Locally Trivial Families of Complex Curves

Daniel C. Epping

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Mathematisches Institut
Fachbereich Mathematik und Informatik
Westfälische Wilhelms-Universität Münster

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DANIEL C. EPPING
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It is possible to contact the author via `eppingdaniel@netscape.net`.

Dean: PROF. DR. DR. h.c. J. CUNTZ
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Introduction

> The whole is more than the sum of its parts.
>
> ARISTOTLE

Let X be a 2-dimensional non-singular affine complex algebraic variety and S be a smooth complex curve. There are two main questions approached in this work. The first is asking, when an algebraic morphism $f\colon X \to S$ defines a differentiable fiber bundle. Analogously, the second one is asking, when $f\colon X \to S$ defines a fiber bundle, which is locally trivial in the holomorphic sense.

The Ehresmann Fibration Theorem and a theorem of HÀ H. V. and LÊ D. T., cf. [HL84], are well known criteria for the existence of differentiable local triviality. Having a key role in this work, both theorems are modified and extended in the process of this thesis. The Ehresmann Fibration Theorem is stating that a mapping between differentiable manifolds, which is proper and has no singular points already defines a $\mathscr{C}^\infty$-fiber bundle. Unit vector fields are lifted from the base space to the total space to construct a globally integrable flow, which yields differentiable local trivializations.

The theorem of HÀ and LÊ is stating that a polynomial mapping from $\mathbb{C}^2$ to $\mathbb{C}$ defines a $\mathscr{C}^\infty$-fiber bundle if and only if the mapping has no singular points and if the Euler-Poincaré characteristic of the fibers is constant. HÀ and LÊ are using a compactification of the graph, controlling its "infinite part" by the constance of the Euler-Poincaré characteristic to construct a submersion, then use methods similar to the Ehresmann Fibration Theorem. To extend this statement to the situation of the scheme $f\colon X \to S$ is in general not possible. Since a compactification of the graph may fiberwise yield irreducible components in the infinite part of the compactification, it is difficult to transfer the methods from the original proof. To conclude differentiable local triviality for the scheme $f : X \to S$ in the main result of this work, it is necessary in addition to require the fibers of f to be pairwise homeomorphic, and to have the same strictly positive geometric genus. Furthermore, the fibers of f above closed points of S are required to be irreducible.

According to the minimal models theorem by S. LICHTENBAUM and I.R. SHAFAREVICH [Sha66] there exists a minimal regular model, a compactification of the morphism, having no exceptional divisors. The resulting total space does not contain any singularities, and the resulting morphism is proper, though not necessarily smooth, i.e. a submersion in the analytic sense. This though is necessary to apply an argument based on the Ehresmann Fibration Theorem.

For a fibered surface having a Henselian base space, there exists the contraction morphism for irreducible components in the special fiber. Using a Henselization locally on the base of a compactification of $X \to S$, irreducible components in the closure are contracted to single points. For any $t \in S$ a non-singular compactification $\bar{X} \times_S \operatorname{Spec}(\mathcal{O}_{S,t}^h)$ is constructed, whose fibers densely contain the fibers of $X \times_S \operatorname{Spec}(\mathcal{O}_{S,t}^h)$. The geometric genus of the fibers is preserved this way. According to a result of T. Sekiguchi, F. Oort, and N. Suwa [SOS89] the minimal regular model of $\bar{X} \times_S \operatorname{Spec}(\mathcal{O}_{S,t}^h)$ is smooth. For all $t \in S$ the smooth minimal model of $\bar{X} \times_S \operatorname{Spec}(\mathcal{O}_{S,t}^h)$ commutes with the minimal regular model $\bar{X} \to S$ of $X \to S$, which in turn is smooth as well.

In case S has a trivial fundamental group, the infinite part of $\bar{X} \to S$ is shown to consist of a finite disjoint union of global holomorphic sections. Analogous to the Ehresmann Fibration Theorem, it is then possible to locally lift integrable vector fields, which are parallel to these sections. The lifted vector fields are then glued together with a partition of unity, hence yielding local trivializations for the pair of spaces $(\bar{X}, X)$, rendering X as a differentiable fiber bundle. Theorem 2.70 is giving the main result.

Furthermore holomorphic local triviality is investigated in three parts. At first, families of hyperbolic Riemann surfaces are pursued. Then families of curves of genus 1 as well as 0 are discussed.

A theorem of W. Fischer and H. Grauert [GF65] ensures local triviality in the holomorphic sense for a proper holomorphic submersion, which is an analytically isotrivial family of connected compact complex manifolds. This theorem is altered and extended as follows, in order to generalize the requirement of isotriviality for the case of the scheme $f : X \to S$. For this main result of Section 3.1.1 on holomorphic local triviality it is necessary to require $S \in \{\mathbb{C}, \mathbb{C}^*, \mathbb{P}^1_{\mathbb{C}}, T\}$, where T is the complex torus, and that the compactification of each fiber is a hyperbolic Riemann surface. Furthermore suppose that all fibers are pairwise homeomorphic. Under these conditions, f defines a holomorphic fiber bundle.

The Rigidity Theorem of S. Ju. Arakelov, A. N. Parshin , Y. Manin, and H. Grauert [Mum99] states in particular that families of smooth projective curves of genus 2 or higher are isotrivial, in case the base space is one of $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{P}^1_{\mathbb{C}}$, or T. This theorem already answers the original question in the case of families of projective curves. In the general case, for the family $X \to S$ there exists a regular minimal model $\bar{X}$, which is isotrivial by the Rigidity Theorem. The infinite part $\bar{X} \setminus X$ of the minimal regular model is shown to be holomorphically trivial. Consequently the minimal regular model defines a holomorphically fibered pair of spaces, including the original bundle. This result is given in Theorem 3.29.

For the investigation of families of elliptic curves, let the fibers of f be complex tori having exactly one puncture. By a theorem of A. Beauville, a family of elliptic curves is isotrivial if the base space is again one of $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{P}^1_{\mathbb{C}}$, or T. The minimal regular model of X is locally trivial in the holomorphic sense by Fischer and Grauert. The former punctures are translated by holomorphic automorphisms of the fibers and deform holomorphically over the base space. Therefore, even outside its infinite part the resulting bundle is locally trivial in the holomorphic sense.

Families of curves of genus 0 are always isotrivial and therefore define a holomorphic fiber bundle by the theorem of Fischer and Grauert. For Section 3.1.3, let the fibers of $f : X \to S$ be biholomorphic to $\mathbb{P}^1_{\mathbb{C}}$ with up to three punctures. After compactification of the graph and removal of finitely many singular fibers, the resulting family is locally trivial in the holomorphic sense by the theorem of Fischer and Grauert. Fiberwise, the infinite part of the compactification consists constantly of up to three points. These are translated such that they deform holomorphically within the bundle. The resulting bundle is consequently trivial in the holomorphic sense outside its infinite part.

Finally, let $f : X \to S$ be a holomorphic fiber bundle having Riemann surfaces as fibers, where X is not necessarily affine. This situation is investigated for global triviality in the holomorphic sense. A result of H. Grauert [Gra58] ensures global triviality for holomorphic bundles having a connected complex Lie group as structure group. The focus is therefore narrowed down to fiber bundles having a non-discrete and non-connected Lie group as structure group. Using non-abelian cohomology, the main theorem, Theorem 3.43, of this section states that such a fiber bundle is globally trivial in the holomorphic sense if and only if its corresponding cocycle is contained in the kernel of a mapping defined in the theorem.

A Note on Reading this Thesis. The reader is advised to start reading the definition of a fiber bundle and only look through the contents of the first chapter without going into much detail. The preliminary chapter should then be skipped, since it only serves to give a collection of mostly algebraic preparatory material used in following chapters. The reader can therefore use it as reference and, if needed, come back to it for later consultation.

With the exception of the usage of a minimal regular model in chapter three, chapters two and three can be read independently. The result for the construction of differentiable triviality in chapter two is mostly using methods from algebraic geometry, and constitutes the main result of this thesis. This algebraic part ranges from Section 2.1 to 2.3, followed by the main theorem, Theorem 2.70. The third chapter is largely using complex analytic methods. The main results for local triviality in the holomorphic sense for fiber genus 2 or higher, Theorem 3.29, genus 1, Theorem 3.33, and fiber genus 0, Theorem 3.34, are located at the end of Sections 3.1.1, 3.1.2, and 3.1.3. A criterion for global triviality in the holomorphic sense, Theorem 3.43, is located in the shorter final section, Section 3.2.

1 Preliminaries

Im großen Garten der Geometrie kann
sich jeder nach seinem Geschmack
einen Strauß pflücken.

D. Hilbert

One of the most important objects used in the following chapters is the notion of a scheme, in particular that of a fibered surface. Some basic definitions are therefore recalled in accordance to [Liu02]. The note on the comparison of algebraic and analytic categories is based on [Har77]. For the following, a scheme X will be called *integral at* $x \in X$ if $\mathcal{O}_{X,x}$ is an integral domain. This is equivalent to saying that X is reduced at x and that there is a single irreducible component of X passing through x. The scheme X is called *integral* if it is reduced and irreducible. This implies that X is integral at all of its points.

1.1 Definition (Affine Scheme, Scheme). An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to the spectrum of some ring. A *scheme* is a locally ringed topological space $(X, \mathcal{O}_X)$ admitting an open covering $\{U_i\}_{i \in I}$ such that $(U_i, \mathcal{O}_X|_U)$ is an affine scheme for every i.

1.2 Definition (Normal Scheme). A scheme X is called *normal* at $x \in X$ or x is called a *normal point* of X if $\mathcal{O}_{X,x}$ is normal. The scheme X is called *normal* if it is irreducible and normal at all points.

1.3 Definition (Zariski Tangent Space). Let X be a scheme and $x \in X$. Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$ and $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ be the residue field. Then $\mathfrak{m}_x/\mathfrak{m}_x^2 = \mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}} k(x)$ defines a $k(x)$-vector space in a natural way. Its dual $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ is called the *Zariski tangent space to X at x*, and is denoted by $T_x^z X$.

1.4 Definition (Regular Scheme). A scheme X is called *Noetherian* if it is a finite union of affine open subsets X_i such that $\mathcal{O}_{X_i}(X_i)$ is a Noetherian ring for all i. It is called *locally Noetherian* if every point possesses a Noetherian neighborhood.

Let X be a locally Noetherian scheme, and let $x \in X$ be a point. The scheme X is *regular* at $x \in X$, or x is a *regular point* of X, if $\mathcal{O}_{X,x}$ is regular, i.e.

$$\dim \mathcal{O}_{X,x} = \dim_{k(x)} T_x^z X.$$

The scheme X is *regular* if it is regular at all of its points. A point $x \in X$ which is not regular is called a *singular point* of X.

1.5 Definition (Reduced Scheme). A ring R is called *reduced* if $\{0\}$ is the only nilpotent element of R. A scheme X is called *reduced at $x \in X$* if the ring $\mathcal{O}_{X,x}$ is reduced. The scheme X is called *reduced* if it is reduced at every point.

1.6 Definition (Dedekind Scheme). A ring is called a *domain* if 0 is a prime ideal. A Noetherian normal domain of dimension 1 is called a *Dedekind domain*. A normal locally Noetherian scheme of dimension 1 is called a *Dedekind scheme*.

1.7 Definition (Universally Catenary Schemes). A Noetherian ring R is called *catenary* if for any triplet of prime ideals $\mathfrak{q} \subseteq \mathfrak{p} \subseteq \mathfrak{m}$, there is

$$\operatorname{ht}(\mathfrak{m}/\mathfrak{q}) = \operatorname{ht}(\mathfrak{m}/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p}/\mathfrak{q}),$$

where the *height* ht is the supremum of the lengths of strictly ascending chains of prime ideals contained in the given prime ideal. A Noetherian ring R is called *universally catenary* if every finitely generated R-algebra is catenary. A finitely generated algebra over a universally catenary ring is universally catenary. A locally Noetherian scheme X is *catenary* if its local rings are catenary, and is *universally catenary* if $\mathbb{A}^n_X$ is catenary for every $n \geq 0$.

1.8 Remark. Any scheme of locally finite type over a regular Noetherian scheme is universally catenary, see [Liu02, Corollary 2.16].

1.9 Definition (Geometrically Integral Scheme). Let X be a scheme of finite type over a field k. Let $\bar{k}$ be the algebraic closure of k. The scheme X is called *geometrically reduced*, resp. *geometrically irreducible*, if $X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\bar{k})$ is reduced, resp. irreducible. The scheme X is called *geometrically integral*, if $X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\bar{k})$ is integral.

1.10 Definition (Formal Fiber). Let $(R, \mathfrak{m})$ be a locally Noetherian ring, and let $\hat{R}$ be its completion for the $\mathfrak{m}$-adic topology. Fibers of the canonical morphism

$$\operatorname{Spec}(\hat{R}) \to \operatorname{Spec}(R)$$

are called the *formal fibers of R*.

The following notion of excellent rings and schemes was introduced by A. Grothendieck in his “Éléments de Géométrie Algébrique” (EGA), see [EGA, IV_2, 7.8.1].

1.11 Definition (Excellent Scheme). A Noetherian ring R is called *excellent* if it verifies the following three properties.

(a) Spec(R) is universally catenary.

(b) For every $\mathfrak{p} \in \mathrm{Spec}(R)$, the formal fibers of $R_\mathfrak{p}$ are geometrically regular.

(c) For every finitely generated R-algebra B, the set of regular points of Spec(B) is open in Spec(B).

Note that conditions (a) and (b) only relate to the localization of R at the prime ideals, which is not the case for condition (c). A locally Noetherian scheme X is called *excellent* if there exists an affine covering $\{U_i\}_{i\in I}$ of X such that $\mathcal{O}_X(U_i)$ is excellent for every i.

1.12 Remark. Two important properties of excellent schemes are recalled from [Liu02, Theorem 8.2.39].

(a) Any complete Noetherian local ring, in particular a field, is excellent.

(b) Moreover, let S be an excellent locally Noetherian scheme. Then any scheme that is locally of finite type over S is excellent.

Consequently, the property of a scheme to be excellent is almost always true for the schemes used in this work. It is only in the construction of a smooth minimal model of a scheme over the spectrum of a local ring in Section 2.3 that the latter is not necessarily excellent. For an example of a discrete valuation ring which is not excellent, see [Liu02, Example 8.2.31].

1.13 Definition (Generic Fiber). Let $f\colon X \to Y$ be a morphism of schemes, where Y is an irreducible scheme with generic point η. The fiber X_η is called the *generic fiber of the morphism* f.

1.14 Proposition. *Let S be a Dedekind scheme, let X be an integral scheme of dimension 2, and $f : X \to S$ be a dominant morphism, i.e. $f(X)$ is dense in X. Then $X \to S$ is flat.*

Proof. Since X is irreducible, there exists an injection $i : \mathcal{O}_X \hookrightarrow K(X)$. The morphism f is dominant, and therefore induces an injection $j : K(S) \to K(X)$ which does not vanish. Since S is a Dedekind scheme, the surface X is flat over S if and only if $\mathcal{O}_X$ does not possess any S-torsion. Let $g \in \mathcal{O}_{X,x}$, $x \in X$, and $r \in \mathcal{O}_{S,g(x)}$. Suppose that $r \cdot g = 0$. It follows that $i(r \cdot g) = 0$ and consequently $i(g) \cdot j(r) = 0$. Then either $i(g) = 0$ or $j(r) = 0$, since $K(X)$ is a field. Therefore if $j(r) \neq 0$ (resp. $i(g) \neq 0$) then $i(g) = 0$ (resp. $j(r) = 0$). Since i (resp. j) is an injection, it follows that $g = 0$ (resp. $r = 0$). □

1.15 Definition (Fibered Surface). Let S be a Dedekind scheme. An integral projective flat S-scheme $f\colon X \to S$ such that $\dim X = 2$ is called a *fibered surface* over S. The scheme X is also called a *projective flat S-curve*. The S-scheme X is a *normal* (resp. *regular*) *fibered surface* if X is normal (resp. regular). A *morphism* (resp. *rational mapping*) between fibered surfaces is a morphism (resp. rational mapping) that is compatible with the structure of S-schemes.

A Note on the Comparison of Algebraic and Analytic Categories

By a ground-breaking result of J.-P.Serre [Ser56] there exists a correspondence between the category of complex analytic spaces with coherent analytic sheaves and complex algebraic projective varieties with coherent algebraic sheaves, generally referred to as the "GAGA"-principle.

First, a few basic facts on cohomology are recalled. The following theorem is a fundamental property of Zariski cohomology, and will be used to define the Euler-Poincaré characteristic.

1.16 Theorem (Serre). *Let A be a Noetherian ring, and X be a projective scheme over A. Let $\mathscr{F}$ be a coherent sheaf on X. Then for any integer $p \geq 0$, the A-module*

$$H^p(X, \mathscr{F})$$

is finitely generated.

See [Liu02, Theorem 5.3.2].

1.17 Theorem. *Let X be an affine scheme. Then for any quasi-coherent sheaf $\mathscr{F}$ on X, and for any integer $i \geq 1$,*

$$H^i(X, \mathscr{F}) = 0.$$

See [Liu02, Proposition 5.2.18].

1.18 Theorem. *Let X be a quasi-projective scheme of dimension d over a Noetherian ring A. Then X admits a covering by $d+1$ affine open subsets. In particular,*

$$H^i(X, \mathscr{F}) = 0, \quad \text{for } i > d.$$

See [Liu02, Proposition 5.2.24].

1.19 Definition (Complex Analytic Space). Let X be a Hausdorff space and $\mathcal{O}_X$ be a sheaf of rings. The space $(X, \mathcal{O}_X)$ is called a *complex analytic space* if every $x \in X$ has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $(U', \mathcal{O}_{U'})$, where U' is an analytic subset of an open set W in some $\mathbb{C}^n$, $\mathcal{O}_{U'}$ is the sheaf of germs of holomorphic functions on U', and $\mathcal{O}_{U'} = (\mathcal{O}_W/\mathscr{I})|_{U'}$, where $\mathscr{I}$ is some ideal, which has the analytic subset U' as set of zeros.

Let X be a scheme of finite type over $\mathbb{C}$. It is possible to cover X with open affine subsets $X_i = \mathrm{Spec}(A_i)$, where each A_i is an algebra of finite type over $\mathbb{C}$. Therefore

$$A_i \cong \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_q),$$

where $f_1, \dots, f_q$ are polynomials in $x_1, \dots, x_n$. As holomorphic functions on $\mathbb{C}^n$ the set of common zeros of these functions define an analytic subset $(X_i)_h$. Since "glueing together" the open sets X_i is giving the scheme X, it is possible to glue the analytic spaces $(X_i)_h$ in the same way. The resulting space is called the *associated complex analytic space of* X. This construction yields a functor h from the category of schemes of finite type over $\mathbb{C}$ to the category of complex analytic spaces.

Let $\mathscr{F}$ be a coherent sheaf on X. For the Zariski topology, $\mathscr{F}$ is locally a cokernel of a morphism φ of free sheaves

$$\mathcal{O}_U^m \xrightarrow{\varphi} \mathcal{O}_U^n \longrightarrow \mathscr{F} \longrightarrow 0.$$

Since the analytic topology is finer than the Zariski topology, U_h is open in X_h. Furthermore, φ is yielding local sections of $\mathcal{O}_{U_h}$. It is possible to define $\mathscr{F}^h$ locally as the cokernel of the corresponding mapping φ_h of free coherent analytic sheaves.

Most properties and theorems are preserved by this functor. A few examples are connectedness, reducedness, and smoothness of spaces or properness of morphisms. For any coherent sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules, it follows that

$$\mathscr{F}_h \cong \varphi^* \mathscr{F}.$$

This induces natural mappings of cohomology groups

$$\psi_i \colon H^i(X, \mathscr{F}) \to H^i(X_h, \mathscr{F}_h).$$

The following theorem of J.-P.Serre is ensuring even an equivalence in cohomology for projective schemes.

1.20 Theorem (Serre). *Let X be a projective scheme over $\mathbb{C}$. Then the functor h induces an equivalence of categories from the category of coherent sheaves on X to the category of coherent analytic sheaves on X_h. Furthermore, for every coherent sheaf $\mathscr{F}$ on X, the natural mappings*

$$\alpha_i \colon H^i(X, \mathscr{F}) \to H^i(X_h, \mathscr{F}_h)$$

are isomorphisms, for all $i \geq 0$.

