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Bianchi-convexity and applications to Ricci flow

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Abstract

Convexity and weaker forms of convexity play a crucial role in many areas of mathematics. In this thesis, we introduce and investigate the new notion of *Bianchi-convexity*, a generalization of convexity inspired by the second Bianchi identity of Riemannian curvature tensors, and give some applications to the Ricci flow: In the setting of algebraic curvature tensors, we generalize Hamilton's maximum principle for Bianchi-convex sets. Using this, in dimension three, we derive a family of non-convex Bianchi-convex sets which are preserved by the Ricci flow. Moreover, we prove rigidity results for compact Ricci solitons respectively complete shrinking gradient Ricci solitons involving the concept of Bianchi-convex functions. As a consequence, we obtain explicit curvature conditions such that complete shrinking gradient Ricci solitons (and as a special case complete Einstein manifolds) satisfying these are locally symmetric. This yields a further step into the direction of a complete classification of shrinking Ricci solitons.

Für Manzi

Contents

Introduction

The notion of convexity plays an important role in many areas of mathematics. It has applications, for instance, in convex minimization, a subfield of optimization theory, where the problem of minimizing convex functions over convex sets is investigated. Since minima of strictly convex functions on vector spaces are unique, these are of particular significance in this area. Concerning convexity of sets, the Brunn-Minkowski inequality gives a connection between the Lebesgue-measure of a certain Minkowski-sum of two convex bodies and their Lebesgue-measure. A direct consequence is the famous isoperimetric inequality, which in three-dimensional space implies that under all bodies with the same surface area, the ball (a convex set) has the largest volume. An analogous statement holds true in arbitrary dimensions. An elementary inequality for convex functions is Jensen's inequality which is the basis of many important results in probability theory, measure theory and analysis. In the study of non-positively curved spaces, convexity is an important tool, since subsets of CAT(0)-spaces are convex if and only if the distance functions from these sets are convex. In Riemannian geometry, the existence of a non-constant convex function f on a complete Riemannian manifold M (i.e. a function that is convex along all geodesics) has strong topological and geometrical consequences. In the case that f is additionally continuous, an obvious implication is that the manifold is non-compact. Moreover, in [Yau74], Yau showed that in this case its volume is infinite. In dimension two, Greene and Shiohama proved that if f is locally non-constant, then M is either diffeomorphic to the plane, the cylinder or the open Möbius strip [GS81b]. Furthermore, in [GS81a], they proved results about the differentiable structure of M depending on whether the set of minimum points of f is empty or not.
In many situations where convexity is used, only a weaker form of convexity is needed. An example for this is the concept of quasiconvex functions (i.e. functions defined on a convex subset of a vector space, the sublevel sets of which are all convex), relevant in quasiconvex programming, a subfield of optimization theory, as well as in game theory, in particular for applications of Sion's minimax theorem. In the study of majorization, one considers Schur-convex functions, i.e. order-preserving functions in the sense that if one argument is majorized by another, then the corresponding values under this functions are ordered in the same way [PPT92, Definition 12.23]. For functions on the space of matrices, the notion of rank-one convexity occurs in the analysis of partial differential equations, calculus of variations and elasticity theory. In the case of twice continuous differentiability, this notion is equivalent to the Legendre-Hadamard condition, which means that in each point the Hessian of the function is positive semidefinite on the space of matrices of rank one.

In the analysis of the Ricci flow, where Riemannian metrics on a manifold evolve according to

$$\frac{\partial}{\partial t} g_t = -2\mathrm{ric}_{g_t}$$

(here ric_{g_t} denotes the Ricci curvature of the metric g_t), convexity is of particular interest via Hamilton's maximum principle. It states that an $O(n)$-invariant, closed and convex subset of the space of algebraic curvature tensors $\mathcal{A}_n := S^2_B(\Lambda^2(\mathbb{R}^n)^*)$, which is invariant under the ordinary differential equation

$$R'(t) = R(t)^2 + R(t)^{\#}, \tag{1}$$

is already invariant under the Ricci flow, i.e. for all n-dimensional compact manifolds M and solutions g_t, $t \in [0, T)$, to the Ricci flow on M with g_0 satisfying Ω, we have that g_t satisfies Ω for all $t \in [0, T)$. Here, a Riemannian metric g satisfies Ω, if the Riemannian curvature operator Rm_g (see Section 1.3) is contained in $\Omega^g \subseteq S^2_B(\Lambda^2 T^*M)$ (i.e. Ω transfered to the fibres via g-isometries) at all points in M, and the map $\#$ will be defined in Definition 1.1.17. Moreover, a set being invariant under a differential equation means that solutions of this differential equation which start in the set stay in it for all times. Following the above ideas, it turns out that this theorem can be generalized by weakening the notion of convexity to what we call *Bianchi-convexity*. We define a closed subset $\Omega \subseteq \mathcal{A}_n$ with smooth boundary to be Bianchi-convex, if for all $R \in \partial\Omega$ and tuples $(T_1, \dots, T_n) \in (T_R\partial\Omega)^n$ which satisfy a certain second Bianchi identity, we have that

$$\sum_{i=1}^{n} \mathbf{II}^{\partial\Omega}_R(T_i, T_i) \leq 0,$$

where $\mathbf{II}^{\partial\Omega}_R$ denotes the second fundamental form of $\partial\Omega$ in R. In the general case that the boundary of Ω is not smooth, we give a definition involving supporting submanifolds. One goal of this thesis is to investigate this new notion, including examples of non-convex Bianchi-convex sets which show that Bianchi-convexity is a real generalization of convexity, and to give a proof of the following maximum principle.

Theorem A. *Let $\Omega \subseteq \mathcal{A}_n$ be $O(n)$-invariant, closed, Bianchi-convex and uniformly transversally star-shaped with respect to λI for some $\lambda \in \mathbb{R}$ (see Definition 4.3.1). If Ω is invariant under the ordinary differential equation* (1)*, then Ω is invariant under the Ricci flow.*

Here, I denotes the identity in $\mathcal{A}_n$. In dimension three, using the maximum principle, Theorem A, we derive a family of non-convex Bianchi-convex sets which are invariant under the Ricci flow.

Proposition. *For $a \in (\frac{1}{3}, \frac{2}{5})$ and $c > 0$, the set*

$$\left\{ R \in \mathcal{A}_3 \;\middle|\; \|R\|^2 - a\,\mathrm{scal}(R)^2 \leq c \quad \textit{and} \quad \mathrm{scal}(R) \geq b_{a,c} \right\}$$

is invariant under (1)*, thus invariant under the Ricci flow. Here,*

$$b_{a,c} := \sqrt{\frac{3c}{3a-1}}\,\sinh\!\left(\frac{3}{2}\right) > 0.$$

The second part of this thesis treats Bianchi-convex functions, i.e. smooth functions $F : U \to \mathbb{R}$, where $U \subseteq \mathcal{A}_n$ is open, such that for all $R \in U$ and tuples $(T_1, \dots, T_n) \in (T_R U)^n$ that satisfy the afore-mentioned second Bianchi identity, we have that

$$\sum_{i=1}^{n} \mathrm{Hess}_R F(T_i, T_i) \geq 0.$$

If the inequality above is strict unless $T_i = 0$ for each $i = 1, \dots, n$, then F is called strictly Bianchi-convex. In dimension $n \geq 3$, a non-constant smooth function F on an open cone $\Omega \subseteq \mathcal{A}_n \setminus \{0\}$, the sublevel sets of which are strictly convex cones, can never be convex. However, we will see that up to an appropriate reparametrization and restriction such a function is Bianchi-convex:

Theorem B. *For each open cone U with $\overline{U} \subset \Omega \cap \mathcal{B}_n$ and such that $\operatorname{Hess}_R F|_{R^\perp}$ is positive definite for all $R \in \overline{U}$ with $dF_R = 0$, there exists a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi' > 0$ such that $\varphi \circ F$ restricted to U is strictly Bianchi-convex.*

Here, the closure of U is taken in $\mathcal{A}_n \setminus \{0\}$ and we define the cone

$$\mathcal{B}_n := \left\{ R \in \mathcal{A}_n \mid R|_{\Lambda^2(v^\perp)} \not\equiv 0 \text{ for all } v \in \mathbb{R}^n \setminus \{0\} \right\}.$$

In the study of singularity formation of the Ricci flow, Ricci solitons are of special significance. These are tuples (M, g, X, λ) consisting of a Riemannian manifold (M, g), a smooth vector field X and a real number λ which satisfy

$$\operatorname{ric}_g + \frac{1}{2}\mathcal{L}_X g = \lambda g,$$

where $\mathcal{L}$ denotes the Lie derivative. In particular, shrinking gradient Ricci solitons (i.e. Ricci solitons with $\lambda > 0$ and $X = \operatorname{grad}_g f$ for some smooth function $f : M \to \mathbb{R}$, denoted by (M, g, f, λ)) arise as singularity models for so-called Type I Ricci flows. However, although several partial results exist, a classification of shrinking gradient Ricci solitons is not yet understood and is an active area of research.
Trivial examples for shrinking gradient Ricci solitons are Einstein manifolds E with positive Einstein constant λ as well as $(E \times \mathbb{R}^k, g_E + dx^2, f, \lambda)$, where $k > 0$ and the potential function is given by $f(e, x) := \frac{\lambda}{2}\|x\|^2$ for $e \in E$ and $x \in \mathbb{R}^k$. It is known that all two-dimensional complete shrinking gradient Ricci solitons are either S^2, $\mathbb{R}P^2$ or $\mathbb{R}^2$ ([Ham88], [BM15, Corollary 1]) and in dimension three they are finite quotients of S^3, $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$ [CCZ08, Proposition 4.7]. In higher dimensions, the classification is more difficult. There exist non-trivial shrinking gradient Ricci solitons (see [Koi90], [Cao96], [WZ04], [FIK03]). However, all known examples so far are Kähler. Yet there are many rigidity results for complete shrinking gradient Ricci solitons satisfying some curvature condition such as bounded non-negative curvature operator (in the four-dimensional non-compact case [Nab07]), non-negative sectional curvature and scalar curvature bounded from above by 2λ [PW09, Theorem 1.4], vanishing Weyl tensor [Zha09a, Theorem 1.2], harmonic Weyl tensor ([FLGR11], [MS13]) and vanishing Bach tensor [CC13] to name just a few.
The notion of Bianchi-convex functions introduced above yields a further step into the direction of a complete classification of shrinking gradient Ricci solitons, namely the following rigidity result.

Theorem C. *Let $\Omega \subseteq \mathcal{A}_n \setminus \{0\}$ be an open and $O(n)$-invariant cone and $F : \Omega \to \mathbb{R}$ a scale- and $O(n)$-invariant, smooth, bounded and strictly Bianchi-convex function, the sublevel sets of which are invariant under the ordinary differential equation* (1)*. Then all n-dimensional complete shrinking gradient Ricci solitons (M, g, f, λ) such that g satisfies Ω are locally symmetric.*

By [FLGR11] and [MS13], such a Ricci soliton (M, g, f, λ) is trivial, i.e. either Einstein or a finite quotient of $E \times \mathbb{R}^k$, where $k > 0$, E is an $(n-k)$-dimensional Einstein manifold and $\mathbb{R}^k$ is the Gaussian shrinking soliton.

Keeping the reparametrization theorem, Theorem B, in mind, a first step into the direction of finding functions that satisfy the assumption of Theorem C is to find a one-parameter family of strictly convex cones in $\mathcal{A}_n$ which are invariant under the ordinary differential equation (1). Given the conjecture of Böhm-Wilking that the cones

$$\Omega_a := \left\{ R \in \mathcal{A}_n \;\middle|\; \left(\frac{n-2}{4} + a\right)\|R\|^2 \leq \|\operatorname{ric}(R)\|^2 \;\text{ and }\; \operatorname{scal}(R) > 0 \right\}$$

are invariant under (1) for $n \geq 12$ and $a \in [0, \frac{n}{4}]$, respectively the conjecture of the author that the cones

$$\Theta_a := \left\{ R \in \mathcal{A}_n \;\middle|\; \sphericalangle(R, I) \leq \arctan\left(\sqrt{\frac{1}{a}}\right) \text{ and } \operatorname{scal}(R) > 0 \right\}$$

are invariant under (1) for $n \geq 3$ and $a \geq d_n$, where $d_n := \frac{(n-2)(n+1)}{2}$, one obtains scale- and $O(n)$-invariant, bounded and smooth functions

$$\Omega \to \mathbb{R} : R \mapsto \frac{\|R\|^2}{\|\operatorname{ric}(R)\|^2}$$
$$\Theta \to \mathbb{R} : R \mapsto \frac{\|R_{\operatorname{ric}_0} + R_W\|^2}{\|R_I\|^2},$$

where $\Omega := \cup_{a>0}\Omega_a$, $\Theta := \cup_{a>\frac{2}{n-2}}\Theta_a$. Above, R_W, R_{ric_0} respectively R_I denote the Weyl, traceless Ricci respectively scalar curvature part of the algebraic curvature tensor R and I is the identity in $\mathcal{A}_n$. The sublevel sets of these functions are strictly convex. Moreover, the sublevel sets of the first function are invariant under (1), for the second function this is only true after restricting it to Θ_{d_n}. With the help of the reparametrization theorem, Theorem B, this yields the following two applications of Theorem C.

Theorem D. *Let $n \geq 12$. Then all n-dimensional complete shrinking gradient Ricci solitons (M, g) with g satisfying Ω_a for some $a > \frac{1}{2}$ are locally symmetric.*

Theorem E. *Let $n \geq 3$. Then all n-dimensional complete shrinking gradient Ricci solitons (M, g) with g satisfying Θ_{d_n} are locally symmetric.*

In dimension $n \geq 5$, we show that the Bryant soliton, a complete non-compact steady gradient Ricci soliton, satisfies $\Omega_a \subseteq \mathcal{B}_n$ for some $a > \frac{1}{2}$. Since it is not locally symmetric, this shows that Theorem D (and consequently Theorem C) is sharp in the sense that it is false, if one drops the assumption "shrinking". Furthermore, as a consequence of Theorem D we obtain the following result.

Theorem F. *Let $n \geq 12$. Then all n-dimensional complete Einstein manifolds (M, g) with g satisfying Ω_a for some $a > 0$ are locally symmetric.*

This thesis is organized as follows. After discussing some preliminary notions, in Chapter 2 we introduce curvature conditions, that is $O(n)$-invariant subsets $\Omega \subseteq \mathcal{A}_n$, and the corresponding subsets Ω^g of the bundle of algebraic curvature tensors over a Riemannian manifold (M, g) being parallel with respect to the Levi-Civita connection ∇^g of (M, g). As in our results, curvature conditions often arise as domains of $O(n)$-invariant functions F, we provide formulas for derivatives of the corresponding parallel functions F^g on Ω^g in terms of those of F. Moreover, we give some properties of tangent cones as well as a connection to subsets of a vector space which are invariant under an ordinary differential equation of the form $f'(t) = \Phi(f(t))$. In Chapter 3, we investigate the notion of Bianchi-convex sets and in dimension three derive examples of such sets of algebraic curvature tensors that are not convex but invariant under (1). The fourth chapter is dedicated to maximum principles. We recall the weak and strong maximum principles for functions and Hamilton's maximum principle. Using the Uhlenbeck trick, we give a reformulation of the latter in the special case of algebraic curvature tensors. As one main result we give a proof of Theorem A. Chapter 5 prefaces the second part of this thesis. Here, the notion of Bianchi-convex functions is introduced and the reparametrization theorem (Theorem B) is proved. The last chapter deals with Ricci solitons and aims at giving rigidity results for compact Ricci solitons as well as complete shrinking gradient Ricci solitons (see Theorem 6.1.16 and Theorem C). Finally, we derive two applications of these classification results, Theorem D and Theorem E, based on conjectures of Böhm-Wilking and the author.

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Chapter 1

Preliminaries

This chapter is dedicated to introducing all known objects, spaces, notions and facts that will be important throughout the whole present work. As we will be interested in Riemannian manifolds which satisfy certain curvature conditions, we define the space of algebraic curvature tensors and give some first properties. We recall notions of curvature of a Riemannian manifold, as for example the Riemannian curvature tensor, which pointwise can be identified with an algebraic curvature tensor. Moreover, we consider time-dependent Riemannian metrics on a manifold and introduce a metric connection on the space-time. In the special case that the time-dependent metrics are solutions to the Ricci flow, we formulate the evolution equation of the Riemannian curvature operator along these metrics.

1.1 Skew-symmetric endomorphisms, two-forms and algebraic curvature tensors

In this section, we introduce the space of algebraic curvature tensors, which plays a crucial role when investigating the curvature of a Riemannian manifold. We give some properties of this space and define the Ricci respectively scalar curvature of an algebraic curvature tensor. Moreover, we recall that there is an orthogonal decomposition of this space into irreducible subspaces and derive formulas for the norms of the components of an algebraic curvature tensor with respect to this splitting. Finally, we provide a definition and some characteristics of the map $\#$, which allows us to formulate the right-hand side of certain differential equations arising when working with the Ricci flow, such as the evolution equation of the Riemannian curvature operator.

Throughout this section, let V be an n-dimensional Euclidean vector space.

Definition 1.1.1. We define the spaces

$$\begin{aligned} \mathfrak{so}(V) &:= \{A : V \to V \mid A \text{ is a skew-symmetric endomorphism}\} \\ \text{and} \quad \Lambda^2 V^* &:= \{\omega : V \times V \to \mathbb{R} \mid \omega \text{ is antisymmetric and bilinear}\}. \end{aligned}$$

As a special case, we set $\mathfrak{so}(n) := \mathfrak{so}(\mathbb{R}^n)$.

Remark 1.1.2. Using the scalar product, we will always freely identify $\mathfrak{so}(V)$ and $\Lambda^2 V^*$ via the isomorphism

$$\begin{aligned} \mathfrak{so}(V) &\to \Lambda^2 V^* \\ A &\mapsto \omega_A, \end{aligned} \tag{1.1}$$

where $\omega_A(v,w) := \langle v, A(w)\rangle$ for $v,w \in V$. On $\mathfrak{so}(V)$, we have the scalar product

$$\langle A, B\rangle := \frac{1}{2}\mathrm{tr}(A^tB) = -\frac{1}{2}\mathrm{tr}(AB).$$

Moreover, we choose the scalar product on Λ^2V^* in such a way that the isomorphism (1.1) is an isometry. This means that for each orthonormal basis $(b^1,\dots,b^n)$ of V^* the vectors $b^i\wedge b^j$, $i<j$, form an orthonormal basis of Λ^2V^*.

Definition 1.1.3. By $S^2(V^*)$, we denote the *space of symmetric bilinear forms on* V.

This space can be identified with the *space* $\mathrm{SymEnd}(V)$ *of self-adjoint endomorphisms of* V via

$$\begin{aligned} \mathrm{SymEnd}(V) &\to S^2(V^*)\\ L &\mapsto \check{L}\\ \hat{\beta} &\leftarrow\!\shortmid \beta, \end{aligned}$$

where $\check{L}(v,w) := \langle L(v), w\rangle$ respectively $\hat{\beta}(v) := \sum_{i=1}^n \beta(v,b_i)b_i$ for $v,w\in V$. Here, $(b_1,\dots,b_n)$ is an orthonormal basis of V. Therefore, we will not always distinguish between symmetric bilinear forms and self-adjoint endomorphisms.

Remark 1.1.4. In the case that $V = \mathfrak{so}(n)$, the identification of $S^2(V^*)$ with $\mathrm{SymEnd}(V)$ leads to a one-to-one correspondence between self-maps $\hat{\Phi}$ on SymEnd(V) satisfying

$$\hat{\Phi}(c(Q)\circ\hat{R}\circ c(Q^t)) = c(Q)\circ\hat{\Phi}(\hat{R})\circ c(Q^t)$$

and self-maps Φ on $S^2(V^*)$ satisfying

$$\Phi(\rho(Q)R) = \rho(Q)\Phi(R)$$

for all $Q\in O(n)$ and $R\in S^2(V^*)$, via $\widehat{\Phi(R)} = \hat{\Phi}(\hat{R})$. Here,

$$c(Q) : \mathfrak{so}(n)\to\mathfrak{so}(n) : A\mapsto QAQ^t$$

denotes the conjugation with Q and $\rho : O(n)\to \mathrm{End}(S^2(V^*))$ the representation of $O(n)$ on $S^2(V^*)$ given by

$$(\rho(Q)R)(A,B) := R(Q^tAQ, Q^tBQ) = R(c(Q^t)A, c(Q^t)B),$$

where $Q\in O(n)$, $R\in S^2(V^*)$ and $A,B\in\mathfrak{so}(n)$.
In the case that $V=\Lambda^2\mathbb{R}^n$, the same is true after replacing $c(Q)$ by the linear map Λ^2Q given by

$$\Lambda^2Q : \Lambda^2\mathbb{R}^n\to\Lambda^2\mathbb{R}^n : v\wedge w\mapsto Qv\wedge Qw$$

for $Q\in O(n)$, and adjusting ρ accordingly.

Definition 1.1.5. The *space of algebraic curvature tensors* $S^2_B(\Lambda^2V^*)$ associated to V is the space of symmetric bilinear forms R on Λ^2V which satisfy the first Bianchi identity, that is

$$R(x\wedge y, z\wedge w) + R(y\wedge z, x\wedge w) + R(z\wedge x, y\wedge w) = 0$$

for all $x,y,z,w\in V$. In the case that $V=\mathbb{R}^n$, we will denote it by $\mathcal{A}_n$.

This space is of special interest when investigating Riemannian manifolds, since the Riemannian curvature tensor can be considered as an algebraic curvature tensor at each point (see Section 1.3).

Remark 1.1.6. From [Str88, Theorem 2.1], it is known that

$$\dim(\mathcal{A}_n) = \frac{n^2(n^2-1)}{12} =: N(n) = N.$$

Therefore, from now on n will always be at least 2.

Remark 1.1.7. The scalar product on $\Lambda^2 V^*$ induces a scalar product on $S^2(\Lambda^2 V^*)$, and hence on $S^2_B(\Lambda^2 V^*)$, given by the formular

$$\langle R, S\rangle = \sum_{\substack{1\le i<j\le n,\\ 1\le k<l\le n}} R(b_i\wedge b_j, b_k\wedge b_l)S(b_i\wedge b_j, b_k\wedge b_l)$$

for $R, S \in S^2(\Lambda^2 V^*)$, where $(b_1,\dots,b_n)$ is an orthonormal basis of V. We notice that $\langle R, S\rangle = \mathrm{tr}(R\circ S)$, if R and S are considered as self-adjoint endomorphisms on $\Lambda^2 V$.

Remark 1.1.8. Let $\mathcal{A}(V)$ be the space of all $(0,4)$-tensors R on V (i.e. of all multilinear maps $R : V\times V\times V\times V \to \mathbb{R}$) which have the following symmetries:

1. $R(x,y,z,w) = -R(y,x,z,w) = -R(x,y,w,z)$,
2. $R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0$

for all $x,y,z,w\in V$. Then the map $\Phi : \mathcal{A}(V) \to S^2_B(\Lambda^2 V^*)$ given by

$$\Phi(R)(x\wedge y, z\wedge w) := 2R(x,y,z,w)$$

for $R\in\mathcal{A}(V)$ is an isometry with respect to the usual tensor norm on $\mathcal{A}(V)$ and the norm on $S^2_B(\Lambda^2 V^*)$ introduced in Remark 1.1.7. Namely, for $R\in\mathcal{A}(V)$ we have that

$$\begin{aligned}\|R\|^2 &= \sum_{i,j,k,l} R(b_i,b_j,b_k,b_l)^2 = \frac{1}{4}\sum_{i,j,k,l}\Phi(R)(b_i\wedge b_j, b_k\wedge b_l)^2\\ &= \sum_{i<j,k<l}\Phi(R)(b_i\wedge b_j, b_k\wedge b_l)^2 = \|\Phi(R)\|^2,\end{aligned}$$

where $(b_1,\dots,b_n)$ is an orthonormal basis of V. Hence, we can and will freely identify the spaces $\mathcal{A}(V)$ and $S^2_B(\Lambda^2 V^*)$.

Lemma 1.1.9. *If V is of dimension $n\le 3$, we have that $S^2_B(\Lambda^2 V^*) = S^2(\Lambda^2 V^*)$.*

Proof. We prove the statement for the case that $n = 3$. The cases $n \le 2$ are trivial. Let $R\in S^2(\Lambda^2 V^*)$ and (b_1,b_2,b_3) be an orthonormal basis of V. In order to show that R satisfies the first Bianchi identity, it suffices to show that

$$R(b_i\wedge b_j, b_k\wedge b_l) + R(b_j\wedge b_k, b_i\wedge b_l) + R(b_k\wedge b_i, b_j\wedge b_l) = 0$$

for all $i,j,k,l\in\{1,2,3\}$. Since the set $\{b_i,b_j,b_k,b_l\}$ consists of at most three elements, at least two are the same. For example, in the case that $i=j$, we compute that

$$\begin{aligned}&R(b_i\wedge b_j, b_k\wedge b_l) + R(b_j\wedge b_k, b_i\wedge b_l) + R(b_k\wedge b_i, b_j\wedge b_l)\\ &= R(\underbrace{b_i\wedge b_i}_{=0}, b_k\wedge b_l) + R(b_i\wedge b_k, b_i\wedge b_l) + R(\underbrace{b_k\wedge b_i}_{=-b_i\wedge b_k}, b_i\wedge b_l) = 0.\end{aligned}$$

The other cases are similar. ☐

Definition 1.1.10. Corresponding to an algebraic curvature tensor $R \in S^2_B(\Lambda^2 V^*)$, we define the *Ricci tensors* $\mathrm{ric}(R) : V \times V \to \mathbb{R}$ and $\mathrm{Ric}(R) : V \to \mathbb{R}$ by

$$\mathrm{ric}(R)(v,w) = \langle \mathrm{Ric}(R)(v), w\rangle = \frac{1}{2}\mathrm{tr}(R(v \wedge \cdot\,, w \wedge \cdot\,)),$$

where $v, w \in V$, and the *scalar curvature* $\mathrm{scal}(R)$ by

$$\mathrm{scal}(R) := \mathrm{tr}(\mathrm{Ric}(R)).$$

Notice, that here we use a different convention for the Ricci tensor and consequently the scalar curvature than e.g. in [BW08] and [CCG+08], where the factor $\frac{1}{2}$ is omitted.

Remark 1.1.11. Let $(b_1, \dots, b_n)$ be an orthonormal basis of V. Then according to Remark 1.1.2 $(b_i \wedge b_j)_{i<j}$ is an orthonormal basis of $\Lambda^2 V$, so that for $R \in S^2_B(\Lambda^2 V^*)$ we find that

$$\begin{aligned}\mathrm{scal}(R) = \mathrm{tr}(\mathrm{Ric}(R)) &= \sum_{i=1}^n \langle \mathrm{Ric}(R)(b_i), b_i\rangle = \frac{1}{2}\sum_{i=1}^n \mathrm{tr}(R(b_i \wedge \cdot, b_i \wedge \cdot)) \\ &= \frac{1}{2}\sum_{i,j=1}^n R(b_i \wedge b_j, b_i \wedge b_j) = \sum_{i<j} R(b_i \wedge b_j, b_i \wedge b_j) \\ &= \mathrm{tr}(R).\end{aligned}$$

By introducing the following symmetric product on symmetric bilinear forms, we can give a decomposition of the space of algebraic curvature tensors into irreducible subspaces with respect to the $O(n)$-representation ρ as discussed in Remark 1.1.4, which will be useful for some applications in Section 6.3.

Definition 1.1.12. Let g, h be symmetric bilinear forms on V. Then the *Kulkarni-Nomizu product* $g \owedge h$ of g and h is defined by

$$(g \owedge h)(x,y,z,w) := g(x,z)h(y,w) - g(x,w)h(y,z) + h(x,z)g(y,w) - h(x,w)g(y,z)$$

for $x, y, z, w \in V$.

Remark 1.1.13. The Kulkarni-Nomizu product of g and h defines an algebraic curvature tensor. Furthermore, we notice that

$$\mathrm{id} \owedge \mathrm{id} = 4I,$$

where id denotes the identity in the space of symmetric bilinear forms on V (i.e. the scalar product on V) and I the identity in $S^2_B(\Lambda^2 V^*)$ (i.e. the induced scalar product on $\Lambda^2 V$ as introduced in Remark 1.1.2). Of course in this equality we used the identification of $\mathcal{A}(V)$ with $S^2_B(\Lambda^2 V^*)$ as in Remark 1.1.8.

Lemma 1.1.14 ([CCG+08])**.** *For $n \geq 4$, the space of algebraic curvature tensors as an $O(n)$-representation has the following orthogonal decomposition into irreducible subrepresentations*

$$\mathcal{A}_n = \mathbb{R}\mathrm{id} \owedge \mathrm{id} \;\oplus\; \mathrm{id} \owedge S^2_0((\mathbb{R}^n)^*) \;\oplus\; \mathcal{W}, \tag{1.2}$$

where id *denotes the given scalar product on $\mathbb{R}^n$, $S^2_0((\mathbb{R}^n)^*)$ the space of traceless symmetric bilinear forms on $\mathbb{R}^n$ and $\mathcal{W} := \ker(\mathrm{Ric})$ the space of Weyl tensors. By R_I, R_{ric_0} and R_W, we denote the components of $R \in \mathcal{A}_n$ with respect to the decomposition* (1.2)*, so that $R = R_I + R_{\mathrm{ric}_0} + R_W$. Then*

$$R_I = \frac{1}{2n(n-1)}\mathrm{scal}(R)\mathrm{id} \owedge \mathrm{id},$$

$$R_{\mathrm{ric}_0} = \frac{1}{n-2}\mathrm{id} \owedge \mathrm{ric}_0(R),$$

where $\mathrm{ric}_0(R) := \mathrm{ric}(R) - \frac{1}{n}\mathrm{scal}(R)\mathrm{id}$ *denotes the traceless part of* $\mathrm{ric}(R)$.

Remark 1.1.15. The factors in the decomposition of R in Lemma 1.1.14 differ from those of [BW08] and [CCG+08, Section 2.2.3] due to the different convention we made for the Ricci tensor and scalar curvature (see above).

Remark 1.1.16. For all bilinear forms β on $\mathbb{R}^n$, we have that

$$\|\beta \owedge \mathrm{id}\|^2 = 4(n-2)\|\beta\|^2 + 4\mathrm{tr}(\beta)^2.$$

This yields for $R \in \mathcal{A}_n$ that

$$\begin{aligned} \|R_I\|^2 &= \frac{2}{n(n-1)}\mathrm{scal}(R)^2, \\ \|R_{\mathrm{ric}_0}\|^2 &= \frac{4}{n-2}\|\mathrm{ric}_0(R)\|^2. \end{aligned} \tag{1.3}$$

Hence, together with

$$\|\mathrm{ric}_0(R)\|^2 = \|\mathrm{ric}(R)\|^2 - \frac{1}{n}\mathrm{scal}(R)^2, \tag{1.4}$$

we obtain that

$$\begin{aligned} \|R\|^2 &= \frac{2}{n(n-1)}\mathrm{scal}(R)^2 + \frac{4}{n-2}\|\mathrm{ric}_0(R)\|^2 + \|R_W\|^2 \\ &= -\frac{2}{(n-1)(n-2)}\mathrm{scal}(R)^2 + \frac{4}{n-2}\|\mathrm{ric}(R)\|^2 + \|R_W\|^2. \end{aligned} \tag{1.5}$$

Moreover, polarization of (1.3) leads to

$$\begin{aligned} \langle R, S\rangle &= \langle R_I, S_I\rangle + \langle R_{\mathrm{ric}_0}, S_{\mathrm{ric}_0}\rangle + \langle R_W, S_W\rangle \\ &= \frac{2}{n(n-1)}\mathrm{scal}(R)\mathrm{scal}(S) + \frac{4}{n-2}\langle \mathrm{ric}_0(R), \mathrm{ric}_0(S)\rangle + \langle R_W, S_W\rangle \end{aligned} \tag{1.6}$$

for all $R, S \in \mathcal{A}_n$.

The following map arises for example in the evolution equation of the Riemannian curvature operator of a Riemannian manifold under the Ricci flow (see Lemma 1.4.2).

Definition 1.1.17. The symmetric bilinear map

$$\# : \mathrm{SymEnd}(\mathfrak{so}(V)) \times \mathrm{SymEnd}(\mathfrak{so}(V)) \to \mathrm{SymEnd}(\mathfrak{so}(V)) : (R, S) \mapsto R\#S$$

is given by

$$\Big\langle (R\#S)(A), B\Big\rangle := -\frac{1}{4}\mathrm{tr}(\mathrm{ad}_A \circ R \circ \mathrm{ad}_B \circ S + \mathrm{ad}_A \circ S \circ \mathrm{ad}_B \circ R)$$

for $A, B \in \mathfrak{so}(V)$. Here, for any $A \in \mathfrak{so}(V)$ the *adjoint representation* $\mathrm{ad}_A : \mathfrak{so}(V) \to \mathfrak{so}(V)$ is given by

$$\mathrm{ad}_A(B) := [A, B] = AB - BA$$

for $B \in \mathfrak{so}(V)$. Note that $R\#S$ is self-adjoint by the trace property. Moreover, we set $R^\# := R\#R$.

Remark 1.1.18. Since $\mathrm{ad}_{c(Q)A} = c(Q)\circ\mathrm{ad}_A\circ c(Q^t)$, hence $\mathrm{ad}_{c(Q)A}\circ\mathrm{ad}_{c(Q)B} = c(Q)\circ\mathrm{ad}_A\circ\mathrm{ad}_B\circ c(Q^t)$ for all $Q \in O(n)$ and $A, B \in \mathfrak{so}(n)$, it is easy to show that

$$c(Q) \circ R^{\#} \circ c(Q^t) = (c(Q) \circ R \circ c(Q^t))^{\#}.$$

Therefore, by Remark 1.1.4 we find that

$$(\widehat{\rho(Q)\check{R}})^{\#} = \rho(Q)\check{R}^{\#}$$

for all $Q \in O(n)$ and $R \in \mathrm{SymEnd}(\mathfrak{so}(n))$. In this sense, $R \mapsto R^{\#}$ is $O(n)$-equivariant.

Remark 1.1.19. Let $M := \dim(\mathfrak{so}(V))$. For an orthonormal basis $(B_1, \ldots, B_M)$ of $\mathfrak{so}(V)$, we have that

$$\langle R^{\#}(B_i), B_j\rangle = -\frac{1}{2}\sum_{k,l,m,p=1}^{M} R_k^l R_m^p c_{jl}^m c_{ip}^k$$

for $i, j = 1, \ldots, M$. Here, the coefficients R_i^j and c_{ij}^k are given by

$$R(B_i) = \sum_{j=1}^{M} R_i^j B_j \quad \text{respectively} \quad [B_i, B_j] = \sum_{k=1}^{M} c_{ij}^k B_k$$

for $i, j = 1, \ldots, M$. We notice that $c_{ij}^k = -c_{ji}^k$.

Example 1.1.20. In dimension 3, $R^{\#}$ can be described as follows. By (E_1, E_2, E_3), we denote the orthonormal basis of $\mathfrak{so}(3)$ defined by

$$E_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\ E_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } E_3 := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since

$$[E_1, E_2] = E_3,\ [E_1, E_3] = -E_2 \text{ and } [E_2, E_3] = E_1,$$

we obtain from Remark 1.1.19 that if R is the self-adjoint endomorphism of $\mathfrak{so}(3)$, which is with respect to (E_1, E_2, E_3) given by the matrix

$$R \mathrel{\widehat{=}} \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix},$$

then with respect to the same basis $R^{\#}$ corresponds the the adjoint matrix, i.e.

$$R^{\#} \mathrel{\widehat{=}} \begin{pmatrix} df - e^2 & ce - bf & be - cd \\ ce - bf & af - c^2 & bc - ae \\ be - cd & bc - ae & ad - b^2 \end{pmatrix}.$$

For R invertible this means that $R^{\#} = \det(R)(R^{-1})^t$.

Remark 1.1.21. One can show that $R^2 + R^{\#}$ is an algebraic curvature tensor.

1.2 The frame bundle and a connection on space-time

Given a one-parameter family of Riemannian metrics on a manifold M, we want to investigate how the geometry changes along this family. For this, a connection on the tangent bundle over the manifold $M \times \mathbb{R}$ plays a central role. In this section, we are particularly interested in the evolution of the orthonormal frame bundle on an initial Riemannian manifold.

Definition 1.2.1. Let (M, g) be an n-dimensional Riemannian manifold with Levi-Civita connection ∇^g. The *orthonormal frame bundle* O^g of (M, g) is the principle bundle with structure group $O(n)$, the fibres over points $x \in M$ of which are given by

$$O_x^g := \{p : (\mathbb{R}^n, \langle\cdot,\cdot\rangle) \to (T_xM, g_x) \mid p \text{ is an linear isometry}\},$$

where $\langle\cdot,\cdot\rangle$ is the standard metric on $\mathbb{R}^n$. Hence, the group $O(n)$ acts freely and transitively on the fibres of O^g from the right.
We say that a smooth curve $p : I \to O^g$, where $I \subseteq \mathbb{R}$ is an interval, is *parallel with respect to* ∇^g along $\gamma := \pi \circ p$, if the vector field $t \mapsto p(t)v$ along γ is parallel with respect to ∇^g for all $v \in \mathbb{R}^n$. Here, $\pi : TM \to M$ denotes the bundle projection on M.

Throughout this section, let $(g_t)_{t\in\mathbb{R}}$ be a smooth family of Riemannian metrics on a manifold M and set

$$h_t := -\frac{1}{2}\frac{\partial}{\partial t}g_t.$$

Definition 1.2.2. On the vector bundle $TM \to M \times \mathbb{R}$, we introduce a linear connection ∇, which on the times slices $M \times \{t\}$ is given by the Levi-Civita connection ∇^{g_t} of the metric g_t and is *not* the product connection: Let $\frac{\partial}{\partial t}$ be the vector field on $M \times \mathbb{R}$ given by $\frac{\partial}{\partial t}|_{(x,t)} := \dot{c}(t)$ for $(x,t) \in M \times \mathbb{R}$, where $c(t) := (x,t)$. Let further $X, Y \in \Gamma(M \times \mathbb{R}, TM)$ be time-dependent vector fields on M. Then ∇ is given by

$$\begin{aligned} (\nabla_Y X)|_{(x,t)} &:= \Big(\nabla^{g_t}_{Y(\cdot,t)} X(\cdot,t)\Big)|_x \\ \Big(\nabla_{\frac{\partial}{\partial t}} X\Big)\Big|_{(x,t)} &:= \frac{\partial}{\partial t}X(x,t) - H_t(X(x,t)) \end{aligned} \tag{1.7}$$

for $(x,t) \in M \times \mathbb{R}$, where by H_t we denote the map

$$H_t : TM \to TM : X \mapsto h_t(X,\cdot)^{\sharp g_t},$$

i.e. for $X \in TM$ the vector H_tX satisfies $g_t(H_tX,\cdot) = h_t(X,\cdot)$.

Remark 1.2.3. The introduced connection extends in the usual way to a connection on the vector bundle $S^2_B(\Lambda^2T^*M) \to M \times \mathbb{R}$. More explicitly, for $R \in \Gamma(M \times \mathbb{R}, S^2_B(\Lambda^2T^*M))$ and $X, Y, Z, V, W \in \Gamma(M \times \mathbb{R}, TM)$ we have that

$$\begin{aligned} (\nabla_Z R)(V \wedge W, X \wedge Y) = \ &\partial_Z(R(V \wedge W, X \wedge Y)) - R(\nabla_Z V \wedge W, X \wedge Y) \\ &- R(V \wedge \nabla_Z W, X \wedge Y) - R(V \wedge W, \nabla_Z X \wedge Y) \\ &- R(V \wedge W, X \wedge \nabla_Z Y) \end{aligned}$$

and

$$\begin{aligned} \Big(\nabla_{\frac{\partial}{\partial t}} R\Big)(V \wedge W, X \wedge Y) = \ &\frac{\partial}{\partial t}\Big(R(V \wedge W, X \wedge Y)\Big) - R\Big(\nabla_{\frac{\partial}{\partial t}} V \wedge W, X \wedge Y\Big) \\ &- R\Big(V \wedge \nabla_{\frac{\partial}{\partial t}} W, X \wedge Y\Big) - R\Big(V \wedge W, \nabla_{\frac{\partial}{\partial t}} X \wedge Y\Big) \\ &- R\Big(V \wedge W, X \wedge \nabla_{\frac{\partial}{\partial t}} Y\Big). \end{aligned}$$

Remark 1.2.4. ∇ is metric with respect to $g(\cdot)$. Namely, for $X, Y \in \Gamma(M \times \mathbb{R}, TM)$ we can compute that

$$\begin{aligned}\frac{\partial}{\partial t}\Big(g_t(X,Y)\Big) &= \left(\frac{\partial}{\partial t}g_t\right)(X,Y) + g_t\left(\frac{\partial}{\partial t}X, Y\right) + g_t\left(X, \frac{\partial}{\partial t}Y\right)\\ &= -2h_t(X,Y) + g_t\Big(\nabla_{\frac{\partial}{\partial t}}X, Y\Big) + g_t(H_tX, Y) + g_t\Big(X, \nabla_{\frac{\partial}{\partial t}}Y\Big) + g_t(X, H_tY)\\ &= g_t\Big(\nabla_{\frac{\partial}{\partial t}}X, Y\Big) + g_t\Big(X, \nabla_{\frac{\partial}{\partial t}}Y\Big).\end{aligned}$$

Definition 1.2.5. Let $x_0 \in M$ and $p_t : \mathbb{R}^n \to T_{x_0}M$ be a one-parameter family of linear maps. Then $t \mapsto p_t$ is *parallel*, if $t \mapsto p_t(v)$ is parallel along the curve $t \mapsto (x_0, t)$ with respect to the connection ∇ (introduced in Definition 1.2.2) for all $v \in \mathbb{R}^n$, i.e. if

$$\frac{d}{dt}p_t(v) = H_t p_t(v)$$

for all $v \in \mathbb{R}^n$.

Lemma 1.2.6. *Let $x_0 \in M$, $R_t \in S^2_B(\Lambda^2 T^*_{x_0}M)$ be a one-parameter family of algebraic curvature tensors and $p_t : \mathbb{R}^n \to T_{x_0}M$ be a one-parameter family of linear maps. If $t \mapsto R_t$ is smooth and $t \mapsto p_t$ is parallel, then*

$$\frac{d}{dt}(p_t^* R_t) = p_t^* \nabla_{\frac{\partial}{\partial t}} R_t.$$

Proof. Let $v, w, x, y \in \mathbb{R}^n$. Then we can compute that

$$\begin{aligned}\frac{d}{dt}(p_t^*R_t)(v\wedge w, x\wedge y) &= \frac{d}{dt}\Big((p_t^*R_t)(v\wedge w, x\wedge y)\Big) = \frac{d}{dt}\Big(R_t(p_t(v)\wedge p_t(w), p_t(x)\wedge p_t(y))\Big)\\ &= \Big(\nabla_{\frac{\partial}{\partial t}}R_t\Big)(p_t(v)\wedge p_t(w), p_t(x)\wedge p_t(y)) + R_t\Big(\underbrace{\nabla_{\frac{\partial}{\partial t}}p_t(v)}_{=0}\wedge p_t(w), p_t(x)\wedge p_t(y)\Big)\\ &\quad + \dots + R_t\Big(p_t(v)\wedge p_t(w), p_t(x)\wedge \underbrace{\nabla_{\frac{\partial}{\partial t}}p_t(y)}_{=0}\Big)\\ &= \Big(\nabla_{\frac{\partial}{\partial t}}R_t\Big)(p_t(v)\wedge p_t(w), p_t(x)\wedge p_t(y))\\ &= \Big(p_t^*\nabla_{\frac{\partial}{\partial t}}R_t\Big)(v\wedge w, x\wedge y).\end{aligned}$$

This finishes the proof. □

Next, we show that if $t \mapsto p_t$ is parallel and p_0 is an isometry with respect to the standard metric $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ and g_0, then p_t is an isometry with respect to $\langle\cdot,\cdot\rangle$ and g_t for all $t \in \mathbb{R}$ as well.

Lemma 1.2.7. *Let $x_0 \in M$ and p be a solution of*

$$\frac{d}{dt}p_t = H_t p_t \tag{1.8}$$

with $p_0 : (\mathbb{R}^n, \langle\cdot,\cdot\rangle) \to (T_{x_0}M, g_0)$ being an isometry. Then $p_t : (\mathbb{R}^n, \langle\cdot,\cdot\rangle) \to (T_{x_0}M, g_t)$ is an isometry for each $t \in \mathbb{R}$.

Proof. Let $v, w \in \mathbb{R}^n$. Then we have that

$$\begin{aligned}\frac{d}{dt}\Big(g_t(p_t(v), p_t(w))\Big) &= \left(\frac{\partial}{\partial t} g_t\right)(p_t(v), p_t(w)) + g_t\left(\frac{d}{dt} p_t(v), p_t(w)\right) + g_t\left(p_t(v), \frac{d}{dt} p_t(w)\right) \\ &= -2h_t(p_t(v), p_t(w)) + g_t(H_t p_t(v), p_t(w)) + g_t(p_t(v), H_t p_t(w)) \\ &= -2h_t(p_t(v), p_t(w)) + h_t(p_t(v), p_t(w)) + h_t(p_t(v), p_t(w)) \\ &= 0.\end{aligned}$$

Therefore, we find for each $t \in \mathbb{R}$ that

$$g_t(p_t(v), p_t(w)) = g_0(p_0(v), p_0(w)) = \langle v, w \rangle.$$ □

Given an initial value, according to the Picard-Lindelöf theorem, there is a solution to the linear ordinary differential equation (1.8). Lemma 1.2.7 therefore shows that given an initial isometry p_0 there is always a parallel curve $t \mapsto p_t \in O^{g_t}$ starting at p_0. This shows that the flow $\frac{\partial}{\partial t} g_t = -2h_t$ preserves the bundle $O^{g_\cdot}$.

1.3 Curvature of Riemannian manifolds

In this section, we recall the common notions of curvature of a Riemannian manifold, give a connection to algebraic curvature tensors and show how the introduced objects behave under rescaling of the metric.

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇^g. Let further $\pi : TM \to M$ be the bundle projection on M, $x \in M$ and $X, Y, Z, W \in T_xM$.

- The *Riemannian curvature tensor* $R_g : TM \times TM \times TM \to TM$ is defined by

  $$R_g(X,Y)Z := \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z.$$

  We may also write

  $$R_g(X, Y, Z, W) := g(R_g(X, Y)W, Z).$$

- The *Riemannian curvature operator* Rm_g is the symmetric bilinear form on $\Lambda^2 TM$, respectively the self-adjoint endomorphism of $\Lambda^2 TM$, defined by

  $$Rm_g(X \wedge Y, Z \wedge W) = g(Rm_g(X \wedge Y), Z \wedge W) := 2R_g(X, Y, Z, W). \tag{1.9}$$

  Hence, Rm_g is a section of the *bundle of algebraic curvature tensors* $S^2_B(\Lambda^2 T^*M)$. Note that $Rm_g = 2I$ for the standard sphere.

- We define the *Ricci curvature* $\mathrm{ric}_g : TM \times TM \to \mathbb{R}$ of g in terms of the Ricci tensor of an algebraic curvature tensor (see Definition 1.1.10) by

  $$\mathrm{ric}_g(X, Y) := \mathrm{ric}(Rm_g(x))(X, Y) = \frac{1}{2}\mathrm{tr}_g(Rm_g(X \wedge \cdot, Y \wedge \cdot)) = \mathrm{tr}_g(R_g(X, \cdot, Y, \cdot))$$

  and $\mathrm{Ric}_g : TM \to TM$ by

  $$g(\mathrm{Ric}_g(X), Y) := \mathrm{ric}_g(X, Y).$$

- The *scalar curvature* $\mathrm{scal}_g : M \to \mathbb{R}$ of g is defined by

$$\mathrm{scal}_g := \mathrm{tr}_g(\mathrm{Ric}(Rm_g)) = \mathrm{tr}_g(\mathrm{Ric}_g).$$

Remark 1.3.1. If $\psi : N \to M$ is a smooth map, then the *pullback* $\psi^* Rm_g$ *of* Rm_g *along* ψ is given by

$$(\psi^* Rm_g)(x)(X \wedge Y, Z \wedge W) := Rm_g(\psi(x))(d\psi(X) \wedge d\psi(Y), d\psi(Z) \wedge d\psi(W))$$

for $x \in M$ and $X, Y, Z, W \in T_x M$.

