

# Detecting orbits along subvarieties via the moment map

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**Abstract.** Let  $G$  be a (real or complex) linear reductive algebraic group acting on an affine variety  $V$ . Let  $W$  be a subvariety. In this work we study how the  $G$ -orbits intersect  $W$ . We develop a criterion to determine when the intersection can be described as a finite union of orbits of a reductive subgroup.

The conditions of the criterion are easily verified in practice and are used to develop techniques to study left-invariant Ricci soliton metrics on nilpotent Lie groups. A nilpotent Lie group is called an Einstein nilradical if it admits a left-invariant Ricci soliton metric. Applying the techniques developed, we show that the classification of Einstein nilradicals can be reduced to the class of so-called indecomposable groups. Among other applications, we construct arbitrarily large continuous families of (non-isomorphic) nilpotent Lie groups which do not admit left-invariant Ricci soliton metrics.

The note finishes by applying our techniques to the adjoint representation of reductive Lie groups. The classical result of finiteness of nilpotent orbits is reproved and it is shown that each of these orbits contains a critical point of the norm squared of the moment map.

## 1. INTRODUCTION

This work addresses the following question. Let  $G$  be a reductive group acting on a variety  $V$  and let  $W$  be a subvariety of interest.

**Question.** *Can we calculate the size of the moduli of  $G$ -orbits intersecting  $W$ ?*

By moduli we mean the set of points up to the equivalence of lying in the same  $G$  orbit. More generally, we are interested in understanding  $W$  up to  $G$ -equivalence; here  $G$  does not preserve  $W$ . We study this question when  $W$  is smooth and there exists a reductive subgroup  $H$  of  $G$  which measures the  $G$ -action along  $W$ . The notion of measuring the  $G$ -action is by means of the moment map and we say that  $G$  is  $H$ -detected along  $W$  if  $W$  is  $H$ -stable and  $m_G(w) \in \mathfrak{h} = \text{Lie } H$  for  $w \in W$ , where  $m_G$  is the moment map of the  $G$  action (see Section 2). Our main technical result is the following.

**Theorem 3.1.** *Let  $G, H, V, W$  be as above. Suppose  $G$  is  $H$ -detected along  $W$ , then, for  $w \in W \subset V$ , the components of  $G \cdot w \cap W$  are  $H_0$ -orbits, where*

$H_0$  is the identity component of  $H$ . Consequently,  $G \cdot w \cap W$  is a finite union of  $H$ -orbits.

We are primarily interested in applying this theorem to study left-invariant metrics on Lie groups. In Section 4 we present applications to the geometry of nilpotent Lie groups.

Since the works of Heber [6] and Lauret [13] (cp. Remark 4.9 in [14]), geometric invariant theory (GIT) has become a powerful tool in the study of Einstein metrics on non-unimodular solvable Lie groups and Ricci soliton metrics on nilpotent Lie groups (called nilsolitons). However, there are few practical tools in the literature for answering questions concerning the geometry of these Lie groups, from the perspective of GIT. Using Theorem 3.1, we produce new tools for studying Lie groups with left-invariant metrics from this perspective.

Recall that a Riemannian manifold  $(M, g)$  is called *Einstein* if its Ricci operator ( $Ric$ ) is a multiple of the identity map. While left-invariant Einstein metrics do exist on some semisimple and solvable Lie groups, they cannot exist on nilpotent Lie groups by a result of Milnor [19]; this is due to the fact that the Ricci tensor has mixed sign for any metric on a nilpotent Lie group. It is natural, then, to ask which metrics on nilpotent Lie groups are ‘closest’ to being Einstein. In a measurable sense, the metrics which are nearest to being Einstein are the so-called *nilsoliton* metrics [13] and are characterized by the property  $Ric = cId + D$  for some  $c \in \mathbb{R}$  and  $D \in Der(\mathfrak{N})$ , where  $\mathfrak{N}$  is the nilpotent Lie algebra in question.

A nilpotent Lie group is called an *Einstein nilradical* if it admits a nilsoliton metric (see [17] for justification of this terminology). The following two questions are central to much of the current research on nilsolitons:

**Question.** *Given a nilpotent Lie group, how can we determine if it is an Einstein nilradical?*

**Question.** *How are the Einstein nilradicals distributed among all nilpotent Lie algebras?*

In response to these questions, we present Theorems 4.2 and 4.4; these theorems describe how to reduce the set of metrics when searching for a nilsoliton metric. More precisely, we study a representation of  $GL_n \mathbb{R}$  on  $V = \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ , the vector space of  $\mathbb{R}^n$ -valued, anti-symmetric maps on  $\mathbb{R}^n \times \mathbb{R}^n$ . Observe that a Lie bracket  $\mu$  on  $\mathbb{R}^n$  can be viewed as an element of  $V$  and from this perspective the orbit  $GL_n \mathbb{R} \cdot \mu$  is the isomorphism class of  $\mu$ . In the notation of Theorem 3.1, the variety  $W$  will be the set of Lie brackets which admit a prescribed set of ‘symmetric derivations’. Theorems 4.2 and 4.4 show that if a nilpotent Lie group admits a soliton metric and its Lie algebra admits a family of symmetric derivations, then there exists a soliton metric preserving those symmetries.

These theorems generalize a result of Heber [6] which applies in the case of having one specific symmetric derivation, the so-called pre-Einstein derivation. In practice, the above theorems are more user friendly when trying to find

a soliton metric on a nilpotent Lie group. Additionally, these theorems are necessary tools for building continuous families of indecomposable (see below), nonisomorphic nilpotent Lie groups which do not admit nilsoliton metrics (see [10]).

**Theorem 4.7.** *There exist continuous families of nonisomorphic, indecomposable two-step nilpotent Lie groups which do not admit Ricci soliton metrics. The size of these families may be made arbitrarily large; more precisely, these families may be chosen so that the dimension of said families grows linearly in the dimension of the underlying Lie group.*

The first examples of the above phenomenon appeared in [31], where Will constructs a curve of such algebras. To our knowledge, the construction given in [10] of these new higher dimensional families is the first general construction of its kind and depends on the technical results of the present work.

In trying to classify Einstein nilradicals, we are able to reduce the question to one concerning its ‘irreducible pieces’. Consider a Lie group  $N$  and write  $N = N_1N_2$  as a product of complementary normal subgroups (i.e.,  $N_1 \cap N_2 = \{e\}$ ). We say  $N$  is *decomposable* if there exists such a decomposition with both  $N_i$  nontrivial; otherwise, the group is called *indecomposable*. Using the above theorems, we obtain

**Theorem 4.5** *Let  $N$  be a nilpotent Lie group such that  $N = N_1N_2$ , a product of complementary normal subgroups. Then  $N$  admits a left-invariant Ricci soliton metric if and only if both  $N_1$  and  $N_2$  admit such a metric.*

This result has previously appeared in [21] where it is proven using different techniques, ones which are specialized to the setting of solitons on nilpotent groups.

In addition to the above, Section 3 also contains a comparison of the techniques used in studying Ricci solitons on  $k$ -step nilpotent Lie groups vs. 2-step nilpotent Lie groups. To study nilpotent Lie groups (of arbitrary step) one uses a particular representation of  $GL_n\mathbb{R}$ . To study 2-step nilpotent Lie groups, it is often helpful to specialize and consider a family of representations of  $SL_q\mathbb{R} \times SL_p\mathbb{R}$  with  $p + q = n$ , as in [4]. On a general  $k$ -step nilpotent Lie group, nilsolitons are realized as critical points of the norm-squared of the moment map for the  $GL_n\mathbb{R}$  representation above [13]. Using ad hoc methods, an analogous result is true in the 2-step setting for the moment map of the  $SL_q\mathbb{R} \times SL_p\mathbb{R}$  above, see [9]. In Corollary 4.3 we demonstrate that this result for 2-step nilpotent Lie groups follows from more general principles which are applicable to all nilpotent groups (cp. Theorems 4.2 and 4.4).

We finish by demonstrating our techniques applied to the adjoint representation in Section 5. Let  $G$  be a semisimple, or reductive, group and consider the adjoint action on  $\mathfrak{g}$ . We reprove the classical result that there are finitely many nilpotent orbits and show that every nilpotent orbit in  $\mathfrak{g}$  is (analytically) distinguished in the sense of Definition 2.6. The mechanics of this proof for the adjoint representation have been known to experts for some time, but not

phrased in the language of moment maps. We present our work for this case to emphasize how these techniques for the adjoint representation can be phrased in a more general language that applies to any representation of a reductive group, with worthwhile applications.

## 2. PRELIMINARIES

We are interested in actions of both real and complex linear reductive groups. As the main results for real groups follow from the special case of complex groups, we will start with complex groups. Let  $G$  be a complex linear reductive group acting (rationally) on an affine algebraic variety  $X$ . The following theorem is well-known.

