On the Asymptotic Distribution of the Dirichlet Eigenvalues of Fractal Chains

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I would like to dedicate this thesis to my wife Malou, my daughter Morag and my son Rhys - You're the best.

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This thesis was brought about by professional requirements for (yet another) scientific work for the completion of my (second) teacher's training. Indeed, Luxemburg Government does not recognise my "mémoire scientifique" from 1997 due to incompatibilities in the legislation regarding private vs. public schools. Thus, instead of writing yet another internationally irrelevant scientific work, I opted for overriding this with a PhD-thesis. It was a personal challenge to complete this academic achievement in parallel to my full-time teaching job as well as the needs of my young family.

Although born from external pressure originally, this work gives me enormous personal satisfaction as I was able to expand my private research into an academic setting.

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Abstract

In this thesis, we offer an investigation of the vibrational properties of discrete one-dimensional systems with an underlying fractal structure. Thus, the primary objects of scrutiny in this work are two types of fractal objects: the first class being quite simple structures with a fractal boundary, the second class having an internal fractal structure but very simple boundaries. By introducing a matrix representation of the related Laplacians, we prove the efficiency of using techniques originally taken from random matrix theory in the area of fractal geometry. Thereby, a unifying framework for the study of these systems has been developed, capable of being extended to higher dimensions.

In dieser Arbeit wird eine Untersuchung der Schwingungseigenschaften von diskreten eindimensionalen Systemen mit einer zugrunde liegenden fraktalen Struktur präsentiert. Hauptsächlich werden in dieser Arbeit zwei Arten von fraktalen Objekten untersucht: die erste Kategorie zeigt sich als recht einfache Struktur mit einer fraktalen Begrenzung, die Zweite mit einer inneren fraktalen Struktur aber einfacher Begrenzung. Durch die Einführung einer Matrixdarstellung der zugehörigen Laplace-Operatoren zeigen wir die Effizienz der Verwendung von aus der Zufallsmatrizentheorie übernommenen Techniken im Bereich der fraktalen Geometrie. Auf diese Weise wird ein vereinheitlichender Rahmen für die Untersuchung dieser Systeme geschaffen, welcher auch auf höherdimensionale Anwendungen erweitert werden kann.

Contents

Contents **vi**

Overview **viii**

1 Introduction **1**

2 Fractal Strings **8**

2.1 Preliminaries 8

2.2 Spectral asymptotics of fractal strings 16

3 Fractal chains **25**

3.1 Monoatomic chains 25

3.2 Fractal chains 29

3.2.1 The dynamic matrix of a fractal chain and its traces . . . 32

3.2.2 The moments of the eigenvalue distribution of a fractal chain 38

3.2.2.1 Example 1: Generalised Cantor chains 40

3.2.2.2 Example 2: The a-chain 41

3.2.3 The moments of the eigenvalue distribution of a fractal chain with cut-off 42

3.2.3.1 Example 1: Generalised Cantor chains 56

3.2.3.2 Example 2: The a-chain 58

3.2.3.3 Minkowski-measurable chains 60

4 Interlude **71**

5 Measure geometric fractal chains **74**
5.1 Preliminaries 74
5.2 A physical interpretation 77
5.3 A physical model 78
5.4 Numerical spectral asymptotics for measure geometric chains . . . 81
5.4.1 Example 1: The measure geometric Cantor chain 82
5.4.2 Example 2: A measure geometric chain with two different scaling ratios 85
5.5 The traces of powers of the dynamic matrix of measure geometric chains 89
5.5.1 Application: Dirichlet eigenvalues of measure geometric strings as zeroes of a generalised trigonometric function . . 94
5.5.1.1 Example 1: The measure geometric triadic Cantor string 98
5.5.1.2 Example 2: A measure geometric string with two different scaling ratios 101

6 Conclusion and Outlook **105**

Appendix A: A different approach for the traces of the measure geometric Cantor chain **110**

Appendix B: The moments of the triadic Cantor chain with and without cut-off: a comparison **117**

References **121**

List of Figures **138**

List of Tables **141**

Overview

Fractal forms are often found in nature. Typical examples are the fractal structures found in green cauliflower, fern leaves, blood vessels, crystal growth processes, chemical oscillators, river systems and coastlines. As a consequence, it is for example impossible to exactly determine the length of the coastline: the more accurately the subtleties of the coastal course are measured, the greater is the length obtained. In the case of a mathematical fractal , such as the Koch curve, it would be unlimited. Although many natural systems exhibit such fractal structure over a finite range of scales, their fractal features disappear at the latest when an atomic scale is reached. It is in this context that we will try to explore the consequence of the discreteness of natural structures on their mathematical description. However, we will limit ourselves here to the most accessible case: fractals in a one-dimensional space.

Thus, the primary objects of scrutiny in this work are two types of fractal chains: the first one being the discrete analogue of fractal strings - bounded subsets of $\mathbb{R}$ with a fractal set as boundary; the second one related to measures on bounded subsets of the real line. It will be shown how important information about these fractal chains may be discovered by combining methods from various areas of physics and mathematics.

This thesis is organised as follows. After an introduction to the history of investigations in the asymptotics of spectra, we review the relevant aspects of the theory of fractal strings in the second chapter. In chapter 3, we introduce the concept of fractal chains as discretised counterpart of fractal strings, together with their underlying physical model. For these it is possible to give a matrix formulation for the Laplacian in the wave equation $-\Delta f = \lambda f$, so that the power of methods and techniques from random matrix theory in the study of fractal strings/chains can be demonstrated. Several examples are shown in more detail and we are able to state a new criterion for the Minkowski-measurability of fractal strings, giving a more precise meaning to a statement by M.L. Lapidus and C. Pomerance concerning the multiplicities of lengths of a string:

> "Intuitively (...) the fact that $N(\lambda)$ does not admit an asymptotic second term is due to the symmetry of the boundary Γ (here, the self-similarity of the Cantor set). Indeed, this symmetry gives rise to high multiplicities in the eigenvalues (equivalently, in the interval lengths $(l_j)_{j=1}^{\infty}$) and thus causes the function $\lambda^{-D/2}((N(\lambda) - \phi(\lambda))$ to oscillate." (see [78], page 67)

Chapter 4 acts as a link to the second part of this work, connecting the two types of fractal chains under scrutiny here by physical considerations. The following chapter is devoted to fractal chains arising from a measure theoretic Laplacian. Again we first provide the necessary background before using random matrix theoretic means for their investigation. In this framework we first present numerical evidence showing the validity of our approach. Subsequently, we show how the characteristic polynomials of the approximations to the matrix Laplacian may be used complementary to other approaches (such as those in [5], for example) for finding the eigenvalues respectively their asymptotics. Although the results presented in this chapter are still at an early stage of development, an in-depth study unfortunately being too complex to fit within the scope of this work, they make clear that the tools exposed here open up new lines of thought, worth further attention. In the final chapter, we provide an exposition of our results together with an outlook on further research to be accomplished through the techniques shown and developed in this thesis. Finally, two short appendices are attached, which give some supplementary material that might be useful for future exploration.

Chapter 1

Introduction

The knowledge of the asymptotic distribution of eigenvalues of the Laplacian is often a prerequisite for model calculations of physical properties in a variety of classical as well as quantum systems. The origins of this problem can be traced back to the Pythagoreans [20, 93] recognising the relation between harmonious vibrations of elastic strings and their relative length - the first natural law ever to be formulated in mathematical terms.

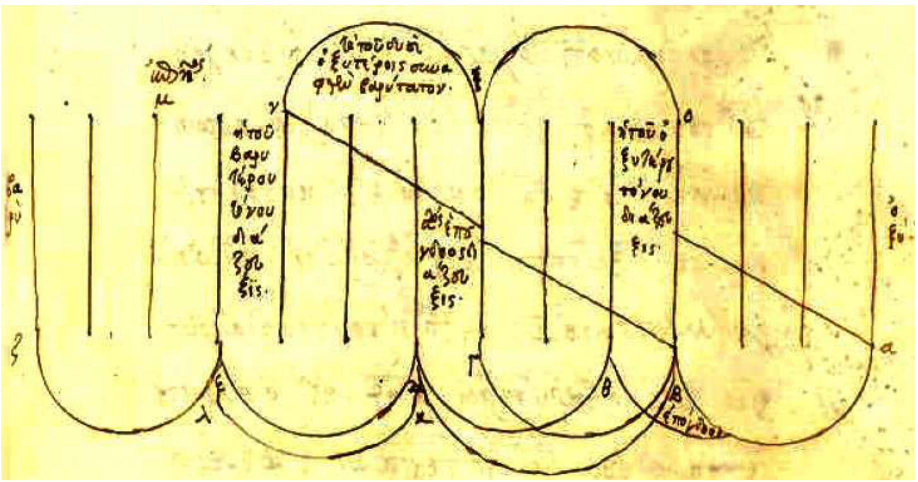

Figure 1.1: Excerpt from a renaissance manuscript of Porphyry's "Eis ta harmonika Ptolemaiou hypomnēma" [93]

The subject came back into focus during the renaissance with the works of Vincentio Galilei (father of Galileo) [47] and especially Marin Mersenne's

"L'harmonie universelle" [87]. In this work he was probably the first to publish what later became known as Mersenne's laws for vibrating strings; their frequency is:

- inversely proportional to their length (also known as Pythagoras' law, see above),
- proportional to the square root of their tension,
- inversely proportional to the square root of their linear mass density.

In 1673, Christian Huygens [56] contributed the concept of forced vibrations from his studies of pendulum oscillations driven by external forces in his "Horologium Oscillatorium".

Several years later, Joseph Sauveur [100] was the first to use beats to determine frequency differences and was thereby able to calculate the absolute frequencies. Since he correctly interpreted beats, it appears that he may have been the first to have an understanding of superposition. Furthermore, he explained the phenomenon of harmonics by arguing that a string can vibrate at additional higher frequencies as it divides itself up into the appropriate number of equal shorter lengths separated by stationary points, which he called noeuds (nodes). Apparently he did not know of the earlier experimental works on the subject by Wallis [115] and Roberts [99]. Later, in work presented in 1713 [101], he derived the fundamental frequency of a string from a theoretical perspective. He treated the string, stretched horizontally and taking the form of a catenary due to the gravitational field, as a compound pendulum and found the frequency of the swinging motion, supposed to have small amplitude.

In the same year, the first description of vibrations of elastic strings in terms of differential equations was given by Brook Taylor [112]. Ten years later, J. Bernoulli [12] reconsidered the question using the - by then familiar - Leibnizian notation and derived Mersenne's laws through mathematical analysis. Bernoulli's treatment of the elements of the string as simple pendulums undergoing small vibrations of identical period is fundamental to his solution, an idea similar to Sauveur's approach and underlying probably all investigations of oscillatory phenomena during this period. Use of this condition tended to be combined with certain restrictions on the motion. Thus both Taylor and Bernoulli assumed in their works that the elements of the string arrive simultaneously from one side at the equilibrium configuration. As a result, they only determined the first fundamental mode. However, this inherently geometric approach appears to have discouraged the investigation of higher modes, thereby concealing Sauveur's insight and acting as an obstacle to the discovery of the principle of superposition [18].

In the 1740's and 1750's, Euler [24–27, 29], d'Alembert [2–4] and D. Bernoulli [8, 9] followed the example of J. Bernoulli regarding the equations as the limit of those for a massless ideally flexible thread (chain) with a finite number of beads as the number of beads approaches infinity while their total mass remains fixed.

The motion of this system of beads being described by a finite system of ordinary differential equations, d'Alembert proposed his method of integrating systems of linear differential equations with constant coefficients. Also starting with this problem, Daniel Bernoulli stated his remarkable hypothesis that the solution of the free oscillations of a string can be represented in the form of a trigonometric series, which lead to a debate raging throughout the following decades on the nature of an "arbitrary" function and its expansion in trigonometric functions, initiating a fundamental discussion of the foundations of mathematical analysis.

Even though this controversy was partially solved by Lagrange [66, 67] (reprinted in [68]), it was brought to a conclusion only in the 19th century by Fourier, Cauchy, Dirichlet and Riemann (for a more complete discussion on this subject, see for example [64] or [98]).

In this context the meaning and relevance of the boundary conditions is especially noteworthy. Over the years it became clear that the description of the relationship between the geometry of a manifold and its spectrum are of utmost importance. In 1910, Hendrik Lorentz' 4th Wolfskehl lecture "Alte und neue Fragen der Physik" - Old and new problems of physics - included the following passage [84]:

> "Zum Schluß soll ein mathematisches Problem Erwähnung finden, das vielleicht bei den anwesenden Mathematikern Interesse erwecken wird. Es stammt aus der Strahlungstheorie von Jeans. In einer vollkommen spiegelnden Hülle können sich stehende elektromagnetische Schwingungen ausbilden, ähnlich den Tönen einer Orgelpfeife; wir wollen nur auf die sehr hohen Obertöne das Augen-

merk richten. Jeans fragt nach der auf ein Frequenzintervall dn fallenden Energie. Dazu berechnet er zuerst die Anzahl der zwischen den Frequenzen n und $n + dn$ liegenden Obertöne und multipliziert die Zahl dann mit der zu jeder Frequenz gehörigen Energie, die nach einem Satze der statistischen Mechanik für alle Frequenzen gleich ist. (...)

Hierbei entsteht das mathematische Problem, zu beweisen, daß die Anzahl der genügend hohen Obertöne zwischen n und $n+dn$ unabhängig von der Gestalt der Hülle und nur ihrem Volumen proportional ist. Für mehrere einfache Formen der Hülle, wo sich die Rechnung durchführen läßt, wird der Satz in einer Leidener Dissertation bestätigt werden. Es ist nicht zu zweifeln, daß er allgemein, auch für mehrfach zusammenhängende Räume, gültig ist. Analoge Sätze werden auch bei andern schwingenden Gebilden, wie elastischen Membranen und Luftmassen etc., bestehen"

"In conclusion there is a mathematical problem which perhaps will arouse the interest of mathematicians who are present. It originates in the radiation theory of Jeans. In an enclosure with a perfectly reflecting surface there can form standing electromagnetic waves analogous to tones of an organ pipe; we shall confine our attention to very high overtones. Jeans asks for the energy in the frequency interval dn. To this end he calculates the number of overtones which lie between the frequencies n and $n + dn$ and multiplies this number by the energy which belogs to the frequency n, and which according to a theorem statistical mechanics is the same for all frequencies. (...)

It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lies between n and $n + dn$ is independent of the shape of the enclosure and is simply proportional to its volume. For many simple shapes for which calculations can be carried out, this theorem has been verified in a Leiden dissertation. There is no doubt that it holds in general even for multiply connected regions. Similar theorems for other vibrating structures like membranes, air masses, etc. should also hold." (translation by M. Kac in [61])

The study of the asmptotics of eigenvalues goes back even further than stated by Lorentz; probably to Friedrich Pockels' 1891 work "Über die partielle Differentialgleichung $\Delta u + k^2 u = 0$ und deren Auftreten in der mathematischen Physik" [92]. It was more than a decade later that Rayleigh calculated the asymptotic number of modes in the case of a rectangular parallelepiped [96] and Jeans tackled the radiation problem [59]. However it was clearly Lorentz (and in a footnote Sommerfeld [106]) who drew attention to the problem of the boundary conditions. In her aforementioned Leiden dissertation [97], Johanna Reudler verified Lorentz' conjecture for several shapes, but it was Hermann Weyl who published several papers [116–118] on the subject where he obtained the asymptotically leading term for the frequency counting function (i.e. the number of eigenvalues not exceeding a certain value) and proved it to be independent of the shape considered and proportional to the n-dimensional volume of the domain. Since then a lot of progress has been made and in the case of the Dirichlet Laplacian, it is now known to hold for an arbitrary bounded open set in $\mathbb{R}^n$ [88].

The question of whether it is possible to determine even more information about the shape of the manifold from its spectrum was elegantly rephrased by M. Kac in his 1966 paper "Can one hear the shape of a drum?" [61] and still remains an area of active research. Indeed, if the boundary is sufficiently smooth, it has been shown that the $(n-1)$-dimensional volume of the boundary determines the second term in the expansion of the eigenvalue counting function [58] (translated in [57])and [69].

However, if the manifold has a fractal boundary, the second term must be modified since the $(n-1)$-dimensional volume of the boundary is then infinite. As an eigenfunction of the negative Laplacian cannot resolve details of the boundary significantly smaller than its wavelength, M.V. Berry [13, 14] conjectured from scaling arguments that this term might depend on the Hausdorff dimension h of the boundary and be proportional to its h-dimensional Hausdorff-measure.

In the 1980's and 1990's, the interest in this topic surged and the effects of fractal boundaries of a region on the solutions of partial differential equations became an active topic of discussion again. By means of counter-examples, J. Brossard and R. Carmona [17] showed that the Minkowski dimension appeared more suitable than the Hausdorff dimension in the formulation of Berry's conjecture. Moreover, it became clear that the second term is not necessarily monotonic but eventually a rather complicated function [35]. Precise remainder estimates for the asymptotics of the eigenvalue counting function then lead to the reformulation and a partial resolution of the conjecture in [71, 73].

In two joint papers in 1990 and 1993, M. Lapidus and C. Pomerance [77, 78] proved this "modified Weyl-Berry conjecture" in the $(n = 1)$-dimensional case (note however that the conjecture is false for the case $n > 1$ [79]). Furthermore it was shown that it is possible not only to recover the Minkowski dimension of the boundary from the spectrum, but also - under certain conditions - its Minkowski measure [32, 72]. Indeed, if the boundary is Minkowski-measurable, the asymptotic second term of the eigenvalue counting function is monotonic (and depends in a simple way from the boundary's Minkowski measure), whereas in the opposite case its behaviour will be oscillatory.

In this context an unexpected connection with the Riemann zeta-function was discovered as well: the converse of the modified Weyl-Berry conjecture is not true in the case where the boundary's Minkowski dimension is $d_M = \frac{1}{2}$ but it is true everywhere else if and only if the Riemann hypothesis is true [75, 76]. This characterisation of the Riemann hypothesis as an inverse spectral problem shows interesting relations between fractal and spectral geometry on one hand, and number theory on the other.

Another important line of research arises through the consideration of intrinsic structures of the vibrating string. After being challenged by Daniel Bernoulli [10], it was probably again Leonhard Euler [28] and Daniel Bernoulli himself [11] who were the first to consider the influence of a varying linear mass density on the vibrational properties of a string. A little later, Euler even tried to obtain the solution for a continuously varying mass density by approximating it through a finite number of composite strings [30]. In the 1830's, Charles Sturm and Joseph Liouville laid the foundations of what was to be known as Sturm-Liouville theory. Their articles [81–83, 110, 111] were the first example of an in-depth study of the solutions of a second order differential equation and included Sturm's famous theorem of oscillation. Later on, in his seminal book "The Theory of Sound" [94] John William Strutt Lord Rayleigh treated a few, by now classical, examples for the mass distribution on a string in 1877, but in 1887, he also studied the case of a string with a periodic mass density variation [95]; an example followed by Horace Lamb in 1898 [70], who simplified the problem by considering a quantised version. This line of thought is also present in another groundbreaking book on the subject, "Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems" [48] by Feliks R. Gantmacher and Mark G. Krein, and especially in their elegant use made of Stieltjes' memoir on continued fractions [107, 108] in supplement II to the revised edition in 1950 [49]. In a joint work, Krein and I. Kac [60] then used a measure geometric approach to investigations of the spectrum of inhomogeneous vibrating strings in 1974. With the surge of interest in fractals in the 1980's, T. Fujita [46] generalised earlier results by T. Uno and I. Hong in 1959 [114] respectively H. P. McKean, Jr. and D. B. Ray in 1962 [86] on the asymptotics of measure geometric Laplacians to self-similar measures. These investigations were continued by a number of researchers in the following years, such as U. Freiberg,

M. Zähle, J. Löbus, P. Arzt [5, 36, 38, 39, 43, 45], A. Teplyaev, E.J. Bird, S. Ngai [16] to name but a few, thereby building a sound basis for the whole subject.

However, there has not been an in-depth consideration of discrete respectively finite systems in both these contexts yet. In this work, we will try to show how the study of these discrete systems leads to interesting links with random matrix theory and its tools.

Chapter 2

Fractal Strings

The necessary background as well as some useful tools for our subsequent studies will be provided in this chapter. Section 2.1 is devoted to the basic facts and definitions, while Section 2.2 will present already known results for fractal strings in some detail. The material presented in this chapter is compiled and reformulated with some added details from references [31, 74] and [33], except for where noted otherwise. Moreover, some of the proofs have been reformulated for our purpose.

2.1 Preliminaries

Definition 2.1 (Fractal strings)**.** *A fractal string $\mathcal{L}$ is a nonempty bounded open subset of $\mathbb{R}$. Such a set consists of countably many pairwise disjoint open intervals, whose lengths will be denoted by $\ell_1, \ell_2, \ell_3, \ldots > 0$, and called lengths of the string.*

Following the usual notation in the literature, a fractal string will be denoted by $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty}$, where $(\ell_j)_{j\in\mathbb{N}}$ is a nonincreasing sequence of positive numbers with $\lim_{j\to\infty} \ell_j = 0$. For the purpose of this work, the listing order of the lengths is irrelevant and it is always possible to define the strictly decreasing sequence $l_1 > l_2 > \cdots > 0$, where the l_j's are all distinct and counted with multiplicity $\omega_j = \omega_{l_j}$, such that $\mathcal{L}$ can be written as $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty} = \{l_n : l_n \text{ has multiplicity } \omega_n\}_{n=1}^{\infty}$. It should also be noted that $\sum_{j=1}^{\infty} \ell_j = \sum_{n=1}^{\infty} \omega_n l_n$ is finite and equal to the Lebesque measure $vol_1(\mathcal{L})$ of $\mathcal{L}$.

Definition 2.2 (Iterated function systems IFS). *An iterated function system (IFS) is a finite collection of contractions* $S = \{D; S_1, S_2, \ldots, S_m\}$*, with* $m \geq 2$*, on a closed subset* D *of* $\mathbb{R}^n$*. For every IFS* S*, there exists a unique nonempty compact subset* F *of* D*, called the attractor of the IFS, such that (see [7, 55]):*

$$F = \bigcup_{i=1}^{m} S_i(F).$$

Example 2.3 (The Triadic Cantor set). *The triadic Cantor set* $\mathcal{C}_T$ *is the attractor of the IFS* $\{D; S_1, S_2\}$ *on* $\mathbb{R}$*, where:*

$$D = [0, 1],\ S_1 : D \to D,\ S_2 : D \to D,$$

with

$$S_1 = \frac{x}{3},\ \text{and}\ S_2 = \frac{x+2}{3}.$$

Definition 2.4 (The Triadic Cantor string). *Consider the standard triadic (or ternary) Cantor set* $\mathcal{C}_T$ *(Figure 2.1), then the triadic Cantor string* $\mathcal{CS}_T$ *is the complement of* $\mathcal{C}_T$ *with respect to the unit interval* $[0, 1]$ *as shown in Figure 2.2.*

Thus:

$$\mathcal{CS}_T = \{3^{\lfloor -\log_2 j \rfloor}\}_{j=2}^{\infty} = \{\tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \ldots\},$$

respectively $\{l_j\}_{j=0}^{\infty} = \{3^{-(j+1)}\}_{j=0}^{\infty}$, where each l_j appears with multiplicity $\omega_{l_j} = 2^j$.

For the purpose of this work, we will define generalised Cantor strings as follows:

Definition 2.5 (Generalised Cantor strings). *A generalised Cantor string* $\mathcal{CS}$ *with parameters* $1 < a \in \mathbb{N}$ *and* $b \in \mathbb{R}$*,* $b > a$ *is the sequence of lengths given by:*

$$\mathcal{CS} := \{b^{\lfloor -\log_a j \rfloor}\}_{j=2}^{\infty} = \{\tfrac{1}{b}, \tfrac{1}{b^2}, \tfrac{1}{b^2}, \ldots\},$$

or alternatively as $\{l_j\}_{j=0}^{\infty} = \{b^{-(j+1)}\}_{j=0}^{\infty}$*, where each of the* l_j*'s appears with multiplicity* $\omega_j = \omega_{b^{-(j+1)}} = a^j$*.*

Note that the standard triadic Cantor string is obtained by setting $a = 2$ and $b = 3$ in the definition above.

Figure 2.1: The triadic Cantor set $\mathcal{C}_T$

Figure 2.2: The triadic Cantor string $\mathcal{CS}_T$

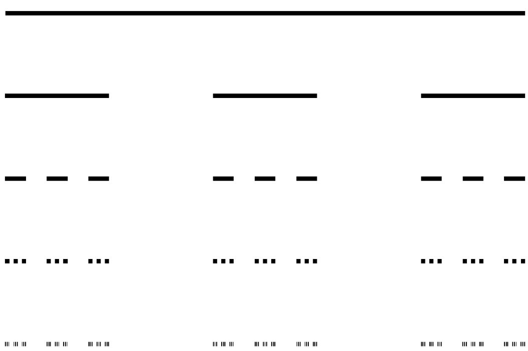

Figure 2.3: A generalised Cantor set $\mathcal{C}$ with parameters $a = 3$ and $b = 5$

Figure 2.4: A generalised Cantor string $\mathcal{CS}$ with parameters $a = 3$ and $b = 5$

Definition 2.6 (a-string)**.** *Given an arbitrary real number $a > 0$, then the fractal string defined by*

$$\mathcal{L}_a := \{l_j\}_{j=1}^{\infty}, \text{ with } l_j = j^{-a} - (j+1)^{-a},$$

is called a-string.

It can be geometrically realised as the open set $\Omega \subset \mathbb{R}$ obtained by removing the points $\{j^{-a}, j \in \mathbb{N}\}$ from the unit interval, that is:

$$\Omega = \bigcup_{j=1}^{\infty} \left((j+1)^{-a}, j^{-a}\right).$$

Hence, its boundary is the (countable) subset of $\mathbb{R}$ given by:

$$\partial\Omega = \{j^{-a}, j \in \mathbb{N}\} \cup \{0\}.$$

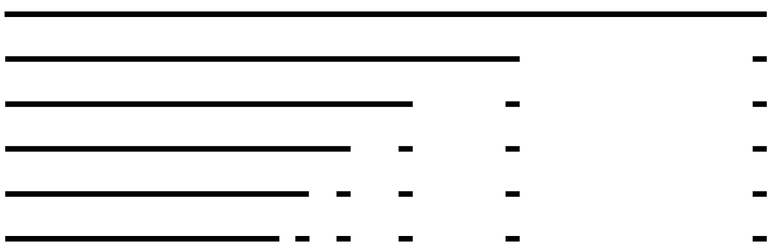

Figure 2.5: The a-set with parameter $a = \frac{\log(3)}{\log(2)} - 1$

Figure 2.6: The a-string with parameter $a = \frac{\log(3)}{\log(2)} - 1$

Definition 2.7 (Distance and ε-neighbourhood)**.** *Let $\varepsilon > 0$ and $B \subset \mathbb{R}$. The distance $d(x, B)$ between a point $x \in \mathbb{R}$ and the set B is given by:*

$$d(x, B) := \inf\{| x - a |: a \in B\},$$

where $| \cdot |$ denotes the one-dimensional Euclidean norm. The (open) ε-neighbourhood of B, denoted as B_ε, is then the set of points that are within a distance ε of B:

$$B_\varepsilon := \{x \in \mathbb{R} : d(x, B) < \varepsilon\},$$

In our case of fractal strings $\mathcal{L}$, we are specifically interested in its boundary respectively the one-dimensional volume, i.e. length, of the set of all points in $\mathcal{L}$ that lie within a distance ε of its boundary $\partial\mathcal{L}$:

$$V(\varepsilon) := vol_1\{x \in \mathcal{L} \mid d(x, \partial\mathcal{L}) < \varepsilon\},$$

where vol_1 designates again the 1-dimensional Lebesgue measure.

Definition 2.8 (Upper and lower Minkowski content)**.** *Let $r \in \mathbb{R}_+$ be given. The upper and lower r-dimensional Minkowski contents of the boundary of a fractal string $\partial\mathcal{L}$ are respectively given by:*

$$\mathscr{M}^*(r, \partial\mathcal{L}) := \limsup_{\varepsilon \to 0^+} \frac{V(\varepsilon)}{\varepsilon^{1-r}},$$

and

$$\mathscr{M}_*(r, \partial\mathcal{L}) := \liminf_{\varepsilon \to 0^+} \frac{V(\varepsilon)}{\varepsilon^{1-r}},$$

It is straightforward to see that if $\mathscr{M}^*(r, \partial\mathcal{L}) < \infty$ for some r, then $\mathscr{M}^*(s, \partial\mathcal{L}) = 0$ for each $s > r$, as:

$$\limsup_{\varepsilon \to 0^+} \frac{V(\varepsilon)}{\varepsilon^{1-s}} = \limsup_{\varepsilon \to 0^+} \frac{V(\varepsilon)}{\varepsilon^{1-r}} \frac{\varepsilon^{1-r}}{\varepsilon^{1-s}} = \limsup_{\varepsilon \to 0^+} \frac{V(\varepsilon)}{\varepsilon^{1-r}} \varepsilon^{s-r}$$

and $\limsup_{\varepsilon \to 0^+} \varepsilon^{s-r} = 0$, if $s - r > 0$. Furthermore, since $\partial\mathcal{L}$ is bounded in $\mathbb{R}$, then clearly $\mathscr{M}^*(r, \partial\mathcal{L}) = 0$ for $r > 1$. On the other hand, akin to the above, if $\mathscr{M}^*(r, \partial\mathcal{L}) > 0$ for some r, then $\mathscr{M}^*(s, \partial\mathcal{L}) = \infty$ for each $s < r$. Therefore, there exists a unique point in $[0, 1]$ at which the function $r \to \mathscr{M}^*(r, \partial\mathcal{L})$ jumps from the value of ∞ to zero. This unique point is called the upper Minkowski dimension of $\partial\mathcal{L}$. The lower Minkowski dimension of $\partial\mathcal{L}$ is defined analogously by using the lower r-dimensional Minkowski content.

Definition 2.9 (Minkowski dimension)**.** *The upper Minkowski dimension is defined by:*

$$\overline{\dim}_M \partial\mathcal{L} := \inf\{r \geq 0 \mid \mathscr{M}^*(r, \partial\mathcal{L}) = 0\} = \sup\{r \geq 0 \mid \mathscr{M}^*(r, \partial\mathcal{L}) = \infty\},$$

and analogously the lower Minkowski dimension by:

$$\underline{\dim}_M \partial\mathcal{L} := \inf\{r \geq 0 \mid \mathscr{M}_*(r, \partial\mathcal{L}) = 0\} = \sup\{r \geq 0 \mid \mathscr{M}_*(r, \partial\mathcal{L}) = \infty\}.$$

When $\overline{\dim}_M \partial\mathcal{L} = \underline{\dim}_M \partial\mathcal{L}$*, the common value is called the Minkowski dimension of* $\partial\mathcal{L}$*, denoted in the following as* $d_M = \dim_M \partial\mathcal{L}$*, where we omit* $\partial\mathcal{L}$ *for sake of notational simplicity.*

Definition 2.10 (Minkowski content)**.** *If the upper and lower* d_M*-dimensional Minkowski contents of* $\partial\mathcal{L}$ *are equal*

$$\mathscr{M}^*(d_M, \partial\mathcal{L}) = \mathscr{M}_*(d_M, \partial\mathcal{L}),$$

then this common value is called Minkowski content $\mathscr{M}(d_M, \partial\mathcal{L}) = \lim_{\varepsilon\to 0^+} V(\varepsilon)\varepsilon^{d_M - 1}$*, and* $\partial\mathcal{L}$ *is called Minkowski-measurable.*

However it should be noted that $\mathscr{M}$ is not a measure, as it fails countable additivity [65].

Remark 2.11. *In the literature on the subject, the Minkowski content and dimension of the string's boundary are in general simply referred to as the Minkowski content and dimension of the string by linguistic imprecision. We will adopt this common agreement at this stage as well. As an example, we will call a fractal string Minkowski-measurable iff upper and lower Minkoski content of its boundary exist and are equal.*

Without proof, we will give here an important result on Minkowski-measurability, first stated by M. L. Lapidus and C. Pomerance as Theorem 2. in [77]:

Remark 2.12 (Criterion for Minkowski-measurability)**.** *A fractal string $\mathcal{L}$ is Minkowski-measurable iff $\ell_j \sim Lj^{-\frac{1}{d_M}}$*

Here the symbol $\sim$ means asymptotic equality in the sense that $a_j \sim b_j$ iff $\lim_{j\to\infty} \frac{a_j}{b_j} = 1$.

Example 2.13. *It is well known that the Minkowski dimension of the triadic Cantor string $\mathcal{CS}_T$ is $d_M = \frac{\log(2)}{\log(3)}$, and as $\mathcal{CS}_T = \{3^{\lfloor -\log_2 j \rfloor}\}_{j=2}^{\infty}$, we are interested in the limit:*

$$\lim_{j\to\infty} 3^{\lfloor -\log_2(j) \rfloor} j^{\frac{\log(3)}{\log(2)}}.$$

Now, for each j in the interval $(2^n, 2^{(n+1)})$, where $n \in \mathbb{N}^*$, we have $3^{\lfloor -\log_2(j) \rfloor} = 3^{-n}$. Thus $3^{\lfloor -\log_2(j) \rfloor} j^{\frac{\log(3)}{\log(2)}}$ is monotonically increasing in the interval and its range is $(1/3, 1)$, regardless of the value given to n. Furthermore, for every j that is an integer power of 2, $3^{\lfloor -\log_2(j) \rfloor} j^{\frac{\log(3)}{\log(2)}} = 1$. Therefore, we have:

$$\limsup_{j\to\infty} 3^{\lfloor -\log_2(j) \rfloor} j^{\frac{\log(3)}{\log(2)}} = 1,$$

and

$$\liminf_{j\to\infty} 3^{\lfloor -\log_2(j) \rfloor} j^{\frac{\log(3)}{\log(2)}} = \frac{1}{3},$$

so that the above limit does not exist and thereby the Cantor string is not Minkowski-measurable.

Example 2.14. *For the a-string, we have:*

$$\ell_j = j^{-a} - (j+1)^{-a} = \frac{\left(1 + \frac{1}{j}\right)^a - 1}{(j+1)^a}.$$

Using the binomial expansion, this may be written as

$$\ell_j = \frac{\left(1 + a\frac{1}{j} + \frac{a(a-1)}{2!}\left(\frac{1}{j}\right)^2 + \frac{a(a-1)(a-2)}{3!}\left(\frac{1}{j}\right)^3 + \dots\right) - 1}{(j+1)^a}$$

$$= \frac{a\frac{1}{j}}{(j+1)^a} + \frac{\frac{a(a-1)}{2!}\left(\frac{1}{j}\right)^2}{(j+1)^a} + \frac{\frac{a(a-1)(a-2)}{3!}\left(\frac{1}{j}\right)^3}{(j+1)^a} + \dots.$$

Therefore:

$$\lim_{j\to\infty} \ell_j j^{a+1} =$$

$$\lim_{j\to\infty}\left(a\left(\frac{j}{j+1}\right)^a + \frac{a(a-1)}{2!}\left(\frac{j}{j+1}\right)^a\frac{1}{j} + \frac{a(a-1)(a-2)}{3!}\left(\frac{j}{j+1}\right)^a\frac{1}{j^2} + \dots\right)$$

$$= a,$$

as $\lim_{j\to\infty}\left(\frac{j}{j+1}\right)^a = 1$ *and* $\lim_{j\to\infty}\frac{1}{j^n} = 0$*, for each* $n \geq 1$*. Thus the a-string is Minkowski-measurable with* $L = a$ *and* $d_M = \frac{1}{a+1}$*.*

Definition 2.15 (Geometric zeta function)**.** *For a fractal string* $\mathcal{L}$*, the geometric zeta function is defined as:*

$$\zeta_{\mathcal{L}}(s) := \sum_{j=1}^{\infty} l_j^s = \sum_{l} \omega_l \cdot l^s,$$

for $s \in \mathbb{C}$ *and* $\mathrm{Re}(s) > d_M$*, where* d_M *is the Minkowski dimension of the string.*

It should be noted that some values of the geometric zeta function have a special meaning, i.e. the total length of the string for example is given by $\zeta_{\mathcal{L}}(1) = \sum_{j=1}^{\infty} l_j$.

Definition 2.16 (Geometric counting function)**.** *The geometric counting function (or alternatively: the counting function of the reciprocal lengths) of a fractal string* $\mathcal{L}$*, is defined by*

$$N_{\mathcal{L}}(x) := \#\left\{j \in \mathbb{N} \mid l_j^{-1} \leq x\right\} = \sum_{n\in\mathbb{N},\ l_n^{-1}\leq x} \omega_n,$$

for $x > 0$ *and where the* ω_n*'s are the multiplicities of the lengths* l_n*.*

Let us consider the following eigenvalue problem on an open bounded set Ω in $\mathbb{R}$ with boundary $\partial\Omega$:

$$-\Delta u = \lambda u$$

in Ω, where Δ denores the Laplacian, with Dirichlet boundary conditions:

$$u|_{\partial\Omega} = 0.$$

Then the eigenvalues form a countable sequence, such that $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \leq \ldots$, with each eigenvalue being repeated according to (algebraic) multiplicity.

Definition 2.17 (Eigenvalue counting function)**.** *For a given positive λ, we define the eigenvalue counting function $N(\lambda)$ as the number of eigenvalues not exceeding λ:*

$$N(\lambda) := \#\{k \in \mathbb{N} \mid \lambda_k \leq \lambda\},\ \lambda > 0.$$

2.2 Spectral asymptotics of fractal strings

In this section, we present the current state of knowledge for the spectral asymptotics of fractal strings, partially with detailed proofs.

The Dirichlet problem on $\mathcal{L}$ may be reformulated as finding the resonant frequencies c fixed at the boundary po :h interval of which $\mathcal{L}$ con:

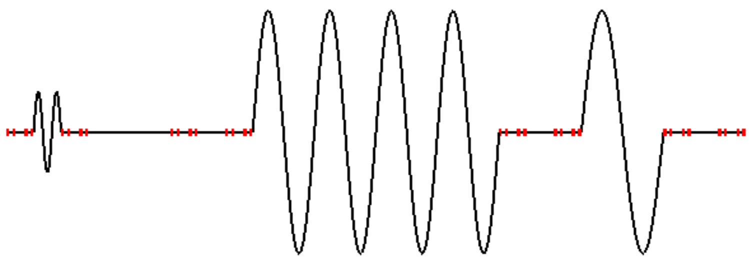

Figure 2.7: An example for eigenfunctions of the Cantor string

Non-trivial solutions occur at the values

$$\lambda = \left(\frac{\pi k}{\ell_j}\right)^2, \text{ with } k = 1, 2, \ldots,$$

for a sinusoidal eigenfunction vanishing outside the interval considered. In other words, if we fix $\lambda > 0$, then the number of eigenvalues k less than λ that are possible for this interval is given by $k = \lfloor \pi^{-1}\lambda^{\frac{1}{2}}\ell_j \rfloor$. Counting these eigenvalues for all lengths ℓ_j of the fractal string $\mathcal{L}$ then leads to:

$$\begin{aligned} N(\lambda) &= \sum_{j=1}^{\infty} \lfloor \pi^{-1}\lambda^{\frac{1}{2}}\ell_j \rfloor \\ &= \sum_{j=1}^{\infty} \pi^{-1}\lambda^{\frac{1}{2}}\ell_j - \sum_{j=1}^{\infty} \left\{ \pi^{-1}\lambda^{\frac{1}{2}}\ell_j \right\} \\ &= \pi^{-1} vol_1(\mathcal{L})\lambda^{\frac{1}{2}} - \varphi(\lambda), \end{aligned} \tag{2.1}$$

where

$$\varphi(\lambda) := \sum_{j=1}^{\infty} \left\{ \pi^{-1}\lambda^{\frac{1}{2}}\ell_j \right\}. \tag{2.2}$$

Note that in the above the symbol $\lfloor x \rfloor$ stands for "the greatest integer less than x" and $\{x\} = x - \lfloor x \rfloor$ means "the fractional part of x".

