Philipp Doebler

Stationary Set Preserving $\mathcal{L}$-Forcings and the Extender Algebra

2010

Mathematik

Stationary Set Preserving $\mathcal{L}$-Forcings and the Extender Algebra

Inaugural-Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften im Fachbereich
Mathematik und Informatik
der Mathematisch-Naturwissenschaftlichen Fakultät
der Westfälischen Wilhelms-Universität Münster

vorgelegt von
Philipp Doebler
aus Duisburg
- 2010 -

Dekan:	Prof. Dr. Dr. h.c. Joachim Cuntz
Erster Gutachter:	Prof. Dr. Ralf Schindler
Zweiter Gutachter:	Prof. Dr. Ilijas Farah
Tag der mündlichen Prüfung:	22.2.2010
Tag der Promotion:	22.2.2010

Abstract

We review the construction of Jensen's $\mathcal{L}$-forcing which we apply to study the Π_2 consequences of the theory $\mathsf{ZFC} + \mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous. Many natural consequences for H_{ω_2} of the theory $\mathsf{ZFC} + \mathsf{MM}$ follow from this weaker theory. We give a new characterization of the axiom (†) by isolating a class of stationary set preserving $\mathcal{L}$-forcings whose semiproperness is equivalent to (†). This characterization is used to generalize work of Todorčević: we show that Rado's Conjecture implies the combinatorial axiom (†).
Furthermore we study genericity iterations beginning with a measurable Woodin cardinal δ. We obtain a generalization of Woodin's Σ^2_1 absoluteness theorem for c.c.c. forcings. We study the class of subsets of ω_1 that extend to a class with unique condensation and develop a genericity iteration for such sets in generic extensions. Such a genericity iteration is used to show absoluteness results. Moreover we show that large cardinals imply that sets that extend to a class with unique condensation are constructible from a real.

Contents

Introduction

One of the main interests of modern set theory is absoluteness. Given two models M, N of set theory a statement ϕ, possibly in a parameter from $N \cap M$, is *absolute* between M and N if

$$M \models \phi \iff N \models \phi.$$

Set theorists are curious about absoluteness phenomena for (at least) two reasons: absoluteness theorems are central results of set theory and have also become important tools in the theory. We give an example: a classic absoluteness result is Shoenfield's Theorem, i.e. given two models M, N of set theory that both contain ω_1 as a subset and a real x, and given a $\Sigma^1_2(x)$ statement ϕ

$$M \models \phi \iff N \models \phi.$$

Shoenfield's Theorem is both, a central result of set theory and an important tool. Often Shoenfield's Theorem is applied in the context of forcing: if $V[G]$ is a forcing extension of V, then

$$V \prec_{\mathbf{\Sigma^1_2}} V[G].$$

Also in this thesis one of the main themes is absoluteness, especially in the sense of forcing absoluteness. We produce absoluteness theorems and we also frequently apply them. The most prominent we apply is the following: Bagaria discovered that bounded forcing axioms can be rephrased as absoluteness axioms.

Theorem (Bagaria [Bag00]) *Let Γ be a class of partial orders. The Bounded Forcing Axiom for Γ holds if, and only if, for all $\mathbb{P} \in \Gamma$*

$$H_{\omega_2} \prec_{\Sigma_1} H_{\omega_2}^{V^{\mathbb{P}}}.$$

We especially consider the case $\Gamma = \{\mathbb{P}\,;\,\mathbb{P}$ preserves stationary subsets of $\omega_1\}$, then the Bounded Forcing Axiom for Γ is Bounded Martin's Maximum, BMM.
We will now go into detail about the structure and contents: this thesis contains work on stationary set preserving $\mathcal{L}$-forcings and their application as well as work on the extender algebra and Σ^2_1-absoluteness. We give an introduction to both parts of the thesis.

$\mathcal{L}$-Forcing

Besides Bagaria's reformulation of BMM, the main tool of the first part of this thesis is R. Jensen's $\mathcal{L}$-forcing. In the early 90's Jensen developed the first example of an $\mathcal{L}$-forcing in the handwritten notes [Jenb]. This first example was a partial order $\mathbb{P}$ with the following properties: if GCH holds, κ is a measurable cardinal and U a normal measure on κ, then $\mathbb{P}$ collapses κ to become ω_1, makes $\mathrm{cof}(\kappa^{+V}) = \omega$ and adds a countable structure $\langle \bar{H}; \bar{U} \rangle$ that iterates in $\omega_1^{V^{\mathbb{P}}} = \kappa$ many times to $\langle H^V_{\kappa^{+V}}; U \rangle$. Since then Jensen has substantially refined his theory of $\mathcal{L}$-forcings: starting with some large cardinal assumption (or possibly just ZFC), GCH, a regular cardinal κ and some regular $\beta > \kappa$, an $\mathcal{L}$-forcing adds a family of countable models M_i, $i < \kappa$ together with a commutative system $\pi_{i,j} : M_i \to M_j$, $i \leq j < \kappa$ of elementary

embeddings such that the direct limit of the system $\langle M_i, \pi_{i,j}; i \leq j < \kappa\rangle$ is H_β^V. In the example of the forcing $\mathbb{P}$ above, the embeddings $\pi_{i,i+1}$ are the ultrapowers formed with the measure $\bar{U}$ and its images respectively. Other $\mathcal{L}$-forcings have different types of embeddings; usually the type of the embedding is closely related to the large cardinal assumption. The model M_0 is not an element of V, so the above type of $\mathcal{L}$-forcing adds reals. Other generalizations of $\mathcal{L}$-forcings do not add reals, see [Jena]. In Chapter 2 of this thesis we will review the construction of $\mathcal{L}$-forcings that add reals; moreover we generalize Jensen's construction to allow for other types of embeddings, especially generic ultrapowers.
The modification of $\mathcal{L}$-forcings towards generic ultrapowers is motivated by the following: B. Claverie and R. Schindler, starting with a precipitous ideal I on ω_1, constructed the first example of an $\mathcal{L}$-forcing $\mathbb{P}(\omega_2, I)$ that adds a direct system $\langle M_i, \pi_{i,j}, I_i, G_i; i \leq j < \kappa\rangle$ such that the direct limit of this system is $\langle H_{\omega_2}, I\rangle$ and for each $i < \omega_1$ $\pi_{i,i+1} : M_i \to M_{i+1} \simeq \mathrm{Ult}(M_i, G_i)$ is a generic ultrapower. The use of generic ultrapowers is the substantial new idea in the construction of Claverie-Schindler forcing; this allows to construct a stationary set preserving $\mathcal{L}$-forcing if I is the nonstationary ideal on ω_1. The first application of (a slight generalization of) $\mathbb{P}(\omega_2, I)$ was the following:

Theorem ([CS09]) *If Bounded Martin's Maximum* (BMM) *holds and* NS_{ω_1} *is precipitous, then* $\delta^1_{\mathbf{2}} = \omega_2$.

Note that it was known that $\delta^1_{\mathbf{2}} = \omega_2$ follows from the much stronger Martin's Maximum (MM). We will construct Claverie-Schindler forcing in Chapter 2 in our setup and study applications of $\mathbb{P}(\omega_2, I)$ and other stationary set preserving $\mathcal{L}$-forcings in Chapter 2 and 3, among them the above theorem, which we generalize in Theorem 3.1.25 by showing that BMM and NS_{ω_1} is precipitous implies Admissible Club Guessing which in turn implies $\delta^1_{\mathbf{2}} = \omega_2$, see Lemma 3.1.9. A rather elementary application we will study, due to S. Todorčević, is that $\mathbb{P}(2^{2^{\omega_1+}}, I)$ seals all antichains of I-positive sets from the ground model, see Theorem 2.4.16.
Chapter 3 is devoted to results in the style of the theorem above: for various Π_2 formula ϕ in parameters from H_{ω_2} we study the following question: if ϕ is a consequence of MM, does $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous imply ϕ? It will turn out that Woodin's Admissible Club Guessing, ϕ_{AC} and ψ_{AC} all follow from $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous, see Theorem 3.1.25 and Corollaries 3.2.11 and 3.2.20 respectively. Woodin had shown this for ψ_{AC} before using a more straightforward forcing construction. To show that ϕ_{AC} follows from $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous we construct a variant of Claverie Schindler forcing.
One can interpret these results as follows: any natural Π_2 consequence of MM for H_{ω_2} follows from BMM and NS_{ω_1} is precipitous. Of course "natural" is a very vague word and moreover there are Π_2 consequences of MM that can provably not follow from BMM and NS_{ω_1} is precipitous, namely certain Gödel sentences that assert the existence of countable models of large cardinals beyond the consistency strength of BMM and NS_{ω_1} is precipitous.
In Chapter 4 of this thesis we use $\mathcal{L}$-forcing to characterize the axiom

$$(\dagger) :\equiv \text{ Every stationary set preserving forcing is semiproper.}$$

We will show:

Theorem (Theorem 4.2.6) *The following are equivalent:*

1. *For every regular cardinal* $\theta \geq \omega_2$ *the Claverie-Schindler forcing* $\mathbb{P}(\theta, \mathsf{NS}_{\omega_1})$ *is semiproper.*

2. (†)

3. CC^{**}

Here CC^{**} is a combinatorial principle that strengthens Chang's Conjecture. The above theorem is in turn applied to show that Rado's Conjecture implies (†) by showing that Rado's Conjecture implies CC^{**}. This generalizes a result by Todorčević, who showed that Rado's Conjecture implies CC^{*}, see [Tod93].

The Extender Algebra

The second part of this thesis, which is identical with Chapter 5, deals with the extender algebra, a construction due to H. Woodin. Given a (fine-structural) sufficiently iterable countable model $\mathcal{M}$ that contains a Woodin cardinal δ, one can construct the extender algebra W_δ in $\mathcal{M}$. This construction then has the following application due to Woodin: given some $x \subset \omega$ and seeing W_δ as a forcing notion, one can find an iteration map $j : \mathcal{M} \to \mathcal{M}^*$ such that x is generic over $\mathcal{M}^*$ for $j(W_\delta)$. This iteration is known as a genericity iteration. It should be stressed that x was arbitrary to begin with and even more than that: if $\mathbb{P}$ is a notion of forcing, $\mathcal{M}$ is highly iterable and τ is a forcing-name for a real in $V^{\mathbb{P}}$, then there is an iterate $\mathcal{M}^*$ of $\mathcal{M}$, such that regardless of the choice of $G \subset \mathbb{P}$ generic over V we have that τ^G is generic over $\mathcal{M}^*$; this result is also due to Woodin. So all possible interpretations of the name τ are generic over $\mathcal{M}^*$.
We will carry out the construction of the extender algebra for a fine-structural model with a Woodin cardinal δ; for reals this has also been done in detail by Steel in [Ste], but we review the construction of the extender algebra, due to Woodin, for subsets of ω_1. The construction for reals starts with ω-many generators, the construction for subsets of ω_1 starts with δ-many generators. For the version with δ-many generators sources besides this thesis are scarce: see [Far] for the coarse case and a slightly different construction of the extender algebra, or see [SS09] for another application of the extender algebra with δ-many generators. These genericity iterations are applied to prove an original Σ^2_1-absoluteness theorem for c.c.c. forcings with ordinal parameters, Theorem 5.5.3.
Additionally we introduce and discuss sets that extend to a class with unique condensation: $A \subset \omega_1$ *extends to* A^* *with unique condensation* if A^* is class of ordinals such that $A^* \cap \omega_1 = A$ and for all uncountable cardinals κ

> if $\lambda > \kappa$ is a sufficiently large regular cardinal, then there is a club $C(A^*, \kappa, \lambda)$ of countable substructures $X \prec H_\lambda$ such that $A^* \cap \kappa \in X$ and
>
> $$A \cap \bar{\kappa} = \overline{A^* \cap \kappa},$$
>
> where π is the inverse of the collapse of X and $\pi(\bar{\kappa}, \overline{A^* \cap \kappa}) = \kappa, A^*$.

We analyse the sets that extend to classes with unique condensation in detail and construct non-trivial examples. We show that these sets can trivialize in the following sense: granted large cardinals in V and an iterability assumption, we show that every set with a uniquely condensing extensions is constructible from a real, see Theorem 5.6.14 for the precise statement.
We mentioned that, given a forcing-name for a real, a genericity iteration exists such that all interpretations of the name are generic over the final model. Given a name τ for a set that extends to a class with unique condensation, we construct a genericity iteration such that all interpretations of τ are generic over the final

model. We show that if τ is in a reasonable forcing extension, then such a genericity iteration behaves like genericity iterations for reals, see Lemma 5.6.16. We apply this Lemma to show two absoluteness results, Theorem 5.6.25 and Theorem 5.6.24.

Danksagung

An dieser Stelle möchte ich mich bei den vielen Leuten bedanken, die mich während meiner Promotion unterstützt haben. Zu größtem Dank bin ich dabei Prof. Dr. Ralf Schindler verpflichtet, der meine Promotion, auch über große Entfernungen hinweg, stets aufmerksam betreut hat. Weiterhin gilt mein Dank meinem Zweitgutachter Prof. Dr. Ilijas Farah und allen Mengentheoretikern am Institut für mathematische Logik und Grundlagenforschung, welche sich oft Zeit für meine Fragen nahmen: Dr. Gunter Fuchs, Benjamin A.P. Claverie, Dominik Adolf, Philipp Schlicht, Daiske Ikegami und Philipp Lücke.
Diese Promotion ist durch das DFG Projekt SCHI 484/3-1 “Anwendungen der Theorie der inneren Modelle auf Fragen der Absolutheit und Korrektheit” ermöglicht worden; an diesem Projekt war ich 3 Jahre angestellt und bin deswegen der DFG zu großem Dank verpflichtet. Bei Prof. Dr. Dr. Katrin Tent möchte ich mich dafür bedanken, dass sie so spontan meine Stelle in den Dezember 2009 hinein verlängert hat und mir so ermöglicht hat meine Promotion ohne Zeitdruck abzuschließen.
Mein persönlichster Dank gilt meiner Frau Anna und meinen Töchtern Clara und Elise.

Philipp Doebler, Münster im Januar 2010

1 Basic Concepts and Definitions

In this chapter we recall basic concepts that we will encounter later. Due to the complexity of what follows it is not possible to give a self-contained exposition in our framework. Definitions and basic results not found in this chapter are part of the standard set theoretic literature, i.e. the reader is advised to consult [Jec03] and [Kan03].
Our notation is standard and follows [Jec03].

1.1 Forcings that preserve ω_1

For the reader's convenience we recall several classes of forcings that do not collapse ω_1.

Definition and Lemma 1.1.1 *A notion of forcing $\mathbb{P}$ is* proper *if it satisfies the following equivalent conditions:*

1. *for every uncountable cardinal λ every stationary $S \subset [\lambda]^\omega$ in V is still stationary in $([\lambda]^\omega)^{V^{\mathbb{P}}}$;*

2. *for every sufficiently large λ, every well-ordering $<$ of H_λ and every countable elementary submodel $X \prec \langle H_\lambda; \in, <\rangle$ the following holds:*

$$\forall p \in X \cap \mathbb{P} : \ \exists q \leq p : q \textit{ is } (X, \mathbb{P})\textit{-generic,}$$

   where q is $(X, \mathbb{P})$-generic if

$$q \Vdash \dot{G} \cap X \textit{ is a filter on } \mathbb{P} \textit{ generic over } X.$$

For a proof of the well know equivalence above see [Jec03, 31.7, 31.16]. Note that all c.c.c. forcings and all ω-closed forcings are proper.

Definition 1.1.2 A notion of forcing $\mathbb{P}$ is *semiproper* if for every sufficiently large λ, every well-ordering $<$ of H_λ and every countable elementary submodel $X \prec \langle H_\lambda; \in, <\rangle$ the following holds:

$$\forall p \in X \cap \mathbb{P} \ \exists q \leq p : q \text{ is } (X, \mathbb{P})\text{-semigeneric,}$$

where q is $(X, \mathbb{P})$-semigeneric if for every name $\dot{\alpha} \in X$ for a countable ordinal

$$q \Vdash \exists \beta \in \check{X} : \dot{\alpha} = \beta.$$

Clearly every proper forcing is also semiproper.