**Theorem 2.1.** *There exists a linear representation  $T : G \rightarrow GL(V)$  and closed imbedding  $I : X \rightarrow V$  which is  $G$ -equivariant; that is,  $I(gx) = T(g)I(x)$  for  $x \in X$ ,  $g \in G$ .*

In this way we can reduce to the setting  $X = V$ , for a proof see [23, Thm. 1.5]. Let  $G$  be a complex reductive group and  $K$  a maximal compact subgroup of  $G$ . Let  $V$  be a complex vector space on which  $G$  acts linearly and rationally. We denote this action by

$$G \times V \rightarrow V \quad (g, v) \mapsto g \cdot v$$

We may endow  $V$  with a positive definite  $i$ -invariant inner product  $\langle \cdot, \cdot \rangle$ ; that is, the real part of a positive definite Hermitian inner product. Such an inner product  $\langle \cdot, \cdot \rangle$  on  $V$  is called  $G$ -compatible if  $\langle \cdot, \cdot \rangle$  is  $K$ -invariant. This definition implies that  $G$  is closed under the transpose (metric adjoint) operation and that  $i\mathfrak{k}$  acts symmetrically, where  $Lie G = \mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$  and  $\mathfrak{k} = Lie K$ . Such inner products always exist on  $V$  by averaging.

**Cartan Involutions.** Let  $G$  be a closed subgroup of  $GL(V)$ . The adjoint with respect to  $\langle \cdot, \cdot \rangle$  is denoted by  $*$ . Any involution of the form  $\theta(g) = (g^*)^{-1}$  for  $g \in GL(V)$  is called a *Cartan involution* of  $GL(V)$ . If the involution  $\theta$  leaves  $G$  stable, then we say that  $\theta$  is a Cartan involution of  $G$ . Such involutions always exist by a result of Mostow (see below).

Now choose  $\langle \cdot, \cdot \rangle$  to be invariant under  $K$ , as above. Then  $G$  is stable under the metric adjoint operation and we have a Cartan involution  $\theta(g) = (g^*)^{-1}$  on  $G$  with  $K = G^\theta$ , the fixed points of  $\theta$ . We denote the corresponding involution on  $\mathfrak{g}$  by  $\theta$  and the following are true:  $\mathfrak{k} = LK = +1$  eigenspace of  $\theta$  and  $\mathfrak{p} = i\mathfrak{k} = -1$  eigenspace of  $\theta$ . We observe that  $\theta$  on  $\mathfrak{g}$  is conjugate linear (by  $i$ -invariance of  $\langle \cdot, \cdot \rangle$ ) and is just complex conjugation with respect to the real form  $\mathfrak{k}$  of  $\mathfrak{g}$ . See [20], [25], and references therein for more information on Cartan involutions and decompositions.

**Proposition 2.2** (Mostow). *Let  $G$  be a complex (linear algebraic) reductive group and  $H$  a reductive subgroup. Then there exists a Cartan involution  $\theta$  which simultaneously preserves  $G$  and  $H$ . The involution  $\theta$  can be chosen so*

that  $K_H = H^\theta$  is a previously chosen maximal compact subgroup of  $H$  and  $K_H \subset K = G^\theta$  where  $K$  is a maximal compact subgroup of  $G$ .

We say that the Cartan decomposition of  $H$  above is *compatible* with the Cartan decomposition of  $G$ . In practice we will be interested in inner products which are both  $G$  and  $H$ -compatible. The above proposition says that such inner products always exist. We observe that if an inner product is  $G$ -compatible and  $H$  is compatible with  $G$ , then the given inner product is  $H$ -compatible.

Similarly, Cartan involutions exist on real algebraic reductive groups and the above proposition is still valid. In the real setting, however, the space  $\mathfrak{p} = -1$  eigenspace of  $\theta$  will not equal  $i\mathfrak{k}$  as  $\mathfrak{g}$  might not be a complex Lie algebra. Here we choose inner products on  $V$  such that  $\mathfrak{p}$  acts symmetrically on  $V$ , such inner products always exist. Cartan involutions and the decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  are good tools for studying the geometry of orbits of real rational representations of real reductive groups, see [25], [18], and [5] for applications to the study of real group orbits.

**Moment maps.** Endow  $\mathfrak{g}$  with an  $Ad K$ -invariant,  $i$ -invariant,  $\theta$ -invariant inner product  $\langle\langle \cdot, \cdot \rangle\rangle$ . This is equivalent to  $\langle\langle \cdot, \cdot \rangle\rangle$  being the real part of an  $Ad K$ -invariant,  $\theta$ -invariant Hermitian inner product on  $\mathfrak{g}$ . We choose  $\langle\langle \cdot, \cdot \rangle\rangle$  to be  $\theta$ -invariant so that  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is an orthogonal decomposition. Such inner products on  $\mathfrak{g}$  always exist as demonstrated by the following examples. Additionally, one could guarantee their existence by averaging.

**Example 2.3.** Let  $G$  be an algebraic reductive subgroup of  $SL(E)$ . Let  $\theta$  denote a  $G$ -stable Cartan involution of  $SL(E)$  and let  $B$  denote the Killing form of  $\mathfrak{sl}(E)$ . Then the inner product  $\langle\langle \cdot, \cdot \rangle\rangle = -B(\cdot, \theta(\cdot))$  satisfies the conditions stated above.

**Example 2.4.** If  $G$  is semisimple then one may use the inner product  $\langle\langle \cdot, \cdot \rangle\rangle = -B_{\mathfrak{g}}(\cdot, \theta(\cdot))$  where  $B_{\mathfrak{g}}$  is the Killing form of  $\mathfrak{g}$ .

We define the  $\mathfrak{g}$ -valued *moment map*  $m : V \rightarrow \mathfrak{g}$  implicitly by

$$\langle\langle m(v), X \rangle\rangle = 2\langle X \cdot v, v \rangle \quad \text{for all } X \in \mathfrak{g}$$

The  $K$ -invariance of  $\langle \cdot, \cdot \rangle$  implies  $m(v) \in \mathfrak{k}^\perp \subset \mathfrak{g}$ . The  $\theta$ -invariance of  $\langle\langle \cdot, \cdot \rangle\rangle$  implies  $\mathfrak{k}^\perp = \mathfrak{p}$  and thence

$$m(v) \in \mathfrak{p} \quad \text{for all } v \in V$$

Having  $\mathfrak{p}$ -valued moment maps is the real reason we required  $\langle\langle \cdot, \cdot \rangle\rangle$  to be  $\theta$ -invariant.

We observe that  $m$  is a (real) homogeneous polynomials of degree 2. Moreover, for  $c \in \mathbb{C}$  we have  $m(cv) = |c|^2 m(v)$ , where  $|c|^2 = c\bar{c}$  is the usual norm square on  $\mathbb{C}$ . This gives rise to a well-defined polynomial on complex projective space

$$m[v] = \frac{m(v)}{|v|^2}$$

In the sequel, we will be interested in  $\|m\|^2$  as a real valued function on both  $V$  and  $\mathbb{P}V$ , the norm-squared being taken with respect to  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ . When  $G$  is a real reductive group the same results hold except that  $\mathfrak{p}$  is not necessarily  $i\mathfrak{k}$ . Moreover, in the real setting our moment map gives rise to a well-defined function on real projective space.

**Closed and distinguished orbits.** Consider the action of  $G$  on  $V$ . It has been shown that the closed orbits are precisely the orbits which intersect the zero set of the moment map  $m : V \rightarrow \mathfrak{p}$ . In the complex setting this theorem was done by Kempf-Ness [12] and in the real setting done by Richardson-Slodowy [25]. We state it below.

**Theorem 2.5.** *Consider the action  $G \times V \rightarrow V$ . Denote the zero set of  $m : V \rightarrow \mathfrak{p}$  by  $\mathfrak{M} = m^{-1}(0)$ . For  $v \in V$ , the orbit  $G \cdot v$  is closed if and only if  $G \cdot v \cap \mathfrak{M} \neq \emptyset$ .*

A nonzero point  $v \in V$  is called *semi-stable* if  $0 \notin \overline{G \cdot v}$ . This set clearly contains all nontrivial closed  $G$ -orbits. In contrast, a point is called *unstable* if zero is contained in the boundary of its  $G$ -orbit. The set of unstable points is called the *null-cone* and can be studied in a more refined way by passing to projective space.

**Definition 2.6.** A point  $v \in V$  or  $[v] \in \mathbb{P}V$  is called  $G$ -distinguished, or just distinguished when the  $G$  action is clear, if  $[v] \in \mathbb{P}V$  is a critical point of  $\|m\|^2 : \mathbb{P}V \rightarrow \mathbb{R}$ . Likewise, we call an orbit  $G \cdot v$ , or  $G \cdot [v]$ , distinguished if it contains a distinguished point.

Closed orbits are always distinguished as zero is an absolute minimum of  $\|m\|^2$ . The nonclosed distinguished orbits all lie in the null-cone.

**Lemma 2.7.** *Let  $v \in V$ . Then  $v$  is distinguished if and only if  $m(v) \cdot v = cv$  for some  $c \in \mathbb{R}$ .*

A proof of this lemma follows immediately from the definitions and can be found in [18]. In the following theorem,  $G$  may be either a real or complex reductive Lie group.