The first term in Equation (2.1) is just Weyl's expression in the one-dimensional case.

Remark 2.18. *We will use in the following the conventions of writing $f(x) \asymp g(x)$, respectively $a_j \asymp b_j$ iff there exist two constants c_1 and c_2, such that*

$$0 < c_1 g(x) \leq f(x) \leq c_2 g(x) < \infty, \text{ for all } x \in \mathbb{R},\ x > 0,$$

respectively

$$0 < c_1 b_j \leq a_j \leq c_2 b_j < \infty, \text{ for all } j \in \mathbb{N}.$$

In the following theorem (see [33]), it will be shown that $\varphi(\lambda) \asymp \lambda^{\frac{d_M}{2}}$ holds under certain conditions.

Theorem 2.19. *If $\ell_j \asymp j^{-\frac{1}{d_M}}$ for some $0 < d_M < 1$ then $\varphi(\lambda) \asymp \lambda^{\frac{d_M}{2}}$.*

We give here a detailed proof, following the sketch given in [33].

Proof. Given a fixed $\lambda > \pi^2 \ell_1^{-2}$, let k be the greatest integer, such that $\pi^{-1}\lambda^{\frac{1}{2}}\ell_k > 1 \Leftrightarrow \ell_k \geq \pi\lambda^{-\frac{1}{2}}$. As $\ell_k \asymp k^{-\frac{1}{d_M}} \Leftrightarrow c_1 k^{-\frac{1}{d_M}} \leq \ell_k \leq c_2 k^{-\frac{1}{d_M}}$ for some constants c_1 and c_2, one has:

$$\ell_k \leq c_2 k^{-\frac{1}{d_M}},$$

with

$$\pi\lambda^{-\frac{1}{2}} \leq \ell_k,$$

so that

$$\begin{aligned}
&\pi\lambda^{-\frac{1}{2}} \leq c_2 k^{-\frac{1}{d_M}} \\
\Leftrightarrow &k^{\frac{1}{d_M}} \leq c_2 \pi^{-1}\lambda^{\frac{1}{2}} \\
\Leftrightarrow &k \leq \left(c_2 \pi^{-1}\lambda^{\frac{1}{2}}\right)^{d_M} \\
\Leftrightarrow &k \leq c_2^{d_M} \pi^{-d_M} \lambda^{\frac{d_M}{2}},
\end{aligned}$$

where c_2 does not depend on λ. Furthermore, as for all j:

$$0 \leq \left\{\pi^{-1}\lambda^{\frac{1}{2}}\ell_j\right\} \leq 1,$$

we have:

$$0 \leq \sum_{j=1}^{k} \left\{\pi^{-1}\lambda^{\frac{1}{2}}\ell_j\right\} \leq k.$$

Thus by Equation (2.2):

$$\sum_{j=k+1}^{\infty} \left\{\pi^{-1}\lambda^{\frac{1}{2}}\ell_j\right\} \leq \varphi(\lambda) \leq k + \sum_{j=k+1}^{\infty} \left\{\pi^{-1}\lambda^{\frac{1}{2}}\ell_j\right\},$$

and as for every $j > k$, one has by the definition of k: $\pi^{-1}\lambda^{\frac{1}{2}}\ell_j < 1$, and therefore

$$\left\{\pi^{-1}\lambda^{\frac{1}{2}}\ell_j\right\} = \pi^{-1}\lambda^{\frac{1}{2}}\ell_j$$

so that:

$$\sum_{j=k+1}^{\infty} \pi^{-1}\lambda^{\frac{1}{2}}\ell_j \leq \varphi(\lambda) \leq k + \sum_{j=k+1}^{\infty} \pi^{-1}\lambda^{\frac{1}{2}}\ell_j.$$

As $c_1 j^{-\frac{1}{d_M}} \leq \ell_j \leq c_2 j^{-\frac{1}{d_M}}$, it is possible to use the integral test estimate to find bounds for $\sum_{j=k+1}^{\infty} \ell_j$. Indeed,

$$c_1 \sum_{j=k+1}^{\infty} j^{-\frac{1}{d_M}} \leq \sum_{j=k+1}^{\infty} \ell_j \leq c_2 \sum_{j=k+1}^{\infty} j^{-\frac{1}{d_M}} \tag{2.3}$$

and

$$\begin{aligned}
&\int_{k+2}^{\infty} j^{-\frac{1}{d_M}}\, dj \leq \sum_{j=k+1}^{\infty} j^{-\frac{1}{d_M}} \leq \int_{k+1}^{\infty} j^{-\frac{1}{d_M}}\, dj \\
\Leftrightarrow\ & \frac{1}{1-\frac{1}{d_M}} j^{1-\frac{1}{d_M}} \Bigg]_{k+2}^{\infty} \leq \sum_{j=k+1}^{\infty} j^{-\frac{1}{d_M}} \leq \frac{1}{1-\frac{1}{d_M}} j^{1-\frac{1}{d_M}} \Bigg]_{k+1}^{\infty} \\
\Leftrightarrow & \frac{1}{\frac{1}{d_M}-1} (k+2)^{1-\frac{1}{d_M}} \leq \sum_{j=k+1}^{\infty} j^{-\frac{1}{d_M}} \leq \frac{1}{\frac{1}{d_M}-1} (k+1)^{1-\frac{1}{d_M}}.
\end{aligned}$$

Now, as

$$k+1 > k \Rightarrow (k+1)^{1-\frac{1}{d_M}} \leq k^{1-\frac{1}{d_M}} = \left(c_2^{d_M} \pi^{-d_M} \lambda^{\frac{d_M}{2}}\right)^{1-\frac{1}{d_M}},$$

and as $\ell_{k+2} \leq \pi \lambda^{-\frac{1}{2}}$ by the definition of k and $\ell_{k+2} \geq c_1 (k+2)^{-\frac{1}{d_M}}$ by the assumption of the theorem, one has:

$$\begin{aligned}
&\pi \lambda^{-\frac{1}{2}} \geq c_1 (k+2)^{-\frac{1}{d_M}} \\
\Leftrightarrow\ & (k+2)^{-\frac{1}{d_M}} \leq \frac{\pi \lambda^{-\frac{1}{2}}}{c_1} \\
\Leftrightarrow\ & (k+2)^{-1} \leq \left(\frac{\pi \lambda^{-\frac{1}{2}}}{c_1}\right)^{d_M} \\
\Leftrightarrow & k+2 \geq \left(\frac{c_1}{\pi \lambda^{-\frac{1}{2}}}\right)^{d_M} = \left(c_1 \pi^{-1} \lambda^{\frac{1}{2}}\right)^{d_M} \\
\Leftrightarrow\ & (k+2)^{1-\frac{1}{d_M}} \geq \left(c_1^{d_M} \pi^{-d_M} \lambda^{\frac{d_M}{2}}\right)^{1-\frac{1}{d_M}}.
\end{aligned}$$

Thus:

$$\frac{1}{\frac{1}{d_M}-1}\left(c_1^{d_M}\pi^{-d_M}\lambda^{\frac{d_M}{2}}\right)^{1-\frac{1}{d_M}} \leq \sum_{j=k+1}^{\infty} j^{-\frac{1}{d_M}} \leq \frac{1}{\frac{1}{d_M}-1}\left(c_2^{d_M}\pi^{-d_M}\lambda^{\frac{d_M}{2}}\right)^{1-\frac{1}{d_M}}$$
$$\Leftrightarrow \frac{d_M}{1-d_M}\left(c_1^{d_M}\pi^{-d_M}\lambda^{\frac{d_M}{2}}\right)^{\frac{d_M-1}{d_M}} \leq \sum_{j=k+1}^{\infty} j^{-\frac{1}{d_M}} \leq \frac{d_M}{1-d_M}\left(c_2^{d_M}\pi^{-d_M}\lambda^{\frac{d_M}{2}}\right)^{\frac{d_M-1}{d_M}}$$
$$\Leftrightarrow \frac{d_M}{1-d_M}c_1^{d_M-1}\pi^{-(d_M-1)}\lambda^{\frac{d_M-1}{2}} \leq \sum_{j=k+1}^{\infty} j^{-\frac{1}{d_M}} \leq \frac{d_M}{1-d_M}c_2^{d_M-1}\pi^{-(d_M-1)}\lambda^{\frac{d_M-1}{2}},$$

allowing us to write Equation (2.3) as:

$$c_1\frac{d_M}{1-d_M}c_1^{d_M-1}\pi^{-(d_M-1)}\lambda^{\frac{d_M-1}{2}} \leq \sum_{j=k+1}^{\infty} \ell_j \leq c_2\frac{d_M}{1-d_M}c_2^{d_M-1}\pi^{-(d_M-1)}\lambda^{\frac{d_M-1}{2}}$$

and thereby:

$$\pi^{-1}\lambda^{\frac{1}{2}}c_1\frac{d_M}{1-d_M}c_1^{d_M-1}\pi^{-(d_M-1)}\lambda^{\frac{d_M-1}{2}} \leq \varphi(\lambda) \leq k+\pi^{-1}\lambda^{\frac{1}{2}}c_2\frac{d_M}{1-d_M}c_2^{d_M-1}\pi^{-(d_M-1)}\lambda^{\frac{d_M-1}{2}}.$$

Using again that $k \leq \pi^{-d_M}c_2^{d_M}\lambda^{\frac{d_M}{2}}$ and simplifying:

$$\frac{d_M}{1-d_M}c_1^{d_M}\pi^{-d_M}\lambda^{\frac{d_M}{2}} \leq \varphi(\lambda) \leq \pi^{-d_M}c_2^{d_M}\lambda^{\frac{d_M}{2}} + \frac{d_M}{1-d_M}c_2^{d_M}\pi^{-d_M}\lambda^{\frac{d_M}{2}}$$
$$\Leftrightarrow \frac{d_M}{1-d_M}c_1^{d_M}\pi^{-d_M}\lambda^{\frac{d_M}{2}} \leq \varphi(\lambda) \leq \left(1+\frac{d_M}{1-d_M}\right)c_2^{d_M}\pi^{-d_M}\lambda^{\frac{d_M}{2}}$$
$$\Leftrightarrow c\lambda^{\frac{d_M}{2}} \leq \varphi(\lambda) \leq c'\lambda^{\frac{d_M}{2}}$$
$$\Leftrightarrow \varphi(\lambda) \asymp \lambda^{\frac{d_M}{2}},$$

which completes the proof. □

Strengthening the condition on the asymptotic behaviour of the lengths ℓ_j of the fractal string to $\ell_j \sim Lj^{-\frac{1}{d_M}} \Leftrightarrow \lim_{j\to\infty} \ell_j j^{\frac{1}{d_M}} = L > 0$, leads to an interesting connection between the concept of Minkowski-measurability (see Remark 2.12) and Riemann's Zeta-function defined below.

Definition 2.20 (Riemann's Zeta-function)**.** *Riemann's Zeta-function is defined* *as:*

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s},$$

for all $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1$. *A meromorphic continuation to* $\mathrm{Re}(s) > 0$ *we will need later is given by:*

$$\zeta(s) := \frac{1}{s-1} + \int_1^{\infty} (t^{-s} - \lfloor t \rfloor^{-s}) dt.$$

We can now state the theorem announced above:

Theorem 2.21. *If* $\ell_j \sim L j^{-\frac{1}{d_M}}$ *for some* $0 < d_M < 1$ *then the following holds:*

$$N(\lambda) = \pi^{-1} vol_1(\mathcal{L}) \lambda^{\frac{1}{2}} + \pi^{-d_M} \zeta(d_M) L^{d_M} \lambda^{\frac{d_M}{2}} + o(\lambda^{\frac{d_M}{2}}),$$

as $\lambda \to \infty$.

Here we will only sketch the proof as in [77], full details may be found in [78] and in [74], where a different approach is chosen.

Sketch of proof. Recall from Equations (2.1) and (2.2) that:

$$\begin{aligned} \varphi(\lambda) &= \sum_{j=1}^{\infty} \left\{ \pi^{-1} \lambda^{\frac{1}{2}} \ell_j \right\} \\ &= \sum_{j=1}^{\infty} \pi^{-1} \lambda^{\frac{1}{2}} \ell_j - \sum_{j=1}^{\infty} \lfloor \pi^{-1} \lambda^{\frac{1}{2}} \ell_j \rfloor \end{aligned}$$

Let $J(\varepsilon) := max\{j \geq 1 : \ell_j \geq \varepsilon\}$, then by the assumption of the theorem:

$$L j^{-\frac{1}{d_M}} \sim \ell_j \geq \varepsilon$$

and thus:

$$j \leq L^{d_M} \varepsilon^{-d_M},$$

so that

$$J(\varepsilon) \sim L^{d_M} \varepsilon^{-d_M}, \text{ as } \varepsilon \to 0^+.$$

Let furthermore $k \geq 2$ be an arbitrary fixed integer, then:

$$\varphi(x) = x \sum_{j > J\left(\frac{1}{x}\right)} \ell_j + \sum_{j \leq J\left(\frac{k}{x}\right)} \{\ell_j x\} + \sum_{p=2}^{k} \sum_{j=J\left(\frac{p}{x}\right)+1}^{J\left(\frac{p-1}{x}\right)} \{\ell_j x\}.$$

Through Abel-summation, this can be rewritten as:

$$\varphi(x) = \alpha(x) + \beta(x) + \gamma(x),$$

where:

$$\begin{aligned}
\alpha(x) &:= x \sum_{j > J\left(\frac{k}{x}\right)} \ell_j, \\
\beta(x) &:= kJ\left(\frac{k}{x}\right) - \sum_{p=1}^{k-1} J\left(\frac{p}{x}\right) \text{ and} \\
\gamma(x) &:= \sum_{j \leq J\left(\frac{k}{x}\right)} \left(\{\ell_j x\} - 1\right).
\end{aligned}$$

By using then that $J(\varepsilon) \sim L^{d_M} \varepsilon^{-d_M}$ and $\ell_j \sim L j^{-\frac{1}{d_M}}$, one obtains for $x \to \infty$:

$$\begin{aligned}
(Lx)^{-d_M} \alpha(x) &\to k^{1-d_M} \frac{d_M}{1-d_M}, \\
(Lx)^{-d_M} \beta(x) &\to k^{1-d_M} - \sum_{p=1}^{k-1} p^{-d_M} \text{ and} \\
(Lx)^{-d_M} \gamma(x) &\leq (Lx)^{-d_M} J\left(\frac{k}{x}\right) \to k^{-d_M}.
\end{aligned}$$

Thus

$$(Lx)^{-d_M} \left(\alpha(x) + \beta(x)\right) \to \frac{1}{1-d_M} k^{1-d_M} - \sum_{p=1}^{k-1} p^{-d_M} = f_k(d_M) + \frac{1}{1-d_M},$$

where

$$f_k(s) := \int_1^k \left(t^{-s} - \lfloor t \rfloor^{-s}\right) dt.$$

For $\mathrm{Re}(s) > 0$, the sequence $\{f_k(s)\}_{k=1}^{\infty}$ converges uniformly to the (analytic) function

$$f(s) := \int_1^{\infty} \left(t^{-s} - \lfloor t \rfloor^{-s}\right) dt,$$

as $k \to \infty$. Furthermore, as stated above, Riemann's Zeta-function admits the meromorphic continuation to $\mathrm{Re} > 0$ given by:

$$\zeta(s) = \frac{1}{s-1} + \int_1^{\infty} \left(\lfloor t \rfloor^{-s} - t^{-s}\right) dt.$$

Thus:

$$f_k(d_M) + \frac{1}{1-d_M} \to -\zeta(d_M), \text{ for } k \to \infty$$

and therefore:

$$(Lx)^{-d_M} \varphi(x) \to -\zeta(d_M), \text{ as } k \to \infty.$$

Finally, setting $x = \pi^{-1}\lambda^{\frac{1}{2}}$ and reassembling all the terms, we obtain

$$N(\lambda) = \pi^{-1} vol_1(\mathcal{L})\lambda^{\frac{1}{2}} + \pi^{-d_M}\zeta(d_M)L^{d_M}\lambda^{\frac{d_M}{2}} + o(\lambda^{\frac{d_M}{2}})$$

for the eigenvalue counting function, as stated. □

The implications of Theorems 2.19 and 2.21 may clearly be seen through the examples of the triadic Cantor string and the a-string with parameter $a = \frac{\log(3)}{\log(2)} - 1$, both having the same Minkowski dimension $d_M = \frac{\log(2)}{\log(3)}$, displayed in Figure 2.8. Indeed, the graph of the eigenvalue counting function for the Cantor string shows oscillations in the spectrum that are typical for strings that are not Minkowski-measurable, while they are absent in the case of the Minkowski-measurable a-string.

These results on the connection between the Riemann Zeta-function and the spectral asymptotics of fractal strings were even taken further by M.L. Lapidus and H. Maier in [75] and [76], where they formulated the Riemann hypothesis in terms of an inverse spectral problem. Being beyond the scope of this work, we will only state the theorem and its corollary without proof here:

Theorem 2.22 (The inverse spectral problem for Riemann's hypothesis)**.** *Let a fractal string $\mathcal{L}$ with Minkowski dimension $d_M \in (0, 1)$ be given. If the eigenvalue*

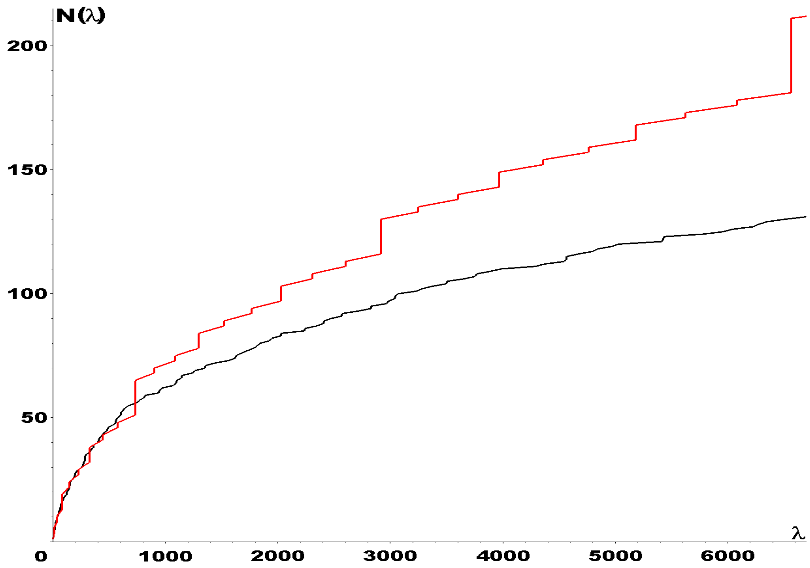

Figure 2.8: Graphs of $N(\lambda)$ for the triadic Cantor string (red) and the a-string with parameter $a = \frac{\log(3)}{\log(2)} - 1$ (black).

counting function is given by $N(\lambda) = \pi^{-1} vol_1(\mathcal{L})\lambda^{\frac{1}{2}} + C\lambda^{\frac{d_M}{2}} + o(\lambda^{\frac{d_M}{2}})$*, with* C *being a constant, then* $\mathcal{L}$ *is Minkowski-measurable if and only if* $\zeta(s)$ *does not have any zero on the vertical line* $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = d_M\}$*.*

From the theorem, it is then easy to deduce the following corollary:

Corollary 2.23. *Since* $\zeta(s)$ *has zeros on the critical line* $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = \frac{1}{2}\}$*, the inverse spectral problem is not true when* $d_M = \frac{1}{2}$*. On the other hand, it is true for every* $d_M \in (0,1) \setminus \{\frac{1}{2}\}$*, if and only if the Riemann hypothesis holds.*

Chapter 3

Fractal chains

In the preceeding chapter, we have rendered the current state of knowledge on the asymptotics of the eigenvalues of fractal strings, one-dimensional drums with a fractal set as boundary. We will now present our results concerning the distribution of these eigenvalues, thereby showing the usefulness of techniques and methods from random matrix theory to the study of vibrating fractals. Particularly, we will establish two new theorems (Theorems 3.8 and 3.9) related to the Minkowski-measurability of one-dimensional fractals. On that account, these fractal drums will be modelled by linear chains of a finite number of discrete masses coupled by springs ("fractal chains", see [21]), thus allowing a description in terms of matrices.

3.1 Monoatomic chains

The monoatomic linear chain of masses coupled by harmonic springs (i.e. obeying Hooke's law) is a textbook example [53, 63] as an introduction into vibrational normal modes (phonons) in solid state physics. Its mathematics are simple and it has many features common to lattice vibrations in general.

The stiffness of each spring shall be K and each atom shall have a mass m. Let u_n be the displacement of the n^{th} atom. The force on the atom n due to the atom at position $n-1$ is then $K(u_{n-1} - u_n)$ and those from the atom at position

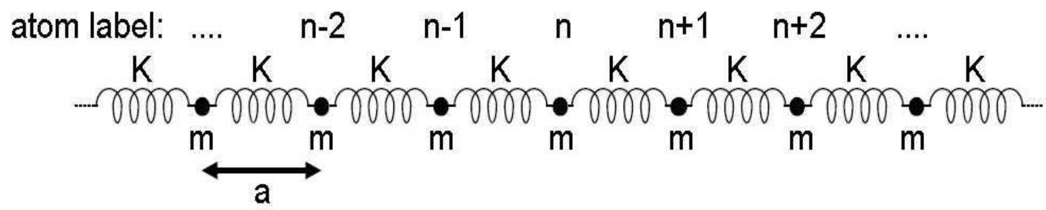

Figure 3.1: Monoatomic chain

$n+1$ is $K(u_{n-1} - u_n)$, so that its equation of motion can be written as:

$$m\frac{d^2u_n}{dt^2} = K\left(u_{n-1} - 2u_n + u_{n+1}\right),$$

respectively

$$m\frac{d^2u_n}{dt^2} + 2Ku_n = K\left(u_{n-1} + u_{n+1}\right).$$

Let κ be the wavevector and a the distance between the atoms, then with the harmonic ansatz:

$$\begin{aligned} u_n &= u_0 e^{i(\omega t - n\kappa a)}, \\ u_{n-1} &= u_0 e^{i(\omega t - (n-1)\kappa a)} = u_n e^{i\kappa a} \text{ and} \\ u_{n+1} &= u_0 e^{i(\omega t - (n+1)\kappa a)} = u_n e^{-i\kappa a}, \end{aligned}$$

one has

$$m\frac{d^2u_n}{dt^2} = -m\omega^2 u_n,$$

and therefore:

$$\left(-m\omega^2 + 2K\right)u_n = Ku_n\left(e^{i\kappa a} + e^{-i\kappa a}\right).$$

Dividing by $Ku_n \neq 0$ and using that $e^{i\kappa a} + e^{-i\kappa a} = 2\cos(\kappa a)$:

$$\begin{aligned}\left(-m\omega^2 + 2K\right)u_n &= Ku_n\left(e^{i\kappa a} + e^{-i\kappa a}\right)\\ \Leftrightarrow -\frac{m}{K}\omega^2 + 2 &= 2\cos(\kappa a)\\ \Leftrightarrow \omega^2 &= 2\frac{K}{m}\left(1 - \cos(\kappa a)\right)\\ \Leftrightarrow \omega^2 &= 4\frac{K}{m}\left(\sin\left(\frac{\kappa a}{2}\right)\right)^2.\end{aligned}$$

Using Dirichlet boundary conditions $u_0(t) = u_{N+1}(t) = 0$, the allowed values for κ are then given by $\kappa = \frac{2\pi}{(N+1)a}n$, with $1 \leq n \leq N+1$ and N being the number of atoms in the chain. The squared frequencies of the chain are therefore given by the so-called dispersion relation:

$$\omega_n^2 = 4\frac{K}{m}\left(\sin\left(\frac{\pi n}{2(N+1)}\right)\right)^2. \tag{3.1}$$

Alternatively, the equations of motion may be reformulated in matrix form, which makes it possible to use tools from random matrix theory (RMT). Indeed, writing Newton's law as:

$$\sum\{F\} = \mathbf{K}\{u\} + \mathbf{\Gamma}\{\dot{u}\} + \mathbf{M}\{\ddot{u}\},$$

where $\{F\}$ denotes the column vector of (external) forces acting on the chain, $\mathbf{K}$ the square matrix of stiffness properties at the atoms (stiffness matrix), $\mathbf{\Gamma}$ the square matrix of damping properties at the atoms, $\mathbf{M}$ the square matrix of inertial properties at the atoms (mass matrix) and $\{u\}, \{\dot{u}\}, \{\ddot{u}\}$ the column vectors of displacements, velocities and accelerations respectively.

In our idealised model, the effects of damping and velocity will be neglected and there will be no external forces acting on the structure, such that the equations of motion reduce to:

$$0 = \mathbf{K}\{u\} + \mathbf{M}\{\ddot{u}\}.$$

For simple harmonic motion, the acceleration is then given as above by:

$$\{\ddot{u}\} = -\omega^2\{u\},$$

where ω^2 is the square of the circular frequency. Thus:

$$\mathbf{K}\{u\} - \omega^2\mathbf{K}\{u\} = 0.$$

Multiplying this equation by $\mathbf{M}^{-1}$, $\mathbf{M}$ being non-singular, from the left yields:

$$\mathbf{M}^{-1}\mathbf{K}\{u\} - \omega^2\mathbf{I}\{u\} = 0, \text{ or}$$
$$\left(\mathbf{D} - \omega^2\mathbf{I}\right)\{u\} = 0,$$

where $\mathbf{D} := \mathbf{M}^{-1}\mathbf{K}$ is called the dynamic matrix and $\mathbf{I}$ is the identity matrix. Now, for a single spring element, the stiffness matrix is given by:

$$\mathbf{K_s} = \begin{pmatrix} K & -K \\ -K & K \end{pmatrix},$$

where K is the stiffness constant of the spring. For an assembly of springs, the total stiffness matrix is given by the following two simple rules [15]:

- A term on the main diagonal $K_{n,n}$ is the sum of the stiffnesses of all spring elements connected to the atom n.
- A term off the main diagonal $K_{n,m}$ is the negative sum of the stiffnesses of all spring elements connecting the atoms n and m.

The mass matrix $\mathbf{M}$ is simply a matrix with the masses of the different atoms on its main diagonal and zero everywhere else. Furthermore, for the monoatomic chain, all the spring stiffnesses and masses are equal, such that the dynamic matrix is given by:

$$\mathbf{D} = \begin{pmatrix} \frac{2K}{m} & -\frac{K}{m} & 0 & 0 & \dots \\ -\frac{K}{m} & \frac{2K}{m} & -\frac{K}{m} & 0 & \dots \\ 0 & -\frac{K}{m} & \frac{2K}{m} & -\frac{K}{m} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} = \frac{K}{m}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots \\ -1 & 2 & -1 & 0 & \dots \\ 0 & -1 & 2 & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

The eigenvalues λ_n of the dynamic matrix are then precisely the squared frequencies ω^2 given in Equation (3.1):

$$\lambda_n = \omega_n^2 = 4\frac{K}{m}\left(\sin\left(\frac{\pi n}{2\,(N+1)}\right)\right)^2.$$

For long wavelengths, i.e. for small n, it is possible to expand the sine-function

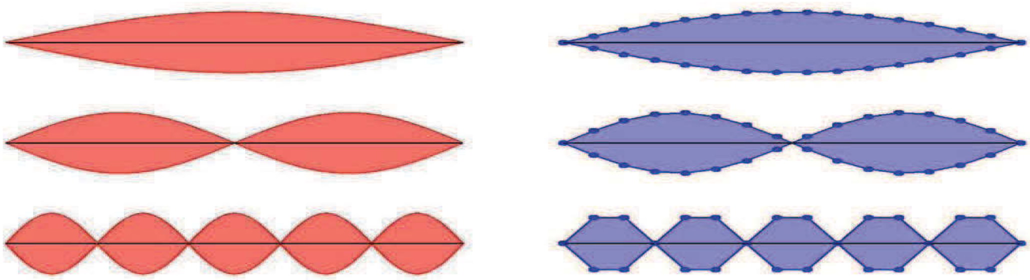

Figure 3.2: Corresponding vibrational modes of a string and a monoatomic chain

and by neglecting the higher order terms, the dispersion relation for the string $\lambda_n \sim n^2$ may be recovered. However, this approximation is no longer valid for the higher normal modes, where the wavelength of the excitation becomes comparable to the length scale of the distances between the masses. Indeed, contrary to the normal homogeneous string, a monoatomic chain posesses a highest possible frequency, a fact that can be easily deduced from Equation 3.1 or Figures 3.2 and 3.3. When the neighbouring masses vibrate in antiphase, there is no higher possible mode, and λ_{max} is given by setting $n = N + 1$ as:

$$\lambda_{max} = 4\frac{K}{m}\left(\sin\left(\frac{\pi\,(N+1)}{2\,(N+1)}\right)\right)^2 = 4\frac{K}{m}.$$

3.2 Fractal chains

The lowest (fundamental) frequency of a monoatomic chain is given by Equation (3.1) with $n = 1$:

$$\lambda_1 = \omega_1^2 = 4\frac{K}{m}\left(\sin\left(\frac{\pi}{2\,(N+1)}\right)\right)^2.$$

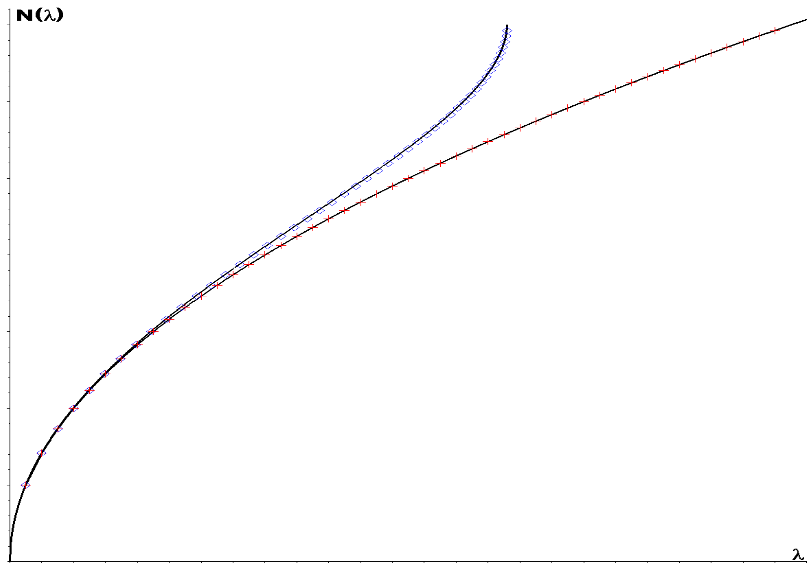

Figure 3.3: Eigenvalue counting functions for a string (crosses, $N(\lambda) \sim \sqrt{\lambda}$) and a monoatomic chain (boxes, $N(\lambda) \sim \arcsin(\sqrt{\lambda})$)

It is then possible to choose K and m, such that the fundamental frequency of the monoatomic chain is the same as that of the length l_1 of the fractal string that is to be modelled. By the same method, monoatomic chains corresponding to any length l_j of the fractal string can then be determined, such that the fractal chain is obtained by combining chains with increasing fundamental frequencies in accordance with the corresponding fractal set construction. The part of the spectrum up to the maximal frequency of the basic chain then allows a comparison with the one of the fractal string as illustrated in Figures 3.4 and 3.5.

However, it is important to note that the number of masses N in the different chains of the obtained pre-fractal has to be chosen such that chains of higher order than the iteration level $\mathfrak{m}$ do not contribute to the spectrum. In other words, the lowest frequency of the chain corresponding to the length $l_{\mathfrak{m}+1}$ of the string has to be larger than the maximal frequency of the basic chain. We will subsequently always choose:

$$N = \left\lceil \frac{\pi}{2} \cdot \frac{1}{\arcsin\left(l_{\mathfrak{m}}\right)} \right\rceil, \tag{3.2}$$

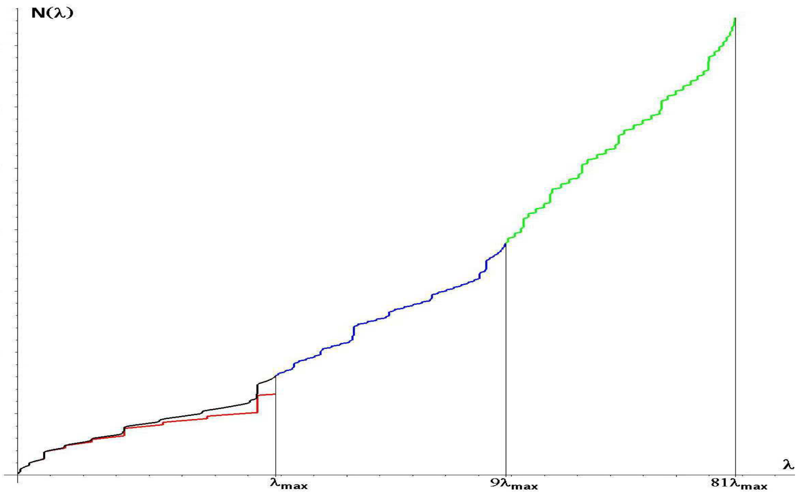

Figure 3.4: Eigenvalue counting functions for the triadic Cantor chain (black, blue, green) and string (red)

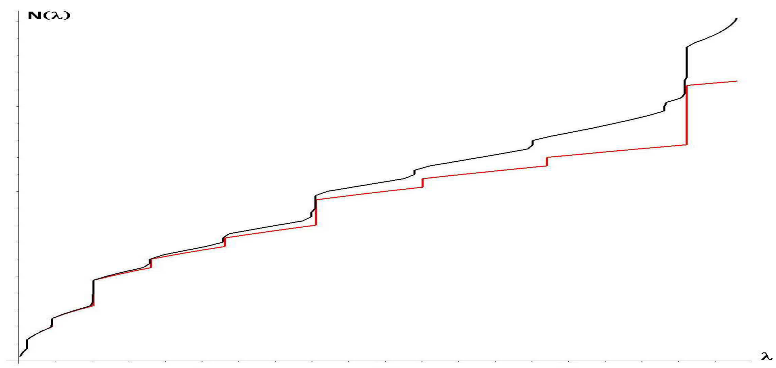

Figure 3.5: Detail of figure (3.4)

where $\lceil \cdot \rceil$ denotes the ceiling function. This choice ensures that at least the fundamental frequency of the $\mathfrak{m}^{th}$-level chain will be part of the spectrum and that all the frequencies of the $(\mathfrak{m}+1)^{th}$-level chain will be above the highest frequency of the basic chain.