Remark 1.3.2. The objects introduced above have the following scaling behaviour under conformal changes of the metric: Let $\alpha > 0$. Then $\nabla^{\alpha g} = \nabla^g$, $R_{\alpha g} = R_g$ (respectively $R_{\alpha g} = \alpha R_g$ when interpreted as (0,4)-tensor), $Rm_{\alpha g} = \alpha Rm_g$ (respectively $Rm_{\alpha g} = \frac{1}{\alpha} Rm_g$ when interpreted as endomorphism of $\Lambda^2 TM$), $\|Rm\|_{\alpha g} = \frac{1}{\alpha^2}\|Rm\|_g$, $\mathrm{ric}_{\alpha g} = \mathrm{ric}_g$, $\mathrm{Ric}_{\alpha g} = \frac{1}{\alpha}\mathrm{Ric}_g$ and $\mathrm{scal}_{\alpha g} = \frac{1}{\alpha}\mathrm{scal}_g$. For a function $f : M \to \mathbb{R}$, we further have that $\mathrm{Hess}_{\alpha g} f = \mathrm{Hess}_g f$.

1.4 A brief introduction to the Ricci flow

Given a manifold equipped with a Riemannian metric g_0, one can consider solutions g_t, $t \in [0, T)$, of the partial differential equation

$$\frac{\partial}{\partial t} g_t = -2\mathrm{ric}_{g_t} \tag{1.10}$$

starting at g_0, so-called *solutions to the Ricci flow*. The Ricci curvature ric_g of a Riemannian manifold (M, g) can be considered as a Laplacian of g. For instance, in harmonic coordinates we have that $(\mathrm{ric}_g)_{ij}$ is given by $-\frac{1}{2}\Delta g_{ij}$ plus lower order terms respectively in normal coordinates by $-\frac{3}{2}\Delta g_{ij}$ in the central point. Therefore, the partial differential equation (1.10) can be seen as a version of the heat equation. Hence roughly speaking, similar to the diffusion of heat the curvature evens out it time such that in the limit one expects a metric of constant curvature. However, there are some problems, since solutions may possibly not be continued for all times as singularities may arise in the flow. In particular, not every manifold can carry a metric of constant curvature.

For compact manifolds, Richard S. Hamilton, who was the first to introduce the Ricci flow in 1982 [Ham82], proved short-time existence and uniqueness of solutions to the Ricci flow to a given initial metric [Ham82]. Provided that the sectional curvature of the initial metric is bounded, Shi [Shi89] showed that one has short-time existence on complete non-compact manifolds. Furthermore, Chen and Zhu [CZ06] proved that complete solutions to the Ricci flow on non-compact manifolds with bounded sectional curvature are unique.

The Ricci flow is the main tool in Perelman's proof of Thurston's geometrization conjecture for three-manifolds [Per02, Per03b, Per03a], which in particular implies the Poincaré conjecture and Thurston's elliptization conjecture. Moreover, the Ricci flows plays a central role in the investigation of manifolds with different curvature conditions as for example in the proofs of Hamilton's theorems for compact 3-manifolds with positive Ricci curvature [Ham82] respectively compact 4-manifolds with positive curvature operator [Ham86], Huisken's result for compact manifolds satisfying an explicit open conical curvature condition in dimension $n \geq 4$ [Hui85], the theorem of Böhm and Wilking for compact manifolds with 2-positive curvature operator [BW08] and the differentiable sphere theorem by Brendle and Schoen [BS09].

Given a solution g_t to the Ricci flow on a manifold M, one is interested in how the Riemannian curvature operator Rm_{g_t} changes in time. By setting $h_t := \mathrm{ric}_{g_t}$, Definition 1.2.2 yields a connection ∇ on the vector bundle $TM \to M \times \mathbb{R}$. In order to formulate the evolution equation of the Riemannian curvature operator, we additionally need to define the Laplace operator acting on sections of the bundle of algebraic curvature tensors.

Definition 1.4.1. Let (M, g) be a Riemannian manifold, $x \in M$ and $\gamma_1, \ldots, \gamma_n$ be geodesics in M such that $\gamma_i(0) = x$ for $i = 1, \ldots, n$ and $(\dot{\gamma_1}(0), \ldots, \dot{\gamma_n}(0))$ is an orthonormal basis of T_xM. The *Laplace operator* Δ_g with respect to g on M is defined as follows.

1. If $f : M \to \mathbb{R}$ is a smooth function, then
$$(\Delta_g f)(x) := \sum_{i=1}^{n} \frac{d^2}{ds^2}\bigg|_{s=0} f(\gamma_i(s)).$$
2. If $R \in \Gamma(M, S^2_B(\Lambda^2 T^*M))$ is a smooth section of the bundle of algebraic curvature tensors, then
$$(\Delta_g R)(x) := \sum_{i=1}^{n} \frac{(\nabla^g)^2}{ds^2}\bigg|_{s=0} R(\gamma_i(s)), \tag{1.11}$$
where ∇^g denotes the Levi-Civita connection of (M, g).
3. More generally, for sections of an arbitrary vector bundle V over M with connection, an associated Laplace operator can be defined by the same formula (1.11) using the connection of V instead of the Levi-Civita connection ∇^g.

We can now formulate the evolution equation of the Riemannian curvature operator under the Ricci flow.

Lemma 1.4.2. ([Ham86, p.155]) *If g_t is a solution to the Ricci flow on a manifold M, then the Riemannian curvature operator of g_t evolves under the partial differential equation*
$$\nabla_{\frac{\partial}{\partial t}} Rm_{g_t} = \Delta_{g_t} Rm_{g_t} + Rm^2_{g_t} + Rm^{\#}_{g_t}. \tag{1.12}$$
Here, for any $x \in M$ we regard $Rm_{g_t}(x)$ as a self-adjoint endomorphism of $\mathfrak{so}(T_xM)$ and define $Rm^2_{g_t}(x) := Rm_{g_t}(x) \circ Rm_{g_t}(x)$ and $Rm^{\#}_{g_t}(x)$ as in Definition 1.1.17.

Chapter 2

Curvature conditions and ODE-invariance

This chapter is dedicated to introducing curvature conditions, that is $O(n)$-invariant subsets Ω of the space of algebraic curvature tensors $\mathcal{A}_n$. To these sets, one can associate subsets Ω^g of the bundle of algebraic curvature tensors over a Riemannian manifold (M,g), which are invariant under parallel transport by the Levi-Civita connection ∇^g of (M,g). Generically, these are exactly the subsets being invariant under ∇^g. Using this notation, we are able to say when a Riemannian metric satisfies a given curvature condition.
Furthermore, we consider subsets of a vector space which are invariant under an ordinary differential equation of the form $f'(t) = \Phi(f(t))$ and give a characterization of these in terms of their tangent cones. In some of our applications, such sets arise as sublevel sets of some function. We show that these are invariant under the mentioned ordinary differential equation if and only if the angle between the gradient of this function and the map Φ is at least $\frac{\pi}{2}$. As an application, we obtain that scalar curvature bounded from below is invariant under the ordinary differential equation $R'(t) = R(t)^2 + R(t)^\#$.

2.1 Curvature conditions

Let (M,g) be an n-dimensional Riemannian manifold and O^g the orthonormal frame bundle on (M,g) (see Definition 1.2.1). There is a left-action of $O(n)$ on the space of algebraic curvature tensors. Remember from Section 1.1 that the representation

$$\rho : O(n) \to \mathrm{End}(\mathcal{A}_n)$$

of $O(n)$ on $\mathcal{A}_n = S^2_B(\Lambda^2(\mathbb{R}^n)^*)$ is given by

$$(\rho(Q)R)(v \wedge w, y \wedge z) := R(Q^{-1}v \wedge Q^{-1}w, Q^{-1}y \wedge Q^{-1}z),$$

where $Q \in O(n)$, $R \in \mathcal{A}_n$ and $v, w, y, z \in \mathbb{R}^n$.

Definition 2.1.1. A subset $\Omega \subseteq \mathcal{A}_n$ is called $O(n)$-*invariant*, if for all $R \in \Omega$ we have that

$$\rho(Q)R \in \Omega$$

for every $Q \in O(n)$.

Definition 2.1.2. To an $O(n)$-invariant subset $\Omega \subseteq \mathcal{A}_n$, we associate the subset $\Omega^g \subseteq S^2_B(\Lambda^2 T^*M)$ defined by

$$\Omega^g := \left\{ R \in S^2_B(\Lambda^2 T^*M) \mid p^*R \in \Omega \text{ for some } p \in O^g_{\pi(R)} \right\},$$

where $\pi : S^2_B(\Lambda^2 T^*M) \to M$ is the projection map and $p^*R \in \mathcal{A}_n$ is the pullback of R along p, i.e.

$$(p^*R)(v \wedge w, x \wedge y) := R(p(v) \wedge p(w), p(x) \wedge p(y))$$

for $v, w, x, y \in \mathbb{R}^n$.

Remark 2.1.3. If $p^*R \in \Omega$ for one $p \in O^g_{\pi(R)}$, then $p^*R \in \Omega$ for every $p \in O^g_{\pi(R)}$. Namely, let $p \in O^g_{\pi(R)}$ with $p^*R \in \Omega$ and $q \in O^g_{\pi(R)}$. Since $O(n)$ acts transitively on $O^g_{\pi(R)}$, there is a $Q \in O(n)$ with $q = p \circ Q$. Therefore, we have that

$$q^*R = (p \circ Q)^*R = \rho(Q^{-1})(p^*R) \in \Omega,$$

since Ω is $O(n)$-invariant.

Lemma 2.1.4. *Ω^g is invariant under parallel transport by ∇^g.*

Proof. Let $s \mapsto R_s$ be a parallel curve in $S^2_B(\Lambda^2(T^*M))$ with respect to ∇^g with $R_0 \in \Omega^g$. Let further $s \mapsto p_s$ be a parallel curve in O^g along $\pi \circ R$ with respect to ∇^g. Then for all $v, w, y, z \in \mathbb{R}^n$, we have that

$$\begin{aligned}
&\frac{d}{ds}\Big(p_s^*R_s\Big)(v \wedge w, y \wedge z) = \frac{d}{ds}\Big((p_s^*R_s)(v \wedge w, y \wedge z)\Big) \\
&\quad = \frac{d}{ds}\Big(R_s\big(p_s(v) \wedge p_s(w), p_s(y) \wedge p_s(z)\big)\Big) \\
&\quad = \underbrace{\Big(\frac{\nabla^g}{ds}R_s\Big)}_{=0}\big(p_s(v) \wedge p_s(w), p_s(y) \wedge p_s(z)\big) + R_s\Big(\underbrace{\frac{\nabla^g}{ds}p_s(v)}_{=0} \wedge p_s(w), p_s(y) \wedge p_s(z)\Big) \\
&\qquad + \cdots + R_s\Big(p_s(v) \wedge p_s(w), p_s(y) \wedge \underbrace{\frac{\nabla^g}{ds}p_s(z)}_{=0}\Big) \\
&\quad = 0,
\end{aligned} \tag{2.1}$$

where the last equality holds, since $s \mapsto p_s$ and $s \mapsto R_s$ are parallel. Therefore,

$$\frac{d}{ds}\Big(p_s^*R_s\Big) = 0$$

and thus $s \mapsto p_s^*R_s$ is constant. Since $R_0 \in \Omega^g$, we have that

$$p_s^*R_s = p_0^*R_0 \in \Omega$$

for all s. Consequently, $R_s \in \Omega^g$ for all s. □

Definition 2.1.5. For $x \in M$, the *holonomy group of ∇^g based at x* is defined by

$$\mathrm{hol}_x(\nabla^g) := \{P_\gamma \mid \gamma : [0,1] \to M \text{ piecewise smooth with } \gamma(0) = \gamma(1) = x\} \subseteq O(T_xM).$$

Here, $P_\gamma : T_xM \to T_xM$ denotes the parallel transport along the loop γ with respect to ∇^g, that is for $X_0 \in T_xM$

$$P_\gamma(X_0) := X(1),$$

where $t \mapsto X(t)$ is the parallel vector field along γ with respect to ∇^g satisfying $X(0) = X_0$.

Remark 2.1.6. If M is connected and $x, y \in M$, then the groups $\mathrm{hol}_x(\nabla^g)$ and $\mathrm{hol}_y(\nabla^g)$ are isomorphic. Namely, let $\gamma : [0,1] \to M$ be a piecewise smooth curve with $\gamma(0) = x$ and $\gamma(1) = y$, then

$$\mathrm{hol}_x(\nabla^g) \to \mathrm{hol}_y(\nabla^g) : Q \mapsto P_\gamma \circ Q \circ P_\gamma^{-1}$$

is an isomorphism.

The holonomy of a generic n-dimensional Riemannian manifold is isomorphic to $O(n)$, or $SO(n)$ if it is orientable. We show that in this case, all subsets of $S^2_B(\Lambda^2 T^*M)$, which are invariant under parallel transport by ∇^g, are of the form Ω^g for a suitable set $\Omega \subseteq \mathcal{A}_n$.

Lemma 2.1.7. *Let $C \subseteq S^2_B(\Lambda^2 T^*M)$ be invariant under parallel transport by ∇^g and suppose that*

$$\mathrm{hol}_x(\nabla^g) = O(T_xM) \quad (\textit{or } SO(T_xM) \textit{ if } M \textit{ is orientable})$$

for an $x \in M$. Then there exists an $O(n)$-invariant (respectively $SO(n)$-invariant) subset $\Omega \subseteq \mathcal{A}_n$ such that $C = \Omega^g$. Here, in the case that M is orientable, we define Ω^g using the orientable frame bundle of M instead of O^g.

Proof. We prove the lemma for the case that $\mathrm{hol}_x(\nabla^g) = O(T_xM)$ for an $x \in M$. The other case works analogously. Choose $p \in O^g_x$ and set

$$\Omega := p^* C_x \subseteq \mathcal{A}_n,$$

where C_x denotes the restriction of C to the fibre over x. Then $\Omega^g_x = C_x$, where again Ω^g_x denotes the restriction of Ω^g to the fibre over x. Since C is invariant under parallel transport, for all $y \in M$ and piecewise smooth paths $\gamma : [0,1] \to M$ with $\gamma(0) = y$ and $\gamma(1) = x$, we have that

$$P^*_\gamma C_x := \{R(P_\gamma(\cdot) \wedge P_\gamma(\cdot), P_\gamma(\cdot) \wedge P_\gamma(\cdot)) \mid R \in C_x\} = C_y.$$

Using the parallel invariance of Ω^g (see Lemma 2.1.4), this leads to

$$C_y = P^*_\gamma C_x = P^*_\gamma \Omega^g_x = \Omega^g_y.$$

Hence, $C = \Omega^g$. It remains to show that Ω is $O(n)$-invariant. To this end, let $Q \in O(n)$. Then $p \circ Q^{-1} \circ p^{-1} \in O(T_xM) = \mathrm{hol}_x(\nabla^g)$. Again using that C is invariant under $\mathrm{hol}_x(\nabla^g)$ yields that

$$C_x = (p \circ Q^{-1} \circ p^{-1})^* C_x = (p^{-1})^*(\rho(Q)(p^* C_x)) = (p^{-1})^*(\rho(Q)\Omega).$$

Thus,

$$\rho(Q)\Omega = p^* C_x = \Omega. \qquad \square$$

Remark 2.1.8. If Ω is open respectively closed, Ω^g is open respectively closed as well.

Example 2.1.9. The set

$$\Omega := \{R \in \mathcal{A}_n \mid \mathrm{Ric}(R) \geq 0\}$$

is $O(n)$-invariant and

$$\Omega^g = \{R \in S^2_B(\Lambda^2 T^*M) \mid \mathrm{Ric}_g(R) \geq 0\},$$

where $\mathrm{Ric}_g(R)$ denotes the Ricci tensor of R with respect to the metric g.

Definition 2.1.10. Let $\Omega \subseteq \mathcal{A}_n$ be $O(n)$-invariant. We say that g *satisfies* Ω, if for all $x \in M$, we have that

$$Rm_g(x) \in \Omega^g.$$

Therefore, we often call such a set Ω *a curvature condition*.

Definition 2.1.11. We define the $O(n)$-invariant set

$$\mathcal{C}(M,g) := \{p^* Rm_g(x) \mid x \in M \text{ and } p \in O^g_x\} \subseteq \mathcal{A}_n.$$

Obviously, g satisfies $\mathcal{C}(M,g)$. Moreover, for curvature conditions $\Omega \subseteq \mathcal{A}_n$, we have that g satisfies Ω if and only if $\mathcal{C}(M,g) \subseteq \Omega$.

Definition 2.1.12. We say that an $O(n)$-invariant set $\Omega \subseteq \mathcal{A}_n$ is *invariant under the Ricci flow*, if for all n-dimensional compact manifolds M and solutions g_t, $t \in [0,T)$, to the Ricci flow on M with g_0 satisfying Ω, we have that g_t satisfies Ω for all $t \in [0,T)$.

2.1.1 As domain of functions

In our results, curvature conditions Ω often arise as domains of $O(n)$-invariant functions F. Again, to these functions one can associate functions F^g which are defined on the associated sets Ω^g and are invariant under parallel transport by the Levi-Civita connection of the Riemannian manifold (M,g). In this section, we provide formulas for derivatives of F^g in terms of those of F.

Throughout this section, let $\Omega \subseteq \mathcal{A}_n$ be an $O(n)$-invariant subset.

Definition 2.1.13. A function $F : \Omega \to \mathbb{R}$ is called $O(n)$*-invariant*, if

$$F(\rho(Q)R) = F(R)$$

for all $R \in \Omega$ and $Q \in O(n)$.
To such a function F, we associate a function $F^g : \Omega^g \to \mathbb{R}$, which is defined via

$$F^g(R) := F(p^* R)$$

for some $p \in O^g_{\pi(R)}$. Remark 2.1.3 shows that this is independent of the choice of p, hence well defined, due to the $O(n)$-invariance of F.

Lemma 2.1.14. *Let $F : \Omega \to \mathbb{R}$ be an $O(n)$-invariant function. Then F^g is invariant under parallel transport with respect to ∇^g.*

Proof. Let $s \mapsto R_s$ be parallel in $S^2_B(\Lambda^2 T^* M)$ with respect to ∇^g with $R_0 \in \Omega^g$, and let $s \mapsto p_s$ be a parallel curve in O^g along $\pi \circ R$ with respect to ∇^g. Then $s \mapsto p_s^* R_s$ is constant as shown in the proof of Lemma 2.1.4, and therefore

$$s \mapsto F^g(R_s) = F(p_s^* R_s)$$

is constant as well. This means that F^g is invariant under parallel transport. □

Remark 2.1.15. A vector space V can be canonically identified with its tangent space T_vV at a point $v \in V$ via the map

$$\mathrm{can}_v : V \to T_vV : \; w \mapsto \frac{d}{dt}\Big|_{t=0}[v + tw].$$

Moreover, for $R \in S^2_B(\Lambda^2 T^*M)$ by ι_R we denote the inclusion

$$\iota_R : T_R S^2_B(\Lambda^2 T^*_{\pi(R)}M) \to T_R S^2_B(\Lambda^2 T^*M) = T_R S^2_B(\Lambda^2 T^*_{\pi(R)}M) \oplus H_R$$

of the vertical part of the tangent space of the bundle at R. Here, H_R denotes the *horizontal space*

$$H_R := \Big\{[\dot{v}(0)] \;\Big|\; v : (-\epsilon, \epsilon) \to S^2_B(\Lambda^2 T^*M) \text{ parallel}$$
$$\text{with respect to } \nabla^g \text{ along } \pi \circ v \text{ with } v(0) = R\Big\}.$$

Setting $V := S^2_B(\Lambda^2 T^*_{\pi(R)}M)$, one can consider the composition of these to maps

$$\mathrm{v}_R := \iota_R \circ \mathrm{can}_R : S^2_B(\Lambda^2 T^*_{\pi(R)}M) \to T_R S^2_B(\Lambda^2 T^*M) : \; S \mapsto S^{\mathrm{v}_R}.$$

For the sake of notational simplicity, we will always use the identifications can_R silently, and omit the base point of v_R and write v instead.

Remark 2.1.16. By Lemma 2.1.14, F^g is invariant under parallel transport with respect to ∇^g. Therefore, dF^g_R vanishes identically on the horizontal space H_R for all $R \in \Omega^g$. Namely, let $[\dot{R}(0)] \in H_R$ such that $s \mapsto R(s)$ is parallel with respect to ∇^g. Then

$$dF^g_R([\dot{R}(0)]) = \frac{d}{ds}\Big|_{s=0} F^g(R(s)) = 0$$

as F^g is invariant under parallel transport, thus $s \mapsto F^g(R(s))$ is constant.

Lemma 2.1.17. *Let Ω be open and $F : \Omega \to \mathbb{R}$ a smooth and $O(n)$-invariant function. Then for all $R \in \Omega^g$, $S \in S^2_B(\Lambda^2 T^*_{\pi(R)}M)$ and $p \in O^g_{\pi(R)}$, we have that*

$$dF^g_R(S^{\mathrm{v}}) = dF_{p^*R}(p^*S).$$

Proof. Let $R \in \Omega^g$, $S \in S^2_B(\Lambda^2 T^*_{\pi(R)}M)$ and $p \in O^g_{\pi(R)}$. Since Ω^g is open, there exists an $\epsilon > 0$ such that $R + tS \in \Omega^g$ for small $t \in (-\epsilon, \epsilon)$. Therefore,

$$\begin{aligned} dF^g_R(S^{\mathrm{v}}) &= \frac{d}{dt}\Big|_{t=0} F^g(R + tS) = \frac{d}{dt}\Big|_{t=0} F(p^*(R + tS)) \\ &= \frac{d}{dt}\Big|_{t=0} F(p^*R + tp^*S) = dF_{p^*R}(p^*S). \end{aligned}$$

□

A key lemma is now the following.

Lemma 2.1.18. *Let Ω be open, $F : \Omega \to \mathbb{R}$ a smooth and $O(n)$-invariant function and $R \in \Gamma(M, \Omega^g)$ a smooth section of Ω^g. Then for each $x \in M$, we have that*

$$(\Delta_g(F^g \circ R))(x) = \sum_{i=1}^n \mathrm{Hess}_{p^*R(x)} F\left(p^*\nabla^g_{e_i} R, p^*\nabla^g_{e_i} R\right) + dF^g_{R(x)}\left((\Delta_g R)(x)^{\mathrm{v}}\right).$$

Here, $(e_1, \ldots, e_n)$ is an orthonormal basis of T_xM with respect to g and $p \in O^g_x$.

Proof. Let $x \in M$, γ a geodesic in M with $\gamma(0) = x$ and $s \mapsto p_s$ a parallel curve in O^g that lies over γ, meaning that $p_s \in O^g_{\gamma(s)}$ for all s. Then we find that

$$\frac{d}{ds}\Big(p_s^* R(\gamma(s))\Big) \overset{(2.1)}{=} p_s^* \frac{\nabla^g}{ds} R(\gamma(s)) = p_s^* \nabla^g_{\dot\gamma(s)} R$$

and therefore that

$$\frac{d}{ds} F^g(R(\gamma(s))) = \frac{d}{ds} F(p_s^* R(\gamma(s))) = dF_{p_s^* R(\gamma(s))}\left(\frac{d}{ds} p_s^* R(\gamma(s))\right) = dF_{p_s^* R(\gamma(s))}\left(p_s^* \frac{\nabla^g}{ds} R(\gamma(s))\right).$$

Differentiating this once more and again using (2.1) yields that

$$\begin{aligned}\frac{d^2}{ds^2} F^g(R(\gamma(s))) &= \mathrm{Hess}_{p_s^* R(\gamma(s))} F\Big(p_s^* \nabla^g_{\dot\gamma(s)} R, p_s^* \nabla^g_{\dot\gamma(s)} R\Big) + dF_{p_s^* R(\gamma(s))}\left(p_s^* \frac{(\nabla^g)^2}{ds^2} R(\gamma(s))\right)\\ &= \mathrm{Hess}_{p_s^* R(\gamma(s))} F\Big(p_s^* \nabla^g_{\dot\gamma(s)} R, p_s^* \nabla^g_{\dot\gamma(s)} R\Big) + dF^g_{R(\gamma(s))}\left(\frac{(\nabla^g)^2}{ds^2} R(\gamma(s))^{\mathrm{v}}\right).\end{aligned}$$

Now, let $(e_1, \ldots, e_n)$ be a g-orthonormal basis of T_xM and $\gamma_1, \ldots, \gamma_n$ geodesics in M with $\gamma_i(0) = x$ and $\dot\gamma_i(0) = e_i$ for $i = 1, \ldots, n$ and let $p \in O^g_x$. Then

$$\begin{aligned}(\Delta_g(F^g \circ R))(x) &= \sum_{i=1}^n \frac{d^2}{ds^2}\Big|_{s=0} F^g(R(\gamma_i(s)))\\ &= \sum_{i=1}^n \left(\mathrm{Hess}_{p^* R(x)} F\Big(p^* \nabla^g_{e_i} R, p^* \nabla^g_{e_i} R\Big) + dF^g_{R(x)}\left(\frac{(\nabla^g)^2}{ds^2}\Big|_{s=0} R(\gamma_i(s))^{\mathrm{v}}\right)\right)\\ &= \sum_{i=1}^n \mathrm{Hess}_{p^* R(x)} F\Big(p^* \nabla^g_{e_i} R, p^* \nabla^g_{e_i} R\Big) + dF^g_{R(x)}\left((\Delta_g R)(x)^{\mathrm{v}}\right).\end{aligned}$$

□

Lemma 2.1.19. *Let Ω be open and $F : \Omega \to \mathbb{R}$ a smooth and $O(n)$-invariant function. Moreover, let (M, g_0) be an n-dimensional Riemannian manifold and $g(t)$, $t \in [0, T)$, be a solution to the Ricci flow with $g(0) = g_0$ such that $g(t)$ satisfies Ω for all $t \in [0, T)$. Then for all $x \in M$ and $t \in [0, T)$, we have that*

$$\frac{\partial}{\partial t} F^{g(t)}(Rm_{g(t)}(x)) = dF^{g(t)}_{Rm_{g(t)}(x)}\left(\nabla_{\frac{\partial}{\partial t}} Rm_{g(t)}(x)^{\mathrm{v}}\right).$$

Proof. Throughout the proof we write $g_t := g(t)$. Let ∇ be the metric connection on the vector bundle $TM \to M \times \mathbb{R}$ introduced in Definition 1.2.2. Let further $x \in M$ and $t \mapsto p_t \in O^{g_t}_x$ be a parallel curve. Since g_t satisfies Ω, the function $F^{g_t} \circ Rm_{g_t} : M \to \mathbb{R}$ is defined for all $t \in [0, T)$ and we have that

$$\begin{aligned}\frac{\partial}{\partial t} F^{g_t}(Rm_{g_t}(x)) &= \frac{\partial}{\partial t} F(p_t^* Rm_{g_t}(x)) = dF_{p_t^* Rm_{g_t}(x)}\left(\frac{\partial}{\partial t}(p_t^* Rm_{g_t}(x))\right)\\ &\overset{1.2.6}{=} dF_{p_t^* Rm_{g_t}(x)}\Big(p_t^* \nabla_{\frac{\partial}{\partial t}} Rm_{g_t}(x)\Big) \overset{2.1.17}{=} dF^{g_t}_{Rm_{g_t}(x)}\Big(\nabla_{\frac{\partial}{\partial t}} Rm_{g_t}(x)^{\mathrm{v}}\Big).\end{aligned}$$

This finishes the proof. □

Similarly to the case of $O(n)$-invariant functions, to $O(n)$-equivariant maps on the space of algebraic curvature tensors, we can associate maps on the bundle of algebraic curvature tensors over a Riemannian manifold. These will enable us to formulate a version of Hamilton's maximum principle for algebraic curvature tensors and finally a generalization of this version to Bianchi-convex sets in Section 4.

Definition 2.1.20. A map $\Phi : \mathcal{A}_n \to \mathcal{A}_n$ is called *$O(n)$-equivariant*, if for each $Q \in O(n)$ and $R \in \mathcal{A}_n$, it satisfies that

$$\Phi(\rho(Q)R) = \rho(Q)\Phi(R),$$

where ρ denotes the representation of $O(n)$ on $\mathcal{A}_n$ as defined in the beginning of Section 2.1. To such a map Φ, we associate a map

$$\begin{aligned} \Phi^g : S^2_B(\Lambda^2 T^*M) &\to S^2_B(\Lambda^2 T^*M) \\ R &\mapsto (p^{-1})^*\Phi(p^*R), \end{aligned}$$

where $p \in O^g_{\pi(R)}$ can be chosen arbitrarily.

Remark 2.1.21. The map Φ^g is well defined. Namely, let $R \in S^2_B(\Lambda^2 T^*M)$ and $p, q \in O^g_{\pi(R)}$. Since $O(n)$ acts transitively on $O^g_{\pi(R)}$, there is a $Q \in O(n)$ with $q = p \circ Q$ and we have that

$$(q^{-1})^*\Phi(q^*R) = (p^{-1})^*\rho(Q)\Phi(\rho(Q^{-1})(p^*R)) = (p^{-1})^*\Phi(p^*R).$$

Moreover, we notice that Φ^g is fibre-preserving.

Remark 2.1.22. Properties of the map Φ such as being locally Lipschitz continuous or bounded transfer to Φ^g.

2.2 Properties of tangent cones

For closed subsets C of a metric space with smooth boundary ∂C, we can linearly approximate the submanifold ∂C at some point $x_0 \in \partial C$, namely by the tangent space $T_{x_0}\partial C$. If the boundary of C is not smooth this concept fails. However, tangent cones of such subsets generalize this notion to arbitrary regularity of the boundary. In this section, we show some properties of tangent cones, which, in the non-smooth case, involves approximating the boundaries of the subsets by certain submanifolds, so-called supporting submanifolds.

Throughout, let V be a metric space and $C \subseteq V$ be a closed subset.

Definition 2.2.1. Let $x_0 \in C$. The *tangent cone of C at x_0* is defined by

$$T_{x_0}C := \overline{\{\dot\gamma(0) \mid \gamma : (-\epsilon, \epsilon) \to V \text{ in } \mathcal{C}^1 \text{ with } \gamma(0) = x_0 \text{ and } \gamma(t) \in C \text{ for all } t \in [0, \epsilon)\}}.$$

From now on, let V be a Euclidean vector space with induced norm $\|\cdot\|$ and let $C \subseteq V$ be a closed subset.

Lemma 2.2.2. *Let $x_0 \in C$. Then we have that*

$$T_{x_0}C \subseteq K_{x_0}C := \{v \in V \mid \forall x \in V \text{ with } d(x, C) = \|x_0 - x\| : \langle x - x_0, v\rangle \le 0\}.$$

Proof. For x_0 being in the interior of C, we have that $K_{x_0}C = V$. Hence, in this case the statement is trivial. Now, let $x_0 \in \partial C$ and $v \in T_{x_0}C$ such that $v = \dot\gamma(0)$, where $\gamma : (-\epsilon, \epsilon) \to V$ is a $\mathcal{C}^1$-curve with $\gamma(0) = x_0$ and $\gamma(t) \in C$ for all $t \in [0, \epsilon)$. Let $x \in V$ with $d(x, C) = \|x_0 - x\|$. If we had that $\langle x - x_0, v\rangle > 0$, then for $t > 0$ small enough, we would find that

$$\|x - \gamma(t)\|^2 = \|x - \gamma(0) - t\dot\gamma(0) + o(t)\|^2 = \|x - x_0\|^2 - 2t \underbrace{\langle x - x_0, v\rangle}_{>0} + o(t) < \|x - x_0\|^2,$$

in contradiction to $\|x - x_0\| = d(x, C) \le \|x - \gamma(t)\|$ since $\gamma(t) \in C$. Thus, $\langle x - x_0, v\rangle \le 0$ and therefore $v \in K_{x_0}C$. The statement follows since $K_{x_0}C$ is closed. □

If the boundary ∂C of C is smooth and of codimension one, then for $x_0 \in \partial C$, we have that

$$T_{x_0}C = \{v \in V \mid \langle v, \mathbf{n}_{x_0} \rangle \leq 0\},$$

where $\mathbf{n}_{x_0}$ denotes the outward pointing unit normal on ∂C at x_0. If the boundary of C is of lower regularity, approximating it pointwise by certain submanifolds of V, a similar, however slightly weaker, result is true (see Lemma 2.2.4). To this end, we introduce the notion of a supporting submanifold.

Definition 2.2.3. Let $x_0 \in \partial C$. A *supporting submanifold of C in x_0* is a submanifold N of V of codimension one that touches C in x_0 such that C locally lies on one side of N, meaning that there is an open neighborhood $U \subseteq V$ of x_0 such that $U \setminus N$ consists of exactly two connected components U_1 and U_2, the closure of one of those (say U_1) containing $C \cap \overline{U}$.
Moreover, by r^N, we will always denote a *signed distance function from a supporting submanifold N of C in x_0*, i.e. a function

$$r^N : U \to \mathbb{R} : x \mapsto \left.\begin{cases} -d(x,N), & x \in U_1 \\ d(x,N), & x \in U_2 \end{cases}\right\} = \begin{cases} -d(x,N), & x \text{ lies on the side of } C \\ d(x,N), & \text{else.} \end{cases}$$

By possibly making U smaller, we can always arrange r^N to be smooth, which we will assume throughout.

Lemma 2.2.4. *Let $x_0 \in \partial C$ and N be a supporting submanifold of C in x_0. Then we have that*

$$T_{x_0}C \subseteq \{v \in V \mid \langle v, \mathbf{n}_{x_0} \rangle \leq 0\} =: H_N,$$

where $\mathbf{n}_{x_0}$ denotes the unit normal on N at x_0 pointing in the opposite direction of C. In particular, the tangent cone $T_{x_0}C$ lies on one side of the tangent space $T_{x_0}N$.

Proof. Let $\gamma : (-\epsilon, \epsilon) \to V$ be once differentiable with $\gamma(0) = x_0$ and $\gamma(t) \in C$ for all $t \in [0, \epsilon)$. Then $\dot{\gamma}(0) \in T_{x_0}C \subseteq V$ and we have that $r^N(\gamma(t)) \leq 0$ for all $t \in [0, \epsilon)$ and $r^N(\gamma(0)) = 0$. Hence,

$$0 \geq \left.\frac{d}{dt}\right|_{t=0} r^N(\gamma(t)) = dr^N_{\gamma(0)}(\dot{\gamma}(0)) = \langle \mathbf{n}_R, \dot{\gamma}(0) \rangle.$$

Passing to the closure yields that $T_{x_0}C \subseteq H_N$. Since the tangent space $T_{x_0}N$ is the boundary of the half space H_N, the tangent cone $T_{x_0}C$ lies on one side of $T_{x_0}N$. □

2.3 Invariance under an ordinary differential equation

We start with the following definition.

Definition 2.3.1. Let V be a vector space and $\Phi : V \to V$ a locally Lipschitz continuous map. A subset $C \subseteq V$ is *invariant under the ordinary differential equation*

$$f'(t) = \Phi(f(t)), \tag{2.2}$$

if for all solutions $f : [0, \delta] \to V$ of (2.2) with $f(0) \in C$, we have that $f(t) \in C$ for all $t \in [0, \delta]$.

Remark 2.3.2. Since Φ is locally Lipschitz continuous, given an initial value, the theorem of Picard-Lindelöf provides existence and uniqueness of such solutions.

In our applications, we will always have that $V = \mathcal{A}_n$ and that $\Phi : \mathcal{A}_n \to \mathcal{A}_n$ is the map corresponding to the self-map $\widehat{\Phi}$ on $\mathrm{SymEnd}(\Lambda^2\mathbb{R}^n)$ given by

$$\widehat{\Phi}(R) := R^2 + R^{\#}$$

for all $R \in \mathrm{SymEnd}(\Lambda^2\mathbb{R}^n)$ as in Remark 1.1.4. Since Φ is a quadratic function, it is locally Lipschitz continuous. Moreover in Remark 1.1.18, we have shown that Φ is $O(n)$-equivariant. From now on, we will freely identify Φ and $\widehat{\Phi}$.

The following proposition is somewhat more general than Hamilton's statement in [Ham86, Lemma 4.1] since the convexity assumption is not required.

Proposition 2.3.3. *Let V be a normed vector space, $\Phi : V \to V$ locally Lipschitz continuous and $C \subseteq V$ a closed set. Then C is invariant under the ordinary differential equation* (2.2) *if and only if for all $v \in \partial C$ we have that $\Phi(v) \in T_vC$.*

Proof. First, assume that C is invariant under (2.2). Let $v \in \partial C$ and $f : [0, \delta] \to V$ be a solution of (2.2) with $f(0) = v$. By the theorem of Picard-Lindelöf, f is defined on the interval $(-\epsilon, \epsilon)$ for an $\epsilon \in (0, \delta)$ as well. Moreover, $f'(0) = \Phi(v)$ and the invariance of C under (2.2) yields that $f(t) \in C$ for all $t \in [0, \delta]$. Hence, $\Phi(v) \in T_vC$.

In order to show the opposite direction, let $f : [0, \delta] \to V$ be a solution of (2.2) with $f(0) \in C$. Let $r : V \to [0, \infty)$ be the distance function from C, i.e. for $v \in V$ let

$$r(v) := d(v, C) = \inf_{w \in C} \|v - w\|.$$

Moreover, for $t \in [0, \delta]$ we set

$$s(t) := r(f(t))^2.$$

In general, the function $s : [0, \delta] \to [0, \infty)$ is not differentiable. Still we can define

$$s'(t) := \limsup_{h \searrow 0} \frac{s(t + h) - s(t)}{h} < \infty$$

for $t \in [0, \delta)$, since r is Lipschitz continuous and f is once continuously differentiable. Let r_0 be the maximum of s on $[0, \delta]$. Then

$$K := \bigcup_{t \in [0,\delta]} B_{\sqrt{r_0}}(f(t))$$

is compact. Since Φ is locally Lipschitz continuous, there exists a constant $L > 0$ such that $\Phi|_K$ is $\frac{L}{2}$-Lipschitz continuous. Our goal is to show that $s'(t) \leq Ls(t)$ for all $t \in [0, \delta)$. Because then for

$$g : [0, \delta] \to [0, \infty) : t \mapsto s(t)e^{-Lt},$$

we find that $g'(t) \leq e^{-Lt}(s'(t) - Ls(t)) \leq 0$ for all $t \in [0, \delta)$ and $g(0) = 0$. Therefore, $g(t) \leq 0$ for all $t \in [0, \delta]$, hence $s(t) \leq 0$ for all $t \in [0, \delta]$. Since s is non-negative, however, this means that $s \equiv 0$, which yields that $f(t) \in C$ for all $t \in [0, \delta]$.

Let now $t \in [0, \delta)$. Since C is closed, there is an $x_t \in C$ with $d(f(t), C) = \|f(t) - x_t\|$. By assumption, $\Phi(x_t) \in T_{x_t}C \subseteq K_{x_t}C$, thus $\langle f(t) - x_t, \Phi(x_t)\rangle \leq 0$. Consequently,

$$\begin{aligned} s'(t) &= \limsup_{h \searrow 0} \frac{d(f(t+h), C)^2 - d(f(t), C)^2}{h} \leq \limsup_{h \searrow 0} \frac{\|f(t+h) - x_t\|^2 - \|f(t) - x_t\|^2}{h} \\ &= \limsup_{h \searrow 0} \frac{\|f(t+h)\|^2 - \|f(t)\|^2 - 2\langle f(t+h) - f(t), x_t\rangle}{h} \\ &= \frac{d}{dt}\|f(t)\|^2 - 2\langle f'(t), x_t\rangle \overset{(2.2)}{=} 2\langle f'(t), f(t)\rangle - 2\langle \Phi(f(t)), x_t\rangle \overset{(2.2)}{=} 2\langle \Phi(f(t)), f(t) - x_t\rangle \\ &\leq 2\langle \Phi(f(t)), f(t) - x_t\rangle - 2\langle \Phi(x_t), f(t) - x_t\rangle = 2\langle \Phi(f(t)) - \Phi(x_t), f(t) - x_t\rangle \\ &\leq 2\|\Phi(f(t)) - \Phi(x_t)\|\|f(t) - x_t\| \leq L\|f(t) - x_t\|^2 = Ls(t), \end{aligned}$$

where the last inequality holds, since

$$\|x_t - f(t)\|^2 = s(t) \leq r_0,$$

thus $x_t \in B_{\sqrt{r_0}}(f(t)) \subseteq K$. □

For a smooth function F, the following lemma gives a connection between the invariance of the sublevel sets of F under the ordinary differential equation (2.2) and the angle between the gradient of F and the map Φ.

Lemma 2.3.4. *Let V be a vector space and $\Phi : V \to V$ a locally Lipschitz continuous map. Let further $C \subseteq V$ be an open set and $F : C \to \mathbb{R}$ a smooth function. If the sublevel sets of F are invariant under* (2.2)*, then for all $v \in C$ we have that*

$$dF_v(\Phi(v)) \leq 0. \tag{2.3}$$

Conversely, if F satisfies (2.3) *for all $v \in C$, then for all $a \in \operatorname{im}(F)$ and solutions $f : [0, \delta] \to C$ with $f(0) \in F^{-1}((-\infty, a])$, we have that $f(t) \in F^{-1}((-\infty, a])$ for all $t \in [0, \delta]$. In particular, if C is additionally invariant under* (2.2)*, then the sublevel sets of F are invariant under* (2.2)*.*

Proof. Under the assumption that the sublevel sets of F are invariant under (2.2), let $v \in C$ and $f : [0, \delta] \to V$ be a solution of (2.2) with $f(0) = v$. Let further $a := F(v)$. Then $v \in F^{-1}((-\infty, a])$. Hence, by assumption $f(t) \in F^{-1}((-\infty, a])$ for all $t \in [0, \delta]$. This means that the function $F \circ f : [0, \delta] \to \mathbb{R}$ is defined and satisfies $F(f(0)) = a$ and $F(f(t)) \leq a$ for all $t \in [0, \delta]$. Therefore, we find that

$$0 \geq \left.\frac{d}{dt}\right|_{t=0} F(f(t)) = dF_v(f'(0)) = dF_v(\Phi(f(0))) = dF_v(\Phi(v)).$$

Conversely, we assume that F satisfies (2.3) for all $v \in C$. Let $f : [0, \delta] \to C$ be a solution of (2.2) with $f(0) \in F^{-1}((-\infty, a])$ for some $a \in \operatorname{im}(F)$. Then

$$\frac{d}{dt}F(f(t)) = dF_{f(t)}(\Phi(f(t))) \leq 0$$

for all $t \in [0, \delta]$. Hence, $F \circ f$ is decreasing, which means that $F(f(t)) \leq F(f(0)) \leq a$, thus $f(t) \in F^{-1}((-\infty, a])$, for all $t \in [0, \delta]$. □

An example of a family of sets which are invariant under the ordinary differential equation

$$R'(t) = R(t)^2 + R(t)^{\#} \tag{2.4}$$

is the following. (Remember that the map # was defined in Definition 1.1.17.)

Example 2.3.5. For $a \in \mathbb{R}$, the sets

$$\{R \in \mathcal{A}_n \mid \mathrm{scal}(R) \geq a\}$$

are invariant under the ordinary differential equation (2.4).

Proof. Let us consider the function

$$F : \mathcal{A}_n \to \mathbb{R} : R \mapsto -\mathrm{scal}(R).$$

Using a formula for the components of $R^2 + R^\#$ (see for example [Ham86]), one can compute that

$$dF_R(R^2 + R^\#) = -\mathrm{scal}(R^2 + R^\#) = -2\|\mathrm{ric}(R)\|^2 \leq 0$$

for all $R \in \mathcal{A}_n$. Therefore, Lemma 2.3.4 implies that for $a \in \mathbb{R}$ the sublevel sets $F^{-1}((-\infty, -a]) = \{R \in \mathcal{A}_n \mid \mathrm{scal}(R) \geq a\}$ are invariant under (2.4). $\square$

Chapter 3

Bianchi-convex sets

In this chapter, we introduce Bianchi-convex sets of algebraic curvature tensors and show some first properties. Bianchi-convexity relaxes the notion of convexity in a certain sense inspired by the second Bianchi identity for the Riemannian curvature tensor of a Riemannian manifold. In dimension 3, we consider Bianchi-convex sets of algebraic curvature tensors whose eigenvalues lie in a sublevel set of some function $f : \mathbb{R}^3 \to \mathbb{R}$ and derive another characterization of Bianchi-convexity for those sets in terms of f. This enables us to find examples for Bianchi-convex sets which are not convex and thereby show that the introduced notion is a real generalization of convexity. Moreover, we show that certain subsets of these Bianchi-convex sets are invariant under the ordinary differential equation (2.4).

3.1 The definition and first properties

First of all, recall that for a submanifold N of codimension one of a Riemannian manifold (M, g) and a point $x \in M$, given the choice of a normal vector $\mathbf{n}_x$ at x, the *second fundamental form of N in x* is defined as the symmetric and bilinear map

$$\mathrm{II}_x^N : T_xN \times T_xN \to \mathbb{R} : (X, Y) \mapsto g_x(\nabla_X^g Y, \mathbf{n}_x),$$

where ∇^g is the Levi-Civita connection of M. For $X, Y \in T_xN$, one can show that

$$g_x(\nabla_X^g Y, \mathbf{n}_x) = -g_x(Y, \nabla_X^g \mathbf{n}). \tag{3.1}$$

Here, $\mathbf{n}$ denotes an extension of $\mathbf{n}_x$ to a neighborhood of x in M.

Remark 3.1.1. Let V be a vector space and $C \subseteq V$ a closed convex set, the boundary ∂C of which is smooth and of codimension one. If one chooses $\mathbf{n}_x$ to be the *outward* pointing unit normal on ∂C at x, then $\mathrm{II}_x^{\partial C}$ is negative semidefinite for all $x \in \partial C$.

Our generalization of the notion of convexity requires a second Bianchi identity for tuples of algebraic curvature tensors.

Definition 3.1.2. An n-tuple $(T_1, \dots, T_n) \in \mathcal{A}_n^n$ *satisfies the second Bianchi identity*, if for some orthonormal basis $(b_1, \dots, b_n)$ of $\mathbb{R}^n$, we have that

$$T_i(b_j \wedge b_k) + T_j(b_k \wedge b_i) + T_k(b_i \wedge b_j) = 0$$

for all $i, j, k \in \{1, \dots, n\}$. Moreover, we can replace $\mathcal{A}_n$ by $S_B^2(\Lambda^2 T_x^* M)$ and $\mathbb{R}^n$ by T_xM for some $x \in M$.

Remark 3.1.3. In dimension $n \leq 2$, the second Bianchi identity is always satisfied. In dimension $n = 3$, $(T_1, T_2, T_3) \in \mathcal{A}_3^3$ satisfies the second Bianchi identity if and only if

$$T_1(b_2 \wedge b_3) + T_2(b_3 \wedge b_1) + T_3(b_1 \wedge b_2) = 0$$

for some orthonormal basis (b_1, b_2, b_3) of $\mathbb{R}^3$.

Example 3.1.4. Let (M, g) be an n-dimensional Riemannian manifold, $x \in M$ and $(b_1, \dots, b_n)$ an orthonormal basis of T_xM. Then $(T_1, \dots, T_n)$, where $T_i := \nabla_{b_i} Rm_g$ for $i = 1, \dots, n$, satisfies the second Bianchi identity with respect to $(b_1, \dots, b_n)$.

For closed subsets of algebraic curvature tensors, we introduce a weaker form of convexity.

Definition 3.1.5. A closed subset $\Omega \subseteq \mathcal{A}_n$ is called *Bianchi-convex*, if for all $\epsilon > 0$ and $R \in \partial\Omega$ there is a supporting submanifold N of Ω in R such that for each $S \in N$ and $(T_1, \dots, T_n) \in (T_S N)^n$ satisfying the second Bianchi identity, we have that

$$\sum_{i=1}^{n} \mathbf{II}_S^N(T_i, T_i) \leq \epsilon \sum_{i=1}^{n} \|T_i\|^2. \tag{3.2}$$

Furthermore, we can replace $\mathcal{A}_n$ by $S_B^2(\Lambda^2 T_x^* M)$ for some $x \in M$.

