Eigenvalues of Measure Theoretic Laplacians on Cantor-like Sets

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Abstract

We study the eigenvalues of the Laplacian $\Delta_\mu = \frac{d}{d\mu}\frac{d}{dx}$. Here, μ is a singular measure on a bounded interval with an irregular recursive structure, which include self-similar measures as a special case. The structure can also be randomly build. For this operator we determine the asymptotic growth behaviour of the eigenvalue counting function.

Furthermore, in the case where μ is self-similar, we give a representation of the eigenvalues of Δ_μ as zero points of generalized sine functions allowing, in particular, an explicit computation. Moreover, we use these functions to describe certain properties of the eigenfunctions.

Zusammenfassung

Wir untersuchen die Eigenwerte des Laplaceoperators $\Delta_\mu = \frac{d}{d\mu}\frac{d}{dx}$. Hierbei ist μ ein singuläres Maß auf einem beschränkten Intervall mit einer irregulären rekursiven Struktur, was selbstähnliche Maße als Spezialfall enthält. Diese Struktur kann auch zufällig sein. Für diesen Operator bestimmen wir das asymptotische Wachstumsverhalten der Eigenwertzählfunktion.

Weiterhin, im Fall eines selbstähnlichen Maßes, stellen wir die Eigenwerte von Δ_μ als Nullstellen von verallgemeinerten Sinusfunktionen dar, was insbesondere eine explizite Berechnung erlaubt. Außerdem benutzen wir diese Funktionen um Eigenschaften der Eigenfunktionen zu beschreiben.

Contents

1. Introduction

1.1. Statement of the problem

Assume that μ is a Borel measure on the interval $[a, b]$ which is singular in the sense, that it has no Radon-Nikodym density with respect to the Lebesgue measure. For example, μ is a measure whose support is the Cantor set. We will define a Laplacian Δ_μ on $[a, b]$ for the measure μ and study the eigenvalue problem

$$\Delta_\mu f = -\lambda f$$

with either homogeneous Dirichlet boundary conditions

$$f(a) = f(b) = 0,$$

or homogeneous Neumann boundary conditions

$$f'(a) = f'(b) = 0.$$

The definition of Δ_μ involves the derivative with respect to the measure μ. If a function $g\colon [a, b] \to \mathbb{R}$ possesses the representation

$$g(x) = g(a) + \int_{[a,x]} \frac{dg}{d\mu}\, d\mu \tag{1.1}$$

for all $x \in [a, b]$, then $\frac{dg}{d\mu}$ is the μ-*derivative* of g. In Freiberg [17] an analytic calculus of the concept of μ derivatives is developed.

The operator Δ_μ is then given by

$$\Delta_\mu f = \frac{d}{d\mu} f'$$

for all f from a suitable domain.

It is well known that if μ is a non-atomic Borel measure, Δ_μ has a pure point spectrum consisting only of eigenvalues with multiplicity one, that accumulate at infinity, see Freiberg [17, Lem. 5.1 and Cor. 6.9] or Bird, Ngai and Teplyaev [5, Th. 2.5].

This operator and the resulting eigenvalue problem has been studied in numerous papers, for example in Feller [15], McKean and Ray [40], Kac and Krein [32], Fujita [24], Naimark and Solomyak [41], Freiberg and Zähle [23], Bird, Ngai and Teplyaev [5], Freiberg [17–20], Freiberg and Löbus [22], Hu, Lau and Ngai [29], and Chen and Ngai [8].

A treatment of the classical theory of boundary problems on the real line can be found for example in Atkinson [2].

1.2. Physical motivation for the Laplacian Δ_μ

Consider a flexible string (e.g. a guitar string) clamped between two points a and b such that, if we deflect it, a tension force drives it back towards its state of equilibrium.

The string shall have a mass distribution given by a function $\rho\colon [a,b] \to [0,\infty)$. We denote the deviation of the string at the point $x \in [a,b]$ at time $t \in [0,\infty)$ by $u(x,t)$ and the tangentially acting tension force by F, where we assume that F does not depend on x and t.

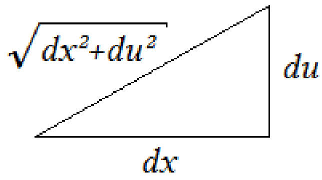
Then the motion of the string, the function $u(x,t)$, is determined by the *wave equation*

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\rho(x)}{F}\frac{\partial^2 u}{\partial t^2}(x,t) \tag{1.2}$$

with Dirichlet boundary conditions $u(a,t) = u(b,t) = 0$ for all t. To model Neumann boundary conditions we need a slightly modified set-up. For that, suppose that the string ends are attached to carriages (without mass) at a and b, freely movable in the direction orthogonal to the string. Since this is a technically difficult construction, we can also think of a narrow (and thus assumed one-dimensional) rectangular basin filled with water with $u(x,t)$ being the water level at the point x at time t. Then $\frac{\partial u}{\partial x}(a,t) = \frac{\partial u}{\partial x}(b,t) = 0$ for all t.

In the following, we give a physical derivation of the one-dimensional wave equation based on Strauss [50, p. 13].

Look at an infinitesimal small interval $[x, x+dx]$ and decompose the force F into a longitudinal component F_{long} and a transversal component F_{trans} in the ratio given in the following diagram.



Then

$$F_{\text{long}} = \frac{F\,dx}{\sqrt{dx^2 + du^2}}$$

and

$$F_{\text{trans}} = \frac{F\,du}{\sqrt{dx^2 + du^2}}.$$

According to Newton's law $F = m \cdot a$, a force gives rise to an acceleration a of a mass m. We assume that there is no motion in longitudinal direction, but the transversal part of the force works on the mass element $\rho(x)dx$ and creates an acceleration $\dfrac{\partial^2 u}{\partial t^2}$ of the string. Newton's law then gives

$$\frac{F\,du}{\sqrt{dx^2 + du^2}} = F\frac{\frac{du}{dx}}{\sqrt{1 + (\frac{du}{dx})^2}} = \rho(x)\,dx \cdot \frac{\partial^2 u}{\partial t^2}.$$

The term $\left(\frac{du}{dx}\right)^2$ is neglectable, because we assume the deviation of the string to be small. Thus, by "dividing by dx", we get the one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\rho(x)}{F}\,\frac{\partial^2 u(x,t)}{\partial t^2}.$$

To solve this equation, we use the technique of separation of variables. For that, we make an ansatz $u(x,t) = f(x)\,g(t)$ leading to

$$f''(x)\,g(t) = \frac{\rho(x)}{F}\,f(x)\,g''(t).$$

It follows that

$$\frac{f''(x)}{\rho(x)\,f(x)} = \frac{1}{F}\,\frac{g''(t)}{g(t)}$$

for all x and t. Therefore, both sides of the equation have to be constant, say $-\lambda$. Thus,

$$f''(x) = -\lambda \rho(x)\,f(x)$$

and

$$g''(t) = -\lambda F\, g(t).$$

The first one of these equations is relevant for our considerations. We integrate it to get

$$\begin{aligned} f'(x) - f'(a) &= -\lambda \int_a^x f(t)\,\rho(t)dt \\ &= -\lambda \int_a^x f(t)\,d\mu(t), \end{aligned}$$

where μ is the measure with density ρ. Now we apply the μ-derivative defined in (1.1) and get

$$\frac{d}{d\mu} f' = -\lambda f.$$

This equation no longer involves the density of μ, meaning that we can use it to formulate the problem for singular (fractal) measures.

A solution f of this eigenvalue equation can be regarded as the shape of the string at a certain fixed time. Then $u(x,t) = g(t)f(x)$ describes a so-called *standing wave*. The square root of the eigenvalues are up to a constant the natural frequencies of the string, that is, the vibrating string generates a sound which is a superposition of overtones with frequencies $c\sqrt{\lambda_n}$.

The same equation arises when considering the *heat equation*

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial u(x,t)}{\partial t}$$

after the separation of variables. Here $u(x,t)$ describes the temperature of some, say, metallic bar at x at the time t and α denotes the thermal diffusivity of the material.

1.3. The Cantor set and generalizations

What we call *Cantor set* was introduced in 1883 as an example for a perfect set by Cantor [7] on page 590 in footnote 11 in the context of investigations on integrability. There, Cantor considers the set of all real numbers that can be expressed as

$$\frac{c_1}{3} + \frac{c_2}{3^2} + \cdots + \frac{c_\nu}{3^\nu} + \cdots\,,$$

where c_ν can take the values 0 or 2.

However, there is an earlier record of a similar construction made in 1875 by Smith [47, p. 147]. There, the interval $[0, 1]$ is divided into m equal parts and the last one is removed. Then this process is repeated on each of the remaining segments and so on. Smith observes that the resulting set is nowhere dense in $[0, 1]$. Subsequently, he varies this construction process to receive a nowhere dense set of positive length. For further information on the history of the Cantor set see also Fleron [16].

In 1919, Hausdorff [28] introduces the dimension and measure concepts that are named after him. Also in this paper, a rather general construction is made, where an interval of varying length is removed step by step from the middle of the previous intervals. Then, investigations are made on the Hausdorff measure and dimension of the resulting set. In particular, the dimension $\frac{\log 2}{\log 3}$ of the classical Cantor set is given on page 172.

Salem [46] gives a geometric construction of a function f whose slope is zero almost everywhere on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$ which is strictly monotonically increasing. This construction starts with a straight line from zero to one that is in the first step replaced by two lines meeting in the middle. Then, each line is transformed in the same fashion. This process is repeated while in each step different transformations are chosen. The resulting function defines a singular measure on $[0, 1]$ where the support is the whole interval $[0, 1]$.

More general constructions and the Hausdorff dimension of the resulting sets are considered for example in Beardon [4] and Falconer [14, p. 61ff.].

There are also numerous different models for random (Cantor) fractals. In Falconer [14, Ch. 15], a random Cantor set is constructed and its Hausdorff dimension determined by an expectation equation. Further papers treating this topic are e.g. Mandelbrot [38], Zähle [54], Falconer [13], Mauldin and Williams [39], Graf [26], and Hutchinson and Rüschendorf [31].

In the present thesis we are not concerned with the Hausdorff dimension of such sets but with the eigenvalues of the corresponding Laplacian and their growth behaviour.

1.4. Outline of the thesis

The thesis consists of two parts more or less independent of each other being formed by Chapters 3 and 4. Before that, in Chapter 2 we give some preliminary considerations which we will need later.

In Chapter 3 we are interested in the asymptotic behaviour of the eigenvalue counting functions N_{Neu} and N_{Dir} given by

$$N_{\mathrm{Neu}}(x) := \#\{\lambda \leq x \colon \lambda \text{ is a Neumann eigenvalue of } \Delta_\mu\}$$

and

$$N_{\mathrm{Dir}}(x) := \#\{\lambda \leq x \colon \lambda \text{ is a Dirichlet eigenvalue of } \Delta_\mu\}$$

for $x \geq 0$. The case where μ is a self-similar measure on $[a, b]$ has been investigated by Fujita [24] and, for a more general operator, Freiberg [19]. For the definition of self-similarity of measures, see Section B.2. In this case, the eigenvalue counting functions grow asymptotically like x^γ, that is, there are constants $c_1, c_2 > 0$ such that

$$c_1 x^\gamma \leq N_{\mathrm{Dir}}(x) \leq N_{\mathrm{Neu}}(x) \leq c_2 x^\gamma$$

for all x greater than some x_0. The *spectral exponent* γ is given as the solution of

$$\sum_{i=1}^{N} (r_i m_i)^\gamma = 1, \tag{1.3}$$

and the number $d_s = 2\gamma$ is often called *spectral dimension* of $\operatorname{supp} \mu$.

In the present work we will consider fractals that are not strictly self-similar, but have an irregular recursive structure. The construction of such sets is done in Section 3.1 but we briefly outline it here. As is very well known, the classical Cantor set can be constructed by iteratively wiping out the middle third of intervals. We do a similar construction as follows. Suppose we are given a collection of iterated function systems $(\mathcal{S}^{(j)})_{j \in J}$ (for the definition of an iterated function system (IFS), see Section B.2) and start with the interval $[a, b]$. In the first step, we choose an arbitrary IFS $\mathcal{S}^{(\xi_1)}$ from our collection and wipe out intervals such that only those intervals remain which are determined by $\mathcal{S}^{(\xi_1)}$. For the second step we choose again an IFS $\mathcal{S}^{(\xi_2)}$ and treat every remaining interval accordingly. In this way we proceed to receive the fractal $K^{(\xi)}$, where ξ denotes the sequence $(\xi_1, \xi_2, \dots)$. Moreover, we construct a measure $\mu^{(\xi)}$ on $[a, b]$ with $\mu^{(\xi)}([a, b]) = 1$. For that we first assign a mass of one to the interval $[a, b]$. Then, in the first wiping step, we divide this mass according to certain weight coefficients at the remaining sub-intervals. This division process is then repeated, where in each step, dependent on ξ, different weights are chosen. In our considerations we assume a given fixed sequence ξ which determines $K^{(\xi)}$ and $\mu^{(\xi)}$ completely. This concept is adopted from Barlow and Hambly [3], where it is used for a construction of *scale irregular Sierpinski gaskets*. Following Hambly [27], we call these measures and sets *homogeneous*, because all “first-level” cells have the same structure.

In Section 3.2 we establish a scaling property that is essential for the proof of our main result. In that we follow Kigami and Lapidus [35, Lem. 2.3].

The main theorem about the asymptotic growth of the eigenvalue counting function is proved in Section 3.3. It states that there are constants $c_1, c_2, C_1, C_2 > 0$ and $x_0 > 0$ such that

$$C_1\, x^\gamma\, e^{-c_1\, g(\log x)} \le N_{\mathrm{Dir}}^{(\xi)}(x) \le N_{\mathrm{Neu}}^{(\xi)}(x) \le C_2\, x^\gamma\, e^{c_2\, g(\log x)}$$

for all $x \geq x_0$, where the function g is determined by the sequence ξ and γ is the solution of an equation similar to (1.3).

In Section 3.4 we go through several examples for $(\mathcal{S}^{(j)})_{j\in J}$ and ξ.

Subsequently, in Section 3.5 we take for ξ a sequence of i.i.d. random variables and apply our results for almost every realization.

In the second part of the thesis, that is Chapter 4, we consider fractal measures on $[0,1]$ that have a strict self-similar structure. For these we give a new technique of determining the eigenvalues and eigenfunctions of Δ_μ that involves a generalization of the sine and cosine functions.

In this we follow the classical case, where μ is the Lebesgue measure. There, the Dirichlet eigenvalue problem reads

$$\begin{aligned} f'' &= -\lambda f \\ f(0) = f(1) &= 0. \end{aligned}$$

Then, for every non-negative λ, $f(x) = \sin(\sqrt{\lambda}x)$ satisfies the equation as well as the boundary condition on the left-hand side. On the right-hand side, the boundary condition is only met if $\sqrt{\lambda}$ is a zero point of the sine function, which are, indeed, very well known.

If we impose Neumann boundary conditions $f'(0) = f'(1) = 0$, we take $f(x) = \cos(\sqrt{\lambda}x)$, because this complies automatically with the left-hand side condition. The right-hand side condition again is satisfied if $\sqrt{\lambda}$ is a zero point of the sine function, which leads to the same eigenvalues as in the Dirichlet case (supplemented by zero). But here sine appears as the derivative of cosine, which will make a difference when we take more general measures.

Now let μ be an arbitrary Borel probability measure on $[0,1]$. For every $z \in \mathbb{R}$ we construct functions sq_z and cp_z depending on μ as a replacement for $\sin(z\,\cdot)$ and $\cos(zx)$. We do that by generalizing the series

$$\sin(zx) = \sum_{n=0}^{\infty}(-1)^n \frac{(zx)^{2n+1}}{(2n+1)!}$$

and

$$\cos(zx) = \sum_{n=0}^{\infty}(-1)^n \frac{(zx)^{2n}}{(2n)!},$$

replacing $\frac{x^n}{n!}$ by appropriate functions $p_n(x)$ or $q_n(x)$, depending on whether we impose Neumann or Dirichlet boundary conditions. These functions fulfill the eigenvalue equation and meet the left-hand side Dirichlet and Neumann boundary condition, respectively.

Putting $p_n := p_n(1)$ and $q_n := q_n(1)$, we define

$$\mathrm{sinq}(z) := \sum_{n=0}^{\infty}(-1)^n q_{2n+1} z^{2n+1}$$

and

$$\mathrm{sinp}(z) := \sum_{n=0}^{\infty}(-1)^n p_{2n+1} z^{2n+1}.$$

For sq_z and cp_z to also match the right-hand side conditions, z has to be chosen as a zero point of sinq in the Dirichlet case and sinp in the Neumann case. All this is described in Section 4.1.

In Section 4.2 we show how to compute the norms in $L_2(\mu)$ of the eigenfunctions by using the sequences p_n and q_n.

The functions cp_z and sq_z satisfy an identity that generalizes the classical trigonometric identity. This is established in Section 4.3.

In Section 4.4 we consider symmetric measures and get some symmetry results.

The main results are established in Section 4.5. We outline these briefly here. Since the functions $p_n(x)$ and $q_n(x)$ are determined in a process of iterative integration alternately with respect to μ and the Lebesgue measure, the coefficients p_n and q_n are difficult to compute in general. But if μ is a self-similar measure with respect to the mappings

$$S_1(x) = r_1 x \qquad \text{and} \qquad S_2(x) = r_2(x-1)+1$$

as well as the weight factors m_1 and m_2, we develop a recursion formula for p_n and q_n involving r_1, r_2, m_1, and m_2.

To illustrate the structure of this recursion formula, we consider again the classical Lebesgue case. There we have

$$p_n = q_n = \frac{1}{n!}$$

which leads to $\operatorname{sinp}(z) = \operatorname{sinq}(z) = \sin(z)$. The sequence $p_n = \frac{1}{n!}$ can be viewed as the solution of the problem

$$\begin{aligned} 2^n p_n &= \sum_{i=0}^{n} p_i \, p_{n-i}, \\ p_1 &= 1, \end{aligned} \tag{1.4}$$

which is derived from the equation $2^n = \sum_{i=0}^{n} \binom{n}{i}$. Our recursion formula for self-similar μ looks more complicated, as it distinguishes between the two different kinds of boundary conditions. Additionally it is different for even and odd values of n, and it involves the parameters r_1, r_2, m_1, m_2 of the measure. However, it has the same basic structure as (1.4).

Moreover, we establish functional equations involving sinp and sinq that can be viewed as generalizations of the classical addition theorems.

In Section 4.6 we consider the especially interesting case where $r_1 m_1 = r_2 m_2$. Then the Neumann eigenvalues fulfill a renormalization formula

$$\lambda_{2n} = R\,\lambda_n,$$

where $1/R = r_1 m_1$. This property has been established in a special case by Volkmer [52] and in our setting by Freiberg [20]. This formula allows us to investigate the growth of subsequences

$$\left(\|\tilde{f}_{k2^n}\|_\infty \right)_{n\in\mathbb{N}}, \qquad \text{for odd } k,$$

where $\tilde{f}_n$ denotes an eigenfunction to the nth Neumann eigenvalue that is normalized to one in $L_2(\mu)$.

We show in Section 4.7 that, if we assume $r_1 + r_2 = 1$ in addition to $r_1 m_1 = r_2 m_2$, the Dirichlet and Neumann eigenvalues coincide.

Finally, by using the formulas we developed in the course of our investigations, we compute approximations of eigenvalues for certain examples in Section 4.8.

Several remarks about possible further investigations are made in Section 4.9.

In Appendix A we give graphs of eigenfunctions that were plotted by using formulas from Section 4.5.

For reference, Appendix B contains some mathematical foundations.

2. Preliminaries

We consider the interval $[a, b]$, where $-\infty < a < b < \infty$. The one-dimensional Lebesgue measure is denoted by λ, the σ-algebra of Borel sets in $[a, b]$ by $\mathcal{B}$. By measurable we will always mean Borel-measurable.

2.1. Derivatives and the Laplacian with respect to a measure

As in Freiberg [17, 19], we define a derivative of a function with respect to a measure.

Definition 2.1.1. Let μ be a non-atomic finite Borel measure on $[a, b]$ and let $f\colon [a, b] \to \mathbb{R}$. A function $h \in L_2([a, b], \mu)$ is called the *μ-derivative of f*, if

$$f(x) = f(a) + \int_a^x h\, d\mu \qquad \text{for all } x \in [a, b].$$

The following lemma gives some technical statements which we will need to prove the uniqueness of the μ-derivative.

Lemma 2.1.2. *Let μ be a finite Borel-measure on $[a, b]$. Then the following holds.*

(i) *For every $A \in \mathcal{B}$ and every $\varepsilon > 0$ there is an open set $V \subseteq [a, b]$ such that $A \subseteq V$ and $\mu(V \setminus A) < \varepsilon$.*

(ii) *Let $f \in L_2([a, b], \mu)$ and $A \in \mathcal{B}$. Then, for each $\varepsilon > 0$, we can find a $\delta > 0$ such that $\int_A |f|\, d\mu < \varepsilon$, provided that $\mu(A) < \delta$.*

(iii) *Every open set $U \subseteq [a, b]$ can be written as a countable union of disjoint open intervals.*

Proof. A proof of (i) can be found in Rudin [43, Th. 2.18] and of (iii) in [43, p. 50]. We prove (ii). Let $f \in L_2(\mu)$, $f \neq 0$ and $\varepsilon > 0$. By the Cauchy-Schwarz inequality follows

$$\int_A |f|\,d\mu = \int_{[a,b]} \mathbf{1}_A |f|\,d\mu \le \Big(\int_{[a,b]} \mathbf{1}_A\,d\mu\Big)^{\frac{1}{2}} \Big(\int_{[a,b]} |f|^2\,d\mu\Big)^{\frac{1}{2}} = \mu(A)^{\frac{1}{2}} \|f\|_{L_2(\mu)}.$$

Thus, if $\mu(A) < \dfrac{\varepsilon^2}{\|f\|^2_{L_2(\mu)}}$, then $\displaystyle\int_A |f|\,d\mu < \varepsilon$. $\square$

Proposition 2.1.3. *The μ-derivative in Definition 2.1.1 is unique in $L_2(\mu)$.*

Proof. Let μ and f be as in Definition 2.1.1. Let h_1 and h_2 be μ-derivatives of f. It follows from the definition that $\int_a^x (h_1 - h_2)\,d\mu = 0$ for all $x \in [a,b]$ and therefore $\int_x^y (h_1 - h_2)\,d\mu = 0$ for all $x, y \in [a,b]$ with $x < y$. By Lemma 2.1.2 (iii) we get $\int_U (h_1 - h_2)\,d\mu = 0$ for any open set $U \subseteq [a,b]$.

As h_1 and h_2 are measurable, the set $\{x \in [a,b] \colon h_1(x) - h_2(x) \neq 0\}$ is a Borel-set. We will show that its measure is zero by contradiction. Assume without loss of generality that $M := \{x \in [a,b] \colon h_1(x) - h_2(x) > 0\}$ has positive measure and let $\varepsilon > 0$. Because of Lemma 2.1.2 (i) and (ii), we can find an open set $V \subseteq [a,b]$ such that $\int_{V \setminus M} |h_1 - h_2|\,d\mu < \varepsilon$. Hence

$$\int_V (h_1 - h_2)\,d\mu = \int_{V\setminus M} (h_1 - h_2)\,d\mu + \int_M (h_1 - h_2)\,d\mu > -\varepsilon + \int_M (h_1 - h_2)\,d\mu,$$

and thus, since $\mu(M) > 0$ and $h_1 - h_2 > 0$ on M, we get $\int_V (h_1 - h_2)\,d\mu > 0$ by choosing ε sufficiently small. This is a contradiction. $\square$

Thus we showed that, if it exists, the μ-derivative of a function f is well defined and so we denote it by $\frac{df}{d\mu}$. The λ-derivative $\frac{df}{d\lambda}$ we denote by f'.

Lemma 2.1.4. *Let $f_1, f_2 \colon [a,b] \to \mathbb{R}$ functions with $f_1 = f_2$ μ-almost everywhere and whose μ-derivatives exist. Then*

$$\frac{df_1}{d\mu} = \frac{df_2}{d\mu}$$

μ-almost everywhere.

Proof. Let $N \in \mathcal{B}$ with $\mu(N) = 0$ such that $f_1(x) = f_2(x)$ for all $x \in [a,b] \setminus N$. Then the proof works as that of Proposition 2.1.3 if we replace the set M by $M \setminus N$. $\square$

Definition 2.1.5. Let μ be a non-atomic, finite Borel-measure on $[a,b]$. We define $H^1([a,b],\mu) = H^1(\mu)$ to be the space of all elements of $L_2(\mu)$ that possess a representative whose μ-derivative exists.

Note that, according to our definition, the μ-derivative is always in $L_2(\mu)$. In case $\mu = \lambda$ is the Lebesgue measure, the definition of $H^1(\lambda)$ is equivalent to the usual definiton of this Sobolev space.

Proposition 2.1.6. *Let μ be a non-atomic, finite Borel-measure on $[a, b]$. All functions in $H^1(\mu)$ are continuous in the sense that each equivalence class in $H^1(\mu)$ has a representative that is continuous on $[a, b]$.*

Proof. Let $f \in H^1(\mu)$. We take f to be the representative whose μ-derivative exists. Let $x \in [a, b]$. For all $y \in [a, b]$ we have by the Cauchy-Schwarz inequality,

$$|f(x) - f(y)| = \left| \int_x^y \frac{df}{d\mu} \, d\mu \right| \leq \int_x^y \left| \frac{df}{d\mu} \right| d\mu \leq \mu([x, y))^{\frac{1}{2}} \left\| \frac{df}{d\mu} \right\|_2 \to 0$$

as $y \downarrow x$, because $\mu(\{x\}) = 0$. Therefore, f is continuous in x. $\square$

Definition 2.1.7. Let μ and ν be atomless, finite Borel-measures on $[a, b]$. The space $H^2([a, b], \nu, \mu)$ is the collection of all functions in $H^1([a, b], \nu)$ whose ν-derivative belongs to $H^1([a, b], \mu)$.

As with $L_2(\mu)$ we will denote the spaces with $H^1(\mu)$ and $H^2(\nu, \mu)$, respectively, if we use the interval $I = [a, b]$.

Remark 2.1.8. According to Proposition 2.1.6 it is clear that, if $f \in H^2(\nu, \mu)$, $\frac{df}{d\nu}$ possesses a continuous representative.

In the following, we take ν to be the Lebesgue measure denoted by λ. Now we define the operator Δ_μ on which our investigations are focused.

Definition 2.1.9. For all $f \in H^2(\lambda, \mu)$ we define

$$\Delta_\mu f := \frac{d}{d\mu} f'.$$

Remark 2.1.10. In Freiberg [17, Cor. 6.4] is shown that $H^2(\lambda, \mu)$ is dense in $L_2(\mu)$.

Similar to the classical derivative, there are several derivation and integration rules. Some are presented in the following.

Proposition 2.1.11. (i) *Let $f, g \in H^1(\mu)$. Then $fg \in H^1(\mu)$ and*

$$\frac{d}{d\mu}(fg) = \frac{df}{d\mu} g + f \frac{dg}{d\mu}.$$

(ii) *Let* $f \in H^2(\lambda,\mu)$ *and* $g \in H^1(\lambda)$. *Then*

$$\int_a^b (\Delta_\mu f) g \, d\mu = f'g\Big|_a^b - \int_a^b f'(x) g'(x) \, dx.$$

(iii) *Let* $g\colon [a,b] \to \mathbb{R}$ *be invertible and* $f \in H^1\big([g(a),g(b)], g\mu\big)$. *Then* $f \circ g \in H^1([a,b],\mu)$ *and*

$$\frac{d}{d\mu}(f \circ g) = \frac{df}{d(g\mu)} \circ g.$$

(iv) *Let* $f \in L_1(\lambda)$ *and* $g \in H^1(\mu)$ *increasing. Then,*

$$\int_a^b f\big(g(t)\big) \, \frac{dg}{d\mu}(t) \, d\mu(t) = \int_{g(a)}^{g(b)} f(x) \, dx.$$

(v) *Suppose that* $f \in H^2(\lambda,\mu)$ *and* $g \in H^1(\lambda)$. *Then, with* $\mathcal{E}(f,g) := \int_a^b f'(t) \, g'(t) \, dt$,

$$\mathcal{E}(f,g) = -\big\langle \Delta_\mu f, g \big\rangle_{L_2(\mu)} + f'(b)g(b) - f'(a)g(a).$$

Proof. (i) is proved in Freiberg [17, Lem. 2.3 (i)] and (ii) in [17, Prop. 3.1]. We first prove (iii). To this end, take $f \in H^1\big([g(a),g(b)], g\mu\big)$ and write for $x \in [a,b]$

$$\begin{aligned} f(g(x)) &= f(g(a)) + \int_{g(a)}^{g(x)} \frac{d}{d(g\mu)} f \, d(g\mu) \\ &= f \circ g(a) + \int_a^x \frac{df}{d(g\mu)} \circ g \, d\mu. \end{aligned}$$

Then the assertion follows by the uniqueness of the μ-derivative.

Next, we show (iv). Let $c,d \in [a,b]$ with $c < d$. We denote $\bar{\mu} = \frac{dg}{d\mu} \cdot \mu$. Then, by the definition of the μ-derivative,

$$\bar{\mu}\big([c,d]\big) = \int_c^d \frac{dg}{d\mu} \, d\mu = g(d) - g(c) = \lambda\big([g(c),g(d)]\big).$$

Since this holds for all intervals in $[a,b]$ it follows by Carathéodory's extension theorem that

$$g\bar{\mu} = \lambda.$$

Then,

$$\int_a^b f\big(g(t)\big)\, d\bar{\mu}(t) = \int_{g(a)}^{g(b)} f(x)\, d(g\bar{\mu})(x) = \int_{g(a)}^{g(b)} f(x)\, dx,$$

which proves the assertion.

(v) follows directly from (ii). □

2.2. Dirichlet forms related to the Laplacian

Throughout this section, we denote by μ a finite atomless Borel measure on $[a, b]$ and by λ the Lebesgue measure.

We define two Dirichlet forms $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F}_0)$ on $L_2(\mu)$ and show that they have the same eigenvalues as Δ_μ with homogeneous Neumann and Dirichlet boundary conditions, respectively. To prove the desired asymptotic properties of the eigenvalue counting functions later, we will work rather with the Dirichlet forms than with the operator Δ_μ itself. The statements and proofs are similar to those given in Kigami and Lapidus [35] and, apart from the compactness of the embeddings that occur, have already been established in Freiberg [18].

In the following we denote by $\mathcal{E}$ the bilinear form given by

$$\mathcal{E}(f, g) := \int_a^b f'(t) g'(t)\, dt$$

for $f, g \in H^1(\lambda)$.

The domain $H^1(\lambda)$ we will denote by $\mathcal{F}$.

Remark 2.2.1. The domain of the operator Δ_μ is contained in $\mathcal{F}$, that is, $H^2(\lambda, \mu) \subseteq \mathcal{F}$. Therefore, since $H^2(\lambda, \mu)$ is dense in $L_2(\mu)$, so is $\mathcal{F}$.

Note, that $\mathcal{F} \subseteq L_2(\lambda)$ in the sense that if $f \in \mathcal{F}$, then there is an equivalence class $\tilde{f} \in L_2(\lambda)$ such that $f \in \tilde{f}$. But furthermore, $\mathcal{F} \subseteq L_2(\mu)$ in the sense that if $f \in \mathcal{F}$, then there is an equivalence class $\bar{f} \in L_2(\mu)$ such that $f \in \bar{f}$.

That means we have to distinguish whether we regard a function $f \in \mathcal{F}$ as an element of $L_2(\lambda)$ or $L_2(\mu)$. For example, if we modify f on a μ-null set, then it does not change in the $L_2(\mu)$ sense, but may change in the $L_2(\lambda)$ sense.

Proposition 2.2.2. *$(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L_2(\mu)$ and the embedding*

$$(\mathcal{F}, \mathcal{E}_1) \hookrightarrow L_2(\mu),$$

is compact, where $(\mathcal{F}, \mathcal{E}_1)$ denotes the Hilbert space $\mathcal{F}$ with the inner product $\mathcal{E}_1$ given by

$$\mathcal{E}_1(f,g) = \mathcal{E}(f,g) + \langle f,g\rangle_{L_2(\mu)}.$$

Proof. It follows as a special case of Theorem 4.1 of Freiberg [18] that $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L_2(\mu)$. It remains to show that the embedding is compact. The proof is similar to that of Lemma 5.4 of Kigami and Lapidus [35].

An operator $T\colon X \to Y$, where X and Y are Banach spaces, is compact, if for every bounded set A in X the image $T(A)$ is precompact in Y.

Let $U \subseteq \mathcal{F}$ be bounded with respect to $\mathcal{E}_1$, that is, there is a $c > 0$ such that for all $f \in U$ holds

$$\int_a^b f(t)^2 \, d\mu(t) + \int_a^b f'(t)^2 \, dt \le c \tag{2.1}$$

or, equivalently, there are constants $c_1, c_2 > 0$ such that for all $f \in U$,

$$\|f\|_{L_2(\mu)} \le c_1$$

and

$$\sqrt{\mathcal{E}(f,f)} \le c_2.$$

We have to show that U is precompact in $L_2(\mu)$. To do that, we show that U is bounded in $C([a,b])$ (with respect to $\|.\|_\infty$) and equicontinuous. Then it follows from Theorem B.4.3 (Arzelà-Ascoli) that U is precompact in $C([a,b])$. Since $C([a,b])$ is continuously embedded in $L_2(\mu)$, U is precompact in $L_2(\mu)$.

For $x \in [a,b]$ we define the function g^x by

$$g^x(t) := \begin{cases} t - x, & t \in [a,x], \\ 0, & t \in (x,b]. \end{cases} \tag{2.2}$$

Then, $g^x \in \mathcal{F}$, and for every $f \in \mathcal{F}$ and $x \in [a,b]$ holds

$$\mathcal{E}(f, g^x) = \int_a^b f'(t) {g^x}'(t) \, dt = \int_a^x f'(t) \, dt = f(x) - f(a).$$

First we show that U is bounded in $C([a,b])$. For every $f \in U$ and $x \in [a,b]$ holds

$$\begin{aligned}
|f(x) - f(a)| &\leq \int_a^b |g^{x\prime}(t)\, f'(t)|\, dt \\
&\leq \sqrt{\mathcal{E}(g^x, g^x)}\, \sqrt{\mathcal{E}(f,f)} \\
&= \sqrt{g^x(x) - g^x(a)}\, \sqrt{\mathcal{E}(f,f)} \\
&= \sqrt{x-a}\, \sqrt{\mathcal{E}(f,f)}
\end{aligned}$$

and therefore

$$\|f - f(a)\|_\infty \leq \sqrt{b-a} \cdot c_2.$$

Furthermore, for $f \in U$,

$$\begin{aligned}
\|f\|_\infty &\leq \|f - f(a)\|_\infty + |f(a)| \\
&\leq \sqrt{b-a}\, c_2 + \frac{1}{\sqrt{\mu([a,b])}} \|f(a)\|_{L_2(\mu)} \\
&\leq \sqrt{b-a}\, c_2 + \frac{1}{\sqrt{\mu([a,b])}} \Big(\|f - f(a)\|_{L_2(\mu)} + \|f\|_{L_2(\mu)} \Big) \\
&\leq \sqrt{b-a}\, c_2 + \|f - f(a)\|_\infty + \frac{1}{\sqrt{\mu([a,b])}} \|f\|_{L_2(\mu)} \\
&\leq 2\sqrt{b-a}\, c_2 + \frac{1}{\sqrt{\mu([a,b])}} c_1,
\end{aligned}$$

and hence, U is bounded in $C([a,b])$.

It remains to show that U is equicontinuous. For $f \in U$ and $x, y \in [a,b]$ holds

$$\mathcal{E}(g^x - g^y, f) = \mathcal{E}(g^x, f) - \mathcal{E}(g^y, f) = f(x) - f(y).$$

Thus,

$$\begin{aligned}
|f(x) - f(y)| &\leq \sqrt{\mathcal{E}(g^x - g^y, g^x - g^y)}\, \sqrt{\mathcal{E}(f,f)} \\
&= \sqrt{|x-y|}\, \sqrt{\mathcal{E}(f,f)} \\
&\leq c_2 \sqrt{|x-y|}.
\end{aligned}$$

Let $\varepsilon > 0$ and $\delta := \left(\frac{\varepsilon}{c_2}\right)^2$. Then, for $|x-y| < \delta$ and $f \in U$,

$$|f(x) - f(y)| \leq c_2 \sqrt{|x-y|} < \varepsilon.$$

$\square$

Lemma 2.2.3. *If f is an eigenfunction of $(\mathcal{E}, \mathcal{F})$ with eigenvalue λ, then*

$$f(x) = f(a) + \int_a^x \int_a^z -\lambda f(t)\, d\mu(t)\, dz \qquad \textit{for all } x \in [a, b]. \tag{2.3}$$

Proof. Let f be an eigenfunction of $(\mathcal{E}, \mathcal{F})$ with eigenvalue λ, that is,

$$\mathcal{E}(f, g) = \lambda \langle f, g \rangle_{L_2(\mu)}, \qquad \text{for all } g \in \mathcal{F}. \tag{2.4}$$

Let $x \in [a, b]$ and let $g^x \in \mathcal{F}$ be as defined in (2.2). Since

$$\mathcal{E}(f, g^x) = f(x) - f(a)$$

it follows that

$$\begin{aligned}
f(x) &= f(a) + \lambda \int_a^b f(t) g^x(t)\, d\mu(t) \\
&= f(a) + \lambda \int_a^x f(t)(t - x)\, d\mu(t) \\
&= f(a) - \lambda \int_a^x \int_a^x \mathbf{1}_{[t,x]}(z) f(t)\, dz\, d\mu(t) \\
&= f(a) - \lambda \int_a^x \int_a^x \mathbf{1}_{[a,z]}(t) f(t)\, d\mu\, dz \\
&= f(a) - \lambda \int_a^x \int_a^z f(t)\, d\mu(t)\, dz.
\end{aligned}$$

$\square$

In the following proposition we show that the Neumann eigenvalues of Δ_μ coincide with those of $(\mathcal{E}, \mathcal{F})$.

Proposition 2.2.4. *For $\lambda \in \mathbb{R}$ and $f \in \mathcal{F}$ holds*

$$\mathcal{E}(f, g) = \lambda \langle f, g \rangle_{L_2(\mu)}$$

for all $g \in \mathcal{F}$ if and only if $f \in H^2(\lambda, \mu)$ and

$$\begin{aligned}
&\Delta_\mu f = -\lambda f, \\
&f'(a) = f'(b) = 0.
\end{aligned}$$

Proof. Assume $f \in \mathcal{F}$ satisfies

$$\mathcal{E}(f,g) = \lambda \langle f, g \rangle_{L_2(\mu)}$$

for all $g \in \mathcal{F}$. Then, according to Lemma 2.2.3,

$$f(x) = f(a) + \int_a^x \int_a^z -\lambda f(t)\, d\mu(t)\, dz \qquad \text{for all } x \in [a,b].$$

Therefore $f \in H^1(\lambda)$ and

$$f'(x) = \int_a^x -\lambda f(t)\, d\mu(t), \qquad \text{for all } x \in [a,b].$$

Thus $f \in H^2(\lambda,\mu)$, $f'(a) = 0$ and $\Delta_\mu f = -\lambda f$. From Lemma 2.1.11 (v) it follows that

$$-\big\langle \Delta_\mu f, g \big\rangle_{L_2(\mu)} + f'(b)g(b) - f'(a)g(a) = \lambda \langle f, g \rangle_{L_2(\mu)} \qquad \text{for all } g \in \mathcal{F}$$

which implies that $f'(b)g(b) = 0$ for all g and consequently $f'(b) = 0$.

Conversely, suppose that $f \in H^2(\mu)$ and $\Delta_\mu f = -\lambda f$ with boundary condition $f'(a) = f'(b) = 0$. Then, as seen in Remark 2.2.1, $f \in \mathcal{F}$ and furthermore, $\langle \Delta_\mu f, g \rangle_{L_2(\mu)} = -\lambda \langle f, g \rangle_{L_2(\mu)}$ for all $g \in \mathcal{F}$. By Lemma 2.1.11 (v) it follows that $\mathcal{E}(f,g) = \lambda \langle f, g \rangle_{L_2(\mu)}$. □

Example 2.2.5. The eigenvalues of the Dirichlet form $\big(\mathcal{E}, H^1(\lambda)\big)$ on $L_2(\lambda)$ are

$$\lambda_n = \frac{n^2 \pi^2}{(b-a)^2}, \quad n \in \mathbb{N}_0.$$

Proof. This is the special case with $\mu = \lambda$. Then, $\Delta_\mu f$ simply is f'' such that we have to solve the Neumann problem $f'' = -\lambda f$ on $[a,b]$ with boundary conditions $f'(a) = f'(b) = 0$. The solutions are, as is easily verified, $f_n(x) = A \cos\big(\sqrt{\lambda_n}(x-a)\big)$, where $A \in \mathbb{R}$ is an arbitrary constant and λ_n is as above. □

Now, we treat the eigenvalue problem with homogeneous Dirichlet boundary conditions by defining a Dirichlet form $(\mathcal{E}, \mathcal{F}_0)$ as follows.

We set

$$\mathcal{F}_0 := \big\{ f \in H^1(\lambda) \colon f(a) = f(b) = 0 \big\}$$

and denote by $\mathcal{E}$ again the restriction of $\mathcal{E}$ to $\mathcal{F}_0$.

Proposition 2.2.6. *$(\mathcal{E}, \mathcal{F}_0)$ is a Dirichlet form on $L_2(\mu)$ and the embedding*

$$(\mathcal{F}_0, \mathcal{E}_1) \hookrightarrow L_2(\mu)$$

is a compact operator.

Proof. $(\mathcal{E}, \mathcal{F}_0)$ is a Dirichlet form according to Proposition 5.3 of Freiberg [18]. Since every bounded set in $(\mathcal{F}_0, \mathcal{E}_1)$ is also bounded in $(\mathcal{F}, \mathcal{E}_1)$, it follows by Proposition 2.2.2 that the embedding is compact. □

Lemma 2.2.7. *Let f be an eigenfunction with eigenvalue λ of the Dirichlet form $(\mathcal{E}, \mathcal{F}_0)$. Then there is a number η such that*

$$f(x) = \eta\,(x-a) + \int_a^x \int_a^z -\lambda f(t)\,d\mu(t)\,dz$$

for all $x \in [a, b]$.

Proof. Let $f \in \mathcal{F}_0$ and $\lambda \in \mathbb{R}$ such that for all $g \in \mathcal{F}_0$,

$$\mathcal{E}(f, g) = \lambda \langle f, g \rangle_{L_2(\mu)}.$$

Let $x \in (a, b]$. For each $n \in \mathbb{N}$ we define a function g_n^x by

$$g_n^x(t) := \begin{cases} -\dfrac{x-y_n}{y_n-a}(t-a), & \text{if } t \in [a, y_n), \\ t-x, & \text{if } t \in [y_n, x), \\ 0, & \text{if } t \in [x, b], \end{cases}$$

where the y_n are chosen such that $a < y_n < x$, and $\lim_{n\to\infty} y_n = a$. Then $g_n^x \in \mathcal{F}_0$ and thus,

$$\mathcal{E}(f, g_n^x) = \lambda \langle f, g_n^x \rangle_{L_2(\mu)} \tag{2.5}$$

for all $n \in \mathbb{N}$.

Further, g_n^x converges in $L_2(\mu)$ to the function g^x defined in (2.2), since

$$\begin{aligned} \|g_n^x - g^x\|_{L_2(\mu)}^2 &= \int_a^{y_n} \left| -\frac{x-y_n}{y_n-a}(t-a) - (t-x) \right|^2 d\mu(t) \\ &\le |x-a|^2 \int_a^{y_n} d\mu(t) \to 0 \quad (n \to \infty). \end{aligned}$$

Thus, $\langle f, g_n^x \rangle_{L_2(\mu)}$ converges to $\langle f, g^x \rangle_{L_2(\mu)}$ as $n \to \infty$.

Now we compute $\mathcal{E}(f, g_n^x)$. We observe that

$$g_n^{x\prime}(t) = \begin{cases} -\dfrac{x-y_n}{y_n-a}, & \text{if } t \in [a, y_n), \\ 1, & \text{if } t \in [y_n, x), \\ 0, & \text{if } t \in [x, b], \end{cases}$$

and hence,

$$\begin{aligned} \mathcal{E}(f, g_n^x) &= -\int_a^{y_n} f'(z)\, \frac{x-y_n}{y_n-a}\, dz + \int_{y_n}^x f'(z)\, dz \\ &= -\frac{x-y_n}{y_n-a}\big(f(y_n) - f(a)\big) + f(x) - f(y_n) \\ &= -(x-a)\frac{f(y_n)}{y_n-a} + f(x). \end{aligned}$$

Since $\frac{f(y_n)}{y_n-a}$ converges to some η as $n \to \infty$, it follows that

$$\mathcal{E}(f, g_n^x) \to f(x) - \eta(x-a)$$

as $n \to \infty$.

