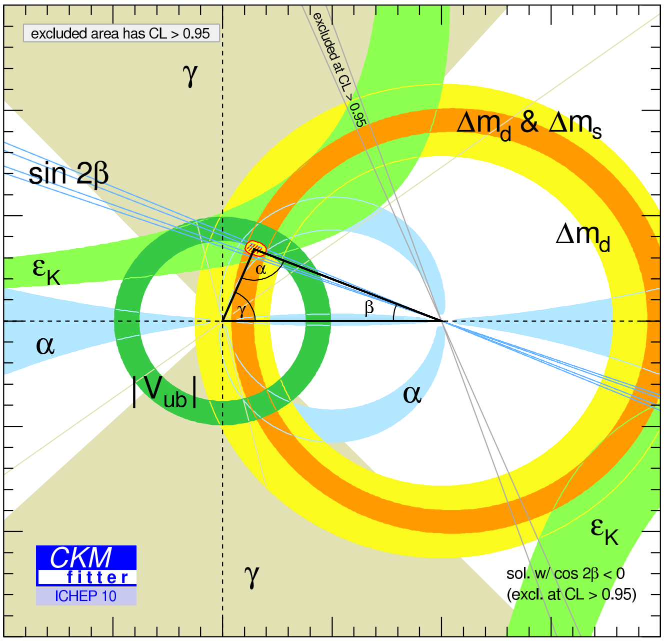


Figure 1.5.: Fit to the Unitary Triangle from the ICHEP conference 2010.

$$\begin{aligned} \lambda &= 0.2253 \pm 0.0007\,, & A &= 0.808^{+0.022}_{0.015} \\ \bar{\rho} &= 0.132^{+0.022}_{-0.014}\,, & \bar{\eta} &= 0.341 \pm 0.013\,. \end{aligned} \tag{1.58}$$

The Jarlskog invariant is given by

$$J = \left(2.91^{+0.19}_{-0.11}\right) \cdot 10^{-5}\,. \tag{1.59}$$

The calculations within this thesis provides the tools to extract $|V_{cb}|$, which is an important input for the so-called "unitary-clock"—the dark green circle around the origin—as well as ϵ_k. The "unitary-clock" is mainly driven by $|V_{ub}/V_{cb}|$. Nevertheless $|V_{cb}|$ is also an important ingredient for the determination of ϵ_K from Kaon physics, which is represented by the bright green hyperbolic function in fig. 1.5. There $|V_{cb}|$ is used as an input with the fourth power and contributes roughly one third to the total uncertainty for this parameter.

We do not want to go in detail to all different measurements, which are implemented in the plot. This is beyond the scope of this thesis, and can be read in references [13,14].

1.3.7. Problems of the Standard Model

The development of the SM has begun over 40 years ago, and a tremendous effort lead to a very successful description of the elementary particles. Improving measurements over decades have shown no large deviations from the predictions of the Standard Model. Therefore nobody doubts the validance of this theory, at least at the energy scales we currently can observe. Nevertheless some theoretical problems arise, which in principle have to be solved. The experiments now running at LHC and the forthcoming super flavour factories should deliver the needed data to search for the new physics (NP) beyond the Standard Model. The problems of the Standard Model, in which people believe, are addressed below. Most of these problems arise due to the comparison with cosmological experiments. In cosmology the relevant force is gravity. Until now there is no consistently quantized theory known for general relativity and therefore if and how the unification with the Standard Model forces can be possible, is unknown. Nevertheless observations from cosmology can be compared with particles physics and some of them lead to theoretical problems.

- **CP Violation**

  Within the SM CP is violated through a complex phase in the CKM matrix[12]. Additionally due to non vanishing neutrino masses we have CP violation in the lepton sector. From cosmological observations it is known, that the galaxies consist of matter, and not anti-matter. Therefore a mechanism after the big bang is responsible for the mismatch between particle and anti-particle. In the process of baryo(lepto)-genesis this could be due to CP violation. However the size of CP violation is too small, to account for that.

- **Number of Generations**

  Experimentally it is well established, that there should be three generations of particles, using the width[13] of the Z^0. This assumptions is essential for the theoretical construction of the SM, but no deeper reason behind this is known.

- **Dark Matter**

  Again from cosmological observations we know the approximate composition of the universe. From the expansion an estimated "dark energy" accounts for roughly 70%, whereas ordinary known matter plays a little role with roughly 2-3%. Additionally from gravitational effects, there should be some matter content in the order of 20-25%, where ordinary matter nor massive neutrinos can be accounted for. This unknown matter is called "dark matter". Many new physics models try to explain this with new heavy particles which have not been discovered, yet. This explanation however needs new physics beyond the Standard Model. Many scientists believe in good reasons, that this has to happen at the TeV scale.

[12] We do not want go into details of strong CP-violation and all that.

[13] Assuming, that there is no fourth generation heavy neutrino with $m_\nu > M_Z/2$.

- **Vacuum Energy**

  The Higgs has a non-zero ground state energy, leading to a vacuum energy. This vacuum energy can be calculated to be $\rho_{\text{Higgs}} = M_H^2 v^2/8 \gtrsim 10^8$ GeV^4. On the other hand in general relativity the cosmological constant has to be introduced. This is also a kind of vacuum energy responsible for an accelerated expansion of the universe, and turns out to be $\rho_{\text{Vac.}} \lesssim 10^{-46}$ GeV^4. In comparison with the vacuum density from particles physics this covers more than 50 orders of magnitude, which cannot be explained. Usually to explain such a difference of orders of magnitude a mechanism is needed, which cancel out huge contributions.

- **Hierarchy Problem**

  In our case the relevant scale for defining hierarchical structures is the energy (or mass) scale. Please note that the masses of the particles differ from the lightest (even if we neglect the neutrinos) to the heaviest by several orders of magnitude, which can be seen by the ratio $m_e/m_t \lesssim 10^{-5}$. But the masses are all created by the same mechanism, and therefore the SM itself does not provide a statement about this fact.

  And even stronger problem is the Higgs mass itself. All particles masses with the exception of the Higgs are created by symmetry breaking. Therefore their size is protected by the symmetry. The mass of the Higgs itself, however, does not have this feature. Even, if we put the value at the GUT scale to a very low value, it would be large at the scales we are observing at the experiments: The Higgs mass has a quadratic divergence with respect to renormalization effects.

- **The Parameters**

  To describe the particles within the Standard Model a lot of parameters are necessary. Additionally it works only, because they take their special values. The model just explains, why the parameters are existing, but now their exact size and number. So the model does not explain any deeper reasons behind it, and from some perspective this might look rather unnatural.

However the Standard Model has been tested to an incredible precision up to know and up to the energies we can measure it. That is the reason, why the SM should at least be seen as an effective theory.

1.4. Motivation and Task

We have learned that the Standard Model exhibits an incredible precision in testing it, yet it has several problems. Basically nowadays there are in principle two approaches to further investigate the structure in the theoritical side.

- We can push computations to its limits, in order to probe the SM in an *indirect* way. In combination with the increasing and advancing experiments all measured data becomes extremely precise. This allows to search for deviations and tensions and in the end gives hints to physics beyond the Standard Model. Besides that, of course, a precise knowledge on the natural parameters is always desired.

- Besides we can *directly* look if new physics (NP) shows up in certain processes. We can either choose a particular model, which may describe the real physical situation, and see if this is applicable. Or we allow for operators, respecting the gauge symmetry of the Standard Model, to see if they contribute. This is an effective theory ansatz and very general. This is very efficient in investigating a structure and also model-independent. Concrete models can be compared to this effective theory by using the renormalization group.

Within this thesis we will pursue both ansatzes in order to clarify the value of $|V_{cb}|$ with respect to the tension between inclusive and exclusive determination.

We will first investigate the non-perturbative structure of the inclusive decay $\bar{B} \to X_c \ell \bar{\nu}_\ell$. It will be shown, that the mathematical tools show the expected behaviour. First we push the computation to higher orders and additionally we treat the charm-quark in different scenarios (different power-counting for the charm mass), and show how the calculation has to be performed in case of light final states, e.g. $\bar{B} \to X_u \ell \bar{\nu}_\ell$. This also gives a deeper theoretical insight into the structure of the expansion, which is not only an expansion in $1/m_b$ but rather in $1/m_b \times 1/m_c$. An estimate of occurring parameters is done in order to perform the numerical estimate of the moments. We show that the theoretical error due to the non-perturbative expansion is at the order of $\mathcal{O}(1\%)$.

In the second part we make some calculations for a model-independent analysis of exclusive $B \to D^{(*)} \ell \bar{\nu}_\ell$ decays. First we use an effective field theory ansatz, to show which operators besides the Standard Model ones can appear to mediate this decay. The conditions that these operators have to fulfill are gauge and Lorentz invariance. In order to perform a solid analysis later on, we first compute the $\mathcal{O}(\alpha_s)$ corrections to all operators and reproduce the known Standard Model result. This will be used later on as an input in the analysis.

2. Phenomenology

The aim of phenomenology is to investigate the structure of the Standard Model with the help of decay or production processes. Using this we can extract the unknown and not calculable parameters of the SM and investigate if the underlying symmetry and principle is assumed correctly. In other words the flavour sector including all parameters related to the different flavours is tested.

Here we pursue two different goals. One is to pinpoint an important parameter, the CKM matrix element $|V_{cb}|$ to a better precision by an improved theory calculation. The other goal is to perform a bottom-up approach to search for deviations of the SM by general assumptions. In doing so we can get a handle of the structure of possible new physics effects, which will give indications how the theory beyond the SM may work. But we do not restrict ourselves to a specific BSM model.

Out of the total 27 parameters in the Standard Model, almost all are related to flavour physics. Therefore it is crucial to test the flavour sector in order to verify or falsify the SM. We will investigate the phenomenology of a bottom quark decaying into a charm quark. This process is a weak decay, and therefore directly related to the CKM matrix element $|V_{cb}|$. As we have already mentioned, this piece itself is required for one side of the unitary triangle. Furthermore it goes in the determination of ϵ_K of Kaon mixing with the fourth power. Therefore we need to have a very precise determination of $|V_{cb}|$.

On the other hand we also try to Figure out the structure behind such processes. Due to the hadronic uncertainty an admixture of different currents for the exclusive transition is not ruled out, yet. If we incorrectly assume only the SM left-handed current this would influence the extraction of $|V_{cb}|$ and may explain the small tension between inclusive and exclusive determination. The same analysis has been performed in the inclusive case, where no sensitivity to right-handed currents has been found [15, 16].

Finally we derive formulae for the different experimental observables. From fits to these formulae it is possible to extract the parameters: $|V_{cb}|$ and quark masses. The necessary theoretical foundations to perform the analysis are described below.

2.1. Particle Decays

We leave out all subtle details, like the quantization of the fields, and mention only some relevant facts. These things are discussed in textbooks [9, 12, 17, 18]. A more elaborate and different approach, but also more difficult, are path integrals. A very good introduction is given by [19].

The decay of particles is in theory always mediated by the appropriate term in the

Lagrangian density. For the description of interactions we need to have higher than bilinear terms, coupling the different Fourier modes of particles to each other. For preserving causality we insist on having local interactions only.

2.1.1. Operators and Scattering Matrix

The operators relevant for the decays have to be read off from the Standard Model Lagrangian, in general we denote them by $\mathcal{L}_{\text{int}}$. The terms in $\mathcal{L}_{\text{int}}$ specify the allowed interactions of the theory. The interaction then is a change of a state in the past to the state in the future. By labeling the "out" states with momenta $\mathbf{p}_i$ and the "in" states with $\mathbf{k}_i$. The transition probability is then given by the time-evolution operator of the interaction term, which defines the scattering matrix S

$$
{}_{\text{out}}\langle \mathbf{p}_1\mathbf{p}_2\ldots|\mathbf{k}_1\mathbf{k}_2\ldots\rangle_{\text{in}} \equiv \langle \mathbf{p}_1\mathbf{p}_2\ldots|S|\mathbf{k}_1\mathbf{k}_2\ldots\rangle \tag{2.1a}
$$

$$
S = \lim_{\substack{x_0^{\text{in}}\to-\infty \\ x_0^{\text{out}}\to\infty}} \mathrm{T}\exp\left[i\int_{x_0^{\text{in}}}^{x_0^{\text{out}}} \mathrm{d}^4x\,\mathcal{L}_{\text{int}} \right]\,. \tag{2.1b}
$$

The S-matrix contains both, all kinds of interactions, and also the probability of no interaction. Therefore it is natural to split the scattering matrix into the unity part and the interaction part, denoted by the matrix T,

$$
S = 1 + iT\,. \tag{2.2}
$$

The invariant matrix element[1] $\mathcal{M}$ is deduced from this by requiring momentum conservation

$$
\langle \mathbf{p}_1\mathbf{p}_2\ldots|iT|\mathbf{k}_1\mathbf{k}_2\ldots\rangle = (2\pi)^4\delta^{(4)}\Big(\sum_{\text{initial}} k_i - \sum_{\text{final}} p_i\Big)\cdot i\mathcal{M}(k_1,k_2,\ldots\to p_1,p_2,\ldots)\,. \tag{2.3}
$$

2.1.2. Matrix Element

As we have seen before, the task is to evaluate the invariant matrix element. Since the time-ordered product contains an infinite series of terms with the interaction terms of increasing power, we usually keep only a finite number of terms. This is valid, as long as the coupling constant, seen as an expansion parameter, is small. This is known as "perturbation theory", which we have encountered already before. Then we are left with matrix elements of the form

$$
\langle \text{Final State}|\mathcal{O}|\text{Initial State}\rangle \tag{2.4}
$$

which have to be computed. $\mathcal{O}$ is a general operator, which is related to the interaction term. In general this is a complicated procedure. However, this can be generalized and for each theory so-called "Feynman rules" can be derived. These are building blocks for those matrix elements, to compute the whole process. The blocks are divided into the parts

[1]It is the analogous part of the scattering amplitude known from quantum mechanics.

- Propagators
- Vertex (interaction of particles)
- External lines.

The rules for the Standard Model are listed in every textbook [9, 12].

In QCD the procedure is a bit more complicated. The matrix element states are bound states of quarks themselves. The coupling constants in this case is no longer in the perturbative regime and therefore the expansion is not applicable any more. We are therefore stuck with a non-perturbative matrix element. We encounter different examples of these within this thesis and explain a way how to deal with them.

2.1.3. Decay Probability

The differential decay probability is in general proportional to the matrix element squared

$$\mathrm{d}\Gamma = \frac{1}{2M_A}\left(\prod_f \frac{\mathrm{d}^3\boldsymbol{p_f}}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathcal{M}(M_A \to p_1,\dots,p_n)|^2\,(2\pi)^4\delta^4(p_A - \sum p_f)\,. \qquad (2.5)$$

$1/2M_A$ is a normalization factor, which defines the reference frame: The decaying particle is assumed to be at rest. Therefore the decay rate is not Lorentz invariant, because all parts but the normalization factor are. Since the momenta of the outgoing particles are not fixed in general, we have to weight the probability by the appropriate phase space.

The total decay rate (or width) is, if summed over all possible final states, inversely proportional to the life time. In case of a specific final state, it is also called partial branching fraction. The total decay rate now by performing the momentum integrals. Furthermore usually one considers decay spectra, which leaves one of the kinematical variables unintegrated, or moments of kinematical variables. The moments M_n, defined as the weighted integral of a kinematical variable p

$$M_n = \int \mathrm{d}p\; p^n \frac{\mathrm{d}\Gamma}{\mathrm{d}p}\,, \qquad \text{with} \qquad \Gamma \equiv M_0 \qquad (2.6)$$

have the advantage, that they are fully integrated, which is required for certain symmetry assumptions.

2.1.4. Application to B Meson Decays

In the full Standard Model the interaction Hamiltonian, which mediates the semileptonic $b \to c$ transition is given by

$$\begin{aligned}\mathcal{H}_W = &\left(\frac{g_2}{\sqrt{2}}\right)^2 V_{\mathrm{cb}}\big(\bar{c}\gamma_\mu\frac{1}{2}(1-\gamma_5)b\big)\big(\bar{e}\gamma_\nu\frac{1}{2}(1-\gamma_5)\nu_\ell\big)\\ &\times \frac{1}{M_W^2-(p_b-p_c)^2}\left[g^{\mu\nu} - \frac{(p_b-p_c)^\mu(p_b-p_c)^\nu}{M_W^2}\right]\,, \end{aligned} \qquad (2.7)$$

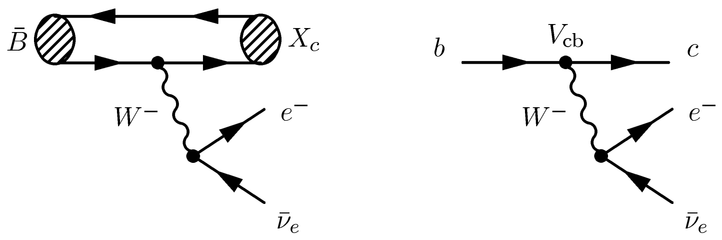


Figure 2.1.: Tree Level Feynman diagrams for a B meson decay.

where only the left-handed fields take part in the process. In Figure 2.1 the process is visualized in a Feynman diagram; the panel on the right shows the interaction at parton level, as described by the operator, and on the left indicates the physical process of colour singlet particles, where the initial and final state quarks are bound into mesons. Since the typical momentum transfer of the interaction is of the order $(p_b - p_c)^2 < p_b^2 = m_b^2 \ll M_W^2$, we can integrate out the heavy W-Boson. Thus we are stuck with an effective Fermi theory with a local weak interaction. This corresponds to a Taylor expansion of the propagator according to

$$\frac{1}{M_W^2 - (p_b - p_c)^2} = \frac{1}{M_W^2}\left[1 + \frac{(p_b - p_c)^2}{M_W^2} + \dots\right]. \tag{2.8}$$

The rapid convergence is good enough that we keep the first term only. The non-locality would be reproduced if we summed up the infinite number of local terms. Therefore the full interaction in eq. (2.7) turns into the simpler term

$$\mathcal{H}_W = \frac{4G_F}{\sqrt{2}} V_{\text{cb}} (\bar{c}\gamma_\mu P_L b)(\bar{e}\gamma^\mu P_L \nu_\ell)\,, \tag{2.9}$$

where we have defined the Fermi constant as $4G_F/\sqrt{2} = g_2^2/(2M_W^2)$, and introduced the abbreviation for the left-handed projector $P_L = 1/2(1-\gamma_5)$. $(\bar{c}\gamma_\mu P_L b)(\bar{e}\gamma^\mu P_L \nu_\ell)$ corresponds to the operator $\mathcal{O}$ in eq. (2.4).

The next step is to compute the transition amplitude relevant for the physical process of a meson decay, which is mediated by the operator at parton level. In quantum field theoretical notation, we derive the invariant matrix element

$$\begin{aligned}\mathcal{M}(\bar{B} \to X_c \ell^- \bar{\nu}_\ell) &= \langle X_c \ell \bar{\nu}_\ell | \mathcal{H}_w | \bar{B} \rangle \\ &= \frac{4G_F}{\sqrt{2}} V_{\text{cb}} \langle X_c \ell \bar{\nu}_\ell | J_{q,\mu} J_\ell^\mu | \bar{B} \rangle\,,\end{aligned} \tag{2.10}$$

and introduce the leptonic $J_\ell^\mu = \bar{\ell}\gamma^\mu P_L \nu_\ell$ and hadronic current $J_{q,\mu} = \bar{c}\gamma_\mu P_L b$. From this we compute the necessary squared matrix element. Please note, that to leading order in electroweak corrections the hadronic and leptonic current factorizes.

$$\begin{aligned}\left|\mathcal{M}(\bar{B} \to X_c \ell^- \bar{\nu}_\ell)\right|^2 &= 8G_F^2 \left|V_{\text{cb}}\right|^2 \left|\langle X_c \ell \bar{\nu}_\ell | J_{q,\mu} J_\ell^\mu | \bar{B} \rangle\right|^2 \\ &= 8G_F^2 \left|V_{\text{cb}}\right|^2 \langle \bar{B} | J_{q,\nu}^\dagger | X_c \rangle \langle X_c | J_{q,\mu} | \bar{B} \rangle \langle 0 | J_\ell^{\nu\,\dagger} | \ell \bar{\nu}_\ell \rangle \langle \ell \bar{\nu}_\ell | J_\ell^\mu | 0 \rangle.\end{aligned} \tag{2.11}$$

In the following we will specify the final hadronic state, and distinguish two different cases:

1. Inclusive case: We sum over all kinematically possible final state hadrons labeled by X_c, which carry one charm quantum number. This has advantages in theory, as we will explain below. Furthermore the experimental analysis is more precise with respect to statistics.

2. Exclusive case: We assume the decay of the pseudo-scalar B meson into either a pseudo-scalar D meson ($X_c \equiv D$), or the vector meson D^* ($X_c \equiv D^*$). Here the decay is totally fixed. In experiment it can be detected very well, but of course with a lower statistic than in the inclusive case. We will later see, that the hadronic uncertainty in theory is more problematic than in the inclusive case.

2.2. Inclusive Decays

We put everything together and use equation (2.5) to compute the differential rate. Please note, that the phase-space integration of the hadronic final state is contained in the sum over the final state hadrons. We sum over the final state spins. Then the differential decay rate is given by

$$\mathrm{d}\Gamma = \sum_{X_c} \sum_{\substack{\text{Spins}\\ \text{leptons}}} \frac{1}{2M_B} \left(\frac{\mathrm{d}^3 \boldsymbol{p}_\ell}{(2\pi)^3} \frac{1}{2E_\ell} \right) \left(\frac{\mathrm{d}^3 \boldsymbol{p}_{\nu_\ell}}{(2\pi)^3} \frac{1}{2E_{\nu_\ell}} \right) \left| \mathcal{M}(\bar{B} \to X_c \ell^- \bar{\nu}_\ell) \right|^2 \\ \times (2\pi)^4 \delta^4(p_B - (p_\ell + p_{\nu_\ell} + p_{X_c})) \,. \tag{2.12}$$

Defining the leptonic tensor $L^{\mu\nu}$ and the hadronic tensor $W_{\mu\nu}$ as

$$L^{\mu\nu} = \sum_{\substack{\text{Spins}\\ \text{leptons}}} \langle 0 | J_\ell^{\mu\,\dagger} | \ell \bar{\nu}_\ell \rangle \langle \ell \bar{\nu}_\ell | J_\ell^\nu | 0 \rangle \tag{2.13}$$

$$W_{\mu\nu} = \frac{1}{2M_B} \sum_{X_c} \langle \bar{B} | J^\dagger_{q,\nu} | X_c \rangle \langle X_c | J_{q,\mu} | \bar{B} \rangle (2\pi)^3 \delta^4(p_B - (p_\ell + p_{\nu_\ell} + p_{X_c})) \,. \tag{2.14}$$

we end up with

$$\mathrm{d}\Gamma = 16\pi G_F^2 \left| V_{\mathrm{cb}} \right|^2 \left(\frac{\mathrm{d}^3 \boldsymbol{p}_\ell}{(2\pi)^3} \frac{1}{2E_\ell} \right) \left(\frac{\mathrm{d}^3 \boldsymbol{p}_{\nu_\ell}}{(2\pi)^3} \frac{1}{2E_{\nu_\ell}} \right) W_{\mu\nu} L^{\mu\nu} \,. \tag{2.15}$$

Depending on the observables we want to analyze, we can calculate the appropriate differential decay rate. Please note, that the non-perturbative input and all quark dynamics is contained in the hadronic tensor. For this we will first apply the optical theorem. Physically this means, that we consider a forward transition amplitude of the form $\bar{B} \to \bar{B}$ via the inclusive charm intermediate state. By using the optical theorem we can relate the absorbative part, meaning the discontinuity, of this transition amplitude to the desired hadronic tensor. The forward matrix element can then be treated within heavy quark effective theory to parameterize the non-perturbative effects. This allows us to relate the binding effects of QCD to a set of basis parameters. We will pursue this ansatz in chapter 4.

2.3. Exclusive Decays

We now turn to a specific hadronic finale state. There we cannot use the optical theorem trick, since this only works for an "inclusive enough" quantity, meaning that we need to sum over enough final states, that are allowed by phase-space and quantum numbers. This guarantees then that enough parts of the phase-space are covered for the validance of the optical theorem. Otherwise we would miss important parts. Here the phase-space is clearly defined since the final state particles are specified. We consider the decays

$$\bar{B} \to D^{(*)} \ell \bar{\nu}_\ell \,,$$

where D is a pseudo-scalar meson containing a charm quark, and D^* is the corresponding vector meson. Therefore the leptonic tensor remains the same, but the hadronic part now turns into,

$$\begin{aligned} W_{\mu\nu} &= \frac{1}{2M_B} \int \frac{\mathrm{d}^3 \boldsymbol{p}_{D^{(*)}}}{(2\pi)^3} \frac{1}{2E_{D^{(*)}}} \langle B|\bar{b}\gamma_\nu \frac{1-\gamma_5}{2} c|D^{(*)}\rangle \langle D^{(*)}|\bar{c}\gamma_\mu \frac{1-\gamma_5}{2} b|\bar{B}\rangle \\ &\quad \times (2\pi)^3 \delta^4(p_B - (p_\ell + p_{\nu_\ell} + p_{D^{(*)}})) \\ &= \frac{1}{2M_B} \int \frac{\mathrm{d}^3 \boldsymbol{p}_{D^{(*)}}}{(2\pi)^3} \frac{1}{2E_{D^{(*)}}} (2\pi)^3 \delta^4(p_B - (p_\ell + p_{\nu_\ell} + p_{D^{(*)}})) \tilde{W}_{\mu\nu} \,. \end{aligned} \tag{2.16}$$

It is convenient to express the hadronic tensor in terms of Lorentz invariant amplitudes, which are called form factors. These form factors are non-perturbative objects describing the binding effects inside the hadron. They correspond to the non-perturbative parameters we encountered in the inclusive case. The Lorentz structure has to be parameterized, obeying the parity and time-reversal invariance of the strong interaction, using the four vectors p_B^μ, $p_{D^{(*)}}^\mu$ and occasionally the polarization vector ϵ^μ. The polarization vector satisfies the constraint $p_{D^{(*)}} \cdot \epsilon = 0$. The Lorentz invariant amplitudes can only depend on $p_B^2 = M_B^2$, $p_{D^{(*)}}^2 = M_{D^{(*)}}^2$ and the invariant scalar product $p_B \cdot p_{D^{(*)}}$. Since the masses are fixed, we choose the momentum transfer to the leptons $q^2 = (p_B - p_{D^{(*)}})^2$ as the only independent variable. Within the Standard Model the decomposition is then commonly done as

$$\langle D|\bar{c}\gamma^\mu b|\bar{B}\rangle = f_+(q^2)(p_B + p_D)^\mu + f_-(q^2)(p_B - pD)^\mu \tag{2.17a}$$

$$\langle D^*|\bar{c}\gamma^\mu b|\bar{B}\rangle = g(q^2)\epsilon^{\mu\alpha\beta\gamma}\epsilon^*_\alpha(p_B + p_{D^*})_\beta(p_B - p_{D^*})_\gamma \tag{2.17b}$$

$$\langle D^*|\bar{c}\gamma^\mu\gamma^5 b|\bar{B}\rangle = -if(q^2)\epsilon^{*\mu} - ip_B \cdot \epsilon^* \left[a_+(q^2)(p_B + p_{D^*})^\mu + a_-(q^2)(p_B - p_{D^*})^\mu\right] . \tag{2.17c}$$

The differential decay rate in q^2 then reads

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}(\bar{B} \to D^{(*)}\ell\bar{\nu}_\ell) = \frac{1}{2M_B} 4G_F^2 |V_{cb}|^2 \int \frac{\mathrm{d}^3\boldsymbol{p}_{D^{(*)}}}{(2\pi)^3} \frac{1}{2E_{D^{(*)}}} \int \frac{\mathrm{d}^3\boldsymbol{p}_\ell}{(2\pi)^3} \frac{1}{2E_\ell} \int \frac{\mathrm{d}^3\boldsymbol{p}_{\nu_\ell}}{(2\pi)^3} \frac{1}{2E_{\nu_\ell}} \times \tilde{W}_{\mu\nu} L^{\mu\nu}\, \delta(q^2 - (p_B - p_{D^{(*)}})^2)(2\pi)^4 \delta^4(p_B - p_{D^{(*)}} - q)) \quad (2.18)$$

where we have to integrate over the appropriate phase-space and the hadronic tensor has to be expressed in terms of the appropriate tensor decomposition (2.17).

Clearly the advantage is, that the decay is fully specified and we can investigate the structure of this specific transition in more detail. It can be extracted more reliably from experiment, although the statistics limit the experimental accuracy. Furthermore the appearing form factors, which depend on kinematical variables, cannot be measured since only the combination with $|V_{cb}|$ appears in the formulae and the total rate cannot be obtained, because the functional dependence is unknown due to its non-perturbative structure. There basically exist two methods to compute the form factor at a specific kinematical point, usually the zero-recoil point defined by $q^2 = (m_b - m_c)^2$. One are QCD sum-rules, and the other is to use lattice QCD

Therefore in total the extraction of $|V_{cb}|$ is less precise compared to the inclusive case. But the form factors are more sensitive to the Dirac structure, and therefore an analysis complementary to [15, 16] seems to be promising in the search of physics beyond the Standard Model. To this end we will extend the Standard Model current in an effective theory model to allow for more Dirac structures. Details will be given in chapter 5.

2.4. Relation to CKM Parameters

A great advantage of such a phenomenological analysis is, besides probing the structure of a quark transition, the access to the CKM matrix element. Therefore most of the Standard Model parameters are related to the flavour sector, which can be accessed and probed through such a phenomenological analysis. Although we deal almost exclusively with $|V_{cb}|$, it is time for a short comment on CP violation. It is manifested in the Standard Model, as we have explained in section 1.3.6, that the only source of CP violation is the phase in the Cabbibo-Kobayashi-Maskawa matrix. CP is the symmetry, which relates particles with their anti-particles. Thus if CP is conserved, for all possible initial states A final states B and the corresponding anti-particle states $\bar{A}$ and $\bar{B}$

$$\Gamma(A \to B) = \Gamma(\bar{A} \to \bar{B}) \quad (2.19)$$

would hold true. It was however noted, that CP is not conserved in nature. Note that the decay time, which is inversly proportional to the total branching fraction, is the same for particle and antiparticle, since CPT is conserved. Thus CP violation in one mode has to be compensated by an additional CP violation in another mode of the partial branching fraction with the opposite sign. Since we measure the absolute values, it can only occur in interference terms of two different amplitudes. There are three different processes, where such CP violation can be observed.

1. *Direct* CP violation in a decay. This is the case when two decay amplitudes interfere. It is defined as the absolute value of the transition matrix element ratio from the process and the decay of the antiparticle in the anti final state, and can be measured via the ratio

$$\left|\langle\bar{f}|\mathcal{H}|\bar{M}\rangle/\langle f|\mathcal{H}|M\rangle\right| \neq 1\,.$$

2. *Indirect* CP violation can occur in neutral meson mixing $M^0 \leftrightarrow \bar{M}^0$, and is induced by the interference of the absorbative and dispersive mixing amplitude. The mixing and its time-evolution is given by a non-hermitian Hamilton operator, which can be diagonalized in terms of a "light" H_L and "heavy" H_H, leading to

$$\begin{aligned} &H = M - \frac{i}{2}\Gamma\,,\\ &|M_L\rangle \propto p|M^0\rangle + q|\bar{M}^0\rangle\,, \qquad |M_H\rangle \propto p|M^0\rangle - q|\bar{M}^0\rangle\,, \qquad |p|^2+|q|^2=1\,,\\ &\left(\frac{q}{p}\right)^2 = \frac{M_{12}^* - (i/2)\Gamma_{12}^*}{M_{12} - (i/2)\Gamma_{12}}\,, \end{aligned}$$

where for a CP conserving quantity $(q/p)^2 = \exp(2i\xi_M)$ is a phase. The physical observable mesons and anti-mesons are then given as a mixing of these quantities

$$\begin{aligned} &|M^0_{\text{phys}}(t)\rangle = g_+(t)|M^0\rangle - \frac{q}{p}g_-(t)|\bar{M}^0\rangle\,, \quad |\bar{M}^0_{\text{phys}}(t)\rangle = g_+(t)|\bar{M}^0\rangle - \frac{p}{q}g_-(t)|M^0\rangle\,,\\ &g_\pm(t) \equiv \frac{1}{2}\left(e^{-iM_Ht-\frac{1}{2}\Gamma_Ht} \pm e^{-iM_Lt-\frac{1}{2}\Gamma_Lt}\right). \end{aligned}$$

The CP violation is measured in the time-dependent asymmetry, defined by

$$A_{\text{SL}} \equiv \frac{\Gamma[\bar{M}^0_{\text{phys}}(t)\to l^+X] - \Gamma[M^0_{\text{phys}}(t)\to l^-X]}{\Gamma[\bar{M}^0_{\text{phys}}(t)\to l^+X] + \Gamma[M^0_{\text{phys}}(t)\to l^-X]} = \frac{1+|q/p|^4}{1-|q/p|^4}\,. \tag{2.20}$$

In fact a deviation from zero indicates CP violation, because $(q/p)^2$ is not only a phase in this case. Note that, also it is build out of time-dependent decay rates, this asymmetry is time-independent.

3. Furthermore CP violation can appear due to the mixing of both cases from above: The interference between a decay with and without mixing: $M^0 \to f$ and $M^0 \to \bar{M}^0 \to f$, where the final state f has to be common to both M^0 and $\bar{M}^0$, including all CP eigenstates. It can be measured via the same asymmetry as in eq. (2.20), but the final state has to be replaced by a CP eigenstate. A prominent example for this is the decay of the neutral Kaon.

Accordingly a lot of phenomenological processes related to the different quark flavours can be used to analyze this important parameters with different quark flavours. Among these are processes related to physics with a strange quark in K mesons, charm quarks in D mesons and bottom quarks in B mesons.

3. Effective Field Theories

Usually a theory has lots of degrees of freedom (d.o.f.), which are not relevant for specific considerations. Therefore reducing the number of d.o.f. to the relevant ones simplifies a problem. A first example of an effective theory (ET) is the Lagrangian in classical mechanics, where some of the coordinates are eliminated by constraints. Then the problem and its equation of motion can be described be generalized coordinates, which are less then the full set of Cartesian coordinates. As a second example we should mention special relativity (SRT) and the low velocity limit, classical mechanics, of it. Here a scale, the velocity, is a kind of separator, which tells if the effective classical theory is applicable. For SRT the relevant scale is the speed of light c and we have to use Lorentz transformation. However if the velocity fulfills $v \ll c$ we can use classical mechanics and the corresponding, much simpler, Galilei transformation. In effective theories often new symmetries arise and these lead to a reduced number of parameters to be fitted by experiment.

In elementary particle physics we use (quantum) field theories. An effective field theory (EFT) is a field theory, which describes the physical problem with only the relevant d.o.f.. These are the relevant particles only. The information on non-relevant particles or modes is absorbed into coefficient functions, which do not contribute to the dynamics of the system. Therefore it is much easier to handle the EFT. A typical distinctive feature is the energy scale at which a process happens. Then only particles with masses below this scale are relevant.

There are basically two different approaches to EFT in phenomenology: On the one hand we can assume, that an underlying and more fundamental theory is known. Then we can "integrate out" the degrees of freedom which are not dynamical for the processes we consider. The coefficient functions, "Wilson coefficients", can be computed explicitly. These techniques, which we will explain below, improve the results. This direct approach is called a "top-down" approach. On the other hand we assume no knowledge about the fundamental theory, we require only a symmetry at the low energy scale, which is usually reasonable. This approach is known as a "bottom-up“ approach. For such cases, we can make a general Ansatz by just requiring this underlying symmetry. The integrated out d.o.f. are then hidden in the Wilson coefficients. An advantage of this approach is, that by a matching procedure every full theory, which respects this symmetry at low energies, can be compared to the effective theory and the corresponding Wilson coefficients can be calculated. Therefore results retrieved by this EFT can be directly used to be compared to a more fundamental theory.

Now we will explain this in more detail with some examples, which are mainly used in this thesis. First we give some general remarks.

3.1. Operator Product Expansion and Perturbative Computations

In this section, the general procedure is briefly outlined, more details can be found in the literature [9, 17, 19, 20]. Therefore an extensive review is out of the scope of this thesis. Generically we are aiming for a separation of scales. Which means the low energy phenomena do not depend on the details of the short-distance physics and therefore the short-distance physics is put into effective couplings. Short-distance corresponds to high energy.

For example the muon decay takes place at the mass of the muon. The intermediate W boson is much heavier $m_\mu \ll M_W$, and therefore the details of the non-local interaction is not probed and we can effectively consider a local four-fermion interaction. The dependence on the high energy scale is separated in non-analytical coefficients of the form $\log m_\mu/M_W$. As we will see in a moment, by resumming these effects we actually can improve the perturbation theory series.

The relevant d.o.f. are given by the particles or modes, which can appear in initial and final state of the process. Then we compute the amplitude to a given order in the perturbative expansion and rewrite it in terms of operators with only these dynamical d.o.f.. The matching between these two steps leads to the result

$$\langle f|\mathcal{L}_{\text{eff.}}|i\rangle = \sum_i C_i \langle f|\mathcal{O}_i|i\rangle \,. \tag{3.1}$$

We now separate the effects by a scale "factorization scale"[1] μ, which discriminates the short- and long-distance effects. Therefore both the Wilson coefficient C and the operator $\mathcal{O}$ depend on this scale

$$C_i \to C_i^{\text{ren}}(\mu)$$
$$\langle \mathcal{O}_i\rangle \to \langle \mathcal{O}_i\rangle^{\text{ren.}}(\mu)\,.$$

This procedure is also called Operator Product Expansion (OPE), since we effectively expand the Lagrangian in terms of different operators with increasing dimension. The Lagrangian has to have mass dimension four, and therefore the operator expansion in terms of the scale Λ, having the dimension of a mass, takes the form

$$\langle f|\mathcal{L}_{\text{eff.}}|i\rangle = \sum_{k=0}^{\infty} \frac{1}{\Lambda^k} \sum_i c_{k,i}(\Lambda/\mu) \langle f|\mathcal{O}_i|i\rangle\big|_\mu \,. \tag{3.2}$$

This is a generalization of Equation (2.8), where Λ takes the role of M_W. This is the scale, up to which the EFT is assumed to be valid. Here the Wilson coefficients $c_{k,i}$ are dimensionless and contain the short-distance contribution, meaning the physics above the scale μ. The matrix elements $\langle f|\mathcal{O}_i|i\rangle\big|_\mu$ contain the long-distance physics below

[1] Actually this is a renormalization scale, since we usually do not apply a cut-off scheme, where the factorization is performed at a hard scale.

this scale. Therefore varying the scale μ corresponds to shifting contributions from the operator to the coefficient and vice versa. In fact no power corrections of the form $1/\mu$ can appear here, because of the renormalizability of the dimension-four part of the effective theory. Furthermore the higher order terms are non renormalizable. The predictive power, however, is ensured since only a *finite* number of counter terms is needed. The only requirement is that the ratio μ/Λ has to be sufficiently small. The matrix element in eq. (3.2) should be μ-independent, however in keeping only a finite number of terms on the right-hand side, a remnant μ-dependence is left over.

3.1.1. Loop Level

As mentioned before, interactions are created by terms of the form

$$T\left[\bar{\psi}\psi \exp\left[i\int \mathrm{d}^4x\mathcal{L}_{\text{int}}\right]\right]$$

and since this is not solvable analytically in four dimensions[2], we have to expand the exponential in terms of a small parameter. Usually this small parameter is given by the coupling constant $\alpha = \frac{g^2}{4\pi}$, where α_s denotes the strong coupling constant and α_{em} the electromagnetic one. As long, as the coupling constant is small, $\alpha \ll 1$, we can do a perturbative expansion. This leads to an increasing number of fields, where we integrate over the coordinates. In momentum space (Feynman diagrams) this is reflected as an integration over an internal momentum, symbolized by loops. Therefore higher order corrections are called loop corrections, whereas the leading order is (usually) a tree-level diagram with no integration over internal momenta.

3.1.2. Renormalization Group and Anomalous Dimension

This loop diagrams lead to the divergence effects that we have already encountered in section 1. Therefore we have to regularize and renormalize additionally the operators within the effective theories. In fact the divergences in effective theories arise due to the integration over all values of the loop momenta, although formally its only valid for small momenta. Thus these additional divergences have to be renormalized by appropriate counter-terms, which then appear in the Lagrangian by rewriting operators in terms of the physical observables. The ultraviolet (UV) region of the full theory has to be reproduced by corrections to the Wilson coefficients, since this is independent of the low-energy parameters. The infrared regime (IR) is obviously the same for both cases.

These additional counter-terms lead to renormalization constants for both the operators and the Wilson coefficients, thus we define the renormalized quantities as

$$\mathcal{O} = Z_{\mathcal{O}}\mathcal{O}^{\text{ren.}} \tag{3.3a}$$

$$C = Z_c C^{\text{ren}}\,. \tag{3.3b}$$

[2]In fact in the two-dimensional 't Hooft model it is exactly solvable and may sometimes give hints to our four dimensional real world.

Furthermore the fact that the un-renormalized bare quantities from the Lagrangian are μ-independent lead to the renormalization group equation for the physical observables. If we have more than one operator the formulae are generalized by introducing the vectors $\boldsymbol{C}$ and $\boldsymbol{\mathcal{O}}$. Then the renormalization constants become matrices, which lead to the renormalization group equation (RGE)

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\boldsymbol{\mathcal{O}}^{\mathrm{ren}}(\mu) = \underbrace{\left[Z_{\mathcal{O}}^{-1}\frac{\mathrm{d}}{\mathrm{d}\ln\mu}Z_{\mathcal{O}}\right]}_{\gamma(\mu)}\boldsymbol{\mathcal{O}}^{\mathrm{ren}}(\mu)\,, \tag{3.4}$$

where γ is the anomalous dimension matrix (ADM). From the fact, that the product is again μ-independent, we can read off

$$Z_c = Z_{\mathcal{O}}^{-1} \tag{3.5a}$$

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\ln\mu}\boldsymbol{C}^{\mathrm{ren.}} = \gamma^T C^{\mathrm{ren.}}\,. \tag{3.5b}$$

From this two RGE equations for the operator (3.4) as well as for the Wilson coefficient (3.5b) we have to mention three important facts:

1. Changing the scale μ corresponds to shift contributions from the operator into the Wilson coefficient and vice versa. This is plausible, because μ acts as a separation scale.

2. For off-diagonal entries in the ADM, furthermore contributions between different operators are shifted by varying the scale.

3. In case of this non-diagonal γ, the reshuffling of contributions between the different operators is compensated by an appropriate shift in the Wilson coefficients, thus the first item is not distorted by the second one.

The anomalous dimension matrix γ also has an expansion in the coupling α

$$\gamma = \frac{\alpha}{4\pi}\gamma^{(1)} + \frac{\alpha^2}{(4\pi)^2}\gamma^{(2)} + \dots \tag{3.6}$$

As we have seen, this matrix describes the "running" of the operators by changing μ. Naturally this means we look with different magnification into this process and the differential equation connects the set of operators with different μ. For that reason, equations of the type (3.4) are called "renormalization group equation" (RGE).

The power of this "group", which relates the set of operators at different μ, can be seen explicitly, if we diagonalize (3.5b). Then with the help of the beta-function (1.46) we get with the eigenvectors $\tilde{\boldsymbol{C}}$

$$\tilde{C}_i(\mu) = \tilde{C}_i(\mu_0)\exp\left[\int\limits_{\alpha(\mu_0)}^{\alpha(\mu)}\mathrm{d}\alpha'\frac{\gamma(\alpha')}{2\beta(\alpha')}\right] \tag{3.7}$$

$$\Rightarrow \qquad \tilde{C}_i(\mu) \approx \tilde{C}_i(\mu_0)\left(\frac{\alpha(\mu)}{\alpha(\mu_0)}\right)^{-\frac{\gamma^{(1)}}{2\beta_0}}\,, \tag{3.8}$$

where the initial condition at μ_0 is calculated from the matching condition to the full theory at the scale, where the heavy particles are integrated out. We see, that this procedure actually resumms the leading logarithms $\left(\alpha \ln \frac{\mu}{\mu_0}\right)^n$. By Taylor expanding this result, we reproduce the computed term to order α

$$\frac{\alpha(\mu)}{\alpha(\mu_0)} \underset{(1.46)}{\approx} \frac{1}{1 - \frac{2\beta_0}{4\pi}\alpha(\mu_0)\ln\frac{\mu_0}{\mu}} \overset{\text{geom.}}{\underset{\text{series}}{=}} \sum_{n=0}^{\infty}\left(\frac{2\beta_0}{4\pi}\alpha(\mu_0)\ln\frac{\mu_0}{\mu}\right)^n . \tag{3.9}$$

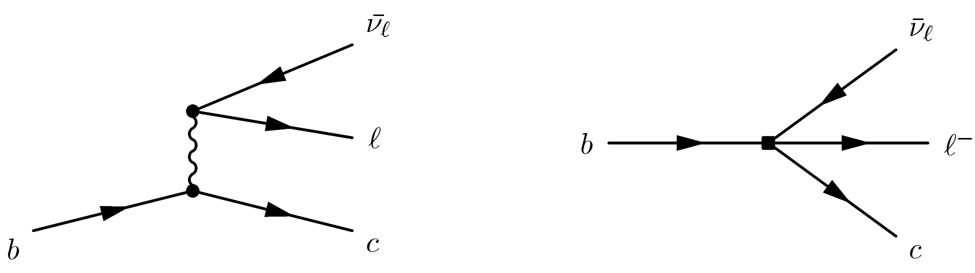
In the next section we will give the set of effective operators of the weak interaction, that we will investigate here.

3.2. Effective Weak Operators

We are dealing with decays of bottom quarks particularly into charm and also the strange quark. This is a weak process mediated by the heavy W-boson. The two very different scales - M_W and m_b, the latter being relevant for the decay, suggest to construct an effective field theory for energies much smaller than the W mass. Therefore we assume that the mass of the exchange boson is much larger than the typical momenta q which are transferred. So we can expand in q/M_W and stop at the constant term. This yields already an amazing precision. Effectively this means we are not able to resolve the non-locality of this interaction and look at the processes as if they were local four-fermion interactions. This means we perform an operator product expansion into local operators, which are responsible for the decays.

In the next two subsections we will derive the effective operators for the two relevant cases within this thesis.

3.2.1. Bottom into Charm Decay: Tree Level



(a) Tree Level Standard Model.

(b) Tree Level effective theory.

Figure 3.1.: Feynman diagrams for semi-leptonic $b \to c$ transition.

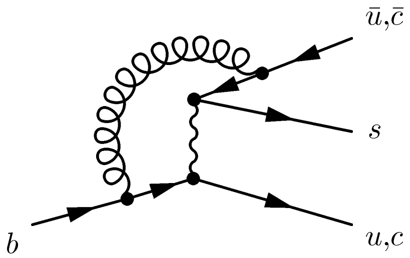
At tree level this is in the full standard model theory displayed in Feynman diagram 3.1(a). In the effective interaction we are left with the operator

$$\mathcal{O} = \bar{c}\gamma_\mu \frac{1-\gamma_5}{2} b \bar{\ell}\gamma^\mu \frac{1-\gamma_5}{2}\nu_\ell \, , \tag{3.10}$$

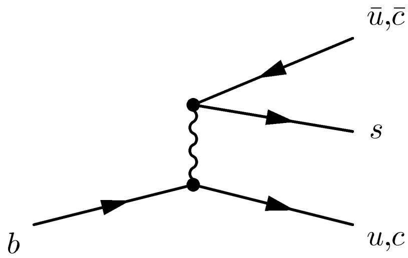
where the left-handedness of this decay is reflected in the projector $P_L = \frac{1-\gamma_5}{2}$. This local interaction is visualized in diagram 3.1(b).

In the section about exclusive decays we will extend the current from V-A Standard Model to arbitrary Dirac structure. However, the diagrams look the same but with a slightly modified vertex Dirac structure.

3.2.2. Bottom into Strange Decay: Loop Level



(a) Full diagram leading to the effective operator $\mathcal{O}_1^{u,c}$. For $b \to s + \gamma$ the up-type quark loop has to get closed.

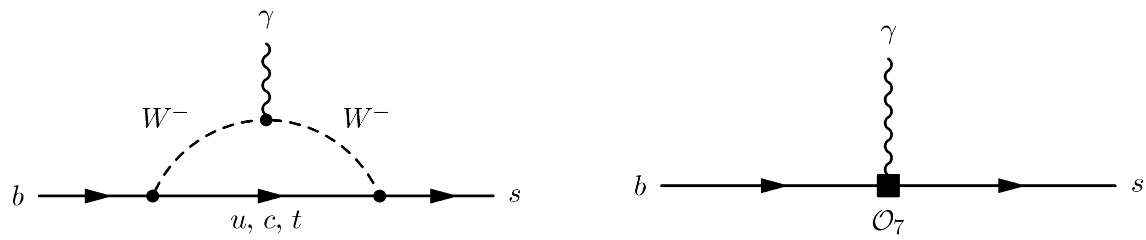


(b) Full diagram leading to the effective operator $\mathcal{O}_2^{u,c}$. For $b \to s + \gamma$ the up-type quark loop has to get closed.

Figure 3.2.: Current-current Feynman diagrams for $b \to s$ transition $\propto V_{qs}^* V_{qb}$.

Now turning to the decay $b \to s$ we notice, that this is a flavour changing neutral current, for which there is no direct (i.e. tree level) transition in the SM. Such transitions can occur only at the loop level, so it is a purely quantum theoretical effect and therefore it is suppressed, see e.g. Figures 3.3: Here the matching between one diagram in full Standard Model and the corresponding effective operator is displayed on the level of a Feynman diagram. Such rare decays are often used for new physics searches, because effects from physics beyond the Standard Model can occur only at loop level in these decays. To this end NP effects are not suppressed a priori with respect to SM contributions. Nevertheless a very good understanding of the SM process is required in order to distinguish such effects. Also the process $B \to X_s\gamma$ is used to pin down some parameters in the $B \to X_c \ell \bar{\nu}_\ell$ decay. The process can only occur if we have an up-type quark running in a loop.