3.2.1 The dynamic matrix of a fractal chain and its traces

Recall that a system of masses coupled by springs obeying Hooke's law can be described in the harmonic approximation by the matrix equation:

$$\left(\mathbf{D} - \omega^2 \mathbf{I}\right) \mathbf{x} = 0$$

where $\mathbf{D}$ is the dynamic matrix; $\mathbf{I}$ the identity matrix and $\mathbf{x}$ the column vector of displacements. The spectrum of the system is thus given by the eigenvalues λ_n of the dynamic matrix. For a fractal chain the dynamic matrix is a block-diagonal matrix, where the block matrices are taken with multiplicity ω_j (not to be confused with the frequencies of the chain) from the set of matrices of type:

$$\mathbf{DM}_j = l_j^{-2} \cdot \frac{K}{m} \begin{pmatrix} 2 & -1 & 0 & 0 & \dots \\ -1 & 2 & -1 & 0 & \dots \\ 0 & -1 & 2 & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

For each sub-chain the squared frequencies of its allowed vibrations are the eigenvalues of the sub-matrices given by:

$$\lambda_{n,j} = l_j^{-2} \cdot 4 \frac{K}{m} \sin\left(\frac{\pi n}{2\left(N+1\right)}\right)^2,$$

where $n = 1 \dots N+1$, $j \in \{1 \dots \mathfrak{m}\}$ and $N = \left\lceil \frac{\pi}{2} \cdot \frac{1}{\arcsin(l_{\mathfrak{m}})} \right\rceil$, so that the traces of the different powers of the complete dynamic matrix are easily accessible. Indeed,

$$\operatorname{tr}\left(\mathbf{DM}_1\right) = \sum_{n=1}^{N+1} \lambda_{n,1} = 4 \frac{K}{m} \sum_{n=1}^{N+1} \sin\left(\frac{\pi n}{2\left(N+1\right)}\right)^2,$$

and

$$\operatorname{tr}\left(\mathbf{DM}_1^k\right) = \sum_{n=1}^{N+1} \lambda_{n,1}^k = \left(4\frac{K}{m}\right)^k \sum_{n=1}^{N+1} \sin\left(\frac{\pi n}{2(N+1)}\right)^{2k}.$$

Furthermore:

$$\operatorname{tr}\left(\mathbf{DM}_j^k\right) = \sum_{n=1}^{N+1} \lambda_{n,j}^k = \sum_{n=1}^{N+1} \left(l_j^{-2}\lambda_{n,1}\right)^k = \left(l_j^{-2}\right)^k \left(4\frac{K}{m}\right)^k \sum_{n=1}^{N+1} \sin\left(\frac{\pi n}{2(N+1)}\right)^{2k},$$

so that the traces of powers of the complete dynamic matrix $\mathbf{D}$ are given by:

$$\operatorname{tr}\left(\mathbf{D}^k\right) = \sum_{j=0}^{\mathfrak{m}} \omega_j \left(l_j^{-2}\right)^k \left(4\frac{K}{m}\right)^k \sum_{n=1}^{N+1} \sin\left(\frac{\pi n}{2(N+1)}\right)^{2k}. \tag{3.3}$$

In order to continue, we need the following proposition, the results of which may be found in the literature for small k, but we will prove here the general case:

Proposition 3.1.

$$\sum_{n=1}^{N+1} \sin\left(\frac{\pi n}{2(N+1)}\right)^{2k} = \binom{2k}{k}\frac{N+1}{2^{2k}} + \frac{1}{2} \tag{3.4}$$

Proof. Following the method of A. F. Timofeev [113], write

$$\begin{aligned}\sin\left(\frac{\pi n}{2(N+1)}\right)^{2k} &= \left(\frac{e^{i\frac{\pi}{2(N+1)}n} - e^{-i\frac{\pi}{2(N+1)}n}}{2i}\right)^{2k} \\ &= \left(\frac{1}{2i}\right)^{2k}\left(e^{i\frac{\pi}{2(N+1)}n} - e^{-i\frac{\pi}{2(N+1)}n}\right)^{2k} \\ &= \frac{(-1)^k}{2^{2k}}\left(e^{i\frac{\pi}{2(N+1)}n} - e^{-i\frac{\pi}{2(N+1)}n}\right)^{2k},\end{aligned}$$

where i is the imaginary unit. Thus, by the binomial theorem and using the symmetry rule $\binom{2k}{l} = \binom{2k}{2k-l}$:

$$\begin{aligned}\sin\left(\frac{\pi n}{2\left(N+1\right)}\right)^{2k} &= \frac{\left(-1\right)^{k}}{2^{2k}}\sum_{l=0}^{2k}\left(-1\right)^{l}\binom{2k}{l}\left(e^{i\frac{\pi}{2(N+1)}n}\right)^{l}\left(e^{-i\frac{\pi}{2(N+1)}n}\right)^{2k-l}\\ &= \frac{1}{2^{2k}}\binom{2k}{k} + \frac{1}{2^{2k-1}}\sum_{l=1}^{k}\left(-1\right)^{l}\binom{2k}{k-l}\cos\left(2\frac{\pi nl}{2\left(N+1\right)}\right),\end{aligned}$$

so that:

$$\begin{aligned}\sum_{n=1}^{N+1}\sin\left(\frac{\pi n}{2\left(N+1\right)}\right)^{2k} &= \sum_{n=1}^{N+1}\frac{1}{2^{2k}}\binom{2k}{k} + \sum_{n=1}^{N+1}\frac{1}{2^{2k-1}}\sum_{l=1}^{k}\left(-1\right)^{l}\binom{2k}{k-l}\cos\left(\frac{\pi nl}{N+1}\right)\\ &= \frac{N+1}{2^{2k}}\binom{2k}{k} + \frac{1}{2^{2k-1}}\sum_{n=1}^{N+1}\sum_{l=1}^{k}\left(-1\right)^{l}\binom{2k}{k-l}\cos\left(\frac{\pi nl}{N+1}\right).\end{aligned}$$

Interchanging the order of summation and rearranging then leads to:

$$\sum_{n=1}^{N+1}\sin\left(\frac{\pi n}{2\left(N+1\right)}\right)^{2k} = \frac{N+1}{2^{2k}}\binom{2k}{k}+\frac{1}{2^{2k-1}}\sum_{l=1}^{k}\left(-1\right)^{l}\binom{2k}{k-l}\sum_{n=1}^{N+1}\cos\left(\frac{\pi nl}{N+1}\right).$$

Using Lagrange's trigonometric inequality

$$\begin{aligned}\sum_{n=1}^{N+1}\cos\left(n\cdot x\right) &= -\cos\left(0\right) + \sum_{n=0}^{N+1}\cos\left(n\cdot x\right)\\ &= -1 + \frac{1}{2}\left(1 + \frac{\sin\left(\left(N+\frac{3}{2}\right)x\right)}{\sin\left(\frac{1}{2}x\right)}\right)\\ &= \frac{1}{2}\left(-1 + \frac{\sin\left(\left(N+\frac{3}{2}\right)x\right)}{\sin\left(\frac{1}{2}x\right)}\right)\end{aligned}$$

on the last sum, one obtains:

$$\sum_{n=1}^{N+1} \cos\left(\frac{\pi n l}{N+1}\right) = \frac{1}{2}\left(-1 + \frac{\sin\left(\frac{\left(N+\frac{3}{2}\right)\pi l}{N+1}\right)}{\sin\left(\frac{\pi l}{2(N+1)}\right)}\right)$$
$$= \frac{1}{2}\left(-1 + \frac{\sin\left(\pi l \frac{N+1+\frac{1}{2}}{N+1}\right)}{\sin\left(\frac{\pi l}{2(N+1)}\right)}\right)$$
$$= \frac{1}{2}\left(-1 + \frac{\sin\left(\pi l + \frac{\pi l}{2(N+1)}\right)}{\sin\left(\frac{\pi l}{2(N+1)}\right)}\right).$$

As $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$:

$$\sum_{n=1}^{N+1} \cos\left(\frac{\pi n l}{N+1}\right) = \frac{1}{2}\left(-1 + \underbrace{\frac{\sin(\pi l)\cos\left(\frac{\pi l}{2(N+1)}\right)}{\sin\left(\frac{\pi l}{2(N+1)}\right)}}_{=0} + \frac{\cos(\pi l)\sin\left(\frac{\pi l}{2(N+1)}\right)}{\sin\left(\frac{\pi l}{2(N+1)}\right)}\right)$$
$$= \frac{1}{2}\left(-1 + \underbrace{\cos(\pi l)}_{=(-1)^l}\right)$$
$$= \frac{1}{2}\left(-1 + (-1)^l\right).$$

Thus:

$$\sum_{n=1}^{N+1} \sin\left(\frac{\pi n}{2(N+1)}\right)^{2k} = \frac{N+1}{2^{2k}}\binom{2k}{k} + \frac{1}{2^{2k}}\sum_{l=1}^{k}(-1)^l\binom{2k}{k-l}\left(-1+(-1)^l\right)$$
$$= \frac{N+1}{2^{2k}}\binom{2k}{k} - \frac{1}{2^{2k}}\sum_{l=1}^{k}(-1)^l\binom{2k}{k-l} + \frac{1}{2^{2k}}\sum_{l=1}^{k}\binom{2k}{k-l}.$$

It remains to evaluate the two sums of binomials. Writing

$$\sum_{l=1}^{k}\binom{2k}{k-l} = \frac{1}{2}\left(\sum_{l=1}^{k}\binom{2k}{k-l} + \sum_{l=1}^{k}\binom{2k}{k-l}\right),$$

and using the symmetry rule $\binom{2k}{k-l} = \binom{2k}{k+l}$ again, we have:

$$\begin{aligned}\sum_{l=1}^{k}\binom{2k}{k-l} &= \frac{1}{2}\left(\sum_{l=1}^{k}\binom{2k}{k-l} + \sum_{l=1}^{k}\binom{2k}{k+l}\right)\\ &= \frac{1}{2}\left(\sum_{l=0}^{k-1}\binom{2k}{l} + \sum_{l=k+1}^{k}\binom{2k}{l}\right)\\ &= \frac{1}{2}\left(\sum_{l=0}^{2k}\binom{2k}{l} - \binom{2k}{k}\right)\\ &= \frac{1}{2}\left(2^{2k} - \binom{2k}{k}\right),\end{aligned}$$

where we used Pascal's fifth identity in the last equality. A similar approach is used for the other sum:

$$\sum_{l=1}^{k}(-1)^l\binom{2k}{k-l} = \frac{1}{2}\left(\sum_{l=1}^{k}(-1)^l\binom{2k}{k-l} + \sum_{l=1}^{k}(-1)^l\binom{2k}{k-l}\right)$$

and by the symmetry rule, we have:

$$\sum_{l=1}^{k}(-1)^l\binom{2k}{k-l} = \frac{1}{2}\left(\sum_{l=1}^{k}(-1)^l\binom{2k}{k-l} + \sum_{l=1}^{k}(-1)^l\binom{2k}{k+l}\right).$$

It is then necessary to consider the cases of k being even or odd:
For k even:

$$\begin{aligned}\sum_{l=1}^{k}(-1)^l\binom{2k}{k-l} &= \frac{1}{2}\left(\sum_{l=0}^{k-1}(-1)^l\binom{2k}{l} + \sum_{l=k+1}^{2k}(-1)^l\binom{2k}{l}\right)\\ &= \frac{1}{2}\left(\sum_{l=0}^{2k}(-1)^l\binom{2k}{l} - \binom{2k}{k}\right).\end{aligned}$$

For k odd:

$$\sum_{l=1}^{k}(-1)^l\binom{2k}{k-l} = \frac{1}{2}\left(-\sum_{l=0}^{k-1}(-1)^l\binom{2k}{l} - \sum_{l=k+1}^{2k}(-1)^l\binom{2k}{l}\right)$$
$$= \frac{1}{2}\left(-\sum_{l=0}^{k}(-1)^l\binom{2k}{l} - \sum_{l=k+1}^{2k}(-1)^l\binom{2k}{l} + (-1)^k\binom{2k}{k}\right)$$
$$= \frac{1}{2}\left(-\sum_{l=0}^{2k}(-1)^l\binom{2k}{l} - \binom{2k}{k}\right).$$

It is well known, that $\sum_{l=0}^{2k}(-1)^l\binom{2k}{l} = 0$, which leaves us with

$$\sum_{l=1}^{k}(-1)^l\binom{2k}{k-l} = -\frac{1}{2}\binom{2k}{k}.$$

Thus,

$$\sum_{n=1}^{N+1}\sin\left(\frac{\pi n}{2(N+1)}\right)^{2k} = \frac{N+1}{2^{2k}}\binom{2k}{k} - \frac{1}{2^{2k}}\left(-\frac{1}{2}\right)\binom{2k}{k} + \frac{1}{2^{2k}}\frac{1}{2}\left(2^{2k} - \binom{2k}{k}\right)$$
$$= \frac{N+1}{2^{2k}}\binom{2k}{k} + \frac{1}{2^{2k}}\frac{1}{2}\binom{2k}{k} + \frac{1}{2^{2k}}\frac{1}{2}2^{2k} - \frac{1}{2^{2k}}\frac{1}{2}\binom{2k}{k}$$
$$= \binom{2k}{k}\frac{N+1}{2^{2k}} + \frac{1}{2},$$

which completes the proof. □

Hence, this proposition allows us to write the traces of the powers of the dynamic matrix in Equation (3.3) as:

$$\operatorname{tr}\left(\mathbf{D}^k\right) = \sum_{j=0}^{\mathfrak{m}}\omega_j\left(l_j^{-2}\right)^k\left(4\frac{K}{m}\right)^k\left(\binom{2k}{k}\frac{N+1}{2^{2k}} + \frac{1}{2}\right)$$
$$= \sum_{j=0}^{\mathfrak{m}}\omega_j l_j^{-2k}\left(4\frac{K}{m}\right)^k\left(\binom{2k}{k}\frac{N+1}{2^{2k}} + \frac{1}{2}\right)$$
$$:= \zeta_{\mathcal{L}}(\mathfrak{m}, -2k)\left(4\frac{K}{m}\right)^k\left(\binom{2k}{k}\frac{N+1}{2^{2k}} + \frac{1}{2}\right),$$

where we defined the "incomplete moment zeta function" $\zeta_{\mathcal{L}}(\mathfrak{m}, -2k)$ in analogy with the geometric zeta function (see Definition 2.15). The knowledge of the incomplete moment zeta function makes it possible to express the traces of the dynamic matrix of the fractal chain under consideration.

Definition 3.2 (Incomplete moment zeta function). *We define the incomplete moment zeta function $\zeta_{\mathcal{L}}(\mathfrak{m}, -2k)$ of a fractal string $\mathcal{L}$ by*

$$\zeta_{\mathcal{L}}(\mathfrak{m}, -2k) = \sum_{j=0}^{\mathfrak{m}} \omega_j l_j^{-2k}$$

for $k \in \mathbb{N}$ and where the ω_j's are the multiplicities of the lengths l_j.

Note that this definition of the incomplete moment zeta function can obviously be extended to the whole complex plane and might then be called "incomplete geometric zeta function", but here we would like to emphasise its relation to the moments of the eigenvalue distribution and thus restrict the definition to integer arguments.

3.2.2 The moments of the eigenvalue distribution of a fractal chain

Important information on the behaviour of the eigenvalues of a matrix may be obtained through the study of the moments of their distribution. It is possible to attach a probability measure, or in other words an eigenvalue probability distribution, $\mu_{\mathbf{D},N'}$ to the dynamic matrix of the fractal chain under scrutiny through the use of the Dirac delta functional in the following way:

$$\mu_{\mathbf{D},N'}(x)dx = \frac{1}{N'} \sum_{i=1}^{N'} \delta\left(x - \frac{\lambda_i(\mathbf{D})}{2}\right) dx,$$

where the normalisation factor in the denominator may be justified by heuristic arguments for the scaling of the eigenvalues [89]. We can then recover the moments $\mathcal{M}_{N,k}$ of the distribution. Indeed:

$$\begin{aligned}
\mathcal{M}_{N,k}(\mathbf{D}) &= \int x^k \mu_{\mathbf{D},N'}(x)dx \\
&= \frac{1}{N'}\sum_{i=1}^{N'}\int_{\mathbb{R}} x^k \delta\left(x - \frac{\lambda_i(\mathbf{D})}{2}\right)dx \\
&= \frac{1}{N'}\sum_{i=1}^{N'}\frac{\lambda_i(\mathbf{D})^k}{(2)^k} = \frac{1}{2^k N'}\sum_{i=1}^{N'}\lambda_i(\mathbf{D})^k \\
&= \frac{\mathrm{tr}(\mathbf{D}^k)}{2^k N'}.
\end{aligned}$$

Therefore, the moments of the eigenvalue distribution of a fractal chain are normalised traces of the corresponding dynamic matrices. First of all, all the entries of the dynamic matrix have to be less than or equal to one, which is achieved by multiplying it by a factor $\frac{1}{2}l_{\mathfrak{m}}^2$, respectively $\left(\frac{1}{2}l_{\mathfrak{m}}^2\right)^k$. In addition we need the number of independent matrix entries, given by:

$$\begin{aligned}
N' = \dim(\mathbf{D}) &= \sum_{j=0}^{\mathfrak{m}}\omega_j\left(\dim(\mathbf{DM_0})+1\right)-1 \\
&= N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)(N+1)-1,
\end{aligned}$$

where $N_{\mathcal{L}}(x)$ is the geometric counting function defined in Definition (2.16), so that:

$$\begin{aligned}
\mathcal{M}_{N,k} &= \frac{\left(\frac{1}{2}l_{\mathfrak{m}}^2\right)^k \mathrm{tr}\left(\mathbf{D}^k\right)}{N'\cdot 2^k} \\
&= \frac{\left(\frac{1}{2}l_{\mathfrak{m}}^2\right)^k \zeta_{\mathcal{L}}(\mathfrak{m},-2k)\left(4\frac{K}{m}\right)^k\left(\binom{2k}{k}\frac{N+1}{2^{2k}}+\frac{1}{2}\right)}{\left(N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)(N+1)-1\right)\cdot 2^k} \\
&= \frac{l_{\mathfrak{m}}^{2k}\zeta_{\mathcal{L}}(\mathfrak{m},-2k)\left(4\frac{K}{m}\right)^k\left(\binom{2k}{k}\frac{N+1}{2^{2k}}+\frac{1}{2}\right)}{\left(N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)(N+1)-1\right)\cdot 2^{2k}}. \qquad (3.5)
\end{aligned}$$

Thus, we have obtained here a general expression for the moments of the eigenvalue distribution of a fractal chain; all the necessary information being encoded in its incomplete moment zeta function and its geometric counting function. For illustration, we will apply these results to the examples of generalised Cantor chains and the a-chain below.

3.2.2.1 Example 1: Generalised Cantor chains

Recall from Definition (2.5), that a generalised Cantor string with parameters a and b consists of the set of lengths $\{l_j\}_{j=0}^{\infty} = \{b^{-(j+1)}\}_{j=0}^{\infty}$, each appearing with multiplicity $\omega_j = \omega_{b^{-(j+1)}} = a^j$. Thus, its incomplete moment zeta function and its geometric counting function are given by:

$$\zeta_{\mathfrak{CS}}(\mathfrak{m}, -2k) = \sum_{j=0}^{\mathfrak{m}} \omega_j l_j^{-2k} = \sum_{j=0}^{\mathfrak{m}} a^j b^{2(j+1)k} = \frac{b^{2k}\left(\left(ab^{2k}\right)^{\mathfrak{m}+1} - 1\right)}{ab^{2k} - 1}$$

and

$$N_{\mathfrak{CS}}(b^{\mathfrak{m}+1}) = \sum_{j=0}^{\mathfrak{m}} \omega_j = \sum_{j=0}^{\mathfrak{m}} a^j = \frac{a^{\mathfrak{m}+1} - 1}{a - 1}$$

respectively. With this, the expression for $\mathcal{M}_{N,k}$ becomes pretty unwieldy:

$$\mathcal{M}_{N,k} = \frac{\left(b^{-2(\mathfrak{m}+1)}\right)^k \frac{b^{2k}\left(\left(ab^{2k}\right)^{\mathfrak{m}+1} - 1\right)}{ab^{2k}-1} \left(4\frac{K}{m}\right)^k \left(\binom{2k}{k}\frac{N+1}{2^{2k}} + \frac{1}{2}\right)}{\left(\frac{a^{\mathfrak{m}+1}-1}{a-1}(N+1) - 1\right) \cdot 2^{2k}}.$$

However, it is possible to get a good impression of the behaviour of the moments of generalised Cantor chains by making a few approximations. We will here not rigorously justify these approximations, as they are only used in the examples and are not of crucial importance for further developments. The approximations used are:

$$\left(ab^{2k}\right)^{\mathfrak{m}+1} - 1 \longmapsto \left(ab^{2k}\right)^{\mathfrak{m}+1},$$
$$ab^{2k} - 1 \longmapsto ab^{2k},$$
$$\binom{2k}{k}\frac{N+1}{2^{2k}} + \frac{1}{2} \longmapsto \binom{2k}{k}\frac{N+1}{2^{2k}}, \text{ and}$$
$$\frac{a^{\mathfrak{m}+1} - 1}{a - 1}(N+1) - 1 \longmapsto \frac{a^{\mathfrak{m}+1}}{a - 1}(N+1).$$

Thus, we obtain after simplification for our approximated moments $\mathcal{M}_{N,k,appx}$ the expression:

$$\mathcal{M}_{N,k,appx} = \left(4\frac{K}{m}\right)^k \binom{2k}{k} \frac{a-1}{a2^{4k}},$$

where it should be noted that due to the normalisations, the parameter b disappears. Furthermore by [109], theorem 2.6:

$$e^{-\frac{1}{8k}} \frac{2^{2k}}{\sqrt{\pi}\sqrt{k}} < \binom{2k}{k} < \frac{2^{2k}}{\sqrt{\pi}\sqrt{k}},$$

so that for large k, we can expect the moments to behave like:

$$\mathcal{M}_{N,k,asymp} = \left(4\frac{K}{m}\right)^k \frac{a-1}{a\sqrt{\pi}} \frac{1}{\sqrt{k}2^{2k}}.$$

The precision of these approximations may be seen in Tables 3.1 and 3.2.

k	$\mathcal{M}_{N,k}$	$\mathcal{M}_{N,k,appx}$	relative error $\frac{\mathcal{M}_{N,k,appx}}{\mathcal{M}_{N,k}} - 1$	$\mathcal{M}_{N,k,asymp}$	relative error $\frac{\mathcal{M}_{N,k,asymp}}{\mathcal{M}_{N,k}} - 1$
1	1	.9425082041	−.0574917959	1.063506622	.063506622
2	.7127559673	.7068811531	−.0082423922	.7520127441	.055077444
3	.5907172921	.5890676275	−.0027926465	.6140158348	.039441105
4	.5165739391	.5154341741	−.0022063927	.5317533112	.029384704
5	.4648942135	.4638907567	−.0021584627	.4756146204	.023059885

Table 3.1: The first five normalised moments and their approximations for the triadic Cantor chain, at approximation level $\mathfrak{m} = 8$.

3.2.2.2 Example 2: The a-chain

For an arbitrary number $a > 0$, the a-string is given by the set of lengths $\{l_j\}_{j=1}^{\infty}$, where $l_j = j^{-a} - (j+1)^{-a}$ and all multiplicities $\omega_j = 1$, see Definition (2.6). Therefore, we can write its incomplete moment zeta function and its geometric counting function as:

$$\zeta_{\mathcal{L}_a}(\mathfrak{m}, -2k) = \sum_{j=1}^{\mathfrak{m}} \omega_j l_j^{-2k} = \sum_{j=1}^{\mathfrak{m}} \left(j^{-a} - (j+1)^{-a}\right)^{-2k}$$

k	$\mathcal{M}_{N,k}$	$\mathcal{M}_{N,k,appx}$	relative error $\frac{\mathcal{M}_{N,k,appx}}{\mathcal{M}_{N,k}} - 1$	$\mathcal{M}_{N,k,asymp}$	relative error $\frac{\mathcal{M}_{N,k,asymp}}{\mathcal{M}_{N,k}} - 1$
1	1	.9866149307	−.0133850693	1.113275734	.113275734
2	.7403952795	.7399611981	−.0005862833	.7872048202	.063222365
3	.6166804256	.6166343316	−.0000747453	.6427500447	.042274115
4	.5395845225	.5395550404	−.0000546386	.5566378669	.031604584
5	.4856258333	.4855995362	−.0000541508	.4978720437	.025217378

Table 3.2: The first five normalised moments and their approximations for a generalised Cantor chain with parameters $a = 3$ and $b = 5$, at approximation level $\mathfrak{m} = 8$.

and

$$N_{\mathcal{L}_a}(l_{\mathfrak{m}}^{-1}) = \sum_{j=1}^{\mathfrak{m}} \omega_j = \sum_{j=1}^{\mathfrak{m}} 1 = \mathfrak{m}$$

respectively, so that:

$$\begin{aligned}\mathcal{M}_{N,k} &= \frac{l_{\mathfrak{m}}^{2k} \zeta_{\mathcal{L}}(\mathfrak{m}, -2k) \left(4\frac{K}{m}\right)^k \left(\binom{2k}{k}\frac{N+1}{2^{2k}} + \frac{1}{2}\right)}{\left(N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)(N+1) - 1\right) \cdot 2^{2k}} \\ &= \frac{\left(\mathfrak{m}^{-a} - (\mathfrak{m}+1)^{-a}\right)^{2k} \sum_{j=1}^{\mathfrak{m}} \left(j^{-a} - (j+1)^{-a}\right)^{-2k} \left(4\frac{K}{m}\right)^k \left(\binom{2k}{k}\frac{N+1}{2^{2k}} + \frac{1}{2}\right)}{\left(\mathfrak{m}(N+1) - 1\right) \cdot 2^{2k}}.\end{aligned}$$

It is quite difficult to approximate this expression, so that we will limit ourselves to compare the moments obtained with those of generalised Cantor chains having the same Minkowski dimension, which leads to the following important remark:

Remark 3.3. *Table 3.3 suggests that the moments of Minkowski-measurable chains decrease at a much faster pace than those of generalised Cantor chains, thereby reflecting the different oscillatory behaviour in the corresponding spectra.*

3.2.3 The moments of the eigenvalue distribution of a fractal chain with cut-off

As already shown in figures (3.2) respectively (3.3), a cut-off frequency has to be introduced in order to allow a comparison to the spectrum of a fractal string, so

k	$\mathcal{CS}_T$, $d_M = \frac{\log(2)}{\log(3)}$	a-chain, $d_M = \frac{\log(2)}{\log(3)}$	$\mathcal{CS}$, $d_M = \frac{\log(3)}{\log(5)}$	a-chain, $d_M = \frac{\log(3)}{\log(5)}$
1	1	1	1	1
2	.7127559673	.5046300428	.7403952795	.5031995168
3	.5907172921	.3437648204	.6166804256	.3405870669
4	.5165739391	.2676078436	.5395845225	.2633965399
5	.4648942135	.2241385614	.4856258333	.2193090535

Table 3.3: Comparison of the first five normalised moments for Cantor chains and the corresponding a-chains of the same Minkowki dimension, at approximation level $\mathfrak{m} = 8$.

that only the part of the spectrum up to the maximal frequency of the fundamental chain is retained. Without loss of generality, it is possible to set the first length of the fractal string to $l_1 = 1$.

The eigenvalues thus to be taken into consideration are:

- Basic chain

$$\lambda_{n,1} = 4\frac{K}{m}\sin\left(\frac{\pi n}{2\left(N+1\right)}\right)^2,$$

with multiplicity ω_1.

- 1^{st} level chain

$$\lambda_{n,2} = l_2^{-2} \cdot 4\frac{K}{m}\sin\left(\frac{\pi n}{2\left(N+1\right)}\right)^2,$$

while

$$l_2^{-2} \cdot \sin\left(\frac{\pi n}{2\left(N+1\right)}\right)^2 \leq 1 \Leftrightarrow n \leq \frac{2\left(N+1\right)}{\pi}\arcsin\left(l_2\right)$$

with multiplicity ω_2.

- j^{th} level chain

$$\lambda_{n,j} = l_j^{-2} \cdot 4\frac{K}{m}\sin\left(\frac{\pi n}{2\left(N+1\right)}\right)^2,$$

while

$$l_j^{-2} \cdot \sin\left(\frac{\pi n}{2\left(N+1\right)}\right)^2 \leq 1 \Leftrightarrow n \leq \frac{2\left(N+1\right)}{\pi}\arcsin\left(l_j\right)$$

with multiplicity ω_j.

This now allows us to define "pseudo-traces" as follows:

$$\operatorname{ptr}\left(\mathbf{DM}_j^k\right) := \sum_{n=1}^{N(j)} \lambda_{n,j}^k = \sum_{n=1}^{N(j)} \left(l_j^{-2}\lambda_{n,1}\right)^k = \left(l_j^{-2}\right)^k \left(4\frac{K}{m}\right)^k \sum_{n=1}^{N(j)} \sin\left(\frac{\pi n}{2\,(N+1)}\right)^{2k},$$

with

$$N(j) = \left\lfloor \frac{2\,(N+1)}{\pi} \arcsin\left(l_j\right) \right\rfloor,$$

so that

$$\operatorname{ptr}\left(\mathbf{D}^k\right) = \sum_{j=0}^{\mathfrak{m}} \omega_j \left(l_j^{-2}\right)^k \left(4\frac{K}{m}\right)^k \sum_{n=1}^{N(j)} \sin\left(\frac{\pi n}{2\,(N+1)}\right)^{2k}. \tag{3.6}$$

The second sum in the above is then calculated by Euler-Maclaurin summation:

$$\sum_{n=1}^{N(j)} \sin\left(\frac{\pi n}{2\,(N+1)}\right)^{2k} = \int_{n=0}^{N(j)} \sin\left(\frac{\pi n}{2\,(N+1)}\right)^{2k} dn + R_k := I_k + R_k.$$

To evaluate the integral I_k, we will need the following proposition:

Proposition 3.4.

$$I_k = \int_{n=0}^{N(j)} \sin\left(\frac{\pi n}{2\,(N+1)}\right)^{2k} dn = \frac{2(N+1)}{\pi} \frac{l_j^{2k+1}\sqrt{1-l_j^2}}{2k+1} \cdot {}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right),$$

where

$${}_2F_1\left(\left.\begin{matrix}a, b\\ c\end{matrix}\right| z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

denotes the Gaussian (or ordinary) hypergeometric function and $(q)_n$ the rising factorial.

Proof. In order to simplify notations, we set $c = \frac{\pi}{2(N+1)}$. It is a well known fact that:

$$\int_{n=0}^{N(j)} \sin\left(cn\right)^{2k} dn = -\frac{1}{c}\cos\left(cn\right) {}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}-k\\ \frac{3}{2}\end{matrix}\right| \cos(cn)^2\right)\Bigg]_{n=0}^{N(j)}.$$

For $n = 0$, and since ${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}-k\\ \frac{3}{2}\end{matrix}\right| z\right)$ converges for every $z \in [0,1]$, we have by Gauss's theorem [51]:

$$-\frac{1}{c}\cos(cn)\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}-k\\ \frac{3}{2}\end{matrix}\right| \cos(cn)^2\right) = -\frac{1}{c}{}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}-k\\ \frac{3}{2}\end{matrix}\right| 1\right) = -\frac{2(N+1)}{\pi}\frac{\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}.$$

Furthermore, we will rewrite the hypergeometric function using one of Barnes' relations [6]:

$$\begin{aligned}{}_2F_1\left(\left.\begin{matrix}a, b\\ c\end{matrix}\right| z\right) =& \frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}{}_2F_1\left(\left.\begin{matrix}c-a, c-b\\ c-a-b+1\end{matrix}\right| 1-z\right)\\ &+\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1\left(\left.\begin{matrix}a, b\\ -c+a+b+1\end{matrix}\right| 1-z\right).\end{aligned}$$

Here, $a = \frac{1}{2}$, $b = \frac{1}{2} - k$ and $c = \frac{3}{2}$ and thus:

$$\begin{aligned}{}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}-k\\ \frac{3}{2}\end{matrix}\right| z\right) =& \frac{\Gamma(-k-\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-k)}(1-z)^{k+\frac{1}{2}}{}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| 1-z\right)\\ &+\frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(1)\Gamma(k+1)}{}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}-k\\ \frac{1}{2}-k\end{matrix}\right| 1-z\right)\\ =& -\frac{1}{2k+1}(1-z)^{k+\frac{1}{2}}{}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| 1-z\right)\\ &+\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}{}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}-k\\ \frac{1}{2}-k\end{matrix}\right| 1-z\right).\end{aligned}$$

As we can express the last hypergeometric function in the above by the simple form

$${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}-k\\ \frac{1}{2}-k\end{matrix}\right| 1-z\right) = \frac{1}{\sqrt{z}},$$

we obtain:

$$
\begin{aligned}
{}_2F_1\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2}-k\\ \frac{3}{2}\end{matrix}\right| z\right) =& -\frac{1}{2k+1}(1-z)^{k+\frac{1}{2}}{}_2F_1\left(\left.\begin{matrix}1,k+1\\ k+\frac{3}{2}\end{matrix}\right| 1-z\right)\\
&+\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\frac{1}{\sqrt{z}},
\end{aligned}
$$

such that

$$
\begin{aligned}
&-\frac{1}{c}\cos(cn){}_2F_1\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2}-k\\ \frac{3}{2}\end{matrix}\right| \cos(cn)^2\right) =\\
&\qquad -\frac{1}{c}\cos(cn)\left(-\frac{1}{2k+1}(1-\cos(cn)^2)^{k+\frac{1}{2}}{}_2F_1\left(\left.\begin{matrix}1,k+1\\ k+\frac{3}{2}\end{matrix}\right| 1-\cos(cn)^2\right)\right.\\
&\qquad \left.+\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\frac{1}{\sqrt{\cos(cn)^2}}\right)\\
&\qquad = -\frac{1}{c}\sqrt{1-\sin(cn)^2}\left(-\frac{1}{2k+1}(\sin(cn))^{2k+1}{}_2F_1\left(\left.\begin{matrix}1,k+1\\ k+\frac{3}{2}\end{matrix}\right| \sin(cn)^2\right)\right.\\
&\qquad \left.+\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\frac{1}{\sqrt{1-\sin(cn)^2}}\right)\\
&\qquad = \frac{1}{c}\frac{1}{2k+1}\sqrt{1-\sin(cn)^2}(\sin(cn))^{2k+1}{}_2F_1\left(\left.\begin{matrix}1,k+1\\ k+\frac{3}{2}\end{matrix}\right| \sin(cn)^2\right) - \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)},
\end{aligned}
$$

as $\cos(cn) \geq 0$ and $\sin(cn) \geq 0$ under the conditions imposed on n. Now, using the approximation $N(j) = \left\lfloor \frac{2(N+1)}{\pi}\arcsin(l_j)\right\rfloor \simeq \frac{2(N+1)}{\pi}\arcsin(l_j)$,

$$
\sin(cN(j)) = \sin\left(\frac{\pi}{2(N+1)}\frac{2(N+1)}{\pi}\arcsin(l_j)\right) = \sin(\arcsin(l_j)) = l_j,
$$

and

$$
\begin{aligned}
&-\frac{1}{c}\cos(cN(j)){}_2F_1\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2}-k\\ \frac{3}{2}\end{matrix}\right| \cos(cN(j))^2\right) =\\
&\qquad \frac{2(N+1)}{\pi}\left(\frac{1}{2k+1}\sqrt{1-l_j^2}\,l_j^{2k+1}{}_2F_1\left(\left.\begin{matrix}1,k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right) - \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\right)
\end{aligned}
$$

. Thus

$$\begin{aligned}
I_k &= \int_{n=0}^{N(j)} \sin{(cn)}^{2k} dn = -\frac{1}{c}\cos{(cn)}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2}-k\\ \frac{3}{2}\end{matrix}\right|\cos(cn)^2\right)\Bigg]_{n=0}^{N(j)} \\
&= \frac{2(N+1)}{\pi}\left(\frac{1}{2k+1}\sqrt{1-l_j^2}\,l_j^{2k+1}{}_2F_1\left(\left.\begin{matrix}1,k+1\\ k+\frac{3}{2}\end{matrix}\right|l_j^2\right) - \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\right) \\
&- \left(-\frac{2(N+1)}{\pi}\frac{\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\right),
\end{aligned}$$

and therefore finally

$$I_k = \frac{2(N+1)}{\pi}\frac{1}{2k+1}\sqrt{1-l_j^2}\,l_j^{2k+1}{}_2F_1\left(\left.\begin{matrix}1,k+1\\ k+\frac{3}{2}\end{matrix}\right|l_j^2\right),$$

which completes the proof. □

Remark 3.5. *The expression for I_k can also be formulated in terms of the incomplete beta function $B_z(a,b)$. Indeed:*

$$\begin{aligned}
I_k &= \frac{2(N+1)}{\pi}\frac{1}{2k+1}\sqrt{1-l_j^2}\,l_j^{2k+1}{}_2F_1\left(\left.\begin{matrix}1,k+1\\ k+\frac{3}{2}\end{matrix}\right|l_j^2\right) \\
&= \frac{1}{2}\frac{2(N+1)}{\pi}B_{l_j^2}\left(k+\frac{1}{2},\frac{1}{2}\right) \\
&= \frac{N+1}{\pi}B_{l_j^2}\left(k+\frac{1}{2},\frac{1}{2}\right).
\end{aligned}$$

From Proposition (3.4) and Remark (3.5) above, we can easily deduce the following corollary:

Corollary 3.6. *I_k can be bounded in the following way:*

$$0 \leq I_k \leq \frac{N+1}{\pi}\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{\Gamma(k+1)},$$

and furthermore:

$$0 \leq \sqrt{1-l_j^2}\,{}_2F_1\left(\left.\begin{matrix}1,k+1\\ k+\frac{3}{2}\end{matrix}\right|l_j^2\right) \leq \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}(2k+1).$$

Proof. The incomplete beta function is differentiable on the interval $[0,1]$ and its derivative

$$\frac{\partial}{\partial z}\left(B_{z^2}\left(k+\frac{1}{2},\frac{1}{2}\right)\right)=\frac{2z\left(z^2\right)^{k-\frac{1}{2}}}{\sqrt{1-z^2}}=\frac{2z^{2k}}{\sqrt{1-z^2}}\geq 0$$

positive on the whole interval. Thus it is a monotonic increasing function on $[0,1]$ and assumes its extremal values on the endpoints. As

$$B_0\left(k+\frac{1}{2},\frac{1}{2}\right)=0,\text{ and } B_1\left(k+\frac{1}{2},\frac{1}{2}\right)=\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{\Gamma(k+1)},$$

we have

$$0\leq I_k\leq\frac{N+1}{\pi}\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}$$

and likewise, as:

$$\sqrt{1-l_j^2}\,{}_2F_1\left(\left.\begin{matrix}1,k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right)=\frac{1}{2}B_{l_j^2}\left(k+\frac{1}{2},\frac{1}{2}\right)l_j^{-2k-1}\left(2k+1\right),$$

we obtain by the product rule for the limits:

$$0\leq\sqrt{1-l_j^2}\,{}_2F_1\left(\left.\begin{matrix}1,k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right)\leq\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\left(2k+1\right)$$

as stated. □

The remainder term R_k is less accessible and needs numerous manipulations in order to formulate and prove the following proposition.