Consequently, it follows from (2.5) that

$$f(x) - \eta(x-a) = \lambda \langle f, g^x \rangle_{L_2(\mu)}.$$

As shown in the proof of Lemma 2.2.3, $\langle f, g^x \rangle_{L_2(\mu)} = -\int_a^x \int_a^z f(t)\, d\mu(t)\, dz$ and thus,

$$f(x) = \eta(x-a) + \int_a^x \int_a^z -\lambda f(t)\, d\mu(t)\, dz, \tag{2.6}$$

which finishes the proof.

$\square$

Proposition 2.2.8. *The following two statements are equivalent:*

(i) $f \in \mathcal{F}_0$ *is an eigenfunction with eigenvalue* λ *of the Dirichlet form* $(\mathcal{E}, \mathcal{F}_0)$*, that is, for all* $g \in \mathcal{F}_0$ *holds*

$$\mathcal{E}(f, g) = \lambda \langle f, g \rangle_{L_2(\mu)}.$$

(ii) $f \in H^2(\lambda, \mu)$ *and* f *is an eigenfunction with eigenvalue* λ *of* Δ_μ *with homoge-*

neous Dirichlet boundary conditions, that is,

$$\Delta_\mu f = -\lambda f$$

and $f(a) = f(b) = 0$.

Proof. Assume $f \in \mathcal{F}_0$ and $\lambda \in \mathbb{R}$ are given such that

$$\mathcal{E}(f,g) = \lambda \langle f, g \rangle_{L_2(\mu)}$$

for all $g \in \mathcal{F}_0$. According to Lemma 2.2.7, f then has the representation

$$f(x) = \int_a^x \left[\eta + \int_a^z -\lambda f(t)\, d\mu(t) \right] dz$$

for all $x \in [a,b]$. Hence, f is differentiable and

$$f'(x) = \eta + \int_a^x -\lambda f(t)\, d\mu(t).$$

Consequently, f' is μ-differentiable, $f'(a) = \eta$, and

$$\Delta_\mu f = -\lambda f.$$

Since $f \in \mathcal{F}_0$ it satisfies homogeneous Dirichlet boundary conditions.

Conversely, assume that $f \in H^2(\lambda, \mu)$ and $\lambda \in \mathbb{R}$ satisfy

$$\Delta_\mu f = -\lambda f$$

and $f(a) = f(b) = 0$. Then, for any $g \in \mathcal{F}_0$,

$$\mathcal{E}(f,g) = -\langle \Delta_\mu f, g \rangle_{L_2(\mu)} = \lambda \langle f, g \rangle_{L_2(\mu)}$$

by Proposition 2.1.11 (ii). □

2.3. A Poincaré inequality

We will need the following Poincaré inequality to estimate the smallest positive eigenvalue.

Lemma 2.3.1. *Let μ be a finite Borel measure on $[a,b]$ and let $f\colon [a,b] \to \mathbb{R}$ with $f \in H^1(\lambda)$ and $f \in L_2(\mu)$. We set $\bar{f} = \frac{1}{\mu([a,b])} \int_a^b f \, d\mu$. Then*

$$\|f - \bar{f}\|_{L_2(\mu)} \leq \frac{\sqrt{(b-a)\mu([a,b])}}{2} \|f'\|_{L_2(\lambda)}. \tag{2.7}$$

Proof. The idea of the proof was found in chapter 6 "Poincaré-type inequalities in dimension one" of Chen [9]. We compute

$$\begin{aligned}
&\int_a^b \int_a^b \big(f(y) - f(x)\big)^2 \, d\mu(x) \, d\mu(y) \\
&= \int_a^b \int_a^b \big[f(y)^2 + f(x)^2 - 2f(x)f(y)\big] \, d\mu(x) \, d\mu(y) \\
&= \mu([a,b]) \int_a^b f(y)^2 \, d\mu(y) + \mu([a,b]) \int_a^b f(x)^2 \, d\mu(x) - 2 \int_a^b f(x) \, d\mu(x) \int_a^b f(y) \, d\mu(y) \\
&= 2\, \mu([a,b]) \Big[\int_a^b f^2 \, d\mu - \mu([a,b]) \bar{f}^2 \Big].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|f - \bar{f}\|^2_{L_2(\mu)} &= \int_a^b f^2 \, d\mu - 2\bar{f} \int_a^b f \, d\mu + \mu([a,b]) \bar{f}^2 \\
&= \int_a^b f^2 \, d\mu - 2\mu([a,b]) \bar{f}^2 + \mu([a,b]) \bar{f}^2 \\
&= \int_a^b f^2 \, d\mu - \mu([a,b]) \bar{f}^2,
\end{aligned}$$

so

$$\int_a^b \int_a^b \big(f(y) - f(x)\big)^2 \, d\mu(x) \, d\mu(y) = 2\mu([a,b]) \, \|f - \bar{f}\|^2_{L_2(\mu)}.$$

On the other hand, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\int_a^b \int_a^b \big(f(y) - f(x)\big)^2 \, d\mu(x) \, d\mu(y) \\
&= 2 \int_a^b \int_a^y \Big(\int_x^y f'(t) \, dt \Big)^2 \, d\mu(x) \, d\mu(y) \\
&\leq 2 \int_a^b \int_a^y \int_x^y f'(t)^2 \, dt \, (y - x) \, d\mu(x) \, d\mu(y),
\end{aligned}$$

then, by interchanging the order of the integrals,

$$
\begin{aligned}
&= 2\int_a^b\int_a^b\int_a^b \mathbf{1}_{[a,y]}(x)\,\mathbf{1}_{[x,y]}(t)\,f'(t)^2(y-x)\,dt\,d\mu(x)\,d\mu(y)\\
&= 2\int_a^b\int_a^b\int_a^b \mathbf{1}_{[a,t]}(x)\,\mathbf{1}_{[t,b]}(y)\,f'(t)^2(y-x)\,d\mu(x)\,d\mu(y)\,dt\\
&= 2\int_a^b f'(t)^2\int_t^b\int_a^t (y-x)\,d\mu(x)\,d\mu(y)\,dt,
\end{aligned}
$$

and finally, since $\mu([a,t]) + \mu([t,b]) = \mu([a,b])$,

$$
\begin{aligned}
&\le 2(b-a)\int_a^b f'(t)^2\,\mu([a,t])\,\mu([t,b])\,dt\\
&\le 2(b-a)\int_a^b f'(t)^2\left(\frac{\mu([a,b])}{2}\right)^2 dt\\
&= \frac{(b-a)\,\mu([a,b])^2}{2}\int_a^b f'(t)^2\,dt.
\end{aligned}
$$

Altogether we showed

$$
\|f-\bar{f}\|^2_{L_2(\mu)} \le \frac{(b-a)\,\mu([a,b])}{4}\|f'\|^2_{L_2(\lambda)}.
$$

□

Note that, if we take $\mu = \delta_a + \delta_b$ and $f(t) = \frac{2}{b-a}(t-a) - 1$, then $\mu([a,b]) = 2$, $\bar{f} = 0$, $\|f\|^2_{L_2(\mu)} = 2$, and $\|f'\|^2_{L_2(\lambda)} = \frac{4}{b-a}$ and therefore (2.7) holds with equality. This suggests that the constant in Lemma 2.3.1 is optimal.

3. Spectral Asymptotics for General Homogeneous Cantor Measures

3.1. Construction of general homogeneous Cantor measures

We follow Barlow and Hambly [3] and use their concept of an *environment sequence*.

Let J be a finite or countably infinite index set. For every $j \in J$ we define an IFS $\mathcal{S}^{(j)}$. To this end, let $N_j \in \mathbb{N}$, $N_j \geq 2$. For $i = 1, \dots, N_j$ let $S_i^{(j)} \colon [a,b] \to [a,b]$ be defined by

$$S_i^{(j)}(x) = r_i^{(j)} x + c_i^{(j)},$$

where $r_i^{(j)} \in (0,1)$ and $c_i^{(j)} \in \mathbb{R}$ are chosen such that

$$a = S_1^{(j)}(a) < S_1^{(j)}(b) \leq S_2^{(j)}(a) < S_2^{(j)}(b) \leq \cdots < S_{N_j}^{(j)}(b) = b. \tag{3.1}$$

Then we set

$$\mathcal{S}^{(j)} := (S_1^{(j)}, \dots, S_{N_j}^{(j)}). \tag{3.2}$$

Let $\xi = (\xi_1, \xi_2, \dots)$ with $\xi_k \in J$ for every $k \in \mathbb{N}$. Then ξ is called *environment sequence*. Each ξ_k stands for an IFS $\mathcal{S}^{(\xi_k)}$.

For each sequence ξ we construct a fractal $K^{(\xi)}$ as follows. We put $K_0 = [a,b]$. Then, we move on to $K_1^{(\xi)}$ by replacing K_0 by the union of the images of K_0 under the functions in $\mathcal{S}^{(\xi_1)}$. The next level, $K_2^{(\xi)}$, is constructed by first mapping K_0 through $S_1^{(\xi_2)}, \dots, S_{N_2}^{(\xi_2)}$ and then taking the images under $S_1^{(\xi_1)}, \dots, S_{N_1}^{(\xi_1)}$.

To describe the nth level set $K_n^{(\xi)}$ we introduce a word space W_n of words of length n by

$$W_n := \{1, \dots, N_{\xi_1}\} \times \{1, \dots, N_{\xi_2}\} \times \cdots \times \{1, \dots, N_{\xi_n}\}.$$

For $n \in \mathbb{N}$ and $w = (w_1, \dots, w_n) \in W_n$ we set

$$S_w^{(\xi)} := S_{w_1}^{(\xi_1)} \circ S_{w_2}^{(\xi_2)} \circ \cdots \circ S_{w_n}^{(\xi_n)}.$$

Now, for $n \in \mathbb{N}$, we put

$$K_n^{(\xi)} := \bigcup_{w \in W_n} S_w^{(\xi)}([a, b]).$$

It follows that $K_{n+1}^{(\xi)} \subseteq K_n^{(\xi)}$ for all $n \in \mathbb{N}$. We define $K^{(\xi)}$ by

$$K^{(\xi)} := \bigcap_{n=1}^{\infty} K_n^{(\xi)}.$$

Proposition 3.1.1. *The set $K^{(\xi)}$ is compact and contains at least countably infinitely many elements, namely $S_w^{(\xi)}(a)$ and $S_w^{(\xi)}(b)$ for all $w \in W_n$ and all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ and let $w = (w_1, \dots, w_n) \in W_n$. Then, by the definition of $K_n^{(\xi)}$, we have $S_w^{(\xi)}(a), S_w^{(\xi)}(b) \in K_n^{(\xi)}$. For any $m \in \mathbb{N}$ we put

$$w' := (w_1, w_2, \dots, w_n, 1, \dots, 1) \in W_{n+m}$$

and

$$w'' := (w_1, w_2, \dots, w_n, N_{\xi_{n+1}}, \dots, N_{\xi_{n+m}}) \in W_{n+m}.$$

Because of (3.1) it is clear that $S_{w'}^{(\xi)}(a) = S_w^{(\xi)}(a)$ and $S_{w''}^{(\xi)}(b) = S_w^{(\xi)}(b)$. Therefore $S_w^{(\xi)}(a)$ and $S_w^{(\xi)}(b)$ belong to $K_{n+m}^{(\xi)}$. Since this holds for all m the assertion is proved. $\square$

The complement $[a, b] \setminus K^{(\xi)}$ is an open set and therefore consists of countably many disjoint open intervals. These intervals we will call *gap intervals* or *gaps* of $K^{(\xi)}$.

$K^{(\xi)}$ is not exactly invariant with respect to some IFS as defined in Section B.2, but we have a suitable replacement for that property. As in Barlow and Hambly [3], we denote a left-shift of ξ by $\theta\xi = (\xi_2, \xi_3, \dots)$. Then we have

$$K^{(\xi)} = \bigcup_{i=1}^{N_{\xi_1}} S_i^{(\xi_1)}\big(K^{(\theta\xi)}\big) \tag{3.3}$$

which follows immediately from the construction of K.

It is readily seen that, if the sequence ξ is periodic, $K^{(\xi)}$ is self-similar. In particular, if we take $J = \{1\}$ and a constant environment sequence $\xi = (1, 1, \dots)$, then $K^{(\xi)}$ is invariant with respect to $\mathcal{S}^{(1)}$.

Next, for any environment sequence ξ, we construct a measure $\mu^{(\xi)}$ on $[a, b]$ with support $K^{(\xi)}$. For each $j \in J$ let $m_1^{(j)}, \dots, m_{N_j}^{(j)} \in (0, 1)$ with

$$\sum_{i=1}^{N_j} m_i^{(j)} = 1.$$

Furthermore, for $w = (w_1, \dots, w_n) \in W_n$ and $n \in \mathbb{N}$ we set

$$m_w^{(\xi)} := m_{w_1}^{(\xi_1)} \cdot \cdot \cdot m_{w_n}^{(\xi_n)}.$$

Let $\mu_0 := \frac{1}{b-a}\lambda$ on $[a, b]$. For $A \in \mathcal{B}([a, b])$ we put

$$\mu_1^{(\xi)}(A) := \sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} \mu_0\big(S_i^{(\xi_1)^{-1}}(A)\big)$$

and for $n \in \mathbb{N}$

$$\mu_n^{(\xi)}(A) := \sum_{w \in W_n} m_w^{(\xi)} \mu_0\big(S_w^{(\xi)^{-1}}(A)\big).$$

Then we define $\mu^{(\xi)}$ by

$$\mu^{(\xi)}(A) := \lim_{n \to \infty} \mu_n^{(\xi)}(A)$$

for all $A \in \mathcal{B}([a, b])$.

Lemma 3.1.2. *For any environment sequence ξ, $\mu^{(\xi)}$ is a probability measure and for all $n \in \mathbb{N}$ and $w \in W_n$ it holds that*

$$\mu^{(\xi)}\big(S_w^{(\xi)}([a, b])\big) = m_w^{(\xi)}.$$

Proof. We fix an environment sequence ξ and omit the superscript (ξ). By Corollary B.3.2 (Vitali-Hahn-Saks) follows that μ is a measure on $\mathcal{B}$.

Let $n \in \mathbb{N}$ and $w \in W_n$. We denote $K_w = S_w([a, b])$. For all $m \in \mathbb{N}$ holds

$$\mu_{n+m}(K_w) = \sum_{w' \in W_{n+m}} m_{w'} \mu_0\big(S_{w'}^{-1}(K_w)\big) = \sum_{v \in W_m} m_{wv} \mu_0\big(S_{wv}^{-1}(K_w)\big)$$

because $K_{w'} \cap K_w = K_{w'}$ if $w' = wv$ for some $v \in W_m$ and else $K_{w'} \cap K_w = \emptyset$. Note that

$$S_{wv}^{-1}(K_w) = S_v^{-1} \circ S_w^{-1}(K_w) = S_v^{-1}([a,b]) = [a,b]$$

for all $v \in W_m$. Consequently, for all $m \in \mathbb{N}$,

$$\mu_{n+m}(K_w) = \sum_{v \in W_m} m_{wv} = \sum_{v \in W_m} m_w m_v = m_w$$

and thus, letting $m \to \infty$,

$$\mu(K_w) = m_w.$$

In particular, choosing $n = 1$ it follows that

$$\mu([a,b]) = \sum_{i=1}^{N_{\xi_1}} \mu(K_i) = \sum_{i=1}^{N_{\xi_1}} m_i = 1.$$

□

Remark 3.1.3. For the case where

$$r := \sup_{j \in J} \sup_{i=1,\dots,N_j} r_i^{(j)} < 1$$

we want to give an alternative way of proving the existence of the measure $\mu^{(\xi)}$ by using the Monge-Kantorovich metric that we define in the following. This metric is also called Kantorovich-Rubinstein metric or Wasserstein metric. It arises in the theory of optimal transportation, for a detailed survey see Bogachev and Kolesnikov [6].

Let $a < b$ and $\mathcal{M}^1\big([a,b]\big)$ be the set of all Borel probability measures on $[a,b]$. For $\mu, \nu \in \mathcal{M}^1$ we set

$$d_{MK}(\mu,\nu) = \sup\bigg\{ \int_{[a,b]} f \, d\mu - \int_{[a,b]} f \, d\nu \colon f \in \mathrm{Lip}_1\big([a,b]\big),\ \sup_{x \in [a,b]} |f(x)| \le 1 \bigg\},$$

where $\mathrm{Lip}_1\big([a,b]\big)$ denotes the set of all Lipschitz-continuous functions $f\colon [a,b] \to \mathbb{R}$ with Lipschitz constant 1.

Then, $\big(\mathcal{M}^1, d_{MK}\big)$ is a complete metric space, see e.g. [6, Theorem 1.1.3].

We show that the sequence $(\mu_n^{(\xi)})_n$ defined above is a Cauchy sequence and therefore converges to a measure $\mu^{(\xi)}$. We take a fixed environment sequence ξ and omit the superscript (ξ). Let $f \in \mathrm{Lip}_1\big([a,b]\big)$ and $n \in \mathbb{N}$. Observe that, since $m_w > 0$,

$$\int_a^b f\, d\mu_n = \sum_{w \in W_n} m_w \int_{S_w(a)}^{S_w(b)} f\, d\big(S_w \mu_0\big).$$

Then, since $\sum\limits_{v=1}^{N_{\xi_{n+1}}} m_{wv} = m_w$,

$$\begin{aligned}
\int_a^b f\, d\mu_{n+1} - \int_a^b f\, d\mu_n &= \sum_{w' \in W_{n+1}} m_{w'} \int_{S_{w'}(a)}^{S_{w'}(b)} f\, dS_{w'}\mu_0 - \sum_{w \in W_n} m_w \int_{S_w(a)}^{S_w(b)} f\, dS_w\mu_0 \\
&= \sum_{w \in W_n} \sum_{v=1}^{N_{\xi_{n+1}}} m_{wv} \int_{S_{wv}(a)}^{S_{wv}(b)} f\, dS_{wv}\mu_0 - \sum_{w \in W_n} m_w \int_{S_w(a)}^{S_w(b)} f\, dS_w\mu_0 \\
&= \sum_{w \in W_n} \sum_{v=1}^{N_{\xi_{n+1}}} m_{wv} \left[\int_{S_{wv}(a)}^{S_{wv}(b)} f\, dS_{wv}\mu_0 - \int_{S_w(a)}^{S_w(b)} f\, dS_w\mu_0 \right] \\
&= \sum_{w \in W_n} \sum_{v=1}^{N_{\xi_{n+1}}} m_{wv} \int_{S_w(a)}^{S_w(b)} \big[f \circ S_v - f\big]\, dS_w\mu_0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left| \int_a^b f\, d\mu_{n+1} - \int_a^b f\, d\mu_n \right| &\le \sum_{w \in W_n} \sum_{v=1}^{N_{\xi_{n+1}}} m_{wv} \int_{S_w(a)}^{S_w(b)} \big| f \circ S_v - f \big|\, dS_w\mu_0 \\
&\le \sum_{w \in W_n} \sum_{v=1}^{N_{\xi_{n+1}}} m_{wv} \int_{S_w(a)}^{S_w(b)} \big| S_v(t) - t \big|\, dS_w\mu_0(t) \\
&\le \sum_{w \in W_n} \sum_{v=1}^{N_{\xi_{n+1}}} m_{wv} (b-a) r_w \\
&\le (b-a) r^n.
\end{aligned}$$

Since this holds for every $f \in \mathrm{Lip}_1$, it follows that $d_{MK}(\mu_{n+1}, \mu_n) \to 0$ as $n \to \infty$.

Proposition 3.1.4. *Analogously to* (3.3) *it holds*

$$\mu^{(\xi)} = \sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)}\, S_i^{(\xi_1)} \mu^{(\theta\xi)}. \tag{3.4}$$

Proof. Let $n \in \mathbb{N}$, $B \in \mathcal{B}$ and $A = S_i^{(\xi_1)}(B)$. By definition, we have

$$\mu_n^{(\theta\xi)}(B) = \sum_{w \in W_n} m_w^{(\theta\xi)} \mu_0\big(S_w^{(\theta\xi)^{-1}}(B)\big).$$

Since $m_i^{(\xi_1)} m_w^{(\theta\xi)} = m_{iw}^{(\xi)}$ it follows that

$$\begin{aligned}
\sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} \mu_n^{(\theta\xi)}(B) &= \sum_{i=1}^{N_{\xi_1}} \sum_{w \in W_n} m_{iw}^{(\xi)} \mu_0\big(S_{iw}^{(\xi)^{-1}}(S_i^{(\xi_i)}(B))\big) \\
&= \sum_{w' \in W_{n+1}} m_{w'}^{(\xi)} \mu_0\big(S_{w'}^{(\xi)^{-1}}(A)\big) \\
&= \mu_{n+1}^{(\xi)}(A).
\end{aligned}$$

By letting n tend to infinity we get

$$\sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} \mu^{(\theta\xi)}\big(S_i^{(\xi_1)^{-1}}(A)\big) = \mu^{(\xi)}(A)$$

proving the assertion. □

Lemma 3.1.5. *Let $i \in \{1, \dots, N_{\xi_1}\}$ and $A \in \mathcal{B}$, $A \subseteq S_i^{(\xi_1)}([a,b])$. Then*

$$\mu^{(\xi)}(A) = m_i^{(\xi_1)} \big(S_i^{(\xi_1)} \mu^{(\theta\xi)}\big)(A).$$

Proof. The proof is immediate by Proposition 3.1.4. □

Remark 3.1.6. Later, we will use derivatives with respect to the measure $\mu^{(\xi)}$. Since $\mu^{(\xi)}$ is zero on the gap intervals of $K^{(\xi)}$, every function in $H^1(\mu^{(\xi)})$ is constant on each gap interval, respectively.

3.2. Scaling of the eigenvalue counting functions

The argumentation in this section relies on Section 6 of Kigami and Lapidus [35]. We fix an environment sequence $\xi = (\xi_1, \xi_2, \dots)$, take a corresponding sequence of non-overlapping IFSs $(\mathcal{S}^{(\xi_i)})_i$ as in Section 3.1 and consider the measure $\mu^{(\xi)}$. Furthermore, as in Section 2.2, let $\mathcal{F} = H^1(\lambda)$ and

$$\mathcal{E}(f,g) = \int_a^b f'(t)\, g'(t)\, dt$$

for $f, g \in \mathcal{F}$. To distinguish between different measures, we will denote a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L_2(\mu)$ by $(\mathcal{E}, \mathcal{F}, \mu)$ in the following.

3.2.1. Scaling of the energy and the scalar product

We will establish a scaling property of $\mathcal{E}$ similar to Kigami and Lapidus [35, Lemma 6.1].

Proposition 3.2.1. *Let $f, g \in \mathcal{F}$. Then $f \circ S_i^{(\xi_1)}, g \circ S_i^{(\xi_1)} \in \mathcal{F}$ and*

$$\mathcal{E}(f,g) = \sum_{i=1}^{N_{\xi_1}} \frac{1}{r_i^{(\xi_1)}} \mathcal{E}(f \circ S_i^{(\xi_1)}, g \circ S_i^{(\xi_1)}) + \sum_{i=1}^{N_{\xi_1}-1} \int_{S_i^{(\xi_1)}(b)}^{S_{i+1}^{(\xi_1)}(a)} f'(t) g'(t)\, dt.$$

Proof. We compute

$$\begin{aligned}
\sum_{i=1}^{N_{\xi_1}} \frac{1}{r_i^{(\xi_1)}} \mathcal{E}(f \circ S_i^{(\xi_1)}, g \circ S_i^{(\xi_1)}) &= \sum_{i=1}^{N_{\xi_1}} \frac{1}{r_i^{(\xi_1)}} \int_a^b (f \circ S_i^{(\xi_1)})'(t) (g \circ S_i^{(\xi_1)})'(t)\, dt \\
&= \sum_{i=1}^{N_{\xi_1}} r_i^{(\xi_1)} \int_a^b f'\big(S_i^{(\xi_1)}(t)\big) g'\big(S_i^{(\xi_1)}(t)\big)\, dt \\
&= \sum_{i=1}^{N_{\xi_1}} \int_{S_i^{(\xi_1)}(a)}^{S_i^{(\xi_1)}(b)} f'(t) g'(t)\, dt \\
&= \mathcal{E}(f,g) - \sum_{i=1}^{N_{\xi_1}-1} \int_{S_i^{(\xi_1)}(b)}^{S_{i+1}^{(\xi_1)}(a)} f'(t) g'(t)\, dt.
\end{aligned}$$

□

Next, we establish a similar scaling property for the $L_2(\mu^{(\xi)})$ scalar product.

Proposition 3.2.2. *Let $f, g \in L_2(\mu^{(\xi)})$. Then*

$$\langle f, g \rangle_{L_2(\mu^{(\xi)})} = \sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} \langle f \circ S_i^{(\xi_1)}, g \circ S_i^{(\xi_1)} \rangle_{L_2(\mu^{(\theta\xi)})}.$$

Proof. We denote $I = [a, b]$. Using $\operatorname{supp} \mu^{(\xi)} = K^{(\xi)}$ and Lemma 3.1.5, we work out that

$$\begin{aligned}\langle f, g\rangle_{L_2(\mu^{(\xi)})} &= \int_I fg\, d\mu^{(\xi)} \\ &= \sum_{i=1}^{N_{\xi_1}} \int_{S_i^{(\xi_1)}(I)} fg\, d\mu^{(\xi)} \\ &= \sum_{i=1}^{N_{\xi_1}} \int_I f \circ S_i^{(\xi_1)} \cdot g \circ S_i^{(\xi_1)}\, d\big(S_i^{(\xi_1)^{-1}} \mu^{(\xi)}\big) \\ &= \sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} \int_I f \circ S_i^{(\xi_1)} \cdot g \circ S_i^{(\xi_1)}\, d\mu^{(\theta\xi)} \\ &= \sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} \big\langle f \circ S_i^{(\xi_1)}, g \circ S_i^{(\xi_1)}\big\rangle_{L_2(\mu^{(\theta\xi)})}.\end{aligned}$$

□

3.2.2. Neumann boundary conditions

We consider the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L_2(\mu^{(\xi)})$, whose eigenvalues coincide with those of $\Delta_{\mu^{(\xi)}}$ with homogeneous Neumann boundary conditions according to Proposition 2.2.4. For shortness and to emphasize the dependence on the measure $\mu^{(\xi)}$, we denote the eigenvalue counting function by $N^{(\xi)}$ instead of $N_{(\mathcal{E},\mathcal{F})}$.

We define a new Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ where $\tilde{\mathcal{F}}$ is the set of all functions $f\colon [a,b] \to \mathbb{R}$ with $f \circ S_i^{(\xi_1)} \in H^1(\lambda)$ for all $i = 1, \dots, N_{\xi_1}$ and the restrictions of f to the first level gap intervals $\big(S_i^{(\xi_1)}(b), S_{i+1}^{(\xi_1)}(a)\big)$ are contained in $H^1\big(\lambda, (S_i^{(\xi_1)}(b), S_{i+1}^{(\xi_1)}(a))\big)$ for all $i = 1, \dots, N_{\xi_1} - 1$.

Then $\mathcal{F} \subseteq \tilde{\mathcal{F}}$, but $\tilde{\mathcal{F}}$ is not contained in $H^1(\lambda)$ because functions in $\tilde{\mathcal{F}}$ need not be continuous in the points $S_1^{(\xi_1)}(a), S_1^{(\xi_1)}(b), \dots, S_{N_{\xi_1}}^{(\xi_1)}(a), S_{N_{\xi_1}}^{(\xi_1)}(b)$.

For functions f and g in $\tilde{\mathcal{F}}$, we define the form $\tilde{\mathcal{E}}$ by

$$\tilde{\mathcal{E}}(f,g) = \sum_{i=1}^{N_{\xi_1}} \frac{1}{r_i^{(\xi_1)}} \mathcal{E}(f \circ S_i^{(\xi_1)}, g \circ S_i^{(\xi_1)}) + \sum_{i=1}^{N_{\xi_1}-1} \int_{S_i^{(\xi_1)}(b)}^{S_{i+1}^{(\xi_1)}(a)} f'(t)\, g'(t)\, dt.$$

Then, Proposition 3.2.1 implies that for $f, g \in \mathcal{F}$ we have $\tilde{\mathcal{E}}(f,g) = \mathcal{E}(f,g)$.

From Proposition 2.2.2 follows that the embedding $\tilde{\mathcal{F}} \hookrightarrow L_2(\mu^{(\xi)})$ is a compact operator and thus we can refer to the eigenvalue counting function of the Dirichlet form $(\tilde{\mathcal{F}}^{(\xi)}, \tilde{\mathcal{E}})$.

Proposition 3.2.3. *For all $x \geq 0$,*

$$N_{(\tilde{\mathcal{F}}^{(\xi)},\tilde{\mathcal{E}})}(x) = \sum_{i=1}^{N_{\xi_1}} N^{(\theta\xi)}(r_i^{(\xi_1)} m_i^{(\xi_1)} x).$$

Proof. Let f be an eigenfunction of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}, \mu^{(\xi)})$ with eigenvalue λ. That is, for all $g \in \tilde{\mathcal{F}}$ holds

$$\tilde{\mathcal{E}}(f,g) = \lambda \langle f, g\rangle_{L_2(\mu^{(\xi)})}.$$

Since f and g are in $L_2(\mu^{(\xi)})$ we can apply Proposition 3.2.2. From this and the definition of $\tilde{\mathcal{E}}$ we get

$$\begin{aligned}
&\sum_{i=1}^{N_{\xi_1}} \frac{1}{r_i^{(\xi_1)}} \mathcal{E}\big(f \circ S_i^{(\xi_1)}, g \circ S_i^{(\xi_1)}\big) + \sum_{i=1}^{N_{\xi_1}-1} \int_{S_i^{(\xi_1)}(b)}^{S_{i+1}^{(\xi_1)}(a)} f'(t)\, g'(t)\, dt \\
&= \lambda \sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} \big\langle f \circ S_i^{(\xi_1)}, g \circ S_i^{(\xi_1)} \big\rangle_{L_2(\mu^{(\theta\xi)})}.
\end{aligned} \tag{3.5}$$

Next we show that the summands of the first sum above coincide with those on the right hand side, respectively. Take $h \in H^1(\lambda)$. For $j = 1, \ldots, N_{\xi_1}$, we define $\tilde{h}_j$ by

$$\tilde{h}_j(x) = \begin{cases} h \circ S_j^{(\xi_1)^{-1}}(x), & \text{if } x \in S_j^{(\xi_1)}(I), \\ 0, & \text{otherwise.} \end{cases}$$

Then $\tilde{h}_j \in \tilde{\mathcal{F}}$, $\tilde{h}_j \circ S_j^{(\xi_1)} = h$ for $j = 1, \ldots, N_{\xi_1}$, and $\tilde{h}_j \circ S_i^{(\xi_1)} = 0$ for $i \neq j$. Moreover, $\tilde{h}_j' = 0$ on all gap intervals. Therefore, if we put $g = \tilde{h}_j$ in (3.5), we get

$$\frac{1}{r_j^{(\xi_1)}} \mathcal{E}(f \circ S_j^{(\xi_1)}, h) = \lambda\, m_j^{(\xi_1)} \langle f \circ S_j^{(\xi_1)}, h \rangle_{L_2(\mu^{(\theta\xi)})}.$$

This equation now holds for all $h \in H^1(\lambda)$ meaning that $r_j^{(\xi_1)} m_j^{(\xi_1)} \lambda$ is an eigenvalue of the form $\big(\mathcal{E}, \mathcal{F}, \mu^{(\theta\xi)}\big)$ to the eigenfunction $f \circ S_j^{(\xi_1)}$ for all $j = 1, \ldots, N_{\xi_1}$.

To prove the converse, assume that λ is a non-negative number such that for every $i = 1, \ldots, N_{\xi_1}$ the number $r_i^{(\xi_1)} m_i^{(\xi_1)} \lambda$ is an eigenvalue of $\big(\mathcal{E}, \mathcal{F}, \mu^{(\theta\xi)}\big)$ to some eigenfunction f_i. Thus,

$$\mathcal{E}(f_i, g) = r_i^{(\xi_1)} m_i^{(\xi_1)} \lambda \langle f_i, g \rangle_{L_2(\mu^{(\theta\xi)})} \tag{3.6}$$

for all $g \in H^1(\lambda)$. Let f be defined by

$$f(x) = \begin{cases} f_i \circ S_i^{(\xi_1)^{-1}}(x), & \text{if } x \in S_i^{(\xi_1)}([a,b]) \text{ for some } i = 1, \dots, N_{\xi_1}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \in \tilde{\mathcal{F}}$ and $f \circ S_i^{(\xi_1)} = f_i$ for every $i = 1, \dots, N_{\xi_1}$. From (3.6) follows that

$$\sum_{i=1}^{N_{\xi_1}} \frac{1}{r_i^{(\xi_1)}} \mathcal{E}\big(f \circ S_i^{(\xi_1)}, g\big) = \lambda \sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} \big\langle f \circ S_i^{(\xi_1)}, g \big\rangle_{L_2(\mu^{\theta\xi})}$$

holds for all $g \in H^1(\lambda)$. Take $\tilde{g} \in \tilde{\mathcal{F}}$. Since $\tilde{g} \circ S_i^{(\xi_1)} \in H^1(\lambda)$ by definition of $\tilde{\mathcal{F}}$ for every $i = 1, \dots, N_{\xi_1}$, we get

$$\sum_{i=1}^{N_{\xi_1}} \frac{1}{r_i^{(\xi_1)}} \mathcal{E}\big(f \circ S_i^{(\xi_1)}, \tilde{g} \circ S_i^{(\xi_1)}\big) = \lambda \sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} \big\langle f \circ S_i^{(\xi_1)}, \tilde{g} \circ S_i^{(\xi_1)} \big\rangle_{L_2(\mu^{\theta\xi})}.$$

Because f is chosen so that $f' = 0$ on the gap intervals $\big(S_i^{(\xi_1)}(b), S_{i+1}^{(\xi_1)}(a)\big)$, the left hand side of the above equation equals the definition of $\tilde{\mathcal{E}}$. Hence, by Proposition 3.2.2 we have

$$\tilde{\mathcal{E}}(f, \tilde{g}) = \lambda \langle f, \tilde{g} \rangle_{L_2(\mu^{(\xi)})}$$

for all $\tilde{g} \in \tilde{\mathcal{F}}$. So λ is an eigenvalue of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}, \mu^{(\xi)})$ with the eigenfunction f.

We thus showed that for each $x \geq 0$ the following statements are equivalent:

(i) $\lambda \leq x$ is an eigenvalue of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}, \mu^{(\xi)})$ with eigenfunction f,

(ii) $r_j^{(\xi_1)} m_j^{(\xi_1)} \lambda \leq r_j^{(\xi_1)} m_j^{(\xi_1)} x$ is an eigenvalue of $(\mathcal{E}, \mathcal{F}, \mu^{(\theta\xi)})$ with eigenfunction $f \circ S_j^{(\xi_1)}$ for each $j = 1, \dots, N_{\xi_1}$.

Thus, λ is of multiplicity N_{ξ_1} and therefore,

$$N_{(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})}(x) = \sum_{i=1}^{N_{\xi_1}} N^{(\theta\xi)}(r_i^{(\xi_1)} m_i^{(\xi_1)} x).$$

$\square$

Corollary 3.2.4. *Since $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is an extension of $(\mathcal{E}, \mathcal{F})$ on $L_2(\mu^{(\xi)})$, it follows from Theorem B.7.4 that*

$$N^{(\xi)}(x) \leq \sum_{i=1}^{N_{\xi_1}} N^{(\theta\xi)}(r_i^{(\xi_1)} m_i^{(\xi_1)} x) \tag{3.7}$$

for all $x \geq 0$.

3.2.3. Dirichlet boundary conditions

Now we prove a similar proposition for Dirichlet boundary conditions. For that we use the Dirichlet form $(\mathcal{E}, \mathcal{F}_0)$ on $L_2(\mu^{(\xi)})$ because according to Proposition 2.2.8, it's eigenvalues coincide with the eigenvalues of $\Delta_{\mu^{(\xi)}}$ with homogeneous Dirichlet boundary conditions. We denote the eigenvalue counting function of $(\mathcal{E}, \mathcal{F}_0)$ on $L_2(\mu^{(\xi)})$ by $N_0^{(\xi)}$.

We define a Dirichlet form $(\mathcal{E}, \tilde{F}_0)$ where the domain $\tilde{\mathcal{F}}_0$ is defined as the set of all $f \in \mathcal{F}_0$ that vanish on all 'first-level' gap intervals, that is

$$\tilde{\mathcal{F}}_0 := \left\{ f \in \mathcal{F}_0 \colon f(x) = 0 \text{ for } x \in \left(S_i^{(\xi_1)}(b), S_{i+1}^{(\xi_1)}(a) \right),\ i = 1, \ldots, N_{\xi_1} - 1 \right\}.$$

We use the notation $\mathcal{E}$ for the restriction of the previously used form $\mathcal{E}$ to $\tilde{F}_0$ as well.

Proposition 3.2.5. *For all $x \geq 0$,*

$$N_{(\mathcal{E}, \tilde{\mathcal{F}}_0, \mu^{(\xi)})}(x) = \sum_{i=1}^{N_{\xi_1}} N_0^{(\theta\xi)}(r_i^{(\xi_1)} m_i^{(\xi_1)} x).$$

Proof. The proof of Proposition 3.2.3 can be adapted here as follows.

Let f be an eigenfunction of $(\mathcal{E}, \tilde{\mathcal{F}}_0, \mu^{(\xi)})$ with eigenvalue λ. That is, for all $g \in \tilde{\mathcal{F}}_0$ holds

$$\mathcal{E}(f, g) = \lambda \langle f, g \rangle_{L_2(\mu^{(\xi)})}.$$

From Propositions 3.2.1 and 3.2.2 we get

$$\begin{aligned} &\sum_{i=1}^{N_{\xi_1}} \frac{1}{r_i^{(\xi_1)}} \mathcal{E}\left(f \circ S_i^{(\xi_1)}, g \circ S_i^{(\xi_1)}\right) + \sum_{i=1}^{N_{\xi_1}-1} \int_{S_i^{(\xi_1)}(b)}^{S_{i+1}^{(\xi_1)}(a)} f'(t)\, g'(t)\, dt \\ &= \lambda \sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} \left\langle f \circ S_i^{(\xi_1)}, g \circ S_i^{(\xi_1)} \right\rangle_{L_2(\mu^{(\theta\xi)})}. \end{aligned} \tag{3.8}$$

Let $h \in \mathcal{F}_0$. For $j = 1, \ldots, N_{\xi_1}$, we define $\tilde{h}_j$ by

$$\tilde{h}_j(x) = \begin{cases} h \circ S_j^{(\xi_1)^{-1}}(x), & \text{if } x \in S_j^{(\xi_1)}(I), \\ 0, & \text{otherwise.} \end{cases}$$

Then, because h is zero at the boundary, $\tilde{h}_j \in \tilde{\mathcal{F}}_0$. Furthermore, $\tilde{h}_j \circ S_j^{(\xi_1)} = h$ for $j = 1, \ldots, N_{\xi_1}$, and $\tilde{h}_j \circ S_i^{(\xi_1)} = 0$ for $i \neq j$. Therefore,

$$\frac{1}{r_j^{(\xi_1)}} \mathcal{E}(f \circ S_j^{(\xi_1)}, h) = \lambda\, m_j^{(\xi_1)} \big\langle f \circ S_j^{(\xi_1)}, h \big\rangle_{L_2(\mu^{(\theta\xi)})}$$

for every $j = 1, \ldots, N_{\xi_1}$. Thus $r_j^{(\xi_1)} m_j^{(\xi_1)} \lambda$ is an eigenvalue of $\big(\mathcal{E}, \mathcal{F}_0, \mu^{(\theta\xi)}\big)$ with eigenfunction $f \circ S_j^{(\xi_1)}$ for all $j = 1, \ldots, N_{\xi_1}$.

Conversely, assume that $r_i^{(\xi_1)} m_i^{(\xi_1)} \lambda$ is an eigenvalue of $\big(\mathcal{E}, \mathcal{F}_0, \mu^{(\theta\xi)}\big)$ with some eigenfunction f_i for every $i = 1, \ldots, N_{\xi_1}$. Then,

$$\mathcal{E}(f_i, g) = r_i^{(\xi_1)} m_i^{(\xi_1)} \lambda \langle f_i, g \rangle_{L_2(\mu^{(\theta\xi)})} \tag{3.9}$$

for all $g \in \mathcal{F}_0$. We define f by

$$f(x) = \begin{cases} f_i \circ S_i^{(\xi_1)^{-1}}(x), & \text{if } x \in S_i^{(\xi_1)}([a, b]) \text{ for some } i = 1, \ldots, N_{\xi_1}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, since f_i is zero at the boundary, $f \in \tilde{\mathcal{F}}_0$ and by (3.9),

$$\sum_{i=1}^{N_{\xi_1}} \frac{1}{r_i^{(\xi_1)}} \mathcal{E}\big(f \circ S_i^{(\xi_1)}, g\big) = \lambda \sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} \big\langle f \circ S_i^{(\xi_1)}, g \big\rangle_{L_2(\mu^{\theta\xi})}$$

holds for all $g \in \mathcal{F}_0$.

Take $\tilde{g} \in \tilde{\mathcal{F}}_0$. Since $\tilde{g} \circ S_i^{(\xi_1)} \in \mathcal{F}_0$, we get as in the proof of Proposition 3.2.3 that

$$\mathcal{E}(f, \tilde{g}) = \lambda \langle f, \tilde{g} \rangle_{L_2(\mu^{(\xi)})}.$$

So, λ is an eigenvalue of $(\mathcal{E}, \tilde{\mathcal{F}}_0, \mu^{(\xi)})$ with eigenfunction f which proves the proposition. □

Corollary 3.2.6. *Since $(\mathcal{E}, \mathcal{F}_0)$ is an extension of $(\mathcal{E}, \tilde{\mathcal{F}}_0)$ and $(\mathcal{E}, \mathcal{F})$ is an extension of $(\mathcal{E}, \mathcal{F}_0)$, we finally have by Theorem B.7.4 and Corollary 3.2.4 that*

$$\sum_{i=1}^{N_{\xi_1}} N_0^{(\theta\xi)}(r_i^{(\xi_1)} m_i^{(\xi_1)} x) \le N_0^{(\xi)}(x) \le N^{(\xi)}(x) \le \sum_{i=1}^{N_{\xi_1}} N^{(\theta\xi)}(r_i^{(\xi_1)} m_i^{(\xi_1)} x) \tag{3.10}$$

for all $x \ge 0$.

3.3. Spectral asymptotics

In this section we prove the main result of the first part of the thesis. Our intention is to show that the eigenvalue counting functions $N^{(\xi)}(x)$ and $N_0^{(\xi)}$ grow asymptotically like x^γ for some $\gamma = \gamma(\xi)$, that is, there are constants $C_1, C_2 > 0$ such that

$$C_1 x^\gamma \le N_{(0)}^{(\xi)}(x) \le C_2 x^\gamma \tag{3.11}$$

for all x greater than some $x_0 > 0$. It turns out, that this is in general not possible.

For this section we will use the following setup and notation. We will refer to Conditions (C1) to (C7) later. Let J be an at most countable set and suppose we are given for each $j \in J$ a non-overlapping IFS $\mathcal{S}^{(j)} = (S^{(j)})_{j\in J}$ with scaling factors $r_1^{(j)}, \dots, r_{N_j}^{(j)}$ and a weight vector $(m_1^{(j)}, \dots, m_{N_j}^{(j)})$ such that

$$\alpha := \inf_{j\in J} \min_{i=1,\dots,N_j} r_i^{(j)} m_i^{(j)} > 0, \tag{C1}$$

$$\beta := \sup_{j\in J} \max_{i=1,\dots,N_j} r_i^{(j)} m_i^{(j)} < 1, \tag{C2}$$

and

$$\sup_{j\in J} N_j < \infty. \tag{C3}$$

Next, let $\xi = (\xi_1, \xi_2, \dots)$ with $\xi_i \in J$, $i = 1, 2, \dots$ be an environment sequence such that for each $j \in J$ there is a limit

$$p_j := \lim_{n\to\infty} h_n^{(j)}, \quad \text{with } p_j > 0 \text{ for at least one } j, \tag{C4}$$

where $h_n^{(j)} := \#\{i \le n \colon \xi_i = j\}$. Further, let g be a monotonically non-decreasing function such that for all $j \in J$ and all $n \in \mathbb{N}$

$$n|h_n^{(j)} - p_j| \leq g(n). \tag{C5}$$

This condition is also imposed by Barlow and Hambly [3] in Section 6.