In Figure 3.2 we display the Feynman diagrams relevant for the effective operators $\mathcal{O}_1^{u,c}$ and $\mathcal{O}_2^{u,c}$. In Figure 3.4(a) we show one example Feynman diagram leading to the operators $\mathcal{O}_{3-6}$. We have not shown the distinction of the colour indices, nor the $V-A$ and $V+A$ structure. The latter arises trough the fact, that a gluonic interaction is purely V and therefore we have this admixture in the interaction with the weak $V-A$ structure. Note, that the corresponding quark loop for the operators $\mathcal{O}_{1-6}$ has to be closed, in order to have a contribution to $b \to s + \gamma$, as indicated in Figure 3.4(b). Furthermore we have the magnetic penguins in Figure 3.4, where either a photon or a gluon can couple to the loop particle.



(a) Full diagram leading to the effective operators $\mathcal{O}_{7\gamma}$.

(b) Effective operator $\mathcal{O}_{7\gamma}$.

Figure 3.3.: One of the full SM diagrams and its corresponding effective operator $\mathcal{O}_{7\gamma}$ contributing to $b \to s + \gamma$ transition..

The list of all effective operators relevant for $b \to s\gamma$ is then given by[3]

$$\mathcal{O}_1^c = (\bar{s}_i \gamma_\mu \frac{1-\gamma_5}{2} c_j)(\bar{c}_i \gamma^\mu \frac{1-\gamma_5}{2} b_j) \qquad \mathcal{O}_1^u = (\bar{s}_i \gamma_\mu \frac{1-\gamma_5}{2} u_j)(\bar{u}_i \gamma^\mu \frac{1-\gamma_5}{2} b_j) \tag{3.11a}$$

$$\mathcal{O}_2^c = (\bar{s}_i \gamma_\mu \frac{1-\gamma_5}{2} c_i)(\bar{c}_j \gamma^\mu \frac{1-\gamma_5}{2} b_j) \qquad \mathcal{O}_2^u = (\bar{s}_i \gamma_\mu \frac{1-\gamma_5}{2} u_i)(\bar{u}_j \gamma^\mu \frac{1-\gamma_5}{2} b_j) \tag{3.11b}$$

$$\mathcal{O}_3 = (\bar{s}_i \gamma_\mu \frac{1-\gamma_5}{2} b_i) \sum_{\substack{q=u,d,s\\c,b}} (\bar{q}_j \gamma^\mu \frac{1-\gamma_5}{2} q_j) \qquad \mathcal{O}_4 = (\bar{s}_i \gamma_\mu \frac{1-\gamma_5}{2} b_j) \sum_{\substack{q=u,d,s\\c,b}} (\bar{q}_j \gamma^\mu \frac{1-\gamma_5}{2} q_i) \tag{3.11c}$$

$$\mathcal{O}_5 = (\bar{s}_i \gamma_\mu \frac{1-\gamma_5}{2} b_i) \sum_{\substack{q=u,d,s\\c,b}} (\bar{q}_j \gamma^\mu \frac{1+\gamma_5}{2} q_j) \qquad \mathcal{O}_6 = (\bar{s}_i \gamma_\mu \frac{1-\gamma_5}{2} b_j) \sum_{\substack{q=u,d,s\\c,b}} (\bar{q}_j \gamma^\mu \frac{1+\gamma_5}{2} q_i) \tag{3.11d}$$

$$\mathcal{O}_{7\gamma}^b = \frac{e}{16\pi^2} m_b (\bar{s}_i \sigma_{\mu\nu} \frac{1+\gamma_5}{2} b_i) F^{\mu\nu} \qquad \mathcal{O}_{7\gamma}^s = \frac{e}{16\pi^2} m_s (\bar{s}_i \sigma_{\mu\nu} \frac{1-\gamma_5}{2} b_i) F^{\mu\nu} \tag{3.11e}$$

$$\mathcal{O}_{8G}^b = \frac{g_s}{16\pi^2} m_b (\bar{s}_i \sigma^{\mu\nu} \frac{1+\gamma_5}{2} b_i) F_{\mu\nu} \qquad \mathcal{O}_{7\gamma}^s = \frac{e}{16\pi^2} m_s (\bar{s}_i \sigma^{\mu\nu} \frac{1-\gamma_5}{2} T_{ij}^a b_j) G_{\mu\nu}^a \tag{3.11f}$$

Here i,j are colour indices. Now we have to sum over all operators with the correct CKM prefactors, where in $\mathcal{O}_{3-6}$ have to consider the sum of all intermediate quarks. as indicated by the sum. Furthermore we impose the unitarity condition

$$V_{ts}^* V_{tb} + V_{cs}^* V_{cb} + V_{us}^* V_{ub} = 0 \tag{3.12}$$

[3]Please note, that the top-quark is integrated out together with the W and Z-boson and therefore it cannot appear in the operators.

and neglect in the end terms containing the u-quark since $|V_{us}^* V_{ub}/V_{ts}^* V_{tb}| \leq 0.025$ such that the effective Hamiltonian is proportional to $|V_{ts}^* V_{tb}|$, only. Of course for precision calculations we should include all terms.

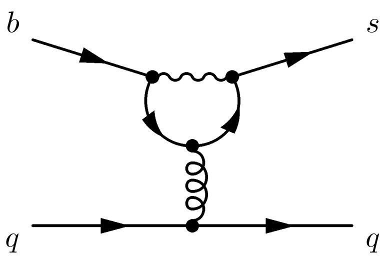
The amplitude in the effective theory for this transition can be written as

$$\begin{aligned}\mathcal{A}(b \to s\gamma) &= C \sum_{i=1,2,3} V_{U_i b} V_{U_i s}^* \; f\Big(\frac{m_{U_i}}{M_W}\Big) \\ &= C \left[V_{cs}^* V_{cb} \Big(f\big(\frac{m_c}{M_W}\big) - f\big(\frac{m_u}{M_W}\big)\Big) + V_{ts}^* V_{tb} \Big(f\big(\frac{m_t}{M_W}\big) - f\big(\frac{m_u}{M_W}\big)\Big) \right] \,, \end{aligned} \tag{3.13}$$

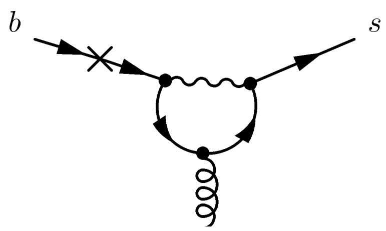
where in the last step the unitarity condition (3.12) has been used. Note that this does not vanish, because the masses of the quarks are not equal. But it leads to an additional suppression and is called GIM mechanism [21], the strength of this suppression (CKM matrix elements, as well as quark mass differences) depends on the process considered. Neglecting also the terms proportional to m_s we end up with the effective Hamiltonian, where we leave out the superscript b and c,

$$\mathcal{H}_{\text{eff}}(b \to s\gamma) = -\frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left[\sum_{i=1}^{6} C_i(\mu)\mathcal{O}_i(\mu) + C_{7\gamma}(\mu)\mathcal{O}_{7\gamma}(\mu) + C_{8G}(\mu)\mathcal{O}_{8G}(\mu) \right] \,. \tag{3.14}$$

The typical scale, where this process takes place is $\mu = \mathcal{O}(m_b)$. Numerically it turns out, that the combination $C_{7\gamma} \cdot \mathcal{O}_{7\gamma}^b$ is the most important one, and therefore we compute later on the corrections to $\mathcal{O}_{7\gamma}^b$, only.



(a) Full diagram leading to the effective operators $\mathcal{O}_{3-6}$.



(b) Full diagram leading to the effective operators $\mathcal{O}_{7\gamma,8G}$.

Figure 3.4.: Penguin Feynman diagrams for $b \to s$ transition in the full theory proportional to $V_{Us}^* V_{Ub}$.

3.3. Non-Perturbative Corrections

At small energies, at which quarks bind to hadrons, the coupling constant of QCD becomes such large, that a perturbative expansion does not make any sense. Then we are in the non-perturbative regime. The QCD scale Λ_{QCD} can be defined, if we equal

(3.9) to

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln \frac{\mu^2}{\Lambda^2_{\mathrm{QCD}}}} \tag{3.15}$$

$$\Rightarrow \qquad \Lambda_{\mathrm{QCD}} = \mu_0 e^{-\frac{4\pi}{2\beta_0 \alpha_s(\mu_0)}} \,. \tag{3.16}$$

Now computing this to three loops [22] we get $\Lambda_{\mathrm{QCD}} = 245^{+20}_{-17}$ MeV. Λ_{QCD} is the scale at which the coupling constant diverges, which is called the Landau pole. At energies below one GeV QCD gets non-perturbative, because $\alpha_s(1\text{ GeV}) \approx 1$ and an expansion in α_s is not possible anymore. On the other hand for higher energies we get $\alpha_s(\mu) \underset{\mu\to\infty}{\to} 0$ and therefore QCD is asymptotically free and we can treat it in the known perturbation theory.

Therefore we have to have methods how to deal with the non-perturbative objects. In the next subsections we will explain how to derive effective theories for hadrons with a heavy quark.

3.3.1. Heavy Quark Expansion

Now in our case we consider b-quarks in B-mesons. The mass of the bottom quark is $m_b \approx 4.2\,\text{GeV}$ and therefore much larger than the typical hadronic binding scale Λ_{QCD}. This allows us to expand in the ratio $\Lambda_{\mathrm{QCD}}/m_b$, and we are left with effective operators describing the interaction of the quark inside the meson. With the help of this procedure we get a handle on the hadronic matrix element. The disadvantage is, that for an increasing precision we need to expand to operators of higher order. Therefore we get more and more non-perturbative parameters in each step of the expansion, which have to be extracted from experiment. These subtle details will be discussed later.

We first derive the Lagrangian for the heavy quark effective theory (HQET). The basic idea is that the b quark is bound inside the B hadron. The momentum of the B meson is given by $P^\mu_B = M_B v^\mu$, where the four-velocity v^μ fulfills $v^2 = 1$. Then the b-quark inside is slightly off-shell

$$p^\mu_b = m_b v^\mu + k^\mu \,, \tag{3.17}$$

where k^μ is a residual momentum describing the off-shellness. To allow the b-quark the binding with a light spectator quark to the hadron, the momentum has to fullfil

$$\frac{|k^\mu|}{m_b} \sim \frac{\Lambda_{\mathrm{QCD}}}{m_b} \ll 1 \,, \tag{3.18}$$

and therefore it gives reason to expand in this ratio. To realize it in a QFT, we notice that v^μ is the only relevant four vector scale. Therefore we project out the large momentum piece of the heavy quark field

$$Q(x) = e^{-im_b v\cdot x} Q_v(x) \tag{3.19}$$

and project onto a "small" and "large" component by the projector $P_\pm = 1/2(1 \pm \not{v})$. This defines the two fields

$$h_v(x) = \frac{1+\not{v}}{2} Q_v(x) \tag{3.20a}$$

$$H_v(x) = \frac{1-\not{v}}{2} Q_v(x) \,. \tag{3.20b}$$

The Dirac field Lagrangian then transforms to

$$\begin{aligned}
\mathcal{L}_Q &= \bar{Q}(x)(i\not{D} - m_Q)Q(x) & (3.21a)\\
&= \bar{Q}_v(x)(i\not{D} - m_Q + m_Q\not{v})Q_v(x) & (3.21b)\\
&= \bar{h}_v iv\cdot D h_v + \bar{H}_v(iv\cdot D + 2m_Q)H_v + \bar{H}_v i\not{D}_\perp h_v + \bar{h}_v i\not{D}_\perp H_v \,. & (3.21c)
\end{aligned}$$

Here we have introduced the perpendicular component to v^μ, defined for any four-vector A^μ by

$$A_\perp^\mu = (g^{\mu\nu} - v^\mu v^\nu)A_\nu$$

with $v \cdot A_\perp = 0$. We find, that the field H_v corresponds to the heavy component with mass $2m_Q$ and h_v corresponds to the light field. Due to the coupling terms, we can eliminate the heavy field by its equation of motion

$$H_v = \frac{1}{iv\cdot D + 2m_Q} i\not{D}_\perp h_v \tag{3.22}$$

and end up with the HQET Lagrangian

$$\mathcal{L}_{\mathrm{HQET}} = \bar{h}_v \left(iv\cdot D + i\not{D}_\perp \frac{1}{2m_Q + iv\cdot D} i\not{D}_\perp \right) h_v \,. \tag{3.23}$$

This Lagrangian has two symmetries

1. Heavy Quark Symmetry (HQS): In the limit $m_Q \to \infty$ the Lagrangian does not distinguish the different quark flavours. Thus charm and bottom quark are treated on equal footings, as long as no corrections to this limit are considered.

2. Heavy Quark Spin Symmetry (HQSS): In the limit of HQS the Lagrangian is additionally independent of the spin of the particle. Therefore the multiplet of mesons, containing the heavy quark, with different spin is degenerate.

Nevertheless, these symmetries are only exact in the limit of infinitely heavy quark masses, and this symmetry is broken by the second term of the Lagrangian. The covariant derivative takes the role of the residual momentum and therefore we can expand in the quantity $\frac{iv\cdot D}{2m_Q}$. The first term of this expansion can be rewritten into the form

$$i\not{D}_\perp \frac{1}{2m_Q} i\not{D}_\perp = \underbrace{\frac{(iD_\perp)^2}{2m_Q}}_{\text{NR kinetic energy}} + \underbrace{\frac{g}{4m_Q}\sigma_{\mu\nu}G^{\mu\nu}}_{\text{chromo-magnetic energy}} \tag{3.24}$$

Please note, that this terms are also known from the non-relativistic reduction of the quantum mechanical hydrogen atom! From these two terms, the breaking of the symmetry becomes obvious. The first term explictly involves the heavy-quark mass, and thus distinguishes between the different heavy-quark flavours. From special interest is the second term, which involves the spin of the particle. This term explains the lifetime difference of a B and B^* meson, due to their different spin.

Heavy quark effective theory introduces a further symmetry of the heavy quark spin symmetry. But in contrast to the usual case integrating out the heavy degree of freedom leaves us with a static b-quark as a remnant source of colour. The first term in the expansion, which describes the spin coupling of the quarks inside the meson explain the life-time difference between B and B^* mesons. For more details, the reader is referred to [20, 23, 24]. A more elaborate access is possible with path integrals, see e.g. [20].

We will now go into more detail for semi-leptonic B-Decays and introduce the techniques we will use later on.

3.3.2. Special Features of Inclusive Decays

We have seen, that in the inclusive case we can relate the decay rate to forward matrix elements of the incoming B-meson. Therefore we are left with matrix elements depending on the HQET fields and a number of covariant derivatives. Now there are several possibilities to define these operators. We can use the HQET fields and expand also the B-meson states into HQET states [25]. The great disadvantage is that we need to compute non-local correlation functions in higher order terms. Therefore using the full states has been proven to be more useful [26].

Within this thesis we will pursue a further slightly modified possibility. We will keep the states as well as the quark fields in full QCD, but rephase the field by the large momentum factor. By the equation of motion for this field it can be seen, that it basically differs from the methods above in higher order corrections. Therefore in principle these operators, including their renormalization, can be transformed into each other. This procedure allows us to perform the $1/m_Q$ expansion in a very systematic way.

3.3.3. Isgur Wise Function and Exclusive Decays

We are dealing with exclusive heavy-to-heavy transitions, in our case $b \to c$. The current mediating this transition is of the form $\bar{c}\Gamma b$ in full QCD, equivalent to $\bar{c}_{v'}\Gamma b_v$ in leading order $1/m_{c,b}$ and α_s. The matrix element relevant for these processes are then of the form

$$\langle H^{(c)}(p')|\bar{c}\Gamma b|H^{(b)}(p)\rangle \,. \tag{3.25}$$

In the next steps we want to represent this in the most general case through HQET operators. Now if the Gamma matrix transforms with the heavy quark spin matrices as $D(R)_c \Gamma D(R)_b^{-1}$, this current is invariant under heavy quark spin symmetry. The most general operators then have to contain the two HQET fields $H_{v'}^{(c)}$ and $H_v^{(b)}$. Finally by demanding also Lorentz covariance we end up with

$$\bar{c}_{v'}\Gamma b_v = \mathrm{Tr}\left[X H_{v'}^{(c)} \Gamma H_v^{(b)}\right] . \tag{3.26}$$

X has to be build out of the most general Dirac matrices depending on the available four vectors: v and v'. Furthermore it has to obey the correct parity and time-reversal properties. Therefore it can be generically parameterized as

$$X = X_0 + X_1 \not{v} + X_2 \not{v}' + X_3 \not{v}\not{v}' . \tag{3.27}$$

Defining $w = v \cdot v'$ and using the equations of motions $\not{v} H_v^{(b)} = H_v^{(b)}$ and $\not{v}' \bar{H}_v^{(c)} = -\bar{H}_v^{(c)}$ we can express everything through the *Isgur-Wise* function [27] $\xi(w)$

$$\bar{c}_{v'}\Gamma b_v = -\xi(w)\mathrm{Tr}\left[H_{v'}^{(c)} \Gamma H_v^{(b)}\right] . \tag{3.28}$$

The heavy quark fields for the different possible quantum numbers of mesons are given by

$$H_v = \begin{cases} H(v) = \frac{1}{2}\sqrt{M_H}(\not{v} - 1) & \text{Pseudo-scalar} \\ H(v,\epsilon) = \frac{1}{2}\sqrt{M_H}\not{\epsilon}(\not{v} - 1) & \text{Vector} \end{cases} \tag{3.29}$$

Heavy quark flavour symmetry implies the normalization condition at the zero-recoil point $v \cdot v' = 1$

$$\xi(1) = 1 . \tag{3.30}$$

In fact in experiment the deviations from unity are measured by the ratio of $B \to D$ and $B \to D^*$, where the CKM matrix element drops out, and the result is proportional to the form-factors squared. To leading order, both form factors coincide with the Isgur-Wise function. Therefore heavy quark symmetry is not realized exactly in nature. Of course this is expected, and there are mainly two corrections to the function

- Non-perturbative corrections to the heavy quark limit $\to$ QCD sum rules.
- Radiative corrections to the operator.

The latter we will investigate for a general new physics ansatz.

4. The Inclusive Decay $B \to X_c \ell \bar{\nu}_\ell$

The inclusive process $B \to X_c \ell \bar{\nu}_\ell$ has very good properties in theory as well as in experiment to study the CKM matrix element $|V_{cb}|$. It has a clean experimental signal, and due to its large branching fraction a low background. Theoretically it is also a clean environment. In this section we investigate the non-perturbative structure in more detail. The aims are to establish the validity of the heavy-quark expansion, improve the uncertainties and to get a better handle on the uncertainties stemming from the non-perturbative input. This will eventually help to better constrain the CKM unitary triangle. To this end, we will first explain the basic setup and determine the operator basis. For the numerical analysis, which we will perform in the following, we give a recipe to estimate the new operators. After we have discussed some details of the formal expansion, we give some comments on a related decay mode, which is also used in the experimental analysis, using the methods presented here.

4.1. The Decay Rate

The interaction is driven by the interaction term of the Lagrangian density. At tree-level the effective interaction for the semileptonic process $b \to c\ell\bar{\nu}_\ell$ is given by

$$\mathcal{L}_{\rm int} = -\mathcal{H}_{\rm int} = -\frac{4G_F}{\sqrt{2}} V_{cb} \left(\bar{c} \frac{\gamma_\mu (1-\gamma_5)}{2} b \right) \left(\bar{e} \frac{\gamma^\mu (1-\gamma_5)}{2} \nu_\ell \right) . \tag{4.1}$$

Here we have already integrated out the heavy W boson to treat this decay within an effective interaction at scales much lower than the W mass. The effective coupling, which is the Fermi constant G_F, is related to the weak isospin coupling via $4G_F/\sqrt{2} = g^2/(2M_W^2)$. From this term we can calculate the matrix element for the inclusive process $B \to X_c \ell \bar{\nu}_\ell$ to be

$$\begin{aligned} \mathcal{M}(\bar{B} \to X_c \ell^- \bar{\nu}_\ell) &= \langle X_c \ell \bar{\nu}_\ell | \mathcal{H}_{\rm int} | \bar{B} \rangle \\ &= \frac{4G_F}{\sqrt{2}} V_{cb} \langle X_c \ell \bar{\nu}_\ell | J_{q,\mu}(0) J_\ell^\mu(0) | \bar{B} \rangle . \end{aligned} \tag{4.2}$$

Since we integrate out the W-boson, the interaction is local. Therefore we leave out the position argument in the following equations. Unless we point out, all operators are taken at $x = 0$. In the matrix element we have also owed the fact, that in nature we observe only quarks bound into hadrons and no free quarks. Thus the initial state is given by the B meson and the final state by some hadron state with a charm-quark quantum number. The non-perturbative information on the binding is hidden in the meson matrix element, which we will investigate in more detail below. Note, that we treat the leptons as massless in the final that and thus $\ell = e, \mu$.

In the last step we have introduced the leptonic and hadronic current, since these two parts do not interfere and can therefore be separated, which can be read off from eq. (4.1)

$$J_\ell^\mu = \bar{\ell}\frac{\gamma^\mu(1-\gamma_5)}{2}\nu_\ell \,, \tag{4.3a}$$

$$J_{q,\mu} = \bar{c}\frac{\gamma_\mu(1-\gamma_5)}{2}b \,. \tag{4.3b}$$

Finally we obtain the differential decay rate out of the general formulae, see e.g. [9]

$$\mathrm{d}\Gamma = \frac{1}{2M_A}\left(\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\left|\mathcal{M}(M_A \to p_1, \ldots, p_n)\right|^2 (2\pi)^4\delta^4(p_A - \sum p_f)\,. \tag{4.4}$$

In the experiment the spins of the initial and final state are not measured. Thus we have to average over the incoming and sum over the outgoing spin states. The initial particle is a (pseudo) scalar and has therefore only one spin direction. Furthermore the phase space integration of the final hadronic state includes a sum over all possible states

$$\mathrm{d}\Gamma = \sum_{X_c}\sum_{\substack{\text{lepton}\\ \text{spins}}} \frac{1}{2M_B}\left(\frac{\mathrm{d}^3 p_\ell}{(2\pi)^3}\frac{1}{2E_\ell}\right)\left(\frac{\mathrm{d}^3 p_{\nu_\ell}}{(2\pi)^3}\frac{1}{2E_{\nu_\ell}}\right)\left|\mathcal{M}(\bar{B} \to X_c \ell^- \bar{\nu}_\ell)\right|^2$$
$$\times (2\pi)^4\delta^4(p_B - (p_\ell + p_{\nu_\ell} + p_{X_c}))\,. \tag{4.5}$$

For the computation of the necessary squared matrix element, we start from eq. (4.2) and use the definition of the leptonic and hadronic tensor

$$L^{\mu\nu} = \sum_{\substack{\text{lepton}\\ \text{spins}}} \langle 0|J_\ell^{\mu\,\dagger}|\ell\bar{\nu}_\ell\rangle\langle\ell\bar{\nu}_\ell|J_\ell^\nu|0\rangle \tag{4.6}$$

$$W_{\mu\nu} = \frac{1}{2M_B}\sum_{X_c}\langle\bar{B}|J_{q,\nu}^\dagger|X_c\rangle\langle X_c|J_{q,\mu}|\bar{B}\rangle(2\pi)^3\delta^4(p_B - (p_\ell + p_{\nu_\ell} + p_{X_c}))\,. \tag{4.7}$$

The phase space integration over the hadronic states is included in the sum over all possible final states X_c. This leads to the intermediate result

$$\begin{aligned}\left|\mathcal{M}(\bar{B} \to X_c \ell^- \bar{\nu}_\ell)\right|^2 &= 8G_F^2\,|V_{\mathrm{cb}}|^2\left|\langle X_c \ell\bar{\nu}_\ell|J_{q,\mu}J_\ell^\mu|\bar{B}\rangle\right|^2\\ &= 8G_F^2\,|V_{\mathrm{cb}}|^2\langle\bar{B}|J_{q,\nu}^\dagger|X_c\rangle\langle X_c|J_{q,\mu}|\bar{B}\rangle\langle 0|J_\ell^{\nu\,\dagger}|\ell\bar{\nu}_\ell\rangle\langle\ell\bar{\nu}_\ell|J_\ell^\mu|0\rangle\,.\end{aligned} \tag{4.8}$$

In total we end up with

$$\mathrm{d}\Gamma = 16\pi G_F^2\,|V_{\mathrm{cb}}|^2\left(\frac{\mathrm{d}^3 p_\ell}{(2\pi)^3}\frac{1}{2E_\ell}\right)\left(\frac{\mathrm{d}^3 p_{\nu_\ell}}{(2\pi)^3}\frac{1}{2E_{\nu_\ell}}\right)W_{\mu\nu}L^{\mu\nu}\,, \tag{4.9}$$

where we will have a closer look on the different parts in turn.

4.1.1. Leptonic Part

As we have seen, the leptonic and hadronic part factorizes. Since there is no confinement for the leptons, we can compute the leptonic tensor easily starting from

$$L^{\mu\nu} = \sum_{\text{spins}} \langle 0|J_\ell^{\dagger\nu}|\ell\bar{\nu}_\ell\rangle\langle\ell\bar{\nu}_\ell|J_\ell^\mu|0\rangle \,. \tag{4.10}$$

With the definition of the V-A weak current (4.3a) this transforms to

$$\begin{aligned} L^{\mu\nu} &= \sum_{\text{spins}} \left(\bar{u}_\ell \frac{1}{2}\gamma^\mu \left(1-\gamma^5\right) v_{\bar{\nu}_\ell}\right)\left(\bar{v}_{\bar{\nu}_\ell}\frac{1}{2}\gamma^\nu \left(1-\gamma^5\right) u_\ell\right) \\ &= \frac{1}{4}\mathrm{Tr}\sum_{\text{spins}}\left(\gamma^\mu\left(1-\gamma^5\right) v_{\bar{\nu}_\ell}\bar{v}_{\bar{\nu}_\ell}\gamma^\nu\left(1-\gamma^5\right)u_\ell\bar{u}_\ell\right) \\ &= \frac{1}{4}\mathrm{Tr}\left(\gamma^\mu\left(1-\gamma^5\right)\not{p}_{\bar{\nu}_\ell}\gamma^\nu\left(1-\gamma^5\right)\not{p}_\ell\right)\,, \end{aligned} \tag{4.11}$$

where we treated the electrons as massless and $\bar{u}$, respectively v are the spinors for outgoing particle (antiparticle). Calculating the trace in the usual way, as presented in textbooks [9], we finally obtain the leptonic tensor

$$L^{\mu\nu} = 2\left(p_\ell^\mu p_{\bar{\nu}_\ell}^\nu + p_\ell^\nu p_{\bar{\nu}_\ell}^\mu - g^{\mu\nu} p_\ell\cdot p_{\bar{\nu}_\ell} - i\epsilon^{\eta\nu\lambda\mu} p_{\ell,\eta} p_{\bar{\nu}_\ell,\lambda}\right)\,, \tag{4.12}$$

where we have used the convention, that $\epsilon^{0123} = -\epsilon_{0123} = +1$.

4.1.2. Hadronic Part

The hadronic tensor contains all information on the non-perturbative binding of the quarks inside the B Meson, which cannot be calculated[1]. The tensor is defined as

$$W_{\mu\nu} = \frac{1}{2M_B}\sum_{X_c}\langle\bar{B}|J_{q,\mu}^\dagger|X_c\rangle\langle X_c|J_{q,\nu}|\bar{B}\rangle(2\pi)^3\delta^4(p_B - q - p_{X_c})\,, \tag{4.13}$$

where $q^\mu = (p_\ell + p_{\nu_\ell})^\mu$ is the momentum transfer to the leptons. We will treat this hadronic tensor in an effective field theory, in order to approximate it to a very high accuracy. To do so, we will transform this matrix element into an expandable and calculable quantity in the sense of heavy-quark expansion. To this end we use the optical theorem, which states that the imaginary part of a forward transition amplitude is related to the production of the intermediate states[2]. We will see later, that the expansions produces higher powers of the intermediate state propagator. Mathematically the optical theorem states, that a propagator is put on-shell and we can use Cauchy's theorem

$$-\frac{1}{\pi}\mathrm{Im}\frac{1}{(x-x'+i\epsilon)^{n+1}} = \frac{(-1)^n}{n!}\delta^{(n)}(x-x')\,. \tag{4.14}$$

[1] Although such objects can in principle be numerically simulated using lattice-QCD methods.

[2] More precisely it is related to the discontinuity of the branch cut of the propagator, but this can be transformed into the imaginary part, which is more useful here.

We already see, that higher orders of a propagator shift the value of the delta distribution by their higher derivatives. This reflects the fact, that the mass of hadrons differ from the mass of their constituent quarks. Yet the higher order terms in the expansion remain local, unless we sum up to infinity. While having in the considered decay some intermediate state with a charm quark inside, we start with a non-local forward matrix element of the form

$$T_{\mu\nu} = -\frac{i}{2M_B} \int \mathrm{d}^4 x e^{-iqx} \langle \bar{B} | \mathrm{T}\left[J^\dagger_{q,\mu}(x), J_{q,\nu}(0) \right] | \bar{B} \rangle \,. \tag{4.15}$$

This can be visualized by the Feynman diagram in Fig. 4.1. The double line denotes

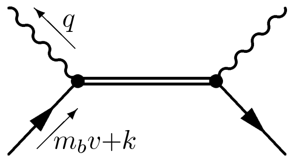


Figure 4.1.: Background field propagator.

the *charm quark* that propagates in the soft background field of the binding gluons. Now by using the optical theorem, we can reproduce the hadronic tensor

$$-\frac{1}{\pi}\,\mathrm{Im}\,T_{\mu\nu} = \frac{1}{2M_B} \sum_{X_c} \langle \bar{B} | J^\dagger_{q,\mu} | X_c \rangle \langle X_c | J_{q,\nu} | \bar{B} \rangle (2\pi)^3 \delta^4 (p_B - q - p_{X_c}) \,. \tag{4.16}$$

To show this, we have to write out the time ordered product and insert the appropriate sum over complete states, which is unity in the according subspace. The time ordering prescription describes the correct expression for the hadronic tensor, since the additional $X_{b\bar{b}c}$ intermediate state cannot become on-shell due to energy conservation. Therefore we notice that

$$-\frac{1}{\pi}\mathrm{Im}\,T_{\mu\nu} = W_{\mu\nu} \,. \tag{4.17}$$

Turning now to the time-ordered product again, where we use the rephased, yet in full QCD, b_v-quark field according to (3.19)

$$T_{\mu\nu} = \frac{1}{2M_B} \langle \bar{B} | \bar{b}_v \Gamma^\dagger_\nu S \Gamma_\mu b_v | \bar{B} \rangle \,, \tag{4.18}$$

where the charm-quark propagator S is given by

$$S = \frac{1}{\not{p}_b - \not{q} - m_c} \,. \tag{4.19}$$

The b quark momentum is now parameterized by the four-velocity of the B meson $P^\mu_B = M_B v^\mu$ according to heavy-quark effective theory

$$p^\mu_b = m_b v^\mu + k^\mu \,. \tag{4.20}$$

There k^μ describes the small off-shell momentum of the b quark that contributes to the binding with soft-gluons to the spectator quark and denotes the relative movement of the quark with respect to the meson. By replacing the momentum k by the covariant derivative, containing the soft background field gluons, this propagator becomes the background-field (BGF) propagator

$$S_{\mathrm{BGF}} = \frac{1}{m_b \not{v} + i\not{D} - \not{q} - m_c} \,. \tag{4.21}$$

This quantity describes the charm-quark propagating in the forward matrix element of the B meson with all the binding gluons and therefore accounts for the difference between quarks and mesons.

In the section on heavy-quark expansion (HQE) we will show, how to deal with this non-perturbative matrix element and how to treat the subtleties. First we will calculate the more trivial part of phase space for the different observables.

4.1.3. Phase Space

In this section we integrate over the phase space, to retrieve the desired differential decay rate. The phase space weights the transition probability with the allowed kinematics of the final state particles. The hadronic and leptonic part do only depend on the scalar products of the kinematic variables, which we keep in mind for computing the phase space integrals. We are interested in different physical observables: The charged lepton energy spectrum, as well as in the kinematical distributions for the partonic final state variables: Energy and invariant mass. The latter ones can only be used for computing moments and not spectra, since we have to integrate over all possible hadronic momenta to accommodate for the assumptions of the heavy-quark expansion. For beeing able to perform cross checks, we consider two different decay rates with different phase space integrations.

Phase-Space for the Lepton Energy Spectrum

First of all we consider the case for the lepton energy spectrum with the triple differential rate in the kinematical variables E_ℓ, $E_{\bar{\nu}_\ell}$ and q^2, where again $q^\mu = p_\ell^\nu + p_{\bar{\nu}_\ell}^\nu$ and $E_{\ell,\bar{\nu}_\ell}$ is the energy of the lepton (neutrino). By differentiating (4.9) we obtain the triple differential rate, where we have defined $\left|\tilde{\mathcal{M}}\right|^2$, which contains the momentum conservation and sum over all X_c states,

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2 \mathrm{d}E_\ell \mathrm{d}E_{\bar{\nu}_\ell}} = \frac{1}{2M_B} \left|\tilde{\mathcal{M}}\right|^2 \left(\frac{\mathrm{d}^3 p_\ell}{(2\pi)^3} \frac{1}{2E_\ell} \right) \left(\frac{\mathrm{d}^3 p_{\nu_\ell}}{(2\pi)^3} \frac{1}{2E_{\nu_\ell}} \right) \times \delta\left(E_\ell - p_\ell^0\right) \delta\left(E_{\bar{\nu}_\ell} - p_{\bar{\nu}_\ell}^0\right) \delta\left(q^2 - (p_\ell + p_{\bar{\nu}_\ell})^2\right) \,. \tag{4.22}$$

The phase-space integral operators are Lorentz invariant

$$\mathrm{d}\tilde{p} = \frac{\mathrm{d}^4 p}{(2\pi)^4} \, (2\pi)\, \delta\left(p^2 - m^2\right) \theta\left(p^0\right) = \frac{\mathrm{d}^3 p}{(2\pi)^3 \, 2E}. \tag{4.23}$$

The integration is purely over the leptonic variables, since the integration over partonic variables is contained in the hadronic tensor. We compute the phase-space in the limit of massless leptons, therefore the kinematics simplifies to

$$p_\ell^2 = 0 \qquad \Rightarrow \qquad |\boldsymbol{p}_\ell|^2 = E_\ell^2 \tag{4.24a}$$

$$p_{\bar\nu_\ell}^2 = 0 \qquad \Rightarrow \qquad \left|\boldsymbol{p}_{\bar\nu_\ell}\right|^2 = E_{\bar\nu_\ell}^2 \,. \tag{4.24b}$$

In the first step, we use the three-dimensional integration operators (4.23) in the massless limit (4.24a) and (4.24b) to perform the phase-space integral. Since the integrand is symmetric we transform to spherical coordinates $\int \mathrm{d}\Omega = \int \mathrm{d}\cos\theta \mathrm{d}\varphi = 4\pi$. The only non-trivial angle is the angle between the two leptons. Thus we align the coordinate system in such a way, that the integration over the independent angles leads to

$$\int \mathrm{d}\cos\theta_{\bar\nu_\ell} \mathrm{d}\varphi_{\bar\nu_\ell} \mathrm{d}\varphi_{\measuredangle_{e,\bar\nu_\ell}} = 2\cdot 2\pi \cdot 2\pi = 8\pi^2 \,. \tag{4.25}$$

Using the abbreviation $\theta = \theta_{\measuredangle_{e,\bar\nu_\ell}}$ we finally compute the phase-space integral to be

$$\begin{aligned}
\mathrm{d}\phi &= \int \frac{\mathrm{d}^3 p_\ell}{(2\pi)^3} \frac{\mathrm{d}^3 p_{\bar\nu_\ell}}{(2\pi)^3} \frac{1}{4E_\ell E_{\bar\nu_\ell}} \delta\left(E_\ell - |\boldsymbol{p}_\ell|\right) \delta\left(E_{\bar\nu_\ell} - \left|\boldsymbol{p}_{\bar\nu_\ell}\right|\right) \delta\left(q^2 - (p_\ell + p_{\bar\nu_\ell})^2\right) \\
&\underset{4.25}{=} \int |\boldsymbol{p}_\ell|^2 \,\mathrm{d}\,|\boldsymbol{p}_\ell| \left|\boldsymbol{p}_{\bar\nu_\ell}\right|^2 \mathrm{d}\left|\boldsymbol{p}_{\bar\nu_\ell}\right| \mathrm{d}\cos\theta \frac{8\pi^2}{4\,(2\pi)^6\, E_\ell E_{\bar\nu_\ell}} \delta\left(E_\ell - |\boldsymbol{p}_\ell|\right) \delta\left(E_{\bar\nu_\ell} - \left|\boldsymbol{p}_{\bar\nu_\ell}\right|\right) \\
&\qquad\qquad \times \delta\left(q^2 - 2E_\ell E_{\bar\nu_\ell} + 2|\boldsymbol{p}_\ell||\boldsymbol{p}_{\bar\nu_\ell}|\cos\theta\right) \\
&= \frac{1}{(2\pi)^4} \int_{-1}^{1} \mathrm{d}\cos\theta \frac{E_\ell^2 E_{\bar\nu_\ell}^2}{2E_\ell E_{\bar\nu_\ell}} \delta\left(q^2 - 2E_\ell E_{\bar\nu_\ell}(1-\cos\theta)\right) \\
&= \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \mathrm{d}\cos\theta \frac{E_\ell E_{\bar\nu_\ell}}{2} \frac{1}{2E_\ell E_{\bar\nu_\ell}} \delta\left(\cos\theta - (1 - \frac{q^2}{2E_\ell E_{\bar\nu_\ell}})\right) \theta\left(1-\cos\theta\right) \theta\left(1+\cos\theta\right) \\
&= \frac{1}{4} \frac{1}{(2\pi)^4} \theta\left(\frac{q^2}{2E_\ell E_{\bar\nu_\ell}}\right) \theta\left(2 - \frac{q^2}{2E_\ell E_{\bar\nu_\ell}}\right) \,.
\end{aligned} \tag{4.26}$$

The unit-step θ functions are important, since they depend on a kinematical variable. While having higher derivatives of delta distributions, we have to integrate by parts and then this unit step functions have to be differentiated as well. We have to keep in mind, that the matrix element (4.8) depends on kinematical variables as $|\mathcal{M}|^2\,(q^2, v\cdot q, E_\ell)$.

Phase-Space Integration for Partonic Observables

We consider additionally the decay in the kinematical variables of the quarks and not the mesons. This has to be done, because we are using HQET. Our aim is to obtain expressions, usually denoted as "building blocks", for computing moments in the invariant mass of the final state hadrons. The differential decay rate is now computed in the kinematical variables Q^2 and $v \cdot Q$, where $Q^\mu = p_b^\mu - q^\mu \equiv p_c^\mu$. Therefore these variables correspond to the partonic invariant mass and partonic energy of the final state quark. In experiment only hadronic variables can be detected, in contrast to theory. The hadronic variables are related to the partonic ones via the binding energy $\lambda = (M_B - m_b) + \mathcal{O}(1/m_b)$ and HQE parameters. Therefore we can write

$$P_x^\mu = (P_B - p_b)^\mu + p_c^\mu \tag{4.27a}$$
$$\Rightarrow \quad M_x^2 = p_c^2 + (M_B - m_b)^2 + 2(M_B - m_b) v \cdot p_c + \mathcal{O}(1/m_b) \tag{4.27b}$$
$$\Rightarrow \quad M_x^2 = \lambda^2 + p_c^2 + 2(M_B - m_b) v \cdot p_c \,, \tag{4.27c}$$

and the decomposition of hadronic moments into partonic ones is straightforward. To this end we redefine the leptonic tensor to include the integration over the leptonic momenta. Then we can also implement a lower cut on the charged lepton energy, which is trivial for the latter case of the lepton energy spectrum. This is required in experiment, since a lower cut should be set there to suppress indistinguishable background. Please note in the following, that we insert unity by the integration over the momentum transfer including the corresponding delta distribution for momentum conservation $\int \frac{\mathrm{d}^4 q}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(q - p_\ell - p_\nu) = 1$. Therefore we end up with

$$\mathrm{d}\Gamma = 16\pi G_F^2 |V_{cb}|^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \underbrace{\left[\int \frac{\mathrm{d}^3 p_\ell \, \mathrm{d}^3 p_\nu}{(2\pi)^6 \, 2E_\ell \, 2E_\nu} L_{\mu\nu} (2\pi)^4 \delta^4 (q - p_\ell - p_\nu) \right]}_{\tilde{L}_{\mu\nu}} W^{\mu\nu} \tag{4.28}$$

$$\tilde{L}^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) L_1 + \left(v^\mu v^\nu + \frac{q^\mu q^\nu (v \cdot q)^2}{q^4} - \frac{v \cdot q (v^\mu q^\nu + q^\mu v^\nu)}{q^2} \right) L_2 - i \epsilon^{\mu\nu\alpha\beta} v_\alpha q_\beta \, L_3 \,. \tag{4.29}$$

The structure functions L_i can be calculated explicitely and are given by

$$L_1 = \frac{q^2}{24\pi [(v \cdot q)^2 - q^2]^{3/2}} \Big[4(v \cdot q)^3 - 3q^2 v \cdot q + 4[(v \cdot q)^2 - q^2]^{3/2} + 6E_{\mathrm{cut}} q^2 - 12 E_{\mathrm{cut}} (v \cdot q)^2 + 12 E_{\mathrm{cut}}^2 v \cdot q - 8 E_{\mathrm{cut}}^3 \Big] \,, \tag{4.30a}$$

$$L_2 = \frac{q^4}{8\pi [(v \cdot q)^2 - q^2]^{5/2}} (2E_{\mathrm{cut}} - v \cdot q)(q^2 - 4E_{\mathrm{cut}} v \cdot q + 4E_{\mathrm{cut}}^2) \,, \tag{4.30b}$$

$$L_3 = \frac{q^2}{8\pi [(v \cdot q)^2 - q^2]^{3/2}} (q^2 - 4E_{\mathrm{cut}} v \cdot q + 4E_{\mathrm{cut}}^2) \,. \tag{4.30c}$$

For the limit of vanishing[3] $E_{\mathrm{cut}} = 1/2 \left(v \cdot q - \sqrt{m_b \, v \cdot q - q^2} \right)$ the structure functions simplify to

$$L_1 = \frac{q^2}{3\pi} \,, \tag{4.31a}$$
$$L_2 = 0 \,, \tag{4.31b}$$
$$L_3 = 0 \,. \tag{4.31c}$$

[3]Please note, that $v \cdot q$ depends on the electron energy, and therefore the limit is non trivial.

Therefore we can write the differential decay rate as

$$
\begin{aligned}
\mathrm{d}\Gamma &= 16\pi G_F^2|V_{cb}|^2 \int \frac{\mathrm{d}s}{2\pi}\frac{\mathrm{d}^4q}{(2\pi)^4} 2\pi\delta(s-q^2)\tilde{L}_{\mu\nu}W^{\mu\nu} && (4.32)\\
&= 16\pi G_F^2|V_{cb}|^2 \int \frac{\mathrm{d}s}{2\pi}\frac{\mathrm{d}^3q}{(2\pi)^3}\tilde{L}_{\mu\nu}W^{\mu\nu} && (4.33)\\
&= \frac{4G_F^2|V_{cb}|^2}{\pi^2}\int \mathrm{d}s\,\frac{|\boldsymbol{q}|^2\mathrm{d}|\boldsymbol{q}|}{2E(\boldsymbol{q})}\,\tilde{L}_{\mu\nu}W^{\mu\nu}\Big|_{E(\boldsymbol{q})=v\cdot q=\sqrt{|\boldsymbol{q}|^2+s}\ ,\, s=q^2} && (4.34)\\
&= \frac{2G_F^2|V_{cb}|^2}{\pi^2}\int \mathrm{d}q^2\mathrm{d}v\cdot q\sqrt{(v\cdot q)^2-q^2}\,\tilde{L}_{\mu\nu}W^{\mu\nu} && (4.35)\\
\Rightarrow \qquad \frac{\mathrm{d}\Gamma}{\mathrm{d}Q^2\mathrm{d}v\cdot Q} &= \frac{2G_F^2|V_{cb}|^2}{\pi^2}\sqrt{(v\cdot Q)^2-Q^2}\tilde{L}_{\mu\nu}W^{\mu\nu}\,, && (4.36)
\end{aligned}
$$

where we have used the abbreviation $Q_\mu = m_b v_\mu - q_\mu$. For later use, please note that Q_μ is identified with the charm quark momentum.

4.2. Heavy Quark Expansion

In this section we will present the tools to derive an approximation for the non-trivial hadronic tensor. Therefore we will perform an operator product expansion in a specific way to expand it into operators of different dimension to the cost of introducing non-perturbative parameters. First we notice, that the actual b quark inside the meson has a small momentum component k_μ relative to the B meson momentum $P_B^\mu = M_B v^\mu$. Thus we can parameterize the b quark momentum with $p_b^\mu = m_b v^\mu + k^\mu$. This residual momentum describes the motion of the b quark within the background field that binds the quarks together. Its magnitude is in the order of the binding energy $\mathcal{O}(\Lambda_{\mathrm{QCD}})$ and therefore the condition $k_\mu \ll m_b$ holds. For this reason we can expand in this quantity which corresponds precisely to heavy quark effective theory (HQET). In contrast to the standard method, we keep the full QCD field, but rephase it according to

$$
b_v(x) = e^{im_b v\cdot x} b(x)\,, \tag{4.37}
$$

such that it only depends on the residual momentum k_μ. The phase redefinition in the time ordered product combines then to the background field propagator (4.21). To circumvent the problem of the ordering of the momenta in the expansion, we replace the momentum k_μ by its covariant derivative iD_μ. This automatically takes care of the ordering of the covariant derivatives and we do not have to take care about x-gluon matrix elements, they are automatically contained in the local expansion. The background field propagator reads with $Q_\mu = m_b v_\mu - q_\mu$:

$$
S_{\mathrm{BGF}} = \frac{1}{\not{Q} + i\not{D} - m_c}\,. \tag{4.38}
$$

This can be written as a geometric series, to yield an expansion[4] in k^μ/m_b

$$S_{\text{BGF}} = \left[\sum_{n=0}^{\infty}(-1)^n \left[(\not{Q}-m_c)^{-1}(i\not{D})\right]^n\right](\not{Q}-m_c)^{-1} \tag{4.39}$$

and the operator product expansion can be cut off at some mass dimension m. Then the expansion reads explicitely

$$S_{\text{BGF}} = \frac{1}{\not{Q}-m_c} - \frac{1}{\not{Q}-m_c}(i\not{D})\frac{1}{\not{Q}-m_c} + \frac{1}{\not{Q}-m_c}(i\not{D})\frac{1}{\not{Q}-m_c}(i\not{D})\frac{1}{\not{Q}-m_c} + \cdots \tag{4.40}$$

Thus we end up with an operator product expansion with operators of the form

$$\langle \bar{B}|\mathcal{O}^{n+3}_{\mu_1\cdots\mu_n}|\bar{B}\rangle = \langle \bar{B}|\bar{b}_v(iD_{\mu_1})\cdots(iD_{\mu_n})b_v|\bar{B}\rangle \tag{4.41}$$

in dimension $n+3$. The remaining task is to evaluate these matrix elements, which contain the non-trivial non-perturbative operators of the OPE. These cannot be calculated from first principles and have to be extracted from experiment. Later on we will give a method, how we estimate higher-order matrix elements. In turn we will explain how to evaluate the expressions, which are called "trace formulae". Therefore, in general, the time-ordered product can be written in the form

$$T_{\mu\nu} = \sum_{n=0}^{m} C^{\mu_1\cdots\mu_n}_{n,\mu\nu}(m_b,v_\mu)\langle \bar{B}|\mathcal{O}^{n+3}_{\mu_1\cdots\mu_n}|\bar{B}\rangle\,, \tag{4.42}$$

where in each dimension n of the non-perturbative expansion the Wilson coefficient $C^{\mu_1\cdots\mu_m}_{\mu\nu}(m_b,v_\mu)$ has a perturbative expansion in the strong coupling α_s.

4.2.1. Non-Perturbative Corrections

We have to compute the time-ordered product in the form (4.42). The steps we will present in detail are basically first to determine the fully contracted basis parameters. In other words we have to choose the basis of all independent non-perturbative parameters and then from this we derive the general tensor decomposition in order to calculate the Wilson coefficients $C^{\mu_1\cdots\mu_n}_{n,\mu\nu}(m_b,v_\mu)$.

We derive the equations of motions for the rephased b-field, where $P_\pm = 1/2(1 \pm \not{v})$ is the projector onto the "large" and "small" components of the full four-component spinor. Using the Dirac equation

$$(i\not{D} - m_b)b(x) = 0 \tag{4.43}$$

$$(i\not{D} + m_b)(i\not{D} - m_b)b(x) = 0 \tag{4.44}$$

with the rephased field (4.37) and keeping track on the ordering of the gamma matrices

$$iD_\mu b_v(x) = e^{im_b v\cdot x}\left(-m_b v_\mu + iD_\mu\right)b(x)\,. \tag{4.45}$$

[4] Actually one sees, that it is only an expansion in the momentum release Q^μ/m_b, only valid if we integrate over all partonic variables.

we get the useful relations

$$\slashed{v} b_v = b_v - \frac{1}{m_b} i\slashed{D} b_v \tag{4.46a}$$
$$P_+ b_v = -\frac{1}{2m_b} i\slashed{D} b_v + b_v \tag{4.46b}$$
$$P_- b_v = \frac{1}{2m_b} i\slashed{D} b_v \tag{4.46c}$$
$$(iv \cdot D) b_v = -\frac{1}{2m_b} i\slashed{D} i\slashed{D} b_v \,. \tag{4.46d}$$

From this special feature it is clear, that depending on the definition of the basis parameters, they contain different pieces which depend non-trivially on m_b. The next step is to define the basis for the tensor decomposition of (4.41).