**Theorem 2.8.** *Let  $G$ ,  $\mathfrak{g}$ ,  $K$ , and  $V$  be as above endowed with inner products as above. Let  $m$  denote the moment map of the representation and, for  $v \in V$ , denote by  $\varphi_t[v]$  the negative gradient flow of  $\|m\|^2$  starting at  $[v]$  in  $\mathbb{P}V$ . Denote the limit point of this flow by  $\varphi_\infty[v]$ . If  $G \cdot v$  is a distinguished orbit with  $v$  such a distinguished point, then for every  $g \in G$ ,  $\varphi_\infty[g \cdot v] \in K \cdot [v]$ .*

This theorem is true in both the real and complex settings, see [7]. A priori, it is not clear that the limit set of the flow is a single point. For this point and more information on moment maps, see [28, Sec. 2.5].

**Detecting  $G$ -orbits along subvarieties.** Let  $G$  be a (real or complex algebraic) reductive group which acts linearly and rationally on  $V$ . Let  $H$  be a reductive subgroup of  $G$  which has a compatible Cartan decomposition (see

Proposition 2.2). Let  $V$  be endowed with a  $G$ -compatible metric (see beginning of Section 2). Recall that the  $G$ -compatible metric on  $V$  is also  $H$ -compatible as  $H$  has a compatible Cartan decomposition. Let  $W$  be an  $H$ -stable smooth subvariety of  $V$  (by smooth we mean  $W$  is also a submanifold).

**Definition 2.9.** We say that the  $G$ -action on  $V$  is ‘ $H$ -detectable along  $W$ ’ if  $m_G(w) \in \mathfrak{h}$  for  $w \in W$ .

Here  $m_G$  denotes the moment map for the  $G$ -action. Analogously, we could state this definition for smooth varieties in projective space.

We observe the following. Let  $V$  be a  $G$ -representation which is  $H$ -detectable on  $W$ , then  $m_G(w) = m_H(w)$  for all  $w \in W$ . The proof follows immediately upon writing out the definitions.

### 3. ORBITS OF COMPATIBLE SUBGROUPS

As above, let  $G$  be a reductive group with Cartan involution  $\theta$  and let  $H$  be a compatible reductive subgroup (cp. Proposition 2.2). Let  $V$  be a  $G$ -representation space endowed with an inner product  $\langle \cdot, \cdot \rangle$  which is  $G$  (and hence  $H$ ) compatible. Let  $m_G$  and  $m_H$  denote the moment maps of the  $G$  and  $H$  actions, resp., on  $V$ . Suppose there exists an  $H$ -stable smooth subvariety  $W \subset V$  on which the  $G$ -action is  $H$ -detectable; that is, such that  $m_G(w) = m_H(w)$  for all  $w \in W$  (see the definition above). The subvariety  $W$  is not required to be closed.

The following theorems are true for real and complex groups. We first give proofs for complex groups and finish the section by explaining how to extend the results over  $\mathbb{C}$  to results over  $\mathbb{R}$ .

**Theorem 3.1.** *Let  $G, H, W, V$  be as above. If  $w \in W \subset V$ , then the components of  $G \cdot w \cap W$  are  $H_0$ -orbits and, consequently,  $G \cdot w \cap W$  is a finite union of  $H$ -orbits.*

**Remark.** In this way we obtain a solution of the original question. The dimension of the moduli of  $G$  orbits which intersect  $W$  is precisely the dimension of the moduli of  $H$  orbits in  $W$ .

While the proof of this theorem is elementary, the criterion of a  $G$ -orbit being  $H$ -detected along some  $W$  is easily verified in practice and can produce information with genuine content. For applications of this theorem to the study of left-invariant metrics on nilpotent Lie groups, see Section 4 and [10]. One general consequence of the above theorem is the following corollary.

**Corollary 3.2.** *Let  $G, H, W, V$  be as in the theorem above. Then for  $w \in W$ , the intersection  $G \cdot w \cap W$  is smooth.*

**Theorem 3.3.** *Let  $G, H, W, V$  be as above but with  $W$  a cone in  $V$ ; that is,  $W$  descends to a projective variety  $\mathbb{P}W$  of  $\mathbb{P}V$ . Consider the induced actions on  $\mathbb{P}V, \mathbb{P}W$ . If  $[w] \in \mathbb{P}W$ , then the components of  $G \cdot [w] \cap \mathbb{P}W$  are  $H_0$ -orbits and, consequently,  $G \cdot [w] \cap \mathbb{P}W$  is a finite union of  $H$ -orbits.*

**Remark.** The proof of the second theorem does not follow immediately from the first. However, the proofs are very similar and we give them simultaneously.

**Corollary 3.4.** *Let  $w \in W \subset V$  where  $W$  is a closed  $H$ -stable smooth subvariety (e.g., a subspace). Then  $G \cdot w$  is distinguished if and only if  $H \cdot w$  is distinguished.*

**Remarks.** (1) Closedness of  $W$  is necessary in this corollary as finding distinguished points involves taking limits. This corollary gives a useful criterion for determining when a specific orbit of  $G$  or  $H$  is distinguished. Applications of this corollary are given in the following section and also in [10].

(2) It is not true, in general, that  $H \cdot w$  must be distinguished if  $G \cdot w$  is distinguished. In the special case of closed orbits, this problem has been explored in [30] and [5]. In [30] it is shown that if  $G \cdot w$  is closed then  $H \cdot gw$  is closed for generic  $g \in G$ . Other criteria to determine closedness of orbits of (sub)groups have been constructed in [5]; at the moment there is no general criterion which completely determines when  $H \cdot w$  is closed, even if  $G \cdot w$  is closed. For an explicit example of a nonclosed  $H$  orbit in a closed  $G$  orbit, see [8].

Before proving the theorem, we use it to deduce Corollary 3.4.

*Proof of the Corollary 3.4.* We apply Theorem 3.3 and Theorem 2.8 to prove the corollary.

Recall that  $[w] \in \mathbb{P}W$ , or  $w \in W$ , is a  $G$ -distinguished point if (by definition)  $[w]$  is a critical point of  $\|m_G\|^2 : \mathbb{P}V \rightarrow \mathbb{R}$  where  $m_G$  is the moment map of the  $G$ -action on  $\mathbb{P}V$ . We denote the moment map on  $V$  and  $\mathbb{P}V$  by the same notation as context should avoid any confusion. It is well-known, see Lemma 2.7, that  $[w] \in \mathbb{P}W \subset \mathbb{P}V$  is  $G$ -distinguished if and only if  $m_G(w) \cdot w = cw$  for some  $c \in \mathbb{R}$ . In this way we see that a point  $w \in W$  is  $H$ -distinguished if and only if  $w \in W$  is  $G$ -distinguished as  $m_G(w) = m_H(w)$ . Thus, if  $H \cdot w$  is  $H$ -distinguished, then  $G \cdot w$  must be  $G$ -distinguished.

Conversely, consider  $w \in W$  such that the orbit  $G \cdot w$  is  $G$ -distinguished. Theorem 2.8 provides the existence of  $g_n \in G_0$  and  $g \in G$  such that  $[g_n \cdot w] \rightarrow [g \cdot w]$  in  $\mathbb{P}W$ , as  $n \rightarrow \infty$ , and  $[g \cdot w]$  is distinguished. Notice that the sequence  $[g_n w]$  can be chosen to lie in  $\mathbb{P}W$  by following the negative gradient flow of  $\|m_G\|^2$  starting at  $[w]$  and our limit point is in  $\mathbb{P}W$  as  $\mathbb{P}W$  is closed. Moreover, observe that  $[g \cdot w]$  is in the same connected component of  $G \cdot [w] \cap \mathbb{P}W$  as  $[w]$ . By Theorem 3.3, there exists  $h \in H$  such that  $g \cdot [w] = h \cdot [w]$ ; that is,  $H \cdot [w]$  contains a distinguished point and hence  $H \cdot w$  is a distinguished orbit.  $\square$

Next we prove Theorems 3.1 and 3.3. The proofs of both are given simultaneously as they are so similar. It suffices to consider  $H$  which is connected as  $H$ , being an algebraic group, has finitely many components.

**Lemma 3.5.** *For  $w \in W$ ,  $T_w(G \cdot w) \cap T_w W = T_w(H \cdot w)$ . When  $W$  is a cone in  $V$  with projection  $\mathbb{P}W \subset \mathbb{P}V$ , we have  $T_{[w]}(G \cdot [w]) \cap T_{[w]}\mathbb{P}W = T_{[w]}(H \cdot [w])$ .*



*Proof of the lemma.* To show equality, we show containment in both directions. One direction is trivial. Since  $H$  preserves  $W$ , we immediately have  $T_w(H \cdot w) \subset T_w(G \cdot w) \cap T_w W$  which then implies  $T_{[w]}(H \cdot [w]) \subset T_{[w]}(G \cdot [w]) \cap T_{[w]}\mathbb{P}W$ . The reverse containments are shown below.