See [Ser56, Théorème 1, p. 19].

1.1 Normalizations and Blowing-ups

1.21 Definition (Analytic Normalization). Let X be a reduced complex analytic space. A finite holomorphic mapping $\pi\colon X' \to X$ is called a *normalization* of X, if X' is normal, and there exists a thin analytic subset A of X such that $\pi^{-1}(A)$ is thin in X', and if $\pi\colon X' \setminus \pi^{-1}(A) \to X \setminus A$ is biholomorphic.

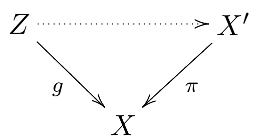
1.22 Proposition. *Let X be a reduced complex analytic space. If $x \in X$ is normal, then*

$$\dim_x S(X) \leq \dim_x X - 2,$$

where $S(X)$ is the singular locus of X.

See [GR84, 5.3].

1.23 Definition (Algebraic Normalization). Let X be an integral scheme. A morphism $\pi\colon X' \to X$ is called a *normalization morphism* if X' is normal, and every dominant morphism $g\colon Z \to X$, where Z is normal, factors uniquely through π.



1.24 Remark. Analytically, outside of singular points, a normalization $\pi\colon X' \to X$ is a one-sheeted analytic covering of a reduced complex space X, where X' is a normal complex space. Since analytic and algebraic normalization coincide, see [KK83, Proposition 71.8], the two will not be distinguished later on.

1.25 Theorem (Normalization Theorem). *Let X be an integral scheme. Then there exists a normalization morphism $\pi\colon X' \to X$, which is unique up to isomorphism of X-schemes. A morphism $f\colon Y \to X$ is the normalization morphism if and only if Y is normal, and f is birational and integral.*

See [Liu02, Proposition 4.1.22].

The birational mapping of a blowing-up of a variety at a point is a main tool in the resolution of singularities of varieties. Let R be a Noetherian ring, and I be an ideal in R. Define the graded R-algebra

$$\tilde{R} := \bigoplus_{d \geq 0} I^d, \text{ where } I^0 := R.$$

This definition is, of course, dependent on I. Let $f_1, \ldots, f_n$ be a system of generators of I. Let $t_i \in I = \tilde{R}_1$ denote the element f_i as a homogeneous element of degree 1. There exists a surjective homomorphism of graded R-algebras

$$\begin{aligned} \varphi\colon R[T_1, \ldots, T_n] &\to \tilde{R} \\ T_i &\mapsto t_i. \end{aligned}$$

Therefore $\tilde{R}$ is a homogeneous R-algebra, i.e. a graded algebra being the quotient of $R[T_1, \ldots, T_n]$ by a homogeneous ideal.

Let $P(t)$ be a homogeneous polynomial with coefficients in R. Then $P(t_1, \ldots, t_n) = 0$ if and only if $P(f_1, \ldots, f_n) = 0$. This motivates the following definition.

1.26 Definition. Let $X = \operatorname{Spec}(R)$ be an affine Noetherian scheme, and let I be an ideal of R. Define $\tilde{X} := \operatorname{Proj}(\tilde{R})$. The canonical morphism $\tilde{X} \to X$ is called the *blowing-up of X with center (or along)* $V(I)$, where $\operatorname{Proj}(\tilde{R})$ is the set of homogeneous prime ideals of $\tilde{R}$, not containing the ideal $\bigoplus_{d>0} \tilde{R}_d$, and $V(I) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subset \mathfrak{p}\}$.

1.27 Remark. It is possible to endow the set $\operatorname{Proj}(\tilde{R})$ with the structure of an R-scheme, see [Liu02, Proposition 2.3.38].

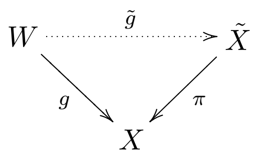
1.28 Definition (Blowing-up of a Scheme). Let X be a scheme. A quasi-coherent sheaf $\mathscr{B}$ of $\mathcal{O}_X$-algebras with grading $\mathscr{B} = \bigoplus_{n\geq 0} \mathscr{B}_n$, where the $\mathscr{B}_n$ are quasi-coherent sub-$\mathcal{O}_X$-modules, is called a *graded $\mathcal{O}_X$-algebra*. If in addition $\mathscr{B}_1$ is finitely generated, and $(\mathscr{B}_1)^n = \mathscr{B}_n$ for every $n \geq 1$, $\mathscr{B}$ is called a *homogeneous $\mathcal{O}_X$-algebra*. For any affine subset U of X, $\mathscr{B}(U)$ is a homogeneous $\mathcal{O}_X$-algebra.

Let $\mathscr{I}$ be a coherent sheaf of ideals on a locally Noetherian scheme X. The X-scheme

$$\operatorname{Proj}\Big(\bigoplus_{n\geq 0} \mathscr{I}^n\Big) \to X$$

is called the *blowing-up of X with center* (or *along*) $V(\mathscr{I})$, the closed subscheme of X corresponding to the sheaf of ideals $\mathscr{I}$. The scheme $\operatorname{Proj}(\bigoplus_{n\geq 0} \mathscr{I}^n)$ will be denoted by $\tilde{X}$. In case X is affine, this definition coincides with the previous definition.

1.29 Theorem (Universal Property of Blowing-ups). *Let X be a locally Noetherian scheme and $\mathscr{I}$ be a coherent sheaf of ideals. Let $\pi\colon \tilde{X} \to X$ be the blowing-up of X along $V(\mathscr{I})$. If $g : W \to X$ is any morphism such that $(g^{-1}\mathscr{I})\mathcal{O}_W$ is an invertible sheaf of* *ideals on W, then there exists a unique morphism $\tilde{g}\colon W \to \tilde{X}$ factoring g.*



See [Liu02, Proposition 8.1.15].

The next theorem shows that projective birational morphisms are blowing-ups.

1.30 Theorem. *Let $f\colon W \to X$ be a projective birational morphism of integral schemes. Suppose that X is quasi-projective over an affine Noetherian scheme. Then f is the blowing-up of X along a closed subscheme.*

See [Liu02, Theorem 8.1.24].

1.31 Remark. In dimension 1 the notion of normality coincides with that of regularity.

It is possible to determine the normalization of an integral projective curve X over a field k of characteristic 0 by successive blowing-ups. Let therefore X be singular, and let $X_1 \to X_0 = X$ be the blowing-up of X_0 along the singular locus of X_0 endowed with the reduced scheme structure. Repeat the process unless the resulting curve is not singular.

1.32 Proposition. *The sequence defined*

$$X = X_0 \longleftarrow X_1 \longleftarrow X_2 \cdots$$

is finite. There exists a desingularization of X by a finite number of successive blowing-ups with regular centers.

See [Liu02, Proposition 8.1.26].

It will be shown in Section 2.2 that there exists a finite number of successive blowing-ups and normalizations to desingularize a Noetherian normal connected and excellent scheme X of dimension 2.

1.2 Introduction to Fiber Bundles

To simplify notation, intersections of sets U_i and U_j may be denoted by U_{ij}.

1.33 Definition (Lie Group). Assume that G is a set that has the structure of a group and at the same time that of an n-dimensional complex manifold. The inverse of $g \in G$ will be denoted by g^{-1}, the identity element by e, and the composition of two elements $g_1,\, g_2 \in G$ by $g_1 g_2$. If G satisfies the following two properties, it is called a *complex Lie group.*

(a) The mapping $g \mapsto g^{-1}$ is holomorphic.

(b) The mapping $(g_1,\, g_2) \mapsto g_1 g_2$ is holomorphic.

One of the most important examples of a complex Lie group is the *general linear group*

$$\mathrm{GL}_n(\mathbb{C}) := \{A \in M_n(\mathbb{C}) \mid \det A \neq 0\},$$

where $M_n(\mathbb{C})$ is the group of complex $(n \times n)$ matrices.

1.34 Definition (Fiber Bundle). Let S and F be complex manifolds, and let G be a complex Lie group acting analytically and faithfully on F. A *topological fiber bundle* over S with structure group G and typical fiber F is given by a topological space X and a continuous mapping $\pi\colon X \to S$, together with

(a) an open covering $\{U_i\}_{i\in I}$ of S,

(b) for any $i \in I$ a topological mapping

$$\varphi_i\colon \pi^{-1}(U_i) \to U_i \times F$$

with $\mathrm{pr}_1 \circ \varphi_i = \pi$,

(c) for any pair of indices $(i, j) \in I \times I$ a continuous mapping $g_{ij}\colon U_i \cap U_j \to G$ with

$$\varphi_i \circ \varphi_j^{-1}(x,\, y) = (x,\, g_{ij}(x)y)$$

for $x \in U_i \cap U_j$ and $y \in F$.

The mappings φ_i are called *local trivializations* and the mappings g_{ij} a *system of transition functions*.

If X is a complex analytic manifold, then analogously $(X,\, \pi,\, S)$ is a *differentiable* (resp. *holomorphic*) *fiber bundle* if f, π, and the g_{ij} are differentiable (resp. *holomorphic*) mappings.

As convention in the following, if the structure group is not explicitly mentioned, the term fiber bundle just refers to the property of being locally trivial.

Fiber bundles, which have vector spaces as fibers, and the general linear group as structure group are called *vector bundles.*

1.35 Remark. Since G is acting faithfully, the following *compatibility condition* holds.

$$g_{ij}g_{jk} = g_{ik}, \text{ for } U_{ijk}.$$

Consequently $g_{ii} = e$ and $g_{ij} = g_{ji}^{-1}$.

1.36 Definition ((Cross) Section). Let $(X,\, f,\, S)$ be a fiber bundle. A *(cross) section* in X over an open subset $U \subset S$ is a mapping $s\colon U \to X$ with $f \circ s = \mathrm{id}_U$. A section, which is defined over the entire base space is called a *global section*.

1.37 Definition (Canonical Line Bundle). Let X be a n-dimensional manifold, $\{U_i\}_{i\in I}$ an open covering of X, and $\{\varphi_i\colon U_i \to \mathbb{C}^n\}$ suitable coordinate charts on X. To construct a fiber bundle, choose $\mathbb{C}$ as the typical fiber and $\mathbb{C}^*$ as structure group, acting on $\mathbb{C}$. It is possible to define transition functions $g_{ij}\colon U_{ij} \to \mathbb{C}^*$ by

$$g_{ij}(x) := \det D_{\varphi_i \circ \varphi_j^{-1}}(\varphi_j(x))^{-1}.$$

By the chain rule and the determinant product theorem, the compatibility conditions are satisfied. "Glueing together", i.e. identifying $(x, p) \in U_i \times \mathbb{C}$ with $(x, g_{ij}(x)(p))$ over U_{ij} yields a fiber bundle over X, called the *canonical line bundle.*

1.38 Example (Tangent Bundle). Let M be an n-dimensional differentiable manifold. Define

$$TM := \bigcup_{a\in M} T_a M,$$

and let $\pi\colon TM \to M$ be the canonical projection such that $\pi(v) = a$, for $v \in T_a M$, where $T_a M$ is the differentiable tangent space at the point a. Through the charts of M, it is possible to define differentiable bundle mappings

$$\varphi\colon \pi^{-1}(U) \to U \times \mathbb{R}^n$$

for any open subset $U \subset M$. Furthermore it is possible to endow TM with the structure of a differentiable manifold. Then $(TM,\, \pi,\, M)$ is a differentiable vector bundle. A section in TM is called a *vector field*.

1.39 Definition (Fiber Bundle Isomorphism). Let $(X,\ f,\ S)$ and $(Y,\ g,\ S)$ topological (resp. differentiable, holomorphic) fiber bundles over a manifold S, with the same fiber F and the same structure group G. Let $\{U_i\}_{i\in I}$ be an open covering of S such that there exist trivializations $\varphi_i\colon f^{-1}(U_i) \to U_i \times F$ and $\psi_i\colon g^{-1}(U_i) \to U_i \times F$.

A *fiber bundle isomorphism* between $(X,\ f,\ S)$ and $(Y,\ g,\ S)$ is a topological (resp. differentiable, holomorphic) mapping $h\colon X \to Y$ with $g \circ h = f$ such that for any $i \in I$ there exists a continuous (resp. differentiable, holomorphic) mapping $h_i\colon U_i \to G$ with

$$\psi_i \circ h \circ \varphi_i^{-1}(x, y) = (x, h_i(x)(y)).$$

The two fiber bundles are called *equivalent* in this case.

2 Differentiable Fiber Bundles

Algebra is the offer made by the devil to the mathematician. The devil says:“I will give you this powerful machine, and it will answer any question you like. All you need to do is to give me your soul: give up geometry and you will have this marvelous machine.”

Sir M. F. Atiyah

A polynomial function $f\colon \mathbb{C}^2 \to \mathbb{C}$ which does not possess any critical points does not necessarily define a differentiable fiber bundle. The following example of S.A. Broughton, see [Bro81], illustrates this fact. The polynomial

$$f(x, y) = x(xy - 1)$$

possesses no critical points. However it does not define a $\mathscr{C}^\infty$-fiber bundle, since the topological type of the fiber at $f = 0$ is different from that of the general fiber. The Euler-Poincaré characteristic of the fiber above 0 is equal to 1, that of the general fiber is equal to 0. The following theorem of Hà H. V. and Lê D. T. states that the Euler-Poincaré characteristic of the fibers being invariant causes a polynomial mapping $f\colon \mathbb{C}^2 \to \mathbb{C}$ to define a differentiable fiber bundle.

Theorem (Hà, Lê). *The polynomial mapping $f\colon \mathbb{C}^2 \to \mathbb{C}$ defines a $\mathscr{C}^\infty$-fiber bundle over a neighborhood of $z \in \mathbb{C}$ if and only if z is not a critical value of f, and if the Euler-Poincaré characteristic of $f^{-1}(z)$ is equal to that of the general fiber of f.*

See [HL84, théorème principal].

In the following chapters the notion of a natural number being positive is always used in the sense of being *strictly positive*. In the present chapter a generalization of the theorem above will be given. The main theorem, Theorem 2.70, extends the mentioned result from $\mathbb{C}^2$ to 2-dimensional non-singular affine complex algebraic varieties. For this, the morphism is required to have irreducible fibers of positive geometric genus and all fibers need to be pairwise homeomorphic. The theorem does not require $\mathbb{C}$ to be the base space, but is instead valid for any smooth complex algebraic curve S.

Usual compactification of the morphism by homogenization with an additional variable, under conditions of HÀ and LÊ's theorem, is compactifying fibers by single points. In the case of a variety as total space this method of compactification may yield 1-dimensional components in the "infinite part" of the closure of single fibers, which make it difficult to transfer HÀ and LÊ's methods.

The main tool in achieving a suitable compactification of a morphism $f\colon X \to S$ will be the following Minimal Models Theorem. A minimal model is a certain compactification of the given morphism. Its existence will be shown below in Section 2.2.

In addition it is necessary to show the minimal model to be smooth. For this, the morphism f will be examined locally over a Henselization $\mathcal{O}^h_{S,z}$, with $z \in S$, in Section 2.3. An arbitrary compactification of the morphism results in components in the "infinite part" of the closure of one fiber. For a Henselian base space there exists the contraction morphism, with which these components can be contracted to isolated points. The resulting space X' with morphism f' is then a relatively minimal model of $X \times_S \operatorname{Spec}(\mathcal{O}^h_{S,z})$, and the fibers of f in X are dense in the fibers of f' in X'. Since the generic fiber of the normalization X'' of X' is smooth, it follows that

$$p_a((X'')_\eta) = p_g((X'')_\eta).$$

It will be shown that the arithmetic genus of the fibers in X'' does not vary over the base $\operatorname{Spec}(\mathcal{O}^h_{S,z})$. The fibers in X are contained densely in the fibers of X''. This is implying the same for X' and X'' in the local Henselian case. Hence arithmetic and geometric genus of the fibers of X'' are equal and constant over the local Henselization, which will prove the morphism to be smooth by a result of T. SEKIGUCHI, F. OORT, and N. SUWA, see [SOS89, Lemma 2.3]. The arithmetic genus being constant and positive makes X'' a minimal regular model of $X \times_S \operatorname{Spec}(\mathcal{O}^h_{S,z})$.

Let $\bar{X}$ be the minimal model of X over S. Due to properties of the local Henselization, the minimal model $\bar{X}$ commutes with the local minimal model X'' which is smooth. The smoothness of $\bar{X}$ follows from the uniqueness of the minimal model.

For this situation the Ehresmann Fibration Theorem, Theorem 2.49, is giving local differentiable trivializations on $\bar{X}$. To show that X also defines a $\mathscr{C}^\infty$-fiber bundle, it will be shown that the unit vector field in S can be lifted locally to a vector field w in $\bar{X}$ such that w is tangential to $\bar{X} \setminus X$. For this, the morphism $\bar{X} \to S$ restricted to $\bar{X} \setminus X$ is shown to be smooth as well. "Glueing" these local vector fields together with a partition of unity is resulting in a globally integrable vector field, whose restriction to $\bar{X} \setminus X$ and X is defining an integrable vector field on $\bar{X} \setminus X$ and X respectively. From this vector field local differentiable trivializations will be constructed using a recursive argument such that the space $(\bar{X},\ X)$ turns out to be a differentiably fibered pair.

2.1 The Minimal Regular Model

2.1 Definition (Monoidal Transformation). Let X be a regular surface. A *monoidal transformation* of X is defined to be the blowing-up of a single point $P \in X$. Let $f\colon \tilde{X} \to X$ be the monoidal transformation with center P. Then f induces an isomorphism of $\tilde{X} \setminus f^{-1}(P)$ onto $X \setminus \{P\}$. The inverse image of P is a curve E, which is called an *exceptional curve* or *exceptional divisor*.

2.2 Definition (Strict Transform). Let X be a locally Noetherian scheme and $f\colon \tilde{X} \to X$ the blowing-up of X along a closed subscheme Y, which has the associated ideal $\mathscr{I}$. Let W be a closed subscheme of X not contained in Y. Let $\tilde{W}$ be the blowing-up of the inverse image ideal sheaf $(j^{-1}\mathscr{I})\mathcal{O}_W$ on W, where j is the imbedding of W into X. The closed subscheme $\tilde{W} \subseteq \tilde{X}$ is called the *strict transform of* W under the blowing up f.

2.3 Remark. Set-theoretically, the strict transform of W is the Zariski closure of $f^{-1}(W \setminus Y)$ in $\tilde{X}$.

2.4 Definition (Regular Model). Let S be a Dedekind scheme with function field K. Let C be a normal projective curve over K. A normal fibered surface $X \to S$ with generic fiber X_η together with an isomorphism $X_\eta \to C$ is called a *model* of C over S. It is called a *regular model* of C if X is regular. A *morphism* $X \to X'$ of two models of C is a morphism of S-schemes that is compatible with the isomorphisms $X_\eta \to C$, $X'_\eta \to C$.

2.5 Definition (Minimal Regular Model). A regular fibered surface $X \to S$ is *relatively minimal* or is a *relatively minimal model of its function field* $K(X)$, if it does not contain any exceptional divisor, i.e. every birational morphism of regular fibered surfaces $X \to Y$ is an isomorphism.

It is called *minimal* or a *minimal model of* $K(X)$, if every birational mapping of regular fibered surfaces $X \dashrightarrow Y$ is a birational morphism. A regular fibered surface $Y \to S$ with this property such that the generic fiber Y_η is isomorphic to X, is called a *minimal regular model of* X.

A minimal model is, of course, relatively minimal. If it exists, the minimal model is unique up to unique isomorphism.

A *morphism* of two (regular) models Y, Z of X_η is a morphism of fibered S-surfaces $Y \to Z$ which is compatible with the birational mappings $Y \dashrightarrow X$, $Z \dashrightarrow X$.

2.6 Remark. Analytically, a regular minimal model over a smooth complex curve is a manifold together with a proper holomorphic mapping.

The following theorem is insuring the existence of minimal regular models of regular curves. The existence will be proven in Section 2.2.