The Non-Perturbative Matrix Elements

By keeping the full QCD field in the OPE we have only local operators and no non-local pieces from expanding the state as well as the field. To coincide with the usually defined parameters in dimension 5, which is equal to expanding up to $1/m_b^2$, we define the operators to be

$$2M_B\,\mu_\pi^2 = -\langle \bar{B}|\bar{b}_v\, iD_\rho iD_\sigma\, b_v|\bar{B}\rangle\ \Pi^{\rho\sigma} \tag{4.47a}$$
$$2M_B\,\mu_G^2 = \frac{1}{2}\langle \bar{B}|\bar{b}_v\,\big[iD_\rho, iD_\sigma\big]\big(-i\sigma_{\alpha\beta}\big)\, b_v|\bar{B}\rangle\ \Pi^{\alpha\rho}\Pi^{\beta\sigma} \,. \tag{4.47b}$$

Here $\Pi_{\mu\nu} \equiv v_\mu v_\nu - g_{\mu\nu}$ is the projector onto the spatial components. Commonly one can give these parameters a physical interpretation. Clearly μ_π^2 is linked to the expectation value of the spatial momentum squared, therefore it is referred to as the kinetic term. Using $[iD_\mu, iD_\nu] = ig_s G_{\mu\nu}$ and the relation

$$\gamma_\mu\gamma_5 \to P_+\gamma_\mu\gamma_5 P_+ \ = s_\mu \tag{4.48}$$
$$\sigma_{\mu\nu} \to P_+\sigma_{\mu\nu}P_+ \ = v^\alpha \epsilon_{\alpha\mu\nu\beta} s^\beta \,, \tag{4.49}$$

where s_μ is the spin vector, we can identify μ_G^2 to leading order in $1/m_b$ as the chromo-magnetic moment $\boldsymbol{s}\cdot\boldsymbol{B}$. In dimension 6, corresponding to $1/m_b^3$ we define the Darwin term ρ_D^3 and the spin-orbit term ρ_{LS}^3 as

$$2M_B\,\rho_D^3 = \frac{1}{2}\langle \bar{B}|\bar{b}_v\,\Big[iD_\rho, \big[iD_\sigma, iD_\lambda\big]\Big]\, b_v|\bar{B}\rangle\ \Pi^{\rho\lambda} v^\sigma \tag{4.50a}$$
$$2M_B\,\rho_{LS}^3 = \frac{1}{2}\langle \bar{B}|\bar{b}_v\,\Big\{iD_\rho, \big[iD_\sigma, iD_\lambda\big]\Big\}\big(-i\sigma_{\mu\nu}\big)\, b_v|\bar{B}\rangle\ \Pi^{\alpha\rho}\Pi^{\beta\lambda} v^\sigma \,. \tag{4.50b}$$

The Darwin term is related to the divergence of the chromo-electric field $\nabla\cdot\boldsymbol{E}$, and the spin-orbit term to the curl of the chromo-electric field $\boldsymbol{s}\cdot\nabla\times\boldsymbol{E}$. These terms are also known from the hydrogen atom, but there we have the commutative QED fields instead of QCD. This non-commutativity plays a role in the higher order matrix elements as we will see in a moment.

Now turning to the next term in the expansion: Dimension seven $(1/m_b^4)$. The operators have already been identified and the corresponding coefficients have been computed in [28, 29], but some of the basis operators were dismissed. Therefore we repeat the derivation of the complete basis.

First we write down all possible combinations of the four covariant derivatives. They can be contracted via the metric tensor $g_{\mu\nu}$ or the momentum direction of the B meson v_μ. For the spin triplet we additionally have the spin matrix $\big(-i\sigma_{\alpha\beta}\big)$. From the equations of motions (4.46) we see, that a time derivative $iv \cdot D \equiv iD_0$ cannot be at the most left or right position, since this can be rewritten into a pure higher order correction. see eq. (4.46d). The four velocity contracted with the spin-matrix is directly related to a higher order correction via the e.o.m. This can be easily seen by rewriting

$$v_\alpha(-i\sigma^{\alpha\beta}) = \frac{1}{2}(\not{v}\gamma^\beta - \gamma^\beta\not{v}) \tag{4.51}$$

and using equation (4.46a). For an even (odd) number of covariant derivatives there can only be an even (odd) number of time derivatives, because we need a fully contracted Lorentz scalar: Besides the four-velocity v_α with one index we can only contract the matrix element with an even number of indices $g_{\alpha\beta}$ and $\big(-i\sigma_{\alpha\beta}\big)$. Thus we are at first glance sticking to eleven non-perturbative parameters out of which four are spin-singlets (4.52a-4.52d) and the others are spin-tripletts.

$$2M_B\,\tilde{m}_1 = \langle\bar{B}|\bar{b}_v\,iD_\rho iD_\sigma iD_\lambda iD_\delta\,b_v|\bar{B}\rangle\,\frac{1}{3}\left(\Pi^{\rho\sigma}\Pi^{\lambda\delta} + \Pi^{\rho\lambda}\Pi^{\sigma\delta} + \Pi^{\rho\delta}\Pi^{\sigma\lambda}\right) \tag{4.52a}$$

$$2M_B\,\tilde{m}_2 = \langle\bar{B}|\bar{b}_v\,\big[iD_\rho, iD_\sigma\big]\big[iD_\lambda, iD_\delta\big]\,b_v|\bar{B}\rangle\;\Pi^{\rho\delta}v^\sigma v^\lambda \tag{4.52b}$$

$$2M_B\,\tilde{m}_3 = \langle\bar{B}|\bar{b}_v\,\big[iD_\rho, iD_\sigma\big]\big[iD_\lambda, iD_\delta\big]\,b_v|\bar{B}\rangle\;\Pi^{\rho\lambda}\Pi^{\sigma\delta} \tag{4.52c}$$

$$2M_B\,\tilde{m}_4 = \langle\bar{B}|\bar{b}_v\,\Big\{iD_\rho, \Big[iD_\sigma, \big[iD_\lambda, iD_\delta\big]\Big]\Big\}\,b_v|\bar{B}\rangle\;\Pi^{\sigma\lambda}\Pi^{\rho\delta} \tag{4.52d}$$

$$2M_B\,\tilde{m}_5 = \langle\bar{B}|\bar{b}_v\,\big\{iD_\rho, D_\sigma\big\}\big[iD_\lambda, iD_\delta\big]\big(-i\sigma_{\alpha\beta}\big)\,b_v|\bar{B}\rangle\;\Pi^{\sigma\lambda}\Pi^{\alpha\rho}\Pi^{\beta\delta} \tag{4.52e}$$

$$2M_B\,\tilde{m}_6 = \langle\bar{B}|\bar{b}_v\,\big\{iD_\rho, iD_\sigma\big\}\big[iD_\lambda, iD_\delta\big]\big(-i\sigma_{\alpha\beta}\big)\,b_v|\bar{B}\rangle\;\Pi^{\rho\sigma}\Pi^{\alpha\lambda}\Pi^{\beta\delta} \tag{4.52f}$$

$$2M_B\,\tilde{m}_7 = \langle\bar{B}|\bar{b}_v\,\big[iD_\rho, iD_\sigma\big]\big\{iD_\lambda, iD_\delta\big\}\big(-i\sigma_{\alpha\beta}\big)\,b_v|\bar{B}\rangle\;\Pi^{\sigma\lambda}\Pi^{\alpha\rho}\Pi^{\beta\delta} \tag{4.52g}$$

$$2M_B\,\tilde{m}_8 = \langle\bar{B}|\bar{b}_v\,\big[iD_\rho, iD_\sigma\big]\big\{iD_\lambda, iD_\delta\big\}\big(-i\sigma_{\alpha\beta}\big)\,b_v|\bar{B}\rangle\;\Pi^{\lambda\delta}\Pi^{\alpha\rho}\Pi^{\beta\sigma} \tag{4.52h}$$

$$2M_B\,\tilde{m}_9 = \langle\bar{B}|\bar{b}_v\,\big[iD_\rho, iD_\sigma\big]\big[iD_\lambda, iD_\delta\big]\big(-i\sigma_{\alpha\beta}\big)\,b_v|B\rangle\;\Pi^{\alpha\sigma}\Pi^{\beta\lambda}\Pi^{\rho\delta} \tag{4.52i}$$

$$2M_B\,\tilde{m}_{10} = \langle\bar{B}|\bar{b}_v\,\bigg[iD_\rho, \Big[iD_\sigma, \big[iD_\lambda, iD_\delta\big]\Big]\bigg]\big(-i\sigma_{\alpha\beta}\big)\,b_v|\bar{B}\rangle\;\Pi^{\rho\beta}\Pi^{\lambda\alpha}\Pi^{\sigma\delta} \tag{4.52j}$$

$$2M_B\,\tilde{m}_{11} = \langle\bar{B}|\bar{b}_v\,\big[iD_\rho, iD_\sigma\big]\big[iD_\lambda, iD_\delta\big]\big(-i\sigma_{\alpha\beta}\big)\,b_v|\bar{B}\rangle\;\Pi^{\alpha\rho}\Pi^{\beta\delta}v^\sigma v^\lambda\,. \tag{4.52k}$$

We reduce the number of independent matrix elements: Please note, that we are dealing with forward matrix elements, where final and initial state have the same momentum. Thus they are invariant under time transformation. Furthermore these non-perturbative parameters should be real, from which follows that the appearing operators can only be hermitian combinations. These conditions reduce the number of independent parameters according to

$$\begin{aligned}
& \langle A_1|\mathcal{O}|A_2\rangle \overset{!}{=} T(\langle A_1|\mathcal{O}|A_2\rangle) \\
\Rightarrow \quad & \langle A_2|\mathcal{O}|A_1\rangle \overset{!}{=} \langle A_1|\mathcal{O}|A_2\rangle^* = \langle A_2|\mathcal{O}^\dagger|A_1\rangle \\
\Rightarrow \quad & \mathcal{O} \overset{!}{=} \mathcal{O}^\dagger \qquad \text{for } |A_1\rangle \equiv |A_2\rangle\,.
\end{aligned} \tag{4.53}$$

From this follows, with $A_1 \equiv A_2$, that

$$2M_B\,\tilde{m}_5 = 2M_B\,\tilde{m}_7 \tag{4.54a}$$
$$2M_B\,\tilde{m}_6 = 2M_B\,\tilde{m}_8\,. \tag{4.54b}$$

Thus we redefine the complete basis, where the unchanged four spin singlet parameters are still given by

$$2M_B\,m_1 = \langle\bar{B}|\bar{b}_v\,iD_\rho iD_\sigma iD_\lambda iD_\delta\,b_v|\bar{B}\rangle\,\frac{1}{3}\left(\Pi^{\rho\sigma}\Pi^{\lambda\delta} + \Pi^{\rho\lambda}\Pi^{\sigma\delta} + \Pi^{\rho\delta}\Pi^{\sigma\lambda}\right) \tag{4.55a}$$
$$2M_B\,m_2 = \langle\bar{B}|\bar{b}_v\left[iD_\rho, iD_\sigma\right]\left[iD_\lambda, iD_\delta\right]b_v|\bar{B}\rangle\,\Pi^{\rho\delta}v^\sigma v^\lambda \tag{4.55b}$$
$$2M_B\,m_3 = \langle\bar{B}|\bar{b}_v\left[iD_\rho, iD_\sigma\right]\left[iD_\lambda, iD_\delta\right]b_v|B\rangle\,\Pi^{\rho\lambda}\Pi^{\sigma\delta} \tag{4.55c}$$
$$2M_B\,m_4 = \langle\bar{B}|\bar{b}_v\left\{iD_\rho, \left[iD_\sigma, \left[iD_\lambda, iD_\delta\right]\right]\right\}b_v|\bar{B}\rangle\Pi^{\sigma\lambda}\Pi^{\rho\delta}\,, \tag{4.55d}$$

whereas the spin-triplett definitions changes. The number of spin dependent matrix elements reduces from seven to five. We therefore have to redefine the basis. As a hermitian basis for the corresponding 5 spin-triplet matrix elements we choose

$$2M_B\,m_5 = \langle\bar{B}|\bar{b}_v\left[iD_\rho, iD_\sigma\right]\left[iD_\lambda, iD_\delta\right]\left(-i\sigma_{\alpha\beta}\right)b_v|\bar{B}\rangle\;\Pi^{\alpha\rho}\Pi^{\beta\delta}v^\sigma v^\lambda \tag{4.56a}$$
$$2M_B\,m_6 = \langle\bar{B}|\bar{b}_v\left[iD_\rho, iD_\sigma\right]\left[iD_\lambda, iD_\delta\right]\left(-i\sigma_{\alpha\beta}\right)b_v|\bar{B}\rangle\;\Pi^{\alpha\sigma}\Pi^{\beta\lambda}\Pi^{\rho\delta} \tag{4.56b}$$
$$2M_B\,m_7 = \langle\bar{B}|\bar{b}_v\left\{\left\{iD_\rho, iD_\sigma\right\}, \left[iD_\lambda, iD_\delta\right]\right\}\left(-i\sigma_{\alpha\beta}\right)b_v|\bar{B}\rangle\;\Pi^{\sigma\lambda}\Pi^{\alpha\rho}\Pi^{\beta\delta} \tag{4.56c}$$
$$2M_B\,m_8 = \langle\bar{B}|\bar{b}_v\left\{\left\{iD_\rho, iD_\sigma\right\}, \left[iD_\lambda, iD_\delta\right]\right\}\left(-i\sigma_{\alpha\beta}\right)b_v|\bar{B}\rangle\Pi^{\rho\sigma}\Pi^{\alpha\lambda}\Pi^{\beta\delta} \tag{4.56d}$$
$$2M_B\,m_9 = \langle\bar{B}|\bar{b}_v\left[iD_\rho, \left[iD_\sigma, \left[iD_\lambda, iD_\delta\right]\right]\right]\left(-i\sigma_{\alpha\beta}\right)b_v|\bar{B}\rangle\;\Pi^{\rho\beta}\Pi^{\lambda\alpha}\Pi^{\sigma\delta}\,, \tag{4.56e}$$

where $m_5 = \tilde{m}_{11}, m_6 = \tilde{m}_9, m_9 = \tilde{m}_{10}$ and the hermitian combinations $m_7 = \tilde{m}_5 + \tilde{m}_7$ as well as $m_8 = \tilde{m}_6 + \tilde{m}_8$. Of course these parameters can also be related to physical quantities. But from this point on it is not trivial anymore. The QCD fields are non commutative and therefore in this dimension the physical interpretations are not straightforward. Naively you can assign the expectation value $\langle \boldsymbol{p}^4\rangle$ to m_1. We will see later, that this is important for the reparametrization-invariance argument. Using the fact, that the commutator of the covariant derivatives is proportional to the gluon field strength, and the scalar product with v_μ gives the time component, we notice that in analogy to electrodynamics the matrix element m_2 is proportional to $\langle \boldsymbol{E}^2\rangle$. Likewise

m_3 is proportional to $\langle \boldsymbol{B}^2 \rangle$, where $\boldsymbol{E}$ and $\boldsymbol{B}$ are again the chromo-electric and chromo-magnetic field operators. The matrix element m_4 has to be interpreted as $\langle \boldsymbol{p} \cdot \nabla \times \boldsymbol{B} \rangle$, where $\nabla\times$ is the covariant version of the curl as in the case of the Darwin term.

The spin dependent operators can be interpreted along the same lines[5]. We find that $m_5 \propto \langle \boldsymbol{s} \cdot \boldsymbol{E} \times \boldsymbol{E} \rangle$ and $m_6 \propto \langle \boldsymbol{s} \cdot \boldsymbol{B} \times \boldsymbol{B} \rangle$. For the other three ones we derive the relations $m_7 \propto \langle (\boldsymbol{s} \cdot \boldsymbol{p})(\boldsymbol{p} \cdot \boldsymbol{E}) \rangle$, $m_8 \propto \langle (\boldsymbol{s} \cdot \boldsymbol{B})(\boldsymbol{p})^2 \rangle$ and $m_9 \propto \langle \Delta(\boldsymbol{s} \cdot \boldsymbol{B}) \rangle$.

In the previous version [28, 29], four out of the nine parameters were omitted. In comparison with the physical interpretation we notice, that the commutativity was neglected as well as the derivative terms.

But in fact we have to be careful, since depending on the exact definition, there might actually be linear combinations. The next step is to identify the new parameters in dimension 8, which means expanding up to $1/m_b^5$. Here we naively have in total 29 parameters, out of which 10 are spin singlets and 19 spin tripletts. Using the above mentioned T-invariance argument, we reduce them to 18 parameters in total, 7 spin singlet and 11 spin triplet ones. Here we stick to the definition with keeping all four Lorentz components and introducing no commutators, since it is beyond any use to go further in the expansion. Redefining the parameters would only make a difference of higher order terms in $1/m_b$. The basis after imposing the T-invariance argument is given by

$$2M_B r_1 = \langle \bar{B} | \bar{b}_v \, iD_\rho \, (iv \cdot D)^3 \, iD^\rho \, b_v | \bar{B} \rangle \tag{4.57a}$$
$$2M_B r_2 = \langle \bar{B} | \bar{b}_v \, iD_\rho \, (iv \cdot D) \, iD^\rho \, iD_\sigma \, iD^\sigma \, b_v | \bar{B} \rangle \tag{4.57b}$$
$$2M_B r_3 = \langle \bar{B} | \bar{b}_v \, iD_\rho \, (iv \cdot D) \, iD_\sigma \, iD^\rho \, iD^\sigma \, b_v | \bar{B} \rangle \tag{4.57c}$$
$$2M_B r_4 = \langle \bar{B} | \bar{b}_v \, iD_\rho \, (iv \cdot D) \, iD_\sigma \, iD^\sigma \, iD^\rho \, b_v | \bar{B} \rangle \tag{4.57d}$$
$$2M_B r_5 = \langle \bar{B} | \bar{b}_v \, iD_\rho \, iD^\rho \, (iv \cdot D) \, iD_\sigma \, iD^\sigma \, b_v | \bar{B} \rangle \tag{4.57e}$$
$$2M_B r_6 = \langle \bar{B} | \bar{b}_v \, iD_\rho \, iD_\sigma \, (iv \cdot D) \, iD^\sigma \, iD^\rho \, b_v | \bar{B} \rangle \tag{4.57f}$$
$$2M_B r_7 = \langle \bar{B} | \bar{b}_v \, iD_\rho \, iD_\sigma \, (iv \cdot D) \, iD^\rho \, iD^\sigma \, b_v | \bar{B} \rangle \tag{4.57g}$$
$$2M_B r_8 = \langle \bar{B} | \bar{b}_v \, iD_\mu \, (iv \cdot D)^3 \, iD_\nu \, (-i\sigma^{\mu\nu}) \, b_v | \bar{B} \rangle \tag{4.57h}$$
$$2M_B r_9 = \langle \bar{B} | \bar{b}_v \, iD_\mu \, (iv \cdot D) \, iD_\nu \, iD_\rho \, iD^\rho \, (-i\sigma^{\mu\nu}) \, b_v | \bar{B} \rangle \tag{4.57i}$$
$$2M_B r_{10} = \langle \bar{B} | \bar{b}_v \, iD_\rho \, (iv \cdot D) \, iD^\rho \, iD_\mu \, iD_\nu \, (-i\sigma^{\mu\nu}) \, b_v | \bar{B} \rangle \tag{4.57j}$$
$$2M_B r_{11} = \langle \bar{B} | \bar{b}_v \, iD_\rho \, (iv \cdot D) \, iD_\mu \, iD^\rho \, iD_\nu \, (-i\sigma^{\mu\nu}) \, b_v | \bar{B} \rangle \tag{4.57k}$$
$$2M_B r_{12} = \langle \bar{B} | \bar{b}_v \, iD_\mu \, (iv \cdot D) \, iD_\rho \, iD_\nu \, iD^\rho \, (-i\sigma^{\mu\nu}) \, b_v | \bar{B} \rangle \tag{4.57l}$$
$$2M_B r_{13} = \langle \bar{B} | \bar{b}_v \, iD_\rho \, (iv \cdot D) \, iD_\mu \, iD_\nu \, iD^\rho \, (-i\sigma^{\mu\nu}) \, b_v | \bar{B} \rangle \tag{4.57m}$$
$$2M_B r_{14} = \langle \bar{B} | \bar{b}_v \, iD_\mu \, (iv \cdot D) \, iD_\rho \, iD^\rho \, iD_\nu \, (-i\sigma^{\mu\nu}) \, b_v | \bar{B} \rangle \tag{4.57n}$$
$$2M_B r_{15} = \langle \bar{B} | \bar{b}_v \, iD_\mu \, iD_\nu \, (iv \cdot D) \, iD_\rho \, iD^\rho \, (-i\sigma^{\mu\nu}) \, b_v | \bar{B} \rangle \tag{4.57o}$$
$$2M_B r_{16} = \langle \bar{B} | \bar{b}_v \, iD_\rho \, iD_\mu \, (iv \cdot D) \, iD_\nu \, iD^\rho \, (-i\sigma^{\mu\nu}) \, b_v | \bar{B} \rangle \tag{4.57p}$$
$$2M_B r_{17} = \langle \bar{B} | \bar{b}_v \, iD_\mu \, iD_\rho \, (iv \cdot D) \, iD^\rho \, iD_\nu \, (-i\sigma^{\mu\nu}) \, b_v | \bar{B} \rangle \tag{4.57q}$$
$$2M_B r_{18} = \langle \bar{B} | \bar{b}_v \, iD_\rho \, iD_\mu \, (iv \cdot D) \, iD^\rho \, iD_\nu \, (-i\sigma^{\mu\nu}) \, b_v | \bar{B} \rangle \, . \tag{4.57r}$$

[5] Please note that the components of the chromo-electric and chromo-magnetic fields do not commute.

Computational Strategy

We express the matrix element through the basis of all 16 gamma matrices. Then we make an usual tensor decomposition for each of the gamma matrix prefactors. Please note, that we have to keep parity conservation in mind! In order to determine the actual prefactors we make use of the orthogonality of the basis gamma matrices.

The decomposition is organized in such a way, that we start with the highest dimension. Then we may neglect all higher order corrections, which makes the matrix element static. Therefore we can pull the P_+ projector out of the fields, which then simplifies the necessary gamma matrix structure. The decomposition for the highest dimension then reads[6], where the spinor indices explicitely read $\alpha,,\beta$

$$\langle\bar{B}|\bar{b}_{v\,\alpha}(iD_\rho)(iD_\sigma)(iD_\lambda)(iD_\delta)(iD_\gamma)b_{v\,\beta}|\bar{B}\rangle = A^{(1)}_{\rho\sigma\lambda\delta\gamma}\,[P_+]_{\alpha\beta} + A^{(4)}_{\rho\sigma\lambda\delta\gamma\zeta\eta}\left[P_+(-i\sigma^{\zeta\eta})P_+\right]_{\alpha\beta}\,. \tag{4.58}$$

Using symmetry arguments, we can write down the tensor coefficients $A^{(1)}_{\rho\sigma\lambda\delta\gamma}$ and $A^{(4)}_{\rho\sigma\lambda\delta\gamma\zeta\eta}$. By considering the trace properties we can relate these coefficients to the basis parameters defined in (4.57). With this, some arbitrary B meson forward matrix-element with the appropriate number of background field covariant derivatives and some tensor combination of gamma matrices $\tilde{\Gamma}^{\rho\sigma\lambda\delta}$ can be calculated by

$$\langle\bar{B}|\bar{b}_v(iD_\rho)(iD_\sigma)(iD_\lambda)(iD_\delta)(iD_\gamma)\tilde{\Gamma}^{\rho\sigma\lambda\delta\gamma}b_v|\bar{B}\rangle = \sum_i \mathrm{Tr}\left\{\tilde{\Gamma}^{\rho\sigma\lambda\delta\gamma}\hat{\Gamma}^{(i)}\right\}A^{(i)}_{\rho\sigma\lambda\delta\gamma}\,, \tag{4.59}$$

where $\hat{\Gamma}^{(i)} = 1, \gamma_\mu, \gamma^5, \gamma_\mu\gamma^5, (-i\sigma^{\mu\nu})$ and $A^{(i)}_{\rho\sigma\lambda\delta}$ is the corresponding tensor structure.

The next step is to consider the next lower matrix elements recursively. These obtain higher order corrections from the equation of motions, therefore we have to consider all 16 gamma matrices as the basis

$$\begin{aligned}\langle\bar{B}|\bar{b}_{v,\alpha}(iD_\rho)(iD_\sigma)(iD_\lambda)(iD_\delta)b_{v,\beta}|\bar{B}\rangle = {}& A^{(1)}_{\rho\sigma\lambda\delta}1_{\alpha\beta} + A^{(2)}_{\rho\sigma\lambda\delta\zeta}\gamma^\zeta_{\alpha\beta} + A^{(3)}_{\rho\sigma\lambda\delta\zeta}\left[\gamma^\zeta\gamma^5\right]_{\alpha\beta}\\ &+ A^{(4)}_{\rho\sigma\lambda\delta\zeta\eta}\big(-i\sigma^{\zeta\eta}\big)_{\alpha\beta} + A^{(5)}_{\rho\sigma\lambda\delta}\gamma^5_{\alpha\beta}\,.\end{aligned} \tag{4.60}$$

We will basically do the same steps as in the previous case. But now by determining the tensor coefficients we get additional terms to the basis parameters, which may have higher order corrections by the equation of motions. Here the strength of this formalism becomes obvious. These corrections can easily be calculated, because we already have the general matrix element for the higher dimensions. Therefore the structure obtained by the e.o.m. is multiplied into (4.58) and after the trace has been taken, we immediately have the result.

In this manner we continue until we have computed the lowest dimension of 3. There the matrix element is fixed by the normalization condition

$$2M_B = \langle\bar{B}|\bar{b}_v\not{v}b_v|\bar{B}\rangle\,. \tag{4.61}$$

[6]This corresponds to the static limit in HQET.

This method is implemented in a MATHEMATICA program. The only thing to put in by hand is the ansatz for the tensor functions $A^{(i)}$ and the definition of the basis parameters. The structure of the higher order terms has to be calculated by hand, by using the e.o.m (4.46).

In this way, implementing new higher orders is straightforward. First step is to determine the basis parameters. Then we have to calculate the static trace formulae for the new highest dimension. The last step is to modify the second-highest term from the static limit to the general case.

Finally we put everything together to compute the forward matrix element

$$\begin{aligned} T_{\mu\nu} = \langle \bar{B}|\bar{b}_v \Gamma_\nu^\dagger i S_{\mathrm{BGF}} \Gamma_\mu b_v|\bar{B}\rangle &= \sum_i \mathrm{Tr}\left\{\Gamma_\nu^\dagger \frac{1}{\not{Q} - m_c}\Gamma_\mu\,\hat{\Gamma}^{(i)}\right\} A^{(i,0)} \\ &- \sum_i \mathrm{Tr}\left\{\Gamma_\mu^\dagger \frac{1}{\not{Q} - m_c}\gamma^{\mu_1}\frac{1}{\not{Q} - m_c}\Gamma_\mu\,\hat{\Gamma}^{(i)}\right\} A^{(i,1)}_{\mu_1} \\ &+ \sum_i \mathrm{Tr}\left\{\Gamma_\nu^\dagger \frac{1}{\not{Q} - m_c}\gamma^{\mu_1}\frac{1}{\not{Q} - m_c}\gamma^{\mu_2}\frac{1}{\not{Q} - m_c}\Gamma_\mu\,\hat{\Gamma}^{(i)}\right\} A^{(i,2)}_{\mu_1\mu_2} \mp \cdots \end{aligned} \tag{4.62}$$

For simplicity here the sum denotes $i = 1,\gamma_\mu,\gamma^5,\gamma_\mu\gamma^5,(-i\sigma^{\mu\nu})$. The imaginary part is then given by Cauchy's theorem, namely

$$-\frac{1}{\pi}\mathrm{Im}\left(\frac{1}{\Delta_0}\right)^{n+1} = \frac{(-1)^n}{n!}\delta^{(n)}\left((m_b v - q)^2 - m_c^2\right) , \tag{4.63}$$

where we have introduced $\Delta_0 = (m_b v - q)^2 - m_c^2 + i\epsilon$. In this way. we finally obtain the hadronic tensor in HQE:

$$\begin{aligned} W_{\mu\nu} &= -\frac{1}{\pi}\mathrm{Im}\langle B(p)|\bar{b}_v\Gamma_\nu^\dagger i S_{\mathrm{BGF}}\Gamma_\mu b_v|B(p)\rangle \\ &= \sum_i \mathrm{Tr}\left\{\Gamma_\nu^\dagger(\not{Q} + m_c)\Gamma_\mu\,\hat{\Gamma}^{(i)}\right\} A^{(i,0)}\delta\big((m_b v - q)^2 - m_c^2\big) \\ &+ \sum_i \mathrm{Tr}\left\{\Gamma_\nu^\dagger(\not{Q} + m_c)\gamma^{\mu_1}(\not{Q} + m_c)\Gamma_\mu\,\hat{\Gamma}^{(i)}\right\} A^{(i,1)}_{\mu_1}\delta^{(1)}\big((m_b v - q)^2 - m_c^2\big) \\ &+ \sum_i \mathrm{Tr}\left\{\Gamma_\nu^\dagger(\not{Q} + m_c)\gamma^{\mu_1}(\not{Q} + m_c)\gamma^{\mu_2}(\not{Q} + m_c)\Gamma_\mu\hat{\Gamma}^{(i)}\right\} A^{(i,2)}_{\mu_1\mu_2}\frac{\delta^{(2)}((m_b v - q)^2 - m_c^2)}{2} \\ &+ \cdots \\ &+ \sum_i \mathrm{Tr}\left\{\Gamma_\nu^\dagger(\not{Q} + m_c)\gamma^{\mu_1}(\not{Q} + m_c)\cdot\ldots\cdot(\not{Q} + m_c)\gamma^{\mu_m}(\not{Q} + m_c)\Gamma_\mu\,\hat{\Gamma}^{(i)}\right\} A^{(i,m)}_{\mu_1\mu_2\cdots\mu_m} \\ &\qquad \times \frac{\delta^{(m)}((m_b v - q)^2 - m_c^2)}{m!} . \end{aligned} \tag{4.64}$$

The bottleneck of this method are the proliferating non-perturbative parameters. The computation up to $1/m_b^5$ has been done explicitely, and to include even higher terms is possible. However, it is not realistic to think of extracting even the $1/m_b^4$ parameters out of the experiment. The situation is even worse, because we will see that a few terms in the total rate of $1/m_b^5$ are enhanced and have to be accounted for already at $\mathcal{O}(1/m_b^4)$. These terms have been dubbed "intrinsic charm". But we can give a quite good estimate on these new parameters in terms of the already known μ_π^2, μ_G^2 and ρ_D^3, ρ_{LS}^3 as well as some excitation energy $\bar{\epsilon}$. We use this later on, to validate the HQE and to proof the convergence explicitely. From this we can infer the uncertainty on the extraction of $|V_{cb}|$ due to dismissing higher orders in the HQE. One might think also to include some bias of the moments, which can be computed from more elaborated estimates of the parameters, in the fit. This could in principle improve the extraction of the lower-dimensional parameters and reduce the error of the fit.

4.2.2. Perturbative Corrections

So far we have only considered non-perturbative corrections which stem from the fact, that we do not observe free quarks in nature. Additionally, we have to take care about the perturbative corrections, which arise due to the expansion in the strong coupling constant α_s.

It is beyond the scope of this thesis, to deal with such corrections. Nevertheless some comments on this are useful.

The computations of the perturbative corrections to the partonic rate, which means the part that is proportional to $1/m_b^0$ in the non-perturbative expansion, is "straightforward", yet complicated. Within this calculation one computes the usual loop diagrams and solves the renormalization group equation. For the higher orders it gets more complicated. Each Wilson coefficient for a specific non-perturbative higher-order parameter has an expansion in α_s. To compute these Wilson coefficients, the loop diagrams including the expansion of the propagator have to be taken into account. Unfortunately the trick to overcome the ordering of the momentum expansion at tree-level does not work at the loop level. Therefore the x-gluon matrix elements have to be considered as well. These calculations are very involving, so only the α_s correction to μ_π^2 has been calculated,yet. In the case of this kinetic energy operator the advantage is, that this operator is completely symmetric and therefore no gluon matrix element appears, which simplifies the problem.

This total symmetric combination allows for an additional trick, which has also been pursuit. By boosting the total rate in some moving frame, we can reparametrize the momentum, which replaces the rest mass, according to the heavy-quark expansion. It has been shown, that with this trick, one can reproduce the non-perturbative expansion for the specific totally symmetric pieces. Now this is not restricted to tree-level only, it works the same way for the α_s corrections on the parton level, since this just modifies the Wilson coefficient. The symmetry guarantees that to all orders in α_s the coefficient in front of the totally symmetric non-perturbative parameters is the same as the parton-level coefficient. This has been used to cross-check the perturbative corrections to μ_π^2.

From this argument it is obvious, that there can only be one totally symmetric combination for an even expansion in $1/m_b$. Therefore in principle the α_s correction to the parameter m_1 is known, as well as the numerical results for α_s^2 for μ_π^2 and m_1.

With a little modification this holds also true for the moments and spectra.

4.2.3. Current Status

As we have seen before, the heavy-quark expansion is both an expansion in $1/m_b$, but also a perturbative expansion in the strong coupling for each Wilson coefficient. Therefore we have the usual α_s expansion for the parton level $1/m_b^0$. For the higher-order terms the computation is more difficult. There each non-perturbative parameter gets renormalized and we have to compute the loop diagrams including the non-perturbative operators. Unfortunately it is not possible to use the same method as described above. Therefore only the correction to μ_π^2 is known completely. The different terms, which have been calculated, are summarized in table 4.1. The filled bullets indicate, that

		$1/m_b^n$				
	n	0	2	3	4	5
α_s^n	0	●	●	●	● [28]	● [30]
	1	●	○ [31]	–	–	–
	2	● [32, 33]	–	–	–	–
	3	○ [34]	–	–	–	–

Table 4.1.: Current status of the calculation of inclusive semileptonic $b \to c$ decays.

these corrections have been computed completely, while for the circle ones some parts are missing.

Additionally we will see, that the expansion is actually a combined one in $1/m_b^k m_c^l$, where $l > 0$ starts with $k = 3$, such that

$$\Gamma_{3+n} \propto \sum_{i=0}^{n} \frac{1}{m_b^{3+n-i}} \left(\frac{1}{m_c^2}\right)^i \cdot \mathrm{NP(i)}\,, \tag{4.65}$$

where NP(i) are non-perturbative parameters. Without additional gluons, there can be only a even power of m_c. Numerically the power counting $m_c \sim \sqrt{\Lambda_{\mathrm{QCD}} m_b}$ suggests to sum up the different contributions for a fixed calculation up to $1/m_b^k$ according to table 4.2. Again all contributions with a filled bullet have been computed, while the other

		$1/m_b^n$					
	n	0	2	3	4	5	6
$1/m_c^n$	0	●	●	●	●	●	○
	2	–	–	●	○	○	○
	4	–	–	○	○	○	○
	6	–	–	○	○	○	○

Table 4.2.: Current status of the non-perturbative inclusive semi-leptonic calculation

ones have not. To have a consistent expansion, one should include all terms along the same color and line. The argument for only even powers of $1/m_c$ appearing is valid for tree-level calculations, only.

From this status it is obvious, that the non-perturbative corrections are well understood and under control. In this thesis we explain the subtle details and results, which came out of this calculations. Furthermore it is not even necessary to compute higher orders, from the reasons we state, than calculated here. It is clear, that a further improvement can only be gained from perturbative corrections. The correction to μ_G^2 should be completed. Furthermore the Darwin term ρ_D^3 plays an important role, so the correction to it would be desirable.

In the next section we will first derive an approximation scheme for the higher-order non-perturbative parameters. This is essential, in order to discuss the role of these corrections—the values of the parameters have not been measured, which might not even be possible. Then we will use this, to argue about the influence on the extraction of $|V_{cb}|$. Therefore we specify the numerical impact of the higher order terms on all observables in experiment. Then we discuss the special role of the charm quark inside this expansion in more detail and how the subtleties we have mentioned here emerge. We also give some results about weak annihilation matrix elements, which can be extracted from this analysis. Finally we give a short comment on non-perturbative $b \to s\gamma$, since some information from this transition is used in actual fits for extracting the value of $|V_{cb}|$ from inclusive decays.

4.3. Estimate of the Non-Perturbative Matrix Elements

As we have seen in the previous section, the independent hadronic matrix elements—the basic parameters—start to proliferate at order $1/m_b^4$. This would make their complete extraction from data by a fit difficult if not impossible. However, not all the corrections may be expected equally important, and neglecting numerically insignificant effects reduces the number of new parameters, possibly even to a manageable level. Yet this requires estimating the scale of the emerging expectation values well beyond naive dimensional guessing. Furthermore such a "guesstimate" provides a useful basis to judge the convergence behaviour of the series. Nevertheless for including the higher order parameters in the fit, using this factorization ansatz as a basis, we suggest to go beyond the leading-order approximation presented here.

In this chapter we are dealing with the heavy-quark expectation values at tree level appearing up to dimension 8, and in the following we present a method for estimating them based on the saturation by intermediate states.

In our approach we begin with the expectation value containing the full number of derivatives. We insert a sum over the full set of zero-recoil intermediate single-heavy-quark states to divide the composite operator into two ones with less derivatives each. Therefore we can express higher order matrix elements through the lower ones, which have already been determined. The approximation scheme we employ, is to retain the lowest allowed and contributing multiplet intermediate state, only. The accuracy of this approximation relies on how strong the lowest state is already saturated. The uncertainty due to this approximation may be estimated and by including the next higher term it may be refined. In the following pages we will restrict ourselves to the lowest state.

4.3.1. Intermediate State Representation for the Matrix Elements

Our goal is to estimate forward matrix elements of the form

$$\langle \bar{B}|\bar{b} iD_{\mu_1} iD_{\mu_2} \dots iD_{\mu_n} \Gamma\, b|\bar{B}\rangle \,, \tag{4.66}$$

where Γ denotes an arbitrary Dirac matrix. The representation is obtained by splitting the full chain $iD_{\mu_1} iD_{\mu_2} \dots iD_{\mu_n}$ into two ones, labeled by A_1^k and C_k^n

$$A_1^k = iD_{\mu_1} iD_{\mu_2} \dots iD_{\mu_k} \tag{4.67}$$
$$C_k^n = iD_{\mu_{k+1}} iD_{\mu_{k+2}} \dots iD_{\mu_n} \,. \tag{4.68}$$

Inserting a full set of intermediate states, we get

$$\langle \bar{B}|\bar{b} A_1^k\, C_k^n\, \Gamma\, b|\bar{B}\rangle = \frac{1}{2M_B} \sum_n \langle \bar{B}|\bar{b}\, A_1^k\, b(0)|n\rangle \cdot \langle n|\bar{b}(0)\, C_k^n\, \Gamma\, b|\bar{B}\rangle \,, \tag{4.69}$$

where we have assumed the B mesons to be static and at rest which means $|\bar{B}\rangle = |\bar{B}(p = (M_B, \mathbf{0}))\rangle$, and $|n\rangle$ are hadronic states with vanishing spatial momentum with a single b-quark content.

Now we will present an OPE-based proof[7] of equation (4.69). Therefore we introduce a ficticious heavy-quark Q which will be treated as static (i.e., $m_Q \to \infty$ and normalization point is much lower than m_Q), and consider a correlator of the form

$$T_{AC}(q_0) = -i \int \mathrm{d}^4 x\, e^{i q_0 x_0} \langle \bar{B}|T\left[\bar{b} A_1^k Q(x)\, \bar{Q} C_k^n \Gamma b(0)\right]|\bar{B}\rangle \,. \tag{4.70}$$

Note that the spatial momentum transfer $\boldsymbol{q}$ has been explicitly set to vanish and thus $v \cdot q \equiv q_0$ holds.

We shall use the static limit for both b and Q, and hence introduce the 'rephased' fields $\tilde{Q}(x) = e^{i m_Q q_0 x_0} Q(x)$ and likewise for b, and omit the tilde in what follows. Thus the b-quark field coincides with the previously defined b_v field. The form of the resulting exponent in eq. (4.70) suggests to define $\omega = q_0 - m_b + m_Q$ as the natural variable for T_{AC}, and $\frac{1}{2M_B} T_{AC}(\omega)$ is assumed to have a heavy-quark limit.

We can perform the OPE for $T_{AC}(\omega)$ at reasonably large m_Q and $|\omega| \gg \Lambda_{\mathrm{QCD}}$. We can still assume that $|\omega| \ll m_Q$ and thereby neglect all power corrections in $1/m_Q$. In this approximation the propagator of the Q field becomes static, which results in

$$T\left[Q(x)\bar{Q}(0)\right] = i \frac{1 + \not{v}}{2} \delta^3(\boldsymbol{x})\, \theta(x_0)\, P \exp\left(-i \int_0^{x_0} A_0(\tilde{x}_0)\, \mathrm{d}\tilde{x}_0 \right) , \tag{4.71}$$

[7] A similar proof based on effective-field-theory results in the same formula.

where A_0 is the time-component of the background field. Plugging this back into (4.70) we can perform the integration

$$\begin{aligned}T_{AC}(\omega) &= \int \mathrm{d}^4x\, e^{i\omega x_0} \langle \bar{B}|\bar{b}A_1^k \left[\frac{1+\not{v}}{2}\delta^3(\boldsymbol{x})\,\theta(x_0)\,P\,\exp\left(-i\int\limits_0^{x_0} A_0\mathrm{d}\tilde{x}_0\right)\right] C_k^n \Gamma b(0)|\bar{B}\rangle \\ &= \int \mathrm{d}x_0\, e^{i\omega x_0}\langle \bar{B}|\bar{b}A_1^k\,\theta(x_0)\,P\,\exp\left(-i\int\limits_0^{x_0} A_0\mathrm{d}\tilde{x}_0\right) C_k^n \frac{1+\not{v}}{2}\Gamma\, b|\bar{B}\rangle \\ &= \langle \bar{B}|\bar{b}\, A_1^k \frac{1}{-\omega-\pi_0-i\epsilon} C_k^n \frac{1+\not{v}}{2}\Gamma\, b|\bar{B}\rangle\,,\end{aligned} \tag{4.72}$$

where $\pi_0 = iD_0$ is the time component of the covariant derivative of the background field and P denotes the path ordering symbol. In the last step the integration over the zero component leads to the propagator with energy ω in the background field. This representation allows an immediate expansion of $iT_{AC}(\omega)$ in a series in $1/\omega$ at large $|\omega|$:

$$T_{AC}(\omega) = -\sum_{j=0}^{\infty} \langle \bar{B}|\bar{b}\, A_1^k \frac{(-\pi_0)^j}{\omega^{j+1}} C_k^n \frac{1+\not{v}}{2}\Gamma\, b|\bar{B}\rangle\,. \tag{4.73}$$

On the other hand we can derive the correlator using its dispersion relation to be able to insert a full set of intermediate states

$$T_{AC}(\omega) = -\frac{1}{\pi}\int\limits_0^{\infty} \mathrm{d}\nu\, \frac{1}{\nu-\omega+i\epsilon}\mathrm{Im}\, T_{AC}(\nu) \tag{4.74}$$

where the imaginary part is directly connected to the discontinuity due to the known causality prescription. To actually verify, that we cut across the correct branch cut, we state here that in the static limit the scattering amplitude has only one "physical" cut, which corresponds to positive ω.

The imaginary part is given by

$$\begin{aligned}\mathrm{Im}\, T_{AC}(\nu) &= -\frac{1}{2}\int \mathrm{d}^4x e^{i\nu x_0}\langle \bar{B}|\bar{b}A_1^k Q(x)\,\bar{Q}C_k^n\Gamma b(0)|\bar{B}\rangle \\ &= -\frac{1}{2}\sum_{n_Q}\int \mathrm{d}^4x\, e^{-i\boldsymbol{p}_n\cdot\boldsymbol{x}} e^{i(\nu-E_n)x_0}\langle \bar{B}|\bar{b}A_1^k Q(0)|n_Q\rangle\langle n_Q|\bar{Q}C_k^n\Gamma b(0)|\bar{B}\rangle\,,\end{aligned} \tag{4.75}$$

where we have inserted a complete set of states in the last step. The first line can be computed from equation (4.70) by inserting the time-ordering in an integral form. The sum runs over the complete set of the intermediate states $|n_Q\rangle$. Their overall spatial momentum is denoted by $\boldsymbol{p}_n$ and the corresponding energy by E_n. Please note, that the last relation is nothing else than the optical theorem for $T_{AC}(\omega)$, which we have already encountered before.

Since all operators do not depend on x we can directly perform the integration, which simply yields a delta distribution. For the spatial integration over d^3x we get $(2\pi)^3\delta^3(\boldsymbol{p}_n)$ and for the integration over time $\mathrm{d}x_0$ the result is $2\pi\,\delta(E_n-\nu)$, respectively. Therefore only the states with vanishing spatial momentum are projected out, and we denote them as $|n\rangle$ to be consistent with the representation we used before

$$\mathrm{Im}\,T_{AC}(\nu) = -\pi \sum_n \delta(\nu - E_n)\,\langle \bar{B}|\bar{b}A_1^k Q(0)|n\rangle\,\langle n|\bar{Q}C_k^n\Gamma b(0)|\bar{B}\rangle\,. \tag{4.76}$$

The normalization to zero spatial moment $(2\pi)^3\delta^3(\boldsymbol{p}_n)$ is put in the corresponding states with vanishing spatial momentum. Inserting the optical theorem relation (4.76) into the dispersion integral (4.74) yields

$$T_{AC}(\omega) = \sum_n \frac{\langle \bar{B}|\bar{b}A_1^k Q(0)|n\rangle\,\langle n|\bar{Q}C_k^n\Gamma b(0)|\bar{B}\rangle}{E_n - \omega + i\epsilon}\,, \tag{4.77}$$

where the ν integration is trivial due to the delta distribution. The large-ω expansion takes the form

$$T_{AC}(\omega) = -\sum_{j=0}^{\infty}\frac{1}{\omega^{j+1}}\sum_n E_n^j\,\langle \bar{B}|\bar{b}A_1^k Q(0)|n\rangle\,\langle n|\bar{Q}C_k^n\Gamma b(0)|\bar{B}\rangle\,. \tag{4.78}$$

Now we put together these two results by equating the leading power in $1/\omega$ of $T_{AC}(\omega)$ in the equations (4.73) and (4.78). We obtain the relation

$$\langle \bar{B}|\bar{b}A_1^k\,C_k^n\,\frac{1+\not{v}}{2}\,\Gamma\,b(0)|\bar{B}\rangle = \sum_n \langle \bar{B}|\bar{b}\,A_1^k\,Q(0)|n\rangle\cdot\langle n|\bar{Q}\;C_k^n\,\Gamma\,b(0)|\bar{B}\rangle\,, \tag{4.79}$$

which is the intermediate state representation (4.69), we wanted to proof. Note that the projector $(1+\not{v})/2$ on the left-hand side can be omitted since the static $\bar{b}$ field satisfies $\bar{b} = \bar{b}(1+\not{v})/2$.

Considering higher values of j in equations (4.73) and (4.78), which describe the sub-leading parts in $1/\omega$ of $T_{AC}(\omega)$, we immediately generalize the saturation relation (4.69)

$$\begin{aligned}\langle \bar{B}|\bar{b}A_1^k\,\pi_0^k\,C_k^n\frac{1+\not{v}}{2}\Gamma\frac{1+\not{v}}{2}\,b(0)|\bar{B}\rangle = \sum_n (E_B - E_n)^k\,\langle \bar{B}|\bar{b}\,A_1^k\,Q(0)|n\rangle \\ \times\langle n|\bar{Q}\,C_k^n\,\frac{1+\not{v}}{2}\Gamma\frac{1+\not{v}}{2}\,b(0)|\bar{B}\rangle\,.\end{aligned} \tag{4.80}$$

Thus each insertion of the operator $(-\pi_0)$, which is translated into $iv\cdot D$, inside a composite operator acts as a factor of the intermediate state excitation energy.

This is also expected, if we look at the equations of motions. On the one hand, for any colour singlet and the static quark field Q, the relation

$$\begin{aligned} i\partial_0\bar{Q}Cb(x) &= \bar{Q}(x)\left(i\overleftarrow{D}_0 C + CiD_0\right)b(x)) \\ &= \bar{Q}\pi_0 Cb(x)\end{aligned}$$

holds. On the other hand, the time-derivative is connected to the energy of the system. Taking the matrix element of the operator, we end up with

$$\begin{aligned} i\partial_0 \langle n|\bar{Q}Cb(x)|\bar{B}\rangle &= \langle n|\bar{Q}\left(i\overleftarrow{\partial}_0 C + Ci\partial_0\right) b(x)|\bar{B}\rangle \\ &= -(E_n - M_B)\langle n|\bar{Q}Cb(x)|\bar{B}\rangle \,. \end{aligned}$$

A couple of comments are in order before closing this subsection. Although we have phrased this consideration for the case of expectation values in B mesons at rest, these assumptions are not mandatory. The very same saturation by complete set of heavy-quark intermediate states of a given spatial momentum holds for matrix elements where initial and final states may be different, and may have non-vanishing momenta. But they do not have to be the ground pseudo-scalar states, but with arbitrary spin flavor content.

Likewise, it is worth noting that even the static approximation for b quarks is actually superfluous. The saturation by physical intermediate states relies solely upon large mass of the Q quarks which belong to this intermediate state. The only modification required for finite m_b is taking care of the static projectors $\frac{1+\not{v}}{2}$ introduced by the Q-quark propagator. Remember: The very same argument holds true in determining the "trace-formulae". Using the identity

$$1 = \frac{1+\not{v}}{2} + \gamma_5 \frac{1+\not{v}}{2}\gamma_5$$

we arrive at the following generalization:

$$\begin{aligned} &\langle\bar{B}|\bar{b}\,A_1^k\,C_k^n\,\Gamma\,b(0)|\bar{B}\rangle = \\ &\sum_n \left(\langle\bar{B}|\bar{b}\,A_1^k\,Q(0)|n\rangle\cdot\langle n|\bar{Q}\,C_k^n\,\Gamma\,b(0)|\bar{B}\rangle + \langle\bar{B}|\bar{b}\,A_1^k\,\gamma_5\,Q(0)|n\rangle\cdot\langle n|\bar{Q}\,C_k^n\,\gamma_5\Gamma\,b(0)|\bar{B}\rangle\right)\,. \end{aligned} \tag{4.81}$$

The similar relation between π_0 and the excitation energy is only modified by the mass shift of the finite-mass B meson:

$$\begin{aligned} &\langle\bar{B}|\bar{b}\,A_1^k\pi_0^j C_k^n\,\Gamma\,b(0)|\bar{B}\rangle = \sum_n\left[(M_B-m_b)-(M_n-m_Q)\right]^j\times \\ &\left(\langle\bar{B}|\bar{b}\,A_1^k\,Q(0)|n\rangle\cdot\langle n|\bar{Q}\,C_k^n\,\Gamma\,b(0)|\bar{B}\rangle + \langle\bar{B}|\bar{b}\,A_1^k\,\gamma_5\,Q(0)|n\rangle\cdot\langle n|\bar{Q}\,C_k^n\,\gamma_5\Gamma\,b(0)|\bar{B}\rangle\right)\,. \end{aligned} \tag{4.82}$$

We assume in the following practical application a generic excitation energy $-\bar{\epsilon} \equiv [(M_B - m_b) - (M_n - m_Q)]$, which has been averaged over the different contributing intermediate states. It is defined such, that $\bar{\epsilon} > 0$.