Definition 1.1.3 A notion of forcing $\mathbb{P}$ *preserves stationary subsets of* ω_1 if every stationary $S \subset \omega_1$ is stationary in $V[G]$ for all $G \subset \mathbb{P}$ generic over V. When it is clear from the context which stationary sets are meant, we will call a forcing from the above class just *stationary set preserving.*

It is well known that all semiproper forcings are stationary set preserving and that all stationary set preserving forcings do not collapse ω_1^V. So we have the following diagram:

$$\text{c.c.c.} \subset \text{proper} \subset \text{semiproper} \subset \text{stationary set preserving} \subset \omega_1\text{-preserving.}$$

There is one more class of forcings that we will encounter:

Definition and Lemma 1.1.4 (Foreman-Magidor [FM95]) *A notion of forcing* $\mathbb{P}$ *is* reasonable *if it satisfies the following equivalent conditions:*

1. *for all ordinals* α $([\alpha]^\omega)^V$ *is stationary in* $([\alpha]^\omega)^{V^{\mathbb{P}}}$,
2. *for all* $p \in \mathbb{P}$ *and all sufficiently large regular* λ *there is an elementary substructure* $N \prec \langle H_\lambda; \in, \mathbb{P}, \{p\}\rangle$ *and an* $(N, \mathbb{P})$*-generic* $q \leq p$.

It is not difficult to see that reasonable forcings preserve ω_1, since any countable sequence cofinal in ω_1^V would witness that $([\omega_1^V]^\omega)^V$ is not stationary in the extension. Further note that every proper forcing is reasonable.

1.2 Forcing Axioms

D.A. Martin formulated the axiom that is known today as MA_{ω_1}. That was the foundation of the theory of forcing axioms. Martin and Solovay studied the slight generalization of MA_{ω_1}, nowadays called Martin's Axiom MA, see [MS70]. Using Shelah's concept of proper forcing, Baumgartner introduced the Proper Forcing Axiom PFA, see [Bau84]. The provably strongest forcing axiom was then isolated by Foreman, Magidor and Shelah and named Martin's Maximum MM, see [FMS88]. The following definition encompasses many forcing axioms.

Definition 1.2.1 Let Γ be a class of partial orders. The *Forcing Axiom for* Γ, $\mathsf{FA}(\Gamma)$, is the following principle: let $\mathbb{P} \in \Gamma$ and let $\langle D_i\,;\, i \in \omega_1\rangle$ denote a collection of sets dense in $\mathbb{P}$. Then there is a filter $F \subset \mathbb{Q}$ meeting every D_i, $i < \omega_1$.

If Γ is the class of c.c.c. partial orderings, then $\mathsf{FA}(\Gamma)$ is MA_{ω_1}. Clearly MM is $\mathsf{FA}(\Gamma)$ for Γ the class of stationary set preserving forcings, and PFA is $\mathsf{FA}(\Gamma)$ for Γ the class of proper forcings.

1.3 Bounded Forcing Axioms

Goldstern and Shelah were the first to study a "bounded" version of a forcing axioms. They weakened the Proper Forcing Axiom PFA to the Bounded Proper Forcing Axiom BPFA and studied it in depth, see [GS95]. The following definition encompasses many different bounded forcing axioms.

Definition 1.3.1 Let Γ be a class of partial orders. The *Bounded Forcing Axiom for* Γ, $\mathsf{BFA}(\Gamma)$, is the following principle: let $\mathbb{Q} = \mathrm{ro}(\mathbb{P}) \setminus \{0\}$ for some $\mathbb{P} \in \Gamma$, here $\mathrm{ro}(\mathbb{P})$ is the Boolean algebra consisting of the regular open subsets of $\mathbb{P}$. Let $\langle D_i\,;\, i \in \omega_1\rangle$ denote a collection of sets dense in $\mathbb{Q}$, each of cardinality at most ω_1. Then there is a filter $F \subset \mathbb{Q}$ meeting every D_i, $i < \omega_1$.

If Γ is the class of c.c.c. forcings, then $\mathsf{BFA}(\Gamma)$ is just MA_{ω_1}. Clearly BMM is $\mathsf{BFA}(\Gamma)$ for Γ the class of stationary set preserving forcings, and BPFA is $\mathsf{BFA}(\Gamma)$ for Γ the class of proper forcings. The following characterization of $\mathsf{BFA}(\Gamma)$ is due to

Bagaria, see [Bag00]. We will use this characterization without further notice. In our context the characterization of Bounded Forcing Axioms below is much more natural and applicable than the original definition.

Theorem 1.3.2 (Bagaria) *$\mathsf{BFA}(\Gamma)$ holds if, and only if, for all $\mathbb{P} \in \Gamma$*

$$H_{\omega_2} \prec_{\Sigma_1} H_{\omega_2}^{V^{\mathbb{P}}}.$$

We will also make use of the following result from [Sch04]:

Theorem 1.3.3 (Schindler) *If BMM holds, then $X^\sharp$ exists for every set X.*

Note that even a stronger closure property holds: if BMM holds, then for every set X there is an inner model with a strong cardinal containing X, see [Sch06]. Nevertheless we have no use for this fact in the following. The last fact that we will use without further notice is the following:

Theorem 1.3.4 ([Moo05]) *If BPFA holds, then $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.*

1.4 Precipitous Ideals

We will often deal with ideals on ω_1. Given an ideal I, we will consider the partial order $(\mathcal{P}(\omega_1) \setminus I, \subset)$. Forcing with this partial order yields an ultrafilter on $\mathcal{P}(\omega_1)^V$; therefore one is able to form a generic ultrapower:

Definition 1.4.1 An ideal $I \subset \mathcal{P}(Z)$ is *precipitous* if for every $G \subset \mathcal{P}(Z) \setminus I$ generic over V the ultrapower $j : V \to \mathrm{Ult}(V, G)$ is well-founded. Here $\mathrm{Ult}(V, G)$ is formed using functions from V. We will call $j : V \to \mathrm{Ult}(V, G)$ a *generic ultrapower* and identify it with its transitive collapse if it is well-founded.

Note that even if an ideal is not precipitous, well-founded ultrapowers might exist. It is well known that the existence of a precipitous ideal on $\mathcal{P}(\omega_1)$ is equiconsistent with a measurable cardinal, see [JMMP80]. Note that Kakuda and Magidor independently showed how to force a precipitous ideal on ω_1 from a measurable cardinal; this result is known as Kakuda's Theorem which we discuss in more depth in a moment.
By NS_{ω_1} we denote the ideal of nonstationary subsets of ω_1. Magidor showed that the precipitousness of NS_{ω_1} can also be forced from a measurable cardinal, see [JMMP80]. One might believe at first glance that precipitousness is a second order property, but precipitousness is equivalent to a first order combinatorial property, see for example [Jec03, 22.19, 22.21]. Hence if I is precipitous and $G \subset \mathcal{P}(\omega_1) \setminus I$ is V-generic, then $j(I)$ is still precipitous in $\mathrm{Ult}(V, G)$, where $j : V \to \mathrm{Ult}(V, G)$ is the generic ultrapower. So we can force over the ultrapower and form another generic ultrapower. This observation leads to *generic iterations* of which we will give a formal definition further below.

1.4.1 Adding a precipitous ideal with κ-c.c. forcing

Theorem 1.4.2 (Kakuda's Theorem) *If δ is a measurable Woodin cardinal and $\mathbb{P}$ a δ-c.c. notion of forcing, then forcing with $\mathbb{P}$ adds a precipitous ideal on δ.*

The above theorem is due to Kakuda and Magidor independently, see [Kak81] and [Mag80] respectively. If forcing with $\mathbb{P}$ is trivial, the precipitous ideal is the complement of the normal measure U witnessing that δ is measurable. Forcing with this ideal will also be trivial.

1.4.2 Stronger Ideals

If I is a precipitous ideal on ω_1, one can study the Boolean algebra $\mathcal{P}(\omega_1) \setminus I$ and ask which properties it has.

Definition 1.4.3 Let I be a normal uniform ideal on ω_1.

1. I is ω_2-*saturated*, or simply *saturated*, if $\mathcal{P}(\omega_1) \setminus I$ satisfies the ω_2-chain condition.

2. I is ω_1-*dense*, if $\mathcal{P}(\omega_1) \setminus I$ has a dense subset of cardinality ω_1.

It is a standard fact that all saturated ideals on ω_1 are precipitous. Clearly every ω_1-dense ideal is saturated and hence precipitous. If I is any normal uniform ideal on ω_1, then forcing with $\mathcal{P}(\omega_1) \setminus I$ collapses ω_1 to ω. This implies that $\mathcal{P}(\omega_1) \setminus I$ is isomorphic to the Boolean algebra generated by $\mathrm{Col}(\omega, \omega_1)$ if I is ω_1-dense. We stress that in this case $\mathcal{P}(\omega_1) \setminus I$ is a homogeneous forcing. If I is ω_2-saturated, then for all generic $G \subset \mathcal{P}(\omega_1) \setminus I$ and all induced generic embedding $j : V \to \mathrm{Ult}(V, G) =: M$ we have that $M^{\omega_1^V} \subset M$, where M^{ω_1} is calculated in $V[G]$.

1.5 Generic Iterations

We will often deal with generic iterations, obtained by iterating the process of forming generic ultrapowers. For this note that generic ultrapowers are a meaningful concept even in (countable) models of (fragments of) ZFC; all that one needs to form a generic ultrapower is that the class of functions with domain ω_1 exists and is reasonably closed, so that we can prove a version of the Łoś' Theorem. The exact choice of the fragment in the definition below is not too important, we could have used a different fragment. The fragment ZFC^* we use in the definition below originates from [Woo99]; note that for example $H_{\omega_2} \models \mathsf{ZFC}^*$. In the definition below we allow I to be a class from the point of view of M.

Definition 1.5.1 Let M be a countable transitive model of ZFC^* + "ω_1 exists," and let $I \subseteq \mathcal{P}(\omega_1^M)$ be such that $\langle M; \in, I\rangle \models$ "I is a uniform and normal ideal on ω_1^M." Let $\gamma \leq \omega_1$. Then

$$\langle \langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \gamma\rangle, \langle G_i; i < \gamma\rangle\rangle \in V$$

is called a *putative generic iteration of* $\langle M; \in, I\rangle$ (*of length* $\gamma + 1$) iff the following hold true:

1. $M_0 = M$ and $I_0 = I$.

2. For all $i \leq j \leq \gamma$, $\pi_{i,j} \colon \langle M_i; \in, I_i\rangle \to \langle M_j; \in, I_j\rangle$ is elementary, $I_i = \pi_{0,i}(I)$, and $\kappa_i = \pi_{0,i}(\omega_1^M) = \omega_1^{M_i}$.

3. For all $i < \gamma$, M_i is transitive and G_i is $(\mathcal{P}(\kappa_i) \setminus I_i, \subset)$-generic over M_i.

4. For all $i + 1 \leq \gamma$, $M_{i+1} = \mathrm{Ult}(M_i; G_i)$ and $\pi_{i,i+1}$ is the associated ultrapower map.

5. $\pi_{j,k} \circ \pi_{i,j} = \pi_{i,k}$ for $i \leq j \leq \gamma$.

6. If $\lambda \leq \gamma$ is a limit ordinal, then $\langle M_\lambda, \pi_{i,\lambda}; i < \lambda\rangle$ is the direct limit of $\langle M_i, \pi_{i,j}; i \leq j < \lambda\rangle$.

We call
$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \gamma\rangle, \langle G_i; i < \gamma\rangle\rangle$$
a *generic iteration of* $\langle M; \in, I\rangle$ (*of length* $\gamma+1$) iff it is a putative generic iteration of $\langle M; \in, I\rangle$ of length $\gamma+1$ and M_γ is transitive. $\langle M; \in, I\rangle$ is *generically* $\gamma+1$ *iterable* iff for any $\gamma' \leq \omega_1$ every putative generic iteration of $\langle M; \in, I\rangle$ of length $\gamma'+1$ is an iteration. We will call $\langle M; \in, I\rangle$ *generically iterable* if $\langle M; \in, I\rangle$ is generically ω_1+1 iterable in all generic extensions of V. We will call $\langle N; E, J\rangle$ a *(generic) iterate of* $\langle M; \in, I\rangle$ if there is a putative generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \gamma\rangle, \langle G_i; i < \gamma\rangle\rangle$$

of $\langle M_0; \in, I_0\rangle = \langle M; \in, I\rangle$ such that $\langle M_\gamma; E_\gamma, I_\gamma\rangle = \langle N; E, J\rangle$, where E_γ is M_γ's $\in$-relation. Sometimes we will say that $j : \langle M; \in, I\rangle \to \langle N; E, J\rangle$ is a generic iteration if we only want to refer to the iteration map but not to the generic filters.

Notice that we want (putative) iterations of a given countable model $\langle M; \in, I\rangle$ to exist in V, which amounts to requiring that the relevant generics G_i may be found in V.
The next lemmas roughly says: the critical points of a generic iteration and a few functions generate the final model of the iteration.

Lemma 1.5.2 (folklore) *Let*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \gamma\rangle, \langle G_i; i < \gamma\rangle\rangle$$

be a generic iteration of M_0. *Let* $\beta < \alpha \leq \gamma$. *All elements of* M_α *are of the form* $\pi_{\beta,\alpha}(f)(\vec{\xi})$ *for some* $f : \kappa_\beta^n \to M_\beta$, $f \in M_\beta$ *and ordinals* $\xi_1, ..., \xi_n < \omega_1^{M_\alpha}$. *Moreover all the ordinals* $\xi_1, ..., \xi_n$ *can be chosen in* $\{\kappa_i\,;\, \beta < i \leq \alpha\}$.

Proof. Fix $\beta < \omega_1$. We show this by induction on α. Let $\alpha = \gamma + 1$. Then M_α is isomorphic to $\mathrm{Ult}(M_\gamma, G_\gamma)$. Hence every element of M_α has the form $\pi_{\gamma,\alpha}(f)(\kappa_\gamma)$ for some $f : \kappa_\gamma \to M_\gamma$, $f \in M_\gamma$. By the inductive hypothesis f is of the form $\pi_{\beta,\gamma}(g)(\vec{\xi})$ for some $g : \kappa_\beta^n \to M_\beta, g \in M_\beta$ and $\vec{\xi} \in \kappa_\gamma^n$. Then

$$\pi_{\gamma,\alpha}(f)(\kappa_\gamma) = \pi_{\gamma,\alpha}(\pi_{\beta,\gamma}(g)(\vec{\xi}))(\kappa_\gamma) = \pi_{\beta,\alpha}(g)(\vec{\xi})(\kappa_\gamma),$$

since the critical point of $\pi_{\gamma,\alpha}$ is κ_γ.
The case $\mathrm{Lim}(\alpha)$ simply uses the fact that M_α is the direct limit of all M_γ for $\gamma < \alpha$: if $x \in M_\alpha$, then $x = \pi_{\gamma,\alpha}(\bar{x})$ for some $\gamma < \alpha$ and some $\bar{x} \in M_\gamma$. Without loss of generality we may assume $\beta < \gamma$. Then $\bar{x}$ is of the form $\pi_{\beta,\gamma}(g)(\vec{\xi})$ for some $g : \kappa_\beta^n \to M_\beta, g \in M_\beta$ and ordinals $\vec{\xi} \in \kappa_\gamma^n$. Then

$$x = \pi_{\gamma,\alpha}(\bar{x}) = \pi_{\gamma,\alpha}(\pi_{\beta,\gamma}(g)(\vec{\xi})) = \pi_{\beta,\alpha}(g)(\vec{\xi}).$$

□

1.6 Canonical Functions

When working with a generic ultrapower $j : V \to \mathrm{Ult}(V, G)$ we often deal with functions $f : \omega_1 \to \omega_1$ representing ordinals $< j(\omega_1)$.

Definition 1.6.1 Let $\eta < \omega_2$ be uncountable. Let $g : \omega_1 \to \eta$ be surjective. Define $f_\eta : \omega_1 \to \omega_1$ by $f_\eta(\alpha) = otp(g``\alpha)$. We call f_η the *canonical function for η derived from g.*

We recall the folklore around canonical functions. First note that we will not always make g explicit when referring to some canonical f_η; the following remark justifies to do so:

Remark 1.6.2 Let $\omega_1 \leq \eta < \omega_2$. If $g, g' : \omega_1 \to \eta$ are both surjective, then there is a club C such that

$$\forall \alpha \in C : f_\eta(\alpha) = f'_\eta(\alpha),$$

where f_η is the canonical functions derived from g and f'_η is the canonical function derived from g'.

Proof. Let $\alpha < \omega_1$ be such that there is some countable $X \prec H_{\omega_2}$ with $g, g' \in X$ such that $X \cap \omega_1 = \alpha$. We compute $f_\eta(\alpha)$:

$$f_\eta(\alpha) = \mathrm{otp}(g``\alpha) = \mathrm{otp}(g``X \cap \omega_1) = \mathrm{otp}(X \cap \eta),$$

here the last equality holds since g is surjective. Clearly the same computation shows $f'_\eta(\alpha) = \mathrm{otp}(X \cap \eta)$. It now suffices to note that there are club many $\alpha < \omega_1$ such that $X \cap \omega_1 = \alpha$ for some countable $X \prec H_{\omega_2}$ with $g, g' \in X$. □

There is a second approach to canonical functions: we inductively define a series of functions and prove that they are in fact canonical functions.