Let $\gamma \in [0,1]$ be defined as solution of the equation

$$\prod_{j\in J} \left(\sum_{i=1}^{N_j} (r_i^{(j)} m_i^{(j)})^\gamma \right)^{p_j} = 1, \tag{3.12}$$

which is equivalent to

$$\sum_{j\in J} p_j \log \left(\sum_{i=1}^{N_j} (r_i^{(j)} m_i^{(j)})^\gamma \right) = 0.$$

Lemma 3.3.1. *γ is uniquely determined by* (3.12) *and furthermore, $\gamma \in (0,1/2]$. If $\mu^{(\xi)}$ is the Lebesgue measure on $[a,b]$ normalized to 1, then $\gamma = \frac{1}{2}$.*

Proof. At first we have to show that the possibly infinite product in (3.12) exists. To this end, we show that

$$\sum_{j\in J} p_j \log\left(\sum_{i=1}^{N_j} \left(r_i^{(j)} m_i^{(j)}\right)^s \right)$$

converges absolutely for every $s \in [0,1]$. Noting that $r_i^{(j)} m_i^{(j)} \in (0,1)$, we see that for $s \in [0,1]$,

$$\sum_{i=1}^{N_j} \left(r_i^{(j)} m_i^{(j)}\right)^s \leq \sum_{i=1}^{N_j} 1 = N_j \leq \sup_{j\in J} N_j < \infty$$

and

$$\sum_{i=1}^{N_j} \left(r_i^{(j)} m_i^{(j)}\right)^s \geq \sum_{i=1}^{N_j} r_i^{(j)} m_i^{(j)} \geq \alpha N_j \geq \alpha > 0$$

and therefore there is a constant C such that for all $j \in J$,

$$\left| \log \sum_{i=1}^{N_j} \left(r_i^{(j)} m_i^{(j)}\right)^s \right| \leq C.$$

With that, we get

$$\sum_{j\in J} p_j \left|\log\left(\sum_{i=1}^{N_j} \left(r_i^{(j)} m_i^{(j)}\right)^s\right)\right| \le C \sum_{j\in J} p_j \le C.$$

With the continuity of the logarithm, it follows that, for $s \in [0,1]$, the product

$$G(s) := \prod_{j\in J} \left(\sum_{i=1}^{N_j} (r_i^{(j)} m_i^{(j)})^s\right)^{p_j}$$

exists.

Since $0 < r_i^{(j)} m_i^{(j)} < 1$ for all j and i, G is strictly monotonically decreasing. Moreover, G is continuous and

$$G(0) = \prod_{j\in J} N_j^{p_j} > 1.$$

As in Freiberg [21, Rem. 2.1] we show that $G(1/2) \le 1$. By the Cauchy-Schwarz inequality, we have for all $j \in J$,

$$\left(\sum_{i=1}^{N_j} \left(r_i^{(j)} m_i^{(j)}\right)^{1/2}\right)^{p_j} \le \left(\sum_{i=1}^{N_j} r_i^{(j)}\right)^{p_j/2} \left(\sum_{i=1}^{N_j} m_i^{(j)}\right)^{p_j/2} = \left(\sum_{i=1}^{N_j} r_i^{(j)}\right)^{p_j/2}$$

and thus, since $\sum\limits_{i=1}^{N_j} r_i^{(j)} \le 1$,

$$\begin{aligned} G(1/2) &= \prod_{j\in J} \left(\sum_{i=1}^{N_j} \left(r_i^{(j)} m_i^{(j)}\right)^{1/2}\right)^{p_j} \\ &\le \prod_{j\in J} \left(\sum_{i=1}^{N_j} r_i^{(j)}\right)^{p_j/2} \\ &\le 1. \end{aligned}$$

Therefore, $G - 1$ must have exactly one zero point and this must be in the interval $(0, 1/2]$. In the case where $r_i^{(j)} = m_i^{(j)}$ for all j and i and thus, μ is the Lebesgue measure, we have $G(1/2) = 1$ and hence, $\gamma = \frac{1}{2}$. $\square$

Finally, we impose that

$$\prod_{\substack{j\in J\\ \sum_{i=1}^{N_j}(r_i^{(j)}m_i^{(j)})^\gamma>1}} \left(\sum_{i=1}^{N_j}(r_i^{(j)}m_i^{(j)})^\gamma\right) < \infty \tag{C6}$$

and

$$\prod_{\substack{j\in J\\ \sum_{i=1}^{N_j}(r_i^{(j)}m_i^{(j)})^\gamma<1}} \left(\sum_{i=1}^{N_j}(r_i^{(j)}m_i^{(j)})^\gamma\right) > 0. \tag{C7}$$

Remark 3.3.2. Note that (C1)-(C3) are conditions only on the parameters of the collection of IFS's and (C4)-(C5) are conditions only on the environment sequence ξ. So α and β depend only on the IFS's and the p_j and g only on ξ, while γ depends on both. Conditions (C6) and (C7) are only relevant if J contains infinitely many elements. Then these demand a certain relationship between the IFS's and the environment sequence.

Remark 3.3.3. To prove an asymptotic law of the form (3.11), we have to assume that g is bounded. If g is not bounded, we prove a growth law similar to (3.11) but with additional correction terms. We get a similar result to that obtained by Barlow and Hambly [3] for the Laplacian on scale irregular Sierpinski gaskets.

Remark 3.3.4. It is possible to weaken condition (C4) by including the case $p_j = 0$ for all $j \in J$. This can only occur if J is infinite, so assume that $J = \mathbb{N}$. Then (3.12) is true for any γ, but (C6) and (C7) demand a certain convergence of the parameters N_j, $r_i^{(j)}$, and $m_i^{(j)}$ as j tends to infinity which determines a value of γ, see Example 3.4.7.

3.3.1. Neumann boundary conditions

Consider the Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L_2(\mu^{(\xi)})$ for an arbitrary environment sequence ξ. Let λ_1 be the smallest positive eigenvalue (spectral gap) of $(\mathcal{E},\mathcal{F})$ and let f_1 be a corresponding eigenfunction with $\|f_1\|_{L_2(\mu^{(\xi)})} = 1$. Then

$$\lambda_1 \int_a^b f_1\, d\mu = \mathcal{E}(f_1, 1) = 0$$

and therefore f_1 satisfies $\int_a^b f_1\, d\mu = 0$. Since $\mu^{(\xi)}([a,b]) = 1$ and

$$\lambda_1 = \mathcal{E}(f_1, f_1) = \int_a^b f_1'(t)^2\, dt,$$

Lemma 2.3.1 implies that

$$\lambda_1 \geq \frac{4}{b-a}. \tag{3.13}$$

Consequently, if $x \in [0, \frac{4}{b-a})$, then

$$N^{(\xi)}(x) = 1 \tag{3.14}$$

for all environment sequences ξ because $\lambda_0 = 0$ is always an eigenvalue.

Theorem 3.3.5. *Let conditions (C1)-(C7) be satisfied and γ be defined by* (3.12)*. Then there are constants $c_1 > 1$ and $c_2, c_3 > 0$ such that*

$$\frac{N^{(\xi)}(x)}{x^\gamma c_1^{g(c_2 \log c_3 x)}} \leq \Big(\frac{b-a}{4\alpha}\Big)^\gamma \tag{3.15}$$

for all $x \geq \frac{4}{b-a}$. The constants are computed explicitly in the proof.

Proof. We take an environment sequence ξ and a family $(\mathcal{S}^{(j)})_{j\in J}$ of IFS's that satisfy conditions (C1) to (C7) and let γ be defined by (3.12). Let $x \geq \frac{4}{b-a}$ be fixed. An n-fold left-shift is denoted by $\theta^n\xi = (\xi_{n+1}, \xi_{n+2}, \dots)$.

We define a sequence $(i_1, i_2, \dots)$ with $i_k \in \{1, \dots, N_{\xi_k}\}$ as follows: let i_1 be a number between 1 and N_{ξ_1} so that

$$\frac{N^{(\theta\xi)}(r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)}{(r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)^\gamma} = \max_{i=1,\dots,N_{\xi_1}} \frac{N^{(\theta\xi)}(r_i^{(\xi_1)} m_i^{(\xi_1)} x)}{(r_i^{(\xi_1)} m_i^{(\xi_1)} x)^\gamma},$$

let i_2 be a number between 1 and N_{ξ_2} such that

$$\frac{N^{(\theta^2\xi)}(r_{i_2}^{(\xi_2)} m_{i_2}^{(\xi_2)} r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)}{(r_{i_2}^{(\xi_2)} m_{i_2}^{(\xi_2)} r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)^\gamma} = \max_{i=1,\dots,N_{\xi_2}} \frac{N^{(\theta^2\xi)}(r_i^{(\xi_2)} m_i^{(\xi_2)} r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)}{(r_i^{(\xi_2)} m_i^{(\xi_2)} r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)^\gamma},$$

and so on. Then, by (3.10),

$$\begin{aligned}
\frac{N^{(\xi)}(x)}{x^\gamma} &\le \frac{\sum_{i=1}^{N_{\xi_1}} N^{(\theta\xi)}(r_i^{(\xi_1)} m_i^{(\xi_1)} x)}{x^\gamma} \\
&= \sum_{i=1}^{N_{\xi_1}} \frac{N^{(\theta\xi)}(r_i^{(\xi_1)} m_i^{(\xi_1)} x)}{(r_i^{(\xi_1)} m_i^{(\xi_1)} x)^\gamma} (r_i^{(\xi_1)} m_i^{(\xi_1)})^\gamma \\
&\le \frac{N^{(\theta\xi)}(r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)}{(r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)^\gamma} \sum_{i=1}^{N_{\xi_1}} (r_i^{(\xi_1)} m_i^{(\xi_1)})^\gamma .
\end{aligned}$$

Applying the same argument n times we get

$$\frac{N^{(\xi)}(x)}{x^\gamma} \le \frac{N^{(\theta^n\xi)}(r_{i_n}^{(\xi_n)} m_{i_n}^{(\xi_n)} \cdots r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)}{(r_{i_n}^{(\xi_n)} m_{i_n}^{(\xi_n)} \cdots r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)^\gamma} \prod_{k=1}^{n} \left(\sum_{i=1}^{N_{\xi_k}} (r_i^{(\xi_k)} m_i^{(\xi_k)})^\gamma \right). \tag{3.16}$$

Now we choose n such that $r_{i_n} m_{i_n} \cdots r_{i_1} m_{i_1} x < \frac{4}{b-a}$ and $r_{i_{n-1}} m_{i_{n-1}} \cdots r_{i_1} m_{i_1} x \ge \frac{4}{b-a}$. Then, $r_{i_n} m_{i_n} \cdots r_{i_1} m_{i_1} x \in [\frac{4\alpha}{b-a}, \frac{4}{b-a})$. Due to (C1), it then follows from (3.16) and (3.14) that

$$\frac{N^{(\xi)}(x)}{x^\gamma} \le \left(\frac{b-a}{4\alpha}\right)^\gamma \prod_{k=1}^{n} \left(\sum_{i=1}^{N_{\xi_k}} (r_i^{(\xi_k)} m_i^{(\xi_k)})^\gamma \right). \tag{3.17}$$

For reasons of clarity, we introduce the abbreviation $\Sigma_j = \sum_{i=1}^{N_j} (m_i^{(j)} r_i^{(j)})^\gamma$ for $j \in J$. To estimate the product $\prod_{k=1}^{n} \Sigma_{\xi_k}$, we first assort the factors depending on the values of ξ_k. Note that the number of factors in the product with $\xi_k = j$ is $n h_n^{(j)}$. Thus,

$$\prod_{k=1}^{n} \Sigma_{\xi_k} = \prod_{j\in J} (\Sigma_j)^{n h_n^{(j)}} = \prod_{j\in J} (\Sigma_j)^{n(h_n^{(j)} - p_j)}.$$

Here we used the fact, that $\prod_{j\in J} \Sigma_j^{p_j} = 1$ by the definition of γ. Note, that only finitely many of the factors in the product are different from 1. Then, using (C5), that is, $n|h_n^{(j)} - p_j| \leq g(n)$, we get

$$\begin{aligned}
\prod_{j\in J}(\Sigma_j)^{n(h_n^{(j)}-p_j)} &= \prod_{\substack{j\in J\\ \Sigma_j>1}}(\Sigma_j)^{n(h_n^{(j)}-p_j)} \prod_{\substack{j\in J\\ \Sigma_j<1}}(\Sigma_j)^{n(h_n^{(j)}-p_j)} \\
&\leq \prod_{\substack{j\in J\\ \Sigma_j>1}}(\Sigma_j)^{n|h_n^{(j)}-p_j|} \prod_{\substack{j\in J\\ \Sigma_j<1}}(\Sigma_j)^{-n|h_n^{(j)}-p_j|} \\
&\leq \prod_{\substack{j\in J\\ \Sigma_j>1}}(\Sigma_j)^{g(n)} \prod_{\substack{j\in J\\ \Sigma_j<1}}(\Sigma_j)^{-g(n)}.
\end{aligned}$$

By (C6) and (C7),

$$\Pi_1 := \prod_{\substack{j\in J\\ \Sigma_j>1}} \Sigma_j < \infty$$

and

$$\Pi_2 := \prod_{\substack{j\in J\\ \Sigma_j<1}} \Sigma_j > 0.$$

With that we get from (3.17)

$$\frac{N^{(\xi)}(x)}{x^\gamma} \leq \left(\frac{b-a}{4\alpha}\right)^\gamma \left(\frac{\Pi_1}{\Pi_2}\right)^{g(n)}. \tag{3.18}$$

Since we chose n such that $r_{i_{n-1}} m_{i_{n-1}} \cdots r_{i_1} m_{i_1} x \geq \frac{4}{b-a}$, it follows that $\beta^{n-1} x \geq \frac{4}{b-a}$. Therefore, $\left(\frac{1}{\beta}\right)^n \leq \frac{b-a}{4\beta} x$ and consequently, in view of (C2), $n \leq \frac{\log \frac{b-a}{4\beta} x}{\log \frac{1}{\beta}}$. Because g is monotonically non-decreasing, it follows that

$$\frac{N^{(\xi)}(x)}{x^\gamma} \leq \left(\frac{b-a}{4\alpha}\right)^\gamma \left(\frac{\Pi_1}{\Pi_2}\right)^{g\left(\frac{\log \frac{b-a}{4\beta} x}{\log \frac{1}{\beta}}\right)}. \tag{3.19}$$

This proves the theorem with $c_1 = \frac{\Pi_1}{\Pi_2}$, $c_2 = \frac{1}{-\log \beta}$ and $c_3 = \frac{b-a}{4\beta}$. $\square$

Since the values of the constants in Theorem 3.3.5 do not matter for the asymptotic behaviour, we can simplify the statement for large x by assuming g to be regularly varying, see Section B.6. This is no significant restriction.

Corollary 3.3.6. *If the function g in Theorem 3.3.5 is regularly varying, there are numbers $c > 0$ and $x_0 > 0$ such that*

$$\frac{N^{(\xi)}(x)}{x^\gamma e^{c\, g(\log x)}} \leq \left(\frac{b-a}{4\alpha}\right)^\gamma \tag{3.20}$$

for all $x \geq x_0$.

Proof. Let c_1, c_2, c_3 be the constants from Theorem 3.3.5. If g is regularly varying, $g(\log c_3 x)$ is regularly varying, too. Then

$$g(c_2 \log c_3 x) \leq \tilde{c}_1 g(\log c_3 x) \leq \tilde{c}_2 g(\log x)$$

for some constants $\tilde{c}_1$, $\tilde{c}_2 > 0$ and all x sufficiently large. Then

$$c_1^{g(c_2 \log c_3 x)} \leq c_1^{\tilde{c}_2 g(\log x)} = e^{c\, g(\log x)}$$

where $c = \tilde{c}_2 \log c_1$. Now the corollary follows by the theorem. □

3.3.2. Dirichlet boundary conditions

In the previous section we needed a lower estimate for the smallest positive Neumann eigenvalue to make sure that the eigenvalue counting function is equal to one in a certain interval. To use the appropriate counterpart argument in the Dirichlet case, we need an upper estimate for the smallest Dirichlet eigenvalue. Because of this we know then that the eigenvalue counting function is at least one in a certain interval.

To prove that estimate we will need the following representation of the smallest eigenvalue λ_1 of a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L_2(\mu)$ (see e.g. Theorem 1.3 in Edmunds and Evans [11, p. 491]). It holds that

$$\lambda_1 = \inf_{\substack{f \in \mathcal{F} \\ f \neq 0}} \frac{\mathcal{E}(f, f)}{\|f\|^2_{L_2(\mu)}}, \tag{3.21}$$

where $f \neq 0$ means that f is not the zero element in $L_2(\mu)$. It can be seen that if we impose homogeneous Dirichlet boundary conditions and take measures μ that are more and more concentrated near the boundary, λ_1 can grow arbitrarily large. This suggests that, for the upper estimate, we need a restriction on μ ensuring that it cannot be concentrated near the boundary. This is accomplished by conditions (C1) and (C2) imposed at the beginning of this section.

Lemma 3.3.7. *Let* ξ *be an environment sequence and let* λ_1 *be the smallest eigenvalue of the Dirichlet form* $(\mathcal{E}, \mathcal{F}_0)$ *on* $L_2(\mu^{(\xi)})$. *Then*

$$\lambda_1 \leq \frac{1}{\alpha^2(1-\beta)(b-a)}.$$

Because of this, the eigenvalue counting function satisfies

$$N_0^{(\xi)}(x) \geq 1$$

for all $x \geq \frac{1}{\alpha^2(1-\beta)(b-a)}$.

Proof. Let $\mu^{(\xi)}$ be the measure corresponding to the given environment sequence ξ. Let $x_1 = S_1^{(\xi_1)}\big(S_{N_{\xi_2}}^{(\xi_2)}(a)\big)$ and $x_2 = S^{(\xi_1)}(b)$ be the left and right endpoints of the second-level cell that lies on the right end of the leftmost first-level cell. Then $x_1 = a + r_1^{(\xi_1)}\big(1 - r_{N_{\xi_2}}^{(\xi_2)}\big)(b-a)$ and $x_2 = a + r_1^{(\xi_1)}(b-a)$. We define a function $\bar{f}$ by

$$\bar{f}(x) = \begin{cases} \dfrac{x-a}{x_1-a}, & \text{if } x \in [a, x_1] \\ 1, & \text{if } x \in (x_1, x_2] \\ \dfrac{b-x}{b-x_2}, & \text{if } x \in (x_2, b]. \end{cases}$$

Then $\bar{f} \in \mathcal{F}_0$ because it is in $H^1(\lambda)$ and $\bar{f}(a) = \bar{f}(b) = 0$. The energy of $\bar{f}$ can be calculated as

$$\begin{aligned} \mathcal{E}(\bar{f}, \bar{f}) &= \int_a^b \bar{f}'(t)^2\, dt = \Big(\frac{1}{x_1-a}\Big)^2 (x_1-a) + \Big(\frac{1}{b-x_2}\Big)^2 (b-x_2) \\ &= \frac{1}{x_1-a} + \frac{1}{b-x_2} = \frac{1}{r_1^{(\xi_1)}\big(1-r_{N_{\xi_2}}^{(\xi_2)}\big)(b-a)} + \frac{1}{(1-r_1^{(\xi_1)})(b-a)} \\ &= \frac{1 - r_1^{(\xi_1)} + r_1^{(\xi_1)}\big(1-r_{N_{\xi_2}}^{(\xi_2)}\big)}{r_1^{(\xi_1)}\big(1-r_{N_{\xi_2}}^{(\xi_2)}\big)(1-r_1^{(\xi_1)})(b-a)} = \frac{1}{r_1^{(\xi_1)}(1-r_1^{(\xi_1)})(b-a)}. \end{aligned}$$

Furthermore we need an estimate of the L_2-norm of $\bar{f}$. It holds

$$\|\bar{f}\|^2_{L_{\mu^{(x)}}} \geq \int_{x_1}^{x_2} \bar{f}^2\, d\mu^{(\xi)} = \mu^{(\xi)}\big([x_1, x_2]\big) = m_1^{(\xi_1)} m_{N_{\xi_2}}^{(\xi_2)}.$$

For the smallest eigenvalue λ_1 we get by (3.21)

$$\lambda_1 \leq \frac{\mathcal{E}(\bar{f}, \bar{f})}{\|\bar{f}\|_{L_2(\mu^{(\xi)})}} \leq \frac{1}{r_1^{(\xi_1)} m_1^{(\xi_1)} (1-r_1^{(\xi_1)}) m_{N_{\xi_2}}^{(\xi_2)}(b-a)} \leq \frac{1}{\alpha^2(1-\beta)(b-a)}.$$

□

Theorem 3.3.8. *Let conditions (C1)-(C7) be satisfied and γ be defined by* (3.12).

Then there are constants $c_1 > 1$ and $c_2, c_4 > 0$ such that

$$\frac{N_0^{(\xi)}(x)}{x^\gamma\, c_1^{-g(c_2 \log c_4 x)}} \geq [\alpha^3(1-\beta)(b-a)]^\gamma \tag{3.22}$$

for all $x \geq \frac{1}{\alpha^2(1-\beta)(b-a)}$. The constants c_1 and c_2 are the same as in Theorem 3.3.5.

Proof. Let $x \geq \frac{1}{\alpha^2(1-\beta)(b-a)}$. Using (3.10) we get

$$\begin{aligned}
\frac{N_0^{(\xi)}(x)}{x^\gamma} &\geq \sum_{i=1}^{N_{\xi_1}} \frac{N_0^{(\theta\xi)}(r_i^{(\xi_1)} m_i^{(\xi_1)} x)}{(r_i^{(\xi_1)} m_i^{(\xi_1)} x)^\gamma} (r_i^{(\xi_1)} m_i^{(\xi_1)})^\gamma \\
&\geq \min_{i=1,\ldots,N_{\xi_1}} \frac{N_0^{(\theta\xi)}(r_i^{(\xi_1)} m_i^{(\xi_1)} x)}{(r_i^{(\xi_1)} m_i^{(\xi_1)} x)^\gamma} \sum_{i=1}^{N_{\xi_1}} (r_i^{(\xi_1)} m_i^{(\xi_1)})^\gamma \\
&= \frac{N_0^{(\theta\xi)}(r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)}{(r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)^\gamma} \sum_{i=1}^{N_{\xi_1}} (r_i^{(\xi_1)} m_i^{(\xi_1)})^\gamma,
\end{aligned}$$

where i_1 denotes the index where the minimum is attained. The same argument applied to $\frac{N_0^{(\theta\xi)}(r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)}{(r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)^\gamma}$ gives

$$\frac{N_0^{(\xi)}(x)}{x^\gamma} \geq \frac{N_0^{(\theta^2\xi)}(r_{i_2}^{(\xi_2)} m_{i_2}^{(\xi_2)} r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)}{(r_{i_2}^{(\xi_2)} m_{i_2}^{(\xi_2)} r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)^\gamma} \left(\sum_{i=1}^{N_{\xi_2}} (r_i^{(\xi_2)} m_i^{(\xi_2)})^\gamma \right) \left(\sum_{i=1}^{N_{\xi_1}} (r_i^{(\xi_1)} m_i^{(\xi_1)})^\gamma \right),$$

where i_2 denotes the index for which the appearing minimum is attained. Repeating this n times and denoting $\sum_{i=1}^{N_j} (r_i^{(j)} m_i^{(j)})^\gamma$ by Σ_j for $j \in J$ as in the proof of Theorem 3.3.5, we get

$$\frac{N_0^{(\xi)}(x)}{x^\gamma} \geq \frac{N_0^{(\theta^n\xi)}(r_{i_n}^{(\xi_n)} m_{i_n}^{(\xi_n)} \ldots r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)}{(r_{i_n}^{(\xi_n)} m_{i_n}^{(\xi_n)} \ldots r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x)^\gamma} \prod_{k=1}^n \Sigma_{\xi_k}. \tag{3.23}$$

Now we choose n such that

$$\frac{1}{\alpha^2(1-\beta)(b-a)} \leq r_{i_n}^{(\xi_n)} m_{i_n}^{(\xi_n)} \cdots r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x$$

and

$$r_{i_{n+1}}^{(\xi_{n+1})} m_{i_{n+1}}^{(\xi_{n+1})} \cdots r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x < \frac{1}{\alpha^2(1-\beta)(b-a)}.$$

Consequently,

$$r_{i_n}^{(\xi_n)} m_{i_n}^{(\xi_n)} \cdots r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x \in \Big[\frac{1}{\alpha^2(1-\beta)(b-a)}, \frac{1}{\alpha^3(1-\beta)(b-a)}\Big)$$

and therefore, by Lemma 3.3.7 and (3.23),

$$\frac{N_0^{(\xi)}(x)}{x^\gamma} \geq [\alpha^3(1-\beta)(b-a)]^\gamma \prod_{k=1}^{n} \Sigma_{\xi_k}.$$

As in the proof of Theorem 3.3.5 we assort the factors in the product $\prod_{k=1}^{n} \Sigma_{\xi_k}$ as follows:

$$\prod_{k=1}^{n} \Sigma_{\xi_k} = \prod_{j \in J} (\Sigma_j)^{n h_n^{(j)}} = \prod_{j \in J} (\Sigma_j)^{n(h_n^{(j)} - p_j)},$$

where we use the definition of γ, namely that $\prod\limits_{j \in J} \Sigma_j^{p_j} = 1$.

Then, by $n|h_n^{(j)} - p_j| \leq g(n)$,

$$\begin{aligned}
\prod_{j \in J} (\Sigma_j)^{n(h_n^{(j)} - p_j)} &= \prod_{\substack{j \in J \\ \Sigma_j > 1}} (\Sigma_j)^{n(h_n^{(j)} - p_j)} \prod_{\substack{j \in J \\ \Sigma_j < 1}} (\Sigma_j)^{n(h_n^{(j)} - p_j)} \\
&\geq \prod_{\substack{j \in J \\ \Sigma_j > 1}} (\Sigma_j)^{-n|h_n^{(j)} - p_j|} \prod_{\substack{j \in J \\ \Sigma_j < 1}} (\Sigma_j)^{n|h_n^{(j)} - p_j|} \\
&\geq \prod_{\substack{j \in J \\ \Sigma_j > 1}} (\Sigma_j)^{-g(n)} \prod_{\substack{j \in J \\ \Sigma_j < 1}} (\Sigma_j)^{g(n)}.
\end{aligned}$$

By using the notation

$$\Pi_1 = \prod_{\substack{j \in J \\ \Sigma_j > 1}} \Sigma_j \qquad \text{and} \qquad \Pi_2 = \prod_{\substack{j \in J \\ \Sigma_j < 1}} \Sigma_j,$$

which was defined in the proof of Theorem 3.3.5, we get

$$\frac{N_0^{(\xi)}(x)}{x^\gamma} \geq \big[\alpha^3(1-\beta)(b-a)\big]^\gamma \Big(\frac{\Pi_1}{\Pi_2}\Big)^{-g(n)}. \tag{3.24}$$

Since we chose n such that

$$\frac{1}{\alpha^2(1-\beta)(b-a)} \leq r_{i_n}^{(\xi_n)} m_{i_n}^{(\xi_n)} \cdots r_{i_1}^{(\xi_1)} m_{i_1}^{(\xi_1)} x$$

it follows that

$$\frac{1}{\alpha^2(1-\beta)(b-a)} \leq \beta^n x$$

and hence

$$n \leq \frac{\log\big(\alpha^2(1-\beta)(b-a)x\big)}{\log\left(\frac{1}{\beta}\right)}.$$

With that, (3.24) gives

$$\frac{N_0^{(\xi)}(x)}{x^\gamma} \geq \left[\alpha^3(1-\beta)(b-a)\right]^\gamma \left(\frac{\Pi_1}{\Pi_2}\right)^{-g\left(\frac{\log(\alpha^2(1-\beta)(b-a)x)}{-\log\beta}\right)}. \tag{3.25}$$

Then, with $c_1 = \frac{\Pi_1}{\Pi_2}$, $c_2 = \frac{1}{-\log\beta}$ and $c_4 = \alpha^2(1-\beta)(b-a)$ the assertion follows. □

As in the case of Neumann boundary conditions, we will now assume that g is regularly varying for a more comprehensible formulation.

Corollary 3.3.9. *If the function g in Theorem 3.3.8 is regularly varying, there are numbers $c > 0$ and $x_0 > 0$ such that*

$$\frac{N_0^{(\xi)}(x)}{x^\gamma\, e^{-cg(\log x)}} \geq [\alpha^3(1-\beta)(b-a)]^\gamma$$

for all $x \geq x_0$.

Proof. We use the same argumentation as in the proof of Corollary 3.3.6. □

3.3.3. Main theorem

We connect Theorems 3.3.5 and 3.3.8 to state the main result of the first chapter of the thesis. To this end, note that from Theorem B.7.4 follows for every environment sequence ξ that

$$N_0^{(\xi)}(x) \leq N^{(\xi)}(x)$$

for all $x \geq 0$.

Theorem 3.3.10. *Let $(\mathcal{S}^{(j)})_{j\in J}$ be a collection of IFS's, $(m_i^{(j)})_{j\in J, i=1,\ldots,N_j}$ weights, and $(\xi_j)_{j\in J}$ an environment sequence that satisfy the conditions (C1) to (C7) with a regularly varying function g. Let $\gamma = \gamma(\xi)$ be defined by*

$$\prod_{j\in J}\left(\sum_{i=1}^{N_j}(r_i^{(j)} m_i^{(j)})^\gamma\right)^{p_j} = 1.$$

Then there are constants $c_1, c_2, C_1, C_2 > 0$ and $x_0 > 0$ such that

$$C_1\, x^{\gamma}\, e^{-c_1\, g(\log x)} \le N_0^{(\xi)}(x) \le N^{(\xi)}(x) \le C_2\, x^{\gamma}\, e^{c_2\, g(\log x)}$$

for all $x \ge x_0$.

Proof. The theorem follows immediately from Corollaries 3.3.6 and 3.3.9 and Theorem B.7.4 as pointed out above. □

Remark 3.3.11. In Barlow and Hambly [3], heat kernel estimates for a Laplacian on "scale irregular" Sierpinski gaskets are established. In the end they lead to certain growth results for the eigenvalue counting function. Although their operator is quite different from the one we use, the results resemble each other.

3.4. Deterministic examples

Example 3.4.1. Suppose $J = \{a, b, c\}$ and, for each $j \in J$, take $\mathcal{S}^{(j)}$ with

$$r_1^{(j)} = \ldots = r_{N_j}^{(j)} = r^{(j)}$$

and let the weights be

$$m_1^{(j)} = \ldots = m_{N_j}^{(j)} = \frac{1}{N_j}.$$

Consider an environment sequence with asymptotic relative frequencies p_a, p_b, and p_c different from zero. Then γ is determined by

$$\left[N_a\Big(r^{(a)}\frac{1}{N_a}\Big)^{\gamma}\right]^{p_a}\left[N_b\Big(r^{(b)}\frac{1}{N_b}\Big)^{\gamma}\right]^{p_b}\left[N_c\Big(r^{(c)}\frac{1}{N_c}\Big)^{\gamma}\right]^{p_c} = 1,$$

which leads to

$$\begin{aligned}\gamma &= \frac{-\log\big(N_a^{p_a}\, N_b^{p_b}\, N_c^{p_c}\big)}{\log\left[\Big(\frac{r^{(a)}}{N_a}\Big)^{p_a}\Big(\frac{r^{(b)}}{N_b}\Big)^{p_b}\Big(\frac{r^{(c)}}{N_c}\Big)^{p_c}\right]}\\ &= \frac{p_a \log N_a + p_b \log N_b + p_c \log N_c}{p_a \log \frac{N_a}{r^{(a)}} + p_b \log \frac{N_b}{r^{(b)}} + p_c \log \frac{N_c}{r^{(c)}}}\end{aligned}$$

In addition to γ, the asymptotic growths of the eigenvalue counting function depends on the function g in Theorem 3.3.10. This is being discussed in the following two examples, where we have the same γ but different functions g.

Example 3.4.2. Take $J = \{a, b\}$ where a and b denote indices, let $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ be corresponding IFS's and $(m_1^{(a)}, \dots, m_{N_a}^{(a)})$ and $(m_1^{(b)}, \dots, m_{N_b}^{(b)})$ weight vectors. We consider the environment sequence

$$\xi = (a, b, a, b, a, b, \dots).$$

Then, for all $n \in \mathbb{N}$,

$$n \left| h_n^{(a)} - \frac{1}{2} \right| = n \left| h_n^{(b)} - \frac{1}{2} \right| = \begin{cases} 0 & \text{, if } n \text{ is even} \\ \frac{1}{2} & \text{, if } n \text{ is odd} \end{cases} \leq \frac{1}{2}$$

which means that $p_a = p_b = \frac{1}{2}$ and we can take $g(n) = \frac{1}{2}$ in Theorem 3.3.10. Therefore,

$$C_1 \, x^\gamma \leq N_0^{(\xi)}(x) \leq N^{(\xi)}(x) \leq C_2 \, x^\gamma.$$

This is not surprising, because ξ is periodic and therefore, $K^{(\xi)}$ and $\mu^{(\xi)}$ are self-similar with respect to the IFS

$$\left\{ S_i^{(a)} \circ S_k^{(b)} : i = 1, \dots, N_a, \; k = 1, \dots, N_b \right\}$$

and vector of weights $(m_i^{(a)} m_k^{(b)})_{i,k}$.

The equation (3.12) defining γ then becomes

$$\left(\sum_{i=1}^{N_a} \left(r_i^{(a)} m_i^{(a)} \right)^\gamma \right) \left(\sum_{i=1}^{N_b} \left(r_i^{(b)} m_i^{(b)} \right)^\gamma \right) = \sum_{i=1}^{N_a} \sum_{k=1}^{N_b} \left(r_i^{(a)} r_k^{(b)} m_i^{(a)} m_k^{(b)} \right)^\gamma = 1.$$

Example 3.4.3. Let J, the IFS's and the weight vectors as in Example 3.4.2 but take the environment sequence

$$\xi = (a, b, a, a, b, b, a, a, a, b, b, b, a, a, a, a, b, b, b, b, \dots).$$

We count the numbers $n \, h_n^{(a)}$ in the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
ξ_n	a	b	a	a	b	b	a	a	a	b	b	b	a	a	a	a
$n \, h_n^{(a)}$	1	1	2	3	3	3	4	5	6	6	6	6	7	8	9	10
$n \, (h_n^{(a)} - \frac{1}{2})$	$\boxed{\frac{1}{2}}$	0	$\frac{1}{2}$	$\boxed{1}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\boxed{\frac{3}{2}}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	$\boxed{2}$

We highlighted the values for square numbers with a box. We see that

$$n\left(h_n^{(a)}-\frac{1}{2}\right)\leq\frac{k}{2},\quad \text{if } k^2\leq n<(k+1)^2, \text{ for } k\in\mathbb{N},$$

and therefore, for all $n\in\mathbb{N}$,

$$n\left(h_n^{(a)}-\frac{1}{2}\right)\leq\frac{\sqrt{n}}{2}.$$

Moreover, since

$$n\left(h_n^{(a)}-\frac{1}{2}\right)=\frac{\sqrt{n}}{2}\quad \text{if } n=k^2 \text{ for } k\in\mathbb{N},$$

and

$$n\left(h_n^{(a)}-\frac{1}{2}\right)=0\quad \text{if } n=k(k+1) \text{ for } k\in\mathbb{N},$$

we have

$$\limsup_{n\to\infty}\frac{n\left|h_n^{(a)}-\frac{1}{2}\right|}{\sqrt{n}}=\frac{1}{2},$$

and

$$\liminf_{n\to\infty}\frac{n\left|h_n^{(a)}-\frac{1}{2}\right|}{\sqrt{n}}=0.$$

Therefore, $p_a=p_b=\frac{1}{2}$, and the optimal choice of g in Theorem 3.3.10 is $g(n)=\frac{1}{2}\sqrt{n}$, $n\in\mathbb{N}$. Thus, we get

$$C_1\,x^\gamma\,e^{-c_1\sqrt{\log x}}\leq N_0^{(\xi)}(x)\leq N^{(\xi)}(x)\leq C_2\,x^\gamma\,e^{c_2\sqrt{\log x}},$$

where γ is the same as in Example 3.4.2.

In the following we consider examples with an countably infinite set J. We will see that in this case the family of IFS's and the environment sequence can not be chosen independently. This is because conditions (C6) and (C7) require a certain interaction of these.

Example 3.4.4. Let $J=\mathbb{N}$. To keep it simple, we take a collection of (non-overlapping) IFS's with the same number of mappings $N\geq 2$, and for all $j\in\mathbb{N}$, $m_1^{(j)}=\cdots=m_N^{(j)}=\frac{1}{N}$, and $r_1^{(j)}=\cdots=r_N^{(j)}=r^{(j)}\in[\alpha,\frac{1}{N})$ for some $\alpha>0$.

Consider the environment sequence

$$\xi=(1,2,1,3,1,4,1,5,1,6,\dots).$$

Then $p_1 = \frac{1}{2}$, whereas $p_j = 0$ for all $j \geq 2$. Since $n(h_n^{(1)} - \frac{1}{2}) \leq \frac{1}{2}$ and $nh_n^{(j)} \leq 1$ for $j \geq 2$, we take $g(n) = 1$.

To determine γ only $r^{(1)}$ plays a role because (3.12) reduces to

$$\left(N\left(\frac{r^{(1)}}{N}\right)^{\gamma}\right)^{\frac{1}{2}} = 1,$$

and therefore

$$\gamma = -\frac{\log N}{\log \frac{r^{(1)}}{N}}.$$

But to satisfy conditions (C6) and (C7) we get conditions on $r^{(j)}$, $j \geq 2$, that demand them to be "asymptotically indistinguishable". We must check

$$\sum_{\substack{j\in\mathbb{N}\\ \log\Sigma_j>0}} \log\Sigma_j < \infty$$

and

$$\sum_{\substack{j\in\mathbb{N}\\ \log\Sigma_j<0}} \log\Sigma_j > -\infty.$$

For every $j \in \mathbb{N}$ we have $\Sigma_j = N\left(\frac{r^{(j)}}{N}\right)^{\gamma}$ and therefore,

$$\begin{aligned}
\log\Sigma_j &= \log N - \frac{\log N}{\log \frac{r^{(1)}}{N}} \log \frac{r^{(j)}}{N} \\
&= \log N \, \frac{\log \frac{r^{(1)}}{N} - \log \frac{r^{(j)}}{N}}{\log \frac{r^{(1)}}{N}} \\
&= \frac{\log N}{\log \frac{r^{(1)}}{N}} \, \log \frac{r^{(1)}}{r^{(j)}}.
\end{aligned}$$

So (C6) and (C7) are satisfied if and only if

$$\sum_{j=1}^{\infty} \left| \log \frac{r^{(1)}}{r^{(j)}} \right| < \infty$$

and that means that in order for Theorem 3.3.10 to be applicable, $r^{(j)}$ must converge to $r^{(1)}$ sufficiently fast. Assuming that this is the case, the theorem gives

$$C_1\, x^{\gamma} \leq N_0^{(\xi)}(x) \leq N^{(\xi)}(x) \leq C_2\, x^{\gamma}.$$

The exponent γ is the same as in the case of a strictly self-similar measure that is constructed only with the IFS $\mathcal{S}^{(1)}$, but $\mu^{(\xi)}$ is not exactly self-similar because is is slightly "disturbed" by the IFS's $\mathcal{S}^{(j)}$, $j \geq 2$. However, this disturbance vanishes asymptotically as $r^{(j)} \to r^{(1)}$.

Example 3.4.5. We would like to give an example of an environment sequence where $J = \mathbb{N}$ and $p_j > 0$ for all $j \in \mathbb{N}$. Consider ξ defined by $\xi_n = v_2(n) + 1$, where $v_2(n)$ denotes the exponent of 2 in the prime decomposition of n. For better comprehensibility we make the following table.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
ξ_n	1	2	1	3	1	2	1	4	1	2	1	3	1	2	1	5
$n\,(h_n^{(1)} - \frac{1}{2})$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$n\,(h_n^{(2)} - \frac{1}{4})$	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$n\,(h_n^{(3)} - \frac{1}{8})$	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0

We have for all $j \in \mathbb{N}$

$$\frac{2^{j-1}-1}{2^j} \leq n\left(h_n^{(j)} - \frac{1}{2^j}\right) \leq \frac{1}{2}$$

which means that $p_j = \frac{1}{2^j}$ and $g(n) = \frac{1}{2}$.

Example 3.4.6. Let $J = \mathbb{N}$ and let ξ be such that $p_j > 0$ for all $j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} p_j = 1$.

As in Example 3.4.4, let for all $j \in \mathbb{N}$ be $N_j = N \geq 2$, $m_1^{(j)} = \cdots = m_N^{(j)} = \frac{1}{N}$, and $r_1^{(j)} = \cdots = r_N^{(j)} = r^{(j)} \in [\alpha, \frac{1}{N})$ for some $\alpha > 0$.

Then γ is defined by

$$\prod_{j=1}^{\infty} \left[N \left(\frac{r^{(j)}}{N}\right)^{\gamma}\right]^{p_j} = 1$$

which is equivalent to

$$\sum_{j=1}^{\infty} p_j \left[\log N + \gamma \log \frac{r^{(j)}}{N}\right] = 0$$

and therefore,

$$\gamma = -\frac{\sum\limits_{j=1}^{\infty} p_j \log N}{\sum\limits_{j=1}^{\infty} p_j \log \frac{r^{(j)}}{N}} = -\frac{\log N}{\sum\limits_{j=1}^{\infty} p_j \log \frac{r^{(j)}}{N}}.$$

Because $r^{(j)} \geq \alpha$, we have $\sum\limits_{j=1}^{\infty} p_j \log \frac{r^{(j)}}{N} \geq \sum\limits_{j=1}^{\infty} p_j \log \frac{\alpha}{N} = \log \frac{\alpha}{N} > -\infty$.

Finally we have to check conditions (C6) and (C7), that is, $\sum_{j=1}^{\infty} |\log \Sigma_j| < \infty$. Since

$$\begin{aligned}
\log \Sigma_j &= \log N + \gamma \log \frac{r^{(j)}}{N} \\
&= \log N \left(1 - \frac{1}{\sum_{k=1}^{\infty} p_k \log \frac{r^{(k)}}{N}} \log \frac{r^{(j)}}{N} \right) \\
&= \frac{\log N}{\sum_{k=1}^{\infty} p_k \log \frac{r^{(k)}}{N}} \left(\sum_{k=1}^{\infty} p_k \log \frac{r^{(k)}}{N} - \log \frac{r^{(j)}}{N} \right) \\
&= \frac{\log N}{\sum_{k=1}^{\infty} p_k \log \frac{r^{(k)}}{N}} \left(\sum_{k=1}^{\infty} p_k \log r^{(k)} - \log r^{(j)} \right),
\end{aligned}$$

this condition is equivalent to

$$\sum_{j=1}^{\infty} \left| \log r^{(j)} - \sum_{k=1}^{\infty} p_k \log r^{(k)} \right| < \infty. \tag{3.26}$$

That means that Theorem 3.3.10 can only be applied, if the sequence of the contraction ratios $r^{(j)}$ converges sufficiently fast to $\prod_{k=1}^{\infty} \left(r^{(k)}\right)^{p_k}$. We give such an example for the $r^{(j)}$ supposing that $N = 2$. Let

$$r^{(1)} = \left(3 \cdot 2^{\frac{1}{p_1} \sum_{k=2}^{\infty} \frac{p_k}{k^2}} \right)^{-1}$$

and

$$r^{(j)} = \frac{1}{3} \, 2^{\frac{1}{j^2}} \qquad \text{for } j \geq 2.$$

Then

$$\begin{aligned}
\sum_{k=1}^{\infty} p_k \log r^{(k)} &= p_1 \log \frac{1}{3} - \sum_{k=2}^{\infty} \frac{p_k}{k^2} \log 2 + \sum_{k=2}^{\infty} p_k \log \left(\frac{1}{3} \, 2^{\frac{1}{k^2}} \right) \\
&= \log \frac{1}{3}
\end{aligned}$$

and therefore, for all $j \geq 2$,

$$\left| \log r^{(j)} - \sum_{k=1}^{\infty} p_k \log r^{(k)} \right| = \frac{1}{j^2} \, \log 2$$

which implies (3.26).

Example 3.4.7. We give an example for the case we considered in Remark 3.3.4, namely that $p_j = 0$ for all $j \in J$. Take $J = \mathbb{N}$ and the parameters of the IFSs as in the previous examples. Then Equation (3.12) does not define a value for γ but (C6) and (C7) demand that

$$\sum_{j=1}^{\infty} \left| \log\Big(N \big(\tfrac{r^{(j)}}{N} \big)^{\gamma} \Big) \right| < \infty.$$

Thus, in order for our theorem to be applicable, $r^{(j)}$ must tend to $N^{1-\frac{1}{\gamma}}$ sufficiently fast, so that γ can be computed if N and $r^{(j)}$ are given.

3.5. Application to random homogeneous measures

In this section we consider fractal measures that result when we take for the environment sequence ξ a sequence of i.i.d. random variables with values in the index set J. That means in every step of the construction of the measure, it is independently chosen which IFS is taken. At that, for each cell the same IFS is used. Such a construction based on the Sierpinski gasket is for example studied in Hambly [27].

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, let J be an at most countable index set, and suppose we are given an IFS $\mathcal{S}^{(j)}$ on $[a, b]$ for each $j \in J$ and weight factors $m_i^{(j)}$.

Let $(p_j)_{j \in J}$ be a distribution on J with $0 < p_j < 1$ for all $j \in J$ and let $\xi = (\xi_1, \xi_2, \dots)$ be a sequence of i.i.d J-valued random variables on Ω with $\mathbb{P}(\xi_i = j) = p_j$ for $j \in J$.