4.3.2. Lowest State Saturation Ansatz

So far the only approximation in our intermediate state representation (4.69), we have assumed, is the static limit for the b quark. It may be used to apply a dynamic QCD approximation. In the following we want to use the dimension 5 and 6 B meson heavy-quark expectation values of equation (4.66) as an input. Concretely they are given by μ_π^2, μ_G^2, ρ_D^3 and ρ_{LS}^3.

All operators with four and more derivatives, meaning dimension 7 and higher, must have an even number of spatial derivatives due to rotational invariance. Thus the operators with four derivatives have either four spatial derivatives, or two time and two spatial derivatives. Likewise, the $D = 8$ operators with five derivatives may have four spatial and a single derivative, or two spatial and three time derivatives.

We start with the $D = 7$ operators with four spatial derivatives, and apply (4.79). Now for convenience we label the "rephased" b-quark field again with b_v to be consistent with the definition of the basis matrix elements

$$\langle \bar{B}|\bar{b}_v\, iD_j iD_k iD_l iD_m \Gamma\, b_v|\bar{B}\rangle = \sum_n \langle \bar{B}|\bar{b}_v iD_j iD_k b_v|n\rangle\, \langle n|\bar{b}_v\, iD_l iD_m \Gamma\, b_v|\bar{B}\rangle\,. \tag{4.83}$$

The intermediate states $|n\rangle$ contributing are either the ground-state multiplet B, B^*, or excited states with the same content of light degrees of freedom. The ground-state factorization approximation assumes that the sum in (4.83) is to a large extent saturated by the ground-state spin-symmetry doublet. Hence we retain only the contribution of the ground state and discard the contribution of higher excitations. In the case of dimension seven operators the result is expressed in terms of the expectation values with two derivatives, i.e. μ_π^2 and μ_G^2. Matrix elements involving B^* are related to them by spin symmetry. We illustrate below the compact result of summation over the multiplet of states. The method we use is most economic and turns out particularly transparent when generalizing the ground-state approximation.

4.3.3. Scheme

Here we derive the concrete expressions for the sum over ground-state heavy-quark symmetry multiplet of meson states, encountered in the factorization approximation for the $D = 7$ and $D = 8$ expectation values (4.83). The trace formalism has been developed in particular to describe the heavy-quark spin multiplets at different velocities, e.g. in exclusive decays.

With the spin of the heavy-quark dynamically decoupled in the heavy-quark limit, the B and B^* meson wavefunctions at rest, M and $M_\lambda^{(*)}$, can be represented as matrices as we have already seen for the derivation of the Isgur-Wise function

$$M = \sqrt{M_B}\,\frac{1+\gamma_0}{2}\, i\gamma_5 \qquad M_\lambda^{(*)} = \sqrt{M_B}\,\frac{1+\gamma_0}{2}\,(\boldsymbol{\gamma}\boldsymbol{\epsilon}_\lambda), \tag{4.84}$$

where we have equated the B and B^* masses in the heavy-quark limit. One of the indices in these matrices corresponds to the heavy-quark spin and another to that of the light degrees of freedom. Spin symmetry yields the well known trace formula for the matrix elements:

$$\langle H'|\bar{b}iD_\mu iD_\nu \Gamma\, b|H\rangle = -\mathrm{Tr}\left[\bar{M}_{H'}\Gamma M_H \Lambda^{\mathrm{light}}_{\mu\nu}\right] \tag{4.85}$$

where H and H' are either B or B^*, $M_{H,H'}$ are the corresponding meson wavefunctions (4.84) and $\Lambda^{\mathrm{light}}_{\mu\nu}$ encodes dynamics associated with light degrees of freedom, e.g. the information on the non-perturbative parameters. While a complicated unknown hadronic tensor in the general situation, $\Lambda^{\mathrm{light}}_{\mu\nu}$ takes a simple form for mesons at rest, already known from the dimension 5 trace formula

$$\Lambda^{\mathrm{light}}_{\mu\nu}\Big|_{v\cdot v'=1} = -\frac{\mu_\pi^2}{3}\Pi_{\mu\nu} + \frac{\mu_G^2}{6} i\sigma_{\mu\nu}\,. \tag{4.86}$$

The time components vanish due to the applied heavy-quark limit.

In order to evaluate the ground-state contribution for the matrix element of dimension seven operators, containing four derivatives, we must sum over the states of the (B,B^*) spin-symmetry doublet. To do this we employ the relation valid for arbitrary R

$$-\frac{1}{2M_B}\mathrm{Tr}\left[R\bar{M}\right]M - \sum_\lambda \frac{1}{2M_B}\mathrm{Tr}\left[R\bar{M}^{(*)}_\lambda\right]M^{(*)}_\lambda = \frac{1+\gamma_0}{2}R\frac{1-\gamma_0}{2} \tag{4.87}$$

expressing the completeness of the B, B^* states in spin space. Note, that we project into the upper component and therefore the basis is reduced to four matrices instead of 16. In the next step we evaluate the right-hand side of (4.83) in the ground-state saturation, where we rely only on the first two states B and B^*. For the first excited state B^* we have to sum over the polarizations. With the trace formula (4.85) we express the two product matrix elements and use the relation (4.87) to simplify the expression

$$\begin{aligned}
&\frac{1}{2M_B}\left[\langle B|\bar{b}\,iD_j iD_k b|B\rangle\,\langle B|\bar{b}\,iD_l iD_m\Gamma\, b|B\rangle + \sum_\lambda \langle B|\bar{b}\,iD_j iD_k b|B^*_\lambda\rangle\,\langle B^*_\lambda|\bar{b}\,iD_l iD_m\Gamma\, b|B\rangle\right]\\
&= \frac{1}{2M_B}\left[\mathrm{Tr}\left[\bar{M}M\Lambda^{\mathrm{light}}_{jk}\right]\mathrm{Tr}\left[\bar{M}\Gamma M\Lambda^{\mathrm{light}}_{lm}\right] + \sum_\lambda \mathrm{Tr}\left[\bar{M}M^{(*)}_\lambda\Lambda^{\mathrm{light}}_{jk}\right]\mathrm{Tr}\left[\bar{M}^{(*)}_\lambda\Gamma M\Lambda^{\mathrm{light}}_{lm}\right]\right]\\
&= -\mathrm{Tr}\left[\bar{M}\frac{1+\gamma_0}{2}\Gamma M\Lambda^{\mathrm{light}}_{lm}\frac{1-\gamma_0}{2}\Lambda^{\mathrm{light}}_{jk}\right] = -\mathrm{Tr}\left[\bar{M}\Gamma M\Lambda^{\mathrm{light}}_{lm}\Lambda^{\mathrm{light}}_{jk}\right]\,.
\end{aligned} \tag{4.88}$$

The result can be verified by defining $R = \Gamma M\Lambda^{\mathrm{light}}_{lm}$ and noticing, that we can write the trace into components $\mathrm{Tr}\left[\bar{M}M^{(*)}\Lambda^{\mathrm{light}}_{jk}\right] \equiv \bar{M}_{\alpha\alpha'}M^{(*)}_{\alpha'\kappa}(\Lambda^{\mathrm{light}}_{jk})_{\kappa\alpha}$, where we assume summation over repeated indices. In the last step the static projectors have been reabsorbed, which is valid in our approximation. The product of the two hadronic tensors Λ^{light} in (4.86) can be decomposed into

$$\begin{aligned}
\Lambda^{\mathrm{light}}_{lm}\Lambda^{\mathrm{light}}_{jk} = &\frac{(\mu_\pi^2)^2}{9}g_{jk}g_{lm} - \frac{\mu_\pi^2\mu_G^2}{18}(g_{jk}i\sigma_{lm} + i\sigma_{jk}g_{lm}) +\\
&\frac{(\mu_G^2)^2}{36}(g_{jm}g_{kl} - g_{jl}g_{km} + g_{jm}i\sigma_{kl} - g_{jl}i\sigma_{km} + i\sigma_{jm}g_{kl} - i\sigma_{jl}g_{km})\,.
\end{aligned} \tag{4.89}$$

In the static limit we therefore obtain for spin-singlet and spin-triplet B expectation values

$$\frac{1}{2M_B}\langle B|\bar{b}\, iD_j iD_k iD_l iD_m\, b|B\rangle = \frac{(\mu_\pi^2)^2}{9} g_{jk}g_{lm} + \frac{(\mu_G^2)^2}{36}\left(g_{jm}g_{kl} - g_{jl}g_{km}\right) \quad (4.90)$$

$$\frac{1}{2M_B}\langle B|\bar{b}\, iD_j iD_k iD_l iD_m\,(-i\sigma_{ab})\, b|B\rangle = -\frac{\mu_\pi^2\mu_G^2}{18}\big(g_{jk}g_{la}g_{mb} - g_{jk}g_{lb}g_{ma} + g_{lm}g_{ja}g_{kb} - g_{lm}g_{jb}g_{ka}\big) + \frac{(\mu_G^2)^2}{36}\big[g_{jm}(g_{lb}g_{ka} - g_{la}g_{kb}) - g_{jl}(g_{ka}g_{mb} - g_{kb}g_{ma}) + g_{kl}(g_{ja}g_{mb} - g_{jb}g_{ma}) - g_{km}(g_{ja}g_{lb} - g_{jb}g_{la})\big]\,, \quad (4.91)$$

from which we can compute the factorization estimate of the dimension seven matrix elements.

For the $D = 8$ operators with four spatial and one time derivative we follow the same route, yet one of the two Λ^{light} now describes the operator of the form $\bar{b}iD_j iD_0 iD_k\Gamma b$. Denoting the corresponding hadronic tensor of light degrees of freedom by $\mathcal{R}^{\text{light}}_{\mu\nu}$, we express it through the Darwin and Spin-Orbit expectation values:

$$\mathcal{R}^{\text{light}}_{\mu\nu}\Big|_{v\cdot v'=1} = \frac{\rho_D^3}{3}\Pi_{\mu\nu} + \frac{\rho_{LS}^3}{6} i\sigma_{\mu\nu}\,. \quad (4.92)$$

Please note, that it is fully symmetric with respect to which Λ^{light} we exchange. This reflects the fact, that the observable matrix elements obey T-parity. Repeating the same steps we end up with

$$\frac{1}{2M_B}\langle B|\bar{b}\, iD_j iD_0 iD_k iD_l iD_m\, b|B\rangle = -\frac{\rho_D^3\mu_\pi^2}{9} g_{jk}g_{lm} + \frac{\rho_{LS}^3\mu_G^2}{36}\left(g_{jm}g_{kl} - g_{jl}g_{km}\right) \quad (4.93)$$

$$\frac{1}{2M_B}\langle B|\bar{b}\, iD_j iD_0 iD_k iD_l iD_m(-i\sigma_{ab})\, b|B\rangle = \frac{\rho_D^3\mu_G^2}{18} g_{jk}(g_{la}g_{mb} - g_{lb}g_{ma}) - \frac{\rho_{LS}^3\mu_\pi^2}{18} g_{lm}(g_{ja}g_{kb} - g_{jb}g_{ka}) + \frac{\mu_G^2\rho_{LS}^3}{36}\big[g_{jm}(g_{lb}g_{ka} - g_{la}g_{kb}) - g_{jl}(g_{ka}g_{mb} - g_{kb}g_{ma}) + g_{kl}(g_{ja}g_{mb} - g_{jb}g_{ma}) - g_{km}(g_{ja}g_{lb} - g_{jb}g_{la})\big]\,. \quad (4.94)$$

We are now able to obtain all required factorization results in the following section, using the input values in Table 4.3.

Parameter	μ_π^2	μ_G^2	ρ_D^3	ρ_{LS}^3	$\bar{\epsilon}$
Value	$0.45\,\text{GeV}^2$	$0.35\,\text{GeV}^2$	$0.15\,\text{GeV}^3$	$-0.15\,\text{GeV}^3$	$0.4\,\text{GeV}$

Table 4.3.: Input values for the factorization estimate.

4.3.4. Summary for $\mathcal{O}(\Lambda^4_{\text{QCD}})$ and $\mathcal{O}(\Lambda^5_{\text{QCD}})$ Expectation Values

Combining the above relations we can evaluate all the required non-perturbative parameters at order $1/m_b^4$ and $1/m_b^5$ in terms of a few parameters μ_π^2, μ_G^2, ρ_D^3, ρ_{LS}^3 and $\bar{\epsilon}$ taken as an input. The input values are listed in table 4.3. Table 4.4 lists the resulting expressions for the expectation values in dimension 7, and give the corresponding numerical estimates. Tables 4.5 and 4.6 list the according numbers and expressions for the dimension 8 singlet and triplet parameters.

Singlet parameter	m_1	m_2	m_3	m_4	
Fact. estimate	$\frac{5}{9}\left(\mu_\pi^2\right)^2$	$-\bar{\epsilon}\rho_D^3$	$-\frac{2}{3}\left(\mu_G^2\right)^2$	$\left(\mu_G^2\right)^2+\frac{4}{3}\left(\mu_\pi^2\right)^2$	
Value / GeV^4	0.113	-0.06	-0.82	0.393	
Triplet parameter	m_5	m_6	m_7	m_8	m_9
Fact. estimate	$-\bar{\epsilon}\rho_{LS}^3$	$\frac{2}{3}\left(\mu_G^2\right)^2$	$-\frac{8}{3}\mu_G^2\mu_\pi^2$	$-8\mu_G^2\mu_\pi^2$	$\left(\mu_G^2\right)^2-\frac{10}{3}\mu_G^2\mu_\pi^2$
Value / GeV^4	0.060	0.082	-0.420	-1.260	-0.403

Table 4.4.: Estimate for the dimension 7 parameters.

The tabulated values are of course no precision predictions. We now point out the limitations of accuracy. First, the expressions depend on μ_π^2, μ_G^2, ρ_D^3, ρ_{LS}^3 which are themselves only known with limited precision. The same holds true for the value of $\bar{\epsilon}$, which should be seen as an effective averaged parameter in our case. This aspect, however, is easy to quantify using the expression in the tables 4.4, 4.5 and 4.6.

Parameter	r_1	r_2	r_3	r_4
Fact. estimate	$\bar{\epsilon}_0^2\rho_D^3$	$-\mu_\pi^2\rho_D^3$	$-\frac{1}{3}\mu_\pi^2\rho_D^3-\frac{1}{6}\mu_G^2\rho_{LS}^3$	$\rho_D^3\bar{\epsilon}_0^2-\frac{1}{3}\mu_\pi^2\rho_D^3+\frac{1}{6}\mu_G^2\rho_{LS}^3$
Value / GeV^4	0.024	-0.068	-0.014	-0.007
Parameter	r_5	r_6	r_7	r_8
Fact. estimate	0	$\bar{\epsilon}^2\rho_D^3$	0	$\bar{\epsilon}^2\rho_{LS}^3$
Value / GeV^4	0	0.024	0	-0.024

Table 4.5.: Estimate for the dimension 8 singlet parameters.

Secondly, we are using heavy-quark effective theory, which means we treat the b quarks as well as the “intermediate” b quarks in the infinite mass limit, while the definition of the parameters in the HQE include the full QCD fields, meaning the full mass dependence. Generically such finite-mass corrections are governed by the parameter $\mu_{\text{hadr}}/2m_b$ and can be sizable up to 15% in B hadrons [35]. However, there is a specific suppression of such pre-asymptotic correction in the ground-state pseudoscalar mesons related to the observed proximity of these states to the so-called ‘BPS’ regime [36]. Since we deal here exclusively with the pseudo-scalar ground state, we expect the finite mass corrections to be substantially smaller.

Parameter	r_9	r_{10}
Fact. estimate	$-\mu_\pi^2 \rho_{LS}^3$	$\mu_G^2 \rho_D^3$
Value / GeV^4	0.068	0.053
Parameter	r_{11}	r_{12}
Fact. estimate	$\frac{1}{3}\mu_G^2\rho_D^3 - \frac{1}{6}\left(\mu_G^2 + 2\mu_\pi^2\right)\rho_{LS}^3$	$-\frac{1}{3}\mu_G^2\rho_D^3 - \frac{1}{6}\left(\mu_G^2 - 2\mu_\pi^2\right)\rho_{LS}^3$
Value / GeV^4	0.004	0.014
Parameter	r_{13}	r_{14}
Fact. estimate	$-\frac{1}{3}\mu_G^2\rho_D^3 + \frac{1}{6}\left(\mu_G^2 + 2\mu_\pi^2\right)\rho_{LS}^3$	$\rho_{\mathrm{LS}}^3\epsilon_0^2 + \frac{1}{3}\mu_G^2\rho_D^3 + \frac{1}{6}\mu_G^2\rho_{\mathrm{LS}}^3 - \frac{1}{3}\mu_\pi^2\rho\_\mathrm{LS}^3$
Value / GeV^4	-0.049	0.007
Parameter	r_{15}	r_{16}
Fact. estimate	0	0
Value / GeV^4	0	0
Parameter	r_{17}	r_{18}
Fact. estimate	$\epsilon_0^2\rho_{\mathrm{LS}}^3$	0
Value / GeV^4	-0.024	0

Table 4.6.: Estimate for the dimension 8 triplet parameters.

In all the following numerical results of terms with dimension higher or equal seven, we will use the estimated for the non-perturbative parameters above. For the lower dimensional terms, we use the values quoted in Table 4.3

Some Comments on the Validity of this Approximation

The major issue is thus the validity of the employed approximation for the matrix elements. In particular the question should be raised, if retaining only to the relevant lowest states, is valid for these operators. The degree to which this ansatz is applicable, depends on the operator in question. It is expected to deteriorate when the number of derivatives, or in other words the dimension of the operator increases.

We now present some argumentation to estimate the relative accuracy for the expectation values. Please note, that in general one would expect these expectation values to scale like $[\Lambda_{\mathrm{QCD}}]^n$, which is indeed correct up to the measured ones in dimension 6. However, the definition of the higher order matrix elements is not physically meaningful per se. Since we include multiple commutators, we had to put the correct normalization factors in front. Because we did not, some of the parameters seem to have an unnaturally large value. Thus the choice of the basis directly influences the shifting of the strength between Wilson coefficient and operator, as it is expected for obvious reasons. More over the sign of a parameter cannot be guessed. The significance, both of the operators themselves and of the uncertainties in their evaluation, must therefore be gauged by the effect on the moments.

The dominance of the lowest state and suppression of transitions into highly excited states typically holds for the bound states with a smooth potential. In field theory for heavy-light mesons this question was studied [37] in two-dimensional QCD, the so called 't Hooft model, which is exactly solvable in the limit of a large number of colors. In particular, the ground-state expectation values for operators with two spatial derivatives were found to be saturated by the first 'P-wave' state to an amazing degree of accuracy. We should remind that there is no spin in $1+1$ dimensions, therefore only one, not two P-wave families. In case this also applies to real QCD we would expect a good accuracy of the employed factorization ansatz for the operators containing two spatial derivatives m_2 and m_5 at order $1/m_b^4$ and a reasonable one for r_1, r_6, r_8, r_{17} at order $1/m_b^5$, which have three spatial derivatives. It might apply also to some other related combinations.

The situation is a priori less clear with the other operators in dimension seven, containing four spatial derivatives. In the 't Hooft model the effects related to deviations from the ground-state factorization were studied and were found to be nearly saturated, again to a very good degree, by the first 'radial' excitation [37].[8] For some of the expectation values at order $1/m_b^4$ they can be estimated following the reasoning of Refs. [35, 38]. The non-factorizable contributions taken at face value appear to be about 50% of the ground-state one [39].

Additionally we have to think about renormalization and corrections to this operators. In reality, the effective interaction in full QCD is singular at short distances due to perturbative physics. Hard gluon corrections lead to a slow decrease of the transitions to the highly excited states—yet they are dual to perturbation theory. This is taken care of in the Wilsonian renormalization procedure, which is assumed in the kinetic scheme and adopted for the common analysis of these decays. From this perspective, one can say that the factorization ansatz yields the expectation values at a low normalization point $\mu < \epsilon_{\rm rad} \approx 0.6\,\text{GeV}$, before the channels to radially excited states open up; the excitation energy $\epsilon_{\rm rad}$ for such lowest resonance states is probably around $600-700\,\text{MeV}$. Of course, the actual μ-dependence at such low scale does not coincide with the one derived from perturbation theory. At intermediate excitation energies it must be in some respect dual to the perturbative one.

Keeping this in mind it should be appreciated that even for definitely positive correlators, or those expectation values where all intermediate states contribute with the same sign, simply adding the first excitation to the ground state contribution may already constitute some overshooting. Indeed, we may consider the excitation energy

[8] The contribution itself turned out quite significant if normalized literally to $(\mu_\pi^2)^2$, apparently since μ_π^2 is anomalously small in this case, due to a factor of 3, which is the number of space dimensions. If normalized to $\overline{\Lambda}^4$ it was about 3/4. In actual B mesons μ_π^2 is close to $\overline{\Lambda}^2$.

of the i-th radially excited state $\epsilon_{\rm rad}^{(i)}$ and compare it to perturbative contributions. A resonance state residing at mass $\epsilon_{\rm rad}^{(1)}$ may be dual to the perturbative contribution over the domain of masses

$$\frac{\epsilon_{\rm rad}^{(1)}}{2} < \varepsilon < \frac{\epsilon_{\rm rad}^{(1)} + \epsilon_{\rm rad}^{(2)}}{2} \tag{4.95}$$

and likewise for higher resonances. Then a better approximation for the expectation value normalized at $\mu = \epsilon_{\rm rad}^{(1)}$ would be to add only half of the first excitation contribution. In practical terms, as long as the power mixing in the perturbative corrections to the conventional Wilson coefficients, most notably of the unit operator, has not been extended to order $1/m_b^4$ and $1/m_b^5$, the effective non-factorizable piece may turn out even less.

An additional feature of actual QCD is the existence of the low-mass continuum contribution beyond pure resonances, most notably states like $B^{(*)}\pi$ and their $SU(3)$ siblings. Their contribution is typically $1/N_c$ suppressed and usually does not produce a prominent effect in quantities which are finite in the chiral limit. This is true for the corrections to factorization for higher-dimension expectation values. They can be expected to contribute up to 25% of the ground state, yet this may be partially offset once the actual QCD conventional expectation values are used in the factorization ansatz, that likewise incorporate such states [38].

Considering all these arguments, we tentatively assign the uncertainty in the factorization estimate to be at the scale of 50%. Yet this should be understood to apply to the "positive" operators where the lowest-state contribution does not vanish and the excited state multiplets yield the same-sign contribution. The corrections to factorization will be addressed in more detail in the forthcoming paper [39].

4.4. Higher Order Corrections in the Rate and Moments

Armed with the numerical estimates of all the required expectation values we are in the position to evaluate the higher-order power corrections to inclusive B decays. The primary quantity of interest is the total semileptonic width $\Gamma_{\rm sl}(b \to c\ell\bar{\nu}_\ell)$ used for the precision extraction of $|V_{cb}|$. The lepton energy spectrum and the lepton energy moments, as well as the hadronic invariant mass moments, which are normalized and thus $|V_{cb}|$ drops out, are used to pin point the non-perturbative parameters and the quark masses. Additionally information from $B \to X_s\gamma$ is used, because these parameters do not depend on the final state [40]. We will apply these corrections also to this decay in a separate section 4.6.

4.4.1. $\Gamma(B \to X_c\,\ell\nu)$

Assuming the fixed values of m_b, m_c and the non-perturbative parameters as tabulated in the last section, we find the following power corrections at different orders in $1/m_b$.

We define $\delta\Gamma_{1/m^k}$ as all terms in the total rate contributing only at the specific order $1/m^k$, including the non-perturbative parameters, in the expansion. The shifts induced by the already known terms, which are included in experiment are given by:

$$\frac{\delta\Gamma_{1/m^2}}{\Gamma_{\rm parton}} = -0.043 \qquad \frac{\delta\Gamma_{1/m^3}}{\Gamma_{\rm parton}} = -0.030\,. \tag{4.96}$$

We estimate the contributions from higher-order terms in the expansion to be

$$\frac{\delta\Gamma_{1/m^4}}{\Gamma_{\rm parton}} = 0.0075 \qquad \frac{\delta\Gamma^{\rm IC}}{\Gamma_{\rm parton}} = 0.007 \qquad \frac{\delta\Gamma_{1/m^5}}{\Gamma_{\rm parton}} = 0.006\,, \tag{4.97}$$

where $\Gamma_{\rm parton}$ includes the phase space suppression factor $(1-8\rho+\ldots)$ of approximately 0.63. We will show separately how the contribution scaling like $1/m_b^3 m_c^2$ arise, which is here denoted by $\delta\Gamma^{\rm IC}$ in section 4.5. As anticipated [41], it dominates the higher-order effects and may even exceed the $1/m_b^4$ correction, yet it is to some extent offset by the regular $1/m_b^5$ terms.

The numerical results (4.97) suggest that the power series for $\Gamma_{\rm sl}(b \to c\ell\bar{\nu}_\ell)$ is well behaved, and is under good numerical control, provided the non-perturbative expectation values are known or at least estimated at a reasonable precision. Higher-order terms induce decreasing corrections except where anticipated on theoretical grounds. The estimated overall shift due to higher-order terms

$$\frac{\delta\Gamma_{1/m^4} + \delta\Gamma_{1/m^5}}{\Gamma_{\rm parton}} \simeq 0.013 \tag{4.98}$$

is well within the interval assessed in [41] and, taken at face value, would yield a 0.65% *direct* reduction in $|V_{cb}|$.

But this is not to be the whole story however, because the quark masses, which determine the partonic width, are not known beforehand with an accuracy required to extract $|V_{cb}|$ with the percent precision. Rather, as we have mentioned earlier, their relevant combination is extracted from the fit to the data on kinematic moments of the $B \to X_c\,\ell\bar{\nu}_\ell$ decay distributions, that are also affected by power corrections[9]. Therefore we investigate these moments in turn.

4.4.2. Moments

The key in the OPE evaluation of $\Gamma_{\rm sl}(b \to c\ell\bar{\nu}_\ell)$ and, therefore, in the extraction of $|V_{cb}|$ are the first moments of lepton energy $\langle E_\ell\rangle$ and of hadron invariant mass squared $\langle M_X^2\rangle$, which pinpoint the precision value of the combination of m_b and m_c, that drives the total decay probability. Moreover, analyzed through order $1/m_b^3$, these two moments turned out to depend on nearly the same combination of the heavy-quark parameters. This allowed for a non-trivial cross check of the OPE-based theory prediction [42]: Essentially, $\langle M_X^2\rangle$ could be predicted in terms of $\langle E_\ell\rangle$ and vice versa, once the heavy-quark parameters were allowed to vary within theoretically acceptable range.

[9]In fact higher moments are more affected by these higher order power-corrections

Therefore, besides the practical question about the shift in the fitted heavy-quark parameters, another important issue emerges of whether the consistency between measured values of $\langle E_\ell \rangle$ and of $\langle M_X^2 \rangle$ persists once higher-order corrections are accounted for. Numerically we find

$$\delta\langle E_\ell \rangle = 0.013 \text{ GeV}, \qquad \delta\langle M_X^2 \rangle = -0.086 \text{ GeV}^2 \tag{4.99}$$

where the changes shown are a combined contribution of corrections to order $1/m_b^4$ and $1/m_b^5$. In our analysis we follow Ref. [43] and evaluate the moments $\mathcal{M}$ as the ratios

$$\frac{\int \mathrm{d}E_\ell \, \mathrm{d}v \cdot q \, \mathrm{d}q^2 K(E_\ell, v \cdot q, q^2) \, C(E_\ell, v \cdot q, q^2) \, \frac{\mathrm{d}^3\Gamma_{\rm sl}}{\mathrm{d}E_\ell \mathrm{d}v \cdot q \mathrm{d}q^2}}{\int \mathrm{d}E_\ell \, \mathrm{d}v \cdot q \, \mathrm{d}q^2 C(E_\ell, v \cdot q, q^2) \, \frac{\mathrm{d}^3\Gamma_{\rm sl}}{\mathrm{d}E_\ell \mathrm{d}v \cdot q \mathrm{d}q^2}} \tag{4.100}$$

without prior expanding the ratio itself in $1/m_b$. Here K is the corresponding kinematic observable in question, meaning powers of E_ℓ or M_X^2 and C is an explicit kinematic cut if imposed. The integrals both in denominator and in numerator are taken directly as obtained in the OPE through the corresponding terms in $1/m_b$. In particular, powers of $(M_B - m_b)$ for hadronic mass moments effectively are *not* treated as power suppressed. Throughout this analysis we perform numeric evaluation of higher-order–induced changes in observables discarding perturbative corrections altogether. This turned out to be a good approximation in the kinetic scheme.

To assess the practical significance of the numerical changes in the moments, we compare the result including all higher order terms with the result obtained with the values of "conventional" heavy-quark parameters used in the OPE so far, only. Thereby we analyze the size of the shift in the heavy quark parameters, that would accommodate for this new effects, if we consider the expansion only up to dimension six.

Specifically, we choose to consider the variations in m_b, μ_π^2 and ρ_D^3 for this purpose, since these turn out to be most relevant in the analysis so far. We assume changes of the order $|\delta m_b| \gtrsim 10$ MeV, $|\delta\mu_\pi^2| \gtrsim 0.1$ GeV^2 and $|\delta\rho_D^3| \gtrsim 0.1$ GeV^3 as significant, bearing in mind the estimated accuracy of the existing OPE predictions [43]. We do not include m_c here[10] for the following reason: The quark mass dependence of the moments is essentially given by the combination $m_b - 0.7m_c$. The interval allowed by the fits on individual values of m_b and m_c separately, is much wider than 10 MeV. Therefore in practice the required variation in m_c is not independent and is derived from the corresponding variation in m_b. The stated significance of $\delta m_b \sim 10$ MeV refers, in fact, to the above combination of masses.

In the actual fits, the effect of newly implemented higher-order corrections has to be compensated by a simultaneous change in all heavy-quark parameters, to accomodate the experimental values. To visualise this, we quote for each moment $\mathcal{M}^{(k)}$, which is computed up to $1/m_b^{k-3}$ and $\delta\mathcal{M}^{(j)} = \mathcal{M}^{(j)} - \mathcal{M}^{(6)}$ denotes the shift due to higher-order terms of dimension 7 to j, the separate values

$$\delta m_b = -\frac{\delta\mathcal{M}^{(8)}}{\frac{\partial\mathcal{M}^{(6)}}{\partial m_b}}, \qquad \delta\mu_\pi^2 = -\frac{\delta\mathcal{M}^{(8)}}{\frac{\partial\mathcal{M}^{(6)}}{\partial\mu_\pi^2}}, \qquad \delta\rho_D^3 = -\frac{\delta\mathcal{M}^{(8)}}{\frac{\partial\mathcal{M}^{(6)}}{\partial\rho_D^3}}. \tag{4.101}$$

[10]This implies we keep it fixed throughout the analysis

This assumes the extreme situation, as if only one of the HQE parameters were responsible for the adjustment and if the higher-order effect had been made up for, completely. These ad hoc assumption shall just indicate the influence of the terms. Clearly, if a particular shift in a heavy-quark parameter comes out abnormally large, it should simply be discarded: This only signals that the moment in question is insensitive to this parameter and the moment rather constrains other OPE parameters. On the contrary, if a shift is small, this generally means that the parameter is well-constrained and, typically, should not be adjusted. In a sense, the values in equation (4.101) would assume that the quality of the fit before including the calculated corrections has been perfect, and this clearly is oversimplification. Yet this is suitable to gauge the significance of the corrections we study.

The dependence of the considered moments upon heavy-quark parameters, which enter the denominators in equation (4.101) is given in a ready-to-use form in Ref. [43], or can be deduced from the analytical results of our calculation. From that we obtain the relative shifts

$$\begin{aligned}\langle E_\ell\rangle : \quad & \delta m_b = -33\text{ MeV}\,, & \delta\mu_\pi^2 &= -0.39\text{ GeV}^2\,, & \delta\rho_D^3 &= 0.15\text{ GeV}^3 \\ & \Rightarrow \frac{\delta|V_{cb}|}{|V_{cb}|} = 0.022 & \Rightarrow \frac{\delta|V_{cb}|}{|V_{cb}|} &= -0.005 & \Rightarrow \frac{\delta|V_{cb}|}{|V_{cb}|} &= 0.014 \end{aligned} \tag{4.102a}$$

$$\begin{aligned}\langle M_X^2\rangle : \quad & \delta m_b = -17\text{ MeV}\,, & \delta\mu_\pi^2 &= -0.12\text{ GeV}^2\,, & \delta\rho_D^3 &= 0.086\text{ GeV}^3 \\ & \Rightarrow \frac{\delta|V_{cb}|}{|V_{cb}|} = 0.011 & \Rightarrow \frac{\delta|V_{cb}|}{|V_{cb}|} &= -0.0015 & \Rightarrow \frac{\delta|V_{cb}|}{|V_{cb}|} &= 0.008\,. \end{aligned} \tag{4.102b}$$

The dependence of $\Gamma_{\rm sl}(B)$ on heavy-quark parameters has also been carefully studied [41, 43, 44]. We have supplemented the corresponding relative shift in $|V_{cb}|$ by the variation of the heavy-quark parameters according to

$$\frac{\delta|V_{cb}|}{|V_{cb}|} = -\frac{1}{2}\frac{1}{\Gamma_{\rm sl}}\frac{\partial\Gamma_{\rm sl}}{\partial\,\text{HQP}}\,\delta\,\text{HQP}\,, \qquad \frac{1}{\Gamma_{\rm sl}}\frac{\partial\Gamma_{\rm sl}}{\partial\,\text{HQP}} = \begin{cases} 0.0013\text{ MeV}^{-1} & \text{HQP} = m_b \\ -0.026\text{ GeV}^{-2} & \text{HQP} = \mu_\pi^2 \\ -0.18\text{ GeV}^{-3} & \text{HQP} = \rho_D^3 \end{cases}. \tag{4.103}$$

The equations (4.102) suggest that one of the possible 'solutions' is an increase in ρ_D^3 by about 0.1 GeV^3. This value may additionally be affected by a variation in μ_π^2 of the scale of 0.05 GeV^2 or in m_b about 10 MeV. The relevance of this solution essentially depends on how precisely the presently fitted standard heavy-quark parameters accommodate both $\langle E_\ell\rangle$ and $\langle M_X^2\rangle$.

In actuality, the most precise measurements coming from the threshold production of B mesons at B-factories require a lower cut on the lepton energy. Therefore, the proper analysis of the moments should include a E_ℓ cut around 1 GeV.

Fig. 4.2 a–f show the size of the non-perturbative OPE terms for the first three central lepton energy and hadron mass squared moments, depending on the lower cut $E_\ell^{\rm cut}$ on the charged lepton energy. Different orders in $1/m_b$ have distinct colours.[11]

[11] As before, the explicit factor $M_B - m_b$ in hadronic mass moments does not count as power suppressed.

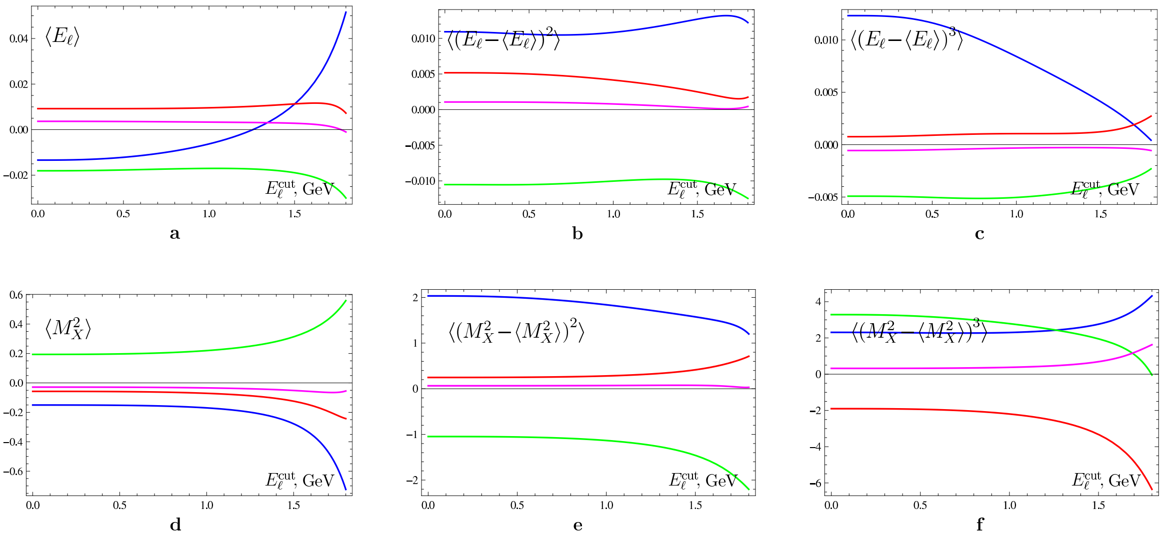


Figure 4.2.: Power corrections to the first three (central) moments of charged lepton energy (upper row) and hadron invariant mass squared (lower row), at different orders in the $1/m_Q$ expansion, in units of GeV in the corresponding power. Blue is the effect of μ_π^2 and μ_G^2 (order $1/m_b^2$), green at order $1/m_b^3$, red at $1/m_b^4$ and magenta finally shows the shift upon including $D = 8$ expectation values at order $1/m_b^5$

Keeping in mind that higher moments must generally be sensitive to higher-order OPE expectation values, we conclude that at moderate cuts $E_\ell^{\rm cut} \lesssim 1.5$ GeV preserving sufficient "hardness" of the inclusive probability, the power expansion is well behaved. The $1/m_b^5$ effects are small compared to the $1/m_b^4$ corrections. This is expected since the intrinsic charm (IC) effects are not parametrically enhanced in the higher moments [41, 45–47], as we will also see in a moment.

At the same time, it is clear that the estimated effects from higher powers in the $1/m_b$ expansion are not negligible, in particular in the second and higher moments. Those are sensitive to μ_π^2 and ρ_D^3, and high-order terms may produce their sizable shift.

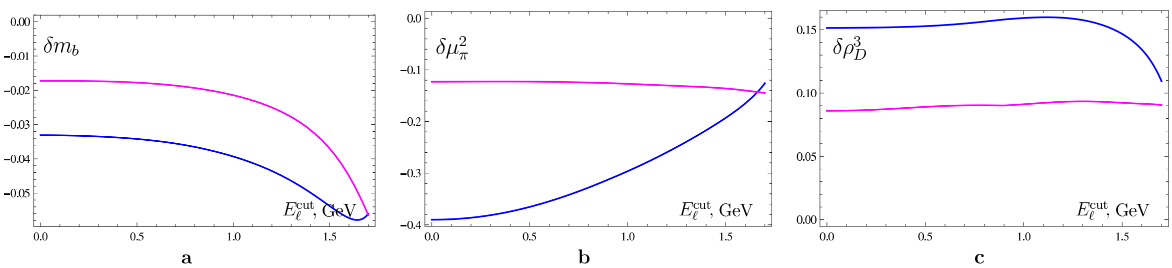


Figure 4.3.: Changes in m_b, μ_π^2 and in ρ_D^3, respectively required alone to literally offset the effect of higher-order power terms in $\langle E_\ell \rangle$ (blue) and in $\langle M_X^2 \rangle$ (magenta), at a given $E_\ell^{\rm cut}$. In units of GeV in the corresponding power.

To visualize the potential effect, we have plotted the analogies of the shifts in equations (4.102), $\delta m_b(E_\ell^{\rm cut})$, $\delta\mu_\pi^2(E_\ell^{\rm cut})$ and $\delta\rho_D^3(E_\ell^{\rm cut})$ for all six moments as functions of the lower cut $E_\ell^{\rm cut}$ and the corresponding $\delta|V_{cb}|/|V_{cb}|$. Here we present these dependences only for $\langle E_\ell\rangle$ and for $\langle M_X^2\rangle$, on single plots, separately for δm_b, $\delta\mu_\pi^2$ and $\delta\rho_D^3$, see Figures 4.3 a–c. The special roles of these two moments have been mentioned earlier in this subsection. The corresponding values (4.101) for all six moments at a representative mild cut $E_\ell^{\rm cut} = 1$ GeV are shown in Tables 4.7 and 4.8 together with the corresponding relative shift $\delta|V_{cb}|/|V_{cb}|$. The latter is easily estimated using Ref. [41], cf. equation (4.103)

$$\frac{\delta|V_{cb}|}{|V_{cb}|} \simeq \begin{cases} -0.0066 & \text{at} \quad \delta m_b \quad = 10 \text{ MeV} \\ 0.0013 & \text{at} \quad \delta\mu_\pi^2 \quad = 0.1 \text{ GeV}^2 \\ 0.009 & \text{at} \quad \delta\rho_D^3 \quad = 0.1 \text{ GeV}^3 \end{cases} . \tag{4.104}$$

As seen from the plots in figs. 4.2, the cut-dependence of higher-order corrections shows a generally expected behavior which qualitatively follows the behavior already found in the $1/m_b^2$ and $1/m_b^3$ effects. The new corrections likewise show mild cut dependence at $E_\ell^{\rm cut} \lesssim 1$ GeV and, typically, sharply change above $E_\ell^{\rm cut} \approx 1.5$ GeV, in line with the overall deterioration of the process hardness with raising cut on E_ℓ.

	$\langle E_\ell\rangle$	$\langle(E_\ell-\langle E_\ell\rangle)^2\rangle$	$\langle(E_\ell-\langle E_\ell\rangle)^3\rangle$
δm_b / MeV $(\delta\lvert V_{cb}\rvert/\lvert V_{cb}\rvert)$	-39 (0.026)	-60 (0.040)	—
$\delta\mu_\pi^2$ / GeV^2 $(\delta\lvert V_{cb}\rvert/\lvert V_{cb}\rvert)$	-0.30 (-0.004)	-0.12 (-0.0016)	-0.04 (-0.0005)
$\delta\rho_D^3$ / GeV^3 $(\delta\lvert V_{cb}\rvert/\lvert V_{cb}\rvert)$	0.16 (0.014)	0.09 (0.008)	0.02 (0.020)

Table 4.7.: Higher order power corrections to the charged lepton energy moments with $E_\ell^{\rm cut} = 1$ GeV translated into the required conventional heavy-quark parameter shifts to offset them. Also shown are relative shifts in $|V_{cb}|$, which would be induced through them, assuming the fixed value of $\Gamma_{\rm sl}(B)$. Entries where δm_b would exceed 100 MeV were left blank

Based on the numeric pattern of the corrections to the moments we anticipate, that the inclusion of higher-order power-suppressed effects will mostly amount to increase the fitted value of ρ_D^3 by about 0.1 GeV^3 compared to the fit where only $D=5$ and $D=6$ non-perturbative expectation values are retained. It could be accompanied with a possible shift in μ_π^2 by about ± 0.05 GeV^2. Figures 4.4 a–f illustrate this assertion, showing the combined effect of the new power corrections for the six moments together with the effect of decreasing ρ_D^3 by 0.12 GeV^3. The similarity of the shifts suggests that the lack of higher-power corrections in the theoretical expressions used so far could be to some extent faked by a lower value of the Darwin expectation value. The third lepton moment is highly sensitive to the Darwin expectation value, and uncalculated α_s-corrections to the latter may be blamed for the mismatch apparent in the plot Figs. 4.4 c. Besides, lepton energy moments are to a large extent saturated by the parton expressions. Therefore their high precision allows to discuss non-perturbative effects, and their value relies on a high degree of cancellation of conventional perturbative corrections. The extent of such cancellation at higher loops is not known beforehand, which warrants a cautious attitude towards theoretical precision of higher lepton moments at the required level, and places more emphasis in this respect on higher moments of the hadronic mass.

	$\langle M_X^2 \rangle$	$\langle (M_X^2 - \langle M_X^2 \rangle)^2 \rangle$	$\langle (M_X^2 - \langle M_X^2 \rangle)^3 \rangle$
δm_b / MeV $(\delta \lvert V_{cb} \rvert / \lvert V_{cb} \rvert)$	-21 (0.014)	—	—
$\delta \mu_\pi^2$ / GeV^2 $(\delta \lvert V_{cb} \rvert / \lvert V_{cb} \rvert)$	-0.13 (-0.0017)	-0.08 (-0.0010)	0.33 (0.0043)
$\delta \rho_D^3$ / GeV^3 $(\delta \lvert V_{cb} \rvert / \lvert V_{cb} \rvert)$	0.09 (0.008)	0.05 (0.005)	0.10 (0.009)

Table 4.8.: The corresponding Table to 4.7 for the hadronic invariant mass moments.

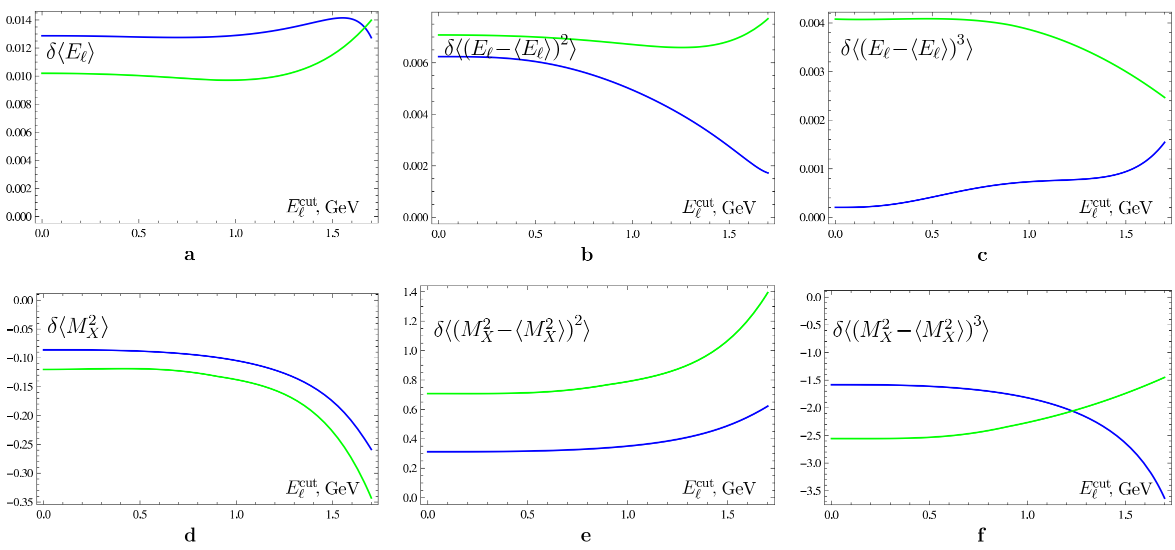


Figure 4.4.: Effect of including higher-order power corrections (blue) to the first three moments of charged lepton energy, upper row, and of hadron invariant mass squared, lower row, and effect of decreasing the Darwin expectation value by 0.12 GeV^3 (green), in GeV to the corresponding power

A good way to experimentally extract information on the Darwin expectation value is the third central hadronic mass squared moment. So far B-factories did not attempt to measure it. It has been extracted with an informative accuracy in the DELPHI analysis [48], however for unspecified reasons this moment was not included into the global fit, according to HFAG. We find that the higher-order power corrections (predominantly $1/m_b^4$) tend to shift this moment by about twice the DELPHI error bar.

Combining the increase in ρ_D^3 with the direct effect on $\Gamma_{\rm sl}$, Eq. (4.98), we expect an overall increase in $|V_{cb}|$ by something like 0.4 percent

$$\frac{\delta|V_{cb}|}{|V_{cb}|} \approx +(0.003 \div 0.005)\,.$$

This estimate is based on the expectation values in the ground-state factorization approximation and would scale with their magnitude. The actual number may be up to a factor of 1.5 larger. We emphasize that this would remain only an educated expectation, since the result strongly depends on the details of the existing fit to the data, on the precision of different data points and on their correlations. The final conclusions should be drawn through incorporating the new corrections in the actual fit to the data.

4.5. Intrinsic Charm

In this section we investigate the behaviour of the charm quark for different scenarios. In the standard calculation, presented above, we have noticed that infrared sensitive (in the vicinity of $m_c \to 0$) terms to the charm quark mass arise in the total rate. The structure of them looks like

$$m_c^2 \log m_c^2 + \frac{m_c^2}{m_b^2} \log m_c^2 + \frac{1}{m_b^3}\left(\log m_c^2 + \frac{1}{m_c^2} + \ldots\right) + \ldots\,, \tag{4.105}$$

from which we infer, that the infrared sensitivity gets stronger in higher orders. But the charm is besides the bottom quark also a heavy-quark and therefore its scale appears within the heavy-quark expansion and no infrared problem occur numerically. However one might assume to investigate the light up quark instead of the charm quark. We will show, how to treat this case properly. In Figure 4.5 we see that in the heavy-quark expansions the bottom quark is integrated out of the theory at the scale of its mass. It is still remnant as a static source of color, since in QCD no quark can decay. In other words we integrate out the hard quantum fluctuations, only. Nevertheless the charm quark, as it is lighter than the bottom quark, still appears as a dynamical quark at that point. To treat this, we may take different points of view, which we will discuss in turn.

Our starting point is again the hadronic tensor (4.7) with $P_L = 1/2(1-\gamma_5)$ being the left-handed projector, where the delta distribution of momentum conservation is

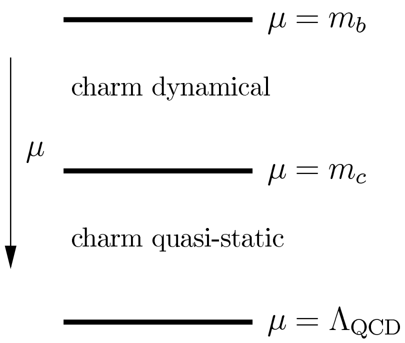


Figure 4.5.: Renormalization group viewpoint of the charm quark.

rewritten into an integral form. Now we transform this using translational invariance

$$2M_B W_{\mu\nu} = \int \frac{\mathrm{d}^4x}{2\pi} \sum_{X_c} e^{i(m_b v - p_x - q)\cdot x} \langle \bar{B}|\bar{b}(0)\gamma_\nu P_L c(0)|X_c\rangle\langle X_c|\bar{c}(0)\gamma_\mu P_L b(0)|\bar{B}\rangle \quad (4.106)$$

$$= \int \frac{\mathrm{d}^4x}{2\pi} e^{-iq\cdot x} \sum_{X_c} \langle \bar{B}|\bar{b}(x)\,\gamma_\nu P_L\, c(x)|X_c\rangle\langle X_c|\bar{c}(0)\,\gamma_\mu P_L\, b(0)|\bar{B}\rangle \quad (4.107)$$

$$= \int \frac{\mathrm{d}^4x}{2\pi} e^{-iq\cdot x} \langle \bar{B}|\bar{b}(x)\,\gamma_\nu P_L\, c(x)\bar{c}(0)\,\gamma_\mu P_L\, b(0)|\bar{B}\rangle \quad (4.108)$$

$$= \frac{1}{2\pi} \int d^4x\, e^{i(m_b v - q)\cdot x} \langle \bar{B}|\bar{b}_v(x)\,\gamma_\nu P_L\, c(x)\bar{c}(0)\,\gamma_\mu P_L\, b_v(0)|\bar{B}\rangle\,. \quad (4.109)$$

Please note that the hadronic tensor in this form is non-local and it does not contain the sum over all different intermediate states. Additionally it is explicitely clear, that the hadronic tensor does not only contain the terms we have presented, but additionally four-quark operators with charm-quark content. These operators have to be considered, but in a safe way, such that double counting is avoided. If we now perform the OPE for the hadronic tensor, this product of the two $b \to c$ currents is matched onto the set of local operators, namely four-fermion operators with charm-quark content and the non-perturbative parameters, at the scale $\mu \sim m_b$. See also again Fig. 4.5. Now, as far as the charm-quark mass is concerned, we may take different points of view. We will explain the general assumptions in the next few abstracts, and then discuss them in detail.