Definition 1.6.3 Let $\eta < \omega_2$. Let $f_0 : \omega_1 \to \omega_1$ be defined by $f_0(\alpha) = 0$ for all $\alpha < \omega_1$. Suppose that f_β has already been defined for all $\beta < \eta$. If $\eta = \beta + 1$, then we set

$$f_\eta(\alpha) := f_\beta(\alpha) + 1.$$

If η is an ordinal of cofinality ω, say $(\beta_i)_{i \in \omega}$ is a sequence cofinal in η, then we set

$$f_\eta(\alpha) := \sup\{f_{\beta_i}(\alpha)\,;\, i \in \omega\}.$$

If η is an ordinal of cofinality ω_1, say $(\beta_i)_{i \in \omega_1}$ is a sequence cofinal in η, then we set

$$f_\eta(\alpha) := \sup\{f_{\beta_i}(\alpha)\,;\, i < \alpha\}.$$

We call the construction in the uncountable cofinality case a *diagonal supremum.*

It is easy to verify the following: if $\eta < \omega_2$ is a limit ordinal, then f_η (modulo a nonstationary set) does not depend on the choice of the cofinal sequence. More generally we have the following notation: if $f, g : \omega_1 \to \omega_1$, then we set $f \leq g$ if and only if the set $\{\alpha \in \omega_1\,;\, f(\alpha) > g(\alpha)\}$ is nonstationary. Clearly we can define relations $=, <$ in the very same fashion. The relation $<$ is also a well founded partial ordering because NS_{ω_1} is σ-complete. Hence each $f : \omega_1 \to \omega_1$ has a rank $||f||$. We need a straightforward generalization. If S is stationary, then we say $f <_S g$ if and only if the set $\{\alpha \in S\,;\, f(\alpha) \geq g(\alpha)\}$ is nonstationary. This leads to a new notion of rank $||f||_S$. Note that if $S \subset T$ then $||f||_T \leq ||f||_S$. The following lemma is then easy to check, see [Jec03, 24.5].

Lemma 1.6.4 *If $\eta < \omega_2$ is a limit ordinal, then f_η is a least upper bound of $\{f_\beta\,;\, \beta < \eta\}$ in $\leq$. Each f_η is unique modulo $=$, and $||f_\eta||_S = \eta$ for all stationary S.* □

We now check that each f_η is in fact a canonical function according to our original definition.

Lemma 1.6.5 *Let $\eta < \omega_2$. Then there is a surjection $g : \omega_1 \to \eta$ and some club $C \subset \omega_1$ such that for all $\alpha \in C$*

$$f_\eta(\alpha) = \operatorname{otp} g``\alpha.$$

Hence f_η is a canonical function

Proof. By induction on η. There are clearly three cases: successor case, countable cofinality and uncountable cofinality. We discuss the uncountable cofinality case, the other cases are simpler. Let $(\beta_i)_{i<\omega_1}$ witness that η has uncountable cofinality. By our inductive hypothesis there exists for each $i < \omega_1$ some surjection $g_{\beta_i} : \omega_1 \to \beta_i$ and some club C_i witnessing that f_{β_i} is a canonical function. We construct $g : \omega_1 \to \eta$ such that if λ is a limit ordinal, then

$$g``\lambda = \bigcup_{i<\omega_1} g_{\beta_i}``\lambda.$$

For this recall that there is a bijection $\pi : \omega_1 \to \omega_1 \times \omega_1$ such that $\pi``\lambda = \lambda \times \lambda$ for all limit ordinals λ. Let $\alpha < \omega_1$ and let $\pi(\alpha) = (\alpha_0, \alpha_1)$. We set

$$g(\alpha) := g_{\beta_{\alpha_0}}(\alpha_1).$$

Clearly for all limit ordinals λ

$$g``\lambda = \bigcup_{i<\lambda} g_{\beta_i}``\lambda.$$

There is a club $D \subset \mathrm{Lim}$ such that for all $\alpha \in D$ if $i, j \in D$, $i < j$, then $g_{\beta_i}``\alpha \subset g_{\beta_j}``\alpha$. So for $\lambda \in D$

$$\operatorname{otp} g``\lambda = \operatorname{otp} \bigcup_{i<\lambda} g_{\beta_i}``\lambda = \sup\{\operatorname{otp} g_{\beta_i}``\lambda\,;\, i < \lambda\}.$$

Let C denote the diagonal intersection of all the C_i. Let $\lambda \in C \cap D$ be some limit ordinal. Then

$$\operatorname{otp} g``\lambda = \sup\{\operatorname{otp} g_{\beta_i}``\lambda\,;\, i < \lambda\} = \sup\{f_{\beta_i}(\lambda)\,;\, i < \lambda\} = f_\eta(\lambda).$$

□

As a corollary to the above lemmata we note:

Corollary 1.6.6 *Let f_η be a canonical function for $\eta < \omega_2$. In every generic ultrapower $j : V \to N$, η is represented by f_η; i.e. if N is well-founded, then $\eta = [f_\eta]$. This clearly holds regardless of the representative.* □

The following lemma describes a general way to obtain canonical functions. Note that a non-transitive X in the hypothesis of the lemma below can be obtained by taking a substructure closed under sequences of length ω_1 of some H_κ, $\kappa > \omega_1$ regular. Moreover note that a canonical function only needs to defined on a subset of ω_1 containing a club.

Lemma 1.6.7 *Let X be a model of ZFC^- of cardinality ω_1 such that $X \cap \omega_2$ is transitive. Let $\vec{X} = \langle X_i; i < \omega_1\rangle$ be a continuous chain of countable elementary* *submodels of X. For $i < \omega_1$ let $\pi_i : M_i \to X_i$ denote the inverse of the transitive collapse of X_i. Let $\beta < \omega_1$ and $\alpha \in X_\beta \cap \omega_2$. Then*

$$f : \omega_1 \setminus \beta \to \omega_1; \gamma \mapsto \pi_\gamma^{-1}(\alpha)$$

is a canonical function for α.

Proof. Let $f_\alpha : \omega_1 \to \omega_1$ be a canonical function for α derived from some surjective $g : \omega_1 \to \alpha$. We have to show that there is a club C such that $f_\alpha(\gamma) = f(\gamma)$ for all $\gamma \in C$. Let κ be large enough so that H_κ contains $\vec{X}$ and g. Let C consist of the ordinals $\gamma < \omega_1$ such that $\gamma = Y \cap \omega_1$ for a countable substructure $Y \prec H_\kappa$ with $\vec{X}, g \in Y$. For $\gamma \in C$ and a $Y \prec H_\kappa$ witnessing this we can then compute:

$$f_\alpha(\gamma) = \mathrm{otp}(g``\gamma) = \mathrm{otp}(Y \cap \alpha).$$

Since $X \cap \omega_2$ is transitive we have: $Y \cap \alpha = Y \cap X \cap \alpha = \bigcup\{X_i \cap \alpha\,;\, i < \gamma\}$; so, since $\vec{X}$ is continuous, $Y \cap \alpha = X_\gamma \cap \alpha$. Hence

$$\mathrm{otp}(Y \cap \alpha) = \mathrm{otp}(X_\gamma \cap \alpha) = \pi_\gamma^{-1}(\alpha) = f(\gamma).$$

This is what we wanted to show. □

1.7 Admissibles, Indiscernibles and Sharps

We will assume that the reader has a basic understanding of the theory of admissible ordinals. Since we do not too frequently encounter admissibles, we hint the reader at [Bar75]. Also we presuppose that the reader is familiar with indiscernibles, say with the exposition in [Kan03]. When referring to the sharp of a set, we do not think of Ehrenfeucht-Mostowski blueprints; we rather have the following in mind:

Definition 1.7.1 For a set X, we let $X^\sharp$ denote the least X-*mouse*, i.e. the least X-premouse $\mathcal{P} = \langle J_\alpha(X); \in, X, E_\alpha\rangle$, such that $E_\alpha \neq \emptyset$, $\mathcal{P}$ is sound above X and $\mathcal{P}$ is iterable.

For all concepts of inner model theory [Ste] is our reference if not otherwise stated; there the reader can find the definition of premouse, soundness and iterability.
Note that the universe of any $X^\sharp$ is a model of ZFC^- and also of ZFC^*+"ω_1. exists".
The next lemma shows how sharps and indiscernibles are related.

Lemma 1.7.2 (Folklore) *Let X be a set and suppose $X^\sharp = \mathcal{P} = \langle J_\alpha(X); \in, X, E_\alpha\rangle$ exists. Let κ denote the critical point of E_α. Then κ is an X-indiscernible. More generally: if $\pi : \mathcal{P} \to \mathcal{M}$ is an iteration map, where $\mathcal{M}$ is a linear iterate of $\mathcal{P}$ that we produced using only E_α and its images, then $\pi(\kappa)$ is also an X-indiscernible.* □

1.8 Universally Baire Sets of Reals

Feng, Magidor and Woodin introduced and analysed the universally Baire sets of reals.

Definition and Lemma 1.8.1 ([FMW92]) *Let $A \subset \omega^\omega$ and let $\kappa \geq \omega$ be a cardinal. The following are equivalent:*

1. *For every topological space X with a regular open basis of cardinality $\leq \kappa$ and for every continuous function $f : X \to \omega^\omega$ the set $f^{-1}``A$ is Baire.*

2. *There are trees T, S on $\omega \times 2^\kappa$ such that $A = p[T]$ and for every forcing $\mathbb{P}$ of cardinality $\leq \kappa$*
$$V^{\mathbb{P}} \models p[\check{T}] = \omega^\omega \setminus p[\check{S}].$$

3. *There are trees T, S such that $A = p[T]$ and*
$$V^{\mathrm{Col}(\omega,\kappa)} \models p[\check{T}] = \omega^\omega \setminus p[\check{S}].$$

*If A satisfies the three equivalent conditions, we will say that A is κ-*universally Baire*. If A is κ-universally Baire for all cardinals $\kappa \geq \omega$, then we will say that A is* universally Baire. □

In [FMW92] it is shown that all analytic sets are universally Baire (and hence also all co-analytic sets). In general large cardinals are required to see that a given projective set is universally Baire. The following lemma says that any set that is $\mathbf{\Delta^1_2}$ in all forcing extensions of V, is also universally Baire. Note that this pointclass contains the sets that are provably $\mathbf{\Delta^1_2}$.

Lemma 1.8.2 (Folklore) *Let ϕ_0 and ϕ_1 denote Σ^1_2 formulae with real parameters and one free variable. Furthermore assume if M is a forcing extension of V, then*
$$M \models \forall x : \phi_0(x) \iff \neg\phi_1(x).$$
Then $\{x \in \omega^\omega\,;\, \phi_0(x)\}^V$ is universally Baire.

Proof. For this let us recall a fact about Shoenfield trees: given a $\mathbf{\Sigma^1_2}$-formula ϕ one usually constructs the Shoenfield tree T for ϕ on $\omega \times \omega_1$; i.e. the canonical tree T such that $p[T] = \{x \in \omega^\omega\,;\, \phi(x)\}$. Nevertheless it is well known that one can also construct a canonical Shoenfield tree T for ϕ on $\omega \times \kappa$ for a regular cardinal $> \omega_1$. If T and the parameters of ϕ are in some transitive $M \models \mathsf{ZFC}$ such that $\omega_1^M \leq \kappa$, then $p[T]^M = \{x \in \omega^\omega\,;\, \phi(x)\}^M$. Furthermore note that such a T is in $L[\vec{p}]$ where $\vec{p}$ are the real parameters of ϕ.
We need to show that $\{x \in \omega^\omega\,;\, \phi_0(x)\}$ is κ universally Baire for all κ. Let T be the Shoenfield tree on $\omega \times \kappa^+$ for ϕ_0 and let S be the Shoenfield tree on $\omega \times \kappa^+$ for ϕ_1. Let M denote a forcing extension of V by a forcing of size $\leq \kappa$. We have $p[T]^M = \{x \in \omega^\omega\,;\, \phi_0(x)\}^M$, $p[S]^M = \{x \in \omega^\omega\,;\, \phi_1(x)\}^M$, so by our hypothesis
$$M \models \forall x \in \omega^\omega : x \in p[T] \iff x \notin p[S].$$
□

1.9 Chang's Conjecture

In this thesis we will encounter Chang's Conjecture and some of its generalizations. Before we give a definition we recall a basic notion: for infinite cardinals $\lambda > \kappa$ a closed unbounded set $C \subset [\lambda]^\kappa$ is *strongly closed unbounded* if there is some $F : [\lambda]^{<\omega} \to \lambda$ such that $C \supset C_F := \{x \in [\lambda]^\kappa\,;\, F``[x]^{<\omega} \subset x\}$. By a theorem of Kueker, for $\kappa = \omega$ the notion of closed unbounded and strongly closed unbounded coincide.

Definition and Lemma 1.9.1 *The following are equivalent*

1. *Every model of type (ω_2, ω_1) has an elementary submodel of type (ω_1, ω).*

2. *The set $\{X \subset \omega_2\,;\, \mathrm{otp}(X) = \omega_1\}$ intersects all strongly closed unbounded sets.*

If one of the equivalent conditions holds, then we say Chang's Conjecture (CC) *holds.*

Proof. Let $F : [\omega_2]^{<\omega} \to \omega_2$ be a function. The set $C_F := \{x \in [\omega_2]^{\omega_1} ; F``[x]^{<\omega} \subset x\}$ is strongly club in $[\omega_2]^{\omega_1}$. Granted 1. we have to show an $X \in C_F$ of ordertype ω_1 exists. For this study the structure $\mathfrak{A} = \langle \omega_2; \in, \omega_1, F, G\rangle$ where $G : \omega_2 \times \omega_1 \to \omega_2$ is such that $G(\alpha, \cdot) : \omega_1 \to \alpha$ is a surjection. By 1. a substructure $\mathfrak{B} = \langle A; \in, B, F, G\rangle \prec \mathfrak{A}$ exists such that A has cardinality ω_1 and B is countable. Let $\langle \alpha; \in, \beta, \bar{F}, \bar{G}\rangle$ denote the transitive collapse of $\mathfrak{B}$. Clearly $\alpha = \mathrm{otp}(\alpha) = \mathrm{otp}(A)$ and A is closed under F. So it remains to show $\alpha = \omega_1$. If $\omega_1 \in \alpha$, then by elementarity $\bar{G}(\omega_1, \cdot) : \beta \to \omega_1$ is a surjection, but this contradicts that β is countable. Since α is uncountable $\omega_1 = \alpha$.
For the converse assume we have a model $\mathfrak{A} = \langle A; R, ...\rangle$ of type (ω_2, ω_1). Without loss of generality we can assume $A = \omega_2$ and also we can add the $\in$ relation, so $\mathfrak{A} = \langle \omega_2; R, \in, ...\rangle$. There is a function $F : [\omega_2]^{<\omega} \to \omega_2$ such that all sets closed under F are elementary substructures of $\mathfrak{A}$. Clearly the sets of cardinality ω_1 closed under F form a club in $[\omega_2]^{\omega_1}$, hence by our hypothesis there is some $B \subset \omega_2$ of ordertype ω_1 such that B is closed under F. Since $R \subset A$ is a bounded subset of A, we have that $R \cap B$ is bounded in B. Any bounded subset of B must have countable ordertype. This shows that $\langle B; R \cap B, ...\rangle$ is of type (ω_1, ω). □

2 $\mathcal{L}$-Forcing

Following Jensen's handwritten notes [Jena] we develop the theory of $\mathcal{L}$-forcing more or less from scratch. In contrast to [Jena] we do not make use of infinitary languages and Barwise theory. This of course changes the proofs; the forcings we can construct with this approach are nevertheless the same. Additionally we discuss $\mathcal{L}$-forcings that add generic iterations and isolate a class of $\mathcal{L}$-forcings that preserve stationary subsets of ω_1. Such a forcing was constructed in [CS09] and a reader familiar with [CS09] will see that our presentation is influenced by [CS09].