Proposition 3.7. *In first order approximation, the remainder term R_k is given by:*

$$R_k=\frac{1}{2}l_j^{2k}+\frac{1}{2}\sqrt{1-l_j^2}\left(l_j\right)^{2k-1}k\frac{\pi}{6\left(N+1\right)}.$$

Proof. Setting $f(n) = \sin\left(\frac{\pi n}{2(N+1)}\right)^{2k}$, the remainder term reads:

$$
\begin{aligned}
R_k =& - B_1\left(f(N(j)) + f(0)\right) + \sum_{l=1}^{\infty} \frac{B_{l+1}}{(l+1)!}\left(f^{(l)}(N(j)) - f^{(l)}(0)\right) \\
:=& R_{k,1} + R_{k,\infty},
\end{aligned}
$$

where the B_i's are the Bernoulli numbers and $f^{(l)}$ denotes the l^{th} derivative of f. Using the same approximation as above for $N(j)$, the first term can easily be evaluated:

$$
\begin{aligned}
R_{k,1} =& - B_1\left(f(N(j)) + f(0)\right) = \frac{1}{2}\sin\left(\frac{\pi}{2(N+1)}N(j)\right)^{2k} \\
\simeq& \frac{1}{2}\sin\left(\frac{\pi}{2(N+1)}\frac{2(N+1)\arcsin(l_j)}{\pi}\right)^{2k} \\
=& \frac{1}{2}l_j^{2k}.
\end{aligned}
$$

The second term however needs numerous manipulations; first of all as $B_{l+1} = 0$ for l even, we need only consider the case of l being odd. Setting $c = \frac{\pi}{2(N+1)}$, we find by induction over l that:

$$
\begin{aligned}
f^{(l)}(n) =& \sum_{m=1}^{k} \frac{(2m)^l c^l \binom{2k}{k-m}(-1)^{\frac{l+1}{2}+m}\sin(2mcx)}{2^{2k-1}} \\
=& \frac{2^l c^l}{2^{2k-1}}(-1)^{\frac{l+1}{2}} \sum_{m=1}^{k} m^l \binom{2k}{k-m}(-1)^m \sin(2mcx),
\end{aligned}
$$

so that:

$$
\begin{aligned}
R_{k,\infty} =& \sum_{l=1}^{\infty} \frac{B_{l+1}}{(l+1)!}(f^{(l)}(N(j)) - \underbrace{f^{(l)}(0)}_{=0}) \\
=& \sum_{l=1}^{\infty} \frac{B_{l+1}}{(l+1)!} f^{(l)}(N(j)) \\
=& \sum_{l=1}^{\infty} \frac{B_{l+1}}{(l+1)!}\frac{2^l c^l}{2^{2k-1}}(-1)^{\frac{l+1}{2}} \sum_{m=1}^{k} m^l \binom{2k}{k-m}(-1)^m \sin(2mcN(j)).
\end{aligned}
$$

Here we can interchange the sums and rearrange to obtain:

$$\begin{aligned} R_{k,\infty} &= \sum_{m=1}^{k}\sum_{l=1}^{\infty} \frac{B_{l+1}}{(l+1)!}\frac{2^l c^l}{2^{2k-1}}(-1)^{\frac{l+1}{2}} m^l \binom{2k}{k-m}(-1)^m \sin(2mcN(j)) \\ &= \frac{1}{2^{2k-1}}\sum_{m=1}^{k}\binom{2k}{k-m}(-1)^m \sin(2mcN(j)) \sum_{l=1}^{\infty}\frac{B_{l+1}}{(l+1)!}(2c)^l(-1)^{\frac{l+1}{2}} m^l. \end{aligned}$$

As:

$$\sum_{l=1}^{\infty}\frac{B_{l+1}}{(l+1)!}(2c)^l(-1)^{\frac{l+1}{2}} m^l = \frac{1}{2}\left(\cot(cm) - \frac{1}{cm}\right),$$

we have

$$R_{k,\infty} = \frac{1}{2^{2k}}\sum_{m=1}^{k}\binom{2k}{k-m}(-1)^{m+1}\sin(2mcN(j))\left(\frac{1}{cm} - \cot(cm)\right). \tag{3.7}$$

This expression is quite cumbersome and tedious to evaluate, but it is possible to give a useful approximation. Setting $\nu = 2m$ and $x = cN(j)$, the sine can be expanded [1], formula 3.173 as:

$$\begin{aligned} \sin(\nu x) &= (-1)^{\frac{\nu}{2}-1}\cos(x)\left\{2^{\nu-1}\sin(x)^{\nu-1} - \tfrac{(\nu-2)}{1!}2^{\nu-3}\sin(x)^{\nu-3} + \tfrac{(\nu-3)(\nu-4)}{2!}2^{\nu-5}\sin(x)^{\nu-5}\cdots\right\} \\ &= (-1)^{\frac{\nu}{2}-1}\cos(x)\sum_{\mu=1}^{\frac{\nu}{2}} 2^{\nu-2\mu+1}\sin(x)^{\nu-2\mu+1}\binom{\nu-\mu}{\nu-2\mu+1}(-1)^{\mu-1}, \end{aligned}$$

and by the consecutive changes of variables $\nu \to 2m$ and $m - \mu + 1 \to \ell$, we obtain:

$$\sin(2mx) = \cos(x)\sum_{\ell=1}^{m} 2^{2\ell-1}\sin(x)^{2\ell-1}\binom{m+\ell-1}{2\ell-1}(-1)^{\ell+1}.$$

Approximating $N(j)$ again as above, one has:

$$\begin{aligned}\sin(x) &= \sin\left(cN(j)\right)\\ &\simeq \sin\left(\frac{\pi}{2(N+1)}\frac{2\left(N+1\right)}{\pi}\arcsin\left(l_j\right)\right)\\ &= \sin\left(\arcsin b^{-j}\right)\\ &= l_j,\end{aligned}$$

and

$$\cos\left(cN(j)\right) \simeq \cos\left(\arcsin\left(l_j\right)\right) = \sqrt{1-l_j^2},$$

so that:

$$\sin\left(2mcN(j)\right) \simeq \sqrt{1-l_j^2}\sum_{\ell=1}^{m}\left(2l_j\right)^{2\ell-1}\binom{m+\ell-1}{2\ell-1}(-1)^{\ell+1}.$$

Furthermore, let us express $\frac{1}{cm} - \cot(cm)$ by its Taylor-series:

$$\frac{1}{cm} - \cot(cm) = \frac{cm}{3} + \mathcal{O}\left((cm)^3\right)$$

In first order approximation, we can then write Equation 3.7 in the form:

$$\begin{aligned}R_{k,\infty} &= \frac{1}{2^{2k}}\sum_{m=1}^{k}\binom{2k}{k-m}(-1)^{m+1}\sqrt{1-l_j^2}\sum_{\ell=1}^{m}\left(2l_j\right)^{2\ell-1}\binom{m+\ell-1}{2\ell-1}(-1)^{\ell+1}\frac{cm}{3}\\ &= \frac{1}{2^{2k}}\sum_{m=1}^{k}\sum_{\ell=1}^{k}\binom{2k}{k-m}(-1)^{m+1}\sqrt{1-l_j^2}\left(2l_j\right)^{2\ell-1}\binom{m+\ell-1}{2\ell-1}(-1)^{\ell+1}\frac{cm}{3}.\end{aligned} \tag{3.8}$$

Using the method of Iverson-bracketing, it is then possible to rearrange the summation limits as to obtain:

$$\begin{aligned}
R_{k,\infty} &= \frac{1}{2^{2k}} \sum [1 \le m \le k][1 \le \ell \le m] \binom{2k}{k-m} \sqrt{1-l_j^2}\, (2l_j)^{2\ell-1} \binom{m+\ell-1}{2\ell-1} (-1)^{m+\ell+2} \frac{cm}{3} \\
&= \frac{1}{2^{2k}} \sum [1 \le \ell \le m \le k] \binom{2k}{k-m} \sqrt{1-l_j^2}\, (2l_j)^{2\ell-1} \binom{m+\ell-1}{2\ell-1} (-1)^{m+\ell} \frac{cm}{3} \\
&= \frac{1}{2^{2k}} \sum_{\ell=1}^{k} \sum_{m=\ell}^{k} (-1)^{\ell} \sqrt{1-l_j^2}\, (2l_j)^{2\ell-1} \frac{c}{3} \binom{2k}{k-m} \binom{m+\ell-1}{2\ell-1} (-1)^m m \\
&= \frac{1}{2^{2k}} \sum_{\ell=1}^{k} (-1)^{\ell} \sqrt{1-l_j^2}\, (2l_j)^{2\ell-1} \frac{c}{3} \sum_{m=\ell}^{k} (-1)^m \binom{2k}{k-m} \binom{m+\ell-1}{2\ell-1} m \\
&= \frac{1}{2^{2k}} \sum_{\ell=1}^{k} (-1)^{\ell} \sqrt{1-l_j^2}\, (2l_j)^{2\ell-1} \frac{c}{3} S,
\end{aligned}$$

with

$$S := \sum_{m=\ell}^{k} (-1)^m \binom{2k}{k-m} \binom{m+\ell-1}{2\ell-1} m.$$

Expanding

$$\binom{m+\ell-1}{2\ell-1} m = \prod_{\mu=1}^{\ell} \frac{m^2-(\mu-1)^2}{(2\ell-1)!},$$

S can be written as:

$$\begin{aligned}
S &= \sum_{m=\ell}^{k} (-1)^m \binom{2k}{k-m} \binom{m+\ell-1}{2\ell-1} m \\
&= \sum_{m=\ell}^{k} (-1)^m \binom{2k}{k-m} \prod_{\mu=1}^{\ell} \frac{m^2-(\mu-1)^2}{(2\ell-1)!},
\end{aligned}$$

or, as $\forall m < \ell : \binom{m+\ell-1}{2\ell-1} = 0$:

$$\begin{aligned} S &= \sum_{m=1}^{k}(-1)^m \binom{2k}{k-m} \prod_{\mu=1}^{\ell} \frac{m^2-(\mu-1)^2}{(2\ell-1)!} \\ &= \sum_{m=1}^{k}(-1)^m \prod_{\mu=1}^{\ell} \frac{m^2-(\mu-1)^2}{(2\ell-1)!} \binom{2k}{k-m} \\ &= \frac{1}{2}\sum_{m=1}^{k}(-1)^m \prod_{\mu=1}^{\ell} \frac{m^2-(\mu-1)^2}{(2\ell-1)!} \left(\binom{2k}{k-m} + \binom{2k}{k+m}\right), \end{aligned}$$

since $\binom{2k}{k-i} = \binom{2k}{k+i}$. By rearranging terms and reindexing:

$$\begin{aligned} S =& \frac{1}{2} \sum_{-k \leq m \leq k,\ m\neq 0} (-1)^m \prod_{\mu=1}^{\ell} \frac{m^2-(\mu-1)^2}{(2\ell-1)!} \binom{2k}{k+m} \\ =& \frac{1}{2} \sum_{-k \leq m \leq k} (-1)^m \prod_{\mu=1}^{\ell} \frac{m^2-(\mu-1)^2}{(2\ell-1)!} \binom{2k}{k+m}, \end{aligned}$$

as

$$\prod_{\mu=1}^{\ell} \frac{m^2-(\mu-1)^2}{(2\ell-1)!} = 0 \text{ for } m = 0.$$

Reindexing again using $\nu = k + m$ then leads to:

$$S = \frac{1}{2}(-1)^k \sum_{\nu=0}^{2k}(-1)^\nu \prod_{\mu=1}^{\ell} \frac{(\nu-k)^2-(\mu-1)^2}{(2\ell-1)!} \binom{2k}{\nu}.$$

Now, for any polynomial $P_{2k}(\nu)$ of degree d in ν, $\sum_{0\leq\nu\leq 2k}(-1)^\nu P_{2k}(\nu)\binom{2k}{\nu}$ will vanish if $2k$ exceeds d. Indeed, P_{2k} can be expressed as a linear combination of the first $d+1$ of the basis polynomials $1, \nu, \binom{\nu}{2}, \ldots, \binom{\nu}{p}, \ldots$, with coefficients which are polynomials in $2k$, and then, for each $p = 0, \dots, d$:

$$\begin{aligned}\sum_{\nu=0}^{2k}(-1)^{\nu}\binom{\nu}{p}\binom{2k}{\nu} &= \binom{2k}{p}\sum_{\nu=p}^{2k}(-1)^{\nu}\binom{2k-p}{\nu-p} \\ &= \binom{2k}{p}(-1)^{p}(1-1)^{2k-p} \\ &= 0,\end{aligned}$$

by the trinomial revision identity and furthermore by the binomial theorem. In the case under consideration here, we have $d = 2\ell$, so the sum will vanish whenever $2k > 2\ell$, i.e., $k > \ell$.
Thus, in S only the term with $m = \ell = k$ remains, so that:

$$S = \sum_{m=\ell}^{k}(-1)^{m}\binom{2k}{k-m}\binom{m+\ell-1}{2\ell-1}m = (-1)^{k}\binom{2k}{0}\binom{2k-1}{2k-1}k = (-1)^{k}k$$

and hence as $S = 0$ for $l < k$,

$$\begin{aligned}R_{k,\infty} &= \frac{1}{2^{2k}}\sum_{\ell=1}^{k}(-1)^{\ell}\sqrt{1-l_j^2}\,(2l_j)^{2\ell-1}\,\frac{c}{3}S \\ &= \underbrace{\frac{1}{2^{2k}}\sum_{\ell=1}^{k-1}(-1)^{\ell}\sqrt{1-l_j^2}\,(2l_j)^{2\ell-1}\,\frac{c}{3}S}_{=0} + \frac{1}{2^{2k}}(-1)^{k}\sqrt{1-l_j^2}\,(2l_j)^{2k-1}\,S \\ &= \frac{1}{2^{2k}}(-1)^{2k}\sqrt{1-l_j^2}\,(2l_j)^{2k-1}\,k\frac{c}{3} \\ &= \frac{1}{2^{2k}}\sqrt{1-l_j^2}\,(2l_j)^{2k-1}\,k\frac{c}{3} \\ &= \frac{1}{2^{2k}}\sqrt{1-l_j^2}\,(2l_j)^{2k-1}\,k\frac{\pi}{6\,(N+1)} \\ &= \frac{1}{2}\sqrt{1-l_j^2}\,(l_j)^{2k-1}\,k\frac{\pi}{6\,(N+1)},\end{aligned}$$

as $c = \frac{\pi}{2(N+1)}$. □

Note that in the following order of approximation for $\frac{1}{cm} - \cot cm$, $\prod_{\mu=1}^{\ell}\frac{m^2-(\mu-1)^2}{(2\ell-1)!}$ has to be replaced by $\prod_{\mu=1}^{\ell} m^2\frac{m^2-(\mu-1)^2}{(2\ell-1)!}$, so that the polynomial $P_{2k}(\nu)$ above is of degree $d' = d+2$, and thus the sum will only vanish for $k > \ell + 1$. However, this correction term tends very rapidly to zero and can thus safely be neglected. The pseudo-traces in Equation (3.6) are hence given by:

$$\begin{aligned}
\operatorname{ptr}\left(\mathbf{D}^k\right) &= \sum_{j=0}^{\mathfrak{m}} \omega_j \left(l_j^{-2}\right)^k \left(4\frac{K}{m}\right)^k \sum_{n=1}^{N(j)} \sin\left(\frac{\pi n}{2\,(N+1)}\right)^{2k} \\
&= \sum_{j=0}^{\mathfrak{m}} \omega_j \left(l_j^{-2}\right)^k \left(4\frac{K}{m}\right)^k \left(\frac{2(N+1)}{\pi}\frac{l_j^{2k+1}\sqrt{1-l_j^2}}{2k+1} \cdot {}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right)\right. \\
&\left. +\frac{1}{2}l_j^{2k} + \frac{1}{2}\sqrt{1-l_j^2}\,(l_j)^{2k-1}\,k\frac{\pi}{6\,(N+1)}\right) \\
&= \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{2(N+1)}{\pi}\frac{l_j\sqrt{1-l_j^2}}{2k+1} \cdot {}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right)\right. \\
&\left. +\frac{1}{2} + \frac{1}{2}\sqrt{1-l_j^2}\,(l_j)^{-1}\,k\frac{\pi}{6\,(N+1)}\right) \\
&= \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{2(N+1)}{\pi}\frac{l_j\sqrt{1-l_j^2}}{2k+1} \cdot {}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right)\right) \\
&+ \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-l_j^2}\,(l_j)^{-1}\,k\frac{\pi}{6\,(N+1)}\right).
\end{aligned} \tag{3.9}$$

Before treating the important case of Minkowski-measurable chains, it seems appropriate to reconsider the two standard examples already used in the previous section for illustration.

3.2.3.1 Example 1: Generalised Cantor chains

For generalised Cantor chains with parameters a and b, the pseudo-traces are obtained as:

$$
\begin{aligned}
\operatorname{ptr}\left(\mathbf{D}^k\right) &= \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{2(N+1)}{\pi} \frac{l_j\sqrt{1-l_j^2}}{2k+1} \cdot {}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right)\right) \\
&+ \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-l_j^2}\,(l_j)^{-1}\, k\frac{\pi}{6\,(N+1)}\right) \\
&= \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} a^j \left(\frac{2(N+1)}{\pi} \frac{b^{-(j+1)}\sqrt{1-b^{-2(j+1)}}}{2k+1} \cdot {}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| b^{-2(j+1)}\right)\right) \\
&+ \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} a^j \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-b^{-2(j+1)}}\,\left(b^{-(j+1)}\right)^{-1} k\frac{\pi}{6\,(N+1)}\right) \\
&:= W + R
\end{aligned}
$$

We can give upper bounds for the "Weyl"-term W by using Corollary (3.6). As:

$$
\sqrt{1-l_j^2}\,{}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right) \leq \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\,(2k+1)\,,
$$

it is bounded by:

$$
\begin{aligned}
W &= \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} a^j \left(\frac{2(N+1)}{\pi} \frac{b^{-(j+1)}\sqrt{1-b^{-2(j+1)}}}{2k+1} \cdot {}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| b^{-2(j+1)}\right)\right) \\
&\leq \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} a^j \left(\frac{2(N+1)}{\pi} \frac{b^{-(j+1)}}{2k+1} \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\,(2k+1)\right) \\
&= \left(4\frac{K}{m}\right)^k \frac{(N+1)}{\pi} \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} \sum_{j=0}^{\mathfrak{m}} a^j b^{-(j+1)} \\
&= \left(4\frac{K}{m}\right)^k \frac{(N+1)}{\pi} \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} \frac{\left(\frac{a}{b}\right)^{\mathfrak{m}+1} - 1}{a-b}.
\end{aligned}
$$

Furthermore, as $\sqrt{1-b^{2(j+1)}} \leq 1$, the second term R may be bounded by:

$$\begin{aligned}
R &= \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} a^j \left(\frac{1}{2}+\frac{1}{2}\sqrt{1-b^{-2(j+1)}}\left(b^{-(j+1)}\right)^{-1} k\frac{\pi}{6\,(N+1)}\right) \\
&\leq \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} a^j \left(\frac{1}{2}+\frac{1}{2}\left(b^{-(j+1)}\right)^{-1} k\frac{\pi}{6\,(N+1)}\right) \\
&= \left(4\frac{K}{m}\right)^k \left(\frac{a^{\mathfrak{m}+1}-1}{2\,(a-1)}+\frac{k\pi}{12(N+1)}\frac{b\left((ab)^{\mathfrak{m}+1}-1\right)}{ab-1}\right).
\end{aligned}$$

Thus, an upper bound for the pseudo-traces is given by:

$$\begin{aligned}
\operatorname{ptr}\left(\mathbf{D}^k\right) \leq \left(4\frac{K}{m}\right)^k \Bigg(&\frac{N+1}{\pi}\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\frac{\left(\frac{a}{b}\right)^{\mathfrak{m}+1}-1}{a-b} \\
&+\frac{a^{\mathfrak{m}+1}-1}{2\,(a-1)}+\frac{k\pi}{12(N+1)}\frac{b\left((ab)^{\mathfrak{m}+1}-1\right)}{ab-1}\Bigg).
\end{aligned}$$

This expression is still quite inaccessible. However, substituting its first order approximation for $N+1 = \lceil \frac{\pi}{2}\frac{1}{\arcsin\left(b^{-(\mathfrak{m}+1)}\right)}\rceil + 1 \longmapsto \frac{\pi b^{\mathfrak{m}+1}}{2}$, there is a substantial gain in transparency:

$$\begin{aligned}
\operatorname{ptr}\left(\mathbf{D}^k\right) &\leq \left(4\frac{K}{m}\right)^k \Bigg(\frac{\frac{\pi b^{\mathfrak{m}+1}}{2}}{\pi}\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\frac{\left(\frac{a}{b}\right)^{\mathfrak{m}+1}-1}{a-b} \\
&\quad+\frac{a^{\mathfrak{m}+1}-1}{2\,(a-1)}+\frac{k\pi}{12(\frac{\pi b^{\mathfrak{m}+1}}{2})}\frac{b\left((ab)^{\mathfrak{m}+1}-1\right)}{ab-1}\Bigg) \\
&\approx \left(4\frac{K}{m}\right)^k \Bigg(\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\frac{b^{\mathfrak{m}+1}\left(\left(\frac{a}{b}\right)^{\mathfrak{m}+1}-1\right)}{a-b} \\
&\quad+\frac{a^{\mathfrak{m}+1}-1}{2\,(a-1)}+\frac{k}{6}\frac{(ab)^{\mathfrak{m}+1}-1}{b^{\mathfrak{m}}\,(ab-1)}\Bigg) \\
&= \left(4\frac{K}{m}\right)^k \left(\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\frac{a^{\mathfrak{m}+1}-b^{\mathfrak{m}+1}}{a-b}+\frac{a^{\mathfrak{m}+1}-1}{2\,(a-1)}+\frac{k}{6}\frac{ba^{\mathfrak{m}+1}-b^{-\mathfrak{m}}}{ab-1}\right).
\end{aligned}$$

Recalling that $a^{n+1}-b^{n+1} = (a-b)\sum_{i=0}^{n} a^{(n-i)}b^i$, it is now easy to see that the first term is $\mathcal{O}\left(b^{\mathfrak{m}}\right)$ and the remaining terms $\mathcal{O}\left(a^{\mathfrak{m}}\right)$. Alternatively, we can state that the "Weyl"-term is $\mathcal{O}(N)$ and the remainder terms are $\mathcal{O}\left(N^{d_M}\right)$, as for a generalised Cantor chain we have $d_M = \frac{\ln(a)}{\ln(b)} \Leftrightarrow a = b^{d_M}$ and $N = \lceil \frac{\pi}{2} \frac{1}{\arcsin\left(b^{-(\mathfrak{m}+1)}\right)} \rceil = \mathcal{O}(b^{\mathfrak{m}+1})$.

3.2.3.2 Example 2: The a-chain

Recall that for the a-chain, the lengths are given by $l_j = j^{-a} - (j+1)^{-a}$, with $j = 1 \ldots \infty$, and the multiplicities are always $\omega_j = 1$. Thus the pseudo-traces are given by:

$$\begin{aligned}
\operatorname{ptr}\left(\mathbf{D}^k\right) &= \left(4\frac{K}{m}\right)^k \sum_{j=1}^{\mathfrak{m}} \omega_j \left(\frac{2(N+1)}{\pi} \frac{l_j\sqrt{1-l_j^2}}{2k+1} \cdot {}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right)\right) \\
&+ \left(4\frac{K}{m}\right)^k \sum_{j=1}^{\mathfrak{m}} \omega_j \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-l_j^2}\,(l_j)^{-1}\, k \frac{\pi}{6\,(N+1)}\right) \\
&= \left(4\frac{K}{m}\right)^k \sum_{j=1}^{\mathfrak{m}} \left(\frac{2(N+1)}{\pi} \frac{l_j\sqrt{1-l_j^2}}{2k+1} \cdot {}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right)\right) \\
&+ \left(4\frac{K}{m}\right)^k \sum_{j=1}^{\mathfrak{m}} \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-l_j^2}\,(l_j)^{-1}\, k \frac{\pi}{6\,(N+1)}\right) \\
&:= W + R.
\end{aligned}$$

Using again Corollary (3.6), the "Weyl"-term W can easily be bounded. Indeed:

$$\begin{aligned}
W &= \left(4\frac{K}{m}\right)^k \sum_{j=1}^{\mathfrak{m}} \left(\frac{2(N+1)}{\pi} \frac{l_j\sqrt{1-l_j^2}}{2k+1} \cdot {}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right)\right) \\
&\leq \left(4\frac{K}{m}\right)^k \sum_{j=1}^{\mathfrak{m}} \left(\frac{2(N+1)}{\pi} \frac{l_j}{2k+1} \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)} (2k+1)\right) \\
&= \left(4\frac{K}{m}\right)^k \frac{2(N+1)}{\pi} \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)} \sum_{j=1}^{\mathfrak{m}} l_j \\
&\leq \left(4\frac{K}{m}\right)^k \frac{2(N+1)}{\pi} \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)},
\end{aligned}$$

as $\sum_{j=1}^{\mathfrak{m}} l_j \leq 1$ by the definition of the a-chain. In order to give an upper bound for the remaining terms, we will again use the fact that $\sqrt{1-l_j^2} \leq 1$, such that:

$$\begin{aligned}
R &= \left(4\frac{K}{m}\right)^k \sum_{j=1}^{\mathfrak{m}} \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-l_j^2}\,(l_j)^{-1}\,k\frac{\pi}{6\,(N+1)}\right) \\
&\leq \left(4\frac{K}{m}\right)^k \sum_{j=1}^{\mathfrak{m}} \left(\frac{1}{2} + \frac{1}{2}\,(l_j)^{-1}\,k\frac{\pi}{6\,(N+1)}\right) \\
&\leq \left(4\frac{K}{m}\right)^k \left(\frac{1}{2}\mathfrak{m} + \frac{1}{6}k\sum_{j=1}^{\mathfrak{m}} \left((l_j)^{-1}\,\frac{\pi}{2\,(N+1)}\right)\right).
\end{aligned}$$

For clarity, it is possible to substitute again its first order approximation for $N+1 = \lceil \frac{\pi}{2}\frac{1}{\arcsin(l_{\mathfrak{m}})} \rceil + 1 \longmapsto \frac{\pi l_{\mathfrak{m}}^{-1}}{2}$, such that:

$$\begin{aligned}
W &\leq \left(4\frac{K}{m}\right)^k \frac{2(N+1)}{\pi} \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)} \\
&\approx \left(4\frac{K}{m}\right)^k \frac{2(\frac{\pi l_{\mathfrak{m}}^{-1}}{2})}{\pi} \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)} \\
&= \left(4\frac{K}{m}\right)^k l_{\mathfrak{m}}^{-1} \frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)},
\end{aligned}$$

and

$$\begin{aligned}
R &\leq \left(4\frac{K}{m}\right)^k \left(\frac{1}{2}\mathfrak{m} + \frac{1}{6}k\sum_{j=1}^{\mathfrak{m}} \left((l_j)^{-1}\,\frac{\pi}{2\,(N+1)}\right)\right) \\
&\approx \left(4\frac{K}{m}\right)^k \left(\frac{1}{2}\mathfrak{m} + \frac{1}{6}k\sum_{j=1}^{\mathfrak{m}} \left((l_j)^{-1}\,\frac{\pi}{2\left(\frac{\pi l_{\mathfrak{m}}^{-1}}{2}\right)}\right)\right) \\
&= \left(4\frac{K}{m}\right)^k \left(\frac{1}{2}\mathfrak{m} + \frac{1}{6}k\sum_{j=1}^{\mathfrak{m}} \frac{l_{\mathfrak{m}}}{l_j}\right).
\end{aligned}$$

Now, as the sequence of lengths is decreasing, i.e., $l_{\mathfrak{m}} \leq l_j \Leftrightarrow \frac{l_{\mathfrak{m}}}{l_j} \leq 1$, we have:

$$\begin{aligned} R &\approx \left(4\frac{K}{m}\right)^k \left(\frac{1}{2}\mathfrak{m} + \frac{1}{6}k\sum_{j=1}^{\mathfrak{m}} \frac{l_{\mathfrak{m}}}{l_j}\right) \\ &\leq \left(4\frac{K}{m}\right)^k \left(\frac{1}{2}\mathfrak{m} + \frac{1}{6}k\sum_{j=1}^{\mathfrak{m}} 1\right) \\ &= \left(4\frac{K}{m}\right)^k \left(\frac{1}{2}\mathfrak{m} + \frac{1}{6}k\mathfrak{m}\right) \\ &= \left(4\frac{K}{m}\right)^k \frac{k+3}{6}\mathfrak{m}. \end{aligned}$$

It should be noted that although not being the best possible upper bound, it is sufficiently accurate to illustrate the example. Hence, we have for the pseudo-traces:

$$\begin{aligned} \operatorname{ptr}\left(\mathbf{D}^k\right) &= W + R \\ &\leq \left(4\frac{K}{m}\right)^k \left(l_{\mathfrak{m}}^{-1}\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)} + \frac{k+3}{6}\mathfrak{m}\right). \end{aligned}$$

As the a-chain is Minkowski-measurable and as $\omega_j = 1 \Rightarrow \ell_j = l_j$, we have by Remark (2.12):

$$\begin{aligned} &l_j \sim L j^{-\frac{1}{d_M}} \\ &\Rightarrow l_{\mathfrak{m}} \sim L\mathfrak{m}^{-\frac{1}{d_M}} \\ &\Leftrightarrow \mathfrak{m} \sim L^{d_M}\left(l_{\mathfrak{m}}^{-1}\right)^{d_M}. \end{aligned}$$

Finally, taking into account that $N = \mathcal{O}\left(l_{\mathfrak{m}}^{-1}\right)$, it becomes obvious that again the first term in the expression for the bound of the pseudo-traces is $\mathcal{O}\left(N\right)$, while the remainder term is $\mathcal{O}\left(N^{d_M}\right)$, as was to be expected.

3.2.3.3 Minkowski-measurable chains

In the previous example, we used the Minkowski-measurability of the a-chain in order to get a neat expression allowing us to study the asymptotic behaviour of its pseudo-traces. However, the a-chain is just one example of Minkowski-measurable chains. Because of the importance of this class of fractal chains, we will try a more in-depth exploration in the following. As before, the starting point for our subsequent study of Minkowski-measurable chains are their pseudo-traces given by Equation (3.9):

$$\begin{aligned}
\operatorname{ptr}\left(\mathbf{D}^k\right) &= \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{2(N+1)}{\pi}\frac{l_j\sqrt{1-l_j^2}}{2k+1}\cdot {}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right)\right) \\
&+ \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{1}{2}+\frac{1}{2}\sqrt{1-l_j^2}\,(l_j)^{-1}\,k\frac{\pi}{6\,(N+1)}\right) \\
&:= W + R.
\end{aligned}$$

The "Weyl"-term W can again be bounded using Corollary (3.6):

$$\begin{aligned}
W &= \left(4\frac{K}{m}\right)^k \sum_{j=1}^{\mathfrak{m}} \omega_j \left(\frac{2(N+1)}{\pi}\frac{l_j\sqrt{1-l_j^2}}{2k+1}\cdot {}_2F_1\left(\left.\begin{matrix}1, k+1\\ k+\frac{3}{2}\end{matrix}\right| l_j^2\right)\right) \\
&\leq \left(4\frac{K}{m}\right)^k \sum_{j=1}^{\mathfrak{m}} \omega_j \left(\frac{2(N+1)}{\pi}\frac{l_j}{2k+1}\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}(2k+1)\right) \\
&= \left(4\frac{K}{m}\right)^k \frac{2(N+1)}{\pi}\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)}\sum_{j=1}^{\mathfrak{m}} \omega_j l_j \\
&= \left(4\frac{K}{m}\right)^k \frac{2(N+1)}{\pi}\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)} vol_1(\mathcal{L}),
\end{aligned}$$

where $vol_1(\mathcal{L})$ is simply the total length of the fractal chain. Using the same approach as in the previous section, i.e. approximating $\sqrt{1-l_j^2} \leq 1$ and $N+1 =$

$\lceil \frac{\pi}{2} \frac{1}{\arcsin(l_{\mathfrak{m}})} \rceil + 1 \longmapsto \frac{\pi l_{\mathfrak{m}}^{-1}}{2}$, the second term may be bounded by:

$$
\begin{aligned}
R &= \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-l_j^2}\,(l_j)^{-1}\, k \frac{\pi}{6\,(N+1)}\right) \\
&\leq \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{1}{2} + \frac{1}{2}\,(l_j)^{-1}\, k \frac{\pi}{6\frac{\pi l_{\mathfrak{m}}^{-1}}{2}}\right) \\
&= \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{1}{2} + \frac{1}{2}\frac{k}{3}\frac{l_{\mathfrak{m}}}{l_j}\right).
\end{aligned}
$$

Finally, the sequence of lengths is decreasing, $l_j \geq l_{\mathfrak{m}} \Leftrightarrow \frac{l_{\mathfrak{m}}}{l_j} \leq 1$, such that:

$$
\begin{aligned}
R &\leq \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{1}{2} + \frac{1}{2}\frac{k}{3}\right) \\
&= \left(4\frac{K}{m}\right)^k \frac{3+k}{6} \sum_{j=0}^{\mathfrak{m}} \omega_j \\
&= \left(4\frac{K}{m}\right)^k \frac{3+k}{6} N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right),
\end{aligned}
$$

by the definition of the geometric counting function $N_{\mathcal{L}}(x)$. Hence the pseudo-traces admit an upper bound:

$$
\begin{aligned}
\operatorname{ptr}\left(\mathbf{D}^k\right) &= W + R \\
&\leq \left(4\frac{K}{m}\right)^k \left(\frac{2(N+1)}{\pi}\frac{\sqrt{\pi}\,\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)} vol_1(\mathcal{L}) + \frac{3+k}{6} N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)\right),
\end{aligned}
$$

where the "Weyl"-term is $\mathcal{O}(N) = \mathcal{O}\left(l_{\mathfrak{m}}^{-1}\right)$ and from our results for the a-chain, we expect the remaining terms to be $\mathcal{O}\left(N^{d_M}\right)$. Indeed, by Remark (2.12), $N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right) = \mathcal{O}\left(l_{\mathfrak{m}}^{-d_M}\right) = \mathcal{O}\left(N^{d_M}\right)$, as $\mathcal{L}$ is Minkowski-measurable. From this, it is possible to deduce the following two theorems:

Theorem 3.8. *A fractal string $\mathcal{L}$ is Minkowski-measurable if and only if:*

$$
N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right) = C\mathfrak{m} + o(\mathfrak{m})
$$

and

$$R = \left(4\frac{K}{m}\right)^k \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-l_j^2}\,(l_j)^{-1}\,k\frac{\pi}{6\,(N+1)}\right) = \mathcal{O}\left(N^{d_M}\right),$$

for $\mathfrak{m} \to \infty$.

Proof. Let us first recall that the following are equivalent:

1. $\mathcal{L}$ is Minkowski-measurable.
2. $\lim_{j\to\infty} l_j j^{\frac{1}{d_M}} = L$, respectively $l_j = Lj^{-\frac{1}{d_M}} + o\left(j^{-\frac{1}{d_M}}\right)$, as $j \to \infty$.
3. $\lim_{x\to\infty} \frac{N_{\mathcal{L}}(x)}{x^{d_M}} = c$, respectively $N_{\mathcal{L}} = cx^{d_M} + o\left(x^{d_M}\right)$, as $x \to \infty$.

"$\Rightarrow$"
If $\mathcal{L}$ is minkowski-measurable, then by point 3. above:

$$\begin{aligned} &\lim_{x\to\infty} \frac{N_{\mathcal{L}}(x)}{x^{d_M}} = c \\ \Rightarrow &\lim_{\mathfrak{m}\to\infty} \frac{N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)}{\left(l_{\mathfrak{m}}^{-1}\right)^{d_M}} = c \\ \Leftrightarrow &\lim_{\mathfrak{m}\to\infty} \frac{N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)\mathfrak{m}^{-1}}{\left(l_{\mathfrak{m}}^{-1}\right)^{d_M}\mathfrak{m}^{-1}} = c. \end{aligned}$$

By point 2. above and the power rule, the limit:

$$\begin{aligned} &\lim_{\mathfrak{m}\to\infty} l_{\mathfrak{m}}^{d_M}\mathfrak{m} = L^{d_M} \\ \Leftrightarrow &\lim_{\mathfrak{m}\to\infty} \left(l_{\mathfrak{m}}^{-1}\right)^{d_M}\mathfrak{m}^{-1} = L^{-d_M}, \end{aligned}$$

exists and is different from zero, such that by the quotient rule:

$$\begin{aligned} &\lim_{\mathfrak{m}\to\infty} \frac{N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)\mathfrak{m}^{-1}}{\left(l_{\mathfrak{m}}^{-1}\right)^{d_M}\mathfrak{m}^{-1}} = c \\ \Leftrightarrow &\lim_{\mathfrak{m}\to\infty} N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)\mathfrak{m}^{-1} = c\lim_{\mathfrak{m}\to\infty}\left(l_{\mathfrak{m}}^{-1}\right)^{d_M}\mathfrak{m}^{-1} \\ \Leftrightarrow &\lim_{\mathfrak{m}\to\infty} N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)\mathfrak{m}^{-1} = cL^{-d_M} = C. \end{aligned}$$

Thus:

$$N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right) = C\mathfrak{m} + o\left(\mathfrak{m}\right),$$

for $\mathfrak{m} \to \infty$.
"$\Leftarrow$"
Assume that the remainder term:

$$R = \left(4\frac{K}{m}\right)^{k} \sum_{j=0}^{\mathfrak{m}} \omega_j \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - l_j^2}\left(l_j\right)^{-1} k \frac{\pi}{6\left(N+1\right)}\right) = \mathcal{O}\left(N^{d_M}\right),$$

then

$$N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right) = \mathcal{O}\left(N^{d_M}\right),$$

and as

$$N^{d_M} \approx \left(\frac{\pi l_{\mathfrak{m}}^{-1}}{2}\right)^{d_M},$$

we have:

$$\begin{aligned} &N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right) = \mathcal{O}\left(l_{\mathfrak{m}}^{d_M}\right) \\ \Leftrightarrow &\lim_{\mathfrak{m}\to\infty} \frac{N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)}{l_{\mathfrak{m}}^{d_M}} = \tilde{c} \neq 0 \\ \Leftrightarrow &\lim_{\mathfrak{m}\to\infty} \frac{l_{\mathfrak{m}}^{d_M}}{N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)} = \frac{1}{\tilde{c}}. \end{aligned}$$

Thus:

$$\lim_{\mathfrak{m}\to\infty} \mathfrak{m}^{-1} l_{\mathfrak{m}}^{-d_M} = \lim_{\mathfrak{m}\to\infty} \left(\frac{N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)}{\mathfrak{m}} \frac{l_{\mathfrak{m}}^{d_M}}{N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)}\right).$$

As $\lim_{\mathfrak{m}\to\infty} \frac{N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)}{\mathfrak{m}} = C$ exists, we have by the product rule:

$$\lim_{\mathfrak{m}\to\infty} \mathfrak{m}^{-1} l_{\mathfrak{m}}^{-d_M} = \lim_{\mathfrak{m}\to\infty} \frac{N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)}{\mathfrak{m}} \lim_{\mathfrak{m}\to\infty} \frac{l_{\mathfrak{m}}^{d_M}}{N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)} = C\frac{1}{\tilde{c}} = L^{-d_M},$$

or equivalently:

$$\lim_{\mathfrak{m}\to\infty} l_{\mathfrak{m}}^{d_M} \mathfrak{m} = L^{d_M},$$

which completes the proof. □

If we have no information on the behaviour of the remainder term R, it is still possible to formulate a slightly weaker version of the theorem above:

Theorem 3.9. *The geometric counting function of a fractal string $\mathcal{L}$ is given by:*

$$N_{\mathcal{L}}\left(l_{\mathfrak{m}}^{-1}\right)h(\mathfrak{m}) = C\mathfrak{m}f(\mathfrak{m}) + o\left(\mathfrak{m}\right),$$

with

$$\lim_{\mathfrak{m}\to\infty}\frac{\log(f(\mathfrak{m}))}{\log(\mathfrak{m})} = 0$$

and

$$\lim_{\mathfrak{m}\to\infty}\frac{\log(h(\mathfrak{m}))}{\log(\mathfrak{m})} = 0$$

if and only if

$$l_{\mathfrak{m}}g(l_{\mathfrak{m}}) = L\left(\mathfrak{m}f(\mathfrak{m})\right)^{-\frac{1}{d_M}} + o\left(\mathfrak{m}^{-\frac{1}{d_M}}\right),$$

with

$$\lim_{l_{\mathfrak{m}}\to 0}\frac{\log(g(l_{\mathfrak{m}}))}{\log(\frac{l_{\mathfrak{m}}}{2})} = 0$$

for $\mathfrak{m}\to\infty$.