We prove the lemma at the affine level first. By translation, we have the following identifications

$$\begin{aligned} T_w(G \cdot w) &\simeq \mathfrak{g} \cdot w \\ T_w(H \cdot w) &\simeq \mathfrak{h} \cdot w \end{aligned}$$

Recall that our Cartan decompositions  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  and  $\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H$  satisfy  $\mathfrak{k}_H \subset \mathfrak{k}$ ,  $\mathfrak{p} = i\mathfrak{k}$ , and  $\mathfrak{p}_H = i\mathfrak{k}_H$ . Take  $X \in \mathfrak{p} \ominus \mathfrak{p}_H$ , the orthogonal compliment of  $\mathfrak{p}_H$  in  $\mathfrak{p}$ , then

$$0 = \langle X, m_G(w) \rangle = \langle X \cdot w, w \rangle \text{ for all } w \in W$$

as  $m_G(w) = m_H(w) \in \mathfrak{h}$ . Consider  $v_w \in T_w W$  and a curve  $\gamma(t) \in W$  tangent to  $v_w$  at  $t = 0$ ; that is  $\gamma(0) = w$  and  $\gamma'(0) = v$ . Applying the above equation to this curve and differentiating we obtain

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \right|_{t=0} \langle X \cdot \gamma(t), \gamma(t) \rangle \\ &= \langle X \cdot v, w \rangle + \langle v, X \cdot w \rangle \\ &= 2\langle X \cdot w, v \rangle \end{aligned}$$

as  $X = X^t \in \mathfrak{p} \ominus \mathfrak{p}_H \subset \mathfrak{p}$  is symmetric with respect to  $\langle \cdot, \cdot \rangle$ . Thus,  $X \cdot w \perp T_w W$  for  $w \in W$  and  $X \in \mathfrak{p} \ominus \mathfrak{p}_H$ .

Since  $T_w W$  is a complex vector space and  $\langle \cdot, \cdot \rangle$  is  $i$ -invariant, for  $w \in W$ ,  $v_w \in T_w W$ , and  $X \in \mathfrak{p} \ominus \mathfrak{p}_H$  we have  $0 = \langle X \cdot w, -iv \rangle = \langle iX \cdot w, v \rangle = \langle (iX) \cdot w, v \rangle$  as  $\mathfrak{g}$  acts  $\mathbb{C}$ -linearly. Since  $\mathfrak{g} \ominus \mathfrak{h} = (\mathfrak{k} \oplus \mathfrak{p}) \ominus (\mathfrak{k}_H \oplus \mathfrak{p}_H) = \mathbb{C} - \text{span}\{\mathfrak{p} \ominus \mathfrak{p}_H\}$  we have

$$(1) \quad X \cdot w \perp T_w W \text{ for } w \in W, X \in \mathfrak{g} \ominus \mathfrak{h}$$

Now pick  $X \in \mathfrak{g}$  such that  $X \cdot w \in \mathfrak{g} \cdot w \cap T_w W$ . Writing  $X = X_1 + X_2 \in \mathfrak{h} \oplus (\mathfrak{g} \ominus \mathfrak{h})$ , we obtain  $X_1 \cdot w + X_2 \cdot w \in T_w W$  which then implies  $X_2 \cdot w \in T_w W$  as  $H$  acts on  $W$  by hypothesis. Here we are using the smoothness of  $W$  to insure that  $X_2 \cdot w = X \cdot w - X_1 \cdot w \in T_w W$ . But then  $X_2 \cdot w = 0$  by Equation (1). Hence,  $X \cdot w = X_1 \cdot w \in \mathfrak{h} \cdot w$  as desired. This proves the lemma in the affine case.

To prove the second part of the lemma, we reduce to the first part. Recall that the map  $\pi : V \rightarrow \mathbb{P}V$  is a submersion and the restrictions  $\pi : G \cdot v \rightarrow G \cdot [v]$  and  $\pi : W \rightarrow \mathbb{P}W$  are surjective. Let  $\pi_* : TV \rightarrow T\mathbb{P}V$  denote the induced map on the tangent bundles and recall that if  $v_w \in T_w V$  is such that  $\pi_*(v_w) = 0 \in T_{[w]}\mathbb{P}V$ , then  $v \in \mathbb{C} \langle w \rangle$ .

Consider  $X_{[w]} \in T_{[w]}(G \cdot [w]) \cap T_{[w]}\mathbb{P}W$ . Then there exist  $X'_w \in T_w(G \cdot w)$  and  $X''_w \in T_w W$  such that  $X_{[w]} = \pi_*(X'_w) = \pi_*(X''_w) \in T_{[w]}\mathbb{P}W$ . This implies  $\pi_*(X'_w - X''_w) = 0 \in T_{[w]}\mathbb{P}W$ , which implies  $X'_w - X''_w \in \mathbb{C} \langle w \rangle$ , which then implies  $X'_w \in X''_w + \mathbb{C} \langle w \rangle \subset T_w W$  as  $W$  is a cone and smooth at  $w \neq 0$ .

That is,  $X'_w \in T_w(G \cdot w) \cap T_w W = T_w(H \cdot w)$  by the first part of the lemma and so  $X_{[w]} = \pi_*(X'_w) \in \pi_*(T_w H \cdot w) = T_{[w]}(H \cdot [w])$ . Thus,  $T_{[w]}(G \cdot [w]) \cap T_{[w]} \mathbb{P}W = T_{[w]}(H \cdot [w])$ .  $\square$

As the lemma is now proven, we continue with the proof of the theorems. If we knew  $G \cdot w \cap W$  were smooth, then by dimension arguments and the lemma above it would be easy to establish that  $G \cdot w \cap W \simeq H \cdot w$  locally near  $w$ . A priori, however, we cannot guarantee smoothness; obviously the theorems show smoothness a posteriori. We recall the following basic proposition on algebraic group actions; for a proof see [3, I.1.8].

**Proposition 3.6.** *Let  $G$  be an algebraic group acting on a variety  $X$ . Then for  $x \in X$ , the boundary  $\partial(G \cdot x) = \overline{G \cdot x} - G \cdot x$  consists of  $G$ -orbits of strictly lesser dimension. Moreover, the orbit  $G \cdot x$  is open and dense in  $\overline{G \cdot x}$ . The Hausdorff and Zariski closures of the orbit coincide.*

**Lemma 3.7.** *Let  $G$  be an algebraic group and let  $H$  be a closed subgroup of  $G$ . Let  $G$  act on a variety  $V$  and assume that  $W$  is an  $H$ -stable subvariety. Suppose the following is true for every  $w \in W$*

$$T_w(G \cdot w) \cap T_w W = T_w(H \cdot w)$$

*Then the intersection of each  $G$ -orbit with  $W$  is a union of finitely many  $H$ -orbits.*

Clearly this lemma combined with the previous lemma completes the proof of the theorems. We would like to thank Chuck Hague for pointing out to us that this lemma previously existed in the literature (see [11, Lem. 2.4]). We include a proof for the sake of completeness and for the nonexpert.

*Proof of the lemma.* Let  $w \in W$ , we can decompose the variety  $G \cdot w \cap W = \cup X_i$  into irreducible components  $X_i$ . As we are interested in the component of  $G \cdot w \cap W$  that contains  $w$ , we only need to consider the irreducible components  $X_i$  that contain  $w$ .

We begin by showing  $H$  preserves  $X_i$ . Take  $p_i \in X_i$  which is a smooth point of  $X_i$ . Then  $G \cdot p_i \cap W$  coincides with  $X_i$  near  $p_i$  (in the Hausdorff sense), and so  $h \cdot p_i \in X_i$  for  $h \in \mathcal{O}_e \subset H$  where  $\mathcal{O}_e$  is a (Hausdorff) open neighborhood of the identity element  $e \in H$ . Let  $\mu : G \times V \rightarrow V$  denote the  $G$ -action on  $V$ . Since  $\overline{\mathcal{O}_e} = H$  and  $\mu$  is continuous we have

$$\mu(H, p_i) = \mu(\overline{\mathcal{O}_e}, p_i) \subset \overline{\mu(\mathcal{O}_e, p_i)} \subset X_i$$

This holds for any smooth point of  $X_i$ . Let  $\mathcal{U}$  denote the set of smooth points of  $X_i$ . Then  $\overline{\mathcal{U}} = X_i$  (cp. [27, Lem. 2.1.1]) and as before

$$\mu(H, X_i) = \mu(H, \overline{\mathcal{U}}) \subset \overline{\mu(H, \mathcal{U})} \subset \bigcup_{p_i \in \mathcal{U}} \overline{\mu(H, p_i)} \subset \bigcup_{p_i \in \mathcal{U}} X_i = X_i$$

This shows  $H$  acts on  $X_i$ .

Now that  $H \cdot p_i \subset X_i$  we may compare the following dimensions

$$\dim H \cdot p_i \leq \dim X_i \text{ at } p_i \leq \dim(T_{p_i}(G \cdot p_i) \cap T_{p_i}W) = \dim T_{p_i}(H \cdot p_i)$$

where the second inequality follows from  $T_{p_i}(G \cdot p_i \cap W) \subset T_{p_i}(G \cdot p_i) \cap T_{p_i}W$  and the last equality is the hypothesis of the present lemma. Thus  $\dim H \cdot p_i = \dim X_i$  and we see that  $H \cdot p_i$  is an (analytic) open neighborhood of  $p_i$  in  $X_i$ . As  $X_i$  is irreducible, we see that  $\overline{H \cdot p_i} = X_i$ . Since  $H \cdot p_i$  is open and dense in  $\overline{H \cdot p_i}$ , the same is true for  $H \cdot p_i \subset X_i$ .

Next we show  $H \cdot p_i = X_i$ . Let  $w' \in X_i$ . First we observe that  $w' \in \overline{H \cdot p_i}$ , as stated in the previous paragraph. The next lemma will use the fact that the Hausdorff and Zariski closure of the orbit coincide (cp. Proposition 3.6).