2.7 Theorem (Minimal Models Theorem). *Let R be a Dedekind domain, and let X be a regular curve over $S := \operatorname{Spec}(R)$. Assume that the fraction field $K(S)$ of R is algebraically closed in $K(X)$, i.e. the generic fiber X_η is geometrically integral. Then there exists a relatively minimal model Y of the function field $K(X)$. If $H^1(X_\eta, \mathcal{O}_{X_\eta}) \neq 0$, then Y is a minimal model of $K(X)$.*

See J. Lipman [Art86] for the existence of a regular model, and T.C.K. Chinburg [Chi86, §7] for the existence of the minimal regular model.

Let X be a projective scheme over a Noetherian ring, and let $\mathscr{F}$ be a coherent sheaf on X. The cohomology groups $H^p(X, \mathscr{F})$ are finitely generated for all $p \geq 0$ by the theorem of J.-P. Serre above, Theorem 1.16, and vanish for $p > \dim X$ by Theorem 1.18. Therefore the Euler-Poincaré characteristic in the following definition is well defined.

2.8 Definition (Euler-Poincaré Characteristic). Let X be a projective variety over a field k. Let $\mathscr{F}$ be coherent sheaf on X. The alternating sum

$$\chi_k(\mathscr{F}) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X,\, \mathscr{F})$$

is called the Euler-Poincaré characteristic of $\mathscr{F}$.

2.9 Definition (Arithmetic Genus). Let X be a projective curve over a field k. The *arithmetic genus p_a of X* is defined to be the integer

$$p_a := 1 - \chi_k(\mathcal{O}_X).$$

If the projective curve X is geometrically connected and geometrically reduced, so that $H^0(X, \mathcal{O}_X) = k$, then

$$p_a = \dim_k H^1(X,\, \mathcal{O}_X).$$

2.10 Definition (Geometric Genus). Let X be a smooth variety of pure dimension n over a field k. The invertible sheaf on X

$$\omega_X := \bigwedge^n \Omega_{X/k}$$

is called the *canonical sheaf* of X, the nth exterior power of the sheaf of differentials, where $n = \dim X$. If X is projective and smooth, define the *geometric genus* p_g of X to be

$$p_g := \dim_k H^0(X,\, \omega_X).$$

However, if X is a curve and either not smooth or not projective, the geometric genus of X is defined to be the geometric genus of its unique geometrically integral and normal compactification. This unique compactification always exists for curves. If $k = \mathbb{C}$ and X is normal, this means that X is topologically equivalent to a sphere with a finite number of handles attached to it, and having a finite number of punctures. If X is smooth, the punctures are filled with points without adding singularities. The geometric genus is then the usual topological genus, simply defined to be the number of handles.

2.11 Remark. In case of a projective smooth curve X over a field, $H^1(X, \mathcal{O}_X)$ and $H^0(X, \omega_X)$ are dual vector spaces as a consequence of Serre Duality, [Har77, III, Theorem 7.6]. Therefore the arithmetic genus and geometric genus coincide.

2.12 Proposition. *Let S be a Dedekind scheme with generic point η and closed point s. Let $f : X \to S$ be a projective morphism and $\mathscr{F}$ a coherent sheaf on X that is flat over S. Then*

$$\chi_{k(s)}(\mathscr{F}_s) = \chi_{k(\eta)}(\mathscr{F}_\eta).$$

See [EGA, III_2, Corollaire 7.9.3].

The fact that the Euler-Poincaré characteristic is constant for flat structure sheaves of fibers of a surface will be used to conclude the constance of the arithmetic genus of these fibers.

2.2 Existence of the Minimal Regular Model in the Global Case

The results of this section can be found in [Art86] and [Chi86]. For the construction of a minimal regular model, a regular fibered surface is needed. The following theorem of J. Lipman is used to construct such a regular surface.

2.13 Theorem (Lipman). *Let X be a Noetherian normal connected and excellent scheme of dimension 2. Define a sequence of schemes*

$$X = X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \cdots$$

inductively as follows: Let $S_i \subset X_i$ be the (reduced) singular locus. Then X_{i+1} is the normalization of the blowing-up of S_i in X_i or equivalently, the normalization of the scheme obtained by blowing up the maximal ideal of S_i in succession. Then each X_i is a Noetherian, normal, connected and excellent scheme of dimension 2, and the mappings f_i are proper. Moreover the scheme X_n is regular if n is sufficiently large.

See M. Artin's presentation of the proof in [Art86].

For the following, all schemes and rings are excellent and Noetherian, R is a Dedekind ring, the curve X over R is connected and normal. Moreover S is always normal.

2.2.1 The Factorization Theorem and the Castelnuovo Criterion

The next theorem shows that proper birational morphisms are blowing-ups.

2.14 Theorem (Factorization Theorem). *Let X, X' be regular surfaces and $\pi : X' \to X$ be a proper birational morphism. Then X' is isomorphic to the scheme obtained from X by a finite number of successive blowing-ups.*

See [Chi86, §2].

Using the Factorization Theorem it is now possible to clarify the uniqueness of blowing-downs of exceptional curves on regular surfaces.

2.15 Lemma. *Let Y be a normal locally Noetherian scheme, let X be an integral scheme, and $f : X \to Y$ a proper birational morphism. Then the following properties hold.*

(a) *The canonical homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.*

(b) *There exists an open subset V of Y such that $f^{-1}(V) \to V$ is an isomorphism, and X_y has no isolated points if $y \notin V$. Moreover, the complement of V has codimension ≥ 2.*

See [Liu02, Corollary 4.4.3].

2.16 Corollary. *Let X be a regular surface and Y be a normal surface. Suppose $\pi : X \to Y$ is a proper birational morphism, a blowing-down of a prime divisor E on X to a point P of Y and an isomorphism outside of $|E|$. Then Y and P determine X and E up to isomorphism over Y, and X and E determine Y up to isomorphism.*

Proof. With E being prime, the Factorization Theorem shows π to be the blowing-up of Y at P. Therefore X and E are determined up to isomorphism over Y. The converse statement is a consequence of Lemma 2.15. □

2.17 Definition (Intersection Product). Let $i : E \to X$ be a closed injection of a proper integral curve E over a field k into a regular scheme X. There is a canonical morphism from the group of Cartier divisors $\mathrm{Div}(X)$ into the group of Weil divisors $Z^1(X)$. Since X is normal, this morphism is an immersion. From the regularity of X follows that the morphism is surjective. It is now possible to define a *positive Cartier divisor on* X to be a closed subscheme F of X such that the sheaf of ideals $\mathscr{I}$ which defines F is invertible. Define $i_k(E, F)$, the *intersection of E and F with respect to k*, to be $\deg(i^*\mathscr{I}^{-1})$, where $i^*\mathscr{I}^{-1}$ is the induced invertible sheaf on E. If $k = H^0(E, \mathcal{O}_E)$, the *self-intersection* $E^{(2)}$ of E is defined to be $i_k(E, E)$. A Cartier divisor on X is a formal integral linear combination of positive Cartier divisors.

The Castelnuovo Criterion is used to decide whether a fibered surface is minimal. It states that an exceptional divisor of a regular fibered surface is contained in a single fiber, is of genus zero, and has only normal self-intersection.

2.18 Theorem (Castelnuovo Criterion). *Let $X \to S$ be a regular fibered surface. A prime divisor E of X is exceptional if and only if*

(a) E is contained in a fiber X over a closed point of S,

(b) $H^1(E, \mathcal{O}_E) = 0$, and

(c) $E^{(2)} = -1$.

In this case E is isomorphic to $\mathbb{P}^1_k$ over the field $k = H^0(E, \mathcal{O}_E)$.

See [Chi86, §6].

2.2.2 The Minimal Models Theorem

2.19 Lemma. *Let R be a Dedekind ring and let X be a regular fibered surface over $S := \mathrm{Spec}(R)$. Assume that the fraction field $K(S)$ of R is algebraically closed in $K(X)$, i.e. the generic fiber X_η is geometrically integral. Then X has the following properties.*

(a) The fibers of X are connected.

(b) Let x be a closed point of S, with residue field $k = k(x)$. Let V be the real vector space whose basis is the set $\{F_i\}$ of irreducible components of the fiber X_x of X over x. Then $i_k(X_x, D) = 0$ for all $D \in V$. The pairing on $V/\mathbb{R}X_x$, induced by $i_k(\, , \,)$, is negative definite.

(c) The exceptional divisors on X lie in reducible fibers of X. The number $\alpha(X)$ of irreducible divisors of X which lie in reducible fibers of X is finite.

By linearity the intersection product i_k can be prolonged to V.

See [Chi86, Lemma 7.1].

2.20 Corollary. *Let X and S be as in Lemma 2.19. Construct a sequence $\{X(n)\}_n$ of regular curves over S in the following way. Let $X(0) = X$. If $X(n)$ has been defined, and there is an exceptional curve on $X(n)$, let $\pi_n : X(n) \to X(n+1)$ be a blowing-down over S of one such curve. Then the sequence $\{X(n)\}_n$ is necessarily finite. If Y is the last term in the sequence, then Y is a relatively minimal model for $K(X)$.*

Proof. The number $\alpha(X(n))$ of irreducible divisors of $X(n)$ which lie in reducible fibers must decrease as n increases. Since the number $\alpha(X(0))$ of components in $X(0)$ is finite, the sequence $\{X(n)\}_n$ must be finite. The final term Y in this sequence can have no exceptional curves, therefore Y is a relatively minimal model for $K(X)$ by the Factorization Theorem. □

The following lemma is the key in showing that Y in Corollary 2.20 is a minimal model if the generic fiber of X has positive genus.

2.21 Lemma. *Let R be a Dedekind ring and let X be a regular fibered surface over $S := \operatorname{Spec}(R)$. Assume that the fraction field $K(S)$ of R is algebraically closed in $K(X)$, i.e. the generic fiber X_η is geometrically integral. Let $X' \to X$ be a birational S-morphism from a regular curve X' over S to X, which is the blowing-down of an exceptional curve E on X' to a point $P \in X$. Assume that*

$$H^1(X_\eta, \mathcal{O}_{X_\eta}) \neq 0.$$

Suppose that C' is an exceptional curve on X'. Then either $C' = E$, or $C = f(C')$ is an exceptional curve in X which does not contain P.

One of the key results used in the proof of this lemma in [Chi86] is Lemma 2.19 (b). First it is shown that, in case $C' \neq E$, the curve C is an exceptional curve if $P \notin C$. Furthermore, in case $P \in C$, Lemma 2.19 (b) provides that C is a rational multiple of a fiber of X over a closed point of S. Hence $C^{(2)} = 0$. It is shown that C is isomorphic to $\mathbb{P}^1_k$, where $k = H^0(C', \mathcal{O}_{C'}) = H^0(C, \mathcal{O}_C)$. Finally it is concluded that

$$H^1(X_\eta, \mathcal{O}_{X_\eta}) = H^1(X, \mathcal{O}_X) \otimes_R K = 0,$$

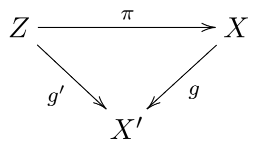
where K is the quotient field of the Dedekind ring R. This is contradicting the hypothesis of $H^1(X_\eta, \mathcal{O}_{X_\eta}) \neq 0$, proving the lemma.

For a detailed proof see [Chi86, Lemma 7.2].

2.22 Definition (S-birational Mapping). Let X and Y be regular surfaces over a base scheme S. An *S-rational mapping* $f\colon X \to Y$ is an equivalence class of S-morphisms from open dense subsets of X to Y, where two such morphisms are equivalent if they agree on the intersection of their domains. By [EGA, I, Proposition 7.2.2], there is a largest dense open set U in X, on which an element in the equivalence class of f is defined. The set U is called *the domain of f*. In particular, f is a morphism if and only if f is defined at each $x \in X$, i.e. an element in the equivalence class of f has x in its domain. By [EGA, II, Proposition 7.3.5], the codimension of $X \setminus U$ in X is at least two.

If f induces an isomorphism between the function fields of X and Y, then f is called an *S-birational mapping.*

2.23 Proposition. *Let $g : X \to X'$ be a proper S-birational mapping between regular 2-dimensional surfaces, which are proper over a base scheme S. Then there is a regular surface Z and a proper birational S-morphism $\pi : Z \to X$ and $g' : Z \to X'$ such that $g' = g \circ \pi$.*



See [Chi86, Proposition 2.2].

2.24 Theorem (Minimal Models Theorem). *Let R be a Dedekind domain, and let X be a regular curve over $S := \mathrm{Spec}(R)$. Assume that the fraction field $K(S)$ of R is algebraically closed in $K(X)$, i.e. the generic fiber X_η is geometrically integral. Construct a sequence $\{X(n)\}_n$ of regular curves over S as done in Corollary 2.20.*
If $H^1(X_\eta, \mathcal{O}_{X_\eta}) \neq 0$, then the final curve Y in this sequence is a minimal model of $K(X)$.

See T.C.K. Chinburg [Chi86, §7]. The proof of the Minimal Models Theorem was first given independently by S. Lichtenbaum and I.R. Shafarevich, cf. [Lic68] and [Sha66].

Proof. According to Corollary 2.20, the final curve Y is a relatively minimal model for $K(X)$. Suppose Y and Y' are relatively minimal models of $K(X)$, and assume that the general fiber of X has positive genus. It follows from Proposition 2.23 that there is a regular curve X' over S for which there are proper S-birational morphisms $\pi : X' \to Y$ and $\pi' : X' \to Y'$. Suppose further that X' has been chosen, so that $\alpha(X')$, the number of irreducible divisors contained in reducible fibers of X', is minimal. It follows from Lemma 2.21 and $H^1(X_\eta, \mathcal{O}_{X_\eta}) \neq 0$ that X' can not have any exceptional curves. From the Factorization Theorem, Theorem 2.14, follows that π and π' are isomorphisms. □

2.3 Existence of the Minimal Regular Model in the Local Henselian Case

For a fibered surface over the spectrum of a Henselian discrete valuation ring there exists the contraction morphism for irreducible components. This contraction is used to construct a fibered surface, which does not contain any exceptional divisors and therefore is a relatively minimal regular model, whose fibers contain the fibers of the original surface densely.

If the fibers of the original surface are irreducible and of constant positive geometric genus, then these attributes are inherited through the construction by the relatively minimal regular model, which in turn is shown to be smooth and minimal. Later, this construction will yield a submersion in the analytic sense, allowing to create a differentiable trivialization in the main theorem.

2.3.1 Henselization

2.25 Definition (Étale Morphism). Let $f : X \to Y$ be a morphism of finite type of locally Noetherian schemes. Let $P \in X$ and $P' = f(P)$. The morphism f is called *unramified* at P if the homomorphism $\mathcal{O}_{Y,P'} \to \mathcal{O}_{X,P}$ verifies $\mathfrak{m}_{P'}\mathcal{O}_{X,P} = \mathfrak{m}_P$, i.e. $\mathcal{O}_{X,P}/\mathfrak{m}_{P'}\mathcal{O}_{X,P} = k(P)$, and if the (finite) extension of residue fields $k(P') \to k(P)$ is separable. The morphism f is called *étale* at P if it is unramified and flat at P. A homomorphism of Noetherian local rings $A \to B$ is called *étale* if it is unramified, flat, and if B is a localization of a finitely generated A-algebra. The morphism is called *unramified* (resp. *étale*) if it is unramified (resp. étale) at every point of X.

For this section let R be a local ring with residue field k. Let S be the affine (local) spectrum of R and let s be the closed point of S.

2.26 Definition (Henselian Ring). The local ring R is called *Henselian* if, for each monic polynomial $P \in R[T]$, all k-rational simple zeros of the residue class $\bar{P} \in k[T]$ lift to R-rational zeros of P.

2.27 Definition (Jacobson Radical). Let R be a ring. The intersection of all maximal ideals in R,

$$\mathfrak{r} := \bigcap_{\mathfrak{m} \in \operatorname{Spm}(R)} \mathfrak{m},$$

is called the *Jacobson radical of* R, where $\operatorname{Spm}(R)$ is the set of maximal ideals of R.

2.28 Theorem (Hensel's Lemma). *Let R be a ring, which possesses only a finite number of maximal ideals. Let R be separated and complete with respect to the $\mathfrak{r}$-adic topology, where $\mathfrak{r}$ is the Jacobson radical of R. Then R is Henselian.*

See A. Grothendieck [EGA, IV_4, 18.5.14].

Equivalently to Definition 2.26 there is the following definition.

2.29 Definition (Henselian Scheme). The local scheme S is called *Henselian* if each étale mapping $S' \to S$ is a local morphism at all points x of S' over s with trivial residue field extension $k(x) = k(s)$.

Since a local Noetherian ring R is always a subring of its $\mathfrak{m}$-adic completion $\hat{R}$, these local rings are a priori subrings of Henselian rings. The "smallest" Henselian ring containing R is called the Henselization of R.

2.30 Definition (Henselization). A *Henselization* of a local ring R is a Henselian local ring R^h together with a local morphism $i : R \to R^h$ such that the following universal property is satisfied: For any local morphism $u : R \to A$ from R to a Henselian local ring A, there exists a unique local morphism $u^h : R^h \to A$ such that $u^h \circ i = u$.

2.31 Remark. If the Henselization exists, it is unique up to unique isomorphism. Moreover, the residue field of R^h must be k. In view of Definition 2.29, the Henselization of R must be the "union"of all local ring $\mathcal{O}_{X,x}$ of étale R-schemes at points x above the closed points s of $S = \operatorname{Spec}(R)$, whose residue fields coincide with k. For the existence of such a "union" in terms of inductive limits, see [BLR90, 2.3, Lemma 7].

2.32 Definition (Vertical curve). Let $f : X \to S$ be a fibered surface over a Dedekind scheme S. A closed curve E in X is called *vertical* if $f(E)$ is reduced to a point.

2.33 Definition (Contraction Morphism). Let $X \to S$ be a normal fibered surface. Let $\mathcal{E}$ be a finite set of integral (projective) curves on X. A normal fibered surface $X' \to S$ together with a projective birational morphism $f : X \to X'$ such that for every integral vertical curve $\mathcal{E}$ on X, the set $f(E)$ is a point if and only if $E \in \mathcal{E}$, is called a *contraction morphism of the* $E \in \mathcal{E}$.

The following theorem is the key result in the construction of a compactification of the morphism of a fibered surface, having no 1-dimensional components in the infinite part.

2.34 Theorem. *Let $X \to S$ be a normal fibered surface over the spectrum of a Henselian discrete valuation ring. Then for any proper subset $\mathcal{E}$ of the set of irreducible components of X_s, where $s \in S$ is a closed point, the contraction morphism of $E \in \mathcal{E}$ exists.*

See [Liu02, Theorem 8.3.36].

2.3.2 The Minimal Regular Model in the Henselian Case

The existence of the contraction morphism over a Henselian base will be used in the following to construct a compactification, whose fibers densely contain the fibers of the original surface. This way, the appearance of positive-dimensional components in the closure will be circumvented.

2.35 Proposition. *Let S be a Dedekind scheme and $f : X \to S$ be a normal integral flat S-scheme, and let f be surjective. Suppose that S is Henselian, and* $\dim X = 2$. *Then there exists a compactification X' of X such that the the fibers of X are dense in the fibers of X'.*

Proof. Let X'' be a compactification of f. Let $\mathcal{E}$ be the set of curves in the "infinite part" $X''_s \setminus X_s$ of the compactification. By Theorem 2.34 there exists the contraction morphism $f : X'' \to X'$, where $f(E)$ is a point for each $E \in \mathcal{E}$. Therefore X' is the desired compactification. □

2.36 Proposition. *Let S be a Dedekind scheme and $f : X \to S$ be an integral projective scheme such that f is dominant and geometrically integral, and* $\dim X = 2$*. Then the function $t \mapsto p_a(X_t)$ with $t \in S$ is constant.*

Proof. By Proposition 1.14, $\mathcal{O}_X$ is flat over S, since X is integral. Then by Proposition 2.12 the Euler-Poincaré characteristic $\chi(\mathcal{O}_{X_t})$ of the fibers X_t is constant. □

2.37 Definition (δ-invariant). Let A be a ring and M be an A-module. The module M is called *simple* if $M \neq \{0\}$, and if 0 and M are the only sub-A-modules. It is called *of finite length* if there exists a chain

$$0 = M_0 \subset \cdots \subset M_n = M$$

of sub-A-modules of M such that M_{i+1}/M_i is simple for every $i \leq n-1$, in particular, $M_{i+1} \neq M$. The *length of* M is then defined to be $\text{length}_A(M) := n$.