Proof. "$\Rightarrow$"
Recall that the Minkowski-dimension d_M of the fractal string $\mathcal{L}$ is given by

$$\begin{aligned} d_M &= 1 - \lim_{\varepsilon\to 0}\frac{\log V(\varepsilon)}{\log\varepsilon} \\ \Leftrightarrow 1 - d_M &= \lim_{\varepsilon\to 0}\frac{\log V(\varepsilon)}{\log\varepsilon}, \end{aligned}$$

where $V(\varepsilon)$ denotes the ε-neighbourhood of the boundary of $\mathcal{L}$, which may be expressed as:

$$V(\varepsilon) = \sum_{j:l_j\geq 2\varepsilon} 2\varepsilon + \sum_{j:l_j<2\varepsilon} l_j = 2\varepsilon\cdot N_{\mathcal{L}}\left(\frac{1}{2\varepsilon}\right) + \sum_{j:l_j<2\varepsilon} l_j.$$

. Setting $2\varepsilon = l_{\mathfrak{m}}$, we have:

$$V\left(\frac{l_{\mathfrak{m}}}{2}\right) = \frac{l_{\mathfrak{m}}}{2} \cdot 2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right) + \sum_{j:l_j<l_{\mathfrak{m}}} l_j,$$

and thus

$$\begin{aligned}
1 - d_M &= \lim_{l_{\mathfrak{m}}\to 0} \frac{\log\left(\frac{l_{\mathfrak{m}}}{2} \cdot 2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right) + \sum_{j:l_j<l_{\mathfrak{m}}} l_j\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)} \\
\Leftrightarrow 1 - d_M &= \lim_{l_{\mathfrak{m}}\to 0} \frac{\log\left(\frac{l_{\mathfrak{m}}}{2} \cdot \left(2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right) + 2\sum_{j:l_j<l_{\mathfrak{m}}} \frac{l_j}{l_{\mathfrak{m}}}\right)\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)} \\
\Leftrightarrow 1 - d_M &= \lim_{l_{\mathfrak{m}}\to 0} \frac{\log\left(\frac{l_{\mathfrak{m}}}{2}\right) + \log\left(2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right) + 2\sum_{j:l_j<l_{\mathfrak{m}}} \frac{l_j}{l_{\mathfrak{m}}}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)} \\
\Leftrightarrow 1 - d_M &= \lim_{l_{\mathfrak{m}}\to 0} \left(1 + \frac{\log\left(2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right) + 2\sum_{j:l_j<l_{\mathfrak{m}}} \frac{l_j}{l_{\mathfrak{m}}}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}\right) \\
\Leftrightarrow 1 - d_M &= 1 + \lim_{l_{\mathfrak{m}}\to 0} \frac{\log\left(2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right) + 2\sum_{j:l_j<l_{\mathfrak{m}}} \frac{l_j}{l_{\mathfrak{m}}}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)} \\
\Leftrightarrow -d_M &= \lim_{l_{\mathfrak{m}}\to 0} \frac{\log\left(2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right) + 2\sum_{j:l_j<l_{\mathfrak{m}}} \frac{l_j}{l_{\mathfrak{m}}}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)} \\
\Leftrightarrow 1 &= \lim_{l_{\mathfrak{m}}\to 0} \frac{\log\left(2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right) + 2\sum_{j:l_j<l_{\mathfrak{m}}} \frac{l_j}{l_{\mathfrak{m}}}\right)}{(-d_M)\log\left(\frac{l_{\mathfrak{m}}}{2}\right)} \\
\Leftrightarrow 1 &= \lim_{l_{\mathfrak{m}}\to 0} \frac{\log\left(2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right) + 2\sum_{j:l_j<l_{\mathfrak{m}}} \frac{l_j}{l_{\mathfrak{m}}}\right)}{\log\left(\left(\frac{l_{\mathfrak{m}}}{2}\right)^{-d_M}\right)}.
\end{aligned}$$

Now as the sequence of lengths is decreasing, we have $l_j > l_{\mathfrak{m}}$ for every $j < \mathfrak{m}$ and thus the second term in the numerator above:

$$2\sum_{j:l_j<l_{\mathfrak{m}}} \frac{l_j}{l_{\mathfrak{m}}} = 0,$$

so that:

$$\lim_{l_{\mathfrak{m}}\to 0}\frac{\log\left(2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right)+2\sum_{j:l_j<l_{\mathfrak{m}}}\frac{l_j}{l_{\mathfrak{m}}}\right)}{\log\left(\left(\frac{l_{\mathfrak{m}}}{2}\right)^{-d_M}\right)}=1$$

$$\Leftrightarrow\lim_{l_{\mathfrak{m}}\to 0}\frac{\log\left(2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right)\right)}{\log\left(\left(\frac{l_{\mathfrak{m}}}{2}\right)^{-d_M}\right)}=1. \tag{3.10}$$

From this we can deduce that:

$$\lim_{l_{\mathfrak{m}}\to 0}\frac{2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right)\mathfrak{h}\left(l_{\mathfrak{m}}\right)}{\left(\frac{l_{\mathfrak{m}}}{2}\right)^{-d_M}}=1$$

$$\Leftrightarrow\lim_{l_{\mathfrak{m}}\to 0}\left(\frac{l_{\mathfrak{m}}}{2}\right)^{d_M}2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right)\mathfrak{h}\left(l_{\mathfrak{m}}\right)=1$$

$$\Leftrightarrow\lim_{l_{\mathfrak{m}}\to 0}2^{1-d_M}l_{\mathfrak{m}}^{d_M}N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right)\mathfrak{h}\left(l_{\mathfrak{m}}\right)=1$$

with:

$$\lim_{l_{\mathfrak{m}}\to 0}\frac{\log\left(\mathfrak{h}\left(l_{\mathfrak{m}}\right)\right)}{\log\left(\left(\frac{l_{\mathfrak{m}}}{2}\right)^{-d_M}\right)}=0.$$

Then, by the assumption of the theorem:

$$\lim_{\mathfrak{m}\to\infty}\mathfrak{m}^{-1}N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right)=Cf(\mathfrak{m}),$$

and noting that as $l_{\mathfrak{m}}\to 0$, $\mathfrak{m}\to\infty$, we have

$$\lim_{\mathfrak{m}\to\infty}2^{1-d_M}l_{\mathfrak{m}}^{d_M}N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right)\mathfrak{h}\left(l_{\mathfrak{m}}\right)=1$$

$$\Leftrightarrow\lim_{\mathfrak{m}\to\infty}2^{1-d_M}l_{\mathfrak{m}}^{d_M}\mathfrak{m}\mathfrak{m}^{-1}N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right)\mathfrak{h}\left(l_{\mathfrak{m}}\right)=1$$

$$\Leftrightarrow\lim_{\mathfrak{m}\to\infty}2^{1-d_M}l_{\mathfrak{m}}^{d_M}\mathfrak{m}Cf(\mathfrak{m})\mathfrak{h}\left(l_{\mathfrak{m}}\right)=1.$$

Putting:

$$g(\mathfrak{m})=\left(\mathfrak{h}(\mathfrak{m})\right)^{-d_M}$$

$$\Leftrightarrow\mathfrak{h}(\mathfrak{m})=\left(g(\mathfrak{m})\right)^{d_M},$$

we obtain that:

$$
\begin{aligned}
&\lim_{\mathfrak{m}\to\infty} 2^{1-d_M} l_{\mathfrak{m}}^{d_M} \mathfrak{m} C f(\mathfrak{m}) \mathfrak{h}\left(l_{\mathfrak{m}}\right) = 1\\
\Leftrightarrow &\lim_{\mathfrak{m}\to\infty} 2^{1-d_M} l_{\mathfrak{m}}^{d_M} \mathfrak{m} C f(\mathfrak{m}) \left(g\left(l_{\mathfrak{m}}\right)\right)^{d_M} = 1\\
\Leftrightarrow &\lim_{\mathfrak{m}\to\infty} 2^{1-d_M} \left(l_{\mathfrak{m}} g\left(l_{\mathfrak{m}}\right)\right)^{d_M} \mathfrak{m} C f(\mathfrak{m}) = 1
\end{aligned}
$$

and finally:

$$
\begin{aligned}
&\lim_{\mathfrak{m}\to\infty} \left(l_{\mathfrak{m}} g\left(l_{\mathfrak{m}}\right)\right)^{d_M} \mathfrak{m} f(\mathfrak{m}) = \frac{2^{d_M-1}}{C}\\
\Leftrightarrow &\lim_{\mathfrak{m}\to\infty} l_{\mathfrak{m}} g\left(l_{\mathfrak{m}}\right)\left(\mathfrak{m} f(\mathfrak{m})\right)^{\frac{1}{d_M}} = \left(\frac{2^{d_M-1}}{C}\right)^{\frac{1}{d_M}}\\
\Leftrightarrow &\lim_{\mathfrak{m}\to\infty} l_{\mathfrak{m}} g\left(l_{\mathfrak{m}}\right)\left(\mathfrak{m} f(\mathfrak{m})\right)^{\frac{1}{d_M}} = L\\
\Leftrightarrow & l_{\mathfrak{m}} g\left(l_{\mathfrak{m}}\right) = L\left(\mathfrak{m} f(\mathfrak{m})\right)^{\frac{1}{-d_M}} + o\left(\mathfrak{m}^{-\frac{1}{d_M}}\right).
\end{aligned}
$$

"$\Leftarrow$"

Assume that:

$$
\begin{aligned}
&\lim_{\mathfrak{m}\to\infty} l_{\mathfrak{m}} g\left(l_{\mathfrak{m}}\right)\left(\mathfrak{m} f(\mathfrak{m})\right)^{\frac{1}{d_M}} = L\\
\Leftrightarrow &\lim_{\mathfrak{m}\to\infty} \frac{1}{L} l_{\mathfrak{m}} g\left(l_{\mathfrak{m}}\right)\left(\mathfrak{m} f(\mathfrak{m})\right)^{\frac{1}{d_M}} = 1,
\end{aligned}
$$

then, as $\mathfrak{m} \to \infty$, $l_{\mathfrak{m}} \to 0$:

$$
\begin{aligned}
&\log\left(\lim_{l_{\mathfrak{m}}\to 0} \frac{1}{L} l_{\mathfrak{m}} g\left(l_{\mathfrak{m}}\right)\left(\mathfrak{m} f(\mathfrak{m})\right)^{\frac{1}{d_M}}\right) = 0\\
\Rightarrow &\lim_{l_{\mathfrak{m}}\to 0} \log\left(\frac{1}{L} l_{\mathfrak{m}} g\left(l_{\mathfrak{m}}\right)\left(\mathfrak{m} f(\mathfrak{m})\right)^{\frac{1}{d_M}}\right) = 0\\
\Leftrightarrow &\lim_{l_{\mathfrak{m}}\to 0} \left(\log\left(\frac{1}{L}\right) + \log\left(l_{\mathfrak{m}}\right) + \log\left(g(l_{\mathfrak{m}})\right) + \frac{1}{d_M}\log\left(\mathfrak{m}\right) + \frac{1}{d_M}\log\left(f\left(\mathfrak{m}\right)\right)\right) = 0\\
\Leftrightarrow &\lim_{l_{\mathfrak{m}}\to 0} \left(\log\left(\frac{l_{\mathfrak{m}}}{2}\right)\left(\frac{\log\left(\frac{1}{L}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)} + \frac{\log\left(l_{\mathfrak{m}}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)} + \frac{\log\left(g(l_{\mathfrak{m}})\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)} + \frac{1}{d_M}\frac{\log\left(\mathfrak{m}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)} + \frac{1}{d_M}\frac{\log\left(f\left(\mathfrak{m}\right)\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}\right)\right) = 0,
\end{aligned}
$$

and as $\log\left(\frac{l_{\mathfrak{m}}}{2}\right)$ is unbounded:

$$\lim_{l_{\mathfrak{m}}\to 0}\left(\log\left(\frac{l_{\mathfrak{m}}}{2}\right)\left(\frac{\log\left(\frac{1}{L}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}+\frac{\log\left(l_{\mathfrak{m}}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}+\frac{\log\left(g(l_{\mathfrak{m}})\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}+\frac{1}{d_M}\frac{\log\left(\mathfrak{m}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}+\frac{1}{d_M}\frac{\log\left(f\left(\mathfrak{m}\right)\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}\right)\right)=0$$
$$\Rightarrow \lim_{l_{\mathfrak{m}}\to 0}\left(\frac{\log\left(\frac{1}{L}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}+\frac{\log\left(l_{\mathfrak{m}}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}+\frac{\log\left(g(l_{\mathfrak{m}})\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}+\frac{1}{d_M}\frac{\log\left(\mathfrak{m}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}+\frac{1}{d_M}\frac{\log\left(f\left(\mathfrak{m}\right)\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}\right)=0.$$

Furthermore, as:

$$\lim_{l_{\mathfrak{m}}\to 0}\left(\frac{\log\left(\frac{1}{L}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}\right)=0,$$

$$\lim_{l_{\mathfrak{m}}\to 0}\left(\frac{\log\left(l_{\mathfrak{m}}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}\right)=1$$

and as by the assumption of the theorem:

$$\lim_{l_{\mathfrak{m}}\to 0}\left(\frac{1}{d_M}\frac{\log\left(g\left(l_{\mathfrak{m}}\right)\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}\right)=0,$$

we have

$$\lim_{l_{\mathfrak{m}}\to 0}\left(1+\frac{1}{d_M}\frac{\log\left(\mathfrak{m}\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}+\frac{1}{d_M}\frac{\log\left(f\left(\mathfrak{m}\right)\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}\right)=0$$
$$\Leftrightarrow \lim_{l_{\mathfrak{m}}\to 0}\left(-\frac{1}{d_M}\frac{\log\left(\mathfrak{m}f\left(\mathfrak{m}\right)\right)}{\log\left(\frac{l_{\mathfrak{m}}}{2}\right)}\right)=1$$
$$\Leftrightarrow \lim_{l_{\mathfrak{m}}\to 0}\left(\frac{\log\left(\mathfrak{m}f\left(\mathfrak{m}\right)\right)}{\log\left(\left(\frac{l_{\mathfrak{m}}}{2}\right)^{-d_M}\right)}\right)=1.$$

We already know by Equation (3.10) that:

$$\lim_{l_{\mathfrak{m}}\to 0}\frac{\log\left(2N_{\mathfrak{L}}\left(l_{\mathfrak{m}}^{-1}\right)\right)}{\log\left(\left(\frac{l_{\mathfrak{m}}}{2}\right)^{-d_M}\right)}=1$$

and hence, using again that $\mathfrak{m} \to \infty$, $l_\mathfrak{m} \to 0$:

$$\lim_{l_\mathfrak{m} \to 0} \frac{\left(\frac{\log\left(2N_\mathfrak{L}\left(l_\mathfrak{m}^{-1}\right)\right)}{\log\left(\left(\frac{l_\mathfrak{m}}{2}\right)^{-d_M}\right)} \right)}{\left(\frac{\log(\mathfrak{m} f(\mathfrak{m}))}{\log\left(\left(\frac{l_\mathfrak{m}}{2}\right)^{-d_M}\right)} \right)} = 1$$

$$\Leftrightarrow \lim_{\mathfrak{m} \to \infty} \frac{\log\left(2N_\mathfrak{L}\left(l_\mathfrak{m}^{-1}\right)\right)}{\log\left(\mathfrak{m} f\left(\mathfrak{m}\right)\right)} = 1,$$

from which we can deduce that:

$$\lim_{\mathfrak{m} \to \infty} \frac{2N_\mathfrak{L}\left(l_\mathfrak{m}^{-1}\right) h(\mathfrak{m})}{\mathfrak{m} f\left(\mathfrak{m}\right)} = 1,$$

respectively

$$N_\mathfrak{L}\left(l_\mathfrak{m}^{-1}\right) h(\mathfrak{m}) = C\mathfrak{m} f(\mathfrak{m}) + o\left(\mathfrak{m}\right),$$

with:

$$\lim_{\mathfrak{m} \to \infty} \frac{\log\left(h(\mathfrak{m})\right)}{\log\left(\mathfrak{m} f\left(\mathfrak{m}\right)\right)} = 0$$

$$\Leftrightarrow \lim_{\mathfrak{m} \to \infty} \frac{\log\left(h(\mathfrak{m})\right)}{\log\left(\mathfrak{m}\right) + \log\left(f\left(\mathfrak{m}\right)\right)} = 0$$

$$\Leftrightarrow \lim_{\mathfrak{m} \to \infty} \frac{\log\left(h(\mathfrak{m})\right)}{\log\left(\mathfrak{m}\right)\left(1 + \frac{\log(f(\mathfrak{m}))}{\log(\mathfrak{m})}\right)} = 0.$$

$$\Rightarrow \lim_{\mathfrak{m} \to \infty} \frac{\log\left(h(\mathfrak{m})\right)}{\log\left(\mathfrak{m}\right)} = 0,$$

as:

$$\lim_{\mathfrak{m} \to \infty} \frac{\log(f(\mathfrak{m}))}{\log(\mathfrak{m})} = 0,$$

which completes the proof. □

The two theorems above provide a new characterisation of Minkowski-measurability through the methods developed in this thesis. Furthermore, as already stated in the overview, we thereby obtain a more precise statement concerning the multiplicities of lengths of a Minkowski-measurable string than the one previously obtained by M.L. Lapidus and C. Pomerance in [78].

Chapter 4

Interlude

As pointed out before, the case of the fractal strings respectively fractal chains investigated in the preceeding chapters may appear a bit artificial. Indeed, such a fractal string may as well be represented as a "fractal harp" [74], as shown in Figure 4.1, thereby emphasizing the disconnectedness of the underlying set.

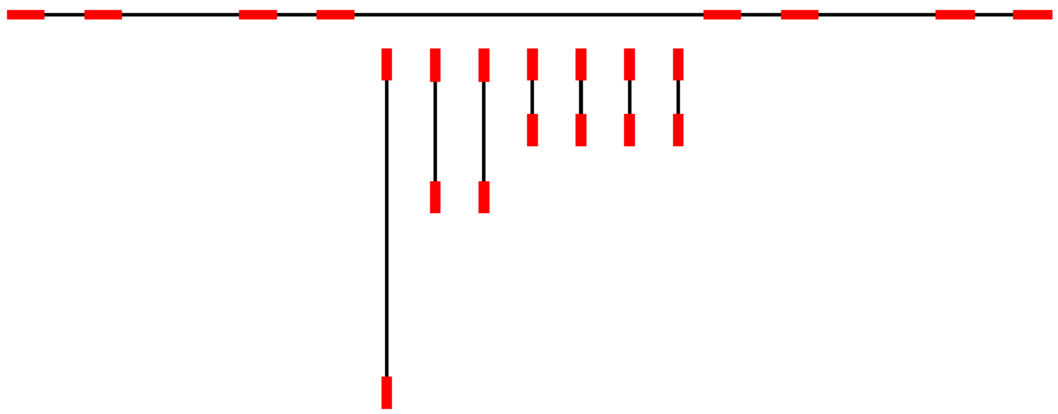

Figure 4.1: An approximation to the triadic Cantor string and the corresponding harp

From a physical viewpoint, the transition from fractal strings to fractal chains is then simply a discretisation of the constituting strings with a constant linear density $\varrho = \frac{m}{\ell}$ to a system with lumped masses coupled by massless springs (Figure 4.2).

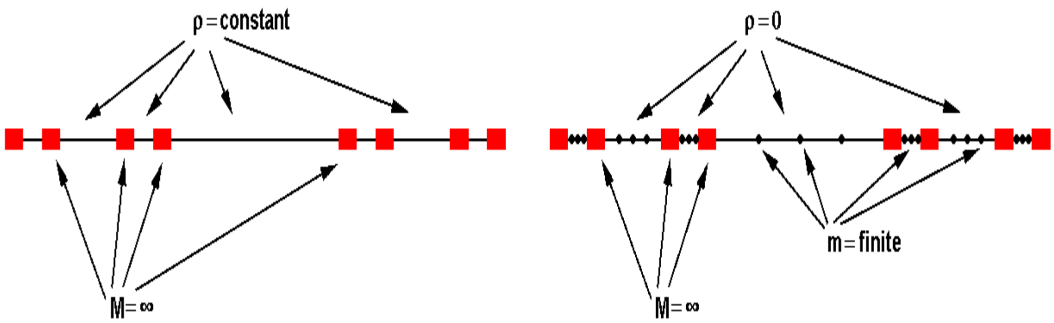

Figure 4.2: The transition from a fractal string (left) to a fractal chain (right)

One may now wonder what happens if one releases the conditions on the nodes (i.e. the boundary of the set) by allowing the masses placed at these nodes to be finite.

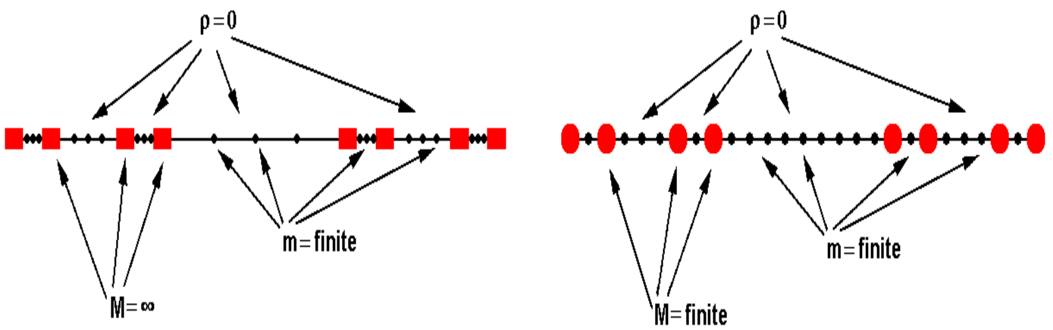

Figure 4.3: From a fractal chain (left) to a fractal-layered chain (right)

The case shown at the right of Figure 4.3 has been treated to some extent, at least numerically in [22] and in [23]. As visualised in Figure 4.4, it appears that a further modified version of this configuration is related to measure geometric operators, a relationship to be explored in detail in the next chapter.

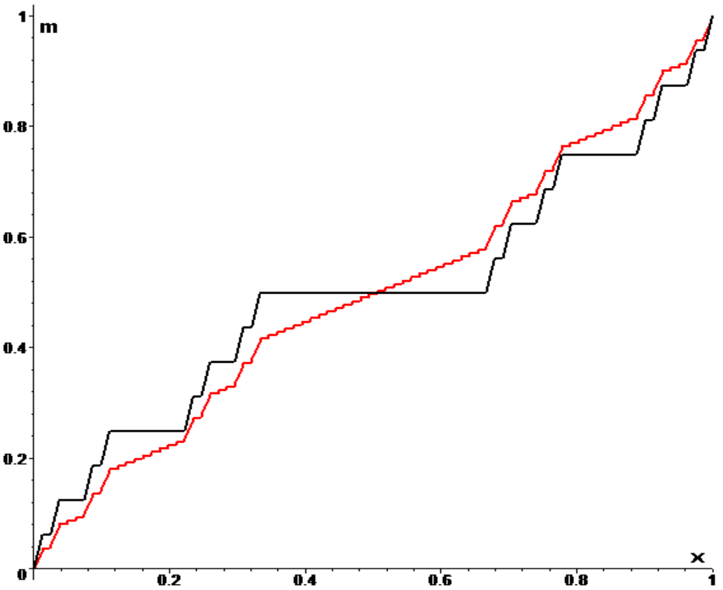

Figure 4.4: The mass distribution of an approximation of a fractal-layered chain (red) compared to that of a measure geometric chain (black)

Chapter 5

Measure geometric fractal chains

Measure geometric fractal chains arise from a different context than those treated in the preceeding chapters. However, it is also possible in this case to develop a physical model that may be described in terms of dynamic matrices. In this framework, the techniques presented earlier appear useful and a few results thereby obtained are included here.

5.1 Preliminaries

Although the involved fractals are very similar or even the same as those already considered, the underlying approach is quite different (see for example [16, 19, 34, 37, 39–43, 45, 46, 54, 60, 86, 90, 91, 102–105, 114] and the references contained therein), so that it is necessary to state several already known results on the subject beforehand. The material presented in this section is compiled from [37, 38, 44]. Consider generalised second order differential operators of the form $\mathbf{A}^\mu = \frac{d}{d\mu}\frac{d}{dx}$, where μ is a finite atomless Borel measure on $[0,1]$ which is compactly supported on $L := \operatorname{supp}\mu \subset [0,1]$ with $\{0,1\} \in L$. This operator may be interpreted as a measure geometric Laplacian with properties analogous to those of a standard Euclidean Laplacian. Denote by $L_2([0,1], dx)$ the Hilbert space of all real-valued and square-integrable functions on the interval $[0,1]$. The Sobolev space $W^{1,2}[0,1]$ is then given by:

$$W^{1,2}[0,1] := \left\{ f : [0,1] \to \mathbb{R} \mid \exists f' \in L_2([0,1], dx) \mid f(x) = f(0) + \int_0^x f'(y)\, dy,\ x \in [0,1] \right\}$$

Note that for any $f \in W^{1,2}[0,1]$, f' is called the weak derivative of f. The second derivative is then defined by repeating this construction with respect to the measure μ instead of the Lebesgue measure. Let $L_2(L, d\mu)$ denote the Hilbert space of all square μ-integrable functions on L. By setting

$$\mathcal{D}(\mathbf{A}^\mu) := \left\{ f \in W^{1,2}[0,1] \mid \exists f'' \in L_2(L, d\mu) \mid f'(x) = f'(0) + \int_0^x f''(y)\, d\mu(y),\ x \in [0,1] \right\}, \tag{5.1}$$

we can define the operator $\mathbf{A}^\mu = \frac{d}{d\mu}\frac{d}{dx}$ on $\mathcal{D}(\mathbf{A}^\mu)$ by:

$$\mathbf{A}^\mu f = \frac{d}{d\mu}\left(\frac{df}{dx}\right) := \begin{cases} f'' & \text{on } L \\ 0 & \text{everywhere else,} \end{cases}$$

where f'' is given by Equation (5.1) above.

In the following, we will only consider Dirichlet boundary conditions, i.e. the restriction $\mathbf{A}_D^\mu$ of $\mathbf{A}^\mu$ on $\mathcal{D}(\mathbf{A}_D^\mu) := \{f \in \mathcal{D}(\mathbf{A}^\mu) \mid f(0) = f(1) = 0\}$. The operator $\mathbf{A}_D^\mu$ is then a negative symmetric operator on $L_2(L, d\mu)$ and we can consider the eigenvalue problem:

$$-\mathbf{A}_D^\mu f = \lambda f, \text{ with } f \in \mathcal{D}(\mathbf{A}_D^\mu).$$

In the self-similar case, i.e. if $L \subset [0,1]$ is the attractor of an IFS with contractions $S = \{[0,1]; S_1, \ldots, S_m\}$, $m \geq 2$ as defined in Definition 2.2, and if for any Borel set A in $[0,1]$, the Borel probability measure μ satisfies

$$\mu(A) = \sum_{i=1}^{m} \varrho_i \mu\left(S_i^{-1}(A)\right)$$

for a given m-dimensional vector of weights $\varrho = (\varrho_1, \ldots, \varrho_m)$, where $\varrho_i \in \mathbb{R}^+$ and $\sum_{i=1}^m \varrho_i = 1$, then it holds that:

$$N_D^\mu(x) \asymp x^\gamma, \text{ as } x \to \infty \tag{5.2}$$

for the eigenvalue counting function $N_D^\mu(x) := \#\{k \in \mathbb{N} \mid \lambda_k \leq x\}$, $x > 0$, defined in a way analogous to Definition (2.17), with the spectral exponent γ being the unique solution of

$$\sum_{i=1}^m (\varrho_i r_i)^\gamma = 1, \tag{5.3}$$

where the r_i are the scaling ratios of the contractions S_i, $i = 1 \ldots m$.

Remark 5.1. *If the weights ϱ_i are chosen such that $\varrho_i = r_i^d$, for $i = 1, \ldots, m$, where d denotes the Hausdorff dimension (which, in the self-similar case, is identical to the Minkowski dimension) of $L = supp\ \mu$, then μ is simply the normalised Hausdorff measure on L and the spectral exponent is given by $\gamma = \frac{d}{d+1}$.*

Furthermore, by applying the renewal theorem, it is possible to establish the following theorem:

Theorem 5.2. *Under the assumptions made above, two cases are to be distinguished for the asymptotic behaviour of the eigenvalue counting function $N_D^\mu(x)$ as x tends to infinity:*

- *The non-arithmetic case:*
  If the additive group $\sum_{i=1}^m \mathbb{Z} log(\varrho_i r_i)$ is a dense subset of $\mathbb{R}$, then $N_D^\mu(x)\, x^{-\gamma}$ converges as $x \to \infty$.

- *The arithmetic case:*
  If $\sum_{i=1}^m \mathbb{Z} log(\varrho_i r_i)$ belongs to a discrete subgroup of $\mathbb{R}$, i.e. if $\sum_{i=1}^m \mathbb{Z} log(\varrho_i r_i) = h\mathbb{Z}$ for some $h \in \mathbb{R}$, then

  $$N_D^\mu(x) = (G(\ln x) + o(1))\, x^\gamma, \text{ as } x \to \infty$$

  holds, where G is a positive, T-periodic function and T the positive generator of the subgroup.

Remark 5.3. *It must be noted that convergence of* $N_D^\mu(x)\, x^{-\gamma}$ *as* $x \to \infty$ *does not necessarily imply the non-arithmetic case. Indeed, even in the arithmetic case, the function* G *may be a constant function and thus the limit* $\lim_{x\to\infty} N_D^\mu(x)$ *can also exist in this case.*

5.2 A physical interpretation

It is possible to give a physical interpretation of the generalised second order differential equations described in the preceeding section (see for example [60] or [5]). In order to do this, we start with the well known one-dimensional wave equation for a string fixed at its endpoints a and b, given as:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\rho(x)}{F_T}\frac{\partial^2 u(x,t)}{\partial t^2}, \text{ with } u(a,t) = u(b,t) = 0,$$

where $u(x,t)$ is the displacement of the string at the point $x \in [a,b]$ at time $t \in [0,\infty)$, $\rho : [a,b] \to [0,\infty)$ the linear mass density (mass distribution) along the string and F_T the constant tension of the string. This differential equation can be solved by the method of separation of variables if we make the ansatz $u(x,t) = v(x)w(t)$, so that:

$$\ddot{v}(x)w(t) = \frac{\rho(x)}{F_T} v(x)\ddot{w}(t)$$

or alternatively:

$$\frac{\ddot{v}(x)}{\rho(x)v(x)} = \frac{1}{F_T}\frac{\ddot{w}(t)}{w(t)}.$$

As this must hold for each x and t, both sides of the equation have to equal a constant, denoted here by $-\lambda$, and thus we have for the left hand side:

$$\frac{\ddot{v}(x)}{\rho(x)v(x)} = -\lambda,$$

respectively

$$\ddot{v}(x) = -\lambda\rho(x)v(x).$$

We can now integrate this equation to obtain:

$$\begin{aligned}\dot{v}(x) - \dot{v}(a) &= -\lambda \int_a^x v(y)\rho(y)dy \\ &= -\lambda \int_a^x v(y)d\mu(y),\end{aligned}$$

with μ being the measure induced by the linear mass density ρ, so that $d\mu(y) = \rho(y)dy$. Using the concepts developed in the preceeding section, we can then state the eigenvalue problem for the string in the form:

$$\frac{d}{d\mu}\dot{v} = \frac{d}{d\mu}\frac{d}{dx}v = \mathbf{A}_D^\mu v = -\lambda v,$$

respectively

$$-\mathbf{A}_D^\mu v = \lambda v.$$

5.3 A physical model

The question now arises on how the kind of string described by a fractal measure could be approximated. We will solve the model following the pioneering works of F.P. Gantmacher and M.G. Krein [48], translated in [50]. For this we consider massless strings of length l loaded with N beads, obtained according to the construction rules of the corresponding fractal set, as shown in Figure 5.1. At each level of approximation j, the configuration of the beads induces a measure μ_j that is not atomless, but as the total mass is kept constant, these measures will tend to the desired atomless measure μ.

It must be noted, that this is not the best possible approximation in terms of the resulting quantisation error (see for example [52, 62]). Indeed the best approximation assigns to each midpoint of the basic intervals of order j a mass 2^{-j} instead of the masses 2^{-j-1} assigned to the two endpoints of these same intervals by the model used here. However, the chosen procedure has the advantage of showing the relationship to the type of fractal chains treated in the preceeding chapters most clearly.

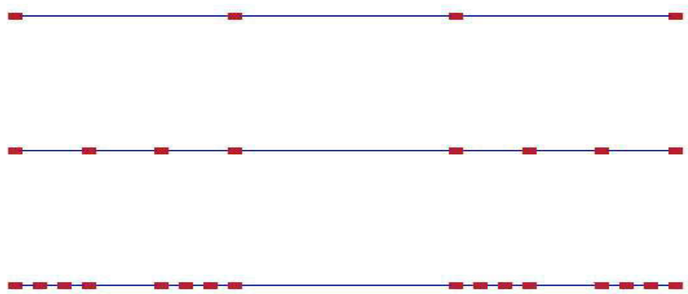

Figure 5.1: The first three iterations of a beaded string loaded according to the classical Cantor set construction

Now, let $u_i(t)$ denote the transverse displacement of the mass m_i at the instant t. Then the formulas for the kinetic and potential energy of this string under constant tension σ take the form

$$T = \sum_{i=0}^{N} \frac{m_i}{2} \dot{u}_i^2$$

and

$$V = \frac{\sigma}{2} \sum_{i=0}^{N} \frac{1}{l_i} \left(u_{i+1} - u_i \right),$$

where l_i denotes the distance between the masses m_i and m_{i+1}. Moreover, we have $y_0 = y_N = 0$ under Dirichlet boundary conditions. We can expand V to obtain

$$V = \sum_{i=1}^{N} \frac{\sigma}{2} \left(\frac{1}{l_{i-1}} + \frac{1}{l_i} \right) u_i^2 - 2 \sum_{i=1}^{N-1} \frac{\sigma}{2} \left(\frac{1}{l_i} \right) u_i u_{i+1}.$$

Notice that T and V fit the template of a Sturm system:

$$T = \sum_{i=0}^{N} c_i \dot{u}_i^2$$

and

$$V = \sum_{i=1}^{N} a_i u_i^2 - 2 \sum_{i=1}^{N-1} b_i u_i u_{i+1}$$

with coefficients:

$$a_i = \frac{\sigma}{2} \left(\frac{1}{l_{i-1}} + \frac{1}{l_i} \right), \tag{5.4}$$

$$b_i = \frac{\sigma}{2} \left(\frac{1}{l_i} \right) \tag{5.5}$$

and

$$c_i = \frac{m_i}{2},$$

from which we can build the mass matrix **M**:

$$\mathbf{M} = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}$$

and the stiffness matrix **K**:

$$\mathbf{K} = \begin{bmatrix} a_1 & -b_1 & & \\ -b_1 & a_2 & \ddots & \\ & \ddots & \ddots & -b_{n-1} \\ & & -b_{n-1} & a_n \end{bmatrix}.$$

The Euler-Lagrange equations describe the evolution of this system according to the differential equation:

$$\mathbf{M}\ddot{u} + \mathbf{K}u = 0.$$

Substituting the ansatz

$$u(t) = sin(\omega t + \theta)$$

into this differential equation, we find, after simplification, that solutions of this form exist, provided that

$$\mathbf{K}u = \omega^2 \mathbf{M}u.$$

We can convert this into a standard eigenvalue problem by premultiplying with $\mathbf{M}^{-\frac{1}{2}}$ and postmultiplying with $\mathrm{Id} = \mathbf{M}^{-\frac{1}{2}}\mathbf{M}^{\frac{1}{2}}$:

$$\mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}\mathbf{M}^{\frac{1}{2}}u = \omega^2 \mathbf{M}^{\frac{1}{2}}u.$$

By relabelling the variables: $\mathbf{D} = \mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}$, $v = \mathbf{M}^{\frac{1}{2}}u$ and $\lambda = \omega^2$, we arrive at the standard eigenvalue problem:

$$\mathbf{D}v = \lambda v.$$

Notice that $\mathbf{M}^{-\frac{1}{2}} = \left(\mathbf{M}^{-\frac{1}{2}}\right)^T$, so $\mathbf{D}$ is a symmetric matrix. It inherits the structure of $\mathbf{K}$,

$$\mathbf{D} = \begin{bmatrix} \alpha_1 & -\beta_1 & & \\ -\beta_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & -\beta_{n-1} \\ & & -\beta_{n-1} & \alpha_n \end{bmatrix}$$

with coefficients; $\alpha_i = c_i a_i$ and $\beta_i = -\sqrt{c_i c_{i+1}} b_i$. The eigenvalues can then be obtained by one of the standard numerical algorithms.

5.4 Numerical spectral asymptotics for measure geometric chains

A few empirical results on the spectra of two typical examples of measure geometric chains will be presented in this section, as these examples will again be used in the next section in order to illustrate the use of the techniques developed therein.

5.4.1 Example 1: The measure geometric Cantor chain

The eigenvalues for different approximation levels to the measure geometric Cantor chain are calculated directly from the corresponding dynamic matrices by the usual numerical techniques. In Figure 5.2, a normalised eigenvalue counting function $N^{\mu_j}_{D,\mathrm{norm}}(x) := \frac{1}{N(j)} N^{\mu_j}_D(\frac{x}{\lambda_{N(j)}(j)})$ for the eigenvalues $0 \le \lambda_1(j) \le \lambda_2(j) \le \ldots \le \lambda_{N(j)}(j)$ of the j^{th} approximation of the measure geometric Cantor chain is depicted and the approximate self-similarity in the resulting spectra is clearly visible (Note that the curves have been shifted for visualisation purposes). At

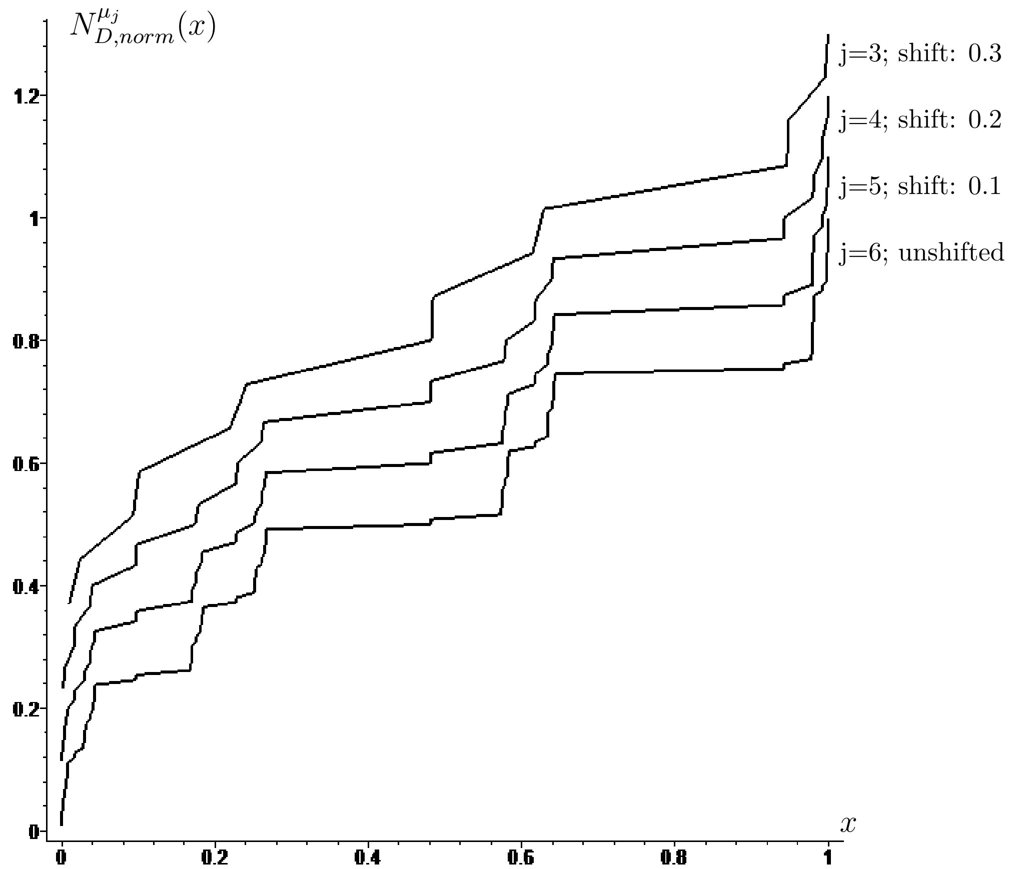

Figure 5.2: Normalised eigenvalue counting functions for different approximation levels j of the measure geometric Cantor chain.

this point, one may thus wonder about the behaviour of the spectral exponent of the consecutive approximations to the eigenvalue counting function. As an example, the empirical eigenvalue counting function thus obtained for the $j = 7^{th}$ approximation to our model of the triadic measure geometric Cantor chain is displayed together with the prediction from Equation 5.3 and a power law fit in Figure 5.3. From the graph, it is obvious that the empirical spectral exponent is

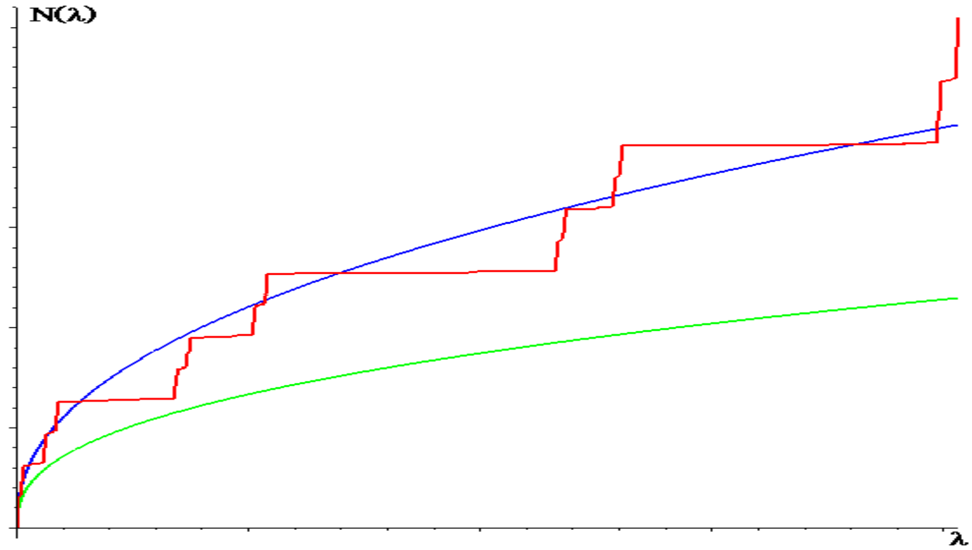

Figure 5.3: $N(\lambda)$ as a function of λ for a seventh order measure geometric Cantor beaded string, blue: power-law fit, green: expectation from Equation 5.3.

larger than expected. Nevertheless it decreases for higher iteration levels towards the theoretical value $\gamma_{th} = \frac{\ln(2)}{\ln(6)} \approx .3868528073$. The eigenvalue counting function was calculated for the first eight approximations, together with power-law fits to the results in order to determine the spectral exponent. However, due to the largeness of the involved dynamic matrices, computation time explodes. We therefore attempt to estimate the spectral exponent for the iteration level j going to infinity by a fit to the data contained in Table 5.1. Using an exponential fitting function

$$\gamma_{exp,j} \approx .3842123042e^{-.4176167871j} + .4144647159,$$

we obtain a spectral exponent of $\gamma_{exp,\infty} = .4144647159$, in excellent agreement with the numerical data ($r^2 = .9999991155$), but notably bigger than predicted on theorical grounds. Another approach would be to assume that for the iteration level going to infinity, the theoretical value will be recovered and then fit an exponential to the differences between empirical and theoretical values, in which case we obtain

$$\gamma_{exp,j} - \gamma_{th} \approx .3389689479e^{-.2916814134j},$$

or equivalently:

$$\gamma_{exp,j} \approx .3868528073 + .3389689479e^{-.2916814134j},$$

for a correlation coefficient of $r^2 = .9906972163$.