**Sublemma 3.8.**  $\dim T_{w'}(G \cdot w') \cap T_{w'}W \geq \dim H \cdot p_i$

*Proof of the sublemma.* By hypothesis  $G \cdot w' = G \cdot p_i$  and there exists  $w_n \in H \cdot p_i$  such that  $w_n \rightarrow w'$  as  $w'$  is in the Hausdorff closure of  $H \cdot p_i$ .

Pick an orthonormal basis  $\{X_j^n\}_{j=1}^r$  of  $T_{w_n}H \cdot w_n = T_{w_n}H \cdot p_i$  where  $r = \dim H \cdot p_i$ . By passing to a subsequence we may assume  $X_j^n \rightarrow X_j$  as  $n \rightarrow \infty$ , for  $j = 1, \dots, r$ . Since  $G \cdot w' = G \cdot p_i$  we have  $X_j \in T_{w'}G \cdot w'$  which then implies  $\{X_j\}_{j=1}^r \subset T_{w'}(G \cdot w') \cap T_{w'}W$ . As this collection of vectors is orthonormal, the sublemma is proven. □

By hypothesis and the above work, we have

$$\dim H \cdot w' = \dim T_{w'}(H \cdot w') = \dim(T_{w'}G \cdot w' \cap T_{w'}W) \geq \dim H \cdot p_i$$

But  $w' \in \overline{H \cdot p_i}$ . Proposition 3.6 implies  $\dim H \cdot w' \leq \dim H \cdot p_i$  with equality if and only if  $H \cdot w' = H \cdot p_i$ . Applying this fact and the inequality above, we see that  $H \cdot w' = H \cdot p_i$ . Since  $w' \in X_i$  was arbitrary we have  $H \cdot p_i = X_i$ . We observe that we have shown  $X_i = H \cdot w'$  for any  $w' \in X_i$ .

Recall that we have decomposed  $G \cdot w \cap W = \cup X_i$  into irreducible components. If an irreducible component  $X_i$  contains  $w$  then  $X_i = H \cdot w$ , as shown above. Thus the topological component of  $G \cdot w \cap W$  containing  $w$  is

$$\bigcup_{\{X_i \mid w \in X_i\}} X_i = \bigcup H \cdot w = H \cdot w. \text{ This proves the lemma and the proofs of } \{X_i \mid w \in X_i\}$$

our theorems are complete.

**Remark.** We point out that this technique of using compatible subgroups to study  $G \cdot w \cap W$  does not completely detect the phenomenon of the intersection being a finite union of  $H$ -orbits. There do exist examples where  $W$  is an  $H$ -stable subspace, each intersection  $G \cdot w \cap W$  is finite union of  $H$ -orbits, but that the  $G$ -action is not  $H$ -detectable along  $W$  for any choice of inner products on  $V$  and  $\mathfrak{g}$ . This is proven by constructing  $H$  and  $W$  so that the  $G$ -orbit through a generic point of  $W$  is closed, but so that the  $H$ -orbit through said points is not closed (cp. Corollary 3.4).

**The question of closed orbits of subgroups.** Let  $G$  be a reductive group acting rationally on  $V$ . Let  $H$  be a reductive subgroup. If  $G \cdot v$  is closed, then it is known that  $H \cdot gv$  is closed for generic  $g \in G$ . This problem has been worked on by many people, see, e.g., [16], [22], [30], or [8]. However, it is not true for all  $g \in G$  that  $H \cdot gv$  must be closed. It is an interesting problem to try and determine when the orbit of  $H$  is closed.

**Corollary 3.9.** *Suppose there exists an  $H$ -stable smooth (closed) subvariety  $W$  along which  $G$  is  $H$ -detected. If  $G \cdot w$  is closed, then so is  $H \cdot w$ .*

This is the special case of Corollary 3.4 when our distinguished orbit is closed. We do not know if the converse is true.

**Question 3.10.** *Let  $H$  be a reductive subgroup of  $G$ . Let  $G$  act rationally on  $V$  and suppose that  $G \cdot v$  and  $H \cdot v$  are closed. Do there exist inner products on  $V$  and  $\mathfrak{g}$ , satisfying the hypothesis of Theorem 3.1, such that  $G$  is  $H$ -detected along the smooth subvariety  $H \cdot v$ ?*

As stated at the end of the previous subsection, there do exist examples of  $V, W, G, H$  such that  $G$  cannot be  $H$ -detected along  $W$  for any choice of inner products on  $V$  and  $\mathfrak{g}$ .

**Real algebraic groups.** Here we explain how to obtain the above results over  $\mathbb{R}$ . Let  $G$  be a real algebraic reductive group. The real group  $G$  can be realized as the real points of a complex algebraic reductive group  $G^{\mathbb{C}}$  such that  $G$  is Zariski dense. This is well-known, see, e.g., [25].

The following result of Borel-Harish-Chandra ([1, Prop. 2.3]) is the standard way of relating the real and complex settings. Let  $V$  be real vector space on which  $G$  acts linearly and rationally. Let  $V^{\mathbb{C}} = V \otimes \mathbb{C}$  denote the complexification, then  $G^{\mathbb{C}}$  acts linearly and rationally on  $V^{\mathbb{C}}$ .

**Theorem 3.11.** *Consider  $v \in V \subset V^{\mathbb{C}}$ . Then  $G^{\mathbb{C}} \cdot v \cap V$  is a finite union of  $G$ -orbits. Moreover, the orbit  $G^{\mathbb{C}} \cdot v$  is closed if and only if  $G \cdot v$  is closed.*

In the last assertion, the only if direction requires the work of either [2] or [25]. In fact, a slightly stronger version of this theorem is true. We state this version below and refer the reader to [7] for a proof.

**Theorem 3.12.** *Consider  $v \in V \subset V^{\mathbb{C}}$ . Then  $G^{\mathbb{C}} \cdot v$  is distinguished if and only if  $G \cdot v$  is distinguished.*

To obtain Theorems 3.1 and 3.3 over  $\mathbb{R}$ , one just needs know that they are true over  $\mathbb{C}$  and apply Theorem 3.11. To obtain Corollary 3.4 over  $\mathbb{R}$ , one just needs to know that it is true over  $\mathbb{C}$  and apply Theorem 3.12.

#### 4. APPLICATIONS TO THE LEFT-INVARIANT GEOMETRY OF LIE GROUPS

The theorems in this section can be viewed as specialized change of basis theorems. We are interested in left-invariant Ricci soliton metrics on nilpotent Lie groups. Such a metric is called a *nilsoliton* and if a nilpotent Lie group

admits such a metric, it is called an *Einstein nilradical* (see [17] for justification of this terminology).

**Question.** *Which nilpotent Lie groups are Einstein nilradicals? If a nilpotent Lie group admits such a metric, how can one find this special metric?*

As a left-invariant metric on a Lie group is equivalent to an inner product on its Lie algebra, we reduce to studying Lie algebras with inner products. The nilsoliton condition can be completely phrased at the algebra level.

There are two points of view that one can take. The first is to fix a Lie bracket on a vector space and vary the inner product, the second is to fix an inner product on a vector space and vary the Lie bracket. We will work from the second perspective as it has produced many results and allows us to exploit tools from Geometric Invariant Theory. This is the perspective taken by J. Lauret and others (see [15] and references therein).

These special metrics are realized as critical points of the norm squared of the moment map corresponding to a particular  $GL(n, \mathbb{R})$  action. Let  $\mathfrak{N}$  be an  $n$ -dimensional real vector space with fixed inner product  $\langle \cdot, \cdot \rangle$ . Consider the space  $V = \wedge^2(\mathfrak{N})^* \otimes \mathfrak{N}$ . This is the space of skew-symmetric bilinear forms from  $\mathfrak{N} \times \mathfrak{N}$  to  $\mathfrak{N}$ . The set of Lie brackets on  $\mathfrak{N}$  is a variety  $\mathcal{V}$  in  $V$ . The change of basis action of  $GL(n, \mathbb{R})$  on  $\mathfrak{N}$  induces the following action on  $V$

$$g \cdot \mu(X, Y) = g\mu(g^{-1}X, g^{-1}Y)$$

for  $g \in GL(n, \mathbb{R})$ ,  $\mu \in V$ , and  $X, Y \in \mathbb{R}^n$ . This action preserves the variety  $\mathcal{V}$  of Lie brackets. Moreover, if  $\mu \in \mathcal{V}$  is a Lie bracket on  $\mathfrak{N}$ , then the orbit  $GL_n \cdot \mu$  is precisely the isomorphism class of  $\mu$  in  $\mathcal{V}$ . The metric nilpotent Lie algebra with bracket  $\mu$  and inner product  $\langle \cdot, \cdot \rangle$  is denoted  $\mathfrak{N}_\mu$  and the simply connected nilpotent Lie group with Lie algebra  $\mathfrak{N}_\mu$  and left-invariant metric corresponding to  $\langle \cdot, \cdot \rangle$  is denoted by  $\{N_\mu, \langle \cdot, \cdot \rangle\}$  or just  $N_\mu$  when  $\langle \cdot, \cdot \rangle$  is understood. Let  $g^*\langle \cdot, \cdot \rangle = \langle g^{-1}\cdot, g^{-1}\cdot \rangle$  for  $g \in GL(n, \mathbb{R})$ , then  $\{N_{g \cdot \mu}, \langle \cdot, \cdot \rangle\}$  is isometric to  $\{N_\mu, g^*\langle \cdot, \cdot \rangle\}$ . Thus  $N_\mu$  admits a nilsoliton metric if and only if  $N_{g \cdot \mu}$  is a nilsoliton for some  $g \in GL(n, \mathbb{R})$ . In this way we can study the set of left-invariant metrics on  $N_\mu$  by studying the orbit  $GL(n, \mathbb{R}) \cdot \mu$  in  $\mathcal{V}$ .