Remark 3.1.6. If the boundary of Ω is smooth, then the supporting submanifolds in the Definition 3.1.5 can be chosen to be $\partial\Omega$ itself, and we obtain that Ω is Bianchi-convex, if and only if for all $R \in \partial\Omega$ and $(T_1, \dots, T_n) \in (T_R \partial\Omega)^n$ satisfying the second Bianchi identity, we have that

$$\sum_{i=1}^{n} \mathbf{II}_R^{\partial\Omega}(T_i, T_i) \leq 0.$$

Roughly speaking, in order for a set of algebraic curvature tensors to be Bianchi-convex, concavity is permitted in certain directions as long as these directions are compensated by the convex ones.

Remark 3.1.7. Closed convex subsets of $\mathcal{A}_n$ are Bianchi-convex.

Remark 3.1.8. Let (M, g) be an n-dimensional Riemannian manifold and $\Omega \subseteq \mathcal{A}_n$ an $O(n)$-invariant set. If Ω is Bianchi-convex, then Ω^g (as defined in Definition 2.1.2) is fibrewise Bianchi-convex.

Lemma 3.1.9. *The intersection of two Bianchi-convex sets is Bianchi-convex.*

Proof. Let $\Omega, \Omega' \subseteq \mathcal{A}_n$ be Bianchi-convex sets. Let $\epsilon > 0$ and $R \in \partial(\Omega \cap \Omega')$. If $R \in \partial\Omega \cap \Omega' \subseteq \partial\Omega$, then, since Ω is Bianchi-convex, there exists a supporting submanifold N of Ω in R such that (3.2) is true for each $S \in N$ and $(T_1, \dots, T_n) \in (T_S N)^n$ that satisfies the second Bianchi identity. Since N is also a supporting submanifold of $\Omega \cap \Omega'$ in R, we are done in this case. The case that $R \in \partial\Omega' \cap \Omega \subseteq \partial\Omega'$ works analogously. □

The following lemma is a crucial step towards generalizing Hamilton's maximum principle [Ham86, Theorem 4.3] to the Bianchi-convex setting (see Section 4.3).

Lemma 3.1.10. *Let (M, g) be an n-dimensional Riemannian manifold, $C \subseteq S_B^2(\Lambda^2 T^* M)$ be closed, invariant under parallel transport with respect to the Levi-Civita connection ∇^g and fibrewise Bianchi-convex. Let $R \in \Gamma(M, C)$ be a smooth section of C and $R(x) \in \partial C_x$ for some point $x \in M$. Moreover, assume that $(\nabla_{b_1} R, \dots, \nabla_{b_n} R)$ satisfies the second Bianchi identity with* *respect to some orthonormal basis* $(b_1, \dots, b_n)$ *of* T_xM. *Let further* $\epsilon > 0$ *and* N *be a supporting submanifold of* C_x *in* $R(x)$ *satisfying* (3.2). *Then we have that*

$$\left\langle (\Delta_g R)(x), \mathbf{n}_{R(x)} \right\rangle_g \leq \epsilon \sum_{i=1}^{n} \|\nabla^g_{b_i} R\|_g^2,$$

where $\mathbf{n}_{R(x)}$ *denotes the unit normal on* N *at* $R(x)$ *pointing in the opposite direction of* C_x.

Above, we identified the normal vector $\mathbf{n}_{R(x)}$, which was a priori a tangent vector to $S^2_B(\Lambda^2 T^*_x M)$ at $R(x)$, with a vector of $S^2_B(\Lambda^2 T^*_x M)$ itself.

Proof. For $i = 1, \dots, n$, let $\gamma_i : (-\delta, \delta) \to M$ be geodesics with $\gamma_i(0) = x$ and $\dot\gamma_i(0) = b_i$. Since C is invariant under parallel transport with respect to ∇^g, for $i = 1, \dots, n$ and $t \in (-\delta, \delta)$ we find that

$$h_i(t) := \left(P_{\gamma_i|_{[0,t]}} \right)^{-1} \underbrace{(R \circ \gamma_i)(t)}_{\in C_{\gamma_i(t)}} \in C_x,$$

where $P_{\gamma_i|_{[0,t]}}$ denotes the parallel transport along $\gamma_i|_{[0,t]}$ with respect to ∇^g. Hence, h_i is a curve in C_x with $h_i(0) = R(x) \in N$. Now, let r^N be a signed distance function from N. Then

$$\begin{aligned} &(r^N \circ h_i)(0) = r^N(R(x)) = 0, \\ \text{and} \quad &(r^N \circ h_i)(t) \leq 0 \quad \text{for all } t \in (-\delta, \delta). \end{aligned}$$

Therefore, 0 is a local maximum of $r^N \circ h_i$. On the one hand, this implies that

$$0 = \left.\frac{d}{dt}\right|_{t=0} (r^N \circ h_i)(t) = dr^N_{h_i(0)}(h_i'(0)) = \langle \mathrm{grad}_{h_i(0)} r^N, h_i'(0) \rangle = \left\langle \mathbf{n}_{h_i(0)}, h_i'(0) \right\rangle,$$

thus

$$\nabla^g_{b_i} R = \left.\frac{\nabla^g}{dt}\right|_{t=0} R(\gamma_i(t)) = h_i'(0) \in T_{h_i(0)} N.$$

On the other hand, we obtain that

$$0 \geq \left.\frac{d^2}{dt^2}\right|_{t=0} (r^N \circ h_i)(t) = \mathrm{Hess}_{h_i(0)} r^N \left(h_i'(0), h_i'(0) \right) + dr^N_{h_i(0)}(h_i''(0)).$$

Since

$$\begin{aligned} \mathrm{Hess}_{h_i(0)} r^N \left(h_i'(0), h_i'(0) \right) &= \left\langle \nabla^g_{h_i'(0)} \mathrm{grad}\, r^N, h_i'(0) \right\rangle \overset{(3.1)}{=} - \left\langle \mathrm{grad}_{R(x)} r^N, \nabla^g_{h_i'(0)} h_i'(0) \right\rangle \\ &= - \left\langle \mathbf{n}_{R(x)}, \nabla^g_{h_i'(0)} h_i'(0) \right\rangle = -\mathrm{II}^N_{R(x)}(h_i'(0), h_i'(0)), \end{aligned}$$

we find that

$$\mathrm{II}^N_{R(x)}(h_i'(0), h_i'(0)) \geq \left\langle \mathbf{n}_{R(x)}, h_i''(0) \right\rangle, \tag{3.3}$$

which leads to

$$\begin{aligned} \left\langle (\Delta_g R)(x), \mathbf{n}_{R(x)} \right\rangle &= \sum_{i=1}^{n} \left\langle \left.\frac{(\nabla^g)^2}{dt^2}\right|_{t=0} R(\gamma_i(t)), \mathbf{n}_{R(x)} \right\rangle = \sum_{i=1}^{n} \left\langle h_i''(0), \mathbf{n}_{R(x)} \right\rangle \\ &\overset{(3.3)}{\leq} \sum_{i=1}^{n} \mathrm{II}^N_{R(x)}(h_i'(0), h_i'(0)) = \sum_{i=1}^{n} \mathrm{II}^N_{R(x)}(\nabla^g_{b_i} R, \nabla^g_{b_i} R) \\ &\overset{(3.2)}{\leq} \epsilon \sum_{i=1}^{n} \|\nabla^g_{b_i} R\|^2, \end{aligned}$$

where the last inequality holds since $(\nabla^g_{b_1} R, \dots, \nabla^g_{b_n} R) \in (T_{R(x)} N)^n$ satisfies the second Bianchi identity (with respect to the orthonormal basis $(b_1, \dots, b_n)$) and C_x is Bianchi-convex. □

3.2 Bianchi-convex sets in dimension 3

In this section, we have a closer look at Bianchi-convex sets in dimension $n = 3$. First of all, the following lemma allows us to reformulate the second Bianchi identity in terms of symmetric 3×3-matrices and oriented orthonormal bases of $\mathbb{R}^3$, which makes explicit calculations much easier.

Lemma 3.2.1. *Let (B_1, B_2, B_3) be a positively oriented orthonormal basis of $\Lambda^2\mathbb{R}^3$. Then there exists an orthonormal basis (b_1, b_2, b_3) of $\mathbb{R}^3$ with $B_1 = b_2 \wedge b_3$, $B_2 = b_3 \wedge b_1$ and $B_3 = b_1 \wedge b_2$. Here, an orthonormal basis of $\Lambda^2\mathbb{R}^3$ is called positively oriented, if it lies in the same connected component as $(e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2)$, where (e_1, e_2, e_3) is the standard basis of $\mathbb{R}^3$.*

Proof. Let $\mathrm{ONB}(\mathbb{R}^3)$ and $\mathrm{ONB}(\Lambda^2\mathbb{R}^3)$ be the sets of orthonormal bases of $\mathbb{R}^3$ and $\Lambda^2\mathbb{R}^3$, respectively. These are compact submanifolds of $\mathbb{R}^{3\times3}$ and $(\Lambda^2\mathbb{R}^3)^3$, respectively. Let us consider the map

$$\Phi : \mathrm{ONB}(\mathbb{R}^3) \to \mathrm{ONB}(\Lambda^2\mathbb{R}^3) : (b_1, b_2, b_3) \mapsto (b_2 \wedge b_3, b_3 \wedge b_1, b_1 \wedge b_2)$$

between these manifolds and show that it is surjective. In order to do so, we first show that Φ is a local diffeomorphism. Let $b = (b_1, b_2, b_3) \in \mathrm{ONB}(\mathbb{R}^3)$. The differential of Φ at b is given by

$$\begin{aligned} d\Phi_b : T_b\mathrm{ONB}(\mathbb{R}^3) &\to T_{\Phi(b)}\mathrm{ONB}(\Lambda^2\mathbb{R}^3) \\ X &\mapsto (X_2 \wedge b_3 - b_2 \wedge X_3, X_3 \wedge b_1 - b_3 \wedge X_1, X_1 \wedge b_2 - b_1 \wedge X_2). \end{aligned}$$

Using that

$$T_b\mathrm{ONB}(\mathbb{R}^3) = \left\{ \left(\sum_{j=1}^3 c_{1j} b_j, \sum_{j=1}^3 c_{2j} b_j, \sum_{j=1}^3 c_{3j} b_j \right) \middle|\, c_{ii} = 0, c_{ij} = -c_{ji} \right\},$$

one shows that $d\Phi_b$ is injective. Thus, as a linear map between equal dimensional spaces, $d\Phi_b$ is bijective. The inverse function theorem yields that there is an open neighborhood U of b such that $\Phi|_U : U \to \Phi(U)$ is a diffeomorphism. Since $b \in \mathrm{ONB}(\mathbb{R}^3)$ was arbitrary, Φ is a local diffeomorphism and therefore an open map. Hence, $\Phi(\mathrm{ONB}(\mathbb{R}^3))$ is open in $\mathrm{ONB}(\Lambda^2\mathbb{R}^3)$, and compact since Φ is continuous and $\mathrm{ONB}(\mathbb{R}^3)$ is compact. In particular, $\mathrm{ONB}(\mathbb{R}^3)$ is open and closed. This results in Φ mapping surjectively onto each connected component of $\mathrm{ONB}(\Lambda^2\mathbb{R}^3)$, which means that if it hits a connected component at all, it hits every point of it. It remains to show which of the two connected components of $\mathrm{ONB}(\Lambda^2\mathbb{R}^3)$ are hit by Φ. Since the standard basis (e_1, e_2, e_3) of $\mathbb{R}^3$ and (e_2, e_1, e_3) are of the opposite orientation but both $\Phi(e_1, e_2, e_3)$ and $\Phi(e_2, e_1, e_3)$ have the same orientation, continuity of Φ yields that Φ is surjective onto the connected component of $\mathrm{ONB}(\Lambda^2\mathbb{R}^3)$ containing $(e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2)$, which we call the positively oriented orthonormal bases of $\Lambda^2\mathbb{R}^3$. □

Lemma 3.2.2. *Let $(T_1, T_2, T_3) \in \mathcal{A}_3^3$. Then the following are equivalent:*

(1) There is an orthonormal basis (b_1, b_2, b_3) of $\mathbb{R}^3$ such that

$$T_1(b_2 \wedge b_3) + T_2(b_3 \wedge b_1) + T_3(b_1 \wedge b_2) = 0.$$

(2) There is a positively oriented orthonormal basis (c_1, c_2, c_3) of $\mathbb{R}^3$, meaning that the matrix $(c_1 c_2 c_3) \in SO(3)$, such that

$$\sum_{i=1}^3 T_i^M c_i = 0,$$

where T_i^M denotes the matrix representation of T_i with respect to $(e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2)$, where (e_1, e_2, e_3) is the standard basis of $\mathbb{R}^3$.

Proof. With respect to the standard scalar product on $\mathbb{R}^3$ and the scalar product on $\Lambda^2\mathbb{R}^3$ introduced in Section 1.1 (using the identification of $\Lambda^2\mathbb{R}^3$ with $\mathfrak{so}(3)$), we consider the following isometry

$$\psi : \mathbb{R}^3 \to \Lambda^2\mathbb{R}^3 : (x_1, x_2, x_3)^t \mapsto x_1 e_2 \wedge e_3 + x_2 e_3 \wedge e_1 + x_3 e_1 \wedge e_2.$$

Now, $T_i^M = \psi^{-1} \circ T_i \circ \psi$. In order to show the implication "$\Rightarrow$", we assume that there is an orthonormal basis (b_1, b_2, b_3) of $\mathbb{R}^3$ such that

$$T_1(b_2 \wedge b_3) + T_2(b_3 \wedge b_1) + T_3(b_1 \wedge b_2) = 0. \tag{3.4}$$

Let

$$C := \begin{pmatrix} c_1^1 & c_1^2 & c_1^3 \\ c_2^1 & c_2^2 & c_2^3 \\ c_3^1 & c_3^2 & c_3^3 \end{pmatrix}$$

be the change of basis matrix between the orthonormal bases $E := (e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2)$ and $B := (b_2 \wedge b_3, b_3 \wedge b_1, b_1 \wedge b_2)$, that is

$$\begin{aligned} b_2 \wedge b_3 &= c_1^1 e_2 \wedge e_3 + c_1^2 e_3 \wedge e_1 + c_1^3 e_1 \wedge e_2, \\ b_3 \wedge b_1 &= c_2^1 e_2 \wedge e_3 + c_2^2 e_3 \wedge e_1 + c_2^3 e_1 \wedge e_2 \\ \text{and} \quad b_1 \wedge b_2 &= c_3^1 e_2 \wedge e_3 + c_3^2 e_3 \wedge e_1 + c_3^3 e_1 \wedge e_2, \end{aligned} \tag{3.5}$$

and set

$$c_i := C^{-1} e_i = \begin{pmatrix} c_i^1 \\ c_i^2 \\ c_i^3 \end{pmatrix} = \sum_{j=1}^{3} c_i^j e_j \tag{3.6}$$

for $i = 1, 2, 3$. Since both E and B are positively oriented, we find that $C \in SO(3)$. Hence, (c_1, c_2, c_3) is a positively oriented orthonormal basis of $\mathbb{R}^3$. It follows that

$$\begin{aligned} \sum_{i=1}^{3} T_i^M c_i &= \sum_{i=1}^{3} \psi^{-1}(T_i(\psi(c_i))) \\ &\overset{(3.6)}{=} \psi^{-1}\left(T_1\left(\sum_{i=1}^{3} c_1^i \psi(e_i)\right) + T_2\left(\sum_{i=1}^{3} c_2^i \psi(e_i)\right) + T_3\left(\sum_{i=1}^{3} c_3^i \psi(e_i)\right)\right) \\ &\overset{(3.5)}{=} \psi^{-1}\left(T_1(b_2 \wedge b_3) + T_2(b_3 \wedge b_1) + T_3(b_1 \wedge b_2)\right) \\ &\overset{(3.4)}{=} 0. \end{aligned}$$

Conversely, let (c_1, c_2, c_3) be a positively oriented orthonormal basis of $\mathbb{R}^3$ with

$$\sum_{i=1}^{3} T_i^M c_i = 0. \tag{3.7}$$

Then $C := (c_1 c_2 c_3)^t \in SO(3)$, hence (B_1, B_2, B_3), where

$$\begin{aligned} B_1 &:= c_1^1 e_2 \wedge e_3 + c_1^2 e_3 \wedge e_1 + c_1^3 e_1 \wedge e_2, \\ B_2 &:= c_2^1 e_2 \wedge e_3 + c_2^2 e_3 \wedge e_1 + c_2^3 e_1 \wedge e_2 \\ \text{and} \quad B_3 &:= c_3^1 e_2 \wedge e_3 + c_3^2 e_3 \wedge e_1 + c_3^3 e_1 \wedge e_2, \end{aligned}$$

is a positively oriented orthonormal basis of $\Lambda^2\mathbb{R}^3$. Therefore, by Lemma 3.2.1 there is an orthonormal basis (b_1, b_2, b_3) of $\mathbb{R}^3$ such that $B_1 = b_2 \wedge b_3$, $B_2 = b_3 \wedge b_1$ and $B_3 = b_1 \wedge b_2$. As above, this yields that

$$T_1(b_2 \wedge b_3) + T_2(b_3 \wedge b_1) + T_3(b_1 \wedge b_2) = \psi\left(\sum_{i=1}^{3} T_i^M c_i\right) \overset{(3.7)}{=} 0. \qquad \square$$

From now on, we will always identify $\mathcal{A}_3$ with the space of symmetric (3×3)-matrices (via mapping each algebraic curvature tensor to its matrix representation with respect to $(e_2\wedge e_3, e_3\wedge e_1, e_1\wedge e_2)$, where (e_1, e_2, e_3) is the standard basis of $\mathbb{R}^3$).

Remark 3.2.3. The triple $(T, 0, 0)$ (respectively $(0, T, 0)$ and $(0, 0, T)$) satisfies the second Bianchi identity if and only if T is singular, that is the kernel of T is non-trivial. In dimensions $n \geq 4$, however, $T \in \mathcal{A}_n$ being singular in general does not imply that the n-tuple $(T, 0, \ldots, 0)$ satisfies the second Bianchi identity, since there is no splitting as in Lemma 3.2.1 anymore.

Corollary 3.2.4. *Let $\Omega \subseteq \mathcal{A}_3$ be a Bianchi-convex set with smooth boundary $\partial\Omega$ of codimension one and $R \in \partial\Omega$. Then the second fundamental form $\mathbf{II}_R^{\partial\Omega}$ is negative semidefinite on the singular part of $T_R\partial\Omega$, i.e. for each $T \in T_R\partial\Omega$ with $\ker(T) \neq \{0\}$, we have that*

$$\mathbf{II}_R^{\partial\Omega}(T, T) \leq 0.$$

Next, we show that in the definition of Bianchi-convexity, it is equivalent to require that the triples satisfy the second Bianchi identity with respect to some fixed positively oriented orthonormal basis of $\mathbb{R}^3$, say the standard basis.

Lemma 3.2.5. *A set $\Omega \subseteq \mathcal{A}_3$ is Bianchi-convex if and only if for all $R \in \partial\Omega$ and $(T_1, T_2, T_3) \in (T_R\partial\Omega)^3$ with*

$$\sum_{i=1}^{3} T_i e_i = 0,$$

we have that

$$\sum_{i=1}^{3} \mathbf{II}_R^{\partial\Omega}(T_i, T_i) \leq 0. \tag{3.8}$$

Here, (e_1, e_2, e_3) denotes the standard basis of $\mathbb{R}^3$.

Remark 3.2.6. It is, however, *not* true that a triple $(T_1, T_2, T_3) \in \mathcal{A}_3$ satisfies the second Bianchi identity if and only if

$$\sum_{i=1}^{3} T_i e_i = 0, \tag{3.9}$$

where (e_1, e_2, e_3) denotes the standard basis of $\mathbb{R}^3$. Of course, only the obvious of both implications is true, namely (3.9) implies that (T_1, T_2, T_3) satisfies the second Bianchi identity.

Proof. That the implication "$\Rightarrow$" holds true is obvious. To show the other direction, let $R \in \partial\Omega$ and take $(T_1, T_2, T_3) \in (T_R\partial\Omega)^3$ satisfying

$$\sum_{i=1}^{3} T_i b_i = 0$$

for some positively oriented orthonormal basis (b_1, b_2, b_3) of $\mathbb{R}^3$. There exists some $Q \in \mathrm{SO}(3)$ with $(b_1, b_2, b_3) = (Qe_1, Qe_2, Qe_3)$ and we have that $b_i = Qe_i = \sum_{j=1}^3 Q_{ji} e_j$. For $i = 1, 2, 3$, we set

$$\widetilde{T}_i := \sum_{j=1}^3 Q_{ij} T_j.$$

Then we find that

$$\sum_{i=1}^3 \widetilde{T}_i e_i = \sum_{i,j=1}^3 Q_{ij} T_j e_i = \sum_{j=1}^3 T_j \underbrace{\sum_{i=1}^3 Q_{ij} e_i}_{=b_j} = \sum_{j=1}^3 T_j b_j = 0.$$

Hence, by assumption, we obtain that

$$\begin{aligned} 0 &\geq \sum_{i=1}^3 \mathbf{II}_R^{\partial\Omega}(\widetilde{T}_i, \widetilde{T}_i) = \sum_{i=1}^3 \mathbf{II}_R^{\partial\Omega}\left(\sum_{j=1}^3 Q_{ij} T_j, \sum_{k=1}^3 Q_{ik} T_k\right) = \sum_{i,j,k=1}^3 Q_{ij} Q_{ik} \mathbf{II}_R^{\partial\Omega}(T_j, T_k) \\ &= \sum_{j,k=1}^3 \Bigg(\underbrace{\sum_{i=1}^3 Q_{ji}^t Q_{ik}}_{=(Q^t Q)_{jk} = \delta_{jk}}\Bigg) \mathbf{II}_R^{\partial\Omega}(T_j, T_k) = \sum_{j=1}^3 \mathbf{II}_R^{\partial\Omega}(T_j, T_j). \end{aligned}$$

This proves that Ω is Bianchi-convex. $\square$

Remark 3.2.7. The formulation of Lemma 3.2.5 is convenient since for $R \in \partial\Omega$ the set

$$(T_R\partial\Omega)^3_{(2\mathrm{BI})} := \left\{ (T_1, T_2, T_3) \in (T_R\partial\Omega)^3 \;\middle|\; \sum_{i=1}^3 T_i e_i = 0 \right\} \tag{3.10}$$

forms a vector space.

3.2.1 Ansatz

Although the reformulation of Bianchi-convexity of sets given in Lemma 3.2.5 is somewhat better to handle, it is still unclear whether there are Bianchi-convex sets which are *not* convex, so whether Bianchi-convexity is at all a reasonable notion. To this end, we develop another characterization of Bianchi-convex sets which have the following concrete form.

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function with $f^{-1}(0) \neq \emptyset$ such that 0 is a regular value of f, i.e. $df_x \neq 0$ for all $x \in f^{-1}(0)$. Let further f be symmetric, meaning that $f(x_1, x_2, x_3) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$ for all $x \in \mathbb{R}^3$ and all permutations $\sigma \in S_3$. We set

$$\Omega_f := \{R \in \mathcal{A}_3 \mid f(\lambda(R)) \leq 0\}.$$

Here,

$$\lambda : \mathcal{A}_3 \to \mathbb{R}^3 : R \mapsto (\lambda_1(R), \lambda_2(R), \lambda_3(R)),$$

where $\lambda_1(R) \leq \lambda_2(R) \leq \lambda_3(R)$ are the eigenvalues of R.

Example 3.2.8. The following functions satisfy the properties mentioned above:

$$\begin{aligned} & x \mapsto x_1 + x_2 + x_3 \\ \text{or} \quad & x \mapsto x_1^2 + x_2^2 + x_3^2 - a(x_1 + x_2 + x_3)^2 - c \end{aligned}$$

for $a \in \mathbb{R}$ and $c > 0$.

Lemma 3.2.9. *The boundary*

$$\partial\Omega_f = \{R \in \mathcal{A}_3 \mid f(\lambda(R)) = 0\} = (f \circ \lambda)^{-1}(0)$$

of Ω_f is smooth and 5-dimensional, that is of codimension one.

Proof. Since f is smooth and symmetric, by Schwarz's theorem [Sch75], f can be written as a smooth function in the elementary symmetric polynomials $\sigma_1, \sigma_2, \sigma_3$ on $\mathbb{R}^3$, i.e. there is a smooth function $g : \mathbb{R}^3 \to \mathbb{R}$ such that $f = g(\sigma_1, \sigma_2, \sigma_3)$. Moreover, it is well known that for $i = 1, 2, 3$ the function $A \mapsto \sigma_i(\lambda(A))$ is smooth on the space of matrices. This yields that

$$\begin{aligned} f \circ \lambda : \mathcal{A}_3 &\to \mathbb{R} \\ A &\mapsto g(\sigma_1(\lambda(A)), \sigma_2(\lambda(A)), \sigma_3(\lambda(A))) \end{aligned}$$

as composition of smooth functions is smooth as well.
Next, we show that 0 is a regular value of $f\circ\lambda$. To this end, let $R \in (f\circ\lambda)^{-1}(0)$ and write $\lambda := \lambda(R)$. Since R is symmetric, there is some $Q \in O(3)$ such that $R = QDQ^t$, where $D := \mathrm{diag}(\lambda)$. Since $f(\lambda) = 0$ and 0 is a regular value of f, there is an $x \in T_\lambda\mathbb{R}^3$ such that $df_\lambda(x) \neq 0$. Set $D(s) := \mathrm{diag}(\lambda + sx)$ and $c(s) := QD(s)Q^t$. Due to the symmetry of f, we obtain that

$$(f \circ \lambda)(c(s)) = f(\lambda(D(s))) = f(\lambda + sx).$$

Differentiating this equation at $s = 0$ yields that

$$d(f \circ \lambda)_R(\dot{c}(0)) = df_\lambda(x) \neq 0.$$

Hence, $d(f \circ \lambda)_R$ is surjective. This shows that 0 is a regular value of $f \circ \lambda$. Now, the submersion theorem [Kli95, Theorem 1.3.3] implies that $(f \circ \lambda)^{-1}(0)$ is a submanifold of $\mathcal{A}_3$ of codimension one. □

Lemma 3.2.10. *Let $R \in \partial\Omega_f$. Then there exists an $\epsilon > 0$ and a smooth curve $c : (-\epsilon, \epsilon) \to \partial\Omega_f$ with $c(0) = R$ and $c(s)$ having pairwise distinct eigenvalues for all $s \neq 0$.*

Proof. Let $R \in \partial\Omega_f$ with eigenvalues λ_1, λ_2 and λ_3. Set $\lambda := (\lambda_1, \lambda_2, \lambda_3)$.

Step 1: We show that there exists a $v \in T_\lambda f^{-1}(0)$ such that for $\epsilon > 0$ small enough $\gamma_v(s) := \lambda + sv$ has pairwise distinct components for $s \in (-\epsilon, \epsilon)$ with $s \neq 0$.

- If λ_1, λ_2 and λ_3 are pairwise distinct, this is clearly true for each $v \in \mathbb{R}^3$, in particular for each $v \in T_\lambda f^{-1}(0)$, and $\epsilon > 0$ small enough.
- If two of the eigenvalues of R coincide, say $\lambda_1 = \lambda_2$, then the symmetry of f gives that

$$\partial_1 f(\lambda_1, \lambda_1, \lambda_3) = \frac{d}{ds}\Big|_{s=0} f(\lambda_1 + s, \lambda_1, \lambda_3) = \frac{d}{ds}\Big|_{s=0} f(\lambda_1, \lambda_1 + s, \lambda_3) = \partial_2 f(\lambda_1, \lambda_1, \lambda_3).$$

  Therefore, $v := (1, -1, 0)^t \in \mathrm{grad}_\lambda f^\perp = T_\lambda f^{-1}(0)$ and it is easy to see that $\gamma_v(s)$ has pairwise distinct components for $s \in (-\epsilon, \epsilon)$ but $s \neq 0$ for $\epsilon > 0$ small enough.

Step 2: Let $v \in T_\lambda f^{-1}(0)$ as in Step 1 and $\tilde{\gamma} : (-\epsilon, \epsilon) \to f^{-1}(0)$ be any smooth curve with $\tilde{\gamma}(0) = \lambda$ and $\dot{\tilde{\gamma}}(0) = v$. Since γ_v and $\tilde{\gamma}$ correspond up to first order, $\tilde{\gamma}(s)$ has pairwise distinct components for each $s \neq 0$ and $\epsilon > 0$ small enough as well.

Step 3: Let $Q \in O(3)$ such that $R = Q\mathrm{diag}(\lambda)Q^t$. Then, $c := Q(\mathrm{diag} \circ \tilde{\gamma})Q^t$ is a smooth curve in $\partial\Omega_f$ with $c(s)$ having pairwise distinct eigenvalues for each $s \in (-\epsilon, \epsilon)$ with $s \neq 0$ for $\epsilon > 0$ small enough. □

Lemma 3.2.11. *$(T\partial\Omega_f)^3_{(2BI)}$ is a smooth vector bundle over $\partial\Omega_f$ of rank 12.*

Proof. For $i = 1, 2, 3$, $R \in \mathcal{A}_3$ and $(T_1, T_2, T_3) \in (T_R\mathcal{A}_3)^3$, we define

$$\widetilde{s}_i(R)(T_1, T_2, T_3) := \left\langle e_i, \sum_{j=1}^{3} T_j e_j \right\rangle,$$

where (e_1, e_2, e_3) denotes the standard basis of $\mathbb{R}^3$. (Here, we canonically identified $\mathcal{A}_3$ with $T_R\mathcal{A}_3$.) Then $\widetilde{s}_i \in \Gamma(\mathcal{A}_3, ((T\mathcal{A}_3)^3)^*)$, $i = 1, 2, 3$, are smooth sections of $((T\mathcal{A}_3)^3)^*$. Since $\partial\Omega_f$ is smooth, their restrictions $s_i \in \Gamma(\partial\Omega_f, ((T\partial\Omega_f)^3)^*)$, $i = 1, 2, 3$, are smooth sections of $((T\partial\Omega_f)^3)^*$. We claim that these are linearly independent in each point $R \in \partial\Omega_f$. In order to see this, let $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\alpha s_1(R) + \beta s_2(R) + \gamma s_3(R) = 0.$$

Then for all $(T_1, T_2, T_3) \in (T_R\partial\Omega_f)^3$, we have that

$$\alpha \left\langle e_1, \sum_{j=1}^{3} T_j e_j \right\rangle + \beta \left\langle e_2, \sum_{j=1}^{3} T_j e_j \right\rangle + \gamma \left\langle e_3, \sum_{j=1}^{3} T_j e_j \right\rangle = 0. \tag{3.11}$$

Since the system of linear equations $\sum_{j=1}^{3} T_j e_j = 0$ has full rank 3 as a system in the T_i, (3.11) leads to $\alpha = \beta = \gamma = 0$. Hence, $\{s_1(R), s_2(R), s_3(R)\}$ is linearly independent. Moreover, we notice that

$$\ker(s_1(R)) \cap \ker(s_2(R)) \cap \ker(s_3(R)) \overset{(3.10)}{=} (T_R\partial\Omega_f)^3_{(2\mathrm{BI})}.$$

Proposition A.0.2 implies that

$$\ker(s_1) \cap \ker(s_2) \cap \ker(s_3) \overset{(3.10)}{=} (T\partial\Omega_f)^3_{(2\mathrm{BI})}$$

is a smooth subbundle of $(T\partial\Omega_f)^3$ of rank 12 since $(T\partial\Omega_f)^3$ has rank 15. □

The following considerations aim at explicitly computing the second fundamental form of $\partial\Omega_f$.

Lemma 3.2.12. *For each $D := \mathrm{diag}(\lambda) \in \partial\Omega_f$, i.e. $\lambda \in f^{-1}(0)$, with $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, we have that*

$$T_D\partial\Omega_f = [\mathfrak{so}(3), D] \oplus \mathrm{diag}(\ker(df_\lambda)). \tag{3.12}$$

Proof. Let $D := \mathrm{diag}(\lambda) \in \partial\Omega_f$ with λ_1, λ_2 and λ_3 being pairwise distinct. In order to prove the inclusion "$\supseteq$", let $S \in \mathfrak{so}(3) = T_{\mathbb{1}_3}O(3)$ and $x \in \ker df_\lambda = T_\lambda f^{-1}(0)$. Let further $\epsilon > 0$, $Q : (-\epsilon, \epsilon) \to O(3)$ be a curve with $Q(0) = \mathbb{1}_3$ and $\dot{Q}(0) = S$ and $c : (-\epsilon, \epsilon) \to f^{-1}(0)$ be a curve with $c(0) = \lambda$ and $\dot{c}(0) = x$. Then for the curve

$$\gamma : (-\epsilon, \epsilon) \to \partial\Omega_f : s \mapsto Q(s)\mathrm{diag}(c(s))Q(s)^t,$$

we have that

$$[S, D] + \mathrm{diag}(x) = \dot{\gamma}(0) \in T_D\partial\Omega_f.$$

This yields that

$$[\mathfrak{so}(3), D] + \mathrm{diag}(\ker(df_\lambda)) \subseteq T_D\partial\Omega_f. \tag{3.13}$$

Since (E_1, E_2, E_3) (as defined in Example 1.1.20) is a basis of $\mathfrak{so}(3)$, the commutator

$$[\mathfrak{so}(3), D] = \operatorname{span}\Big\{[E_i, D] \mid i = 1, 2, 3\Big\},$$

where

$$\begin{aligned} [E_1, D] &= (\lambda_3 - \lambda_2)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ [E_2, D] &= (\lambda_3 - \lambda_1)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \text{and} \quad [E_3, D] &= (\lambda_2 - \lambda_1)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.14}$$

Since λ_1, λ_2 and λ_3 are pairwise distinct, this shows that $[\mathfrak{so}(3), D]$ is the space of all symmetric matrices with vanishing diagonal, hence 3-dimensional and orthogonal to the space of diagonal matrices. Therefore the sum in (3.13) is direct. Since 0 is a regular value of f, the submersion theorem implies that $f^{-1}(0)$ is 2-dimensional. Consequently,

$$[\mathfrak{so}(3), D] \oplus \operatorname{diag}(\ker(df_\lambda)) = [\mathfrak{so}(3), D] \oplus \operatorname{diag}(T_\lambda f^{-1}(0))$$

is a 5-dimensional space. By Lemma 3.2.9, we have equality in (3.13). □

Remark 3.2.13. Since for each $Q \in O(3)$ the map

$$\varphi_Q : \partial\Omega_f \to \partial\Omega_f : R \mapsto QRQ^t$$

is an isometric diffeomorphism, for all $R \in \partial\Omega_f$

$$d\varphi_Q|_R : T_R\partial\Omega_f \to T_{QRQ^t}\partial\Omega_f : X \mapsto X \mapsto QXQ^t$$

is an isomorphism. Hence, we have that

$$T_{QRQ^t}\partial\Omega_f = d\varphi_Q|_R(T_R\partial\Omega_f) = QT_R\partial\Omega_f Q^t.$$

Therefore, Lemma 3.2.12 yields an explicit formula for the tangent space of $\partial\Omega_f$ at an arbitrary point $R \in \partial\Omega_f$ with pairwise distinct eigenvalues.

Lemma 3.2.14. *For each $D := \operatorname{diag}(\lambda) \in \partial\Omega_f$, we have that the outward pointing unit normal on $\partial\Omega_f$ at D is given by*

$$\mathbf{n}_D = \frac{1}{\|\operatorname{grad}_\lambda f\|}\operatorname{diag}(\operatorname{grad}_\lambda f). \tag{3.15}$$

Proof. First of all, let $D := \operatorname{diag}(\lambda) \in \partial\Omega_f$ with λ_1, λ_2 and λ_3 being pairwise distinct. Using Lemma 3.2.12, we observe that

$$T_D\mathcal{A}_3 = T_D\partial\Omega_f \oplus N_D\partial\Omega_f = [\mathfrak{so}(3), D] \oplus \operatorname{diag}(\ker(df_\lambda)) \oplus N_D\partial\Omega_f.$$

Since $[\mathfrak{so}(3), D]$ is the space of all symmetric matrices, the diagonal of which is zero, we obtain that

$$\operatorname{diag}(\ker(df_\lambda)) \oplus N_D\partial\Omega_f = \operatorname{diag}(\mathbb{R}^3).$$

It follows that the normal space of $\partial\Omega_f$ at D is given by

$$N_D\partial\Omega_f = \mathbb{R}\operatorname{diag}(\operatorname{grad}_\lambda f).$$

Recalling that $\|\operatorname{diag}(\operatorname{grad}_\lambda f)\| = \|\operatorname{grad}_\lambda f\|$ yields that $\mathbf{n}_D$ is given by (3.15) in this case. Here, the norm on the left-hand side is the one induced by the scalar product on $\mathcal{A}_3$ introduced in Remark 1.1.7, which is given by

$$\langle A, B\rangle = \sum_{i,j=1}^{3} A_{ij}B_{ij}$$

for symmetric 3×3-matrices A and B. Since Lemma 3.2.10 implies that the set

$$\{\lambda \in f^{-1}(0) \mid \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1\}$$

is dense in $f^{-1}(0)$ and the right-hand side of (3.15) depends continuously on λ, we obtain that (3.15) is also true for general $\lambda \in f^{-1}(0)$. □

Remark 3.2.15. For $Q \in O(3)$, let φ_Q be as in Remark 3.2.13 and $R \in \partial\Omega_f$. For all $X \in T_{QRQ^t}\partial\Omega_f$, we have that

$$0 = \langle \mathbf{n}_R, d\varphi_Q|_R{}^{-1}(X)\rangle = \langle d\varphi_Q|_R(\mathbf{n}_R), X\rangle.$$

Thus, $d\varphi_Q|_R(\mathbf{n}_R) \in N_{QRQ^t}\partial\Omega_f$. Since

$$\|d\varphi_Q|_R(\mathbf{n}_R)\| = \|\mathbf{n}_R\| = 1$$

and $\mathbb{1}_3 \in O(3)$, it follows that

$$\mathbf{n}_{QRQ^t} = d\varphi_Q|_R(\mathbf{n}_R) = Q\mathbf{n}_R Q^t.$$

Combining this with Lemma 3.2.14 gives an explicit formula for the unit normal on $\partial\Omega_f$ at every point $R \in \partial\Omega_f$.

Remark 3.2.16. For all $Q \in O(3)$, $R \in \partial\Omega_f$ and $T_1, T_2 \in T_R\partial\Omega_f$, we have that

$$\begin{aligned}\mathrm{II}_R^{\partial\Omega_f}(T_1, T_2) &= -\langle T_2, \nabla_{T_1}\mathbf{n}\rangle = -\langle d\varphi_Q|_R(T_2), d\varphi_Q|_R(\nabla_{T_1}\mathbf{n})\rangle \\ &= -\left\langle QT_2Q^t, \nabla_{QT_1Q^t}\mathbf{n}\right\rangle = \mathrm{II}_{QRQ^t}^{\partial\Omega_f}(QT_1Q^t, QT_2Q^t).\end{aligned}$$

Here, φ_Q denotes the isometric diffeomorphism introduced in Remark 3.2.13.

Lemma 3.2.17. *For each $D := \operatorname{diag}(\lambda) \in \partial\Omega_f$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, the second fundamental form $\mathrm{II}_D^{\partial\Omega_f}$ is diagonal with respect to the decomposition $T_D\partial\Omega_f = [\mathfrak{so}(3), D] \oplus \operatorname{diag}(\ker(df_\lambda))$, i.e.*

$$\mathrm{II}_D^{\partial\Omega_f} \mathrel{\widehat{=}} \begin{pmatrix} \mathrm{II}_D^{\partial\Omega_f}|_{[\mathfrak{so}(3),D]} & 0 \\ 0 & \mathrm{II}_D^{\partial\Omega_f}|_{\operatorname{diag}(\ker(df_\lambda))} \end{pmatrix}.$$

Proof. Let $D := \operatorname{diag}(\lambda) \in \partial\Omega_f$ with λ_i, $i = 1, 2, 3$, being pairwise distinct. Let further $[S, D] \in [\mathfrak{so}(3), D]$ and $Q : (-\epsilon, \epsilon) \to O(3)$ be a curve with $Q(0) = \mathbb{1}_3$ and $\dot{Q}(0) = S$. For $s \in (-\epsilon, \epsilon)$, we set $c(s) := Q(s)DQ(s)^t$. Since

$$f(\lambda(c(s))) = f(\lambda) = 0$$

for all $s \in (-\epsilon, \epsilon)$, c is a curve in $\partial\Omega_f$ with $c(0) = D$ and $\dot{c}(0) = [S, D]$. Now, by Remark 3.2.15, we find that

$$\begin{aligned} \nabla_{[S,D]}\mathbf{n} &= \frac{d}{ds}\Big|_{s=0} (\mathbf{n} \circ c)(s) = \frac{d}{ds}\Big|_{s=0} Q(s)\mathbf{n}_D Q(s)^t \\ &= S\mathbf{n}_D + \mathbf{n}_D S^t = S\mathbf{n}_D - \mathbf{n}_D S = [S, \mathbf{n}_D]. \end{aligned} \tag{3.16}$$

Since $\mathbf{n}_D$ is diagonal, $[S, \mathbf{n}_D]$ has vanishing diagonal by (3.14). As a consequence,

$$\mathbf{II}_D^{\partial\Omega_f}([S, D], X) = -\langle X, \nabla_{[S,D]}\mathbf{n}\rangle = -\langle X, [S, \mathbf{n}_D]\rangle = 0$$

for $X \in \operatorname{diag}(\ker(df_\lambda))$, where again $\langle\cdot,\cdot\rangle$ denotes the scalar product on $\mathcal{A}_3$ introduced in Remark 1.1.7. □

In the next lemma, we give explicit formulas for the second fundamental form of $\partial\Omega_f$ on the two factors of the above decomposition.

Lemma 3.2.18. *Let $D := \operatorname{diag}(\lambda) \in \partial\Omega_f$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. Then for all $X = \operatorname{diag}(x), Y = \operatorname{diag}(y)$ with $x, y \in \ker(df_\lambda)$, we have that*

$$\mathbf{II}_D^{\partial\Omega_f}(X, Y) = -\frac{1}{\|\operatorname{grad}_\lambda f\|}\operatorname{Hess}_\lambda f(x, y).$$

For all $S_1, S_2 \in \mathfrak{so}(3)$, we have that

$$\mathbf{II}_D^{\partial\Omega_f}([S_1, D], [S_2, D]) = -\langle [S_2, D], [S_1, \mathbf{n}_D]\rangle.$$

Proof. Let $X = \operatorname{diag}(x), Y = \operatorname{diag}(y) \in \operatorname{diag}(\ker(df_\lambda))$. Let further $g : (-\epsilon, \epsilon) \to f^{-1}(0)$, $\epsilon > 0$, be a curve with $g(0) = \lambda$ and $\dot{g}(0) = x$. Now, set $c(s) := \operatorname{diag}(g(s))$ for every $s \in (-\epsilon, \epsilon)$. Then c is a curve in $\partial\Omega_f$ with $c(0) = D$ and $\dot{c}(0) = X$ and we have that

$$\begin{aligned} \nabla_X\mathbf{n} &= \frac{d}{ds}\Big|_{s=0} (\mathbf{n} \circ c)(s) = \frac{d}{ds}\Big|_{s=0} \frac{1}{\|\operatorname{grad}_{g(s)} f\|}\operatorname{diag}(\operatorname{grad}_{g(s)} f) \\ &= \frac{d}{ds}\Big|_{s=0} \left(\frac{1}{\|\operatorname{grad}_{g(s)} f\|}\right)\operatorname{diag}(\operatorname{grad}_\lambda f) + \frac{1}{\|\operatorname{grad}_\lambda f\|}\operatorname{diag}\left(\nabla_x \operatorname{grad} f\right). \end{aligned}$$

Since $y \in \ker df_\lambda$, it follows that

$$\begin{aligned} \mathbf{II}_D^{\partial\Omega_f}(X, Y) &= -\langle Y, \nabla_X\mathbf{n}\rangle \\ &= -\frac{d}{ds}\Big|_{s=0} \left(\frac{1}{\|\operatorname{grad}_{g(s)} f\|}\right)\underbrace{\langle y, \operatorname{grad}_\lambda f\rangle}_{=0} - \frac{1}{\|\operatorname{grad}_\lambda f\|}\langle y, \nabla_x \operatorname{grad} f\rangle \\ &= -\frac{1}{\|\operatorname{grad}_\lambda f\|}\operatorname{Hess}_\lambda f(x, y). \end{aligned}$$

Using (3.16), for $S_1, S_2 \in \mathfrak{so}(3)$, we can compute that

$$\mathbf{II}_D^{\partial\Omega_f}([S_1, D], [S_2, D]) = -\langle [S_2, D], \nabla_{[S_1,D]}\mathbf{n}\rangle \overset{(3.16)}{=} -\langle [S_2, D], [S_1, \mathbf{n}_D]\rangle.$$ □

3.2.1.1 The first factor of the decomposition (3.12)

The following Lemma shows that Bianchi-convex sets can only be concave in directions coming from the second factor $\operatorname{diag}(\ker(df_\lambda))$ of the decomposition (3.12). Since, by Lemma 3.2.18, the second fundamental form on this space is essentially given by the Hessian of f, the degree of concavity of Ω_f is determined by that of $f^{-1}((-\infty, 0])$.

Lemma 3.2.19. *Let Ω_f be Bianchi-convex and $D := \operatorname{diag}(\lambda) \in \partial\Omega_f$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. Then $\mathbf{II}_D^{\partial\Omega_f}|_{[\mathfrak{so}(3),D]}$ is negative semidefinite.*

Proof. In the proof of Lemma 3.2.12, we have seen that $([E_1, D], [E_2, D], [E_3, D])$ is a basis of $[\mathfrak{so}(3), D]$ which only consists of singular matrices. According to Remark 3.2.3, for $i = 1, 2, 3$, the triples $([E_i, D], 0, 0)$ therefore satisfy the second Bianchi identity. Since Ω_f is Bianchi-convex,

$$\mathbf{II}_D^{\partial\Omega_f}([E_i, D], [E_i, D]) \leq 0$$

for $i = 1, 2, 3$. Moreover, one can compute that

$$\begin{aligned} [E_1, \mathbf{n}_D] &= \frac{\partial_3 f(\lambda) - \partial_2 f(\lambda)}{\|\operatorname{grad}_\lambda f\|}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ [E_2, \mathbf{n}_D] &= \frac{\partial_3 f(\lambda) - \partial_1 f(\lambda)}{\|\operatorname{grad}_\lambda f\|}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \text{and} \quad [E_3, \mathbf{n}_D] &= \frac{\partial_2 f(\lambda) - \partial_1 f(\lambda)}{\|\operatorname{grad}_\lambda f\|}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.17}$$

Together with (3.14), we find for $i \neq j$ that

$$\mathbf{II}_D^{\partial\Omega_f}([E_i, D], [E_j, D]) = -\langle [E_j, D], [E_i, \mathbf{n}_D] \rangle \overset{(3.14)}{=} 0.$$

Therefore, $\mathbf{II}_D^{\partial\Omega_f}|_{[\mathfrak{so}(3),D]}$ is diagonal with respect to the basis $([E_1, D], [E_2, D], [E_3, D])$ and has non-positive diagonal elements, hence it is negative semidefinite. □

Corollary 3.2.20. *If Ω_f is Bianchi-convex, then for all $\lambda \in f^{-1}(0)$ with $\lambda_1 < \lambda_2 < \lambda_3$, we have that*

$$\partial_1 f(\lambda) \leq \partial_2 f(\lambda) \leq \partial_3 f(\lambda). \tag{3.18}$$

Proof. Let $\lambda \in f^{-1}(0)$ with $\lambda_1 < \lambda_2 < \lambda_3$. Set $D := \operatorname{diag}(\lambda)$. Again using that the triples $([E_i, D], 0, 0)$, $i = 1, 2, 3$, satisfy the second Bianchi identity, the Bianchi-convexity of Ω_f implies that

$$\begin{aligned} 0 \geq \mathbf{II}_D^{\partial\Omega_f}([E_1, D], [E_1, D]) = -\langle [E_1, D], [E_1, \mathbf{n}_D] \rangle &\overset{(3.14),(3.17)}{=} -2\frac{(\lambda_3 - \lambda_2)(\partial_3 f(\lambda) - \partial_2 f(\lambda))}{\|\operatorname{grad}_\lambda f\|}, \\ 0 \geq \mathbf{II}_D^{\partial\Omega_f}([E_3, D], [E_3, D]) = -\langle [E_3, D], [E_3, \mathbf{n}_D] \rangle &\overset{(3.14),(3.17)}{=} -2\frac{(\lambda_2 - \lambda_1)(\partial_2 f(\lambda) - \partial_1 f(\lambda))}{\|\operatorname{grad}_\lambda f\|}. \end{aligned}$$

Therefore, $\partial_3 f(\lambda) - \partial_2 f(\lambda)) \geq 0$ and $\partial_2 f(\lambda) - \partial_1 f(\lambda)) \geq 0$. □

Remark 3.2.21. Let $D := \operatorname{diag}(\lambda)$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$ and define

$$Z_1(\lambda) := \frac{\partial_3 f(\lambda) - \partial_2 f(\lambda)}{\lambda_3 - \lambda_2}, \ Z_2(\lambda) := \frac{\partial_3 f(\lambda) - \partial_1 f(\lambda)}{\lambda_3 - \lambda_1} \ \text{ and } \ Z_3(\lambda) := \frac{\partial_2 f(\lambda) - \partial_1 f(\lambda)}{\lambda_2 - \lambda_1}.$$

For $i = 1, 2, 3$, let further

$$F_i := \frac{[E_i, D]}{\|[E_i, D]\|}.$$

Then (F_1, F_2, F_3) is an orthonormal basis of $[\mathfrak{so}(3), D]$, and with respect to this basis, we find that

$$\mathrm{II}_D^{\partial\Omega_f}\Big|_{[\mathfrak{so}(3),D]} \mathrel{\widehat{=}} -\frac{1}{\|\operatorname{grad}_\lambda f\|} \operatorname{diag}(Z_1(\lambda), Z_2(\lambda), Z_3(\lambda)).$$

If Ω_f is Bianchi-convex and additionally $\lambda \in f^{-1}(0)$, then $Z_i(\lambda) \geq 0$ for $i = 1, 2, 3$.