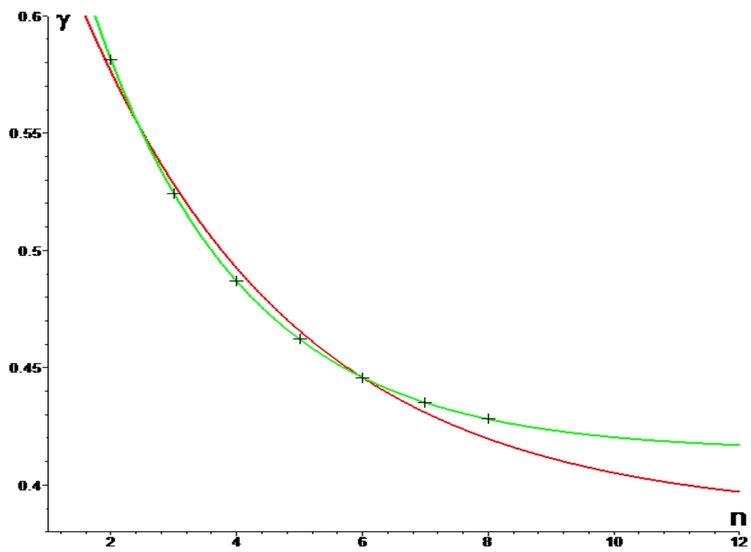

Figure 5.4: Fits to the empirical value for γ: direct exponential fit (green), by an exponential fit for $\gamma_{exp,j} - \gamma_{th}$ (red).

Taking into consideration the higher correlation coefficient for the direct fit, it appears likely that the difference between the empirical spectral exponent $\gamma_{exp,\infty}$ and the theoretical value γ_{th} is not an artefact but relates to higher terms for the spectral asymptotics for the eigenvalue counting function not contained in the theory yet, but it appears premature at this point to make any conjectures.

Iteration level j	Spectral exp. $\gamma_{exp,j}$	Rel. error $\frac{\gamma_{exp,j}}{\gamma_{th}}$	Corr. coeff. r^2
2	.5811456978	1.502239836	.9874801293
3	.5241536656	1.354917570	.9911948532
4	.4868316531	1.258441567	.9927419688
5	.4621031627	1.194519347	.9924434198
6	.4457886491	1.152346941	.9918004560
7	.4350772540	1.124658386	.9912320096
8	.4281003266	1.106623291	.9908043079

Table 5.1: Spectral exponents and correlations for the second to eight iterations

5.4.2 Example 2: A measure geometric chain with two different scaling ratios

A straightforward extension of the results for the triadic Cantor set is the application of the methods presented to more general sets obtained through an iterated function system (IFS), as the next example will show.

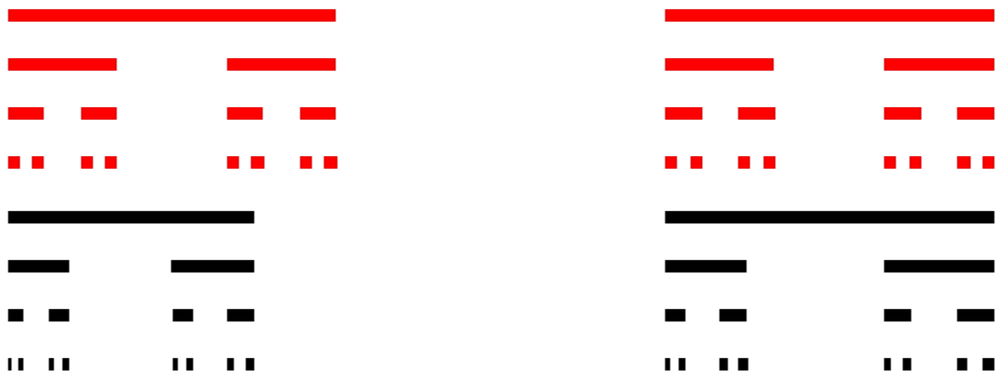

Figure 5.5: The first four stages in the construction of the Cantor set (above) and its analogue with two different scaling ratios (below).

We will now consider a set analogous to the triadic Cantor set, but with two different scaling ratios $r_1 = \frac{1}{4}$ and $r_2 = \frac{1}{3}$ instead of $r_1 = r_2 = \frac{1}{3}$ (see Figure 5.5). Furthermore, the mass matrix is not a simple scalar matrix as in the case of the Cantor set. Indeed the weights ϱ_i are not identical, but we have $\varrho_1 = \left(\frac{1}{4}\right)^d$ and $\varrho_2 = \left(\frac{1}{3}\right)^d$, with d being the (unique) solution to $\left(\frac{1}{4}\right)^d + \left(\frac{1}{3}\right)^d = 1$, leading to a more complicated structure of the matrix.

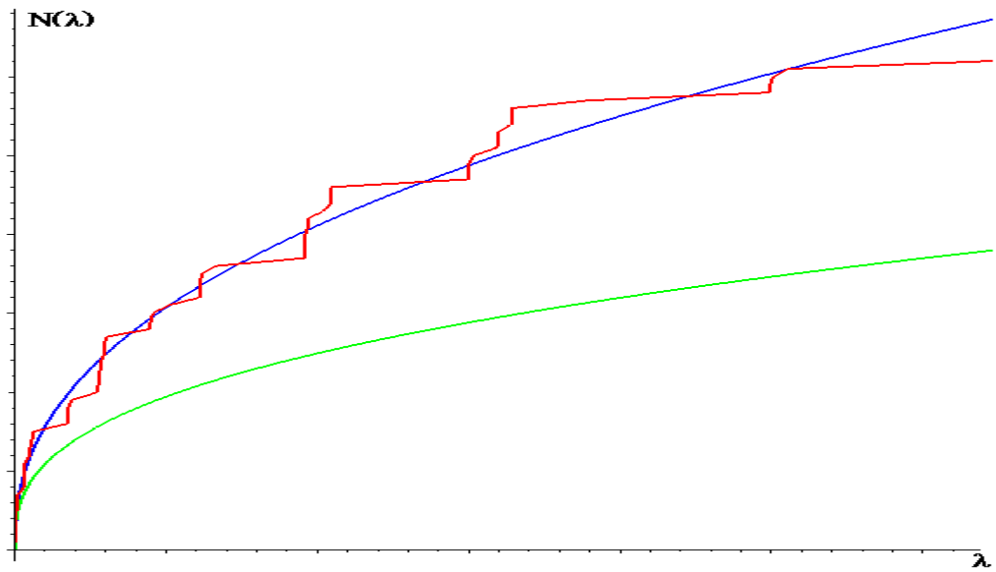

Figure 5.6: $N(\lambda)$ as a function of λ for the fifth order measure geometric beaded string with scaling ratios $r_1 = \frac{1}{4}$ and $r_2 = \frac{1}{3}$, blue: power-law fit, green: expectation from Equation 5.3.

Although this set is still self-similar, we are in the non-arithmetic case as detailed above. Thus, we expect a different behaviour of the eigenvalue counting function, which should converge as $x^\gamma = x^{\frac{d}{d+1}}$ for $x \to \infty$. The numerical results are displayed in Figure 5.6 for the eigenvalue counting function of an approximation to this measure geometric chain, again together with the prediction from Equation 5.3 and a power law fit. Note that here as well the empirical spectral exponent is larger than expected, but decreasing for higher iteration levels towards the theoretical value $\gamma_{th} \approx .3591792841$, as shown in Table 5.2. A direct exponential fit to the numerical data leads to:

$$\gamma_{exp,j} = .3944339298 + 1.397516812e^{-.6973230553j}$$

and thus a spectral exponent of $\gamma_{exp,\infty} = .3944339298$, for a correlation coefficient of $r^2 = .9971452711$, while we obtain through an exponential fit to the differences:

$$\gamma_{exp,j} - \gamma_{th} \approx 1.035154523e^{-.5122042344j},$$

Iteration level j	Spectral exp. $\gamma_{exp,j}$	Rel. error $\frac{\gamma_{exp,j}}{\gamma_{th}}$	Corr. coeff. r^2
1	.7439261802	2.071183426	1.000000000
2	.5564274578	1.549163558	.9904905534
3	.4853204184	1.351192677	.9938516799
4	.4445257935	1.237615345	.9942803053
5	.4195608533	1.168109832	.9953937077
6	.4034406621	1.123229206	.9964191165
7	.3926714797	1.093246457	.9970909198

Table 5.2: Spectral exponents and correlations for the first seven iterations

or equivalently:

$$\gamma_{exp,j} \approx .3591792841 + 1.035154523 e^{-.5122042344 j},$$

with a correlation coefficient of $r^2 = .9845580382$.

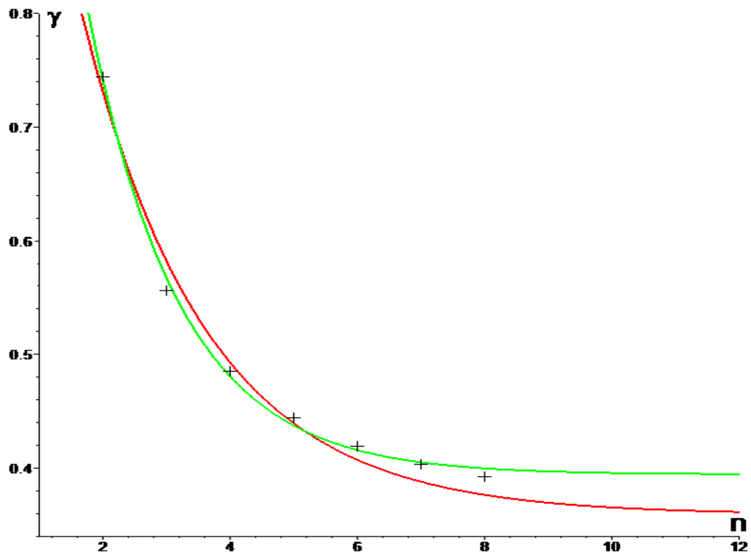

Figure 5.7: Fits to the empirical value for γ: direct exponential fit (green), by an exponential fit for $\gamma_{exp,j} - \gamma_{th}$ (red).

The same considerations as for the measure geometric Cantor chain apply in this case as well; the difference between the empirical spectral exponent and the theoretical value probably being not an artefact but possibly related to higher terms for the spectral asymptotics for the eigenvalue counting function.

Analogously to the measure geometric Cantor chain, the spectrum of the chain under consideration also displays an approximate self-similarity as shown in Figure 5.8, but moreover, the influence of the symmetries of the underlying fractal becomes obvious when comparing the graphs (Figures 5.3 and 5.6) of the corresponding eigenvalue counting functions in each case. Indeed, as predicted by the theory, in the case of the measure geometric Cantor chain (i.e. the arithmetic case) strong oscillations are visible, whereas in the second example (non-arithmetic case) the graph shows much weaker oscillations.

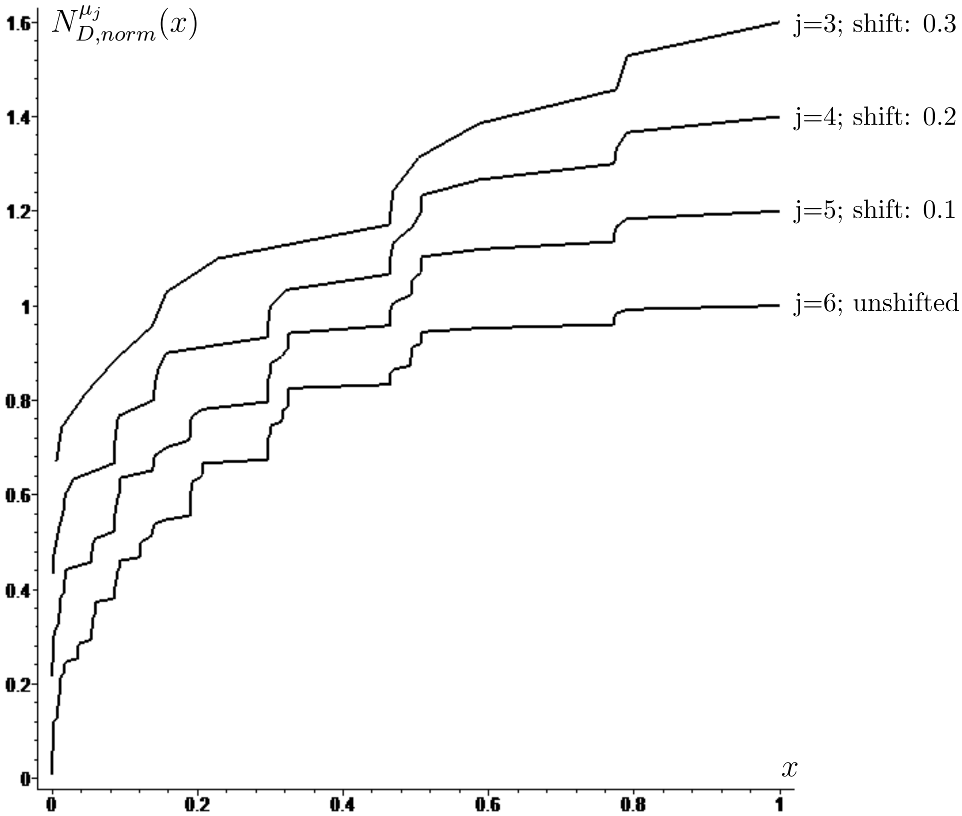

Figure 5.8: Normalised eigenvalue counting functions for different approximation levels j of the measure geometric chain with two scaling ratios.

5.5 The traces of powers of the dynamic matrix of measure geometric chains

Let $\{l_i(j)\}_{i=1}^{N(j)}$ denote the sequence of lengths separating the masses in our physical model of measure geometric chains and $\mathbf{K}_j$ the corresponding stiffness matrix, constructed as detailed in Section 5.3, where j is the approximation level under consideration and $N(j)$ the number of masses. Since the stiffness matrix $\mathbf{K}_j$ of a measure theoretic chain at any approximation level j is a tridiagonal matrix, it is easily possible to establish the following quite simple relations for the traces of powers of the stiffness matrix by careful bookkeeping:

Lemma 5.4. *The traces of the stiffness matrix $\mathbf{K}_j$ of a measure geometric chain and of its square $\mathbf{K}_j^2$ are given by:*

- $\mathrm{tr}(\mathbf{K}_j) = \frac{\sigma}{2}\left(-\frac{1}{l_0(j)} - \frac{1}{l_{N(j)}(j)} + 2\sum_{i=0}^{N(j)} \frac{1}{l_i(j)}\right)$, *and*
- $\mathrm{tr}(\mathbf{K}_j^2) = \left(\frac{\sigma}{2}\right)^2 \left(4\sum_{i=0}^{N(j)} \left(\frac{1}{l_i(j)}\right)^2 + 2\sum_{i=0}^{N(j)-1} \frac{1}{l_i(j)l_{i+1}(j)} - 3l_0^2(j) - 3l_{N(j)}^2(j)\right)$,

where $N(j) = 2^{j+1} - 2$ is the size of the matrix.

Proof. By the construction rule for the stiffness matrix, the elements $a_{i,i}(j)$ on its diagonal are given by Equation 5.4 and thus:

$$\mathrm{tr}\left(\mathbf{K}_j\right) = \sum_{i=1}^{N(j)} a_{i,i}(j) = \frac{\sigma}{2}\sum_{i=1}^{N(j)}\left(\frac{1}{l_{i-1}(j)} + \frac{1}{l_i(j)}\right) = \frac{\sigma}{2}\left(-\frac{1}{l_0(j)} - \frac{1}{l_{N(j)}(j)} + 2\sum_{i=0}^{N(j)}\frac{1}{l_i(j)}\right).$$

Furthermore, we have:

$$\mathrm{tr}\left(\mathbf{K}_j^2\right) = \sum_{i=1}^{N(j)}\sum_{k=1}^{N(j)} a_{i,k}(j)a_{k,i}(j).$$

As the matrix is symmetric, $a_{i,k}(j) = a_{k,i}(j)$ and moreover $a_{i,k}(j) = 0, \forall |i-k| > 1$, such that:

$$\mathrm{tr}\left(\mathbf{K}_j^2\right) = \sum_{i=1}^{N(j)}\sum_{k=1}^{N(j)} a_{i,k}(j)a_{k,i}(j) = \sum_{i=1}^{N(j)} a_{i,i}^2(j) + \sum_{i=2}^{N(j)} a_{i-1,i}^2(j) + \sum_{i=1}^{N(j)-1} a_{i+1,i}^2(j).$$

With;

$$a_{i,i}^2(j) = \left(\frac{\sigma}{2}\right)^2 \left(\frac{1}{l_{i-1}(j)} + \frac{1}{l_i(j)}\right)^2 = \left(\frac{\sigma}{2}\right)^2 \left(\left(\frac{1}{l_{i-1}(j)}\right)^2 + \left(\frac{1}{l_i(j)}\right)^2 + \left(\frac{2}{l_{i-1}(j)l_i(j)}\right)\right),$$

and

$$a_{i-1,i}^2(j) = a_{i+1,i}^2(j) = \left(\frac{\sigma}{2}\right)^2 \left(\frac{1}{l_i(j)}\right)^2,$$

we finally obtain:

$$\operatorname{tr}(\mathbf{K}_j^2) = \left(\frac{\sigma}{2}\right)^2 \left(4\sum_{i=0}^{N(j)} \left(\frac{1}{l_i(j)}\right)^2 + 2\sum_{i=0}^{N(j)-1} \frac{1}{l_i(j)l_{i+1}(j)} - 3l_0^2(j) - 3l_{N(j)}^2\right),$$

thereby completing the proof of the lemma. □

Although it is possible to continue in this manner, the expressions for higher powers become very cumbersome. However, the use of only the first two already allows us to give upper and lower bounds for the trace of all powers of the stiffness matrix $\mathbf{K}_j$:

Proposition 5.5. *The traces of powers of the stiffness matrix $\mathbf{K}_j$ are bounded from above and from below in the following way:*

$$\left(\frac{\operatorname{tr}(\mathbf{K}_j^2)}{(\operatorname{tr}(\mathbf{K}_j))^2}\right)^{k-1} \leq \frac{\operatorname{tr}(\mathbf{K}_j^k)}{(\operatorname{tr}(\mathbf{K}_j))^k} \leq \left(\frac{\max(\lambda(\mathbf{K}_j))}{\operatorname{tr}(\mathbf{K}_j)}\right)^{k-1},$$

with $\lambda(\mathbf{K}_j)$ denoting the set of the eigenvalues of $\mathbf{K}_j$.

Proof. We have:

$$\frac{\frac{\operatorname{tr}(\mathbf{K}_j^k)}{(\operatorname{tr}(\mathbf{K}_j))^k}}{\frac{\operatorname{tr}(\mathbf{K}_j^{k-1})}{(\operatorname{tr}(\mathbf{K}_j))^{k-1}}} = \frac{1}{\operatorname{tr}(\mathbf{K}_j)} \frac{\operatorname{tr}(\mathbf{K}_j^k)}{\operatorname{tr}(\mathbf{K}_j^{k-1})} = \frac{1}{\operatorname{tr}(\mathbf{K}_j)} \mathcal{L}(k, \lambda(\mathbf{K}_j)),$$

where $\mathcal{L}(\cdot,\cdot)$ denotes the Lehmer mean [80], defined by:

$$\mathcal{L}(p, \{x_i\}_{i=1}^n) := \frac{\sum_{i=1}^n x_i^p}{\sum_{i=1}^n x_i^{p-1}}.$$

Thus:

$$\frac{\operatorname{tr}(\mathbf{K}_j^k)}{(\operatorname{tr}(\mathbf{K}_j))^k} = \frac{1}{\operatorname{tr}(\mathbf{K}_j)}\mathcal{L}(k, \lambda(\mathbf{K}_j))\frac{\operatorname{tr}(\mathbf{K}_j^{k-1})}{(\operatorname{tr}(\mathbf{K}_j))^{k-1}}$$

and by induction:

$$\frac{\operatorname{tr}(\mathbf{K}_j^k)}{(\operatorname{tr}(\mathbf{K}_j))^k} = \left(\frac{1}{\operatorname{tr}(\mathbf{K}_j)}\right)^{k-1} \mathcal{L}(k, \lambda(\mathbf{K}_j)) \cdot \mathcal{L}(k-1, \lambda(\mathbf{K}_j)) \cdots \mathcal{L}(2, \lambda(\mathbf{K}_j)).$$

Now, as by the properties of the Lehmer mean:

$$\frac{\operatorname{tr}(\mathbf{K}_j^2)}{\operatorname{tr}(\mathbf{K}_j)} = \mathcal{L}(2, \lambda(\mathbf{K}_j)) \leq \mathcal{L}(k, \lambda(\mathbf{K}_j)) \leq \mathcal{L}(\infty, \lambda(\mathbf{K}_j)) = \max(\lambda(\mathbf{K}_j)),\ \forall k \geq 2,$$

we have:

$$\left(\frac{\mathcal{L}(2, \lambda(\mathbf{K}_j))}{\operatorname{tr}(\mathbf{K}_j)}\right)^{k-1} \leq \frac{\operatorname{tr}(\mathbf{K}_j^k)}{(\operatorname{tr}(K_j))^k} \leq \left(\frac{\max(\lambda(\mathbf{K}_j))}{\operatorname{tr}(\mathbf{K}_j)}\right)^{k-1},$$

or equivalently

$$\left(\frac{\operatorname{tr}(\mathbf{K}_j^2)}{(\operatorname{tr}(\mathbf{K}_j))^2}\right)^{k-1} \leq \frac{\operatorname{tr}(\mathbf{K}_j^k)}{(\operatorname{tr}(\mathbf{K}_j))^k} \leq \left(\frac{\max(\lambda(\mathbf{K}_j))}{\operatorname{tr}(\mathbf{K}_j)}\right)^{k-1}.$$

□

The knowledge of the behaviour of the powers of the traces of the stiffness matrices allows us to give bounds for those of the dynamic matrice $\mathbf{D}_j$ as well.

Proposition 5.6. *The traces of powers of the dynamic matrix $\mathbf{D}_j$ are bounded by:*

$$\left(\frac{\min\left(c_i^{-1}(j)\right)}{\max\left(c_i^{-1}(j)\right)}\right)^k \left(\frac{\operatorname{tr}(\mathbf{K}_j^2)}{(\operatorname{tr}(\mathbf{K}_j))^2}\right)^{k-1} \leq \frac{\operatorname{tr}(\mathbf{D}_j^k)}{(\operatorname{tr}(\mathbf{D}_j))^k} \leq \left(\frac{\max(\lambda(\mathbf{D}_j))}{\min\left(c_i^{-1}(j)\right)\operatorname{tr}(\mathbf{K}_j)}\right)^{k-1},$$

where $\min\left(c_i^{-1}(j)\right)$ *and* $\max\left(c_i^{-1}(j)\right)$ *are the minimal resp. maximal entry of the corresponding inverse mass matrix* $\mathbf{M}_j^{-1}$.

Proof. As the trace of a product is invariant under cyclic permutations, we have:

$$\mathrm{tr}\left(\mathbf{D}_j\right) = \mathrm{tr}\left(\mathbf{M}_j^{-\frac{1}{2}}\mathbf{K}_j\mathbf{M}_j^{-\frac{1}{2}}\right) = \mathrm{tr}\left(\mathbf{M}_j^{-\frac{1}{2}}\mathbf{M}_j^{-\frac{1}{2}}\mathbf{K}_j\right) = \mathrm{tr}\left(\mathbf{M}_j^{-1}\mathbf{K}_j\right).$$

Now as $\mathbf{M}_j$ is a diagonal matrix, the trace of $\left(\mathbf{M}_j^{-1}\mathbf{K}_j\right)^k = \mathbf{M}_j^{-k}\mathbf{K}_j^k$ is the sum of the products of the diagonal entries and thus:

$$\min\left(c_i^{-1}(j)\right)\mathrm{tr}(\mathbf{K}_j) \leq \mathrm{tr}(\mathbf{D}_j) \leq \max\left(c_i^{-1}(j)\right)\mathrm{tr}(\mathbf{K}_j),$$

respectively

$$\frac{1}{\left(\max\left(c_i^{-1}(j)\right)\right)^k(\mathrm{tr}(\mathbf{K}_j))^k} \leq \frac{1}{(\mathrm{tr}(\mathbf{D}_j))^k} \leq \frac{1}{\left(\min\left(c_i^{-1}(j)\right)\right)^k(\mathrm{tr}(\mathbf{K}_j))^k},$$

and

$$\left(\min\left(c_i^{-1}(j)\right)\right)^k\mathrm{tr}(\mathbf{K}_j^k) \leq \mathrm{tr}(\mathbf{D}_j^k) \leq \left(\max\left(c_i^{-1}(j)\right)\right)^k\mathrm{tr}(\mathbf{K}_j^k),$$

such that:

$$\frac{\left(\min\left(c_i^{-1}(j)\right)\right)^k\mathrm{tr}(\mathbf{K}_j^k)}{\left(\max\left(c_i^{-1}(j)\right)\right)^k(\mathrm{tr}(\mathbf{K}_j))^k} \leq \frac{\mathrm{tr}(\mathbf{D}_j^k)}{(\mathrm{tr}(\mathbf{D}_j))^k},$$

and thus by Proposition (5.5):

$$\left(\frac{\min\left(c_i^{-1}(j)\right)}{\max\left(c_i^{-1}(j)\right)}\right)^k\left(\frac{\mathrm{tr}(\mathbf{K}_j^2)}{(\mathrm{tr}(\mathbf{K}_j))^2}\right)^{k-1} \leq \frac{\mathrm{tr}(\mathbf{D}_j^k)}{(\mathrm{tr}(\mathbf{D}_j))^k}.$$

Although the relations above would also allow to obtain an upper bound, it is more convenient to use the same approach as in Proposition (5.5) again and use the properties of the Lehmer mean to establish that:

$$\frac{\mathrm{tr}(\mathbf{D}_j^k)}{(\mathrm{tr}(\mathbf{D}_j))^k} \leq \left(\frac{\max(\lambda(\mathbf{D}_j))}{\mathrm{tr}(\mathbf{D}_j)}\right)^{k-1}.$$

Then, as

$$\frac{1}{\mathrm{tr}(\mathbf{D}_j)} \leq \frac{1}{\min\left(c_i^{-1}(j)\right)\mathrm{tr}(\mathbf{K}_j)},$$

we have:

$$\frac{\operatorname{tr}(\mathbf{D}_j^k)}{(\operatorname{tr}(\mathbf{D}_j))^k} \leq \left(\frac{\max(\lambda(\mathbf{D}_j))}{\min\left(c_i^{-1}(j)\right) tr(\mathbf{K}_j)} \right)^{k-1},$$

which concludes the proof of the proposition. □

From this proposition it is easily possible to derive the following corollary for the upper bound of the traces of powers of the dynamic matrix:

Corollary 5.7. *The following inequality holds:*

$$\frac{\operatorname{tr}(\mathbf{D}_j^k)}{(\operatorname{tr}(\mathbf{D}_j))^k} \leq \left(\frac{CN(j)^{\frac{1}{\gamma}}}{\min\left(c_i^{-1}(j)\right) \operatorname{tr}(\mathbf{K}_j)} \right)^{k-1}, \text{ for } j \to \infty,$$

where $N(j) = 2^{j+1} - 2$ is the number of eigenvalues, respectively the size of the dynamic matrix.

Proof. By Equation (5.2), the eigenvalue counting function $N_D^\mu(x)$ is monotonously increasing and fulfils

$$N_D^\mu(x) \asymp x^\gamma, \text{ for } x \to \infty,$$

respectively

$$x \asymp (N_D^\mu(x))^{\frac{1}{\gamma}}, \text{ for } x \to \infty.$$

Let us now consider the monotonously increasing sequence of the sorted eigenvalues λ_n, then by the definition of $N_D^\mu(x)$, $N_D^\mu(\lambda_n) = \#\{k \in \mathbb{N} \mid \lambda_k \leq \lambda_n\} = n$, and thus:

$$\lambda_n \asymp n^{\frac{1}{\gamma}}, \text{ for } n \to \infty.$$

Hence, as the eigenvalues are ordered according to their magnitude, $\max(\lambda(\mathbf{D}_j)) = \lambda_{N(j)}$ and as $N(j) \to \infty$ for $j \to \infty$, we obtain hereby:

$$\max(\lambda(\mathbf{D}_j)) = \lambda_{N(j)} \asymp N(j)^{\frac{1}{\gamma}}, \text{ for } j \to \infty,$$

so that

$$\max(\lambda(\mathbf{D}_j)) \leq C \cdot N(j)^{\frac{1}{\gamma}}, \text{ for } j \to \infty,$$

for some constant C and finally, by Proposition (5.6):

$$\frac{\operatorname{tr}(\mathbf{D}_j^k)}{(\operatorname{tr}(\mathbf{D}_j))^k} \leq \left(\frac{CN(j)^{\frac{1}{\gamma}}}{\min\left(c_i^{-1}(j)\right) \operatorname{tr}(\mathbf{K}_j)} \right)^{k-1}, \text{ for } j \to \infty.$$

□

5.5.1 Application: Dirichlet eigenvalues of measure geometric strings as zeroes of a generalised trigonometric function

In a recent thesis, P. Arzt [5] defined analogues of the sine and cosine functions such that their squared zeroes are the eigenvalues of the measure geometric Laplacian on self-similar sets. In our context of the Dirichlet Laplacian, the function of interest is the sinq-function, defined as:

$$\operatorname{sinq}(z) := \sum_{n=0}^{\infty} (-1)^n q_{2n+1} z^{2n+1}, \text{ for } z \in \mathbb{R},$$

where the coefficients q_{n+1} may be obtained through a recursive procedure (details in [5]). Such an infinite series represents a traditional instrument in the representation of functions, where their approximation, as well as their termwise differentiation and integration are classical applications. Although infinite products have also been known and developed for centuries, their usefulness in the same applications has often been overseen. The two forms share a lot of common features, but an important difference is the fact that the partial products of an infinite product representation share the same zeroes with the original function, whereas the Maclaurin expansion does not; a property that might be crucial in further applications. Therefore, we will now show that an infinite product representation, analoguous to the standard Euler product formula for the sine function, does also exist for the sinq function.

Lemma 5.8. *The* sinq *function has a representation of the form:*

$$\mathrm{sinq}(z) = \exp(h(z)) z \prod_{n=1}^{\infty} (1 - \frac{z^2}{\lambda_n}),$$

with $h(z)$ being some entire function and $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ the (real) eigenvalues of the measure geometric Laplacian.

Proof. Indeed, by Weierstraß's factorisation theorem, such a product representation converges if the sum

$$\sum_{n=1}^{\infty} \left(\frac{r}{\lambda_n^{\frac{1}{2}}} \right)^{k_n+1} = r^{k_n+1} \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n^{\frac{1}{2}}} \right)^{k_n+1} = r^{k_n+1} \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} \right)^{\frac{k_n}{2}}$$

converges for each $r > 0$ and some $k_n \in \mathbb{N}^*$. We will show in the following that the sum converges for $k_n = 1$ (and therefore for all $k_n \geq 1$). By Equation 5.2, we know that

$$N_D^{\mu}(x) \asymp x^{\gamma}, \text{ for } x \to \infty,$$

or equivalently

$$\begin{aligned}
&c_1 \lambda_n^{\gamma} \leq N_D^{\mu}(\lambda_n) \leq c_2 \lambda_n^{\gamma} \\
\Leftrightarrow &c_1^{\frac{1}{\gamma}} \lambda_n \leq (N_D^{\mu}(\lambda_n))^{\frac{1}{\gamma}} \leq c_2^{\frac{1}{\gamma}} \lambda_n \\
\Leftrightarrow &c_1^{\frac{1}{\gamma}} \leq (N_D^{\mu}(\lambda_n))^{\frac{1}{\gamma}} \lambda_n^{-1} \leq c_2^{\frac{1}{\gamma}} \\
\Leftrightarrow &\left(\frac{c_1}{N_D^{\mu}(\lambda_n)} \right)^{\frac{1}{\gamma}} \leq \lambda_n^{-1} \leq \left(\frac{c_2}{N_D^{\mu}(\lambda_n)} \right)^{\frac{1}{\gamma}} \\
\Rightarrow &\frac{1}{\lambda_n} \leq \left(\frac{c_2}{N_D^{\mu}(\lambda_n)} \right)^{\frac{1}{\gamma}}.
\end{aligned}$$

Thus:

$$r^2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \leq r^2 \sum_{n=1}^{\infty} \left(\frac{c_2}{N_D^{\mu}(\lambda_n)} \right)^{\frac{1}{\gamma}} = r^2 c_2^{\frac{1}{\gamma}} \sum_{n=1}^{\infty} \left(\frac{1}{N_D^{\mu}(\lambda_n)} \right)^{\frac{1}{\gamma}},$$

and as by the definition of $N_D^\mu(x)$, $N_D^\mu(\lambda_n) = n$, we obtain

$$r^2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \leq r^2 c_2^{\frac{1}{\gamma}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{\gamma}}}.$$

Now, as the Haussdorff dimension d of the fractal sets under consideration is always $d \in (0,1)$ and as we choose the weights in the "natural way" (see Remark 5.1 above), the spectral exponent $\gamma = \frac{d}{d+1} < \frac{1}{2}$ always satisfies $\frac{1}{\gamma} > 2$ and thus a fortiori $\frac{1}{\gamma} > 1$. Therefore the above sum

$$r^2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \leq r^2 c_2^{\frac{1}{\gamma}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{\gamma}}} \leq r^2 c_2^{\frac{1}{\gamma}} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

always converges, which proves the assertion. □

Although valid approximations to the sinq function may be obtained by both approaches, - either by the partial products of the infinite product representation or by the partial sums of the Maclaurin expansion given by P. Arzt -, we will push our strategy here a little further by the use of the characteristic polynomials of the dynamic matrices $\mathbf{D}_j$ as approximations to the partial products in question. The characteristic polynomial of $\mathbf{D}_j$ may be written as:

$$p_{\mathbf{D}_j}(\lambda) = \lambda^{N-1} - \mathrm{tr}(\mathbf{D}_j)\lambda^{N-2} + \frac{\mathrm{tr}(\mathbf{D}_j)^2 - \mathrm{tr}(\mathbf{D}_j^2)}{2!}\lambda^{N-3} - \ldots,$$

and thus, conjecturing that $\exp(h(z)) = 1$, the approximation to the MacLaurin series by:

$$\begin{aligned}\mathrm{sinq}_j(\lambda) &:= \lambda \cdot p_{\mathbf{D}_j}(\lambda) = \lambda^N - \mathrm{tr}(\mathbf{D}_j)\lambda^{N-1} + \frac{\mathrm{tr}(\mathbf{D}_j)^2 - \mathrm{tr}(\mathbf{D}_j^2)}{2!}\lambda^{N-2} - \ldots \\ &:= r_{j,N}\lambda^N - r_{j,N-1}\lambda^{N-1} + r_{j,N-2}\lambda^{N-2} - \ldots\end{aligned}$$

where $r_{j,N} := 1$, and the coefficients:

$$
\begin{aligned}
r_{j,N-1} &:= \operatorname{tr}(\mathbf{D}_j) \\
r_{j,N-2} &:= \frac{\operatorname{tr}(\mathbf{D}_j)^2 - \operatorname{tr}(\mathbf{D}_j^2)}{2!} = \frac{\operatorname{tr}(\mathbf{D}_j)^2}{2!}\left(1 - \frac{\operatorname{tr}(\mathbf{D}_j^2)}{\operatorname{tr}(\mathbf{D}_j)^2}\right) \\
r_{j,N-3} &:= \frac{\operatorname{tr}(\mathbf{D}_j)^3 - 3\operatorname{tr}(\mathbf{D}_j)\operatorname{tr}(\mathbf{D}_j^2) + 2\operatorname{tr}(\mathbf{D}_j^3)}{3!} = \frac{\operatorname{tr}(\mathbf{D}_j)^3}{3!}\left(1 - \frac{3\operatorname{tr}(\mathbf{D}_j^2)}{\operatorname{tr}(\mathbf{D}_j)^2} + \frac{2\operatorname{tr}(\mathbf{D}_j^3)}{\operatorname{tr}(\mathbf{D}_j)^3}\right), \\
&\ \vdots
\end{aligned}
$$

may be obtained through the Newton-Girard identities, given here in the form of a determinant as:

$$
q_k = \frac{1}{k!}\begin{vmatrix}
T_1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
T_2 & T_1 & 2 & 0 & \cdots & \cdots & 0 \\
T_3 & T_2 & T_1 & 3 & 0 & \cdots & 0 \\
T_4 & T_3 & T_2 & T_1 & \ddots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
T_{k-1} & T_{k-2} & T_{k-3} & T_{k-4} & \cdots & \ddots & k-1 \\
T_k & T_{k-1} & T_{k-2} & T_{k-3} & T_{k-4} & \cdots & T_1
\end{vmatrix}
$$

with $k! := 1\cdot 2\cdot \ldots \cdot k$ denoting the factorial of k and the abbreviation $T_k := \operatorname{tr}(\mathbf{D}_j^k)$, for a fixed $j \in \mathbb{N}$.