The inner product on  $\mathfrak{N}$  extends naturally to an inner product on  $V$  defined by

$$\langle \lambda, \mu \rangle = \sum_{ijk} \langle \lambda(X_i, X_j), X_k \rangle \langle \mu(X_i, X_j), X_k \rangle$$

where  $\{X_i\}$  is any orthonormal basis of  $\mathfrak{N}$ . This inner product on  $V$  is  $O(n)$ -invariant, where  $O(n)$  is the orthogonal group relative to  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{N}$ .

A Cartan decomposition of  $\mathfrak{gl}(n)$  is given by  $\mathfrak{so}(n) \oplus \mathit{symm}(n)$  where  $\mathfrak{so}(n)$  is the Lie algebra of  $O(n)$ , the skew-symmetric endomorphisms relative to  $\langle \cdot, \cdot \rangle$ , and  $\mathit{symm}(n)$  is the set of symmetric endomorphisms relative to  $\langle \cdot, \cdot \rangle$ . We consider the inner product  $\langle\langle A, B \rangle\rangle = \mathit{tr}(AB^t)$  on  $\mathfrak{gl}(n)$ , where the transpose is determined by  $\langle \cdot, \cdot \rangle$ . This gives rise to the following moment map (see [15,

Prop. 3.5])

$$(2) \quad m(\mu) = -4 \sum_i (ad_\mu X_i)^t ad_\mu X_i + 2 \sum_i ad_\mu X_i (ad_\mu X_i)^t$$

where  $\{X_i\}$  is any orthonormal basis of  $\mathfrak{N}$ . Or equivalently, for  $X, Y \in \mathfrak{N}$ ,

$$(3) \quad \begin{aligned} \langle m(\mu)X, Y \rangle &= -4 \sum_{ij} \langle \mu(X, X_i), X_j \rangle \langle \mu(Y, X_i), X_j \rangle \\ &\quad + 2 \sum_{ij} \langle \mu(X_i, X_j), X \rangle \langle \mu(X_i, X_j), Y \rangle \end{aligned}$$

**Theorem 4.1** (Lauret). *The Lie group  $N_\mu$  (with left-invariant metric  $\langle \cdot, \cdot \rangle$ ) is a left-invariant Ricci soliton if and only if  $[\mu]$  is a critical point of  $\|m\|^2 : \mathbb{P}\mathcal{V} \rightarrow \mathbb{R}$ ; here  $[\mu]$  denotes the class of  $\mu$  in  $\mathbb{P}\mathcal{V}$ . Equivalently,  $N_\mu$  is an Einstein nilradical if and only if the orbit  $GL(n, \mathbb{R}) \cdot \mu$  is distinguished.*

Given  $\mu \in \mathcal{V}$ , we are interested in the problem of finding  $g \in GL(n, \mathbb{R})$  such that  $N_{g \cdot \mu}$  is a nilsoliton. The following theorems say that  $g$  can be chosen from a subgroup of  $GL(n, \mathbb{R})$  that reflects natural symmetries in the metric algebra  $\mathfrak{N}_\mu$ . In this way, the following are considered specialized change of basis theorems.

**Theorem 4.2.** *Let  $\{N, \langle \cdot, \cdot \rangle\}$  be a nilpotent Lie group with left-invariant metric  $\langle \cdot, \cdot \rangle$ . Denote by  $\{\mathfrak{N}, \langle \cdot, \cdot \rangle\}$  the Lie algebra of  $N$  with inner product  $\langle \cdot, \cdot \rangle$  corresponding to the left-invariant metric on  $N$ .*

*Suppose  $N$  admits a left-invariant Ricci soliton metric; that is, there exists  $g \in GL(\mathfrak{N})$  such that  $g^*\langle \cdot, \cdot \rangle = \langle g^{-1} \cdot, g^{-1} \cdot \rangle$  is a nilsoliton. Moreover, suppose that  $\{\mathfrak{N}, \langle \cdot, \cdot \rangle\}$  admits a symmetric derivation  $D \in Der(\mathfrak{N})$  ( $D$  is symmetric with respect to  $\langle \cdot, \cdot \rangle$ ).*

*Let  $\mathfrak{N} = \bigoplus \mathfrak{N}_\lambda$  denote the eigenspace decomposition of  $D$ . Then the element  $g \in GL(\mathfrak{N})$  such that  $g^*\langle \cdot, \cdot \rangle$  is nilsoliton can actually be chosen from the subgroup  $GL(\mathfrak{N}_{\lambda_1}) \times \cdots \times GL(\mathfrak{N}_{\lambda_k})$ .*

**Remark.** This theorem was known in the special case that the symmetric derivation  $D$  is the unique derivation such that the rank 1 extension  $\mathfrak{s} = \mathbb{R} \langle D \rangle \oplus \mathfrak{N}$  admits a left-invariant Einstein metric [6, Prop. 6.8]. In fact, there it is shown that a slightly smaller group can be used. However, our theorem is very useful in practice when it is not known which symmetric derivation should be used to uniquely extend to an Einstein solvmanifold.

As a practical application, we have the following.

**Corollary 4.3.** *Let  $\{N, \langle \cdot, \cdot \rangle\}$  be a two-step nilpotent Lie group with Lie algebra  $\mathfrak{N}$ . Denote the commutator of  $\mathfrak{N}$  by  $\mathfrak{Z} = [\mathfrak{N}, \mathfrak{N}]$ . Then  $N$  is an Einstein nilradical if and only if there exist  $g \in GL(\mathfrak{Z}^\perp) \times GL(\mathfrak{Z})$  such that  $\{N, g^*\langle \cdot, \cdot \rangle\}$  is a nilsoliton; here  $\mathfrak{Z}^\perp \subset \mathfrak{N}$  is taken relative to  $\langle \cdot, \cdot \rangle$ .*

*Proof of Corollary.* This corollary follows immediately from the theorem as every two-step nilpotent Lie algebra admits a symmetric derivation defined by  $Id$  on  $\mathfrak{Z}^\perp$  and  $2Id$  on  $\mathfrak{Z}$ .  $\square$

**Remark.** In [4] and [9], 2-step nilpotent Lie groups, and their left-invariant metrics, are studied using a particular representation of  $SL_q\mathbb{R} \times SL_p\mathbb{R}$  on  $\mathfrak{so}(q)^p$ . Adapting the techniques of [13] to this setting, one is able to characterize the 2-step nilpotent Lie groups with left-invariant metrics which are Ricci solitons as distinguished points of the action of  $SL_q\mathbb{R} \times SL_p\mathbb{R}$  on  $\mathfrak{so}(q)^p$ . The above corollary demonstrates that the ad hoc methods for characterizing nilsolitons on 2-step nilpotent Lie groups can be replaced by a more general procedure which applies to all nilpotent Lie groups.

*Proof of Theorem 4.2.* Let  $N = N_\mu$  for some  $\mu \in \mathcal{V}$ . We will apply Corollary 3.4 for the particular representation at hand.

By hypothesis, our nilpotent Lie algebra is an Einstein nilradical and so Theorem 4.1 implies that the  $G = GL(n, \mathbb{R})$  orbit is distinguished. Consider the subspace  $W = \{\lambda \in V \mid D\lambda(X, Y) = \lambda(DX, Y) + \lambda(X, DY) \text{ for } X, Y \in \mathfrak{N}\}$ . This is a vector subspace which contains  $\mu$  as  $D \in \text{Der}(\mathfrak{N}_\mu)$ .

**Lemma.** Let  $\alpha, \beta$  be eigenvalues  $D$  with corresponding eigenspaces  $V_\alpha, V_\beta$ , then for  $\lambda \in W$ ,  $\lambda(V_\alpha, V_\beta) \subset V_{\alpha+\beta}$ , the eigenspace corresponding to  $\alpha + \beta$ .

The proof is immediate.

Define  $H$  to be the group  $GL(\mathfrak{N}_{\lambda_1}) \times \dots \times GL(V_{\lambda_k})$ . First observe that  $H$  is closed under the metric adjoint with respect to  $\langle \cdot, \cdot \rangle$ , hence  $H$  is reductive. Also,  $W$  is  $H$ -stable as the elements of  $H$  are precisely the elements of  $G$  which commute with  $D$ .