The sublevel sets of a convex function are convex, as is well known. Since Ω_f is a sublevel set of $f \circ \lambda$, it is natural to ask whether f being convex implies that Ω_f is already convex. Since by assumption f is symmetric, the answer is yes.

Lemma 3.2.22. *If f is convex, then each connected component of Ω_f is convex.*

Proof. Step 1: By assumption, f is symmetric and convex, hence Schur-convex (for a reference see [PPT92, Def. 12.23, Thm. 12.27]). Therefore, the Schur-Ostrowski criterion [PPT92, Thm. 12.25] yields that

$$(x_i - x_j)\,(\partial_i f(x) - \partial_j f(x)) \geq 0$$

for all $x \in \mathbb{R}^3$ and $i, j = 1, 2, 3$. In particular, for all $\lambda \in f^{-1}(0)$ with λ_1, λ_2 and λ_3 being pairwise distinct, we have that $Z_i(\lambda) \geq 0$ for $i = 1, 2, 3$. Remark 3.2.21 implies that $\mathrm{II}_D^{\partial\Omega_f}|_{[\mathfrak{so}(3),D]}$ is negative semidefinite, where $D := \operatorname{diag}(\lambda)$. Moreover, f being convex together with Lemma 3.2.18 yields that $\mathrm{II}_D^{\partial\Omega_f}|_{\operatorname{diag}(\ker(df_\lambda))}$ is negative semidefinite as well. Thus, Lemma 3.2.17 gives that $\mathrm{II}_D^{\partial\Omega_f}$ is negative semidefinite. Using Remark 3.2.16, all in all we have shown that $\mathrm{II}_R^{\partial\Omega_f}$ is negative semidefinite in all points $R \in \partial\Omega_f$ with pairwise different eigenvalues.

Step 2: Let now $R \in \partial\Omega_f$, the eigenvalues of which are not pairwise distinct. By Lemma 3.2.10, there exists a smooth curve $c : (-\epsilon, \epsilon) \to \partial\Omega_f$ with $c(0) = R$ and $c(s)$ has pairwise distinct eigenvalues for all $s \neq 0$. Let $T \in T_R\partial\Omega_f$ and $s \mapsto T(s) \in T_{c(s)}\partial\Omega_f$ a smooth vector field along c with $T(0) = T$. From Step 1, we obtain for all $s \neq 0$ that

$$\mathrm{II}_{c(s)}^{\partial\Omega_f}(T(s), T(s)) \leq 0.$$

By continuity, this is also true for $s = 0$, hence $\mathrm{II}_R^{\partial\Omega_f}$ is negative semidefinite.

Taking both steps together yields that $\mathrm{II}_R^{\partial\Omega_f}$ is negative semidefinite for all $R \in \partial\Omega_f$. Consequently, the connected components of Ω_f are convex. □

Remark 3.2.23. However, Ω_f being (component-wise) convex in general does not imply that f is convex. To see this, for $a \in (0, 1)$ consider the symmetric smooth function

$$f_a : \mathbb{R}^3 \to \mathbb{R} : x \mapsto a - e^{-\|x\|^2}.$$

If $x \in f_a^{-1}(0)$, then $\|x\|^2 = -\log(a) > 0$, hence $x \neq 0$, which implies that $\operatorname{grad}_x f_a = 2ax \neq 0$. It follows that 0 is a regular value of f_a. Moreover,

$$D^2 f(e_1) = \frac{2}{e}\operatorname{diag}(-1, 1, 1)$$

is indefinite. Because of this, f_a is not convex. Since for $R \in \mathcal{A}_3$ we have that $\|\lambda(R)\| = \|R\|$, we observe that

$$\Omega_{f_a} = \{R \in \mathcal{A}_3 \mid f_a(\lambda(R)) \leq 0\} = \left\{ R \in \mathcal{A}_3 \,\middle|\, \|R\| \leq \sqrt{-\log(a)} \right\}$$

is a ball in $\mathcal{A}_3$, hence convex, while f_a is not convex.

3.2.1.2 The second factor of the decomposition (3.12)

As we have seen in the previous section, the second factor $\operatorname{diag}(\ker(df_\lambda))$ of the decompositon (3.12) is the more interesting one concerning the degree of concavity a Bianchi-convex set can have, meaning that concave directions can only come from this space. In this section, we investigate how positive the second fundamental form of a Bianchi-convex set restricted to this space can be.

Let Ω_f be Bianchi-convex and $D := \operatorname{diag}(\lambda) \in \partial\Omega_f$ with λ_1, λ_2 and λ_3 being pairwise distinct. Then for all $(T_1, T_2, T_3) \in T_D\partial\Omega_f$, there are unique $S_i \in [\mathfrak{so}(3), D]$ and $x_i \in \ker(df_\lambda)$ such that $T_i = S_i + \operatorname{diag}(x_i)$ for $i = 1, 2, 3$. If $\sum_{i=1}^3 T_i e_i = 0$, where (e_1, e_2, e_3) denotes the standard basis of $\mathbb{R}^3$, we have that

$$\sum_{i=1}^{3} \mathrm{II}_D^{\partial\Omega_f}(\operatorname{diag}(x_i), \operatorname{diag}(x_i)) \leq -\sum_{i=1}^{3} \mathrm{II}_D^{\partial\Omega_f}(S_i, S_i),$$

where we used Lemma 3.2.17. As a consequence, for all $x_1, x_2, x_3 \in \ker(df_\lambda)$, we have that

$$\sum_{i=1}^{3} \mathrm{II}_D^{\partial\Omega_f}(\operatorname{diag}(x_i), \operatorname{diag}(x_i)) \leq \min \underbrace{-\sum_{i=1}^{3} \mathrm{II}_D^{\partial\Omega_f}(S_i, S_i)}_{\overset{3.2.19}{\geq} 0}, \tag{3.19}$$

where the minimum is taken over all $S_1, S_2, S_3 \in [\mathfrak{so}(3), D]$ with $\sum_{i=1}^{3}(S_i + \operatorname{diag}(x_i))e_i = 0$.

Before determining the right-hand side of the inequality (3.19) more specifically, we make sure that we are not taking the minimum over an empty set.

Lemma 3.2.24. *Let $D := \operatorname{diag}(\lambda)$, where $\lambda \in \mathbb{R}^3$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. For all $x_1, x_2, x_3 \in \mathbb{R}^3$, there exist $S_1, S_2, S_3 \in [\mathfrak{so}(3), D]$ such that*

$$\sum_{i=1}^{3}(S_i + \operatorname{diag}(x_i))e_i = 0.$$

Here, (e_1, e_2, e_3) denotes the standard basis of $\mathbb{R}^3$.

Proof. Let $x_1, x_2, x_3 \in \mathbb{R}^3$. Then for all $a, b, c \in \mathbb{R}$,

$$S_1 := \begin{pmatrix} 0 & -x_{22} & -x_{33} \\ -x_{22} & 0 & a \\ -x_{33} & a & 0 \end{pmatrix}, \; S_2 := \begin{pmatrix} 0 & -x_{11} & b \\ -x_{11} & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \text{ and } S_3 := \begin{pmatrix} 0 & c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are elements of $[\mathfrak{so}(3), D]$, where $x_i = (x_{i1}, x_{i2}, x_{i3})$ for $i = 1, 2, 3$, and we have that

$$\sum_{i=1}^{3}(S_i + \operatorname{diag}(x_i))e_i = 0. \qquad \square$$

Lemma 3.2.25. *Let $D := \operatorname{diag}(\lambda)$, where $\lambda \in \mathbb{R}^3$ with $\lambda_1 < \lambda_2 < \lambda_3$ and $\partial_1 f(\lambda) \leq \partial_2 f(\lambda) \leq \partial_3 f(\lambda)$. Let further $Z_i := Z_i(\lambda)$, $i = 1, 2, 3$, be defined as in Remark 3.2.21. Then, unless $Z_1 = Z_2 = Z_3 = 0$, for all $x_1, x_2, x_3 \in \mathbb{R}^3$ we have that*

$$\min -\sum_{i=1}^{3} \mathrm{II}_D^{\partial\Omega_f}(S_i, S_i) = \frac{2}{\|\operatorname{grad}_\lambda f\|}\left(\frac{Z_2 Z_3}{Z_2 + Z_3}x_{11}^2 + \frac{Z_1 Z_3}{Z_1 + Z_3}x_{22}^2 + \frac{Z_1 Z_2}{Z_1 + Z_2}x_{33}^2\right), \tag{3.20}$$

where the minimum is taken over all $S_1, S_2, S_3 \in [\mathfrak{so}(3), D]$ with $\sum_{i=1}^{3}(S_i + \operatorname{diag}(x_i))e_i = 0$. If $Z_1 = Z_2 = Z_3 = 0$, the left-hand side of (3.20) is zero.

Remark 3.2.26. Notice that $Z_i + Z_j \neq 0$ for all $i \neq j$, unless $Z_1 = Z_2 = Z_3 = 0$.

Proof. By assumption, $Z_i \geq 0$ for $i = 1, 2, 3$. The case that $Z_1 = Z_2 = Z_3 = 0$ is trivial since then $\mathrm{II}_D^{\partial\Omega_f}|_{[\mathfrak{so}(3),D]} \equiv 0$. In the other case, fix $x_1, x_2, x_3 \in \mathbb{R}^3$. Let further

$$S_i := \begin{pmatrix} 0 & a_i & b_i \\ a_i & 0 & c_i \\ b_i & c_i & 0 \end{pmatrix} \in [\mathfrak{so}(3), D],$$

where $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, 2, 3$, such that

$$\sum_{i=1}^{3}(S_i + \operatorname{diag}(x_i))e_i = 0.$$

Then

$$\begin{aligned} x_{11} + a_2 + b_3 &= 0, \\ a_1 + x_{22} + c_3 &= 0, \\ b_1 + c_2 + x_{33} &= 0 \end{aligned} \tag{3.21}$$

and $S_i = \sqrt{2}(c_i F_1 + b_i F_2 + a_i F_3)$, $i = 1, 2, 3$, where (F_1, F_2, F_3) denotes the orthonormal basis of $[\mathfrak{so}(3), D]$ introduced in Remark 3.2.21. Consequently,

$$-\sum_{i=1}^{3}\mathrm{II}_D^{\partial\Omega_f}(S_i, S_i) \overset{3.2.21}{=} \frac{2}{\|\operatorname{grad}_\lambda f\|}\sum_{i=1}^{3}(c_i^2 Z_1 + b_i^2 Z_2 + a_i^2 Z_3).$$

Since $Z_i \geq 0$ for $i = 1, 2, 3$, the function

$$g : \mathbb{R}^9 \to \mathbb{R} : (a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) \mapsto \sum_{i=1}^{3}(c_i^2 Z_1 + b_i^2 Z_2 + a_i^2 Z_3)$$

is bounded from below and homogeneous of degree two. Using the method of Langrange multipliers, one can compute that the minimum of g subject to the equality constraints (3.21) is given by

$$\frac{Z_2 Z_3}{Z_2 + Z_3}x_{11}^2 + \frac{Z_1 Z_3}{Z_1 + Z_3}x_{22}^2 + \frac{Z_1 Z_2}{Z_1 + Z_2}x_{33}^2.$$

This proves the statement. $\square$

Lemma 3.2.25 directly provides how concave a Bianchi-convex set Ω_f can at most be.

Corollary 3.2.27. *Let Ω_f be Bianchi-convex and $D := \mathrm{diag}(\lambda) \in \partial\Omega_f$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. Let further $Z_i := Z_i(\lambda)$, $i = 1, 2, 3$, be defined as in Remark 3.2.21. Then, unless $Z_1 = Z_2 = Z_3 = 0$, for all $x \in \ker(df_\lambda)$ we have that*

$$\mathbf{II}_D^{\partial\Omega_f}(\mathrm{diag}(x), \mathrm{diag}(x)) \leq \frac{2}{\|\mathrm{grad}_\lambda f\|} \min_{\substack{\{i,j,k\} \\ =\{1,2,3\}}} \frac{Z_i Z_j}{Z_i + Z_j} x_k^2. \tag{3.22}$$

In the case that $Z_1 = Z_2 = Z_3 = 0$, the left-hand side of (3.22) is non-positive.

Proof. Using Corollary 3.2.20, Lemma 3.2.25 shows that for all $x_1, x_2, x_3 \in \ker(df_\lambda)$, we have that

$$\sum_{i=1}^{3} \mathbf{II}_D^{\partial\Omega_f}(\mathrm{diag}(x_i), \mathrm{diag}(x_i)) \leq \begin{cases} 0, & Z_1 = Z_2 = Z_3 = 0 \\ \frac{2}{\|\mathrm{grad}_\lambda f\|} \left(\frac{Z_2 Z_3}{Z_2 + Z_3} x_{11}^2 + \frac{Z_1 Z_3}{Z_1 + Z_3} x_{22}^2 + \frac{Z_1 Z_2}{Z_1 + Z_2} x_{33}^2 \right), & \text{else} \end{cases}.$$

In particular, this holds true for $x_1, x_2, x_3 \in \ker(df_\lambda)$ with in each case two of these vectors being zero. ☐

Remark 3.2.28. As we have seen in Lemma 3.2.18, the inequality (3.22) is equivalent to

$$\mathrm{Hess}_\lambda f(x, x) \geq -2 \min_{\substack{\{i,j,k\} \\ =\{1,2,3\}}} \frac{Z_i Z_j}{Z_i + Z_j} x_k^2.$$

An immediate consequence of Corollary 3.2.27 is the following statement.

Lemma 3.2.29. *Let Ω_f be Bianchi-convex. If for all $\lambda \in f^{-1}(0)$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$ there is an $i \in \{1, 2, 3\}$ such that $Z_i(\lambda) = 0$, then the connected components of Ω_f are convex.*

Proof. Let $\lambda \in f^{-1}(0)$ with λ_1, λ_2 and λ_3 being pairwise distinct and set $D := \mathrm{diag}(\lambda)$. Since Ω_f is Bianchi-convex, $Z_i(\lambda) \geq 0$ for $i = 1, 2, 3$. By assumption, there is a $j \in \{1, 2, 3\}$ such that $Z_j(\lambda) = 0$. Therefore, from Corollary 3.2.27, we obtain that $\mathbf{II}_D^{\partial\Omega_f}|_{\mathrm{diag}(\ker(df_\lambda))}$ is negative semidefinite, and by Lemma 3.2.19 so is $\mathbf{II}_D^{\partial\Omega_f}|_{[\mathfrak{so}(3),D]}$. Arguing as in the proof of Lemma 3.2.22 finishes the proof. ☐

Remark 3.2.30. Lemma 3.2.29 shows that in order for Ω_f to be Bianchi-convex but *not* convex, $\partial_1 f(\lambda) < \partial_2 f(\lambda) < \partial_3 f(\lambda)$ must hold true for all $\lambda \in f^{-1}(0)$ with $\lambda_1 < \lambda_2 < \lambda_3$.

3.2.1.3 Another characterization of Bianchi-convex sets

The previous considerations finally enable us to prove another characterization of Bianchi-convex sets of the form Ω_f for a function $f : \mathbb{R}^3 \to \mathbb{R}$ as above.

Proposition 3.2.31. *The set Ω_f is Bianchi-convex if and only if for all $\lambda \in f^{-1}(0)$ with $\lambda_1 < \lambda_2 < \lambda_3$ the following are true.*

1.) $\partial_1 f(\lambda) \leq \partial_2 f(\lambda) \leq \partial_3 f(\lambda)$ and,

2.) unless $Z_1 = Z_2 = Z_3 = 0$, we have for all $x \in \ker(df_\lambda)$ that

$$\mathrm{Hess}_\lambda f(x, x) \geq -2 \min_{\substack{\{i,j,k\} \\ =\{1,2,3\}}} \frac{Z_i Z_j}{Z_i + Z_j} x_k^2,$$

while if $Z_1 = Z_2 = Z_3 = 0$, we have that $\mathrm{Hess}_\lambda f(x, x) \geq 0$ for all $x \in \ker(df_\lambda)$. Here, $Z_i := Z_i(\lambda)$, $i = 1, 2, 3$, are defined as in Remark 3.2.21.

Proof. The implication "⇒" follows immediately from Corollary 3.2.20 and Corollary 3.2.27. To show the converse implication, let first of all $R \in \partial\Omega_f$ with pairwise different eigenvalues. Then there exists a $Q \in O(3)$ such that $R = QDQ^t$, where $D := \operatorname{diag}(\lambda)$ for some $\lambda \in f^{-1}(0)$ with $\lambda_1 < \lambda_2 < \lambda_3$.

Step 1: Let $(T_1, T_2, T_3) \in (T_R\partial\Omega_f)^3$ with $\sum_{i=1}^3 T_i Q e_i = 0$. Recalling Lemma 3.2.12 and Remark 3.2.13, for $i = 1, 2, 3$ there are unique $S_i \in [\mathfrak{so}(3), D]$ and $x_i \in \ker(df_\lambda)$ such that $T_i = Q(S_i + X_i)Q^t$, where $X_i := \operatorname{diag}(x_i)$. Then

$$0 = \sum_{i=1}^3 T_i Q e_i = \sum_{i=1}^3 Q(S_i + X_i)e_i,$$

which implies that

$$\sum_{i=1}^3 (S_i + X_i)e_i = 0. \tag{3.23}$$

Now, we can compute

$$\sum_{i=1}^3 \mathrm{II}_R^{\partial\Omega_f}(T_i, T_i) \overset{3.2.16}{=} \sum_{i=1}^3 \mathrm{II}_D^{\partial\Omega_f}(S_i + X_i, S_i + X_i) \overset{3.2.17}{=} \sum_{i=1}^3 \mathrm{II}_D^{\partial\Omega_f}(S_i, S_i) + \sum_{i=1}^3 \mathrm{II}_D^{\partial\Omega_f}(X_i, X_i).$$

If $Z_1 = Z_2 = Z_3 = 0$, from Remark 3.2.21, we obtain that $\mathrm{II}_D^{\partial\Omega_f}|_{[\mathfrak{so}(3),D]} \equiv 0$ and the second assumption together with Lemma 3.2.18 yields that $\mathrm{II}_D^{\partial\Omega_f}|_{\operatorname{diag}(\ker(df_\lambda))} \leq 0$. Hence, in this case, we immediately obtain that

$$\sum_{i=1}^3 \mathrm{II}_R^{\partial\Omega_f}(T_i, T_i) \leq 0.$$

In the other cases, both assumptions together with Lemma 3.2.25 provide that

$$\begin{aligned}
\sum_{i=1}^3 \mathrm{II}_R^{\partial\Omega_f}(T_i, T_i) &\overset{2.}{\leq} \sum_{i=1}^3 \mathrm{II}_D^{\partial\Omega_f}(S_i, S_i) + \frac{2}{\|\operatorname{grad}_\lambda f\|} \sum_{i=1}^3 \min_{\substack{\{j,k,l\} \\ =\{1,2,3\}}} \frac{Z_j Z_k}{Z_j + Z_k} x_{il}^2 \\
&\leq \sum_{i=1}^3 \mathrm{II}_D^{\partial\Omega_f}(S_i, S_i) + \frac{2}{\|\operatorname{grad}_\lambda f\|} \left(\frac{Z_2 Z_3}{Z_2 + Z_3} x_{11}^2 + \frac{Z_1 Z_3}{Z_1 + Z_3} x_{22}^2 + \frac{Z_1 Z_2}{Z_1 + Z_2} x_{33}^2 \right) \\
&\overset{\substack{1.), \\ 3.2.25}}{=} \sum_{i=1}^3 \mathrm{II}_D^{\partial\Omega_f}(S_i, S_i) + \min_{\substack{\widetilde{S}_1, \widetilde{S}_2, \widetilde{S}_3 \in [\mathfrak{so}(3), D]: \\ \sum_{i=1}^3 (\widetilde{S}_i + X_i)e_i = 0}} - \sum_{i=1}^3 \mathrm{II}_D^{\partial\Omega_f}(\widetilde{S}_i, \widetilde{S}_i) \\
&\overset{(3.23)}{\leq} \sum_{i=1}^3 \mathrm{II}_D^{\partial\Omega_f}(S_i, S_i) - \sum_{i=1}^3 \mathrm{II}_D^{\partial\Omega_f}(S_i, S_i) = 0.
\end{aligned}$$

Step 2: Let now $(T_1, T_2, T_3) \in (T_R\partial\Omega_f)^3$ with $\sum_{i=1}^3 T_i e_i = 0$. Then

$$\sum_{i=1}^3 \widetilde{T}_i Q e_i = 0,$$

where $\widetilde{T}_i := \sum_{j=1}^3 Q_{ji} T_j$ for $i = 1, 2, 3$. Similarly to the proof of Lemma 3.2.5, one computes that

$$\sum_{i=1}^3 \mathbf{II}_R^{\partial\Omega_f}(T_i, T_i) = \sum_{i=1}^3 \mathbf{II}_R^{\partial\Omega_f}(\widetilde{T}_i, \widetilde{T}_i) \overset{\text{Step 1}}{\leq} 0.$$

Thus, by Lemma 3.2.5, we have shown that Ω_f is Bianchi-convex in all $R \in \partial\Omega_f$ with pairwise distinct eigenvalues.

To finish the proof, let $R \in \partial\Omega_f$, the eigenvalues of which are *not* pairwise distinct. From Lemma 3.2.10, we know that there exists a smooth curve $c : (-\epsilon, \epsilon) \to \partial\Omega_f$ with $c(0) = R$ and $c(s)$ having pairwise distinct eigenvalues for all $s \neq 0$. Let further $(T_1, T_2, T_3) \in (T_R\partial\Omega_f)^3$ with $\sum_{i=1}^3 T_i e_i = 0$. Then $(T_1, T_2, T_3) \in (T_R\partial\Omega_f)^3_{(2\mathrm{BI})}$. Since $(T\partial\Omega_f)^3_{(2\mathrm{BI})}$ is a vector bundle, as we have shown in Lemma 3.2.11, there exists a smooth section $s \mapsto T(s) \in (T_{c(s)}\partial\Omega_f)^3_{(2\mathrm{BI})}$ along c such that $T(0) = (T_1, T_2, T_3)$. Hence, $T(s) = (T_1(s), T_2(s), T_3(s))$, where $s \mapsto T_i(s) \in T_{c(s)}\partial\Omega_f$ are smooth vector fields along c with $T_i(0) = T_i$ for $i = 1, 2, 3$, satisfying $\sum_{i=1}^3 T_i(s)e_i = 0$. Since we already know that Ω_f is Bianchi-convex in all points in $\partial\Omega_f$ having pairwise distinct eigenvalues, we find that

$$\sum_{i=1}^3 \mathbf{II}_{c(s)}^{\partial\Omega_f}(T_i(s), T_i(s)) \leq 0$$

for all $s \neq 0$. By continuity, this holds true for $s = 0$ as well. This proves that Ω_f is Bianchi-convex. □

3.2.1.4 Application

In order to prove the subsequent proposition, we need the following lemma.

Lemma 3.2.32. *In each two-dimensional linear subspace of* $\mathbb{R}^3$ *there are two linearly independent vectors with one vanishing component.*

Proof. Let U be a two-dimensional subspace of $\mathbb{R}^3$ not being one of the coordinate planes, since we are done otherwise. Then $U \cap \{x \in \mathbb{R}^3 \mid x_1 = 0\}$, $U \cap \{x \in \mathbb{R}^3 \mid x_2 = 0\}$ and $U \cap \{x \in \mathbb{R}^3 \mid x_3 = 0\}$, as intersections of transversal subspaces, are one-dimensional. Moreover, these three intersections cannot coincide since elements x in their intersection have $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, hence $x = 0$. This yields that at least two of these three subspaces are distinct. Choosing a non-zero vector in each of them provides two vectors as desired. □

Proposition 3.2.33. *If* Ω_f *is Bianchi-convex and scale-invariant, then the connected components of* Ω_f *are convex.*

Proof. As we have seen in the proof of Lemma 3.2.22, it suffices to show that $\mathbf{II}_D^{\partial\Omega_f}$ is negative semidefinite for all $D := \operatorname{diag}(\lambda)$, where $\lambda \in f^{-1}(0)$ with λ_1, λ_2 and λ_3 being pairwise distinct. Since Ω_f is Bianchi-convex, from Lemma 3.2.19, we already know that $\mathbf{II}_D^{\partial\Omega_f}|_{[\mathfrak{so}(3),D]}$ is negative semidefinite. Thus, by Lemma 3.2.17, it remains to show that this is also true for $\mathbf{II}_D^{\partial\Omega_f}|_{\operatorname{diag}(\ker(df_\lambda))}$, which by Lemma 3.2.18 is equivalent to $\operatorname{Hess}_\lambda f|_{T_\lambda f^{-1}(0)}$ being positive semidefinite. To this end, let λ and D be as above. Then $D \in \partial\Omega_f$. Since Ω_f is scale-invariant, so is $\partial\Omega_f$ as we know from Lemma A.0.1. Therefore, $\alpha D \in \partial\Omega_f$, i.e. $\alpha\lambda \in f^{-1}(0)$, for all $\alpha > 0$. Hence, $f^{-1}(0)$ is scale-invariant as well. Because of this, for $s \in (-\epsilon, \epsilon)$ with $0 < \epsilon < 1$, the curve $s \mapsto c(s) := \lambda + s\lambda$ is contained in $f^{-1}(0)$ and we find that

$$df_\lambda(\lambda) = \frac{d}{ds}\Big|_{s=0} (f \circ c)(s) = 0.$$

This yields that $\lambda \in \ker(df_\lambda) = T_\lambda f^{-1}(0)$. Furthermore,

$$0 = \frac{d^2}{ds^2}\Big|_{s=0} (f \circ c)(s) = \mathrm{Hess}_\lambda f(\lambda, \lambda).$$

Now, let $x \in T_\lambda f^{-1}(0)$. If $d : (-\delta, \delta) \to f^{-1}(0)$, $0 < \delta < 1$, is a curve with $d(0) = \lambda$ and $\dot{d}(0) = x$ and $\gamma(s,t) := d(t) + sd(t)$ for $s, t \in (-\delta, \delta)$, then γ is a curve in $f^{-1}(0)$ as well and we find that

$$0 = \frac{d^2}{ds\,dt}\Big|_{s=t=0} f(\gamma(s,t)) = \frac{d}{ds}\Big|_{s=0} df_{\lambda+s\lambda}(x + sx) = \mathrm{Hess}_\lambda f(\lambda, x) + df_\lambda(x) = \mathrm{Hess}_\lambda f(\lambda, x).$$

Moreover, $x = r\lambda + v$ for a unique $r \in \mathbb{R}$ and $v \in \lambda^\perp \cap T_\lambda f^{-1}(0)$ and we have that

$$\mathrm{Hess}_\lambda f(x,x) = r^2 \underbrace{\mathrm{Hess}_\lambda f(\lambda,\lambda)}_{=0} + 2r \underbrace{\mathrm{Hess}_\lambda f(\lambda, v)}_{=0} + \mathrm{Hess}_\lambda f(v,v) = \mathrm{Hess}_\lambda f(v,v). \tag{3.24}$$

Hence, it remains to show that $\mathrm{Hess}_\lambda f$ restricted to $\lambda^\perp \cap T_\lambda f^{-1}(0)$ is positive semidefinite. Since $T_\lambda f^{-1}(0) \subseteq \mathbb{R}^3$ is a two-dimensional vector space, by Lemma 3.2.32, there are two linearly independent vectors having at least one vanishing component. Thus, $T_\lambda f^{-1}(0) \setminus \mathbb{R}\lambda$ contains a vector x_0 having at least one vanishing component. Writing $x_0 = r\lambda + v_0$ for unique $r \in \mathbb{R}$ and $0 \neq v_0 \in \lambda^\perp \cap T_\lambda f^{-1}(0)$, by Proposition 3.2.31, we therefore find that

$$\mathrm{Hess}_\lambda f(v_0, v_0) \overset{(3.24)}{=} \mathrm{Hess}_\lambda f(x_0, x_0) \geq 0.$$

Since the space $\lambda^\perp \cap T_\lambda f^{-1}(0)$ is one-dimensional, it follows that $\mathrm{Hess}_\lambda f$ restricted to $\lambda^\perp \cap T_\lambda f^{-1}(0)$ is positive semidefinite. □

3.2.2 Example of ODE-invariant non-convex Bianchi-convex sets

We are now in the position to show that Bianchi-convexity is a genuine generalization of convexity, i.e. there are Bianchi-convex sets which are *not* convex.

Proposition 3.2.34. *For $a \in \left(\frac{1}{3}, \frac{2}{5}\right)$ and $c > 0$, the set*

$$\Omega_{a,c} := \{R \in \mathcal{A}_3 \mid \|R\|^2 - a\,\mathrm{scal}(R)^2 \leq c\}$$

is Bianchi-convex but not convex.

Proof. For $a \in \left(\frac{1}{3}, \frac{2}{5}\right)$ and $c > 0$, we consider the function

$$f_{a,c} : \mathbb{R}^3 \to \mathbb{R} : x \mapsto \|x\|^2 - a\langle x, I\rangle^2 - c,$$

where $I := (1,1,1)^t$. Obviously, $f_{a,c}$ is smooth and symmetric. For $x := \left(\sqrt{\frac{c}{1-a}}, 0, 0\right)^t \in \mathbb{R}^3$, we have that $f_{a,c}(x) = 0$, thus $f_{a,c}^{-1}(0) \neq \emptyset$. Moreover, 0 is a regular value of $f_{a,c}$, since $c \neq 0$. Remembering Remark 1.1.11, we obtain that $\Omega_{a,c} = \Omega_{f_{a,c}}$.

Step 1: Let $\lambda \in \mathbb{R}^3$ with $\lambda_1 < \lambda_2 < \lambda_3$. Because

$$\mathrm{grad}_\lambda f_{a,c} = \begin{pmatrix} 2\lambda_1 - 2a(\lambda_1 + \lambda_2 + \lambda_3) \\ 2\lambda_2 - 2a(\lambda_1 + \lambda_2 + \lambda_3) \\ 2\lambda_3 - 2a(\lambda_1 + \lambda_2 + \lambda_3) \end{pmatrix},$$

we obtain that

$$\partial_1 f_{a,c}(\lambda) < \partial_2 f_{a,c}(\lambda) < \partial_3 f_{a,c}(\lambda). \tag{3.25}$$

Let

$$E_{11} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_{22} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } E_{33} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$A := \frac{1}{2} D^2 f_{a,c}(\lambda) = \mathbb{1}_3 - a \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The matrices $A + E_{11}$, $A + E_{22}$ and $A + E_{33}$ all have the same eigenvalues

$$1, \ \lambda_1(a) := \frac{1}{2}\left(3 - 3a + \sqrt{9a^2 + 2a + 1}\right) \text{ and } \lambda_2(a) := \frac{1}{2}\left(3 - 3a - \sqrt{9a^2 + 2a + 1}\right).$$

Since $\lambda_1(a) > 0$ for all $a \in \mathbb{R}$ and $\lambda_2(a) > 0$ for all $a < \frac{2}{5}$, we find that $A + E_{11}$, $A + E_{22}$ and $A + E_{33}$ are positive definite. Therefore, for each $x \in \mathbb{R}^3 \setminus \{0\}$ and $k \in \{1, 2, 3\}$, we have that

$$\langle (A + E_{kk})x, x \rangle > 0,$$

which is equivalent to

$$\frac{1}{2} \mathrm{Hess}_\lambda f_{a,c}(x, x) = \frac{1}{2} \langle D^2 f_{a,c}(\lambda) x, x \rangle = \langle Ax, x \rangle > -\langle E_{kk} x, x \rangle = -x_k^2.$$

Since $Z_1(\lambda) = Z_2(\lambda) = Z_3(\lambda) = 2$, this results in

$$\mathrm{Hess}_\lambda f_{a,c}(x, x) > \max_{k \in \{1,2,3\}} (-2x_k^2) = -2 \min_{k \in \{1,2,3\}} x_k^2 = -2 \min_{\substack{\{i,j,k\} \\ = \{1,2,3\}}} \frac{Z_i Z_j}{Z_i + Z_j} x_k^2 \tag{3.26}$$

for all $x \in \mathbb{R}^3 \setminus \{0\}$. With a view on (3.25) and (3.26), Proposition 3.2.31 yields that $\Omega_{a,c}$ is Bianchi-convex. (For this we only used that $a < \frac{2}{5}$ and $c \neq 0$.)

Step 2: First, we observe that we can write the function $f_{a,c}$ as follows: For all $x \in \mathbb{R}^3$, we have that

$$f_{a,c}(x) = \langle Ax, x \rangle - c.$$

Furthermore, there is some $Q \in O(3)$ such that $A = QDQ^t$, where $D := \mathrm{diag}(1 - 3a, 1, 1)$. Computing for all $x \in \mathbb{R}^3$ that

$$f_{a,c}(Qx) = \langle QDx, Qx \rangle - c = \langle Dx, x \rangle - c = (1 - 3a)x_1^2 + x_2^2 + x_3^2 - c,$$

we find that

$$\begin{aligned} f_{a,c}^{-1}((-\infty, 0]) &= \{x \in \mathbb{R}^3 \mid f_{a,c}(x) \leq 0\} = \{Qx \mid x \in \mathbb{R}^3, f_{a,c}(Qx) \leq 0\} \\ &= Q\{x \in \mathbb{R}^3 \mid (1 - 3a)x_1^2 + x_2^2 + x_3^2 \leq c\}. \end{aligned} \tag{3.27}$$

From this, it is easy to read off that $f_{a,c}^{-1}((-\infty,0])$ is *not* convex given that $1-3a<0$, i.e. $a>\frac{1}{3}$, and $c>0$, but a one-sheeted hyperboloid. In this case, let

$$R := \operatorname{diag}\left(Q\begin{pmatrix}\sqrt{c}\\ \sqrt{3ac}\\ 0\end{pmatrix}\right) \quad \text{and} \quad S := \operatorname{diag}\left(Q\begin{pmatrix}-\sqrt{c}\\ \sqrt{3ac}\\ 0\end{pmatrix}\right).$$

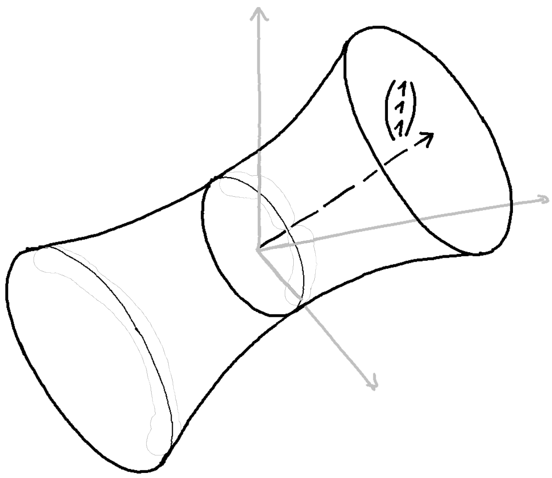
Keeping in mind that

$$\Omega_{a,c} = \Omega_{f_{a,c}} = \{R\in\mathcal{A}_3 \mid \lambda(R)\in f_{a,c}^{-1}((-\infty,0])\}$$

yields that $R,S\in\partial\Omega_{a,c}$. However, the connecting straight line $t\mapsto c(t):=tR+(1-t)S$, where $t\in[0,1]$, is not contained in $\Omega_{a,c}$. For example,

$$c\left(\frac{1}{2}\right) = \operatorname{diag}\left(Q\begin{pmatrix}0\\ \sqrt{3ac}\\ 0\end{pmatrix}\right) \notin \Omega_{a,c}$$

since $3ac>c$. Consequently, $\Omega_{a,c}$ is not convex. □



$f_{a,c}^{-1}(0)$ for $a\in\left(\frac{1}{3},\frac{2}{5}\right)$ and $c>0$

Remark 3.2.35. From the proof of Proposition 3.2.34, it even follows that for $a\in\left(\frac{1}{3},\frac{2}{5}\right)$ and $c>0$, the set $\Omega_{a,c}$ is *strictly* Bianchi-convex, that is for all $R\in\partial\Omega_{a,c}$ and $(T_1,T_2,T_3)\in(T_R\partial\Omega_{a,c})^3\setminus\{0\}$ satisfying the second Bianchi identity, the inequality

$$\sum_{i=1}^{3}\mathbf{II}_R^{\partial\Omega_{a,c}}(T_i,T_i)<0$$

is strict. Consequently, the sets $\Omega_{a,c}$ can be deformed a little bit in such a way that they remain Bianchi-convex.

Next, we show that suitable subsets of the sets $\Omega_{a,c}$ are invariant under the ordinary differential equation (2.4).

Proposition 3.2.36. *For every $a \in \left(\frac{1}{3}, \frac{2}{5}\right)$ and $c > 0$, there exists a constant $b_{a,c} > 0$ such that the intersection*

$$\widetilde{\Omega}_{a,c} := \Omega_{a,c} \cap \{R \in \mathcal{A}_3 \mid \operatorname{scal}(R) \geq b_{a,c}\}$$

is invariant under the ordinary differential equation (2.4)*. Moreover, a possible choice of $b_{a,c}$ is given in* (3.31) *below.*

Remark 3.2.37. Since $\{R \in \mathcal{A}_3 \mid \operatorname{scal}(R) \geq b_{a,c}\}$ as a half space is convex and hence also Bianchi-convex, with a view on Lemma 3.1.9, the set $\widetilde{\Omega}_{a,c}$ as intersection of Bianchi-convex sets is still Bianchi-convex.

Proof. Fix $a \in \left(\frac{1}{3}, \frac{2}{5}\right)$ and $c > 0$. Firstly, in three steps, we show that the set

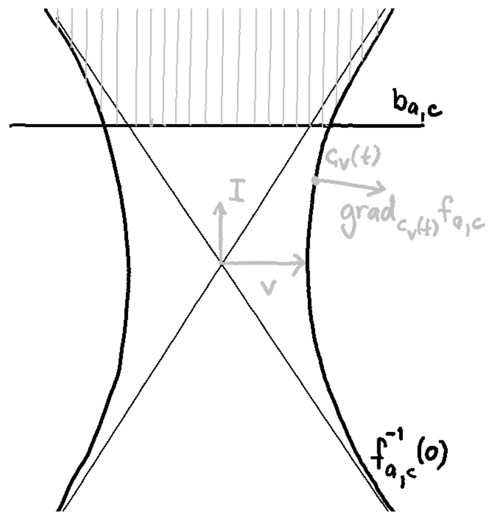
$$C_{a,c} := f_{a,c}^{-1}((-\infty, 0]) \cap \{y \in \mathbb{R}^3 \mid y_1 + y_2 + y_3 \geq b_{a,c}\}$$

is invariant under the ordinary differential equation

$$f'(t) = \varphi(f(t)). \tag{3.28}$$

Here, $f_{a,c}$ denotes the function introduced in the proof of Proposition 3.2.34 and the map φ is defined as follows

$$\varphi : \mathbb{R}^3 \to \mathbb{R}^3 : x \mapsto \begin{pmatrix} x_1^2 + x_2 x_3 \\ x_2^2 + x_1 x_3 \\ x_3^2 + x_1 x_2 \end{pmatrix}.$$



Step 1: For $v \in \mathbb{R}^3$ with $\|v\| = \sqrt{c}$ and $v \perp I$, where $I := (1, 1, 1)^t$, we consider the curve

$$c_v : \mathbb{R} \to f_{a,c}^{-1}(0) : t \mapsto \cosh(t)v + \beta \sinh(t) I$$

in $f_{a,c}^{-1}(0)$ with $c_v(0) = v$, where

$$\beta := \sqrt{\frac{c}{9a - 3}},$$

and show that

$$\langle \varphi(c_v(t)), \operatorname{grad}_{c_v(t)} f_{a,c} \rangle \leq 0$$

for all $t \geq t_0 := \frac{3}{2}$. First, since $v \perp I$, i.e. $v_1 + v_2 + v_3 = 0$, we notice that

$$0 = (v_1 + v_2 + v_3)^2 = v_1^2 + v_2^2 + v_3^2 + 2(v_1 v_2 + v_1 v_3 + v_2 v_3) = c + 2(v_1 v_2 + v_1 v_3 + v_2 v_3),$$

thus

$$v_1 v_2 + v_1 v_3 + v_2 v_3 = -\frac{c}{2} \tag{3.29}$$

and

$$0 = (v_1 + v_2 + v_3)^3 = v_1^3 + v_2^3 + v_3^3 + 6 v_1 v_2 v_3 + 3(\underbrace{v_1^2 v_2 + v_1^2 v_3 + v_1 v_2^2 + v_1 v_3^2 + v_2^2 v_3 + v_2 v_3^2}_{\substack{=(v_1+v_2+v_3)(v_1v_2+v_1v_3+v_2v_3)-3v_1v_2v_3 \\ =-3v_1v_2v_3}})$$

$$= v_1^3 + v_2^3 + v_3^3 - 3 v_1 v_2 v_3,$$

hence

$$v_1^3 + v_2^3 + v_3^3 = 3 v_1 v_2 v_3. \tag{3.30}$$

Using (3.29) and (3.30), one can calculate that

$$\begin{aligned} \langle \varphi(c_v(t)), \operatorname{grad}_{c_v(t)} f_{a,c} \rangle &= 12 v_1 v_2 v_3 \cosh(t)^3 + 12\beta^3(1-3a)\sinh(t)^3 + 3c\beta(1-a)\cosh(t)^2\sinh(t) \\ &= 12 v_1 v_2 v_3 \cosh(t)^3 + 3\beta^3(1-9a^2)\sinh(t)^3 + 3c\beta(1-a)\sinh(t). \end{aligned}$$

Moreover, using the method of Langrange multipliers, one can show that the maximum of the function $g : \mathbb{R}^3 \to \mathbb{R} : x \mapsto x_1 x_2 x_3$ subject to the equality constraints $x_1 + x_2 + x_3 = 0$ and $x_1^2 + x_2^2 + x_3^2 = c$ is given by $\frac{1}{3\sqrt{6}} c^{\frac{3}{2}}$. Therefore,

$$\begin{aligned} \langle \varphi(c_v(t)), \operatorname{grad}_{c_v(t)} f_{a,c} \rangle &\leq \frac{4}{\sqrt{6}} c^{\frac{3}{2}} \cosh(t)^3 + 3\beta^3(1-9a^2)\sinh(t)^3 + 3c\beta(1-a)\sinh(t) \\ &= \left(\frac{4}{\sqrt{6}} \cosh(t)^3 + \frac{3(1-9a^2)}{(9a-3)^{\frac{3}{2}}} \sinh(t)^3 + \frac{3(1-a)}{\sqrt{9a-3}} \sinh(t) \right) c^{\frac{3}{2}} \\ &< 0 \end{aligned}$$

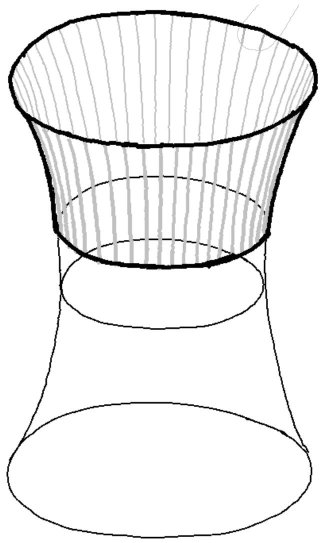
for all $a \in \left(\frac{1}{3}, \frac{2}{5}\right)$, $c > 0$ and $t \geq t_0 := \frac{3}{2}$. We set

$$b_{a,c} := \sqrt{\frac{3c}{3a-1}} \sinh\left(\frac{3}{2}\right). \tag{3.31}$$

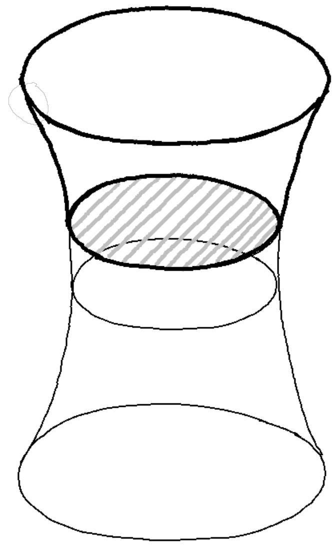
Then $c_v(t_0) \in \{ y \in \mathbb{R}^3 \mid y_1 + y_2 + y_3 = b_{a,c} \}$. Consequently, since v (as in the beginning of Step 1) was arbitrary, we have shown that

$$\langle \varphi(x), \operatorname{grad}_x f_{a,c} \rangle < 0$$

for any $x \in f_{a,c}^{-1}(0) \cap \{ y \in \mathbb{R}^3 \mid y_1 + y_2 + y_3 \geq b_{a,c} \}$.



Step 1



Step 2

Step 2: For $x \neq 0$, we have that

$$\begin{aligned}\langle \varphi(x), (-I) \rangle &= -(x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_2 + x_2 x_3) \\ &= -\frac{1}{2}\left(\|x\|^2 + (x_1 + x_2 + x_3)^2\right) \\ &< 0.\end{aligned}$$

Notice that, in particular, this is true for $x \in f_{a,c}^{-1}((-\infty, 0]) \cap \{y \in \mathbb{R}^3 \mid y_1 + y_2 + y_3 = b_{a,c}\}$ and that $-I$ is the outward pointing unit normal on this part of the boundary of $C_{a,c}$.

Away from the set

$$S_{a,c} := f_{a,c}^{-1}(0) \cap \{y \in \mathbb{R}^3 \mid y_1 + y_2 + y_3 = b_{a,c}\},$$

the boundary of the set $C_{a,c}$ is smooth. Hence, all things considered, in steps 1 and 2 we have shown that

$$\langle \varphi(x), \mathbf{n}_x \rangle < 0,$$

i.e. $\varphi(x)$ is contained in the tangent cone $T_x C_{a,c}$, for each $x \in \partial C_{a,c} \setminus S_{a,c}$.

Step 3: Now, let $x \in S_{a,c}$ and define

$$\gamma(t) := x + t\varphi(x)$$

for $t \in \mathbb{R}$. Then, by Step 1, we have that

$$\left.\frac{d}{dt}\right|_{t=0} f_{a,c}(\gamma(t)) = \langle \mathrm{grad}_x f_{a,c}, \varphi(x) \rangle < 0.$$

Since $f_{a,c}(\gamma(0)) = f_{a,c}(x) = 0$, this implies that there is an $\epsilon_1 > 0$ such that $f_{a,c}(\gamma(t)) \leq 0$, i.e. $\gamma(t) \in f_{a,c}^{-1}((-\infty, 0])$, for all $t \in [0, \epsilon_1)$. Moreover, from Step 2, we know that

$$\left.\frac{d}{dt}\right|_{t=0} \langle \gamma(t), I \rangle = \langle \varphi(x), I \rangle > 0.$$

Since $\langle\gamma(0), I\rangle = \langle x, I\rangle = b_{a,c}$, this yields that there is an $\epsilon_2 > 0$ such that $\langle\gamma(t), I\rangle \geq b_{a,c}$, i.e. $\gamma(t) \in \{y \in \mathbb{R}^3 \mid y_1 + y_2 + y_3 \geq b_{a,c}\}$, for all $t \in [0, \epsilon_2)$. In total, $\gamma(t) \in C_{a,c}$ for all $t \in [0, \epsilon)$, where $\epsilon := \min\{\epsilon_1, \epsilon_2\}$. Due to $\gamma(0) = x$ and $\dot\gamma(0) = \varphi(x)$, we have shown that $\varphi(x) \in T_x C_{a,c}$ as well. Proposition 2.3.3 together with the previous considerations yields that $C_{a,c}$ is invariant under the ordinary differential equation (3.28).