Let X be an integral projective scheme of dimension 1 over a field k, and let $f : \tilde{X} \to X$ be its normalization. Define the δ-invariant δ_P for $P \in X$ as

$$\delta_P := \text{length}_{\mathcal{O}_{X,P}}(\tilde{\mathcal{O}}_{X,P}/\mathcal{O}_{X,P}),$$

where $\tilde{\mathcal{O}}_{X,P}$ is the integral closure of $\mathcal{O}_{X,P}$. Then $\delta_P = 0$ if and only if P is a normal, hence regular point of X. The δ-*invariant* of X is then defined as

$$\delta := \sum_{P \in X} \delta_P.$$

Hence $\delta = 0$ if and only if X is regular.

2.38 Remark. Let X be an integral projective scheme of dimension 1 over a curve over a field k. It follows from the definition that, between the δ-invariant and the geometric genus, there is the close relation

$$p_g(X_\eta) - p_g(X_s) = \delta.$$

Using the constance of the two genera, the following proposition is insuring the smoothness of the morphism of the minimal regular model constructed later in Theorem 2.44.

2.39 Proposition (Sekiguchi, Oort, Suwa). *Let R be a discrete valuation ring and $S := \mathrm{Spec}(R)$. By η and s denote the generic and special points of S, respectively. Let $f : X \to S$ be a projective flat morphism with geometrically integral curves as fibers. Assume $p_g(X_\eta) = p_g(X_s)$. Then the mapping $\tilde{f} : \tilde{X} \to S$ induced by the normalization of X has the non-singular models of X_η and X_s as fibers, i.e. the morphism $\tilde{f}$ is smooth.*

The proof of this proposition in [SOS89] is written very short and may consequently not be transparent. Therefore, a proof is given at this point.

Proof. Since the fiber X_s is geometrically reduced, the mapping $\mathcal{O}_{X_s} \to \mathcal{O}_{\tilde{X}_s}$ is injective and therefore the following sequence of coherent sheaves,

$$0 \to \mathcal{O}_{X_s} \to \mathcal{O}_{\tilde{X}_s} \to \mathcal{O}_{\tilde{X}_s}/\mathcal{O}_{X_s} \to 0,$$

is exact. Hence there is a long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(X_s, \mathcal{O}_{X_s}) \to H^0(\tilde{X}, \mathcal{O}_{\tilde{X}_s}) \to H^0(X_s, \mathcal{O}_{\tilde{X}_s}/\mathcal{O}_{X_s}) \\ &\to H^1(X_s, \mathcal{O}_{X_s}) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}_s}) \to H^1(X_s, \mathcal{O}_{\tilde{X}_s}/\mathcal{O}_{X_s}) \to 0. \end{aligned}$$

In this case, since a 0-dimensional scheme is affine,

$$H^1(X_s, \mathcal{O}_{\tilde{X}_s}/\mathcal{O}_{X_s}) = 0$$

by Theorem 1.17. Since X_s and $\tilde{X}_s$ are geometrically reduced and connected, both $H^0(X_s, \mathcal{O}_{X_s})$ and $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}_s})$ are of dimension 1. By Proposition 2.12, it follows from the flatness of f that the arithmetic genus of the fibers of $\tilde{X} \to S$ is constant.

Since the generic fiber of S, $\tilde{X}_\eta \to \mathrm{Spec}(K(\eta))$ is geometrically reduced and normal, it is therefore smooth. Consequently

$$p_a(\tilde{X}_\eta) = p_g(\tilde{X}_\eta) = p_g(X_\eta) = p_a(X_s).$$

By hypothesis the geometric genus is constant. It follows that

$$\dim H^1(X_s, \mathcal{O}_{X_s}) = \dim H^1(\tilde{X}_s, \mathcal{O}_{\tilde{X}_s}).$$

From the exactness of the long exact cohomology sequence follows that

$$H^0(X_s, \mathcal{O}_{\tilde{X}_s}/\mathcal{O}_{X_s}) = 0.$$

□

For the original proof, cf. [SOS89, Lemma 2.3].

2.40 Remark. The δ-invariant is a measure for the difference of the arithmetic and geometric genus of a curve. The δ-invariant being 0 is equivalent to the non-existence of vanishing cycles in the fibers.

2.41 Definition (Depth). Let A be ring. Let M be an A-module. An element $a \in A$ is said to be *M-regular*, if the mapping $M \to M$ defined by multiplication by a is injective. A sequence of elements $a_1, \dots, a_n$ of A is called *M-regular*, if a_1 is regular for M, and if a_{i+1} is regular for $M/(a_1 M + \cdots a_i M)$ for every $1 \leq i \leq n-1$. If I is an ideal of A such that $IM \neq M$, and if the $a_i \in I$, it is called an *M-regular sequence in I*. The $I -$ depth *of* M, denoted by $\operatorname{depth}_I M$, is the maximal number of elements of an M-regular sequence in I. When A is a Noetherian local ring with maximal ideal $\mathfrak{m}$, and M is finitely generated over A, $\operatorname{depth}_{\mathfrak{m}} M$ is denoted by $\operatorname{depth} M$.

2.42 Definition (Serre Condition). Let $n \geq 0$ be an integer. A locally Noetherian scheme X is said to *verify Serre condition* (S_n), if for any $P \in X$ it holds

$$(S_n) \qquad \operatorname{depth} \mathcal{O}_{X,P} \geq \inf\{n, \dim \mathcal{O}_{X,P}\}.$$

2.43 Remark. A locally Noetherian scheme X always verifies property (S_0). Property (S_1) is equivalent to X having no embedded points.

2.44 Theorem. *Let S be a complete scheme of dimension 1 and let $X \to S$ be a geometrically integral quasi-projective flat S-scheme such that its fibers are of positive geometric genus such that X is of dimension 2. Furthermore let the function $t \mapsto p_g(X_t)$ with $t \in S$ be constant. Then the minimal model of X is smooth.*

Proof. Since S is complete, it is also Henselian by Theorem 2.28. Then by Proposition 2.35, there exists a compactification X' of X such that the fibers of X are dense in the fibers of X'. Since S is the spectrum of a complete discrete valuation ring, S is excellent, and in particular the normalization $f : X'' \to X$ of X'' is finite. The generic fiber over S, $X''_\eta \to \operatorname{Spec}(K(\eta))$, is geometrically reduced, it is therefore smooth. Consequently $p_g(X''_\eta) = p_a(X''_\eta)$, cf. Remark 2.11. By Proposition 2.36 the arithmetic genus of the fibers of $X'' \to S$ is constant. Since the fibers of X are dense in the fibers of X', the fibers of X and X' and therefore X'' have the same geometric genus. Hence both geometric and arithmetic genus of the fibers of X'' are constant and positive.

Since X'' is normal, it satisfies Serre condition (S_2), cf. [Liu02, Lemma 8.2.21]. From the flatness of the morphism follows that the special fiber X''_s satisfies Serre condition (S_1), i.e. possesses no embedded points. By construction, the geometrically reduced scheme X_s is densely contained in X''_s. Since X''_s satisfies (S_1), it follows that X''_s is reduced as well.

Therefore $X'' \to S$ is smooth by Proposition 2.39, and as a consequence X'' is regular. Already $X'' \to S$ is relatively minimal, and since in particular the arithmetic genus of the generic fiber is positive, it follows from Theorem 2.7 that $X' \to S$ is the desired smooth minimal model. □

2.45 Remark. The fibers of the original fibered surface are dense in the fibers of the minimal regular model constructed above. The condition on the geometric genus to be positive is necessary, since otherwise the relatively minimal model constructed above will generally not be minimal and may therefore not be unique up to isomorphism.
Let $X = \mathbb{P}^1_S$ and $\bar{X}$ be the blowing-up of X with center a closed point $x \in X(k(s))$. In $\bar{X}_s$ the strict transform D of X_s is an exceptional divisor. Let $\bar{X} \to X'$ be the contraction of D. Then the models X and X' of $\bar{X}$ are relatively minimal, but not isomorphic as models of $\bar{X}$. The identity on the generic fiber induces a birational map $X \dashrightarrow X'$, which does not extend to a morphism, since the generic points of the fiber X_s and X'_s induce distinct valuations in $K(\bar{X})$.

2.4 Local Differentiable Triviality

The following proposition is insuring the compatibility of the minimal model with étale base change, and base change resulting from the completion of a local ring. With suitable conditions and after localization and Henselization in the base, the minimal regular model of a surface is smooth as shown in Section 2.3. Since the minimal regular model commutes with this type of base change, the global minimal regular model over the entire base is smooth.

2.46 Proposition. *Let $X \to S$ be a regular fibered surface over a Dedekind scheme such that $p_a(X_\eta) \geq 1$. Let $S' \to S$ be an étale surjective morphism, or let S be the spectrum of a discrete valuation ring R and $S' = \mathrm{Spec}(\hat{R})$, where $\hat{R}$ is the completion of the local ring R. Then $X \to S$ is minimal if and only if $X \times_S S' \to S'$ is minimal.*

See [Liu02, Proposition 9.3.28].

2.47 Theorem. *Let R be an excellent Dedekind domain, and let X be a geometrically integral quasi-projective curve over $S := \mathrm{Spec}(R)$. Suppose that the fibers of $X \to S$ are of positive genus. Furthermore suppose that the function $t \mapsto p_g(X_t)$ with $t \in S$ is constant. Then the (minimal) model of X is smooth.*

Proof. Let $X' \to X$ be the normalization of X. Since X' is Noetherian, normal, connected and excellent, there exists a regular surface X'' and a proper mapping $X'' \to X'$ due to Theorem 2.13. Since the fraction field $K(\mathrm{Spec}(R))$ is algebraically closed in $K(X'')$, by Theorem 2.7 there exists a relatively minimal regular model $\bar{X}$ of X''. It

follows from the geometric integrality of X_η that X''_η is also smooth, hence $p_a(X''_\eta) = p_g(X''_\eta) \geq 1$. Consequently $\bar{X}$ is the minimal model of X'' and therefore of X.
Let $t \in S$. Note that $\hat{\mathcal{O}}^h_{S,t} = \hat{\mathcal{O}}_{S,t}$. It follows from Theorem 2.44 that the minimal model of $X \times_S \operatorname{Spec}(\hat{\mathcal{O}}_{S,t})$ is smooth.

It follows from Proposition 2.46 that $\bar{X} \times_S \operatorname{Spec}(\mathcal{O}_{S,t})$ commutes with the local smooth minimal model of $X \times_S \operatorname{Spec}(\hat{\mathcal{O}}_{S,t})$. From the uniqueness of minimal regular models follows now that there exists an open neighborhood U of t such that the minimal regular model $\bar{X}$ is smooth over U, hence $\bar{X} \times_S U$ is smooth. Consequently $\bar{X}$ is smooth over the entire base space S. □

2.4.1 A Variation of the Ehresmann Fibration Theorem

The existence of a smooth minimal regular model is yielding a proper smooth morphism of the compactification of a suitable surface. The projective space is compact in the analytic category and the pendent to the proper algebraic morphism is therefore proper in the analytic sense. All following results are developed for real manifolds. To develop differentiable trivializations of the minimal regular model, and moreover the original surface, unit vector fields are lifted from the base space.

2.48 Definition (Submersion). A smooth mapping between differentiable manifolds $f\colon M \to N$ is called a *submersion* if and only if the tangential mapping $T_x f : T_x M \to T_{f(x)} N$ is surjective for all $x \in M$.

2.49 Theorem (Ehresmann). *Let X and S be differentiable manifolds, and $f : X \to S$ a proper (surjective) submersion. Then $(X,\, f,\, S)$ is a $\mathscr{C}^\infty$-fiber bundle.*

See [Kod86, §2.3].

2.50 Definition (Fibered pair). Let X be a smooth manifold , let S be smooth connected manifold, and Y be a submanifold of X. Let $f : X \to S$ be a differentiable fiber bundle. The pair $(X,\, Y)_f$ is called a *differentiably fibered pair* with projection f, if there exist local trivializations for f, which induce local trivializations for $f|_Y$. Thus there exists a pair $(F,\, E)$, where E is a submanifold of a manifold F. Furthermore for every $y \in S$ there exists a neighborhood U such that there exists a fiber preserving diffeomorphism $\tau : f^{-1}(U) \to U \times F$, which induces a diffeomorphism $\tau|_{(f|_Y)^{-1}(U)} \to U \times E$. This causes $(Y,\, f,\, S)$ and $(X \setminus Y,\, f|_{X\setminus Y},\, S)$ to be differentiable fiber bundles as well.

2.51 Definition (Partition of Unity). Let X be a topological space and $\{U_i\}_{i\in I}$ an open covering of X. A smooth *partition of unity*, subordinate to the covering $\{U_i\}_{i\in I}$, is a family $\{\rho_j\}_{j\in J}$ of smooth real-valued functions $\rho_j : X \to [0, 1]$ such that

(a) $\rho_j \geq 0$ everywhere.

(b) For all $j \in J$ there exists a $i \in I$ and a mapping $\tau \colon J \to I$ such that $\operatorname{supp}(\rho_j) := \overline{\{x \in X \mid \rho_j(x) = 0\}} \subset U_{\tau(j)}$.

(c) The systems of sets $\operatorname{supp}(\rho_j)$ is locally finite.

(d) $\sum\limits_{j \in J} \rho_j = 1$.

2.52 Remark. The sum in (d) is well defined because of the local finiteness in (c). The partition of unity is a tool for "glueing together" local vector fields to obtain a globally integrable flow.

2.53 Theorem. *Let X be a smooth manifold. For every open covering of X there exists a smooth partition of unity.*

See [BJ73, 7.3].

In the following, a differentiable local trivialization realizing a differentiably fibered pair of spaces is constructed by lifting vector fields and glueing them together with a partition of unity. This part is analogous to the proof of Thom's isotopy Lemma (cf. [GM80, 1.5]) without the use of controlled vector fields.

2.54 Definition (Flow). Let X be a differentiable manifold, and let $A \subset \mathbb{R} \times X$ be an open subset such that $\{0\} \times X \subset A$ and $A \cap (\mathbb{R} \times \{x\})$ is connected for all $x \in X$. A differentiable mapping

$$\Phi : A \to X$$

such that $\Phi(0, x) = x$ and $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$, whenever both sides are defined for all parameters is called a *(local) flow* on X. A flow with $A = \mathbb{R} \times X$ is called a *global flow.* For $x \in X$ write $A \cap (\mathbb{R} \times \{x\})$ as $I_x \times \{x\}$, where $I_x :=]a_x, b_x[$ with possible infinite a_x and b_x. The mapping

$$\begin{aligned} \gamma_x : I_x &\to X \\ t &\mapsto \Phi(t, x) \end{aligned}$$

is called the *flow line* of Φ through x. A flow is called *maximal*, if for all $x \in X$ the interval I_x is maximal.

Every flow Φ on X defines a vector field $\dot{\Phi}(0, x) := \dot{\gamma_x}(0) \in \Gamma(X, TX)$ on X. For the reverse statement see the following theorem.

2.55 Theorem. *Let X be a smooth manifold and $v \in \Gamma(X, TX)$ a vector field on X. Then there exists a maximal flow Φ_v, such that $\dot{\Phi}_v(0, x) = v(x)$.*

Proof. Every γ_x solves the differential equation $\dot{\gamma}(t) = v(\gamma(t))$. The theorem is therefore a restatement of the Existence and Uniqueness Theorem in the theory of ordinary differential equations. □

2.56 Theorem (Rank Theorem). *Let X be a smooth manifold of dimension n, and let S be a smooth manifold of dimension s. Let Y be a smooth submanifold of X, let $f\colon X \to S$ be a differentiable mapping and $x \in Y$. Suppose that f and $f|_Y$ are of constant rank r in a neighborhood of x. Then there exist charts (U, φ) of X with $x \in U$ and (V, ψ) of S with $f(x) \in V$ such that*

$$\varphi(U \cap Y) = \{(a_1, \dots, a_n) \in \varphi(U) \mid a_{m+1} = \cdots = a_n = 0\}$$
$$\psi \circ f \circ \varphi^{-1}(a_1, \dots, a_n) = (a_1, \dots, a_r, 0, \dots, 0).$$

Proof. Without loss of generality it is possible to assume $X = \mathbb{R}^n$, $Y = \{a \in \mathbb{R}^n \mid a_{m+1} = \cdots = a_n = 0\}$, $S = \mathbb{R}^s$, $x = 0$, and $f(0) = 0$. The germ of a function g is denoted by $\hat{g}$. After a change of coordinates, and since $\operatorname{rank}_0(f|_Y) \geq r$, it is possible to assume the matrix

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \leq i,j \leq r}$$

to be regular in 0. Define a function $h\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$h(a) := (f_1(a), \dots, f_r(a), a_{r+1}, \dots, a_n).$$

The Jacobian matrix Dh is regular in 0, and the germ $\hat{h}$ of h in 0 is invertible. Furthermore the functions h and h^{-1}, where defined, map points of Y into Y. The germ $\hat{g} := \hat{f} \circ \hat{h}^{-1}$ is represented by the mapping

$$b \mapsto (b_1, \dots, b_r, g_{r+1}(b), \dots, g_n(b)).$$

Its Jacobian matrix is of the form

$$Dg(b) = \begin{pmatrix} E_r & 0 \\ * & A(b) \end{pmatrix}, \text{ where } A(b) := \left(\frac{\partial g_i}{\partial x_j}\right)_{r+1 \leq i,j \leq n},$$

and E_r is the $(r \times r)$ unit matrix. Since $\operatorname{rank} f = r$ close to 0, $\hat{A} = 0$, hence $\partial g_i / \partial x_j = 0$ for $r + 1 \leq i$ and $j \leq n$. Hence, the g_i do not depend on the variables $b_{r+1}, \dots, b_n$. Let the germ $\hat{k}\colon (\mathbb{R}^s, 0) \to (\mathbb{R}^s, 0)$ be defined by

$$c \mapsto (c_1, \dots, c_r, c_{r+1} - g_{r+1}(c_1, \dots, c_r, 0, \dots, 0), \dots, c_s - g_s(c_1, \dots, c_r, 0, \dots, 0)).$$

It follows that

$$Dk = \begin{pmatrix} E_r & 0 \\ * & E_{s-r} \end{pmatrix},$$

hence $\hat{k}$ is invertible. Therefore $\hat{k} \circ \hat{f} \circ \hat{h}^{-1} = \hat{k} \circ \hat{g}$ is represented by

$$\begin{aligned} b \mapsto \hat{k}(b_1, \dots, b_r, g_{r+1}(b), \dots, g_s(b)) \\ = (b_1, \dots, b_r, 0, \dots, 0), \end{aligned}$$

since the g_i do not depend on $b_{r+1}, \dots b_n$, and is of the desired form. Furthermore Y is invariant under the coordinate change h^{-1}. □

2.57 Corollary. *Let X and S be smooth manifolds. Let X be n-dimensional, S be s-dimensional, and Y be a submanifold of X of codimension 1. Let $x \in Y$ and $f \colon X \to S$ be a differentiable mapping such that $f|_Y$ is a submersion in x. Then there exists a chart (U, φ) near x with $\varphi(U) \subset \mathbb{R}^n$ and $\varphi(U \cap Y) = \{(a_1, \dots, a_n) \in \varphi(U) \mid a_{m+1} = \dots = a_n = 0\}$ and a chart (V, ψ) near $f(x)$ such that*

$$\psi \circ f \circ \varphi^{-1} = (a_1, \dots, a_s).$$

Proof. Without loss of generality assume $X = \mathbb{R}^n$, $S = \mathbb{R}^s$, and $x = 0$. Since the rank of f is semicontinuous, the mapping f is still a submersion in a neighborhood of 0. Using the Rank Theorem, Theorem 2.56, on Δ^n and $\Delta^n \cap \{x \in \mathbb{R}^n \mid x_n = 0\}$ completes the proof. □

The following lemmata are insuring the liftability of vector fields to construct a differentiably fibered pair of spaces.