For each $\omega \in \Omega$ let $K^{(\xi(\omega))}$ be the set and $\mu^{(\xi(\omega))}$ the measure constructed in Section 3.1. Then $N^{(\xi(\omega))}$ and $N_0^{(\xi(\omega))}$ denote the eigenvalue counting functions of the Neumann and Dirichlet Laplacian, respectively.

Corollary 3.5.1. *Let J, $\mathcal{S}^{(j)}$ and $m_i^{(j)}$ be such that (C1), (C2) and (C3) are satisfied, and let γ be defined by*

$$\prod_{j \in J} \left(\sum_{i=1}^{N_j} (r_i^{(j)} m_i^{(j)})^{\gamma} \right)^{p_j} = 1.$$

Suppose, if J is infinite, that (C6) and (C7) are satisfied.

Then there exist $C_1, C_2 > 0$, $x_0 > 0$, and $c_1(\omega), c_2(\omega) > 0$ such that

$$C_1 x^\gamma e^{-c_1(\omega)\sqrt{\log x \log\log\log x}} \leq N_0^{(\xi(\omega))}(x) \leq N^{(\xi(\omega))}(x) \leq C_2 x^\gamma e^{c_2(\omega)\sqrt{\log x \log\log\log x}} \tag{3.27}$$

for $x \geq x_0$ and almost every $\omega \in \Omega$.

Proof. Let $\xi_1, \xi_2, \dots$ be i.i.d J-valued random variables with $\mathbb{P}(\xi_i = j) = p_j$ for $j \in J$.

For each $j \in J$ and $k \in \mathbb{N}$ we set

$$X_k^{(j)} := \frac{1}{\sqrt{p_j(1-p_j)}} \left[\mathbf{1}_{\{\xi_i = j\}} - p_j \right]$$

and

$$S_n^{(j)} := \sum_{k=1}^{n} X_k^{(j)}.$$

Since $\mathbb{E}\, X_k^{(j)} = 0$ and $\operatorname{Var} X_k^{(j)} = 1$ we can apply the law of the iterated logarithm stated in Theorem B.5.1, and thus there is a set $A \in \mathcal{A}$ with $\mathbb{P}(A) = 1$ such that for all $\omega \in A$ there is a number $c(\omega) > 1$ with

$$\frac{|S_n^{(j)}(\omega)|}{\sqrt{2n \log\log n}} \leq c(\omega) \qquad \text{for all } n \in \mathbb{N}.$$

Because

$$\begin{aligned} S_n^{(j)} &= \frac{1}{\sqrt{p_j(1-p_j)}} \Big(\sum_{k=1}^{n} \mathbf{1}_{\{\xi_i = j\}} - n p_j \Big) \\ &= \frac{1}{\sqrt{p_j(1-p_j)}} n \big(h_n^{(j)} - p_j \big), \end{aligned}$$

it follows that

$$n(h_n^{(j)} - p_j) \leq c(\omega) \sqrt{p_j(1-p_j)} \sqrt{2n \log\log n} \leq c(\omega) \sqrt{n \log\log n}$$

and thus conditions (C4) and (C5) are satisfied with $g(\omega)(n) = c(\omega)\sqrt{n \log\log n}$.

Therefore, for each $\omega \in A$ the conditions (C1)- (C7) are satisfied and we can apply Theorem 3.3.10. This gives (3.27). □

In the case where $r_i^{(j)} = r^{(j)}$ and $m_i^{(j)} = \frac{1}{N_j}$ for all $i = 1, \dots, N_j$, we will reformulate the equation defining γ to give it a more stochastic interpretation. For this, suppose we have random variables N and r with values in $\{N_j \colon j \in J\}$ and $\{r^{(j)} \colon j \in J\}$, respectively, and

$$\mathbb{P}\big(N = N_j, r = r^{(j)}\big) = \mathbb{P}\big(N = N_j\big) = \mathbb{P}\big(r = r^{(j)}\big) = p_j.$$

Then (3.12) leads to

$$\gamma = \frac{\sum\limits_{j \in J} p_j \log N_j}{\sum\limits_{j \in J} p_j \log \frac{N_j}{r^{(j)}}} = \frac{\mathbb{E} \log N}{\mathbb{E} \log \frac{N}{r}}.$$

Remark 3.5.2. As mentioned in Section 1.3 there are many other random models of fractals. For example, in contrast to our construction the one made in Falconer [14, Ch. 15] allows a continuous distribution of the parameters $r^{(j)}$.

4. Eigenvalues of the Laplacian as Zeros of Generalized Sine Functions

As outlined in Section 1.4, we now want to approach the eigenvalue problem for Δ_μ by following the basic lines of the classical case.

In Section 4.1 we introduce functions that can be viewed as generalizations of trigonometric functions. We investigate these in the succeeding sections, imposing more conditions on the measure μ in each section.

4.1. Generalized trigonometric functions

Let μ be an atomless Borel probability measure on $[0,1]$. We construct sequences of functions $p_n(x)$ and $q_n(x)$ depending on μ.

Definition 4.1.1. For $x \in [0,1]$ we set $p_0(x) = q_0(x) = 1$ and, for $n \in \mathbb{N}$,

$$p_n(x) := \begin{cases} \int_0^x p_{n-1}(t)\,d\mu(t) & \text{, if } n \text{ is odd,} \\ \int_0^x p_{n-1}(t)\,dt & \text{, if } n \text{ is even,} \end{cases}$$

and

$$q_n(x) := \begin{cases} \int_0^x q_{n-1}(t)\,dt & \text{, if } n \text{ is odd,} \\ \int_0^x q_{n-1}(t)\,d\mu(t) & \text{, if } n \text{ is even.} \end{cases}$$

Then, for $n \in \mathbb{N}$, we have by definition $p_{2n}, q_{2n+1} \in H^1(\lambda)$, $p_{2n+1}, q_{2n} \in H^1(\mu)$ and

$$\frac{d}{d\mu} p_{2n+1} = p_{2n}, \quad q'_{2n+1} = q_{2n}, \quad p'_{2n} = p_{2n-1} \quad \text{and} \quad \frac{d}{d\mu} q_{2n} = q_{2n-1}.$$

Remark 4.1.2. If we take μ to be the Lebesgue measure, then

$$p_n(x) = q_n(x) = \frac{x^n}{n!}.$$

In the following, we will transfer classical concepts and techniques to a general measure μ by replacing $\frac{x^n}{n!}$ by $p_n(x)$ or $q_n(x)$. In this sense, we can look at $p_n(x)$ or $q_n(x)$ as a kind of generalized monomials.

To prove convergence of the series defined below, we will need the following lemma.

Lemma 4.1.3. *For all $x \in [0,1]$, $z \in \mathbb{R}$ and $n \in \mathbb{N}_0$ holds*

$$\begin{aligned} p_{2n+1}(x) &\le \frac{1}{n!}\, q_2(x)^n, & q_{2n+1}(x) &\le \frac{1}{n!}\, p_2(x)^n, \\ p_{2n}(x) &\le \frac{1}{n!}\, p_2(x)^n, & q_{2n}(x) &\le \frac{1}{n!}\, q_2(x)^n. \end{aligned}$$

Proof. The proof is taken from [22, Lemma 2.3] and works with complete induction. First we show the inequality involving p_{2n+1}. For $n = 0$ the inequality reduces to $p_1(x) \le 1$, which is true. Assume the assertion holds for some $n \in \mathbb{N}_0$, then

$$\begin{aligned} p_{2n+3}(x) &= \int_0^x \int_0^t p_{2n+1}(s)\, ds\, d\mu(t) \le \frac{1}{n!} \int_0^x \int_0^t q_2(s)^n\, ds\, d\mu(t) \\ &\le \frac{1}{n!} \int_0^x q_2(t)^n \int_0^t ds\, d\mu(t) = \frac{1}{n!} \int_0^x q_2(t)^n\, q_1(t)\, d\mu(t). \end{aligned}$$

Since $\frac{d}{d\mu} q_2 = q_1$, we can apply Proposition 2.1.11 (iv) to transform the last integral, so that we get

$$p_{2n+3}(x) \le \frac{1}{n!} \int_0^{q_2(x)} w^n\, dw = \frac{1}{(n+1)!}\, q_2(x)^{n+1}.$$

Next we show the inequality for the odd-numbered q. For $n = 0$ we have $q_1(x) \le 1$ which is true for $x \in [0,1]$. Let $n \in \mathbb{N}_0$ and suppose the assertion holds for n, then

$$\begin{aligned} q_{2n+3}(x) &= \int_0^x \int_0^t q_{2n+1}(s)\, d\mu(s)\, dt \le \frac{1}{n!} \int_0^x \int_0^t p_2(s)^n\, d\mu(s)\, dt \\ &\le \frac{1}{n!} \int_0^x p_2(t)^n \int_0^t d\mu(s)\, dt = \frac{1}{n!} \int_0^x p_2(t)^n\, p_1(t)\, dt \\ &= \frac{1}{n!} \int_0^{p_2(x)} w^n\, dw = \frac{1}{(n+1)!}\, p_2(x)^{n+1}. \end{aligned}$$

The proof for the even-numbered q is the same as for the odd-numbered p and for the even p as for the odd q. $\square$

Definition 4.1.4. Using the functions $p_n(x)$ and $q_n(x)$ we now define for $x \in [0,1]$ and $z \in \mathbb{R}$:

$$\mathrm{sp}_z(x) := \sum_{n=0}^{\infty} (-1)^n \, z^{2n+1} p_{2n+1}(x), \qquad \mathrm{sq}_z(x) := \sum_{n=0}^{\infty} (-1)^n \, z^{2n+1} q_{2n+1}(x),$$
$$\mathrm{cp}_z(x) := \sum_{n=0}^{\infty} (-1)^n \, z^{2n} p_{2n}(x), \qquad \mathrm{cq}_z(x) := \sum_{n=0}^{\infty} (-1)^n \, z^{2n} q_{2n}(x).$$

Example 4.1.5. If μ is the Lebesgue measure, then

$$\mathrm{sp}_z(x) = \mathrm{sq}_z(x) = \sin(zx), \qquad \mathrm{cp}_z(x) = \mathrm{cq}_z(x) = \cos(zx).$$

Lemma 4.1.6. *For every $z \in \mathbb{R}$ the series in Definition 4.1.4 converge uniformly absolutely on $[0,1]$ and the following differentiation rules hold:*

$$\frac{d}{d\mu} \mathrm{sp}_z = z \, \mathrm{cp}_z, \qquad \mathrm{sq}_z' = z \, \mathrm{cq}_z,$$
$$\mathrm{cp}_z' = -z \, \mathrm{sp}_z, \qquad \frac{d}{d\mu} \mathrm{cq}_z = -z \, \mathrm{sq}_z \, .$$

Proof. Let $z \in \mathbb{R}$. By Lemma 4.1.3 we get for all $N \in \mathbb{N}$

$$\sup_{x\in[0,1]} \sum_{n=N}^{\infty} |z|^{2n+1} p_{2n+1}(x) \le \sup_{x\in[0,1]} \sum_{n=N}^{\infty} \frac{|z|^{2n+1} \, q_2(x)^n}{n!} \le \sum_{n=N}^{\infty} \frac{|z|^{2n+1}}{n!}.$$

Hence, for every $z \in \mathbb{R}$ the series $\sum_{n=0}^{\infty} |z|^{2n+1} p_{2n+1}(x)$ converges uniformly in x. The proof for the other series works analogously with the estimates in Lemma 4.1.3. Thus, we can differentiate term by term and get the above rules. $\square$

Now we show the relation between cp_z and sq_z to the eigenvalue problem for Δ_μ. Consider the Neumann problem

$$\frac{d}{d\mu} f' = -\lambda f$$
$$f'(0) = f'(1) = 0.$$

It is well known that the eigenvalues can be sorted according to size such that

$$\lambda_{N,0} < \lambda_{N,1} < \lambda_{N,2} < \cdots ,$$

where $\lambda_{N,0} = 0$ and $\lim\limits_{m\to\infty} \lambda_{N,m} = \infty$.

Proposition 4.1.7. *The Neumann eigenvalues $\lambda_{N,m}$, $m \in \mathbb{N}_0$, are the squares of the non-negative zeros of the function* sinp *given by*

$$\operatorname{sinp}(z) := \operatorname{sp}_z(1) = \sum_{n=0}^{\infty} (-1)^n p_{2n+1} z^{2n+1}, \qquad \textit{for } z \in \mathbb{R},$$

where we write p_n instead of $p_n(1)$ for simplicity. The corresponding eigenfunctions $f_{N,m}$ are given by

$$f_{N,m}(x) := \operatorname{cp}_{\lambda_{N,m}^{1/2}}(x) = \sum_{n=0}^{\infty} (-1)^n \, \lambda_{N,m}^n \, p_{2n}(x), \quad x \in [0,1].$$

Proof. Using the differentiation rules from Lemma 4.1.6 it is easy to see that cp_z satisfies the eigenvalue equation if $\lambda = z^2$, while it also fulfills the left boundary condition $\operatorname{cp}_z'(0) = -z \operatorname{sp}_z(0) = 0$. In order that cp_z satisfies the right boundary condition, too, z has to be zero itself or it must be chosen such that $\operatorname{sp}_z(1) = 0$. It is known (see Freiberg [17] p.40) that the solution of the above problem is unique up to a multiplicative constant. So z is a zero point of sinp if and only if z^2 is a Neumann eigenvalue of $-\frac{d}{d\mu}\frac{d}{dx}$.

Thus, for $m \in \mathbb{N}_0$, $f_{N,m} = \operatorname{cp}_{\lambda_{N,m}^{1/2}}(x)$ is an eigenfunction to the mth Neumann eigenvalue $\lambda_{N,m}$. □

We treat the Dirichlet eigenvalue problem

$$\frac{d}{d\mu} f' = -\lambda f$$
$$f(0) = f(1) = 0$$

similarly. We denote the Dirichlet eigenvalues such that

$$\lambda_{D,1} < \lambda_{D,2} < \lambda_{D,3} < \cdots$$

where $\lambda_{D,1} > 0$ and $\lim\limits_{n\to\infty} \lambda_{D,n} = \infty$.

Proposition 4.1.8. *The Dirichlet eigenvalues $\lambda_{D,m}$, $m \in \mathbb{N}$, are the squares of the positive zeros of the function* sinq *given by*

$$\operatorname{sinq}(z) := \operatorname{sq}_z(1) = \sum_{n=0}^{\infty} (-1)^n q_{2n+1} z^{2n+1}, \qquad \text{for } z \in \mathbb{R}$$

where, as above, q_n stands for $q_n(1)$. The corresponding eigenfunctions $f_{D,m}$ are given by

$$f_{D,m}(x) = \operatorname{sq}_{\lambda_{D,m}^{1/2}}(x) = \sqrt{\lambda_{D,m}} \sum_{n=0}^{\infty} (-1)^n \, \lambda_{D,m}^n \, q_{2n+1}(x), \quad x \in [0,1].$$

Proof. The function sq_z satisfies the equation if $\lambda = z^2$ and also the left boundary condition $\operatorname{sq}_z(0) = 0$. The right boundary condition gives $\operatorname{sq}_z(1) = 0$. So z^2 is a Dirichlet eigenvalue of $-\frac{d}{d\mu}\frac{d}{dx}$ if and only if z is a zero point of sinq and $z \neq 0$.

Thus, for $m \in \mathbb{N}$, the function $f_{D,m} = \operatorname{sq}_{\lambda_{D,m}^{1/2}}(x)$ is an eigenfunction to the mth Dirichlet eigenvalue $\lambda_{D,m}$. This construction has also been used in Freiberg and Löbus [22]. □

Remark 4.1.9. An eigenfunction is only unique up to a multiplicative constant. Throughout the chapter we will use the notations $f_{N,m}$ and $f_{D,m}$ for the eigenfunctions as constructed above. One would also get these by imposing the additional conditions $f_{N,m}(0) = 1$ and $f'_{D,m}(0) = \sqrt{\lambda_{D,m}}$.

So if we only know the sequences $\big(p_n(1)\big)_n$ and $\big(q_n(1)\big)_n$, we can determine the Neumann and Dirichlet eigenvalues by means of the functions sinp and sinq.

Analogously to sinp and sinq we define

$$\operatorname{cosp}(z) := \operatorname{cp}_z(1) = \sum_{n=0}^{\infty} (-1)^n p_{2n} z^{2n}$$

and

$$\operatorname{cosq}(z) := \operatorname{cq}_z(1) = \sum_{n=0}^{\infty} (-1)^n q_{2n} z^{2n}$$

for $z \in \mathbb{R}$.

These functions are linked with the eigenvalue problems with mixed boundary conditions

$$(ND) \qquad \begin{gathered} \frac{d}{d\mu} f' = -\lambda f \\ f'(0) = 0, \quad f(1) = 0, \end{gathered}$$

and

$$(DN) \qquad \begin{gathered} \frac{d}{d\mu} f' = -\lambda f \\ f(0) = 0, \quad f'(1) = 0. \end{gathered}$$

We treat these problems as the problems in the above Propositions 4.1.7 and 4.1.8. If $\operatorname{cosp}\big(\sqrt{\lambda}\big) = 0$, the solutions to (ND) are multiples of $\mathrm{cp}_{\sqrt{\lambda}}$, because

$$\mathrm{cp}'_{\sqrt{\lambda}}(0) = -\sqrt{\lambda}\,\mathrm{sp}_{\sqrt{\lambda}}(0) = 0$$

and

$$\mathrm{cp}_{\sqrt{\lambda}}(1) = \operatorname{cosp}\big(\sqrt{\lambda}\big).$$

Similarly, if $\operatorname{cosq}\big(\sqrt{\lambda}\big) = 0$, the solutions to (DN) are multiples of $\mathrm{sq}_{\sqrt{\lambda}}$, because

$$\mathrm{sq}_{\sqrt{\lambda}}(0) = 0$$

and

$$\mathrm{sq}'_{\sqrt{\lambda}}(1) = \sqrt{\lambda}\,\mathrm{cq}_{\sqrt{\lambda}}(1) = \sqrt{\lambda}\operatorname{cosq}\big(\sqrt{\lambda}\big).$$

Therefore, the (ND) eigenvalues are the squares of the zeros of cosp and the (DN) eigenvalues are the squares of the zeros of cosq.

4.2. Calculation of L_2-norms

In the course of the following sections we will often use some of the following easy to prove multiplication formulas which we state here for easy reference. For absolutely summable sequences $(a_n)_n$ and $(b_n)_n$ holds

$$\left(\sum_{j=0}^{\infty} a_{2j}\right) \cdot \left(\sum_{k=0}^{\infty} b_{2k}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{2k}\, b_{2n-2k}, \tag{4.1}$$

$$\left(\sum_{j=0}^{\infty} a_{2j}\right) \cdot \left(\sum_{k=0}^{\infty} b_{2k+1}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{2k}\, b_{2n+1-2k}, \tag{4.2}$$

$$\left(\sum_{j=0}^{\infty} a_{2j+1}\right) \cdot \left(\sum_{k=0}^{\infty} b_{2k+1}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{2k+1}\, b_{2n+1-2k}. \tag{4.3}$$

In this section we consider again an arbitrary atomless Borel probability measure μ on $[0,1]$. The following useful lemma is an analogue to integration by parts and can be found in Freiberg [17] as Proposition 3.1.

Lemma 4.2.1. *For $c, d \in [0,1]$ with $c < d$ and functions $f \in H^1(\mu)$ and $g \in H^1(\lambda)$ we have*

$$\int_c^d \frac{df}{d\mu}(t)\, g(t)\, d\mu(t) = f\, g\Big|_c^d - \int_c^d f(t)\, g'(t)\, dt.$$

We will develop some properties of $p_n(x)$ and $q_n(x)$ and therefore of the eigenfunctions in the Neumann and Dirichlet case. In this section, μ is an arbitrary atomless Borel probability measure on $[0,1]$.

It turns out that by knowing the sequences $(p_n)_n$ and $(q_n)_n$ we can not only determine the Neumann and Dirichlet eigenvalues, but also the $L_2(\mu)$-norms of the eigenfunctions $f_{N,m}$ and $f_{D,m}$. We will need the following lemma to achieve that.

Lemma 4.2.2. *For $k, n \in \mathbb{N}_0$ with $k \leq n$ and for all $x \in [0,1]$ we have*

$$\int_0^x p_{2k}(t)\, p_{2n-2k}(t)\, d\mu(t) = \sum_{j=0}^{2k} (-1)^j p_j(x)\, p_{2n+1-j}(x) \tag{4.4}$$

and

$$\int_0^x q_{2k+1}(t)\, q_{2n+1-2k}(t)\, d\mu(t) = \sum_{j=0}^{2k+1} (-1)^{j+1} q_j(x)\, q_{2n+3-j}(x). \tag{4.5}$$

Proof. We prove (4.4) by induction on k. If $k = 0$ and $n \geq 0$, we have

$$\int_0^x p_0(t)\, p_{2n}(t)\, d\mu(t) = p_{2n+1}(x)$$

and so the assertion holds. Now, take $k \in \mathbb{N}_0$ and assume that the assertion holds for k and all $n \geq k$. Then, for all $n \geq k+1$,

$$\begin{aligned}\int_0^x p_{2k+2}(t)\, p_{2n-2k-2}(t)\, d\mu(t) &= p_{2k+2}(x)\, p_{2n-2k-1}(x) - \int_0^x p_{2k+1}(t)\, p_{2n-2k-1}(t)\, dt \\ &= p_{2k+2}(x)\, p_{2n-2k-1}(x) - p_{2k+1}(x)\, p_{2n-2k}(x) + \int_0^x p_{2k}(t)\, p_{2n-2k}(t)\, d\mu(t),\end{aligned}$$

by Lemma 4.2.1. Thus, by the induction hypothesis, we have for all $n \geq k+1$

$$\int_0^x p_{2k+2}(t)\, p_{2n-2k-2}(t)\, d\mu(t) = \sum_{j=0}^{2k+2} (-1)^j p_j(x)\, p_{2n+1-j}(x),$$

which proves (4.4).

The proof of (4.5) works the same way, first, let $k = 0$ and $n \geq 0$. Then, by Lemma 4.2.1,

$$\int_0^x q_1(t)\, q_{2n+1}(t)\, d\mu(t) = q_1(x)\, q_{2n+2}(x) - q_{2n+3}(x)$$

which is the induction basis. Now, let $k \in \mathbb{N}_0$ and assume that the assertion holds for k and all $n \geq k$. Then, for all $n \geq k+1$,

$$\begin{aligned}\int_0^x q_{2k+3}(t)\, q_{2n-2k-1}(t)\, d\mu(t) &= q_{2k+3}(x)\, q_{2n-2k}(x) - \int_0^x q_{2k+2}(t)\, q_{2n-2k}(t)\, dt \\ &= q_{2k+3}(x)\, q_{2n-2k}(x) - q_{2k+2}(x)\, q_{2n-2k+1}(x) + \int_0^x q_{2k+1}(t)\, q_{2n-2k+1}(t)\, d\mu(t),\end{aligned}$$

again by Lemma 4.2.1. Thus, by the induction hypothesis, we have for all $n \geq k+1$

$$\int_0^x q_{2k+3}(t)\, q_{2n-2k-1}(t)\, d\mu(t) = \sum_{j=0}^{2k+3} (-1)^{j+1} q_j(x)\, q_{2n+3-j}(x).$$

□

Proposition 4.2.3. *Let $z \in \mathbb{R}$. Then*

$$\|\mathrm{cp}_z\|^2_{L_2(\mu)} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \sum_{k=0}^{n} (n+1-2k)\, p_{2k}\, p_{2n+1-2k}, \tag{4.6}$$

and

$$\|\mathrm{sq}_z\|^2_{L_2(\mu)} = \sum_{n=0}^{\infty} (-1)^n z^{2n+2} \sum_{k=0}^{n+1} (n+1-2k)\, q_{2k+1}\, q_{2n+2-2k}, \tag{4.7}$$

where $p_j = p_j(1)$ *and* $q_j = q_j(1)$.

Proof. First we prove (4.6). Using (4.1) we get for all $x \in [0,1]$ and $z \in \mathbb{R}$

$$\begin{aligned}
\mathrm{cp}_z(x)^2 &= \Big(\sum_{j=0}^{\infty}(-1)^j z^{2j} p_{2j}(x)\Big)\Big(\sum_{k=0}^{\infty}(-1)^k z^{2k} p_{2k}(x)\Big) \\
&= \sum_{n=0}^{\infty}(-1)^n\, z^{2n} \sum_{k=0}^{n} p_{2k}(x)\, p_{2n-2k}(x).
\end{aligned}$$

Consequently, applying (4.4),

$$\begin{aligned}
\|\mathrm{cp}_z\|^2_{L_2(\mu)} &= \int_0^1 \mathrm{cp}_z(t)^2\, d\mu(t) \\
&= \sum_{n=0}^{\infty}(-1)^n\, z^{2n} \sum_{k=0}^{n} \int_0^1 p_{2k}(t)\, p_{2n-2k}(t)\, d\mu(t) \\
&= \sum_{n=0}^{\infty}(-1)^n\, z^{2n} \sum_{k=0}^{n} \sum_{j=0}^{2k}(-1)^j p_j\, p_{2n+1-j}.
\end{aligned}$$

Note, that for any sequence $a = (a_j)_{j\in\mathbb{N}_0}$ holds

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{j=0}^{2k} a_j &= \sum_{k=0}^{n}\sum_{j=0}^{k} a_{2j} + \sum_{k=1}^{n}\sum_{j=0}^{k-1} a_{2j+1} \\
&= \sum_{j=0}^{n}\sum_{k=j}^{n} a_{2j} + \sum_{j=0}^{n-1}\sum_{k=j+1}^{n} a_{2j+1} \\
&= \sum_{j=0}^{n}(n-j+1)\, a_{2j} + \sum_{j=1}^{n}(n-j+1)\, a_{2j-1}
\end{aligned}$$

and thus,

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{j=0}^{2k}(-1)^j p_j\, p_{2n+1-j} &= \sum_{k=0}^{n}(n-k+1)\, p_{2k}\, p_{2n+1-2k} - \sum_{k=1}^{n}(n-k+1)\, p_{2k-1}\, p_{2n+2-2k} \\
&= \sum_{k=0}^{n}(n-k+1)\, p_{2k}\, p_{2n+1-2k} - \sum_{k=1}^{n} k\, p_{2k}\, p_{2n+1-2k} \\
&= \sum_{k=0}^{n}(n+1-2k)\, p_{2k}\, p_{2n+1-2k},
\end{aligned}$$

which proves (4.6).

Now we show (4.7), which works similarly. We have

$$\begin{aligned}
\mathrm{sq}_z(x)^2 &= \Big(\sum_{j=0}^{\infty}(-1)^j z^{2j+1} q_{2j+1}(x)\Big)\Big(\sum_{k=0}^{\infty}(-1)^k z^{2k+1} q_{2k+1}(x)\Big) \\
&= \sum_{n=0}^{\infty}(-1)^n\, z^{2n+2} \sum_{k=0}^{n} q_{2k+1}(x)\, q_{2n+1-2k}(x).
\end{aligned}$$

Therefore, by (4.5),

$$\begin{aligned}
\|\mathrm{sq}_z\|^2_{L_2(\mu)} &= \int_0^1 \mathrm{sq}_z(t)^2\, d\mu(t) \\
&= \sum_{n=0}^{\infty}(-1)^n\, z^{2n+2} \sum_{k=0}^{n} \int_0^1 q_{2k+1}(t)\, q_{2n+1-2k}(t)\, d\mu(t) \\
&= \sum_{n=0}^{\infty}(-1)^n\, z^{2n+2} \sum_{k=0}^{n} \sum_{j=0}^{2k+1} (-1)^{j+1} q_j\, q_{2n+3-j}.
\end{aligned}$$

For any sequence $a = (a_j)_{j\in\mathbb{N}_0}$ holds

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{j=0}^{2k+1} a_j &= \sum_{k=0}^{n}\sum_{j=0}^{k} a_{2j} + \sum_{k=0}^{n}\sum_{j=0}^{k} a_{2j+1} \\
&= \sum_{j=0}^{n}\sum_{k=j}^{n} a_{2j} + \sum_{j=0}^{n}\sum_{k=j}^{n} a_{2j+1} \\
&= \sum_{j=0}^{n}(n-j+1)\, a_{2j} + \sum_{j=0}^{n}(n-j+1)\, a_{2j+1}
\end{aligned}$$

and thus,

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{j=0}^{2k+1}(-1)^{j+1} q_j\, q_{2n+3-j} &= -\sum_{k=0}^{n}(n-k+1)\, q_{2k}\, q_{2n+3-2k} + \sum_{k=0}^{n}(n-k+1)\, q_{2k+1}\, q_{2n+2-2k} \\
&= -\sum_{k=1}^{n+1} k\, q_{2k+1}\, q_{2n+2-2k} + \sum_{k=0}^{n}(n-k+1)\, q_{2k+1}\, q_{2n+2-2k} \\
&= \sum_{k=0}^{n+1}(n+1-2k)\, q_{2k+1}\, q_{2n+2-2k},
\end{aligned}$$

which proves (4.7). □

Corollary 4.2.4. *The $L_2(\mu)$-norm of the Neumann eigenfunction $f_{N,m}$ is given by*

$$\|f_{N,m}\|_{L_2(\mu)}^2 = \sum_{n=0}^{\infty}(-1)^n \lambda_{N,m}^n \sum_{k=0}^{n}(n+1-2k)\, p_{2k}\, p_{2n+1-2k}$$

and of the Dirichlet eigenfunction $f_{D,m}$ by

$$\|f_{D,m}\|_{L_2(\mu)}^2 = \sum_{n=0}^{\infty}(-1)^n \lambda_{D,m}^{n+1} \sum_{k=0}^{n+1}(n+1-2k)\, q_{2k+1}\, q_{2n+2-2k}.$$

4.3. A trigonometric identity

As in the previous section, we consider an atomless Borel probability measure μ on $[0,1]$. We prove a formula that links the functions $\mathrm{cp}_z, \mathrm{cq}_z, \mathrm{sp}_z$, and sq_z generalizing the trigonometric identity $\sin^2 + \cos^2 = 1$. For this we need the following lemma.

Lemma 4.3.1. *For $k, n \in \mathbb{N}$ with $k \leq n$ and for all $x \in [0,1]$ we have*

$$\int_0^x q_{2k-1}(t)\, p_{2n-2k}(t)\, d\mu(t) = \sum_{j=0}^{2k-1}(-1)^{j+1} q_j(x)\, p_{2n-j}(x).$$

Proof. We prove this by induction on k. For $k = 1$ and $n \geq 1$, we get by Lemma 4.2.1

$$\int_0^x q_1(t)\, p_{2n-2}(t)\, d\mu(t) = q_1(x)\, p_{2n-1}(x) - \int_0^x p_{2n-1}(t)\, dt = q_1(x)\, p_{2n-1}(x) - p_{2n}(x),$$

and so the assertion holds. Now, take $k \in \mathbb{N}$, and assume that the assertion holds for k and all $n \geq k$. Then, again by using Lemma 4.2.1, we get

$$\begin{aligned}\int_0^x q_{2k+1}(t)\, p_{2n-2k-2}(t)\, d\mu(t) &= q_{2k+1}(x)\, p_{2n-2k-1}(x) - \int_0^x q_{2k}(t)\, p_{2n-2k-1}(t)\, dt \\ &= q_{2k+1}(x)\, p_{2n-2k-1}(x) - q_{2k}(x)\, p_{2n-2k}(x) + \int_0^x q_{2k-1}(t)\, p_{2n-2k}(t)\, d\mu(t).\end{aligned}$$

Thus, by the induction hypothesis, for all $n \geq k+1$,

$$\int_0^x q_{2k+1}(t)\, p_{2n-2k-2}(t)\, d\mu(t) = \sum_{j=0}^{2k+1}(-1)^{j+1} q_j(x)\, p_{2n-j}(x),$$

which finishes the proof. $\square$

Corollary 4.3.2. *If we set $n = k$ in Lemma 4.3.1, we get the formula*

$$\sum_{j=0}^{2n} (-1)^j \, q_j(x) \, p_{2n-j}(x) = 0,$$

which holds for all $n \in \mathbb{N}$ and $x \in [0, 1]$.

With the above corollary we can prove the following theorem.

Theorem 4.3.3. *For all $x \in [0, 1]$ and $z \in \mathbb{R}$ holds*

$$\mathrm{cq}_z(x) \ \mathrm{cp}_z(x) + \mathrm{sq}_z(x) \ \mathrm{sp}_z(x) = 1.$$

Proof. Take $x \in [0, 1]$ and $z \in \mathbb{R}$. Then, by Corollary 4.3.2,

$$\begin{aligned}
&\mathrm{cq}_z(x) \ \mathrm{cp}_z(x) + \mathrm{sq}_z(x) \ \mathrm{sp}_z(x) \\
&= \sum_{n=0}^{\infty} (-1)^n z^{2n} \sum_{k=0}^{n} q_{2k}(x) \, p_{2n-2k}(x) + \sum_{n=0}^{\infty} (-1)^n z^{2n+2} \sum_{k=0}^{n} q_{2k+1}(x) \, p_{2n+1-2k}(x) \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n z^{2n} \left[\sum_{k=0}^{n} q_{2k}(x) \, p_{2n-2k}(x) - \sum_{k=0}^{n-1} q_{2k+1}(x) \, p_{2n-(2k+1)}(x) \right] \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n z^{2n} \sum_{k=0}^{2n} (-1)^k \, q_k(x) \, p_{2n-k}(x) \\
&= 1.
\end{aligned}$$

$\square$

4.4. Symmetric measures

In this section we consider symmetric measures μ on $[0,1]$, meaning that, additionally to being an atomless Borel probability measure, μ shall satisfy

$$\mu\big([0,x]\big) = \mu\big([1-x,1]\big)$$

for all $x \in [0,1]$.

Proposition 4.4.1. *Let μ be symmetric and let $x \in [0,1]$. Then, for $n \in \mathbb{N}_0$ holds*

$$p_{2n+1}(x) = \sum_{k=0}^{n} p_{2k+1}\, q_{2n-2k}(x) - \sum_{k=1}^{n} p_{2k}\, p_{2n-2k+1}(x) - p_{2n+1}(1-x), \tag{4.8}$$

and for $n \in \mathbb{N}$

$$p_{2n}(x) = \sum_{k=0}^{n-1} p_{2k+1}\, q_{2n-2k-1}(x) - \sum_{k=1}^{n} p_{2k}\, p_{2n-2k}(x) + p_{2n}(1-x). \tag{4.9}$$

Proof. For $p_1(x)$ the formula reduces to $p_1(x) = p_1 - p_1(1-x)$. This holds since

$$p_1(x) = \mu\big([0,x]\big) = \mu\big([1-x,1]\big) = \int_0^1 d\mu - \int_0^{1-x} d\mu = p_1 - p_1(1-x).$$

Assume $p_{2n+1}(x)$ satisfies the above formula for some $n \in \mathbb{N}_0$. Then

$$\begin{aligned}
p_{2n+2}(x) &= \int_0^x p_{2n+1}(t)\, dt \\
&= \sum_{k=0}^{n} p_{2k+1} \int_0^x q_{2n-2k}(t)\, dt - \sum_{k=1}^{n} p_{2k} \int_0^x p_{2n-2k+1}(t)\, dt - \int_0^x p_{2n+1}(1-t)\, dt \\
&= \sum_{k=0}^{n} p_{2k+1}\, q_{2n-2k+1}(x) - \sum_{k=1}^{n} p_{2k}\, p_{2n-2k+2}(x) - \int_{1-x}^1 p_{2n+1}(t)\, dt \\
&= \sum_{k=0}^{n} p_{2k+1}\, q_{2n-2k+1}(x) - \sum_{k=1}^{n} p_{2k}\, p_{2n-2k+2}(x) - p_{2n+2}(1) + p_{2n+2}(1-x) \\
&= \sum_{k=0}^{n} p_{2k+1}\, q_{2n-2k+1}(x) - \sum_{k=1}^{n+1} p_{2k}\, p_{2n-2k+2}(x) + p_{2n+2}(1-x).
\end{aligned}$$

Now, let the assertion be true for some $2n$, $n \in \mathbb{N}$. Since μ is symmetric, we have that $d\mu(t) = d\mu(1-t)$. Thus,

$$\begin{aligned}
p_{2n+1}(x) &= \int_0^x p_{2n}(t)\, d\mu(t) \\
&= \sum_{k=0}^{n-1} p_{2k+1} \int_0^x q_{2n-2k-1}(t)\, d\mu(t) - \sum_{k=1}^{n} p_{2k} \int_0^x p_{2n-2k}(t)\, d\mu(t) + \int_0^x p_{2n}(1-t)\, d\mu(t) \\
&= \sum_{k=0}^{n-1} p_{2k+1}\, q_{2n-2k}(x) - \sum_{k=1}^{n} p_{2k}\, p_{2n-2k+1}(x) + \int_{1-x}^1 p_{2n}(t)\, d\mu(t) \\
&= \sum_{k=0}^{n-1} p_{2k+1}\, q_{2n-2k}(x) - \sum_{k=1}^{n} p_{2k}\, p_{2n-2k+1}(x) + p_{2n+1}(1) - p_{2n+1}(1-x) \\
&= \sum_{k=0}^{n} p_{2k+1}\, q_{2n-2k}(x) - \sum_{k=1}^{n} p_{2k}\, p_{2n-2k+1}(x) - p_{2n+1}(1-x).
\end{aligned}$$

□

Corollary 4.4.2. *Let μ be symmetric. Then, for $n \in \mathbb{N}$,*

$$\sum_{k=0}^{n} p_{2k}\, p_{2n-2k+1} = \sum_{k=0}^{n} p_{2k+1}\, q_{2n-2k}. \tag{4.10}$$

Proof. This follows from Proposition 4.4.1 by putting $x = 1$ in (4.8). □

Remark 4.4.3. In the special case where μ is the Lebesgue measure, the above formulas reduce to $\sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Corollary 4.4.4. *Let μ be symmetric. Then the following statements hold.*

(i) $p_{2n} = q_{2n}$ *for all $n \in \mathbb{N}$.*

(ii) $\mathrm{cosp}(z) = \mathrm{cosq}(z)$ *for all $z \in \mathbb{R}$.*

(iii) $\mathrm{cosp}^2(z) + \mathrm{sinp}(z)\,\mathrm{sinq}(z) = 1$ *for all $z \in \mathbb{R}$.*

(iv) *We have the recursion formula*

$$p_{2n} = \frac{1}{2} \sum_{k=1}^{2n-1} (-1)^{k+1}\, p_k\, q_{2n-k}. \tag{4.11}$$

Proof. We prove (i) by induction. By putting $n = 1$ in (4.10), we find that

$$p_3 + p_2 p_1 = p_1 q_2 + p_3,$$

which implies $p_2 = q_2$. Assume that $p_{2k} = q_{2k}$ for all k smaller than some $n \in \mathbb{N}$, $n \geq 2$. We reverse the order of the summands in the second sum of (4.10) to get

$$\sum_{k=0}^{n-1} p_{2k}\, p_{2n-2k+1} + p_{2n}\, p_1 = \sum_{k=0}^{n-1} p_{2n-2k+1}\, q_{2k} + p_1\, q_{2n}.$$

Now it follows from the induction hypothesis that $p_{2n} = q_{2n}$. Then, (ii) follows immediately and by Proposition 4.3.3 also (iii).

Clearly, (iv) follows from (i) and Corollary 4.3.2. □

Proposition 4.4.5. *Let μ be symmetric. Then, for all $z \in \mathbb{R}$ and $x \in [0,1]$,*

$$\mathrm{cp}_z(1-x) = \mathrm{cosp}(z)\,\mathrm{cp}_z(x) + \mathrm{sinp}(z)\,\mathrm{sq}_z(x).$$

Proof. Rearranging (4.9) gives

$$p_{2n}(1-x) = \sum_{k=0}^{n} p_{2k}\, p_{2n-2k}(x) - \sum_{k=0}^{n-1} p_{2k+1}\, q_{2n-2k-1}(x).$$

We multiply the equation with $(-1)^n z^{2n}$ and sum from $n = 0$ to infinity to get

$$\begin{aligned}
\mathrm{cp}_z(1-x) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} (iz)^{2k} p_{2k} \cdot (iz)^{2n-2k} p_{2n-2k}(x) \\
&\qquad - \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} (iz)^{2k+1} p_{2k+1} \cdot (iz)^{2n-2k-1} q_{2n-2k-1}(x) \\
&= \sum_{n=0}^{\infty} (-1)^n z^{2n} p_{2n} \cdot \sum_{k=0}^{\infty} (-1)^k z^{2k} p_{2k}(x) \\
&\qquad + \sum_{n=1}^{\infty} (-1)^n z^{2n+1} p_{2n+1} \cdot \sum_{k=0}^{\infty} (-1)^k z^{2k+1} q_{2k+1}(x) \\
&= \mathrm{cosp}(z)\,\mathrm{cp}_z(x) + \mathrm{sinp}(z)\,\mathrm{sq}_z(x).
\end{aligned}$$

□

Corollary 4.4.6. *Let μ be symmetric. Then the Neumann eigenfunctions $f_{N,m}$ are either symmetric or antisymmetric, that is, either*

$$f_{N,m}(x) = f_{N,m}(1-x) \qquad \text{or} \qquad f_{N,m}(x) = -f_{N,m}(1-x)$$

for all $x \in [0,1]$.

Proof. Let z^2 be a Neumann eigenvalue. Then, by Proposition 4.1.7, $\operatorname{sinp}(z) = 0$ and hence, by Corollary 4.4.4 (iii), $|\operatorname{cosp}(z)| = 1$. Thus, by Proposition 4.4.5, we get

$$\mathrm{cp}_z(1-x) = \pm\,\mathrm{cp}_z(x).$$

Since $\mathrm{cp}_z = f_{N,m}$ for $z^2 = \lambda_m$ the corollary is proved. □

Analogous to (4.8) there is a formula relating $q_{2n+1}(x)$ to $q_{2n+1}(1-x)$, namely

$$q_{2n+1}(x) = \sum_{k=0}^{n} q_{2k+1}\, p_{2n-2k}(x) - \sum_{k=1}^{n} q_{2k}\, q_{2n+2k+1}(x) - q_{2n+1}(1-x). \tag{4.12}$$

The proof is exactly like the proof of Proposition 4.4.1. As in the proof of Proposition 4.4.5, we rearrange, multiply with $(-1)^n z^{2n+1}$, and sum up to get

$$\mathrm{sq}_z(1-x) = \operatorname{sinq}(z)\,\mathrm{cp}_z(x) - \operatorname{cosq}(z)\,\mathrm{sq}_z(x).$$

If now z^2 is a Dirichlet eigenvalue, then $\operatorname{sinq}(z) = 0$ and $\operatorname{cosq}(z) = \operatorname{cosp}(z) = \pm 1$ and it follows that

$$\mathrm{sq}_z(1-x) = \mp\,\mathrm{sq}_z(x).$$

Thus, we have the following proposition.

Proposition 4.4.7. *Let μ be symmetric. Then the Dirichlet eigenfunctions $f_{D,m}$ are either symmetric or antisymmetric, that is, either*

$$f_{D,m}(x) = f_{D,m}(1-x) \qquad \text{or} \qquad f_{D,m}(x) = -f_{D,m}(1-x)$$

for all $x \in [0,1]$.

4.5. Self-similar measures

In this section we impose that the measure μ has a self-similar structure. For definitions of the concept of iterated function systems and self-similar measures, see Section B.2. For reasons of simplicity, we take an IFS consisting only of two mappings, but it does not raise considerable problems to generalize this to an arbitrary number.

Let r_1, r_2, m_1 and m_2 be positive numbers satisfying $r_1+r_2 \le 1$ and $m_1+m_2 = 1$. Let $\mathcal{S} = (S_1, S_2)$ be the IFS given by

$$S_1(x) = r_1 x \qquad \text{and} \qquad S_2(x) = r_2 x + 1 - r_2, \qquad x \in [0,1].$$

By K we denote the invariant set of $\mathcal{S}$ and by μ its invariant measure with vector of weights (m_1, m_2).

In this case we are able to prove several properties of the functions $p_n(x)$ and $q_n(x)$ that resemble corresponding ones of $\frac{x^n}{n!}$. These we will employ to examine the Neumann and Dirichlet eigenfunctions and eigenvalues of $-\frac{d}{d\mu}\frac{d}{dx}$. In particular, we will develop a recursion law for $p_n(1)$ and $q_n(1)$.

The self-similar structure of the measure can be used in integral transformations to receive derivation rules like the following.

Lemma 4.5.1. *Let $F \in H^1(\mu)$ and $f = \frac{dF}{d\mu}$. Then*

$$\frac{d}{d\mu} F(r_1 x) = m_1 f(r_1 x)$$

and

$$\frac{d}{d\mu} F(1 - r_2 + r_2 x) = m_2 f(1 - r_2 + r_2 x).$$

Proof. Since $F \in H^1(\mu)$, it can be written as

$$F(r_1 x) = F(0) + \int_0^{r_1 x} f(t)\, d\mu(t).$$

The measure μ is invariant with respect to S_1 and S_2 which means that

$$\mu = m_1(S_1\mu) + m_2(S_2\mu).$$

Consequently, if restricted to $[0, r_1]$, we have

$$\mu = m_1(S_1\mu),$$

and hence

$$F(r_1 x) = F(0) + \int_0^{r_1 x} m_1 f(t)\, d(S_1\mu)(t) = F(0) + \int_0^{x} m_1 f(r_1 t)\, d\mu(t).$$

Thus, the first assertion follows.