2.1 Definition of $\mathcal{L}$-Forcing

We will now describe what all $\mathcal{L}$-forcings have in common: given some cardinal $\theta > \omega_1$, an $\mathcal{L}$-forcing $\mathbb{P}$ adds a system $\langle M_i; i \leq \kappa\rangle$ of models and a commutative, continuous system of elementary embeddings $\pi_{i,j} : M_i \to M_j$ for $i \leq j \leq \kappa$ such that all M_i are countable for $i < \kappa$ and $M_\kappa = H_\theta^V$. Also $\mathbb{P}$ preserves the regularity of κ, hence $\kappa = \omega_1^{V^{\mathbb{P}}}$. Note that the cardinality of H_θ^V is $\aleph_1$ in the extension, since $H_\theta^V = M_\kappa$ is a direct limit of countable structures and κ is the ω_1 of the extension. Different $\mathcal{L}$-forcings are not only constructed by choosing different κ and θ but also by discussing different types of elementary embeddings $\langle \pi_{i,j}; i \leq j \leq \kappa\rangle$; for example $\pi_{i,i+1}$ could be an ultrapower or a generic ultrapower. Also there are $\mathcal{L}$-forcings such that subsets of κ are coded into the system of elementary embeddings $\langle \pi_{i,j}; i \leq j \leq \kappa\rangle$, see for example Theorem 3.2.5 and Theorem 3.2.15. Also note that in the definitions below we want to include the possibility that M_κ is slightly larger than H_θ^V, say $M_\kappa = (H_\theta^V)^\sharp$.
In the following we will often consider models of a language $\mathcal{L}$ of set theory with two additional constants $\dot{\pi}$ and $\dot{M}$, where the constant $\dot{M}$ will be interpreted as a continuous system of models and the constant $\dot{\pi}$ will be interpreted as a commuting system of elementary embeddings. Note that we do not restrict ourselves to a language with only these two constants; it might be convenient to consider models of languages with additional non-logical symbols to build other $\mathcal{L}$-forcings.
We outline our cardinal setup: in the following we fix three regular cardinals $\rho = 2^{<\rho} > 2^\theta > \theta = 2^{<\theta} > \kappa > \omega$. We are aware of the fact that such ρ and θ might not always exist. Nevertheless in the applications we have in mind it is not relevant: one forces $\rho = 2^{<\rho}$ and $\theta = 2^{<\theta}$ and then proceeds with the construction we will outline, see the proof of Theorem 2.4.8 for a rigorous treatment of this problem.
Let us fix a model $\mathcal{M}^0$ of cardinality θ such that $\theta \subset |\mathcal{M}|$ and let $<_0$ be a well-ordering of $\mathcal{M}^0$ of ordertype $\mathsf{OR} \cap |\mathcal{M}^0|$. End-extend $<_0$ to a well ordering $<$ of H_ρ of order type ρ such that $<\restriction |\mathcal{M}^0| = <_0$. In an abuse of notation we will write $<$ for $<_0$ if talking about $\mathcal{M}^0$. All Skolem-hulls will be calculated with the help of $<$; i.e. we choose $<$-least witnesses. We set

$$\mathcal{M} = \langle |\mathcal{M}|; \in, <, ...\rangle,$$

i.e. $\mathcal{M}^0$ extended by $<$, and

$$\mathcal{H} = \langle H_\rho; \in, <, \mathcal{M}, ...\rangle.$$

For the next few definitions we fix θ, κ, $\mathcal{M}$ and $\mathcal{H}$ as above.

Definition 2.1.1 Let $\mathfrak{A} = \langle |\mathfrak{A}|; \in, \dot{\pi}^{\mathfrak{A}}, \dot{M}^{\mathfrak{A}}, ...\rangle$ be a transitive model. Let Φ be a (possibly uncountable) collection of statements in the language of $\mathfrak{A}$ with parameters from $H_{\theta+}$. We call $\mathfrak{A}$ a *certifying structure for Φ with respect to $H_{\theta+}$, κ and $\mathcal{M}$* if the following conditions are met:

1. $H^V_{\theta+} \subset \mathfrak{A}$ and $\mathcal{M} \in \mathfrak{A}$,
2. $\mathfrak{A} \models \mathsf{ZFC}^-$, where ZFC^- is ZFC without the power set axiom,
3. $\mathfrak{A} \models \kappa = \omega_1$,
4. $\mathfrak{A} \models \dot{M}^{\mathfrak{A}} = \langle \dot{M}^{\mathfrak{A}}_i; i \leq \kappa\rangle$ and if $i < \kappa$ then $\dot{M}^{\mathfrak{A}}_i$ is countable in $\mathfrak{A}$ and $\dot{M}^{\mathfrak{A}}_\kappa = \mathcal{M}$,
5. $\mathfrak{A} \models \dot{\pi}^{\mathfrak{A}} = \langle \dot{\pi}^{\mathfrak{A}}_{i,j}; i \leq j \leq \kappa\rangle$ is a commutative, continuous system of elementary embeddings $\dot{\pi}^{\mathfrak{A}}_{i,j} : \dot{M}^{\mathfrak{A}}_i \to \dot{M}^{\mathfrak{A}}_j$,
6. $\mathfrak{A} \models \mathrm{crit}(\dot{\pi}^{\mathfrak{A}}_{i,\kappa}) < \mathrm{crit}(\dot{\pi}^{\mathfrak{A}}_{i+1,\kappa})$ for all $i < \kappa$,
7. $\mathfrak{A} \models \kappa \in \mathrm{ran}(\dot{\pi}^{\mathfrak{A}}_{i,\kappa})$ for all $i \leq \kappa$,
8. $\mathfrak{A} \models \phi$ for every $\phi \in \Phi$.

We stress that every ϕ in item 8. above contains the symbols from the language of $\mathfrak{A}$, especially every ϕ may contain the symbols $\dot{\pi}$ and $\dot{M}$. If it is clear from the context which $H_{\theta+}$, κ and $\mathcal{M}$ are meant we will drop them. Clearly if $\kappa > \omega_1$ then a certiyfing structure cannot exist in V. There is a slightly more subtle reason why certifying structures do not exist in V: since $\kappa = \omega_1^{\mathfrak{A}}$ and $\mathcal{M}$ is a limit of countable structures the model $\mathcal{M}$ has at most cardinality $\aleph_1$ in $\mathfrak{A}$; since the cardinality of $\mathcal{M}$ is at least $\aleph_2$ in V a certifying structure can not exist in V. We will hence always consider certifying structures in some forcing extension.
The choice of Φ and $\mathcal{M}$ determines what kind of $\mathcal{L}$-forcing we construct. Of course we only want to consider reasonable choices of Φ, i.e. those that are consistent in some certifying structure:

Definition 2.1.2 Let $\mathcal{M}$, θ and κ be as above. Let Φ be a collection of statements in the language of set theory with two additional constants $\dot{\pi}$, $\dot{M}$ (and maybe other relation and function symbols) with parameters from V. We call Φ *consistent* if in $V^{\mathrm{Col}(\omega,2^\theta)}$ there is a certifying structure for Φ with respect to $H_{\theta+}, \kappa$ and $\mathcal{M}$.

Jensen isolated the following concept of resectionability; we merely made minor changes to adopt it to our situation. Resectionability is crucial in the proof that $\mathcal{L}$-forcings preserve the regularity of κ and that certain $\mathcal{L}$-forcings preserve stationary subsets of ω_1, see Theorem 2.2.3 and Theorem 2.3.2 respectively. Roughly resectionability states the following: given a certifying structure $\mathfrak{A}$ for Φ and a small certifying structure $\bar{\mathfrak{A}} \in \mathfrak{A}$ that certifies an "initial segment" of $\dot{\pi}^{\mathfrak{A}}$ and $\dot{M}^{\mathfrak{A}}$ we can change $\dot{\pi}^{\mathfrak{A}}$ and $\dot{M}^{\mathfrak{A}}$ according to $\dot{\pi}^{\bar{\mathfrak{A}}}$ and $\dot{M}^{\bar{\mathfrak{A}}}$ and still have that the changed structure is a certifying structure for Φ. We try to explain why resectionability is crucial in the theory of $\mathcal{L}$-forcing: all conditions of a $\mathcal{L}$-forcing will be certified by a certifying structure. In the proof for the central Theorem 2.3.2 resectionability is used to find certifying structures for a certain $\mathcal{L}$-forcing condition q that appears in the proof; without resectionability it is not clear that q is certified by some structure, i.e. it is not clear that q is a condition. Note that in all cases we know resectionability is trivial to check, albeit it is a lengthy concept. We believe that only pathological examples of a non-resectionable forcings exist.

Definition 2.1.3 Let θ, κ and $\mathcal{M}$ be as above. Let Φ be a collection of statements in the language of set theory with two additional constants $\dot\pi$, $\dot M$ with parameters from H_{θ^+}. We call Φ *resectionable* if the following holds: let $\mathfrak{A} = \langle |\mathfrak{A}|; \in, \dot\pi^{\mathfrak{A}}, \dot M^{\mathfrak{A}}, ...\rangle \in \mathcal{H}^{\mathrm{Col}(\omega,2^\theta)}$ be a certifying structure for Φ with respect to H_{θ^+}, κ and $\mathcal{M}$. Let $Y \prec \mathcal{H}$ such that $\Phi \in Y$ and $H \in H^{\mathfrak{A}}_{\omega_1}$ where H denotes the transitive collapse of Y. Let $\bar\Phi, \bar N, \bar\pi, \bar{\mathcal{M}}, \sigma$ such that

1. $\bar\Phi$ is the transitive collapse of Φ,
2. $\bar N$ is transitive and $\bar N \subset H$,
3. $\bar\pi, \bar{\mathcal{M}} \in H^{\mathfrak{A}}_{\omega_1}$ and $\sigma \in \mathfrak{A}$,
4. $\sigma : \langle \bar N; \bar\pi, \bar{\mathcal{M}}\rangle \prec \langle H^V_{\theta^+}; \dot\pi^{\mathfrak{A}}, \mathcal{M}\rangle$.

Letting $\bar\kappa = \sigma^{-1}(\kappa)$, we have: $\bar{\mathcal{M}} = \dot M^{\mathfrak{A}}_{\bar\kappa}$, where $\sigma(\bar{\mathcal{M}}) = \mathcal{M}$, and $\sigma \restriction \bar{\mathcal{M}} = \dot\pi^{\mathfrak{A}}_{\alpha,\kappa}$. Let $\bar\theta = \sigma^{-1}(\theta)$. Let $\bar{\mathfrak{A}} \in H^{\mathfrak{A}}_{\omega_1}$ be a certifying structure for $\bar\Phi$ with respect to $\bar N$, $\bar\kappa$ and $\bar{\mathcal{M}}$. Define $\tilde M = \langle \tilde M_i; i \leq \kappa\rangle$ and $\tilde\pi = \langle \tilde\pi_{i,j}; i \leq j \leq \kappa\rangle$ by:

$$\tilde M_i = \begin{cases} \dot M^{\bar{\mathfrak{A}}}_i & \text{if } i \leq \bar\kappa, \\ \dot M^{\mathfrak{A}}_i & \text{if } i \geq \bar\kappa; \end{cases}$$

$$\tilde\pi_{i,j} = \begin{cases} \dot\pi^{\bar{\mathfrak{A}}}_{i,j} & \text{if } i \leq j \leq \bar\kappa, \\ \dot\pi^{\mathfrak{A}}_{\bar\kappa,j} \circ \dot\pi^{\bar{\mathfrak{A}}}_{i,\bar\kappa} & \text{if } i \leq \bar\kappa \leq j, \\ \dot\pi^{\mathfrak{A}}_{i,j} & \text{if } \bar\kappa \leq i \leq j. \end{cases}$$

Form $\tilde{\mathfrak{A}}$ by interpreting $\dot M$ and $\dot\pi$ by $\tilde M$ and $\tilde\pi$. THEN $\tilde{\mathfrak{A}}$ is a certifying structure for Φ with respect to θ, κ and $\mathcal{M}$.

Note that in the above definition, since $H \in H^{\mathfrak{A}}_{\omega_1}$, it is clear by elementarity of $H \to \mathcal{H}$ that a countable certifying structure $\bar{\mathfrak{A}}$ exists in $H^{\mathfrak{A}}_{\omega_1}$. Also, since $\bar{\mathfrak{A}} \in \mathfrak{A}$, we can calculate $\tilde\pi$ and $\tilde M$ in $\mathfrak{A}$.
We will now define $\mathcal{L}$-forcing in a very general form. We first define a set of preconditions $\tilde{\mathbb{P}}$; the forcing we are going to define will be a subset of $\tilde{\mathbb{P}}$. The first and the second component of a condition should be seen as finite attempts to describe a system of models and a commutative system of elementary embeddings: the first component gives finitely many heights of the models, the second component gives finite approximations to the maps of the commutative system restricted to the ordinals. Though these maps only act on ordinals we will be able to extend them using the well-ordering $<$, see Lemma 2.2.2. The third and the fourth component help to control the embeddings. The third component locally bounds the heights of the models and the fourth component allows us to extend some of the elementary embeddings to embeddings into $\mathcal{H}$; both components are only important in the proof of 2.3.2.

Definition 2.1.4 Let θ, κ and $\mathcal{M}$ be as above. Let Φ be a collection of statements in the language of set theory with two additional constants $\dot\pi$, $\dot M$ with parameters from H_{θ^+} such that Φ is consistent and resectionable. Then $\tilde{\mathbb{P}} := \tilde{\mathbb{P}}(\theta, \kappa, \mathcal{M})$ is the collection of all quadruples[1] p of the form

$$p = \langle \langle \beta^p_i; i \in \mathrm{dom}(p)\rangle, \langle \pi^p_i; i \in \mathrm{dom}(p)\rangle, c^p, \langle \tau^p_i; i \in \mathrm{dom}_-(p)\rangle\rangle$$

where

[1] This is in constrast to [Jena] and [CS09] where triples are used.

1. All three $\mathrm{dom}(p)$, c^p and $\mathrm{dom}_-(p)$ are finite and $\mathrm{dom}_-(p) \subset \mathrm{dom}(p) \subset \kappa$ and $c^p \subset \mathrm{dom}(p)$.
2. $\langle \beta_i^p ; i \in \mathrm{dom}(p)\rangle$ is a sequence of ordinals $< \kappa$.
3. For $i \in \mathrm{dom}(p)$ π_i^p is a finite partial map from β_i^p to θ.
4. For $i \in \mathrm{dom}_-(p)$ $\tau_i^p \subset H_\theta$.

Let $p \in \tilde{\mathbb{P}}$. By $\Phi(p)$ we denote Φ augmented by the further statements:

1. $\beta_i^p = \mathsf{OR} \cap \dot{M}_i$ and $\pi_i^p \subset \dot{\pi}_{i,\kappa}$ for $i \in \mathrm{dom}(p)$,[2]
2. $\exists \bar{a}$: $\dot{\pi}_{i,\kappa} : \langle \dot{M}_i, \bar{a}\rangle \to \langle \mathcal{M}, \tau_i^p\rangle$ for $i \in \mathrm{dom}_-(p)$,
3. $\mathsf{OR} \cap \dot{M}_i < \gamma = \mathrm{crit}(\dot{\pi}_\gamma)$ for $\gamma \in c^p$ and $i < \gamma$.[3]

If Φ is clear from the context a certifying structure for $\Phi(p)$ will also be called a certifying structure for p. Set $\tau_i^p(n) = \{x \,;\, \langle n, x\rangle \in \tau_i^p\}$ for $n < \omega$. A condition $p \in \tilde{\mathbb{P}}$ will be called *good* if $\Phi(p)$ is consistent. We will call p *neat* if for all $i < j \in \mathrm{dom}_-(p)$ there is some n and finitely many ordinals $\vec{u} \in \mathrm{ran}(\pi_j^p)$ such that

$$\tau_i^p = \{\langle m, x\rangle \,;\, \vec{u}^\frown m^\frown x \in \tau_j^p(n)\}.$$

Finally we are ready to define $\mathcal{L}$-forcing. By $\mathbb{P}_\Phi$ we will denote the collection of all good and neat $p \in \tilde{\mathbb{P}}$ ordered as follows: if $p, q \in \mathbb{P}_\Phi$ then $p \le q$ if and only $\langle \beta_i^q ; i \in \mathrm{dom}(q)\rangle \subset \langle \beta_i^p ; i \in \mathrm{dom}(p)\rangle$ and $\pi_i^q \subset \pi_i^p$ for $i \in \mathrm{dom}(q)$ and $\tau_i^q = \tau_i^p$ for $i \in \mathrm{dom}_-(q)$ and $c^q \subset c^p$.

In general it is nontrivial to see that a given $\mathcal{L}$-forcing is nonempty. We will first carry out a basic analysis and later show that nonempty $\mathcal{L}$-forcings exist.

2.2 The Basic Analysis of $\mathcal{L}$-Forcing

Here is a lemma on the extendability of conditions in $\mathcal{L}$-forcings.