2.58 Lemma. *Let X and S be smooth manifolds, Y be a smooth submanifold of X of codimension 1 and $f \colon X \to S$ be a submersion such that $f|_Y$ is a submersion. For a vector field $v \in \Gamma(S, TS)$ on S, there exists a vector field $w \in \Gamma(X, TX)$ such that*

$$\begin{aligned} (T_x f)(w(x)) = v(f(x)), \quad \forall x \in X \text{ and} \\ w(x) \in T_x Y, \quad \forall x \in Y. \end{aligned}$$

Proof. Let $\{U_i\}_{i \in I}$ be an open covering of X and $v \in \Gamma(S, TS)$. The local existence of suitable vector fields $w_i \in \Gamma(U_i, TU_i)$ is a direct consequence of Corollary 2.57. For the global existence of a suitable vector field the local vector fields w_i are glued together using a partition of unity as follows. There exists a smooth partition of unity $\{\rho_i\}_{i \in I}$ subordinate to the covering $\{U_i\}_{i \in I}$ by Theorem 2.53. Define a global vector field on X as

$$w := \sum_{j \in J} \rho_j w_{\tau(j)},$$

where τ is a mapping in accordance to Definition 2.51. The differential of f at $x \in X$ is of the form

$$\begin{aligned}(T_x f)(w_x) &= (T_x f)\left(\sum_{i\in I} \rho_i(x) w_{ix}\right) \\ &= \sum_{i\in I} \rho_i(x)(T_x f)(w_{ix}) \\ &= (T_x f)(w_{ix}) \\ &= v_{f(x)}.\end{aligned}$$

Moreover, $w(x) \in T_x Y$ for all $x \in Y$, therefore w is the desired global vector field. Notice that $w|_Y$ is a vector field on Y. □

2.59 Lemma. *Let X be a smooth manifold and Φ be a flow on X. Let γ_x be the flow line of Φ through x. Suppose that $b_x < \infty$ with $I_x :=]a_x, b_x[$ being the maximal interval. Then $\lim_{t\to b_x} \gamma_x(t) = \infty$, with respect to the one-point compactification of X. The analogous statement is true for $a_x > -\infty$.*

Proof. Assume the negation of the conclusion. Let (t_n) be a sequence in $]a_x, b_x[$ such that $t_n \to b_x$. Since X is a locally compact Hausdorff space, the sequence (y_n), where $y_n := \gamma_x(t_n)$, possesses a converging subsequence in X. Let y be the limit of this subsequence, and γ_y be the flow line through y. Define

$$\tilde{\gamma}_x(t) := \begin{cases} \gamma_x(t), & a_x < t < b_x \\ \gamma_y(t - b_x), & b_x \le t < b_x + b_y \end{cases}.$$

Then $\tilde{\gamma}_x$ is a flow line through x, which is in contradiction to b_x being maximal. □

2.60 Lemma. *Let X and S be smooth manifolds, let Y be a smooth submanifold of X of codimension 1, and $f\colon X \to S$ be a proper differentiable mapping. Let w be a vector field on X and v be a vector field on S such that*

$$\begin{gathered}(T_x f)(w(x)) = v(f(x)), \quad \forall x \in X \text{ and} \\ w(x) \in T_x Y, \quad \forall x \in Y.\end{gathered}$$

Suppose that v is globally integrable, then so are $w|_{X\setminus Y}$ and $w|_Y$.

Proof. By assumption v is globally integrable. Let $x \in w|_{X\setminus Y}$ and γ_x be the solution curve of $w|_{X\setminus Y}$ through x with maximal interval $I_x :=]a_x, b_x[$. Assume that γ_x is not defined on all of $\mathbb{R}$, so without loss of generality let $b_x < \infty$. From Lemma 2.59 follows that $\lim_{t\to b_x} \gamma_x(t) = \infty$ in $X \setminus Y$, i.e. with respect to the one-point compactification of

$X \setminus Y$. This is equivalent to either $\lim_{t \to b_x} \gamma_x(t) = \infty$ in all of X, or $(\gamma_x(t_n))$ possesses a subsequence, which converges in Y, where (t_n) is a sequence with $t_n \to b_x$.

First consider $\lim_{t \to b_x} \gamma_x(t) = \infty$ in all of X. Since f is proper and continuous, it follows that $\lim_{t \to b_x} f(\gamma_x(t)) = \infty$ with respect to the one-point compactification of S. This is in contradiction to v being globally integrable in S, since $f \circ \gamma_x$ is a solution curve of v through $f(x)$. The second case is in contradiction to $w(x') \in T_{x'}Y$ for all $x' \in Y$.
With f, also $f|_Y$ is proper. Using the same argument on $f|_Y$ is giving the analogous result for $w|_Y$. □

2.61 Remark. The maximal flow Φ_w of the globally integrable vector field w on X constructed in Lemma 2.58 consists of the flows of $w|_{X \setminus Y}$ on $X \setminus Y$ and $w|_Y$ on Y. It still possesses all properties of a flow, since the flow lines on $X \setminus Y$ do not reach Y.

The following theorem is extending the statement of the Ehresmann Fibration Theorem from a differentiably fibered manifold to a differentiably fibered pair of manifolds.

2.62 Theorem. *Let X and S be smooth manifolds, Y be a smooth submanifold of X of codimension 1 and $f\colon X \to S$ be a proper submersion such that $f|_Y$ is a submersion. Then $(X,\, Y)_f$ is a differentiably fibered pair.*

Proof. Without loss of generality let $S = \mathbb{R}^s$. Let $e_1, \ldots, e_s$ be the unit vector fields on S. From Lemma 2.58 and Lemma 2.60 follows the existence of globally integrable vector fields $w_1, \ldots, w_s$ such that

$$(T_x f)(w_i(x)) = e_i, \quad \forall x \in X, \forall i \in \{1, \ldots, s\} \text{ and}$$
$$w_i(x) \in T_x Y, \quad \forall x \in Y, \forall i \in \{1, \ldots, s\}.$$

For $i \in \{1, \ldots, s\}$ let $\Phi_i\colon \mathbb{R} \times X \to X$ be the global flow for w_i. Define a differentiable mapping $\sigma\colon S \times f^{-1}(0) \to X$ with

$$\sigma((t_1, \ldots, t_s), x) = \Phi_1(t_1, \Phi_2(t_2, \ldots, \Phi_{s-1}(t_{s-1}, \Phi_s(t_s, x)) \ldots)).$$

It follows from the construction that $f(\sigma((t_1, \ldots, t_s), x)) = (t_1, \ldots, t_s)$. The mapping $\tau\colon X \to S \times f^{-1}(0)$ with

$$\tau(x) = (t_1, \ldots, t_s, \Phi_s(-t_s, \Phi_{s-1}(-t_{s-1}, \ldots, \Phi_2(-t_2, \Phi_1(-t_1, x)) \ldots))),$$

for $f(x) = (t_1, \ldots, t_s)$ is differentiable as well and inverse to σ. Therefore σ is the desired trivialization which causes $(X,\, Y)_f$ to be a differentiably fibered pair. □

2.4.2 The Main Theorem

Before proving the main result, it will be shown that the requirement of the total space to be geometrically integral over the base is already satisfied if the fibers above analytic, i.e. closed, points of the base space are irreducible.

2.63 Definition (Geometric Number of Irreducible Components). Let X be a scheme over a field k, and let $\bar{k}$ be the algebraic closure of k. The number n of irreducible components of $X \times_k \bar{k}$ is called the *geometric number of irreducible components of* X.

2.64 Theorem. *Let S be an irreducible variety, and let $f : X \to S$ be a morphism of varieties. Let n_η be the geometric number of irreducible components of X_η, where η is the generic point of S. Then there exists a neighborhood U of η in S such that the geometric number of irreducible components $n(X_s)$ of the fiber over s is equal to n_η for all $s \in U$.*

See [EGA, IV_3, Proposition 9.7.8].

2.65 Theorem. *Let X be a geometrically reduced algebraic variety over a field k. Then there exists a point in X with residue field a finite separable extension of k.*

See [Liu02, Proposition 3.2.20]

2.66 Theorem. *Let X be an algebraic variety over k, and let K/k be an algebraic extension. If X is reduced, and K/k is separable, then X_K is reduced.*

See [Liu02, Proposition 3.2.7].

2.67 Theorem. *Let X be a regular algebraic curve over a complex algebraic curve S such that the fibers of closed points are irreducible. Then X is geometrically integral over S.*

Proof. Since field extensions in characteristic 0 are separable, it follows with Theorem 2.66 that X is geometrically reduced over S.

Therefore, it is sufficient to prove X to be geometrically irreducible over S. Let η be the generic point in S, and let $\bar{\eta}$ be the algebraic closure of η. By hypothesis, the fibers above closed points are irreducible. Consequently, the geometric fiber above any closed point of S is irreducible. Hence, it is sufficient to show that the geometric generic fiber $X_{\bar{\eta}}$ is irreducible. It follows by Theorem 2.64 that there exists a neighborhood U of η in S such that the geometric number $n(X_s)$ of irreducible components of the fibers is constant for all $s \in U$. Since U is geometrically reduced and of finite type over a field, it follows by Theorem 2.65 that U contains a closed point. Fibers above closed points are geometrically irreducible. Therefore, all geometric fibers above U are geometrically irreducible. □

2.68 Remark. Under otherwise equal conditions, the proof of Theorem 2.67 also works in case X is a scheme of finite type over a reduced complex variety.

The additional hypothesis of the fibers being homeomorphic allows to show that the infinite part of the smooth minimal regular model constructed before is a smooth submanifold over S. With the results of Section 2.4.1 it is now possible to lift the unit vector field from the base space to construct a stratum-preserving differentiable trivialization resulting in a differentiably fibered pair of spaces.

2.69 Lemma. *Let X be a normal locally Noetherian scheme. Let F be a closed subset of X of codimension* $\operatorname{codim}(F, X) \geq 2$. *Then the restriction*

$$\mathcal{O}_X(X) \to \mathcal{O}_X(X \setminus F)$$

is an isomorphism. In other words, every regular function on $X \setminus F$ extends uniquely to a regular function on X.

See [Liu02, Theorem 4.1.4].

It is now possible to prove the main result.

2.70 Theorem (Main Theorem). *Let S be a smooth complex curve, and let X be a regular affine curve over S. Suppose that the fibers of $X \to S$ are of positive geometric genus and that the fibers above closed points of S are irreducible. Furthermore suppose that all fibers are pairwise homeomorphic. Then f defines a $\mathscr{C}^\infty$-fiber bundle.*

Proof. By Theorem 2.67, X is geometrically integral over S. All fibers of $X \to S$ have the same geometric genus, since all fibers of f are pairwise homeomorphic. Since S is a curve over $\mathbb{C}$, it is excellent. By Theorem 2.24 and Theorem 2.47 there exists a minimal model X' of X which is smooth. The morphism $f' : X' \to S$ is proper in the algebraic category. By GAGA, f' is also proper in the analytic sense. It follows from Theorem 2.49 that (X', f', S) defines a $\mathscr{C}^\infty$-fiber bundle. The "infinite part" $X' \setminus X$ of the minimal model X' consists of a set E of finitely many curves and, due to dimensional reasons, a set F of finitely many isolated points. The infinite part $X' \setminus X$ will be considered a scheme with the reduced structure. By construction, the infinite part $X'_\eta \setminus X_\eta$ of the generic fiber consists of a finite number of points $\{P_1, \ldots, P_n\}$. Since E does not possess any vertical part, it is evident that $E \subseteq \overline{\{P_1, \ldots, P_n\}}$.

To prove that $X' \setminus X = E$, it is sufficient to show that $F = \emptyset$. Without loss of generality let S be affine, hence X is affine. By assumption $X' \setminus \overline{\{P_1, \ldots, P_n\}}$ is 2-dimensional and normal. By Lemma 2.69, $X' \setminus (\overline{\{P_1, \ldots, P_n\}} \cup F) = X$ would not be affine. Hence $F = \emptyset$. Since all fibers are homeomorphic, the number of points in the infinite parts $X'_t \setminus X_t$ of a fiber does not vary with $t \in S$. Therefore the curves $\bar{P}_i$ are pairwise disjoint for $i \in \{1, \ldots, n\}$. As an additional consequence, the curves $\bar{P}_i$ cannot possess any self-intersections, since the cardinality of the set $X'_t \setminus X_t$ is constant.

The next step in the proof is to show that $f'|_E$ is a submersion. Since the morphism $f'|_E : E \to S$ is dominant on every irreducible component, and E is reduced, it is also flat. Therefore it is sufficient to show that it is unramified. Since field extensions in characteristic 0 are separable, $f'|_E$ is generically unramified. In addition, $f'|_E$ is finite and its fibers consist of $d := \deg f'$ or less points, where d is counted without multiplicities. After hypothesis the fibers are pairwise homeomorphic, therefore the number of points in one fiber of $f'|_E$ is constant, hence equal to d. Consequently the fibers are reduced. It follows that the mapping $f'|_E : E \to S$ is étale and in particular a submersion. Since E is a smooth submanifold of X' of codimension 1, and $f'|_E$ is a proper submersion, it follows from Theorem 2.62 that $(X', E)_{f'}$ is a differentiably fibered pair, which makes (X, f, S) a $\mathscr{C}^\infty$-fiber bundle. □

2.71 Remark. In particular, the existence of a smooth compactification of the affine morphism given in the theorem above is proven. Moreover, it is shown that the resulting morphism of the compactification is even smooth when restricted to the "infinite part", $X' \setminus X$, of the compactification.

With notation from the theorem above, let S possess a trivial fundamental group. In this case, for every curve $\bar{P}_i \subset E$, the mapping $f|_{\bar{P}_i} : \bar{P}_i \to S$ is even an isomorphism, and all curves in E therefore define global holomorphic sections.

2.72 Corollary. *Let X be a 2-dimensional non-singular complex affine algebraic variety and $f\colon X \to \mathbb{C}$ a polynomial mapping with irreducible fibers. The mapping f defines a $\mathscr{C}^\infty$-fiber bundle over a neighborhood of $z \in \mathbb{C}$ if the geometric genus of $f^{-1}(z)$ is positive, and $f^{-1}(z)$ is homeomorphic to the general fiber of f.*

Proof. Considering $\mathbb{C}$ as an affine line, this Corollary is a direct consequence of the previous theorem. □

3 Holomorphic Fiber Bundles

The construction of local triviality in the holomorphic sense will be divided into three sections. The first part is dealing with fiber bundles having an algebraic morphism as projection, and the compactification of the typical fiber is a hyperbolic Riemann surface. The Rigidity Theorem of S. Ju. Arakelov, A. N. Parshin,Y. Manin and H. Grauert ensures isotriviality for this type of family, and a result of W. Fischer and H. Grauert ensures local triviality for the minimal model of such a surface. For the base space being $\mathbb{C}, \mathbb{C}^*, \mathbb{P}^1_{\mathbb{C}}$, or the complex torus T, the infinite part of the minimal model is shown to be holomorphically trivial, rendering the original fibered surface locally trivial in the holomorphic sense.

A theorem of A. Beauville ensures isotriviality for families of elliptic curves, in case the base space is again one of $\mathbb{C}, \mathbb{C}^*, \mathbb{P}^1_{\mathbb{C}}$, or T. Local triviality in the holomorphic sense is constructed for families whose typical fiber is a torus with exactly one puncture. The regular minimal model of this family is locally trivial in the holomorphic sense, again by the result of W. Fischer and H. Grauert. The former punctures are translated on the compactified fibers such that they deform holomorphically over the base space, yielding holomorphic local trivializations on the original family of punctured tori.

At last fiber bundles having a sphere with up to three punctures as typical fiber are investigated in part three. The theorem of Fischer and Grauert is providing a local trivialization in the holomorphic sense for the bundle of the compactified punctured spheres. With Möbius transformations the punctures are translated on the surface such that they are transported holomorphically by deformations over the base space. This is resulting in holomorphic local trivialization on the original bundle of punctured spheres.

Non-Abelian Cohomology

To investigate the triviality of holomorphic fiber bundles with Lie groups as structure groups, it is necessary to define a cohomology theory for sheaves with values in non-abelian groups. Instead of cohomology groups it is only possible to construct cohomology sets. Without strong restrictions, it is only possible though to define these sets up to the first cohomology set. For a topological space X and a non-abelian sheaf $\mathscr{F}$, the set $H^0(X, \mathscr{F})$ is defined analogously to the abelian case as the set of global sections on X. The cohomological set $H^1(X, \mathscr{F})$ is defined using the relation on the 1-cochains of being pairwise cohomologous. In the following, indices will again be simplified in the sense of $U_{ij} := U_i \cap U_j$.

3.1 Definition. Let $\mathscr{F}$ be a sheaf of groups on a topological space X, and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of X. A 1-cochain $f : I^2 \to \prod_{(i,j)\in I^2} \Gamma(U_{ij}, \mathscr{F})$, denoted shorter by $\{f_{ij}\}$ on $\mathfrak{U}$ with values in $\mathscr{F}$ is called a 1-*cocycle*, if it satisfies the cochain condition

$$f_{ij}(x) f_{jk}(x) = f_{ik}(x) \quad \text{for all } x \in U_{ijk}.$$

Two 1-cochains $\{f_{ij}\}$ and $\{g_{ij}\}$ are called *cohomologous*, if there is a 0-cochain $\{h_i\}$ on $\mathfrak{U}$ with values in $\mathscr{F}$ such that

$$f_{ij}(x) = h_i^{-1}(x) g_{ij}(x) h_j(x) \quad \text{for all } x \in U_{ij}.$$

Therefore, the property of being cohomologous defined above defines an equivalence relation, which is denoted by $\sim_c$. Let $Z^1(\mathfrak{U}, \mathscr{F})$ be the set of 1-cocycles in regards to $\mathfrak{U}$ with values in $\mathscr{F}$. If $\mathscr{F}$ is a sheaf of abelian groups, $Z^1(\mathfrak{U}, \mathscr{F})$ possesses a group structure and the cohomology relation defined here is congruent with the usual cohomology theories.

3.2 Definition. Let $\mathscr{F}$ be a sheaf of groups on a topological space X. The *first cohomology set* of an open covering $\mathfrak{U}$ of X *with values in* $\mathscr{F}$ is defined to be the set of 1-cocycles in regard to $\mathfrak{U}$ modulo the relation of being cohomologous:

$$H^1(\mathfrak{U}, \mathscr{F}) := Z^1(\mathfrak{U}, \mathscr{F}) / \sim_c .$$

Generally $H^1(\mathfrak{U}, \mathscr{F})$ does not possess a group structure, since already $Z^1(\mathfrak{U}, \mathscr{F})$ generally does not possess one.

3.3 Definition. Let $\mathscr{F}$ be a sheaf of groups on a topological space X. The sets $H^1(\mathfrak{U}, \mathscr{F})$ are generating a directed system on the open coverings of X. Define *the first cohomology set of X with values in* $\mathscr{F}$ to be

$$H^1(X, \mathscr{F}) := \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, \mathscr{F}).$$

3.4 Remark. For a non-abelian sheaf $\mathscr{F}$, the set $H^1(X, \mathscr{F})$ possesses a distinguished element. Let p be a homomorphism between sheaves of groups $\mathscr{G}$ and $\mathscr{F}$, therefore inducing a mapping from $H^0(\mathfrak{U}, \mathscr{G})$ to $H^0(\mathfrak{U}, \mathscr{F})$, where U is an open subset of X, and furthermore a mapping from $C^1(\mathfrak{U}, \mathscr{G})$ to $C^1(\mathfrak{U}, \mathscr{F})$, preserving the cohomology relation of chains, where $C^1(\mathfrak{U}, \mathscr{F})$ is the set of 1-cochains in regards to $\mathfrak{U}$ with values in $\mathscr{F}$. This mapping is compatible with inductive limits, hence p defines a mapping $p^* : H^1(X, \mathscr{G}) \to H^1(X, \mathscr{F})$ leaving the neutral element invariant.

For a more detailed construction of the existence of long exact cohomology sequences in the non-abelian case see J. FRENKEL's work [Fre57], or [Epp03, Kap. 3].