Analogously, it follows that on $[1 - r_2, 1]$ we have

$$\mu = m_2(S_2\mu)$$

and thus,

$$\begin{aligned} F(1 - r_2 + r_2x) &= F(0) + \int_0^{1-r_2} f(t)\,d\mu(t) + \int_{1-r_2}^{1-r_2+r_2x} f(t)\,d\mu(t) \\ &= F(1 - r_2) + \int_{1-r_2}^{1-r_2+r_2x} m_2 f(t)\,d(S_2\mu)(t) \\ &= F(1 - r_2) + \int_0^x m_2 f(1 - r_2 + r_2t)\,d\mu(t), \end{aligned}$$

which proves the second assertion. □

In the following proposition we present a formula that can be viewed as an analogue of the binomial theorem, adapted to the self-similar measure μ. It relates the left part, contained in $[0, r_1]$, to the right part, contained in $[1 - r_2, 1]$.

Proposition 4.5.2. *For $x \in [0, 1]$ and $n \in \mathbb{N}_0$,*

$$\begin{aligned} p_{2n+1}(1 - r_2 + r_2x) &= \sum_{i=0}^{n} p_{2i+1}(r_1)\left(\tfrac{r_2m_2}{r_1m_1}\right)^{n-i} q_{2n-2i}(r_1x) \\ &\quad + \sum_{i=0}^{n} p_{2i}(r_1)\left(\tfrac{r_2}{r_1}\right)^{n-i}\left(\tfrac{m_2}{m_1}\right)^{n-i+1} p_{2n-2i+1}(r_1x) \\ &\quad + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} p_{2i+1}(r_1)\left(\tfrac{r_2}{r_1}\right)^{n-i-1}\left(\tfrac{m_2}{m_1}\right)^{n-i} p_{2n-2i-1}(r_1x), \end{aligned} \tag{4.13}$$

where a sum from 0 to -1 is regarded as zero, and, for $n \in \mathbb{N}$,

$$\begin{aligned} p_{2n}(1 - r_2 + r_2x) &= \sum_{i=0}^{n} p_{2i}(r_1)\left(\tfrac{r_2m_2}{r_1m_1}\right)^{n-i} p_{2n-2i}(r_1x) \\ &\quad + \sum_{i=0}^{n-1} p_{2i+1}(r_1)\left(\tfrac{r_2}{r_1}\right)^{n-i}\left(\tfrac{m_2}{m_1}\right)^{n-i-1} q_{2n-2i-1}(r_1x) \\ &\quad + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} p_{2i+1}(r_1)\left(\tfrac{r_2m_2}{r_1m_1}\right)^{n-i-1} p_{2n-2i-2}(r_1x). \end{aligned} \tag{4.14}$$

Remark 4.5.3. If $r_1 = m_1$ and $r_2 = m_2$ and $r_1 + r_2 = 1$ (and hence, μ is the Lebesgue measure), the above formulas reduce to

$$\Big(r_1 + r_2 x\Big)^n = \sum_{i=0}^{n} \binom{n}{i} r_1^i \, (r_2 x)^{n-i}, \qquad n \in \mathbb{N}.$$

Proof. We prove the proposition by induction. As seen in the proof of Lemma 4.5.1 we have $\mu = m_1(S_1\mu)$ on $[0, r_1]$ and $\mu = m_2(S_2\mu)$ on $[1 - r_2, 1]$. Therefore,

$$\begin{aligned}
p_1(1 - r_2 + r_2 x) &= \int_0^{1-r_2+r_2x} d\mu = \int_0^{r_1} d\mu + \int_{1-r_2}^{1-r_2+r_2x} d\mu \\
&= p_1(r_1) + m_2 \int_{1-r_2}^{1-r_2+r_2x} d(S_2\mu) = p_1(r_1) + m_2 \int_0^{x} d\mu \\
&= p_1(r_1) + m_2 \int_0^{r_1 x} d(S_1\mu) = p_1(r_1) + \frac{m_2}{m_1} \int_0^{r_1 x} d\mu \\
&= p_1(r_1) + \frac{m_2}{m_1} p_1(r_1 x),
\end{aligned}$$

which proves the assertion for p_1.

Assume that the formula for p_{2n+1} holds for some $n \in \mathbb{N}_0$. Then

$$\begin{aligned}
p_{2n+2}(1 - r_2 + r_2 x) &= \int_0^{r_1} p_{2n+1}(t)\, dt + \int_{r_1}^{1-r_2} p_{2n+1}(t)\, dt + \int_{1-r_2}^{1-r_2+r_2x} p_{2n+1}(t)\, dt \\
&= p_{2n+2}(r_1) + [1 - (r_1 + r_2)] p_{2n+1}(r_1) + r_2 \int_0^{x} p_{2n+1}(1 - r_2 + r_2 t)\, dt.
\end{aligned}$$

Applying the induction hypothesis, we get

$$\begin{aligned}
p_{2n+2}(1 - r_2 + r_2 x) &= p_{2n+2}(r_1) + [1 - (r_1 + r_2)] p_{2n+1}(r_1) \\
&\quad + \sum_{i=0}^{n} p_{2i+1}(r_1) (\tfrac{r_2 m_2}{r_1 m_1})^{n-i} r_2 \int_0^x q_{2n-2i}(r_1 t)\, dt \\
&\quad + \sum_{i=0}^{n} p_{2i}(r_1) (\tfrac{r_2}{r_1})^{n-i} (\tfrac{m_2}{m_1})^{n-i+1} r_2 \int_0^x p_{2n-2i+1}(r_1 t)\, dt \\
&\quad + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} p_{2i+1}(r_1) (\tfrac{r_2}{r_1})^{n-i-1} (\tfrac{m_2}{m_1})^{n-i} r_2 \int_0^x p_{2n-2i-1}(r_1 t)\, dt
\end{aligned}$$

$$
\begin{aligned}
&= p_{2n+2}(r_1) + [1-(r_1+r_2)]p_{2n+1}(r_1) \\
&\qquad + \sum_{i=0}^{n} p_{2i+1}(r_1)(\tfrac{r_2}{r_1})^{n-i+1}(\tfrac{m_2}{m_1})^{n-i} q_{2n-2i+1}(r_1x) \\
&\qquad + \sum_{i=0}^{n} p_{2i}(r_1)(\tfrac{r_2m_2}{r_1m_1})^{n-i+1} p_{2n-2i+2}(r_1x) \\
&\qquad + [1-(r_1+r_2)] \sum_{i=0}^{n-1} p_{2i+1}(r_1)(\tfrac{r_2m_2}{r_1m_1})^{n-i} p_{2n-2i}(r_1x) \\
&= \sum_{i=0}^{n+1} p_{2i}(r_1)(\tfrac{r_2m_2}{r_1m_1})^{n-i+1} p_{2n-2i+2}(r_1x) + \sum_{i=0}^{n} p_{2i+1}(r_1)(\tfrac{r_2}{r_1})^{n-i+1}(\tfrac{m_2}{m_1})^{n-i} q_{2n-2i+1}(r_1x) \\
&\quad + [1-(r_1+r_2)] \sum_{i=0}^{n} p_{2i+1}(r_1)(\tfrac{r_2m_2}{r_1m_1})^{n-i} p_{2n-2i}(r_1x),
\end{aligned}
$$

which is the formula for p_{2n+2}.

Furthermore, suppose that the assertion is true for p_{2n} for some $n \in \mathbb{N}$. Then, transforming μ as in the proof of the initial step and applying the induction hypothesis in the same way as above,

$$
\begin{aligned}
&p_{2n+1}(1-r_2+r_2x) = \int_0^{r_1} p_{2n}(t)\,d\mu(t) + \int_{r_1}^{1-r_2} p_{2n}(t)\,d\mu(t) + \int_{1-r_2}^{1-r_2+r_2x} p_{2n}(t)\,d\mu(t) \\
&= p_{2n+1}(r_1) + m_2 \int_0^{x} p_{2n}(1-r_2+r_2t)\,d\mu(t) \\
&= p_{2n+1}(r_1) + \sum_{i=0}^{n} p_{2i}(r_1)(\tfrac{r_2}{r_1})^{n-i}(\tfrac{m_2}{m_1})^{n-i+1} p_{2n-2i+1}(r_1x) + \sum_{i=0}^{n-1} p_{2i+1}(r_1)(\tfrac{r_2m_2}{r_1m_1})^{n-i} q_{2n-2i}(r_1x) \\
&\quad + [1-(r_1+r_2)] \sum_{i=0}^{n-1} p_{2i+1}(r_1)(\tfrac{r_2}{r_1})^{n-i-1}(\tfrac{m_2}{m_1})^{n-i} p_{2n-2i-1}(r_1x) \\
&= \sum_{i=0}^{n} p_{2i+1}(r_1)(\tfrac{r_2m_2}{r_1m_1})^{n-i} q_{2n-2i}(r_1x) + \sum_{i=0}^{n} p_{2i}(r_1)(\tfrac{r_2}{r_1})^{n-i}(\tfrac{m_2}{m_1})^{n-i+1} p_{2n-2i+1}(r_1x) \\
&\quad + [1-(r_1+r_2)] \sum_{i=0}^{n-1} p_{2i+1}(r_1)(\tfrac{r_2}{r_1})^{n-i-1}(\tfrac{m_2}{m_1})^{n-i} p_{2n-2i-1}(r_1x),
\end{aligned}
$$

which is the formula for p_{2n+1}. $\square$

Analogous formulas hold for the functions q_n.

Proposition 4.5.4. *For $x \in [0,1]$ and $n \in \mathbb{N}_0$,*

$$\begin{aligned} q_{2n+1}(1-r_2+r_2x) &= \sum_{i=0}^{n} q_{2i+1}(r_1)(\tfrac{r_2m_2}{r_1m_1})^{n-i} p_{2n-2i}(r_1x) \\ &\quad + \sum_{i=0}^{n} q_{2i}(r_1)(\tfrac{r_2}{r_1})^{n-i+1}(\tfrac{m_2}{m_1})^{n-i} q_{2n-2i+1}(r_1x) \\ &\quad + [1-(r_1+r_2)] \sum_{i=0}^{n} q_{2i}(r_1)(\tfrac{r_2m_2}{r_1m_1})^{n-i} p_{2n-2i}(r_1x), \end{aligned} \tag{4.15}$$

and, for $n \in \mathbb{N}$,

$$\begin{aligned} q_{2n}(1-r_2+r_2x) &= \sum_{i=0}^{n} q_{2i}(r_1)(\tfrac{r_2m_2}{r_1m_1})^{n-i} q_{2n-2i}(r_1x) \\ &\quad + \sum_{i=0}^{n-1} q_{2i+1}(r_1)(\tfrac{r_2}{r_1})^{n-i-1}(\tfrac{m_2}{m_1})^{n-i} p_{2n-2i-1}(r_1x) \\ &\quad + [1-(r_1+r_2)] \sum_{i=0}^{n-1} q_{2i}(r_1)(\tfrac{r_2}{r_1})^{n-i-1}(\tfrac{m_2}{m_1})^{n-i} p_{2n-2i-1}(r_1x). \end{aligned} \tag{4.16}$$

Proof. The proof works by induction analogously to that of Proposition 4.5.2. □

We translate the formulas about the functions $p_n(x)$ and $q_n(x)$ into formulas about $\mathrm{cp}_z(x)$ and $\mathrm{sq}_z(x)$. In the Lebesgue case, these are the usual addition theorems for $\cos(r_1z + r_2xz)$ and $\sin(r_1z + r_2xz)$.

Corollary 4.5.5. *Let $z \in \mathbb{R}$ and $x \in [0,1]$. With the abbreviation $\bar{z} := \sqrt{\frac{r_2m_2}{r_1m_1}}z$ we get*

$$\begin{aligned} \mathrm{cp}_z(1-r_2+r_2x) &= \mathrm{cp}_z(r_1)\,\mathrm{cp}_{\bar{z}}(r_1x) - \sqrt{\tfrac{r_2m_1}{r_1m_2}}\,\mathrm{sp}_z(r_1)\,\mathrm{sq}_{\bar{z}}(r_1x) \\ &\quad - [1-(r_1+r_2)]z\,\mathrm{sp}_z(r_1)\,\mathrm{cp}_{\bar{z}}(r_1x) \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \mathrm{sq}_z(1-r_2+r_2x) &= \mathrm{sq}_z(r_1)\,\mathrm{cp}_{\bar{z}}(r_1x) + \sqrt{\tfrac{r_2m_1}{r_1m_2}}\,\mathrm{cq}_z(r_1)\,\mathrm{sq}_{\bar{z}}(r_1x) \\ &\quad + [1-(r_1+r_2)]z\,\mathrm{cq}_z(r_1)\,\mathrm{cp}_{\bar{z}}(r_1x). \end{aligned} \tag{4.18}$$

Proof. We prove (4.18). We multiply (4.15) with $(-1)^n z^{2n+1} = \frac{1}{i}(iz)^{2n+1}$, sum from

$n = 0$ to infinity and get

$$
\begin{aligned}
\operatorname{sq}_z(1 - r_2 + r_2 x) &= \frac{1}{i} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (iz)^{2k+1} q_{2k+1}(r_1) \left(i \sqrt{\tfrac{r_2 m_2}{r_1 m_1}} z \right)^{2n-2k} p_{2n-2k}(r_1 x) \\
&+ \sqrt{\tfrac{r_2 m_1}{r_1 m_2}} \frac{1}{i} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (iz)^{2k} q_{2k}(r_1) \left(i \sqrt{\tfrac{r_2 m_2}{r_1 m_1}} z \right)^{2n-2k+1} q_{2n-2k+1}(r_1 x) \\
&+ [1 - (r_1 + r_2)] z \sum_{n=0}^{\infty} \sum_{k=0}^{n} (iz)^{2k} q_{2k}(r_1) \left(i \sqrt{\tfrac{r_2 m_2}{r_1 m_1}} z \right)^{2n-2k} p_{2n-2k}(r_1 x) \\
&= \frac{1}{i} \left(\sum_{n=0}^{\infty} (iz)^{2n+1} q_{2n+1}(r_1) \right) \left(\sum_{k=0}^{\infty} \left(i \sqrt{\tfrac{r_2 m_2}{r_1 m_1}} z \right)^{2k} p_{2k}(r_1 x) \right) \\
&+ \sqrt{\tfrac{r_2 m_1}{r_1 m_2}} \frac{1}{i} \left(\sum_{n=0}^{\infty} (iz)^{2n} q_{2n}(r_1) \right) \left(\sum_{k=0}^{\infty} \left(i \sqrt{\tfrac{r_2 m_2}{r_1 m_1}} z \right)^{2k+1} q_{2k+1}(r_1 x) \right) \\
&+ [1 - (r_1 + r_2)] z \left(\sum_{n=0}^{\infty} (iz)^{2n} q_{2n}(r_1) \right) \left(\sum_{k=0}^{\infty} \left(i \sqrt{\tfrac{r_2 m_2}{r_1 m_1}} z \right)^{2k} p_{2k}(r_1 x) \right) \\
&= \operatorname{sq}_z(r_1) \operatorname{cp}_{\bar{z}}(r_1 x) + \sqrt{\tfrac{r_2 m_1}{r_1 m_2}} \operatorname{cq}_z(r_1) \operatorname{sq}_{\bar{z}}(r_1 x) + [1 - (r_1 + r_2)] z \operatorname{cq}_z(r_1) \operatorname{cp}_{\bar{z}}(r_1 x).
\end{aligned}
$$

By multiplying (4.14) with $(-1)^n z^{2n}$ and summing up, (4.17) is proved in the same way. $\square$

The following scaling properties hold that are a replacement of the property $\left(\frac{1}{2}x\right)^n = \frac{1}{2^n} x^n$ for p_n and q_n.

Proposition 4.5.6. *For $x \in [0,1]$ and $n \in \mathbb{N}_0$ we have*

$$
p_{2n+1}\big(r_1 x\big) = r_1^n m_1^{n+1}\, p_{2n+1}(x), \qquad q_{2n+1}\big(r_1 x\big) = r_1^{n+1} m_1^n\, q_{2n+1}(x),
$$

and, for $n \in \mathbb{N}$,

$$
p_{2n}\big(r_1 x\big) = (r_1 m_1)^n\, p_{2n}(x), \qquad q_{2n}\big(r_1 x\big) = (r_1 m_1)^n\, q_{2n}(x).
$$

Proof. We prove the asserted property for p_n by induction on $n \in \mathbb{N}$. Since μ satisfies $\mu(B) = m_1 (S_1 \mu)(B)$ for all Borel sets $B \subseteq [0, r_1]$, we have

$$
p_1(r_1 x) = \int_0^{r_1 x} d\mu = m_1 \int_0^{r_1 x} d(S_1 \mu) = m_1 \int_0^x d\mu = m_1 p_1(x).
$$

Suppose the assertion is true for p_{2n+1} for some $n \in \mathbb{N}_0$. Then

$$p_{2n+2}(r_1 x) = \int_0^{r_1 x} p_{2n+1}(t)\,dt = r_1 \int_0^x p_{2n+1}(r_1 t)\,dt = (r_1 m_1)^{n+1} p_{2n+2}(x).$$

If we assume that the formula holds for p_{2n} for some $n \in \mathbb{N}$, then, transforming μ as above,

$$p_{2n+1}(r_1 x) = \int_0^{r_1 x} p_{2n}(t)\,d\mu(t) = m_1 \int_0^x p_{2n}(r_1 t)\,d\mu(t) = r_1^n m_1^{n+1} p_{2n+1}(x).$$

The formula for q_n is proved analogously. □

Next, we deduce formulas corresponding to those in Proposition 4.5.6 that relate values of cp_z and sq_z at $S_1(x) = r_1 x$ to values of $\mathrm{cp}_{\sqrt{r_1 m_1} z}$ and $\mathrm{sq}_{\sqrt{r_1 m_1} z}$ at x.

Proposition 4.5.7. *For all $x \in [0,1]$ and $z \in \mathbb{R}$ we have*

$$\mathrm{cp}_z\big(S_1(x)\big) = \mathrm{cp}_{\sqrt{r_1 m_1} z}(x) \tag{4.19}$$

and

$$\mathrm{sq}_z\big(S_1(x)\big) = \sqrt{\frac{r_1}{m_1}}\,\mathrm{sq}_{\sqrt{r_1 m_1} z}(x). \tag{4.20}$$

Furthermore, we have

$$\mathrm{sp}_z\big(S_1(x)\big) = \sqrt{\frac{m_1}{r_1}}\,\mathrm{sp}_{\sqrt{r_1 m_1} z}(x)$$

and

$$\mathrm{cq}_z\big(S_1(x)\big) = \mathrm{cp}_{\sqrt{r_1 m_1} z}(x).$$

Proof. With Proposition 4.5.6 we get

$$\mathrm{cp}_z(r_1 x) = \sum_{n=0}^{\infty} (-1)^n z^{2n} p_{2n}(r_1 x) = \sum_{n=0}^{\infty} (-1)^n (\sqrt{r_1 m_1} z)^{2n} p_{2n}(x) = \mathrm{cp}_{\sqrt{r_1 m_1} z}(x)$$

and

$$\begin{aligned}\mathrm{sq}_z(r_1 x) &= \sum_{n=0}^{\infty} (-1)^n z^{2n+1} q_{2n+1}(r_1 x) = \sqrt{\frac{r_1}{m_1}} \sum_{n=0}^{\infty} (-1)^n (\sqrt{r_1 m_1} z)^{2n+1} q_{2n+1}(x) \\ &= \sqrt{\frac{r_1}{m_1}}\,\mathrm{sq}_{\sqrt{r_1 m_1} z}(x).\end{aligned}$$

The other two equations are obtained by deriving. □

The counterparts of (4.19) and (4.20) are the following formulas for $\mathrm{cp}_z\left(S_2(x)\right)$ and $\mathrm{sq}_z\left(S_2(x)\right)$.

Proposition 4.5.8. *For all $x \in [0,1]$ and $z \in \mathbb{R}$ we have*

$$\begin{aligned}\mathrm{cp}_z\left(S_2(x)\right) = {} & \operatorname{cosp}(\sqrt{r_1 m_1} z) \operatorname{cp}_{\sqrt{r_2 m_2} z}(x) - \sqrt{\frac{r_2 m_1}{r_1 m_2}} \operatorname{sinp}(\sqrt{r_1 m_1} z) \operatorname{sq}_{\sqrt{r_2 m_2} z}(x) \\ & - [1-(r_1+r_2)] \sqrt{\frac{m_1}{r_1}} z \operatorname{sinp}(\sqrt{r_1 m_1} z) \operatorname{cp}_{\sqrt{r_2 m_2} z}(x)\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\mathrm{sq}_z\left(S_2(x)\right) = {} & \sqrt{\frac{r_1}{m_1}} \operatorname{sinq}(\sqrt{r_1 m_1} z) \operatorname{cp}_{\sqrt{r_2 m_2} z}(x) + \sqrt{\frac{r_2}{m_2}} \operatorname{cosq}(\sqrt{r_1 m_1} z) \operatorname{sq}_{\sqrt{r_2 m_2} z}(x) \\ & + [1-(r_1+r_2)] z \operatorname{cosq}(\sqrt{r_1 m_1} z) \operatorname{cp}_{\sqrt{r_2 m_2} z}(x).\end{aligned} \tag{4.22}$$

Furthermore, we have

$$\begin{aligned}\mathrm{sp}_z\left(S_2(x)\right) = {} & \sqrt{\frac{m_1}{r_1}} \operatorname{sinp}(\sqrt{r_1 m_1} z) \operatorname{cq}_{\sqrt{r_2 m_2} z}(x) + \sqrt{\frac{m_2}{r_2}} \operatorname{cosp}(\sqrt{r_1 m_1} z) \operatorname{sp}_{\sqrt{r_2 m_2} z}(x) \\ & - [1-(r_1+r_2)] \sqrt{\frac{m_1 m_2}{r_1 r_2}}\, z \operatorname{sinp}(\sqrt{r_1 m_1} z) \operatorname{sp}_{\sqrt{r_2 m_2} z}(x)\end{aligned}$$

and

$$\begin{aligned}\mathrm{cq}_z\left(S_2(x)\right) = {} & \operatorname{cosq}(\sqrt{r_1 m_1} z) \operatorname{cq}_{\sqrt{r_2 m_2} z}(x) - \sqrt{\frac{r_1 m_2}{r_2 m_1}} \operatorname{sinq}(\sqrt{r_1 m_1} z) \operatorname{sp}_{\sqrt{r_2 m_2} z}(x) \\ & - [1-(r_1+r_2)]\, z \sqrt{\frac{m_2}{r_2}} \operatorname{cosq}(\sqrt{r_1 m_1} z) \operatorname{sp}_{\sqrt{r_2 m_2} z}(x).\end{aligned}$$

Proof. By (4.17) and Proposition 4.5.7 we get

$$\begin{aligned}
\mathrm{cp}_z(1-r_2+r_2x) &= \mathrm{cp}_z(r_1)\,\mathrm{cp}_{\sqrt{\frac{r_2m_2}{r_1m_1}}z}(r_1x) - \sqrt{\tfrac{r_2m_1}{r_1m_2}}\,\mathrm{sp}_z(r_1)\,\mathrm{sq}_{\sqrt{\frac{r_2m_2}{r_1m_1}}z}(r_1x)\\
&\quad - [1-(r_1+r_2)]z\,\mathrm{sp}_z(r_1)\,\mathrm{cp}_{\sqrt{\frac{r_2m_2}{r_1m_1}}z}(r_1x)\\
&= \mathrm{cosp}(\sqrt{r_1m_1}z)\,\mathrm{cp}_{\sqrt{r_2m_2}z}(x) - \sqrt{\frac{r_2m_1}{r_1m_2}}\,\mathrm{sinp}(\sqrt{r_1m_1}z)\,\mathrm{sq}_{\sqrt{r_2m_2}z}(x)\\
&\quad - [1-(r_1+r_2)]\sqrt{\frac{m_1}{r_1}}z\,\mathrm{sinp}(\sqrt{r_1m_1}z)\,\mathrm{cp}_{\sqrt{r_2m_2}z}(x).
\end{aligned}$$

Analogously, (4.22) is proved using (4.18).

The other two equations are obtained by deriving. □

If the functions cosp, sinp and sinq are assumed to be known, then equations (4.19) and (4.21) allow to compute basically all relevant values of the function cp_z. If, namely, x is a point in the invariant set K, then there is a sequence $(x_n)_n$ that converges to x and takes only values of the form

$$S_{w_1}\circ S_{w_2}\circ\cdots\circ S_{w_n}(0) \quad \text{or} \quad S_{w_1}\circ S_{w_2}\circ\cdots\circ S_{w_n}(1),$$

where $n\in\mathbb{N}$ and $w_1,\dots w_n\in\{1,2\}$. For each of these values, (4.19) and (4.21) can be applied n times to get a formula containing only values of cosp, sinp and sinq. For example

$$\begin{aligned}
\mathrm{cp}_z\left(S_2(S_1(1))\right) &= \mathrm{cosp}(\sqrt{r_1m_1}z)\,\mathrm{cosp}(\sqrt{r_2m_2r_1m_1}z)\\
&\quad - \sqrt{\frac{r_2m_1}{r_1m_2}}\,\mathrm{sinp}(\sqrt{r_1m_1}z)\,\mathrm{sinq}(\sqrt{r_2m_2r_1m_1}z)\\
&\quad - [1-(r_1+r_2)]\sqrt{\frac{m_1}{r_1}}z\,\mathrm{sinp}(\sqrt{r_1m_1}z)\,\mathrm{cosp}(\sqrt{r_2m_2r_1m_1}z).
\end{aligned}$$

The same holds for sq_z and formulas (4.20) and (4.22). This procedure we will use to compute numerically the maximum value of eigenfunctions in Section 4.8 as well as to produce images of eigenfunctions shown in the appendix.

Therefore we are interested in the functions sinq, sinp, cosp, and cosq. These have power series representations with coefficients $p_n = p_n(1)$ and $q_n = q_n(1)$. For these numerical sequences we prove a recursion formula in the following.

Proposition 4.5.9. (i) *For* $n \in \mathbb{N}_0$,

$$\begin{aligned} p_{2n+1} &= \sum_{i=0}^{n} r_1^i m_1^{i+1} (r_2 m_2)^{n-i} p_{2i+1}\, q_{2n-2i} \\ &+ \sum_{i=0}^{n} (r_1 m_1)^i r_2^{n-i} m_2^{n-i+1} p_{2i}\, p_{2n-2i+1} \\ &+ [1-(r_1+r_2)] \sum_{i=0}^{n-1} r_1^i m_1^{i+1} r_2^{n-i-1} m_2^{n-i} p_{2i+1}\, p_{2n-2i-1}. \end{aligned} \tag{4.23}$$

(ii) *For* $n \in \mathbb{N}$,

$$\begin{aligned} p_{2n} &= \sum_{i=0}^{n} (r_1 m_1)^i (r_2 m_2)^{n-i} p_{2i}\, p_{2n-2i} \\ &+ \sum_{i=0}^{n-1} r_1^i m_1^{i+1} r_2^{n-i} m_2^{n-i-1} p_{2i+1}\, q_{2n-2i-1} \\ &+ [1-(r_1+r_2)] \sum_{i=0}^{n-1} r_1^i m_1^{i+1} (r_2 m_2)^{n-i-1} p_{2i+1}\, p_{2n-2i-2}. \end{aligned} \tag{4.24}$$

(iii) *For* $n \in \mathbb{N}_0$,

$$\begin{aligned} q_{2n+1} &= \sum_{i=0}^{n} r_1^{i+1} m_1^i (r_2 m_2)^{n-i} q_{2i+1}\, p_{2n-2i} \\ &+ \sum_{i=0}^{n} (r_1 m_1)^i r_2^{n-i+1} m_2^{n-i} q_{2i}\, q_{2n-2i+1} \\ &+ [1-(r_1+r_2)] \sum_{i=0}^{n} (r_1 m_1)^i (r_2 m_2)^{n-i} q_{2i}\, p_{2n-2i}. \end{aligned} \tag{4.25}$$

(iv) *For* $n \in \mathbb{N}$,

$$\begin{aligned} q_{2n} &= \sum_{i=0}^{n} (r_1 m_1)^i (r_2 m_2)^{n-i} q_{2i}\, q_{2n-2i} \\ &+ \sum_{i=0}^{n-1} r_1^{i+1} m_1^i r_2^{n-i-1} m_2^{n-i} q_{2i+1}\, p_{2n-2i-1} \\ &+ [1-(r_1+r_2)] \sum_{i=0}^{n-1} (r_1 m_1)^i r_2^{n-i-1} m_2^{n-i} q_{2i}\, p_{2n-2i-1}. \end{aligned} \tag{4.26}$$

Remark 4.5.10. If we take $r_1 = m_1$ and $r_2 = m_2$ (and thus $r_1 + r_2 = 1$ and μ is the Lebesgue measure), the above formulas reduce to $\sum_{i=0}^{n} \binom{n}{i} r_1^i r_2^{n-i} = 1$.

Proof. We put $x = 1$ in Propositions 4.5.2, 4.5.4 and 4.5.6. Then we eliminate all terms of the form $p_n(r_1)$ and $q_n(r_1)$ to obtain formulas that contain only the members of the sequences $(p_n)_n$ and $(q_n)_n$ (as well as r_1, r_2, m_1 and m_2). $\square$

To get the desired recursion formulas, we solve the above formulas for the highest order terms.

Corollary 4.5.11. (i) *For $n \in \mathbb{N}$,*

$$\begin{aligned} p_{2n+1} = & \frac{1}{1 - r_1^n m_1^{n+1} - r_2^n m_2^{n+1}} \Bigg(\sum_{i=0}^{n-1} r_1^i m_1^{i+1} (r_2 m_2)^{n-i} p_{2i+1}\, q_{2n-2i} \\ & + \sum_{i=1}^{n} (r_1 m_1)^i r_2^{n-i} m_2^{n-i+1} p_{2i}\, p_{2n-2i+1} \\ & + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} r_1^i m_1^{i+1} r_2^{n-i-1} m_2^{n-i} p_{2i+1}\, p_{2n-2i-1} \Bigg). \end{aligned} \tag{4.27}$$

(ii) *For $n \in \mathbb{N}$,*

$$\begin{aligned} p_{2n} = & \frac{1}{1 - (r_1 m_1)^n - (r_2 m_2)^n} \Bigg(\sum_{i=1}^{n-1} (r_1 m_1)^i (r_2 m_2)^{n-i} p_{2i}\, p_{2n-2i} \\ & + \sum_{i=0}^{n-1} r_1^i m_1^{i+1} r_2^{n-i} m_2^{n-i-1} p_{2i+1}\, q_{2n-2i-1} \\ & + [1 - (r_1 + r_2)] \sum_{i=0}^{n-1} r_1^i m_1^{i+1} (r_2 m_2)^{n-i-1} p_{2i+1}\, p_{2n-2i-2} \Bigg). \end{aligned} \tag{4.28}$$

(iii) *For $n \in \mathbb{N}$,*

$$\begin{aligned} q_{2n+1} = & \frac{1}{1 - r_1^{n+1} m_1^n - r_2^{n+1} m_2^n} \Bigg(\sum_{i=0}^{n-1} r_1^{i+1} m_1^i (r_2 m_2)^{n-i} q_{2i+1}\, p_{2n-2i} \\ & + \sum_{i=1}^{n} (r_1 m_1)^i r_2^{n-i+1} m_2^{n-i} q_{2i}\, q_{2n-2i+1} \\ & + [1 - (r_1 + r_2)] \sum_{i=0}^{n} (r_1 m_1)^i (r_2 m_2)^{n-i} q_{2i}\, p_{2n-2i} \Bigg). \end{aligned} \tag{4.29}$$

(iv) *For* $n \in \mathbb{N}$,

$$\begin{aligned} q_{2n} =& \frac{1}{1-(r_1m_1)^n-(r_2m_2)^n}\Bigg(\sum_{i=1}^{n-1}(r_1m_1)^i(r_2m_2)^{n-i}q_{2i}\,q_{2n-2i} \\ &+\sum_{i=0}^{n-1} r_1^{i+1}m_1^i r_2^{n-i-1}m_2^{n-i}q_{2i+1}\,p_{2n-2i-1} \\ &+[1-(r_1+r_2)]\sum_{i=0}^{n-1}(r_1m_1)^i r_2^{n-i-1}m_2^{n-i}q_{2i}\,p_{2n-2i-1}\Bigg). \end{aligned} \tag{4.30}$$

Remark 4.5.12. Consider two self-similar measures μ and μ^* on $[0,1]$, where μ^* is the reflection of μ with respect to the point $\frac{1}{2}$. Thus, μ^* is described as invariant measure by interchanging the parameters r_1, m_1 and r_2, m_2 in the IFS defining μ. Then the above recursive formulas show that the associated p- and q-sequences satisfy $p^*_{2n} = q_{2n}$, $q^*_{2n} = p_{2n}$, $p^*_{2n+1} = p_{2n+1}$ and $q^*_{2n+1} = q_{2n+1}$ for all $n \in \mathbb{N}$. Hence, $\mathrm{cosp}^* = \mathrm{cosq}$, $\mathrm{cosq}^* = \mathrm{cosp}$, $\mathrm{sinp}^* = \mathrm{sinp}$ and $\mathrm{sinq}^* = \mathrm{sinq}$. This is consistent with the physical intuition that the Neumann as well as the Dirichlet eigenfrequencies do not change when the vibrating string producing them is reversed.

Example 4.5.13. We take $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$. Then, K is the middle third Cantor set and μ is the normalized $\frac{\log 2}{\log 3}$-dimensional Hausdorff measure restricted to K. We calculate the first members of the sequences $(p_n)_n$ and $(q_n)_n$ using formulas (4.27) and (4.29) for p_{2n+1} and q_{2n+1}, which simplify to

$$p_{2n+1} = \frac{1}{2 \cdot 6^n - 2}\Big(\sum_{i=1}^{2n} p_i\, p_{2n+1-i} + \sum_{i=0}^{n-1} p_{2i+1}\, p_{2n-2i-1}\Big)$$

$$q_{2n+1} = \frac{1}{3 \cdot 6^n - 2}\Big(\sum_{i=1}^{2n} q_i\, q_{2n+1-i} + \sum_{i=0}^{n} q_{2i}\, q_{2n-2i}\Big).$$

Since μ is symmetric, we can use for p_{2n} and q_{2n} the simpler formula (4.11)

$$p_{2n} = q_{2n} = \frac{1}{2}\sum_{i=1}^{2n-1} (-1)^{i+1} p_i\, q_{2n-i}$$

from Corollary 4.4.4. Then,

$$\begin{aligned}
p_1 &= 1, & q_1 &= 1, & p_2 &= \frac{1}{2},\\
p_3 &= \frac{1}{5}, & q_3 &= \frac{1}{8}, & p_4 &= \frac{3}{80},\\
p_5 &= \frac{27}{2\,800}, & q_5 &= \frac{21}{4\,240}, & p_6 &= \frac{311}{296\,800},\\
p_7 &= \frac{6\,383}{31\,906\,000}, & q_7 &= \frac{33\,253}{383\,465\,600}, & p_8 &= \frac{4\,716\,349}{329\,780\,416\,000}
\end{aligned}$$

and therefore

$$\operatorname{sinp}(z) = z - \frac{6}{5}\frac{z^3}{3!} + \frac{81}{70}\frac{z^5}{5!} - \frac{57\,447}{56\,975}\frac{z^7}{7!} + \cdots$$

$$\operatorname{sinq}(z) = z - \frac{3}{4}\frac{z^3}{3!} + \frac{63}{106}\frac{z^5}{5!} - \frac{299\,277}{684\,760}\frac{z^7}{7!} + \cdots$$

and

$$\operatorname{cosp}(z) = \operatorname{cosq}(z) = 1 - \frac{z^2}{2!} + \frac{9}{10}\frac{z^4}{4!} - \frac{2\,799}{3\,710}\frac{z^6}{6!} + \frac{42\,447\,141}{73\,611\,700}\frac{z^8}{8!} - \ldots.$$

The functions sinp, sinq, cosp and cosq can be characterized by the following system of functional equations.

Theorem 4.5.14. *For $z \in \mathbb{R}$ we have*

$$\begin{aligned}
\operatorname{sinp}(z) &= \sqrt{\tfrac{m_1}{r_1}}\operatorname{sinp}(\sqrt{r_1 m_1} z)\operatorname{cosq}(\sqrt{r_2 m_2} z) \\
&\quad + \sqrt{\tfrac{m_2}{r_2}}\operatorname{cosp}(\sqrt{r_1 m_1} z)\operatorname{sinp}(\sqrt{r_2 m_2} z) \\
&\quad - [1-(r_1+r_2)]\sqrt{\tfrac{m_1 m_2}{r_1 r_2}}\, z\operatorname{sinp}(\sqrt{r_1 m_1} z)\operatorname{sinp}(\sqrt{r_2 m_2} z)
\end{aligned} \tag{4.31}$$

$$\begin{aligned}
\operatorname{sinq}(z) &= \sqrt{\tfrac{r_1}{m_1}}\operatorname{sinq}(\sqrt{r_1 m_1} z)\operatorname{cosp}(\sqrt{r_2 m_2} z) \\
&\quad + \sqrt{\tfrac{r_2}{m_2}}\operatorname{cosq}(\sqrt{r_1 m_1} z)\operatorname{sinq}(\sqrt{r_2 m_2} z) \\
&\quad + [1-(r_1+r_2)]\, z\operatorname{cosq}(\sqrt{r_1 m_1} z)\operatorname{cosp}(\sqrt{r_2 m_2} z)
\end{aligned} \tag{4.32}$$

$$\begin{aligned}
\operatorname{cosp}(z) &= \operatorname{cosp}(\sqrt{r_1 m_1} z)\operatorname{cosp}(\sqrt{r_2 m_2} z) \\
&\quad - \sqrt{\tfrac{r_2 m_1}{r_1 m_2}}\operatorname{sinp}(\sqrt{r_1 m_1} z)\operatorname{sinq}(\sqrt{r_2 m_2} z) \\
&\quad - [1-(r_1+r_2)]\sqrt{\tfrac{m_1}{r_1}} z\operatorname{sinp}(\sqrt{r_1 m_1} z)\operatorname{cosp}(\sqrt{r_2 m_2} z)
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
\operatorname{cosq}(z) &= \operatorname{cosq}(\sqrt{r_1 m_1} z)\operatorname{cosq}(\sqrt{r_2 m_2} z) \\
&\quad - \sqrt{\tfrac{r_1 m_2}{r_2 m_1}}\operatorname{sinq}(\sqrt{r_1 m_1} z)\operatorname{sinp}(\sqrt{r_2 m_2} z) \\
&\quad - [1-(r_1+r_2)]\sqrt{\tfrac{m_2}{r_2}} z\operatorname{cosq}(\sqrt{r_1 m_1} z)\operatorname{sinp}(\sqrt{r_2 m_2} z).
\end{aligned} \tag{4.34}$$

Furthermore, the functions sinp, sinq, cosp *and* cosq *are the only analytic functions that solve the above system of functional equations and satisfy the conditions that* sinp *and* sinq *are odd,* cosp *and* cosq *are even, and*

$$\lim_{z\to 0}\frac{\operatorname{sinp}(z)}{z} = \lim_{z\to 0}\frac{\operatorname{sinq}(z)}{z} = 1$$

and

$$\operatorname{cosp}(0) = \operatorname{cosq}(0) = 1.$$

Remark 4.5.15. If we would know all the values of all four functions on a given interval, say, $[0,a]$, then, using the formulas above, we could calculate all values of all four functions on $[0,(\max_i \sqrt{r_i m_i})^{-1}a]$. Then, iteratively, we get the values on $[0,(\max_i \sqrt{r_i m_i})^{-2}a]$ and so on. So, the functions are determined on $[0,\infty)$ by their values on an arbitrary small interval $[0,a]$.

Furthermore, the theorem describes a kind of "self-similarity" of our four functions.

Proof. To show that sinp, sinq, cosp and cosq satisfy the equations, put $x = 1$ in Proposition 4.5.8.

Suppose that f_1, f_2, g_1 and g_2 are real analytic functions that satisfy the above equations, and that f_1, f_2 are odd, g_1, g_2 are even, $\lim\limits_{z\to 0} \frac{f_1(z)}{z} = \lim\limits_{z\to 0} \frac{f_2(z)}{z} = 1$, and $g_1(0) = g_2(0) = 1$. Then, power series representations exist, that is, there are real sequences (a_n), (b_n), (c_n) and (d_n) such that for all $z \in \mathbb{R}$ holds

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^{2n+1}, \quad f_2(z) = \sum_{n=0}^{\infty} b_n z^{2n+1}, \quad g_1(z) = \sum_{n=0}^{\infty} c_n z^{2n}, \quad g_2(z) = \sum_{n=0}^{\infty} d_n z^{2n},$$

where $a_0 = b_0 = c_0 = d_0 = 1$. Since these functions satisfy (4.31), we get for all $z \in \mathbb{R}$

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^{2n+1} = &\sqrt{\frac{m_1}{r_1}} \sum_{n=0}^{\infty} z^{2n+1} \sum_{k=0}^{n} a_k \sqrt{r_1 m_1}^{2k+1} d_{n-k} \sqrt{r_2 m_2}^{2n-2k} \\ &+ \sqrt{\frac{m_2}{r_2}} \sum_{n=0}^{\infty} z^{2n+1} \sum_{k=0}^{n} c_k \sqrt{r_1 m_1}^{2k} a_{n-k} \sqrt{r_2 m_2}^{2n+1-2k} \\ &- [1-(r_1+r_2)] \sqrt{\frac{m_1 m_2}{r_1 r_2}} \sum_{n=0}^{\infty} z^{2n+3} \sum_{k=0}^{n} a_k \sqrt{r_1 m_1}^{2k+1} a_{n-k} \sqrt{r_2 m_2}^{2n+1-2k}.\end{aligned}$$

If we derive this equation $2j+1$ times and put $z = 0$, we receive formula (4.23) for a_j. Analogously, one can show that b_j satisfies (4.25), c_j satisfies (4.24) and d_j satisfies (4.26). Together with the initial condition $a_0 = b_0 = c_0 = d_0 = 1$ it follows that $a_j = p_{2j+1}$, $b_j = q_{2j+1}$, $c_j = p_{2j}$ and $d_j = q_{2j}$ for all $j \in \mathbb{N}$. Thus, $f_1 = \mathrm{sinp}$, $f_2 = \mathrm{sinq}$, $g_1 = \mathrm{cosp}$ and $g_2 = \mathrm{cosq}$. □

Example 4.5.16. (i) If we take $r_1 = m_1$ and $r_2 = m_2$ and $r_1 + r_2 = 1$, then K is the unit interval and μ the Lebesgue measure. The functions sinp, sinq, cosp and cosq equal the usual sine and cosine functions, and the formulas in Theorem 4.5.14 simplify to

$$\begin{aligned}\sin(z) &= \sin(r_1 z + r_2 z) = \sin(r_1 z)\cos(r_2 z) + \cos(r_1 z)\sin(r_2 z), \\ \cos(z) &= \cos(r_1 z + r_2 z) = \cos(r_1 z)\cos(r_2 z) - \sin(r_1 z)\sin(r_2 z).\end{aligned}$$

(ii) Let $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$. Then μ is the Cantor measure and the formulas in Theorem 4.5.14 can be rewritten as

$$\operatorname{sinp}(\sqrt{6}z) = \frac{\sqrt{6}}{2}\operatorname{sinp}(z)\Big(2\operatorname{cosp}(z) - z\operatorname{sinp}(z)\Big) \tag{4.35}$$

$$\operatorname{sinq}(\sqrt{6}z) = \frac{\sqrt{6}}{3}\operatorname{cosp}(z)\Big(2\operatorname{sinq}(z) + z\operatorname{cosp}(z)\Big) \tag{4.36}$$

$$\operatorname{cosp}(\sqrt{6}z) = \operatorname{cosp}(z)^2 - \operatorname{sinp}(z)\operatorname{sinq}(z) - z\operatorname{cosp}(z)\operatorname{sinp}(z). \tag{4.37}$$

Since K is symmetric, cosp = cosq.

Observe that Theorem 4.5.14 in combination with the recursive rules in Corollary 4.5.11 supply a technique for investigation of further properties of the eigenvalues, for example for numerical computation. On a given interval $[0, a]$ we can approximate the functions sinp, sinq, cosp and cosq arbitrarily exact by polynomials consisting of sufficiently many members of the corresponding power series. Then, by Theorem 4.5.14, we can extend all four functions successively to larger intervals.

4.6. Self-similar measures with $r_1m_1 = r_2m_2$

In this section we suppose μ is a self-similar measure as in the last section but with parameters additionally satisfying $r_1m_1 = r_2m_2$. This case is particularly interesting because there we have the following property.