Lemma 2.2.1 *Let $\mathbb{P} = \mathbb{P}_\Phi$ be an $\mathcal{L}$-forcing. Let $p \in \mathbb{P}$.*

1. *Let $u \subset \kappa$ be finite such that $\mathrm{dom}(p) \subset u$. There is $p' \le p$ such that $u \subset \mathrm{dom}(p')$.*
2. *Let $i \in \mathrm{dom}(p)$ and let $u \subset \beta_i^p$ be finite. There is $p' \le p$ such that $u \subset \mathrm{dom}(\pi_i^{p'})$.*
3. *Let $u \subset \mathsf{OR} \cap |\mathcal{M}|$ be finite. There is $p' \le p$ such that $u \subset \mathrm{ran}(\pi_j^{p'})$ for some $j \in \mathrm{dom}(p')$.*
4. *Let $u \subset \pi_\lambda^p$ be finite where λ is a limit ordinal. There is $p' \le p$ and $j \in \mathrm{dom}(p')$ such that $u \subset \mathrm{ran}(\pi_j^{p'}) \cap \lambda$.*
5. *There is $p' \le p$ such that $\mathrm{ran}(\pi_i^{p'}) \subset ran(\pi_j^{p'})$ whenever $i, j \in \mathrm{dom}(p')$, $i < j$.*

[2]Readers of [CS09] will note that the first component of the conditions in [CS09] are finite approximations to the critical points of the maps $\dot{\pi}_{i,\kappa}$, where our first component is a finite approximation to the sequence of heights of model $\mathsf{OR} \cap \dot{M}_i$, like in [Jena].

[3]This is in contrast to the approach in [Jena]. Jensen uses the stronger statement $\beta_i^p < \mathrm{crit}(\dot{\pi}_{i+1})$ which would exclude all examples involving generic ultrapowers.

6. *Let $i \in \mathrm{dom}(p)$ and let $\xi_1, ..., \xi_m \in \mathrm{ran}(\pi_i^p)$ such that $\mathcal{M} \models \exists\eta\phi(\eta, \vec{\xi})$. There is $p' \leq p$ with some $\eta \in \mathrm{ran}(\pi_i^{p'})$ such that $\mathcal{M} \models \phi(\eta, \vec{\xi})$.*

7. *Let $\alpha < \kappa$. There is $p' \leq p$ and some $\alpha' \in c^{p'}$ such that $\alpha' \geq \alpha$.*

Since proofs of all but the last statement above can be found in [Jena], we only prove the last statement. Nevertheless the proofs are all of the same structure and basically follow this pattern: look at a certifying structure $\mathfrak{A}$ for p and extend the condition according to $\dot{\pi}^{\mathfrak{A}}$ and $\dot{M}^{\mathfrak{A}}$. Note also that [CS09] contains similar lemmata.

Proof. Assume w.l.o.g. that $\mathrm{dom}(p) \subset \alpha < \kappa$. Let $\mathfrak{A}$ be a certifying structure for p. Clearly $\kappa \subset \mathfrak{A}$. Pick some $X \prec H_{\theta^+}^{\mathfrak{A}}$ of cardinality less than $\kappa = \omega_1^{\mathfrak{A}}$ such that $\alpha \cup \{\dot{\pi}^{\mathfrak{A}}, \dot{M}^{\mathfrak{A}}\} \subset X$. Let $\bar{X}$ denote the transitive collapse of X and let $\bar{\mathcal{M}}$ denote the collapse of $\mathcal{M}$. Then

$$\bar{X} \models \bar{\mathcal{M}} \text{ is uncountable and a limit of countable, transitive structures.}$$

Hence for all $i < \kappa \cap X = \omega_1^{\bar{X}}$

$$\mathsf{OR} \cap \dot{M}_i^{\mathfrak{A}} < \kappa \cap X.$$

Define p' by $\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{\kappa \cap X\}$, $\beta_{\kappa\cap X}^{p'} = \mathsf{OR} \cap \bar{\mathcal{M}}$, $\pi_{\kappa\cap X}^{p'} = \emptyset$, $c^{p'} = c^p \cup \{\kappa \cap X\}$ and leaving all other components of p unchanged. So, noting that $\bar{\mathcal{M}} = \dot{M}_{\kappa\cap X}^{\mathfrak{A}}$, the condition p' is certified by $\mathfrak{A}$. Clearly p' is neat and $p' \leq p$. □

Lemma 2.2.2 *Let $\mathbb{P} = \mathbb{P}_\Phi$ be an $\mathcal{L}$-forcing. Let $G \subset \mathbb{P}$ be generic.*

1. *Then $\bigcup\{\langle\beta_i^p; i \in \mathrm{dom}(p)\rangle\,;\, p \in G\} = \langle\beta_i; i < \kappa\rangle$, where $\beta_i < \kappa$ for $i < \kappa$. Set $\beta_\kappa = \mathsf{OR} \cap |\mathcal{M}|$.*

2. *Let $i < \kappa$. Set $\bar{\pi}_i = \bigcup\{\pi_i^p\,;\, p \in G \wedge i \in \mathrm{dom}(p)\}$. Then $\bar{\pi}_i : \beta_i \to \beta_\kappa$ is monotone with $\kappa_i = \mathrm{crit}(\bar{\pi}_i)$ where $\kappa_i := \bar{\pi}_i^{-1}(\kappa)$.*

3. *If $i \leq j \leq \kappa$, then $\mathrm{ran}(\bar{\pi}_i) \subset \mathrm{ran}(\bar{\pi}_j)$ (letting $\bar{\pi}_\kappa = \mathrm{id}\restriction\beta_\kappa$).*

4. *If $\lambda \leq \kappa$ is a limit ordinal, then $\mathrm{ran}(\bar{\pi}_\lambda) = \cup\{\mathrm{ran}(\bar{\pi}_i)\,;\, i < \lambda\}$.*

5. *Let $i \leq \kappa$. Set $X_i := \mathrm{Hull}^{\mathcal{M}}(\mathrm{ran}(\bar{\pi}_i))$. Then $X_i \cap \beta_\kappa = \mathrm{ran}(\bar{\pi}_i)$.*

6. *Let $\pi_i : M_i \to X_i$ be the inverse of the transitive collapse of X_i for $i \leq \kappa$. Then $\pi_i : M_i \prec \mathcal{M}$, $\pi_i\restriction\beta_i = \bar{\pi}_i$, $\beta_i = \mathsf{OR} \cap M_i$ and $\pi_i\restriction\kappa_i = \mathrm{id}\restriction\kappa_i$.*

7. *Set $\pi_{i,j} = \pi_j^{-1} \circ \pi_i$ for $i \leq j \leq \kappa$. Then $\pi_{i,j} : M_i \prec M_j$ and $\langle\pi_{i,j}; i \leq j \leq \kappa\rangle$ is a commutative continuous system of embeddings.* □

A proof of the above lemma and the next theorem can be found in [Jena].

Theorem 2.2.3 *Let $\mathbb{P} = \mathbb{P}_\Phi$ be an $\mathcal{L}$-forcing. Let $G \subset \mathbb{P}$ be generic. Then κ is regular in $V[G]$.* □

2.3 Stationary Set Preserving $\mathcal{L}$-Forcings

To discuss stationary set preserving forcings we need to discuss certifying structures that are close to V in the following sense:

Definition 2.3.1 Let $I \in V$ be a normal, uniform ideal on a regular uncountable cardinal λ. A ZF^- model $\mathfrak{A}$ such that $\mathcal{P}(\lambda)^V \subset |\mathfrak{A}|$ is *I-close to V* if every $S \in \mathcal{P}(\lambda)^V \setminus I$ is stationary in $\mathfrak{A}$.

Clearly the previous definition is only interesting if $\mathfrak{A}$ lives in a forcing extension of V. The proof of the following theorem is modeled after Jensen's proof of Theorem 2.2.3 and the proof of [CS09, Lemma 17].

Theorem 2.3.2 *Let $I \in V$ be a normal, uniform ideal on κ. Let $\rho > \theta > \kappa$ and $<, \mathcal{M}, \mathcal{H}$ as above. Let Φ be a collection of statements in a language containing constants $\dot{\pi}$ and $\dot{M}$ such that for every $S \in \mathcal{P}(\lambda)^V \setminus I$ the statement*

"S is stationary in κ"

is contained in Φ. Let $\mathbb{P} = \mathbb{P}_\Phi$ be the resulting $\mathcal{L}$-forcing. Then all certifying structures for Φ are I-close to V and in $V^{\mathbb{P}}$ every $S \in \mathcal{P}(\omega_1)^V \setminus I$ is stationary in κ.

Proof. Let us assume that Φ is consistent and resectionable, else $\mathbb{P} = \emptyset$ and the theorem trivializes. Let $S \subset \kappa$ be I positive and in V and let $\dot{C}$ be a $\mathbb{P}$-name such that for some $p \in \mathbb{P}$

$$p \Vdash \dot{C} \text{ is club in } \omega_1 = \check{\kappa}.$$

We want to find some $\alpha < \omega_1$ and some $p' \leq p$ such that

$$p' \Vdash \check{\alpha} \in \dot{C} \cap \check{S}.$$

Recall that $\rho > 2^\theta$, hence without loss of generality we can assume that $\dot{C} \in H_\rho$ by replacing $\dot{C}$ with the set

$$\{\langle \check{\beta}, r\rangle \,;\, r \leq p \wedge r \Vdash \check{\beta} \in \dot{C}\}.$$

Let $\langle \tau(n); n < \omega\rangle$ enumerate the $\tau \subset H_\theta$ that are $\mathcal{H}$-definable from $\dot{C}, p, \kappa, \mathcal{M}, \leq_{\mathbb{P}}, \Phi$ and set

$$\tau := \{\langle x, n\rangle \,;\, n < \omega \wedge x \in \tau(n)\}.$$

If $\mathcal{A}$ is a model and X a set, we write $X \prec \mathcal{A}$ to mean: $X \subset |\mathcal{A}|$ and $\mathcal{A}|X \prec A$.
Claim 1. For any $X \subset |\mathcal{M}|$ the following are equivalent:

1. $X \prec \langle \mathcal{M}, \tau(n)\rangle$ for all $n < \omega$.
2. Let $Y = \mathrm{Hull}^{\mathcal{H}}(X \cup \{\dot{C}, p, \kappa, \mathcal{M}, \leq_{\mathbb{P}}, \Phi\})$. Then $Y \cap |\mathcal{M}| = X$.

Proof of Claim 1. Let us assume $X \prec \langle \mathcal{M}, \tau\rangle$. Each $z \in Y$ is $\mathcal{H}$-definable in parameters from $X \cup \{\dot{C}, p, \kappa, \mathcal{M}, \leq_{\mathbb{P}}, \Phi\}$, hence especially each $z \in Y \cap |\mathcal{M}|$ is definable in this fashion. So by the choice of τ each such z is in X.
For the converse again recall the definition of τ. □(Claim 1)

Let $G \subset \mathrm{Col}(\omega, 2^\theta)$ be generic over V. Let us work in $V[G]$ for a while. Let $\mathfrak{B}$ be a certifying structure for p. Then $\langle \mathcal{M}, \tau\rangle \in H_{\theta^+} \subset \mathfrak{B}$. Since $\kappa = \omega_1^{\mathfrak{B}}$ is regular in $\mathfrak{B}$ and S is stationary in $\mathfrak{B}$ there is $\alpha < \kappa$ such that

- $\alpha = \mathrm{crit}(\dot{\pi}^{\mathfrak{B}}_{\alpha,\kappa}) \supset \mathrm{dom}(p)$,
- for all $i < \alpha$: $\mathsf{OR} \cap \dot{M}^{\mathfrak{B}}_i < \alpha$,
- $\alpha \in S$,
- there is $\bar{\tau} \subset \dot{M}^{\mathfrak{B}}_\alpha$ such that

$$\langle \dot{M}^{\mathfrak{B}}_\alpha, \bar{\tau}\rangle \prec \langle \mathcal{M}, \tau\rangle.$$

We define $p' = \langle\langle\beta_i^{p'}; i \in \mathrm{dom}(p')\rangle, \langle\pi_i^{p'}; i \in \mathrm{dom}(p')\rangle, c^{p'}, \langle\tau_i^{p'}; i \in \mathrm{dom}_-(p')\rangle\rangle$ as follows:

- $\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{\alpha\}$,
- $\beta_i^{p'} = \beta_i^p$ and $\pi_i^{p'} = \pi_i^p$ for $i \in \mathrm{dom}(p)$,
- $\tau_i^{p'} = \tau_i^p$ for $i \in \mathrm{dom}_-(p)$,
- $\beta_\alpha^{p'} = \mathsf{OR} \cap \dot{M}_\alpha^{\mathfrak{B}}$,
- $\pi_\alpha^{p'} = \{\langle\alpha, \kappa\rangle\}$,
- $\tau_\alpha^{p'} = \tau$,
- $c^{p'} = c^p \cup \{\alpha\}$.

Then p' is good since $\mathfrak{B}$ is a certifying structure for p'; also p' is neat since each $\tau_i^p = \tau(n)$ for some n. Thus $p' \leq p$ and it suffices to show: $p' \Vdash \alpha \in \dot{C}$. Suppose this is not the case and work towards a contradiction. Hence there is $q \leq p'$ and $\xi < \alpha$ such that

$$q \Vdash \dot{C} \cap \alpha \subset \xi;$$

else $\dot{C}$ would be unbounded in α and hence, by closedness, $\dot{C}$ would contain α. Pick a certifying structure $\mathfrak{A}$ for q. Then $\tau_\alpha^q = \tau$. In $\mathfrak{A}$ there is some $\bar{\tau}$ such that

$$\dot{\pi}_{\alpha,\kappa}^{\mathfrak{A}} : \langle\dot{M}_\alpha^{\mathfrak{A}}, \bar{\tau}\rangle \prec \langle\mathcal{M}, \tau\rangle,$$

and the only choice for $\bar{\tau}$ is $\bar{\tau} = (\dot{\pi}_{\alpha,\kappa}^{\mathfrak{A}})^{-1}``\tau$. Let $X := \mathrm{ran}(\dot{\pi}_{\alpha,\kappa}^{\mathfrak{A}})$ and $Y := \mathrm{Hull}^{\mathcal{H}}(X \cup \{\dot{C}, p, \kappa, \mathcal{M}, \leq_{\mathbb{P}}, \Phi\})$. Then $Y \cap \mathcal{M} = X$ by the above claim. Let $\pi : H \to Y$ denote the inverse of the transitive collapse of Y. Then $\dot{\pi}_{\alpha,\kappa}^{\mathfrak{A}} \subset \pi$ and $\pi(\dot{M}_\alpha^{\mathfrak{A}}) = \mathcal{M}$.

Claim 2. $H \in \mathfrak{A}$ and there is an elementary map $\sigma : H' \to H_{\theta^+}$ from some transitive $H' \subset H$ such that $\sigma \in \mathfrak{A}$.

Proof of Claim 2. Let $\tilde{Y} := \mathrm{Hull}^{\mathcal{H}}(\mathcal{M} \cup \{\dot{C}, p, \kappa, \mathcal{M}, \leq_{\mathbb{P}}, \Phi\})$ and let $\tilde{\pi} : \tilde{H} \to \tilde{Y}$ denote the inverse of the transitive collapse of Y. Clearly $\tilde{\pi} \in V$ and $\tilde{H} \in H_{\theta^+}$. Hence $\tilde{H} \in \mathfrak{A}$. Note that $\tilde{\pi} \restriction \mathcal{M} = \mathrm{id} \restriction \mathcal{M}$. Let

$$\tilde{\pi}(\tilde{C}, \tilde{p}, \tilde{\kappa}, \tilde{\mathcal{M}}, \tilde{\leq}, \tilde{\Phi}) = \dot{C}, p, \kappa, \mathcal{M}, \leq_{\mathbb{P}}, \Phi,$$

and let

$$Y^* = \mathrm{Hull}^{\tilde{H}}(X \cup \{\tilde{C}, \tilde{p}, \tilde{\kappa}, \tilde{\mathcal{M}}, \tilde{\leq}, \tilde{\Phi}\}).$$

Since $X \in \mathfrak{A}$ also $Y^* \in \mathfrak{A}$ and hence the transitive collapse of Y^* is in $\mathfrak{A}$. This transitive collapse is H. Since $H_{\theta^+} \in \mathfrak{A}$ and $H_{\theta^+} \in \tilde{Y}$ it is straightforward to define σ. □(Claim 2)

Now let

$$\pi(\bar{C}, \bar{p}, \bar{\kappa}, \bar{\mathcal{M}}, \leq_{\bar{\mathbb{P}}}, \bar{\Phi}, \bar{\theta}) = \dot{C}, p, \kappa, \mathcal{M}, \leq_{\mathbb{P}}, \Phi, \theta.$$

Set $q' = q \restriction \alpha$ i.e. the condition that results by intersecting $\mathrm{dom}(q)$, c^q and $\mathrm{dom}_-(q)$ with α. Then $q \leq q'$.