Step 4: Now, we are in the position to show that $\widetilde\Omega_{a,c}$ is invariant under (2.4), i.e. that for each $R \in \partial\widetilde\Omega_{a,c}$, we have that $R^2 + R^\# \in T_R\widetilde\Omega_{a,c}$ (see Proposition 2.3.3).
To this end, let $R \in \partial\widetilde\Omega_{a,c}$ and $Q \in O(3)$ such that $R = QDQ^t$, where $D := \operatorname{diag}(\lambda)$. Then of course $\lambda \in \partial C_{a,c}$. Above we have shown that therefore $\varphi(\lambda) \in T_\lambda C_{a,c}$. Hence, by Example 1.1.20 together with the fact that $\widetilde\Omega_{a,c}$ is $O(3)$-invariant, we obtain that

$$D^2 + D^\# \overset{1.1.20}{=} \operatorname{diag}(\varphi(\lambda)) \in \operatorname{diag}(T_\lambda C_{a,c}) \subseteq T_{\operatorname{diag}(\lambda)}\widetilde\Omega_{a,c} = T_D\widetilde\Omega_{a,c} = Q^t(T_R\widetilde\Omega_{a,c})Q$$

and consequently

$$R^2 + R^\# \overset{1.1.20}{=} Q(D^2 + D^\#)Q^t \in T_R\widetilde\Omega_{a,c}. \qquad \square$$

From (3.27), we immediately see that for $c = 0$ and $a > \frac{1}{3}$ the sets $\Omega_{a,c}$ are convex double cones. The following proposition shows that the upper cone, that is the subset of $\Omega_{a,0}$ which lies in the half-space of non-negative scalar curvature, is invariant under the ordinary differential equation (2.4).

Proposition 3.2.38. *For all $a \geq \frac{1}{3}$, the cones*

$$\widetilde\Omega_{a,0} = \Omega_{a,0} \cap \{R \in \mathcal{A}_3 \mid \operatorname{scal}(R) \geq 0\}$$

are invariant under the ordinary differential equation (2.4)*.*

Proof. Throughout the proof, we will use the notations as in the proof of Proposition 3.2.36. First of all, let $a > \frac{1}{3}$. It is obvious that $0 = \varphi(0) \in T_0 C_{a,0}$. Now, let $0 \neq x \in f_{a,0}^{-1}(0) \cap \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \geq 0\}$. Then there are $\alpha, \beta > 0$ and $v \in \mathbb{R}^3$ with $\|v\| = 1$ and $v \perp I$ such that $x = \alpha v + \beta I$. Firstly, let $\alpha = 1$. Since $x \in f_{a,0}^{-1}(0)$, we find that

$$0 = f_{a,0}(x) = \|v + \beta I\|^2 - a\langle v + \beta I, I\rangle^2 = 1 + 3\beta^2(1 - 3a),$$

hence $\beta = \frac{1}{\sqrt{9a-3}}$. Similar to the proof of Proposition 3.2.36, one can show that

$$\begin{aligned}
\langle\varphi(x), \operatorname{grad}_x f_{a,0}\rangle &= 12v_1v_2v_3 + 3(1-a)\beta + 12(1-3a)\beta^3 \\
&\leq \frac{4}{\sqrt{6}} + 3(1-a)\beta + 12(1-3a)\beta^3 \\
&= \frac{4}{\sqrt{6}} + \frac{1 - 9a^2}{\sqrt{3}(3a-1)^{\frac{3}{2}}} \\
&\leq 0
\end{aligned}$$

for all $a > \frac{1}{3}$. Now, for $\alpha \neq 1$, by setting $y := v + \frac{\beta}{\alpha}I$, we find that $x = \alpha y$, hence

$$\langle\varphi(x), \operatorname{grad}_x f_{a,0}\rangle = \langle\varphi(\alpha y), \operatorname{grad}_{\alpha y} f_{a,0}\rangle = \alpha^3\langle\varphi(y), \operatorname{grad}_y f_{a,0}\rangle \leq 0.$$

Consequently, we have shown that $\varphi(x) \in T_x C_{a,0}$ for all $x \in \partial C_{a,0}$. Arguing as in the proof of Proposition 3.2.36 shows the statement in the case that $a > \frac{1}{3}$. Since (with a view on (3.27)) $\widetilde\Omega_{\frac{1}{3},0}$ is the set of non-negative multiples of the identity in $\mathcal{A}_3$, this finishes the proof. $\square$

Chapter 4

Maximum principles

In this chapter, we recall the weak and strong maximum principles for functions, introduce Hamilton's maximum principle and give a reformulation in the special case of algebraic curvature tensors and the Ricci flow. The aim is to generalize this version to Bianchi-convex sets. As an application, we obtain new curvature conditions which are preserved by the Ricci flow in dimension three.

4.1 Statements for functions

Theorem 4.1.1 (Weak parabolic maximum principle for scalars [Top06, p. 35])**.** *For $t \in [0,T)$, where $0 < T < \infty$, let g_t be a smooth family of metrics on a closed manifold M. Suppose that $u \in \mathcal{C}^\infty(M \times [0,T), \mathbb{R})$ solves*

$$\frac{\partial u}{\partial t} \leq \Delta_{g_t} u.$$

Let $\alpha \in \mathbb{R}$. If $u(\cdot, 0) \leq \alpha$, then $u(\cdot, t) \leq \alpha$ for all $t \in [0,T)$.

Remark 4.1.2. By setting $\alpha := \max_M u(\cdot, 0)$, it follows immediately that u attains its supremum on $M \times \{0\}$, that is

$$\sup_{M \times [0,T)} u = \max_M u(\cdot, 0).$$

Remark 4.1.3. The *strong* parabolic maximum principle for scalars [Top06, p. 36] tells us that under the same assumptions as in Theorem 4.1.1, $u(\cdot, t) < \alpha$ for all $t \in (0,T)$, unless $u \equiv \alpha$. In particular, if u attains its maximum in the interior of $M \times [0,T)$, i.e. at a point (x,t) with $x \in M$ and $t \in (0,T)$, then u is constant.

Remark 4.1.4. Theorem 4.1.1 is also true after replacing all three $\leq$ by $\geq$ and is called the *weak parabolic minimum principle*. Applying this, we obtain that if g_t is a solution to the Ricci flow and $\mathrm{scal}_{g_0} \geq \alpha$ for some $\alpha \in \mathbb{R}$, then $\mathrm{scal}_{g_t} \geq \alpha$ for all $t \in [0,T)$. Moreover, strictly positive scalar curvature is preserved by the Ricci flow as well.

4.2 Hamilton's maximum principle and the Uhlenbeck trick

For $t \in [0,T)$, let g_t be a smooth family of Riemannian metrics on a compact manifold M and V a vector bundle over M with a time-independent fibre metric h and connections $\widetilde{\nabla}^t$ which are compatible with h. Let further $U \subset V$ be open, $\Phi : U \to U$ be a fibre-preserving smooth map and $C \subseteq U$ a closed subset which is invariant under parallel transport by $\widetilde{\nabla}^t$ for all $t \in [0,T)$ and fibre-wise convex. Then the maximum principle of Hamilton [Ham86, Theorem 4.3] is as follows.

Theorem 4.2.1 (Hamilton). *If C is invariant under the ordinary differential equation*

$$f'(t) = \Phi(f(t))$$

in each fibre, then C is invariant under the partial differential equation

$$\frac{\partial}{\partial t} f_t = \widetilde{\Delta}_{g_t} f_t + \Phi(f_t), \tag{4.1}$$

meaning that each solution of (4.1), i.e. time-dependent section $f_t \in \Gamma(M,V)$, with $f_0(x) \in C$ for all $x \in M$ satisfies that $f_t(x) \in C$ for all $x \in M$ and $t \in [0,T_0)$. Here, $T_0 \leq T$ denotes the maximal existence time of the solution f.

Here, the Laplacian $\widetilde{\Delta}_{g_t}$ of a section $f \in \Gamma(M,V)$ at time t is defined as in Definition 1.4.1 and formed using the metric g_t and the connection $\widetilde{\nabla}^t$.

Remark 4.2.2. In the special case that $V = \mathbb{R}$ is the line bundle equipped with the standard metric h and the usual connection $\widetilde{\nabla}^t = \partial$ and that $\Phi \equiv 0$, Hamilton's maximum principle implies that all intervals $[a,b]$ are invariant under the heat equation. This leads to a version of the weak parabolic maximum principle (see Theorem 4.1.1) and thereby gives an explanation why Theorem 4.2.1 is called a maximum principle even though no maximum appears in its formulation.

As a special case, let us consider the bundle of algebraic curvature tensors $S^2_B(\Lambda^2 T^*M)$ together with the fibre metrics g_t and the induced Levi-Civita connections ∇^{g_t}. In order to reformulate Hamilton's maximum principle in this setting, we use the so-called *Uhlenbeck trick* to get rid of the time-dependence of the metric. More precisely, we fix a vector bundle V which is isomorphic to TM via a bundle isomorphism $u_0 : V \to TM$ with fibre metric $h := u_0^* g_0$. Hence, $u_0 : (V,h) \to (TM, g_0)$ is a bundle isometry. Now, for $t \in [0,T]$, let $u_t = u(t) : V \to TM$ be a family of bundle homomorphisms with $u(0) = u_0$ solving

$$\frac{\partial}{\partial t} u_t = H_t(u_t).$$

Here, H_t is defined as in Definition 1.2.2. One can show that $u_t^* g_t = h$ for all $t \in [0,T]$. Thus, $u_t : (V,h) \to (TM, g_t)$ is a bundle isometry for all $t \in [0,T]$. By pulling back the Levi-Civita connections ∇^{g_t} on TM with the isometries u_t, we obtain compatible connections $\widetilde{\nabla}^t$ on V and similarly for $S^2_B(\Lambda^2 T^*M)$ and $S^2_B(\Lambda^2 V^*)$. More precisely, for sections X in TM and R in $S^2_B(\Lambda^2 T^*M)$, we have that

$$\widetilde{\nabla}^t_X (u_t^* R) = u_t^* (\nabla^{g_t}_X R).$$

Moreover, let $n := \dim(M)$ and $\Phi : \mathcal{A}_n \to \mathcal{A}_n$ be an $O(n)$-equivariant locally Lipschitz continuous map. We define the fibre-preserving map

$$\widetilde{\Phi} := u_0^* \circ \Phi^{g_0} \circ (u_0^{-1})^* : S^2_B(\Lambda^2 V^*) \to S^2_B(\Lambda^2 V^*),$$

where $\Phi^{g_t} : S^2_B(\Lambda^2 T^*M) \to S^2_B(\Lambda^2 T^*M)$ is defined as in Definition 2.1.20. One easily checks that $\widetilde{\Phi} = u_t^* \circ \Phi^{g_t} \circ (u_t^{-1})^*$ for all $t \in [0,T]$ since $u_t^* O^{g_t} = u_0^* O^{g_0}$ for all $t \in [0,T]$. Then each time-dependent section $R_t \in \Gamma(M, S^2_B(\Lambda^2 T^*M))$ is a solution to

$$\nabla_{\frac{\partial}{\partial t}} R_t = \Delta_{g_t} R_t + \Phi^{g_t}(R_t), \tag{4.2}$$

if and only if $\widetilde{R}_t := u_t^* R_t \in \Gamma(M, S_B^2(\Lambda^2 V^*))$ solves

$$\frac{\partial}{\partial t}\widetilde{R}_t = \widetilde{\Delta}_{g_t}\widetilde{R}_t + \widetilde{\Phi}(\widetilde{R}_t). \tag{4.3}$$

In (4.2), ∇ is the connection on the vector bundle $S_B^2(\Lambda^2 T^*M) \to M \times \mathbb{R}$ as introduced in Definition 1.2.2 respectively Remark 1.2.3.

This leads to the following reformulation of Hamilton's maximum principle in this setting.

Theorem 4.2.3. *Let $\Omega \subseteq \mathcal{A}_n$ be an $O(n)$-invariant, closed and convex set. If Ω is invariant under the ordinary differential equation*

$$R'(t) = \Phi(R(t)),$$

then the family of sets $\Omega^{g_t} \subseteq S_B^2(\Lambda^2 T^*M)$ is invariant under the partial differential equation

$$\nabla_{\frac{\partial}{\partial t}} R_t = \Delta_{g_t} R_t + \Phi^{g_t}(R_t), \tag{4.4}$$

i.e. for solutions R to (4.4) with $R_0(x) \in \Omega_x^{g_0}$ for all $x \in M$, we have that $R_t(x) \in \Omega_x^{g_t}$ for all $x \in M$ and $t \in [0, T_0)$. Here, $T_0 \leq T$ denotes the maximal existence time of R.

Proof (using Theorem 4.2.1). Let R be a solution to (4.4) with $R_0(x) \in \Omega^{g_0}$ for all $x \in M$. Then $\widetilde{R} := u^* R$ is a solution to (4.3) with $\widetilde{R}_0(x) \in u_0^* \Omega^{g_0}$ for all $x \in M$. By the assumptions on Ω, we find that $u_0^* \Omega^{g_0}$ is closed, fibre-wise convex and invariant under the ordinary differential equation $S'(t) = \widetilde{\Phi}(S(t))$. Moreover by Lemma 2.1.4, the set Ω^{g_0} is invariant under parallel transport by ∇^{g_0}. Since $u_0^* \Omega^{g_0} = u_t^* \Omega^{g_t}$ for all t (where again we used that $u_t^* O^{g_t} = u_0^* O^{g_0}$ for all t), this yields that this set is invariant under parallel transport by $u_t^* \nabla^{g_t} = \widetilde{\nabla}^t$ for all t. Consequently, Theorem 4.2.1 implies that $\widetilde{R}_t(x) \in u_0^* \Omega^{g_0} = u_t^* \Omega^{g_t}$ for all t and $x \in M$. Thus, $R_t(x) \in \Omega^{g_t}$ for all t and $x \in M$. □

Corollary 4.2.4. *Let $\Omega \subseteq \mathcal{A}_n$ be an $O(n)$-invariant, closed and convex set. If Ω is invariant under the ordinary differential equation* (2.4)*, then Ω is invariant under the Ricci flow (see Definition 2.1.12).*

In the next section, we will generalize this corollary to Bianchi-convex sets.

Remark 4.2.5. Hamilton's maximum principle implies for example that in dimension $n = 3$ non-negative sectional curvature as well as non-negative and positive Ricci curvature are preserved by the Ricci flow, i.e. the sets

$$\{R \in \mathcal{A}_n \mid \lambda_1(R) \geq 0\},\ \{R \in \mathcal{A}_n \mid \lambda_1(R) + \lambda_2(R) \geq 0\} \text{ and } \{R \in \mathcal{A}_n \mid \lambda_1(R) + \lambda_2(R) > 0\},$$

are invariant under the Ricci flow. Here, $\lambda_i(R)$ denotes the i-th eigenvalue of R.

4.3 Generalization for tensors in the Bianchi-convex setting

In this section, we prove a generalization of Hamilton's maximum principle in the setting of Corollary 4.2.4 to Bianchi-convex sets and apply it to the sets $\widetilde{\Omega}_{a,c}$ discussed in Section 3.2.2.

Definition 4.3.1. A closed set $\Omega \subseteq \mathcal{A}_n$ is called *uniformly transversally star-shaped with respect to* $S \in \mathcal{A}_n$, if for each compact set $K \subseteq \mathcal{A}_n$ there exists a constant $r > 0$ such that for each $R \in K \cap \partial\Omega$ there is an $\varepsilon_0 > 0$ such that

$$R + \varepsilon B_r(S - R) \subseteq \Omega$$

for all $\varepsilon \in [0, \varepsilon_0)$.

Remark 4.3.2. Notice that if Ω is uniformly transversally star-shaped with respect to S, then for all $R \in \Omega$ we have that $S - R$ is in the interior of the tangent cone $T_R\Omega$.

Theorem 4.3.3. *Let $\Omega \subseteq \mathcal{A}_n$ be $O(n)$-invariant, closed, Bianchi-convex and uniformly transversally star-shaped with respect to λI for some $\lambda \in \mathbb{R}$. If Ω is invariant under the ordinary differential equation* (2.4)*, then Ω is invariant under the Ricci flow (see Definition 2.1.12)*

In Theorem 4.3.3, I denotes the identity in $\mathcal{A}_n$. Before we prove this maximum principle, we show two auxiliary lemmas.

Lemma 4.3.4. *If a closed set $\Omega \subseteq \mathcal{A}_n$ is uniformly transversally star-shaped with respect to $S \in \mathcal{A}_n$, then for each $R \in \Omega$, we have that $R + \alpha(S - R)$ is in the interior of Ω for all $\alpha \in (0, 1)$.*

Proof. From the assumption, it immediately follows that $S - R \in T_R\Omega$ for all $R \in \partial\Omega$. Therefore, by Proposition 2.3.3, Ω is invariant under the ordinary differential equation

$$R'(t) = S - R(t). \tag{4.5}$$

Let $R \in \Omega$. Then $R(t) := R + (1 - e^{-t})(S - R)$ is a solution to (4.5) with $R(0) = R$. Therefore, $R(t) \in \Omega$ for all $t \in [0, \infty)$, which by the closedness of Ω means that $R + \alpha(S - R) \in \Omega$ for all $\alpha \in [0, 1]$.

Now, let $R \in \partial\Omega$, $K \subseteq \mathcal{A}_n$ be a compact set containing R and $r > 0$ be as in Definition 4.3.1. By assumption, there is an $\varepsilon_0 > 0$ such that $R + \varepsilon B_r(S - R) \subseteq \Omega$ for all $\varepsilon \in [0, \varepsilon_0)$. It follows that $(1 - \alpha) \cup_{\varepsilon \in [0,\varepsilon_0)} (R + \varepsilon B_r(S - R)) + \alpha S \subseteq \Omega$ for all $\alpha \in [0, 1]$. Therefore, $(1 - \alpha)R + \alpha S$ is contained in the interior of Ω for all $\alpha \in (0, 1)$.

If R is in the interior of Ω, then there is a neighborhood U of R which is contained in the interior of Ω as well. Therefore, $(1 - \alpha)U + \alpha S \subseteq \Omega$ for all $\alpha \in [0, 1]$, which shows that $(1 - \alpha)R + \alpha S$ is in the interior of Ω for all $\alpha \in [0, 1)$. □

Lemma 4.3.5. *If a closed set $\Omega \subseteq \mathcal{A}_n$ is uniformly transversally star-shaped with respect to $S \in \mathcal{A}_n$, then for each compact set $K \subseteq \mathcal{A}_n$, we have that*

$$-a := \sup\langle n, S - R\rangle < 0, \tag{4.6}$$

where the supremum is taken over all $R \in K \cap \partial\Omega$ and $n \in \mathcal{A}_n$ with $\langle n, v\rangle \leq 0$ for all $v \in T_R\Omega$ and $\|n\| = 1$ (i.e. outward pointing generalized normal vectors n on $\partial\Omega$ at R).

Proof. Let $K \subseteq \mathcal{A}_n$ be compact, $r > 0$ as in Definition 4.3.1, $R \in K \cap \partial\Omega$ and $n \in \mathcal{A}_n$ with and $\|n\| = 1$ and $\langle n, v\rangle \leq 0$ for all $v \in T_R\Omega$. By assumption, there is an $\varepsilon_0 > 0$ such that $R + \varepsilon B_r(S - R) \subseteq \Omega$ for all $\varepsilon \in [0, \varepsilon_0)$. Hence, $B_r(S - R) \subseteq T_R\Omega$ and by scale-invariance of $T_R\Omega$, it follows that $\mathbb{R}_{>0}B_r(S - R) \subseteq T_R\Omega$. Let $\alpha \in (0, \pi)$ denote the opening angle of the cone $\mathbb{R}_{>0}B_r(S - R)$. Then

$$\arccos\left(\frac{\langle n, S - R\rangle}{\|S - R\|}\right) = \sphericalangle(n, S - R) > \frac{\alpha}{2} + \frac{\pi}{2}$$

and since $\frac{\alpha}{2} + \frac{\pi}{2} \in (\frac{\pi}{2}, \pi)$, we find that

$$\frac{\langle n, S - R\rangle}{\|S - R\|} < \cos\left(\frac{\alpha}{2} + \frac{\pi}{2}\right).$$

Hence,

$$\langle n, S - R\rangle < \|S - R\| \cos\left(\frac{\alpha}{2} + \frac{\pi}{2}\right) \leq \max_{\widetilde{R} \in K \cap \partial\Omega} \|S - \widetilde{R}\| \cos\left(\frac{\alpha}{2} + \frac{\pi}{2}\right) < 0.$$

Since α only depends on K, this finishes the proof. □

Now, we are in the position to prove the maximum principle in the Bianchi-convex setting.

Proof of Theorem 4.3.3. Let M be an n-dimensional compact manifold and g_t, $t \in [0, T)$, be a solution to the Ricci flow with g_0 satisfying Ω, i.e. with $Rm_{g_0}(x) \in \Omega_x^{g_0}$ for all $x \in M$. Let $T_1 \in (0, T)$. We will show that $Rm_{g_t} \in \Omega^{g_t}$ for all $t \in [0, T_1]$. To this end, let $a > 0$ be defined by (4.6) as in Lemma 4.3.5 with $K = B_r(0)$, were $r > 0$ is so large that $\mathcal{C}(M, g_t) \subseteq B_r(0)$ for all $t \in [0, T_1]$, and set

$$\begin{aligned} L &:= \max_{(x,t) \in M \times [0,T_1]} \|Rm_{g_t}(x) + Rm_{g_t}(x) \# I_{g_t}(x)\|_{g_t} \\ \text{and} \quad P &:= \max_{(x,t) \in M \times [0,T_1]} \|\lambda I_{g_t}(x) - Rm_{g_t}(x)\|_{g_t}, \end{aligned}$$

where $I_{g_t} \in \Gamma(M, S_B^2(\Lambda^2 T^* M))$ with $I_{g_t}(x)$ being the identity in $S_B^2(\Lambda^2 T_x^* M)$ with respect to g_t for all $x \in M$. (Recall that the set of possible algebraic curvature tensors $\mathcal{C}(M, g)$ of a Riemannian manifold (M, g) was defined in Definition 2.1.11.) By the compactness of $M \times [0, T_1]$, the constants r, L and P are finite. Moreover, we choose $b > \frac{2|\lambda| L}{a}$ and $\varepsilon_0 \in (0, 1)$ such that

$$\varepsilon_0 e^{bT_1} < \min\left\{\frac{1}{2}, \frac{ab - 2|\lambda| L}{\lambda^2 \sqrt{2n(n-1)^3} + bP}\right\}.$$

For $\varepsilon \in (0, \varepsilon_0)$, we define

$$R_t^\varepsilon := Rm_{g_t} + \varepsilon e^{bt}(\lambda I_{g_t} - Rm_{g_t}) = (1 - \varepsilon e^{bt}) Rm_{g_t} + \varepsilon e^{bt} \lambda I_{g_t}$$

for $t \in [0, T)$. Using Lemma 1.4.2 and that $I + I^\# = (n-1)I$, by [BW08, Lemma 2.1], we find that

$$\begin{aligned} \nabla_{\frac{\partial}{\partial t}} R_t^\varepsilon &\overset{1.4.2}{=} -\varepsilon b e^{bt} Rm_{g_t} + (1 - \varepsilon e^{bt}) \left(\Delta_{g_t} Rm_{g_t} + Rm_{g_t}^2 + Rm_{g_t}^\#\right) + \varepsilon b e^{bt} \lambda I_{g_t} \\ &= \Delta_{g_t} R_t^\varepsilon + \varepsilon b e^{bt} \left(\lambda I_{g_t} - Rm_{g_t}\right) \\ &\quad + \frac{1}{1 - \varepsilon e^{bt}} \left((R_t^\varepsilon)^2 + (R_t^\varepsilon)^\# - \varepsilon^2 e^{2bt} \lambda^2 (I_{g_t} + I_{g_t}^\#) - 2(1 - \varepsilon e^{bt}) \varepsilon e^{bt} \lambda (Rm_{g_t} + Rm_{g_t} \# I_{g_t})\right) \\ &= \Delta_{g_t} R_t^\varepsilon + \varepsilon b e^{bt} \left(\lambda I_{g_t} - R_t^\varepsilon\right) + \varepsilon^2 b e^{2bt} \left(\lambda I_{g_t} - Rm_{g_t}\right) \\ &\quad + \frac{1}{1 - \varepsilon e^{bt}} \left((R_t^\varepsilon)^2 + (R_t^\varepsilon)^\#\right) - \frac{\varepsilon^2 e^{2bt} \lambda^2 (n-1)}{1 - \varepsilon e^{bt}} I_{g_t} - 2\varepsilon e^{bt} \lambda (Rm_{g_t} + Rm_{g_t} \# I_{g_t}). \end{aligned} \tag{4.7}$$

By assumption and Lemma 4.3.4, $R_0^\varepsilon = Rm_{g_0} + \varepsilon(\lambda I_{g_0} - Rm_{g_0})$ is in the interior of Ω^{g_0}. We claim that R_t^ε is in the interior of Ω^{g_t} for all $t \in [0, T_1]$. Suppose this is not true. Then there is a minimal time $t_0 \in (0, T_1]$ such that $R_{t_0}^\varepsilon(x_0) \in \partial\Omega_{x_0}^{g_{t_0}}$ for some $x_0 \in M$, since M is compact. Hence,

$R_t^\varepsilon(x) \in \Omega_x^{g_t}$ for all $t \in [0, t_0]$ and $x \in M$, and in particular $R_{t_0}^\varepsilon \in \Gamma(M, \Omega^{g_{t_0}})$. Let $t \mapsto p_t \in O_{x_0}^{g_t}$ be parallel and set $S_0 := p_{t_0}^* R_{t_0}^\varepsilon(x_0) \in \partial\Omega$. Furthermore, for $i = 1, \ldots, n$, set $b_i := p_{t_0}(e_i)$, where $(e_1, \ldots, e_n)$ denotes the standard basis of $\mathbb{R}^n$. Then $(b_1, \ldots, b_n)$ is an orthonormal basis of $T_{x_0}M$ with respect to g_{t_0}. We choose $\delta > 0$ such that

$$\delta \sum_{i=1}^n \|\nabla_{b_i}^{g_{t_0}} R_{t_0}^\varepsilon\|_{g_{t_0}}^2 < \varepsilon e^{bt_0} \left(ab - 2|\lambda| L - \varepsilon_0 e^{bT_1} \lambda^2 \sqrt{2n(n-1)^3} + \varepsilon_0 b e^{bT_1} P \right).$$

By the choice of the constants b and ϵ_0, the right-hand side of this inequality is positive. Since Ω is Bianchi-convex, there is a supporting submanifold N of Ω in S_0 with

$$\sum_{i=1}^n \mathbf{II}_S^N(T_i, T_i) \le \delta \sum_{i=1}^n \|T_i\|^2$$

for all $S \in N$ and $(T_1, \ldots, T_n) \in (T_S N)^n$ that satisfy the second Bianchi identity. It follows that $(p_{t_0}^{-1})^* N$ is a supporting submanifold of $\Omega_{x_0}^{g_{t_0}}$ in $R_{t_0}^\varepsilon(x_0)$ with

$$\sum_{i=1}^n \mathbf{II}_S^{(p_{t_0}^{-1})^* N}(T_i, T_i) \le \delta \sum_{i=1}^n \|T_i\|_{g_{t_0}}^2$$

for all $S \in (p_{t_0}^{-1})^* N$ and $(T_1, \ldots, T_n) \in (T_S (p_{t_0}^{-1})^* N)^n$ that satisfy the second Bianchi identity.
By $\mathbf{n}_{S_0}$, we denote the unit normal on N at S_0 pointing in the opposite direction of Ω. Then $\mathbf{n}_{R_{t_0}^\varepsilon(x_0)} := (p_{t_0}^{-1})^* \mathbf{n}_{S_0}$ is the unit normal on $(p_{t_0}^{-1})^* N$ at $R_{t_0}^\varepsilon(x_0)$ pointing in the opposite direction of $\Omega_{x_0}^{g_{t_0}}$.
Let further r^N be a signed distance function from N. Now, using that Ω is invariant under the ordinary differential equation (2.4) in combination with Proposition 2.3.3 and Lemma 2.2.4, and applying Lemma 3.1.10, we can compute that

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_0} r^N(p_t^* R_t^\varepsilon(x_0)) =\ & dr_{S_0}^N \left(\left.\frac{d}{dt}\right|_{t=t_0} p_t^* R_t^\varepsilon(x_0) \right) \\
\overset{1.2.6}{=}\ & dr_{S_0}^N \left(p_{t_0}^* \nabla_{\frac{\partial}{\partial t}} R_t^\varepsilon(x_0) \Big|_{t=t_0} \right) \\
\overset{(4.7)}{=}\ & \left\langle \mathbf{n}_{S_0}, p_{t_0}^* \Delta_{g_{t_0}} R_{t_0}^\varepsilon(x_0) \right\rangle + \frac{1}{1 - \varepsilon e^{bt_0}} \left\langle \mathbf{n}_{S_0}, p_{t_0}^* \left(R_{t_0}^\varepsilon(x_0)^2 + R_{t_0}^\varepsilon(x_0)^\# \right) \right\rangle \\
& - 2\varepsilon e^{bt_0} \lambda \left\langle \mathbf{n}_{S_0}, p_{t_0}^* \left(Rm_{g_{t_0}}(x_0) + Rm_{g_{t_0}}(x_0) \# I_{g_{t_0}}(x_0) \right) \right\rangle \\
& - \frac{\varepsilon^2 e^{2bt_0} \lambda^2 (n-1)}{1 - \varepsilon e^{bt_0}} \langle \mathbf{n}_{S_0}, I \rangle + \varepsilon b e^{bt_0} \left\langle \mathbf{n}_{S_0}, p_{t_0}^* \left(\lambda I_{g_{t_0}}(x_0) - R_{t_0}^\varepsilon(x_0) \right) \right\rangle \\
& + \varepsilon^2 b e^{2bt_0} \left\langle \mathbf{n}_{S_0}, p_{t_0}^* \left(\lambda I_{g_{t_0}}(x_0) - Rm_{g_{t_0}}(x_0) \right) \right\rangle \\
\le\ & \left\langle (\mathbf{n}_{R_{t_0}^\varepsilon(x_0)}, \Delta_{g_{t_0}} R_{t_0}^\varepsilon(x_0) \right\rangle_{g_{t_0}} + 2 \underbrace{\left\langle \mathbf{n}_{S_0}, S_0^2 + S_0^\# \right\rangle}_{\overset{2.3.3, 2.2.4}{\le} 0} + \frac{\varepsilon^2 e^{2bt_0} \lambda^2 (n-1)}{1 - \varepsilon e^{bt_0}} \|I\| \\
& - 2\varepsilon e^{bt_0} \lambda \left\langle \mathbf{n}_{R_{t_0}^\varepsilon(x_0)}, \left(Rm_{g_{t_0}}(x_0) + Rm_{g_{t_0}}(x_0) \# I_{g_{t_0}}(x_0) \right) \right\rangle_{g_{t_0}} \\
& + \varepsilon b e^{bt_0} \underbrace{\left\langle \mathbf{n}_{R_{t_0}^\varepsilon(x_0)}, \left(\lambda I_{g_{t_0}}(x_0) - R_{t_0}^\varepsilon(x_0) \right) \right\rangle_{g_{t_0}}}_{\le -a} \\
& + \varepsilon^2 b e^{2bt_0} \left\langle \mathbf{n}_{R_{t_0}^\varepsilon(x_0)}, \left(\lambda I_{g_{t_0}}(x_0) - Rm_{g_{t_0}}(x_0) \right) \right\rangle_{g_{t_0}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\langle \mathbf{n}_{R^{\varepsilon}_{t_0}(x_0)}, \Delta_{g_{t_0}} R^{\varepsilon}_{t_0}(x_0) \right\rangle_{g_{t_0}} + \varepsilon^2 e^{2bt_0} \lambda^2 \sqrt{2n(n-1)^3} \\
&\quad + 2\varepsilon e^{bt_0} |\lambda| \underbrace{\left\| Rm_{g_{t_0}}(x_0) + Rm_{g_{t_0}}(x_0) \# I_{g_{t_0}}(x_0) \right\|_{g_{t_0}}}_{\leq L} \\
&\quad - \varepsilon b e^{bt_0} a + \varepsilon^2 b e^{2bt_0} \underbrace{\| \lambda I_{g_{t_0}}(x_0) - R^{\varepsilon}_{t_0}(x_0) \|_{g_{t_0}}}_{\leq P} \\
&\overset{3.1.10}{\leq} \delta \sum_{i=1}^{n} \| \nabla^{g_{t_0}}_{b_i} R^{\varepsilon}_{t_0} \|^2_{g_{t_0}} + \varepsilon e^{bt_0} \Big(2|\lambda| L - ab + \varepsilon_0 e^{bT_1} \lambda^2 \sqrt{2n(n-1)^3} + \varepsilon_0 e^{bT_1} P \Big) \\
&< 0.
\end{aligned}
$$

Here, we could apply Lemma 3.1.10 for $C = \Omega^{g_{t_0}}$ since with $Rm_{g_{t_0}}$ also $R^{\varepsilon}_{t_0} = Rm_{g_{t_0}} + \varepsilon e^{bt_0}(\lambda I_{g_{t_0}} - Rm_{g_{t_0}})$ satisfies the second Bianchi identity and, in addition, $R^{\varepsilon}_{t_0} \in \Gamma(M, \Omega^{g_{t_0}})$. This together with the fact that $r^N(p^*_{t_0} R^{\varepsilon}_{t_0}(x_0)) = r^N(S_0) = 0$ yields that there is a $\mu > 0$ such that

$$r^N(p^*_t R^{\varepsilon}_t(x_0)) > 0$$

for all $t \in (t_0 - \mu, t_0)$. Therefore, $p^*_t R^{\varepsilon}_t(x_0) \notin \Omega$, thus $R^{\varepsilon}_t(x_0) \notin \Omega^{g_t}_{x_0}$ for all $t \in (t_0 - \mu, t_0)$. This, however, is a contradiction to $R^{\varepsilon}_t(x_0) \in \Omega^{g_t}_{x_0}$ for all $t \in [0, t_0]$ as assumed in the beginning of the proof. Hence, as we claimed, $R^{\varepsilon}_t(x)$ is in the interior of $\Omega^{g_t}_x$ for all $x \in M$ and $t \in [0, T_1]$.

Since Ω^{g_t} is closed and $R^{\varepsilon}_t(x)$ converges to $Rm_{g_t}(x)$ as ε tends to zero for each $t \in [0, T_1]$ and $x \in M$, we have that $Rm_{g_t}(x) \in \Omega^{g_t}_x$ for all $t \in [0, T_1]$ and $x \in M$. Since T_1 was chosen arbitrarily, this is true for all $t \in [0, T)$. Consequently, we have shown that Ω is invariant under the Ricci flow. □

Remark 4.3.6. Let us consider the special case that $n = 3$. As we have seen in Section 3.2.2, for $a \in (\frac{1}{3}, \frac{2}{5})$ and $c > 0$, the sets $\widetilde{\Omega}_{a,c} \subset \mathcal{A}_3$ are $O(3)$-invariant, closed, Bianchi-convex and invariant under the ordinary differential equation (2.4). Moreover, it is easy to varify that $\widetilde{\Omega}_{a,c}$ satisfies the cone condition with respect to I. Therefore, Theorem 4.3.3 yields that $\widetilde{\Omega}_{a,c}$ is invariant under the Ricci flow.

Chapter 5

Bianchi-convex functions

In this chapter, we introduce the notion of a Bianchi-convex function and give a first connection to the Ricci flow on a compact manifold. The sublevel sets of such functions are Bianchi-convex sets. In order to obtain examples of Bianchi-convex functions, we show that smooth functions, the sublevel sets of which are strictly convex cones, can be reparametrized in such a way that (restricted appropriately) they become strictly Bianchi-convex. This will lead to rigidity results of complete shrinking gradient Ricci solitons as an application of Theorem 6.2.9 in the subsequent chapter.

5.1 The definition and first properties

We start with the definition of a Bianchi-convex function.

Definition 5.1.1. Let $U \subseteq \mathcal{A}_n$ be open. A smooth function $F : U \to \mathbb{R}$ is called *Bianchi-convex at* $R \in U$, if for all $(T_1, \ldots, T_n) \in (T_R U)^n$ satisfying the second Bianchi identity, we have that

$$\sum_{i=1}^{n} \mathrm{Hess}_R F(T_i, T_i) \geq 0.$$

Moreover, we say that F is *strictly Bianchi-convex at* $R \in U$, if the inequality above is strict unless $T_i = 0$ for each i. If F is (strictly) Bianchi-convex at all $R \in U$, we call F *(strictly) Bianchi-convex.*

Remark 5.1.2. Let V be a vector space, $F : V \to \mathbb{R}$ a smooth function and $c \in \mathbb{R}$ a regular value of F. By the submersion theorem [Kli95, Theorem 1.3.3], we have that $F^{-1}(c)$ is a submanifold of V of codimension one, and that for all $v \in F^{-1}(c)$ and $X, Y \in T_v F^{-1}(c) = \ker(dF_v)$,

$$\mathbf{n}_v = \frac{\mathrm{grad}_v F}{\|\mathrm{grad}_v F\|}$$

is the outward pointing unit normal on $F^{-1}(c) = \partial F^{-1}((-\infty, c])$ at v. Therefore,

$$\begin{aligned} \mathrm{II}_v^{F^{-1}(c)}(X,Y) &= -\langle Y, \nabla_X \mathbf{n}\rangle = -\left\langle Y, \nabla_X \frac{\mathrm{grad} F}{\|\mathrm{grad} F\|}\right\rangle \\ &= -\frac{1}{\|\mathrm{grad}_v F\|}\langle Y, \nabla_X \mathrm{grad} F\rangle - \partial_X\left(\frac{1}{\|\mathrm{grad} F\|}\right)\underbrace{\langle Y, \mathrm{grad}_v F\rangle}_{=0} \\ &= -\frac{1}{\|\mathrm{grad}_v F\|}\mathrm{Hess}_v F(X,Y). \end{aligned}$$

Lemma 5.1.3. *Let $F : \mathcal{A}_n \to \mathbb{R}$ be a Bianchi-convex function with $dF_R \neq 0$ for all $R \in \mathcal{A}_n$. Then all sublevel sets of F are Bianchi-convex.*

Proof. Let $c \in \mathbb{R}$. We show that the sublevel set $\{F \leq c\}$ is Bianchi-convex. To this end, let $R \in \partial\{F \leq c\} = F^{-1}(c)$ and $(T_1, \dots, T_n) \in (T_R F^{-1}(c))^n$ satisfying the second Bianchi identity. Since F is Bianchi-convex and $T_R F^{-1}(c) \subseteq T_R \mathcal{A}_n$, Remark 5.1.2 yields that

$$-\|\mathrm{grad}_R F\| \sum_{i=1}^{n} \mathbf{II}_R^{F^{-1}(c)}(T_i, T_i) = \sum_{i=1}^{n} \mathrm{Hess}_R F(T_i, T_i) \geq 0.$$

Hence,

$$\sum_{i=1}^{n} \mathbf{II}_R^{F^{-1}(c)}(T_i, T_i) \leq 0,$$

which proves the statement. □

Considering Bianchi-convex functions along the Riemannian curvature tensor of a Ricci flow, we obtain the following result.

Remark 5.1.4. Throughout, a *sublevel set of a function F* will always be a non-trivial sublevel set of F, that is a set $F^{-1}((-\infty, c])$ for some $c \in \mathrm{im}(F)$.

Proposition 5.1.5. *Let $\Omega \subseteq \mathcal{A}_n$ be an open and $O(n)$-invariant set and $F : \Omega \to \mathbb{R}$ a smooth, $O(n)$-invariant and Bianchi-convex function, the sublevel sets of which are closed and invariant under the ordinary differential equation* (2.4). *Moreover, let (M, g_0) be a compact n-dimensional Riemannian manifold and $g(t)$, $t \in [0, T)$, be the solution to the Ricci flow with $g(0) = g_0$. If g_0 satisfies Ω, then $g(t)$ satisfies Ω for all $t \in [0, T)$.*
Moreover, the function $F^g \circ Rm_g : M \times [0, T) \to \mathbb{R}$ satisfies the heat inequality

$$\frac{\partial}{\partial t}\left(F^{g(t)} \circ Rm_{g(t)}\right) \leq \Delta_{g(t)}\left(F^{g(t)} \circ Rm_{g(t)}\right).$$

Hence, by the parabolic maximum principle either $F^g \circ Rm_g$ is constant or $\max_M(F^{g(t)} \circ Rm_{g(t)})$ *is strictly decreasing in t. Moreover, if F is strictly Bianchi-convex and $F^g \circ Rm_g$ is constant, we have that (M, g_0) is locally symmetric, i.e. $\nabla^{g_0} Rm_{g_0} \equiv 0$.*

Remark 5.1.6. In fact, instead of being closed (in $\mathcal{A}_n$), it suffices to assume that the sublevel sets of F are closed in a set $U \subseteq \mathcal{A}_n$, which is invariant under the Ricci flow, i.e. if g_0 satisfies U, then g_t satisfies U for all $t \in [0, T)$. For example, this is true for $U := \{R \in \mathcal{A}_n \mid \mathrm{scal}(R) > 0\}$.

Proof. Throughout the proof, we write $g_t := g(t)$. The total space of the orthonormal frame bundle O^{g_0} on (M, g_0) is compact, because (M, g_0) is compact. Therefore, $\mathcal{C}(M, g_0)$ is compact as it is the image of the compact set O^{g_0} under the continuous function $O^{g_0} \to \mathcal{A}_n : p \mapsto p^* Rm_{g_0}(\pi(p))$. Moreover, since g_0 satisfies Ω, we have that $\mathcal{C}(M, g_0) \subset \Omega$. Both together implies that

$$a := \max_{R \in \mathcal{C}(M, g_0)} F(R)$$

exists, thus $\mathcal{C}(M, g_0) \subseteq F^{-1}((-\infty, a])$. For continuity reasons there is a maximal $t_0 > 0$ such that $\mathcal{C}(M, g_t) \subset \Omega$ for all $t \in [0, t_0)$. This gives that the function $F^g \circ Rm_g$ is well defined on $M \times [0, t_0)$.

Therefore, we can calculate that

$$\begin{aligned}\frac{\partial}{\partial t}F^{g_t}(Rm_{g_t}(x)) &\overset{2.1.19}{=} dF^{g_t}_{Rm_{g_t}(x)}\left(\nabla_{\frac{\partial}{\partial t}}Rm_{g_t}(x)^{\mathrm{v}}\right)\\ &\overset{(1.12)}{=} dF^{g_t}_{Rm_{g_t}(x)}\left(\Delta_{g_t}Rm_{g_t}(x)^{\mathrm{v}}\right) + dF^{g_t}_{Rm_{g_t}(x)}\left(Rm^2_{g_t}(x)^{\mathrm{v}} + Rm^{\#}_{g_t}(x)^{\mathrm{v}}\right)\\ &\overset{2.1.18}{=} \left(\Delta_{g_t}(F^{g_t}\circ Rm_{g_t})\right)(x) + dF^{g_t}_{Rm_{g_t}(x)}\left(Rm^2_{g_t}(x)^{\mathrm{v}} + Rm^{\#}_{g_t}(x)^{\mathrm{v}}\right)\\ &\quad - \sum_{i=1}^{n}\mathrm{Hess}_{p_t^* Rm_{g_t}(x)}F\left(p_t^*\nabla^{g_t}_{e_i^t}Rm_{g_t}(x), p_t^*\nabla^{g_t}_{e_i^t}Rm_{g_t}(x)\right)\end{aligned} \tag{5.1}$$

for $t \in [0, t_0)$ and $x \in M$, where $(e_1^t, \ldots, e_n^t)$ is a g_t-orthonormal basis of T_xM and $p_t \in O_x^{g_t}$. Here, we applied Lemma 2.1.19 in the first and the evolution equation of the Riemannian curvator operator (1.12) in the second step. The last equality holds due to Lemma 2.1.18. Since $p_t : (\mathbb{R}^n, \langle\cdot,\cdot\rangle) \to (T_xM, g_t(x))$ is an isometry, $(b_1^t, \ldots, b_n^t)$, where $b_i^t := p_t^{-1}(e_i^t)$, is an orthonormal basis of $\mathbb{R}^n$. We set $T_i := p_t^*\nabla^{g(t)}_{e_i^t}Rm_{g_t}(x) \in \mathcal{A}_n$. Then

$$\begin{aligned}&T_i(b_j\wedge b_k) + T_j(b_k\wedge b_i) + T_k(b_i\wedge b_j)\\ &\quad= \nabla^{g_t}_{e_i^t}Rm_{g_t}(x)(e_j\wedge e_k) + \nabla^{g_t}_{e_j^t}Rm_{g_t}(x)(e_k\wedge e_i) + \nabla^{g_t}_{e_k^t}Rm_{g_t}(x)(e_i\wedge e_j) = 0.\end{aligned}$$

Consequently, $(T_1, \ldots, T_n) \in \mathcal{A}_n^n$ satisfies the second Bianchi identity, and therefore the last summand on the right-hand side of (5.1) is non-positive due to the Bianchi-convexity of F.
Since the sublevel sets of F are invariant under (2.4), from the lemmas 2.3.4 and 2.1.17 it follows that

$$dF^{g_t}_{Rm_{g_t}(x)}\left(Rm^2_{g_t}(x)^{\mathrm{v}} + Rm^{\#}_{g_t}(x)^{\mathrm{v}}\right) \overset{2.1.17}{=} dF_{p_t^* Rm_{g_t}(x)}\left(p_t^*\left(Rm^2_{g_t}(x) + Rm^{\#}_{g_t}(x)\right)\right) \overset{2.3.4}{\leq} 0.$$

Thus, the second summand on the right-hand sight of (5.1) is also non-positive. Put together, we obtain that $F^g \circ Rm_g : M \times [0, t_0) \to \mathbb{R}$ satisfies the heat inequality

$$\frac{\partial}{\partial t}(F^{g_t}\circ Rm_{g_t}) \leq \Delta_{g_t}(F^{g_t}\circ Rm_{g_t}). \tag{5.2}$$

The weak parabolic maximum principle (see Remark (4.1.2)) yields that

$$\sup_{M\times[0,t_0)}(F^g\circ Rm_g) = \max_M(F^{g_0}\circ Rm_{g_0}) = a$$

and therefore $\mathcal{C}(M, g_t) \subseteq F^{-1}((-\infty, a])$ for all $t \in [0, t_0)$. Suppose that $t_0 < T$. Then by continuity reasons we find that

$$\mathcal{C}(M, g_{t_0}) \subseteq F^{-1}((-\infty, a]) \subset \Omega,$$

since the sublevel sets of F are closed. As above, this yields that there is a $t_1 > t_0$ such that $\mathcal{C}(M, g_t) \subset \Omega$ for all $t \in [t_0, t_1)$, in contradiction to the maximality of t_0. Consequently, $t_0 = T$ and thus g_t satisfies Ω for all $t \in [0, T)$.

Thus, we have shown that the function $F^g \circ Rm_g$ is defined on all of $M \times [0, T)$ and satisfies the heat inequality (5.2).