Theorem 4.6.1. *Let $r_1m_1 = r_2m_2$. If λ is the mth Neumann eigenvalue of $-\frac{d}{d\mu}\frac{d}{dx}$, then $\frac{1}{r_1m_1}\lambda$ is the $2m$th Neumann eigenvalue, that is, for all $m \in \mathbb{N}$,*

$$r_1m_1\,\lambda_{N,2m} = \lambda_{N,m}.$$

This Theorem has been proved with the method of Prüfer angles by Volkmer [52] for the case $r_1 = r_2 = \frac{1}{3}$, $m_1 = m_2 = \frac{1}{2}$ and by Freiberg [20] in a more general setting. It delivers the foundation for the statements in this section. An analogous property for Dirichlet eigenvalues does not seem to hold. However, in the symmetric case there is a similar relation between Dirichlet eigenvalues and eigenvalues of the problems (DN) or (ND) posed in Section 4.1. Remember, (DN) has boundary conditions $f(0) = f'(1) = 0$ and (ND) has $f'(0) = f(1) = 0$.

Proposition 4.6.2. *Let μ be symmetric, that is $r := r_1 = r_2$ and $m_1 = m_2 = \frac{1}{2}$ and let λ be an eigenvalue of (DN) or (ND). Then $\frac{2}{r}\lambda$ is a Dirichlet eigenvalue and if f*

is a $\frac{2}{r}\lambda$-Dirichlet eigenfunction, then $f \circ S_1$ is a λ-(DN) eigenfunction, and $f \circ S_2$ is a λ-(ND) eigenfunction.

Proof. In Corollary 4.4.4 we showed that since μ is symmetric, we have $\mathrm{cosp} = \mathrm{cosq}$. Then we can factorize (4.32) and get

$$\mathrm{sinq}\big(\sqrt{\tfrac{2}{r}}z\big) = \mathrm{cosp}(z) \cdot \Big[2\sqrt{2r}\,\mathrm{sinq}(z) + (1-2r)\,z\,\mathrm{cosp}(z)\Big].$$

Since λ is an eigenvalue of the (DN) and the (ND) problem, $\mathrm{cosp}(\sqrt{\lambda}) = 0$. Then, $\mathrm{sinq}(\sqrt{\frac{2}{r}\lambda}) = 0$ and thus, $\frac{2}{r}\lambda$ is a Dirichlet eigenvalue. From Propositions 4.5.7 and 4.5.8 we get

$$\mathrm{sq}_{\sqrt{2r^{-1}\lambda}} \circ S_1 = \sqrt{2r}\,\mathrm{sq}_{\sqrt{\lambda}}$$

and

$$\mathrm{sq}_{\sqrt{2r^{-1}\lambda}} \circ S_2 = \sqrt{2r}\,\mathrm{sinq}\big(\sqrt{\lambda}\big)\,\mathrm{cp}_{\sqrt{\lambda}},$$

which proves the proposition. $\square$

In the following we treat only the Neumann eigenvalue problem using Theorem 4.6.1. With the formula

$$\mathrm{cosq}(z)\ \mathrm{cosp}(z) + \mathrm{sinq}(z)\ \mathrm{sinp}(z) = 1, \tag{4.38}$$

which follows from Theorem 4.3.3 by setting $x = 1$, we rearrange the functional equations from Theorem 4.5.14. With the abbreviation

$$h(z) := r_1\,\mathrm{cosp}(z) + r_2\,\mathrm{cosq}(z) - \big[1-(r_1+r_2)\big]\,z\,\mathrm{sinp}(z) \tag{4.39}$$

we can write

$$\mathrm{sinp}(z) = \frac{\sqrt{r_1 m_1}}{r_1 r_2}\ \mathrm{sinp}\big(\sqrt{r_1 m_1}z\big)\ h\big(\sqrt{r_1 m_1}z\big), \tag{4.40}$$

$$\mathrm{cosp}(z) = -\frac{r_2}{r_1} + \frac{1}{r_1}\ \mathrm{cosp}\big(\sqrt{r_1 m_1}z\big)\ h\big(\sqrt{r_1 m_1}z\big), \tag{4.41}$$

and

$$\mathrm{sinq}(z) = \big[1-(r_1+r_2)\big]z + \frac{1}{\sqrt{r_1 m_1}}\ \mathrm{sinq}\big(\sqrt{r_1 m_1}z\big)\ h\big(\sqrt{r_1 m_1}z\big), \tag{4.42}$$

$$\mathrm{cosq}(z) = -\frac{r_1}{r_2} + \frac{1}{r_2}\ \mathrm{cosq}\big(\sqrt{r_1 m_1}z\big)\ h\big(\sqrt{r_1 m_1}z\big). \tag{4.43}$$

Employing the above formulas we can calculate the values of cosp, cosq and sinq at the zero points of sinp.

Lemma 4.6.3. *Let $m \in \mathbb{N}$ and let $v(m)$ be the multiplicity of the prime factor 2 in m. Let $z_m := \sqrt{\lambda_{N,m}}$ be the square root of the mth Neumann eigenvalue, that is, the mth zero point of* sinp. *Then*

$$\operatorname{cosp}(z_m) = \Big(-\frac{r_2}{r_1}\Big)^{2^{v(m)}} \tag{4.44}$$

$$\operatorname{cosq}(z_m) = \Big(-\frac{r_1}{r_2}\Big)^{2^{v(m)}} \tag{4.45}$$

$$\operatorname{sinq}(z_m) = a_{v(m)} \cdot z_m \tag{4.46}$$

where $(a_k)_k$ is determined by

$$a_0 = 1 - (r_1 + r_2)$$

and, for $k \in \mathbb{N}$,

$$a_k = 1 - (r_1 + r_2) + a_{k-1}\bigg(r_1\Big(-\frac{r_2}{r_1}\Big)^{2^{k-1}} + r_2\Big(-\frac{r_1}{r_2}\Big)^{2^{k-1}}\bigg).$$

Proof. Suppose m is odd. Then $\operatorname{sinp}(z_m) = 0$ and $\operatorname{sinp}\big(\sqrt{r_1 m_1} z_m\big) \neq 0$. To see this, suppose $\operatorname{sinp}\big(\sqrt{r_1 m_1} z_m\big) = 0$. Then $r_1 m_1 z_m^2$ would be a Neumann eigenvalue, say $r_1 m_1 z_m^2 = \lambda_{N,l}$ for some $l \in \mathbb{N}$, and because of Theorem 4.6.1, z_m^2 would be the eigenvalue $\lambda_{N,2l}$. Thus, $m = 2l$, which is a contradiction.

Hence, it follows by (4.40) that $h\big(\sqrt{r_1 m_1} z_m\big) = 0$. Then, by (4.41), we see that $\operatorname{cosp}(z_m) = -\frac{r_2}{r_1}$.

By (4.33) follows that, for all $z \in \mathbb{R}$, if

$$\operatorname{sinp}\big(\sqrt{r_1 m_1} z\big) = 0,$$

then

$$\operatorname{cosp}(z) = \operatorname{cosp}^2\big(\sqrt{r_1 m_1} z\big).$$

Thus, if $m = 2l$ for some odd l, then $\sqrt{r_1 m_1} z_m = z_l$ and hence,

$$\operatorname{cosp}(z_m) = \operatorname{cosp}^2(z_l) = \Big(-\frac{r_2}{r_1}\Big)^2.$$

Iteratively, we get that, if $m = 2^k l$ for some odd l,

$$\operatorname{cosp}(z_m) = \Big(-\frac{r_2}{r_1}\Big)^{2^k},$$

which proves (4.44).

Since $\operatorname{sinp}(z_m) = 0$ for all $m \in \mathbb{N}$ we get by (4.38) that

$$\operatorname{cosq}(z_m) = \frac{1}{\operatorname{cosp}(z_m)} = \Big(-\frac{r_1}{r_2}\Big)^{2^{v(m)}},$$

which is (4.45).

Now we show (4.46). At first, suppose $v(m) = 0$, that is, m is odd. Then, as above, $h\big(\sqrt{r_1 m_1} z_m\big) = 0$ and thus, by (4.42),

$$\operatorname{sinq}(z_m) = \big[1 - (r_1 + r_2)\big] z_m.$$

Observe that we have for all m

$$h(z_m) = r_1\Big(-\frac{r_2}{r_1}\Big)^{2^{v(m)}} + r_2\Big(-\frac{r_1}{r_2}\Big)^{2^{v(m)}}. \tag{4.47}$$

Suppose $v(m) \geq 1$. Then $\sqrt{r_1 m_1} z_m = z_{\frac{m}{2}}$ and thus,

$$\begin{aligned}
\frac{\operatorname{sinq}(z_m)}{z_m} &= 1 - (r_1 + r_2) + \frac{\operatorname{sinq}\big(\sqrt{r_1 m_1} z_m\big)}{\sqrt{r_1 m_1} z_m} h\big(\sqrt{r_1 m_1} z_m\big) \\
&= 1 - (r_1 + r_2) + \frac{\operatorname{sinq}\big(z_{\frac{m}{2}}\big)}{z_{\frac{m}{2}}} h\big(z_{\frac{m}{2}}\big) \\
&= 1 - (r_1 + r_2) + \frac{\operatorname{sinq}\big(z_{\frac{m}{2}}\big)}{z_{\frac{m}{2}}} \bigg(r_1\Big(-\frac{r_2}{r_1}\Big)^{2^{v(m)-1}} + r_2\Big(-\frac{r_1}{r_2}\Big)^{2^{v(m)-1}}\bigg).
\end{aligned}$$

Hence, $\dfrac{\operatorname{sinq}(z_m)}{z_m}$ depends only on $v(m)$ and so, with $a_{v(m)} = \dfrac{\operatorname{sinq}(z_m)}{z_m}$, we get

$$a_{v(m)} = 1 - (r_1 + r_2) + a_{v(m)-1}\bigg(r_1\Big(-\frac{r_2}{r_1}\Big)^{2^{v(m)-1}} + r_2\Big(-\frac{r_1}{r_2}\Big)^{2^{v(m)-1}}\bigg),$$

which proves the assertion. □

We use the above computed values of $\operatorname{cosp}(z_m)$ and Propositions 4.5.7 and 4.5.8 to get a relation between the mth and the $2m$th Neumann eigenfunction as defined in Proposition 4.1.7.

Proposition 4.6.4. *Let $m \in \mathbb{N}$ and $v(m)$ be the 2-multiplicity of m. We denote the mth Neumann eigenfunction by $f_m := \mathrm{cp}_{\sqrt{\lambda_m}}$. Then, for all $x \in [0,1]$,*

$$f_{2m}\big(S_1(x)\big) = f_m(x) \tag{4.48}$$

and

$$f_{2m}\big(S_2(x)\big) = \Big(-\frac{m_1}{m_2}\Big)^{2^{v(m)}} f_m(x). \tag{4.49}$$

Proof. Because of Theorem 4.6.1 we have $\lambda_m = r_1 m_1 \lambda_{2m}$ and thus,

$$\mathrm{sinp}\big(\sqrt{r_1 m_1 \lambda_{2m}}\big) = 0.$$

Since $f_m = \mathrm{cp}_{\sqrt{\lambda_m}}$, Propositions 4.5.7 and 4.5.8 give for $x \in [0,1]$

$$f_{2m}\big(S_1(x)\big) = f_m(x)$$

and

$$f_{2m}\big(S_2(x)\big) = \mathrm{cosp}\big(\sqrt{\lambda_m}\big)\, f_m(x).$$

Noting that $\dfrac{r_2}{r_1} = \dfrac{m_1}{m_2}$, we get with (4.44) that

$$f_{2m}\big(S_2(x)\big) = \Big(-\frac{m_1}{m_2}\Big)^{2^{v(m)}} f_m(x).$$

□

The above proposition can be employed to work out the relationship between the suprema and the $L_2(\mu)$ norms of f_m and f_{2m}.

Proposition 4.6.5. *Let $m \in \mathbb{N}$ and $v(m)$ the 2-multiplicity of m. Then*

$$\|f_{2m}\|^2_{L_2(\mu)} = \left(m_1 + m_2\Big(\frac{m_1}{m_2}\Big)^{2^{v(m)+1}}\right)\|f_m\|^2_{L_2(\mu)} \tag{4.50}$$

and

$$\|f_{2m}\|_\infty = \max\Big\{1, \Big(\frac{m_1}{m_2}\Big)^{2^{v(m)}}\Big\}\|f_m\|_\infty. \tag{4.51}$$

Proof. At first we prove (4.50). For $m \in \mathbb{N}$ we have

$$\begin{aligned}
\|f_{2m}\|^2_{L_2(\mu)} &= \int_{S_1(0)}^{S_1(1)} f_{2m}(t)^2 \, d\mu(t) + \int_{S_2(0)}^{S_2(1)} f_{2m}(t)^2 \, d\mu(t) \\
&= m_1 \int_{S_1(0)}^{S_1(1)} f_{2m}(t)^2 \, d(S_1\mu)(t) + m_2 \int_{S_2(0)}^{S_2(1)} f_{2m}(t)^2 \, d(S_2\mu)(t) \\
&= m_1 \int_0^1 f_{2m}\big(S_1(t)\big)^2 \, d\mu(t) + m_2 \int_0^1 f_{2m}\big(S_2(t)\big)^2 \, d\mu(t).
\end{aligned}$$

By (4.48) and (4.49) we get

$$\begin{aligned}
\|f_{2m}\|^2_{L_2(\mu)} &= m_1 \int_0^1 f_m(t)^2 \, d\mu(t) + m_2 \Big(-\frac{m_1}{m_2}\Big)^{2^{v(m)+1}} \int_0^1 f_m(t)^2 \, d\mu(t) \\
&= \Big[m_1 + m_2 \Big(\frac{m_1}{m_2}\Big)^{2^{v(m)+1}}\Big] \|f_m\|^2_{L_2(\mu)}.
\end{aligned}$$

Now we show (4.51). With (4.48) and (4.49) we have

$$\sup_{x \in [S_1(0), S_1(1)]} |f_{2m}(x)| = \sup_{x \in [0,1]} \big|f_{2m}\big(S_1(x)\big)\big| = \sup_{x \in [0,1]} |f_m(x)| = \|f_m\|_\infty$$

and

$$\sup_{x \in [S_2(0), S_2(1)]} |f_{2m}(x)| = \sup_{x \in [0,1]} \big|f_{2m}\big(S_2(x)\big)\big| = \Big(\frac{m_1}{m_2}\Big)^{2^{v(m)}} \|f_m\|_\infty.$$

Therefore, since f_{2m} is linear on $[S_1(1), S_2(0)]$ and continuous,

$$\sup_{x \in [0,1]} |f_{2m}(x)| = \max\Big\{1, \Big(\frac{m_1}{m_2}\Big)^{2^{v(m)}}\Big\} \|f_m\|_\infty.$$

□

Now we consider the normalized Neumann eigenfunctions. For $m \in \mathbb{N}_0$ we set

$$\tilde{f}_m := \|f_m\|^{-1}_{L_2(\mu)} f_m.$$

We are interested in the asymptotic behaviour of the sequence $(\|\tilde{f}_m\|_\infty)_m$. With Proposition 4.6.5 we get some information about certain subsequences stated in the following theorem.

Theorem 4.6.6. *Let μ be a self-similar measure with $r_1 m_1 = r_2 m_2$. Then, for all $m \in \mathbb{N}_0$,*

$$\|\tilde{f}_{2m}\|_\infty = \frac{\max\Big\{1, \big(\frac{m_1}{m_2}\big)^{2^{v(m)}}\Big\}}{\sqrt{m_1 + m_2\big(\frac{m_1}{m_2}\big)^{2^{v(m)+1}}}} \|\tilde{f}_m\|_\infty. \tag{4.52}$$

Suppose $m_1 \leq m_2$ and let l be an odd number. Then, for all $k \in \mathbb{N}$,

$$\|\tilde{f}_{2^k l}\|_\infty = m_1^{-\frac{k}{2}} \prod_{j=1}^{k} \Big(1 + \Big(\frac{m_1}{m_2}\Big)^{2^j - 1}\Big)^{-\frac{1}{2}} \|\tilde{f}_l\|_\infty. \tag{4.53}$$

Proof. (4.52) follows directly from (4.50) and (4.51). Suppose $m_1 \leq m_2$ and $l \in \mathbb{N}$ is odd. Then iterative application of (4.52) gives

$$\|\tilde{f}_{2l}\|_\infty = \frac{1}{\sqrt{m_1}} \frac{1}{\sqrt{1 + \frac{m_1}{m_2}}} \|\tilde{f}_l\|_\infty,$$

$$\|\tilde{f}_{2^2 l}\|_\infty = \frac{1}{\sqrt{m_1}} \frac{1}{\sqrt{1 + \big(\frac{m_1}{m_2}\big)^3}} \frac{1}{\sqrt{m_1}} \frac{1}{\sqrt{1 + \frac{m_1}{m_2}}} \|\tilde{f}_l\|_\infty,$$

and so on, and therefore (4.53) holds. □

Corollary 4.6.7. *Let $l \in \mathbb{N}$ be odd. Then the following statements hold.*

(i) *If $m_1 = m_2$, then for all $k \in \mathbb{N}$,*

$$\|\tilde{f}_{2^k l}\|_\infty = \|\tilde{f}_l\|_\infty.$$

(ii) *If $m_1 < m_2$, then $C := \dfrac{1}{\sqrt{m_1\big(1 + \frac{m_1}{m_2}\big)}} > 1$, and we have for all $k \in \mathbb{N}$,*

$$\|\tilde{f}_{2^k l}\|_\infty \geq C^k \|\tilde{f}_l\|_\infty.$$

Additionally, for all $k \in \mathbb{N}$,

$$\|\tilde{f}_{2^k l}\|_\infty \leq m_1^{-\frac{k}{2}} \left(\frac{m_2}{m_1}\right)^{\frac{k}{2}(2^k - 1)} \|\tilde{f}_l\|_\infty.$$

Proof. (i) follows directly from (4.53) by putting $m_1 = m_2 = \frac{1}{2}$.

If $m_1 < m_2$, then, for all $j \in \mathbb{N}$,

$$1 + \Big(\frac{m_1}{m_2}\Big)^{2^j - 1} \leq 1 + \frac{m_1}{m_2}.$$

Then,

$$\|\tilde{f}_{2^k l}\|_\infty \geq m_1^{-\frac{k}{2}} \Big(1 + \frac{m_1}{m_2}\Big)^{-\frac{k}{2}} \|\tilde{f}_l\|_\infty,$$

and since $m_1 < m_2$ implies $m_1 < \frac{1}{2}$, we have $m_1\big(1 + \frac{m_1}{m_2}\big) < 1$.

For the upper estimate, we write

$$\prod_{j=1}^{k}\bigg(1 + \Big(\frac{m_1}{m_2}\Big)^{2^j-1}\bigg) \geq \bigg(1 + \Big(\frac{m_1}{m_2}\Big)^{2^k-1}\bigg)^k \geq \Big(\frac{m_1}{m_2}\Big)^{k(2^k-1)},$$

which proves (ii). □

4.7. Self-similar measures with $r_1 m_1 = r_2 m_2$ and $r_1 + r_2 = 1$

As in the previous section we have the condition $r_1 m_1 = r_2 m_2$. We treat the special case where $r_1 + r_2 = 1$ from which follows that $r_1 = m_2$ and $r_2 = m_1$. Such measures have been investigated e.g. by Sabot [44] and [45].

Theorem 4.7.1. *Let μ be a self-similar measure where $r_1 = m_2$ and $r_2 = m_1$ (and therefore $r_1 + r_2 = 1$). Then the positive eigenvalues of $-\frac{d}{d\mu}\frac{d}{x}$ with Neumann boundary conditions coincide with those with Dirichlet boundary conditions.*

Proof. Since the eigenvalues are the squares of the zeros of sinp and sinq, respectively, it is sufficient to show that sinp = sinq. To do that we show that for all $n \in \mathbb{N}_0$

$$p_{2n+1} = q_{2n+1}.$$

We do this by complete induction using the recursion formulas from Corollary 4.5.11. By Definition 4.1.1 we have

$$p_1 = \int_0^1 d\mu = 1$$

and

$$q_1 = \int_0^1 dt = 1.$$

Now, let $n \in \mathbb{N}$ and suppose that for $i = 0, \dots, n-1$ holds $p_{2i+1} = q_{2i+1}$. By (4.27) and rearrangement of the order of the terms in the sums we get

$$\begin{aligned}
p_{2n+1} &= \frac{1}{1 - m_2^n r_2^{n+1} - m_1^n r_1^{n+1}} \Bigg(\sum_{i=0}^{n-1} m_2^i r_2^{i+1} (r_1 m_1)^{n-i} p_{2i+1}\, q_{2n-2i} \\
&\qquad + \sum_{i=1}^{n} (r_2 m_2)^i m_1^{n-i} r_1^{n-i+1} p_{2i}\, p_{2n-2i+1} \Bigg) \\
&= \frac{1}{1 - m_2^n r_2^{n+1} - m_1^n r_1^{n+1}} \Bigg(\sum_{i=1}^{n} m_2^{n-i} r_2^{n+1-i} (r_1 m_1)^i p_{2n+1-2i}\, q_{2i} \\
&\qquad + \sum_{i=0}^{n-1} (r_2 m_2)^{n-i} m_1^i r_1^{i+1} p_{2n-2i}\, p_{2i+1} \Bigg).
\end{aligned}$$

Then, by the induction hypothesis and (4.29),

$$\begin{aligned}
p_{2n+1} &= \frac{1}{1 - m_2^n r_2^{n+1} - m_1^n r_1^{n+1}} \Bigg(\sum_{i=1}^{n} m_2^{n-i} r_2^{n+1-i} (r_1 m_1)^i q_{2n+1-2i}\, q_{2i} \\
&\qquad + \sum_{i=0}^{n-1} (r_2 m_2)^{n-i} m_1^i r_1^{i+1} p_{2n-2i}\, q_{2i+1} \Bigg) \\
&= q_{2n+1}.
\end{aligned}$$

□

With the above theorem we can reformulate Theorem 4.3.3 to get a property of the Wronskian of $f_{N,m}$ and $f_{D,m}$.

Corollary 4.7.2. *Let μ be as above, let λ_m be the mth eigenvalue, let $f_{N,m} = \mathrm{cp}_{\sqrt{\lambda_m}}$ and $f_{D,m} = \mathrm{sq}_{\sqrt{\lambda_m}}$ be the corresponding Neumann and Dirichlet eigenfunctions constructed in Section 4.1. Then, for all $x \in [0,1]$,*

$$f_{N,m}(x)\, f'_{D,m}(x) - f_{D,m}(x)\, f'_{N,m}(x) = \sqrt{\lambda_m}.$$

Proof. We put $z = \sqrt{\lambda_m}$ in Theorem 4.3.3 and observe that

$$f'_{N,m}(x) = \mathrm{cp}'_{\sqrt{\lambda_m}}(x) = -\sqrt{\lambda_m}\, \mathrm{sp}_{\sqrt{\lambda_m}}(x)$$

and

$$f'_{D,m}(x) = \mathrm{sq}'_{\sqrt{\lambda_m}}(x) = \sqrt{\lambda_m}\, \mathrm{cq}_{\sqrt{\lambda_m}}(x).$$

□

Since eigenfunctions can be multiplied with any non-zero number, the above equation states basically that the Wronskian is constant. A similar property of a slightly different Wronskian has been established in Freiberg [17, p. 41].

4.8. Figures and numbers

In this section we give some explicit results and figures calculated by using formulas we developed in the preceding sections for several examples of self-similar measures. For the calculations we used Sagemath cloud [49].

Example 4.8.1. Tables 4.1, 4.2 and 4.3 collect the first few values of the sequences $(p_n)_n$ and $(q_n)_n$ for the classical Cantor set with evenly distributed measure, that is, for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$. We computed these values with the recursion formulas in Corollary 4.5.11 that we implemented for that purpose in Sagemath.

Figures 4.1 and 4.2 show plots of the functions sinp and sinq for $x \in (0, 50)$ and for $x \in (0, 120)$, respectively, where the first 100 terms of the series are taken into account. In the figures, sinq is drawn in a dash-dot line and sinp in a solid line. The zero points of these functions squared give the Dirichlet and Neumann eigenvalues, respectively. Observe that the pictures suggest that the eigenvalues are in the order

$$\lambda_{N,0} < \lambda_{N,1} < \lambda_{D,1} < \lambda_{D,2} < \lambda_{N,2} < \lambda_{N,3} < \lambda_{D,3} < \lambda_{D,4} < \dots .$$

Table 4.4 contains the first 32 positive Neumann eigenvalues correct to 15 decimal places (rounded down). These values have been calculated as zero points of the polynomial

$$\sum_{n=0}^{a} (-1)^n \, p_{2n+1} \, z^n \quad \left(\approx \frac{\mathrm{sinp}(\sqrt{z})}{\sqrt{z}} \right),$$

using certain numerical methods. For that we used the command `findroot` from the `mpmath` library in Sagemath cloud ([49]) with a starting value that we took from a plot in each case. This way, we computed each zero point of the above polynomial with an accuracy of 100 digits, where we chose a in each case such that the first 15 decimals of the zero point remain fixed against any further increase of the number of terms. Note that by Lemma 4.1.3 we have

$$p_{2n+1} \leq \frac{1}{n!} q_2(1)^n = \frac{1}{n! \cdot 2^n},$$

from which a more detailed error estimate can be obtained.

Observe that, as stated in Theorem 4.6.1, we have that $\lambda_{N,2m} = 6 \cdot \lambda_{N,m}$ for all m. The distances between eigenvalues differ very much, there are several groups that lie very close together while there are big gaps as well.

In Table 4.5 we give approximate values of the $L_2(\mu)$ norms and the sup norms of the eigenfunctions $f_{N,m} = \mathrm{cp}_{\sqrt{\lambda_m}}$.

The L_2 norms have been calculated with the formula in Corollary 4.2.4 where we put in the values for λ from Table 4.4. The number of summands had to be chosen higher with bigger eigenvalues, so that the limit value could be approximated with sufficient accuracy.

For the supremum norms we calculated $f_{N,m}\big(S_w(0)\big)$ and $f_{N,m}\big(S_w(1)\big)$ for all words $w \in \{1,2\}^n$ for a certain iteration level n and determined the biggest of these values. We varied n between 5 and 8 to get the values. These calculations were made with the formulas in Proposition 4.5.8. For that, the eigenvalue λ_m and values of the functions sinp, sinq and cosp were needed. Note that the sup norm values are rather rough approximations.

Then we determined the sup norm of the normalized eigenfunctions

$$\|\tilde{f}_{N,m}\|_\infty = \frac{\|f_{N,m}\|_\infty}{\|f_{N,m}\|_{L_2(\mu)}}.$$

Observe that, as stated in Equation (4.52), the values for even m are the same as for $\frac{m}{2}$, respectively.

In Table 4.6 we state the first 32 eigenvalues with Dirichlet boundary conditions exact to 15 decimals. The procedure for the calculations is the same as with the Neumann eigenvalues explained above. Note that two values at a time lie close together, namely $\lambda_{D,2m-1}$ and $\lambda_{D,2m}$. Especially close together are pairs of the form $\lambda_{D,2^n-1}$ and $\lambda_{D,2^n}$. Therefore we had to increase the accuracy of $\lambda_{D,31}$ and $\lambda_{D,32}$ to 25 digits to make the difference visible.

Approximations of the Dirichlet eigenvalues have been calculated before by Etienne [12] by approximating μ by finitely many point masses.

As in the Neumann case, we calculated norms of Dirichlet eigenfunctions, see Table 4.7.

n	p_{2n+1}
0	1
1	$\frac{1}{5}$
2	$\frac{27}{2\,800}$
3	$\frac{6\,383}{31\,906\,000}$
4	$\frac{928\,046\,087}{427\,065\,638\,720\,000}$
5	$\frac{18\,312\,146\,532\,699}{1\,290\,321\,173\,531\,252\,800\,000}$
6	$\frac{36\,205\,626\,974\,761\,334\,065\,053}{595\,390\,835\,517\,679\,574\,442\,022\,016\,000\,000}$
7	$\frac{4\,976\,934\,962\,986\,304\,441\,117\,658\,183}{27\,444\,983\,400\,881\,701\,904\,144\,720\,110\,742\,041\,600\,000}$
8	$\frac{9\,554\,109\,968\,352\,546\,557\,662\,907\,330\,504\,773\,561\,465\,623}{24\,293\,779\,244\,421\,488\,801\,231\,482\,393\,897\,413\,175\,652\,507\,508\,121\,600\,000\,000}$
9	$\frac{146\,991\,787\,616\,583\,137\,720\,984\,325\,054\,111\,289\,057\,094\,244\,281\,881\,523\,497}{228\,839\,658\,236\,344\,563\,453\,452\,927\,437\,095\,017\,291\,959\,177\,590\,164\,358\,527\,465\,655\,296\,000\,000\,000}$

Table 4.1.: The first ten odd members of (p_n) for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

n	q_{2n+1}
0	1
1	$\frac{1}{8}$
2	$\frac{21}{4\,240}$
3	$\frac{33\,253}{383\,465\,600}$
4	$\frac{76\,118\,969}{91\,537\,621\,184\,000}$
5	$\frac{20\,165\,083\,798\,890\,939}{4\,103\,397\,246\,999\,022\,891\,520\,000}$
6	$\frac{129\,726\,498\,389\,261\,896\,497}{6\,714\,982\,210\,971\,717\,632\,658\,867\,200\,000}$
7	$\frac{2\,413\,673\,468\,793\,966\,201\,825\,434\,809\,368\,471}{45\,210\,174\,990\,342\,427\,454\,327\,995\,801\,851\,920\,608\,256\,000\,000}$
8	$\frac{1\,194\,381\,655\,935\,980\,000\,421\,990\,244\,022\,269\,580\,561\,517}{11\,036\,319\,046\,998\,816\,108\,771\,342\,849\,627\,021\,590\,229\,476\,137\,440\,051\,200\,000}$
9	$\frac{126\,866\,175\,828\,333\,349\,955\,887\,526\,100\,988\,154\,691\,317\,901\,447\,037\,378\,112\,773}{762\,232\,235\,417\,372\,510\,271\,600\,164\,875\,680\,211\,782\,266\,161\,937\,386\,279\,477\,493\,896\,522\,956\,800\,000\,000}$

Table 4.2.: The first ten odd members of (q_n) for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

n	p_{2n}, q_{2n}
0	1
1	$\frac{1}{2}$
2	$\frac{3}{80}$
3	$\frac{311}{296\,800}$
4	$\frac{4\,716\,349}{329\,780\,416\,000}$
5	$\frac{186\,511\,983\,201}{1\,659\,577\,072\,065\,920\,000}$
6	$\frac{7\,179\,455\,540\,679\,158\,013}{12\,761\,565\,438\,166\,961\,192\,627\,200\,000}$
7	$\frac{159\,906\,376\,968\,352\,543\,502\,900\,259}{83\,334\,473\,684\,067\,539\,316\,352\,053\,491\,456\,000\,000}$
8	$\frac{60\,996\,703\,846\,644\,308\,894\,938\,372\,985\,688\,873}{13\,022\,158\,544\,999\,621\,792\,336\,779\,426\,151\,940\,728\,460\,083\,200\,000}$
9	$\frac{55\,173\,436\,475\,334\,110\,717\,731\,416\,972\,957\,128\,310\,218\,371\,151\,677}{6\,487\,868\,455\,643\,720\,781\,486\,892\,657\,131\,701\,453\,895\,648\,546\,905\,230\,058\,700\,800\,000\,000}$

Table 4.3.: The first ten even members of (p_n) and (q_n) for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

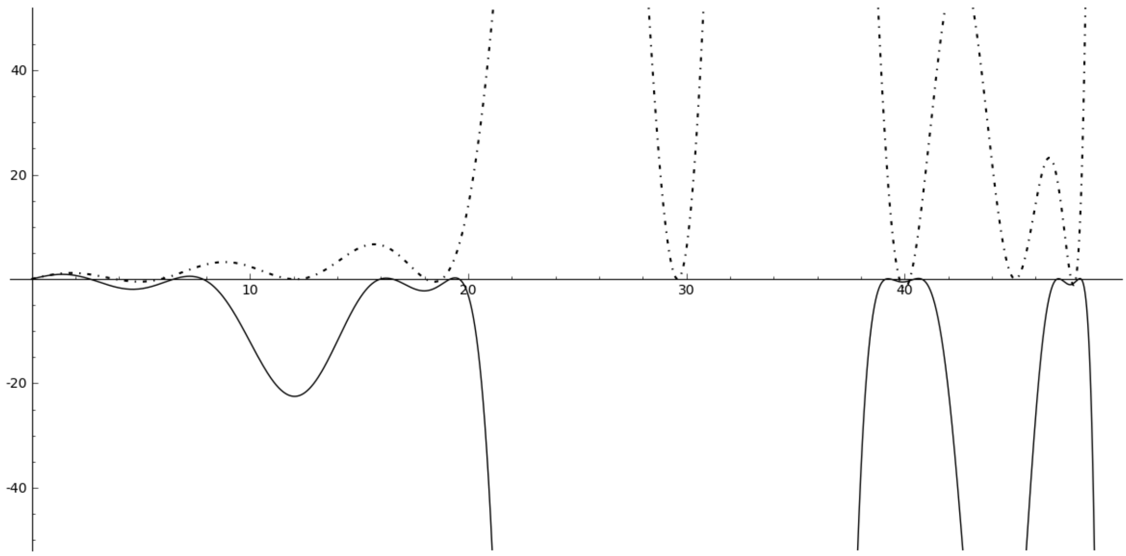


Figure 4.1.: sinp (solid, Neumann) and sinq (dash-dot, Dirichlet) for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

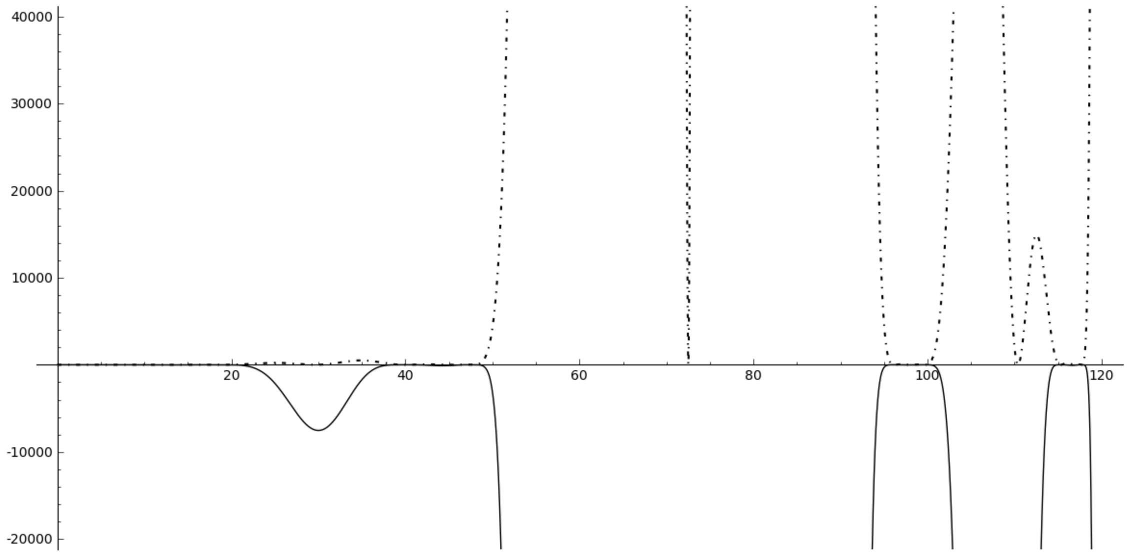


Figure 4.2.: sinp (solid, Neumann) and sinq (dash-dot, Dirichlet) for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

m	$\lambda_{N,m}$	a	m	$\lambda_{N,m}$	a
1	7.09743 10981 41122	12	17	9211.73939 77562 51229	86
2	42.58458 65888 46733	19	18	9288.33494 53277 71442	85
3	61.34420 39227 01662	19	19	9316.34702 24100 75024	85
4	255.50751 95330 80403	28	20	9827.40854 94892 99413	87
5	272.98357 08191 47205	28	21	9847.99008 31996 68501	87
6	368.06522 35362 09975	30	22	9975.76460 05394 58261	87
7	383.55288 31276 93176	31	23	9994.03735 25970 68208	87
8	1533.04511 71984 82423	47	24	13250.34804 73035 59112	97
9	1548.05582 42212 95240	47	25	13260.71659 87844 44965	96
10	1637.90142 49148 83235	48	26	13324.61669 98067 78407	97
11	1662.62743 34232 43043	48	27	13342.22766 88915 03102	97
12	2208.39134 12172 59852	53	28	13807.90379 25969 54355	97
13	2220.76944 99677 96401	53	29	13816.72725 09206 34538	98
14	2301.31729 87661 59059	53	30	13875.49272 43655 06427	98
15	2312.58212 07275 84404	53	31	13883.67238 03565 18424	97
16	9198.27070 31908 94542	85	32	55189.62421 91453 67256	160

Table 4.4.: Neumann eigenvalues of $-\frac{d}{d\mu}\frac{d}{dx}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

m	$\|f_{N,m}\|_2$	$\|f_{N,m}\|_\infty$	$\|\tilde{f}_{N,m}\|_\infty$	m	$\|f_{N,m}\|_2$	$\|f_{N,m}\|_\infty$	$\|\tilde{f}_{N,m}\|_\infty$
1	0.801	1.000	1.248	17	0.666	1.001	1.503
2	0.801	1.000	1.248	18	0.687	1.007	1.467
3	0.966	1.261	1.306	19	0.829	1.307	1.577
4	0.801	1.000	1.248	20	0.746	1.049	1.405
5	0.746	1.049	1.405	21	0.688	1.093	1.588
6	0.966	1.261	1.306	22	0.897	1.356	1.512
7	1.145	1.604	1.401	23	1.057	1.703	1.612
8	0.801	1.000	1.248	24	0.966	1.261	1.306
9	0.687	1.007	1.467	25	0.826	1.262	1.529
10	0.746	1.049	1.405	26	0.886	1.306	1.474
11	0.897	1.356	1.512	27	1.063	1.694	1.594
12	0.966	1.261	1.306	28	1.145	1.604	1.401
13	0.886	1.306	1.474	29	1.049	1.656	1.579
14	1.145	1.604	1.401	30	1.346	2.029	1.508
15	1.346	2.029	1.508	31	1.579	2.563	1.625
16	0.801	1.000	1.248	32	0.801	1.000	1.248

Table 4.5.: Norms of Neumann eigenfunctions for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

m	$\lambda_{D,m}$	a	m	$\lambda_{D,m}$	a
1	14.43524 05120 53874	13	17	9233.86793 80086 63779	84
2	35.26023 80242 77225	16	18	9271.62879 27212 74161	83
3	140.78105 33845 56059	24	19	9589.26839 61415 98781	85
4	151.29061 60550 19631	23	20	9598.24041 25849 12727	85
5	326.05732 83577 53770	29	21	9923.46445 25858 18608	85
6	353.41692 07675 57756	29	22	9957.06520 21538 29857	87
7	876.27445 96020 73755	39	23	12190.28558 35702 41470	93
8	876.50531 85096 60313	39	24	12190.29241 90995 34112	94
9	1581.17702 42871 45662	46	25	13284.12682 48731 70732	94
10	1619.40072 91584 24238	46	26	13311.27448 40460 62950	95
11	2029.61356 34510 19039	51	27	13668.53690 39463 19748	96
12	2033.85281 30577 61437	51	28	13671.26816 68726 11762	96
13	2268.79163 36445 60767	53	29	13851.83951 26643 76419	96
14	2289.60406 94424 69130	52	30	13866.93782 41331 73771	96
15	5258.33939 69212 17309	71	31	31550.03640 02815 21874 64223 25788	139
16	5258.33940 31726 23308	71	32	31550.03640 02815 21874 89689 65410	139

Table 4.6.: Dirichlet eigenvalues of $-\frac{d}{d\mu}\frac{d}{dx}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

m	$\|f_{D,m}\|_2$	$\|f_{D,m}\|_\infty$	$\|\tilde{f}_{D,m}\|_\infty$	m	$\|f_{D,m}\|_2$	$\|f_{D,m}\|_\infty$	$\|\tilde{f}_{D,m}\|_\infty$
1	0.627	0.920	1.469	17	8.492	15.635	1.841
2	0.711	0.985	1.387	18	7.115	12.394	1.742
3	0.446	0.790	1.770	19	1.685	3.996	2.372
4	0.457	0.793	1.734	20	1.679	3.985	2.374
5	1.115	1.628	1.461	21	5.110	10.266	2.009
6	1.273	2.105	1.654	22	5.787	11.252	1.944
7	0.262	0.646	2.469	23	0.415	1.195	2.883
8	0.262	0.646	2.468	24	0.415	1.306	3.151
9	2.798	5.034	1.799	25	9.565	18.428	1.927
10	2.656	4.694	1.767	26	8.944	16.597	1.856
11	0.719	1.460	2.032	27	1.950	4.852	2.489
12	0.717	1.602	2.233	28	1.945	4.863	2.501
13	3.048	5.661	1.857	29	8.777	15.403	1.755
14	3.481	6.296	1.809	30	9.995	19.451	1.946
15	0.151	0.509	3.369	31	0.087	0.416	4.765
16	0.151	0.528	3.491	32	0.087	0.416	4.765

Table 4.7.: Norms of Dirichlet eigenfunctions for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

Example 4.8.2. For the next example, we take the asymmetric self-similar measure with $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = \dfrac{1}{3^{d_H}}$ and $m_2 = \dfrac{1}{4^{d_H}}$ where d_H is the Hausdorff dimension of the invariant set. That is, d_H is the solution of the equation

$$\frac{1}{3^{d_H}} + \frac{1}{4^{d_H}} = 1.$$

For the calculations we used 0.56049886522386387883902233 for d_H. Variation of this value led to no change in the first 15 digits of the eigenvalues. Plots of sinp and sinq are shown in Figure 4.3 and the first eigenvalues exact to 15 decimal places are displayed in Table 4.8. Note that here $m_1 r_1 \neq m_2 r_2$. There seem to be no fixed order of Neumann and Dirichlet eigenvalues as in Example 4.8.1 and there are no clear pairings of the values.

Example 4.8.3. Figure 4.4 shows plots of sinp and sinq for $r_1 = \frac{1}{3}$, $r_2 = \frac{1}{4}$ and $m_1 = \frac{3}{7}$, $m_2 = \frac{4}{7}$. The invariant set is geometrically the same as in Example 4.8.2, but m_1 and m_2 are chosen such that $r_1 m_1 = r_2 m_2 = \frac{1}{7}$ and thus, $\lambda_{N,2m} = 7 \cdot \lambda_{N,m}$.

Comparing with Example 4.8.1, we observe that the Neumann eigenvalues behave qualitatively similar, but the Dirichlet eigenvalues do not appear in such close pairs. However, it seems to hold again, that two Neumann and two Dirichlet eigenvalues appear in turns.

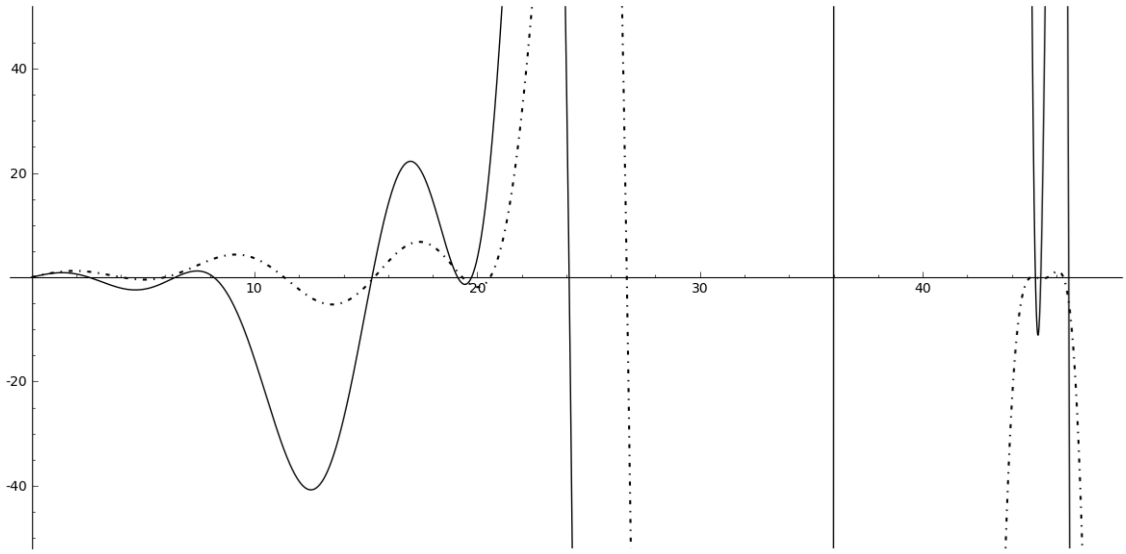


Figure 4.3.: sinp (solid, Neumann) and sinq (dash-dot, Dirichlet) for $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = \frac{1}{3^{d_H}}$ and $m_2 = \frac{1}{4^{d_H}}$

m	$\lambda_{N,m}$	a	m	$\lambda_{D,m}$	a
1	6.567037965687942	11	1	16.107849410419070	12
2	41.632795946820830	16	2	35.907601066462638	15
3	66.822767372091789	19	3	128.330447556120622	21
4	233.355013145153884	24	4	236.463676343561213	24
5	365.584215801794021	27	5	373.701929431216995	27
6	389.945618826510339	28	6	423.638157028808414	28
7	582.138208794906725	30	7	713.786986198043209	31
8	1295.888937033626505	37	8	2013.164883016581104	44

Table 4.8.: Neumann and Dirichlet eigenvalues for $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = \frac{1}{3^{d_H}}$ and $m_2 = \frac{1}{4^{d_H}}$.