The zeroes $\lambda(\mathbf{D}_j)$ of $\operatorname{sinq}_j(\lambda) = \lambda p_{\mathbf{D}_j}(\lambda)$ thereby approximate the zeroes of the sinq function, rapidly gaining accuracy as the iteration level increases. Unfortunately, the coefficients in the Newton-Girard identities rise too quickly, so that our bounds for the traces of the powers of the dynamic matrix cannot be used to obtain reasonable bounds on the coefficients of the characteristic polynomial and the MacLaurin expansion of the sinq function. However, the characteristic polynomials of the dynamic matrices still provide an efficient way to approximate the coefficients in the MacLaurin expansion of the sinq function as well as the eigenvalues of the Dirichlet Laplacian, as will be illustrated in the examples below.

5.5.1.1 Example 1: The measure geometric triadic Cantor string

For the measure geometric triadic Cantor string, the coefficients q_{2n+1} are given in Table 5.3.

n	q_{2n+1}	Decimal value of q_{2n+1}
0	1	1
1	$\frac{1}{8}$	.125
2	$\frac{21}{4240}$	$.4952830189\ldots \cdot 10^{-2}$
3	$\frac{33253}{383465600}$	$.8671703537\ldots \cdot 10^{-4}$
4	$\frac{76118969}{91537621184000}$	$.8315593962\ldots \cdot 10^{-6}$
5	$\frac{20165083798890939}{4103397246999022891520000}$	$.4914241197\ldots \cdot 10^{-8}$
6	$\frac{1297264983892618964 97}{67149822109717176326588672000000}$	$.1931896382\ldots \cdot 10^{-10}$
7	$\frac{241367346879396620182543480936847 1}{45210174990342427454327995801851920608256000000}$	$.5338783735\ldots \cdot 10^{-13}$

Table 5.3: The first coefficients of the MacLaurin expansion of $\text{sinq}(z)$ for the measure geometric triadic Cantor string [5].

The sinq function is thus given in this case by:

$$\begin{aligned} \text{sinq}(z) : &= \sum_{n=0}^{\infty} (-1)^n q_{2n+1} z^{2n+1} \\ &= z - \frac{1}{8} z^3 + \frac{21}{4240} z^5 - \frac{33253}{383465600} z^7 + \cdots . \end{aligned}$$

This function may now be approximated by the characteristic polynomials of the dynamic matrices $\mathbf{D}_j$ as exposed above. In Table 5.4, the results of this approximation procedure are compiled for different iteration levels j, showing an excellent agreement with the exact values.

Furthermore, the convergence behaviour of the Euler partial products and Maclaurin partial sums are depicted in Figure 5.9, illustrating the difference between the two approaches. As expected, the Euler partial products are much better behaving than the Maclaurin expansion in the sense that much less terms

n	q_{2n+1} (from [5])	$r_{2,n}$	$r_{5,n}$
0	1	1	1
1	.125	.12345...	.12499...
2	$.49528\ldots \cdot 10^{-2}$	$.46724\ldots \cdot 10^{-2}$	$.49529\ldots \cdot 10^{-2}$
3	$.86717\ldots \cdot 10^{-4}$	$.72635\ldots \cdot 10^{-4}$	$.86740\ldots \cdot 10^{-4}$
4	$.83155\ldots \cdot 10^{-6}$	$.53335\ldots \cdot 10^{-6}$	$.83222\ldots \cdot 10^{-6}$
5	$.49142\ldots \cdot 10^{-8}$	$.18375\ldots \cdot 10^{-8}$	$.49227\ldots \cdot 10^{-8}$
6	$.19318\ldots \cdot 10^{-10}$	$.23926\ldots \cdot 10^{-11}$	$.19380\ldots \cdot 10^{-10}$

Table 5.4: Approximations for the first coefficients in the expansion of sinq(z) for the measure geometric triadic Cantor string

are needed for an acceptable precision in the determination of the location of the zeroes. However, it seems that the Maclaurin partial products are superior in reproducing the precise location of the zeroes, so that both approaches should be used in a complementary way, see Table 5.5, where the values obtained by the different procedures are compiled for the first 14 eigenvalues.

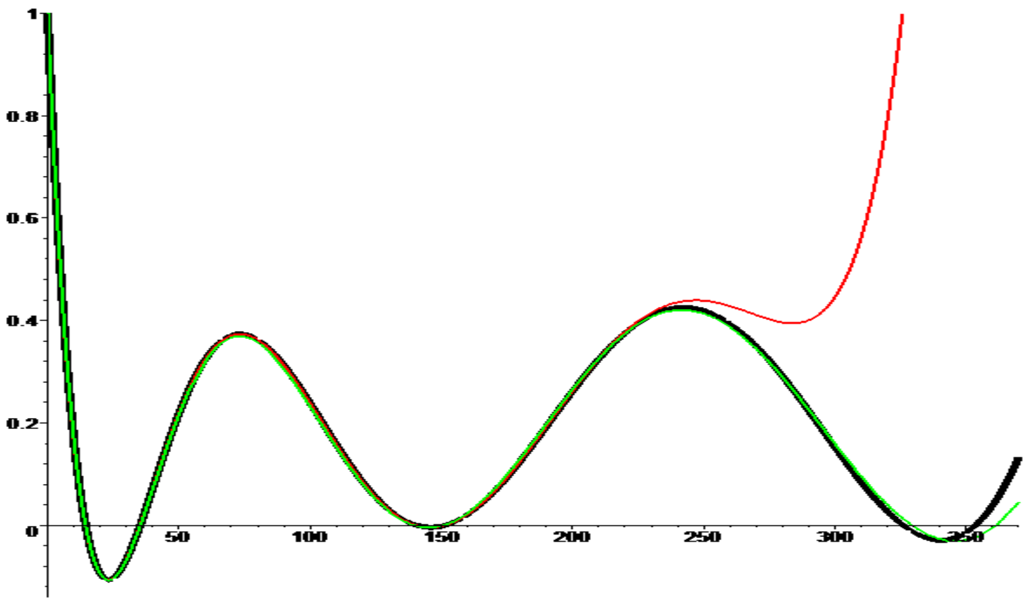

Figure 5.9: Comparison of Euler (green) and Maclaurin (red) expansions with 15 terms of the sinq function for the measure geometric triadic Cantor string with the exact function (black).

i	Euler expansion		Maclaurin expansion		
	15 terms (j=3)	31 terms (j=4)	15 terms	31 terms	λ_i from [5]
1	14.43255178	14.43762586	14.43524052	14.43524052	14.43524051
2	35.35117654	35.28609386	35.26023798	35.26023798	35.26023802
3	139.5766208	140.9112455	140.7409942	140.7810639	140.7810534
4	150.8057878	151.5317597	151.4033955	151.2906053	151.2906161
5	329.1786428	327.7153593	264.7423721	326.0567532	326.0573284
6	361.3572084	355.6811463	-	353.4177464	353.4169208
7	722.6493481	871.1442351	-	-	876.2744596
8	725.8900454	871.4268943	-	-	876.5053185
9	921.5659867	1571.090688	-	-	1581.177024
10	942.8877239	1613.225029	-	-	1619.400729
11	1419.506205	2060.242648	-	-	2029.613563
12	1420.106796	2065.507130	-	-	2033.852813
13	1493.090646	2349.243103	-	-	2268.791634
14	1499.601264	2376.048024	-	-	2289.604069

Table 5.5: Goodness of appoximation for the zeroes of the sinq function.

For sake of completeness, we will also give our bounds on the traces of the dynamic matrices here. In the case of the measure geometric Cantor string with unit length, tension and mass, the traces of the stiffness matrix $\mathbf{K}_j$, respectively its square $\mathbf{K}_j^2$ are given by:

$$\operatorname{tr}(\mathbf{K}_j) = \frac{1}{5}\left(8 \cdot 6^j - 5 \cdot 3^j - 3\right),$$

respectively

$$\operatorname{tr}(\mathbf{K}_j^2) = \frac{1}{170}\left(362 \cdot 18^j - 255 \cdot 9^j - 102 \cdot 3^j - 90\right).$$

Thus

$$\frac{\operatorname{tr}(\mathbf{K}_j^2)}{(\operatorname{tr}(\mathbf{K}_j))^2} \geq \frac{905}{1088} 2^{-j},$$

so that the higher powers of the dynamic matrix are bounded (see Proposition

5.6) from below by:

$$\left(\frac{905}{1088}2^{-j}\right)^{k-1} \leq \left(\frac{\mathrm{tr}(\mathbf{K}_j^2)}{(\mathrm{tr}(\mathbf{K}_j))^2}\right)^{k-1} \leq \frac{\mathrm{tr}(\mathbf{D}_j^k)}{(\mathrm{tr}(\mathbf{D}_j))^k}, \text{ and by:}$$

$$\frac{\mathrm{tr}(\mathbf{D}_j^k)}{(\mathrm{tr}(\mathbf{D}_j))^k} \leq \left(\frac{CN(j)^{\frac{1}{\gamma}}}{\min\left(c_i^{-1}(j)\right)\mathrm{tr}(\mathbf{K}_j)}\right)^{k-1}$$

from above (Corollary 5.7), with $N(j) = 2^{j+1} - 2$ and $\frac{1}{\gamma} = \frac{\ln(6)}{\ln(2)}$. This can be simplified even further as in this case the mass matrix is a scalar matrix with $\min\left(c_i^{-1}(j)\right) = \max\left(c_i^{-1}(j)\right) = 2^{j+2}$:

$$\frac{\mathrm{tr}(\mathbf{D}_j^k)}{(\mathrm{tr}(\mathbf{D}_j))^k} \leq \left(\frac{C\left(2^{j+1}-2\right)^{\frac{\ln(6)}{\ln(2)}}}{2^{j+2}\frac{1}{5}\left(8\cdot 6^j - 5\cdot 3^j - 3\right)}\right)^{k-1} \leq \left(\frac{15C}{16}2^{-j}\right)^{k-1}.$$

5.5.1.2 Example 2: A measure geometric string with two different scaling ratios

For the measure geometric string with two different scaling ratios introduced above, the coefficients q_{2n+1} are summarised in Table 5.6.

n	Decimal value of q_{2n+1}[5]	$r_{2,n}$	$r_{5,n}$
0	1	1	1
1	$.1127708838\ldots$	$.11202\ldots$	$.11277\ldots$
2	$.3996475470\ldots\cdot 10^{-2}$	$.38490\ldots\cdot 10^{-2}$	$.39969\ldots\cdot 10^{-2}$
3	$.5979114624\ldots\cdot 10^{-4}$	$.52741\ldots\cdot 10^{-4}$	$.59812\ldots\cdot 10^{-4}$
4	$.4716361707\ldots\cdot 10^{-6}$	$.33249\ldots\cdot 10^{-6}$	$.47202\ldots\cdot 10^{-6}$
5	$.2228258258\ldots\cdot 10^{-8}$	$.83372\ldots\cdot 10^{-9}$	$.22318\ldots\cdot 10^{-8}$
6	$.6830278639\ldots\cdot 10^{-11}$	$.79994\ldots\cdot 10^{-11}$	$.68494\ldots\cdot 10^{-11}$

Table 5.6: The first coefficients of the MacLaurin expansion of $\mathrm{sinq}(z)$ for the measure geometric string with two different scaling ratios

In this case, the sinq function is thus given by:

$$\begin{aligned}\mathrm{sinq}(z) :&= \sum_{n=0}^{\infty}(-1)^n q_{2n+1} z^{2n+1} \\ &= z - .1127708838z^3 + .399647547 \cdot 10^{-2} z^5 - .5979114624 \cdot 10^{-4} z^7 + \cdots .\end{aligned}$$

Again it is possible to approximate this function by the characteristic polynomials of the dynamic matrices $\mathbf{D}_j$, with the results (see again Table 5.6, columns 3 and 4) in excellent agreement with the exact values. The convergence behaviour of the Euler partial products and Maclaurin partial sums is very similar to that already observed in the case of the triadic Cantor string as can be seen in Figure 5.10. Furthermore, the complementary nature of both approaches is also reflected here in Table 5.7.

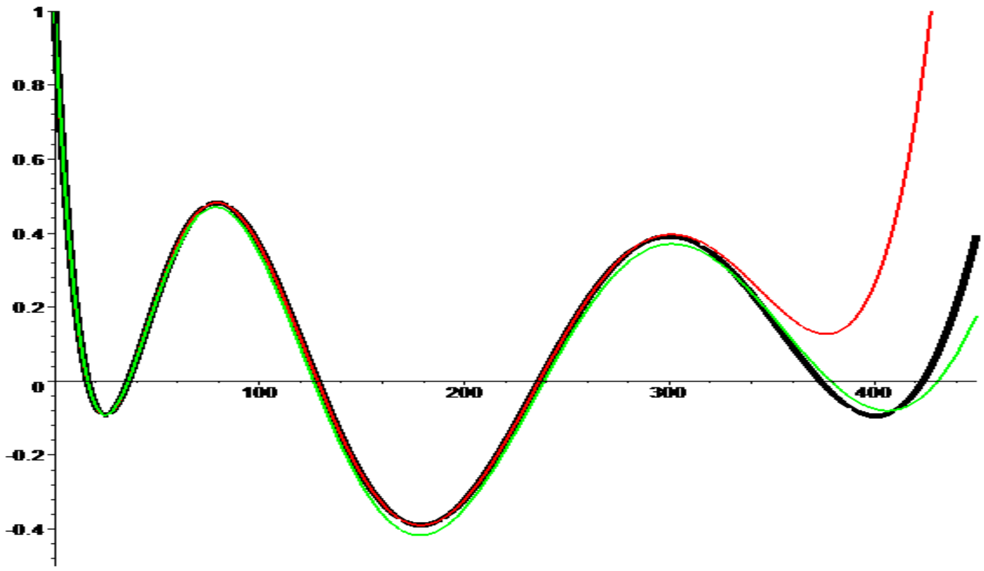

Figure 5.10: Comparison of Euler (green) and Maclaurin (red) expansions with 15 terms of the sinq function for the measure geometric fractal string with two scaling ratios with the exact function (black).

Once more, we state our bounds on the traces of the dynamic matrices here for sake of completeness. For unit length, tension and mass, the traces of the

i	Euler expansion		Maclaurin expansion		
	15 terms (j=3)	31 terms (j=4)	15 terms	31 terms	λ_i from [5]
1	16.10040819	16.10800303	16.10784937	16.10784937	16.10784941
2	36.00493193	35.92811605	35.90760124	35.90760124	35.90760106
3	126.3184029	128.2017000	128.3304467	128.3304456	128.3304475
4	238.7840729	237.2626229	236.4499727	236.4636999	236.4636763
5	378.5291318	375.4260909	-	373.7010994	373.7019294
6	431.1612225	425.8377868	-	423.6396377	423.6381570
7	627.4593717	702.7281198	-	713.7835520	713.7869861
8	1273.909249	2025.528203	-	1916.839156	2013.164883

Table 5.7: Goodness of appoximation for the zeroes of the sinq function.

stiffness matrix $\mathbf{K}_j$, respectively its square $\mathbf{K}_j^2$ are given by:

$$\operatorname{tr}(\mathbf{K}_j) = \frac{1}{10}\left(14 \cdot 7^j - 5 \cdot 4^j - 5 \cdot 3^j - 4\right),$$

respectively

$$\operatorname{tr}(\mathbf{K}_j^2) = \frac{1}{7700}\left(12548 \cdot 25^j - 1320 \cdot 4^j - 1680 \cdot 3^j - 5775 \cdot 9^j - 5775 \cdot 16^j - 1848\right).$$

Thus

$$\frac{\operatorname{tr}(\mathbf{K}_j^2)}{(\operatorname{tr}(\mathbf{K}_j))^2} \geq \frac{3773}{3137}\left(\frac{49}{25}\right)^{-j}.$$

Furthermore, as:

$$\min\left(c_i^{-1}(j)\right) = (3^\gamma)^j,$$

and

$$\max\left(c_i^{-1}(j)\right) = (4^\gamma)^j,$$

with $\gamma = \frac{d}{d+1} \approx .3591792841$, and $d \approx .5604988652$ being the solution of the Moran equation $\left(\frac{1}{4}\right)^d + \left(\frac{1}{3}\right)^d = 1$. Hence, we obtain the following upper and lower bounds for the powers of the traces by Proposition (5.6) and Corollary (5.7):

$$\left(\left(\frac{3}{4}\right)^{\gamma j}\right)^k \left(\frac{3773}{3137}\left(\frac{49}{25}\right)^{-j}\right)^{k-1} \leq \frac{\operatorname{tr}(\mathbf{D}_j^k)}{(\operatorname{tr}(\mathbf{D}_j))^k} \leq \left(\frac{10C\left(2^{2j+1}-2\right)^{\frac{1}{\gamma}}}{3^{\gamma j}\left(14 \cdot 7^j - 5 \cdot 4^j - 5 \cdot 3^j - 4\right)}\right)^{k-1}.$$

Through our investigations, we have thus shown the complementarity of our method to the one used in [5], both approaches being inherently different but leading to the same results and having their advantages and disadvantages depending on goal and situation.

Chapter 6

Conclusion and Outlook

In this thesis, we offer an investigation of discrete respectively finite systems through the study of the moments of the eigenvalue distribution for fractal chains:

- We introduce a matrix representation of the related Laplacians, thereby suggesting links to random matrix and graph theory.
- Exact results as well as lower and upper bounds for the moments of the eigenvalue distribution are obtained for the chains under consideration, as well as a new criterion for Minkowski-measurability.
- Further extensions are then made to fractal measure geometric Laplacians in the one-dimensional case, where we show the usefulness of the methods and techniques developed.
- Euler-expansions of generalised trigonometric function whose squared zeroes are the eigenvalues of the corresponding measure geometric Laplacian are approximated.
- The most unexpected result of this work is the exposition of an important and fascinating relation between the two, at first glance very different, types of fractal objects studied; the first class being quite simple structures with a fractal boundary, the second class having an internal fractal structure but very simple boundaries (see Figure 6.1). This discovery clearly proves the efficiency of using the techniques originally taken from random matrix theory in the area of fractal geometry as a unifying framework.

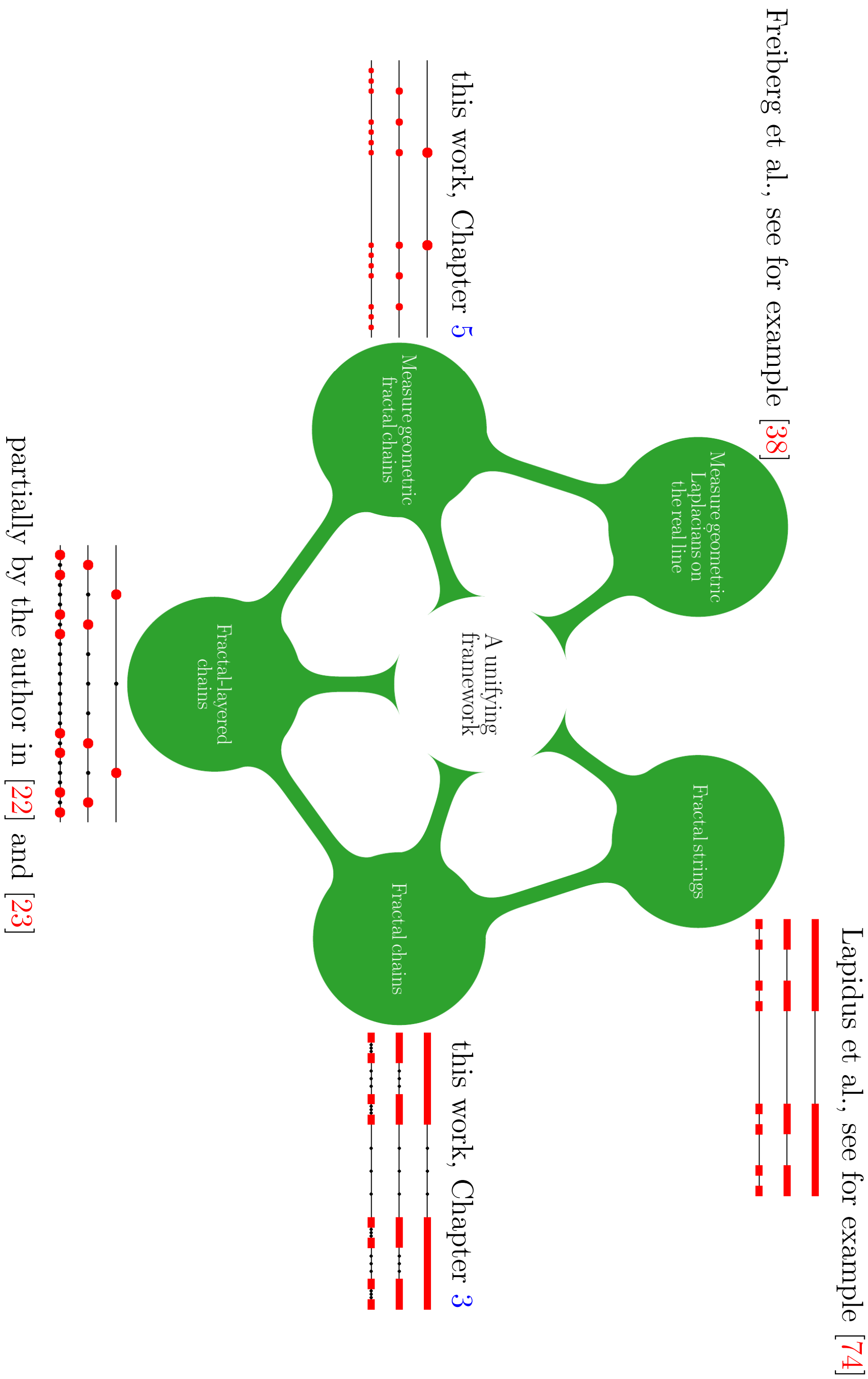

Figure 6.1: The unifying framework

Originally unforeseen, our approach revealed itself to be extremely fertile. Thus, in the course of its writing, this thesis has become not a mere statement or presentation of results obtained but has evolved into a draft programme for future research, opening up new ways certainly worth exploring.

We will close this chapter with a non-exhaustive list of thoughts and questions of interest arising from within this work.

- What sense can be given to the moments of fractal chains without cut-off in Section 3.2.2? Table 3.3 suggests that the moments of Minkowski-measurable chains decrease at a much faster pace than those of generalised Cantor chains. Other types of chains, of Minkowski-measurable and not Minkowski-measurable type should be investigated, maybe generalising this observation.

- The connection between the moments of the eigenvalue distribution and oscillations in the spectrum is not absolutely clear yet. In this context it appears interesting to find lower bounds for the "Berry-term" in the moments of Minkowski-measurable chains.

- Are the differences between the spectral exponents γ_{exp} and γ_{th} in the asymptotics of the eigenvalue counting function (Section 5.4) of measure geometric chains an artefact due to the discretisation or are they indicative of contributions of higher order terms?

- What lessons could be learned from an investigation of adjacent neighbour or n^{th}-nearest neigbour spacings of the eigenvalues of measure geometric chains?

- Is it possible to extend the approach used in the appendix to more complex chains such as chains with multiple scaling ratios, as we only covered the case of μ being the homogeneous middle third Cantor measure, and can the bounds on the traces somehow be improved?

- What is the meaning of the growth factor c for the powers of traces suggested by the results compiled in the appendix? Would it be possible to estimate its value through numerical experience, thereby potentially allowing us to find approximations to the sinq-function, or conversely, could it be possible to recover a power law for the traces from just the knowledge of the first coefficients in the sinq-function? In this context, an investigation of the asymptotic behaviour of the Newton-Girard coefficients might simplify the task.

- In view of the approximation of the sinq-function by the characteristic polynomials of the matrix Laplacians, it would certainly be worth studying the spectral asymptotics of measure geometric strings corresponding to distributions with known moments. Furthermore, is it possible to deduce the spectrum of random strings from the knowledge of their statistical parameters only?

- An in-depth study of fractal-layered chains (see References [22, 23]) should also be addressed in pursuit of a better understanding of the links (see Figure 6.1) between the different types of fractal strings.

- Is it possible within this framework to establish a direct link between the two main types of fractal strings, thereby also elucidating the connection between the arithmetic/non-arithmetic and Minkowski measurable/non-Minkowski measurable dichotomies?

- It would be interesting to apply respectively transfer the techniques and methods developed here for the one-dimensional case to higher dimensional settings.

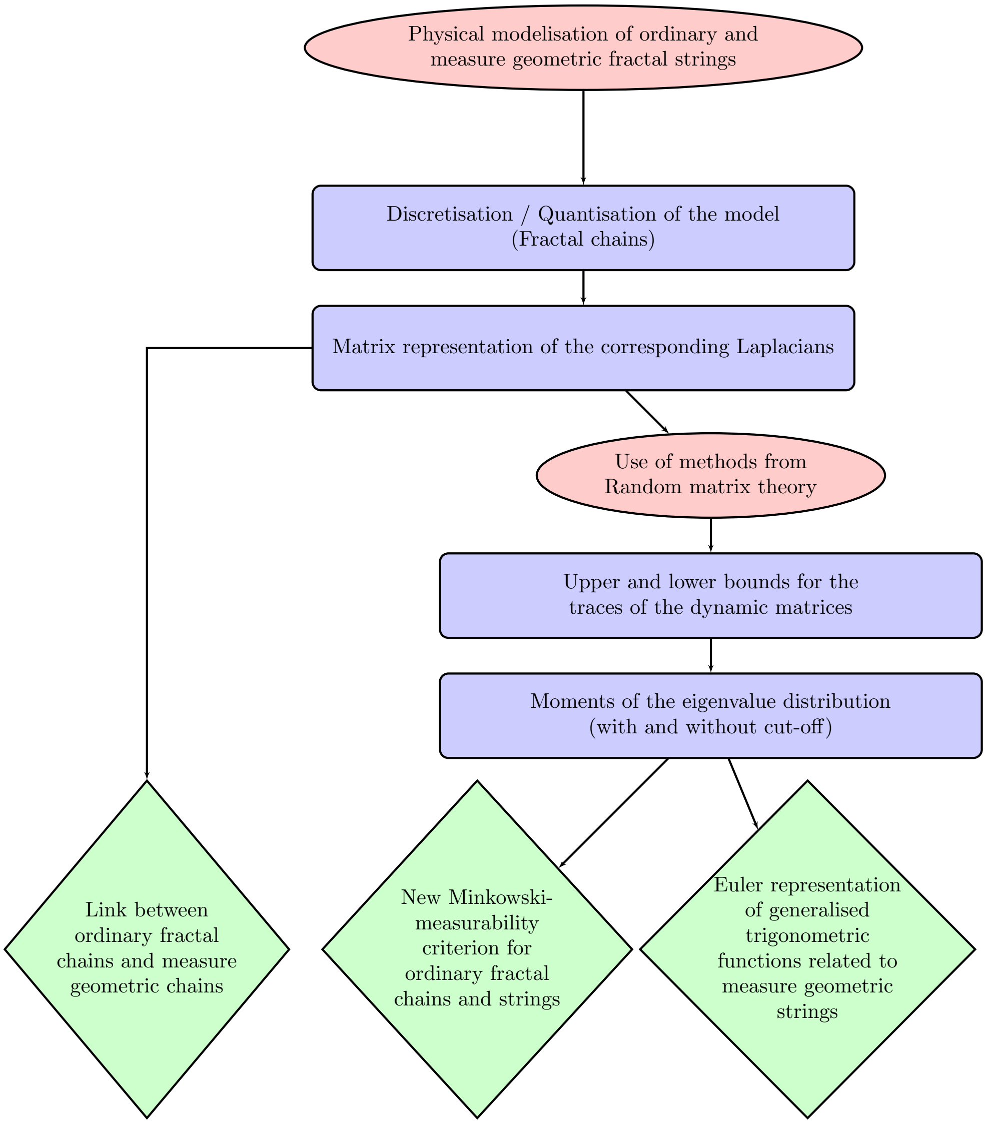

Figure 6.2: Explanatory chart of methods and results of this thesis; ellipses: strategies, rectangles: methods, diamonds: results.

Appendix A: A different approach for the traces of the measure geometric Cantor chain

This alternative approach for bounding the traces of the measure geometric Cantor chain is inspired by [16].

The matrix Laplacian

In the case of the Cantor measure, the dynamic matrix (the Laplacian in matrix form) at approximation level j is given by

$$\mathbf{D}_j = \mathbf{M}_j^{-\frac{1}{2}} \mathbf{K}_j \mathbf{M}_j^{-\frac{1}{2}},$$

where $\mathbf{M}_j$ is a multiple of the identity matrix and $\mathbf{K}_j$ is given by:

$$\mathbf{K}_j = 3^j \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \frac{4}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{4}{3} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \frac{10}{9} & -\frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{9} & \frac{10}{9} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{4}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{4}{3} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

The stiffness matrix $\mathbf{K}_j$ can then be decomposed in two more accessible matrices:

$$
\begin{aligned}
\mathbf{K}_j &:= 3^j \left(\mathbf{K}_{main,j} + \mathbf{E}_j\right) \\
&:= 3^j \left(\begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & -\frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{9} & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \right) \\
&:= 3^j \left(\begin{bmatrix}
\mathbf{K}'_{base} & 0 & 0 & 0 & 0 \\
0 & \mathbf{K}_{base} & 0 & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & 0 & \mathbf{K}_{base} & 0 \\
0 & 0 & 0 & 0 & \mathbf{K}''_{base}
\end{bmatrix} + \mathbf{E}_j \right),
\end{aligned}
$$

with

$$
\mathbf{K}'_{base} := \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \; \mathbf{K}_{base} := \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \text{ and } \mathbf{K}''_{base} := \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.
$$

The traces of powers of the stiffness matrix

Both matrices $\mathbf{K}_{main,j}$ and $\mathbf{E}_j$ are positive definite, so that by Raleigh's principle [95], we have:

$$\operatorname{tr}(\mathbf{K}_j^k) \geq \operatorname{tr}(3^j \mathbf{K}_{main,j}^k) = 3^j \operatorname{tr}(\mathbf{K}_{main,j}^k). \tag{1}$$

Furthermore, by a result of J. Magnus and H. Neudecker [85]:

$$\operatorname{tr}(\mathbf{K}_j^k) = 3^j \operatorname{tr}\left((\mathbf{K}_{main,j} + \mathbf{E}_j)^k\right) \leq 3^j \left(\operatorname{tr}(\mathbf{K}_{main,j}^k)^{\frac{1}{k}} + \operatorname{tr}(\mathbf{E}_j^k)^{\frac{1}{k}}\right)^k. \tag{2}$$

Now, by simple induction, the trace of $\mathbf{E}_j$ is given by:

$$\mathrm{tr}(\mathbf{E}_j^k) = \frac{2^k}{3^k}\frac{(2\cdot 3^k)^j - 1}{(2\cdot 3^k - 1)3^{k(j-1)}} \leq \frac{2^k}{3^k}2^j.$$

The apparent self-similarity of the matrices $\mathbf{K}_{main,j}$ can then be exploited to obtain the traces of their powers, as they can be expressed through the constituting submatrices $\mathbf{K'}_{base}$, $\mathbf{K''}_{base}$ and $\mathbf{K}_{base}$ defined above. Indeed:

- $\mathrm{tr}(\mathbf{K'}^k_{base}) = \mathrm{tr}(\mathbf{K''}^k_{base})$.
- The eigenvalues of $\mathbf{K'}_{base}$, respectively $\mathbf{K''}_{base}$ are given by $S = 4\cos^2(\frac{\pi}{7})$, $\frac{2S-3}{S-1}$ and $\frac{S-3}{S-2}$.
- The eigenvalues of $\mathbf{K}_{base}$ are given by $4\cos^2(\frac{\pi}{8}) = 2+\sqrt{2}$, 2, $4\sin^2(\frac{\pi}{8}) = 2-\sqrt{2}$ and 0.
- Each of the matrices $\mathbf{K}_{main,j}$ consists of one submatrix $\mathbf{K'}_{base}$, one submatrix $\mathbf{K''}_{base}$ and $2^{j-1}-2$ submatrices $\mathbf{K}_{base}$.

Proposition 1. The trace of $\mathbf{K}^k_{main,j}$ is bounded by:

$$2^{j-1}S^k\left(1+\frac{328^k+41^k}{687^k}\right) \leq \mathrm{tr}\left(\mathbf{K}^k_{main,j}\right),$$

from below, and by:

$$\mathrm{tr}\left(\mathbf{K}^k_{main,j}\right) \leq 2^{j-1}\left(2+\sqrt{(2)}\right)^k\left(1+\frac{577^k+169^k}{985^k}\right)$$

from above, where $S = 4\cos\left(\frac{\pi}{7}\right)^2$ is the silver constant.

Proof. The traces of $\mathbf{K'}^k_{base}$ and $\mathbf{K''}^k_{base}$ are given by the sums of their eigenvalues:

$$\begin{aligned}\operatorname{tr}(\mathbf{K'}^k_{base}) = \operatorname{tr}(\mathbf{K''}^k_{base}) &= S^k + \left(\frac{2S-3}{S-1}\right)^k + \left(\frac{S-3}{S-2}\right)^k \\ &= S^k \left(1 + \left(\frac{2S-3}{S(S-1)}\right)^k + \left(\frac{S-3}{S(S-2)}\right)^k\right).\end{aligned}$$

In order to obtain upper and lower bounds for this expression, we use the continued fraction expansion of $\frac{2S-3}{S(S-1)}$ and use for example:

$$\frac{328}{687} \leq \frac{2S-3}{S(S-1)} \leq \frac{329}{687}.$$

Using the same denominator, we have furthermore:

$$\frac{41}{687} \leq \frac{S-3}{S(S-2)} \leq \frac{42}{687},$$

so that

$$S^k \left(1 + \frac{328^k + 41^k}{687^k}\right) \leq \operatorname{tr}(\mathbf{K'}^k_{base}) = \operatorname{tr}(\mathbf{K''}^k_{base}) \leq S^k \left(1 + \frac{329^k + 42^k}{687^k}\right).$$

Using the same approach for $\mathbf{K}^k_{base}$ leads to:

$$\left(2+\sqrt{2}\right)^k \left(1 + \frac{576^k + 168^k}{985^k}\right) \leq \operatorname{tr}(\mathbf{K}^k_{base}) \leq \left(2+\sqrt{2}\right)^k \left(1 + \frac{577^k + 169^k}{985^k}\right).$$

Finally, by its block-diagonal structure, the trace of $\mathbf{K}^k_{main,j}$ is the sum of the traces of its submatrices $\operatorname{tr}(\mathbf{K}^k_{main,j}) = 2\operatorname{tr}(\mathbf{K'}^k_{base}) + (2^{j-1} - 2)\operatorname{tr}(\mathbf{K}^k_{base})$, with

$\mathrm{tr}(\mathbf{K'}^k_{base}) \leq \mathrm{tr}(\mathbf{K}^k_{base})$ and thus, using the bounds given above:

$$2^{j-1}S^k\left(1+\frac{328^k+41^k}{687^k}\right) \leq \mathrm{tr}\left(\mathbf{K}^k_{main,j}\right) \leq 2^{j-1}\left(2+\sqrt{(2)}\right)^k\left(1+\frac{577^k+169^k}{985^k}\right),$$

which concludes the proof. □

Note that these bounds are not the best possible, but sufficient here.