Now, let F be strictly Bianchi-convex and $F^g \circ Rm_g$ be constant, in particular a solution to the heat equation

$$\frac{\partial}{\partial t}(F^{g_t}\circ Rm_{g_t}) = \Delta_{g_t}(F^{g_t}\circ Rm_{g_t}).$$

Then combining this with (5.1) results in

$$\sum_{i=1}^{n} \mathrm{Hess}_{p_t^* Rm_{g_t}(x)} F \left(p_t^* \nabla^{g_t}_{e_i^t} Rm_{g_t}(x), p_t^* \nabla^{g_t}_{e_i^t} Rm_{g_t}(x) \right) = 0$$

for all $t \in [0,T)$, $x \in M$, $p_t \in O_x^{g_t}$ and g_t-orthonormal bases $(e_1^t, \dots, e_n^t)$ of T_xM. Since F is strictly Bianchi-convex, this means that

$$p_t^* \nabla^{g_t}_{e_i^t} Rm_{g_t}(x) = 0$$

for $i = 1, \dots n$. As a consequence, $\nabla^{g_t} Rm_{g_t} \equiv 0$, that is (M, g_t) is locally symmetric, for all $t \in [0,T)$. □

5.2 A reparametrization theorem

The sublevel sets of convex functions are convex as is well known. However, a function, the sublevel sets of which are convex, need not to be convex. The following theorem shows that (in the case that the sublevel sets are strictly convex cones) up to reparametrization and restriction such a function is at least Bianchi-convex. The goal of this section is to prove this statement. Here, the cone $\mathcal{B}_n$ defined by

$$\mathcal{B}_n := \left\{ R \in \mathcal{A}_n \mid R|_{\Lambda^2(v^\perp)} \not\equiv 0 \text{ for all } v \in \mathbb{R}^n \setminus \{0\} \right\},$$

where the restriction on $\Lambda^2(v^\perp)$ is meant in the endomorphism sense, plays an important role.

Theorem 5.2.1. *Let $n \geq 3$, $\Omega \subseteq \mathcal{A}_n \setminus \{0\}$ an open cone and $F : \Omega \to \mathbb{R}$ a smooth function, the sublevel sets of which are strictly convex cones. Let further U be an open cone with $\overline{U} \subset \Omega \cap \mathcal{B}_n$ (where the closure is taken in $\mathcal{A}_n \setminus \{0\}$) and assume that* $\mathrm{Hess}_R F|_{R^\perp}$ *is positive definite for all $R \in \overline{U}$ with $dF_R = 0$. Then there exists a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi' > 0$ such that $\varphi \circ F|_U$ is strictly Bianchi-convex.*

Remark 5.2.2. By a *strictly convex cone* we mean a cone, i.e. a scale-invariant set, the base of which is strictly convex.

Example 5.2.3. As an anticipation on the next chapter, examples for functions F as in Theorem 5.2.1 are the following:

$$\Omega \to \mathbb{R} : R \mapsto \frac{\|R\|^2}{\|\mathrm{ric}(R)\|^2}$$
$$\Theta \to \mathbb{R} : R \mapsto \frac{\|R_{\mathrm{ric}_0} + R_W\|^2}{\|R_I\|^2},$$

where Ω and Θ will be defined in Section 6.3.2 respectively Section 6.3.3. Moreover, in these sections, using Theorem 5.2.1 and Lemma 5.2.13, we will show that reparametrizing and restricting these functions appropriately provides strictly Bianchi-convex functions which are not convex.

Remark 5.2.4. In order to facilitate following the somewhat technical proofs of the subsequent auxiliary lemmas, we sketch the idea of the proof of Theorem 5.2.1: The wanted reparametrization $\varphi : \mathbb{R} \to \mathbb{R}$ is going to be defined by $\varphi(s) := e^{\kappa s}$ for $s \in \mathbb{R}$, where the main task is to show that $\kappa > 0$ can be chosen appropriately. As we will see in Lemma 5.2.13, for all $R \in U$, we have that $\mathrm{Hess}_R(\varphi \circ F)$ has exactly one negative eigenvalue, all other eigenvalues are positive (unless

$dF_R = 0$ which turns out to be a rather trivial situation). The set of all "negative directions" in T_RU, that is the set of all $S \in T_RU$ with $\mathrm{Hess}_R(\varphi \circ F)(S,S) < 0$, is therefore a double cone. Given a tuple $T := (T_1, \dots, T_n) \in (T_RU)^n \setminus \{0\}$ satisfying the second Bianchi identity, we will show that there is at least one index $i \in \{1, \dots, n\}$ such that T_i has a certain uniform angle to R and a non-negligible length in comparison to the length of T (see Lemma 5.2.16; this uses $U \subset \mathcal{B}_n$). By making κ large enough, we are able to shrink the opening angle of the double cone mentioned above such that T_i becomes a "positive direction" (in the sense analogously to above), i.e. gives a positive contribution to the sum $\sum_{i=1}^n \mathrm{Hess}_R(\varphi \circ F)(T_i, T_i)$ (see Lemma 5.2.14). In order to have that the sum becomes positive as well, we use the fact about the length of T_i (as indicated above).

5.2.1 Scale-invariant functions

In this section, we collect some first properties of scale-invariant functions on Euclidean vector spaces. In particular, we study how their gradient and Hessian behave under rescaling and how the Hessian looks like in a matrix representation. Moreover, we start investigating certain reparametrizations of these functions in the case that their sublevel sets are strictly convex.

Throughout, let V be a Euclidean vector space, $\Omega \subseteq V \setminus \{0\}$ an open cone and $F : \Omega \to \mathbb{R}$ a smooth function, the sublevel sets of which are strictly convex cones.

Lemma 5.2.5. *F is scale-invariant, i.e. $F(\alpha v) = F(v)$ for all $v \in \Omega$ and $\alpha > 0$.*

In particular, the lemma shows that F cannot be defined in 0 unless it is constant.

Proof. Let $v \in \Omega$ and set $F(v) =: c$. Then $v \in F^{-1}(c) = \partial\{F \le c\}$. Since, by assumption, $\{F \le c\}$ is scale-invariant, due to Lemma A.0.1 this is also true for $\partial\{F \le c\}$. Therefore, $\alpha v \in \partial\{F \le c\}$ for all $\alpha > 0$. This, however, implies that $F(v) = c = F(\alpha v)$ for all $\alpha > 0$. □

Lemma 5.2.6. *For all $v \in \Omega$ and $\alpha > 0$, we have that $\mathrm{grad}_v F = \alpha \cdot \mathrm{grad}_{\alpha v} F$.*

Proof. For $\alpha > 0$, we define the map $\mu_\alpha : V \to V : v \mapsto \alpha v$. For all $v \in V$, the differential $d\mu_\alpha|_v$ is given by multiplication with α as well. Since F is scale-invariant, we have that $F \circ \mu_\alpha = F$ for all $\alpha > 0$. This yields that

$$\begin{aligned} \mathrm{grad}_v F &= \mathrm{grad}_v (F \circ \mu_\alpha) = D_v(F \circ \mu_\alpha)^t \\ &= (D_{\mu_\alpha(v)} F \cdot D_v \mu_\alpha)^t = D_v {\mu_\alpha}^t \cdot D_{\alpha v} F^t = \alpha \cdot \mathrm{grad}_{\alpha v} F. \end{aligned}$$ □

Lemma 5.2.7. *For all $v \in \Omega$ and $\alpha > 0$, we have that $\mathrm{Hess}_v F = \alpha^2 \cdot \mathrm{Hess}_{\alpha v} F$, where the Hessian is considered as bilinear form.*

Proof. Let $v \in \Omega$, $X, Y \in T_v\Omega$ and $\alpha > 0$. Then

$$\begin{aligned} \mathrm{Hess}_v F(X,Y) &= \langle (D_X \mathrm{grad} F)(v), Y \rangle \overset{5.2.6}{=} \alpha \cdot \langle (D_X \mathrm{grad}_{\alpha \cdot} F)(v), Y \rangle = \alpha^2 \cdot \langle (D_X \mathrm{grad} F)(\alpha v), Y \rangle \\ &= \alpha^2 \cdot \mathrm{Hess}_{\alpha v} F(X,Y). \end{aligned}$$ □

Lemma 5.2.8. *For all $v \in \Omega$ and $w \in V$, we have that*

$$\mathrm{Hess}_v F(v,w) = -dF_v(w).$$

Proof. Let $w \in V$. Then

$$\begin{aligned}\mathrm{Hess}_v F(v,w) &= \frac{d}{ds}\Big|_{s=0} dF_{v+sw}(v) \\ &= \frac{d}{ds}\Big|_{s=0} \underbrace{dF_{v+sw}(v+sw)}_{=0} - \frac{d}{ds}\Big|_{s=0} dF_{v+sw}(sw) \\ &= -\frac{d}{ds}\Big|_{s=0} s \cdot dF_{v+sw}(w) \\ &= -dF_v(w).\end{aligned}$$

Here, in the third equality, we used that $F(u+su)$ is constant in s for all $u \in \Omega$ by scale-invariance of F (Lemma 5.2.5). Thus, $dF_u(u) = 0$ for all $u \in \Omega$. □

Let $v \in \Omega$ with $dF_v \neq 0$ and set $F(v) := c$. Near v, $F^{-1}(c)$ is a smooth submanifold and we can split

$$T_vU = T_vV = T_vF^{-1}(c) \oplus \mathbb{R}\mathrm{grad}_v F = (v^\perp \cap T_vF^{-1}(c)) \oplus \mathbb{R}v \oplus \mathbb{R}\mathrm{grad}_v F.$$

Let $(b_1, ..., b_{N-2})$ be an orthonormal basis of $v^\perp \cap T_vF^{-1}(c)$ such that

$$\left(b_1, ..., b_{N-2}, b_{N-1} := \frac{v}{\|v\|}, b_N := \frac{\mathrm{grad}_v F}{\|\mathrm{grad}_v F\|}\right)$$

is an orthonormal basis of T_vU, where by N we denote the dimension of V. Since $F(v) = c$, we have that $v \in \partial\{F \leq c\} = F^{-1}(c)$ and since $\{F \leq c\}$ is a strictly convex cone, we have that $\mathrm{Hess}_v F$ restricted to $T_vF^{-1}(c)$ is positive semidefinite and restricted to $v^\perp \cap T_vF^{-1}(c)$ is even positive definite. Together with Lemma 5.2.8, this observation yields the following lemma.

Lemma 5.2.9. *For all $v \in \Omega$ with $dF_v \neq 0$, we have with respect to the basis $(b_1, \ldots, b_N)$ above that*

$$\mathrm{Hess}_v F \mathrel{\widehat{=}} \Big(\mathrm{Hess}_v F(b_i, b_j)\Big)_{i,j} = \begin{pmatrix} A & 0 & b \\ 0 & 0 & e \\ b^t & e & f \end{pmatrix},$$

where $A = A(v) \in \mathrm{SymMat}(N-2 \times N-2, \mathbb{R})$ is positive definite, $b = b(v) \in \mathbb{R}^{N-2}$ and $e = e(v)$, $f = f(v) \in \mathbb{R}$.

Remark 5.2.10. Notice that, by Lemma 5.2.8, we have that

$$e(v) = \mathrm{Hess}_v F\left(\frac{v}{\|v\|}, \frac{\mathrm{grad}_v F}{\|\mathrm{grad}_v F\|}\right) = -\frac{\|\mathrm{grad}_v F\|}{\|v\|}$$

for $v \in \Omega$ with $dF_v \neq 0$.

Notation 5.2.11. For every vector space V, bilinear form $H : V \times V \to \mathbb{R}$ and linear subspace W of V, we will denote $H|_W := H|_{W \times W}$ and by $\lambda_{\min}(H)$ respectively $\lambda_{\max}(H)$ the smallest respectively largest eigenvalue of H, calculated with respect to the metric.

Lemma 5.2.12. *For $s \in \mathbb{R}$, let $\varphi(s) := e^{\kappa s}$ for some $\kappa \in \mathbb{R}$. Then for all $v \in \Omega$ with $dF_v \neq 0$, we have that*

$$\mathrm{Hess}_v(\varphi \circ F) \mathrel{\widehat{=}} \varphi'(F(v)) \begin{pmatrix} A & 0 & b \\ 0 & 0 & e \\ b^t & e & f + \kappa e^2 \|v\|^2 \end{pmatrix}$$

with respect to the orthonormal basis $(b_1, ..., b_N)$, where $A = A(v)$, $b = b(v)$, $e = e(v)$ and $f = f(v)$ are as in Lemma 5.2.9.

Proof. For all $v \in \Omega$, we have that

$$\mathrm{Hess}_v(\varphi \circ F) = \varphi''(F(v)) \cdot dF_v \otimes dF_v + \varphi'(F(v)) \cdot \mathrm{Hess}_v F, \tag{5.3}$$

which immediately implies the statement. □

Lemma 5.2.13. *Let $\varphi : \mathbb{R} \to \mathbb{R} : s \mapsto e^{\kappa s}$, where $\kappa > 0$. Then for all $v \in \Omega$ with $dF_v \neq 0$, the Hessian* $\mathrm{Hess}_v(\varphi \circ F)$ *has exactly one negative eigenvalue, all other eigenvalues are positive.*

Proof. Let $v \in \Omega$ with $dF_v \neq 0$. Then, by Remark 5.2.10, $e(v) \neq 0$. Therefore, in the notation of Lemma 5.2.12, we have that

$$\begin{aligned}
\det(\mathrm{Hess}_v(\varphi \circ F)) &= \varphi'(F(v))^N \det \begin{pmatrix} A & 0 & b \\ 0 & 0 & e \\ b^t & e & f + \kappa e^2 \|v\|^2 \end{pmatrix} \\
&= \varphi'(F(v))^N \det(A) \det \left(\begin{pmatrix} 0 & e \\ e & f + \kappa e^2 \|v\|^2 \end{pmatrix} - \begin{pmatrix} 0 \\ b^t \end{pmatrix} A^{-1} \begin{pmatrix} 0 & b \end{pmatrix} \right) \\
&= \varphi'(F(v))^N \det(A) \det \begin{pmatrix} 0 & e \\ e & f + \kappa e^2 \|v\|^2 - b^t A^{-1} b \end{pmatrix} \\
&= -\varphi'(F(v))^N \det(A) e^2 \\
&< 0.
\end{aligned}$$

It follows that $\mathrm{Hess}_v(\varphi \circ F)$ is invertible and in particular has no vanishing but at least one negative eigenvalue. Since $\mathrm{Hess}_v(\varphi \circ F)|_{(\mathrm{grad}_v F)^\perp}$ is positive semidefinite and $(\mathrm{grad}_v F)^\perp$ has codimension one, we find that $\mathrm{Hess}_v(\varphi \circ F)$ has exactly one negative eigenvalue and all other eigenvalues are positive. □

The following lemma shows that having some uniform lower bound for $\lambda_{\min}(\mathrm{Hess}_v F|_{v^\perp})$, the negative eigenvalue of $\mathrm{Hess}_v F$ can be pushed arbitrarily close to zero using a suitable reparametrization.

Lemma 5.2.14. *Assume that there exists some $\rho > 0$ such that*

$$\lambda_{\min}(\mathrm{Hess}_v F|_{v^\perp}) \geq \frac{\rho}{\|v\|^2} \tag{5.4}$$

for all $v \in \Omega$. Then for each $\kappa > 0$, we have that

$$-\lambda_{\min}(\mathrm{Hess}_v(\varphi \circ F)) \leq \frac{1}{\rho\kappa} \lambda_{\min}(\mathrm{Hess}_v(\varphi \circ F)|_{v^\perp})$$

for all $v \in \Omega$, where $\varphi(s) := e^{\kappa s}$ for $s \in \mathbb{R}$.

Note that the sublevel sets of F being strictly convex cones is equivalent to

$$\lambda_{\min}\left(\mathrm{Hess}_v F|_{v^\perp \cap (\mathrm{grad}_v F)^\perp}\right) > 0$$

for all $v \in \Omega$, which is implied by (5.4). Therefore in Lemma 5.2.14, it suffices to assume that F is scale-invariant together with (5.4).

Proof. Let $\kappa > 0$ and $v \in \Omega$. Due to the scale-invariance of F (see Lemma 5.2.7), we may assume that $\|v\| = 1$. In the case that $dF_v = 0$, let $w \in V$ and write $w = \alpha v + w'$, where $\alpha \in \mathbb{R}$ and $w' \in v^\perp$. Then, using Lemma 5.2.8, we can compute that

$$\begin{aligned}
\mathrm{Hess}_v(\varphi \circ F)(w,w) &\overset{(5.3)}{=} \varphi'(F(v))\mathrm{Hess}_v F(w,w) \\
&= \varphi'(F(v))\left(\alpha^2 \mathrm{Hess}_v F(v,v) + 2\alpha \mathrm{Hess}_v F(v,w') + \mathrm{Hess}_v F(w',w')\right) \\
&\overset{5.2.8}{=} \varphi'(F(v))\mathrm{Hess}_v F(w',w') \\
&\geq \varphi'(F(v))\lambda_{\min}(\mathrm{Hess}_v F|_{v^\perp})\,\|w'\|^2 \\
&\overset{(5.4)}{\geq} 0.
\end{aligned} \tag{5.5}$$

This shows that $\mathrm{Hess}_v(\varphi \circ F)$ is positive semidefinite. Hence,

$$\lambda_{\min}(\mathrm{Hess}_v(\varphi \circ F)) \geq 0 \geq -\frac{1}{\rho\kappa}\lambda_{\min}(\mathrm{Hess}_v(\varphi \circ F)|_{v^\perp}).$$

Now, we consider the case that $dF_v \neq 0$. By Lemma 5.2.12, we have that

$$\mathrm{Hess}_v(\varphi \circ F) \mathrel{\widehat{=}} \varphi'(F(v)) \begin{pmatrix} A & 0 & b \\ 0 & 0 & e \\ b^t & e & f + \kappa e^2 \end{pmatrix}$$

with respect to the orthonormal basis $(b_1, ..., b_N)$, where $A = A(v)$, $b = b(v)$, $e = e(v)$ and $f = f(v)$ are as in Lemma 5.2.9. Hence, by assumption

$$\begin{aligned}
\lambda_{\min}(\mathrm{Hess}_v(\varphi \circ F)|_{v^\perp}) &= \varphi'(F(v))\lambda_{\min}\begin{pmatrix} A & b \\ b^t & f + \kappa e^2 \end{pmatrix} \\
&\geq \varphi'(F(v))\lambda_{\min}\begin{pmatrix} A & b \\ b^t & f \end{pmatrix} + \varphi'(F(v))\underbrace{\lambda_{\min}\begin{pmatrix} 0 & 0 \\ 0 & \kappa e^2 \end{pmatrix}}_{=0} \\
&= \varphi'(F(v))\lambda_{\min}(\mathrm{Hess}_v F|_{v^\perp}) \\
&\overset{(5.4)}{\geq} \varphi'(F(v))\rho.
\end{aligned} \tag{5.6}$$

On the other hand, we find that

$$\begin{aligned}
-\lambda_{\min}(\mathrm{Hess}_v(\varphi \circ F)) &= -\varphi'(F(v))\lambda_{\min}\begin{pmatrix} A & 0 & b \\ 0 & 0 & e \\ b^t & e & f + \kappa e^2 \end{pmatrix} \\
&\leq -\varphi'(F(v))\underbrace{\lambda_{\min}\begin{pmatrix} A & 0 & b \\ 0 & 0 & 0 \\ b^t & 0 & f \end{pmatrix}}_{=0} - \varphi'(F(v))\lambda_{\min}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & e & \kappa e^2 \end{pmatrix} \\
&\overset{(5.4)}{=} -\varphi'(F(v))\lambda_{\min}\begin{pmatrix} 0 & e \\ e & \kappa e^2 \end{pmatrix} \overset{5.2.10}{=} \varphi'(F(v))\|\mathrm{grad}_v F\|\lambda_{\max}\begin{pmatrix} 0 & 1 \\ 1 & \kappa e \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \varphi'(F(v))\|\mathrm{grad}_v F\| \underbrace{\left(\frac{\kappa e}{2} + \sqrt{\frac{\kappa^2 e^2}{4} + 1}\right)}_{<-\frac{1}{\kappa e}} \\
&< \frac{\varphi'(F(v))}{\kappa} \\
&\overset{(5.6)}{\leq} \frac{1}{\kappa\rho}\lambda_{\min}(\mathrm{Hess}_v(\varphi \circ F)|_{v^\perp}),
\end{aligned}
$$

which is what we wanted to show. □

The next lemma shows that the lower bound from Lemma 5.2.14 can actually by achieved using the assumptions of Theorem 5.2.1.

Lemma 5.2.15. *Let U be an open cone with $\overline{U} \subset \Omega$, where the closure is taken in $V \setminus \{0\}$. Assume that $\mathrm{Hess}_v F|_{v^\perp}$ is positive definite for all $v \in \overline{U}$ with $dF_v = 0$. Then there exist constants $\kappa, \rho > 0$ such that*

$$\lambda_{\min}(\mathrm{Hess}_v(\varphi \circ F)|_{v^\perp}) \geq \frac{\rho}{\|v\|^2}$$

for all $v \in \overline{U}$, where $\varphi(s) := e^{\kappa s}$ for $s \in \mathbb{R}$.

Proof. Since $\lambda_{\min}(\mathrm{Hess}_v F|_{v^\perp})$ depends continuously on v, there is some open scale-invariant neighborhood $W \subseteq \Omega$ of $\{v \in \Omega \mid dF_v = 0\}$ such that

$$\lambda_{\min}(\mathrm{Hess}_v F|_{v^\perp}) > 0 \tag{5.7}$$

for all $v \in W$.
Choose $\kappa > \max\{\kappa_0, 0\}$, where we set

$$\kappa_0 := \max_{\substack{v \in \overline{U} \setminus W: \\ \|v\|=1}} \frac{1}{\|\mathrm{grad}_v F\|^2}\Big(b^t(v)A^{-1}(v)b(v) - f(v)\Big),$$

using the notation of Lemma 5.2.9. Notice that the maximum exists since it is taken over a compact set. Using Lemma 5.2.12, we find that with this choice of κ,

$$
\begin{aligned}
\det\left(\mathrm{Hess}_v(\varphi \circ F)|_{v^\perp}\right) &= \varphi'(F(v))^{N-1} \det\begin{pmatrix} A & b \\ b^t & f + \kappa e^2\|v\|^2 \end{pmatrix} \\
&= \varphi'(F(v))^{N-1} \det(A)\Big(f + \kappa e^2\|v\|^2 - b^t A^{-1} b\Big) > 0
\end{aligned}
$$

for each $v \in \overline{U} \setminus W$, since $e^2\|v\|^2 = \|\mathrm{grad}_v F\|^2$ (see Remark 5.2.10). Because additionally A is positive definite, Sylvester's criterion [Gil91] gives that $\mathrm{Hess}_v(\varphi \circ F)|_{v^\perp}$ is positive definite for $v \in \overline{U} \setminus W$. Hence, together with (5.7), we obtain that

$$\lambda_{\min}(\mathrm{Hess}_v(\varphi \circ F)|_{v^\perp}) > 0$$

for all $v \in \overline{U}$. Since $\overline{U} \cap \{v \in V \mid \|v\| = 1\}$ is compact, there exists some $\rho' > 0$ such that

$$\lambda_{\min}(\mathrm{Hess}_v(\varphi \circ F)|_{v^\perp}) > \rho'$$

for all $v \in \overline{U} \cap \{v \in V \mid \|v\| = 1\}$. By scale-invariance of F (see Lemma 5.2.7), this finishes the proof. □

5.2.2 Two further ingredients for the proof of the theorem

The following lemma explains the relation between the sets $\mathcal{B}_n$ and the second Bianchi identity.

Lemma 5.2.16. *For all cones $C \subset \mathcal{B}_n$ which are closed in $\mathcal{A}_n \setminus \{0\}$, there are constants $L \in (0,1)$ and $\theta \in (0, \frac{\pi}{2}]$ such that the following is true. For all $R \in C$ and $T := (T_1, \dots, T_n) \in \mathcal{A}_n^n$ satisfying the second Bianchi identity, there exists an $i \in \{1, \dots, n\}$ such that $\sphericalangle(T_i, R) \in [\theta, \pi - \theta]$ and $\|T_i\| \geq L\|T\|$.*

Proof. We assume that the statement is not true. Then there exists a cone $C \subset \mathcal{B}_n$, which is closed in $\mathcal{A}_n \setminus \{0\}$, such that for all $m \in \mathbb{N}$ with $m > 1$ there exists an $R_m \in C$ and a $T^m = (T_1^m, \dots, T_n^m) \in \mathcal{A}_n^n$ satisfying the second Bianchi identity with respect to some orthonormal basis $e^m = (e_1^m, \dots, e_n^m)$ of $\mathbb{R}^n$ such that for all $i \in \{1, \dots, n\}$, we have that

$$\sphericalangle(T_i^m, R_m) > \pi - \frac{1}{m} \quad \text{or} \quad \sphericalangle(T_i^m, R_m) < \frac{1}{m} \quad \text{or} \quad \|T_i^m\| < \frac{1}{m}\|T^m\|.$$

Due to the scale-invariance of C, we may assume without loss of generality that $\|T^m\| = 1$ and $\|R_m\| = 1$. Since $\{R \in C \mid \|R\| = 1\}$ and $\{T \in \mathcal{A}_n^n \mid \|T\| = 1\}$ are compact, the sequences $(R_m)_{m\in\mathbb{N}}$ and $(T^m)_{m\in\mathbb{N}}$ subconverge to an $R_\infty \in C$ with $\|R_\infty\| = 1$ respectively a $T^\infty \in \mathcal{A}_n^n$ with $\|T^\infty\| = 1$, i.e. there is a sequence $(m_l)_{l\in\mathbb{N}} \subseteq \mathbb{N}$ with

$$\lim_{l\to\infty} R_{m_l} = R_\infty,$$

a subsequence $(m_{l_p})_{p\in\mathbb{N}} \subseteq (m_l)_{l\in\mathbb{N}}$ with

$$\lim_{p\to\infty} T^{m_{l_p}} = T^\infty$$

and for all $i \in \{1, \dots, n\}$ we have that

$$\sphericalangle(T_i^\infty, R_\infty) = \pi \quad \text{or} \quad \sphericalangle(T_i^\infty, R_\infty) = 0 \quad \text{or} \quad \|T_i^\infty\| = 0.$$

Therefore, we find that for all $i \in \{1, \dots, n\}$

$$T_i^\infty = \alpha_i R_\infty \tag{5.8}$$

for an $\alpha_i \in \mathbb{R}$. We do not have that $\alpha_i = 0$ for all $i \in \{1, \dots, n\}$, since otherwise we had that $T^\infty = 0$ in contradiction to $\|T^\infty\| = 1$. Without loss of generality, let $\alpha_1 \neq 0$.

Next, we show that T^∞ satisfies the second Bianchi identity as well: Since the space of orthonormal bases of $\mathbb{R}^n$ is compact, there is a subsequence $(b_q)_{q\in\mathbb{N}} \subseteq (m_{l_p})_{p\in\mathbb{N}}$ such that e^{b_q} converges to an orthonormal basis e^∞ of $\mathbb{R}^n$ for $q \to \infty$. Together with the fact that T^m satisfies the second Bianchi identity for all $m \in \mathbb{N}$, this implies that

$$\begin{aligned} 0 &= \lim_{q\to\infty} T_i^{b_q}(e_j^{b_q} \wedge e_k^{b_q}) + T_j^{b_q}(e_k^{b_q} \wedge e_i^{b_q}) + T_k^{b_q}(e_i^{b_q} \wedge e_j^{b_q}) \\ &= T_i^\infty(e_j^\infty \wedge e_k^\infty) + T_j^\infty(e_k^\infty \wedge e_i^\infty) + T_k^\infty(e_i^\infty \wedge e_j^\infty) \end{aligned}$$

for all $i, j, k \in \{1, \dots, n\}$. Hence, T^∞ satisfies the second Bianchi identity with respect to e^∞. Moreover, by (5.8) this yields that

$$R_\infty(\underbrace{\alpha_i e_j^\infty \wedge e_k^\infty + \alpha_j e_k^\infty \wedge e_i^\infty + \alpha_k e_i^\infty \wedge e_j^\infty}_{:= v_{ijk}}) = 0$$

for all $i, j, k \in \{1, \dots, n\}$ and due to the linearity of R_∞ that

$$R_\infty|_{\operatorname{span}\{v_{ijk} | i,j,k \in \{1,\dots,n\}\}} \equiv 0.$$

Let now $(e^1_\infty, \dots, e^n_\infty)$ be the dual basis corresponding to $(e^\infty_1, \dots, e^\infty_n)$, i.e. $e^\infty_i(e^j_\infty) = \delta^j_i$ for $i, j \in \{1, \dots, n\}$, where we identified $\mathbb{R}^n$ with $((\mathbb{R}^n)^*)^*$. We set

$$\alpha := \sum_{l=1}^n \alpha_l e^l_\infty.$$

Then for all $i, j, k \in \{1, \dots, n\}$, we have that

$$\begin{aligned}
v_{ijk}(\alpha, \cdot) &= \sum_{l=1}^n \alpha_l \Big(\alpha_i(\delta^l_j e^\infty_k - \delta^l_k e^\infty_j) + \alpha_j(\delta^l_k e^\infty_i - \delta^l_i e^\infty_k) + \alpha_k(\delta^l_i e^\infty_j - \delta^l_j e^\infty_i) \Big) \\
&= \alpha_i(\alpha_j e^\infty_k - \alpha_k e^\infty_j) + \alpha_j(\alpha_k e^\infty_i - \alpha_i e^\infty_k) + \alpha_k(\alpha_i e^\infty_j - \alpha_j e^\infty_i) \\
&= 0
\end{aligned}$$

and thus that

$$\operatorname{span}\{v_{ijk} \mid i, j, k \in \{1, \dots, n\}\} \subseteq \Lambda^2(\alpha^\perp). \tag{5.9}$$

For $2 \le i < j \le n$, let $\beta_{ij} \in \mathbb{R}$ with

$$0 = \sum_{2 \le i < j \le n} \beta_{ij} v_{1ij} = \sum_{2 \le i < j \le n} \beta_{ij}(\alpha_1 e^\infty_i \wedge e^\infty_j + \alpha_i e^\infty_j \wedge e^\infty_1 + \alpha_j e^\infty_1 \wedge e^\infty_i).$$

Since $\{e^\infty_i \wedge e^\infty_j \mid 1 \le i < j \le n\}$ is a basis of $\Lambda^2\mathbb{R}^n$ and $\alpha_1 \ne 0$, we find that $\beta_{ij} = 0$ for $2 \le i < j \le n$. It follows that $\{v_{1ij} \mid 2 \le i < j \le n\}$ is linearly independent. Consequently,

$$\begin{aligned}
\dim(\operatorname{span}\{v_{ijk} \mid i, j, k \in \{1, \dots, n\}\}) &\ge \#\{v_{1ij} \mid 2 \le i < j \le n\} = \frac{(n-1)(n-2)}{2} = \dim(\Lambda^2(\alpha^\perp)) \\
&\overset{(5.9)}{\ge} \dim(\operatorname{span}\{v_{ijk} \mid i, j, k \in \{1, \dots, n\}\}).
\end{aligned}$$

This implies that

$$\operatorname{span}\{v_{ijk} \mid i, j, k \in \{1, \dots, n\}\} = \Lambda^2(\alpha^\perp)$$

and therefore we have shown that $R_\infty|_{\Lambda^2(\alpha^\perp)} \equiv 0$. This, however, is a contradiction to our assumption. Consequently, the statement of the lemma is true. □

Directly from the proof of Lemma 5.2.16, we obtain the following result.

Corollary 5.2.17. *Let $R \in \mathcal{A}_n$. If $(\alpha_1 R, \dots, \alpha_n R)$ satisfies the second Bianchi identity, where $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \setminus \{0\}$, then $R \in \mathcal{B}^c_n$.*

The final ingredient will be the next linear algebra lemma.

Lemma 5.2.18. *Let $H : V \times V \to \mathbb{R}$ be a symmetric bilinear form on an N-dimensional Euclidean vector space V with one negative and $N-1$ positive eigenvalues. Let further $0 \ne R \in V$ such that $H(R, R) = 0$ and $H|_{R^\perp}$ is positive definite. Then for all $w \in V \setminus \{0\}$, the following is true:*

a) If $2\arctan\left(\sqrt{\frac{-\lambda_{\min}(H)}{\lambda_{\min}(H|_{R^\perp})}}\right) < \sphericalangle(w, R) < \pi - 2\arctan\left(\sqrt{\frac{-\lambda_{\min}(H)}{\lambda_{\min}(H|_{R^\perp})}}\right)$, then $H(w, w) > 0$.

b) *If* $2\arctan(\epsilon) \leq \sphericalangle(w, R) \leq \pi - 2\arctan(\epsilon)$, *where* $\epsilon > \sqrt{\frac{-\lambda_{\min}(H)}{\lambda_{\min}(H|_{R^\perp})}}$, *then*

$$H(w, w) \geq C(H, R, \epsilon)\|w\|^2.$$

Here,

$$C(H, R, \epsilon) := \frac{\epsilon^2 \lambda_{\min}(H|_{R^\perp}) + \lambda_{\min}(H)}{1 + \epsilon^2} > 0.$$

Remark 5.2.19. Of course, statement *a*) follows from *b*). However, in the proof of *b*) it is convenient to use *a*), which is why we prove *a*) beforehand.

Remark 5.2.20. The zero-directions of a bilinear form H as in Lemma 5.2.18, i.e. those vectors $v \in V$ with $H(v, v) = 0$, form the boundary of a double cone with axis $\mathrm{Eig}(H, \lambda_{\min}(H))$ and an $(N-2)$-dimensional ellipse as base. For the vectors $v \in V$ in the interior of the cone, we have that $H(v, v) < 0$ and for those outside the cone that $H(v, v) > 0$.

Proof. Let $e_1 \in V$ be an eigenvector to the negative eigenvalue of H with $\|e_1\| = 1$ and set $\lambda := -\lambda_{\min}(H) > 0$.

a) For $v \in V$ with $\|v\| = 1$ and $H(v, e_1) = 0$, that is $v \perp e_1$ and $H(v, v) > 0$, let α_v be the opening angle of the double cone (around the axis $\mathbb{R}e_1$) of the non-positive directions of H in $\mathbb{R}e_1 \oplus \mathbb{R}v$, i.e. the double cone of those vectors $w \in \mathbb{R}e_1 \oplus \mathbb{R}v$ with $H(w, w) \leq 0$. In particular, we have that $\alpha_v = 2\sphericalangle(e_1, u)$, where u with $\sphericalangle(e_1, u) \leq \frac{\pi}{2}$ is on the boundary of this cone.

   We claim that

$$\alpha_v = 2\arctan\left(\sqrt{\frac{\lambda}{H(v, v)}}\right). \tag{5.10}$$

   Namely, let $0 \neq u = ae_1 + bv$, $a, b \in \mathbb{R}$, be an element of the boundary of the cone with $\sphericalangle(e_1, u) \leq \frac{\pi}{2}$, i.e.

$$0 = H(u, u) = -\lambda a^2 + b^2 H(v, v). \tag{5.11}$$

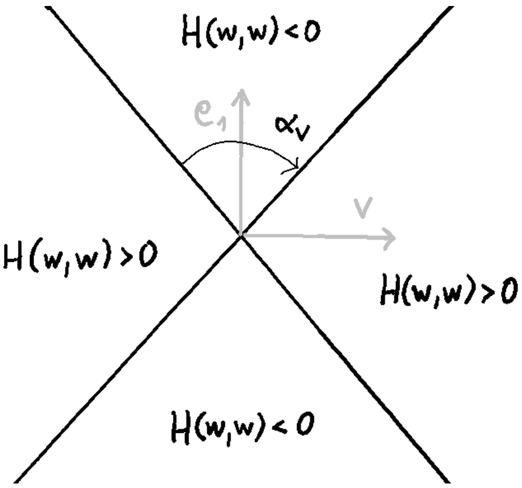
   Then, due to $H(v, v) > 0$ and $\lambda > 0$, it follows that $a, b \neq 0$. In particular, we have that

$$0 < \sphericalangle(e_1, u) < \frac{\pi}{2}.$$

   Therefore, (5.11) yields that

$$\sqrt{\frac{\lambda}{H(v, v)}} = \left|\frac{b}{a}\right| = \tan(\sphericalangle(e_1, u)) = \tan\left(\frac{\alpha_v}{2}\right),$$

   which is the claim.



Moreover, we notice that

$$\lambda_{\min}(H|_{e_1{}^\perp}) \geq \lambda_{\min}(H|_{u^\perp}) \tag{5.12}$$

for all $u \in V$, as $\mathbb{R}e_1$ is the eigenspace of the smallest eigenvalue of H.

Now, let $w \in V \setminus \{0\}$ with $H(w,w) \leq 0$. In the case that $\sphericalangle(w,R) \leq \frac{\pi}{2}$, we obtain that

$$\begin{aligned}\sphericalangle(w,R) &\leq \sup_{\substack{v\in V:\\ H(v,v)\leq 0,\\ \langle v,R\rangle \geq 0}} \sphericalangle(v,R) \leq \max_{v\in e_1^\perp:\|v\|=1} \alpha_v \overset{(5.10)}{=} 2\arctan\left(\sqrt{\frac{\lambda}{\min\limits_{v\in e_1^\perp:\|v\|=1} H(v,v)}}\right) \\ &= 2\arctan\left(\sqrt{\frac{\lambda}{\lambda_{\min}(H|_{e_1{}^\perp})}}\right) \overset{(5.12)}{\leq} 2\arctan\left(\sqrt{\frac{\lambda}{\lambda_{\min}(H|_{R^\perp})}}\right).\end{aligned} \tag{5.13}$$

If $\sphericalangle(w,R) \geq \frac{\pi}{2}$, we find that

$$\sphericalangle(w,R) = \pi - \sphericalangle(-w,R) \overset{(5.13)}{\geq} \pi - 2\arctan\left(\sqrt{\frac{\lambda}{\lambda_{\min}(H|_{R^\perp})}}\right).$$

This proves a).

b) Step 1: Let $v \in V$ with $\|v\| = 1$ and $H(v,e_1) = 0$, i.e. $v \perp e_1$. First, we show that for $w \in \mathbb{R}e_1 \oplus \mathbb{R}v$, $w \neq 0$, with $\arctan(\epsilon) \leq \sphericalangle(w,e_1) \leq \pi - \arctan(\epsilon)$, where $\epsilon > \sqrt{\frac{\lambda}{H(v,v)}}$, we have that

$$H(w,w) \geq \underbrace{\frac{\epsilon^2 H(v,v) - \lambda}{1+\epsilon^2}}_{>0} \|w\|^2.$$

To this end, write $w = ae_1 + bv$, where $a, b \in \mathbb{R}$ and suppose that $\arctan(\epsilon) \leq \sphericalangle(w,e_1) \leq \pi - \arctan(\epsilon)$ and $\epsilon > \sqrt{\frac{\lambda}{H(v,v)}}$. Clearly, $b \neq 0$.

In the case that $a = 0$, we find that

$$\sphericalangle(w,e_1) = \frac{\pi}{2}.$$

Thus, the assumption is satisfied and we have that

$$H(w,w) = b^2 H(v,v) = H(v,v)\|w\|^2 \geq \frac{\epsilon^2 H(v,v) - \lambda}{1+\epsilon^2}\|w\|^2$$

as desired.

Let now $a > 0$. Then $\tan(\sphericalangle(w, e_1)) = \left|\frac{b}{a}\right|$ and therefore, by assumption,

$$\frac{b^2}{a^2} \geq \epsilon^2 > \frac{\lambda}{H(v,v)}. \tag{5.14}$$

Consequently,

$$\delta := \frac{\lambda}{\epsilon^2 H(v,v)} \in (0,1) \tag{5.15}$$

and we obtain that

$$\begin{aligned}
H(w,w) = &\ -\lambda a^2 + b^2 H(v,v) \\
= &\ -\lambda a^2 + \delta H(v,v) b^2 + (1-\delta) H(v,v) b^2 \\
\overset{(5.14)}{\geq} &\ -\lambda a^2 + \delta H(v,v)\epsilon^2 a^2 + (1-\delta)H(v,v)b^2 \\
\overset{(5.15)}{=} &\ (1-\delta)H(v,v)b^2 \\
= &\ (1-\delta)H(v,v)\left(\frac{1}{1+\epsilon^2}b^2 + \frac{\epsilon^2}{1+\epsilon^2}b^2\right) \\
\overset{(5.14),(5.15)}{\geq} &\ \left(1 - \frac{\lambda}{\epsilon^2 H(v,v)}\right) H(v,v)\frac{\epsilon^2}{1+\epsilon^2}(a^2+b^2) \\
= &\ \frac{\epsilon^2 H(v,v) - \lambda}{1+\epsilon^2}\|w\|^2.
\end{aligned} \tag{5.16}$$

Since with w also $-w$ satisfies the assumption on the angle $\sphericalangle(-w, e_1)$, in the case that $a < 0$ we find that

$$H(w,w) = H(-w,-w) \overset{(5.16)}{\geq} \frac{\epsilon^2 H(v,v) - \lambda}{1+\epsilon^2}\|-w\|^2 = \frac{\epsilon^2 H(v,v) - \lambda}{1+\epsilon^2}\|w\|^2.$$

Step 2: Let $w \in V\setminus\{0\}$ with $2\arctan(\epsilon) \leq \sphericalangle(w,R) \leq \pi - 2\arctan(\epsilon)$, where $\epsilon > \sqrt{\frac{\lambda}{\lambda_{\min}(H|_{R^\perp})}}$. First, we show that w satisfies the assumptions of Step 1. To this end, we claim that

$$\arctan(\epsilon) \leq \sphericalangle(w, e_1) \leq \pi - \arctan(\epsilon). \tag{5.17}$$

Namely, if $\sphericalangle(e_1, R) \leq \frac{\pi}{2}$, then similar to (5.13) one shows that

$$\sphericalangle(e_1, R) \leq \frac{1}{2}\max_{\substack{v \in e_1^\perp: \\ \|v\|=1}} \alpha_v \leq \arctan\left(\sqrt{\frac{\lambda}{\lambda_{\min}(H|_{R^\perp})}}\right) < \arctan(\epsilon).$$

Hence, by assumption,

$$\sphericalangle(w, e_1) \geq \sphericalangle(w,R) - \sphericalangle(R, e_1) \geq 2\arctan(\epsilon) - \arctan(\epsilon) = \arctan(\epsilon)$$

and

$$\sphericalangle(w, e_1) \leq \sphericalangle(w, R) + \sphericalangle(R, e_1) \leq \pi - 2\arctan(\epsilon) + \arctan(\epsilon) = \pi - \arctan(\epsilon).$$

Since $-R$ still satisfies the assumption on the angle $\sphericalangle(w, -R)$, exchanging R by $-R$, in the same way one shows that the inequalities (5.17) also hold in the case that $\sphericalangle(e_1, R) \geq \frac{\pi}{2}$.

Moreover, we compute that

$$\epsilon > \sqrt{\frac{\lambda}{\lambda_{\min}(H|_{R^\perp})}} \overset{(5.12)}{\geq} \sqrt{\frac{\lambda}{\lambda_{\min}(H|_{e_1{}^\perp})}} = \sqrt{\frac{\lambda}{\min\limits_{\|v\|=1, v\perp e_1} H(v,v)}}$$
$$= \max_{\|v\|=1, v\perp e_1} \sqrt{\frac{\lambda}{H(v,v)}} \geq \sqrt{\frac{\lambda}{H(w^\perp, w^\perp)}}.$$

Here, we denote $w^\perp := \frac{\pi(w)}{\|\pi(w)\|} \perp e_1$, where $\pi : V \to e_1{}^\perp$ is the orthogonal projection on $e_1{}^\perp$. This is well defined, since $\pi(w) \neq 0$. Namely, if we had that $\pi(w) = 0$, that is $w \in \mathbb{R}e_1$, we would obtain that $H(w,w) \leq 0$ and thus due to a) that

$$\pi - 2\arctan\left(\sqrt{\frac{\lambda}{\lambda_{\min}(H|_{R^\perp})}}\right) \leq \sphericalangle(w, R) \leq 2\arctan\left(\sqrt{\frac{\lambda}{\lambda_{\min}(H|_{R^\perp})}}\right).$$

This, however, is a contradiction to the assumption.

Now, by Step 1, it follows that

$$\begin{aligned} H(w,w) &\geq \frac{\epsilon^2 H(w^\perp, w^\perp) - \lambda}{1+\epsilon^2}\|w\|^2 \\ &\geq \frac{\epsilon^2 \lambda_{\min}(H|_{e_1{}^\perp}) - \lambda}{1+\epsilon^2}\|w\|^2 \\ &\overset{(5.12)}{\geq} \underbrace{\frac{\epsilon^2 \lambda_{\min}(H|_{R^\perp}) - \lambda}{1+\epsilon^2}}_{=:C(H,R,\epsilon)}\|w\|^2. \end{aligned}$$

Here, $C(H, R, \epsilon) > 0$ by the choice of ϵ. □

5.2.3 Proof of the theorem

Now, we are in the position to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. By Lemma 5.2.15, there are constants $\kappa', \rho > 0$ such that

$$\lambda_{\min}(\mathrm{Hess}_R(\psi \circ F)|_{R^\perp}) \geq \frac{\rho}{\|R\|^2} \tag{5.18}$$

for all $R \in \overline{U}$, where $\psi(s) := e^{\kappa' s}$ for $s \in \mathbb{R}$. Throughout the proof, write $\widetilde{F} := \psi \circ F$.

Let $L \in (0,1)$, $\theta \in (0, \frac{\pi}{2}]$ be constants corresponding to the cone $\overline{U} \subset \mathcal{B}_n$ as in Lemma 5.2.16. Then for $s \in \mathbb{R}$, we set

$$\varphi(s) := e^{\kappa s} \qquad \text{with} \quad \kappa \geq \frac{1}{\rho\delta^2},$$
$$\text{where} \quad \delta \in \left(0, \sqrt{\frac{L^2\epsilon^2}{1+(1-L^2)\epsilon^2}}\right)$$
$$\text{and} \quad \epsilon := \tan\left(\frac{\theta}{2}\right).$$

Let $R \in U$ and let $T := (T_1, \ldots, T_n) \in (T_R U)^n \setminus \{0\} \subset \mathcal{A}_n^n$ satisfy the second Bianchi identity.

If $dF_R = 0$, then $d\widetilde{F}_R = 0$ and, by (5.5), $\mathrm{Hess}_R(\varphi \circ \widetilde{F})$ is positive semidefinite. For $k = 1, \ldots, n$, write $T_k = \alpha_k R + S_k$, where $\alpha_k \in \mathbb{R}$ and $S_k \in R^\perp$. Then,

$$\sum_{k=1}^{n} \mathrm{Hess}_R(\varphi \circ \widetilde{F})(T_k, T_k) \geq 0. \tag{5.19}$$

Suppose we have equality in (5.19). Then, by Lemma 5.2.8,

$$\begin{aligned} 0 = & \sum_{k=1}^{n} \Big(\mathrm{Hess}_R(\varphi \circ \widetilde{F})(S_k, S_k) + 2\alpha_k \underbrace{\mathrm{Hess}_R(\varphi \circ \widetilde{F})(S_k, R)}_{=0} + \alpha_k^2 \underbrace{\mathrm{Hess}_R(\varphi \circ \widetilde{F})(R, R)}_{=0} \Big) \\ \overset{5.2.8}{=} & \sum_{k=1}^{n} \mathrm{Hess}_R(\varphi \circ \widetilde{F})(S_k, S_k). \end{aligned}$$

Since

$$\mathrm{Hess}_R(\varphi \circ \widetilde{F})|_{R^\perp} \overset{(5.3)}{=} \varphi'(\widetilde{F}(R)) \mathrm{Hess}_R \widetilde{F}|_{R^\perp}$$

is positive definite by (5.18) and $\kappa > 0$, this implies that $S_k = 0$ for each k. Hence, $T_k = \alpha_k R$ for each k, which implies that $R \in \mathcal{B}_n^c$ by Corollary 5.2.17, in contradiction to $R \in U \subset \mathcal{B}_n$. Therefore, the inequality in (5.19) is strict, i.e. $\varphi \circ \widetilde{F}$ is strictly Bianchi-convex in all $R \in U$ with $dF_R = 0$.