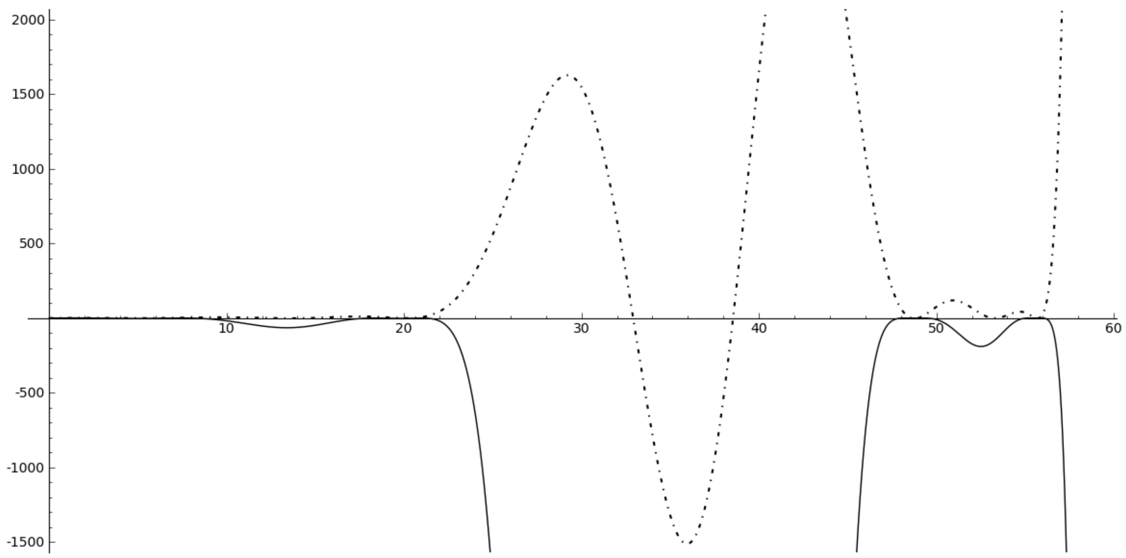


Figure 4.4.: sinp (solid, Neumann) and sinq (dash-dot, Dirichlet) for$r_1 = 1/3$, $r_2 = 1/4$, $m_1 = \frac{3}{7}$ and $m_2 = \frac{4}{7}$.

m	$\lambda_{N,m}$	a	m	$\lambda_{D,m}$	a
1	6.752284245618646	10	1	16.452512161464721	12
2	47.265989719330522	16	2	36.904245287406090	15
3	62.066872795561511	18	3	154.577520453343494	21
4	330.861928035313659	26	4	212.376524344704458	23
5	345.194670941772007	27	5	395.526819249411977	27
6	434.468109568930577	28	6	417.532700806716224	27
7	446.407999438501248	28	7	1083.253271255975735	34
8	2316.033496247195616	45	8	1485.470110503836517	37
9	2332.825185220436900	46	9	2360.481274606702758	44
10	2416.362696592404055	46	10	2397.801276276128236	44
11	2434.484694248270572	46	11	2830.491432008378221	47
12	3041.276766982514042	50	12	2850.987710468049166	47
13	3051.736543145083444	50	13	3093.525406096403347	48
14	3124.855996069508739	50	14	3111.593713450879200	48
15	3133.914016082441210	49	15	7582.772906434721944	58
16	16212.234473730369315	82	16	10398.290767742394136	64

Table 4.9.: Neumann and Dirichlet eigenvalues for $r_1 = 1/3$, $r_2 = 1/4$, $m_1 = \frac{3}{7}$ and $m_2 = \frac{4}{7}$.

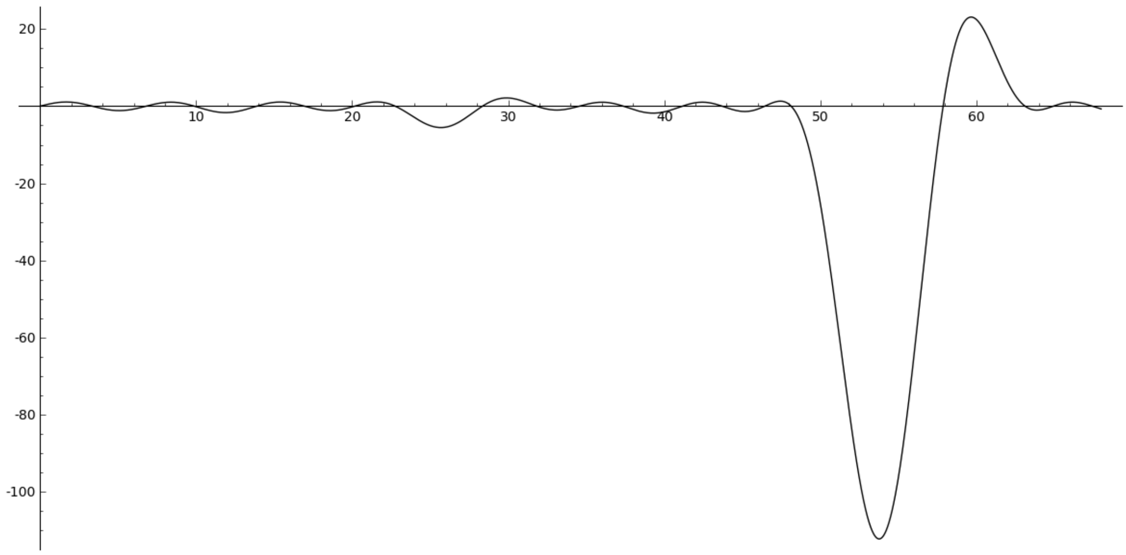


Figure 4.5.: sinp (coincides with sinq) for $r_1 = 0.6$, $r_2 = 0.4$, $m_1 = 0.4$ and $m_2 = 0.6$.

m	λ_m	a	m	λ_m	a
1	11.113238313123921	13	9	1012.173153820335730	54
2	46.305159638016340	19	10	1194.689209582619744	57
3	97.600761284513435	24	11	1396.721102656337624	60
4	192.938165158401419	29	12	1694.457661189469363	65
5	286.725410299828738	34	13	1910.161155469398890	67
6	406.669838685472647	38	14	2157.157626864968439	70
7	517.717830447592425	41	15	2316.668360848575284	73
8	803.909021493339246	48	16	3349.620922888913526	83

Table 4.10.: Neumann (and Dirichlet) eigenvalues for $r_1 = 0.6$, $r_2 = 0.4$, $m_1 = 0.4$ and $m_2 = 0.6$.

Example 4.8.4. We choose the measure with $r_1 = 0.6$, $r_2 = 0.4$, $m_1 = 0.4$ and $m_2 = 0.6$. This measure is supported on the whole interval $[0, 1]$, yet is singular to the Lebesgue measure. In Theorem 4.7.1 we showed that sinp and sinq and thus the Dirichlet and Neumann eigenvalues coincide. In Figure 4.5 a plot of sinp is displayed. It is comparable to the sine function, which we would get for $r_1 = r_2 = m_1 = m_2 = 0.5$. Table 4.10 contains the first 16 eigenvalues. Plots of the Neumann and Dirichlet eigenfunctions can be found in Appendix A.

Example 4.8.5. We take $r_1 = 0.9$, $r_2 = 0.1$, $m_1 = 0.1$ and $m_2 = 0.9$. The resulting measure is supported on $[0, 1]$ as in Example 4.8.4, but in Figure 4.6 we see that sinp looks very different from the sine function. Table 4.11 contains the first 16

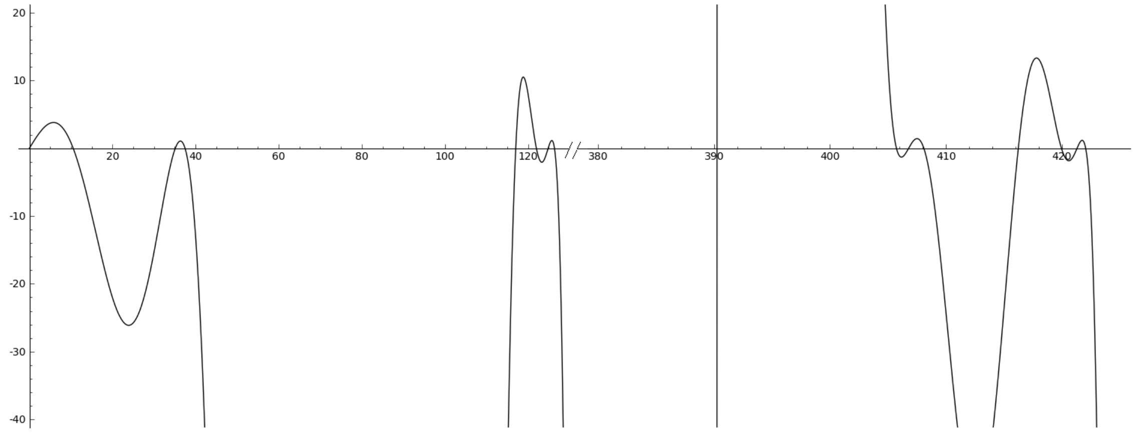


Figure 4.6.: sinp (coincides with sinq) for $r_1 = 0.9$, $r_2 = 0.1$, $m_1 = 0.1$ and $m_2 = 0.9$.

m	λ_m	a	m	λ_m	a
1	111.021168159382246	9	9	164619.744662988877161	44
2	1233.568535104247184	15	10	165477.638319057818349	43
3	1403.454381590697316	15	11	166543.871983977116823	43
4	13706.317056713857601	24	12	173265.973035888557594	43
5	14892.987448715203651	24	13	176376.608221577384951	43
6	15593.937573229970183	25	14	177512.483919664028463	44
7	15976.123552769762561	25	15	178137.700783469958606	44
8	152292.411741265084466	40	16	1692137.908236278716292	72

Table 4.11.: Neumann (and Dirichlet) eigenvalues for $r_1 = 0.9$, $r_2 = 0.1$, $m_1 = 0.1$ and $m_2 = 0.9$.

eigenvalues up to 15 decimal places.

4.9. Remarks and outlook

In this section we state several remarks and thoughts we could not pursue any further in the scope of this thesis.

Conjecture 1. Due to the examination of several examples (see e.g. Examples 4.8.1, 4.8.3, 4.8.4 and 4.8.5) we conjecture that in case of a self-similar measure μ with $r_1 m_1 = r_2 m_2$ the Neumann and Dirichlet eigenvalues satisfy

$$\lambda_{N,0} < \lambda_{N,1} < \lambda_{D,1} < \lambda_{D,2} < \lambda_{N,2} < \lambda_{N,3} < \lambda_{D,3} < \lambda_{D,4} < \dots .$$

Remark 2. It would be very interesting to find out, if there was a relation between our sequences $(p_n)_n$ and $(q_n)_n$ to any known number sequences as e.g. Bernoulli or Euler numbers. Indeed, the definition of $p_n(x)$ or $q_n(x)$ (Definition 4.1.1) is reminiscent of the recursive definition of the Euler polynomials $E_n(x)$ by $E_0(x) := 1$ and

$$E_n(x) := \int_c^x n E_{n-1}(t)\, dt,$$

where $c = \frac{1}{2}$ if n is odd and $c = 0$ for even n. Then the nth Euler number is $E_n = 2^n E_n(1/2)$.

Furthermore, Equation (4.11) has a similar structure as the recursion rule

$$\alpha_n = \frac{1}{2n} \sum_{j=0}^{n-1} \alpha_j\, \alpha_{n-1-j}$$

with $\alpha_0 = \alpha_1 = 1$. With that we have $\alpha_n = \frac{1}{n!}|E_{2n}|$.

Remark 3. One could investigate the functional equations in Theorem 4.5.14 further. In the simple case where $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$, for instance, we can transform them (after eliminating terms containing sinq by using $\operatorname{cosp}^2(z) + \operatorname{sinp}(z)\operatorname{sinq}(z) = 1$) with the abbreviations $u(z) = z \operatorname{sinp}(z)$ and $v(z) = 2\operatorname{cosp}(z)$ to

$$\begin{aligned} u\big(\sqrt{6}z\big) &= 3\,u(z)\,v(z) - 3\,u(z)^2 \\ v\big(\sqrt{6}z\big) &= v(z)^2 - v(z)\,u(z) - 2. \end{aligned}$$

From this one can derive recursion formulas for the sequence $(p_n)_n$ that contain only members of p_n and not, as in Corollary 4.5.11, both p_n and q_n.

Furthermore, it could be possible to somehow solve these functional equations to get a more direct representation of sinp and cosp.

Remark 4. We defined our functions sinp, sinq, cosp and cosq only for real arguments. However, one can just allow the argument to be complex. Then these power series can be treated with methods of complex analysis.

Remark 5. Our recursion law for p_n and q_n works only for self-similar measures with $r_1 + r_2 \leq 1$. It would be interesting to know if one could develop similar formulas for measures with overlaps, i.e. if $r_1 + r_2 > 1$. Such measures are treated for example in [42] and [8], which contains, in particular, numerical solutions of the eigenvalue problem by the finite elements method.

Remark 6. In this work, we examined the eigenvalues of $-\frac{d}{d\mu}\frac{d}{dx}$ by following the basic lines of the treatment of the classical second derivative operator on the interval. In this classical case all eigenvalues are multiples of π^2 and have therefore direct representations in many forms, e.g. by using the series expansion of arctan. Maybe one can find a series representation of eigenvalues of the generalized operator, too, by using such functions as sinp, sinq, cosp and cosq.

Remark 7. In Corollary 4.6.7 we stated upper and lower estimates for subsequences $\left(\|\tilde{f}_{2^k l}\|_\infty\right)_k$, l odd, of the suprema of the normed eigenfunctions. We have no information about the growth of the sequence $\left(\|\tilde{f}_{2k+1}\|_\infty\right)_k$, though.

Such estimates could be used to prove estimates of the heat kernel

$$K(t,x,y) = \sum_{m=1}^{\infty} e^{-\lambda_m t} \tilde{f}_m(x)\, \tilde{f}_m(y)$$

for the corresponding quasi-diffusion process. This process has been investigated for example in [37], [33] and [34].

Remark 8. We used the functions $p_n(x)$ and $q_n(x)$, $x \in [0,1]$, defined in Definition 4.1.1 to replace monomials $\frac{1}{n!}x^n$ in the classical case. One could use these functions to build a kind of generalized polynomials that are adjusted to the measure μ. For instance, we take the sequence

$$\tilde{P}_0(x) = 1, \quad \tilde{P}_1(x) = q_1(x), \quad \tilde{P}_2(x) = p_2(x), \quad \tilde{P}_3(x) = q_3(x), \quad \ldots$$

and orthogonalize it in $L_2(\mu)$ by using the Gram-Schmidt process. We take odd numbered $q_n(x)$ and even numbered $p_n(x)$, because they are the building blocks for the eigenfunctions sq_z and cp_z and they are continuously Lebesgue-differentiable, namely

$$q_n'(x) = q_{n-1}(x) \quad \text{for odd } n$$

and

$$p_n'(x) = p_{n-1}(x) \quad \text{for even } n.$$

Furthermore, we can μ-integrate them and get

$$\int_0^x q_n(t)\, d\mu(t) = q_{n+1}(x) \quad \text{for odd } n$$

and

$$\int_0^x p_n(t)\, d\mu(t) = p_{n+1}(x) \quad \text{for even } n.$$

With that we can apply the generalized integration by parts rule from Lemma 4.2.1 to do the calculations in the Gram-Schmidt algorithm. Note again that we use the notation $p_n := p_n(1)$ and $q_n := q_n(1)$ and assume that those numbers are given since we have a recursion rule in the self-similar case. Then we get

$$P_0(x) := 1,$$

$$P_1(x) := q_1(x) - \int_0^1 q_1(t)\, d\mu(t) = q_1(x) - q_2,$$

$$P_2(x) := p_2(x) - \int_0^1 p_2(t)\, d\mu(t) - \frac{\int_0^1 \big(q_1(t) - q_2\big)\, p_2(t)\, d\mu(t)}{\int_0^1 \big(q_1(t) - q_2\big)^2\, d\mu(t)} \big(q_1(x) - q_2\big).$$

We calculate

$$\begin{aligned}\int_0^1 \big(q_1(t) - q_2\big)\, p_2(t)\, d\mu(t) &= \int_0^1 q_1(t)\, p_2(t)\, d\mu(t) - \int_0^1 q_2\, p_2(t)\, d\mu(t)\\ &= q_1(t)\, p_3(t)\Big|_0^1 - \int_0^1 p_3(t)\, dt - q_2 p_3\\ &= q_1 p_3 - p_4 - q_2 p_3\end{aligned}$$

and

$$\int_0^1 \big(q_1(t)-q_2\big)^2\,d\mu(t) = \int_0^1 q_1(t)^2\,d\mu(t) - 2q_2\int_0^1 q_1(t)\,d\mu(t) + q_2^2\int_0^1 d\mu(t)$$
$$= q_1q_2 - \int_0^1 q_2(t)\,dt - 2q_2^2 + q_2^2p_1$$
$$= q_1q_2 - q_3 - 2q_2^2 + q_2^2p_1.$$

To simplify these expressions a bit we utilize $p_1 = q_1 = 1$ which follows from the definition and $p_2+q_2 = 1$ which follows from Corollary 4.3.2 by putting $n = 1$. Then we get

$$P_2(x) = p_2(x) - \frac{p_4 - p_2\,p_3}{q_3 - p_2\,q_2}\,q_1(x) + \frac{q_2\,p_4 - q_3\,p_3}{q_3 - p_2\,q_2}.$$

In this fashion one can calculate a sequence of $L_2(\mu)$-orthogonal “polynomials”.

As an example we take the Lebesgue measure for μ and put $p_n(x) = q_n(x) = \frac{1}{n!}x^n$. Then

$$P_0(x) = 1, \quad P_1(x) = x - \frac{1}{2}, \quad P_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$$

which are the first Legendre polynomials on $[0, 1]$ (not normed).

If μ is the standard Cantor measure, then $p_2 = q_2 = \frac{1}{2}$, $p_3 = \frac{1}{5}$ and $q_3 = \frac{1}{8}$ and we get

$$P_0(x) = 1, \quad P_1(x) = q_1(x) - \frac{1}{2}, \quad P_2(x) = p_2(x) - \frac{1}{2}q_1(x) + \frac{1}{20}.$$

Maybe one can use these functions for further analytical studies.

Remark 9. With the presented methods one could investigate not only the equation $\frac{d}{d\mu}f' = -\lambda f$, but maybe other differential equations on the interval $[0, 1]$ that are generalized involving a self-similar measure μ.

Remark 10 (Fourier series)**.** It is well known that the normed eigenfunctions $(\tilde{f}_{N,k})_{k=0}^\infty$ and $(\tilde{f}_{D,k})_{k=1}^\infty$ form orthonormal bases in $L_2(\mu)$ (see [17]).

We denote $c_{N,k} := \|\mathrm{cp}_{\sqrt{\lambda_{N,k}}}\|_{L_2(\mu)}$ and $c_{D,k} := \|\mathrm{sq}_{\sqrt{\lambda_{D,k}}}\|_{L_2(\mu)}$ so that

$$\tilde{f}_{N,k}(x) = \frac{1}{c_{N,k}}\,\mathrm{cp}_{\sqrt{\lambda_{N,k}}}$$

and

$$\tilde{f}_{D,k}(x) = \frac{1}{c_{D,k}}\,\mathrm{sq}_{\sqrt{\lambda_{D,k}}}.$$

As an example, we decompose some functions $f \in L_2(\mu)$ into series of eigenfunctions

(Fourier series), ignoring questions about convergence for the moment. Assume that for $x \in [0,1]$

$$f(x) = \sum_{k=0}^{\infty} a_k \tilde{f}_{N,k}(x)$$

with

$$a_k = \int_0^1 f(t)\, \tilde{f}_{N,k}(t)\, d\mu(t).$$

For reasons of simplicity, we take μ to be a symmetric measure. Then cosp = cosq and we have $\operatorname{cosp}^2(z) + \operatorname{sinp}(z)\ \operatorname{sinq}(z) = 1$. From that follows that $\operatorname{cosp}^2(\sqrt{\lambda_{N,k}}) = 1$ and it is heuristically clear that $\operatorname{cosp}(\sqrt{\lambda_{N,k}}) = (-1)^k$. Employing this fact and Lemma 4.2.1, the computations can be made explicitly, following the lines of the classical (Euclidean) case.

As a first example, take $f(x) = x$. Then, for $k \in \mathbb{N}$,

$$\begin{aligned}
a_k &= \frac{1}{c_{N,k}} \int_0^1 t \cdot \operatorname{cp}_{\sqrt{\lambda_k}}(t)\, d\mu(t) \\
&= \frac{1}{c_{N,k}} \left[\frac{1}{\sqrt{\lambda_{N,k}}}\, t\ \operatorname{sp}_{\sqrt{\lambda_{N,k}}}(t) \Big|_0^1 - \frac{1}{\sqrt{\lambda_{N,k}}} \int_0^1 \operatorname{sp}_{\sqrt{\lambda_{N,k}}}(t)\, dt \right] \\
&= \frac{1}{c_{N,k}\lambda_{N,k}} \operatorname{cp}_{\sqrt{\lambda_{N,k}}}(t) \Big|_0^1 \\
&= \frac{1}{c_{N,k}\lambda_{N,k}} \Big(\operatorname{cosp}(\sqrt{\lambda_{N,k}}) - 1 \Big).
\end{aligned}$$

Thus, $a_k = 0$ for even $k \geq 1$ and $a_k = -\dfrac{2}{c_{N,k}\lambda_{N,k}}$ for odd k. Furthermore, we have

$$a_0 = \int_0^1 t\, d\mu(t) = q_2(1) = q_2.$$

Therefore, we have the decomposition into Neumann eigenfunctions

$$x = q_2 - 2 \sum_{k=0}^{\infty} \frac{1}{c_{N,2k+1}\lambda_{N,2k+1}}\, \tilde{f}_{N,2k+1}(x).$$

Note, that the required norms $c_{N,k}$ can be computed with Corollary 4.2.4.

We apply Parseval's identity to this series. This gives

$$\int_0^1 t^2\, d\mu(t) = q_2^2 + \sum_{k=0}^{\infty} \frac{4}{c_{N,2k+1}^2\, \lambda_{N,2k+1}^2},$$

and with

$$\int_0^1 t^2\, d\mu(t) = t\, q_2(t)\Big|_0^1 - \int_0^1 q_2(t)\, dt = q_2 - q_3$$

and $1 - q_2 = p_2$ we get

$$\sum_{k=0}^{\infty} \frac{1}{c_{N,2k+1}^2\, \lambda_{N,2k+1}^2} = \frac{1}{4}(p_2\, q_2 - q_3).$$

If we choose the Lebesgue measure for μ (then $p_2 = q_2 = \frac{1}{2}$, $q_3 = \frac{1}{6}$ and $c_{N,2k+1}^2 = \frac{1}{2}$), the above equation becomes the well known identity

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}.$$

In the same fashion we compute the decomposition of some more examples (μ symmetric):

$$\begin{aligned}
x &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{c_{D,k}\sqrt{\lambda_{D,k}}} \tilde{f}_{D,k}(x) \\
1 &= \sum_{k=0}^{\infty} \frac{2}{c_{D,2k+1}\sqrt{\lambda_{D,2k+1}}}\, \tilde{f}_{D,2k+1}(x) \\
f_{D,2n+1}(x) &= \frac{2}{\sqrt{\lambda_{D,2n+1}}} - 2\sqrt{\lambda_{D,2n+1}} \sum_{k=1}^{\infty} \frac{1}{\big(\lambda_{N,2k} - \lambda_{D,2n+1}\big) c_{N,2k}}\, \tilde{f}_{N,2k}(x), \\
&\quad \text{for every } n \in \mathbb{N}_0
\end{aligned}$$

which are plotted for the standard middle third Cantor measure in Figures 4.7 and 4.8. For the images we used the eigenfunctions as computed and shown in the appendix and the norms $c_{N,k}$ and $c_{D,k}$ shown in Tables 4.5 and 4.7.

Applying Parseval's identity to these decompositions leads, as above, to

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{c_{D,k}^2 \lambda_{D,k}} &= q_2 - q_3 \\
\sum_{k=0}^{\infty} \frac{1}{c_{D,2k+1}^2\, \lambda_{D,2k+1}} &= \frac{1}{4} \\
\sum_{k=1}^{\infty} \frac{1}{\big(\lambda_{N,2k} - \lambda_{D,2n+1}\big)^2 c_{N,2k}^2} &= \frac{c_{D,2n+1}^2}{4\lambda_{D,2n+1}} - \frac{1}{\lambda_{D,2n+1}^2}.
\end{aligned}$$

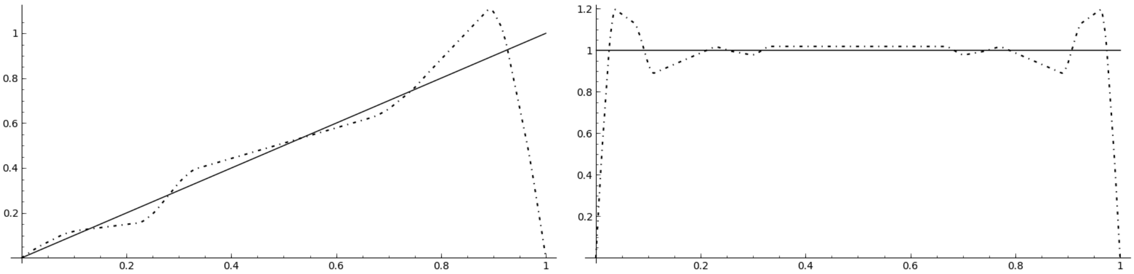


Figure 4.7.: Left side: approximation of $f(x) = x$ by the first 5 terms of the Dirichlet Fourier series for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.
Right side: approximation of $f(x) = 1$ by the first 5 terms of the Dirichlet Fourier series for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

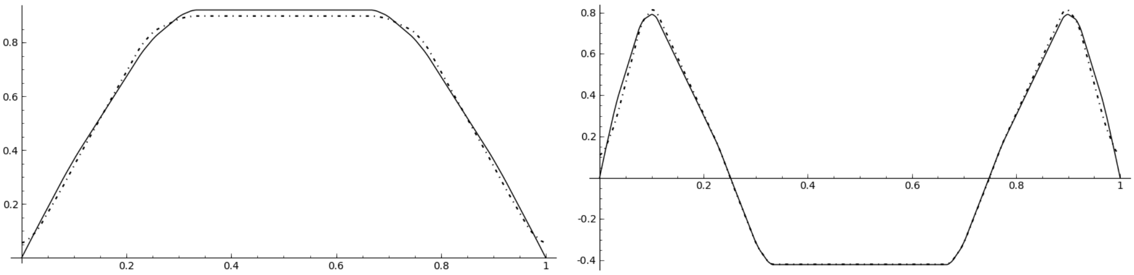


Figure 4.8.: Left side: approximation of $f_{D,1}$ by the first 3 terms of the Neumann Fourier series for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.
Right side: approximation of $f_{D,3}$ by the first 3 terms of the Neumann Fourier series for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

If we again take the Lebesgue measure for μ, we receive the well known identities

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

$$\sum_{k=1}^{\infty} \frac{1}{\left(4k^2 - (2n+1)^2\right)^2} = \frac{\pi^2}{16(2n+1)^2} - \frac{1}{2(2n+1)^4}.$$

Remark 11. The definition of the operator $-\frac{d}{d\mu}\frac{d}{dx}$ can be extended to subsets of $\mathbb{R}^d$, $d \in \mathbb{N}$, see, for example, [48], [41] and [29]. This case, however, is substantially more difficult and the techniques presented here can probably not be readily extended to it.

A. Plots of Eigenfunctions

Figures A.1 to A.4 show plots of the first 32 $L_2(\mu)$-normalized Neumann eigenfunctions for the measure μ with $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$, that is, Example 4.8.1. The eigenfunction corresponding to $\lambda_{N,0}$ is just a constant and is therefore not imaged. The pictures have been created by calculating $f_{N,m}\big(S_w(0)\big)$ and $f_{N,m}\big(S_w(1)\big)$ for all words $w \in \{1,2\}^n$ by iterative application of the formulas in Proposition 4.5.8. Then, since the eigenfunctions are linear on all gap intervals, we connected the points with straight lines. For the images we chose an iteration level of $n = 8$.

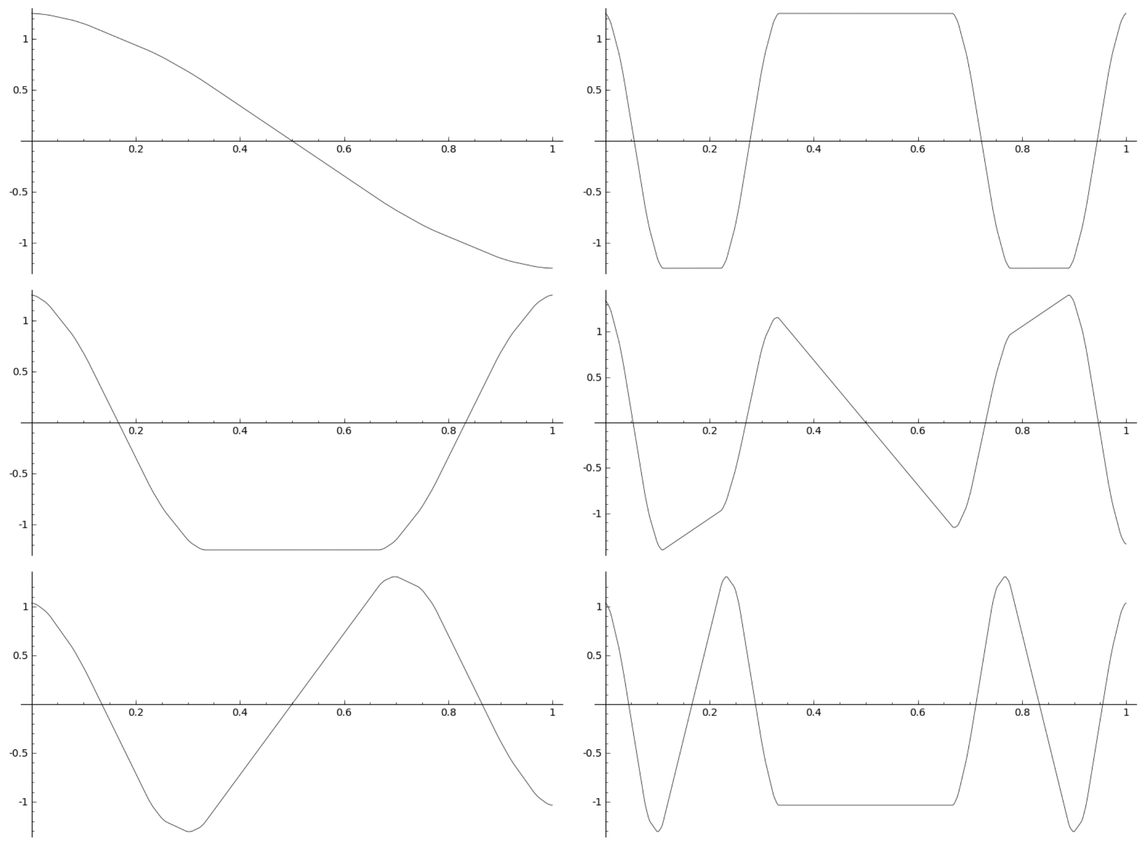


Figure A.1.: $\tilde{f}_{N,1}$ to $\tilde{f}_{N,6}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

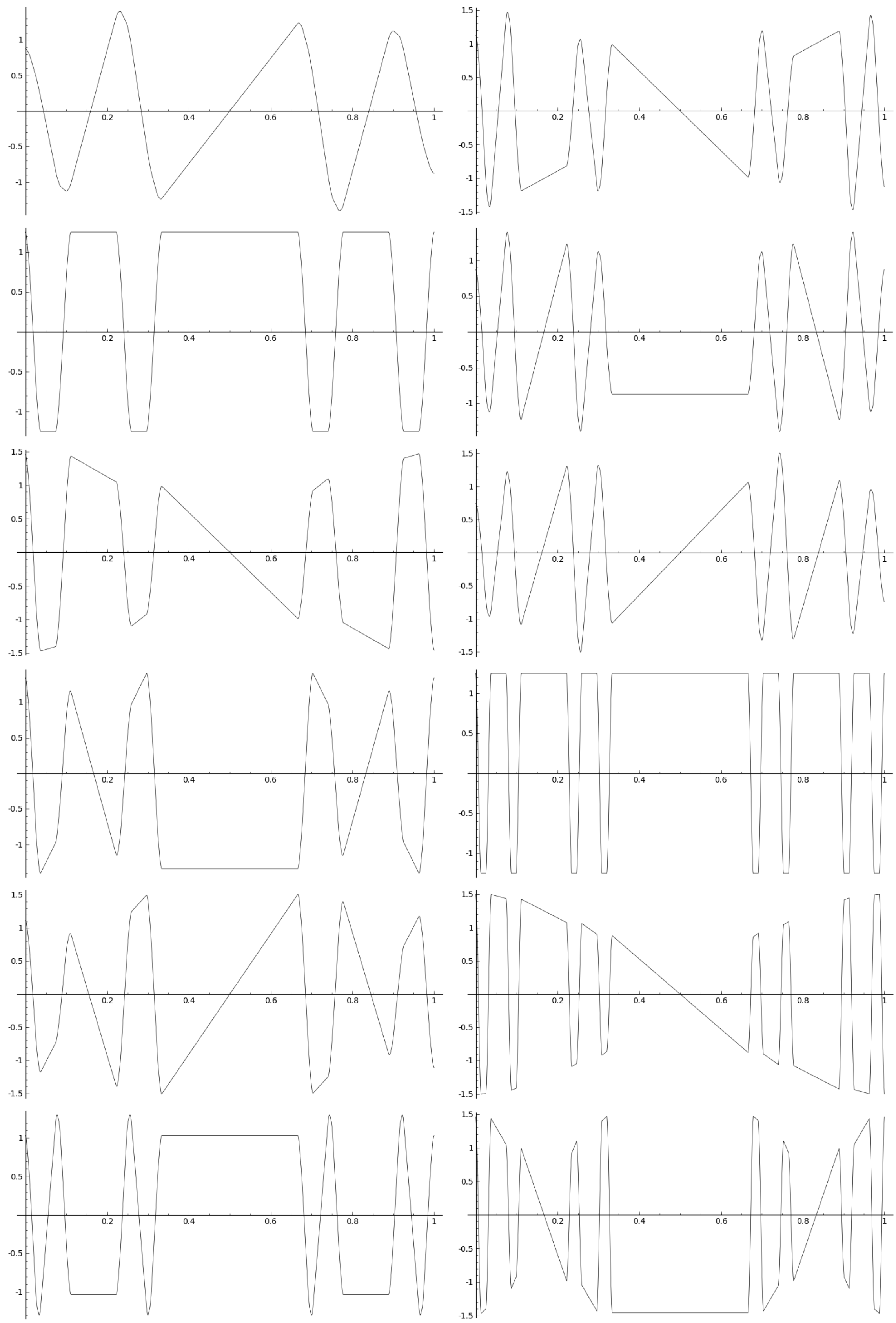


Figure A.2.: $\tilde{f}_{N,7}$ to $\tilde{f}_{N,18}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

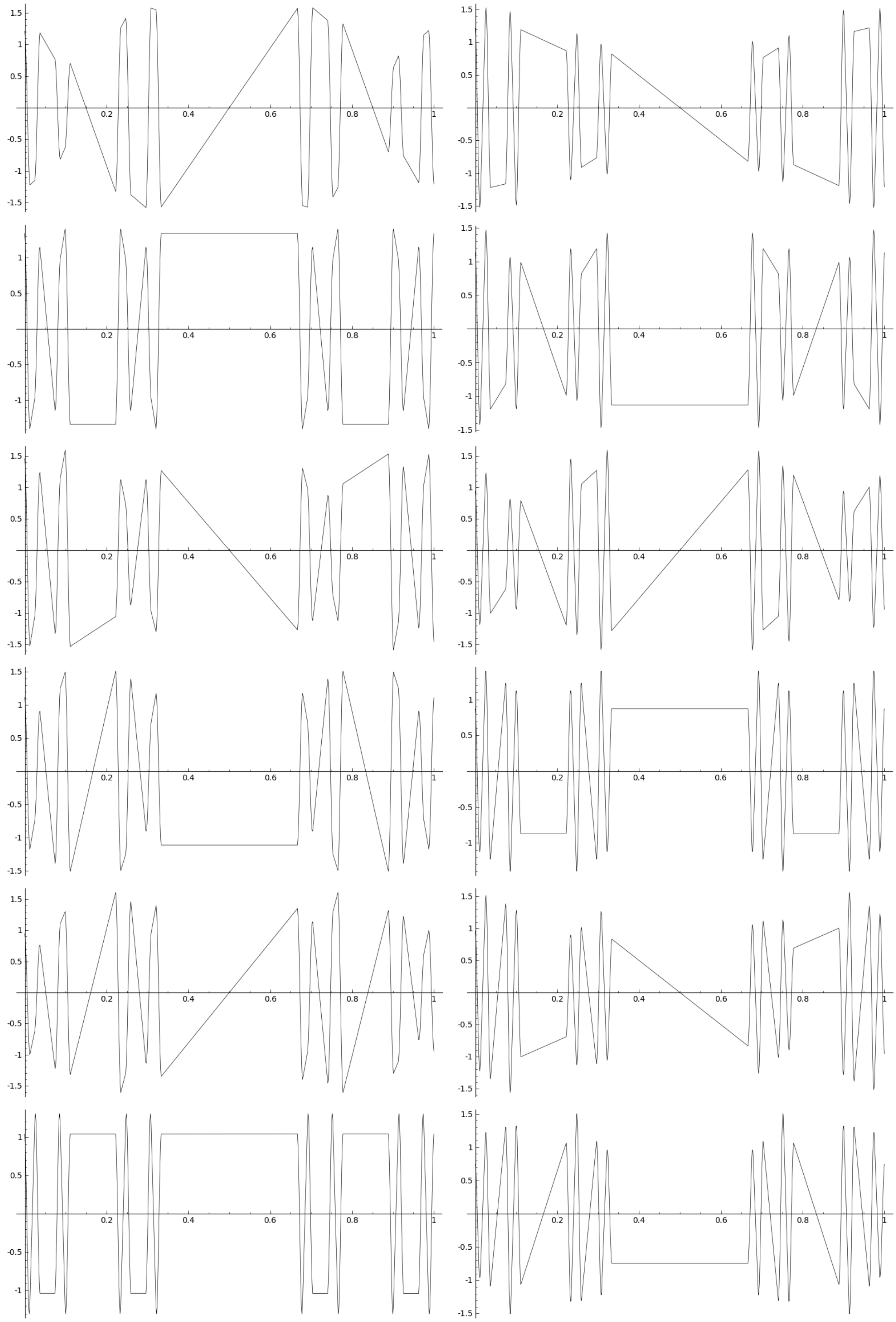


Figure A.3.: $\tilde{f}_{N,19}$ to $\tilde{f}_{N,30}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

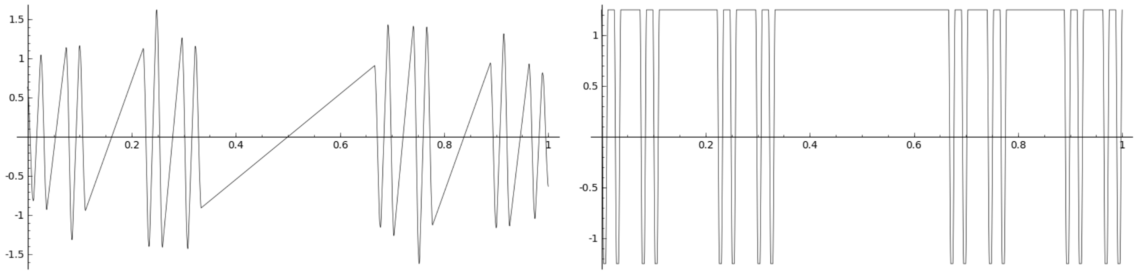


Figure A.4.: $\tilde{f}_{N,31}$ to $\tilde{f}_{N,32}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

Figures A.5 to A.7 show the first 32 normalized Dirichlet eigenfunctions for the same measure as above. The images were created in the same way as the ones with Neumann eigenfunctions above.