1. First we may assume the standard case, which states that $m_b \sim m_c \gg \Lambda_{\mathrm{QCD}}$. This means that the short-distance matching coefficients, the Wilson coefficients, and the phase space integrals are functions of the fixed ratio $\rho = m_c^2/m_b^2$. In other words we integrate out all (hard) quantum fluctuations with virtualities of order $m_{b,c}^2$. Thus we integrate out both the bottom quark, as well as the charm quark at the same time. In the end we are left with light degrees of freedom, only: Light quarks and gluons, together with the quasi-static b-quark field in HQET. In a standard renormalization scheme like $\overline{\mathrm{MS}}$, operators that contain

charm quark fields do not appear at scales below $\mu < m_c$. More precisely, such operators would correspond to quasi-static charm quarks, as in the case of the bottom quark. The difference here is, that the bottom quark is the initial state, but the charm quark would be some quasi-static virtual state. Thus it cannot contribute to the considered forward matrix elements

$$\langle \bar{B}|\bar{b}_v \dots c_{\text{static}}\,\bar{c}_{\text{static}} \dots b_v|\bar{B}\rangle\Big|_{\mu<m_c} \equiv 0\,, \tag{4.110}$$

because of energy conservation: $m_b + 2m_c + \Delta E_{\text{soft}} > m_B$. This case corresponds to have in Fig. 4.5 the upper two lines at one scale and no running in between.

That is in fact the point of view, which is usually considered in the precision determination of $|V_{cb}|$.

2. As a second case we consider the power counting $m_b \gg m_c \gg \Lambda_{\text{QCD}}$, and integrate out hard b-quark fluctuations at a different scale than the charm quark. This is exactly shown in Fig. 4.5. In this case, for the first matching at the high scale $\mu_h \sim m_b$ we still keep the charm quark dynamical, and the corresponding "intrinsic-charm" operators appear in the OPE[12]. We will derive and use the renormalization-group for these operators, to scale down to the semi-hard scale $\mu_{sh} \sim m_c$, where we finally integrate out the charm-quark. As before, the "intrinsic-charm" operators then match onto local operators built from light fields. Obviously, the main difference compared to case 1 is, that logarithmic terms $\ln(m_c/m_b)$ are created by the RGE and thus they can be resumed into short-distance coefficient functions [51], while the analytic terms should now be expanded in powers of $m_c/m_b \sim \sqrt{\Lambda_{\text{QCD}}/m_b} \sim 0.3$.

3. Finally, we may assume that $m_b \gg m_c \gtrsim \Lambda_{\text{QCD}}$. This corresponds to have in Fig. 4.5 the lower two lines at the same scale and therefore no running in between. Furthermore we cannot integrate out the charm-quark effects perturbatively in this case, and we are thus left with genuine intrinsic-charm operators, whose hadronic matrix elements have to be defined at a sufficiently high scale μ_0, satisfying $m_b \geq \mu_0 \gg m_c$. Notice that the matrix elements of the intrinsic-charm operators contain the non-analytic dependence on the charm-quark mass m_c, and consequently the partonic phase-space integration for the calculation of various moments of the differential rate has to be modified accordingly, in order to avoid double counting. This case would correspond to consider the decay $b \to u$ properly. We show, that in this case the HQE is safely defined for all orders, although from the standard calculation in case 1 infrared sensitive terms appear, which on first glance indicate some breakdown of the expansion. In discussing this scenario, we show how to setup the HQE properly to high orders.

[12] More precisely, the "intrinsic-charm" operators correspond to local operators for semi-hard charm quarks, i.e. quarks with all momentum components of order m_c. This is to be distinguished from the hard-collinear (jet) modes for the charm-quark which appear in non-local operators describing the shape-function region [49, 50] for $b \to c\ell\nu$.

In the following we shall discuss the different cases in turn. In each of the scenarios we have to distinguish two cases. For the rise of the non-analytic logarithmic terms, we consider everything in a dimensional regularization scheme. There the logarithms are created by the RGE running. For the power-like infrared sensitivities it is more intuitive to impose a cut-off scheme.

4.5.1. $m_b \sim m_c \gg \Lambda_{\mathrm{QCD}}$

Origin of the logarithmic terms

As explained above, we first integrate out both, the hard bottom quark fluctuations and the charm quark, at a common scale $\mu \sim m_b$. Thus we are left with operators built from soft fields, only. At order $1/m_b^3$ the only matrix elements appearing are the Darwin term ρ_D^3 and the spin-orbit term ρ_{LS}^3 defined previously in eq. (4.50). We slightly modify this definition, but the only difference is due to higher orders in $1/m_b$. Here we are interested in leading effects, only. Therefore these two non-perturbative parameters are given within this section by

$$2M_B\,\rho_D^3 = \langle \bar{B}|\bar{b}_v\,(iD_\mu)(iv\cdot D)(iD^\mu)\,b_v|\bar{B}\rangle \tag{4.111a}$$
$$2M_B\,\rho_{LS}^3 = \langle \bar{B}|\bar{b}_v\,(iD_\mu)(iv\cdot D)(iD_\nu)(-i\sigma^{\mu\nu})\,b_v|\bar{B}\rangle\,. \tag{4.111b}$$

We will restrict ourselves to the charged-lepton energy spectrum and the related moments in this special analysis. In its spectrum, the rise and treatment of the infrared terms is traced easily. The same arguments hold true for the partonic observables, however the strategy has to be slightly modified in this case. We present the analysis with the dimensionless variables

$$\rho = \frac{m_c^2}{m_b^2} \tag{4.112a}$$
$$y = \frac{2E_\ell}{m_b} \qquad , \quad 0 \le y \le 1-\rho\,. \tag{4.112b}$$

In the charged-lepton energy spectrum, which is presented completely in the appendix, we obtain a contribution, among others, of the form

$$\left.\frac{d\Gamma}{dy}\right|_{\rho_D^3} = \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2\,\frac{\rho_D^3}{m_b^3}\left\{-\frac{8\,\theta(1-y-\rho)}{1-y} + \ldots\right\}. \tag{4.113}$$

For later use, we have only quoted the most singular term in the limit $\rho \to 0$, and therefore $y \to 1$. Upon integration it yields a logarithmically enhanced contribution to the total rate

$$\Gamma\Big|_{\rho_D^3} = \frac{G_F^2 m_b^5}{192\pi^3}\,|V_{cb}|^2\,\frac{\rho_D^3}{m_b^3}\,\{8\,\ln\rho + \ldots\}\,, \tag{4.114}$$

where the ellipses denote the contributions from the sub-leading terms in (4.113) which are of order $\rho\ln\rho$. Similarly, we identify the leading terms in the moments (see also appendix)

$$\langle (y-y_0)^n\rangle\Big|_{\rho_D^3} = \frac{G_F^2 m_b^5}{192\pi^3}\,|V_{cb}|^2\,\frac{\rho_D^3}{m_b^3}\,\{8\,(1-y_0)^n\,\ln\rho+\ldots\}\ . \tag{4.115}$$

Note that for $m_b \sim m_c$ the logarithm is actually of order one and represents a regular contribution to the Wilson coefficient. Therefore the remaining terms in curly brackets enter on the same level for the assumed power counting. The phase space boundary for y is given by $0 < y < 1-\rho$. Since we treat $\rho \sim \mathcal{O}(1)$ the boundary is away from $y = 1$, which is the sensitive part[13], by an amount of order one.

As we have mentioned before, a similar logarithmically enhanced term also appears in the partonic rate,

$$\Gamma\Big|_{\text{partonic}} = \frac{G_F^2 m_b^5}{192\pi^3}\,|V_{cb}|^2\,\left\{1-12\,\rho^2\,\ln\rho+\ldots\right\}\ , \tag{4.116}$$

and in the related moment,

$$\langle 1-y\rangle\Big|_{\text{partonic}} = \frac{G_F^2 m_b^5}{192\pi^3}\,|V_{cb}|^2\,\left\{6\,\rho^2\,\ln\rho+\ldots\right\}\ . \tag{4.117}$$

In contrast to the Darwin-term contribution, the logarithmic term vanishes in the limit $\rho \to 0$. Nevertheless, as has been shown in [51], such "phase-space logs" can be resumed into short-distance coefficients, as we are going to discuss in scenario two.

Power-like infrared sensitive terms

Here we scrutinize the standard way of setting up the OPE, treating both bottom and charm quark as equally heavy, and show how the power-like infrared sensitive terms emerge. Thereby we restrict ourselves to the partonic observables, in contrast to the lepton energy observable for the logarithms. However, the argumentation is valid for both cases, and will go along the same line. As explained in the higher orders section before, only operators with non-relativistic b quarks and their (covariant) derivatives appear.[14]

The OPE is constructed in the standard way, described before and it yields the HQE parameters. We start by considering the doubly differential rate for $B \to X_c\,\ell\bar{\nu}_\ell$

$$\begin{aligned}\frac{\mathrm{d}^2\Gamma}{\mathrm{d}(v\cdot Q)\mathrm{d}Q^2} =& \frac{G_F^2|V_{cb}|^2}{24\pi^3}\sqrt{(v\cdot Q)^2-Q^2}\,\theta\left((v\cdot Q)^2-Q^2\right)\theta\left(m_b - v\cdot Q\right)\\ &\theta\left(m_b^2+Q^2-2m_b v\cdot Q\right)L_{\mu\nu}W^{\mu\nu}(Q)\,,\end{aligned} \tag{4.118}$$

[13]The kinematical relevant region of the infrared sensitive corresponds to a soft collinear charm, therefore it amounts to the case of an electron with maximal energy.

[14]We do not discuss here operators involving light quarks appearing additionally at $\mathcal{O}(\alpha_s)$ level, their contribution is small.

where $Q = m_b v - q$ with q denoting the momentum transfer to the lepton pair and v the B-meson velocity as before. $L_{\mu\nu}$ is the leptonic tensor and contains the phase-space integration with

$$L_{\mu\nu}(Q) = -m_b^2[g_{\mu\nu} - v_\mu v_\nu] + m_b[2g_{\mu\nu}(v\cdot Q) - v_\mu Q_\nu - Q_\mu v_\nu] - [g_{\mu\nu}Q^2 - Q_\mu Q_\nu] \quad (4.119)$$

and $W^{\mu\nu}$ is the hadronic counter-part defined as above. The expansion of the hadronic tensor is done by external field methods, which is particularly economical at tree level, see e.g. [28] and above in this thesis. This yields

$$\begin{aligned} 2M_B W_{\mu\nu}(Q) &= \langle \bar{B}|\bar{b}_v \Gamma_\nu (\not{Q} + m_c)\Gamma_\mu b_v|\bar{B}\rangle\, \delta(Q^2 - m_c^2) \\ &+ \langle \bar{B}|\bar{b}_v \Gamma_\nu (\not{Q} + m_c)(i\not{D})(\not{Q} + m_c)\Gamma_\mu b_v|\bar{B}\rangle\, \delta'(Q^2 - m_c^2) \\ &+ \frac{1}{2}\langle \bar{B}|\bar{b}_v \Gamma_\nu (\not{Q} + m_c)(i\not{D})(\not{Q} + m_c)(i\not{D})(\not{Q} + m_c)\Gamma_\mu b_v|\bar{B}\rangle\, \delta''(Q^2 - m_c^2) \\ &+ \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\langle \bar{B}|\bar{b}_v \Gamma_\nu \left[(\not{Q} + m_c)(i\not{D})\right]^n (\not{Q} + m_c)\Gamma_\mu b_v|\bar{B}\rangle\, \delta^{(n)}(Q^2 - m_c^2)\,. \end{aligned} \quad (4.120)$$

The main challenge in evaluating higher order contributions lies in identifying the independent hadronic parameters controlling the "string" of matrix elements

$$\langle \bar{B}|\bar{b}_v (iD_{\mu_1})(iD_{\mu_2})\cdots(iD_{\mu_n}) b_v|\bar{B}\rangle \quad (4.121)$$

entering at order n. The general procedure has been shown before. It should be noted that these quantities do not depend on Q.

Contracting the hadronic and leptonic tensors for the pseudo-scalar B mesons and integrating over phase-space leads to function of $(v\cdot Q)$ and Q^2 with the general form

$$L_{\mu\nu}W^{\mu\nu} = \sum_{n=0}^{\infty} \sqrt{(v\cdot Q)^2 - Q^2}\, P_n(v\cdot Q, Q^2)\delta^{(n)}(Q^2 - m_c^2) \quad (4.122)$$

with $P_n(v\cdot Q, Q^2)$ denoting a polynomial in $v\cdot Q$ and Q^2. The analysis so far referred to a fully differential distribution in general kinematics. To proceed to the inverse mass expansion we need to consider partially integrated probabilities.

The integration over the variable $v\cdot Q$ has the limits $\sqrt{Q^2} \le v\cdot Q \le (m_b^2 + Q^2)/(2m_b)$, which yields terms logarithmic in Q^2 from the lower end of integration. Focusing on these logarithms we get for $l = 0, 1, \ldots$

$$\begin{aligned} &\int_{\sqrt{Q^2}} \mathrm{d}(v\cdot Q)\, \sqrt{(v\cdot Q)^2 - Q^2}(v\cdot Q)^{2l} = C_l\, (Q^2)^{l+1} \ln\frac{Q^2}{m_b^2} + \cdots\,,\quad C_l = \frac{\Gamma(l+\frac{1}{2})}{4\sqrt{\pi}\,\Gamma(l+2)} \\ &\int_{\sqrt{Q^2}} \mathrm{d}(v\cdot Q)\, \sqrt{(v\cdot Q)^2 - Q^2}\,(v\cdot Q)^{2l+1} = 0 + \cdots \end{aligned} \quad (4.123)$$

where the ellipses denote polynomial terms in Q^2, and the coefficients C_l are simple fractions: $C_0 = 1/4,\ C_1 = 1/16,\ C_2 = 1/32,\ C_3 = 5/256$ ...

These logarithms are the source of the IR sensitivity of the coefficient functions to the charm mass in this approach. This becomes manifest when they are combined with the derivatives of the δ-function in Eq.(4.122):

$$(Q^2)^k \ln Q^2\ \delta^{(k)}(Q^2-m_c^2) = (-1)^k\, k!\ \ln m_c^2\ \delta(Q^2-m_c^2) + \cdots \tag{4.124}$$

$$(Q^2)^k \ln Q^2\ \delta^{(n)}(Q^2-m_c^2) = (-1)^{n-k-1}\, k!(n-k-1)!\left(\frac{1}{m_c^2}\right)^{n-k} \delta(Q^2-m_c^2) \quad \text{for } n > k \tag{4.125}$$

where k is some integer power and the ellipses point to less singular terms as $m_c \to 0$. Note that at the lower limit of integration in Eq. (4.123) we have $Q_0^2 = Q^2$, i.e. $\boldsymbol{Q} \to 0$. This is the infrared regime for the charm quark in the final state.

For this analysis it is therefore useful to write the leptonic tensor, neglecting here the lepton energy cut, as

$$L_{\mu\nu} = L_{\mu\nu}^{\text{lead}} + L_{\mu\nu}^{\text{sub}} + L_{\mu\nu}^{\text{subsub}}\,, \tag{4.126}$$

where we sort it after powers of Q.

The leading IR sensitive terms in the integrated rates arise from the leading term in the leptonic tensor

$$L_{\mu\nu}^{\text{lead}} = -m_b^2[g_{\mu\nu} - v_\mu v_\nu] \tag{4.127}$$

already in the expression for the differential rate. The sub-leading parts of the leptonic tensor, containing the vector Q, would lead to additional powers of $v \cdot Q$ and Q^2 and hence do not contribute to the leading infrared sensitivities. To obtain non-leading in $1/m_b$ terms, we consider this sub-leading terms

$$L_{\mu\nu}^{\text{sub}} = m_b[2g_{\mu\nu}(v \cdot Q) - v_\mu Q_\nu - Q_\mu v_\nu] \tag{4.128}$$

$$L_{\mu\nu}^{\text{subsub}} = -[g_{\mu\nu}Q^2 - Q_\mu Q_\nu]\,. \tag{4.129}$$

On the other hand, the n^{th} term in the sum (4.120) for the hadronic tensor contains

$$P_n \propto \Gamma_\nu(\not{Q} + m_c)\gamma_{\mu_1}(\not{Q} + m_c)\gamma_{\mu_2}\cdots(\not{Q} + m_c)\gamma_{\mu_n}(\not{Q} + m_c)\Gamma_\mu \tag{4.130}$$

which yields upon contraction with the leptonic tensor (4.127) and with the non-perturbative matrix elements (4.121) a contribution of the form (see (4.122))

$$P_n = \sum_{ijl} a_{ijl}\ (v \cdot Q)^i (Q^2)^j (m_c^2)^l \quad \text{with} \quad i + 2j + 2l = n + 1\,. \tag{4.131}$$

Note that due to the purely left-handed structure of the current Γ, we can have only even powers of m_c here. Furthermore to allow for a fully contracted Lorentz scalar it is necessary, that if n is even, i necessarily is odd (and vice versa). Hence from (4.123) we conclude that IR sensitive terms can appear only if n is odd, which means that the number of covariant derivatives in the hadronic matrix element has to be odd as well[15]. Therefore all operators which contribute to intrinsic charm have to match onto partonic matrix elements with at least one gluon. In turn, it implies that there is no intrinsic charm contribution to operators of the form $\langle (\boldsymbol{k}^2)^n \rangle$ where $\boldsymbol{k}$ are the spatial components of the residual b-quark momentum. This is also ensured by reparametrization invariance.

Now we can trace how singular terms actually emerge in leading order. Putting everything together with $i = 2m$ (remember we need to have an even power in $v \cdot Q$!)

$$\int \mathrm{d}(v\cdot Q)\sqrt{(v\cdot Q)^2 - Q^2}P_n((v\cdot Q), Q^2)\delta^{(n)}(Q^2 - m_c^2) \tag{4.132}$$

$$= \sum_{ijl} a_{ijl} \int \mathrm{d}(v\cdot Q)\sqrt{(v\cdot Q)^2 - Q^2}\left[(v\cdot Q)^i (Q^2)^j (m_c^2)^l\right]\delta^{(n)}(Q^2 - m_c^2) \tag{4.133}$$

$$\Longrightarrow \sum_{ijl} C_l\, a_{ijl}\, (Q^2)^{m+j+1}(m_c^2)^l \ln\left(\frac{Q^2}{m_b^2}\right)\delta^{(n)}(Q^2 - m_c^2)\,. \tag{4.134}$$

Terms with odd i, on the other hand, do not contain a logarithm. Note that we have $2m + 2j + 2l = n + 1$, which can be satisfied at any odd n. Thus we arrive at

$$\int \mathrm{d}(v\cdot Q)\sqrt{(v\cdot Q)^2 - Q^2}P_n((v\cdot Q), Q^2)\delta^{(n)}(Q^2 - m_c^2) \tag{4.135}$$

$$\overset{\text{sing}}{=} \sum_{ijl} C_l\, a_{ijl}\, (Q^2)^{(n+3-2l)/2}(m_c^2)^l \ln\left(\frac{Q^2}{m_b^2}\right)\delta^{(n)}(Q^2 - m_c^2)\,. \tag{4.136}$$

Hence IR sensitive terms can appear starting at $n = 3$ with the logarithmic dependence on m_c of the Darwin term, as we have shown in the last section. For $n > 3$ we obtain from these terms a tree-level contribution to the total rate of the form

$$\Gamma_n \propto \frac{1}{m_b^3}\left(\frac{1}{m_c^2}\right)^{(n-3)/2} \qquad \text{with } n = 5,7,9\ldots\,, \tag{4.137}$$

i.e., even a power-like singularity for $m_c \to 0$.

Now we are in a position to present the leading polynoms to identify the IR sensitivities.

For $n = 0$ the leading term reads

$$P_0^{\text{lead}} = \frac{3}{2}m_b^2(v\cdot Q) \tag{4.138}$$

which is an odd power of $v \cdot Q$ and hence does not lead to IR sensitive terms as just explained. One should note that the sub-leading contribution to the leptonic tensor $L_{\mu\nu}$ yields—upon contraction with the partonic hadronic tensor—a (sub-leading) term of the form

$$P_0^{\text{sub}} = -2m_b(v\cdot Q)^2 - Q^2 m_b\,. \tag{4.139}$$

[15]This argument is valid for tree level and leading IR sensitive terms, only.

It leads to a contribution of the form $m_c^4 \ln(m_c^2)$ in the phase-space factor of the partonic total rate. To low orders in Λ/m_b it is easy to keep as well the terms stemming from the sub-leading pieces in $L_{\mu\nu}$, which we present in turn. We will then focus on those novel terms $\propto 1/m_c^k$ that arise in leading order in $1/m_b$.

The next term with $n = 1$ is given by

$$P_1 = \frac{\mu_G^2 - \mu_\pi^2}{12 m_b} \left(5Q^4 + 7m_c^2 Q^2 - 20(v \cdot Q)^2 Q^2 - 10 m_c^2 (v \cdot Q)^2 \right) , \tag{4.140}$$

where the hadronic tensor is contracted with the subsub-leading part of the leptonic tensor $L_{\mu\nu}^{\text{subsub}}$. This yields again an $m_c^4 \ln(m_c^2)$ term upon integration over the phase space.

For $n = 2$ the leading term of the leptonic tensor again contains only odd powers of $(v \cdot Q)$ which do not generate any logarithms.

At $n = 3$ the IR sensitive contribution is the Darwin term. Explicitly, we have

$$\langle \bar{B} | \bar{b}_v (iD_\alpha)(iD_\gamma)(iD_\beta) b_v | \bar{B} \rangle = \frac{1}{6} M_B \rho_D^3 (g_{\alpha\beta} - v_\alpha v_\beta) v_\gamma (\not{v} + 1) + \cdots \tag{4.141}$$

from which we obtain

$$P_3^{\text{Dar}} = -\frac{\rho_D^3}{12} m_b^2 \left(3(Q^2 - m_c^2)^2 + 8(v \cdot Q)^4 - 8Q^2 (v \cdot Q)^2 \right) . \tag{4.142}$$

Upon integration over $(v \cdot Q)$ we arrive at terms with three types of prefactors:

$$(Q^2)^3 \ln\left(\frac{Q^2}{m_b^2}\right) , \quad m_c^2 (Q^2)^2 \ln\left(\frac{Q^2}{m_b^2}\right) , \quad m_c^4 Q^2 \ln\left(\frac{Q^2}{m_b^2}\right) . \tag{4.143}$$

They have three derivatives with respect to Q^2 from the $\delta^{(3)}$-function, and hence the first term yields a $\ln(m_c^2)$ factor which is the first infrared sensitive contribution. In the other two terms explicit factors of m_c^2 kill the infrared singularity and thus they remain finite for $m_c \to 0$. It is straightforward to check that in this way we end up with the correct prefactor for the infrared log in the Darwin contribution.

The terms with $n = 4$ create again only odd powers of $(v \cdot Q)$ and would yield infrared singularity for the $1/m_b$-sub-leading piece. Finally at $n = 5$ the following nine structures arise

$$\begin{aligned} P_5 \propto \quad & (v \cdot Q)^6 , \quad (v \cdot Q)^4 Q^2 , \quad (v \cdot Q)^4 m_c^2 , \quad (v \cdot Q)^2 (Q^2)^2 , \quad (v \cdot Q)^2 m_c^4 , \\ & (v \cdot Q)^2 Q^2 m_c^2 , \quad (Q^2)^3 , \quad (Q^2)^2 m_c^2 , \quad Q^2 m_c^4 . \end{aligned} \tag{4.144}$$

Upon integration over $(v \cdot Q)$ we obtain terms of the form

$$(Q^2)^4 \ln\left(\frac{Q^2}{m_b^2}\right) , \quad (Q^2)^3 m_c^2 \ln\left(\frac{Q^2}{m_b^2}\right) , \quad (Q^2)^2 m_c^4 \ln\left(\frac{Q^2}{m_b^2}\right) , \quad Q^2 m_c^6 \ln\left(\frac{Q^2}{m_b^2}\right) \tag{4.145}$$

coming with five derivatives of the δ-function. All, on contrast to the Darwin case, thus yield contributions of order $1/m_c^2$ in the total rate. They will be addressed in detail for the case $m_b \gg m_c \gg \Lambda_{\rm QCD}$. A similar consideration extends in a straightforward way to higher orders where $n > 5$ emerge. These would generate terms inversely proportional to even larger powers of inverse charm mass.

As a first resumé we state that IR sensitive contributions—i.e., those singular for $m_c \to 0$—unequivocally arise from the lower end of the integration over $v{\cdot}Q$ (i.e. $\boldsymbol{Q} \to 0$) due to the presence of the non-analytic factor $\sqrt{(v \cdot Q)^2 - Q^2} = |\boldsymbol{Q}\,|$ in the integrand. For the dimension-six Darwin term they are of the form $\ln{(m_c^2/m_b^2)}$. Higher-dimension contributions exhibit an even stronger singularity, viz. powers of $1/m_c$. Without extra gluon loops this happens first for dimension eight.

The discussion above showed that such IC effects emerge for the fully integrated width. The same arguments apply for the moments of the decay distributions. As an example, the partonic energy spectrum related to $\mathrm{d}\Gamma/\mathrm{d}(v{\cdot}Q)$ likewise exhibits a singular behaviour at $(v \cdot Q) \to 0$, it yields the IC terms in the total rate upon integration over the variable $(v \cdot Q)$, see Eq. (4.123). In the partonic energy moments $\langle (v \cdot Q)^n \rangle$ the additional powers of $(v \cdot Q)$ delay the emergence of singular IC terms to higher orders, see (4.136). For a moment with a given n IC terms always appear at sufficiently high orders in the OPE. It therefore depends on the kinematical variable at which order IC terms emerge.

4.5.2. $m_b \gg m_c \gg \Lambda_{\mathrm{QCD}}$

Now we investigate the case, where we can treat the charm as a second heavy scale. Therefore we have to integrate out the charm at a different scale than the bottom quark. This demonstrates how to treat the light final state quark properly. First of all we show how to treat the logarithmic sensitive terms in the lepton energy spectrum. For this we can restrict ourselves up to the dimension 6 operators. Then we extend this to the power-like IR-sensitive terms and the partonic kinematical variables. There we have to include everything up to $1/m_b^5$.

Treatment of the logarithmic terms in the lepton-energy spectrum

When we integrate out the b quark first at a scale $\mu_h \sim m_b$ and still keep the charm quark dynamical, we have to take into account operators with explicit charm quarks until those are finally integrated out at the semi-hard scale $\mu_{sh} \sim m_c$. In addition to the dimension-5 and dimension-6 operators, defining μ_π^2, μ_G^2 and ρ_D^3, ρ_{LS}^3, one thus finds (at tree level) matrix elements of the local "intrinsic-charm" operators by expanding (4.109) into local four-fermion operators

$$\begin{aligned} 2M_B\, W_{\mu\nu}^{IC} &= (2\pi)^3\, \delta^4(q - m_b v)\, \langle \bar{B}|(\bar{b}_v\, \gamma_\nu P_L\, c)\, (\bar{c}\, \gamma_\mu P_L\, b_v)|\bar{B}\rangle \\ &+ (2\pi)^3 \left(\frac{\partial}{\partial q_\alpha}\, \delta^4(q - m_b v) \right) \langle \bar{B}| i\partial_\alpha\, \bar{b}_v\, \gamma_\nu P_L\, c)\, (\bar{c}\, \gamma_\mu P_L\, b_v)|\bar{B}\rangle \\ &+ \dots\,, \end{aligned} \tag{4.146}$$

which can be interpreted as the probability to find semi-hard (i.e. off-shell) charm quarks inside the heavy $\bar{B}$-meson.

Notice, that the power-counting for the semi-hard charm fields $[c] = (m_c)^{3/2}$ is now different from the ones for soft HQET fields $[b_v] = \Lambda^{3/2}$, and therefore it may be convenient to use a notation as in [50], where the "intrinsic-charm" operators in the first line of (4.146) are suppressed by $\lambda^3 \equiv (m_c/m_b)^3$, the ones in the second line by λ^4, the kinetic and chromo-magnetic operators by $\lambda^4 \equiv (\Lambda/m_b)^2$, and the Darwin and spin-orbit term by λ^6. Due to chiral symmetry, only the λ^4 "intrinsic-charm" operators contribute to the partonic rate for $b \to c\ell\nu$, related to the $\rho^2 \ln\rho$ term in (4.116). Additional soft-gluon couplings to semi-hard charm quarks are further suppressed, and this will give rise to the λ^6 suppressed terms $\rho_D^3 \ln\rho$ in (4.114), descending from the λ^3 "intrinsic-charm" operators.

Let us consider first the matrix elements of the operator in the first line of (4.146). They may be decomposed in terms of two hadronic parameters, $T_1(\mu)$ and $T_2(\mu)$,

$$(4\pi)^2 \langle \bar{B}|\bar{b}_v \,\gamma_\nu P_L \, c \; \bar{c}\, \gamma_\mu P_L \, b_v|\bar{B}\rangle = 2M_B \left(T_1(\mu)\, g_{\mu\nu} + T_2(\mu)\, v_\mu v_\nu\right) . \tag{4.147}$$

The contribution to the rate of the matrix element of the local "intrinsic-charm" operators is concentrated at small hadronic mass m_X and in the endpoint of the lepton energy spectrum, the infrared space-space regime of the charm quark. Performing the tree-level matching at $\mu = m_b$, we have

$$\frac{\mathrm{d}^2\Gamma^{IC}}{\mathrm{d}m_X^2\,\mathrm{d}y} = \delta(m_X^2)\,\delta(1-y)\,\Gamma^{IC} \quad \text{and} \quad \frac{\mathrm{d}\Gamma^{IC}}{\mathrm{d}y} = \delta(1-y)\,\Gamma^{IC} , \tag{4.148}$$

with

$$\Gamma^{IC} = -\frac{G_F^2 m_b^5}{24\pi^3}\,|V_{cb}|^2 \frac{3\,T_1(m_b)}{m_b^3} . \tag{4.149}$$

On the other hand, the calculation of the matching coefficients for the contribution of ρ_D^3 and ρ_{LS}^3 to the total rate now has to be performed in the limit $m_c \ll m_b$. Notice, that the naive limit $\rho \to 0$ in (4.113) would give ill-defined expressions. In particular, the integral over

$$\mathrm{d}y\,\frac{\theta(1-y)}{1-y}$$

would be infrared divergent in the lepton-energy endpoint. As we will see, the new IR divergence in the phase-space integration, appearing in the limit $\rho \to 0$, is related to the UV renormalization of the "intrinsic-charm" operators (4.147). Defining the hadronic parameters $T_{1,2}(\mu)$ in the $\overline{\text{MS}}$ scheme, we also have to perform the phase-space integral in $D = 4-2\epsilon$ dimensions. As a result, the contribution of the Darwin term to the total rate is regularized by plus-distributions,

$$\frac{\theta(1-y)}{1-y} \to \left[\frac{\theta(1-y)}{1-y}\right]_+ - \delta(1-y)\,\ln\left(\frac{\mu^2}{m_b^2}\right) , \tag{4.150}$$

which exactly subtracts the effects of semi-hard charm quarks, that would otherwise be double-counted when adding (4.149) to the decay rate. The plus-distribution is defined

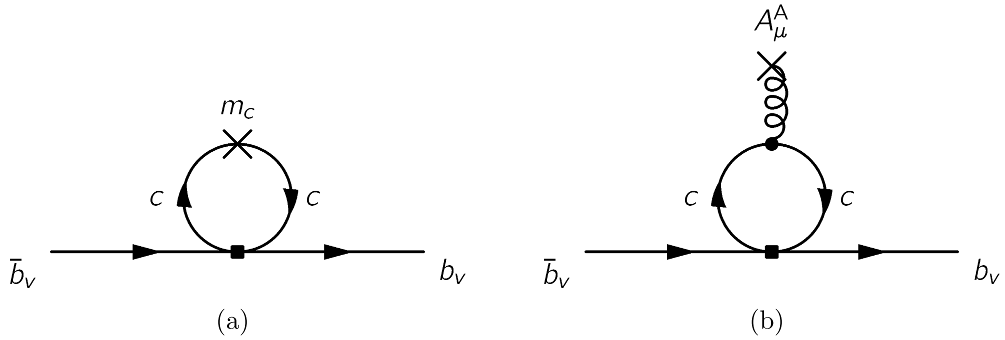


Figure 4.6.: Leading diagrams determining the mixing of four-quark into two-quark operators.

as

$$\int_0^1 \mathrm{d}x \, \frac{f(x)}{[1-x]_+} = \int_0^1 \mathrm{d}x \, \frac{f(x) - f(1)}{1-x} \,,$$

such that the divergence is canceled out and the integral remains finite.

The final expression for the combined contributions of the Darwin term and the "intrinsic-charm" operators to the lepton-energy spectrum at order $1/m_b^3$ can be written as

$$\frac{\mathrm{d}\Gamma^{(3)}}{\mathrm{d}y}\bigg|_{\rho_D^3 + \mathrm{IC}} = \frac{G_F^2 m_b^5}{24\pi^3} \, |V_{cb}|^2 \left\{ \frac{C_{\rho_D}(y,\mu) \, \rho_D^3(\mu)}{m_b^3} + \frac{C_{T_1}(y,\mu) \, T_1(\mu)}{m_b^3} \right\} , \tag{4.151}$$

which should be used for $m_c \le \mu \le m_b$. The matching conditions for the short-distance coefficient functions—including the limit $\rho \to 0$ for the sub-leading terms in (4.113) as given in the appendix—are given by

$$\begin{aligned} C_{\rho_D}(y,m_b) &= -\left[\frac{y^2(9-5y+2y^2)\,\theta(1-y)}{6(1-y)}\right]_+ + \frac{17}{12}\,\delta(y-1) \\ &\quad + \frac{5}{24}\,\delta'(y-1) - \frac{1}{72}\,\delta''(y-1) + \mathcal{O}(\alpha_s) \,, \\ C_{T_1}(y,m_b) &= -3\,\delta(y-1) + \mathcal{O}(\alpha_s) \,, \\ C_{T_2}(y,m_b) &= \mathcal{O}(\alpha_s) \,. \end{aligned} \tag{4.152}$$

In appendix A.2, we derive the leading terms in the anomalous-dimension matrix that describe the mixing of the "intrinsic-charm" operators $\{T_1(\mu), T_2(\mu)\}$ into the Darwin term $\rho_D(\mu)$, see also Fig. 4.6(b),

$$\frac{d}{d\ln\mu} \begin{pmatrix} \rho_D(\mu) \\ T_1(\mu) \\ T_2(\mu) \end{pmatrix} = -\left\{ \begin{pmatrix} 0 & 0 & 0 \\ -2/3 & 0 & 0 \\ 4/3 & 0 & 0 \end{pmatrix} + \mathcal{O}(\alpha_s) \right\} \begin{pmatrix} \rho_D(\mu) \\ T_1(\mu) \\ T_2(\mu) \end{pmatrix} . \tag{4.153}$$

Neglecting the $\mathcal{O}(\alpha_s)$ contributions to the anomalous-dimension matrix, we only determine the leading-logarithmic terms,[16] which are generated by the renormalization-group equation for the short-distance coefficients

$$\begin{aligned} C_{T_i}(y,\mu) &\simeq C_{T_i}(y,m_b)\,, \\ C_{\rho_D}(y,\mu) &\simeq C_{\rho_D}(y,m_b) - \frac{1}{3}\ln\frac{\mu^2}{m_b^2}\left(C_{T_1}(y,m_b) - 2\,C_{T_2}(y,m_b)\right)\,. \end{aligned} \tag{4.154}$$

Now, integrating out the semi-hard charm quarks at $\mu_{\text{sh}} = m_c$ implies that due to energy conservation no charm quark can appear in any occurring operator. Therefore this is equivalent to setting

$$T_i(\mu \le m_c) = 0\,. \tag{4.155}$$

In this case, the expression for the lepton-energy spectrum (4.151) simplifies to

$$\left.\frac{d\Gamma^{(3)}}{dy}\right|_{\rho_D^3+\text{IC}} = \frac{G_F^2 m_b^5}{24\pi^3}\,|V_{cb}|^2\,\frac{C_{\rho_D}(y,m_c)\,\rho_D^3(m_c)}{m_b^3}\,, \tag{4.156}$$

and the information on "intrinsic charm", i.e. the non-analytic dependence on the charm-quark mass, has been completely absorbed into the short-distance function $C_{\rho_D}(y,m_c)$. This can be made explicit by inserting the leading-order matching conditions (4.152) for $C_{T_i}(y,m_b)$, which results in

$$C_{\rho_D}(y,m_c) \simeq C_{\rho_D}(y,m_b) + \ln\frac{m_c^2}{m_b^2}\,\delta(y-1)\,. \tag{4.157}$$

In this way (4.156) reproduces the logarithmic term in the lepton-energy moments in (4.115) as well as the finite terms (given by the limit $\rho \to 0$ of Eq. (A.6) in the Appendix).

Similar considerations can be made for the $\rho^2 \ln\rho$ term in the partonic rate. To this end we decompose the matrix elements of the operators in the second line of (4.146) as

$$\begin{aligned} &(4\pi)^2\,\langle\bar{B}|(i\partial_\alpha\,\bar{b}_v\,\gamma_\nu P_L\,c)\,(\bar{c}\,\gamma_\mu P_L\,b_v)|\bar{B}\rangle \\ &= 2M_B\left(T_3(\mu)\,g_{\mu\nu}\,v_\alpha + T_4(\mu)\,g_{\mu\alpha}\,v_\nu + T_5(\mu)\,g_{\nu\alpha}\,v_\mu + T_6(\mu)\,v_\mu v_\nu v_\alpha - T_7(\mu)\,i\epsilon_{\mu\nu\alpha\beta}v^\beta\right)\,. \end{aligned} \tag{4.158}$$

Note that in unpolarized observables, only the sum $T_4(\mu)+T_5(\mu)$ appears. Generalizing the results for the total rate in [51] to the lepton-energy spectrum, and concentrating again on the leading logarithmic terms, we find

$$\begin{aligned} \left.\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right|_{\text{partonic + IC}} = \frac{G_F^2 m_b^5}{192\pi^3}\,|V_{cb}|^2 \Bigg\{ & C_0(y,\mu) + \rho\,C_1(y,\mu) + \rho^2\,C_2(y,\mu) \\ & + \frac{\sum_{i=3}^{7} C_{T_i}(y,\mu)\,T_i(\mu)}{m_b^4}\Bigg\}\,, \end{aligned} \tag{4.159}$$

[16] Strictly speaking, these are N^{-1}LL. To resum the leading-logarithms of the order $\alpha_s^n \times \ln^n\rho$, we would need the $\mathcal{O}(\alpha_s)$ mixing of the Darwin term into itself and the complete set of "intrinsic-charm" operators into themselves, which goes beyond the scope of this work (see however [51] for a complete leading-log analysis for the dimension-7 contributions to the total rate).

with

$$\begin{aligned}
C_0(y,m_b) &= \left(6y^2 - 4y^3\right)\theta(1-y) + \mathcal{O}(\alpha_s)\,, \\
C_1(y,m_b) &= -6y^2\,\theta(1-y) - 6\,\delta(y-1) + \mathcal{O}(\alpha_s)\,, \\
C_2(y,m_b) &= \left[\frac{12\,\theta(1-y)}{1-y}\right]_+ - \left[\frac{6\,\theta(1-y)}{(1-y)^2}\right]_{++} \\
&\quad - 6\,\theta(1-y) + 6\,\delta(y-1) + 3\,\delta'(y-1) + \mathcal{O}(\alpha_s)\,,
\end{aligned} \tag{4.160}$$

and

$$\begin{aligned}
C_{T_3}(y,m_b) &= -24\,\delta'(y-1) + 48\,\delta(y-1) + \mathcal{O}(\alpha_s)\,, \\
C_{T_4,T_5}(y,m_b) &= -24\,\delta(y-1) + \mathcal{O}(\alpha_s)\,, \\
C_{T_6}(y,m_b) &= \mathcal{O}(\alpha_s)\,, \\
C_{T_7}(y,m_b) &= 24\,\delta'(y-1) + \mathcal{O}(\alpha_s)\,.
\end{aligned} \tag{4.161}$$

Again, the "intrinsic-charm" operators T_{3-7} mix into the 2-particle operator, namely $m_c^4\,\bar{b}\,\not{v}\,b$ (see appendix), and consequently, the coefficient $C_2(y,\mu)$ evolves as

$$C_2(y,m_c) \simeq C_2(y,m_b) - \frac{1}{8}\,\ln\frac{\mu^2}{m_b^2}\left(C_{T_3}(y,m_b) - C_{T_4}(y,m_b) - C_{T_5}(y,m_b) - C_{T_7}(y,m_b)\right)\,. \tag{4.162}$$

Inserting the leading-order matching conditions, one has

$$-\frac{1}{8}\left(C_{T_3}(y,m_b) - C_{T_4}(y,m_b) - C_{T_5}(y,m_b) - C_{T_7}(y,m_b)\right) = 6\,\delta'(y-1) - 12\,\delta(y-1)\,, \tag{4.163}$$

and one reproduces the logarithmic terms $-12\rho^2\ln\rho$ in $\Gamma_{\rm part}$ and $6\rho^2\ln\rho$ in $\langle 1-y\rangle_{\rm part}$, respectively, see (4.116, 4.117).

The power-like terms

Now turning to an alternative way to describe IC effects. We now choose to integrate out the 'heavy' degrees of freedom only above the scale m_c and concentrate on the partonic kinematical variables. As in the last section this leaves the charm quark as a dynamical entity, much in the same way as would be required for light quarks in QCD, e.g. in $b \to u\,\ell\nu$. Again we have now to include four-quark operators containing charm-quark fields explicitly.

The charm quarks in b decay can act as both a hard and a soft degree of freedom. The hard component is treated the same way as described in the previous section. The additional matrix elements of the four-quark operators contain the "soft" part of the still dynamical quarks, which for now we treat non-perturbatively. Accordingly, we have to write the original QCD product of currents as

$$\begin{aligned}
&\langle\bar{B}|\bar{b}(x)\Gamma_\nu c(x)\,\bar{c}(0)\Gamma_\mu b(0)|\bar{B}\rangle \\
&= \langle\bar{B}|\bar{b}(x)\Gamma_\nu\,\langle c(x)\bar{c}(0)\rangle\,\Gamma_\mu b(0)|\bar{B}\rangle_{>\mu} + \langle\bar{B}|\bar{b}(x)\Gamma_\nu c(x)\,\bar{c}(0)\Gamma_\mu b(0)|\bar{B}\rangle_{<\mu}\,.
\end{aligned} \tag{4.164}$$

The product $c(x)\bar{c}(0)$ can be viewed as the Green's function of the charm quark inside the external gluon field in the B meson, averaged over the field configurations present in the meson. This is an exact relation as long as the bottom meson has no charm content. This applies to the l.h.s. as well as to each of the two terms on the r.h.s. The decomposition on the r.h.s. merely reflects the different treatment used to describe these terms.

The first term with "large" μ corresponds to the "perturbative" (in $1/m_c$) standard calculation. The second term has to be added now, since the charm quark is still a dynamic quark controlled by non-perturbative dynamics. The role of the intermediate scale μ is to draw the demarcation between the two dynamical regimes. Possibly integrating out the charm corresponds to matching the second term to zero, as in the previous section.

Even though the first term in equation (4.164) is evaluated in the "direct" way depicted in detail in the beginning of this chapter, the result differs due to the introduction of the cutoff μ. The precise form of how μ enters depends on the concrete way the (hard) separation is implemented. Let us mention that for tree-level calculations without extra perturbative loops it is sufficient and convenient to simply integrate the distribution with the constraint

$$0 \leq q^2 < (m_b - \mu)^2 \,, \tag{4.165}$$

where the charm mass is replaced by the cut-off μ. The upper bound above effectively introduces the separation scale in eq. (4.164) when the correlator is integrated over the phase space to obtain the inclusive width or its moments.

Evaluating the first term in equation (4.164) gave contributions that are IR sensitive to the charm mass. Taking the formal limit $m_c \to 0$ separately term by term in the expansion would have yielded divergent expressions. With a nonzero $\mu \gg \Lambda_{\mathrm{QCD}}$ this changes: All individual terms remain regular by themselves at $m_c \to 0$. The role of the infrared regulator, which previously the physical large charm mass has played, is now taken over by the Wilsonian cutoff.

This is evident on general grounds: The terms IR-singular for $m_c \to 0$ came only from the soft charm configuration with momentum $Q \lesssim m_c$—the domain now excluded by introducing the cut-off. Alternatively, this can be traced explicitly in the formalism of Sect. 4.5.1. For instance, with the constraint of eq. (4.165) the lower limit of integration in $(v \cdot Q)$ rises from $\sqrt{Q^2}$ to $\frac{Q^2+2m_b\mu-\mu^2}{2m_b} \simeq \mu$. Therefore the infrared sensitive term in the Q^2 spectrum, $\log\frac{Q^2}{m_b^2}$ in eq. (4.124), turns into $\log\frac{\mu^2}{m_b^2}$. All the integrals become analytic functions of m_c at $m_c \ll \mu$ and the logs and inverse powers of m_c are replaced by those of the cutoff scale μ, playing the role of a mass.

The second term in the r.h.s. of equation (4.164) has a smooth $m_c \to 0$ limit. Although it may have soft "chiral" singularities when both m_c and one of the light quarks become massless, the expectation values should remain finite; only higher derivatives with respect to the charm mass may have singularities if m_u or m_d vanish. We note that this expectation value, being regularized in the ultraviolet, is well-defined and these conclusions hold with no reservations.

We have already seen, that as long as $m_c \gg \Lambda_{\rm QCD}$ holds, we have non-analytic non-perturbative terms at order $1/m_b^3$ scaling like $\frac{\Lambda_{\rm QCD}^3}{m_b^3} \ln \frac{m_b^2}{m_c^2}$ and $\frac{\Lambda_{\rm QCD}^3}{m_b^3} \left(\frac{\Lambda_{\rm QCD}^2}{m_c^2}\right)^k$ with $k > 0$: In general, odd powers of m_c also emerge if perturbative corrections are considered. Each of these terms separately are singular at $m_c \to 0$. The leading term is driven by the Darwin expectation value, it represents an IR singularity in m_c/m_b which, in principle, is observable, at least at large m_b and sufficient accuracy. This we have analyzed in the last section considering the lepton energy spectrum.

The expansion in $\Lambda_{\rm QCD}/m_c$ makes, however, sense only as long as charm remains heavy on the scale of QCD dynamics. At lower m_c the successive terms with higher k would formally dominate. The whole function of m_c, which is then to be seen as a non-perturbative quantity, stabilizes at $m_c \lesssim \Lambda_{\rm QCD}$ and approaches a finite value at $m_c \to 0$. A model for such a behavior can be given, for instance, by

$$\frac{\rho_D^3}{m_b^3} \ln \frac{m_b^2}{m_c^2 + \Lambda^2} \,. \tag{4.166}$$

Here Λ is a strong interaction mass scale parameter of the order of $\Lambda_{\rm QCD}$ which, expanded in $1/m_c^2$, would yield the whole series in $1/m_c^2$. The actual coefficients for the $1/m_c^{2k}$ terms may, of course, be different, and they can be calculated in the OPE along either road. We will give a brief numerical discussion on this, at the end of the section.

Returning to the OPE analysis proper, we can bridge the two ways of accounting for the non-perturbative charm effects by looking at the μ-dependence of both terms in eq. (4.164). More accurately we have to integrate over the phase space to obtain the inclusive probability. Their sum must be μ independent which provides useful relations. The following note should be kept in mind.

The form of the μ-dependence is determined by the regularization scheme. In the Wilsonian procedure with a hard cutoff the leading log dependence is accompanied by power terms. The resulting dependence is qualitatively different for $\mu \ll m_c$ and for $\mu \gg m_c$. The first assumption corresponds to a heavy charm quark, whereas the second to a light charm quark. We will discuss these two cases in turn.

1. When μ is taken small compared to m_c, it enters only as a small power correction, $\sim (\mu/m_c)^l$ in the first term of equation (4.164). In the formally possible limit $\mu \to 0$ ($m_c \gg \Lambda_{\rm QCD}$ fixed) the last term would vanish. Then the whole correction to the width is given by the first term. Raising μ up to m_c and above moves the correction to the width from the first term to the second. The calculation of Sect. 4.5.1 represents the evaluation of the first term contribution in the $\Lambda_{\rm QCD}/m_c$-expansion for $\mu \ll m_c$ in the limit $\mu \to 0$. Such a calculation makes sense only as long as charm is sufficiently heavy. The matching of the second term to zero, as in the case of dimensional regularization, is therefore reproduced.

2. At $\mu \gg m_c$ the situation is different. Beyond the leading term $\mathcal{O}_D \ln \frac{\mu^2}{m_c^2}$ the μ dependence is suppressed by powers of m_c^2/μ^2 in the first term. In the second term in equation (4.164) these appear as a residual dependence of the higher-dimensional terms on the ultraviolet cutoff for integrals intrinsically convergent

at the scale m_c. In the first term it shows now up as small coefficient functions of higher-dimension operators—which would normally be saturated at soft charm configurations $Q \sim m_c$—proportional to $1/m_b^3\, 1/\mu^{2k}$, instead of $1/m_b^3\, 1/m_c^{2k}$ without a cutoff. It is then convenient to assume a large renormalization point $\mu \gg m_c$ and neglect these terms altogether. We will focus on this on the next pages, to investigate the behaviour in the case $m_b \gg m_c \gg \Lambda_{\rm QCD}$. This scenario can also be applied for the case $m_c \sim \Lambda_{\rm QCD}$.