For  $\lambda \in W$ , we will show  $m(\lambda) \in \mathfrak{h} = LH$  by means of Equation (3). Let  $X_\alpha$  (resp.  $X_\beta$ )  $\in \mathfrak{N}$  be in the  $\lambda_\alpha$  (resp.  $\lambda_\beta$ ) eigenspace of  $D$ ,  $\lambda_\alpha \neq \lambda_\beta$ . Choose an orthonormal basis  $\{X_i\}$  consisting of eigenvectors of  $D$ , and apply Equation (3). This gives

$$\begin{aligned} \langle \langle m(\lambda)X_\alpha, X_\beta \rangle \rangle &= -4 \sum_{ij} \langle \lambda(X_\alpha, X_i), X_j \rangle \langle \lambda(X_\beta, X_i), X_j \rangle \\ &\quad + 2 \sum_{ij} \langle \lambda(X_i, X_j), X_\alpha \rangle \langle \lambda(X_i, X_j), X_\beta \rangle \\ &= 0 \end{aligned}$$

To see that this is zero, we observe that each summand is zero. In the first summation, we have

$$\langle \mu(X_\alpha, X_i), X_j \rangle \langle \mu(X_\beta, X_i), X_j \rangle = 0$$

since either  $\lambda_\alpha + \lambda_i \neq \lambda_j$  or  $\lambda_\beta + \lambda_i \neq \lambda_j$  as  $\lambda_\alpha \neq \lambda_\beta$ ; here we are applying the lemma above. Similarly, all the terms in the second summation are zero. Thus  $m(\lambda) \in \mathfrak{h}$  for  $\lambda \in W$ .

Applying Corollary 3.4 together with Theorem 4.1 completes the proof.  $\square$

**Theorem 4.4.** Consider  $\{N, \langle \cdot, \cdot \rangle\}$  and a collection of symmetric derivations  $\{D_\alpha\}$  of  $\mathfrak{N}$  (relative to  $\langle \cdot, \cdot \rangle$ ). Denote by  $H^\alpha$  the group which preserves the eigenspace decomposition of  $D_\alpha$ , as in Theorem 4.2. Define  $H = \cap H^\alpha$ . The

group  $H$  is a reductive algebraic group and  $N$  admits a nilsoliton metric if and only if  $\{N, h^*(\cdot, \cdot)\}$  is such a metric for some  $h \in H$ .

**Remark.** It is not always true that the intersection of reductive groups is reductive, although such intersections are ‘generically’ reductive. See, e.g., [8] for an explicit example of this nongeneric phenomenon.

*Proof.* It was shown in the proof of the previous theorem that each  $H^\alpha$  is closed under the metric adjoint. Hence, the same is true for the intersection  $H$ . Moreover,  $H$  is a variety as it is the intersection of varieties. Thus,  $H$  is a reductive algebraic group. Let  $\mathfrak{h}$  be the Lie algebra of  $H$ , then  $\mathfrak{h} = \cap \mathfrak{h}^\alpha$ , where  $\mathfrak{h}^\alpha$  is the Lie algebra of  $H^\alpha$ .

Define a subspace  $W_\alpha = \{\lambda \in W \mid D_\alpha \lambda(X, Y) = \lambda(D_\alpha X, Y) + \lambda(X, D_\alpha Y)\}$  for  $X, Y \in \mathfrak{N}$ . Define  $W = \cap W_\alpha$ , this subspace is  $H$ -stable as each  $W_\alpha$  is  $H^\alpha$ -stable. The proof of the previous theorem shows that for  $\lambda \in W \subset W_\alpha$  we have  $m(\lambda) \in \mathfrak{h}^\alpha$ . Hence for  $\lambda \in W$ ,  $m(\lambda) \in \mathfrak{h}$ .

As before, applying Corollary 3.4 together with Theorem 4.1 completes the proof.  $\square$

**Theorem 4.5.** *Suppose that  $\mathfrak{N}$  is a nilpotent algebra that can be written as a sum of ideals  $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2$ . Then  $N$  is an Einstein nilradical if and only if both  $\mathfrak{N}_1, \mathfrak{N}_2$  are Einstein nilradicals.*

**Remark.** The property of being an Einstein nilradical is a property of the Lie group (or algebra) and does not depend on the choice of inner product on  $\mathfrak{N}$ .

The above result has previously appeared in a preprint of Nikolayevky [21]. In that work the author uses special subgroups and relates Ricci solitons to closed orbits of said subgroups. These subgroups are determined by first identifying a so-called ‘pre-Einstein derivation’. While the pre-Einstein derivation is very interesting in its own right, our work does not need to consider such derivations and associated subgroups, instead we use distinguished orbits giving a new proof of the theorem.

*Proof.* We choose to endow  $\mathfrak{N}$  with an inner product so that  $\mathfrak{N}_1$  and  $\mathfrak{N}_2$  are orthogonal. With this choice of inner product we have  $\mathfrak{N} = \mathfrak{N}_\mu = \mathfrak{N}_{\mu_1} \oplus \mathfrak{N}_{\mu_2}$  where  $\mu = \mu_1 + \mu_2 \in W := (\wedge^2 \mathfrak{N}_1^* \otimes \mathfrak{N}_1) \oplus (\wedge^2 \mathfrak{N}_2^* \otimes \mathfrak{N}_2) \subset \wedge^2 \mathfrak{N}^* \otimes \mathfrak{N}$ . As in the previous theorems, we will apply Corollary 3.4 together with Theorem 4.1.

We will show for  $\lambda \in W$  that  $m(\lambda) \in \mathfrak{g}(\mathfrak{N}_1) \times \mathfrak{g}(\mathfrak{N}_2)$ . We apply Equation (3) by choosing an orthonormal basis which respects  $\mathfrak{N}_1 \oplus \mathfrak{N}_2$ . For  $X \in \mathfrak{N}_1, Y \in \mathfrak{N}_2$  we have

$$\begin{aligned} \langle \langle m(\lambda)X, Y \rangle \rangle &= -4 \sum_{ij} \langle \lambda(X, X_i), X_j \rangle \langle \lambda(Y, X_i), X_j \rangle \\ &\quad + 2 \sum_{ij} \langle \lambda(X_i, X_j), X \rangle \langle \lambda(X_i, X_j), Y \rangle \end{aligned}$$



In the first summand, if  $X_i \in \mathfrak{N}_1$ , then  $\lambda(Y, X_i) = 0$ ; if  $X_i \in \mathfrak{N}_2$ , then  $\lambda(X, X_i) = 0$ . In the second summand, if  $X_i, X_j$  are not in the same subspace  $\mathfrak{N}_1$  or  $\mathfrak{N}_2$ , then  $\lambda(X_i, X_j) = 0$ . Now suppose, without loss of generality, that  $X_i, X_j \in \mathfrak{N}_1$ . Then we have  $\lambda(X_i, X_j) \in \mathfrak{N}_1$  with is orthogonal to  $\mathfrak{N}_2$ , hence  $\langle \lambda(X_i, X_j), Y \rangle = 0$ . In this way, we see that  $\langle \langle m(\lambda)X, Y \rangle \rangle = 0$  for  $X \in \mathfrak{N}_1$ ,  $Y \in \mathfrak{N}_2$ . Thus  $m(\lambda) \in \mathfrak{gl}(\mathfrak{N}_1) \times \mathfrak{gl}(\mathfrak{N}_2)$  and the proof is complete.  $\square$

**Proposition 4.6.** *Consider  $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2$ . Let  $\mathcal{V}_i$  denote the variety of Lie brackets on  $\mathfrak{N}_i$  (see the remarks at the beginning of this section for more information on the set of Lie brackets). For  $\mu_i \in \mathcal{V}_i$ , consider the set of Lie brackets on  $\mathfrak{N}$  that can be written in the form  $\mu = \mu_1 \oplus \mu_2$ . The moduli of such is precisely the moduli of  $GL(\mathfrak{N}_1) \times GL(\mathfrak{N}_2)$  orbits in  $\mathcal{V}_1 \times \mathcal{V}_2 \subset \mathcal{V}$ .*

*Proof.* This follows immediately from the proof of the previous theorem and Theorem 3.1, see also the remarks following Theorem 3.1.  $\square$

**Remark.** In this way we can construct moduli (i.e., nonisomorphic continuous families) of Einstein and non-Einstein nilradicals. For example, pick one non-Einstein nilradical  $\mathfrak{N}_1$  (which are known to exist, see [17] or [10]) and add on the vector space  $\mathfrak{N}_2$ . Letting the bracket vary on  $\mathfrak{N}_2$  will produce moduli of algebras  $\mathfrak{N}_1 \oplus \mathfrak{N}_2$  which cannot be Einstein nilradical by the previous proposition. However, the moduli of non-Einstein nilradicals constructed in this way is somewhat trivial. In [10], there are given constructions of non-Einstein nilradicals which do not arise as direct sums, that is, the examples given are indecomposable (cp. Corollary 4.3).

**Theorem 4.7.** *There exist continuous families of nonisomorphic, indecomposable two-step nilpotent Lie groups which do not admit Ricci soliton metrics. The size of these families may be made arbitrarily large; more precisely, these families may be chosen so that the dimension of said families grows linearly in the dimension of the underlying Lie group.*

These are not the first examples of moduli of non-Einstein nilradicals. In [31], C. Will produces a curve of (pairwise) nonisomorphic, non-Einstein nilradicals. However, this is the first general construction for building many examples of non-Einstein nilradicals. To our knowledge, Will’s examples are the only other known examples of nondiscrete families of this phenomenon.