3.5 Definition (Exactness). A sequence of sets A_i containing a distinguished element e and mappings $f_i : A_i \to A_{i+1}$ is called *exact*, if

$$f_i^{-1}(\{e\}) = f_{i-1}(A_{i-1}).$$

Define the kernel of f_i to be $\ker f_i := f_i^{-1}(\{e\})$.

Let $\mathscr{F}$ be a sheaf of groups and $\mathscr{G}$ a subsheaf of groups of $\mathscr{F}$. The sheaf of cosets $\mathscr{F}/\mathscr{G}$ possesses a neutral section e. Let $i : \mathscr{G} \to \mathscr{F}$ be the canonical injection, and $p : \mathscr{F} \to \mathscr{F}/\mathscr{G}$ the canonical projection. The sequence of sheaves

$$\{e\} \to \mathscr{G} \xrightarrow{i} \mathscr{F} \xrightarrow{p} \mathscr{F}/\mathscr{G} \to \{e\}$$

is exact, since on the stalks there is $\operatorname{im} i|_{\mathscr{G}_x} = \ker p|_{\mathscr{F}_x}$.

3.6 Theorem (Frenkel). *Let $\mathscr{G}$ be a subsheaf of normal divisors of the sheaf of groups $\mathscr{F}$. There exists an exact sequence*

$$\begin{aligned}\{e\} \to H^0(X,\mathscr{G}) \to H^0(X,\mathscr{F}) \to H^0(X,\mathscr{F}/\mathscr{G})&\\ \to H^1(X,\mathscr{G}) \to H^1(X,\mathscr{F}) \to H^1(X,\mathscr{F}/\mathscr{G}).&\end{aligned}$$

See [Fre57, p. 156].

3.7 Remark. If $\mathscr{G}$ is a central subsheaf of the sheaf of groups $\mathscr{F}$, i.e. $\mathscr{G}_x$ is contained in the center of $\mathscr{F}_x$ for all $x \in X$, the exact sequence in the previous Theorem extends to $H^2(X,\mathscr{G})$.

Let L be a complex Lie group, let B be a complex space, and let $\mathscr{L}^c$ (resp. $\mathscr{L}^a$) be the sheaf of germs of continuous (resp. holomorphic) mappings with values in L. The sheaves $\mathscr{L}^c$ and $\mathscr{L}^a$ are sheaves of groups which are abelian if and only if L is an abelian Lie group. Obviously $\mathscr{L}^a$ is a subsheaf of $\mathscr{L}^c$.

3.8 Proposition. *Let L be a complex Lie group, and let S be a complex space. Let $\mathscr{K}^c$ (resp. $\mathscr{K}^a$) be the set of isomorphism classes of topological (resp. holomorphic) fiber bundles with base S, structure group L and fiber F. Then $\mathscr{K}^c$ (resp. $\mathscr{K}^a$) is isomorphic to $H^1(S,\mathscr{L}^c)$ (resp. $H^1(S,\mathscr{L}^a)$).*

See F. Hirzebruch [Hir62, §3.2].

3.9 Definition (Stein Space). A complex space X is called *Stein space*, if it satisfies the following two properties.

(a) It is *holomorphically convex*: For any compact set $M \subset X$ the *holomorphically convex hull*

$$\hat{M} := \{x \in X \mid |f(x)| \leq \sup_M |f|,\ f \in \mathcal{O}(X)\},$$

is likewise a compact subset of X.

(b) It is *holomorphically spreadable*: For every $x \in X$ there is a neighborhood $U(x)$ and a holomorphic mapping $f : X \to \mathbb{C}^k$, which does not degenerate $U(x)$, i.e. the set $f^{-1}(z) \cap U(x)$ is discrete in $U(x)$ for all $z \in \mathbb{C}^k$.

The second property ensures the existence of sufficiently many holomorphic functions in X to provide a complex function theory. Stein spaces are called *holomorphically complete spaces.*

3.10 Theorem (Grauert). *With the same conditions as Proposition 3.8 let S be a Stein space. Then*

$$H^1(S, \mathscr{L}^c) \cong H^1(S, \mathscr{L}^a).$$

This theorem is the main result in [Gra57].

3.1 Holomorphic Local Triviality

3.11 Definition (Hyperbolic Riemann surface). A Riemann surface X is called *elliptic, parabolic*, or *hyperbolic*, if its universal covering is isomorphic to $\mathbb{P}^1_{\mathbb{C}}, \mathbb{C}$, or Δ respectively.

3.12 Theorem.

(a) The Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ is an elliptic Riemann surface.

(b) All complex tori, as well as $\mathbb{C}$ and $\mathbb{C}^*$ are parabolic Riemann surfaces.

(c) Any Riemann surface not being isomorphic to one of the surfaces in (a) or (b) is hyperbolic.

In particular, a compact Riemann surface is elliptic, parabolic, or hyperbolic, if its genus is zero, one, or greater than one respectively.

See [For77, III, Satz 27.12].

3.13 Remark. Since smooth projective curves of genus one are called elliptic curves, the term *elliptic Riemann surface* for $\mathbb{P}^1_{\mathbb{C}}$ will not be used further.

3.1.1 Families of Hyperbolic Riemann surfaces

3.14 Definition (Family of curves). Let S be a smooth curve of geometric genus p_g and $\Sigma \subset S$ be a finite set of points. A *family over* $S \setminus \Sigma$, $f : X \to S \setminus \Sigma$, is a surjective flat mapping with equidimensional fibers.

3.15 Definition (Isotriviality). A family $f : X \to S$ is called *isotrivial* if $f^{-1}(P) \cong f^{-1}(P')$ for all P, $P' \in S$.

From the theorem of Fischer and Grauert, Theorem 3.28, it follows that proper smooth isotrivial families of compact Riemann surfaces are already locally trivial in the holomorphic sense.

At the ICM in Stockholm in 1962 I. R. Shafarevich, conjectured: "There exists only a finite number of fields of algebraic functions $K/\mathbb{C}$ of a given genus $g \geq 1$, the critical prime divisors of which belong to a given finite set Σ." The conjecture can be modified and made more precise as follows, where the mentioned genus g will be denoted by g'.

3.16 Shafarevich Conjecture. *Let S be a smooth projective curve of geometric genus g and $\Sigma \subset S$ be a set of n points. Let $q \in \mathbb{Z}$ and $g' \geq 2$. Then*

(I) (Boundedness, Rigidity) there exists, up to isomorphism, only finitely many non-isotrivial smooth families of curves of geometric genus g' over $S \setminus \Sigma$. These are called "admissible families".

(II) (Hyperbolicity) If

$$2g - 2 + n \leq 0,$$

then there are no admissible families.

This conjecture was confirmed by A. N. Parshin for the case $\Sigma = \emptyset$, cf. [Par68]. Part (I) of the conjecture was confirmed by S. Ju. Arakelov in general.

3.17 Theorem (Arakelov). *Let S be a curve and $\Sigma \subset S$ a finite set of points. Then there exist only finitely many non-isomorphic non-constant curves of fixed genus over $k(S)$ for which Σ is the set of points of degeneracy.*

See [Ara71, Theorem 1].

Results of S. Ju. Arakelov, A. N. Parshin, Y. Manin and H. Grauert led to the following theorem, also called the "*Shafarevich-Mordell conjecture in the function field case*", see [Ara71, Par68, Man63] and [Gra65].

3.18 Theorem (Rigidity Theorem of Arakelov-Parshin-Manin-Grauert). *Let S be a smooth projective curve of (geometric) genus g and Σ be a set of n points in S. Then there are only finitely many families of non-isomorphic curves of geometric genus $g' \geq 2$ over $S \setminus \Sigma$,*

$$f : X \to S \setminus \Sigma,$$

which are non-isotrivial. If

$$2g - 2 + n \leq 0,$$

there are none at all.

The theorem can be found in the second edition of D. B. Mumford's lecture notes [Mum99, A. II, p. 253].

3.19 Definition (Principal Fiber Bundle). A fiber bundle with fiber F such that $F \cong \mathrm{Aut}(F)$ is called a *principal fiber bundle*.

The following theorem is well known in the case of global triviality in the topological sense, cf. [Ste51, §8, 8.3]. The proof for the holomorphic case in the following is done in analogy.

3.20 Theorem (Section Theorem). *Let X and S be complex manifolds. A holomorphic principal fiber bundle (X, f, S) is globally trivial in the holomorphic sense if and only if it admits a global holomorphic section.*

Proof. Let F be the fiber of (X, f, S), and let $\{U_i\}_{i\in I}$ be an open covering of S. For any $i \in I$ there exists a biholomorphic mapping

$$\varphi_i : U_i \times F \to f^{-1}(U_i)$$

with $\mathrm{pr}_1 \circ \varphi_i^{-1} = f$. For any pair of indices $(i, j) \in I \times I$, the mapping

$$g_{ji} : U_i \cap U_j \to \mathrm{Aut}(F),$$

given by $\varphi_i^{-1} \circ \varphi_j(x, P) = (x, g_{ij}(x)(P))$ for $x \in U_i \cap U_j$ and $P \in F$ is biholomorphic.

If the mapping $\varphi_{i,x}\colon F \to f^{-1}(x)$ is defined by setting

$$\varphi_{i,x}(P) = \varphi_i(x, P),$$

then, for each pair $(i, j) \in I$, and each $x \in U_i \cap U_j$, the mapping

$$\varphi_{j,x}^{-1} \circ \varphi_{i,x}\colon F \to F$$

coincides with the operation of an element of $\mathrm{Aut}(F)$, which is unique since $\mathrm{Aut}(F)$ is acting faithfully.

It is convenient to introduce the mapping

$$f_j : f^{-1}(U_j) \to F$$

defined by $f_j(P) = \varphi_{j,x}^{-1}(P)$, where $x = f(P)$. Then f_i satisfies the identities

$$\begin{aligned} &f_i \circ \varphi_i(x, y) = y, \\ &\varphi_j(f(z), f_j(z)) = z, \\ &g_{ij}(f(z))(f_j(z)) = f_i(z), \quad \text{for } f(z) \in U_i \cap U_j. \end{aligned} \tag{3.1}$$

Suppose a global section $s : S \to X$ is given. Define $h_i(x) := f_i(s(x))$ for $x \in U_i$. From (3.1) follows immediately

$$g_{ij}(x)(h_j(x)) = h_i(x), \quad x \in U_i \cap U_j. \tag{3.2}$$

Therefore the cocycle splits, and the fiber bundle is trivial.

Conversely, suppose (X, f, S) is globally trivial. By Proposition 3.8 there exist biholomorphic functions h_i satisfying (3.2). Define

$$s_i := \varphi_i(x, h_i(x)), \ x \in U_i.$$

Then s_i is holomorphic. From (3.2) follows $s_i(x) = s_j(x)$ for $x \in U_i \cap U_j$. Hence $s(x) = s_i(x)$ for $x \in U_i$ defines a global holomorphic section. □

3.21 Definition (Ramification Divisor). Let $f : X \to Y$ be a finite, separable morphism of complete non-singular projective curves. Then the *ramification divisor of f* is

$$\mathcal{R} := \sum_{P \in X} \mathrm{length}(\Omega_{X/Y})_P \cdot P.$$

3.22 Theorem (Hurwitz). *Let $f : X \to Y$ be a finite separable morphism of nonsingular projective integral seperated curves. Let $n = \deg f$. Then*

$$2p_a(X) - 2 = n \cdot (2p_a(Y) - 2) + \deg \mathcal{R}.$$

See [Har77, IV, Corollary 2.4].

3.23 Theorem (Schwarz). *If X is a smooth integral projective complex curve of (geometric) genus 2 or higher, its automorphism group $\mathrm{Aut}(X)$ is finite.*

See [FK92, V.1.2, Corollary 2].

The next four propositions will be used to control the infinite part of a minimal model of a given fiber bundle with base space $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{P}^1_{\mathbb{C}}$, and T in order to prove the existence of a holomorphic structure.

3.24 Proposition. *Let X be a complex manifold, let $f : X \to \mathbb{C}$ be a holomorphic mapping and $(X, f, \mathbb{C})$ be a holomorphic fiber bundle such that the fibers of f are biholomorphic to a hyperbolic Riemann surface F. If the bundle $(X, f, \mathbb{C})$ possesses a global holomorphic section $s : \mathbb{C} \to X$, then $(X, s(\mathbb{C}))_f$ is a holomorphically fibered pair.*

Proof. The fiber bundle $(X, f, \mathbb{C})$ is classified by its 1-cocycle in the first cohomology set and is therefore 1-cocycle of a suitable principal bundle. Since by Schwarz's Theorem, Theorem 3.23, the automorphism group of curves of genus 2 or higher is finite, the principal bundle is an unbounded and unramified covering $(X', f', \mathbb{C})$ with a finite number of sheets. Since $\mathbb{C}$ is simply connected, the total space consists of exactly this number of connected components, and a single connected component $X'_k \subset X'$ is homeomorphic to $\mathbb{C}$ via f'. Since $f|_{X'_k}$ is holomorphic and $f'^{-1}|_{X'_k}$ is continuous, the latter is also holomorphic by the criterion for biholomorphicy. Hence $f'^{-1}|_{X'_k}$ defines a global holomorphic section in X'. By the Section Theorem, Theorem 3.20, the principal fiber bundle $(X', f', \mathbb{C})$, and therefore the bundle $(X, f, \mathbb{C})$ is globally trivial. With respect to a trivialization of the bundle, the section s is of the form

$$\begin{aligned} s : \mathbb{C} &\to \mathbb{C} \times F \\ x &\mapsto (x, s'(x)), \end{aligned}$$

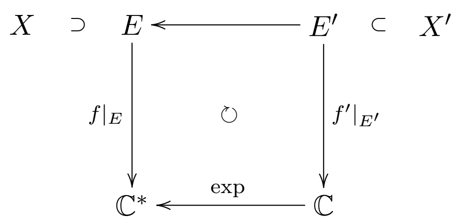
where $s' : \mathbb{C} \to F$ is a holomorphic mapping. Since F is hyperbolic, the mapping s' can be lifted to the universal covering of F, the open unit disc Δ, and is therefore constant by Liouville's Theorem. Hence $(X, s(\mathbb{C}))_f$ is a holomorphically fibered pair. □

3.25 Proposition. *Let X be a complex manifold, let $f : X \to \mathbb{C}^*$ be a holomorphic mapping and $(X,\, f,\, \mathbb{C}^*)$ be a holomorphic fiber bundle such that the fibers of f are biholomorphic to a hyperbolic Riemann surface F. If there exists an unbounded and unramified holomorphic covering*

$$f : E \to \mathbb{C}^*$$

with a complex submanifold $E \subset X$, then $(X,\, E)_f$ is a holomorphically fibered pair.

Proof. The fiber bundle $f : X \to \mathbb{C}^*$ can be pulled back via the universal covering mapping $\exp\colon \mathbb{C} \to \mathbb{C}^*$ to a holomorphic fiber bundle $(X',\, f',\, \mathbb{C})$. This way, the covering $f : E \to \mathbb{C}^*$ is lifted to an unbounded and unramified covering $f' : E' \to \mathbb{C}$ in X' with $E' \subset X'$.



The fiber bundle $(X',\, f',\, \mathbb{C})$ is classified by its 1-cocycle in the first cohomology set and is therefore the 1-cocycle of a suitable principal bundle $(X'',\, g,\, \mathbb{C})$. By Schwarz's Theorem, Theorem 3.23 the automorphism group of curves of genus 2 or higher is finite. Therefore the principal bundle is an unbounded and unramified covering, which decomposes over $\mathbb{C}$ into finitely many connected components $X''_1, \ldots, X''_n$, each of which is homeomorphic to $\mathbb{C}$. Since $g|_{X''_i}$ is holomorphic and $(g|_{X''_i})^{-1}$ is continuous for all i, the latter is also holomorphic by the criterion for biholomorphicy, and defines a global holomorphic section. By the Section Theorem, Theorem 3.20, the principal fiber bundle $(X'',\, g,\, \mathbb{C})$ is globally trivial in the holomorphic sense. Therefore the fiber bundle $(X',\, f',\, \mathbb{C})$ is also globally trivial in the holomorphic sense.

Since $\mathbb{C}$ is simply connected, the covering space E' decomposes into connected components $E'_1, \ldots, E'_n$. Each component E'_i is homeomorphic to $\mathbb{C}$ via f'. Since $f'|_{E'_i}$ is holomorphic, and $(f'|_{E'_i})^{-1}$ is continuous for all i, the latter is also holomorphic by the criterion of biholomorphicy. Hence $(f'|_{E'_i})^{-1}$ defines a holomorphic section

$$t\colon \mathbb{C} \to X'.$$

The fiber bundle $(X',\, f',\, \mathbb{C})$ is globally trivial in the holomorphic sense. Therefore, with respect to a trivialization of the fiber bundle, the section t is of the form

$$\begin{aligned} t : \mathbb{C} &\to \mathbb{C} \times F \\ x &\mapsto (x,\, t'(x)), \end{aligned}$$

where $t' : \mathbb{C} \to F$ is a holomorphic mapping. Since F is hyperbolic, the mapping t' can be lifted to the universal covering of F, the open unit disc Δ, and is therefore constant by Liouville's Theorem. Therefore the section t is constant with respect to the chosen trivialization, and $(X',\, E')_{f'}$ is a holomorphically fibered pair.

$$\begin{array}{ccccccc}
X & \supset & f^{-1}(U) & \longleftarrow & f'^{-1}(U') & \subset & X' \\
 & & s \uparrow\downarrow f|_{f^{-1}(U)} & \circlearrowright & f'|_{f'^{-1}(U')} \downarrow\uparrow t & & \\
\mathbb{C}^* & \supset & U & \xleftarrow{\exp} & U' & \subset & \mathbb{C}
\end{array}$$

Let U be a simply connected subset of $\mathbb{C}^*$, and let U' be a subset of $\mathbb{C}$ such that $\exp(U') = U$. The section t induces a holomorphic section $s\colon U \to f^{-1}(U)$. The diagram above commutes, thus rendering the section s constant over U with respect to the chosen trivialization. Therefore $(X,\, E)_f$ is also a holomorphically fibered pair. □

3.26 Proposition. *Let X be a complex manifold, let $f : X \to \mathbb{P}^1_{\mathbb{C}}$ be a holomorphic mapping and $(X,\, f,\, \mathbb{P}^1_{\mathbb{C}})$ be a holomorphic fiber bundle such that the fibers of f are biholomorphic to a hyperbolic Riemann surface F. If there exists an unbounded and unramified holomorphic covering*

$$f : E \to \mathbb{P}^1_{\mathbb{C}}$$

with a complex submanifold $E \subset X$, then $(X,\, E)_f$ is a holomorphically fibered pair.

Proof. Without loss of generality let $\{U_1,\, U_2\}$ be an open covering of $\mathbb{P}^1_{\mathbb{C}}$ such that $U_i \cong \mathbb{C}$ for $i = 1, 2$. The proof for each U_i as base space is then analogous to the proof of Proposition 3.24. □

3.27 Proposition. *Let X be a complex manifold and T be the complex torus. Let $f : X \to T$ be a holomorphic mapping and $(X,\, f,\, T)$ be a holomorphic fiber bundle such that the fibers of f are biholomorphic to a hyperbolic Riemann surface F. If there exists an unbounded and unramified holomorphic covering*

$$f : E \to T$$

with a complex submanifold $E \subset X$, then $(X,\, E)_f$ is a holomorphically fibered pair.

Proof. Identify the complex torus T with $\mathbb{C}/\Gamma$, where Γ is a complex lattice. The bundle $(X,\, f,\, T)$ can be pulled back via the universal covering mapping to a holomorphic fiber bundle $(X',\, f',\, \mathbb{C})$.

$$\begin{array}{ccc} X & \longleftarrow & X' \\ {\scriptstyle f}\big\downarrow & \circlearrowright & \big\downarrow{\scriptstyle f'} \\ \mathbb{C}/\Gamma & \longleftarrow & \mathbb{C} \end{array}$$

From this point, the proof is analogous to the proof of Proposition 3.25, where the open subset $U \subset \mathbb{C}/\Gamma$ has to be chosen such that $U \cap \Gamma = \emptyset$, and U is contained in one period parallelogram.