A short comment is in order on how the separation would look like in the often adopted dimensional regularization scheme (DR), where no power-like dependence on the renormalization scale ever arises. For small m_c the only μ-dependence within DR enters through $\ln\mu$ in the Darwin operator, as we have seen before. In such a scheme, however, charm quarks must be treated as massless. The dynamic four-quark operators have to be renormalized in the UV likewise in the way of DR. The requirement to set $m_c = 0$ then is rather clear. Keeping m_c finite would destroy naive DR in a direct calculation: it yields a finite result for the first term around $D = 4$ because the potential IR singularity at $D = 4$ dimensions is regularized by a non-zero charm mass. As a result, this requires an additional matching procedure, usually order by order in m_c if the charm mass dependence is important. This has been used to show how the $\rho^2 \log\rho$ term in the total rate arises and has to be extended for the power-like IR sensitivities. This is a rather typical feature of dimensional regularization.

In logarithmic terms, as for the Darwin operator in the integrated width, the charm mass plays the role of a renormalization point, especially seen in DR. Therefore, as discussed in Ref. [46], the infrared dependence on m_c in the 'conventional' calculation along the 'first' road must match the ultraviolet (UV) dependence of the corresponding four-quark expectation value given by $\rho_D^3 \ln\frac{\Lambda_{\rm UV}^2}{m_c^2}$. A similarly dual description applies also for the terms scaling like inverse powers of m_c. They can be calculated conventionally assuming charm to be heavy following the route of Sect. 4.2.1. Alternatively, they can be obtained as the corresponding pieces of the four-quark expectation value, normalized at $\mu \gg m_c, \Lambda_{\rm QCD}$. The results over either road must be identical, whenever one is in the domain where the expansion can be applied. The first road, without implementing a cutoff is, of course, justified only for $m_c \gg \Lambda_{\rm QCD}$. The second route is formally valid for an arbitrary hierarchy between m_c and $\Lambda_{\rm QCD}$. However, we do not have the means to calculate the expectation value through gluon operators without charm fields, when charm becomes light.

In order to make the above mentioned correspondence explicit, we will address the $1/m_b^3\, 1/m_c^n$ terms where at tree level only even n emerge. To this end, we consider the contribution of the second term in eq. (4.164) involving the explicit charm quark operators. Inserting it into the hadronic tensor we get as in eq. (4.146)

$$2M_B W_{\mu\nu}^{\rm IC} = (2\pi)^3 \delta^4(Q)\, \langle \bar{B}|\bar{b}_v \Gamma_\nu c\, \bar{c}\Gamma_\mu b_v|\bar{B}\rangle_\mu + (2\pi)^3 \left(-i\frac{\partial}{\partial Q_\alpha}\right) \delta^4(Q) \langle \bar{B}|\partial_\alpha \bar{b}_v \Gamma_\nu c\, \bar{c}\Gamma_\mu b|\bar{B}\rangle_\mu + \ldots \tag{4.167}$$

and we retain only the first term. The extra derivative for (renormalized) expectation values can bring in a factor of m_c or μ at most, it can be traced that powers of x_α translate into powers of $1/m_b$. As we have seen before, this term therefore matches in dimensional regularization into the term proportional to the mass operator. Therefore, the higher terms in this expansion yield higher powers in the $1/m_b$ expansion with additional powers of m_c. We then can focus in this analysis on the first term which is the conventional $D = 6$ four-quark operator. Note that the δ-function projects out the leading term of the leptonic tensor (4.127). Furthermore, this contribution is localized at $Q = 0$—hence at $Q^2 = 0$ and $v \cdot Q = 0$, in agreement with the findings of the last section. As a consequence, step-functions in equation (4.118) become superfluous. The last relation completing the arithmetic part is

$$\int \mathrm{d}(v \cdot Q)\mathrm{d}Q^2 \sqrt{(v \cdot Q)^2 - Q^2}\delta^4(Q) = \frac{1}{2\pi} \,. \tag{4.168}$$

Omitting QCD corrections, the relevant diagrams for calculating $W_{\mu\nu}^{\mathrm{IC}}$ in eq. (4.167) are the one loop diagrams involving the charm-quark loop with an arbitrary number of external gluons. These diagrams have been considered already in [45]. Examples of the Feynman diagrams are shown in Figure 4.7. It is advantageous to first perform a Fierz rearrangement of the four quark operator according to

$$\bar{b}\,\Gamma^\nu c\;\bar{c}\,\Gamma^\mu b = -\frac{1}{2}\bar{b}_\alpha \Gamma_\rho b_\beta\;\bar{c}_\beta \Gamma_\sigma c_\alpha \left[-i\epsilon^{\mu\nu\rho\sigma} + g^{\sigma\mu}g^{\rho\nu} + g^{\sigma\nu}g^{\rho\mu} - g^{\rho\sigma}g^{\mu\nu}\right], \tag{4.169}$$

where α, β are color indices, and the minus sign comes from anti-commutativity of the quark fermion fields.

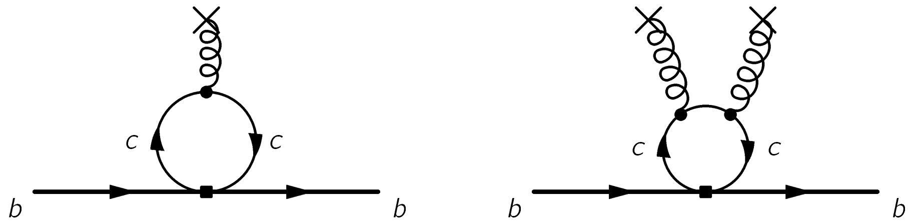


Figure 4.7.: Diagrams illustrating calculation of the charm loop in the external field. The curly lines generically reflect insertions of the external gluon field.

We may construct the charm mass expansion of this charm loop in the external gauge field and take the average over the B meson state. There are a few subtleties related to this procedure, since the two charm-quark operators are taken at coinciding space-time points. As in [45] we may start from the time-ordered product of two charm-quark operators at displaced points, which amounts to consider the conventional charm propagator in an external field. This propagator is generally gauge dependent, yet in the end this is compensated by the same displacement in the b-quark fields. For constructing the short-distance expansion of the Green's function the fixed-point gauge is convenient.

The limit of coinciding points in the charm Green's function is formally divergent, yet gauge independent. Subtracting the free Green's function, this piece is accounted for in the purely partonic width, we end up with a mild log divergence present in the vector current, proportional to $[D_\mu, G_{\mu\nu}]$:

$$\langle \bar{c}_\alpha \gamma_\nu c_\beta \rangle_A = \frac{2}{3}\frac{1}{(4\pi)^2} \ln\left(\frac{\Lambda_{\rm UV}^2}{m_c^2}\right) [D^\kappa\,,\,G_{\kappa\nu}]_{\beta\alpha} + \cdots \tag{4.170}$$

where $G_{\mu\nu}$ is the QCD gauge-field strength tensor. In this context $\Lambda_{\rm UV}$ should be identified with μ, and the ellipses denote finite terms to be considered below. As discussed before, this ultraviolet-singular log matches onto the infrared piece of the conventionally calculated Darwin coefficient function. The contributions from the axial-vector $\bar{c}\gamma_\nu\gamma^5 c$ current are convergent ab initio.

Now we turn to the finite terms, which—in cut-off scheme—account for the power-like infrared sensitive terms. The lowest finite terms yield $1/m_b^3\, 1/m_c^2$ contributions given by

$$\langle \bar{c}_\alpha \gamma_\nu \gamma_5 c_\beta \rangle_A = \frac{1}{48\pi^2 m_c^2}\left(2\left\{\left[D_\kappa, G^{\kappa\lambda}\right], \tilde{G}_{\nu\lambda}\right\} + \left\{\left[D_\kappa, \tilde{G}_{\nu\lambda}\right], G^{\kappa\lambda}\right\}\right)_{\beta\alpha} + \cdots \tag{4.171}$$

$$\begin{aligned}\langle \bar{c}_\alpha \gamma_\nu c_\beta \rangle_A = \frac{i}{240\pi^2 m_c^2}\Big(&13\left[D_\kappa, \left[G_{\lambda\nu}, G^{\lambda\kappa}\right]\right] + 8i\left[D^\kappa, \left[D^\lambda, [D_\lambda, G_{\kappa\nu}]\right]\right] \\ &-4i\left[D^\lambda, [D^\kappa, [D_\lambda, G_{\kappa\nu}]]\right]\Big)_{\beta\alpha} + \cdots\end{aligned} \tag{4.172}$$

The computation assumes $\mu \gg m_c$ and neglects power terms $\sim (m_c/\mu)^k$, according to the scenarios we have presented. Inserting this into (4.169) we end up with dimension-eight $\bar{b}_v \ldots b_v$ operators with soft-gluon fields. Their coefficient functions compared to the partonic $D = 3$ operator $\bar{b}_v\, b_v$ are proportional to $1/m_b^3\, 1/m_c^2$.

A closer look into the calculations of the charm loop in the external field reveals that the resulting expressions exactly parallel those in section 4.5.1 once the leading-order approximation for the leptonic tensor (4.127) is adopted and only non-analytic terms according to eq. (4.123) are retained. This occurs before the full integration is performed, when one takes the integral over the time-like component of the loop momentum by the residues at $Q^2 = m_c^2$, for each power term in the expanded propagator.

In fact, the technique of the loop calculation in the external field itself allows to derive a number of relations which strongly constrain the form of the operators, which can appear in such an expansion. It has been presented in detail in the first part of Ref. [52]. These relations ensure that the result has always the form of multiple commutators. This sharpens one observation made already in Sec. 4.5.1. There we noted that the partonic matrix elements of IC contributions necessarily have to be at least one-gluon matrix elements. From the arguments given in this section we conclude that intrinsic charm contributions involve only gluon fields and their derivatives. There will be no derivatives acting on the bottom quark fields that would generate a dependence on the 'residual momentum' of the decaying b-quark.

We have verified through explicit calculations of the $1/m_b^3\, 1/m_c^2$ terms that these two ways to calculate $1/m_c$-singular terms yield the same result for the operator expansion.

At order $1/m_b^3\,1/m_c^2$ the result can thus be expressed through five operators, which are determined by two contributions to the axial vector current and three contributions to the vector current:

$$2M_B\tilde{r}_1 = \langle B|\bar{b}_v\Big[iD_\kappa, \big[iD_\lambda, [iD^\lambda, iG^{\kappa\alpha}]\big]\Big]b_v|B\rangle\; v_\alpha \tag{4.173a}$$

$$2M_B\tilde{r}_2 = \langle B|\bar{b}_v\Big[iD_\lambda, \big[iD_\kappa, [iD^\lambda, iG^{\kappa\alpha}]\big]\Big]b_v|B\rangle\; v_\alpha \tag{4.173b}$$

$$2M_B\tilde{r}_3 = \langle B|\bar{b}_v\Big[iD_\kappa, \big[iG_{\lambda\alpha}, iG^{\lambda\kappa}\big]\Big]b_v|B\rangle\; v^\alpha \tag{4.173c}$$

$$\begin{aligned}2M_B\tilde{r}_4 = {} & \langle B|\bar{b}_v\big\{\big[iD^\rho, iG_{\rho\lambda}\big], iG_{\delta\gamma}\big\}(-i\sigma_{\alpha\beta})b_v|B\rangle \\ & \times \frac{1}{2}\left(g^{\lambda\alpha}g^{\delta\beta}v^\gamma - g^{\lambda\alpha}g^{\gamma\beta}v^\delta + g^{\delta\alpha}g^{\gamma\beta}v^\lambda\right)\end{aligned} \tag{4.173d}$$

$$\begin{aligned}2M_B\tilde{r}_5 = {} & \langle B|\bar{b}_v\big\{\big[iD^\rho, iG_{\sigma\lambda}\big], iG_{\rho\gamma}\big\}(-i\sigma_{\alpha\beta})b_v|B\rangle \\ & \times \frac{1}{2}\left(g^{\sigma\alpha}g^{\lambda\beta}v^\gamma - g^{\sigma\alpha}g^{\gamma\beta}v^\lambda + g^{\lambda\alpha}g^{\gamma\beta}v^\sigma\right)\,.\end{aligned} \tag{4.173e}$$

The contributions originating from the axial current yield spin-triplet operators, while those of the vector current yield spin singlet operators. These operators are in fact related to a linear-combination of the basis parameters (4.57).

4.5.3. $m_b \gg m_c \sim \Lambda_{\mathrm{QCD}}$

Now we consider the case of a very light charm-quark. This corresponds to the $b \to u$ transition. In one respect we should be careful: The B meson containing $b\bar{u}$ has to be treated differently from $b\bar{d}$, since in the first case there is a spectator quark contribution.

Treatment of the Darwin Term

If we consider the dynamics at the charm-quark mass scale to be in the non-perturbative regime, we cannot exploit the condition (4.155) and are left with the general formula for the leptonic-energy spectrum in (4.151), which should be evaluated at a scale μ_0 that satisfies $m_c \ll \mu_0 \leq m_b$. Moreover, we have to take seriously the new power counting. In dimensional regularization terms proportional to ρ have to be considered as operators with mass insertion, as we have seen before. Therefore this power counting implies that terms of order ρ^2 now count as $(\Lambda/m_b)^4$ and should be neglected to the order that we are considering, we are thus left with

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\bigg|_{\mathrm{partonic}} = \frac{G_F^2 m_b^5}{192\pi^3}\,|V_{cb}|^2\Big\{C_0(y,\mu_0) + \rho\, C_1(y,\mu_0) + \mathcal{O}(\rho^2)\Big\}\,, \tag{4.174a}$$

$$\frac{\mathrm{d}\Gamma^{(3)}}{\mathrm{d}y}\bigg|_{\rho_D^3+\mathrm{IC}} = \frac{G_F^2 m_b^5}{24\pi^3}\,|V_{cb}|^2\left\{\frac{C_{\rho_D}(y,\mu_0)\,\rho_D^3(\mu_0)}{m_b^3} + \frac{C_{T_1}(y,\mu_0)\,T_1(\mu_0)}{m_b^3}\right\}, \tag{4.174b}$$

together with the contributions to the lepton-energy spectrum from μ_π^2,μ_G^2 and ρ_{LS}^3 (see e.g. [28]), where the limit $\rho \to 0$ to the considered order $(1/m_b^3)$ is trivial. Due to the power-counting we therefore have to consider $\mathcal{O}(\rho)$ terms in the partonic rate, only.

In that order, the genuinely intrinsic-charm contribution comes together with the Darwin term, only. In particular, to leading logarithmic accuracy (4.154), the contributions to the total rate, and the moments $\langle y \rangle$ and $\langle y^2 \rangle$ can be obtained, exactly, as

$$\Gamma^{(3)}\Big|_{\rho_D^3+\mathrm{IC}} = \frac{G_F^2 m_b^5}{24\pi^3}\,|V_{cb}|^2 \left\{ X(\mu_0) + \frac{\rho_D^3(\mu_0)}{m_b^3}\left[\frac{17}{12}\right] \right\}, \tag{4.175a}$$

$$\langle y \rangle\Big|_{\rho_D^3+\mathrm{IC}} = \frac{G_F^2 m_b^5}{24\pi^3}\,|V_{cb}|^2 \left\{ X(\mu_0) + \frac{\rho_D^3(\mu_0)}{m_b^3}\left[\frac{47}{30}\right] \right\}, \tag{4.175b}$$

$$\langle y^2 \rangle\Big|_{\rho_D^3+\mathrm{IC}} = \frac{G_F^2 m_b^5}{24\pi^3}\,|V_{cb}|^2 \left\{ X(\mu_0) + \frac{\rho_D^3(\mu_0)}{m_b^3}\left[\frac{287}{180}\right] \right\}, \tag{4.175c}$$

where we defined the parameter combination

$$X(\mu_0) = -\frac{3\,T_1(\mu_0)}{m_b^3} + \ln\frac{\mu_0^2}{m_b^2}\,\frac{\rho_D^3(\mu_0)}{m_b^3}\,. \tag{4.176}$$

In this case, inevitably the "new" hadronic parameter T_1 has to be introduced in contrast to the heavy charm quark. Nevertheless some statements can be made at this point. Considering a sizeable value for $T_1(\mu_0)$ at small hadronic scales μ_0, in contrast to the perturbative situation considered in the previous subsection, and taking into account that the ρ_D^3 contribution in $X(\mu_0)$ is formally enhanced by $\ln \mu_0^2/m_b^2$, we may ignore the (small) differences between the individual moments induced by the numbers in square brackets in (4.175), to first approximation. Therefore, even in this genuine intrinsic-charm scenario, the inclusion of a large non-perturbative intrinsic-charm effect, basically amounts to treating the Darwin term ρ_D^3 for the effective parameter X. In any case, one may consider the limit $m_c \sim \Lambda_{\mathrm{QCD}}$ rather academic, and would prefer the scenario with semi-hard charm-quarks as in the previous subsection for the discussion of "intrinsic-charm" effects in inclusive semi-leptonic B decays.

We should also mention that (4.174a) and (4.174b) provide the appropriate formulas for the massless limit, relevant to $b \to u\ell\nu$ decays, after appropriate changes $V_{cb} \to V_{ub}$ and re-interpretation of the intrinsic-charm operators as so-called weak annihilation operators [53–55]. Notice that the (local) annihilation operators enter at order $1/m_b^3$ in the standard OPE, whereas their non-local counterparts, necessary to describe the shape-function region, already enter at (relative) order Λ/m_b [56–59].

Treatment of the Power-like Infrared Sensitive Terms

Our preceding discussion can suggest a dual description of IR charm effects that allows insights into a higher-order OPE calculation of the heavy-to-light case $B \to X_u \ell \bar{\nu}_\ell$. More specifically the limiting case $m_c \to 0$ is of relevance when treating the heavy-to-light case beyond order $1/m_b^2$. The inherent IR divergence of the standard calculation for $m_c/m_b \to 0$ underlies the importance of the four quark operator matrix element in the heavy-to-light case, which is usually called weak-annihilation (WA) contribution. More precisely, here it corresponds to its valence-quark insensitive piece which affects semileptonic B^+ and B^0 decays equally. We refer to it as non-valence WA. This we already have seen for the logarithmic dependence of the Darwin term.

In the next few pages, we want to give an estimate of the size of this contribution. Therefore we will investigate the power-like terms in the total rate, in order to make use of the model (4.166). Then we will argue, on how the order of magnitude of weak annihilation can be inferred from experimental data. In the end, we will compare these two approaches for the size of these effects.

In fact, the total rate, expressed in terms of the operators given in (4.173) reads as

$$\frac{m_b^3 m_c^2}{\Gamma_0}\,\Gamma\Big|_{\frac{1}{m_c^2}} = -\frac{3}{2}\frac{2}{15}\left(-8\tilde{r}_1 + 4\tilde{r}_2 - 13\tilde{r}_3\right) + \frac{1}{2}\frac{2}{3}\left(-2\tilde{r}_4 - \tilde{r}_5\right). \tag{4.177}$$

With the given numerical estimates of the non-perturbative parameters from tables 4.5 and 4.5 we find the intrinsic-charm operators[17] numerically to be

$$\tilde{r}_1 \approx 0.20\,\mathrm{GeV}^5, \qquad \tilde{r}_2 \approx 0.25\,\mathrm{GeV}^5, \qquad \tilde{r}_3 \approx 0.14\,\mathrm{GeV}^5, \tag{4.178a}$$

$$\tilde{r}_4 \approx -0.30\,\mathrm{GeV}^5, \qquad \tilde{r}_5 \approx 0.29\,\mathrm{GeV}^5\,. \tag{4.178b}$$

which lead to a shift for the total rate of[18]

$$\delta\Gamma\Big|_{\frac{1}{m_c^2}} \approx (0.75\%) \times \Gamma_{\mathrm{Parton}}\,. \tag{4.179}$$

This numerical estimate should not be considered bullet-proof. While the individual matrix elements are predicted with reasonable confidence in their signs and magnitudes, we are faced with a set of terms with different signs. Thus cancellations will in general occur among them. Their degree may depend on the numerical accuracy of the applied ground-state factorization, as well as on the precise values of the lower-dimension expectation values μ_π^2, ρ_D^3 and ρ_{LS}^3.

Finally we note that for the moments the situation is different. Contributions that introduce an IR sensitivity to the charm quark mass in the total rate may become regular for the moments. This becomes evident if we consider moments of the partonic invariant mass such as $\langle (p^2 - m_c^2)^n \rangle$ which remain regular as $m_c \to 0$ till higher orders in Λ_{QCD}. We have given a numerical analysis on this in section 4.4.

Having at hand the numerical estimates of the higher-dimension expectation values allows us to refine the model (4.166) for the charm-mass dependence in the IR regime. Since the $1/(m_b^3 m_c^2)$ corrections calculated in this ansatz are fixed in terms of a non-perturbative parameter Λ, we assign the latter the value which would reproduce these

[17] We have found a discrepancy with Ref. [45] in the overall factor for one of the expectation values. It does not produce a noticeable numerical change for the correction to the width, however.

[18] Ref. [45] included an additional phase-space suppression factor for the IC kinematic of $(1 - m_c/m_b)^2$. Based on the operator analysis we can show that actually it is absent in the case at hand.

leading corrections. This yields by expanding the model in m_c and comparing it to the direct calculation

$$\Lambda^2 \equiv M_*^2 = \frac{\tilde{f}_1}{5\rho_D^3} - \frac{\tilde{f}_2}{10\rho_D^3} + \frac{13\tilde{f}_3}{40\rho_D^3} - \frac{\tilde{f}_4}{12\rho_D^3} - \frac{\tilde{f}_5}{24\rho_D^3} \simeq (0.7\ \text{GeV})^2\,, \tag{4.180}$$

and thus the model predicts

$$(-\delta^{\alpha\beta} + v^\alpha v^\beta)\,\frac{1}{2M_B}\langle \bar{B}|\bar{b}\gamma_\alpha(1-\gamma_5)c\,\bar{c}\gamma_\beta(1-\gamma_5)b|\bar{B}\rangle\bigg|_\mu \simeq -\frac{\rho_D^3}{4\pi^2}\ln\frac{m_b^2}{m_c^2+M_*^2}\,, \tag{4.181}$$

see also in plot 4.8. In this case the correction to the width at $m_c^2 \ll m_b^2$ takes the form

$$\frac{\delta\,\Gamma_{\rm sl}}{\Gamma_{\rm sl}} \simeq -\frac{8\rho_D^3}{m_b^3}\left(\ln\frac{m_b^2}{m_c^2+M_*^2} - \frac{77}{48}\right)\,, \tag{4.182}$$

where the constant term accounts for the explicit UV contribution in this limit. This model illustrates to which extent the charm quark may be considered heavy in this context.

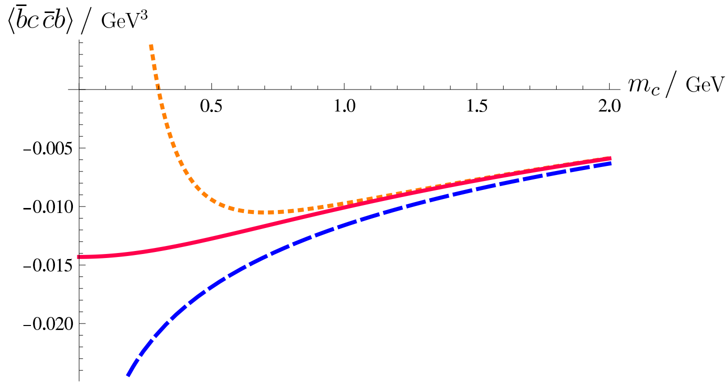


Figure 4.8.: The WA expectation value $\frac{1}{2M_B}\langle B|\bar{b}\gamma^k(1-\gamma_5)c\,\bar{c}\gamma^k(1-\gamma_5)b|B\rangle_\mu$ as a function of the charm mass: The leading logarithmic contribution in m_c (blue dashed), including additionally the $1/m_c^2$ power correction (orange dotted), and the complete behavior according to ansatz (4.181) (red solid). We assume the value $\rho_D^3 = 0.15$ GeV^3 and m_b is 4.59 GeV.

It should be acknowledged that the equations (4.166), (4.180) and (4.182) are only a reasonable model for the effects at intermediate to small m_c. In fact, the size of the effective mass scale M_* as determined by matching the leading charm power corrections may not be fully universal: It depends on the particular Lorentz structure of the weak vertices. It would likewise differ, say, in weak B^* decays—although numerically such a variation in M_* would be insignificant.

The important question here is the potential scale of the non-valence WA contributions, once the effect of the Darwin operator has been separated out, which we have presented before[19]. These effects are directly related to the pieces with $1/m_c$ contribution. A direct computation based on the method discussed above is not possible. Yet one may try to estimate the natural scale expected for WA by approaching the massless case from the heavy-quark side, as it was done for the Darwin expectation value using dimensional regularization. For this purpose we use the model suggested above. In spite of its admitted oversimplification, we can be sure that the difference between this ansatz and the actual QCD contribution remains finite at any m_c including the limit $m_c \to 0$. The model has the advantage of being sufficiently accurate when extrapolating down from the side of intermediately heavy charm quarks.

It is therefore plausible that setting $m_c = 0$ and applying this model we do not stray far away from the leading effect of the non-valence component of WA in $b \to u\,\ell\nu$ decays (which is its only contribution in B_d decays);[20] this "guesstimate", though, cannot be validated in the context of the $1/m_c$ expansion examined here. In some sense such an assumption implies that no 'phase transition' occurs when going down in mass from heavy to light quarks. We know such a phase transition takes place in the QCD vacuum, yet it may not necessarily be important for the expectation values over B-meson states. One may a priori expect larger WA effects in $b \to u\,\ell\nu$ coming from its flavor-specific piece manifesting itself in the decays of charged B-mesons.

Using the model as presented above, we interpolate between the regimes of heavy and light charm we get an estimate

$$\frac{1}{2M_B}\langle B_d|\bar{b}\boldsymbol{\gamma}(1-\gamma_5)u\,\bar{u}\boldsymbol{\gamma}(1-\gamma_5)b|B_d\rangle \simeq -\frac{\rho_D^3}{4\pi^2}\ln\frac{m_b^2}{\mu^2+M_*^2} \approx -0.01\,\mathrm{GeV}^3\,, \tag{4.183}$$

with $\mu \approx 0.5$ GeV denoting the normalization scale, which replaces the charm mass, and M_* is the hadronic scale parameterizing the non-perturbative effect. This is an educated guess and cannot guarantee to yield even the correct sign of the effect at $\mu \sim 1\,\mathrm{GeV}$. The negative sign physically means that the propagation of the soft u quark (projected onto the spin state specified by the Lorentz structure in question) is suppressed compared to free propagation. Taken at face value, the WA expectation value in Eq. (4.183) would yield the iso-scalar shift in the semileptonic B decay width

$$\frac{\delta\Gamma_{\rm sl}^{\rm WA}(b\to u)}{\Gamma_{\rm sl}(b\to u)} \simeq -\frac{8\rho_D^3}{m_b^3}\ln\frac{m_b^2}{\mu^2+M_*^2} \approx -0.038\,. \tag{4.184}$$

[19] The difference due to the valence part of WA in $b \to u\,\ell\nu$ can be experimentally probed by analyzing the difference in the semileptonic spectra spectra of charged and neutral B decays.

[20] Reference [53] which first analyzed the effects of generalized WA in semileptonic decays, focused on the differences between mesons with different spectators and therefore explicitly subtracted the $\bar{b}u\,\bar{u}b$ expectation value in B_d from that in B^-. Following an earlier classification of the pre-asymptotic power corrections to the inclusive widths it referred to this as the spectator-dependent correction, a terminology continued to a number of later publications. The valence and non-valence effects separately were considered in Ref. [60] where valence effects were sometimes also called 'spectator' contributions. The importance of the non-valence WA for light quarks, although conjectured already back in 1994, was emphasized in Ref. [61]. There the "singlet" and the WA proper pieces referred to the average and the difference of the valence and non-valence components.

A more refined way to assess non-valence WA for $b \to u\,\ell\nu$ would be to consider the real massless case for the final-state quark, $m_u = 0$, thereby introduce an IR cutoff via the kinematic restriction equation (4.165). The first term in equation (4.164) representing the "UV" piece of the $\bar{b}u\,\bar{u}b$ expectation value from the domain of quark momenta above μ is then calculated in the direct way of section 4.2.1 with the limit $m_c \to 0$. The result is expressed in terms of the same five expectation values, with the coefficients scaling as $1/\mu^2$. Yet they would combine to yield in general a different combination of the operators and, consequently, a different number. Evaluating the result with $\mu \approx 0.6\,\text{GeV}$ would provide an estimate of the minimal natural scale of non-valence WA.

Even such an estimate would be admittedly incomplete. Beyond its lack of precision in evolving the μ dependence to as low a value as 0.6 GeV, the total WA should also include $\langle B|\bar{b}u\,\bar{u}b|B\rangle_{<0.6\,\text{GeV}}$, which is the last term in Eq. (4.164). The contribution from the low momenta plausibly exceeds the numerical estimate above, and may even change the overall sign. However, it would be unnatural to allow the contributions from physically distinct domains of low and high momenta to show significant cancellations. Therefore, we would view the thus obtained estimate as a firmer lower bound on the scale of non-valence WA in $b \to u\,\ell\bar{\nu}_\ell$.

One may anticipate a potentially more significant effect for the "valence" part of WA which describes this effect in $\Gamma(B^+ \to X_u\,\ell\bar{\nu}_\ell)$. On physical grounds we expect this contribution to contain a piece independent of the non-valence WA and not related to something that can be traced from the $b \to c\,\ell\nu$ decays in the limit of small charm mass. In the formal derivation of relating $c(x)\bar{c}(0)$ in equation (4.164) to charm Green's functions it would be associated with an additional term. That term appears through the contraction of the quark fields with those of the same flavor whose presence is required in the interpolating currents to produce the initial and to annihilate the final B meson state.

On the other hand, as pointed out in Refs. [53, 62] and exploited later in the publications [60] and [61], one may infer certain information about WA, and in particular about valence WA from the D-meson decays. The charm quark is marginally heavy enough to apply precision heavy-quark expansions to its decays. Extrapolation from charm to bottom may thus be semi-quantitative at best, yet it should provide some constraint on the expected scale of WA's physical implementation. In particular, the recent CLEO-c [63] data on the semileptonic branching fraction of D_s indicate the presence of a *destructive* spectator-related WA contribution of around 20%; it must be the result of non-factorizable four-quark expectation values. In interpreting this observation one has to address the question of $SU(3)$-breaking in the leading non-perturbative corrections described by the kinetic and chromo-magnetic operators and, possibly, in the Darwin expectation value.

The situation for D mesons is different, than in the case of B mesons with respect to the spectator quarks. The charm quark is an up-type quark, and therefore tree level transitions in weak annihilation processes can occur with a down quark, only. While semileptonic "valence" WA cannot occur at all in D^0 decays, it can contribute in D^+ and D_s decays on the Cabbibo suppressed and Cabbibo allowed levels, respectively.

Assuming possible $SU(3)$ breaking to be under control properly in WA, we conclude that the observed difference of the total semileptonic widths for D_s and D^0

$$\frac{\Gamma_{\rm sl}(D_s)}{\Gamma_{\rm sl}(D^0)} = \frac{{\rm Br}_{\rm sl}(D_s)}{{\rm Br}_{\rm sl}(D^0)} \frac{\tau_{D^0}}{\tau_{D_s}} \simeq 0.83 \qquad (4.185)$$

must be dominated by valence WA effects in D_s. To describe the pattern on the WA effects in D mesons we may adopt the nomenclature for the generalized "annihilation" correction following reference [53]: The *valence* weak annihilation ${\rm WA}_q^{\rm val}$ for a particular transition $c \to q\,\ell\nu$, where $q = s$ or d, refers to the *difference* in the matrix elements between D_q and D^0. The *non-valence* ${\rm WA}_d^{\rm n\,val}$ is directly the expectation value in the (non-strange) state D^0, containing no valence d quark. To allow for $SU(3)$ asymmetry we also have to distinguish ${\rm WA}_d^{\rm n\,val}$ from ${\rm WA}_d^{\rm n\,val\,(s)}$ where it is considered in the strange D_s state. The WA operators above may be general products like $\bar{c}q\,\bar{q}c$, either local or nonlocal. For the decaying quark being heavy enough, like in B mesons, it would be sufficient to include only the leading local four-quark operators. With this convention, we in general have in D mesons

$$\Gamma_{\rm sl}(D^+) - \Gamma_{\rm sl}(D^0) = \sin^2\theta_c \cdot {\rm WA}_d^{\rm val} \qquad (4.186a)$$

$$\Gamma_{\rm sl}(D_s) - \Gamma_{\rm sl}(D^0) = \cos^2\theta_c \cdot {\rm WA}_s^{\rm val} - \sin^2\theta_c \left[{\rm WA}_d^{\rm n\,val} - {\rm WA}_d^{\rm n\,val\,(s)}\right] + \Delta_{SU(3)}\,. \qquad (4.186b)$$

By introducing the subscript marking the d or s flavor, we have explicitly allowed for the $SU(3)$ breaking in the expectation values due to the different spectator in a meson or in the light quark field flavor in the corresponding operator. We have still neglected the explicit short-distance $SU(3)$-breaking $\propto m_s^2$ in the coefficient functions emerging due to the larger strange mass in the hard quark Green's functions. It is expected to be strongly suppressed. $\Delta_{SU(3)}$ in Eq. (4.186) therefore denotes only the shift related to the $SU(3)$ violation in the (flavor-singlet) non-perturbative expectation values between the strange and non-strange heavy-meson states. The analysis suggests that these effects are numerically suppressed for the kinetic and the chromo-magnetic operators and should not exceed 5% level in the widths. Then the bulk of the difference in equation (4.185) should be equated with the valence component of WA, at least if $SU(3)$ violation in it is not too strong.

Translating these relations for WA from charm to bottom is associated with significant uncertainties due to a potentially poor representation of the contributions to the inclusive width for charm by the (truncated) OPE. The charm mass is manifestly not large enough for a precision treatment. Relating WA for B decays to the expectation values of the $D = 6$ operators can be done with acceptable theoretical accuracy. Yet expressing the WA contributions in (4.186) through the analogous local expectation values in D mesons is expected to have large corrections – first of all from the corresponding higher-dimension operators with additional derivatives. Related to this is the short-distance "hybrid" [64–66] renormalization of the operators in question from the scale of charm to bottom. In this case it may be even not fully perturbative.

Bearing in mind these potential caveats, we nevertheless use this line of reasoning to estimate the expected size of the valence WA in the semileptonic $b \to u$ width of charged B meson:

$$\frac{\Gamma(B^+ \to X_{\text{light}}\,\ell\bar{\nu}_\ell) - \Gamma(B^0 \to X_{\text{light}}\,\ell\bar{\nu}_\ell)}{\Gamma(B \to X_{\text{light}}\,\ell\nu)} \approx -(0.005 \div 0.01) \tag{4.187}$$

which is similar in magnitude, yet still below our estimates for the minimal scale of the non-valence WA.

4.6. Power Corrections in $b \to s + \gamma$

As another application of the technique described in section 4.2.1 we have considered higher-order power corrections to the decay rate and to the photon energy moments in radiative $b \to s + \gamma$ decays. They have been treated in the approximation of the local weak vertex, derived in section 3.2.2, where no operators with charm quarks or chromo-magnetic $b \to s$ vertex were considered. The analysis parallels that of the semileptonic case, except that the simplicity of the kinematics (corresponding to $q^2 = 0$ in the latter) leads to reasonably compact analytic expressions, even in higher orders. We have performed the calculations for an arbitrary mass ratio m_s/m_b, however quote here the results only at $m_s = 0$.

Therefore, we assume the $b \to s + \gamma$ transition to be mediated by the effective vertex

$$\frac{\lambda}{2}\,\bar{s}\sigma_{\mu\nu}(1-\gamma_5)b\,F^{\mu\nu}\,, \tag{4.188}$$

which is the electromagnetic penguin contribution and due to the large Wilson coefficient the leading part in the decay. In this approximation power corrections to the integrated decay rate become

$$\begin{aligned}\Gamma_{\text{bsg}}(B) = \frac{\lambda^2 m_b^3}{4\pi}\Bigg[1 &- \frac{\mu_\pi^2 + 3\mu_G^2}{2m_b^2} - \frac{11\rho_D^3 - 9\rho_{LS}^3}{6m_b^3}\\ &+ \frac{1}{m_b^4}\Big(\frac{1}{8}m_1 + \frac{7}{12}m_2 + \frac{1}{3}m_3 + \frac{7}{8}m_4 - \frac{17}{12}m_5 + \frac{11}{24}m_6 + \frac{1}{32}m_8 - \frac{1}{6}m_9\Big)\\ &+ \frac{1}{m_b^5}\Big(- \frac{28}{15}r_1 - \frac{69}{20}r_2 + \frac{21}{20}r_3 - \frac{107}{60}r_4 + \frac{41}{40}r_5 + \frac{41}{40}r_6 + \frac{173}{120}r_7 + \frac{4}{3}r_8 + \frac{31}{24}r_9\\ &+ \frac{7}{24}r_{10} - \frac{13}{8}r_{11} + \frac{1}{24}r_{12} - \frac{1}{6}r_{13} - \frac{7}{12}r_{14} + \frac{17}{24}r_{15} - \frac{1}{4}r_{16} - \frac{3}{8}r_{17} + \frac{7}{24}r_{18}\Big)\Bigg]\,.\end{aligned} \tag{4.189}$$

Next we consider the important moments of the observable photon energy, which are normalized to the rate itself, in order to let the prefactors, e.g. the CKM factors, drop out. Therefore we have expanded the normalization denominator, in order to have the non-perturbative parameters in the numerator, only. Noticing, that we expand up to $1/m_b^5$, we will therefore have products of dimension five and six non-perturbative parameters. The first moment of the average photon energy in the decay is given by

$$\begin{aligned}
2\langle E_\gamma \rangle &= m_b + \frac{\mu_\pi^2 - \mu_G^2}{2m_b} - \frac{5\rho_D^3 - 7\rho_{LS}^3}{6m_b^2} \\
&\quad + \frac{1}{m_b^3}\Big(- \frac{1}{8}m_1 - \frac{23}{12}m_2 - \frac{5}{6}m_3 + \frac{7}{24}m_4 - \frac{17}{12}m_5 + \frac{13}{24}m_6 + \frac{1}{6}m_7 \\
&\quad + \frac{3}{32}m_8 - \frac{1}{3}m_9 + \frac{1}{4}(\mu_\pi^2)^2 + \frac{1}{2}\mu_\pi^2\mu_G^2 - \frac{3}{4}(\mu_G^2)^2\Big) \\
&\quad + \frac{1}{m_b^4}\Big(- \frac{5}{4}r_2 + \frac{23}{12}r_3 - \frac{7}{4}r_4 - \frac{5}{8}r_5 + \frac{39}{8}r_6 - \frac{13}{24}r_7 + \frac{25}{24}r_9 - \frac{1}{8}r_{10} \\
&\quad + \frac{19}{24}r_{11} + \frac{7}{24}r_{12} - \frac{7}{6}r_{13} + \frac{19}{12}r_{14} - \frac{25}{24}r_{15} + \frac{7}{12}r_{16} + \frac{61}{24}r_{17} - \frac{17}{8}r_{18} \\
&\quad + \frac{1}{2}\mu_\pi^2\rho_D^3 - \frac{13}{6}\mu_G^2\rho_D^3 + \frac{5}{2}\mu_G^2\rho_{LS}^3 - \frac{1}{6}\mu_\pi^2\rho_{LS}^3\Big) \,. \qquad (4.190)
\end{aligned}$$

For the second and third moments we quote for simplicity the corrections to the moments with respect to $m_b/2$ rather than for the usually considered central moments

$$\begin{aligned}
3\langle (2E_\gamma - m_b)^2 \rangle &= \mu_\pi^2 - \frac{2\rho_D^3 - \rho_{LS}^3}{m_b} + \frac{1}{m_b^2}\Big(\frac{3}{4}m_1 - \frac{7}{2}m_2 - m_3 \\
&\quad + \frac{5}{2}m_5 + \frac{3}{4}m_6 - \frac{1}{2}m_7 + \frac{11}{16}m_8 - \frac{1}{4}m_9 + \frac{1}{2}(\mu_\pi^2)^2 + \frac{3}{2}\mu_\pi^2\mu_G^2\Big) \\
&\quad + \frac{1}{m_b^3}\Big(- 4r_1 + 5r_2 + \frac{3}{2}r_3 - 5r_4 - \frac{21}{4}r_5 + \frac{63}{4}r_6 - \frac{23}{4}r_7 - \frac{13}{4}r_9 + \frac{9}{4}r_{10} \\
&\quad + \frac{25}{4}r_{11} + \frac{21}{4}r_{12} - 8r_{13} + r_{14} + \frac{1}{4}r_{15} + \frac{9}{2}r_{16} + \frac{27}{4}r_{17} - \frac{37}{4}r_{18} \\
&\quad + \frac{5}{6}\mu_\pi^2\rho_D^3 - 3\mu_G^2\rho_D^3 - \mu_\pi^2\rho_{LS}^3 + \frac{3}{2}\mu_G^2\rho_{LS}^3\Big) \,, \qquad (4.191)
\end{aligned}$$

$$\begin{aligned}
3\langle (2E_\gamma - m_b)^3 \rangle &= -\rho_D^3 + \frac{1}{m_b}\Big(\frac{3}{2}m_1 - 2m_2 + \frac{1}{4}m_3 + \frac{8}{5}m_5 \\
&\quad + \frac{1}{10}m_6 - \frac{9}{20}m_7 + \frac{27}{80}m_8 + \frac{1}{10}m_9\Big) \\
&\quad + \frac{1}{m_b^2}\Big(- \frac{16}{5}r_1 + \frac{51}{10}r_2 + \frac{9}{10}r_3 - \frac{17}{5}r_4 - \frac{27}{10}r_5 + \frac{63}{10}r_6 - \frac{8}{5}r_7 + 4r_8 \\
&\quad - \frac{11}{5}r_9 + \frac{71}{10}r_{10} + \frac{9}{10}r_{11} + \frac{13}{5}r_{12} - \frac{7}{5}r_{13} - \frac{11}{5}r_{14} + \frac{23}{5}r_{15} \\
&\quad + \frac{27}{10}r_{16} + \frac{9}{10}r_{17} - \frac{37}{10}r_{18} - \frac{1}{2}(\mu_\pi^2 + 3\mu_G^2)\rho_D^3\Big) \,. \qquad (4.192)
\end{aligned}$$

The terms through $D = 6$ in equations (4.190) and (4.191) coincide with the known ones, cf. Ref. [67].

Numerical aspects have been analyzed similarly to the semileptonic moments, employing the factorization approximation of section 4.3 for the expectation values. The corrections turn out to be rather small, not only in the integrated width and in the average photon energy, but also in the second and even in the third moments corresponding, in the heavy-quark limit, to the kinetic and Darwin expectation values, respectively. Moreover, accounting for the $D = 8$ expectation values yields small effect compared to the $D = 7$ one, except for the second moment where both are quite suppressed. Direct evaluation results in

$$\frac{\delta\Gamma(B \to X_s + \gamma)}{\Gamma(B \to X_s + \gamma)} = -0.036_{1/m_b^2} - 0.0053_{1/m_b^3} + 0.00064_{1/m_b^4} + 0.00015_{1/m_b^5} \quad (4.193a)$$

$$\delta\langle 2E_\gamma\rangle = 11\ \text{MeV}_{1/m_b^2} - 14\ \text{MeV}_{1/m_b^3} + 3\ \text{MeV}_{1/m_b^4} + 0.7\ \text{MeV}_{1/m_b^5} \quad (4.193b)$$

$$12\delta\langle(E_\gamma - \langle E_\gamma\rangle)^2\rangle = -0.106\ \text{GeV}^2_{1/m_b^3} + 0.002\ \text{GeV}^2_{1/m_b^4} + 0.0025\ \text{GeV}^2_{1/m_b^5} \quad (4.193c)$$

$$24\delta\langle(E_\gamma - \langle E_\gamma\rangle)^3\rangle = 0.02\ \text{GeV}^3_{1/m_b^4} - 0.0025\ \text{GeV}^3_{1/m_b^5}\,. \quad (4.193d)$$

The shifts in the second, third and fourth line here may be interpreted as an apparent change, up to the sign, in m_b, μ_π^2 and $-\rho_D^3$, according to the similar analysis in the case of $b \to c\ell\bar{\nu}_\ell$ transitions.

The small effect on μ_π^2 and ρ_D^3, far below the level anticipated in reference [67], is somewhat surprising. The above numbers probably are smaller than the corrections due to the non-valence four-quark operators of the form $\bar{b}s\,\bar{s}b$ appearing at order $\mathcal{O}(\alpha_s)$. They were discussed in reference [67], section 4. It is conceivable that the numerical suppression we obtain, is partially accidental or is an artifact of the factorization approximation for the expectation values. We do not dwell further on this here and plan to look into the issue in the subsequent studies.

5. The Exclusive Decay $B \to D^{(*)}\ell\bar{\nu}_\ell$

As we have explained in the introductory chapters, the second option to extract the CKM matrix element $|V_{cb}|$ is to consider the exclusive process related to the quark transition $b \to c$, namely $B \to D\ell\bar{\nu}_\ell$ and $B \to D^*\ell\bar{\nu}_\ell$. Different final states are sensitive to other combination of form factors, describing the non-perturbative input. In experiment only the combination form factor times CKM matrix element is measured, while the form factor is computed by theoretical methods. Therefore contributions from new physics operators to the form factor may influence the extraction of the CKM matrix element, and may explain the small tension between the inclusive and exclusive measurement. It has already been pointed out, that allowing right handed quark currents in inclusive processes can lead to a 30% admixture of right handed quark currents [16]. But the error is huge, such that it is also compatible with the Standard Model. Therefore the inclusive analysis turns out to be insensitive to decide weather new physics contribution can enter. We will present in the next sections some parts of the calculation to repeat the analysis for the case of exclusive final states. With the distinction of a pseudo-scalar D and vector D^* meson, the analysis is more promising.

5.1. The Decay Rate

At first we start with the same interaction Hamiltonian (4.1) as in the case of inclusive decays. On parton level, the same process takes place. The only difference lies in the non-perturbative hadronic matrix element, which defines the final state. Now we cannot apply the optical theorem and therefore the non-perturbative input depends on the final state. In the Standard Model the underlying interaction for the semileptonic bottom to charm quark decay $b \to c\ell\bar{\nu}_\ell$ is given by

$$\mathcal{H}_{\text{int}} = -\frac{4G_F}{\sqrt{2}} V_{cb} \left(\bar{c} \frac{\gamma_\mu(1-\gamma_5)}{2} b \right) \left(\bar{\ell} \frac{\gamma^\mu(1-\gamma_5)}{2} \nu_\ell \right) , \tag{5.1}$$

but for the new physics analysis, we will slightly modify this to allow for non Standard Model currents, e.g. extending the V-A structure. Furthermore the phase space has a slightly modified form, since now the final state is completely defined, and not indirectly computed by the optical theorem. Neglecting again higher-order electroweak corrections, the matrix element for $B \to D^{(*)}e\bar{\nu}_e$ factorizes into a leptonic and a hadronic part

$$\begin{aligned} \mathcal{M}(\bar{B} \to D^{(*)}e^-\bar{\nu}_e) &= \langle D^{(*)}e\bar{\nu}_e|\mathcal{H}_{\text{int}}|\bar{B}\rangle \\ &= \frac{4G_F}{\sqrt{2}} V_{\text{cb}} \langle D^{(*)}e\bar{\nu}_e|J_{q,\mu}(0)J_e^\mu(0)|\bar{B}\rangle \, , \end{aligned} \tag{5.2}$$

where the leptonic current is defined as in (4.3a). The kinematical variables for the differential rate is given by the velocities of the initial $P_B^\mu = M_B v^\mu$ and final state $P_{D^{(*)}}^\mu = M_D^{(*)} v'^\mu$. From this we define the only independent kinematical quantity

$$w = v \cdot v' = \frac{M_B^2 + M_{D^{(*)}}^2 - q^2}{2 M_B M_{D^{(*)}}} \, . \tag{5.3}$$

This variable is directly related to the momentum transfer $q^\nu = p_e^\nu + p_{\bar{\nu}_e}^\nu = P_B^\nu - P_{D^{(*)}}^\nu$ to the leptons. The differential decay rate is then given by

$$\mathrm{d}\Gamma = \frac{1}{2M_B} \sum_{\text{Spins}} |\mathcal{M}(\bar{B} \to D^{(*)} e^- \bar{\nu}_e)|^2 \mathrm{d}\phi (2\pi)^4 \delta^{(4)}(P_B - P_{D^{(*)}} - p_e - p_{\bar{\nu}_e}) \, , \tag{5.4}$$

where the squared matrix element reads

$$\begin{aligned} \sum_{\text{Spins}} |\mathcal{M}(\bar{B} \to D^{(*)} e^- \bar{\nu}_e)|^2 = 8 G_F^2 |V_{cb}|^2 & \underbrace{\sum_{\text{Spins}} \langle 0 | J_e^{\dagger\nu} | e \bar{\nu}_e \rangle \langle e \bar{\nu}_e | J_e^\mu | 0 \rangle}_{= L^{\mu\nu}} \\ & \times \underbrace{\sum_{\substack{\text{Spins} \\ \text{Pols.}}} \langle \bar{B} | J_{q,\nu}^\dagger | D^{(*)} \rangle \langle D^{(*)} | J_{q,\mu} | \bar{B} \rangle}_{= W_{\mu\nu}} \, . \end{aligned} \tag{5.5}$$

For the comparison with the experimental measurement, we have to average over the incoming $\bar{B}$ spins, but since this is a pseudo-scalar particle the normalization factor is unity. For the final states we have to sum over all spins, and possibly polarizations of a final state vector meson.

First we will compute the calculable parts in the next subsections, and then parametrize the hadronic part $W^{\mu\nu}$ in order to obtain the differential decay rate in w.

5.1.1. Leptonic Part

The leptonic tensor is defined as in (4.12), to remind the reader literally it is given by

$$L^{\mu\nu} = 2 \left(p_e^\mu p_{\bar{\nu}_e}^\nu + p_e^\nu p_{\bar{\nu}_e}^\mu - g^{\mu\nu} p_e \cdot p_{\bar{\nu}_e} - i \epsilon^{\eta\nu\lambda\mu} p_{e\eta} p_{\bar{\nu}_e \lambda} \right) \, . \tag{5.6}$$

In the next step we combine it with the phase space integration over the leptons.