Claim 3. $q' \in Y$.

Proof of Claim 3. Since $\alpha \in c^q$ clearly $\beta_i^{q'} < \alpha \subset Y$ for every $i \in \mathrm{dom}(q')$. The set $\mathrm{dom}(q')$ is finite, hence $\langle\beta_i^{q'}; i \in \mathrm{dom}(q')\rangle \in Y$. Since $\mathrm{ran}(\pi_i^{q'})$ is a finite subset of $X \subset Y$, also $\langle\pi_i^{q'}; i \in \mathrm{dom}(q')\rangle \in Y$. Again by finiteness $c^{q'} \in Y$. If $i \in \mathrm{dom}_-(q')$, then $\tau_i^{q'}$ is $\langle M, \tau(n)\rangle$-definable in parameters from $\mathrm{ran}(\pi_\alpha^q) \subset X$ for some $n < \omega$. Hence $\tau_i^{q'} \in Y$, since $\tau \in Y$. So $\langle\tau_i^{q'}; i \in \mathrm{dom}_-(q')\rangle \in Y$ by finiteness. □(Claim 3)

So there is some $\bar{q}'$ such that $\pi(\bar{q}') = q'$. Then by elementarity $\bar{q}' \leq_{\bar{\mathbb{P}}} \bar{p}$. Since

$$\bar{q}' \Vdash^H_{\bar{\mathbb{P}}} \bar{C} \text{ is club in } \alpha,$$

there is $\bar{r} < \bar{q}'$ in $\bar{\mathbb{P}}$ and some $\beta > \xi$ such that

$$\bar{r} \Vdash^H_{\bar{\mathbb{P}}} \beta \in \bar{C}.$$

Let $r := \pi(\bar{r})$. Then by elementarity $r \in \mathbb{P}$ and r is incompatible with q since $q \Vdash \dot{C} \cap \alpha \subset \xi$. We derive a contradiction by showing that q is in fact compatible with r.
Since $\bar{r}$ is good in H there is some $\bar{\mathfrak{A}} \in H^{\mathrm{Col}(\omega, 2^{\bar{\theta}})}$ that is a certifying structure for $\bar{\Phi}(\bar{r})$. This can be reformulated as a Σ^1_1 statement in a parameter from $H^{\mathfrak{A}}_{\omega_1}$, so by absoluteness such a certifying structure is also in $\mathfrak{A}$; we will again denote this structure by $\bar{\mathfrak{A}}$. By resectionability we can then define a new model $\tilde{\mathfrak{A}}$ with the same universe as $\mathfrak{A}$, by interpreting $\dot{M}$ and $\dot{\pi}$ as $\tilde{M}$ and $\tilde{\pi}$ where:

$$\tilde{M}_i = \begin{cases} \dot{M}_i^{\bar{\mathfrak{A}}} & \text{if } i \leq \bar{\kappa}, \\ \dot{M}_i^{\mathfrak{A}} & \text{if } i \geq \bar{\kappa}; \end{cases}$$

$$\tilde{\pi}_{i,j} = \begin{cases} \dot{\pi}^{\bar{\mathfrak{A}}}_{i,j} & \text{if } i \leq j \leq \bar{\kappa}, \\ \dot{\pi}^{\mathfrak{A}}_{\bar{\kappa},j} \circ \dot{\pi}^{\bar{\mathfrak{A}}}_{i,\bar{\kappa}} & \text{if } i \leq \bar{\kappa} \leq j, \\ \dot{\pi}^{\mathfrak{A}}_{i,j} & \text{if } \bar{\kappa} \leq i \leq j. \end{cases}$$

For this note, that $\bar{\mathfrak{A}} \in \mathfrak{A}$ implies that $\tilde{\pi}$, $\tilde{M} \in \mathfrak{A}$. Then $\tilde{\mathfrak{A}}$ is a certifying structure for Φ. Also $\tilde{\mathfrak{A}}$ is a certifying structure for r: if $i \in \mathrm{dom}(r)$, then $\beta^r_i = \beta^{\tilde{\mathfrak{A}}}_i$ and $\pi^r_i \subset \tilde{\pi}_{i,\kappa}$ and if $i \in \mathrm{dom}_-(r)$, $n < \omega$, then

$$\tilde{\pi}_{i,\kappa} : \langle \dot{M}^{\bar{\mathfrak{A}}}_i, \bar{a} \rangle \to \langle \dot{M}, a^r_i(n) \rangle,$$

where $\bar{a}$ is such that

$$\pi^{\bar{\mathfrak{A}}}_{i,\kappa} : \langle \dot{M}^{\bar{\mathfrak{A}}}_i, \bar{a} \rangle \to \langle \dot{M}^{\mathfrak{A}}_{\bar{\kappa}}, a^r_i(n) \rangle.$$

Since $r \leq_{\mathbb{P}} q'$ it follows that $\tilde{\mathfrak{A}}$ certifies q'; also $\tilde{\mathfrak{A}}$ certifies $q \restriction (\kappa \setminus \alpha)$. Hence $\tilde{\mathfrak{A}}$ certifies $q = q' \cup q \restriction (\kappa \setminus \alpha)$ and also $r \cup q$, where $r \cup q$ is the condition one obtains by setting $c^{r \cup q} = c^r \cup c^q$ and joining all other components in the same way. Clearly $r \cup q$ is good. However $r \cup q$ might not be neat, so we have to modify the partial maps $\pi^{r \cup q}_i$ to obtain an $s \leq_{\mathbb{P}} r, q$.
First note that $r = \pi(\bar{r}) \in Y$. By the definition of τ there is a parameter $w \in X = \mathrm{ran}(\dot{\pi}^{\mathfrak{A}}_{\alpha,\kappa}) = \mathrm{ran}(\dot{\pi}^{\tilde{\mathfrak{A}}}_{\alpha,\kappa})$ and some $n < \omega$ such that r is $\langle M, \tau(n) \rangle$ definable in w. Hence for every $i \in \mathrm{dom}_-(r)$ the set τ^r_i is $\langle M, \tau(n) \rangle$ definable in w. For each $\gamma \in \mathrm{dom}(q) \setminus \alpha$ we pick some w_γ such that $\dot{\pi}^{\mathfrak{A}}_{\gamma,\kappa}(w_\gamma) = w$. For $i \in \mathrm{dom}(r \cup q)$ set

$$\pi^s_i = \begin{cases} \pi^r_i & \text{for } i < \alpha, \\ \pi^q_i \cup \{\langle w_\gamma, w \rangle\} & \text{for } i \geq \alpha; \end{cases}$$

we leave the other components of $r \cup q$ unchanged. Hence s is certified by $\tilde{\mathfrak{A}}$ and is neat by construction. Also clearly $s \leq_{\mathbb{P}} r, q$. But the existence of such an s is a contradiction to the incompatibility of r and q. □

2.4 Examples of Stationary Set Preserving $\mathcal{L}$-Forcings

The first example of a stationary set preserving $\mathcal{L}$-forcing appeared in [CS09]. Readers familiar with [CS09] will note several small differences to our approach. Nevertheless we will construct a forcing very similar to the one from [CS09] in our setup.

2.4.1 Claverie-Schindler forcing

Given a precipitous ideal I on ω_1 and a cardinal $\theta > \omega_1$ we will construct a notion of forcing $\mathbb{P}(I,\theta)$ that adds a generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that $M_{\omega_1} = \langle H_\theta; \in, I\rangle$ and $I_{\omega_1} = I$; here $\kappa_i = \omega_1^{M_i}$ and all M_i with $i < \omega_1$ will be countable. Of course $\mathbb{P}(I,\theta)$ will be an $\mathcal{L}$-forcing. Let us fix a (normal, uniform and) precipitous ideal I on ω_1; note that $\mathsf{NS}_{\omega_1} \subset I$.
We set $\kappa = \omega_1$. Recall the cardinal setup

$$\rho = 2^{<\rho} > 2^\theta > \theta = 2^{<\theta}$$

and fix a well-ordering $<$ of H_ρ such that $<\restriction H_\theta$ is a well-ordering of H_θ of ordertype θ. Set $\mathcal{M} = \langle H_\theta; \in, I, <\rangle$ and $\mathcal{H} = \langle H_\rho; \in, \mathcal{M}, <\rangle$. Note that we are in the situation of Defintion 2.1.4. It is convenient, but not necessary, to add the constants $\dot{\vec{G}}$ and $\dot{I}$.

Definition 2.4.1 By Φ we denote the collection of statements in the language of set theory with the additional constants $\dot{\pi}$, $\dot{M}$,$\dot{I}$ and $\dot{\vec{G}}$ in parameters from H_{θ^+} that contains the following statements:

1. "S is stationary" for every $S \in \mathcal{P}(\omega_1) \setminus I$,
2. "$\dot{\vec{G}} = \langle G_i; i < \omega_1\rangle$ is a sequence",
3. "$\dot{I} = \langle I_i; i \geq \omega_1\rangle$ is a sequence",
4. "G_i is $(\mathcal{P}(\omega_1) \setminus I_i)^{\dot{M}_i}$-generic over $\dot{M}_i$, and $\dot{\pi}_{i,i+1} : \dot{M}_i \to \dot{M}_{i+1} \simeq \mathrm{Ult}(\dot{M}_i, G_i)$ is a generic ultrapower",
5. "$I_{\omega_1} = I$".

Note that the constant $\dot{I}$ is obsolete if $I = \mathsf{NS}_{\omega_1}$. Later we will discuss collections containing Φ, see Definition 3.2.6 and Definition 3.2.16. It will be clear that the results on $\mathbb{P}_\Phi$ we are about to show will also hold for these larger collections.
To see that the forcing $\mathbb{P}_\Phi$ is nonempty we need to see that Φ is consistent, since this will certify $\mathbf{1}_{\mathbb{P}(I,\theta)} = \langle\langle\rangle, \langle\rangle, \emptyset, \langle\rangle\rangle$.

Lemma 2.4.2 ([CS09, Lemma 5]) *Φ is consistent.*

Proof. We need to see that in $V^{\mathrm{Col}(\omega,2^\theta)}$ there a certifying structure for Φ with respect to H_{θ^+}, ω_1 and $\mathcal{M}$. Let g be $\mathrm{Col}(\omega,<\rho)$-generic over V. We work in $V[g]$ until further notice. So $\langle V; \in, I\rangle$ is $\rho+1$ iterable, by [Woo99, 3.11]. Hence $\langle H_\theta; \in, I\rangle$ is also $\rho+1$ iterable. We prepare a book-keeping device: pick a bijection $g : [\rho]^{<\rho} \to \rho$ and a family $\langle S_\nu, \nu < \rho\rangle$ of pairwise disjoint stationary subsets of ρ. Now define $f : \rho \to [\rho]^{<\rho}$ by

$$f(i) = s \iff i \in S_{g(s)}.$$

Note that f enumerates each $s \in [\rho]^{<\rho}$ stationarily often. We recursively construct a generic iteration

$$\mathcal{J} := \langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \rho\rangle, \langle G_i; i < \rho\rangle\rangle$$

of $M_0 = \langle H_\theta; \in, I\rangle$, where $I_i = \pi_{0,i}(I)$. Suppose we are at stage $i < \rho$. If there is a (unique) $j \leq i$ such that $f(i)$ is stationary in M_j, i.e. $\pi_{j,i}(f(i))$ is stationary in M_i, then we choose G_i such that $\pi_{j,i}(f(i)) \in G_i$. If there is no such $j \leq i$ we choose G_i arbitrarily. This defines the generic iteration.
Let S be stationary in M_ρ. Let $j < \rho$ and s be such that $\pi_{j,\rho}(s) = S$. Whenever $j \leq i \leq \rho$ and $f(i) = s$, then $\pi_{j,i}(s) \in G_i$, i.e. $\mathrm{crit}(\pi_{i,i+1}) \in \pi_{i,i+1}(\pi_{j,i}(s)) = \pi_{j,i+1}(s) \subset \pi_{j,\rho}(s) = S$. This shows that

$$S_{g(s)} \setminus j \subset \{i < \rho\,;\, \mathrm{crit}(\pi_{i,i+1}) \in S\},$$

so that in fact S is stationary in $V[g]$.
The map $\pi_{0,\rho} : H_\theta \to M_\rho$ admits a canonical extension $\pi : V \to N$, where N is transitive and $\pi(H_\theta) = M_\rho$. Let us now leave $V[g]$ and pick some h which is $\mathrm{Col}(\omega, \pi(2^\theta))$-generic over $V[g]$. Hence h is also $\mathrm{Col}(\omega, \pi(2^\theta))$-generic over N. Let $x \in N[h]$ be a real that codes $\pi((H_{\theta^+})^V)$ in a natural way. The existence of a countable, well-founded certifying structure for $\pi(\Phi)$ with respect to $\pi(H_{\theta^+})$, $\pi(\omega_1)$ and $\pi(\mathcal{M})$ then clearly a $\Sigma^1_2(x)$ statement. This statement is true in $V[g,h]$: the witness is an initial segment of $V[g]$ that contains the generic iteration we constructed. Hence by Shoenfield absoluteness this statement holds in $N[h]$. So in $N^{\mathrm{Col}(\omega,\pi(2^\theta))}$ there is a certifying structure for $\pi(\Phi)$ with respect to $\pi(H_{\theta^+})$, $\pi(\omega_1)$ and $\pi(\mathcal{M})$. So by elementarity, in $V^{\mathrm{Col}(\omega,2^\theta)}$ there is a certifying structure for Φ with respect to H_{θ^+}, ω_1 and $\mathcal{M}$. □

Note the following trivial fact: if two generic iterations of countable length have the same last model $\langle M; \in, I\rangle$, where I is the ideal, then any continuation of the first generic iteration is a continuation of the second. Hence:

Remark 2.4.3 Φ is resectionable.

The basic analysis of $\mathcal{L}$-forcing we have outlined yields that $\mathbb{P}_\Phi$ adds system of elementary embeddings and a system $\langle M_i; i \leq \omega_1\rangle$ of models such that $M_{\omega_1} = \mathcal{M}$. By Theorem 2.3.2 $\mathbb{P}_\Phi$ will "spare" the I-positive sets, i.e. each $S \in I^+$ is stationary in the forcing extension; in the special case $I = \mathsf{NS}_{\omega_1}$ this means that $\mathbb{P}_\Phi$ preserves stationary sets of ω_1. We do not know yet that the elementary embeddings are in fact a generic iteration. For this we show four lemmata, the first being an easy observation:

Lemma 2.4.4 *Let $\langle \pi_{i,j}; i \leq j \leq \kappa\rangle$ be a system of elementary embeddings and let $\langle M_i; i \leq \kappa\rangle$ be a sequence of transitive models of* ZFC^* + *"ω_1 exists", where $\kappa = \omega_1^V$ and M_i is countable for $i < \kappa$. Additionally assume that*

$$M_\kappa \models \omega_1 = \kappa.$$

Then $\mathrm{crit}(\pi_{i,\kappa}) = \kappa \cap \mathrm{ran}(\pi_{i,\kappa})$.

Proof. By elementarity there is some $\kappa_i \in M_i$ such that $\pi_{i,\kappa}(\kappa_i) = \kappa$; i.e.

$$M_i \models \omega_1 = \kappa_i.$$

Hence $\kappa \cap \mathrm{ran}(\pi_{i,\kappa}) = \kappa_i$ is transitive and countable. Hence $\mathrm{crit}(\pi_{i,\kappa}) = \kappa_i$. □

Lemma 2.4.5 *Let $p \in \mathbb{P}_\Phi$, $i, i+1 \in \mathrm{dom}(p)$. Let $\xi \in \mathrm{ran}(\pi^p_{i+1})$ and suppose $\pi^p_i(\kappa_i) = \kappa$ for some $\kappa_i \in \mathrm{dom}(\pi^p_i)$. There is some $q \leq p$ such that ξ is definable over $\mathcal{M}$ from parameters in $\mathrm{ran}(\pi^q_i) \cup \{\kappa_i\}$.*

Proof. Let $\mathfrak{A}$ certify p. Since $\dot{M}^{\mathfrak{A}}_{i+1} = \text{Ult}(\dot{M}^{\mathfrak{A}}_i, G_i)$ for some $\dot{M}^{\mathfrak{A}}_i$-generic G_i, there is an $f : \kappa_i \to \dot{M}^{\mathfrak{A}}_i$, $f \in \dot{M}^{\mathfrak{A}}_i$ such that $(\pi^p_{i+1})^{-1}(\xi) = \dot{\pi}^{\mathfrak{A}}_{i,i+1}(f)(\kappa_i)$, i.e. $\xi = \dot{\pi}^{\mathfrak{A}}_{i,\kappa}(f)(\kappa_i)$. By the presence of $<$, the function $\dot{\pi}^{\mathfrak{A}}_{i,\kappa}(f)$ is definable over $\mathcal{M}$ in some ordinal parameter $\lambda \in \text{ran}(\dot{\pi}^{\mathfrak{A}}_{i,\kappa})$, say $\dot{\pi}^{\mathfrak{A}}_{i,\kappa}(\bar{\lambda}) = \lambda$. By 2.2.1 we find some $q \leq p$ such that $\text{dom}(p) = \text{dom}(q)$, $\text{dom}_-(p) = \text{dom}_-(q)$, $\beta^q_i = \beta^p_i$ for $i \in \text{dom}(p)$, $c^p = c^q$, $\pi^q_j = \pi^p_j$ for $j \in \text{dom}(p) \setminus \{i\}$, $\pi^q_i = \pi^p_i \cup \{\langle \bar{\lambda}, \lambda \rangle\}$, and $\tau^q_i = \tau^p_i$ for $i \in \text{dom}_-(p)$. Clearly $\mathfrak{A}$ also certifies q and $q \leq p$. □

Note that in the next lemma we discuss definable sets rather than elements to handle the case $\theta = \omega_2$; i.e. the situation where $\mathcal{P}(\omega_1) \setminus I$ is a class from the point of view of H_θ.