$\square$

The next theorem is a standard result for ensuring a holomorphic structure on complex fiber bundles.

3.28 Theorem (Fischer, Grauert). *Let X and S be connected complex manifolds, $f : X \to S$ a surjective proper holomorphic submersion such that the fibers $X_t := f^{-1}(t)$ are connected compact submanifolds of X. Suppose that $f : X \to S$ is an analytically isotrivial family. Then $(X,\, f,\, S)$ is locally trivial in the holomorphic sense.*

See [GF65, p. 89].

3.29 Theorem. *Let X be a 2-dimensional non-singular affine complex algebraic variety, and let $f : X \to S$ be a morphism with $S \in \{\mathbb{C},\, \mathbb{C}^*,\, \mathbb{P}^1_{\mathbb{C}},\, T\}$ such that the compactification of each fiber $f^{-1}(t)$ is a hyperbolic Riemann surface for all $t \in S$. Furthermore suppose that all fibers are pairwise homeomorphic. Then f defines a holomorphic fiber bundle.*

Proof. By Theorem 2.47 there exist a smooth minimal model X' of X together with a proper morphism $f' : X' \to S$. As a consequence of the Rigidity Theorem, Theorem 3.18, $(X',\, f',\, S)$ is an isotrivial family. According to Theorem 3.28, $(X',\, f',\, S)$ is locally trivial in the holomorphic sense.

In the case of $S = \mathbb{C}$, as seen in the proof of Theorem 2.70, the infinite part E of X' consists of a finite number of disjoint curves, which define global holomorphic sections, cf. 2.71. According to Proposition 3.24, $(X',\, E)$ is a holomorphically fibered pair and therefore $(X,\, f,\, \mathbb{C})$ defines a holomorphic fiber bundle.

In case of $S \in \{\mathbb{C}^*,\, \mathbb{P}^1_{\mathbb{C}},\, T\}$ the infinite part E of X also consists of curves such that the morphism f'_E is submersive. But since the fundamental group of the base space is not trivial, these curves only define unbounded and unramified coverings, cf. proof of Theorem 2.70. According to Propositions 3.25–3.27, (X', E) is a holomorphically fibered pair over $S \in \{\mathbb{C}^*, \mathbb{P}^1_{\mathbb{C}}, T\}$. Therefore $(X,\, f,\, \mathbb{C}^*), (X,\, f,\, \mathbb{P}^1_{\mathbb{C}})$ and $(X,\, f,\, T)$ define holomorphic fiber bundles. $\square$

3.30 Remark. A fiber bundle $f : X \to C$ of the compact surface X over the smooth curve C is called a *Kodaira fibration* if f is a submersion but *not* a holomorphic fiber bundle mapping. In view of the Fischer-Grauert Theorem, Theorem 3.28, this means that, though all fibers are smooth curves, their complex structure varies. For a further treatment of Kodaira fibrations see [BPVdV84].

3.1.2 Families of Elliptic Curves

It is possible to construct local triviality in the holomorphic sense for families of elliptic curves in analogy to the result of the previous section. This result is restricted to families of curves, which are isomorphic to one elliptic curve having one or no punctures. A theorem of A. Beauville is stating that families of elliptic curves are isotrivial, in case the base space is either $\mathbb{P}^1_{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}^*$, or T. The theorem of Fischer and Grauert again yields local triviality in the holomorphic sense. For an affine scheme over one of these base spaces, having pairwise homeomorphic curves of geometric genus one as fibers, there exists a smooth minimal model. Since the fibers are not hyperbolic, it is generally not possible to argue that the infinite part of the minimal model is holomorphically trivial, as done in the previous section. Requiring that each fiber is compactified with exactly one point, these points can be translated to the same point by holomorphic automorphisms of the fibers, yielding a holomorphically fibered pair of spaces.

Let E be an elliptic curve and $\mathrm{Aut}(E)$ the group of its biholomorphic automorphisms. The group E, acting on itself by translations, forms a normal subgroup of $\mathrm{Aut}(E)$, and the quotient $\mathrm{Aut}(E)/E$ can be identified with the group of automorphisms leaving 0 fixed. This quotient is the cyclic group $\mathbb{Z}_n$ of order

$$\begin{aligned} n &= 4 \quad \text{if } E \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i), \\ n &= 6 \quad \text{if } E \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\exp(\pi i/3)), \\ n &= 2 \quad \text{in all other cases.} \end{aligned}$$

Then $\mathrm{Aut}(E)$ is the semi-direct product $E \times \mathbb{Z}_n$. Its elements are $(e, z), e \in E, z \in \mathbb{Z}_n$ being the group of the nth root of unity,

$$(e, z) : x \mapsto e + zx, \quad x \in E.$$

The operation in the group is

$$(e, z), (e', z') = (e + ze', zz').$$

The translation group E is described by the universal covering sequence

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{C} \to E \to 0.$$

3.31 Theorem (Beauville). *Any non-isotrivial family of projective curves of genus at least* 1 *over* $\mathbb{P}^1_{\mathbb{C}}$ *has at least* 3 *singular fibers. Over an elliptic curve, there must be at least* 1 *singular fiber.*

See [Bea81, Proposition 1].

3.32 Remark. The statement on non-isotrivial families over a curve of genus 1 is not part of the original proposition of Beauville, but is a result of the proof. Therefore, the theorem of Beauville extends the second statement of the Rigidity Theorem of Arakelov-Parshin-Manin-Grauert, Theorem 3.18.

To construct local triviality in the holomorphic sense it is not possible to proceed with the previous methods from Section 3.1.1. Since the global sections defined by the infinite part of the minimal regular model of the given scheme have values in an elliptic curve instead of a hyperbolic Riemann surface, they are in general not constant with respect to a holomorphic trivialization. It is possible to prove a slightly weaker result as follows.

3.33 Theorem. *Let* X *be a* 2*-dimensional non-singular complex algebraic variety, and let* $f : X \to S$ *be an affine morphism with* $S \in \{\mathbb{C}, \mathbb{C}^*, \mathbb{P}^1_{\mathbb{C}}, T\}$*, where* T *is a complex torus such that each fiber* $f^{-1}(t)$ *is a complex torus having exactly one puncture. Then* f *defines a holomorphic fiber bundle.*

Proof. By Theorem 2.47 there exists a smooth minimal regular model $\bar{f} : \bar{X} \to S$ of X. According to the theorem of Beauville, Theorem 3.31, the family of elliptic curves $\bar{f} : \bar{X} \to S$ is isotrivial. This way, a fiber $f^{-1}(t)$ of f is compactified with one point. As seen in the proof of Theorem 2.70, the infinite part $\bar{X} \setminus X$ of $\bar{X}$ consists of an unbounded and unramified covering E. There exists an open covering $\{U_i\}_{i \in I}$ of S such that for any U_i the covering E defines a holomorphic section, denoted by

$$s_{U_i} : U_i \to \bar{X}|_{\bar{f}^{-1}(U_i)}.$$

By the theorem of Fischer and Grauert, Theorem 3.28, the family $\bar{f} : \bar{X} \to S$ is locally trivial in the holomorphic sense. Choosing local holomorphic trivializations for an open covering $\{V_i\}_{i \in I}$ subordinate to the covering $\{U_i\}_{i \in I}$, the fiber bundle $(\bar{X}, \bar{f}, S)$ is locally of the form

$$\bar{X}|_{f^{-1}(V_i)} \cong V_i \times \mathbb{C}/\Gamma,$$

where Γ is a suitable lattice corresponding to the torus of the fibers.

For all V_i, $V_j \in \{V_i\}_{i \in I}$, and for all $t \in V_{ij}$ there exist biholomorphic mappings

$$\begin{aligned} \psi_{ij} : V_{ij} \times \mathbb{C}/\Gamma &\to V_{ij} \times \mathbb{C}/\Gamma \\ (t,\, z) &\mapsto (t,\, z - s_{U_i}(t)). \end{aligned}$$

These mappings yield local holomorphic trivializations on E and on $\bar{X}$. Therefore, $(\bar{X},\, X)_{\bar{f}}$ is a holomorphically fibered pair of spaces. □

3.1.3 Families of Spheres

Since there exists, up to isomorphism, only one compact Riemann surface of genus 0, there do not exist any admissible families with fibers of genus 0. The construction of local triviality in the holomorphic sense in this section aims at fiber bundles having fibers isomorphic to $\mathbb{C}^*$. Analogously the construction allows equal results for the fibers being isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ having up to three punctures, including of course none at all.
The following theorem is stating that an analytically isotrivial family with fibers being isomorphic to $\mathbb{C}^*$ is locally trivial in the analytic sense outside of a finite number fibers.

3.34 Theorem. *Let X be a 2-dimensional non-singular complex algebraic variety, and S be a smooth irreducible complex algebraic curve. Furthermore let $f \colon X \to S$ be a surjective affine morphism such that the fibers $f^{-1}(t)$ are biholomorphic to $\mathbb{C}^*$. Then there exists a finite set $\Sigma \subset S$ such that*

$$(X \setminus f^{-1}(\Sigma),\, f|_{f^{-1}(S \setminus \Sigma)},\, S \setminus \Sigma)$$

is locally trivial in the holomorphic sense.

Proof. Let $\bar{X}$ be a compactification of X. The product $\bar{X} \times S$ contains the closure $\bar{X}'$ of the graph of f. This product therefore yields a mapping $t \colon \bar{X}' \to \bar{S}$, which fits into the commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad i \quad} & \bar{X} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle t} & \\ & S & \end{array} ,$$

where i denotes the inclusion $x \mapsto (x, f(x))$, and t is the projection on the second factor. Singularities may only appear in the closure $\bar{X}' \setminus X \times S$. Let $\tilde{X} \to \bar{X}$ be the normalization of $\bar{X}$. This normalization induces a proper mapping $\tilde{f} : \tilde{X} \to S$ which commutes with the original mapping f. Since $\tilde{X}$ is compact, it possesses only a finite number of singularities. Proposition 1.22 yields $\dim S(\tilde{X}) = 0$. Denote by $\Sigma_{\tilde{f}}$ the set of critical values of $\tilde{f}$. The sets $\Sigma_{\tilde{f}}$ and $\Sigma_{\tilde{f}|_{\tilde{X} \setminus X}}$, and therefore their union are algebraically closed sets in $\tilde{X}$. Since $\tilde{f}$

is proper in the algebraic sense, the set

$$\Sigma := \tilde{f}(\Sigma_{\tilde{f}} \cup \Sigma_{\tilde{f}|_{\tilde{X}\setminus X}})$$

is algebraically closed in S. Since S is 1-dimensional, the set Σ is finite. Therefore

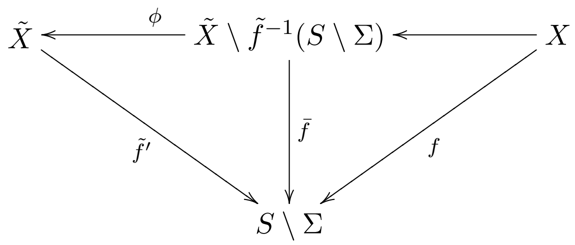
$$\tilde{f}_\Sigma := \tilde{f} : \tilde{X} \setminus \tilde{f}^{-1}(\Sigma) \to S \setminus \Sigma$$

is a proper holomorphic submersion. In particular, $\tilde{f}|_{\tilde{X}\setminus(X \cup \tilde{f}^{-1}(\Sigma))}$ is a holomorphic submersion. The theorem of Fischer and Grauert, Theorem 3.28, gives local trivializations in the holomorphic sense for $(\tilde{X}\setminus\tilde{f}^{-1}(\Sigma),\ \tilde{f}_\Sigma,\ S\setminus\Sigma)$. Fix one of these trivializations.

Since all fibers $f^{-1}(t)$ are biholomorphic to $\mathbb{C}^*$ and $\tilde{f}^{-1}(S \setminus \Sigma)$ is regular, the genus of the fibers $\tilde{f}_\Sigma^{-1}(t)$ is constant. This way, fibers of f over $S \setminus \Sigma$ are compactified with two points ω_{tn}, $\omega_{ts} \in \tilde{X}_t$, referred to as north and south poles of the fibers, which vary holomorphically with t above the base space. All such fibers are biholomorphic by trivialization to $\mathbb{P}^1_{\mathbb{C}}$. Let $t_0 \in S \setminus \Sigma$. For all $t \in S \setminus \Sigma$ there exist Möbius transformations $\phi_t : \tilde{X}_{t_0} \to \tilde{X}_t$, such that $\phi_t(\omega_{t_0 n}) = \omega_{tn}$ and $\phi_t(\omega_{t_0 s}) = \omega_{ts}$, namely

$$z \mapsto \frac{(z - \omega_{t_0 n})(\omega_{ts} - \omega_{tn})}{\omega_{t_0 s} - \omega_{t_0 n}} + \omega_{tn}.$$

The Möbius transformations of all $t \in S \setminus \Sigma$ combined yield a biholomorphic mapping $\phi : \tilde{X} \to \tilde{X}'$, where $\tilde{X}' := \dot{\bigcup}_{t \in S\setminus\Sigma} \phi_t(\tilde{X}_t)$. This mapping induces a proper holomorphic submersion $\tilde{f}'$ which commutes with f over $S \setminus \Sigma$.



North and south poles of the fibers $\tilde{X}'_t$ deform holomorphically. Theorem 3.28 gives holomorphic local trivializations for $(\tilde{X}', \tilde{f}', S \setminus \Sigma)$. Let U_i and U_j be subsets of $S \setminus \Sigma$, such that $U_{ij} := U_i \cap U_j \neq \emptyset$, and let $\tilde{f}'^{-1}(t)$ (resp. $\bar{f}'^{-1}(t')$) be a fiber above $t \in U_i$ (resp. $t' \in U_j$.) For U_i and U_j choose a biholomorphic mapping

$$\psi_{ij} : U_{ij} \times \tilde{f}'^{-1}(t) \to U_{ij} \times \tilde{f}'^{-1}(t')$$

from the holomorphic trivializations given by Theorem 3.28. By construction, the mapping ψ_{ij} commutes with f. Therefore the restriction

$$\psi_{ij}|_X : U_{ij} \times (\tilde{f}'^{-1}(t) \cap X) \to U_{ij} \times (\tilde{f}'^{-1}(t') \cap X)$$
$$= \psi_{ij} : U_{ij} \times \tilde{f}'^{-1}(t) \to U_{ij} \times \tilde{f}'^{-1}(t')$$

defines a holomorphic local trivialization for $(X, f, S\backslash\Sigma)$, which in turn is a biholomorphic fiber bundle.

□

3.35 Remark. For the construction of local triviality as in Theorem 3.34, it is possible to assume fibers to be biholomorphic to $\mathbb{C}$ or $\mathbb{C}^* \setminus \{P\}$, where P is an arbitrary point in $\mathbb{C}^*$. The construction of local triviality is analogous, since there exist biholomorphic automorphisms on $\mathbb{P}^1_{\mathbb{C}}$, Möbius transformations, mapping up to three punctures simultaneously to points, which are transported holomorphically through all fibers. Since there generally do not exist automorphisms on $\mathbb{P}^1_{\mathbb{C}}$ that permute more than three given points, it is not possible to generally construct local triviality for fibers being isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ having more than three punctures.

3.2 Gobal Triviality of Holomorphic Fiber Bundles

Let X be a 2-dimensional non-singular and non-compact complex algebraic variety and S be a smooth non-compact complex curve. In this section, fiber bundles $(X,\, f,\, S)$, which are already locally trivial in the holomorphic sense are investigated for global triviality in the holomorphic sense. Here, f is assumed to be an algebraic morphism such that its fibers are Riemann surfaces. All considered fiber bundles possess a typical fiber, and a complex Lie group as structure group. A strong result of H. Grauert is ensuring global triviality in the holomorphic sense for fiber bundles. The theorem is only ensuring it though for fiber bundles having a connected complex Lie group as structure group. In particular, the previously constructed case of a fiber bundle with typical fiber $\mathbb{C}^*$ and finitely punctured base space is treated. The problem is reduced to a cohomological treatment of sheaves with values in the structure group of the bundle. The cohomological problem can be reduced to the topological case, where the base space consists of a bouquet of spheres. Criteria for global triviality in the holomorphic sense are worked out this way.

The following theorem was first proven by H. Röhrl (see [Röh57]).

3.36 Theorem (Grauert). *Let X be a holomorphic fiber bundle over a non-compact Riemann surface B. If the structure group G of X is a connected compact Lie group, then X is holomorphically trivial.*

See [Gra58, Satz 7].

The following theorem of S. Bochner and D. Montgomery is stating that automorphism groups of compact complex analytic manifolds are compact Lie groups. According to Theorem 3.36 above, a fiber bundle having such a structure group, which is connected in addition, would therefore be globally trivial in the holomorphic case.

Let M be a compact complex manifold. The group $\mathrm{Aut}_c(M)$ of homeomorphisms of M onto itself has a natural topology which can be defined as follows. Since M can be considered a metric space, the distance between any two homeomorphisms h_1 and h_2 is defined, in the usual way, as

$$\mathrm{dist}(h_1,\, h_2) := \sup_{x\in M} d(h_1(x), h_2(x)),$$

where d is the metric on M.

3.37 Theorem (Bochner, Montgomery). *Let M be compact complex analytic manifold, and* $\mathrm{Aut}(M)$ *be the group of all complex holomorphic automorphisms of M where* $\mathrm{Aut}(M)$ *has been topologized as above. Then the group of automorphisms* $\mathrm{Aut}(M)$ *is a complex Lie group.*

See [BM47, Theorem 1].

There exists only a very limited number of Riemann surfaces with non-discrete automorphism group.

3.38 Theorem. *Let M be a Riemann surface. The automorphism group* $\mathrm{Aut}(M)$ *is a non-discrete Lie group if and only if M is biholomorphically equivalent to one of the following Riemann surfaces.*

(a) $\hat{\mathbb{C}}$,

(b) $\mathbb{C}$,

(c) $\mathbb{C}^*$,

(d) T, the complex torus.

See [FK92, V.4.1].

3.39 Remark. Notice that the automorphism groups of the (non-algebraic) Riemann surfaces $\Delta := \{z \in \mathbb{C} \mid |z| < 1\}$, $\Delta^* := \Delta \setminus \{0\}$, and $\Delta_r := \{z \in \mathbb{C} \mid r < |z| < 1\}$, $0 < r < 1$, are non-discrete real Lie groups. The only Riemann surfaces with a non-discrete and non-connected complex Lie group as automorphism group are therefore $\mathbb{C}^*$, and T. These will be studied in more detail.

In addition, the base space will be assumed to be $\mathbb{C} \setminus \Sigma$, where Σ is a finite set of points such that the investigation is narrowed down to holomorphic fiber bundles as constructed in Section 3.1. Instead of choosing $\mathbb{C} \setminus \Sigma$ as base space, it is possible to choose any non-compact Riemann surface. The cycles generating the first homology group are then the cycles of the bouquet of spheres, used for the reduction to the topological situation.

The following results also work for discrete automorphism groups, which constitute a simpler special case of the present situation.

The following theorem is stating that the connected component of a Lie group containing the neutral element is a normal subgroup of its group. This important fact is enabling the construction of a non-abelian exact cohomology sequence to deduce a criterion for global triviality.

3.40 Theorem. *Let G be a Lie group, and let G_0 be the connected component of the neutral element of G. Then the Lie group G_0 is a normal subgroup in G. Furthermore the cardinality of G/G_0 is equal to the number of connected components of G.*

Proof. Since G is a manifold, it satisfies the second axiom of countability and has therefore countably many connected components, which are all open in G and are open submanifolds of G.

For G_0 to be a Lie group, it is necessary to show that if $x, y \in G_0$, also $xy^{-1} \in G_0$. Let γ_1 and γ_2 be continuous paths in G such that $\gamma_1(0) = \gamma_2(0) = e$, and $\gamma_1(1) = x$, $\gamma_2(1) = y$, where e is the neutral element of G. Then $\gamma_1\gamma_2^{-1} : [0,1] \to G$ is a path such that $\gamma_1\gamma_2^{-1}(0) = \gamma_1(0)\gamma_2^{-1}(0) = e$, and $\gamma_1\gamma_2^{-1} = xy^{-1}$. It follows that $xy^{-1} \in G_0$.