5.1.2. Phase Space

Introducing the momentum transfer q_μ, we have to integrate over electron and anti-neutrino momenta

$$\tilde{L}^{\mu\nu} = \int \frac{\mathrm{d}^3 p_e}{(2\pi)^3 2 p_e^0} \int \frac{\mathrm{d}^3 p_{\bar{\nu}_e}}{(2\pi)^3 2 p_{\bar{\nu}_e}^0} L^{\mu\nu} (2\pi)^4 \delta^4 \left[q - (p_e + p_{\bar{\nu}_e}) \right] \, .$$

Obviously this integration is (i) symmetric in the momenta p_e^μ and $p_{\bar{\nu}_e}^\mu$, and (ii) it depends on the momentum transfer q^μ, only. Therefore we can parameterize the result as

$$\tilde{L}^{\mu\nu} = \tilde{L}_1 g^{\mu\nu} + \tilde{L}_2 q^\mu q^\nu \,.$$

Using momentum conservation and the on-shell condition of the final state leptons, we can express the relevant scalar products through the momentum transferred squared to the leptons q^2, namely

$$p_e^2 = 0\,, \qquad p_{\bar{\nu}_e}^2 = 0\,, \tag{5.7a}$$

$$q^2 = 2p_{\bar{\nu}_e} \cdot p_e\,, \tag{5.7b}$$

$$p_e \cdot q = \frac{q^2}{2}\,, \qquad p_{\bar{\nu}_e} \cdot q = \frac{q^2}{2}\,. \tag{5.7c}$$

Now using the result for the scalar integration, and keeping in mind that all tensor reductions

$$g^{\mu\nu} L_{\mu\nu} = 4L_1 + q^2 L_2\,, \qquad q^\mu q^\nu L_{\mu\nu} = L_1 q^2 + L_2 q^4 \tag{5.8a}$$

$$\Rightarrow \qquad L_1 = -\frac{2}{3} q^2 \tilde{L}\,, \qquad L_2 = \frac{2}{3}\tilde{L}\,, \tag{5.8b}$$

do not depend on the integration momenta[1] and using Lorentz invariance we get

$$\begin{aligned} \tilde{L} &= \int \frac{\mathrm{d}^3 p_e}{(2\pi)^3 2p_e^0} \int \frac{\mathrm{d}^3 p_{\bar{\nu}_e}}{(2\pi)^3 2p_{\bar{\nu}_e}^0} (2\pi)^4 \delta^4 \left[q - (p_e + p_{\bar{\nu}_e}\right] \\ &= \int \frac{\mathrm{d}^3 p_e}{(2\pi)^3 2p_e^0}\, 2\pi\delta\left[(q - p_e)^2\right] \\ &= \frac{1}{2\pi} \int \mathrm{d}|\boldsymbol{p}_e| \frac{|\boldsymbol{p}_e|^2}{|\boldsymbol{p}_e|} \delta\left[q_0^2 - 2q_0|\boldsymbol{p}_e|\right]_{q\,\text{rest system}} \end{aligned} \tag{5.9}$$

$$= \frac{1}{4}\frac{1}{2\pi}\,. \tag{5.10}$$

Finally we can combine these results to the phase-space integrated lepton tensor

$$\tilde{L}^{\mu\nu} = \frac{1}{4}\frac{1}{3\pi}\left[q^\mu q^\nu - g^{\mu\nu} q^2\right]\,. \tag{5.11}$$

The next step is to compute the integration over the hadronic variables to establish the differential decay rate, first in the invariant lepton momentum transfer q^2. Then we can easily transform this result in the desired form

$$\mathrm{d}w = -\frac{1}{2M_B M_D}\mathrm{d}q^2\,, \qquad 1 \le w \le 0\,. \tag{5.12}$$

Usually the minus sign is absorbed, such that the integration condition

$$0 \le w \le 1 \tag{5.13}$$

[1] Note, that due to the massless leptons, the current conservation $q_\mu L^{\mu\nu} = 0$ is automatically fulfilled.

holds true, and $w = 1$ corresponds to the case of zero-recoil, whereas $w = 0$ is the case for maximal transfer of momentum to the lepton system.

Therefore we have for the hadronic phase-space integral in terms of the momentum transfer squared

$$\begin{aligned}
&\int \frac{\mathrm{d}^3 P_D}{(2\pi)^3 2P_D^0}\delta\left[q^2 - (P_B - P_D)^2\right] \\
= &\frac{1}{4\pi^2}\int \mathrm{d}|\boldsymbol{p}_D|\frac{|\boldsymbol{p}_D|^2}{\sqrt{M_D^2 + |\boldsymbol{p}_D|^2}}\delta\left[q^2 - M_B^2 - M_D^2 + 2MB\sqrt{M_D^2 + |\boldsymbol{p}_D|^2}\right] \Bigg|_{x=|\boldsymbol{p}_D|^2} \\
= &\frac{1}{4\pi^2}\int \frac{\mathrm{d}x}{2}\frac{\sqrt{x}}{\sqrt{M_D^2 + x}}\frac{\sqrt{M_D^2 + x}}{M_B}\delta\left[x - \frac{(q^2 - M_B^2 - M_D^2)^2 - 4M_B^2 M_D^2}{4M_B^2}\right] \\
= &\frac{\sqrt{(q^2 - M_B^2 - M_D^2)^2 - 4M_B^2 M_D^2}}{16\pi^2 M_B^2} \\
= &\frac{1}{8\pi^2}\frac{M_D}{M_B}\sqrt{w^2 - 1}\,.
\end{aligned} \tag{5.14}$$

Note that the result for the D^* meson phase space is the same with $M_D \to M_{D^*}$. We left out the superscript for a better readability.

5.1.3. Differential Decay Rate

Now we have to combine everything, where the hadronic part is not specified yet. For the differential decay rate with respect to the invariant mass of the leptonic system and respectively to w we have

$$\begin{aligned}
\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2} &= \frac{G_F^2 |V_{cb}|^2}{48\pi^3 M_B^3}\sqrt{(q^2 - M_B^2 - M_{D^*}^2)^2 - 4M_B^2 M_{D^*}^2}\, W_{\mu\nu}\left[q^\mu q^\nu - g^{\mu\nu} q^2\right]\,, \\
\frac{\mathrm{d}\Gamma}{\mathrm{d}w} &= \frac{G_F^2 |V_{cb}|^2}{12\pi^3}\frac{M_{D^*}^2}{M_B}\sqrt{w^2 - 1}\, W_{\mu\nu}\left[q^\mu q^\nu - g^{\mu\nu} q^2\right]\Bigg|_{q^2 = M_B^2 + M_{D^*}^2 - 2M_B M_{D^*} w}^{v\cdot q = M_B - w M_{D^*},\, v'\cdot q = M_B w - M_{D^*}}\,.
\end{aligned} \tag{5.15}$$

In the next section we derive a general basis, which respects the gauge symmetry of the SM, for the hadronic tensor $W_{\mu\nu}$. In the end we present the differential rate in terms of the different hadronic contributions.

5.2. Effective Theory Ansatz for New Physics

As we have already mentioned earlier, we can treat the SM as an effective field theory at low scales, where low scale is at the order of the b-quark mass. Therefore any new physics effects can be expanded into higher dimensional operators, which respect the Standard Model gauge symmetry. Since these effects are stemming from higher scales, they have been integrated out and therefore cannot appear as parts of the low energy effective operators. They contribute to the coefficient function in front of the operator - which is the "coupling constant" of the operator term in the Lagrangian. The corresponding Wilson coefficient for any new physic models can in principle be matched to it. The expansion parameter is then naturally given by the scale Λ_{NP}, at which the effects are integrated out

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \frac{1}{\Lambda_{NP}}\mathcal{L}_5 + \frac{1}{\Lambda_{NP}^2}\mathcal{L}_6 + \dots \tag{5.16}$$

The next step is to construct the allowed operator structures, where the SM fields can be grouped into doublets under the

$$SU(3)_C \otimes SU(2)_W \otimes U(1)_Y \tag{5.17}$$

symmetry. The QCD degree of freedom $SU(3)_C$ is not relevant in this case, such that

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad \text{Left-handed quark fields}, \tag{5.18a}$$

$$q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \qquad \text{Right-handed quark fields}, \tag{5.18b}$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} v + h_0 + i\chi_0 & \sqrt{2}\phi_+ \\ -\sqrt{2}\phi_- & v + h_0 - i\chi_0 \end{pmatrix}, \qquad \text{Higgs field matrix}, \tag{5.18c}$$

where the right-handed symmetry has been explicitly broken and the q_R are singlets under $SU(2)$. The vacuum expectation value of the Higgs is denoted by v. Inserting the appropriate number of Higgs and gauge fields from $SU(2)_W \otimes U(1)_Y$, we will now construct all possible gauge invariant operators. The possible - gauge invariant - combinations can be read off from the Lagrangian. But in the end we only need the part responsible for the $b \to c\ell\bar{\nu}_\ell$ transition. Therefore the Higgs field gets replaced by its VEV, which can be absorbed into the generic coupling constant. From this a minimal flavour violation analysis can be performed in order to estimate the suppression factor of each term, as it was done e.g. in [15,16,68,69]. Please note, that we have to insert one Higgs field for a possible (pseudo-)scalar term, and the tensor term. Therefore in this case, the transition from a left- to a right-handed particle and vice versa is mediated. We keep the leptonic tensor as in the Standard Model, since for this case a very precise Michel parameter analysis showed, that it behaves as expected [70, 71]. However, in case of mesons, we do not have a purely partonic interaction and then effects can arise, which can distort this symmetry.

5.2.1. Operator Basis

We can restrict ourselves to the W-gauge boson coupling of a bottom and charm quark, since we consider only the $b \to c$ transition. The W-boson then decays into a lepton anti-neutrino pair. Without additional new physic operators or loop diagrams we can have only a charged current transition for different flavour. While integrating out the

W-Boson, we derive the general quark current, where in SM we are left with a V-A structure. The coupling to the leptons is very well measured and therefore we keep this part Standard Model like

$$\begin{aligned} J_{q,\mu} =& c_L\,\bar{c}_L\gamma_\mu b_L + c_R\,\bar{c}_R\gamma_\mu b_R + g_L\,\bar{c}_R\frac{i\overleftrightarrow{D}_\mu}{m_b}b_L + g_R\,\bar{c}_L\frac{i\overleftrightarrow{D}_\mu}{m_b}b_R \\ &+ d_L\,\frac{i\partial^\nu}{m_b}\bar{c}_R(i\sigma_{\mu\nu})b_L + d_R\,\frac{i\partial^\nu}{m_b}\bar{c}_L(i\sigma_{\mu\nu})b_R\,. \end{aligned} \tag{5.19}$$

$P_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$ is the left/right handed projector and iD_μ the covariant derivative. Here we have put general coupling constants for the different appearing Dirac structure. Another way of constructing the operator basis, is to simply write down all possible Dirac structures and then insert appropriate derivatives in order to form gauge-invariant objects. In principle we have two different possibilities to insert the derivative and we have to form Lorentz vectors in order to couple it to the leptonic current.

The only possible way for a scalar operator is to insert a derivative. First of all please note that $\bar{c}\Gamma b$ is a colour singlet. Therefore the usual derivative of the complete term is already gauge invariant. This derivative transforms into the momentum transfer to the leptons. Therefore $i\partial_\mu\bar{c}b$ vanishes, if coupled to a pair of massless leptons. The other possibility is to form the left-right derivative resulting in the sum of incoming and outgoing momenta. Due to the additional minus sign of the Dirac equation the partial derivative is equivalent to the covariant derivative. For that reason $\bar{c}i\overleftrightarrow{D}_\mu b$ is the only gauge invariant possible operator structure.

Turning to the tensor contribution $\bar{c}\sigma_{\mu\nu}b$ we need to contract one index with a Lorentz vector, where again the only possible one is given by the derivative. Here in contrast to the scalar case the partial derivative is not directly contracted to the leptonic tensor and there does not vanish. But due to the commutator structure of $\bar{c}i\partial_\nu(-i\sigma^{\mu\nu})b = \frac{1}{2}\bar{c}[\gamma^\mu(i\partial\!\!\!/) - (i\partial\!\!\!/)\gamma^\mu]b$ we cannot form a gauge invariant object in the same manner as above. Therefore the only contribution is given by $i\partial^\nu\bar{c}(-i\sigma_{\mu\nu})b$.

In the following we will define the form factors corresponding to this enhanced current.

5.2.2. Hadronic Part

The relevant matrix elements of the hadronic currents can be described in terms of hadronic form factors. For each operator we define a Lorentz tensor decomposition with scalar form factors in the following. We do not distinguish between the left and right-handed fields in here and the following, but rather refer to the basis of 16 Dirac matrices. For example a vector current with left-handed fields is decomposed into a vector form factor minus an axial-vector form factor. The external four vectors for the decomposition are given by the four velocities of the incoming and outgoing mesons, and additionally the polarization vector in case of the D^* meson in the final state. We have to obey the correct parity and charge conjugation properties of the matrix elements for this decomposition.

In case of the pseudo-scalar to pseudo-scalar $B \to D\,\ell\,\bar{\nu}$ transition the appearing matrix elements can be decomposed into

$$\frac{\langle D(v')|\bar{c}b|B(v)\rangle}{\sqrt{M_B M_D}} = s(w)\,, \tag{5.20a}$$
$$\frac{\langle D(v')|\bar{c}\gamma^\mu b|B(v)\rangle}{\sqrt{M_B M_D}} = h_+(w)(v+v')^\mu + h_-(w)(v-v')^\mu\,, \tag{5.20b}$$
$$\frac{\langle D(v')|\bar{c}\sigma^{\mu\nu}b|B(v)\rangle\rangle}{\sqrt{M_B M_D}} = i(v'^\mu v^\nu - v'^\nu v^\mu)\; t(w)\,, \tag{5.20c}$$

and for the for the $B \to D^*\,\ell\,\bar{\nu}$ decays we have

$$\frac{\langle D^*(v',\epsilon)|\bar{c}\gamma_5 b|B(v)\rangle}{\sqrt{M_B m_{D^*}}} = p_+(w)\epsilon^*_\alpha(v+v')^\alpha + p_-(w)\epsilon^*_\alpha(v-v')^\alpha\,, \tag{5.21a}$$
$$\frac{\langle D^*(v',\epsilon)|\bar{c}\gamma^\mu b|B(v)\rangle}{\sqrt{M_B m_{D^*}}} = i h_V(w)\varepsilon^{\mu\nu\alpha\beta}\epsilon^*_\nu v'_\alpha v_\beta\,, \tag{5.21b}$$
$$\frac{\langle D^*(v',\epsilon)|\bar{c}\gamma^\mu\gamma_5 b|B(v)\rangle}{\sqrt{M_B m_{D^*}}} = h_{A_1}(w)(w+1)\epsilon^{*\mu} - [h_{A_2}(w)v^\mu + h_{A_3}(w)v'^\mu](v\cdot\epsilon^*)\,, \tag{5.21c}$$
$$\frac{\langle D^*(v',\epsilon)|\bar{c}\sigma^{\mu\nu}b|B(v)\rangle}{\sqrt{M_B m_{D^*}}} = \varepsilon^{\mu\nu\kappa\tau}\left[\epsilon^*_\kappa(v'+v)_\tau t_+(w) + \epsilon^*_\kappa(v'-v)_\tau t_-(w) + v'_\kappa v_\tau(v\epsilon^*)t'(w)\right]\,. \tag{5.21d}$$

The matrix elements which we have not listed here vanish because of parity.

Standard Model Limit

In the Standard Model it was noticed, that the charged weak current, connects left-handed fields only. Turning to the current in equation (5.19) therefore in this case the coupling constants are given by $c_L = 1$, and all others vanish. Furthermore the form factors simplify to only three different matrix elements and are simply parameterized by

$$\frac{\langle D(v')|\bar{c}\gamma^\mu b|B(v)\rangle}{\sqrt{M_B M_D}} = \left[h_+(w)(v+v')^\mu + h_-(v-v')^\mu\right]\,, \tag{5.22a}$$
$$\frac{\langle D^*(v',\epsilon)|\bar{c}\gamma^\mu b|B(v)\rangle}{\sqrt{M_B m_{D^*}}} = i h_V(w)\varepsilon^{\mu\nu\alpha\beta}\epsilon^*_\nu v'_\alpha v_\beta\,, \tag{5.22b}$$
$$\frac{\langle D^*(v',\epsilon)|\bar{c}\gamma^\mu\gamma_5 b|B(v)\rangle}{\sqrt{M_B m_{D^*}}} = h_{A_1}(w)(w+1)\epsilon^{*\mu} - \left[h_{A_2}(w)v^\mu + h_{A_3}(w)v'^\mu\right]\epsilon^*\cdot v\,. \tag{5.22c}$$

Heavy Quark Limit

We will work in the heavy quark effective theory, where we assume the heavy quark limit for both the bottom and the charm quark. Within this limit, we can derive the current (5.19) in terms of the heavy quark field $h_{v',c}$ with velocity v' for the charm and $h_{v,b}$ with velocity v for the bottom

$$
\begin{aligned}
J_{h,\mu} = & \, c_L \bar{h}_{v',c}\gamma_\mu P_L h_{v,b} + c_R \bar{h}_{v',c}\gamma_\mu P_R h_{v,b} + g_L(m_b v_\mu + m_c v'_\mu)\bar{h}_{v',c} P_L h_{v,b} \\
& + g_R(m_b v_\mu + m_c v'_\mu)\bar{h}_{v',c} P_R h_{v,b} + d_L(m_b v^\nu - m_c v'^\nu)(\bar{h}_{v',c} i\sigma_{\mu\nu} P_L h_{v,b}) \\
& + d_R(m_b v^\nu - m_c v'^\nu)(\bar{h}_{v',c} i\sigma_{\mu\nu} P_R h_{v,b}) \,.
\end{aligned} \tag{5.23}
$$

As we have seen in the introduction, in this heavy quark limit all form factors can be reduced to the Isgur-Wise function [23, 27]

$$
\begin{aligned}
& h_+(w) = h_V(w) = h_{A_1}(w) = h_{A_3}(w) = \xi(w) \,, \\
& h_-(w) = h_{A_2}(w) = 0 \,,
\end{aligned} \tag{5.24}
$$

which is normalized at zero-recoil to $\xi(1) \equiv 1$.

5.2.3. Differential Decay Rate

We can thus express the decay rate for the semileptonic $\bar{B} \to D^{(*)}\ell\bar{\nu}_\ell$ decays in terms of the Isgur-Wise function by putting together the previous results with equation (5.15)

$$
\frac{\mathrm{d}\Gamma^{B\to D\ell\bar{\nu}_\ell}}{\mathrm{d}w} = G_0(w)|V_{cb}|^2 \mathcal{A}(w)|\xi(w)|^2 \,, \tag{5.25a}
$$

$$
\frac{\mathrm{d}\Gamma^{B\to D_T^*\ell\bar{\nu}_\ell}}{\mathrm{d}w} = G_0^*(w)|V_{cb}|^2 \mathcal{B}^T(w)|\xi(w)|^2 \,, \tag{5.25b}
$$

$$
\frac{\mathrm{d}\Gamma^{B\to D_L^*\ell\bar{\nu}_\ell}}{\mathrm{d}w} = G_0^*(w)|V_{cb}|^2 \mathcal{B}^L(w)|\xi(w)|^2 \,, \tag{5.25c}
$$

$$
\frac{\mathrm{d}\Gamma^{B\to D^*\ell\bar{\nu}_\ell}}{\mathrm{d}w} = G_0^*(w)|V_{cb}|^2 \left[\mathcal{B}^T(w) + \mathcal{B}^L(w)\right] |\xi(w)|^2 \,, \tag{5.25d}
$$

where the kinematical and normalization prefactors are given by

$$
G_0(w) = \frac{G_F^2 M_B^5}{48\pi^3} r^3 \sqrt{w^2-1}(w+1)^2 \,, \tag{5.26a}
$$

$$
G_0^*(w) = \frac{G_F^2 M_B^5}{48\pi^3} (r^*)^3 \sqrt{w^2-1}(w+1)^2 \,. \tag{5.26b}
$$

The non-perturbative input is contained in the Isgur-Wise function, and the information on the different vertex structure is encoded in the different coefficient functions. Therefore we have for each decay one coefficient function depending on the kinematical variable w. Additionally we have split the contribution of D^* into the **L**ongitudinal and **T**ransveral polarization, because these two do not interfere. These three functions are explicitly given by

$$\mathcal{A}(w) = \frac{w-1}{w+1}\left[c_+(1+r) - M_B d_+(r^2 - 2rw + 1) + 2M_B r g_+(w+1)\right]^2 , \tag{5.27a}$$

$$\mathcal{B}^T(w) = 2\left[1 - 2r^* w + (r^*)^2\right]\Big\{[c_- + d_- M_B(r^* - 1)]^2 + \frac{w-1}{w+1}[c_+ + d_+ M_B(r^* + 1)]^2\Big\} , \tag{5.27b}$$

$$\mathcal{B}^L(w) = \left\{c_-(r^* - 1) + 2g_- M_B r^*(w-1) + d_- M_B[(r^*)^2 - 2r^* w + 1]\right\}^2 , \tag{5.27c}$$

with $r = M_D/M_B$, $r^* = m_{D^*}/M_B$. Please note that we have combined the left and right handedness projectors from the current (5.19) into the coupling constants necessary for the definition of the form factors (5.21) and (5.20) as

$$c_\pm = (c_L \pm c_R) , \quad d_\pm = (d_L \pm d_R) \quad \text{and} \quad g_\pm = (g_L \pm g_R) . \tag{5.28}$$

The expressions of the Standard Model are recovered by setting $c_\pm = 1$ while all other couplings are zero. Then the non-perturbative functions (5.27) become

$$\mathcal{A}_{\text{SM}}(w) = \frac{w-1}{w+1}(1+r)^2 , \tag{5.29a}$$

$$\mathcal{B}^T_{\text{SM}}(w) = \frac{4w}{w+1}[1 - 2r^* w + (r^*)^2] , \tag{5.29b}$$

$$\mathcal{B}^L_{\text{SM}}(w) = (r^* - 1)^2 . \tag{5.29c}$$

The branching ratio cannot be determined from this, because the analytical dependence on w of the form factors is not known. Therefore the extrapolation of the slope in the kinematical variable w is used in experiment to extract the product of CKM matrix element times form factor. Therefore by defining the quantities

$$M(w) = \frac{\mathrm{d}\Gamma^{B\to D\ell\bar{\nu}_\ell}}{\mathrm{d}w}\,\frac{1}{G_0}\,\frac{1}{\mathcal{A}(w)} , \tag{5.30a}$$

$$M^*(w) = \frac{\mathrm{d}\Gamma^{B\to D^*\ell\bar{\nu}_\ell}}{\mathrm{d}w}\,\frac{1}{G_0^*}\,\frac{1}{\mathcal{B}^L(w) + \mathcal{B}^T(w)} , \tag{5.30b}$$

we can extract the product form factor times CKM matrix element from experiment. Now at the specific kinematical point of non-recoil $w = 1$ the form factor can be computed by lattice QCD or sum rules. Using these calculated values the extraction of $|V_{cb}|$ becomes possible by comparing the limit

$$\lim_{w\to 1} M^{(*)}(w) = |V_{cb}|^2 \mathcal{F}(1)^2 , \tag{5.31}$$

where $\mathcal{F}(1)$ denotes the corresponding form factor, using the computed form factor at this kinematical point with experiment. But the resulting $|V_{cb}|$ strongly depends on the value for the form factors. In the usual extraction only the Standard Model form factors are considered and therefore possible new physics effects can influence the extraction of the value $|V_{cb}|$ as it can be seen from (5.30). The aim of this work is to quantify this potential effect.

In order to implement the effects of new form factors consistently we compute the vertex corrections at the zero-recoil point to this new form factors in the next section. This will give a numerical hint, on how the prefactor in front of the Isgur-Wise function for the different structure behaves.

5.3. Radiative Corrections to Decay Operators

We compute the vertex corrections for every occurring current. To this end, we have to consider the three diagrams in fig 5.1. In the end we will use this as an estimate for the Isgur-Wise limit normalization. Therefore we consider the renormalization of the vertex in the very special kinematic point of zero-recoil. The zero-recoil point is defined, such that $v \cdot v' = 1$, meaning the velocity of the hadron is conserved. Therefore the momentum transfer to the leptons is fixed as $q_\mu = (m_b - m_c)v_\mu$. First of all we

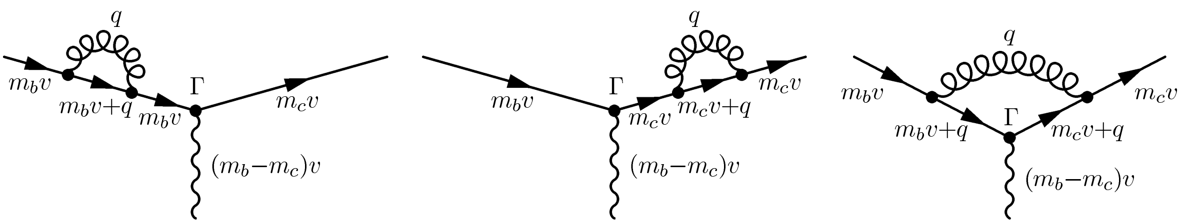


Figure 5.1.: One loop diagrams for the renormalization of the different currents.

determine the renormalization constant of the mass and wave function, which we will put together with the vertex renormalization on the end, to obtain the complete answer. In all calculations we assume naive anti-commuting γ_5 in D dimensions.

5.3.1. Wave Function Renormalization

The wave function renormalization, which is needed to calculate the appropriate counter terms, is independent of the vertex current. So we have to compute this only once. Since we express everything in the Lagrangian through the physical parameters, we have

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + (Z_2 - 1)\bar{\psi} i\partial\!\!\!/ \psi - (Z_2 Z_m - 1)\bar{\psi} m_q \psi \tag{5.32}$$

with the new counter-term vertex, where the bare parameters have a subscript 0,

$$\mathcal{L}_{\text{counter}} = (Z_2 - 1)\bar{\psi} i\partial\!\!\!/ \psi - (Z_2 Z_m - 1)\bar{\psi} m_q \psi \qquad \text{with} \qquad Z_m m_q = m_0\,, Z_2^{\frac{1}{2}}\psi = \psi_0\,. \tag{5.33}$$

For simplicity we have used a generic ψ-quark field with mass m_q in the calculation, which has to be replaced with a bottom-quark and a charm-quark and the corresponding masses in the end. In order to calculate the renormalization constants for the wave function and the mass, defined by the counter terms

$$i(Z_2-1)\not{q} - i(Z_m Z_q - 1)m_q\,, \tag{5.34}$$

we first perform the loop integration for fig. 5.2(b). Thereby we restrict ourselves to first order in $1/\epsilon$, because we perform only a one loop calculation

$$i\Pi_{1\text{ loop}} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(-ig_s\gamma_\alpha T^a)i\left(\not{q}-\not{k}-m_q\right)(-ig_s\gamma_\beta T^b)}{(q-k)^2-m_q^2+i0}\,\frac{-ig^{\alpha\beta}\delta^{ab}}{k^2+i0} \tag{5.35}$$

$$\to (-g_s^2)C_F\mu^{4-D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\,\frac{Dm+(2-D)(\not{q}-\not{k})}{[(q-k)^2-m_q^2]k^2+i0} \tag{5.36}$$

$$\underset{\epsilon\to 0}{\approx} i\frac{\alpha_s C_F}{4\pi}\Bigg[\left(\not{q}-4m_q\right)\frac{1}{\epsilon}+2m_b\left(-\frac{2m_q^2}{q^2}\log\frac{m_q^2-q^2}{m_q^2}+2\log\frac{m_q^2-q^2}{\mu^2}-3\right) \tag{5.37}$$

$$+\not{q}\left(-\log\frac{m_q^2-q^2}{\mu^2}+1+\frac{m_q^2}{q^2}+\frac{m_q^4}{q^4}\log\frac{m_q^2-q^2}{m_q^2}\right)\Bigg] \tag{5.38}$$

$$:= i(\Pi^{(q)}_{1\text{ loop}}(q^2)\,\not{q}+\Pi^{(m)}_{1\text{ loop}}(q^2)\,m_q)\,, \tag{5.39}$$

where ϵ is in $\overline{MS}$ scheme and $D=4-2\epsilon$. In principle we have to be careful, we compute these constants in the pole scheme for the mass definition in the propagator and kept the divergent terms in ϵ, only. To first order in α_s there is no difference. The complete one-loop result is now received by summing up the three diagrams in fig. 5.2 The

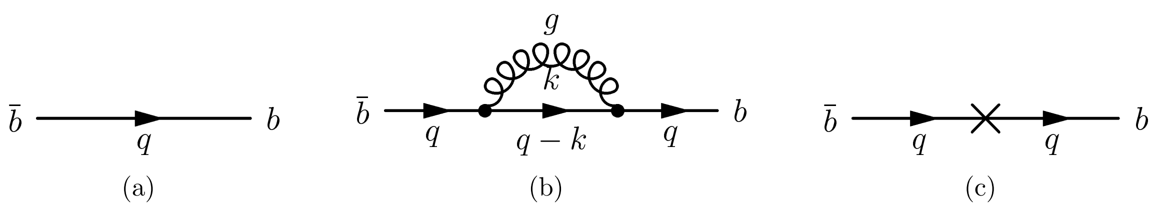


Figure 5.2.: One loop wave-function renormalization diagrams.

counter-term vertex is simply the usual propagator times the renormalization constants, which can be directly read off from eq. (5.33). Thus we end up with

$$i\tilde{\Pi}_{1\text{ loop}} = i\Pi_{1\text{ loop}} + i\left[(Z_2-1)\not{q}-(Z_2Z_m-1)m_q\right]\,. \tag{5.40}$$

The renormalization constants to order α_s are now determined by the requirement, that the pole and residue of the summed one-particle irreducible diagrams at the special on-shell point $\not{q}=m_q$,

$$i\Pi = \frac{i}{\not{q}-m_q-\tilde{\Pi}_{1\text{ loop}}+i0}\,, \tag{5.41}$$

are the same as for the free case. This leads to the two conditions, where in our case $q = b, c$ and $m_q = m_b, m_c$,

$$\Pi\Big|_{\not q = m_q} \stackrel{!}{=} 0\,, \tag{5.42a}$$

$$R_q := \frac{\mathrm{d}}{\mathrm{d}\not q}\Pi\Big|_{\not q = m_q} \stackrel{!}{=} 0\,. \tag{5.42b}$$

From this we compute the renormalization constants to be

$$Z_q = 1 - \frac{1}{\epsilon}\frac{\alpha_s}{4\pi}C_F\,,$$
$$Z_m = 1 - \frac{3}{\epsilon}\frac{\alpha_s}{4\pi}C_F\,.$$

In the end, we will need the correction part to the wave function, which is directly given by the residue of the propagator R_q in eq. (5.42b).

5.3.2. Vertex Renormalization

Now we look at the vertex correction at the zero-recoil point, see fig 5.1. Here the vertex is given by the different operators of the current, namely

- Scalar: $p_b^\mu + p_c^\mu$
- Pseudo-scalar: $(p_b^\mu + p_c^\mu)\gamma^5$
- Vector: γ^μ
- Axial-vector: $\gamma^\mu\gamma^5$
- Tensor: $((p_b)_\nu - (p_c)_\nu)(-i\sigma^{\mu\nu})$

whereas in the Standard Model we have only a vector γ_μ and axial-vector $\gamma_\mu\gamma_5$ part. At zero-recoil the loop integral is given by, middle diagram of fig 5.1,

$$i\Gamma_{1\text{ loop}} = \int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{(-ig_s\gamma_\alpha T^a)i\big(\not p_c + \not q + m_c\big)\,\Gamma i\Big(\not p_b + \not q + m_b\Big)\,(-ig_s\gamma_\beta T^b)}{[(p_c+q)^2 - m_c^2]\,[(p_b+q)^2 - m_b^2] + i0}\,\frac{-ig^{\alpha\beta}\delta^{ab}}{q^2 + i0}\,. \tag{5.43}$$

We use the equation of motion $\frac{1}{2}(1 + \frac{\not p_q}{m_q})q = q$ for q either a charm or bottom quark. Furthermore after the integration we can use the zero-recoil condition, which states that $p_b/m_b = p_c/m_c$. For this reason we can relate all vertex corrections in total to four independent components instead of all sixteen, because we project onto the upper component. Using the Dirac equation we derive the following relations

$$\bar{c}p_q^\mu b = m_q\bar{c}\gamma^\mu b\,, \tag{5.44a}$$

$$\bar{c}p_q^\mu\gamma^5 b = 0 \tag{5.44b}$$

such that we can relate the different occuring vertices to the structures

$$\bar{c}(m_b + m_c)\gamma^\mu b = \bar{c}(m_b + m_c)v^\mu b\,, \tag{5.45a}$$
$$\bar{c}(m_b - m_c)\gamma^\mu\gamma^5 b = \bar{c}(m_b - m_c)s^\mu b\,. \tag{5.45b}$$

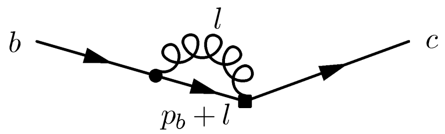
The four velocity has one component, and the spin vector $s^\mu \equiv 1/2(1+\not{v})\gamma^\mu\gamma^5 1/2(1+\not{v})$ has three components [72], since $v \cdot s = 0$.

Now we use the zero-recoil Passarino-Veltmann (PV) functions, to compute the loop integral. To this end, we compute the master integrals for $(1,l^\mu,l^\mu l^\nu)$ and $l^2 l^\mu$ in terms of the Passarino-Veltmann functions and put this into eq. (5.43). The decomposition is given by

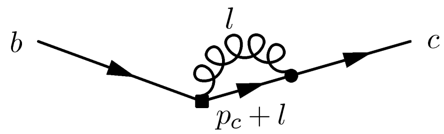
$$B^\mu = b_1 p_b^\mu + b_2 p_c^\mu \tag{5.46a}$$
$$C^\mu = c_1 p_b^\mu + c_2 p_c^\mu \tag{5.46b}$$
$$C^{\mu\nu} = C_1 g^{\mu\nu} + C_2 p_b^\mu p_b^\nu + C_3 p_c^\mu p_c^\nu + C_4 p_b^\mu p_c^\nu + C_5 p_c^\mu p_b^\nu\,. \tag{5.46c}$$

The expressions for the scalar coefficients can be found in Appendix A.4. Finally we obtain the results in terms of the PV functions for the different vertex structures at zero-recoil. This we then have to combine with the wave function renormalization.

Quark-Quark-Gluon-Boson Vertex



(a) Quark-quark-gluon-boson vertex, where the gluon is absorbed from the bottom quark.



(b) Quark-quark-gluon-boson vertex, where the gluon is absorbed from the charm quark.

Figure 5.3.: Two additional Feynman diagrams, which are neccessary for the (pseudo-)scalar vertex correction.

At order α_s we have an additional term appearing for the (pseudo)-scalar current: We can derive a quark-quark-gluon-boson vertex from the covariant derivative, which is of order $\mathcal{O}(g_s)$ and therefore not relevant at tree level. But it has to be computed at $\mathcal{O}(\alpha_s)$ in order to render the (pseudo)-scalar correction gauge invariant. The vertex constructed from eq. (5.19) reads

$$\mathcal{L}_{\text{QQGB}} = \frac{g_{L/R}}{m_b}\bar{c} g_s A_\mu^a T^a P_{L/R} b \tag{5.47}$$
$$\Rightarrow \qquad i\Gamma_{\text{QQGB}} = \frac{g_{L/R}}{m_b}\, g_s\, g_{\mu\nu} T^a P_{L/R}\,, \tag{5.48}$$

where μ is the index coupling to the W-boson, in this case it becomes the leptonic current, and ν is the vector index of the outgoing gluon. Therefore we have to compute the two Feynman diagrams 5.3, where we restrict ourselves to the scalar and pseudo-scalar case in stead of the left- or right-handed contribution in the current, since the form factors are defined with this. We define the generic coupling constant g_{SC} for the scalar case and g_{PSC} for the pseudo-scalar case, respectively.

We denote the corresponding integral of the left diagram 5.3(a) with the superscript $q = b$, and of the right diagram 5.3(b) with $q = c$. For both diagrams 5.3, we additionally have two choices $\Gamma = 1,\gamma_5$, depending on weather we consider the scalar or (pseudo)-scalar case. Then the loop integrals become

$$\begin{aligned} i\Gamma^b_{\mathrm{(P)SC}} &= \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\left[g_s\, g_{\mathrm{(P)SC}} g_{\mu\nu} T^a \Gamma \frac{i(\not{p}_b + \not{l} + m_b)}{(p_b+l)^2 - m_b^2} i g_s T^b \gamma_\rho \frac{-i}{l^2} g^{\nu\rho}\delta^{ab}\right] \\ &= i C_F g_s^2 g_{\mathrm{(P)SC}} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \Gamma \frac{\not{p}_b + \not{l} + m_b}{[(p_b+l)^2 - m_b^2][l^2]}\gamma_\mu \end{aligned} \tag{5.49}$$

$$\begin{aligned} i\Gamma^c_{\mathrm{(P)SC}} &= \int \frac{\mathrm{d}^4 l}{(2\pi)^4} i g_s T^b \gamma_\rho \left[\frac{i(\not{p}_c + \not{l} + m_c)}{(p_c+l)^2 - m_c^2} \Gamma g_s\, g_{\mathrm{SC}} g_{\mu\nu} T^a \frac{-i}{l^2} g^{\nu\rho}\delta^{ab}\right] \\ &= i C_F g_s^2 g_{\mathrm{(P)SC}} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \gamma_\mu \frac{\not{p}_c + \not{l} + m_c}{[(p_c+l)^2 - m_c^2][l^2]}\Gamma\,. \end{aligned} \tag{5.50}$$

Again we assume naive anti-commuting γ^5 in the calculation. Furthermore we make use of the equation of motion for the QCD quark fields $\not{p}_b b = m_b b$, respectively $\bar{c}\not{p}_c = \bar{c} m_c$, and restrict ourselves to the non-recoil point. Using the same methods as before, we can reduce the integrals above to two different types of master integrals

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{[(p_q+l)^2 - m_q^2][l^2]} = B_0(m_q^2, 0, m_q^2) \tag{5.51a}$$

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{l^\alpha}{[(p_q+l)^2 - m_q^2][l^2]} = -\frac{p_q^\alpha}{2m_q^2} A_0(m_q^2)\,, \tag{5.51b}$$

where we either have $q = b$ or $q = c$, depending on the diagram to be calculated. In total we then obtain the results quoted below.

Scalar Vertex with b-Quark

We reduce the D-dimensional Dirac algebra and use the master integrals (5.51). Then the result in $\overline{\mathrm{MS}}$ scheme, with the according definition of $1/\epsilon$ is given by

$$i\Gamma^b_{\mathrm{SC}} = -\frac{3 C_F m_b \alpha_s}{8\pi}\gamma_\mu \left[\frac{1}{\epsilon} - \log\frac{m_b^2}{\mu^2} + \frac{7}{3}\right]. \tag{5.52}$$

Scalar Vertex with c-Quark

Using the same definitions as above, the result is the same as above by replacing m_b with the charm mass m_c

$$i\Gamma^c_{\mathrm{SC}} = -\frac{3 C_F m_c \alpha_s}{8\pi}\gamma_\mu \left[\frac{1}{\epsilon} - \log\frac{m_c^2}{\mu^2} + \frac{7}{3}\right]. \tag{5.53}$$

This two results have to be added to the scalar vertex loop integral, to compute the total vertex correction. Now turning to the pseudo-scalar case.

Pseudo-scalar Vertex with b-Quark

We perform the same steps as above, but now we have $\Gamma = \gamma_5$, which we treat as anti-commuting

$$i\Gamma^b_{\text{PSC}} = \frac{C_F m_b \alpha_s}{8\pi}\gamma_\mu\gamma^5\left[\frac{1}{\epsilon} - \log\frac{m_b^2}{\mu^2} + 1\right]. \tag{5.54}$$

Pseudo-scalar Vertex with c-Quark

Due to the anti-commuting behaviour of the γ_5 matrix, we cannot obtain the result from the bottom quark case by just replacing m_b with m_c. Here we have two different cases, one where we have to commutate the γ_5 through a Dirac matrix, and therefore picking up a sign in one term.

$$i\Gamma^c_{\text{PSC}} = -\frac{C_F m_c \alpha_s}{8\pi}\gamma_\mu\gamma^5\left[\frac{1}{\epsilon} - \log\frac{m_b^2}{\mu^2} + 1\right]. \tag{5.55}$$

We are in the position to present the total result for the vertex correction. In the next section we will put together all individual results.

5.3.3. Result

We are now in the position to calculate the whole vertex correction by putting together all the different ingredients: External leg contribution for the quarks, vertex correction and for the (pseudo-)scalar case the additional vertex. For the external leg contribution we have to obey the requirement, that the residue of the quark propagator remains one. Therefore we have to add the residue of the one loop quark self-energy for the correct result of the one loop vertex function, which we have already calculated by the wave function renormalization. The residue (5.42b) can explicitly be computed by

$$\begin{aligned}
i\Pi^{\text{counter}}_{1\text{ loop}}(m^2) &= \frac{\mathrm{d}}{\mathrm{d}\not{p}} i\Pi_{1\text{ loop}}\Big|_{\not{p}=m} \\
&= i\frac{\mathrm{d}}{\mathrm{d}\not{p}}(\Pi^{(p)}_{1\text{ loop}}\not{p} + \Pi^{(m)}_{1\text{ loop}} m)\Big|_{\not{p}=m} \\
&= i\Big[\Pi^{(p)}_{1\text{ loop}} + \not{p}\frac{\mathrm{d}}{\mathrm{d}\not{p}}\Pi^{(p)}_{1\text{ loop}} + m\frac{\mathrm{d}}{\mathrm{d}\not{p}}\Pi^{(m)}_{1\text{ loop}}\Big]_{\not{p}=m} \\
&= i\Pi^{(p)}_{1\text{ loop}}\Big|_{p^2=m^2} + \not{p}\big(2\not{p}\frac{\mathrm{d}}{\mathrm{d}\,p^2}\big) i\Pi^{(p)}_{1\text{ loop}} + m\big(2\not{p}\frac{\mathrm{d}}{\mathrm{d}\,p^2}\big) i\Pi^{(m)}_{1\text{ loop}}\Big|_{\not{p}=m} \\
&= i\Pi^{(p)}_{1\text{ loop}}\Big|_{p^2=m^2} + 2m^2\frac{\mathrm{d}}{\mathrm{d}\,p^2}\left(i\Pi^{(p)}_{1\text{ loop}} + i\Pi^{(m)}_{1\text{ loop}}\right)_{p^2=m^2}.
\end{aligned} \tag{5.56}$$

Now we add up all terms including this residue, to get the full one loop vertex correction result at zero-recoil. From this we can read off the corresponding renormalization constant. We have to keep in mind, that the renormalization constant for the wave function has a square root, because the field appears only in linear form in the vertex correction. Therefore we have a factor of 1/2 for them. In total we therefore have

$$i\Gamma^{\Gamma} = \Gamma + \Gamma^{\Gamma}_{1\text{ loop}} + \Gamma\frac{1}{2}\left(i\Pi^{\text{counter}}_{1\text{ loop}}(m_b^2) + i\Pi^{\text{counter}}_{1\text{ loop}}(m_c^2)\right) . \tag{5.57}$$

Plugging everything in, the results are

$$\begin{aligned} i\Gamma^{(1)} &= (m_b+m_c)\gamma^\mu + \frac{\alpha_s}{4\pi}\frac{C_F\,(m_b+m_c)\,\gamma^\mu}{2\,(m_b^2-m_c^2)}\Bigg[-\frac{3(m_b^2-m_c^2)}{\epsilon} - 3m_c m_b \log\frac{m_c^2}{m_b^2} \\ &\quad + \left(6\log\frac{m_b^2}{\mu^2} - 3\log\frac{m_c^2}{\mu^2} - 13\right)m_b^2 - \left(-3\log\frac{m_b^2}{\mu^2} + 6\log\frac{m_c^2}{\mu^2} - 13\right)m_c^2\Bigg] , \end{aligned} \tag{5.58a}$$

$$i\Gamma^{(\gamma^\mu)} = \gamma^\mu + \frac{\alpha_s}{4\pi}C_F\gamma^\mu\left[-6 - 3\frac{m_b+m_c}{m_b-m_c}\log\frac{m_c}{m_b}\right] , \tag{5.58b}$$

$$i\Gamma^{(\gamma^\mu\gamma_5)} = \gamma^\mu\gamma^5 + \frac{\alpha_s}{4\pi}C_F\gamma^\mu\gamma^5\left[-8 - 3\frac{m_b+m_c}{m_b-m_c}\log\frac{m_c}{m_b}\right] , \tag{5.58c}$$

$$i\Gamma^{(\gamma_5)} = \frac{\alpha_s}{4\pi}\frac{C_F\gamma^\mu\gamma^5}{2}\left[\frac{3\,(m_b-m_c)}{\epsilon} + \left(-3\log\frac{m_b^2}{\mu^2} + 7\right)m_b + \left(3\log\frac{m_c^2}{\mu^2} - 7\right)m_c\right] , \tag{5.58d}$$

$$i\Gamma^{(-i\sigma^{\mu\nu})} = 0\,. \tag{5.58e}$$

Please remember, that we work in the HQET limit and at the zero-recoil point, where $p_b = m_b v^\mu$ and $p_c = m_c v^\mu$. Therefore there are only 4 independent operator structures possible, where

$$\bar{c}\gamma^\mu b = \bar{c}\frac{1+\not{v}}{2}\gamma^\mu\frac{1+\not{v}}{2}b = \bar{c}v^\mu b \tag{5.59a}$$

$$\bar{c}\gamma^\mu\gamma^5 b = \bar{c}\frac{1+\not{v}}{2}\gamma^\mu\gamma^5\frac{1+\not{v}}{2}b := \bar{c}s^\mu b\,. \tag{5.59b}$$

The four-velocity has one component, and the spin vector s^μ has three independent components, since $v\cdot s = 0$ can be directly verified. Therefore we express the result in terms of γ^μ and $\gamma^\mu\gamma^5$. Please note, that the tensor structure vanishes at this specific kinematical point. Furthermore we can derive a few relations, which are valid at the non-recoil point using $\not{v} = \frac{\not{p}_b}{m_b} = \frac{\not{p}_b}{m_b}$. Using the projector properties as in eq. (5.59) we derive the relations, we already presented in (5.44) again

$$\bar{c}p^\mu_{b,c}b = m_{b,c}\bar{c}\gamma^\mu b\,, \tag{5.60a}$$

$$\bar{c}p^\mu_{b,c}\gamma^5 b = 0\,. \tag{5.60b}$$

These relations have already been used to derive (5.58). From the finite parts of (5.58) we can compute the corrections η_Γ to the form factors near the zero-recoil point $w = 1$.

Up to corrections suppressed by powers of m_Q this η_Γ is a good approximation for the form factors. Our result for the vector case reads

$$\eta_V = 1 + \frac{\alpha_s}{4\pi} C_F \left[-6 - 3\frac{m_b + m_c}{m_b - m_c} \log\frac{m_c}{m_b} \right] , \tag{5.61}$$

and the corresponding axial-vector part reproduces the known analytical results [23],

$$\eta_A = 1 + \frac{\alpha_s}{4\pi} C_F \left[-8 - 3\frac{m_b + m_c}{m_b - m_c} \log\frac{m_c}{m_b} \right] . \tag{5.62}$$

The same corrections factors for the other Dirac combinations, which are not present in the Standard Model, can be read off analogously from Equation (5.58):

$$\begin{aligned}\eta_{\mathrm{SC}}(\mu) = 1 + \frac{\alpha_s}{4\pi}\frac{C_F}{2\,(m_b^2 - m_c^2)}\Bigg[&- 3 m_c m_b \log\frac{m_c^2}{m_b^2} + \left(6\log\frac{m_b^2}{\mu^2} - 3\log\frac{m_c^2}{\mu^2} - 13\right) m_b^2 \\ &- \left(-3\log\frac{m_b^2}{\mu^2} + 6\log\frac{m_c^2}{\mu^2} - 13\right) m_c^2 \Bigg] ,\end{aligned} \tag{5.63a}$$

$$\eta_{\mathrm{PS}}(\mu) = \frac{\alpha_s}{4\pi}\frac{C_F}{2(m_b + m_c)}\left[\left(-3\log\frac{m_b^2}{\mu^2} + 7\right) m_b + \left(3\log\frac{m_c^2}{\mu^2} - 7\right) m_c \right] , \tag{5.63b}$$

$$\eta_{\mathrm{T}}(\mu) = 0\,. \tag{5.63c}$$

In the following we will analyze the results numerically using $m_b = 4.2$ GeV, $m_c = 1.3$ GeV and $C_F = 4/3$. Additionally we assume the scale to be at the bottom mass, such that $\alpha_s(m_b) \approx 0.22$ and $\alpha_s(m_c) \approx 0.30$ [73]. We obtain

$$\eta_V \approx 1.02\,, \tag{5.64a}$$

$$\eta_A \approx 0.97\,, \tag{5.64b}$$

which is compatible with the lattice values [74, 75] for

$$\mathcal{G}^2(1) = 1.074 \pm 0.018 \pm 0.016 \tag{5.65a}$$

$$\mathcal{F}^2(1) = 0.927 \pm 0.024\,, \tag{5.65b}$$

where the form factors are defined as

$$\frac{\mathrm{d}\Gamma^{B\to D\ell\bar{\nu}_\ell}}{\mathrm{d}w} = \frac{G_F^2 |V_{cb}^2 M_B^5}{48\pi^3}(w^2 - 1)3/2r^3(1 + r)^2\mathcal{G}^2(w)\,, \tag{5.66a}$$

$$\begin{aligned}\frac{\mathrm{d}\Gamma^{B\to D^*\ell\bar{\nu}_\ell}}{\mathrm{d}w} &= \frac{G_F^2 |V_{cb}^2 M_B^5}{48\pi^3}(w^2 - 1)^{1/2}(w + 1)^2(r^*)^3(1 - r^*)^2 \\ &\times \left[1 + \frac{4w}{w+1}\frac{1 - 2wr^* + (r^*)^2}{(1 - r^*)^2}\right]\mathcal{F}^2(w)\,,\end{aligned} \tag{5.66b}$$

and coincide with the Isgur-Wise function in the infinite quark mass limit. As a short comment it should be said, that using non lattice methods, e.g. sum rules the values for these form factors seem to be too high [36, 38, 76, 77]. Lower values would prefer a higher value for the extracted $|V_{cb}|$, which then would reduce the tension with the exclusive determination [13]

$$|V_{cb}|^{\text{excl.}} = (38.7 \pm 1.1) \cdot 10^{-3}\,, \tag{5.67a}$$
$$|V_{cb}|^{\text{incl.}} = (41.5 \pm 0.7) \cdot 10^{-3}\,. \tag{5.67b}$$

To get a handle on the new form factors, we define analougiously the scalar η_{sc}, pseudo-scalar η_{ps} and tensor η_{t} coefficient function. We take these factors dimensionless and factor out the $m_b + m_c$ structure in case of the scalar current, and $m_b - m_c$ for the tensor coefficient.