Lemma 2.4.6 *Let $p \in \mathbb{P}_\Phi$, $i \in \text{dom}(p)$ and suppose $D \in H_\theta$ is definable over $\mathcal{M}$ from parameters in $\text{ran}(\pi^p_i)$ and also suppose*

$$\mathcal{M} \models D \text{ is dense in the partial order } \mathcal{P}(\omega_1) \setminus I.$$

There is some $p' \leq p$ and some $X \in D$ that is definable over $\mathcal{M}$ from parameters in $\text{ran}(\pi^{p'}_i)$ such that

$$p' \Vdash_{\mathbb{P}_\Phi} \text{crit}(\pi^{\dot{G}}_{i,i+1}) \in \check{X},$$

where $\pi^{\dot{G}}$ is a $\mathbb{P}_\Phi$-name for the system of elementary embeddings added by $\mathbb{P}_\Phi$.

Proof. By Lemma 2.2.1 it is safe to assume that there is some $\kappa_i \in \text{dom}(\pi^p_i)$ such that $\pi^p_i(\kappa_i) = \omega_1$. By Lemma 2.4.4 $\kappa_i = \text{crit}(\dot{\pi}^{\mathfrak{A}}_{i,\omega_1})$ for all $\mathfrak{A}$ that certify p. This, by Lemma 2.2.2, implies

$$p \Vdash \text{crit}(\pi^{\dot{G}}_{i,i+1}) = \check{\kappa}_i.$$

Let $\mathfrak{A}$ certify p and let $\bar{D} \in \dot{M}^{\mathfrak{A}}_i$ be such that $\dot{\pi}^{\mathfrak{A}}_{i,\omega_1}(\bar{D}) = D$. Since $\mathfrak{A}$ believes that $\dot{\pi}^{\mathfrak{A}}_{i,i+1}$ is a generic ultrapower there is some $\bar{X} \in \bar{D}$ such that $\kappa_i = \text{crit}(\dot{\pi}^{\mathfrak{A}}_{i,i+1}) \in \dot{\pi}^{\mathfrak{A}}_{i,i+1}(\bar{X}) \subset \dot{\pi}^{\mathfrak{A}}_{i,\omega_1}(\bar{X}) =: X$. By the presence of the well-ordering $<$, we know that X is definable; for this note that X is the $<$-least member of D such that $\kappa_i \in X$. Exploiting the well-ordering $<$ and the elementarity of $\text{ran}(\pi^{\mathfrak{A}}_{i,\omega_1}) \prec \mathcal{M}$ we find a single ordinal $\xi \in \text{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$ such that X is definable from ξ. Now another application of 2.2.1 yields some $p' \leq p$ such that $\xi \in \text{ran}(\pi^{p'}_i)$. Hence the lemma follows. □

Lemma 2.4.7 *Let $G \subset \mathbb{P}_\Phi$ be generic over V and let $\langle \pi_{i,j}; i \leq j \leq \kappa \rangle$ be the system of elementary embeddings added by G and let $\langle M_i; i \leq \kappa \rangle$ be the sequence of transitive models added by G. Then:*

1. *$M_{i+1} \simeq \text{Ult}(M_i, G_i)$ for $i < \omega_1$, where $G_i := \{X \in \mathcal{P}(\omega_1^{M_i})^{M_i}; \text{crit}(\pi_{i,i+1} \in \pi_{i,i+1}(X)\}$,*
2. *$\langle \langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1 \rangle, \langle G_i; i < \omega_1 \rangle \rangle$ is a generic iteration, where $\kappa_i = \pi^{-1}_{i,\omega_1}(\kappa)$ and $I_i = \pi^{-1}_{i,\omega_1}(I)$.*

Proof. Fix some $i < \omega_1$. Let $\bar{D}$ be a dense subset of $\mathcal{P}(\omega_1^{M_i})^{M_i}$ that is definable over M_i. Hence $D := \pi_{i,\omega_1}(\bar{D})$ is definable over $\mathcal{M}$ from parameters in $\text{ran}(\pi_{i,\omega_1})$. By the previous lemma and some straightforward density argument, there is some $X \in D$ that is definable over $\mathcal{M}$ from parameters in $\text{ran}(\pi_{i,\omega_1})$ such that

$$\kappa_i = \text{crit}(\pi_{i,i+1}) \in X.$$

Hence by elementarity, there is some $\bar{X}$ definable over M_i such that $\bar{X} \in \bar{D}$ and $\bar{X} \in G_i$. The elements of $\text{Ult}(M_i, G_i)$ are all of the form $[f]_{G_i}$ for some $f \in M_i, f : \kappa_i \to M_i$. It is straightforward to verify that

$$\sigma : \text{Ult}(M_i, G_i) \to \text{Hull}^{\mathcal{M}}(\text{ran}(\pi_{i,\omega_1}) \cup \{\kappa_i\}); [f]_{G_i} \mapsto \pi_{i,\omega_1}(f)(\kappa_i)$$

is an isomorphism. By Lemma 2.4.5

$$M_{i+1} \simeq \text{Hull}^{\mathcal{M}}(\text{ran}(\pi_{i,\omega_1}) \cup \{\kappa_i\}).$$

This shows $M_{i+1} \simeq \text{Ult}(M_i, G_i)$ for $i < \omega_1$. The rest follows easily since one forms direct limits at limit stages of generic iterations and also at limits of the system of elementary embeddings $\langle \pi_{i,j}; i \leq j \leq \kappa \rangle$. □

Combining these results we get:

Theorem 2.4.8 ([CS09]) *Let I be a precipitous ideal on ω_1. Let $\theta > \omega_1$ be a regular cardinal and let $\mathcal{M} = \langle H_\theta; \in, \mathsf{NS}_{\omega_1}, < \rangle$. Then there is an ω_1-preserving $\mathcal{L}$-forcing $\mathbb{P}(I, \theta)$ that adds a generic iteration*

$$\langle \langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1 \rangle, \langle G_i; i < \omega_1 \rangle \rangle$$

such that $M_{\omega_1} = \langle H_\theta^V; \in, I \rangle$ and $I_{\omega_1} = I$; here $\kappa_i = \omega_1^{M_i}$ and all M_i with $i < \omega_1$ are countable. All I-positive $S \in V$ are stationary in $V^{\mathbb{P}(I,\theta)}$. Especially if $I = \mathsf{NS}_{\omega_1}$, then $\mathbb{P}(I, \theta)$ preserves stationary subsets of ω_1.

Proof. If the cardinal setup $\rho = 2^{<\rho} > 2^\theta > \theta = 2^{<\theta}$ holds in V for some regular ρ we set $\mathbb{P}(I, \theta) = \mathbb{P}_\Phi$. If $\text{Col}(\theta, \theta)$ preserves the precipitousness of I we set $\mathbb{P}(I, \theta) = \text{Col}(\rho, \rho) \times \text{Col}(\theta, \theta) \times \dot{\mathbb{P}}_\Phi$, where $\dot{\mathbb{P}}_\Phi$ is a name for $\mathbb{P}_\Phi$ calculated in $V^{\text{Col}(\rho,\rho) \times \text{Col}(\theta,\theta)}$. If $\text{Col}(\theta, \theta)$ does not preserve the precipitousness of I let $\theta' = (2^{2^\theta})^+$ and set $\mathbb{P}(I, \theta) = \text{Col}(\rho', \rho') \times \text{Col}(\theta', \theta') \times \dot{\mathbb{P}}_\Phi$ for some regular $\rho' >> \theta'$ and restrict the resulting generic iteration. This will be possible since $H_\theta \in H_{\theta'}$. Note that θ' is sufficiently large so that the precipitousness of I is preserved by forcing with $\text{Col}(\theta', \theta')$; there are no new strategies in $V^{\text{Col}(\theta', \theta')}$ for the players Empty and Nonempty in the precipitousness game, see [Jec03, 22.21].
In any of the three cases we can apply the previous lemmata and Theorem 2.3.2 to get the desired result. □

The previous theorem does not tell us whether M_0 is generically iterable. The following theorem shows that under an additional large cardinal assumption this is possible.

Theorem 2.4.9 ([CS09]) *Let I be a precipitous ideal on ω_1. Let $\theta > \omega_1$ be a regular cardinal and suppose $H_\theta^\sharp$ exists. Let $\mathcal{M} = \langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}, < \rangle$. Then there is a stationary set preserving $\mathcal{L}$-forcing $\mathbb{P}'(I, \theta)$ that adds a generic iteration*

$$\langle \langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1 \rangle, \langle G_i; i < \omega_1 \rangle \rangle$$

such that $M_{\omega_1} = \mathcal{M}$ and $I_{\omega_1} = I$; here $\kappa_i = \omega_1^{M_i}$ and all M_i with $i < \omega_1$ are countable. Additionally M_0 is generically iterable. All I-positive $S \in V$ are stationary in $V^{\mathbb{P}'(I,\theta)}$. Especially if $I = \mathsf{NS}_{\omega_1}$, then $\mathbb{P}'(I, \theta)$ preserves stationary subsets of ω_1.

Proof. By the previous theorem there is a forcing $\mathbb{P}$ that adds a generic iteration

$$\langle \langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1 \rangle, \langle G_i; i < \omega_1 \rangle \rangle$$

such that $H_\theta \in M_{\omega_1}$. Without loss of generality assume that $H_\theta \in \text{ran}(\pi_{0,\omega_1})$. Set $N_i = \pi_{i,\omega_1}^{-1}(H_\theta)$. Then $N_i = \pi_{i,\omega_1}^{-1}(H_\theta^\sharp) = N_i^\sharp$, so we can restrict the above generic iteration such that the last model is $\langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}, <\rangle$. It remains to show the generic iterability of $\langle N_0^\sharp; \in, I_0\rangle$. For this note that $\langle N_0^\sharp; \in, I_0\rangle$ is generically ω_1+1-iterable if and only if $\langle L[N_0]; \in, I_0\rangle$ is generically ω_1+1-iterable, see [Woo99, 3.8] and also note that $\langle L[N_0]; \in, I_0\rangle$ is generically ω_1+1-iterable by [Woo99, 3.10, 3.11]. □

Question 2.4.10 Ketchersid, Larson and Zapletal also developed a notion of forcing that adds a generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

with countable M_i for $i < \omega_1$ and $M_{\omega_1} \supset H_\theta$ for some given θ, see [KLZ07]. We do not know if this forcing construction can be recast as an $\mathcal{L}$-forcing, nor do we know if it is equivalent to Claverie-Schindler forcing.

2.4.2 Variants of Claverie-Schindler forcing

The forcing $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ can be modified to force (a single instance of) ψ_{AC} and ϕ_{AC}, combinatorial principles isolated by Woodin. These modified forcing constructions are carried out later, see Theorem 3.2.5 and Theorem 3.2.15.

2.4.3 A First Application: Sealing Antichains

We show that Claverie-Schindler seals all antichains in $\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}$. This observation (i.e. what we call Theorem 2.4.16) is due to S. Todorčević. Let us first have a look at the classic approach to sealing antichains:

Definition 2.4.11 Let $I \subset \mathcal{P}(\omega_1)$ be a normal and uniform ideal. Let $\mathcal{A}$ be a maximal antichain in $\mathcal{P}(\omega_1) \setminus I =: I^+$, i.e. if $S, T \in \mathcal{A}$, then $S \cap T \in I$ and for all $S \in I^+$ there is some $T \in \mathcal{A}$ such that $S \cap T \in I^+$. We say $\mathcal{A}$ is *sealed* if there is a surjection $F : \omega_1 \to \mathcal{A}$ and a club $C \subset \omega_1$ such that

$$C \subset \nabla_{\alpha\in\omega_1} F(\alpha) := \{\alpha\,;\, \alpha \in \cup_{\beta<\alpha} F(\beta)\}.$$

We call $\nabla_{\alpha\in\omega_1} F(\alpha)$ the *diagonal union* of F.

The following lemma explains the term sealed.

Lemma 2.4.12 *Let $\mathcal{A}$ be a maximal antichain in $\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}$ that is sealed. Let $F : \omega_1 \to \mathcal{A}$ and C be witnesses. Then in all outer models W such that $\omega_1^V = \omega_1^W$ the set $\mathcal{A}$ contains a maximal antichain in $(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1})^W$.*
Especially if W is a stationary set preserving extension of V, then $\mathcal{A}$ is a maximal antichain in $(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1})^W$.

Proof. Let $S \subset \omega_1$ be some stationary set of W. The set C is also club in W, so $S \cap C$ is stationary. Let us assume S witnesses that $\mathcal{A}$ does not contain a maximal antichain, i.e. if $T \in \mathcal{A}$, then $S \cap T \in \mathsf{NS}_{\omega_1}^W$, and work toward a contradiction. For each $\alpha \in \omega_1$ fix a club D_α such that $F(\alpha) \cap S \cap D_\alpha = \emptyset$. Let $D := \triangle D_\alpha$, the diagonal intersection of the D_α. Then $S \cap C \cap D \neq \emptyset$, say α_0 is in this intersection. Since $C \subset \nabla F(\alpha)$

$$\alpha_0 \in \bigcup_{\beta<\alpha_0} F(\beta).$$

On the other hand by the definition of D

$$\alpha_0 \in \bigcap_{\beta<\alpha_0} D_\beta.$$

So there is some $\beta_0 < \alpha_0$ such that $\alpha_0 \in F(\beta_0)$ and $\alpha_0 \in D_{\beta_0}$. But this is absurd since $D_{\beta_0} \cap F(\beta_0) \cap S = \emptyset$. □

Antichains can be forced to be sealed.

Definition 2.4.13 (Foreman, Magidor, Shelah) Let $\mathcal{A} \subset \mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}$ be a nonempty set. Let $\mathbb{P}_\mathcal{A}$ be the following partial order: elements are pairs (f, c) such that $c \subset \omega_1$ is closed and bounded in ω_1 and

$$f : \max(c) + 1 \to \mathcal{A}$$

and for all $\alpha \in c$ there is some $\beta < \alpha$ such that $\alpha \in f(\beta)$. For $(f, c), (g, d) \in \mathbb{P}_\mathcal{A}$ we set $(g, d) \leq (f, c)$ if and only if $g \supset f$ and $d \supset c$ and $d \cap (\max(c) + 1) = c$.