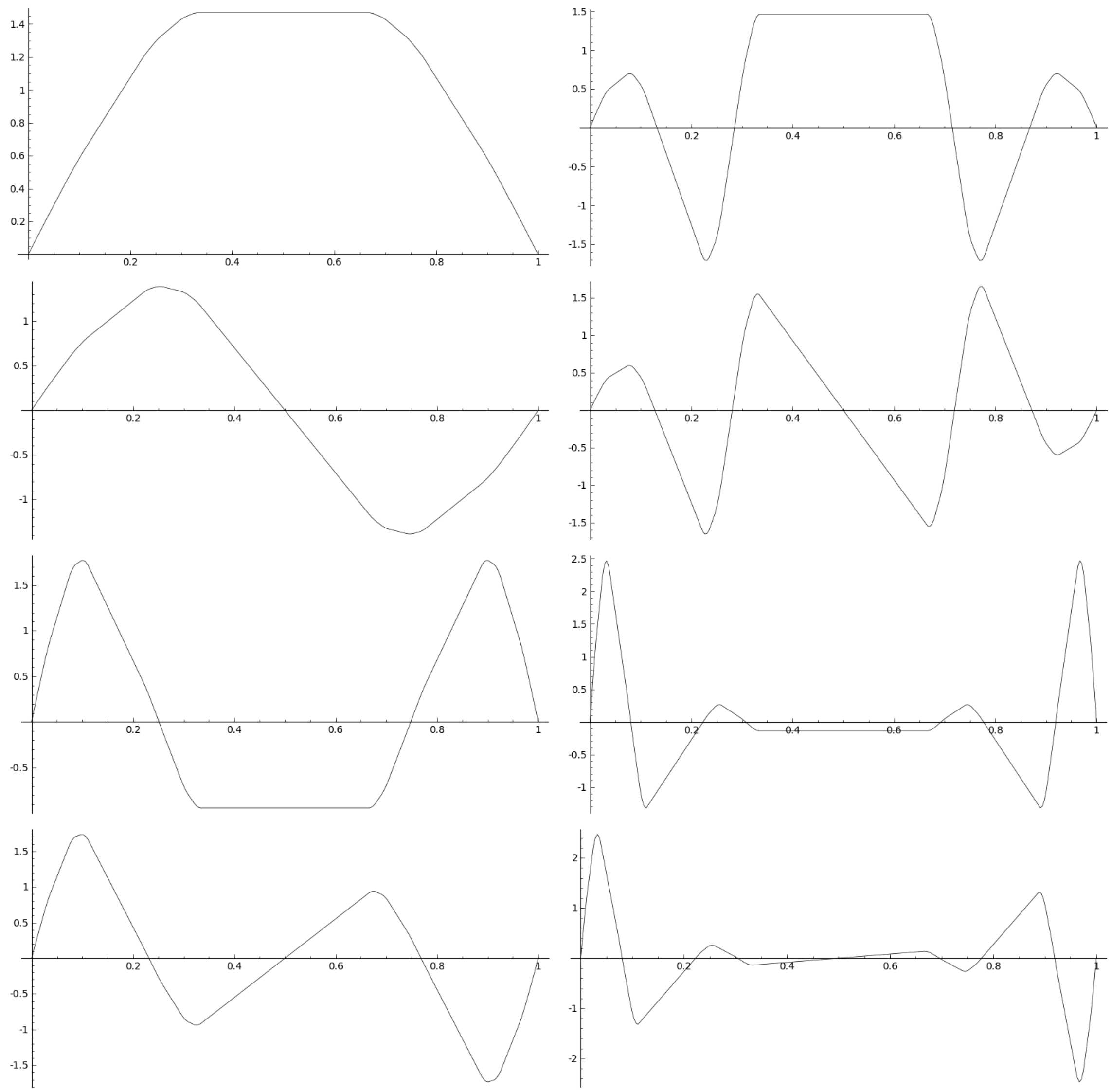


Figure A.5.: $\tilde{f}_{D,1}$ to $\tilde{f}_{D,8}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

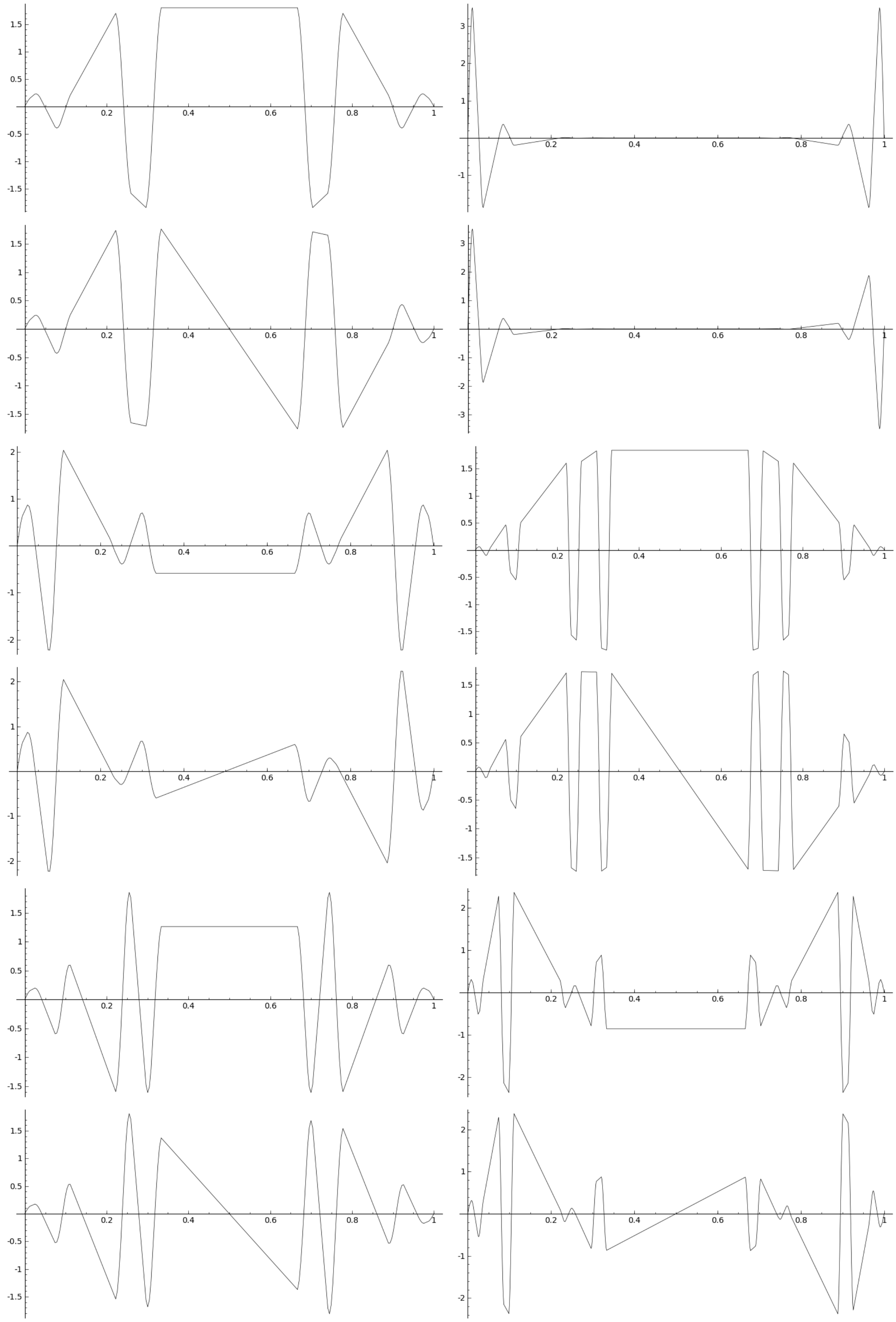


Figure A.6.: $\tilde{f}_{D,9}$ to $\tilde{f}_{D,20}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

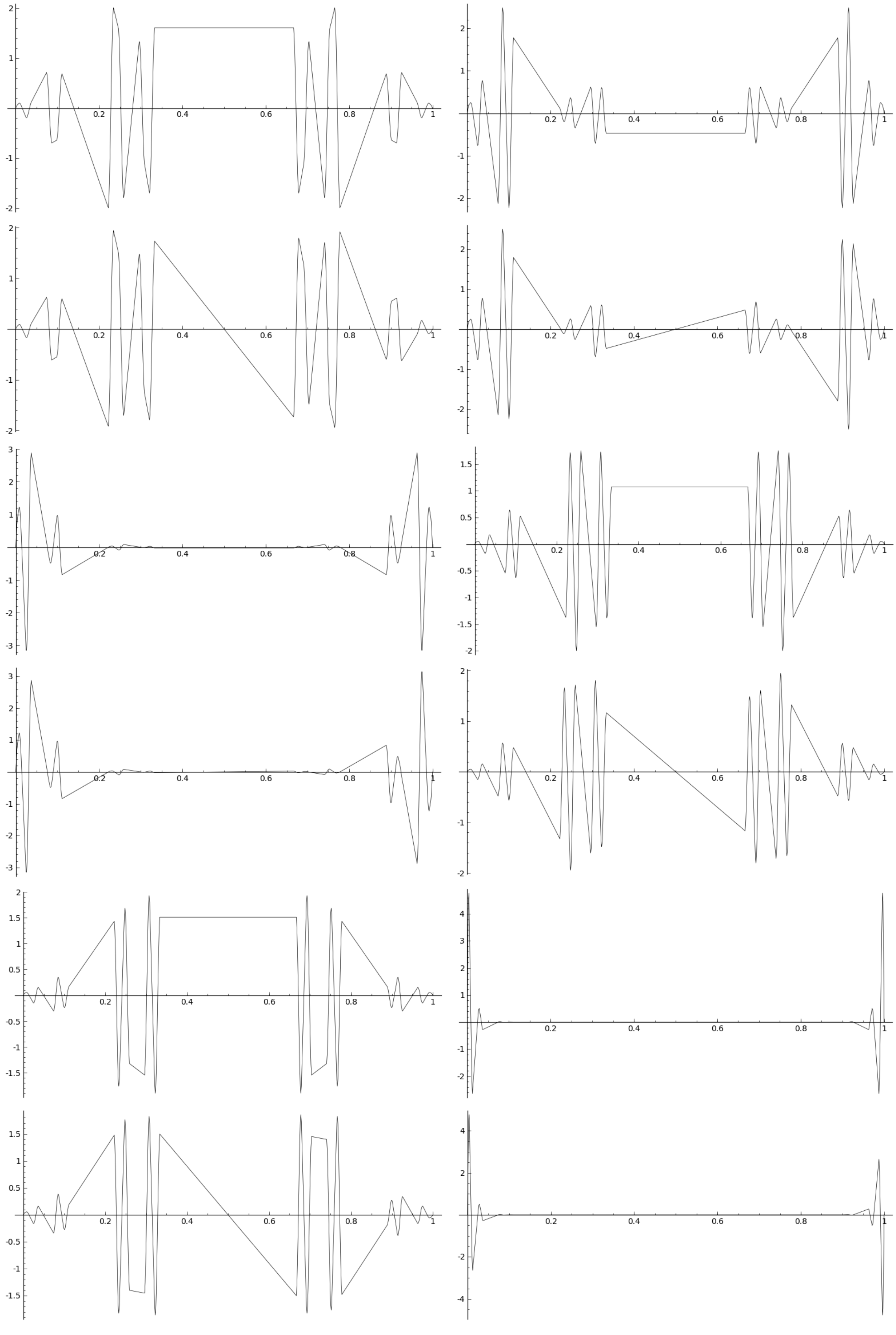


Figure A.7.: $\tilde{f}_{D,21}$ to $\tilde{f}_{D,32}$ for $r_1 = r_2 = \frac{1}{3}$ and $m_1 = m_2 = \frac{1}{2}$.

We want to state some observations made at the above images. First, one can see why the eigenvalues appear in pairs. Namely, we have that, roughly spoken, eigenfunctions $\tilde{f}_{N,2m}$ and $\tilde{f}_{N,2m+1}$ differ only by a twist of the second half. In the Dirichlet case this is even more clearly visible, but with the difference that pairs are formed by an odd and the following even numbered value. Pairs prior to a power of two are hereby especially closely related, which means that the corresponding eigenvalues lie very close together, as seen in Table 4.6.

Then one finds that the Dirichlet and Neumann eigenfunctions are qualitatively very different. The Neumann eigenfunctions seem relatively regular while the Dirichlet eigenfunctions appear more irregularly. This reflects the fact that, physically speaking, Neumann boundary conditions are "natural" while Dirichlet conditions are "forced".

In Figures A.8 to A.11 we consider the self-similar measure from Example 4.8.4, that is with $r_1 = 0.6$, $r_2 = 0.4$, $m_1 = 0.4$ and $m_2 = 0.6$. In Theorem 4.7.1 we showed that Neumann and Dirichlet eigenvalues coincide and the first eigenvalues are displayed in Table 4.10. To get a plot of the eigenfunctions, we calculated $f_{N,m}\big(S_w(0)\big)$ and $f_{N,m}\big(S_w(1)\big)$ as in the previous example, but now, since the measure is supported on the whole interval, we have no gaps where the functions are linear. Therefore we just plotted the separate points for iteration level 8.

The functions are normed in $L_2(\mu)$. Numerical plots of similar eigenfunctions have been done in Bird, Ngai and Teplyaev [5].

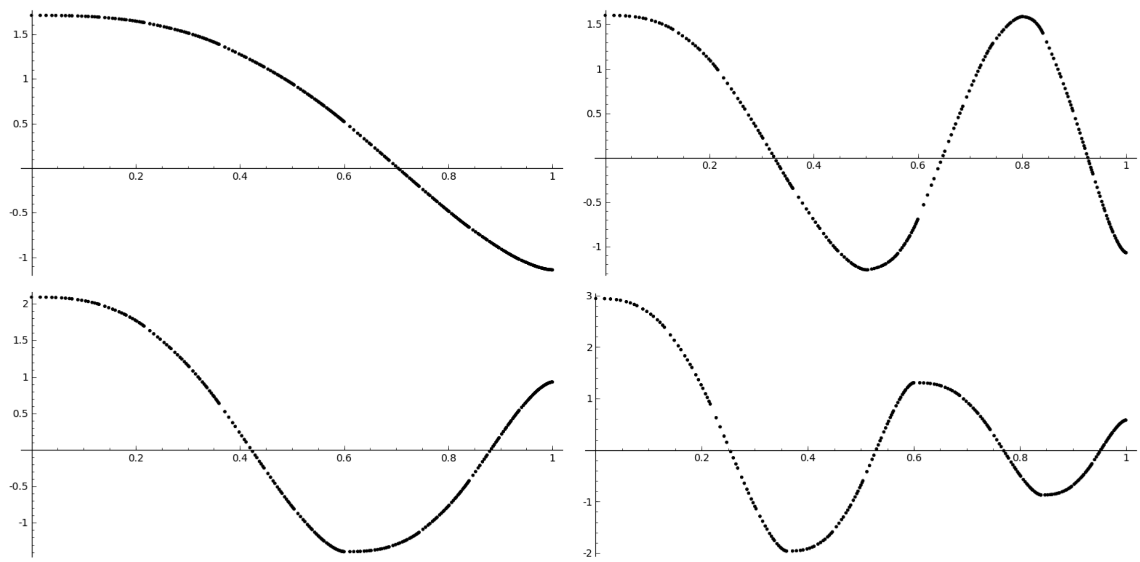


Figure A.8.: $\tilde{f}_{N,1}$ to $\tilde{f}_{N,4}$ for $r_1 = 0.6$, $r_2 = 0.4$, $m_1 = 0.4$ and $m_2 = 0.6$.

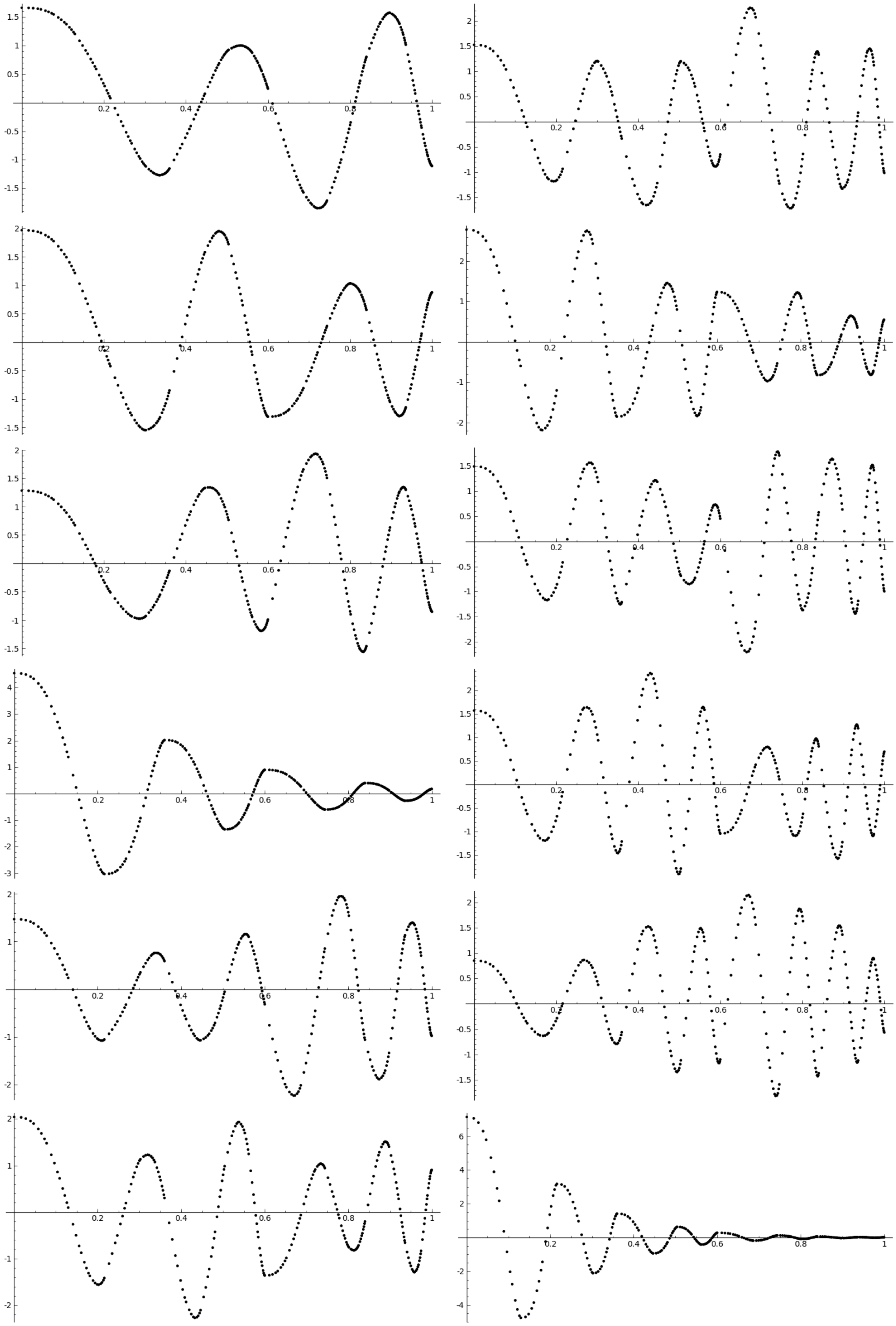


Figure A.9.: $\tilde{f}_{N,5}$ to $\tilde{f}_{N,16}$ for $r_1 = 0.6$, $r_2 = 0.4$, $m_1 = 0.4$ and $m_2 = 0.6$.

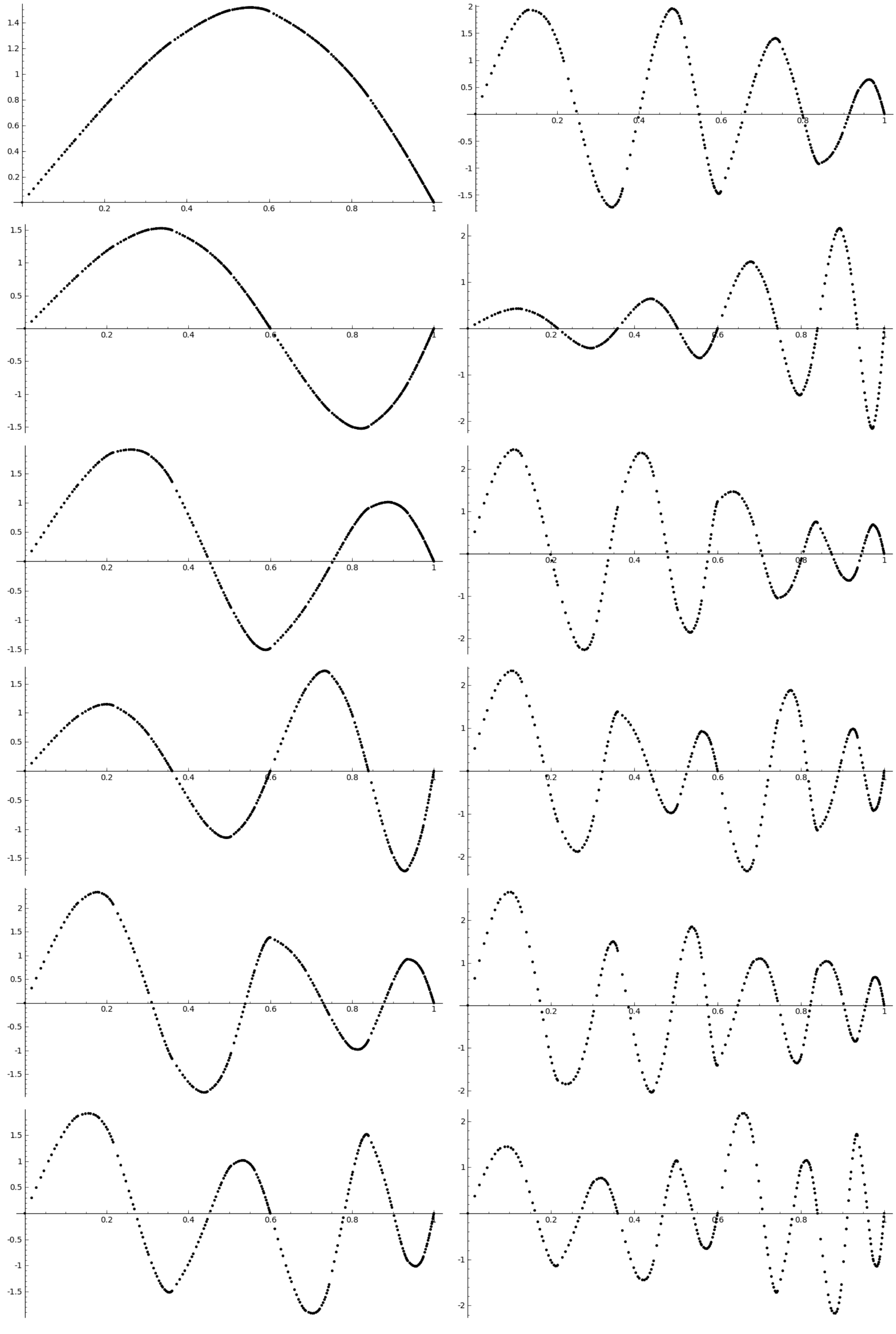


Figure A.10.: $\tilde{f}_{D,1}$ to $\tilde{f}_{D,12}$ for $r_1 = 0.6$, $r_2 = 0.4$, $m_1 = 0.4$ and $m_2 = 0.6$.

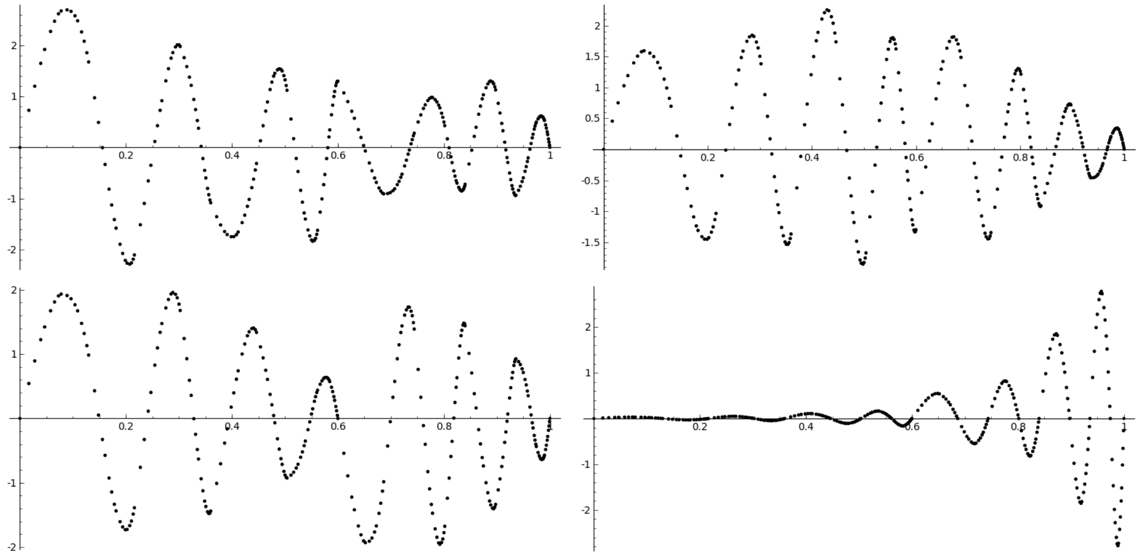


Figure A.11.: $\tilde{f}_{D,13}$ to $\tilde{f}_{D,16}$ for $r_1 = 0.6$, $r_2 = 0.4$, $m_1 = 0.4$ and $m_2 = 0.6$.

B. Mathematical Foundations

B.1. L_2 spaces

Suppose $E = E(x)$ is a property which depends on $x \in [a, b]$ and μ is a Borel-measure on $[a, b]$. Then we say E holds μ-almost everywhere on $[a, b]$, if there is a set $N \in \mathcal{B}$ of measure zero such that $\{x \in [a, b] \colon E(x) \text{ does not hold}\} \subseteq N$.

We denote the image measure of a measure μ through a map S by $S\mu$.

Definition B.1.1. Let μ be a Borel-measure on $[a, b]$.

(i) We define $\tilde{L}_2([a, b], \mu)$ to be the space of all real-valued, Borel-measurable functions on $[a, b]$ with

$$\|f\|_{L_2([a,b],\mu)} := \left(\int_{[a,b]} |f|^2 \, d\mu \right)^{\frac{1}{2}} < \infty.$$

(ii) For $f, g \in \tilde{L}_2([a, b], \mu)$ we set

$$\langle f, g \rangle_{L_2([a,b],\mu)} := \int_{[a,b]} f \, g \, d\mu.$$

(iii) For $f, g \in \tilde{L}_2([a, b], \mu)$ let f be equivalent to g ($f \sim g$) if and only if $f = g$ μ-almost everywhere.

(iv) $L_2([a, b], \mu)$ is defined as the set of all equivalence classes in $\tilde{L}_2([a, b], \mu)$ with respect to $\sim$.

In the following we will speak mostly of *functions* in $L_2([a, b], \mu)$ instead of equivalence classes. By that, we mean either an arbitrary or a specific representative of the corresponding equivalence class, depending on the context. This is justifiable since the integrals of all functions of the same equivalence class coincide. Consequently, equalities or inequalities involving functions in $L_2([a,b],\mu)$ should be interpreted to hold μ-almost everywhere.

Since we will mostly use the interval $I = [a,b]$, we will in this case omit $[a,b]$ and call the space simply $L_2(\mu)$.

Note, that if μ is finite, that is, there is a number M such that $\mu([a,b]) = M$, then the space $C([a,b])$ of all continuous functions on $[a,b]$ is continuously embedded in $L_2(\mu)$, because for all $f \in C([a,b])$,

$$\|f\|_{L_2(\mu)} = \left(\int_a^b f(t)^2 \, d\mu(t) \right)^{1/2} \leq \sqrt{M} \, \|f\|_\infty.$$

Definition B.1.2. We say μ is *non-atomic* or *atomless*, if $\mu(\{x\}) = 0$ for every $x \in [a,b]$.

If μ is non-atomic, we will write $\int_x^y f \, d\mu$ instead of $\int_{[x,y]} f \, d\mu$, $\int_{(x,y]} f \, d\mu$, $\int_{[x,y)} f \, d\mu$ or $\int_{(x,y)} f \, d\mu$.

B.2. Self-similar sets and measures

We give a short introduction to iterated function systems and self-similar sets based on Falconer [14, Ch. 9]. The first systematic treatment of iterated function systems is due to Hutchinson [30].

A mapping $S\colon D \to D$ for a closed subset $D \subseteq \mathbb{R}^n$ is called a contraction on D if there is a number r with $0 < r < 1$ such that

$$|S(x) - S(y)| \leq r|x - y|$$

for all $x, y \in D$. If

$$|S(x) - S(y)| = r|x - y|,$$

then we call S a contracting similarity.

An iterated function system (IFS) is a finite collection of contractions $\mathcal{S} = (S_1, \ldots, S_N)$ on D. A non-empty compact set $K \subseteq D$ with

$$K = \bigcup_{i=1}^N S_i(K)$$

is called an attractor for the IFS $\mathcal{S}$ or invariant with respect to $\mathcal{S}$.

The following theorem is Theorem 9.1 in Falconer [14].

Theorem B.2.1. *Consider the iterated function system given by the contractions* $S_1, \dots, S_N$ *on* $D \subseteq \mathbb{R}^n$*, so that*

$$|S_i(x) - S_i(y)| \leq r_i|x - y|, \quad x, y \in D,$$

with $r_i < 1$ *for each* i*. Then there is a unique attractor* K*, i.e. a non-empty compact set such that*

$$K = \bigcup_{i=1}^{N} S_i(K).$$

Moreover, if we define a transformation F *on the class* $\mathcal{D}$ *of non-empty compact sets in* D *by*

$$F(E) := \bigcup_{i=1}^{N} S_i(E)$$

for $E \in \mathcal{D}$*, and write* F^k *for the* k*th iterate of* F *(so* $F^0(E) = E$ *and* $F^k(E) = F\big(F^{k-1}(E)\big)$ *for* $k \geq 1$*), then*

$$K = \bigcap_{k=0}^{\infty} F^k(E)$$

for every set $E \in \mathcal{D}$ *such that* $S_i(E) \subseteq E$ *for all* i*.*

We say that an IFS $\mathcal{S}$ satisfies the *open set condition* if a non-empty bounded open set V exists such that

$$\bigcup_{i=1}^{N} S_i(V) \subseteq V$$

with the union disjoint.

If the open set condition holds for an IFS $\mathcal{S}$ consisting of similarities with ratios $r_i \in (0, 1)$ then the Hausdorff dimension of the attractor K is given as the solution d of the equation

$$\sum_{i=1}^{N} r_i^d = 1.$$

Example B.2.2. Let $\mathcal{S} = \{S_1, S_2\}$ where $S_1, S_2 \colon [0, 1] \to [0, 1]$ given by

$$S_1(x) = \frac{1}{3}x \quad \text{and} \quad S_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Then the attractor K is the classical *Cantor set* and its Hausdorff dimension is $\frac{\log 2}{\log 3}$.

Furthermore, we need a self-similar measure μ with support $\operatorname{supp}\mu = K$. We denote

$$\mathcal{M}_1 := \{\mu\colon \mu \text{ is a Borel measure on } \mathbb{R}^n \text{ with compact support and } \mu(\mathbb{R}^n) = 1\}.$$

The following definition is taken from Hutchinson [30, Sect. 4.2]. By $S\mu$ we denote the image measure of μ through a map S given by $S\mu(A) = \mu\big(S^{-1}(A)\big)$ for all $A \in \mathcal{B}(\mathbb{R}^n)$.

Definition B.2.3. Let $\mathcal{S} = (S_1, \ldots, S_N)$ be an IFS on $\mathbb{R}^n$ and let $m = (m_1, \ldots, m_N)$ be a weight vector with $0 < m_i < 1$ and $\sum_{i=1}^{N} m_i = 1$. A measure $\mu \in \mathcal{M}_1$ is called *invariant with respect to* $\mathcal{S}$ *and* m if

$$\mu(A) = \sum_{i=1}^{N} m_i \cdot \big(S_i\mu\big)(A)$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$.

In Theorem 4.4.1 of Hutchinson [30, Sect. 4.4] the existence and uniqueness of an invariant measure for a given IFS and a weight vector is proved by using a fixed point argument.

If the contractions S_i are similarities, we call the invariant measure *self-similar measure*. In that case, the particular choice of the weights as $m_i = r_i^d$ is considered natural, where d is the Hausdorff dimension of the attractor.

B.3. The Vitali-Hahn-Saks theorem

We briefly review the theorem of Vitali-Hahn-Saks, which we use for the construction of the measure $\mu^{(\xi)}$. We state the theorem in a form that is sufficient for our case, though it holds also true in a considerably more general setting. Further details can be found e.g. in Alt [1, p. 279], Dunford and Schwartz [10, Sect. III.7], or Yosida [53, p. 70].

Theorem B.3.1. *Let $(\Omega, \mathcal{A}, \nu)$ be a finite measure space, and for every $n \in \mathbb{N}$ let μ_n be a finite measure on $(\Omega, \mathcal{A})$ such that $\mu_n \ll \nu$ and $\lim_{n\to\infty} \mu_n(A)$ exists for every $A \in \mathcal{A}$.*

Then the ν-absolute continuity of μ_n is uniform in n, that is, for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $n \in \mathbb{N}$ and every $A \in \mathcal{A}$ holds

$$\nu(A) < \delta \Rightarrow \mu_n(A) < \varepsilon.$$

Proof. The proof, in which the Baire category theorem is used, can be found in each of the citations mentioned above. □

The name of the theorem is also often associated with the following corollary.

Corollary B.3.2. *Let $(\mu_n)_n$ be a sequence of finite measures on a measurable space $(\Omega, \mathcal{A})$ such that $\mu(A) := \lim\limits_{n\to\infty} \mu_n(A)$ exists for all $A \in \mathcal{A}$.*

Then μ is also a measure on $(\Omega, \mathcal{A})$.

Proof. In order to apply Theorem B.3.1 we construct a measure ν that satisfies the requirements.

For each $A \in \mathcal{A}$ we set

$$\nu(A) := \sum_{n=1}^{\infty} 2^{-n} \frac{\mu_n(A)}{\mu_n(\Omega)}$$

(assuming w.l.o.g. that $\mu_n(\Omega) > 0$ for all $n \in \mathbb{N}$). Then ν is a measure on $(\Omega, \mathcal{A})$. Indeed, if $A_1, A_2, \ldots \in \mathcal{A}$ are pairwise disjoint, we can say that, since all summands are positive,

$$\nu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{n=1}^{\infty} 2^{-n} \sum_{j=1}^{\infty} \frac{\mu_n(A_j)}{\mu_n(\Omega)} = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} 2^{-n} \frac{\mu_n(A_j)}{\mu_n(\Omega)} = \sum_{j=1}^{\infty} \nu(A_j).$$

Furthermore, ν is finite and $\mu_n \ll \nu$ for all $n \in \mathbb{N}$.

We show that μ is continuous in $\emptyset$, which is equivalent to the σ-additivity of μ because it is finite. Let $(A_j)_j$ be a decreasing sequence in $\mathcal{A}$ with $\bigcap\limits_{j=1}^{\infty} A_j = \emptyset$. Let $\varepsilon > 0$. By Theorem B.3.1 there is a $\delta > 0$ such that for all $n \in \mathbb{N}$ and for every A_j holds that $\nu(A_j) < \delta$ implies $\mu_n(A_j) < \varepsilon$. Since ν is $\emptyset$-continuous, we can take j_0 such that $\nu(A_j) < \delta$ for all $j \geq j_0$. Then $\mu_n(A_j) < \varepsilon$ for all $j \geq j_0$ and all $n \in \mathbb{N}$. Thus, $\mu(A_j) \leq \varepsilon$ for all $j \geq j_0$ and hence $\lim\limits_{j\to\infty} \mu(A_j) \leq \varepsilon$. Since this holds for all $\varepsilon > 0$, we have $\lim\limits_{j\to\infty} \mu(A_j) = 0$ and therefore, μ is σ-additive.

The other properties of a measure are very easy to check. □

B.4. The Arzelà-Ascoli theorem

We need the Arzelà-Ascoli theorem to prove the compactness of an embedding. Its statement and proof can be found in Triebel [51, p. 32-33].

Definition B.4.1. Let $a < b$. A set $A \subseteq C([a, b])$ of continuous functions is called equicontinuous if for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that for all $x, y \in [a, b]$ and all $f \in A$,

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Definition B.4.2. Let $(X, \|.\|)$ be a Banach space. A set $A \subseteq X$ is called precompact in X if for every sequence in A there is a convergent subsequence.

Theorem B.4.3. *Let $a < b$. A set $A \subseteq C([a, b])$ is precompact in $\big(C([a, b]), \|.\|_\infty\big)$ if and only if A is bounded and equicontinuous.*

B.5. The law of the iterated logarithm

In Section 3.5 we need the law of the iterated logarithm to apply our results in the random case, that is, where the measure is determined by a sequence of independent, identically distributed random variables.

This theorem has been found by Khinchin in 1923 for certain special cases and extended by Kolmogoroff in 1929. The general form stated below has first been proved by Hartman and Wintner in 1941. Later, further generalizations and several different proofs have been established by various authors. For further references and the proof, see Klenke [36, p. 517ff.].

Theorem B.5.1. *Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with $\mathbb{E}\, X_1 = 0$ and $\operatorname{Var} X_1 = 1$. For $n \in \mathbb{N}$ let*

$$S_n := X_1 + \cdots X_n.$$

Then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = 1 \quad a.s.$$

Remark B.5.2. Analogously holds

$$\liminf_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = -1 \quad \text{a.s.}$$

Therefore, there is a random variable C with $\mathbb{P}\big(0 < C < \infty\big) = 1$ such that for all $n \in \mathbb{N}$,

$$|S_n| \le C\sqrt{n \log\log n} \quad \text{a.s.}$$

B.6. Regularly varying functions

Definition B.6.1. A function $g\colon (0,\infty) \to \mathbb{R}$ is *regularly varying* if, for all $c > 0$,

$$\lim_{t\to\infty} \frac{g(ct)}{g(t)} \in (0,\infty).$$

The notion of regular varying functions is due to Karamata, for further reading, see, for example, Galambos and Seneta [25].

B.7. Dirichlet forms

Definition B.7.1. Let μ be an atomless Borel measure on $[a,b]$ and let $\mathcal{F}$ be a set of functions on $[a,b]$ that forms a dense subspace of $L_2(\mu)$. Let $\mathcal{E}$ be a non-negative symmetric bilinear form on $\mathcal{F}$ and let $\mathcal{E}_\alpha(f,g) = \mathcal{E}(f,g) + \alpha\langle f,g\rangle_{L_2(\mu)}$. For $f \in \mathcal{F}$ let $\tilde{f}$ be defined by

$$\tilde{f}(x) := \begin{cases} 1, & \text{if } f(x) > 1 \\ 0, & \text{if } f(x) < 0 \\ f(x), & \text{otherwise.} \end{cases}$$

Then $(\mathcal{E},\mathcal{F})$ is called a *Dirichlet form on* $L_2(\mu)$ if $(\mathcal{F},\mathcal{E}_\alpha)$ is a Hilbert space for all $\alpha > 0$ and if for all $f \in \mathcal{F}$ we have $\tilde{f} \in \mathcal{F}$ and $\mathcal{E}(\tilde{f},\tilde{f}) \le \mathcal{E}(f,f)$.

In the following we define the concept of eigenvalues for a Dirichlet form.

Definition B.7.2. Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on $L_2(\mu)$. We call $\lambda \in \mathbb{R}$ an *eigenvalue* of $(\mathcal{E},\mathcal{F})$ if there is a $f \in \mathcal{F}, f \ne 0$ such that for all $g \in \mathcal{F}$ holds

$$\mathcal{E}(f,g) = \lambda\langle f,g\rangle_{L_2(\mu)}.$$

As in Kigami and Lapidus [35] we will consider a Dirichlet form where the inclusion map from the Hilbert space $(\mathcal{F},\mathcal{E}_\alpha)$ into $L_2(\mu)$ is a compact operator. Then it follows that the eigenvalues have finite multiplicity and accumulate only at infinity. This can be found in Triebel [51, Th. 4.5.1 and p. 258], see also Edmunds and Evans [11, Sect.

IV.2]. We assort the eigenvalues by size taking in account their multiplicities and denote this sequence by

$$\lambda_1 \leq \lambda_2 \leq \cdots .$$

Definition B.7.3. Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L_2(\mu)$. Then we define the *eigenvalue counting function* $N_{(\mathcal{E},\mathcal{F})}$ by

$$N_{(\mathcal{E},\mathcal{F})}(x) := \#\{k \leq x\colon\ k \text{ is eigenvalue of } (\mathcal{E}, \mathcal{F})\}.$$

The following theorem can be found in Kigami and Lapidus [35, Th. 4.5].

Theorem B.7.4. *Let $(\mathcal{F}, \mathcal{E})$ and $(\mathcal{F}', \mathcal{E}')$ Dirichlet forms on $L_2(\mu)$ with $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{E}|_{\mathcal{F}' \times \mathcal{F}'} = \mathcal{E}'$. Then*

$$N_{(\mathcal{E}',\mathcal{F}')}(x) \leq N_{(\mathcal{E},\mathcal{F})}(x), \qquad \text{for all } x \geq 0.$$

Bibliography

[1] H. W. Alt, *Lineare Funktionalanalysis*, 5th ed., Springer, Berlin, Heidelberg, New York, 2006.

[2] F. V. Atkinson, *Discrete and continuous boundary problems*, Academic Press, Inc., New York, London, 1964.

[3] M. T. Barlow and B. M. Hambly, *Transition density estimates for Brownian motion on scale irregular Sierpinski gaskets*, Ann. Inst. Henri Poincaré, **33** (1997), no. 5, 531–557.

[4] A. F. Beardon, *On the Hausdorff dimension of general Cantor sets*, Proc. Camb. Phil. Soc. **61** (1965), 679–694.

[5] E. J. Bird, S.-M. Ngai, and A. Teplyaev, *Fractal Laplacians on the unit interval*, Ann. Sci. Math. Québec **27** (2003), 135–168.

[6] V. I. Bogachev and A. V. Kolesnikov, *The Monge-Kantorovich problem: achievements, connections, and perspectives*, Russian Math. Surveys **67** (2012), no. 5, 785–890.

[7] G. Cantor, *Ueber unendliche, lineare Punktmannichfaltigkeiten 5*, Math. Ann. **21** (1883), 545–591.

[8] J. Chen and S.-M. Ngai, *Eigenvalues and eigenfunctions of one-dimensional fractal Laplacians defined by iterated function systems with overlaps*, J. Math. Anal. Appl. **364** (2010), 222–241.

[9] M.-F. Chen, *Eigenvalues, Inequalites, and Ergodic Theory*, Springer, London, 2005.

[10] N. Dunford and J. T. Schwartz, *Linear Operators. Part I: General Theory*, Wiley Classics Library, John Wiley & Sons, Inc., New Jersey, 1988.

[11] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Oxford University Press, New York, 1987.

[12] R. J. Etienne, *On the asymptotic distribution of the Dirichlet eigenvalues of Fractal Chains*, Ph.D. Thesis, Universität Siegen, 2014.

[13] K. Falconer, *Random fractals*, Math. Proc. Camb. Phil. Soc. **100** (1986), 559–582.

[14] K. Falconer, *Fractal Geometry, Mathematical Foundations and Applications*, 2nd ed., John Wiley & Sons, Chichester, 2003.

[15] W. Feller, *Generalized second order differential operators and their lateral conditions*, Illinois J. Math. **1** (1957), 459–504.

[16] J. F. Fleron, *A note on the history of the Cantor set and Cantor function*, Math. Mag. **67** (1994), 136–140.

[17] U. Freiberg, *Analytical properties of measure geometric Krein-Feller-operators on the real line*, Math. Nachr. **260** (2003), 34–47.

[18] U. Freiberg, *Dirichlet forms on fractal subsets of the real line*, Real Anal. Exchange **30** (2004/05), no. 2, 589–603.

[19] U. Freiberg, *Spectral asymptotics of generalized measure geometric Laplacians on Cantor like sets*, Forum Math. **17** (2005), 87–104.

[20] U. Freiberg, *Prüfer angle methods in spectral analysis of Krein-Feller-operators*, RIMS Kôkyûroku Bessatsu **B6** (2008), 74–81.

[21] U. Freiberg, *Refinement of the spectral asymptotics of generalized Krein Feller operators*, Forum Math. **23** (2011), 427–445.

[22] U. Freiberg and J.-U. Löbus, *Zeros of eigenfunctions of a class of generalized second order differential operators on the Cantor set*, Math. Nachr. **265** (2004), 3–14.

[23] U. Freiberg and M. Zähle, *Harmonic calculus on fractals – a measure geometric approach I*, Potential Anal. **16** (2002), 265–277.

[24] T. Fujita, *A fractional dimension, self similarity and a generalized diffusion operator*, Probabilistic methods in mathematical physics, Proceedings of Taniguchi International Symposium Katata and Kyoto, 1985, Kinokuniya (1987), 83–90.

[25] J. Galambos and E. Seneta, *Regularly varying sequences*, Proc. amer. math. soc. **41** (1973), no. 1, 110–116.

[26] S. Graf, *Statistically self-similar sets*, Prob. Theor. Related Fields **74** (1987), 357–392.

[27] B. M. Hambly, *Brownian motion on a homogenous random fractal*, Probab. Theory Relat. Fields **94** (1992), 1–38.

[28] F. Hausdorff, *Dimension und äußeres Maß*, Math. Ann. **79** (1919), 157–179.

[29] J. Hu, K.-S. Lau, and S.-M. Ngai, *Laplace operators related to self-similar measures on* $\mathbb{R}^d$, J. Funct. Anal. **239** (2006), 542–565.

[30] J. E. Hutchinson, *Fractals and self similarity*, Indiana University Mathematics Journal **30** (1981), 713–747.

[31] J. E. Hutchinson and L. Rüschendorf, *Random fractals and probability metrics*, Adv. in Appl. Probab. **32** (2000), 925–947.

[32] I. S. Kac and M. G. Krein, *On the spectral functions of the string*, Amer. Math. Soc. Transl. **103** (1974), 19–102.

[33] U. Küchler, *Some asymptotic properties of the transition densities of one-dimensional quasidiffusions*, Publ. RIMS, Kyoto Univ. **16** (1980), 245–268.

[34] U. Küchler, *On sojourn times, excursions and spectral measures connected with quasidiffusions*, J. Math. Kyoto Univ. **26** (1986), 403–421.

[35] J. Kigami and M. L. Lapidus, *Weyl's problem for the spectral distribution of Laplacians on P.C.F. self-similar fractals*, Commun. Math. Phys. **158** (1993), 93–125.

[36] A. Klenke, *Probability Theory*, 2nd ed., Springer, Berlin, 2013.

[37] J.-U. Löbus, *Constructions and generators of one-dimensional quasidiffusions with applications to selfaffine diffusions and Brownian motion on the Cantor set*, Stochast. Stochast. Rep. **42** (1993), 93–114.

[38] B. B. Mandelbrot, *Renewal sets and random cutouts*, Z. Warsch. Verw. Geb. **22** (1972), 145–157.

[39] R. D. Mauldin and S. C. Williams, *Random recursive constructions: asymptotic geometric and topological properties*, Trans. Am. Math. Soc. **295** (1986), 325–346.

[40] H. P. McKean and D. B. Ray, *Spectral distribution of a differential operator*, Duke Math. J. **29** (1962), 281–292.

[41] K. Naimark and M. Solomyak, *The eigenvalue behaviour for the boundary value problems related to self-similar measures on* $\mathbb{R}^d$, Math. Res. Lett. **2** (1995), no. 3, 279–298.

[42] S.-M. Ngai, *Spectral asymptotics of Laplacians associated to one-dimensional iterated function systems with overlaps*, Canad. J. Math. **63** (2011), 648–688.

[43] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, London, 1970.

[44] C. Sabot, *Density of states of diffusions on self-similar sets and holomorphic dynamics in* $\mathbb{P}^k$*: the example of the interval* $[0,1]$, C. R. Acad. Sci. Paris Sér I: Math **327** (1998), 359–364.

[45] C. Sabot, *Integrated density of states of self-similar Sturm-Liouville operators and holomorphic dynamics in higher dimension*, Ann. I. H. Poincaré **37** (2001), 275–311.

[46] R. Salem, *On singular monotonic functions of the Cantor type*, J. Math. Phys. Mass. Inst. Tech. **21** (1942), 69–82.

[47] H. J. S. Smith, *On the integration of discontinuous functions*, Proc. London Math. Soc. **6** (1875), 140–153.

[48] M. Solomyak and E. Verbitsky, *On a spectral problem related to self-similar measures*, Bull. London Math. Soc. **27** (1995), no. 3, 242–248.

[49] W. A. Stein et al., *Sage Mathematics Software (Version 5.12)*, The Sage Development Team, 2013. `http://cloud.sagemath.com`.

[50] W. A. Strauss, *Partielle Differentialgleichungen*, Vieweg, Braunschweig, 1995.

[51] H. Triebel, *Higher Analysis*, 2nd ed., Johann Ambrosius Barth, Leipzig, 1992.

[52] H. Volkmer, *Eigenvalues associated with Borel sets*, Real Analysis Exchange **31** (2005/2006), no. 1, 111–124.

[53] K. Yosida, *Functional analysis*, 6th ed., Springer, Berlin, 1980.

[54] U. Zähle, *Random fractals generated by random cutouts*, Math. Nachr. **116** (1984), 27–52.