Since the anomalous dimension does not vanish for the (pseudo-)scalar cases, we choose $\mu = \sqrt{m_b m_c}$ and take the values for $\mu = m_b$ and $\mu = m_c$ as an indicator for the error. Then we obtain the numerical values

$$\eta_{\text{SC}} = 1.03^{+0.06}_{-0.05}\,. \tag{5.68}$$
$$\eta_{\text{PS}} = 0.002^{+0.021}_{-0.028}\,, \tag{5.69}$$
$$\eta_{\text{T}} = 0\,. \tag{5.70}$$

Since the pseudo-scalar contribution vanishes at non-recoil it only receives a small correction of the order of $\mathcal{O}(2\%)$. However the scalar vertex does not vanish and gets corrections in the order of $\mathcal{O}(10\%)$. From our analysis the sign of the corrections cannot be judged. Therefore the admixture of a scalar current can in principle lead to measurable deviations, however, the handedness of this quark-current admixture cannot be judged from this analysis.

The full and more elaborate analysis, which will answer all these subtle details, using these computations and additionally experimental data as an input will be published in a forthcoming paper [78].

6. Discussion

We will summarize and comment on the results we have achieved. Basically we have presented two different types of analysis, regarding the semi-leptonic bottom into charm quark transition. We applied this underyling quark transition to the exclusive and inclusive B meson decay, which is the observable process at experiments. At the moment from both decay measurements the CKM matrix element $|V_{cb}|$ is extracted. There are a few important things related to it, which should be stressed, again. First there is a small tension between the exclusive and inclusive determination. The matrix element is not only important for these decays, but also to pinpoint down the tip of the unitarity triangle: $|V_{cb}|$ goes into the ϵ_K parameter with the fourth power and therefore with 33% it has the largest contribution to the total error budget of this parameter [81]. Thus a very precise and also reliable extraction is mandatory.

6.1. Summary

The first three chapters are dedicated to an introduction of the theoretical foundations and the techniques used within this thesis, and so set up the notation.

The first calculational part has been related to inclusive decays in chapter four. We have computed the non-perturbative corrections to a high order, thereby defining the new appearing operators (non-perturbative parameters). We have given a prescription, on how these higher dimensional matrix elements can be factorized into known parameters on the basis of a ground-state factorization theorem. From these a numerical estimate has been performed. With this, we analyzed the impact of the new computed higher-order terms for the moments of kinemtical variables and the total rate. Furthermore we had a closer look on the behaviour of the expansion related to the charm quark. We have found, that infrared sensitive terms with respect to the charm quark mass arise, which have been dubbed "intrinsic-charm". Therefore the expansion expected purely in $1/m_b$ actually is an expansion in both $1/m_b$ and also $1/m_c$, starting at $1/m_b^3$. There we investigated several scenarios with respect to the charm, and gave some comments on so-called "weak-annihilaton". The last part has been dedicated to $b \to s + \gamma$ decays. These are used in the experimental fits to extract $|V_{cb}|$ from inclusive semileptonic $b \to c$ transitions. We repeated the numerical analysis with the new computed higher-order matrix elements for the most relevant operator mediating this decay.

In the sixth chapter we have investigated the corresponding quark transition, but for the exclusive semi-leptonic decay $B \to D^{(*)} \ell \bar{\nu}_\ell$ in more detail. Especially we have allowed for physics beyond the Standard Model in an effective field theory approach for the quark transition. There we first have derived the new operators appearing and parameterized them in terms of Lorentz scalar form factors. In order to estimate the influence of a possible admixture to the exclusive determination in $|V_{cb}|$, we have shown in which way the CKM matrix element is actually extracted. In order to re-perform this analysis with real data and allowing for these new rising form factors in a forthcoming publication, we have estimated the strength of these form factors at zero recoil. Therefore we have computed the vertex corrections at this special kinematical point, which has been a quite good estimate for the known vector and axial-vector form factors.

6.2. Inclusive Decay

6.2.1. Higher Orders

We report a detailed study of higher-order power correction in inclusive weak decays of heavy flavor hadrons, focused on the semileptonic B meson decays. The calculations existed since the mid 1990s and included $1/m_b^2$ and $1/m_b^3$ corrections. We extended the analysis to order $1/m_b^4$ and $1/m_b^5$ at tree level, without computing α_s-corrections to the power suppressed Wilson coefficients, thereby confirming the previous known result.

The computed structure functions for this decay are expressed through nine expectation values in dimension seven $(1/m_b^4)$ and 18 additional ones in dimension eight $(1/m_b^5)$. More operators appear with $\mathcal{O}(\alpha_s)$ corrections, cf. Ref. [45,51]. The structure functions allows to compute any differential distribution with arbitrary constraints on the lepton kinematics. Since the expressions are pretty lengthy, especially to order $1/m_b^5$, we do not quote them here. They are generated with MATHEMATICA and used to compute the analytical results for the various moments of the distributions with and without a cut on the charged lepton energy.

The operator basis of the new heavy quark parameters is growing very fast with successive higher orders in $1/m$. Therefore applying the formulae in experiment would be of little practice. The proliferating number cannot be extracted reliably for various and obvious reasons. Therefore we presented a derivation for an estimate of these matrix elements. This has already been used earlier [45]. We deduce a factorization theorem, which is still exact. The next step is to postulate a saturation of the ground-state, such that we keep only the leading term of the factorization. Based on this, we present the factorization result for all dimension seven and eight parameters in terms of the dimension five, μ_π^2 and μ_G^2 as well as dimension six parameters, ρ_D^3 and ρ_{LS}^3 and an excitation energy $\bar{\epsilon}$. This excitation energy can be traced back to time derivatives in the operators.

The calculated structure functions with the factorization estimate input allows us, to derive meaningful numerical estimates. We have analyzed the important question on the accuracy of the OPE related to the truncation to a fixed order in $1/m_b$ as well as the numerical convergence of this series.

From the numerical values we have presented, we conclude that the power expansion in inclusive $\bar{B}$-meson decays is numerically in a good shape, at least if the cut on the leptony energy is small enough $E_{\rm cut} \leq 1.5$ GeV. Furthermore we find that the $1/m_b^4$ corrections are somewhat smaller than those at $1/m_b^3$ and typically tend to partially offset the latter. The $1/m_b^5$ terms are again a bit less important than the $1/m_b^4$ - as expected - although the number of parameters is much higher. The notable exception is the effect of 'Intrinsic Charm' (IC) on the total decay rate which had been argued in [41] to be a potentially significant effect being driven, to higher orders, by an expansion in $1/m_c$ rather than in $1/m_b$. The powerlike terms arise first in $1/m_b^5$ with the power $1/m_b^3 \times 1/m_c^2$, and we have dedicated a separate part of the analysis to this. The situation is expected to be different for high moments of hadronic invariant mass. In usually considered moments the effects turn out to be noticeably smaller and this suggests that the truncation error at this order is largely negligible in practice.

We have also presented the analytic expressions for the corresponding higher-order corrections in the $B \to X_s + \gamma$ decays. A pilot evaluation showed that here the effects are numerically insignificant. A more reliable conclusion, however should include estimates for a wider range of hadronic parameters and eventually for the complete operator basis.

There are two different sources of changes of $|V_{cb}|$ within the analysis. One is related with the direct shift of it by the higher order terms in the total rate, which amounts to -0.65%. The other source stems from the shift in the HQE parameters due to the change in the moments. We expect this to be slightly larger, and in the opposite direction. Alltogether we anticipate a moderate overall upward shift by about half a percent.

The overall scale of the calculated higher-order power corrections shows, as a rule, that they are not negligible at the attained level of precision and are in line with the expectations laid down in Ref. [43]. Practically speaking, we expect the $1/m_b^4$ corrections to be of the same scale as the terms from not yet calculated α_s-corrections to the Wilson coefficients of the chromo-magnetic and of the Darwin operators.

We expect that the main effect of the estimated power corrections in the fit to the semileptonic data will be an increase in the Darwin expectation value ρ_D^3 by about 0.1 GeV^3, while μ_π^2 would not change significantly. Neither will the main combination of quark masses $m_b - 0.7 m_c$, shaping the mass dependence, change essentially. However, this does not apply to m_b or m_c separately. The residual dependence on the absolute values is subtle and the changes in the central value of m_b as large as 50 MeV may not have real significance.

Extracting the sub-leading heavy quark parameters from the $E_\ell^{\rm cut}$-dependence of the moments, a procedure which is effectively introduced in the fits to the data by assuming a strong correlation of theoretical uncertainties at different $E_\ell^{\rm cut}$ for a given moment, looks as an unsafe option. The cut-dependence of the moments is rather mild until the cut is placed relatively high, and there all corrections start to inflate degrading the theoretical accuracy of the OPE predictions. This is seen in the power expansion, and the similar behavior is expected from the uncalculated perturbative effects. The correction begin to blow up apparently somewhere near $E_\ell^{\rm cut} \gtrsim 1.65$ GeV. To remain on the safe side we suggest to rely on theory at $E_\ell^{\rm cut} \leq 1.5$ GeV.

Of course this is only a numerical analysis based upon a ground-state factorization theorem. This result should be taken as an estimate of the error in the $|V_{cb}|$ determination stemming from the power corrections.

6.2.2. Intrinsic Charm Part

A furtherresult of this study is that the OPE for inclusive $B \to X_c\, \ell \bar{\nu}_\ell$ contains terms with an infrared sensitivity to the charm quark mass. Although this has been known, a complete discussion of these so-called "intrinsic charm" contributions had not been presented. We have given such a discussion here in the context of two theoretical frameworks or 'roads' for removing charm quarks from the dynamical degrees of freedom, that a priori appear different, yet in the end yield identical results.

Now turning to the systematic analysis of the expansion, where we have noticed that the expansion is a combined expansion in both m_b and m_c starting at order $1/m_b^3$.

First we have shown how the "intrinsic-charm" contribution in semi-leptonic B-meson decays is related to the renormalization of sub-leading operators (like $m_c^4\, \bar{b}_v b_v$ and the Darwin term) appearing in the operator product expansion for the lepton-energy spectrum and the total rate. Within this analysis, we have distinguished three different cases which correspond to different power counting for the charm-quark mass. In the first case, one assumes $m_b \sim m_c$, i.e. the charm-quark is already integrated out at the hard scale, set by the large b-quark mass in the OPE. Consequently, all dependence on the charm-quark dynamics is already encoded in the matching conditions for the hard coefficient functions, and no "intrinsic-charm" operators should be introduced below the hard scale. The only remnant of "intrinsic charm" is the non-analytic dependence of the coefficient functions on the ratio $\rho = m_c^2/m_b^2$.

A long this way, we have shown that starting at $1/m_b^3$ the standard OPE for $B \to X_c\, \ell \bar{\nu}_\ell$ exhibits not only terms of the form $1/m_b^m \times \log m_c$ from the renormalization mixing, but also $1/m_b^m \times 1/m_c^n$ where at tree level only even n and $m \geq 3$ appear. The matrix elements of local operators, parameterizing their non-perturbative input, appearing with these m_c infrared sensitive coefficients always contain gluon-field-strength operators and their covariant derivatives. In turn the residual momentum of the b quark does not enter here.

We have performed a detailed analysis of the contributions of the form $1/m_b^3 \times 1/m_c^2$ at tree level, which is needed to complete the OPE calculation of $B \to X_c \ell \bar{\nu}_\ell$ up to order $1/m_b^4$, since parametrically $1/m_b^3 \times \Lambda_{\rm QCD}/m_c^2$ is of the same order. The numerical estimates, using the ground-state factorization theorem, confirm the results presented in Ref. [45].

The conclusion for $B \to X_c\, \ell \bar{\nu}_\ell$ is that a calculation to order $1/m_b^n$ has to include also the terms of order $1/m_b^{n-k} \times 1/(m_c^2)^k$ with $n \geq 3$ and $k = 1, \ldots, n-2$. Furthermore, including radiative corrections one obtains also contributions of the order $1/m_b^m \times \alpha_s(m_c)/m_c^k$ where k can also be odd. The lowest terms of this kind are of order $1/m_b^3 \times \alpha_s(m_c)/m_c$ and have been considered in Ref. [45].

These effects are of considerable theoretical interest with respect to subtleties that can arise in non-perturbative dynamics, yet they go beyond it towards more pragmatic goals: They help to validate the goal of reducing the theoretical uncertainty in extracting $|V_{cb}|$ from $B \to X_c\, \ell\bar{\nu}_\ell$ to the 1% level. It is not only an obvious goal to reach this precision goal for the theoretical treatment of B decays, yet also for proper interpretation in Kaon physics. It is required for the proper interpretation of the ultra-rare decays $K \to \pi\, \nu\bar{\nu}$. Their amplitudes have been calculated with high accuracy in terms of m_c and $V_{ts}^* V_{td}$ [79]. Their widths thus scale with V_{cb}^4, and the error on the latter is at least a large component in the stated overall 2% uncertainty, recently more precisely specified to be more than one third [81].

Another viable scenario treats the charm quark mass as intermediate between the hard and the soft scale in the OPE, $m_b \gg m_c \gg \Lambda_{\rm QCD}$. In that case, four-quark operators including soft b-quark fields and semi-hard charm quarks have to be included in the OPE. At the same time, in order to avoid double counting, the semi-hard region has to be subtracted from phase-space integrals by a suitable regularization of the decay spectra in the limit $m_c \ll m_b$. We have shown by explicit calculation how the mixing between the "intrinsic-charm" operators and the Darwin term generates the logarithmically enhanced terms entering the OPE at order $1/m_b^3$. Similarly, extending the results of [51], we could reproduce terms of order $\rho^2 \ln \rho$ in the partonic rate. After integrating out the charm quark at the semi-hard scale, the moments of the lepton-energy spectrum can be entirely described in terms of the standard hadronic input parameters, whereas – again – the complete charm-quark dependence enters via (eventually renormalization-group improved) short-distance coefficients, multiplying, for instance, the Darwin term.

The two approaches to the logarithmic and power-like sensitivities also show the advantages and disadvantages of the different renormalization schemes. For the analysis of this terms, we have used dimensional regularization, as well as a Wilsonian cut-off approach. Within dimensional regularization, the mixing of the four-quark operator into the Darwin term is computed easily. But the explicit appearing powers of the charm quark have to be treated as separate operators with a mass insertion, since in DR the charm mass is treated as vanishing. Within a cut-off scheme the charm mass gets replaced by the clear separation scale μ, and therefore the phase-space integration has to be modified. For treating the power-like terms this is clearly an advantage. However, in the analysis with a charm-quark we can stick to the first road, where charm and bottom are treated as equally heavy.

A somewhat more exotic approach would treat the charm quark as light, i.e. of order $\Lambda_{\rm QCD}$. Only in this case genuine intrinsic-charm (i.e. non-perturbative) effects have to be taken into account. Still, we have found that on the level of a few lepton-energy moments, the experimental data basically constrain a particular combination of the intrinsic-charm contribution and the Darwin term, such that to order $1/m_b^3$ the number of independent hadronic parameters effectively remains the same.

Therefore in $b \to u\,\ell\bar{\nu}_\ell$ decays the straightforward calculation of the higher-order power corrections to the total width beyond order $1/m_b^3$ would yield terms which diverge power-like in the infrared, but this does not mean that one cannot go beyond $1/m_b^3$ order here. The analysis shows that to calculate $\Gamma_{\rm sl}(b \to u)$ without extra α_s-corrections it is sufficient to introduce the corresponding WA four-quark operators. For practical reasons one should simply discard then all the terms, which formally have inverse powers of m_u. These terms stemming from the phase-space integration, which would have to be modified, are negligable and therefore it is easier to discard them.

We have found in this process that the analysis of IC effects give hints about the possible impact of WA in the heavy-to-light transitions $B \to X_u \ell\bar{\nu}_\ell$ – and the extraction of $|V_{ub}|$ there – and on its relation to charm decays. The IC effects for the inclusive distributions are conceptually similar to (generalized) WA corrections extensively discussed in connection to the lifetimes of heavy flavors since the late 1970s. Since in this thesis the usual bottom mesons, which do not contain valence charm quarks, are discussed, we deal here with the case of non-valence WA contributions first noted in Ref. [53]. A profound difference with the conventional WA for light quarks is that charm quarks, even soft ones, to the leading approximation can be viewed perturbatively. The non-trivial strong dynamics affect its propagation at the level of power corrections $(\Lambda_{\rm QCD}/m_c)^k$, while for light quarks this expansion would not be applicable. Nevertheless, we approach the case of conventional WA with light flavours from the heavy-mass side and use a model which naturally interpolates between the regimes of heavy and light quark we get an estimate

$$\frac{\delta\Gamma_{\rm sl}^{\rm n\,val}(b\to u)}{\Gamma_{\rm sl}(b\to u)} \approx -0.032\Big|_{\mu\approx 1\ {\rm GeV}}\,. \tag{6.1}$$

This result should be viewed as an educated guess rather than a real evaluation, from which we cannot count even on the firm prediction of the sign. It also leaves out the more intuitive valence WA which, as a matter of fact, historically gave the phenomenon its name. Yet, as clarified in Ref. [66], its interpretation in the presence of strong interactions is more subtle and may include interference-type contributions, which allow for the net correction to the width even to become negative.

We have used the recently reported [80] measurements of the D_s semileptonic fraction to estimate the significance of spectator-related WA in the CKM-suppressed semileptonic width of B^+. Taken at face, the correction turns out to be somewhat smaller then (6.1) but still destructive

$$\frac{\delta\Gamma_{\rm sl}^{\rm val}(b\to u)}{\Gamma_{\rm sl}(b\to u)} \approx -(0.005 \text{ to } 0.01)\,. \tag{6.2}$$

Since these contributions populate the kinematic domain of small hadronic invariant mass and energy of the final hadron state, they could have an amplified impact on the existing determinations of $|V_{ub}|$ from $B \to X_u \ell\bar{\nu}_\ell$.

Note added P. Gambino informed us about a pilot fit to the semileptonic data including the higher-order corrections to the moments quoted here. The preliminary result did not follow our expectation that the bulk of the effects reduces to the increase in ρ_D^3, although the suppression of the shift in the extracted value of $|V_{cb}|$ was observed. We note, however, that the reported outcome was largely dependent on the assumed strong correlations in the theoretical uncertainties at different $E_\ell^{\rm cut}$. More reliable conclusions can be drawn once a more general fit analysis is performed.

The main conclusion to be drawn is that, as long as the strong dynamics at the charm-quark scale can be treated perturbatively, "intrinsic-charm" effects do not induce an additional source of hadronic uncertainties at the level of $1/m_b^3$ power corrections, apart from the usually considered Darwin and spin-orbit terms. The same will be true for higher orders in the $1/m_b$ expansion as classified in [28]. The issue of whether to resum logarithms $\ln(m_c^2/m_b^2)$ by introducing the above 2-step matching procedure, or sticking to the standard 1-step matching has to be decided by considering radiative corrections to the $1/m_b^3$ expressions which is beyond the scope of this work (see also [33]).

6.3. Exclusive Decay

In the SM the weak charged current is mediated by a left-handed interaction, which was experimentally verified already in the late 50's [71]. The situation in the hadronic sector is more complicated, since we do not observe the direct quark transition, only the decays of the bound objects. This might influence the structure of the underlying decay, however up to now there is no evidence for a contribution from beyond the Standard Model physics. In an analysis using inclusive semileptonic decays no sensitivity to right-handed interaction has been found [16].

The difference in exclusive decays are the form factors, which are sensitive to the different appearing operator structures, whereas in inclusive decays the non-perturbative parameters depend only on the incoming state. For this reasons a repeated analysis using exclusive semileptonic decays is promissing.

We have defined the new occuring matrix elements and computed the necessary differential rate in terms of these matrix elements, where we have used the heavy quark limit. In order to prepare the analysis with real data, we have computed the radiative corrections to the new vertices. These are used to estimate the deviation from the heavy quark normalization of unity at the zero-recoil point. We computed for each vertex Γ the corrections η_Γ analytically and confirmed the known result [23] for the Standard Model contributions $\Gamma = V, A$. We found, that at this specific kinematical point the vertex correction factors are numericaly given by

$$\eta_V \approx 1.02\,, \tag{6.3a}$$
$$\eta_A \approx 0.97\,, \tag{6.3b}$$
$$\eta_{\rm sc} = 1.03^{+0.06}_{-0.05}\,, \tag{6.3c}$$
$$\eta_{\rm ps} = 0.002^{+0.021}_{-0.028}\,, \tag{6.3d}$$
$$\eta_{\rm t} = 0\,. \tag{6.3e}$$

From the values it is obvious, that due to the special kinematical point a tensor does not contribute and a pseudo-scalar one only to a little percentage - yet with a huge relative error. Therefore we conclude that only an admixture of a scalar interaction could be measured, in principle.

From (5.27) it can be seen, that $B \to D\,\ell\bar{\nu}_\ell$ is sensitive to a scalar admixture. Especially the w dependence is different from the one in front of the SM coefficient.

Therefore a careful analysis of the mixing term between these two structures might reveal the underlying structure.

In contrast the semileptonic decay into the vector meson D^*. Here only the longitudinal polarization is sensitive to a possible scalar admixture. Therefore a polarized analysis might even be more sensitive to new physics.

The analysis using real data from B-factories will be topic of a forthcomming publication [78].

6.4. Outlook

With the upcomming new experiments LHC(b) and the super flavour factories, the experimental data becomes better and better. The experiments and accelerators increase the luminosity and therefore statistics, but on the other hand also improve their detectors and analytical methods. In order to improve the measurement of SM parameters along with these experiments, we also have to advance in theory to reduce the theoretical errors together with the experimental one to about the same level.

As we have demonstrated the non-perturbative corrections for the inclusive determination on $|V_{cb}|$ is in a very good shape. Therefore at the moment from the theoretical point ov view, the largest uncertainties are stemming from the unknown perturbative corrections to the Wilson coefficients of μ_G^2 and ρ_D^3. These are - besides μ_π^2, where the corrections are already known [31] - the most relevant non-perturbative parameters. These corrections should be, as we have argued at the level of the non-perturbative corrections to order $1/m_b^4$. Hence it is worthwile to determine these α_s corrections, but to go beyond this seems to be not useful. From a combined fit, using the new theory input and improved data, the matrix element $|V_{cb}|$ could in principle be extracted with an error at the level of $\mathcal{O}(1\%)$.

Besides this, the matrix element $|V_{ub}|$ is from theoretical interest, too. It is extracted from a similar process as, we have considered in this thesis, where the charm gets replaced by an up quark. Therefore the inclusive analysis goes along the same line. But in that case we have to be careful at order $1/m_b^3$ and higher, as we have explained in the context of intrinsic charm. On the other side since $|V_{cb}/V_{ub}| \approx 10$, the charm background is presumably 100 times larger and thus prevents the clean extraction of $|V_{ub}|$ from experiment. To circumvent this problem, the charm background is dismissed by kinematical cuts. But this procedure introduces additional theoretical difficulties, which make the extraction more challenging. Yet from the exclusive analysis with $\bar{B} \to \pi \ell \bar{\nu}_\ell$ the CKM matrix element is also determined. As in the case of a charm final state quark, these two measurements show a slight tension. The exclusive measurement central value is roughly three standard deviations below the inclusive measurement.

We can use the methods described here to improve the inclusive determination of $|V_{ub}|$. Especially the role of weak annihilation should be investigated more carefully. Aside from this, the influence of the experimental cuts to the validity of the theoretical tools should be checked.

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A. Appendix

A.1. Higher Order Trace Formulae

We have analytical results for the trace formulae

$$\langle \bar{B}|\bar{b}_v(iD_{j_0})\dots(iD_{j_n})b_v|\bar{B}\rangle \tag{A.1}$$

up to dimension eight, corresponding to $n = 5$ covariant derivatives. Using this, we have obtained the hadronic tensor $W_{\mu\nu}$ including all of the in total 31 non-perturbative parameters. However, these terms are too lengthy to be diplayed literally here. These expressions are stored in MATHEMATICA text files.

A.2. Lepton Energy Spectrum

The complete expression for the partonic contribution to the lepton-energy spectrum with $m_b \sim m_c \geq \mu$ is given by [28]

$$\begin{aligned}\frac{d\Gamma}{dy}\Big|_{\text{partonic}} =& \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2\Big\{\Big(-\frac{4\rho^3}{(y-1)^3}-\frac{6\,(\rho^3+\rho^2)}{(y-1)^2}-\frac{12\rho^2}{y-1}\\ &-4y^3-6(\rho-1)y^2+2(\rho-3)\rho^2\Big)\theta(1-y-\rho)\Big\}\,.\end{aligned} \tag{A.2}$$

From this one can obtain closed expressions for the $(1-y)^n$ moments,

$$\begin{aligned}\langle(1-y)^n\rangle\Big|_{\text{partonic}} =& \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2\Big\{-\frac{12\,(\rho^n-1)\,\rho^2}{n}-\frac{4\,(\rho^n-\rho^2)\,\rho}{n-2}-\frac{12\,(\rho^{n+2}-1)\,\rho}{n+2}\\ &+\frac{6\,(\rho^2+\rho)\,(\rho^n-\rho)}{n-1}-\frac{2\,(\rho^3-3\rho^2-3\rho+1)\,(\rho^{n+1}-1)}{n+1}\\ &+\frac{6(\rho+1)\,(\rho^{n+3}-1)}{n+3}-\frac{4\,(\rho^{n+4}-1)}{n+4}\Big\}\,.\end{aligned} \tag{A.3}$$

Expanding (A.3) in the small parameter ρ, the logarithmically enhanced terms at order ρ^2 appear only in the total rate and the first moment

$$\langle(1-y)^n\rangle\Big|_{\text{partonic}} = \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2\left(12\,\delta^{n0}-6\,\delta^{n1}\right)\rho^2\ln\rho + \text{analytic/higher-order terms in } \rho. \tag{A.4}$$

The first infrared singularity appears at $\mathcal{O}(1/m_b^3)$ proportional to the Darwin term. The full contribution related to the Darwin term in the lepton-energy spectrum for the case $m_b \sim m_c \geq \mu$ is given by [28]

$$\begin{aligned}
\left.\frac{d\Gamma^{(3)}}{dy}\right|_{\rho_D^3} =& \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2 \frac{\rho_D^3}{m_b^3}\Bigg\{ \\
&\left(\frac{40\rho^3}{3(y-1)^6} + \frac{8\rho^2(3\rho+1)}{(y-1)^5} + \frac{6\rho^2(3\rho+1)}{(y-1)^4} + \frac{16\rho\left(2\rho^2-\rho-1\right)}{3(y-1)^3}\right. \\
&-\frac{28\rho}{3(y-1)^2} + \frac{8}{y-1} + \frac{2}{3}\left(5\rho^3 - 5\rho^2 + 10\rho + 22\right) \\
&\left.+\frac{8}{3}(\rho+3)(y-1) + 4(y-1)^2 + \frac{8}{3}(y-1)^3\right)\theta(1-y-\rho) \\
&-\left(\frac{2(\rho-1)^4(\rho+1)^2}{3\rho^2}\right)\delta(1-y-\rho)\Bigg\}\,. \qquad \text{(A.5)}
\end{aligned}$$

From this we obtain closed expressions for the moments,

$$\begin{aligned}
\left.\langle(1-y)^n\rangle\right|_{\rho_D^3} =& \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2 \frac{\rho_D^3}{m_b^3}\Bigg\{\frac{8\left(\rho^n-1\right)}{n} - \frac{2}{3}(\rho-1)^4(\rho+1)^2\rho^{n-2} + \frac{28\left(\rho^n-\rho\right)}{3(n-1)} \\
&-\frac{2\left(5\rho^3-5\rho^2+10\rho+22\right)\left(\rho^{n+1}-1\right)}{3(n+1)} + \frac{8(\rho+3)\left(\rho^{n+2}-1\right)}{3(n+2)} \\
&-\frac{4\left(\rho^{n+3}-1\right)}{n+3} + \frac{8\left(\rho^{n+4}-1\right)}{3(n+4)} - \frac{16\left(2\rho^2-\rho-1\right)\left(\rho^2-\rho^n\right)}{3(n-2)\rho} \\
&+\frac{6(3\rho+1)\left(\rho^3-\rho^n\right)}{(n-3)\rho} - \frac{8(3\rho+1)\left(\rho^4-\rho^n\right)}{(n-4)\rho^2} + \frac{40\left(\rho^5-\rho^n\right)}{3(n-5)\rho^2}\Bigg\}\,. \qquad \text{(A.6)}
\end{aligned}$$

It turns out, that the lepton-energy spectrum is a polynomial in $1/(1-y)$ and therefore closed expressions for the moments in $(1-y)^n$ are obtained easily. Arbitrary moments can then be derived via

$$\langle(y-y_0)^n\rangle = \sum_{k=0}^{n}\binom{n}{k}(1-y_0)^{n-k}(-1)^k\langle(1-y)^k\rangle\,. \qquad \text{(A.7)}$$

Taking the limit $\rho \to 0$ in (A.6), the logarithmically enhanced terms appear only in the total rate

$$\left.\langle(1-y)^n\rangle\right|_{\rho_D^3} = \frac{G_F^2 m_b^5}{24\pi^3}|V_{cb}|^2\frac{\rho_D^3}{m_b^3}\left(\delta^{n0}\ln\rho + \mathcal{O}(\rho\ln\rho)\right)\,. \qquad \text{(A.8)}$$

A.3. Operator Mixing

A.3.1. Dimension-6

In the following we briefly sketch the derivation of the elements of the anomalous-dimension matrix that govern the mixing of the four-quark ("intrinsic-charm") operators into the Darwin term. For simplicity, we do not construct the complete set of independent operators that would be needed to describe the full one-loop anomalous-dimension matrix, but rather focus on the effect of the charm-loop diagram in figure 4.6(a). For this purpose it is sufficient to consider the two operator structures which enter the hadronic tensor at tree-level (4.146):

$$2M_B T_1(\mu) = \frac{(4\pi)^2}{3}\Big(\langle \bar B(p)|\bar b_v\, \gamma_\mu P_L c\; \bar c\,\gamma^\mu P_L\, b_v|\bar B(p)\rangle - \langle \bar B(p)|\bar b_v\, \not{v} P_L\, c\; \bar c\, \not{v} P_L\, b_v|\bar B(p)\rangle\Big)\,, \tag{A.9a}$$

$$2M_B T_2(\mu) = \frac{(4\pi)^2}{3}\Big(4\,\langle \bar B(p)|\bar b_v\, \not{v} P_L\, c\; \bar c\, \not{v} P_L\, b_v|\bar B(p)\rangle - \langle \bar B(p)|\bar b_v\, \gamma_\mu P_L\, c\; \bar c\,\gamma^\mu P_L\, b_v|\bar B(p)\rangle\Big)\,. \tag{A.9b}$$

Together with the Darwin term they are used to define a simplified operator basis

$$\mathcal{O}_{\rho_D} = \bar b_v\,(iD_\mu)(ivD)(iD^\mu)\, b_v\,, \tag{A.10a}$$

$$\mathcal{O}_{T_1} = (4\pi)^2\,\mu^{2\epsilon}\,\frac{1}{3}\left(\bar b_v\, \gamma_\mu P_L\, c\; \bar c\,\gamma^\mu P_L\, b_v - \bar b_v\, \not{v} P_L\, c\; \bar c\, \not{v} P_L\, b_v\right)\,, \tag{A.10b}$$

$$\mathcal{O}_{T_2} = (4\pi)^2\,\mu^{2\epsilon}\,\frac{1}{3}\left(4\,\bar b_v\, \not{v} P_L\, c\; \bar c\, \not{v} P_L\, b_v - \bar b_v\, \gamma_\mu P_L\, c\; \bar c\,\gamma^\mu P_L\, b_v\right)\,. \tag{A.10c}$$

Notice that for convenience, we have extracted a factor $(4\pi)^2\,\mu^{2\epsilon}$, in order to have a simple, dimensionless anomalous-dimension matrix.[1]

Calculating the one-loop matrix elements of the operators $\mathcal{O}_{T_{1,2}}$ for the partonic transition $b \to b$ in the presence of a soft background field $A_\mu(k)$, see Fig. 4.6(a), and comparing with the tree-level matrix element of the Darwin-term operator, we obtain the following results in $D = 4 - 2\epsilon$ dimensions,

$$\langle b|\mathcal{O}_{T_1}|b\rangle^{(0)} = +\frac{1}{3}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m_c^2}\right)\langle b|\mathcal{O}_{\rho_D}|b\rangle_{\text{tree}}\,, \tag{A.11a}$$

$$\langle b|\mathcal{O}_{T_2}|b\rangle^{(0)} = -\frac{2}{3}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m_c^2}\right)\langle b|\mathcal{O}_{\rho_D}|b\rangle_{\text{tree}}\,, \tag{A.11b}$$

where the one-gluon matrix element of the Darwin-term operator on parton level is given by

$$\langle b|\mathcal{O}_{\rho_D}|b\rangle_{\text{tree}} = \frac{1}{2}\,\langle b|\bar b_v\,[iD_\mu,[(iv\cdot D), iD^\mu]]\, b_v|b\rangle_{\text{tree}} + \mathcal{O}(1/m_b) \tag{A.12a}$$

$$= \frac{g}{2}\Big((v\cdot k)(k\cdot A) - k^2\,(v\cdot A)\Big)\,\bar u_b\, u_b + \ldots \tag{A.12b}$$

From (A.11b) we read off the desired elements of the anomalous dimension matrix

$$\gamma = \begin{pmatrix} 0 & 0 & 0 \\ -2/3 & 0 & 0 \\ 4/3 & 0 & 0 \end{pmatrix} + \mathcal{O}(\alpha_s)\,, \tag{A.13}$$

[1] With this convention, the anomalous-dimension matrix is of order $(\alpha_s)^0$. In order to have it in the standard form, one would have to extract a factor $g_s^2 = 4\pi\alpha_s\,\mu^{2\epsilon}$, instead.

where the neglected higher-order terms describe the mixing of "intrinsic-charm" operators into themselves and of the Darwin term into itself, which are not explicitly needed for the discussion in the body of the text.

A.3.2. Dimension-7

A similarly simplified analysis can be performed for the mixing of the dimension-7 "intrinsic-charm" operators into the dimension-7 two-quark operator $m_c^4\,\bar{b}_v\,\slashed{v}\,b_v$. As before, defining

$$\mathcal{O}_2 = m_c^4\,\bar{b}_v\,\slashed{v}\,b_v\,, \tag{A.14a}$$

$$\mathcal{O}_{T_3} = (4\pi)^2\,\mu^{2\epsilon}\,\frac{1}{3}\left((iv\cdot\partial\,\bar{b}_v\,\gamma_\mu P_L\,c)\,(\bar{c}\,\gamma^\mu P_L\,b_v) - (iv\cdot\partial\,\bar{b}_v\,\slashed{v}P_L\,c)\,(\bar{c}\,\slashed{v}P_L\,b_v)\right)\,, \tag{A.14b}$$

$$\mathcal{O}_{T_4} = (4\pi)^2\,\mu^{2\epsilon}\,\frac{1}{3}\left((i\partial_\alpha\,\bar{b}_v\,\slashed{v}P_L\,c)\,(\bar{c}\,\gamma^\alpha P_L\,b_v) - (iv\cdot\partial\,\bar{b}_v\,\slashed{v}P_L\,c)\,(\bar{c}\,\slashed{v}P_L\,b_v)\right)\,, \tag{A.14c}$$

$$\mathcal{O}_{T_5} = (4\pi)^2\,\mu^{2\epsilon}\,\frac{1}{3}\left((i\partial_\alpha\,\bar{b}_v\,\gamma^\alpha P_L\,c)\,(\bar{c}\,\slashed{v}P_L\,b_v) - (iv\cdot\partial\,\bar{b}_v\,\slashed{v}P_L\,c)\,(\bar{c}\,\slashed{v}P_L\,b_v)\right)\,, \tag{A.14d}$$

$$\mathcal{O}_{T_6} = (4\pi)^2\,\mu^{2\epsilon}\,\frac{1}{3}\left(6\,(iv\cdot\partial\,\bar{b}_v\,\slashed{v}P_L\,c)\,(\bar{c}\,\slashed{v}P_L\,b_v) - (iv\cdot\partial\,\bar{b}_v\,\gamma_\mu P_L\,c)\,(\bar{c}\,\gamma^\mu P_L\,b_v)\right) \tag{A.14e}$$

$$\qquad - (4\pi)^2\,\mu^{2\epsilon}\,\frac{1}{3}\left((i\partial_\alpha\,\bar{b}_v\,\slashed{v}P_L\,c)\,(\bar{c}\,\gamma^\alpha P_L\,b_v) + (i\partial_\alpha\,\bar{b}_v\,\gamma^\alpha P_L\,c)\,(\bar{c}\,\slashed{v}P_L\,b_v)\right)\,, \tag{A.14f}$$

$$\mathcal{O}_{T_7} = (4\pi)^2\,\mu^{2\epsilon}\,\frac{1}{6}\,\epsilon^{\mu\nu\alpha\beta}\,v_\beta\,(i\partial_\alpha\,\bar{b}_v\,\gamma_\nu P_L\,c)\,(\bar{c}\,\gamma_\mu P_L\,b_v)\,, \tag{A.14g}$$

we calculate the contributions to the 2-parton matrix elements from the tadpole diagram in Fig. 4.6(b) as

$$\langle b|\mathcal{O}_{T_3}|b\rangle^{(0)} = +\frac{1}{8}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m_c^2} + \ldots\right)\langle b|\mathcal{O}_2|b\rangle_{\text{tree}}\,, \tag{A.15a}$$

$$\langle b|\mathcal{O}_{T_4}|b\rangle^{(0)} = \langle b|\mathcal{O}_{T_5}|b\rangle^{(0)} = -\frac{1}{8}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m_c^2} + \ldots\right)\langle b|\mathcal{O}_2|b\rangle_{\text{tree}}\,, \tag{A.15b}$$

$$\langle b|\mathcal{O}_{T_6}|b\rangle^{(0)} = 0\,, \tag{A.15c}$$

$$\langle b|\mathcal{O}_{T_7}|b\rangle^{(0)} = -\frac{1}{8}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m_c^2} + \ldots\right)\langle b|\mathcal{O}_2|b\rangle_{\text{tree}}\,, \tag{A.15d}$$

from which we read off the elements of the anomalous dimension matrix entering (4.162).

A.4. Zero Recoil Passarino-Veltmann functions

We calculate every diagram for arbitrary momenta p_b and p_c, but fixed the zero-recoil point $p_b \cdot p_c = m_b m_c$. The results are obtained and expressed in the $\overline{MS}$-scheme with

$$D = 4 - 2\epsilon$$
$$\frac{1}{\epsilon} \equiv \frac{1}{\bar{\epsilon}} = \frac{1}{\epsilon} + \ln 4\pi - \gamma_{\mathrm{E}}$$

First we derive the scalar integrals up to three denominators. The neccessary Passarino-Veltmann functions are given explictely in the $\overline{MS}$-scheme

A.4.1. One Point Function

For one denominator we have a simple integral with only one scale. The result can thus be computed easily

$$A_0^{\mathrm{ZR}}(m) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2 - m^2 + i0} \tag{A.16}$$
$$\to \mu^{4-D} \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{q^2 - m^2 + i0} \tag{A.17}$$
$$\underset{\epsilon\to 0}{\approx} \frac{i}{(4\pi)^2} m^2 \left(\frac{1}{\epsilon} - \log \frac{m^2}{\mu^2} + 1 \right) . \tag{A.18}$$

A.4.2. Two Point Function

In the case of two denominators we can have different kinematical regions at the zero recoil point, and we have two different mass scales in solving this integral. The general formula, where the integration over the Feynman parameters cannot be performed explictly, is given by

$$B_0(p^2, m_1, m_2) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{[(p+q)^2 - m_2^2][q^2 - m_1^2] + i0} \tag{A.19}$$
$$\to \mu^{4-D} \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{[(p+q)^2 - m_2^2][q^2 - m_1^2] + i0} \tag{A.20}$$
$$\underset{\epsilon\to 0}{\approx} \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} - \int_0^1 \mathrm{d}x \, \ln \left[\frac{(m_2^2 + p_1^2(x-1))\, x - m_1^2(x-1)}{\mu^2} \right] \right) . \tag{A.21}$$

The integration can be performed for either a special kinematic situation, or if a hierarchy between the mass scales is given. In the latter case the integrand can be expanded in a Taylor series. From this intermediate result, we derive the needed special zero-recoil functions

$$
\begin{aligned}
B_0^{\mathrm{ZR}}(m^2,0,m) &= \frac{1}{\epsilon} - \log\frac{m^2}{\mu^2} + 2\,, \\
B_0^{\mathrm{ZR}}(0,0,m) &= \frac{1}{\epsilon} - \log\frac{m^2}{\mu^2} + 1 \\
&= B_0^{\mathrm{ZR}}(m^2,0,m) - 1\,, \\
B_0^{\mathrm{ZR}}(0,m,m) &= B_0^{\mathrm{ZR}}(m^2,0,m) - 2 \\
&= B_0^{\mathrm{ZR}}(0,0,m) - 1\,, \\
B_0^{\mathrm{ZR}}((m_b-m_c)^2,m_b,m_c) &= \frac{1}{\epsilon} - \frac{1}{m_b-m_c}\left[m_b\big(\ln\frac{m_b^2}{\mu^2} - 1\big) - m_c\big(\ln\frac{m_c^2}{\mu^2} - 1\big)\right], \\
B_0^{\mathrm{ZR}}(0,m_b,m_c) &= \frac{1}{\epsilon} - \frac{m_b^2}{m_b^2-m_c^2}\big(\ln\frac{m_b^2}{\mu^2} - 1\big) + \frac{m_c^2}{m_b^2-m_c^2}\big(\ln\frac{m_c^2}{\mu^2} - 1\big) \\
&= \frac{1}{m_b-m_c}\big(m_b B_0^{\mathrm{ZR}}(m_b^2,0,m_b) - m_c B_0^{\mathrm{ZR}}(m_c^2,0,m_c)\big)\,.
\end{aligned}
$$

A.4.3. Three Point Function

For the three point function, we need

$$
C_0(p_1^2,p_2^2,m_0,m_1,m_2) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{[(p_1+q)^2-m_1^2][(p_2+q)^2-m_2^2][q^2-m_0^2]+i0} \tag{A.22}
$$

$$
\to \mu^{4-D}\frac{\mathrm{d}^D q}{(2\pi)^D} \frac{1}{[(p_1+q)^2-m_1^2][(p_2+q)^2-m_2^2][q^2-m_0^2]+i0} \tag{A.23}
$$

$$
\begin{aligned}
&= -\frac{i}{(4\pi)^2} 2^{2-D}(D-4)(D-2)\pi^{3-\frac{D}{2}} \csc\frac{D\pi}{2} \int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x} \mathrm{d}y \\
&\times \frac{[(1-x-y)m_0^2 + m_1^2 x + p_1^2(x^2-x) + 2p_1p_2 xy + y m_2^2 + p_2^2(y^2-y)]^{\frac{D-6}{2}}}{\mu^{D-4}\,\Gamma\left(\frac{D}{2}\right)}\,.
\end{aligned} \tag{A.24}
$$

But here we have to be carefull. Since the gluon is massless, we have to deal with an infrared singularity. Therefore the expansion around $D=4$ has to be done, after the Feynman parameter integration. In the end we receive a simple function, which in principle can be related to the one and two point functions

$$
C_0(m_b^2,m_c^2,0,m_b,m_c) = \frac{i}{2(4\pi)^2 m_b m_c}\left[\frac{1}{\epsilon} + \frac{\left(\log\frac{m_b^2}{\mu^2}+2\right)m_c - \left(\log\frac{m_c^2}{\mu^2}+2\right)m_b}{m_b-m_c}\right]. \tag{A.25}
$$

A.4.4. Tensor Functions

We need the results for tensor loop integrals up to three indices. But luckily the integral with three positive powers of the loop integration momentum can be reduced to a simpler one, because two of them are contracted and cancel the gluon propagator. Therefore in total we have to compute

$$B_\alpha(p_1^2,p_2^2.m_1^2,m_2^2) = \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{q_\alpha}{[(p_1+q)^2-m_1^2][(p_2+q)^2-m_2^2]+i0}\,, \tag{A.26a}$$

$$C_\alpha(p_1^2,p_2^2,0,m_1^2,m_2^2) = \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{q_\alpha}{[(p_1+q)^2-m_1^2][(p_2+q)^2-m_2^2][q^2]+i0}\,, \tag{A.26b}$$

$$C_{\alpha\beta}(p_1^2,p_2^2,0,m_1^2,m_2^2) = \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{q_\alpha q_\beta}{[(p_1+q)^2-m_1^2][(p_2+q)^2-m_2^2][q^2]+i0}\,. \tag{A.26c}$$

In the next steps we will perform the tensor decomposition and give the results for the scalar integrals.

A.4.5. Two Point Function with one Index

At a first step we compute an easier master integral of the form

$$B_\alpha(p_1^2,0,m_1^2,m_2^2) = \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{q_\alpha}{[(p_1+q)^2-m_1^2][q^2-m_2^2]+i0}\,. \tag{A.27}$$

We can parametrize this by the external momentum $(p_1)_\alpha$,

$$B_\alpha(q^2,0,m_1^2,m_2^2) = \tilde{B}_1\,(q)_\alpha\,. \tag{A.28}$$

Using the condition $p_1 \cdot q = \frac{1}{2}\left[[(p_1+q)^2-m_1^2] - [q^2-m_2^2] - [q^2-m_1^2+m_2^2]\right]$ we derive the relation

$$\tilde{B}_1 = \frac{1}{2q^2}\left(A_0(m_2^2)^2 - A_0(m_1^2)^2 + (m_1^2-m_2^2-q^2)B_0(q^2,m_1^2,m_2^2)\right)\,. \tag{A.29}$$

Using this result we can reparametrize the momentum and derive additionally

$$\begin{aligned} B_\alpha(p_b^2,p_c^2,m_b^2,m_c^2) =& (p_b-p_c)_\alpha \frac{1}{2(m_b-m_c)^2}\Big[A_0(m_c^2)^2 - A_0(m_b^2)^2 \\ &+ (m_b^2-m_c^2-(m_b-m_c)^2)B_0((m_b-m_c)^2,m_b^2,m_c^2)\Big] \\ &+ (p_c)_\alpha\; B_0((m_b-m_c)^2,m_b^2,m_c^2)\,. \end{aligned} \tag{A.30}$$

A.4.6. Three Point Function with one Index

We have additionally a three point tensor function with one index

$$\begin{aligned} C^\alpha(p_1^2,p_2^2,m_0,m_1,m_2) =& \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{q^\alpha}{[(p_1+q)^2-m_1^2][(p_2+q)^2-m_2^2][q^2-m_0^2]+i0} \\ =& C_1(p_1^2,p_2^2,m_0,m_1,m_2)\,p_1^\alpha + C_2(p_1^2,p_2^2,m_0,m_1,m_2)\,p_2^\alpha\,. \end{aligned} \tag{A.31}$$

With the same methods as above, we can compute the coefficient functions, but in this case we have to solve a system of equations, because the loop integration momentum has to be parameterized by two different momenta. The scalar functions are given by

$$C_1(p_1^2,p_2^2,m_0,m_1,m_2) = \frac{B_0\left(m_b^2,0,m_b^2\right) - B_0\left(\left(m_b-m_c\right)^2,m_b^2,m_c^2\right)}{4m_bm_c}\,, \tag{A.32}$$

$$C_2(p_1^2,p_2^2,m_0,m_1,m_2) = \frac{B_0\left(m_b^2,0,m_b^2\right) - B_0\left(\left(m_b-m_c\right)^2,m_b^2,m_c^2\right)}{4m_c^2}\,. \tag{A.33}$$

A.4.7. Three Point Function with two Indices

The last needed tensor functions is $C^{\alpha\beta}$. It can be parametrized as[2]

$$C^{\alpha\beta}(p_1^2,p_2^2,m_0,m_1,m_2) = \int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{q^\alpha q^\beta}{[(p_1+q)^2-m_1^2][(p_2+q)^2-m_2^2][q^2-m_0^2]+i0} \tag{A.34}$$

$$= C_0 g^{\alpha\beta} + C_1\, p_b^\alpha p_b^\beta + C_2\, p_c^\alpha p_c^\beta + C_3\, p_b^\alpha p_c^\beta + C_4\, p_b^\beta p_c^\alpha\,. \tag{A.35}$$

Using the same scalar product trick from above we get with a slightly more complicated system of equations the results for the five scalar functions

$$C_0 = \frac{1}{4}\left(B_0((m_b-m_c)^2,m_b^2,m_c^2)+1\right)\,, \tag{A.36a}$$

$$C_1 = \frac{1}{8m_b(m_b-m_c)^2}\Big[m_b\left(3B_0(m_b^2,0,m_b^2) - 3B_0((m_b-m_c)^2,m_b^2,m_c^2) - 1\right) + m_c\left(B_0((m_b-m_c)^2,m_b^2,m_c^2) - B_0(m_b^2,0,m_b^2)\right)\Big]\,, \tag{A.36b}$$

$$C_2 = \frac{1}{8m_c^2(m_b-m_c)^2}\Big[m_b(m_b-3m_c)\left(B_0((m_b-m_c)^2,m_b^2,m_c^2) - B_0(m_b^2,0,m_b^2)\right) - 4m_c^2\Big]\,, \tag{A.36c}$$

$$C_3 = \frac{1}{8m_c(m_b-m_c)^2}\Big[4m_c + (m_b+m_c)\left(B_0((m_b-m_c)^2,m_b^2,m_c^2) - B_0(m_b^2,0,m_b^2)\right)\Big]\,, \tag{A.36d}$$

$$C_4 = \frac{1}{8m_c(m_b-m_c)^2}\Big[4m_c + (m_b+m_c)\left(B_0((m_b-m_c)^2,m_b^2,m_c^2) - B_0(m_b^2,0,m_b^2)\right)\Big]\,. \tag{A.36e}$$

[2]We leave out the momentum and mass dependence on the coefficients for readability reasons.

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