Proposition 2.

$$3^j\,\mathrm{tr}\left(\mathbf{K}^k_{main,j}\right) \leq \mathrm{tr}\left(\mathbf{K}^k_j\right) \leq 3^j\,\mathrm{tr}\left(\mathbf{K}^k_{main,j}\right)\left(1+\left(\frac{\mathrm{tr}\left(\mathbf{E}^k_j\right)}{\mathrm{tr}\left(\mathbf{K}^k_{main,j}\right)}\right)^{\frac{1}{k}}\right)^k,$$

Proof. Using Equations 1 and 2 above, the assertion follows immediately. □

Thus, the following corollary holds:

Corollary 3.

$$3^j2^{j-1}S^k\left(1+\frac{328^k+41^k}{687^k}\right) \leq \mathrm{tr}\left(\mathbf{K}^k_j\right)$$

and

$$\mathrm{tr}\left(\mathbf{K}^k_j\right) \leq 3^j2^{j-1}\left(2+\sqrt{(2)}\right)^k\left(1+\frac{2}{3S}2^{\frac{1}{k}}\right)^k\left(1+\frac{577^k+169^k}{985^k}\right).$$

Proof. Using the facts that $\mathrm{tr}(\mathbf{E}^k_j) \leq \frac{2^k}{3^k}2^j$ and $\mathrm{tr}\left(\mathbf{K}^k_{main,j}\right) \geq 2^{j-1}S^k\left(1+\frac{328^k+41^k}{687^k}\right)$, we have

$$\left(\frac{\mathrm{tr}\left(\mathbf{E}^k_j\right)}{\mathrm{tr}\left(\mathbf{K}^k_{main,j}\right)}\right)^{\frac{1}{k}} \leq \left(\frac{\frac{2^k}{3^k}2^j}{2^{j-1}S^k\left(1+\frac{328^k+41^k}{687^k}\right)}\right)^{\frac{1}{k}} \leq \frac{2}{3S}2^{\frac{1}{k}}$$

and the statement follows immediately from the proposition above. □

Finally, this allows us to formulate the following theorem:

Theorem 4. As the trace of $\mathbf{K}_j^k$ is bounded by:

$$2^{j-1}c_1^k\left(1+\frac{328^k+41^k}{687^k}\right) \leq \operatorname{tr}\left(\mathbf{K}_j^k\right) \leq 2^{j-1}c_2^k\left(1+\frac{577^k+169^k}{985^k}\right),$$

there exists $c \in [c_1, c_2]$, such that for all $\varepsilon > 0$, we have:

$$\lim_{k\to\infty} \operatorname{tr}\left(\mathbf{K}_j^k\right)(c-\varepsilon)^k = \infty, \text{ and}$$

$$\lim_{k\to\infty} \operatorname{tr}\left(\mathbf{K}_j^k\right)(c+\varepsilon)^k = 0.$$

Proof. As $\operatorname{tr}\left(\mathbf{K}_j^k\right)$ is bounded by $2^{j-1}c_1^k\left(1+\frac{328^k+41^k}{687^k}\right)$ from below, we have either

$$\lim_{k\to\infty} \frac{\operatorname{tr}\left(\mathbf{K}_j^k\right)}{2^{j-1}c_1^k\left(1+\frac{328^k+41^k}{687^k}\right)} = a,$$

for some finite $a \geq 0$, in which case we set $c = c_1$, or

$$\lim_{k\to\infty} \frac{\operatorname{tr}\left(\mathbf{K}_j^k\right)}{2^{j-1}c_1^k\left(1+\frac{328^k+41^k}{687^k}\right)} = \infty.$$

In this case, consider the fact that $\operatorname{tr}\left(\mathbf{K}_j^k\right)$ is bounded by $2^{j-1}c_1^k\left(1+\frac{577^k+169^k}{985^k}\right)$ from above. Then we either have:

$$\lim_{k\to\infty} \frac{\operatorname{tr}\left(\mathbf{K}_j^k\right)}{2^{j-1}c_2^k\left(1+\frac{577^k+169^k}{985^k}\right)} = b,$$

for some finite $b > 0$, in which case we set $c = c_2$, or

$$\lim_{k\to\infty} \frac{\operatorname{tr}\left(\mathbf{K}_j^k\right)}{2^{j-1}c_2^k\left(1+\frac{577^k+169^k}{985^k}\right)} = 0.$$

As we have in this case simultaneously

$$\lim_{k\to\infty} \frac{\operatorname{tr}\left(\mathbf{K}_j^k\right)}{2^{j-1}c_1^k\left(1+\frac{328^k+41^k}{687^k}\right)} = \infty$$

and

$$\lim_{k\to\infty} \frac{\operatorname{tr}\left(\mathbf{K}_j^k\right)}{2^{j-1}c_2^k\left(1+\frac{577^k+169^k}{985^k}\right)} = 0,$$

there exists a unique point $c_1 < c < c_2$, where the value of $\lim_{k\to\infty} \frac{\operatorname{tr}\left(\mathbf{K}_j^k\right)}{c^k}$ jumps from ∞ to zero, which concludes the proof. $\square$

Appendix B: The moments of the triadic Cantor chain with and without cut-off: a comparison

We present here a comparison between the moments of the triadic Cantor chain before and after introducing a cut-off. Recall that a cut-off eigenvalue had to be introduced in order to allow a direct connection between Cantor chains and strings. In Section 3.2.2.1, we obtained the general expression for the moments of generalised Cantor chains as:

$$\mathcal{M}_{N,k} = \frac{\left(b^{-2(\mathfrak{m}+1)}\right)^k \frac{b^{2k}\left(\left(ab^{2k}\right)^{\mathfrak{m}+1}-1\right)}{ab^{2k}-1} \left(4\frac{K}{m}\right)^k \left(\binom{2k}{k}\frac{N+1}{2^{2k}} + \frac{1}{2}\right)}{\left(\frac{a^{\mathfrak{m}+1}-1}{a-1}(N+1)-1\right) \cdot 2^{2k}}.$$

Furthermore, in Section 3.2.3.1, we deduced the upper bound:

$$\operatorname{ptr}\left(\mathbf{D}^k\right) \leq \left(4\frac{K}{m}\right)^k \left(\frac{N+1}{\pi}\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\frac{\left(\frac{a}{b}\right)^{\mathfrak{m}+1}-1}{a-b} + \frac{a^{\mathfrak{m}+1}-1}{2(a-1)} + \frac{k\pi}{12(N+1)}\frac{b\left((ab)^{\mathfrak{m}+1}-1\right)}{ab-1}\right)$$

for the pseudo-traces related to generalised Cantor chains. In the case of the triadic Cantor chain, $a = 2$ and $b = 3$, so that these expressions simplify to:

$$\mathcal{M}_{N,k} = \left(4\frac{K}{m}\right)^k \frac{\left(2^{\mathfrak{m}+1}3^{2k} - (3^{-2m})^k\right)\left(\binom{2k}{k}\frac{N+1}{2^{2k}} + \frac{1}{2}\right)}{(2 \cdot 3^{2k} - 1)\left((2^{\mathfrak{m}+1} - 1)(N+1) - 1\right) \cdot 2^{2k}},$$

and

$$\begin{aligned}\operatorname{ptr}\left(\mathbf{D}^k\right) \leq \left(4\frac{K}{m}\right)^k \Bigg(& \frac{N+1}{\pi}\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})\left(1-\left(\frac{2}{3}\right)^{\mathfrak{m}+1}\right)}{\Gamma(k+1)} \\ & + 2^{\mathfrak{m}} - 1 + \frac{k\pi\left(6^{\mathfrak{m}+1}-1\right)}{20(N+1)}\Bigg)\end{aligned}$$

respectively. Now, normalising the pseudo-traces in the same manner as in Section 3.2.2, we obtain an upper bound for the moments of the Cantor chain with cut-off:

$$\begin{aligned}\mathcal{M}'_{N,k} &\leq \frac{\operatorname{ptr}\left(\mathbf{D}^k\right)}{2^{2k+1}(N+1)} \\ &= \frac{1}{2^{2k+1}(N+1)}\left(4\frac{K}{m}\right)^k \Bigg(\frac{N+1}{\pi}\frac{\sqrt{\pi}\Gamma(k+\frac{1}{2})\left(1-\left(\frac{2}{3}\right)^{\mathfrak{m}+1}\right)}{\Gamma(k+1)} \\ &+ 2^{\mathfrak{m}} - 1 + \frac{k\pi\left(6^{\mathfrak{m}+1}-1\right)}{20(N+1)}\Bigg) \\ &= \frac{1}{2^{2k+1}}\left(4\frac{K}{m}\right)^k \left(\frac{\Gamma(k+\frac{1}{2})\left(1-\left(\frac{2}{3}\right)^{\mathfrak{m}+1}\right)}{\sqrt{\pi}\Gamma(k+1)} + \frac{2^{\mathfrak{m}}-1}{N+1} + \frac{k\pi\left(6^{\mathfrak{m}+1}-1\right)}{20(N+1)^2}\right).\end{aligned}$$

Using the fact that (Equation 3.2):

$$N = \left\lceil \frac{\pi}{2} \cdot \frac{1}{\arcsin\left(3^{-(\mathfrak{m}+1)}\right)} \right\rceil,$$

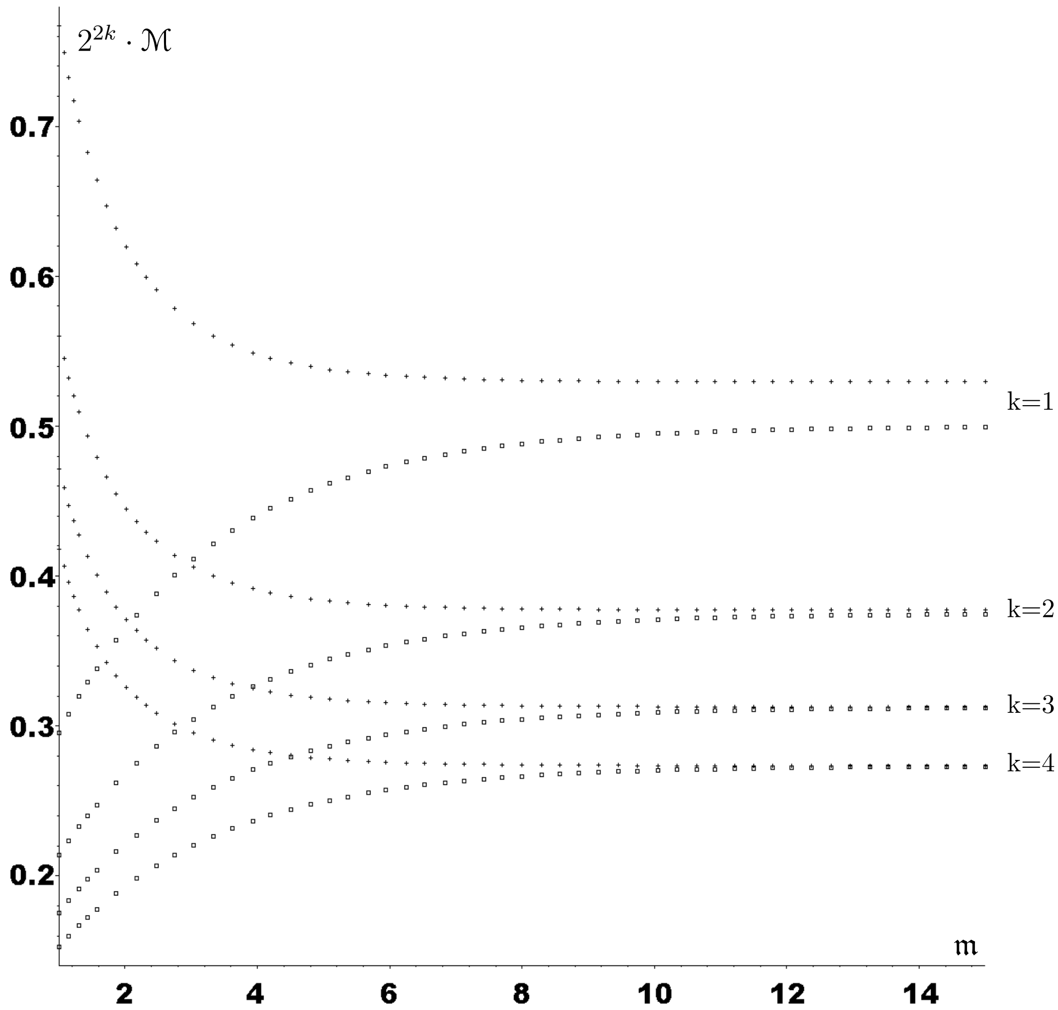

Figure A.1: Behaviour of $\mathcal{M}_{N,k}$ (crosses) and $\mathcal{M}'_{N,k}$ (boxes) as a function of the iteration level $\mathfrak{m}$ for different values of k.

and noting that $\mathfrak{m} \to \infty \Rightarrow N \to \infty$, it is then easy to calculate the limits:

$$\lim_{\mathfrak{m}\to\infty} \mathcal{M}_{N,k} = \left(4\frac{K}{m}\right)^k \frac{3^{2k}}{2 \cdot 3^{2k} - 1} \binom{2k}{k} \frac{1}{2^{4k}}$$

and

$$\begin{aligned}
\lim_{\mathfrak{m}\to\infty} \mathcal{M}'_{N,k} &= \left(4\frac{K}{m}\right)^k \frac{1}{2} \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)} \frac{1}{2^{2k}} \\
&= \left(4\frac{K}{m}\right)^k \frac{1}{2} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{2^{2k}} \\
&= \left(4\frac{K}{m}\right)^k \frac{1}{2} \binom{2k}{k} \frac{1}{2^{4k}},
\end{aligned}$$

where we used the well known fact that $\frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+1)} = \frac{\binom{2k}{k}}{2^{2k}}$. Comparing the limits above for $\mathcal{M}_{N,k}$ and $\mathcal{M}'_{N,k}$, it becomes clear that they are highly similar (see also Figure A.1). Thus, we conjecture that the moments of the Cantor string are given by the above limits, up to some factor depending on k:

$$\mathcal{M}_{\mathcal{CS}_T} = C(k) \left(4\frac{K}{m}\right)^k \binom{2k}{k} \frac{1}{2^{4k}},$$

with $\frac{1}{2} \leq C(k) \leq 1$. However, we must note that the information on the second term in the asymptotic expansion of the eigenvalue counting function, contained in the unnormalised pseudo traces, is lost in the process of passing to the limit.

References

[1] Edwin P. Adams and Richard L. Hippisley. Smithsonian Mathematical Formulae and Tables of Elliptic Functions. volume 73, No. 1 of *Smithsonian Miscellaneous Collections*. Smithsonian Institution, Washington, 1922. 50

[2] Jean-Baptiste le Rond d' Alembert. Recherches sur la courbe que forme une corde tenduë mise en vibration. *Histoire de l'Académie Royale des Sciences et Belles Lettres de Berlin*, 3:214–219, 1747. 3

[3] Jean-Baptiste le Rond d' Alembert. Suite des recherches sur la courbe que forme une corde tenduë mise en vibration. *Histoire de l'Académie Royale des Sciences et Belles Lettres de Berlin*, 3:220–249, 1747.

[4] Jean-Baptiste le Rond d' Alembert. Addition au mémoire sur la courbe que forme une corde tenduë mise en vibration. *Histoire de l'Académie Royale des Sciences et Belles Lettres de Berlin*, 6:355–360, 1750. 3

[5] Peter Arzt. *Eigenvalues of Measure Theoretic Laplacians on Cantor-like Sets*. Phd thesis, University of Siegen, 2014. ix, 7, 77, 94, 98, 99, 100, 101, 103, 104, 141

[6] Ernest W. Barnes. A New Development of the Theory of the Hypergeo-

metric Functions. *Proceedings of the London Mathematical Society, Second Series*, 6:141–177, 1908. 45

[7] Michael F. Barnsley and Stephen G. Demko. Iterated function systems and the global construction of fractals. *Proceedings of the Royal Society London*, A 399(1817):243–275, 1985. 9

[8] Daniel Bernoulli. Réflexions et éclaircissemens sur les nouvelles vibrations des cordes exposées dans les mémoires de l'Académie de 1747 & 1748. *Histoire de l'Académie Royale des Sciences et Belles Lettres de Berlin*, 9:147–172, 1753. 3

[9] Daniel Bernoulli. Sur le mélange de plusieurs espèces de vibrations simples isochrones, qui peuvent coexister dans un même système de corps. *Histoire de l'Académie Royale des Sciences et Belles Lettres de Berlin*, 9:173–195, 1753. 3

[10] Daniel Bernoulli. Lettre de Monsieur Daniel Bernoulli de l'Académie Royale des Sciences, à M. Clairaut de la même académie, au sujet des nouvelles découvertes faites sur les vibrations des cordes tendues. *Le Journal des Sçavans*, mars 1758:157–166, 1758. 6

[11] Daniel Bernoulli. De vibrationibus chordarum, ex duabus partibus tam longitudine quam crassitie ab invicem diversis compositarum. *Novii Commentarii Academiae Scientiarum Imperialis Petropolitanae*, 16:257–280, 1771. 6

[12] John Bernoulli. Theoremata selecta pro sonservatione virium vivarum

demonstranda et experimentis confirmanda. *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, II:200–207, 1727. 2

[13] Michael V. Berry. Distribution of Modes in Fractal Resonators. In *Structural Stability in Physics (Proc. Internat. Symposia Appl. Catastrophe Theory and Topological Concepts in Phys., Inst. Inform. Sci., Univ. Tübingen, Tübingen,1978)*, volume 4 of *Springer Ser. Synergetics*, pages 51–53. Springer, Berlin, 1979. 5

[14] Michael V. Berry. Some Geometric Aspects of Wave Motion: Wavefront Dislocations, Diffraction Catastrophes, Diffractals. In *Geometry of the Laplace Operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii,1979)*, volume 36, page 1328. Amer. Math. Soc., Providence, R.I., 1980. 5

[15] Josef Betten. *Finite Elemente für Ingenieure 1: Grundlagen, Matrixmethoden, Elastisches Kontinuum*. Springer, Berlin, Heidelberg, 1997. 28

[16] Erik J. Bird, Sze-Man Ngai, and Alexander Teplyaev. Fractal Laplacians on the unit interval. *Ann. Sci. Math. Québec*, 27:135–168, 2003. 7, 74, 110

[17] Jean Brossard and René Carmona. Can one hear the dimension of a fractal? *Commun. Math. Phys.*, 104:103–122, 1986. 5

[18] John T. Cannon and Sigalia Dostrovsky. *The Evolution of Dynamics: Vibration Theory from 1687 to 1742*, volume 6 of *Studies in the History of Mathematics and Physical Sciences*. Springer-Verlag, New York, Heidelberg, Berlin, 1981. 3

[19] Jie Chen and Sze-Man Ngai. Eigenvalues and eigenfunctions of one-dimensional fractal Laplacians defined by iterated function systems with overlaps. *J. Math. Anal. Appl.*, 364:222–241, 2010. 74

[20] Ingemar Düring. *Ptolemaios und Porphyrios über die Musik*. Georg Olms Verlag, Hildesheim, Zürich, New York, 1987. 1

[21] Roland J. Etienne. Some numerical results for the spectrum of monoatomic Cantor chains. 2^{nd} Conference on Analysis and Probability on Fractals, Cornell University, Ithaca NY, 2005. 25

[22] Roland J. Etienne. Numerical investigations in the vibrational spectrum of a Cantor-layered chain. In *Physics, Chemistry and Application of Nanostructures, Proceedings of the International Conference Nanomeeting-2007*, pages 117–120. World Scientific, Singapore, 2007. 72, 106, 108

[23] Roland J. Etienne. Numerical investigations in the vibrational spectrum of a Cantor-layered chain II. In *Physics, Chemistry and Application of Nanostructures, Proceedings of the International Conference Nanomeeting-2009*, pages 44–47. World Scientific, Singapore, 2009. 72, 106, 108

[24] Leonhard Euler. Sur la vibration des cordes. *Histoire de l'Académie Royale des Sciences et Belles Lettres de Berlin*, 4:69–85, 1748. 3

[25] Leonhard Euler. De vibratione chordarum exercitatio. *Nova Acta Eruditorum*, pages 512–527, 1749.

[26] Leonhard Euler. De motu corporum flexibilium. *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, 14:182–197, 1751.

[27] Leonhard Euler. Remarques sur les mémoires précedens de M. Bernoulli. *Histoire de l'Académie Royale des Sciences et Belles Lettres de Berlin*, 9:196–222, 1753. 3

[28] Leonhard Euler. De motu vibratorio cordarum inaequaliter crassarum. *Novi Commentarii academiae scientiarum Imperialis Petropolitanae*, 9:246–304, 1764. 6

[29] Leonhard Euler. Sur le mouvement d'une corde qui au commencement n'a été ébranlée que dans une partie. *Histoire de l'Académie Royale des Sciences et Belles Lettres de Berlin*, 21:307–334, 1765. 3

[30] Leonhard Euler. De motu vibratorio chordarum ex partibus quotcunque diversae crassitiei compositarum. *Novi Commentarii academiae scientiarum Imperialis Petropolitanae*, 17:422–431, 1773. 6

[31] Kenneth J. Falconer. *Fractal Geometry : Mathematical Foundations and Applications*. John Wiley & Sons, Chichester, New York, Weinheim, Brisbane, Singapore, Toronto, 1990. 8

[32] Kenneth J. Falconer. On the Minkowski measurability of fractals. *Proceedings of the American Mathematical Society*, 123(4):1115–1124, 1995. 5

[33] Kenneth J. Falconer. *Techniques in Fractal Geometry*. John Wiley & Sons, Chichester, New York, Weinheim, Brisbane, Singapore, Toronto, 1997. 8, 17

[34] William Feller. Generalized second order differential operators and their lateral conditions. *Illinois J. Math.*, 1:459–504, 1957. 74

[35] Jacqueline Fleckinger and Dmitri G. Vasil'ev. Tambour fractal: exemple d'une formule asymptotique à deux termes pour la fonction de comptage. *Comptes Rendus de l'Académie des sciences Paris*, 311, série I:867–872, 1990. 5

[36] Uta R. Freiberg. *Maßgeometrische Laplaceoperatoren für fraktale Teilmengen der reellen Achse*. Phd thesis, Friedrich-Schiller-Universität Jena, 2000. 7

[37] Uta R. Freiberg. Analytical properties of measure geometric Krein-Feller-operators on the real line. *Math. Nachr.*, 260:34–47, 2003. 74

[38] Uta R. Freiberg. A survey on measure geometric Laplacians on Cantor like sets. *Arab. J. Sci. Eng.*, 28(1C):189–198, 2003. 7, 74, 106

[39] Uta R. Freiberg. Dirichlet forms on fractal subsets of the real line. *Real Anal. Exchange*, 30(2):589–603, 2004/5. 7, 74

[40] Uta R. Freiberg. Spectral asymptotics of generalized measure geometric Laplacians on Cantor like sets. *Forum Math.*, 17:87–104, 2005.

[41] Uta R. Freiberg. Prüfer angle methods in spectral analysis of Krein-Feller-operators. In *Proceedings of RIMS Workshop on Stochastic Analysis and Applications*, volume B6 of *RIMS Kôkyûroku Bessatsu*, pages 74–81. Research Institute for Mathematical Sciences, Kyoto, 2008.

[42] Uta R. Freiberg. Refinement of the spectral asymptotics of generalized Krein Feller operators. *Forum Math.*, 23:427–445, 2011.

[43] Uta R. Freiberg and Jörg-Uwe Löbus. Zeros of eigenfunctions of a class of generalized second order differential operators on the cantor set. *Math. Nachr.*, 265:3–14, 2004. 7, 74

[44] Uta R. Freiberg and Jörg-Uwe Löbus. Zeros of eigenfunctions of the generator of a gap diffusion. In *Recent Developments in Stochastic Analysis and Related Topics, Proceedings of the First Sino-German Conference on Stochastic Analysis (A satellite Conference of ICM 2002)*, pages 182–198. World Scientific, Singapore, 2004. 74

[45] Uta R. Freiberg and Martina Zähle. Harmonic calculus on fractals - a measure geometric approach i. *Potential Anal.*, 16:265–277, 2002. 7, 74

[46] Takahiko Fujita. A fractional dimension, self similarity and a generalized diffusion operator. In *Probabilistic methods in mathematical physics, Proceedings of Taniguchi International Symposium, Katata and Kyoto*, pages 83–90. Kinokuniya, Tokyo, 1987. 6, 74

[47] Vincentio Galilei. *Discorso di Vincentio Galilei nobile fiorentino, intorno all' opere di messer Gioseffo Zarlino da Chioggia, et altri importanti particolari attenenti alla musica*. Appresso Giorgio Marescotti, Fiorenza, 1589. 1

[48] Feliks R. Gantmacher and Mark G. Kreĭn. *Oscillyacionye Matricy i Yadra i Malye Kolebaniya Mehaničeskih Sistem*. GITTL, Moscow, Leningrad, 1941. 6, 78

[49] Feliks R. Gantmacher and Mark G. Kreĭn. *Oscillyacionye Matricy i Yadra i*

Malye Kolebaniya Mehaničeskih Sistem, second and revised edition. GITTL, Moscow, Leningrad, 1950. 6

[50] Feliks R. Gantmacher and Mark G. Kreĭn. Oscillation matrices and kernels and small vibrations of mechanical systems. volume 345 of *AMS Chelsea*. American Mathematical Society, Providence, Rhode Island, 2002. 78

[51] Carl F. Gauß. Disquisitiones generales circa seriem infinitam $1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\ .\ 2\ .\ \gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\ .\ 2\ .\ 3\ .\ \gamma(\gamma+1)(\gamma+2)}x^3 + etc.$ *Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores*, pages 3–46, 1813. 45

[52] Siegfried Graf and Harald Luschgy. The quantization of the Cantor distribution. *Math. Nachr.*, 183:113–133, 1997. 78

[53] Karl H. Hellwege. Einführung in die Festkörperphysik I. volume 33 of *Heidelberger Taschenbücher*. Springer, Berlin, Heidelberg, 1968. 25

[54] Jiaxin Hu, Ka-Sing Lau, and Sze-Man Ngai. Laplace operators related to self-similar measures on $\mathbb{R}^d$. *Funct. Anal.*, 239:542–565, 2006. 74

[55] John E. Hutchinson. Fractals and self similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981. 9

[56] Christian Huygens. *Horologium Oscillatorium sive de Motu Pendulorum ad Horologia Aptato Demonstrationes Geometricae.* F. Muguet, Paris, 1673. 2

[57] Victor Ya. Ivriĭ. Second term of the spectral asymptotic expansion of the Laplace - Beltrami operator on manifolds with boundary. *Functional Analysis and its Applications*, 14(2):98–106, 1980. 5

[58] Victor Ya. Ivriĭ. The second term of the spectral asymptotics for the Laplace-Beltrami operator on manifolds with boundary. *Akademiya Nauk SSSR. Funktsional'nyĭ Analiz i Ego Prilozheniya*, 14(2):25–34, 1980. 5

[59] James H. Jeans. On the partition of energy between matter and æther. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, X, sixth series:91–98, 1905. 4

[60] Israel S. Kac and Mark G. Kreĭn. On the spectral functions of the string. In *Nine Papers in Analysis*, volume 103 of *American Mathematical Society Translations, series 2*, pages 19–102. American Mathematical Society, Providence, Rhode Island, 1974. 6, 74, 77

[61] Mark Kac. Can one hear the shape of a drum? *American Mathematical Monthly (Slaught Memorial Papers Nr.11)*, 73:1–23, 1966. 4, 5

[62] Marc Kesseböhmer and Sanguo Zhu. Stability of quantization dimension and quantization for homogeneous Cantor measures. *Math. Nachr.*, 280:866–881, 2007. 78

[63] Charles Kittel. *Introduction to Solid State Physics*. John Wiley & Sons, Chichester, New York, Weinheim, Brisbane, Singapore, Toronto, 1996. 25

[64] Israel Kleiner. Evolution of the function concept: a brief survey. *The College Mathematics Journal*, 20(4):282–300, 1989. 3

[65] Steven G. Krantz and Harold R. Parks. The geometry of domains in space. In *Birkhäuser Advanced Texts*, Basler Lehrbücher. Birkhäuser, Boston, Basel, Berlin, 1999. 13

[66] Joseph Louis de Lagrange. Recherches sur la nature et la propagation du son. *Miscellanea Philosophico-Mathematica Societatis Privatae Taurinensis*, Tomus I:1–112, 1759. 3

[67] Joseph Louis de Lagrange. Nouvelles recherches sur la nature et la propagation du son. *Miscellanea Philosophico-Mathematica Societatis Privatae Taurinensis*, Tomus II:11–172, 1761. 3

[68] Joseph Louis de Lagrange. *Œuvres complètes*, volume Tome I. Gauthier-Villars, Paris, 1867. 3

[69] Pham The Lai. Meilleures estimations asymptotiques des restes de la fonction spectrale et des valeurs propres relatifs. *Mathematica Scandinavica*, 48:5–38, 1981. 5

[70] Horace Lamb. On waves in a medium having a periodic discontinuity of structure. *Memoirs and Proceedings of the Manchester Literary & Philosophical Society*, 42(3):1–20, 1898. 6

[71] Michel L. Lapidus. Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture. *Transactions of the American Mathematical Society*, 325(2):465–529, 1991. 5

[72] Michel L. Lapidus. Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media, and the Weyl-Berry conjecture. In *Ordinary and Partial Differential Equations*, volume IV of *Pitman Research Notes in Mathematics*, pages 126–209. Longman Scientific & Technical, Harlow, Essex, 1993. 5

[73] Michel L. Lapidus and Jacqueline Fleckinger-Pellé. Tambour fractal: vers une résolution de la conjecture de Weyl-Berry pour les valeurs propres du laplacien. *Comptes Rendus de l'Académie des Sciences Paris*, 306, série I:171–175, 1988. 5

[74] Michel L. Lapidus and Machiel van Frankenhuysen. *Fractal Geometry and Number Theory: Complex Dimensions of Fractal Strings and Zeros of Zeta Functions*. Birkhäuser, Boston, Basel, Berlin, 2000. 8, 21, 71, 106

[75] Michel L. Lapidus and Helmut Maier. Hypothèse de Riemann, cordes fractales vibrantes et conjecture de Weyl-Berry modifiée. *Comptes Rendus de l'Académie des Sciences Paris*, 313, série I:19–24, 1991. 6, 23

[76] Michel L. Lapidus and Helmut Maier. The Riemann hypothesis and inverse spectral problems for fractal strings. *Journal of the London Mathematical Society*, 52(2):15–34, 1995. 6, 23

[77] Michel L. Lapidus and Carl Pomerance. Fonction zêta de Riemann et conjecture de Weyl-Berry pour les tambours fractals. *Comptes Rendus de l'Académie des Sciences Paris*, 310, série I:343–348, 1990. 5, 13, 21

[78] Michel L. Lapidus and Carl Pomerance. The Riemann Zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums. *Proceedings of the London Mathematical Society*, 66(3):41–69, 1993. ix, 5, 21, 70

[79] Michel L. Lapidus and Carl Pomerance. Counterexamples to the modified Weyl-Berry conjecture. *Mathematical Proceedings of the Cambridge Philosophical Society*, 119:167–178, 1996. 5

[80] Derrick H. Lehmer. On the compounding of certain means. *Journal of Mathematical Analysis and Applications*, 36:183–200, 1971. 90

[81] Joseph Liouville. Mémoire sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujétis à satisfaire à une même équation différentielle du second ordre, contenant un paramètre variable. *Journal de Mathématiques Pures et Appliquées*, 1:253–265, 1836. 6

[82] Joseph Liouville. Second mémoire sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujétis à satisfaire à une même équation différentielle du second ordre, contenant un paramètre variable. *Journal de Mathématiques Pures et Appliquées*, 2:16–35, 1837.

[83] Joseph Liouville. Troisième mémoire sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujétis à satisfaire à une même équation différentielle du second ordre, contenant un paramètre variable. *Journal de Mathématiques Pures et Appliquées*, 2:418–436, 1837. 6

[84] Hendrik A. Lorentz. Alte und neue Fragen der Physik. *Physikalische Zeitschrift*, XI:1234–1257, 1910. 3

[85] Jan R. Magnus and Heinz Neudecker. Matrix differential calculus with applications in statistics and econometrics, third edition. Wiley Series in Probability and Statistics: Texts and References Section. John Wiley & Sons, Chichester,New York, Weinheim, Brisbane, Singapore, Toronto, 2007. 111

[86] Henry P. Jr. McKean and Daniel B. Ray. Spectral distribution of a differential operator. *Duke Math. J.*, 29:281–292, 1962. 6, 74

[87] Marin Mersenne. *Harmonie Universelle, contenant la Théorie et la Pratique de la Musique,*. Sébastien Cramoisy, Paris, 1636. 2

[88] Guy Métivier. Valeurs propres de problèmes aux limites elliptiques irréguliers. *Bulletin de la société mathématique de France*, 51-52:125–219, 1977. 5

[89] Steven J. Miller and Ramin Takloo-Bighash. *An Invitation to Modern Number Theory*. Princeton University Press, Princeton, Oxford, 2006. 38

[90] Katherine Naimark and Michail Solomyak. The eigenvalue behaviour for the boundary value problems related to self-similar measures on $\mathbb{R}^d$. *Math. Res. Lett. 2*, 2(3):279–298, 1995. 74

[91] Alexandr I. Nazarov and Igor A. Sheipak. Degenerate self-similar measures, spectral asymptotics and small deviations of gaussian processes. *Bulletin of the London Mathematical Society*, 44(1):12–24, 2012. 74

[92] Friedrich Pockels. *Über die partielle Differentialgleichung* $\Delta u + k^2 u = 0$ *und deren Auftreten in der mathematischen Physik*. B.G. Teubner Verlag, Leipzig, 1891. 4

[93] Porphyry. *Eis ta harmonika Ptolemaiou hypomnēma*. Manuscript, Spain or Italy, between 1560-1570. Location: Rare Book & Manuscript Library, University of Pennsylvania, LJS 97. 1, 138

[94] John W. Strutt Baron Rayleigh. *The Theory of Sound, Vol. I.* Macmillan and Co., London, 1877. 6

[95] John W. Strutt Baron Rayleigh. On the maintenance of vibrations by forces of double frequency, and on the propagation of waves through a medium endowed with a periodic structure. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, fifth series*, 24(147):145–159, 1887. 6, 111

[96] John W. Strutt Baron Rayleigh. The dynamical theory of gases and of radiation. *Nature*, LXXI, LXXII:559, 54–55, 243–244, 1905. republished in: Scientific Papers by John William Strutt Baron Rayleigh, Vol. V, Cambridge: at the University Press,1912. 4

[97] Johanna Reudler. *Over de Zwarte Straling in Ruimten van Verschillenden Vorm.* Phd thesis, Rijks-Universiteit te Leiden, 1912. 4

[98] Bernhard Riemann. Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, 13:87–132, 1854. transcribed by D.R. Wilkins, 2000. 3

[99] Francis Roberts. A discourse concerning the musical notes of the trumpet, and trumpet-marine, and of the defects of the same. *Philosophical Transactions of the Royal Society London*, XI:559–563, 1692. 2

[100] Joseph Sauveur. Système général des intervalles des sons, et son application à tous les systêmes et à tous les instruments de musique. *Histoire de l'Académie Royale des Sciences*, pages 299–366, 1701. 2

[101] Joseph Sauveur. Rapport des sons des cordes d'instruments de musique, aux flèches des cordes; et nouvelle détermination des sons fixes. *Histoire de l'Académie Royale des Sciences*, pages 324–350, 1713. 2

[102] Igor A. Sheipak and Anton A. Vladimirov. Indefinite Sturm-Liouville problem for some classes of self-similar singular weights. *Proceedings of the Steklov Institute of Mathematics*, 255(1):82–91, 2006. 74

[103] Igor A. Sheipak and Anton A. Vladimirov. Self-similar functions in $L_2[0,1]$ and the Sturm-Liouville problem with a singular indefinite weight. *Sbornik: Mathematics*, 197(11):13–30, 2006.

[104] Igor A. Sheipak and Anton A. Vladimirov. Asymptotics of the eigenvalues of the Sturm-Liouville problem with discrete self-similar weight. *Mathematical Notes*, 88(5-6):637–646, 2010.

[105] Michail Solomyak and Evgeny Verbitsky. On a spectral problem related to self-similar measures. *Bulletin of the London Mathematical Society*, 27(3):242–248, 1995. 74

[106] Arnold Sommerfeld. Die Greensche Funktion der Schwingungsgleichung für ein beliebiges Gebiet. *Physikalische Zeitschrift*, 11:1057–1066, 1910. 4

[107] Thomas J. Stieltjes. Recherches sur les fractions continues. *Annales de la Faculté des Sciences de Toulouse, 1*[re] *série*, 8(4):J1–J122, 1894. 6

[108] Thomas J. Stieltjes. Recherches sur les fractions continues. *Annales de la Faculté des Sciences de Toulouse, 1*[re] *série*, 9(1):A5–A47, 1895. 6

[109] Pantelimon Stănică. Good lower and upper bounds on binomial coefficients. *Journal of Inequalities in Pure and Applied Mathematics*, 2 (3):1–5, 2001. 41

[110] Charles Sturm. Mémoire sur les équations différentielles linéaires du second ordre. *Journal de Mathématiques Pures et Appliquées*, 1:106–186, 1836. 6

[111] Charles Sturm and Joseph Liouville. Extrait d'un mémoire sur le développement des fonctions en séries dont les différents termes sont assujétis à satisfaire à une même équation différentielle linéaire, contenant un paramètre variable. *Journal de Mathématiques Pures et Appliquées*, 2:220–223, 1837. 6

[112] Brook Taylor. De motu nervi tensi. *Philosophical Transactions of the Royal Society London*, XXVIII:26–32, 1713. 2

[113] Alekseĭ F. Timofeev. *Integrirovanie Funktsij*. OGIZ, State Publishing, Technical and Theoretical Literature, Moskow, Leningrad, 1948. 33

[114] Toshio Uno and Imsik Hong. Some consideration of asymptotic distribution of eigenvalues for the equation $\frac{d^2u}{dx^2} + \lambda\rho(x)u = 0$. *Japanese Journal of Mathematics: Transactions and Abstracts*, 29:152–164, 1959. 6, 74

[115] John Wallis. On the trembling of consonant strings, a new musical discovery. *Philosophical transactions of the Royal Society London*, XII:839–842, 1677. 2

[116] Hermann H. K. Weyl. Ueber die asymptotische Verteilung der Eigen-

werte. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pages 110–117, 1911. 4

[117] Hermann H. K. Weyl. Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). *Mathematische Annalen*, 71:441–479, 1912.

[118] Hermann H. K. Weyl. Über die Abhängigkeit der Eigenschwingungen einer Membran von deren Begrenzung. *Journal für die reine und angewandte Mathematik*, 141:1–11, 1912. 4

List of Figures

1.1 Excerpt from a renaissance manuscript of Porphyry's "Eis ta harmonika Ptolemaiou hypomnēma" [93] 1

2.1 The triadic Cantor set $\mathcal{C}_T$ 10
2.2 The triadic Cantor string $\mathcal{CS}_T$ 10
2.3 A generalised Cantor set $\mathcal{C}$ with parameters $a = 3$ and $b = 5$ . . . 10
2.4 A generalised Cantor string $\mathcal{CS}$ with parameters $a = 3$ and $b = 5$. 10
2.5 The a-set with parameter $a = \frac{\log(3)}{\log(2)} - 1$ 11
2.6 The a-string with parameter $a = \frac{\log(3)}{\log(2)} - 1$ 11
2.7 An example for eigenfunctions of the Cantor string 16
2.8 Graphs of $N(\lambda)$ for the triadic Cantor string (red) and the a-string with parameter $a = \frac{\log(3)}{\log(2)} - 1$ (black). 24

3.1 Monoatomic chain 26
3.2 Corresponding vibrational modes of a string and a monoatomic chain 29
3.3 Eigenvalue counting functions for a string (crosses, $N(\lambda) \sim \sqrt{\lambda}$) and a monoatomic chain (boxes, $N(\lambda) \sim \arcsin(\sqrt{\lambda})$) 30
3.4 Eigenvalue counting functions for the triadic Cantor chain (black, blue, green) and string (red) 31

3.5 Detail of figure (3.4) 31

4.1 An approximation to the triadic Cantor string and the corresponding harp 71
4.2 The transition from a fractal string (left) to a fractal chain (right) 72
4.3 From a fractal chain (left) to a fractal-layered chain (right) 72
4.4 The mass distribution of an approximation of a fractal-layered chain (red) compared to that of a measure geometric chain (black) 73

5.1 The first three iterations of a beaded string loaded according to the classical Cantor set construction 79
5.2 Normalised eigenvalue counting functions for different approximation levels j of the measure geometric Cantor chain. 82
5.3 $N(\lambda)$ as a function of λ for a seventh order measure geometric Cantor beaded string, blue: power-law fit, green: expectation from Equation 5.3. 83
5.4 Fits to the empirical value for γ: direct exponential fit (green), by an exponential fit for $\gamma_{exp,j} - \gamma_{th}$ (red). 84
5.5 The first four stages in the construction of the Cantor set (above) and its analogue with two different scaling ratios (below). 85
5.6 $N(\lambda)$ as a function of λ for the fifth order measure geometric beaded string with scaling ratios $r_1 = \frac{1}{4}$ and $r_2 = \frac{1}{3}$, blue: power-law fit, green: expectation from Equation 5.3. 86
5.7 Fits to the empirical value for γ: direct exponential fit (green), by an exponential fit for $\gamma_{exp,j} - \gamma_{th}$ (red). 87

5.8 Normalised eigenvalue counting functions for different approximation levels j of the measure geometric chain with two scaling ratios. 88
5.9 Comparison of Euler (green) and Maclaurin (red) expansions with 15 terms of the sinq function for the measure geometric triadic Cantor string with the exact function (black). 99
5.10 Comparison of Euler (green) and Maclaurin (red) expansions with 15 terms of the sinq function for the measure geometric fractal string with two scaling ratios with the exact function (black). . . 102

6.1 The unifying framework 106
6.2 Explanatory chart of methods and results of this thesis; ellipses: strategies, rectangles: methods, diamonds: results. 109

A.1 Behaviour of $\mathcal{M}_{N,k}$ (crosses) and $\mathcal{M}'_{N,k}$ (boxes) as a function of the iteration level $\mathfrak{m}$ for different values of k. 119

List of Tables

3.1 The first five normalised moments and their approximations for the triadic Cantor chain, at approximation level $\mathfrak{m} = 8$. 41

3.2 The first five normalised moments and their approximations for a generalised Cantor chain with parameters $a = 3$ and $b = 5$, at approximation level $\mathfrak{m} = 8$. 42

3.3 Comparison of the first five normalised moments for Cantor chains and the corresponding a-chains of the same Minkowki dimension, at approximation level $\mathfrak{m} = 8$. 43

5.1 Spectral exponents and correlations for the second to eight iterations 85

5.2 Spectral exponents and correlations for the first seven iterations . 87

5.3 The first coefficients of the MacLaurin expansion of sinq(z) for the measure geometric triadic Cantor string [5]. 98

5.4 Approximations for the first coefficients in the expansion of sinq(z) for the measure geometric triadic Cantor string 99

5.5 Goodness of appoximation for the zeroes of the sinq function. . . 100

5.6 The first coefficients of the MacLaurin expansion of sinq(z) for the measure geometric string with two different scaling ratios 101

5.7 Goodness of appoximation for the zeroes of the sinq function. . . 103