Let $x \in G_0$ and $\gamma : [0,1] \to G_0$ be a path such that $\gamma(0) = e$, and $\gamma(1) = x$. Let $y \in G$. Then

$$\gamma' := y\gamma y^{-1} : [0,1] \to G$$

is a path with $\gamma'(0) = e$ and $\gamma'(1) = yxy^{-1} \in G_0$. Therefore G_0 is a normal subgroup in G.

With the same argument it follows that, for an open and connected set $U \subset G$, the right (resp. left) cosets, which are diffeomorphic to U, yU (resp. Uy), with fixed $y \in G$, are open and connected as well. Let $G/G_0 = \{[y_i]\}_{i \in I}$, then

$$G = \dot{\bigcup_{i \in I}} y_i G_0.$$

$\square$

3.41 Definition (Wedge Product). Let X and Y be two topological spaces with $x_0 \in X$ and $y_0 \in Y$. The *wedge product* is the subspace

$$X \vee Y := \{(x, y) \in X \times Y \mid x = x_0 \text{ or } y = y_0\}$$

of $X \times Y$.

3.42 Theorem. *Let $\mathcal{K}_\Sigma$ be the set of isomorphisms classes of holomorphic fiber bundles with base space $\mathbb{C} \setminus \Sigma$ and fiber F such that the structure group* $\mathrm{Aut}(F)$ *is a finite (Lie) group, where $\Sigma \subset \mathbb{C}$ is a set of n distinct points. Then there exists a bijection*

$$\mathcal{K}_\Sigma \to \prod^n \mathrm{Aut}(F),$$

preserving the neutral element.

Proof. According to Theorem 3.10 it is sufficient to consider isomorphism classes of topological fiber bundles under otherwise equal conditions. Therefore a holomorphic bundle over the base $\mathbb{C} \setminus \Sigma$ is equivalent to a topological bundle over a bouquet of n spheres $\bigvee^n \mathbb{S}^1$. Let $\mathscr{F}$ be the sheaf of germs of continuous functions with values in the group $\mathrm{Aut}(F)$. Then

$$H^1(\mathbb{C} \setminus \Sigma, \mathscr{F}) \cong H^1\Big(\bigvee^n \mathbb{S}^1, \mathscr{F}\Big).$$

Let P be the point connecting all n spheres.

Define an open covering $\mathscr{V} = \{V_i\}_{i\in I}$ of $\bigvee^n \mathbb{S}^1$ consisting of $n+1$ open sets such that

$$V_0 := \bigvee^n \mathbb{S}^1 \cap \Delta(P).$$

The set $\Delta(P)$ is a disc of radius r with center P such that the boundary $\partial\Delta(P)$ has exactly two intersections with each sphere. Number the spheres $\mathbb{S}^1_1$ to $\mathbb{S}^1_n$. For all $i > 0$, define

$$V_i := \mathbb{S}^1_i \setminus \bar{\Delta}'(P),$$

where $\bar{\Delta}'(P)$ is the closed disc of radius $r - \varepsilon$ with center P and $r > \varepsilon > 0$ such that the boundary $\partial\bar{\Delta}'(P)$ has exactly two intersections with $\mathbb{S}^1_i$. For $i > 0$, all V_i are pairwise disjoint, and each possesses two non-discrete disjoint intersections with V_1. For an example with $n = 3$, cf. Figure 3.1.

Consider a 1-cocycle $\{f_{ij}\}$ in $H^1(\bigvee^n \mathbb{S}^1, \mathscr{F})$ for the open covering $\mathscr{V}$. The cocycle is only defined, modulo inversion, for $i = 0$ and $j \in \{1, \ldots, n\}$. Since all f_{ij} are continuous and the group $\mathrm{Aut}(F)$ is finite, f_{ij} can only take one value in $\mathrm{Aut}(F)$ for each of the two components of every $V_0 \cap V_i$, $i > 0$, and is trivial for $i = j$. There exists a 0-cochain $\{g_i\}_{i\in I}$ on $\mathscr{V}$ with values in $\mathscr{F}$ such that $g_i^{-1} f_{ij} g_j$ is the identity on one of the components of every $V_0 \cap V_i$, $i > 0$. The resulting 1-cocycle is cohomologous to $\{f_{ij}\}$.

Since for each of the n intersections $V_0 \cap V_i$ there exists exactly one element in the automorphism group $\mathrm{Aut}(F)$ besides the identity, it is possible to define an injective mapping

$$\varphi : H^1\Big(\bigvee^n \mathbb{S}^1, \mathscr{F}\Big) \to \prod^n \mathrm{Aut}(F).$$

Conversely, each $V_i \in \mathscr{V}$ is simply connected and contractible. Therefore any topological bundle over any V_i is topologically globally trivial. Since $\mathrm{Aut}(F)$ is finite, it only takes two automorphisms in $\mathrm{Aut}(F)$ to glue together two bundles over an intersection $V_0 \cap V_i$, $i > 0$, one of which can be chosen without restriction to be the identity. Since there are n intersections, it is possible to choose n automorphisms, besides the identity on one of the connected components of each intersection, to construct a topological fiber bundle over $\bigvee^n \mathbb{S}^1$. This yields a mapping

$$\psi : \prod^n \mathrm{Aut}(F) \to H^1\Big(\bigvee^n \mathbb{S}^1, \mathscr{F}\Big).$$

Obviously, $\varphi \circ \psi = \mathrm{id}$. □

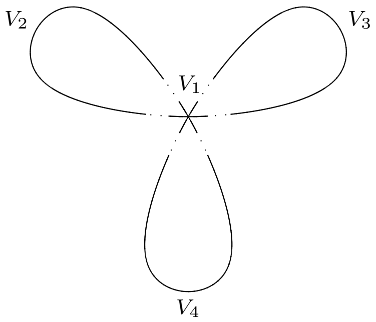


Figure 3.1: Open covering of a bouquet of 3 spheres

3.43 Theorem. *Let $(X, f, \mathbb{C} \setminus \Sigma)$ be a holomorphic fiber bundle with fiber F, having a non-connected Lie group as structure group, where $\Sigma \subset \mathbb{C}$ is a set of n distinct points. Let $\operatorname{Aut}_0(F)$ be the connected component in $\operatorname{Aut}(F)$ containing the neutral element. The bundle $(X, f, \mathbb{C} \setminus \Sigma)$ is globally trivial in the holomorphic sense, if and only if the corresponding 1-cocycle $\{f_{ij}\} \in H^1(\mathbb{C} \setminus \Sigma, \mathscr{F})$ is contained in the kernel of the mapping*

$$H^1(\mathbb{C} \setminus \Sigma, \mathscr{F}) \to \prod^n \operatorname{Aut}(F)/\operatorname{Aut}_0(F)$$

defined below, where $\mathscr{F}$ is the sheaf of germs of holomorphic functions with values in the structure group.

Proof. Let $\mathscr{F}_0$ be the sheaf of germs of holomorphic functions with values in $\operatorname{Aut}_0(F)$. According to Theorem 3.40, $\operatorname{Aut}_0(F)$ is a normal subgroup in $\operatorname{Aut}(F)$. By Theorem 3.6, there exists an exact sequence

$$H^1(\mathbb{C} \setminus \Sigma, \mathscr{F}_0) \to H^1(\mathbb{C} \setminus \Sigma, \mathscr{F}) \to H^1(\mathbb{C} \setminus \Sigma, \mathscr{F}/\mathscr{F}_0).$$

Since $\mathscr{F}_0$ is a connected Lie group, $H^1(\mathbb{C} \setminus \Sigma, \mathscr{F}_0)$ is trivial according to Theorem 3.36. Since $\operatorname{Aut}(F)/\operatorname{Aut}_0(F)$ is a finite group according to Theorem 3.40, there exists a bijection

$$H^1(\mathbb{C} \setminus \Sigma, \mathscr{F}/\mathscr{F}_0) \to \prod^n \operatorname{Aut}(F)/\operatorname{Aut}_0(F),$$

preserving the neutral element as a consequence of Theorem 3.42. Altogether there exists an exact sequence

$$\{e\} \to H^1(\mathbb{C} \setminus \Sigma, \mathscr{F}) \to \prod^n \operatorname{Aut}(F)/\operatorname{Aut}_0(F).$$

□

3.44 Remark. In Theorem 3.38, the only Riemann surfaces having a non-discrete automorphism group are mentioned. The only surfaces among theses, which in addition have a non-connected complex Lie group as automorphism group are $\mathbb{C}^*$ and the complex torus T. Let $\mathscr{F}$, resp. $\mathscr{G}$ be the sheaf of germs of holomorphic functions with values in the group $\mathrm{Aut}(\mathbb{C}^*)$, resp. $\mathrm{Aut}(T)$. Since $\mathrm{Aut}_0(\mathbb{C}^*) \cong \mathbb{C}^*$ and $\mathrm{Aut}_0(T) \cong T$, according to Theorem 3.43 there exist mappings

(a) $H^1(\mathbb{C} \setminus \Sigma, \mathscr{F}) \to \prod^n \mathbb{Z}_2$,

(b) $H^1(\mathbb{C} \setminus \Sigma, \mathscr{G}) \to \prod^n \mathbb{Z}_k$, where $k = 2, 4$, or 6, depending on the torus T.

In view of Theorem 3.43, holomorphic fiber bundles over $\mathbb{C} \setminus \Sigma$ having one of the two mentioned Riemann surfaces as typical fiber are globally-trivial in the holomorphic sense if and only if their corresponding 1-cocycle is contained in the kernel of the corresponding mapping (a) or (b).

Bibliography

[Ara71] ARAKELOV, J. SU., *Families of algebraic curves with fixed degeneracies.* Izv. Akad. Nauk SSSR Ser. Math., 35(6):1277–1302, 1971. 42

[Art86] ARTIN, M., *Lipman's Proof of Resolution of Singularities for Surfaces.* In CORNELL, G. and J. H. SILVERMAN (editors): *Arithmetic Geometry*, Storrs Conference, Connecticut, chapter XI, pages 267–287. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1986. 16, 17

[Bea81] BEAUVILLE, A., *Le nombre minimum de fibres singulières d'une courbe stable sur* $\mathbb{P}^1$. Volume 86 of *Astérisque,* Exposé 6, pages 97–108. Société Mathématique de France, 1981. 49

[BJ73] BRÖCKER, TH. and K. JÄNICH, *Einführung in die Differentialtopologie*, volume 143 of *Heidelberger Taschenbücher.* Springer Verlag, Heidelberg, 1973. 29

[BLR90] BOSCH, S., W. LÜTKEBOHMERT and M. RAYNAUD, *Néron Models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, New York, Berlin, Heidelberg, 1990. 23

[BM47] BOCHNER, S. and D. MONTGOMERY, *Groups on analytic manifolds.* Ann. Math., 48:659–669, 1947. 54

[BPVdV84] BARTH, W., C. PETERS and A. VAN DE VEN, *Compact Complex Surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984. 48

[Bro81] BROUGHTON, S. A., *On the topology of polynomial hypersurfaces.* Proc. Symp. Pure Math., 40(1):167–178, 1981. 13

[Chi86] CHINBURG, T., *Minimal Models for Curves over Dedekind Rings.* In CORNELL, G. and J. H. SILVERMAN (editors): *Arithmetic Geometry*, Storrs Conference, Connecticut, chapter XIII, pages 309–326. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1986. 16, 17, 18, 19, 20, 21

[EGA] GROTHENDIECK, A. and J. DIEUDONNÉ, *Éléments de géométrie algébrique* (EGA). Publ. Math. IHES, 4, 8, 11, 17, 20, 24, 28, 32, 1960–1967. 2, 17, 20, 22, 34

[Epp03] EPPING, D. C., *Nichtabelsche Kohomologie und Vektorbündel auf Riemannschen Flächen.* Westfälische Wilhelms-Universität, Münster, 2003. 39

[FK92] FARKAS, H. M. and I. KRA, *Riemann Surfaces.* Springer-Verlag, Berlin, Heidelberg, New York, 2^{nd} edition, 1992. 44, 54

[For77] FORSTER, O., *Riemannsche Flächen.* Springer-Verlag, Berlin-Heidelberg, 1977. 41

[Fre57] FRENKEL, J., *Cohomologie non abélienne et espaces fibrés.* Bulletin de la Société Mathématique de France, 85:135–220, 1957. 39

[GF65] GRAUERT, H. and W. FISCHER, *Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten.* Nachr. Akad. Wiss. Göttingen, II:89–94, 1965. x, 47

[GM80] GORESKY, M. and R. MACPHERSON, *Stratified Morse Theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, New York, Berlin, Heidelberg, 1980. 29

[GR84] GRAUERT, H. and R. REMMERT, *Coherent Analytic Sheaves.* Springer-Verlag, Berlin-Heidelberg-New York, 1984. 6

[Gra57] GRAUERT, H., *Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen.* Math. Ann., 133:450–472, 1957. 40

[Gra58] ———, *Analytische Faserungen über holomorph-vollständigen Räumen.* Math. Ann., 135:263–273, 1958. xi, 53

[Gra65] ———, *Mordells Vermutung über Punkte auf algebraischen Kurven und Funktionenkörper.* Publ. Math. IHES, 25, 1965. 42

[Har77] HARTSHORNE, R., *Algebraic Geometry.* Springer-Verlag, Berlin-Heidelberg-New York, 1977. 1, 17, 44

[Hir62] HIRZEBRUCH, F., *Neue topologische Methoden in der algebraischen Geometrie*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, Berlin-Göttingen-Heidelberg, 2^{nd} edition, 1962. 40

[HL84] HÀ H. V. and LÊ D. T., *Sur la topologie des polynômes complexes.* Acta Math. Vietnamica, 9:21–32, 1984. ix, 13

[KK83] KAUP, L. and B. KAUP, *Holomorphic Functions of Several Variables.* Number 3 in *Studies in Mathematics.* Walter de Gruyter, Berlin-New York, 1983. 6

[Kod86] KODAIRA, K., *Complex Manifolds and Deformation of Complex Structures*, volume 283 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1986. 28

[Lic68] LICHTENBAUM, S., *Curves over discrete valuation rings*. Am. J. Math., 90:380–405, 1968. 21

[Liu02] LIU, Q., *Algebraic Geometry and Arithmetic Curves*. Oxford University Press, 2002. 1, 2, 3, 4, 6, 7, 8, 18, 23, 26, 27, 34, 35

[Man63] MANIN, Y., *Rational Points on algebraic curves over function fields*. Izv. Akad. Nauk SSSR Ser. Math., 27:1397–1442, 1963. 42

[Mum99] MUMFORD, D. B., *The Red Book of Varieties and Schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, 2^{nd} edition, 1999. x, 42

[Par68] PARSHIN, A. N., *Algebraic curves over function fields*. Izv. Akad. Nauk SSSR Ser. Math., 32:1191–1219, 1968. 41, 42

[Röh57] RÖHRL, H., *Das Riemann-Hilbertsche Problem der Theorie der linearen Differentialgleichungen*. Math. Ann., 133:1–25, 1957. 53

[Ser56] SERRE, J.-P., *Géométrie algébrique et géométrie analytique* (GAGA). Ann. Inst. Fourier, 6:1–42, 1956. 4, 5

[Sha66] SHAFAREVICH, I. R., *Lectures on minimal models and birational transformations of twodimensional schemes*. Tata Inst. Fundam. Res., 1966. ix, 21

[SOS89] SEKIGUCHI, T., F. OORT and N. SUWA, *On the Deformation of Artin-Schreier to Kummer*. Annales Scientifiques de l'École Normale Supérieure, 22:345–375, 1989. x, 14, 25

[Ste51] STEENROD, N., *The Topology of Fiber Bundles*, volume 14 of *Princeton Mathematical Series*. Princeton University Press, Princeton, New Jersey, 1951. 42

Glossary of Notation

$\sim_c$	cohomology as equivalence relation, p. 38.
$X \vee Y$	wedge product $\{(x, y) \in X \times Y \mid x = x_0 \text{ or } y = y_0\}$ of topological spaces X, Y, p. 55.
$\alpha(X)$	number of irreducible divisors of a scheme X, p. 19.
$C^1(\mathfrak{U}, \mathscr{F})$	set of 1-cochains in regards to an open covering $\mathfrak{U}$ with values in a sheaf $\mathscr{F}$, p. 38.
$\operatorname{codim}(F, X)$	codimension of a subset F of X in X, p. 35.
δ	δ-invariant of integral projective schemes of dimension 1, p. 24.
Δ	complex open unit disc, p. 40.
Δ^n	n-dimensional unit sphere, p. 31.
Δ_r	$\{z \in \mathbb{C} \mid r < \lvert t\rvert < 1\}$, p. 54.
$\operatorname{depth} M$	depth of a module M, p. 26.
Df	Jacobian matrix of a function f, p. 30.
$\operatorname{Div}(X)$	group of Cartier divisors on a scheme X, p. 18.
$E^{(2)}$	self-intersection of a curve E, p. 18.
E_n	$(n \times n)$ unit matrix, p. 30.
f	germ of a function f, p. 30.
$\mathrm{GL}_n(\mathbb{C})$	general linear group of complex invertible $(n \times n)$ matrices, p. 9.
$\operatorname{ht}(I)$	height of an ideal I, p. 2.
$\mathscr{I}(V)$	sheaf of ideals of a variety V, p. 4.
$i_k(E, F)$	intersection of a curve E with a positive Cartier divisor F with respect to a field k, p. 18.
K	canonical divisor.
$l(D)$	$\dim_k H^0(X, \mathscr{L}(D))$.
$\operatorname{length}_A(M)$	length of an A-module M, p. 24.
$\mathscr{L}^a$	sheaf of germs of holomorphic maps in a Lie group L, p. 39.
$\mathscr{L}^c$	sheaf of germs of continuous maps in a Lie group L, p. 39.
$\hat{M}$	holomorphically convex hull of a set M, p. 40.
$M_n(\mathbb{C})$	group of complex $(n \times n)$ matrices, p. 9.
$n(X)$	geometric number of irreducible components of a scheme X, p. 34.
$N(X)$	set of non-normal points of a reduced complex space X.
ω_X	canonical sheaf of a smooth variety X, the nth exterior power of the sheaf of differentials, p. 16.
$\Omega_{X/k}$	sheaf of differentials of a variety X over a field k, p. 16.
Φ	flow on a differentiable manifold, p. 29.

p_a	arithmetic genus, p. 16.
p_g	geometric genus, p. 16.
$\mathrm{Proj}(R)$	scheme associated to a graded algebra, p. 7.
R	ramification divisor, p. 43.
$\hat{R}$	completion of a ring R, p. 22.
R^h	Henselization of a ring R, p. 23.
$\mathrm{rank}_a f$	$\mathrm{rank}\, T_a f$, rank of a function f in a, p. 30.
$\bar{s}$	algebraic closure of a point s of a scheme, p. 34.
(S_n)	Serre condition for locally Noetherian schemes, p. 26.
$\mathrm{Spec}(R)$	spectrum of a ring R, p. 3.
$\mathrm{Spm}(R)$	set of maximal ideals of a ring R, p. 22.
$\mathrm{supp}(f)$	$\overline{\{x \in (f) \mid f(x) = 0\}}$, p. 29.
$S(X)$	singular locus of a reduced complex space X, p. 6.
T	complex torus, p. 46.
$T_x M$	analytic tangent space to a manifold M at x, p. 10.
$T_x^z X$	Zariski tangent space to a scheme X at x, p. 1.
U_{ij}	intersection $U_i \cap U_j$ of two sets U_i and U_j, p. 9.
$V(I)$	$\{p \in \mathrm{Spec}(R) \mid I \subseteq \mathfrak{p}\}$, p. 7.
$V(\mathscr{I})$	closed subscheme associated to a quasi-coherent sheaf of ideals, p. 7.
X_η	generic fiber of a morphism, p. 3.
X_h	associated complex analytic space, p. 5.
$(X, Y)_f$	fibered pair with projection f, p. 28.
$Z^1(\mathfrak{U}, \mathscr{F})$	set of 1-cocycles in regards to a covering $\mathfrak{U}$ with values in $\mathscr{F}$, p. 38.
$Z^1(X)$	group of Weil divisors on a scheme X, p. 18.

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