Now, consider the case that $dF_R \neq 0$. Since then also $d\widetilde{F}_R \neq 0$, from (5.6), it follows that

$$\lambda_{\min}\Big(\mathrm{Hess}_R(\varphi \circ \widetilde{F})|_{R^\perp}\Big) > 0. \tag{5.20}$$

Therefore, by Lemma 5.2.14, we have that

$$-\lambda_{\min}\Big(\mathrm{Hess}_R(\varphi \circ \widetilde{F})\Big) \leq \frac{1}{\rho\kappa} \lambda_{\min}\Big(\mathrm{Hess}_R(\varphi \circ \widetilde{F})|_{R^\perp}\Big) \leq \delta^2 \lambda_{\min}\Big(\mathrm{Hess}_R(\varphi \circ \widetilde{F})|_{R^\perp}\Big). \tag{5.21}$$

Thus, using that $L < 1$, we find that

$$\sqrt{\frac{-\lambda_{\min}\Big(\mathrm{Hess}_R(\varphi \circ \widetilde{F})\Big)}{\lambda_{\min}\Big(\mathrm{Hess}_R(\varphi \circ \widetilde{F})|_{R^\perp}\Big)}} \leq \delta < \epsilon.$$

From the choice of L and θ, it follows that there is an $i \in \{1, \ldots, n\}$ with $\sphericalangle(T_i, R) \in [\theta, \pi - \theta]$ and $\|T_i\| \geq L\|T\|$. Without loss of generality, let $i = 1$. This yields that

$$2\arctan(\epsilon) = \theta \leq \sphericalangle(T_1, R) \leq \pi - \theta = \pi - 2\arctan(\epsilon).$$

Due to $R \neq 0$ (because $R \in \mathcal{B}_n$), Lemma 5.2.13, Lemma 5.2.14 and (5.20), we can apply Lemma 5.2.18 and obtain that

$$\mathrm{Hess}_R(\varphi \circ F)(T_1, T_1) \geq C\|T_1\|^2$$

with

$$C := \frac{\epsilon^2 \lambda_{\min}(H|_{R^\perp}) + \lambda_{\min}(H)}{1 + \epsilon^2} > 0,$$

where we used the abbreviation $H := \mathrm{Hess}_R(\varphi \circ \widetilde{F})$. Since $\|T\|^2 = \sum_{k=1}^{n} \|T_k\|^2$, this implies that

$$\begin{aligned}
\sum_{k=1}^{n} \mathrm{Hess}_R(\varphi \circ \widetilde{F})(T_k, T_k) &\geq C\|T_1\|^2 + \lambda_{\min}(H)(\|T_2\|^2 + \cdots + \|T_n\|^2) \\
&= \Big(C - \lambda_{\min}(H)\Big)\|T_1\|^2 + \lambda_{\min}(H)\|T\|^2 \\
&= \frac{\epsilon^2}{1+\epsilon^2}\Big(\lambda_{\min}(H|_{R^\perp}) - \lambda_{\min}(H)\Big)\underbrace{\|T_1\|^2}_{\geq L^2\|T\|^2} + \lambda_{\min}(H)\|T\|^2 \\
&\geq \left(\frac{\epsilon^2 L^2}{1+\epsilon^2}\lambda_{\min}(H|_{R^\perp}) + \left(1 - \frac{\epsilon^2 L^2}{1+\epsilon^2}\right)\lambda_{\min}(H)\right)\|T\|^2 \\
&\overset{(5.21)}{\geq} \frac{\delta^2+1}{\delta^2}\underbrace{\left(\frac{\delta^2}{\delta^2+1} - \frac{\epsilon^2 L^2}{1+\epsilon^2}\right)}_{<0}\underbrace{\lambda_{\min}(H)}_{<0}\|T\|^2 \\
&> 0,
\end{aligned}$$

by Lemma 5.2.13 and the choice of δ. Consequently, this results in $\varphi \circ \widetilde{F} = \varphi \circ \psi \circ F$ restricted to U being strictly Bianchi-convex. $\square$

Chapter 6

Ricci solitons, curvature conditions and local symmetry

In the study of singularity formation of solutions to the Ricci flow, so-called Ricci solitons play an important role. They are natural generalizations of Einstein metrics and correspond to self-similar solutions to the Ricci flow. In this chapter, we give a short introduction to this concept and as a special case to gradient shrinking Ricci solitons. Moreover, we consider curvature conditions that are the domains of certain scale-invariant and strictly Bianchi-convex functions, the sublevel sets of which are invariant under the ordinary differential equation (2.4). As a main result, we show that complete shrinking gradient Ricci solitons (respectively compact Ricci solitons) which satisfy such curvature conditions are already locally symmetric. Finally, we derive two applications of these rigidity results based on conjectures of Böhm-Wilking and the author.

6.1 Ricci solitons

A metric g on a manifold is called *Einstein*, if there exists a constant $\lambda \in \mathbb{R}$ such that $\mathrm{ric}_g = \lambda g$. In this section, we want to investigate a natural generalization of Einstein metrics, so-called Ricci solitons.

Definition 6.1.1. A *Ricci soliton* is a quadruple (M, g, X, λ) consisting of a Riemannian manifold (M, g), a smooth vector field $X \in \Gamma(M, TM)$ and $\lambda \in \mathbb{R}$ such that

$$\mathrm{ric}_g + \frac{1}{2}\mathcal{L}_X g = \lambda g. \tag{6.1}$$

Here, $\mathcal{L}$ denotes the Lie derivative. If X and λ need not to be specified, sometimes we just write (M, g) for a Ricci soliton. A Ricci soliton is called *expanding*, *steady* or *shrinking* depending on whether $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively. Furthermore, we say a Ricci soliton is *complete*, if (M, g) is complete and the vector field X is complete, i.e. if all of its flow curves exist for all times. If the vector field X can be written as the gradient of some smooth function $f : M \to \mathbb{R}$ with respect to g, i.e. $X = \mathrm{grad}_g f$, then the Ricci soliton is called *gradient* and denoted by (M, g, f, λ). The function f is referred to as *potential function.* Since in this case one can compute

$$\mathcal{L}_X g = 2\mathrm{Hess}_g f,$$

the equation (6.1) becomes

$$\mathrm{ric}_g + \mathrm{Hess}_g f = \lambda g. \tag{6.2}$$

Taking the trace of equation (6.2) gives

$$\mathrm{scal}_g + \Delta_g f = n\lambda. \tag{6.3}$$

Remark 6.1.2. If the vector field $X \equiv 0$, which in the gradient Ricci soliton case means that the function f is constant, then g is an Einstein metric. Therefore, Ricci solitons are natural generalizations of Einstein metrics.

Remark 6.1.3. In [Zha09b], Z.-H. Zhang shows that if (M, g, f, λ) is a gradient Ricci soliton, the metric g of which is complete, then $\mathrm{grad}_g f$ is already complete, i.e. (M, g, f, λ) is complete as Ricci soliton.

Remark 6.1.4. Equivalently to the above definition, (M, g_0, X, λ) is a Ricci soliton, if there exists a function $\sigma : [0, T] \to (0, \infty)$ and a one-parameter family of diffeomorphisms $\psi_t : M \to M$ such that

$$g(t) := \sigma(t)\psi_t^* g_0 \tag{6.4}$$

is a solution to the Ricci flow. Namely, given a metric g_0 satisfying (6.1) one can show that

$$g(t) := (1 - 2\lambda t)\psi_t^* g_0$$

is a solution to the Ricci flow (to which we will refer as *the solution to the Ricci flow corresponding to* (M, g_0)). Here, ψ_t is the one-parameter family of diffeomorphisms with $\psi_0 = \mathrm{id}$ given by

$$\frac{\partial \psi_t(x)}{\partial t} = \frac{1}{1 - 2\lambda t} X|_{\psi_t(x)}$$

for $x \in M$. Conversely, if we have a solution to the Ricci flow of the form (6.4), differentiating (6.4) at $t = 0$ yields that (M, g_0, X, λ) satisfies (6.1) for appropriate λ and X.
Therefore, Ricci solitons correspond to the self-similar solutions to the Ricci flow.

Remark 6.1.5. Since $\psi_t : (M, \psi_t^* g_0) \to (M, g_0)$ is an isometry, the evolution of the Riemannian curvature operator (interpreted as bilinear form) of a Ricci soliton in time is very explicit, namely

$$Rm_{g(t)} \overset{1.3.2}{=} \sigma(t) Rm_{\psi_t^* g_0} = \sigma(t)\psi_t^* Rm_{g_0}. \tag{6.5}$$

The following observation is crucial when working with gradient Ricci solitons.

Proposition 6.1.6 ([CLN06, Thm. 4.1])**.** *Let (M, g_0, f_0, λ) be a complete gradient Ricci soliton. Then for all $t \in \mathbb{R}$ with*

$$\sigma(t) := 1 - 2\lambda t > 0,$$

there exist a solution $g(t)$ to the Ricci flow with $g(0) = g_0$, diffeomorphisms $\psi_t : M \to M$ with $\psi_0 = \mathrm{id}|_M$ and functions $f(t) : M \to \mathbb{R}$ with $f(0) = f_0$ such that

1. *ψ_t is the one-parameter family of diffeomorphisms generated by the vector field $X(t) := \frac{1}{\sigma(t)} \mathrm{grad}_{g_0} f_0$. That is*

$$\frac{\partial}{\partial t}\psi_t(x) = \frac{1}{\sigma(t)} \mathrm{grad}_{g_0} f_0|_{\psi_t(x)}. \tag{6.6}$$

2. $g(t)$ is the pullback of g_0 by ψ_t up to the scale factor $\sigma(t)$:

$$g(t) = \sigma(t)\psi_t^* g_0.$$

3. $f(t)$ is the pullback of f_0 by ψ_t:

$$f(t) = \psi_t^* f_0 = f_0 \circ \psi_t.$$

Moreover,

$$\mathrm{ric}_{g(t)} + \mathrm{Hess}_{g(t)} f(t) = \frac{\lambda}{\sigma(t)} g(t) \tag{6.7}$$

and

$$\frac{\partial}{\partial t} f(t) = \|\mathrm{grad}_{g(t)} f(t)\|^2_{g(t)}. \tag{6.8}$$

Remark 6.1.7. Below, we often write g_t and f_t instead of $g(t)$ and $f(t)$. Moreover, whenever we speak of the solution $g(t)$ to the Ricci flow corresponding to (M, g_0, f_0, λ), the same notation as in Proposition 6.1.6 is implied.

Remark 6.1.8. As an immediate consequence, in the case of a shrinking gradient soliton, i.e. $\lambda > 0$, we observe that $g(t)$ is an ancient solution, that is defined for all $t \in (-\infty, \frac{1}{2\lambda})$.

Remark 6.1.9. Taking the trace with respect to $g(t)$ in (6.7), we obtain that

$$\mathrm{scal}_{g(t)} + \Delta_{g(t)} f(t) = \frac{\lambda n}{1 - 2\lambda t}. \tag{6.9}$$

Remark 6.1.10. Perelman [Per02] showed that any compact Ricci soliton is a gradient Ricci soliton.

In Lemma 1.4.2, we reminded of the evolution equation of the Riemannian curvature operator under the Ricci flow. In the special case that the Ricci flow corresponds to a gradient Ricci soliton, the evolution equation is as follows.

Lemma 6.1.11. *Let (M, g_0, f_0, λ) be a gradient Ricci soliton and let $g(t) = \sigma(t)\psi_t^* g_0$ be the corresponding solution to the Ricci flow with $g(0) = g_0$. Then the Riemannian curvature operator $Rm_{g(t)}$ of $g(t)$ evolves under the partial differential equation*

$$\nabla_{\frac{\partial}{\partial t}} Rm_{g(t)} = \frac{2\lambda}{\sigma(t)} Rm_{g(t)} + \nabla^{g(t)}_{\mathrm{grad}_{g(t)} f(t)} Rm_{g(t)},$$

where $f(t) := \psi_t^* f_0$.

Proof. Let $x \in M$ and $(e_1^0, \ldots, e_n^0)$ be a g_0-orthonormal basis of $T_x M$, where n is the dimension of M. For $i = 1, \ldots, n$, let e_i be the solution to the partial differential equation

$$\frac{\partial}{\partial t} e_i(t) = \mathrm{Ric}_{g(t)}(e_i(t)) \tag{6.10}$$

with $e_i(0) = e_i^0$. Note, that (6.10) is equivalent to $\nabla_{\frac{\partial}{\partial t}} e_i(t) = 0$. Then $(e_1(t), \ldots, e_n(t))$ is a $g(t)$-orthonormal basis of $T_x M$ for all t. Extending $(e_1(t), \ldots, e_n(t))$ for each t to a neighborhood of x via parallel transport along geodesics starting at x and using the abbreviations $e_i^t := e_i(t)$ and $g_t := g(t)$, we can compute that

$$\begin{aligned}\Big(\nabla_{\frac{\partial}{\partial t}} Rm_{g_t}(x)\Big)&(e_i^t \wedge e_j^t, e_k^t \wedge e_l^t)\\ &= \frac{\partial}{\partial t}\Big(Rm_{g_t}(x)(e_i^t \wedge e_j^t, e_k^t \wedge e_l^t)\Big) - Rm_{g_t}(x)\left(\nabla_{\frac{\partial}{\partial t}} e_i^t \wedge e_j^t, e_k^t \wedge e_l^t\right)\\ &\quad - \cdots - Rm_{g_t}(x)\left(e_i^t \wedge e_j^t, e_k^t \wedge \nabla_{\frac{\partial}{\partial t}} e_l^t\right)\\ &\overset{(1.7)}{=} \frac{\partial}{\partial t}\Big(Rm_{g_t}(x)(e_i^t \wedge e_j^t, e_k^t \wedge e_l^t)\Big)\\ &= \left(\frac{\partial}{\partial t} Rm_{g_t}(x)\right)(e_i^t \wedge e_j^t, e_k^t \wedge e_l^t) + Rm_{g_t}(x)(\mathrm{Ric}_{g_t}(e_i^t) \wedge e_j^t, e_k^t \wedge e_l^t)\\ &\quad + \cdots + Rm_{g_t}(x)(e_i^t \wedge e_j^t, e_k^t \wedge \mathrm{Ric}_{g_t}(e_l^t)).\end{aligned} \tag{6.11}$$

Using (6.5), the first term of the right-hand side of (6.11) can be rewritten as follows

$$\begin{aligned}&\left(\frac{\partial}{\partial t} Rm_{g_t}(x)\right)(e_i^t \wedge e_j^t, e_k^t \wedge e_l^t)\\ &\overset{(6.5)}{=} -2\lambda(\psi_t^* Rm_{g_0})(x)(e_i^t \wedge e_j^t, e_k^t \wedge e_l^t) + \sigma(t)\left(\frac{\partial}{\partial t}(\psi_t^* Rm_{g_0})(x)\right)(e_i^t \wedge e_j^t, e_k^t \wedge e_l^t)\\ &= -\frac{2\lambda}{\sigma(t)} Rm_{g_t}(x)(e_i^t \wedge e_j^t, e_k^t \wedge e_l^t)\\ &\quad + \sigma(t)\Bigg(\frac{\partial}{\partial t}\Big((\psi_t^* Rm_{g_0})(x)(e_i^t \wedge e_j^t, e_k^t \wedge e_l^t)\Big) - (\psi_t^* Rm_{g_0})(x)(\mathrm{Ric}_{g_t}(e_i^t) \wedge e_j^t, e_k^t \wedge e_l^t)\\ &\qquad - \cdots - (\psi_t^* Rm_{g_0})(x)(e_i^t \wedge e_j^t, e_k^t \wedge \mathrm{Ric}_{g_t}(e_l^t))\Bigg).\end{aligned} \tag{6.12}$$

Since for all $Y \in T_xM$ we have that

$$\frac{\nabla^{g_0}}{dt} d\psi_t(Y) = \nabla^{g_0}_{\dot{\psi}_t(x)} d\psi_t(Y) = \nabla^{g_0}_{d\psi_t(Y)} \dot{\psi}_t(x) \tag{6.13}$$

and due to (6.2) for all $X \in TM$ that

$$\mathrm{Ric}_{g_0}(X) + \nabla^{g_0}_X \mathrm{grad}_{g_0} f_0 = \lambda X, \tag{6.14}$$

we find for all $i = 1, \ldots, n$ that

$$\begin{aligned}\frac{\nabla^{g_0}}{dt} d\psi_t(e_i^t) &\overset{(6.13)}{=} \nabla^{g_0}_{d\psi_t(e_i^t)} \dot{\psi}_t(x) + d\psi_t(\mathrm{Ric}_{g_t}(e_i^t)) \overset{(6.6)}{=} \frac{1}{\sigma(t)} \nabla^{g_0}_{d\psi_t(e_i^t)} \mathrm{grad}_{g_0} f_0|_{\psi_t(x)} + d\psi_t(\mathrm{Ric}_{g_t}(e_i^t))\\ &\overset{(6.14)}{=} \frac{\lambda}{\sigma(t)} d\psi_t(e_i^t) - \frac{1}{\sigma(t)} \mathrm{Ric}_{g_0}(d\psi_t(e_i^t)) + d\psi_t(\mathrm{Ric}_{g_t}(e_i^t))\\ &= \frac{\lambda}{\sigma(t)} d\psi_t(e_i^t) - \frac{1}{\sigma(t)} d\psi_t(\mathrm{Ric}_{\psi_t^* g_0}(e_i^t)) + d\psi_t(\mathrm{Ric}_{g_t}(e_i^t))\\ &\overset{1.3.2}{=} \frac{\lambda}{\sigma(t)} d\psi_t(e_i^t).\end{aligned} \tag{6.15}$$

Hence, we obtain that the first summand in the bracket on the right-hand side of (6.12) is equal to

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(Rm_{g_0}(\psi_t(x))\left(d\psi_t(e_i^t)\wedge d\psi_t(e_j^t), d\psi_t(e_k^t)\wedge d\psi_t(e_l^t)\right)\Big)\\
&\quad= \left(\nabla^{g_0}_{\dot\psi_t(x)} Rm_{g_0}\right)(\psi_t(x))\left(d\psi_t(e_i^t)\wedge d\psi_t(e_j^t), d\psi_t(e_k^t)\wedge d\psi_t(e_l^t)\right)\\
&\qquad + Rm_{g_0}(\psi_t(x))\left(\frac{\nabla^{g_0}}{dt} d\psi_t(e_i^t)\wedge d\psi_t(e_j^t), d\psi_t(e_k^t)\wedge d\psi_t(e_l^t)\right)\\
&\qquad + \cdots + Rm_{g_0}(\psi_t(x))\left(d\psi_t(e_i^t)\wedge d\psi_t(e_j^t), d\psi_t(e_k^t)\wedge \frac{\nabla^{g_0}}{dt} d\psi_t(e_l^t)\right)\\
&\overset{(6.15)}{=} \left(\psi_t^*\nabla^{g_0}_{\dot\psi_t(x)} Rm_{g_0}\right)(x)(e_i^t\wedge e_j^t, e_k^t\wedge e_l^t) + \frac{4\lambda}{\sigma(t)}(\psi_t^* Rm_{g_0})(x)(e_i^t\wedge e_j^t, e_k^t\wedge e_l^t)\\
&\overset{(6.6),(6.5)}{=} \frac{1}{\sigma(t)}\left(\psi_t^*\nabla^{g_0}_{\operatorname{grad}_{g_0} f_0|_{\psi_t(x)}} Rm_{g_0}\right)(x)(e_i^t\wedge e_j^t, e_k^t\wedge e_l^t) + \frac{4\lambda}{\sigma(t)^2} Rm_{g_t}(x)(e_i^t\wedge e_j^t, e_k^t\wedge e_l^t).
\end{aligned}$$

Plugging this into (6.12) yields that

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} Rm_{g_t}(x)\right)(e_i^t\wedge e_j^t, e_k^t\wedge e_l^t)\\
&\quad = \frac{2\lambda}{\sigma(t)} Rm_{g_t}(x)(e_i^t\wedge e_j^t, e_k^t\wedge e_l^t) + \left(\psi_t^*\nabla^{g_0}_{\operatorname{grad}_{g_0} f_0|_{\psi_t(x)}} Rm_{g_0}\right)(x)(e_i^t\wedge e_j^t, e_k^t\wedge e_l^t)\\
&\qquad - \Big((Rm_{g_t})(x)(\operatorname{Ric}_{g_t}(e_i^t)\wedge e_j^t, e_k^t\wedge e_l^t) + \cdots + (Rm_{g_t})(x)(e_i^t\wedge e_j^t, e_k^t\wedge \operatorname{Ric}_{g_t}(e_l^t))\Big).
\end{aligned}$$

From this, we obtain that

$$\begin{aligned}
&\left(\nabla_{\frac{\partial}{\partial t}} Rm_{g_t}(x)\right)(e_i^t\wedge e_j^t, e_k^t\wedge e_l^t)\\
&\quad = \frac{2\lambda}{\sigma(t)} Rm_{g_t}(x)(e_i^t\wedge e_j^t, e_k^t\wedge e_l^t) + \left(\psi_t^*\nabla^{g_0}_{\operatorname{grad}_{g_0} f_0|_{\psi_t(x)}} Rm_{g_0}\right)(x)(e_i^t\wedge e_j^t, e_k^t\wedge e_l^t).
\end{aligned}$$

It remains to show that

$$\psi_t^*\nabla^{g_0}_{\operatorname{grad}_{g_0} f_0|_{\psi_t(x)}} Rm_{g_0} = \nabla^{g_t}_{\operatorname{grad}_{g_t} f_t|_x} Rm_{g_t}.$$

From $f_t = f_0\circ\psi_t$, it follows that for all $Y\in T_xM$ we have that

$$\begin{aligned}
g_0(\operatorname{grad}_{g_0} f_0|_{\psi_t(x)}, d\psi_t(Y)) &= df_0(d\psi_t(Y)) = df_t(Y) = g_t(\operatorname{grad}_{g_t} f_t|_x, Y)\\
&= \sigma(t)(\psi_t^* g_0)(\operatorname{grad}_{g_t} f_t|_x, Y) = \sigma(t) g_0(d\psi_t(\operatorname{grad}_{g_t} f_t|_x), d\psi_t(Y)).
\end{aligned}$$

Since $d\psi_t|_x$ is an isomorphism, this provides that

$$\operatorname{grad}_{g_0} f_0|_{\psi_t(x)} = \sigma(t) d\psi_t(\operatorname{grad}_{g_t} f_t|_x). \tag{6.16}$$

Moreover, we have that $\psi_t : (M, \psi_t^* g_0) \to (M, g_0)$ is an isometry. Thus,

$$\psi_t^*\nabla^{g_0}_{\operatorname{grad}_{g_0} f_0|_{\psi_t(x)}} Rm_{g_0} = \nabla^{\psi_t^* g_0}_{d\psi_t^{-1}(\operatorname{grad}_{g_0} f_0|_{\psi_t(x)})}\psi_t^* Rm_{g_0} \overset{\substack{1.3.2,\\(6.16),(6.5)}}{=} \nabla^{g_t}_{\operatorname{grad}_{g_t} f_t|_x} Rm_{g_t}.$$

as we wanted to show. □

6.1.1 Ricci solitons and curvature conditions

In this section, we show that conical curvature conditions are invariant under self-similar solutions to the Ricci flow. If such a curvature condition is the domain of a certain scale-invariant and Bianchi-convex function F, the sublevel sets of which are invariant under the ordinary differential equation (2.4), and if, moreover, the curvature condition is satisfied by a gradient Ricci soliton, then F along the Riemannian curvature tensor of the Ricci soliton is f-subharmonic, where f denotes the potential function of the Ricci soliton.

Remark 6.1.12. Let (M, g_0) be a Ricci soliton and $g(t) = \sigma(t)\psi_t^* g_0$, $t \in [0, T)$, be the corresponding solution to the Ricci flow with $g(0) = g_0$. Then, with a view on Remark 1.3.2 and remembering Definition 2.1.11, in the case that Rm_g is considered as bilinear form on $\Lambda^2 TM$, we have that $\mathcal{C}(M, g_0) = \sigma(t)\mathcal{C}(M, g(t))$ for all $t \in [0, T)$. In the case that Rm_g is interpreted as endomorphism of $\Lambda^2 TM$, we have that $\mathcal{C}(M, g_0) = \sigma(t)^3\mathcal{C}(M, g(t))$ for all $t \in [0, T)$.

Recall that for a Riemannian manifold (M, g), by definition, g satisfies $\mathcal{C}(M, g)$ and for curvature conditions $\Omega \subseteq \mathcal{A}_n$ (where n is the dimension of M), we have that g satisfies Ω if and only if $\mathcal{C}(M, g) \subseteq \Omega$. Therefore, Remark 6.1.12 implies the following lemma.

Lemma 6.1.13. *Let $\Omega \subseteq \mathcal{A}_n$ be an $O(n)$-invariant cone, (M, g_0) an n-dimensional Ricci soliton and $g(t) = \sigma(t)\psi_t^* g_0$, $t \in [0, T)$, be the corresponding solution to the Ricci flow with $g(0) = g_0$. If g_0 satisfies Ω, then $g(t)$ satisfies Ω for all $t \in [0, T)$.*

Corollary 6.1.14. *Let $\Omega \subseteq \mathcal{A}_n$ be an open and $O(n)$-invariant cone and $F : \Omega \to \mathbb{R}$ be a smooth and $O(n)$-invariant function. Moreover, let (M, g_0) be an n-dimensional Ricci soliton such that g_0 satisfies Ω and let $g(t) = \sigma(t)\psi_t^* g_0$, $t \in [0, T)$, be the corresponding solution to the Ricci flow with $g(0) = g_0$. Then for all $t \in [0, T)$ and $x \in M$, the following is true*

$$\frac{\partial}{\partial t} F^{g(t)}(Rm_{g(t)}(x)) = dF^{g(t)}_{Rm_{g(t)}(x)}\left(\nabla_{\frac{\partial}{\partial t}} Rm_{g(t)}(x)^{\mathrm{v}}\right).$$

Proof. This follows directly from Lemma 2.1.19 and Lemma 6.1.13. ☐

Lemma 6.1.15. *Let $\Omega \subseteq \mathcal{A}_n \setminus \{0\}$ be an open and $O(n)$-invariant cone and $F : \Omega \to \mathbb{R}$ be a scale- and $O(n)$-invariant, smooth and Bianchi-convex function, the sublevel sets of which are invariant under the ordinary differential equation (2.4). Then for any n-dimensional gradient Ricci soliton (M, g_0, f_0, λ) such that g_0 satisfies Ω, we have that*

$$\Delta_{f_0}(F^{g_0} \circ Rm_{g_0}) \geq 0.$$

Here, we define the f_0-Laplacian $\Delta_{f_0} := \Delta_{g_0} - \partial_{\mathrm{grad}_{g_0} f_0}$.

Proof. Let $g(t) = \sigma(t)\psi_t^* g_0$, $t \in [0, T)$, be the solution to the Ricci flow corresponding to (M, g_0) with $g(0) = g_0$ and write $g_t := g(t)$ throughout the proof. Since g_0 satisfies Ω, so does g_t for all $t \in [0, T)$ (see Lemma 6.1.13), thus the function $F^{g_t} \circ Rm_{g_t} : M \to \mathbb{R}$ is defined for all $t \in [0, T)$. As in the proof of Proposition 5.1.5, one can show that

$$\frac{\partial}{\partial t}(F^{g_t} \circ Rm_{g_t}) \leq \Delta_{g_t}(F^{g_t} \circ Rm_{g_t}). \tag{6.17}$$

Furthermore, the scale-invariance of F yields for all $R \in \Omega$ that

$$dF_R(R) = 0. \tag{6.18}$$

Namely, let $R \in \Omega$ and $F(R) =: c$. Then R is contained in $\partial F^{-1}((-\infty, c])$, which is a cone, since F is scale-invariant, and therefore contains the curve $s \mapsto c(s) := R + sR$ for $s \in (-\epsilon, \epsilon)$ with $\epsilon < 1$. Now,

$$dF_R(R) = \frac{d}{ds}\Big|_{s=0} F(c(s)) = \frac{d}{ds}\Big|_{s=0} F((1+s)R) = \frac{d}{ds}\Big|_{s=0} F(R) = 0.$$

Therefore, on the other hand, this together with the evolution equation of a gradient Ricci soliton (see Lemma 6.1.11) and the fact that F^{g_t} is invariant under parallel transport for all $t \in [0, T)$ (see Lemma 2.1.14) gives for $t \in [0, T)$, $x \in M$ and $p \in O_x^{g_t}$ that

$$\begin{aligned} \frac{\partial}{\partial t} F^{g_t}(Rm_{g_t}(x)) &\overset{6.1.14}{=} dF^{g_t}_{Rm_{g_t}(x)} \left(\nabla_{\frac{\partial}{\partial t}} Rm_{g_t}(x)^{\mathrm{v}} \right) \\ &\overset{6.1.11}{=} \frac{2\lambda}{\sigma(t)} dF^{g_t}_{Rm_{g_t}(x)}(Rm_{g_t}(x)^{\mathrm{v}}) + dF^{g_t}_{Rm_{g_t}(x)}(\nabla^{g_t}_{\mathrm{grad}_{g_t} f_t|_x} Rm_{g_t}{}^{\mathrm{v}}) \\ &= \frac{2\lambda}{\sigma(t)} dF_{p^* Rm_{g_t}(x)}(p^* Rm_{g_t}(x)) + dF^{g_t}_{Rm_{g_t}(x)}(\nabla^{g_t}_{\mathrm{grad}_{g_t} f_t|_x} Rm_{g_t}{}^{\mathrm{v}}) \\ &\overset{(6.18)}{=} dF^{g_t}_{Rm_{g_t}(x)}(\nabla^{g_t}_{\mathrm{grad}_{g_t} f_t|_x} Rm_{g_t}{}^{\mathrm{v}}) \\ &= \partial_{\mathrm{grad}_{g_t} f_t|_x}(F^{g_t} \circ Rm_{g_t}). \end{aligned} \tag{6.19}$$

Consequently, for $t = 0$ the equations (6.17) and (6.19) provide that

$$\Delta_{g_0}(F^{g_0} \circ Rm_{g_0}) \geq \partial_{\mathrm{grad}_{g_0} f_0}(F^{g_0} \circ Rm_{g_0}). \qquad \square$$

6.1.2 Rigidity of compact Ricci solitons

Using the weak and strong parabolic maximum principles for functions, in this section we show a first rigidity result for compact Ricci solitons.

Theorem 6.1.16. *Let $\Omega \subseteq \mathcal{A}_n \setminus \{0\}$ be an open and $O(n)$-invariant cone and $F : \Omega \to \mathbb{R}$ a scale- and $O(n)$-invariant, smooth and strictly Bianchi-convex function, the sublevel sets of which are invariant under the ordinary differential equation* (2.4)*. Then all n-dimensional compact Ricci solitons (M, g_0) such that g_0 satisfies Ω are locally symmetric.*

Proof. Let $g(t) = \sigma(t)\psi_t^* g_0$, $t \in [0, T)$, be the solution to the Ricci flow corresponding to (M, g_0) with $g(0) = g_0$ and write $g_t := g(t)$ throughout the proof. As in the proof of Lemma 6.1.15, we find that $F^g \circ Rm_g : M \times [0, T) \to \mathbb{R}$ satisfies the heat inequality

$$\frac{\partial}{\partial t}(F^{g_t} \circ Rm_{g_t}) \leq \Delta_{g_t}(F^{g_t} \circ Rm_{g_t}).$$

Since M is compact and $F^g \circ Rm_g \in \mathcal{C}^\infty(M \times [0, T), \mathbb{R})$, we can exhaust the weak and strong parabolic maximum principles in order to show that the function $F^g \circ Rm_g$ is actually a solution to the heat equation. For this, let $x \in M$, $t \in [0, T)$ and $p \in O_x^{g_t}$. Moreover, set $q := \sqrt{\sigma(t)} d\psi_t|_x \circ p$. Then $q \in O^{g_0}_{\psi_t(x)}$ and the scale-invariance of F gives

$$\begin{aligned} F^{g_t}(Rm_{g_t}(x)) = F(p^* Rm_{g_t}(x)) &\overset{(6.5)}{=} F(\sigma(t)(d\psi_t|_x \circ p)^* Rm_{g_0}(\psi_t(x))) \\ &= F\left(\frac{1}{\sigma(t)} q^* Rm_{g_0}(\psi_t(x))\right) = F(q^* Rm_{g_0}(\psi_t(x))) = F^{g_0}(Rm_{g_0}(\psi_t(x))). \end{aligned}$$

Maximizing this over M yields for all $t \in [0,T)$ that

$$\max_{x\in M} F^{g_t}(Rm_{g_t}(x)) = \max_{x\in M} F^{g_0}(Rm_{g_0}(\psi_t(x))) = \max_{x\in M} F^{g_0}(Rm_{g_0}(x)) = \sup_{[0,T)\times M} F^g \circ Rm_g,$$

where in the last step we applied the weak parabolic maximum principle (see Remark 4.1.2), that is the supremum of $F^g \circ Rm_g$ is attained in $M \times \{0\}$. In particular, this is true for all $t \in (0,T)$. Thus, the strong maximum principle (see Remark 4.1.3) implies that $F^g \circ Rm_g$ is constant and in particular a solution to the heat equation. As in the proof of Proposition 5.1.5, this leads to $\nabla^{g_t} Rm_{g_t} \equiv 0$, hence (M, g_t) being locally symmetric, for all $t \in [0,T)$. □

Remark 6.1.17. In Lemma 6.1.15 and consequently Theorem 6.1.16, the condition that F is strictly Bianchi-convex and its sublevel sets are invariant under the ordinary differential equation (2.4) can be relaxed as follows: Due to the scale-invariance of F and Remark 6.1.12, the function F needs only to be strictly Bianchi-convex when restricted to $\mathcal{C}(M, g_0)$ and we only need to have that

$$dF_R(R^2 + R^\#) \leq 0$$

for $R \in \mathcal{C}(M, g_0)$.

In the next section, we want to prove a similar result for non-compact Ricci solitons.

6.2 Shrinking gradient Ricci solitons

In this section, we have a closer look at shrinking gradient Ricci solitons. They are of special interest as they arise as possible singularity models for the Ricci flow. More precisely, suppose that $g(t)$, $t \in [0,T)$ with $T < \infty$, is a maximal solution to the Ricci flow and that there exists a constant $C > 0$ such that for all $t \in [0,T)$

$$\sup_{x\in M} \|Rm_{g(t)}(x)\|_{g(t)} \leq \frac{C}{T-t},$$

i.e. $g(t)$ is a so-called *Type I Ricci flow*, then for every "singular" point $p \in M$, there exists a sequence $(\lambda_i)_{i\in\mathbb{N}}$ tending to infinity as $i \to \infty$ such that the rescaled Ricci flows $(M, g_i(t), p)$, $t \in [-\lambda_i T, 0)$, where

$$g_i(t) := \lambda_i g\left(T + \frac{t}{\lambda_i}\right),$$

subconverge to a non-flat shrinking gradient Ricci soliton as $i \to \infty$ [EMT11].

Remark 6.2.1. If (M, g, f, λ) is a shrinking gradient Ricci soliton, then $\left(M, 2\lambda g, f, \frac{1}{2}\right)$ is a shrinking gradient Ricci soliton with the same potential function. Namely,

$$\mathrm{ric}_{2\lambda g} + \mathrm{Hess}_{2\lambda g} f = \mathrm{ric}_g + \mathrm{Hess}_g f = \lambda g = \frac{1}{2}(2\lambda g).$$

Lemma 6.2.2 ([Zha09b]). *Let (M, g, f, λ) be a complete shrinking gradient Ricci soliton. Then* $\mathrm{scal}_g \geq 0$.

Lemma 6.2.3 ([Ham95]). *Let (M, g, f, λ) be a complete shrinking gradient Ricci soliton. Then*

$$\mathrm{scal}_g + \|\mathrm{grad}_g f\|_g^2 - 2\lambda f \equiv c$$

for a constant c.

For the proof, see for example [Cao10, Lemma 1.1].

For $\tilde{f} := f + \frac{c}{2\lambda}$, we have that

$$\mathrm{scal}_g + \|\mathrm{grad}_g \tilde{f}\|_g^2 - 2\lambda \tilde{f} \equiv 0. \tag{6.20}$$

Due to Lemma 6.2.2, the *normalization* $\tilde{f}$ of f is non-negative.

The asymptotic behaviour of the potential function of a complete non-compact shrinking gradient Ricci soliton is as follows.

Lemma 6.2.4 ([CZ10, Thm. 1.1])**.** *Let $(M^n, g, f, \frac{1}{2})$ be a complete non-compact shrinking gradient Ricci soliton satisfying* (6.20)*. Then*

$$\frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2,$$

where $c_1, c_2 > 0$ depend only on n and the geometry of g on $B_1(x_0)$. Here, $r := d(x_0, \cdot)$ denotes the distance function from a fixed point $x_0 \in M$.

6.2.1 Rigidity in the general case

In this section, we prove a second rigidity result for complete shrinking gradient Ricci solitons.

Remark 6.2.5. For the convenience of the reader, we briefly recall the divergence theorem: For compact sets B with smooth boundary, continuously differentiable functions f, g and continuously differentiable vector fields X defined on a neighborhood of B, we have that

$$\int_{\partial B} fg\langle X, \mathbf{n}\rangle dA = \int_B \mathrm{div}(fgX) dV = \int_B g\partial_X f + f\partial_X g + fg\mathrm{div}(X) dV,$$

where $\mathbf{n}$ denotes the outward pointing unit normal on ∂B.

Lemma 6.2.6. *Let (M, g, f, λ) be a complete shrinking gradient Ricci soliton. Then for all smooth functions $u : M \to \mathbb{R}$ with $|u(x)| \leq C$ for all $x \in M$ for some constant $C \geq 0$, we have that $\Delta_f u \geq 0$ implies that $\Delta_f u = 0$. Here, the f-Laplacian Δ_f is defined as in Lemma 6.1.15.*

Proof. By $\tilde{f} = f + d$, we denote the normalization of the potential function f. Moreover, for $r \in \mathbb{R}$ we define the set $D(r) := \{x \in M \mid \tilde{f}(x) \leq r\}$, which is compact due to Lemma 6.2.4 and Remark 6.2.1. Let $u : M \to \mathbb{R}$ be as in the statement. Then for $r \in \mathbb{R}$, we have that

$$\begin{aligned} h(r) &:= \int_{D(r)} \langle \mathrm{grad} u, \mathrm{grad} \tilde{f}\rangle e^{-\tilde{f}} dV \\ &= \int_0^r \int_{\partial D(s)} \langle \mathrm{grad} u, \mathbf{n}\rangle e^{-\tilde{f}} dA ds \end{aligned}$$

by the co-area formula [SY94], where $\mathbf{n} := \frac{\mathrm{grad}\tilde{f}}{\|\mathrm{grad}\tilde{f}\|}$ denotes the outward pointing unit normal on the boundary $\partial D(s)$. Since $\Delta_f = \Delta_{\tilde{f}}$, by assumption we have that

$$h'(r) = \int_{\partial D(r)} \langle \mathrm{grad} u, \mathbf{n}\rangle e^{-\tilde{f}} dA \overset{6.2.5}{=} \int_{D(r)} \mathrm{div}(e^{-\tilde{f}} \mathrm{grad} u) dV = \int_{D(r)} \underbrace{\Delta_{\tilde{f}} u}_{\geq 0} e^{-\tilde{f}} dV \geq 0.$$

Combining the soliton equations (6.3) and (6.20) yields that $\lambda n - 2\lambda\tilde{f} = \Delta\tilde{f} - \|\mathrm{grad}\tilde{f}\|^2$. Hence, on the other hand

$$\begin{aligned} h(r) &= \int_{D(r)} \left(\mathrm{div}\left(ue^{-\tilde{f}}\mathrm{grad}\tilde{f}\right) - u\left(\Delta\tilde{f} - \|\mathrm{grad}\tilde{f}\|^2\right)e^{-\tilde{f}}\right) dV \\ &= \int_{D(r)} \mathrm{div}\left(ue^{-\tilde{f}}\mathrm{grad}\tilde{f}\right) dV - \lambda \int_{D(r)} u\left(n - 2\tilde{f}\right) e^{-\tilde{f}} dV \\ &\overset{6.2.5}{=} \int_{\partial D(r)} u\langle \mathrm{grad}\tilde{f}, \mathbf{n}\rangle e^{-\tilde{f}} dA - \lambda \int_{D(r)} u\left(n - 2\tilde{f}\right) e^{-\tilde{f}} dV. \end{aligned}$$

For the first term on the right-hand side, we obtain that

$$\begin{aligned} \left|\int_{\partial D(r)} u\langle \mathrm{grad}\tilde{f}, \mathbf{n}\rangle e^{-\tilde{f}} dA\right| &\le C \int_{\partial D(r)} \langle \mathrm{grad}\tilde{f}, \mathbf{n}\rangle e^{-\tilde{f}} dA \\ &\overset{6.2.5}{=} C \int_{D(r)} \mathrm{div}(e^{-\tilde{f}}\mathrm{grad}\tilde{f}) dV \\ &= C \int_{D(r)} \left(\Delta\tilde{f} - \|\mathrm{grad}\tilde{f}\|^2\right) e^{-\tilde{f}} dV \\ &= C\lambda \int_{D(r)} \left(n - 2\tilde{f}\right) e^{-\tilde{f}} dV. \end{aligned}$$

In total, using that $\tilde{f} \ge 0$, we find that

$$\begin{aligned} h(r) \le |h(r)| &\le 2C\lambda \int_{D(r)} \left(n - 2\tilde{f}\right) e^{-\tilde{f}} dV \\ &\le 2C\lambda n e^{-d} \int_{D(r)} e^{-f} dV \\ &\le 2C\lambda n e^{-d} \int_M e^{-f} dV. \end{aligned}$$

By [CZ10, Corollary 1.1], the weighted volume $\int_M e^{-f} dV$ of M is finite, hence h is bounded from above. Since additionally h is monotonously increasing, there exists a sequence $(r_i)_{i\in\mathbb{N}}$ with $\lim_{i\to\infty} r_i = \infty$ such that

$$0 = \lim_{i\to\infty} h'(r_i) = \int_M \Delta_{\tilde{f}} u e^{-\tilde{f}} dV.$$

This implies that $\Delta_f u = \Delta_{\tilde{f}} u = 0$, since the integrand is non-negative. □

Remark 6.2.7. Due to Sard's theorem and the preimage theorem, the boundary $\partial D(r)$ is smooth for almost all $r \in \mathbb{R}$. In the proof of Lemma 6.2.6, the equations derived by applying the divergence theorem therefore only hold true for almost all r. However, by continuity reasons, they are correct for all r.

Corollary 6.2.8. *Let $\Omega \subseteq \mathcal{A}_n \setminus \{0\}$ be an open and $O(n)$-invariant cone and $F : \Omega \to \mathbb{R}$ a scale- and $O(n)$-invariant, smooth, bounded and Bianchi-convex function, the sublevel sets of which are invariant under the ordinary differential equation* (2.4)*. Then for any n-dimensional complete shrinking gradient Ricci soliton (M, g, f, λ) such that g satisfies Ω we have that*

$$\Delta_f(F^g \circ Rm_g) = 0.$$

Proof. This follows directly from Lemma 6.1.15 and Lemma 6.2.6. □

Theorem 6.2.9. *Let $\Omega \subseteq \mathcal{A}_n \backslash \{0\}$ be an open and $O(n)$-invariant cone and $F : \Omega \to \mathbb{R}$ a scale- and $O(n)$-invariant, smooth, bounded and strictly Bianchi-convex function, the sublevel sets of which are invariant under the ordinary differential equation* (2.4). *Then all n-dimensional complete shrinking gradient Ricci solitons (M, g, f, λ) such that g satisfies Ω are locally symmetric.*

Remark 6.2.10. Note that the assumptions in Theorem 6.2.9 can be relaxed as in Remark 6.1.17 and, in addition, F needs only to be bounded on $\mathcal{C}(M, g)$. Furthermore, if $\mathcal{C}(M, g)$ is contained in a closed non-trivial sublevel set $F^{-1}((-\infty, c]) \neq \Omega$ of F, then $F^{-1}((-\infty, c]) \cap \{R \in \mathcal{A}_n \mid \|R\|_g = 1\}$ is compact, hence due to its scale-invariance, F is bounded on $F^{-1}((-\infty, c])$. Therefore, in this case, the boundedness assumption on F in Theorem 6.2.9 can be removed as well.

Proof. From Corollary 6.2.8, we know that $\Delta_f(F^g \circ Rm_g) = 0$. Therefore, the proof of Lemma 6.1.15 yields that

$$\sum_{i=1}^{n} \mathrm{Hess}_{p^* Rm_g(x)} F \left(p^* \nabla^g_{e_i} Rm_g(x), p^* \nabla^g_{e_i} Rm_g(x) \right) = 0$$

for $x \in M$, where $(e_1, \ldots, e_n)$ is an orthonormal basis of $T_x M$ and $p \in O^g_x$. Since F is strictly Bianchi-convex, this shows that $\nabla^g Rm_g \equiv 0$. □

Remark 6.2.11. If a Riemannian manifold (M, g) is locally symmetric, we find that in particular the covariant derivative of the Weyl part of the Riemannian curvature tensor vanishes, which implies that its divergence is zero. By [FLGR11] and [MS13], this shows that all n-dimensional locally symmetric complete shrinking gradient Ricci solitons are either Einstein or finite quotients of $E \times \mathbb{R}^k$, where $k > 0$, E is an $(n-k)$-dimensional Einstein manifold and $\mathbb{R}^k$ is the Gaussian shrinking soliton.

Remark 6.2.12. In Theorem 6.2.9, one can relax the condition that Ω and F are $O(n)$-invariant and require invariance under $SO(n)$ or $U(\frac{n}{2})$ (if n is even) instead. In that case, the Ricci soliton (M, g) in question needs to be orientable respectively Kähler. Here, one uses the fact that Kähler manifolds stay Kähler under the Ricci flow.

6.3 Application

A first step into the direction of finding functions that satisfy the assumptions of Theorem 6.1.16 respectively Theorem 6.2.9 is to find one-parameter families of strictly convex cones in $\mathcal{A}_n$ which are invariant under the ordinary differential equation (2.4). Constructing a function which has the sets of such a family as sublevel sets, reparametrizing and restricting it appropriately provides a scale-invariant strictly Bianchi-convex function (see Theorem 5.2.1). In this section, we give two examples of such families and thereby derive two explicit rigidity results for complete shrinking gradient Ricci solitons, and as a special case for complete Einstein manifolds.

6.3.1 The cone $\mathcal{B}_n$

To obtain explicit applications of theorems 6.1.16 and 6.2.9, we will need to apply the reparametrization theorem 5.2.1. Therefore in this section, we want to have a closer look at the cone

$$\mathcal{B}_n = \left\{ R \in \mathcal{A}_n \mid R|_{\Lambda^2(v^\perp)} \not\equiv 0 \text{ for all } v \in \mathbb{R}^n \setminus \{0\} \right\},$$

where the restriction on $\Lambda^2(v^\perp)$ is meant in the endomorphism sense.

Remark 6.3.1. The cone $\mathcal{B}_n$ is open and dense in $\mathcal{A}_n$.

Remark 6.3.2. Notice that, by Lemma 3.2.1, the complement $\mathcal{B}_3^c$ of $\mathcal{B}_3$ is exactly the set of singular symmetric 3×3-matrices. Moreover, analogously to the three-dimensional case (see Remark 3.2.3), we have that the tuple $(T, 0, \ldots, 0) \in \mathcal{A}_n^n$ satisfies the second Bianchi identity if and only if T is contained in $\mathcal{B}_n^c$. In this sense, in arbitrary dimensions, elements of $\mathcal{B}_n^c$ are a natural generalization of singular matrices in $\mathcal{A}_3$.