In the study of two-step nilpotent Lie algebras there are different representations that are very useful for obtaining concrete results. We will not go into the details of setting these up and refer the reader to [10] for the applications of this work in the two-step case. The work of that paper, loc. cit., allows one to concretely construct moduli of Einstein and non-Einstein nilradicals in the set of “nongeneric” two-step nilalgebras.

## 5. NILPOTENT ORBITS IN THE ADJOINT REPRESENTATION

In this section we present the case of the adjoint representation. We focus our attention on the null-cone of this representation. Recall that the null-cone

of a representation  $V$  consists of points  $v \in V$  such that  $0 \in \overline{G \cdot v}$ . When  $V = \mathfrak{g}$  is the adjoint representation,  $X \in \mathfrak{g}$  is in the null-cone if and only if  $X$  is nilpotent, see, e.g., Remark 10.2 of [1].

It is well-known that there exist only finitely many orbits in the null-cone of an adjoint representation of a reductive group. We reprove this result by applying Theorem 3.1. Moreover, we show that each of these orbits is distinguished (in the sense of Definition 2.6).

The known general proofs of the finiteness result are essentially the same as ours, without the language of the moment map. The existing proofs use Lemma 3.7 and an additional theorem on the reducibility of representations of reductive groups; these ideas originally go back to Richardson [24].

**Proposition 5.1.** *Let  $G = SL_n(\mathbb{C})$ . The null-cone of the adjoint action of  $G$  on  $\mathfrak{g}$  has finitely many orbits.*

These orbits are in one-to-one correspondence with the partitions of  $n$ ; they correspond to the different Jordan normal forms with  $0$ 's along the diagonal (the nilpotent endomorphisms). See [11, Sec. 1.1] for more details.

**Theorem 5.2** (Dynkin-Kostant). *Let  $G$  be a (real or complex algebraic) linear reductive group. There exist only finitely many orbits in the null-cone of the adjoint representation.*

We prove this theorem in the complex setting and apply Theorem 3.11 to immediately obtain the theorem for real groups once known for complex groups (cp. the end of Section 3).

*Proof.* It suffices to consider connected reductive groups  $G$ . As  $G$  is a linear group,  $G \subset SL_n(\mathbb{C})$  for some  $n \in \mathbb{N}$ . Let  $\theta$  be a  $G$ -stable Cartan involution of  $SL_n(\mathbb{C})$ , such exists by Proposition 2.2. Then  $\mathfrak{g} \subset \mathfrak{sl}_n(\mathbb{C})$  and we will endow  $\mathfrak{sl}_n(\mathbb{C})$  with the inner product from Example 2.3. That is,  $\langle \cdot, \cdot \rangle = -B(\cdot, \theta(\cdot))$  where  $B$  is the Killing form of  $\mathfrak{sl}_n(\mathbb{C})$ . This inner product is  $SU(n)$ ,  $K$ , and  $i$ -invariant; and so, this inner product satisfies the requirements of Section 2. We endow  $\mathfrak{sl}_n(\mathbb{C}) = Lie\ SL_n(\mathbb{C})$  and  $\mathfrak{g} = Lie\ G$  with the same inner product inner product  $\langle\langle \cdot, \cdot \rangle\rangle = \langle \cdot, \cdot \rangle$ .

From this one readily computes

$$(4) \quad m(X) = -[X, \theta(X)]$$

for  $X \in \mathfrak{sl}_n(\mathbb{C})$ . Observe that the  $SL_n(\mathbb{C})$  orbits are  $G$ -detected along  $\mathfrak{g}$  (see Definition 2.9); that is, the moment map takes values in  $\mathfrak{g}$  when evaluated along  $\mathfrak{g}$ .

We also observe that the  $G$ -null-cone in  $\mathfrak{g}$  is contained in the  $SL_n(\mathbb{C})$ -null-cone in  $\mathfrak{sl}_n(\mathbb{C})$ . Applying Theorem 3.1 we see that each  $SL_n(\mathbb{C})$  orbit through the  $G$ -null-cone consists of finitely many  $G$ -orbits. However, there are only finitely many such  $SL_n(\mathbb{C})$ -orbits (see Proposition 5.1). Hence, there are only finitely many  $G$ -orbits in the  $G$ -null-cone.  $\square$

**Theorem 5.3.** *Let  $G$  be a semisimple group acting on  $\mathfrak{g}$  by the adjoint representation. Every orbit in the null-cone of  $\mathfrak{g}$  is distinguished (in the sense of Definition 2.6).*

**Remark.** This result has previously appeared in the literature, see Lemma 2.11 of [29]. The proof presented in that work uses the work of [26]. Our proof relies on constructing such critical points for the case of  $\mathfrak{sl}_n(\mathbb{C})$  and then applying Corollary 3.4.

**Proposition 5.4.** *Every nilpotent orbit in the adjoint representation of  $SL_n$  is distinguished.*

*Proof.* We first prove that the principal orbit in  $\mathfrak{sl}_n$  is distinguished. Consider the element

$$J_n = \begin{bmatrix} 0 & \lambda_1 & & & \\ & 0 & \lambda_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & \lambda_{n-1} \\ & & & & 0 \end{bmatrix}$$

We point out that this element is in the same  $SL_n$ -orbit as the standard nilpotent Jordan block as long as the  $\lambda_i$  are nonzero.

By Equation (4),  $m(X) = -[X, -X^*] = XX^* - X^*X$  and

$$m(X) \cdot X = [m(X), X] = m(X)X - Xm(X) = 2XX^*X - X^*X^2 - X^2X^*$$

We claim that  $\lambda_i = \sqrt{\frac{i}{2}(n-i)}$  satisfies the condition  $m(J_n) \cdot J_n = J_n$  (cp. Definition 2.6 and Lemma thereafter with  $c = 1$ ) and hence  $SL_n \cdot J_n$  will be distinguished. This is easy to verify by direct computation.

To see that every nilpotent orbit contains a distinguished point, observe that every nilpotent endomorphism can be conjugated to be of the form

$$X = \begin{bmatrix} J_{k_1} & & & \\ & J_{k_2} & & \\ & & \ddots & \\ & & & J_{k_j} \end{bmatrix}$$

where  $\sum k_i = n$ . This is the Jordan form where the super diagonal has nonzero entries which are rescaled. The matrix  $m(X)$  will have the same block decomposition and hence

$$\begin{aligned} m(X) \cdot X &= \begin{bmatrix} m(J_{k_1}) \cdot J_{k_1} & & & & \\ & m(J_{k_2}) \cdot J_{k_2} & & & \\ & & \ddots & & \\ & & & m(J_{k_j}) \cdot J_{k_j} & \\ & & & & \ddots \end{bmatrix} \\ &= \begin{bmatrix} J_{k_1} & & & & \\ & J_{k_2} & & & \\ & & \ddots & & \\ & & & J_{k_j} & \\ & & & & \ddots \end{bmatrix} = X \end{aligned}$$

□

We continue now with the proof of the theorem for all semisimple groups. We use the following observation without proof.

**Lemma 5.5.** *Let  $X \in \mathfrak{g}$  be in the null-cone. Then  $cX \in G \cdot X$  for all  $c \in \mathbb{K}$ , where  $\mathbb{K} = \mathbb{R}$ , respectively  $\mathbb{C}$ , if  $\mathfrak{g}$  is a real, respectively complex, Lie algebra.*

First, we reduce to the case that  $G$  is simple. Suppose that the result is known for simple groups and decompose  $\mathfrak{g} = \mathfrak{g}_1 \times \dots \times \mathfrak{g}_k$  as a product of simple factors. Let  $X$  be a nilpotent element of  $\mathfrak{g}$ , then  $X = \sum X_i$  where  $X_i \in \mathfrak{g}_i$  are all nilpotent elements. Observe that  $m(X) = \sum m(X_i)$ . Using the lemma above and the hypothesis, we may assume that each  $X_i$  satisfies  $m(X_i) \cdot X_i = X_i$ . But now  $m(X) \cdot X = \sum m(X_i) \cdot X_i = \sum X_i = X$ .

All that remains to be shown is that the theorem is true in the case  $G$  is simple. Firstly, we embed  $G$  into  $SL_n$  so that  $G$  is selfadjoint, as in the beginning of the section. As  $G$  is simple, any  $K$ -invariant inner product on  $\mathfrak{p}$  is unique up to scaling. This is due to the fact that  $K$  acts irreducibly on  $\mathfrak{p}$ . Thus, given any inner product on  $\mathfrak{g}$ , as in Section 2, our moment map is uniquely defined up to rescaling. As rescaling the moment map does not affect the property of a point being distinguished, we may assume that our inner product on  $\mathfrak{g}$  is the restriction of the inner product on  $\mathfrak{sl}_n$ .

Recall that the  $SL_n$ -orbits are  $G$ -detected along  $\mathfrak{g}$ . As every nilpotent element of  $\mathfrak{g}$  is a nilpotent element of  $\mathfrak{sl}_n$  and every nilpotent orbit of  $SL_n$  is  $SL_n$ -distinguished, applying Corollary 3.4 we see that every nilpotent orbit of  $G$  is  $G$ -distinguished.

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