Definition 6.3.3. For $\omega, \eta \in \Lambda^2(\mathbb{R}^n)^*$, the *symmetric tensor product* $\omega \odot \eta$ of ω and η is defined as follows:

$$\omega \odot \eta := \omega \otimes \eta + \eta \otimes \omega,$$

that is for $\mu, \nu \in \Lambda^2\mathbb{R}^n$, we have that

$$(\omega \odot \eta)(\mu, \nu) = \omega(\mu)\eta(\nu) + \eta(\mu)\omega(\nu).$$

Remark 6.3.4. Let $R \in \mathcal{B}_n^c$ and $v \in \mathbb{R}^n \setminus \{0\}$ with $R|_{\Lambda^2(v^\perp)} \equiv 0$. Let further $(e_1, \ldots, e_n)$ be an orthonormal basis of $\mathbb{R}^n$ with $e_1 = v$ and $(e^1, \ldots, e^n)$ the corresponding dual basis. Then $R = \sum_{i<j,k<l} R_{ijkl} e^i \wedge e^j \odot e^k \wedge e^l$ and for $p, q \neq 1$, we obtain that

$$\begin{aligned} 0 = R(e_p \wedge e_q) &= \sum_{i<j,k<l} R_{ijkl} \left((\delta_p^i \delta_q^j - \delta_q^i \delta_p^j) e^k \wedge e^l + (\delta_p^k \delta_q^l - \delta_q^k \delta_p^l) e^i \wedge e^j \right) \\ &= 2 \sum_{i<j} (R_{pqij} - R_{qpij}) e^i \wedge e^j = 4 \sum_{i<j} R_{pqij} e^i \wedge e^j. \end{aligned}$$

Since $(e^i \wedge e^j)_{i<j}$ is a basis of $\Lambda^2(\mathbb{R}^n)^*$, this yields that $R_{pqij} = 0$ for $i < j$. Due to the symmetry of R, all components R_{ijkl} with $1 \notin \{i, j\}$ or $1 \notin \{k, l\}$ are zero. Consequently, we have that

$$R = \sum_{i,j=2}^{n} R_{1i1j} e^1 \wedge e^i \odot e^1 \wedge e^j.$$

Definition 6.3.5. We call an algebraic curvature tensor $R \in \mathcal{A}_n$ to be *of type (D)*, if

$$R = c \sum_{i=2}^{n} e^1 \wedge e^i \odot e^1 \wedge e^i$$

for some suitable orthonormal basis $(e^1, \ldots, e^n)$ of $(\mathbb{R}^n)^*$ and $c \in \mathbb{R}$.

In order to understand the geometric meaning of the cone $\mathcal{B}_n$ a bit better, subsequently we give some properties of the complement $\mathcal{B}_n^c$ of $\mathcal{B}_n$.

Lemma 6.3.6. *For $R \in \mathcal{B}_n^c$ with* $\operatorname{scal}(R) > 0$*, we have that*

$$\sphericalangle(R, I) \geq \arctan\left(\sqrt{\frac{n-2}{2}}\right).$$

Equality holds if and only if R is of type (D). Here, I denotes the identity in $\mathcal{A}_n$.

Proof. Let $R \in \mathcal{B}_n^c$ and $v \in \mathbb{R}^n \setminus \{0\}$ with $R|_{\Lambda^2(v^\perp)} \equiv 0$. Let further $(e_1, \ldots, e_n)$ be an orthonormal basis of $\mathbb{R}^n$ with $e_1 = v$ and $(e^1, \ldots, e^n)$ the corresponding dual basis. In the following computation, we consider R as (0,4)-tensor, that is $R = \sum_{i,j,k,l=1}^{n} R_{ijkl} e^i \otimes e^j \otimes e^j \otimes e^k$. Remembering Section 1.1 and Remark 6.3.4, we find that

$$\operatorname{scal}(R) \overset{\substack{1.1.11,\\ 1.1.8}}{=} \sum_{i,j=1}^{n} R_{ijij} \overset{6.3.4}{=} 2 \sum_{i=2}^{n} R_{1i1i} \tag{6.21}$$

and therefore

$$\begin{aligned} \|R - R_I\|^2 &= \|R\|^2 - \|R_I\|^2 \\ &\overset{1.1.16}{=} \sum_{i,j,k,l=1}^{n} {R_{ijkl}}^2 - \frac{2}{n(n-1)} \operatorname{scal}(R)^2 \\ &\overset{6.3.4}{=} 4 \sum_{i,j=2}^{n} {R_{1i1j}}^2 - \frac{8}{n(n-1)} \left(\sum_{i=2}^{n} R_{1i1i} \right)^2 \\ &\geq 4 \sum_{i=2}^{n} {R_{1i1i}}^2 - \frac{8}{n(n-1)} \left(\sum_{i=2}^{n} R_{1i1i} \right)^2 \\ &\geq \frac{4}{n-1} \left(\sum_{i=2}^{n} R_{1i1i} \right)^2 - \frac{8}{n(n-1)} \left(\sum_{i=2}^{n} R_{1i1i} \right)^2 \\ &= \frac{4n-8}{n(n-1)} \left(\sum_{i=2}^{n} R_{1i1i} \right)^2 \\ &= \frac{n-2}{n(n-1)} \operatorname{scal}(R)^2 \\ &\overset{1.1.16}{=} \frac{n-2}{2} \|R_I\|^2. \end{aligned} \tag{6.22}$$

Since $\operatorname{scal}(R) > 0$, i.e. $\sphericalangle(R, I) < \frac{\pi}{2}$, it follows that

$$\tan(\sphericalangle(R, I)) = \frac{\|R - R_I\|}{\|R_I\|} \geq \sqrt{\frac{n-2}{2}}. \tag{6.23}$$

The first inequality in (6.22) is an equality if and only if $R_{1i1j} = 0$ for all $i \neq j$, whereas by Cauchy-Schwarz, the second inequality is an equality if and only if there is a constant $c \in \mathbb{R}$ such that $R_{1i1i} = c$ for all $i = 2, \ldots, n$. Hence, in (6.23) equality holds for

$$R = c \sum_{i=2}^{n} e^1 \wedge e^i \odot e^1 \wedge e^i. \qquad \square$$

Lemma 6.3.7. *For $R \in \mathcal{B}_n^c$, we have that*

$$\|R\|^2 \geq \frac{4}{n} \|\operatorname{ric}(R)\|^2.$$

Equality holds if and only if R is of type (D).

Proof. Let $R \in \mathcal{B}_n^c$ be as in the proof of Lemma 6.3.6. Then, by Remark 6.3.4, we have that

$$\begin{aligned}
\|\mathrm{ric}(R)\|^2 &= \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} R_{ikjk} \right)^2 \\
&= \left(\sum_{k=2}^{n} R_{1k1k} \right)^2 + 2 \underbrace{\sum_{i=2}^{n} \left(\sum_{k=2}^{n} R_{ik1k} \right)^2}_{=0} + \sum_{i,j=2}^{n} \left(\sum_{k=1}^{n} R_{ikjk} \right)^2 \\
&= \left(\sum_{k=2}^{n} R_{1k1k} \right)^2 + \sum_{i,j=2}^{n} {R_{i1j1}}^2,
\end{aligned}$$

hence by setting $\theta = \frac{n-1}{n}$ that

$$\begin{aligned}
\|R\|^2 &= \sum_{i,j,k,l=1}^{n} {R_{ijkl}}^2 = 4 \sum_{i,j=2}^{n} {R_{1i1j}}^2 = 4 \sum_{i=2}^{n} {R_{1i1i}}^2 + 4 \sum_{i \neq j} {R_{1i1j}}^2 \\
&= 4\theta \sum_{i=2}^{n} {R_{1i1i}}^2 + 4(1-\theta) \sum_{i=2}^{n} {R_{1i1i}}^2 + 4 \sum_{i \neq j} {R_{1i1j}}^2 \\
&\geq \frac{4\theta}{n-1} \left(\sum_{i=2}^{n} R_{1i1i} \right)^2 + 4(1-\theta) \sum_{i,j=2}^{n} {R_{1i1j}}^2 \\
&= \frac{4}{n} \|\mathrm{ric}(R)\|^2.
\end{aligned}$$

Here, equality holds if and only if $R_{1i1j} = 0$ for $i \neq j$ and $R_{1i1i} = c$ for $i = 2, \ldots, n$ for some constant $c \in \mathbb{R}$, i.e. if and only if R is of type (D). □

An algebraic curvature tensor $R \in \mathcal{A}_n$ is called *Einstein*, if there is some $\lambda \in \mathbb{R}$ such that $\mathrm{ric}(R) = \lambda \cdot \mathrm{id}$, that is if $\mathrm{ric}_0(R) = 0$. We refer to λ as the *Einstein constant of R*.

Lemma 6.3.8. *For $n \geq 3$, we have that*

$$\{R \in \mathcal{A}_n \setminus \{0\} \mid R \text{ is Einstein}\} \subseteq \mathcal{B}_n.$$

Proof. Suppose there is an $R \in \mathcal{B}_n^c$ with $R \neq 0$ and $\mathrm{ric}(R) = \lambda \mathrm{id}$ for some $\lambda \in \mathbb{R}$. Let R be as in the proof of Lemma 6.3.6. Since $\mathrm{ric}(R)_{ij} = \sum_{k=1}^{n} R_{ikjk}$ for $i, j = 1, \ldots, n$ and $R \in \mathcal{B}_n^c$, we obtain that

$$\lambda = \mathrm{ric}(R)_{ii} = R_{i1i1} = R_{1i1i}$$

for $i = 2, \ldots, n$ and therefore that

$$\lambda = \mathrm{ric}(R)_{11} = \sum_{k=2}^{n} R_{1k1k} = (n-1)\lambda.$$

This, however, is a contradiction, since $n \geq 3$ and $\lambda \neq 0$. Indeed, if we had that $\lambda = 0$, then $\mathrm{ric}(R) = 0$, which would imply that $0 = \mathrm{ric}(R)_{ij} = R_{i1j1}$ for $i, j \neq 1$. Hence, by Remark 6.3.4, this would lead to $R = 0$. □

Remark 6.3.9. All algebraic curvature tensors in $\mathcal{A}_2$ are Einstein, whereas $\mathcal{B}_2 = \emptyset$.

6.3.2 Application 1

For $n \geq 3$ and $a \in [0, \frac{n}{4}]$, we consider the family of scale-invariant sets

$$\Omega_a := \left\{ R \in \mathcal{A}_n \;\middle|\; \left(\frac{n-2}{4} + a\right) \|R\|^2 \leq \|\mathrm{ric}(R)\|^2 \;\text{ and }\; \mathrm{scal}(R) > 0 \right\}.$$

Remark 6.3.10. For $a > b$, we have that $\Omega_a \subseteq \Omega_b$. More precisely, from Remark 1.1.16, we obtain that

$$\left(\frac{n-2}{4} + a\right) \|R\|^2 \leq \|\mathrm{ric}(R)\|^2$$

is equivalent to

$$\frac{4a-n}{4}\|R_I\|^2 + a\|R_{\mathrm{ric}_0}\|^2 + \left(\frac{n-2}{4} + a\right) \|R_W\|^2 \leq 0.$$

This immediately yields that for $a > \frac{n}{4}$ the sets Ω_a are empty. For $a = \frac{n}{4}$, we have that $\Omega_a = \mathbb{R}_{>0} \cdot I$, where I denotes the identity in $\mathcal{A}_n$. Moreover, for $a \geq 0$ the cones Ω_a are convex and in the case that $a > 0$ even strictly convex. For $a = 0$, we have that

$$\begin{aligned} \Omega_0 &\overset{1.1.16}{=} \left\{ R \in \mathcal{A}_n \;\middle|\; \frac{n-2}{n}\|R_W\|^2 \leq \|R_I\|^2 \;\text{ and }\; \mathrm{tr}(R) > 0 \right\} \\ &= \left\{ R \in \mathcal{A}_n \;\middle|\; \sphericalangle(R_I + R_W, I) \leq \arctan\left(\sqrt{\frac{n-2}{n}}\right) \;\text{ and }\; \mathrm{tr}(R) > 0 \right\}. \end{aligned}$$

Lemma 6.3.11. *For $a \in (\frac{1}{2}, \frac{n}{4}]$, we have that $\Omega_a \subseteq \mathcal{B}_n$. Moreover, if $R \in \mathcal{A}_n$ is of type (D) with* $\mathrm{scal}(R) > 0$*, then we have that $R \in \partial\Omega_{\frac{1}{2}} \cap \mathcal{B}_n^c$.*

Proof. Let $a \in (\frac{1}{2}, \frac{n}{4}]$ and assume that there is an $R \in \Omega_a \cap \mathcal{B}_n^c$. Then we have that

$$\|\mathrm{ric}(R)\|^2 \geq \left(\frac{n-2}{4} + a\right) \|R\|^2 \overset{6.3.7}{\geq} \left(\frac{n-2}{4} + a\right) \frac{4}{n}\|\mathrm{ric}(R)\|^2.$$

Therefore, either $\|\mathrm{ric}(R)\| = 0$, which is a contradiction to $\mathrm{scal}(R) > 0$, or $a \leq \frac{1}{2}$, in contradiction to the assumption. Hence, $\Omega_a \subseteq \mathcal{B}_n$.
Moreover, due to Lemma 6.3.7, we find that R as in the statement of Lemma 6.3.11 is contained in $\mathcal{B}_n^c$ as well as in $\partial\Omega_{\frac{1}{2}}$, since $\mathrm{scal}(R) > 0$. □

On the set $\mathrm{scal}_+ := \{R \in \mathcal{A}_n \mid \mathrm{scal}(R) > 0\}$, we can define the $O(n)$-invariant function

$$F : \mathrm{scal}_+ \to \mathbb{R} : R \mapsto \frac{\|R\|^2}{\|\mathrm{ric}(R)\|^2}. \tag{6.24}$$

Due to $\mathrm{scal}(R) > 0$, thus $\mathrm{ric}(R) \neq 0$, for all $R \in \mathrm{scal}_+$, this function is well defined. Restricted to the open cone $\Omega := \cup_{a>0}\Omega_a$, F is bounded, more specifically

$$0 \leq F(R) < \frac{4}{n-2}$$

for $R \in \Omega$, and its sublevel sets are given by $\Omega_a = F^{-1}((-\infty, \frac{4}{n-2+4a}])$ for $a > 0$ and thereby are strictly convex cones. Moreover, F is smooth and for each $R \in \mathrm{scal}_+$ its differential is given by

$$\begin{aligned} dF_R(S) = &\ \frac{2}{\|\mathrm{ric}(R)\|^2}\big(\langle R, S\rangle - F(R)\langle \mathrm{ric}(R), \mathrm{ric}(S)\rangle\big) \\ \overset{(1.6)}{=} &\ \frac{2}{\|\mathrm{ric}(R)\|^2}\Bigg(\left(\frac{2}{n(n-1)} - \frac{F(R)}{n}\right)\mathrm{scal}(R)\mathrm{scal}(S) \\ &\qquad + \left(\frac{4}{n-2} - F(R)\right)\langle \mathrm{ric}_0(R), \mathrm{ric}_0(S)\rangle + \langle R_W, S_W\rangle\Bigg) \end{aligned} \tag{6.25}$$

for all $S \in T_R\mathrm{scal}_+ \cong \mathcal{A}_n$. Here, we used Remark 1.1.16, $\mathrm{ric}(R) = \mathrm{ric}_0(R) + \frac{1}{n}\mathrm{scal}(R)\mathrm{id}$ and $\langle \mathrm{ric}_0(R), \mathrm{id}\rangle = \mathrm{tr}(\mathrm{ric}_0(R)) = 0$. For the identity I, we find that $F(I) = \frac{2}{n-1}$, since $\|I\|^2 = \frac{n(n-1)}{2}$ and $\mathrm{ric}(I) = \frac{n-1}{2}\mathrm{id}$. Using (6.25), this directly leads to $dF_I = 0$. The next lemma shows that the positive multiples of I are the only points where the differential of F vanishes.

Lemma 6.3.12. *For $R \in \mathrm{scal}_+$ with $R \notin \mathbb{R}_{>0}I$, we have that $dF_R \neq 0$.*

Proof. Let $R \in \mathrm{scal}_+$ with $R \notin \mathbb{R}_{>0}I$. Then either $R_W \neq 0$ or $R_{\mathrm{ric}_0} \neq 0$. If $R_W \neq 0$, we set $S := R_W$. Then by definition, we have that $\mathrm{ric}(S) = 0$, hence $\mathrm{scal}(S) = 0$ and therefore $\mathrm{ric}_0(S) = 0$. This yields that

$$dF_R(S) = \frac{2\|R_W\|^2}{\|\mathrm{ric}(R)\|^2} > 0.$$

Conversely, if $R_{\mathrm{ric}_0} \neq 0$, we put $S := R_{\mathrm{ric}_0}$. Then $S_W = 0$ and since $\mathrm{ric}(R_{\mathrm{ric}_0}) = \mathrm{ric}_0(R)$, we again find that $\mathrm{scal}(S) = 0$ and thus that $\mathrm{ric}_0(S) = \mathrm{ric}(S) = \mathrm{ric}_0(R)$. Consequently,

$$\begin{aligned} dF_R(S) = &\ \frac{2}{\|\mathrm{ric}(R)\|^2}\left(\frac{4}{n-2} - F(R)\right)\|\mathrm{ric}_0(R)\|^2 \\ \overset{(1.3)}{=} &\ \frac{2}{\|\mathrm{ric}(R)\|^2}\underbrace{\left(1 - \frac{n-2}{4}F(R)\right)}_{>0}\|R_{\mathrm{ric}_0}\|^2 > 0. \end{aligned}$$

This leads to $dF_R \neq 0$ in any case. □

Furthermore, we notice that the Hessian of F at R is given by

$$\begin{aligned} \mathrm{Hess}_R F(S,S) &= \frac{d}{dt}\Big|_{t=0} dF_{R+tS}(S) \\ &= \frac{2}{\|\mathrm{ric}(R)\|^2}\left(\|S\|^2 - 2dF_R(S)\langle \mathrm{ric}(R), \mathrm{ric}(S)\rangle - F(R)\|\mathrm{ric}(S)\|^2\right) \end{aligned} \tag{6.26}$$

for $S \in T_R\mathrm{scal}_+$.

As we have already seen in Lemma 5.2.8, the Hessian of a scale-invariant function at a point R is always zero restricted to the space $\mathbb{R}R$. However, the following lemma shows that $\mathrm{Hess}_I F$ is "as positive definite as possible".

Lemma 6.3.13. *For all $S \in \mathcal{A}_n$ with $S \perp I$, we have that*

$$\mathrm{Hess}_I F(S,S) \geq \frac{4}{(n-1)^3}\|S\|^2.$$

Moreover, $\mathrm{Hess}_I F$ is diagonal with respect to the decomposition (1.2) of $\mathcal{A}_n$.

Proof. Let $S \in \mathcal{A}_n$ with $S \perp I$. Then $S_I = 0$, hence $\mathrm{scal}(S) = 0$ and therefore

$$\|S\|^2 \overset{(1.5)}{=} \frac{4}{n-2}\|\mathrm{ric}_0(S)\|^2 + \|S_W\|^2. \tag{6.27}$$

Since $dF_I = 0$ and $F(I) = \frac{2}{n-1}$, we obtain that

$$\begin{aligned}
\mathrm{Hess}_I F(S,S) &\overset{(6.26)}{=} \frac{2}{\|\mathrm{ric}(I)\|^2}\left(\|S\|^2 - \frac{2}{n-1}\|\mathrm{ric}(S)\|^2\right) \\
&\overset{(1.4),(1.5)}{=} \frac{8}{n(n-1)^2}\left(\|S_W\|^2 + \frac{2n}{(n-1)(n-2)}\|\mathrm{ric}_0(S)\|^2\right) \\
&\overset{(6.27)}{=} \frac{8}{n(n-1)^2}\left(\frac{n-2}{2(n-1)}\|S_W\|^2 + \frac{n}{2(n-1)}\|S\|^2\right) \\
&\geq \frac{4}{(n-1)^3}\|S\|^2.
\end{aligned} \tag{6.28}$$

By polarization of the last equality in (6.28), we immediately obtain that $\mathrm{Hess}_I F(I,S) = 0$ and polarizing the first equality in (6.28) yields that $\mathrm{Hess}_I F(R,W) = 0$ for $W \in \mathcal{W}$ and $R \perp W$. □

Remark 6.3.14. In dimension $n = 3$, the closures of the sets Ω_a coincide with the sets discussed in Section 3.2.2 for $c = 0$. More precisely, $\overline{\Omega}_a = \Omega_a \cup \{0\} = \widetilde{\Omega}_{\frac{1}{4a},0}$. (To show this, notice that $\mathcal{W} = \{0\}$ in dimension 3.) In Proposition 3.2.38, we have already seen that for $a \leq \frac{3}{4}$ these sets are invariant under the ordinary differential equation (2.4).
Analogously to Proposition 3.2.34, it should be possible to show that for certain a and $c \in \mathbb{R}$ the sets

$$\left\{R \in \mathcal{A}_n \;\middle|\; \left(\frac{n-2}{4} + a\right)\|R\|^2 - \|\mathrm{ric}(R)\|^2 \leq c \ \text{ and } \ \mathrm{scal}(R) \geq 0\right\}$$

are strictly Bianchi-convex.

Remark 6.3.15. An unpublished conjecture of Christoph Böhm and Burkhard Wilking is that for $n \geq 12$ the cones Ω_a are invariant under the ordinary differential equation (2.4) for $a \in [0, \frac{n}{4}]$.

Some evidence for this is the following.

Lemma 6.3.16. *There is some $\epsilon > 0$ such that $\overline{\Omega}_a$ respectively Ω_a are invariant under the ordinary differential equation* (2.4) *for $a \in [\frac{n}{4} - \epsilon, \frac{n}{4}]$.*

Proof. First of all, we consider the function

$$G : \mathrm{scal}_+ \to \mathbb{R} : R \mapsto dF_R(\Phi(R)),$$

where $\Phi(R) := R^2 + R^\#$, and show that restricted to Ω_a it is non-positive for a near $\frac{n}{4}$. To this end, we notice that $I^2 + I^\# = (n-1)I$ (see [BW08, Lemma 2.1]. Therefore, $G(I) = (n-1)dF_I(I) = 0$, since F is scale-invariant. Moreover, since $dF_I = 0$, for $R \in \mathrm{scal}_+$ we find that

$$\begin{aligned}
dG_I(R) &= \frac{d}{ds}\Big|_{s=0} G(I+sR) = \frac{d}{ds}\Big|_{s=0} dF_{I+sR}(\Phi(I+sR)) \\
&= \mathrm{Hess}_I F(R, \Phi(I)) = (n-1)\mathrm{Hess}_I F(R,I) \overset{6.3.13}{=} 0,
\end{aligned}$$

where in the last step we applied Lemma 6.3.13. Hence, $dG_I = 0$, which means that I is a critical point of G. Again using [BW08, Lemma 2.1], we observe that

$$\frac{d}{ds}\Big|_{s=0} \Phi(I+sR) = 2(R + I\#R) = 2(n-1)R_I + (n-2)R_{\mathrm{ric}_0}. \tag{6.29}$$

Moreover, (6.26) yields that

$$\mathrm{Hess}_I F(R_W, R_W) \overset{(6.26)}{=} \frac{2}{\|\mathrm{ric}(I)\|^2}\|R_W\|^2 = \frac{8}{n(n-1)^2}\|R_W\|^2 \tag{6.30}$$

and since $\mathrm{ric}(R_{\mathrm{ric}_0}) = \mathrm{ric}_0(R)$ that

$$\mathrm{Hess}_I F(R_{\mathrm{ric}_0}, R_{\mathrm{ric}_0}) \overset{(6.26)}{=} \frac{8}{n(n-1)^2}\left(\|R_{\mathrm{ric}_0}\|^2 - \frac{2}{n-1}\|\mathrm{ric}(R_{\mathrm{ric}_0})\|^2\right) \overset{(1.3)}{=} \frac{4}{(n-1)^3}\|R_{\mathrm{ric}_0}\|^2. \tag{6.31}$$

This together with Lemma 6.3.13 yields that for $R \in \mathcal{A}_n$, the Hessian of G at I is given by

$$\begin{aligned}
\mathrm{Hess}_I G(R,R) &= \frac{d^2}{ds^2}\Big|_{s=0} G(I+sR) = \frac{d^2}{ds^2}\Big|_{s=0} dF_{I+sR}(\Phi(I+sR))\\
&= D^3F_I(R,R,\Phi(I)) + 2\mathrm{Hess}_I F\left(R, \frac{d}{ds}\Big|_{s=0}\Phi(I+sR)\right)\\
&\overset{(6.29),6.3.13}{=} \frac{d}{ds}\Big|_{s=0} \mathrm{Hess}_{I+s\Phi(I)} F(R,R) + 2(n-2)\mathrm{Hess}_I F(R_{\mathrm{ric}_0}, R_{\mathrm{ric}_0})\\
&= \frac{d}{ds}\Big|_{s=0} \frac{1}{(1+(n-1)s)^2}\mathrm{Hess}_I F(R,R) + 2(n-2)\mathrm{Hess}_I F(R_{\mathrm{ric}_0}, R_{\mathrm{ric}_0})\\
&= -2(n-1)\mathrm{Hess}_I F(R,R) + 2(n-2)\mathrm{Hess}_I F(R_{\mathrm{ric}_0}, R_{\mathrm{ric}_0})\\
&\overset{6.3.13}{=} -2\mathrm{Hess}_I F(R_{\mathrm{ric}_0}, R_{\mathrm{ric}_0}) - 2(n-1)\mathrm{Hess}_I F(R_W, R_W)\\
&\overset{(6.30),(6.31)}{=} -\frac{8}{(n-1)^3}\|R_{\mathrm{ric}_0}\|^2 - \frac{16}{n(n-1)}\|R_W\|^2\\
&\leq 0.
\end{aligned}$$

In total, we have shown that $G(I) = 0$, $dG_I = 0$ and that $\mathrm{Hess}_I G$ is negative definite on $I^\perp$. Consequently, I is a local maximum of G, which means that there is a neighborhood $U \subseteq \mathrm{scal}_+$ of I such that $G|_U \leq 0$. Since G is scale- and $O(n)$-invariant, without loss of generality we may assume that U is scale- and $O(n)$-invariant as well. It follows that there is a $\epsilon > 0$ such that $G|_{\Omega_{\frac{n}{4}-\epsilon}} \leq 0$.

Now, let $a \in [\frac{n}{4}-\epsilon, \frac{n}{4}]$, $R_0 \in \partial\Omega_a = \partial F^{-1}((-\infty, \frac{4}{n-2+4a}]) \subseteq \mathrm{scal}_+$ and $R : [0,\delta] \to \mathcal{A}_n$ be a solution to the ordinary differential equation (2.4) with $R(0) = R_0$. From Example 2.3.5 we know that scal_+ is invariant under (2.4), hence $R(t) \in \mathrm{scal}_+$ for all $t \in [0,\delta]$. It follows that $F \circ R$ is defined and we have that

$$\frac{d}{dt}\Big|_{t=0} F(R(t)) = dF_{R_0}(\Phi(R_0)) = G(R_0) \leq 0,$$

which means that $F(R(t)) \leq F(R_0) = \frac{4}{n-2+4a}$ and equivalently

$$R(t) \in F^{-1}\left(\left(-\infty, \frac{4}{n-2+4a}\right]\right) = \Omega_a \subseteq \overline{\Omega}_a = \Omega_a \cup \{0\}$$

for small t. Comparing Definition 2.2.1, this yields that

$$\Phi(R_0) = R'(0) \in T_{R_0}\overline{\Omega}_a. \tag{6.32}$$

Since clearly $\Phi(0) = 0 \in T_0\overline{\Omega}_a$, (6.32) holds true for all $R_0 \in \partial\overline{\Omega}_a$. Consequently, Proposition 2.3.3 implies that $\overline{\Omega}_a$ is invariant under (2.4). Moreover, Ω_a is invariant under (2.4). Namely, if there was a solutions $R : [0, \delta] \to \mathcal{A}_n$ of (2.4) with $R(0) \in \Omega_a$ and $R(t_0) = 0$ for some $t_0 \in [0, \delta]$, then by uniqueness of solutions of ordinary differential equations we had that $R \equiv 0$, in contradiction to $R(0) \in \Omega_a$, thus $R(0) \neq 0$. □

The following rigidity result is based on the conjecture of Böhm and Wilking (see Remark 6.3.15).

Theorem 6.3.17. *For $n \geq 12$, let (M, g) be an n-dimensional complete shrinking gradient Ricci soliton with $\overline{\mathbb{R}_{>0} \cdot \mathcal{C}(M, g)} \subset \Omega \cap \mathcal{B}_n$, where $\Omega = \cup_{a>0}\Omega_a$ and the closure is taken in $\mathcal{A}_n \setminus \{0\}$. Then (M, g) is locally symmetric.*

Proof. By assumption, in $\mathcal{A}_n \setminus \{0\}$, the closed set $\overline{\mathbb{R}_{>0} \cdot \mathcal{C}(M, g)}$ is contained in the open set $\Omega \cap \mathcal{B}_n$. Therefore, there exists an open neighborhood U of $\overline{\mathbb{R}_{>0} \cdot \mathcal{C}(M, g)}$ such that $\overline{U} \subset \Omega \cap \mathcal{B}_n$. Since $\Omega \cap \mathcal{B}_n$ is an $O(n)$-invariant cone, without loss of generality we may assume that U is $O(n)$- and scale-invariant as well.

By Lemma 6.3.12, the only points where the differential of F vanishes are the positive multiples of I. From Lemma 6.3.13, we know that $\text{Hess}_I F|_{I^\perp}$ is positive definite. Therefore, by the scale-invariance of F (see Lemma 5.2.7), $\text{Hess}_R F|_{R^\perp}$ is positive definite for all $R \in \mathbb{R}_{>0} I$. Since the function $F : \Omega \to \mathbb{R}$ is smooth and its sublevel sets are strictly convex cones, according to Theorem 5.2.1 there exists a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi' > 0$ such that $\varphi \circ F|_U$ is strictly Bianchi-convex.

As it satisfies $\mathcal{C}(M, g)$, g also satisfies U, that is $\mathcal{C}(M, g) \subset U \subset \Omega$. Since the sublevel sets of $F : \Omega \to \mathbb{R}$ are conjectured to be invariant under the ordinary differential equation (2.4), Lemma 2.3.4 shows that in particular we have that $dF_R(R^2 + R^\#) \leq 0$, hence $d(\varphi \circ F)_R(R^2 + R^\#) \leq 0$ (since $\varphi' > 0$), for all $R \in \mathcal{C}(M, g)$. Consequently, we can apply Theorem 6.2.9 together with Remark 6.1.17 to the smooth, bounded, strictly Bianchi-convex, scale- and $O(n)$-invariant function $\varphi \circ F : U \to \mathbb{R}$ and obtain that (M, g) is locally symmetric. □

Remark 6.3.18. For $n \geq 12$, let (M, g) be an n-dimensional complete shrinking gradient Ricci soliton with g satisfying Ω_a for some $a > \frac{1}{2}$. Then according to Lemma 6.3.11, we automatically have that $\overline{\mathbb{R}_{>0} \cdot \mathcal{C}(M, g)} \subset \Omega \cap \mathcal{B}_n$. Thus, by Theorem 6.3.17, (M, g) is locally symmetric.

A direct consequence of Theorem 6.3.17 is the following.

Theorem 6.3.19. *Let $n \geq 12$. Then all n-dimensional complete Einstein manifolds (M, g) with g satisfying Ω_a for some $a > 0$ are locally symmetric.*

Proof. Clearly, (M, g) is a complete shrinking gradient Ricci soliton. Let $a > 0$. Since $\mathcal{C}(M, g) \subseteq \Omega_a$ and, moreover, Ω_a and $\{R \in \mathcal{A}_n \setminus \{0\} \mid R \text{ is Einstein}\}$ are closed cones (in $\mathcal{A}_n \setminus \{0\}$), according to Lemma 6.3.8, we have that

$$\overline{\mathbb{R}_{>0} \cdot \mathcal{C}(M, g)} \subseteq \Omega_a \cap \{R \in \mathcal{A}_n \setminus \{0\} \mid R \text{ is Einstein}\} \subset \Omega \cap \mathcal{B}_n.$$

Therefore, Theorem 6.3.17 yields that (M, g) is locally symmetric. □

6.3.2.1 The Bryant soliton

In this section, we show that the assumption "shrinking" cannot be dropped in Theorem 6.3.17 (and consequently in Theorem 6.2.9 in dimension $n \geq 12$).

The *Bryant soliton* $(M, g, f, 0)$ is a complete, non-compact and steady gradient Ricci soliton, where $M = \mathbb{R}^n$ and on $M \setminus \{0\} \cong (0, \infty) \times S^{n-1}$ (via polar coordinates) g is the warped product metric $g = dt^2 + w(t) g_{S^{n-1}}$, where t denotes the standard coordinate on $(0, \infty)$, $(S^{n-1}, g_{S^{n-1}})$ is the unit sphere with constant sectional curvature 1 and $w : (0, \infty) \to \mathbb{R}$ satisfies $w > 0$ and is given by a system of ordinary differential equations involving derivatives of the potential function f. Moreover, $\lim_{t\to 0} w(t) = 0$ and $\lim_{t\to 0} w'(t) = 1$ ensure that g can be smoothly extended to all of $\mathbb{R}^n$. For $x \in S^{n-1}$, with respect to an orthonormal basis $(b_1 = \frac{\partial}{\partial t}, b_2, \ldots, b_n)$, where $(b_2, \ldots, b_n)$ is an orthonormal basis of $T_x S^{n-1}$, we have that

$$Rm_g(t, x)(b_1 \wedge b_i, b_1 \wedge b_i) = -2\frac{w''(t)}{w(t)} =: a(t)$$
$$Rm_g(t, x)(b_i \wedge b_j, b_i \wedge b_j) = 2\frac{1 - w'(t)^2}{w(t)^2} =: b(t)$$

for all $t > 0$ and $i \neq j \in \{2, \ldots, n\}$. In other words, the sectional curvature of the Bryant soliton for planes tangent to the radial direction is given by $\frac{a}{2}$ and for planes tangent to S^{n-1} by $\frac{b}{2}$. This yields that for all $(t, x) \in (0, \infty) \times S^{n-1}$ and isometries $p : \mathbb{R}^n \to T_{(t,x)}M$ with $p(e_1) = \frac{\partial}{\partial t}$, where $(e_1, \ldots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$, we have that

$$p^* Rm_g(t, x) = a(t)D + b(t)(I - D),$$

where I denotes the identity in $\mathcal{A}_n$ (as in Remark 1.1.13) and D is of type (D) with respect to $(e_1, \ldots, e_n)$ (with $c = 1$ in Definition 6.3.5). Note that $2(I - D)$ is the Riemannian curvature operator of $\mathbb{R} \times S^{n-1}$ together with the product metric, if one identifies $\frac{\partial}{\partial t}$ with e_1 and $T_x S^{n-1}$ with $e_1^\perp$ for $x \in S^{n-1}$. Since the sectional curvatures of the Bryant soliton are known to be positive, we find that $a(t), b(t) > 0$ for $t > 0$. Furthermore, $a_0 := \lim_{t\to 0} a(t)$ and $b_0 := \lim_{t\to 0} b(t)$ exist and are positive.
The Bryant soliton is asymptotically cylindrical in the following sense: Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in (M, g) tending to infinity. Then there exists a sequence $(\lambda_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ and isometries $p_i : \mathbb{R}^n \to T_{x_i}M$, $i \in \mathbb{N}$, such that

$$\lim_{i\to\infty} \lambda_i p_i^* Rm_g(x_i) = 2(I - D), \tag{6.33}$$

the Riemannian curvature operator of the cylinder $\mathbb{R} \times S^{n-1}$. Of course, this requires that $p_i(e_1) = \frac{\partial}{\partial t}$ for $i = 1, \ldots, n$. (For references of the above see for example [CCG+07].)

Lemma 6.3.20. *There is an $\tilde{\epsilon} > 0$ such that $\frac{b(t)}{a(t)} > \tilde{\epsilon}$ for all $t > 0$.*

Proof. We suppose the opposite. Then there exists a sequence $(t_i)_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} \frac{b(t_i)}{a(t_i)} = 0.$$

Case 1: The sequence $(t_i)_{i\in\mathbb{N}}$ is bounded. Then there exists a convergent subsequence $(t_{i_k})_{k\in\mathbb{N}}$. Set $t := \lim_{k\to\infty} t_{i_k}$. If $t = 0$, we have that

$$0 = \lim_{k\to\infty} \frac{b(t_{i_k})}{a(t_{i_k})} = \frac{\lim_{k\to\infty} b(t_{i_k})}{\lim_{k\to\infty} a(t_{i_k})} > 0,$$

since the limits in the numerator and denominator exist and are positive. This is a contradiction. If $t > 0$, then due to continuity we have that

$$0 = \lim_{k\to\infty} \frac{b(t_{i_k})}{a(t_{i_k})} = \frac{b(t)}{a(t)},$$

which implies that $b(t) = 0$. This a contradiction to $b > 0$.

Case 2: The sequence $(t_i)_{i\in\mathbb{N}}$ diverges. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in S^{n-1}. Then $(t_i, x_i)_{i\in\mathbb{N}}$ tends to infinity in $\mathbb{R}\times S^{n-1}$. Since the Bryant soliton is asymptotically cylindrical, there exists a sequence $(\lambda_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ and isometries $p_i : \mathbb{R}^n \to T_{(t_i,x_i)}M$, $i \in \mathbb{N}$ with (6.33). Hence,

$$\begin{aligned}
2(I - D) &= \lim_{i\to\infty} \lambda_i p_i^* Rm_g(t_i, x_i)\\
&= \lim_{i\to\infty} \lambda_i \Big(a(t_i)D + b(t_i)(I - D)\Big)\\
&= \lim_{i\to\infty} \lambda_i a(t_i) \left(D + \frac{b(t_i)}{a(t_i)}(I - D)\right)\\
&= \left(\lim_{i\to\infty} \lambda_i a(t_i)\right)\left(D + \lim_{i\to\infty} \frac{b(t_i)}{a(t_i)}(I - D)\right)\\
&= cD,
\end{aligned}$$

where the forth equality holds, since the sequence $(x_i y_i)_{i\in\mathbb{N}}$, where $x_i := \lambda_i a(t_i)$ and $y_i := D + \frac{b(t_i)}{a(t_i)}(I - D)$, is convergent (with limit $I - D$) and $(y_i)_{i\in\mathbb{N}}$ converges by assumption to $D \neq 0$, hence $\lim_{i\to\infty} x_i =: c$ exists. This is a contradiction, since I, D are linearly independent (in the case that $c \neq -2$) respectively to $I \neq 0$ (if $c = -2$).

Consequently, the statement is true. □

Lemma 6.3.21. *Let $n \geq 5$. Then g satisfies Ω_a for some $a > \frac{1}{2}$.*

Proof. We consider the function

$$g : [0, \infty) \to \mathbb{R} : t \mapsto F(D + t(I - D)) = \frac{4 + 2(n-2)t^2}{n + 2(n-2)t + (n-2)^2t^2}$$

with F being the function defined in (6.24). Obviously, $g(0) = \frac{4}{n}$. Since $n \geq 4$, we have that $g(t) < \frac{4}{n}$ for $t > 0$ and since $n \geq 5$, that $\lim_{t\to\infty} g(t) = \frac{2}{n-2} < \frac{4}{n}$. Moreover, g has one local minimum (at $t = 1$) and this is even global. Therefore, for all $\epsilon > 0$ there exists some $\delta > 0$ such that $g(t) \leq \frac{4}{n} - \delta$ for all $t \geq \epsilon$. (Namely, $\delta = \frac{4}{n} - g(\epsilon)$.) In particular, there is some $\tilde{\delta} > 0$ such that

$$g\left(\frac{b(t)}{a(t)}\right) \leq \frac{4}{n} - \tilde{\delta}$$

for all $t > 0$, since by Lemma 6.3.20 we have that $\frac{b(t)}{a(t)} > \tilde{\epsilon}$ for $t > 0$.

Now, set $\epsilon := \frac{n^2\tilde{\delta}}{16-4n\tilde{\delta}} > 0$. Let $(t, x) \in (0, \infty) \times S^{n-1}$ and $p : \mathbb{R}^n \to T_{(t,x)}M$ be an isometry. Then the scale-invariance of F and $a > 0$ imply that

$$\begin{aligned}
F(p^*Rm_g(t, x)) &= F(a(t)D + b(t)(I - D)) = F\left(D + \frac{b(t)}{a(t)}(I - D)\right) = g\left(\frac{b(t)}{a(t)}\right)\\
&\leq \frac{4}{n} - \tilde{\delta} = \frac{4}{n + 4\epsilon}.
\end{aligned}$$

Since $\frac{b_0}{a_0} \geq \tilde{\epsilon}$, the same is true for t tending to 0. Together with the fact that the scalar curvature of the Bryant soliton is positive for $n \geq 2$, this shows that $p^* Rm_g(x) \in \Omega_{\frac{1}{2}+\epsilon}$ for all $x \in M$, thus $\mathcal{C}(M,g) \subseteq \Omega_{\frac{1}{2}+\epsilon}$, which means that g satisfies $\Omega_{\frac{1}{2}+\epsilon}$. □

Remark 6.3.22. Since the Bryant soliton is not locally symmetric, Lemma 6.3.21 shows that in general Theorem 6.3.17 (and consequently Theorem 6.2.9 in dimension $n \geq 12$) is false, if one drops the assumption "shrinking" (see Remark 6.3.18).

6.3.3 Application 2

For $n \geq 3$ and $a \geq 0$, we consider the family of scale-invariant sets

$$\begin{aligned}\Theta_a :&= \left\{ R \in \mathcal{A}_n \,\middle|\, a\|R_{\mathrm{ric}_0} + R_W\|^2 \leq \|R_I\|^2 \text{ and } \mathrm{scal}(R) > 0 \right\} \\ &= \left\{ R \in \mathcal{A}_n \,\middle|\, \sphericalangle(R, I) \leq \arctan\left(\sqrt{\frac{1}{a}}\right) \text{ and } \mathrm{scal}(R) > 0 \right\},\end{aligned}$$

where I denotes the identity in $\mathcal{A}_n$. Note that $\Theta_0 = \mathrm{scal}_+$ and $\Theta_\infty = \mathbb{R}_{>0} I$. For $a > 0$, the cones Θ_a are strictly convex. In [Hui85, Theorem 3.1], Huisken shows that Θ_a is preserved by the Ricci flow for $a \geq d_n$, where $d_n := \frac{(n-2)(n+1)}{2}$ (although his proof works only in dimension $n \geq 4$ and for $n = 5$, he needs that $d_5 = 10$). The constant d_n is the smallest value a such that $\Theta_a \subseteq \{R \in \mathcal{A}_n \mid R \geq 0\}$, the cone of positive curvature operators. In dimension $n \geq 4$, the author expects that Θ_a is even invariant under the ordinary differential equation (2.4) for all $a \geq d_n$ (in dimension $n = 3$, see Remark 6.3.24 below). Some indication for this is the following.

Lemma 6.3.23. *There exists an $a_0 > 0$ such that Θ_a is invariant under the ordinary differential equation* (2.4) *for $a \geq a_0$.*

Proof. The proof works analogously to the proof of Lemma 6.3.16. □

Remark 6.3.24. Notice that in dimension $n = 3$, the closures of the sets Θ_a coincide with the sets discussed in Section 3.2.2 for $c = 0$. More precisely, $\overline{\Theta}_a = \Theta_a \cup \{0\} = \widetilde{\Omega}_{\frac{1+a}{3a},0}$. In Proposition 3.2.38, we have already seen that these sets are invariant under the ordinary differential equation (2.4) for all $a \geq 0$.

Lemma 6.3.25. *For $a > \frac{2}{n-2}$, we have that $\Theta_a \subseteq \mathcal{B}_n$.*

Proof. Let $R \in \mathcal{B}_n^c$. Then, by Lemma 6.3.6, we have that

$$\sphericalangle(R, I) \geq \arctan\left(\sqrt{\frac{n-2}{2}}\right).$$

Hence, $R \notin \Theta_a$, which finishes the proof. □

Next, we define the smooth and $O(n)$-invariant function

$$F : \mathrm{scal}_+ \to \mathbb{R} : R \mapsto \frac{\|R_{\mathrm{ric}_0} + R_W\|^2}{\|R_I\|^2}. \tag{6.34}$$

For $a > 0$, its sublevel sets are the strictly convex cones $F^{-1}((-\infty, a]) = \Theta_{\frac{1}{a}}$. The differential of F at $R \in \mathrm{scal}_+$ is given by

$$\begin{aligned} dF_R(S) &= \frac{2}{\|R_I\|^2}\Big(\langle R_{\mathrm{ric}_0} + R_W, S_{\mathrm{ric}_0} + S_W\rangle - F(R)\langle R_I, S_I\rangle\Big) \\ &= \frac{2}{\|R_I\|^2}\Big(\langle R, S\rangle - (1 + F(R))\langle R_I, S_I\rangle\Big) \end{aligned}$$

for all $S \in T_R\mathrm{scal}_+$. Since $F(I) = 0$, we immediately obtain that $dF_I = 0$, which is clearly contained in scal_+, since $\mathrm{scal}(I) = \frac{n(n-1)}{2}$. The next lemma shows that the positive multiples of I are the only points where the differential of F vanishes.

Lemma 6.3.26. *For $R \in \mathrm{scal}_+$ with $R \notin \mathbb{R}_{>0}I$, we have that $dF_R \neq 0$.*

Proof. Let $R \in \mathrm{scal}_+$ with $R \notin \mathbb{R}_{>0}I$. Then either $R_{\mathrm{ric}_0} \neq 0$ or $R_W \neq 0$. If $R_{\mathrm{ric}_0} \neq 0$, we find that

$$dF_R(R_{\mathrm{ric}_0}) = 2\frac{\|R_{\mathrm{ric}_0}\|^2}{\|R_I\|^2} > 0$$

and if $R_W \neq 0$, we have that

$$dF_R(R_W) = 2\frac{\|R_W\|^2}{\|R_I\|^2} > 0.$$

Hence, $dF_R \neq 0$. □

Moreover, one can calculate the Hessian of F at R to be

$$\mathrm{Hess}_R F(S,S) = \frac{2}{\|R_I\|^2}\left(\|S\|^2 - 2dF_R(S)\langle R_I, S_I\rangle - (1 + F(R))\|S_I\|^2\right)$$

for $S \in T_R\mathrm{scal}_+$. For $R = I$, we obtain that

$$\mathrm{Hess}_I F(S,S) = \frac{2}{\|R_I\|^2}\left(\|S_{\mathrm{ric}_0} + S_W\|^2\right).$$

For $S \perp I$, this yields that

$$\mathrm{Hess}_I F(S,S) = \frac{2}{\|R_I\|^2}\|S\|^2.$$

Under the assumption that in dimension $n \geq 3$ the cone Θ_a is invariant under the ordinary differential equation (2.4) for $a \geq d_n$, the following theorem holds.

Theorem 6.3.27. *Let $n \geq 3$. Then all n-dimensional complete shrinking gradient Ricci solitons (M, g) with g satisfying Θ_{d_n} are locally symmetric.*

Proof. Using the observations above, the proof works similarly to the proof of Theorem 6.3.17 after exchanging Ω by $\Theta := \cup_{a > \frac{2}{n-2}}\Theta_a$ and U by the interior of Θ_a for some $a \in (\frac{2}{n-2}, d_n)$. However, we apply a slightly more general version of Theorem 6.2.9 (see Remark 6.1.17). □

Appendix A

In order not to disturb the reading flow, we have outsourced a collection of some statements we needed in the previous work. In this chapter, we finally give the results and corresponding references respectively proofs.

Lemma A.0.1. *Let $(V, \|\cdot\|)$ be a normed vector space and $\Omega \subseteq V$ a closed scale-invariant subset. Then $\partial\Omega$ is scale-invariant as well.*

Proof. If $v = 0 \in \partial\Omega$, then trivially $tv = 0 \in \partial\Omega$ for all $t > 0$. Now, let $v \in \partial\Omega$ with $v \neq 0$. Suppose there is a $t_0 > 0$ with $t_0 v \notin \partial\Omega$. Since Ω is scale-invariant, we have that $t_0 v$ is in the interior of Ω. Hence, there exists an $\epsilon > 0$ such that $B_\epsilon(t_0 v)$ is contained in the interior of Ω. Since Ω is scale-invariant, it follows that $\mathbb{R}_{>0} \cdot B_\epsilon(t_0 v)$ is contained in Ω. In particular, we have that

$$B_{\frac{\epsilon}{t_0}}(v) = \frac{1}{t_0} B_\epsilon(t_0 v) \subseteq \Omega$$

and therefore v lies in the interior of Ω. This, however, is a contradiction to $v \in \partial\Omega$. □

From [Die76, 16.17.5] the following Proposition is known.

Proposition A.0.2. *Let V be a vector bundle over a manifold M and let $s_1, \dots, s_k \in \Gamma(M, V^*)$ be smooth sections of V^* being linear independent in each point $p \in M$. Then*

$$\ker(s_1) \cap \cdots \cap \ker(s_k)$$

is a subbundle of V.

Remark A.0.3. Since $s_i(p) : V_p \to \mathbb{R}$ is linear and non-zero for $i = 1, 2, 3$ and $p \in M$, the dimension of its kernel is $\dim(V_p) - 1$. Since $\{s_1(p), \dots, s_k(p)\}$ is linear independent,

$$\dim\Big(\ker(s_1(p)) \cap \cdots \cap \ker(s_k(p))\Big) = \dim(V_p) - k.$$

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