Lemma 2.4.14 *The forcing $\mathbb{P}_\mathcal{A}$ preserves stationary subsets of ω_1 if $\mathcal{A}$ is predense in $\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}$. So especially for all antichains $\mathcal{A}$ the forcing $\mathbb{P}_\mathcal{A}$ preserves stationary subsets of ω_1. Furthermore the forcing is ω-distributive.*

Proof. Let $S \subset \omega_1$ be a stationary set in V and let $\dot{C} \in V^{\mathbb{P}_\mathcal{A}}$ be a name for a club. Since $\mathcal{A}$ is predense, there is some $T \in \mathcal{A}$ such that $S \cap T$ is stationary. Let λ be large enough so that all the dense subsets of $\mathbb{P}_\mathcal{A}$ are contained in H_λ and $\dot{C} \in H_\lambda$. Pick a countable $X \prec H_\lambda$ such that $\{\mathcal{A}, \mathbb{P}_\mathcal{A}, T, S, \dot{C}\} \subset X$ and $\alpha := \omega_1 \cap X \in S \cap T$. Let $\pi : M \to X$ be the inverse of the transitive collapse of X. Let g be generic for $\pi^{-1}(\mathbb{P}_\mathcal{A})$ over M. Let

$$c := \bigcup\{c' ; \exists f' : (f', c') \in g\}; f := \bigcup\{\pi(f') ; \exists c' : (f', c') \in g\}.$$

Clearly c is club in α and $f : \alpha \to \mathcal{A} \cap X$ is surjective. Note that (f, c) is not a condition, since c is not closed in ω_1. It is straightforward to check that $p := (f^\frown(\alpha, T), c \cup \{\alpha\}) \in \mathbb{P}_\mathcal{A}$; the key fact is that by the genericity of g over M there is some $\beta < \alpha$ such that $f(\beta) = T$. Since

$$M[g] \models \pi^{-1}(\dot{C})^g \text{ is club in } \alpha,$$

we have $p \Vdash \dot{C}$ is unbounded in α. Then $p \Vdash \alpha \in \dot{C}$. So $p \Vdash \alpha \in \check{S} \cap \dot{C}$.
For the ω-distributivity we look at some name ρ for a function with domain ω and range in V. We only have to modify the argument above slightly. Pick an X as above with the additional property that $\rho \in X$. Then construct p as before. By genericity p decides $\rho(n)$ for all $n \in \omega$, i.e. for all $n \in \omega$ there is some $r \geq p$ and some x such that

$$r \Vdash \rho(n) = \check{x}.$$

□

Lemma 2.4.15 (Foreman, Magidor, Shelah) *Let $\mathcal{A}$ be a maximal antichain in $\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}$. Let $G \subset \mathbb{P}_\mathcal{A}$ be generic over V. Let*

$$C := \bigcup\{c ; \exists f : (f, c) \in G\}; F := \bigcup\{f ; \exists c : (f, c) \in G\}.$$

Then $C \subset \omega_1$ is closed unbounded and $F : \omega_1 \to \mathcal{A}$ is surjective and C, F witness that $\mathcal{A}$ is sealed in $V[G]$.

Proof. By the previous lemma we know that $V[G]$ is a stationary set preserving extension of V. Obvious density arguments imply that C is club and that F is surjective. We show that $C \subset \nabla_{\alpha\in\omega_1} F(\alpha)$. Fix some $\alpha \in C$. Hence there is some $(f,c) \in G$ such that $\alpha \in c$. Hence by the definition of $\mathbb{P}_{\mathcal{A}}$ we have that $\alpha \in f(\beta)$ for some $\beta < \alpha$. Then clearly $\alpha \in \bigcup_{\beta<\alpha} F(\beta)$ which suffices to show. □

We now show how to seal all antichains in V by a single notion of forcing.

Theorem 2.4.16 *Let $\theta > 2^{2^{\omega_1}}$ and regular. Let $I \subset \mathcal{P}(\omega_1)$ be a precipitous ideal. Let $\mathbb{P}$ be a notion of forcing that preserves all I-positive sets; i.e. all I-positive sets are stationary in $V^{\mathbb{P}}$. Furthermore suppose that $\mathbb{P}$ adds a generic iteration*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \le j \le \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that $\langle M_{\omega_1}, I_{\omega_1}\rangle = \langle H_\theta; I\rangle$. Then all maximal antichains $\mathcal{A} \subset (\mathcal{P}(\omega_1)\setminus I)^V$ in V are sealed in $V^{\mathbb{P}}$.

We state the following obvious corollary before proving the theorem.

Corollary 2.4.17 *Let $\theta > 2^{2^{\omega_1}}$ and regular. Let I be precipitous. Let $\mathbb{P} = \mathbb{P}(I,\theta)$. Then all maximal antichains $\mathcal{A} \subset (\mathcal{P}(\omega_1)\setminus I)^V$ in V are sealed in $V^{\mathbb{P}}$.*

We now prove the theorem.

Proof. Let $\mathcal{A} \in V$ be a maximal antichain in $\mathcal{P}(\omega_1)\setminus I$. Let

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \le j \le \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

be a generic iteration with last model $\langle H_\theta; I\rangle$. Then clearly $\mathcal{A} \in H_\theta$. Let us assume without loss of generality that there is some $A \in M_0$ such that $\pi_{0,\omega_1}(A) = \mathcal{A}$. Let $\Gamma : \omega_1 \to \omega_1 \times \omega$ denote a function such that for all limit ordinals $\delta < \omega_1$ the function $\Gamma\restriction\delta : \delta \to \delta\times\omega$ is bijective. We also fix for all $\alpha < \omega_1$ enumerations

$$\sigma_\alpha : \omega \to M_\alpha$$

of all stationary subsets of κ_α in M_α. We define a function F that will witness that $\mathcal{A}$ is sealed in $V^{\mathbb{P}}$. We set

$$F : \omega_1 \to \mathcal{A}; \alpha \mapsto \begin{cases} \pi_{\beta,\omega_1}(\sigma_\beta(n)) & \text{if } \Gamma(\alpha) = (\beta,n) \text{ and } \pi_{\beta,\omega_1}(\sigma_\beta(n)) \in \mathcal{A}; \\ S_0 & \text{else, for some fixed } S_0 \in \mathcal{A}. \end{cases}$$

Let $C := \{\alpha < \omega_1\,;\, \alpha = \kappa_\alpha\}$. It clearly remains to show

$$C \subset \nabla_{\alpha\in\omega_1} F(\alpha).$$

Fix some $\alpha \in C$. The set $\pi_{0,\alpha}(A)$ is a maximal antichain in $(\mathcal{P}(\kappa_\alpha)\setminus I_\alpha)^{M_\alpha}$. Hence by the genericity of G_α over M_α there is a unique $t \in \pi_{0,\alpha}(A)$ such that $t \in G_\alpha$. This implies that $\alpha \in \pi_{\alpha,\alpha+1}(t) \subset \pi_{\alpha,\omega_1}(t)$. Since α is a limit ordinal, there is a $\beta < \alpha$ and a $\bar{t} \in M_\beta$ such that $\pi_{\beta,\alpha}(\bar{t}) = t$. So there is some $n \in \omega$ such that $\sigma_\beta(n) = \bar{t}$ an hence some $\gamma < \alpha$ such that $\Gamma(\gamma) = (\beta,n)$. Note that $\gamma < \alpha$ since α is a limit ordinal. So we have $F(\gamma) = \pi_{\alpha,\omega_1}(\bar{t})$. Hence $\alpha \in \bigcup_{\alpha'<\alpha} F(\alpha')$, which is what we needed to show. □

Theorem 2.4.16 yields a characterization of precipitousness:

Corollary 2.4.18 *Let $I \subset \mathcal{P}(\omega_1)$ be a normal and uniform ideal. The following are equivalent:*

1. *I is precipitous.*

2. *There is a notion of forcing $\mathbb{P}$ such that all maximal antichains $\mathcal{A} \subset \mathcal{P}(\omega_1) \setminus I$, $\mathcal{A} \in V$ are sealed in $V^{\mathbb{P}}$ and all $S \in \mathcal{P}(\omega_1) \setminus I$, $S \in V$ are stationary in $V^{\mathbb{P}}$.*

3. *For all cardinals $\theta > \omega_1$ there is a notion of forcing $\mathbb{P}$ such that $\mathbb{P}$ adds a generic iteration of length $\omega_1 + 1$ with last model $\langle H_\theta; I\rangle$ and all $S \in \mathcal{P}(\omega_1) \setminus I$, $S \in V$ are stationary in $V^{\mathbb{P}}$.*

Proof. Let us assume 1. holds. Then $\mathbb{P}(I, \theta)$ witnesses that 3. holds. Clearly 3. implies 2. by the previous theorem. It remains to check that 2. implies 1.
Assume there is some $S \in \mathcal{P}(\omega_1) \setminus I$ such that

$$S \Vdash \mathrm{Ult}(V, \dot{G}) \text{ is ill-founded,}$$

and work towards a contradiction. So there is some $\theta > 2^{2^{\omega_1}}$ that is a cardinal in $V^{\mathbb{P}}$ such that

$$S \Vdash \mathrm{Ult}(\check{H}_\theta^V, \dot{G}) \text{ is ill-founded.}$$

Let $X \prec \langle H_\theta^{V^{\mathbb{P}}}; \in, I, H_\theta^V\rangle$, $X \in V^{\mathbb{P}}$ such that $S \in X$ and $\alpha := X \cap \omega \in S$. This is possible, since S is stationary in $V^{\mathbb{P}}$. Let $\mathfrak{A}$ denote the collection of all maximal antichains of $\mathcal{P}(\omega_1) \setminus I$. So for every $\mathcal{A} \in \mathfrak{A} \cap X$, there is a surjection $F : \omega_1 \to \mathcal{A}$ and a club $C \subset \omega_1$, $F, C \in X$, such that F, C witness that $\mathcal{A}$ is sealed. Let $\pi : \langle \bar{H}; \in, \bar{I}, \bar{H}_\theta^V\rangle \to X$ denote the inverse of the transitive collapse and let $\pi(\bar{S}, \bar{\mathfrak{A}}) = S, \mathfrak{A}$. Set

$$g = \{a \in M \,;\, a \in \mathcal{P}(\alpha) \setminus \bar{I} \wedge \alpha \in \pi(a)\}.$$

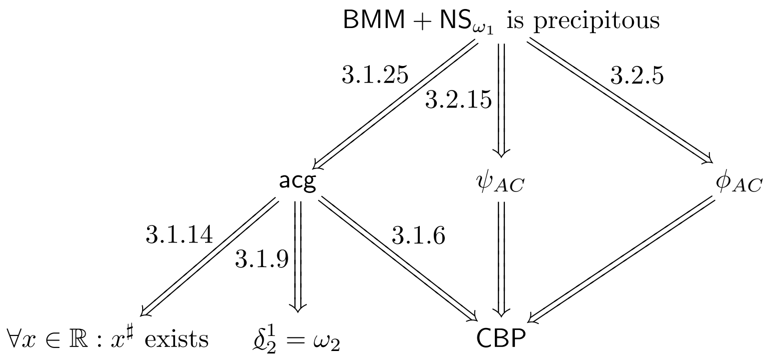
Clearly $\bar{S} \in g$. We claim that g is generic over $\bar{H}_\theta^V$. So let $\bar{\mathcal{A}} \in \bar{\mathfrak{A}}$ and let $F, C \in X$ witness that $\pi(\bar{\mathcal{A}})$ is sealed. Then $\alpha \in C$, since $C \cap \alpha$ is unbounded in α by elementarity. This implies that there is some $\beta < \alpha$ such that $\alpha \in F(\beta)$. So $\pi^{-1}(F(\beta)) \in g$ (in fact this β is unique). That shows that g intersects all members of $\bar{\mathfrak{A}}$, hence g is generic over $\bar{H}_\theta^V$. Since $\bar{S} \in g$, the generic ultrapower $j : \bar{H}_\theta^V \to \mathrm{Ult}(\bar{H}_\theta^V, g)$ is ill-founded. This contradicts the fact that the map

$$j(f)(\alpha) \mapsto \pi(f)(\alpha)$$

witnesses the well-foundedness of j. This finishes the proof. $\square$

3 The Theory $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous

We will now investigate which Π_2 sentences (over H_{ω_2}) that are consequences of $\mathsf{ZFC} + \mathsf{MM}$ also follow from $\mathsf{ZFC} + \mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous. It turns out that admissible club guessing (acg), $\underset{\sim}{\delta}^1_2 = \omega_2$, the club bounding principle (CBP), and ψ_{AC} as well as ϕ_{AC} follow from this weaker theory. This was known for $\underset{\sim}{\delta}^1_2 = \omega_2$ and ψ_{AC} but not for ϕ_{AC} and acg. We will define and study the principles mentioned above. First we will study acg and its various direct consequences and show that acg follows from $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous. Then we will define and study ϕ_{AC} and ψ_{AC} and show that they are consequences of $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous.
Here is an outline of the strategy of our investigation. Given a Π_2 statement $\phi = \forall x \bar{\phi}(x, p)$, where $p \in H_{\omega_2}$ and $\bar{\phi}$ is Σ_1 we will make use of the theory $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous as follows: the precipitousness of NS_{ω_1} will allow us to construct a stationary set preserving $\mathcal{L}$-forcing that forces $\bar{\phi}(x, p)$ for a fixed $x \in H_{\omega_2}$. Then we will apply BMM to see that $\bar{\phi}(x, p)$ holds in the ground model. The forcing will be Claverie-Schindler forcing or a variant.
The theory $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous does nevertheless not imply that NS_{ω_1} is saturated, in contrast to MM: it is well known that MM implies that NS_{ω_1} is ω_2-saturated, see [FMS88], but by [Woo99, 10.103, 10.99] $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous does not[1].
The following diagram illustrates the logical structure of the various statements; here the superscripts of the implications refer to the respective result:



The implication from ψ_{AC} to CBP is due to Aspero and Welch, see [AW02]. Claverie and Schindler have shown that $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous implies $\underset{\sim}{\delta}^1_2 = \omega_2$, see [CS09]. The implication from ϕ_{AC} to CBP follows easily from Lemma 3.1.5. All implications from acg are due to Woodin, see [Woo99, (proof of) 3.19]; nevertheless we will discuss acg in greater detail and review the implications from acg.

3.1 The Principle acg

Let $x \subset \omega$. Recall that an ordinal α is *x-admissible*, or simply admissible if x is clear from the context, if $L_\alpha[x] \models \mathsf{KP}$, where KP is Kripke-Platek set theory.

[1] In the situation of [Woo99, 10.103] one considers a ${}^2\mathbb{P}_{\max}$ extension; there NS_{ω_1} is not saturated but one can check that it is precipitous using the ${}^2\mathbb{P}_{\max}$ analysis in [Woo99, 6.14].

Definition 3.1.1 We call the following principle *admissible club guessing* (acg). For all clubs $C \subseteq \omega_1$ there exists a real x such that

$$A_x := \{\alpha < \omega_1 \,;\, L_\alpha[x] \text{ is admissible}\} \subset C.$$

Note that acg is a Π_2 statement in no parameters over H_{ω_2}.

Remark 3.1.2 The principle acg was isolated by Woodin. If MM holds, then NS_{ω_1} is ω_2-saturated and the universe is closed under the sharp operation (the closure under the sharp operation is already a consequence of BMM, see [Sch04]). So by [Woo99, 3.17] $\underset{\sim}{\delta}^1_2 = \omega_2$ and hence by [Woo99, 3.16, 3.19] acg holds.

3.1.1 Consequences of acg

We discuss several interesting consequences of acg. Recall the definition of the Club Bounding Principle (CBP).

Definition 3.1.3 The *Club Bounding Principle* (CBP) is the following axiom: For all $f : \omega_1 \to \omega_1$ there is some $\eta < \omega_2$ and some club C such that for all $\alpha \in C : f(\alpha) < f_\eta(\alpha)$ where f_η is a canonical function for η.

Clearly CBP can be recast as a Π_2 statement in no parameters over H_{ω_2}.

Definition 3.1.4 (Tilde Operation) Let $T \subset \omega_1$. Then we set

$$\tilde{T} := \{\alpha < \omega_2 \,;\, \omega_1 \leq \alpha \wedge \mathbf{1}_{\mathcal{P}(\omega_1)\setminus \mathsf{NS}_{\omega_1}} \Vdash \alpha \in j(T)\},$$

where j is a name for the generic ultrapower added by $\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}$.

Note that $\tilde{C}$ contains ω_1 if and only if C contains a club.

Lemma 3.1.5 (Folklore) *The following are equivalent:*

1. CBP
2. *For every club $C \subset \omega_1$ there is a club $D \subset \omega_1$, $\alpha < \omega_2$, $\omega_1 < \alpha$ and a canonical function f_α for α such that f_α“$D \subset C$.*
3. *For every club $C \subset \omega_1$ there is some $\alpha \in \tilde{C}$ such that $\omega_1 < \alpha$.*

Proof. Unraveling the definition of $\tilde{C}$ it is not difficult to see that 2. and 3. are equivalent. It remains to show 1. if and only if 2. We assume CBP. Let $C \subset \omega_1$ be club. Inductively we construct a sequence $\langle f_i; i < \omega\rangle$ and a sequence $\langle \alpha_i; i < \omega\rangle$ of ordinals $< \omega_2$ such that $\operatorname{ran}(f_i) \subset C$ and such that there is a club D_i such that $f_i(\xi) < f_{\alpha_i}(\xi) < f_{i+1}(\xi)$ for all $\xi \in D_i$ where f_{α_i} is a canonical function for α_i. Set $f_0(\xi) = \min(C \setminus (\xi + 1))$ for $\xi < \omega_1$. Then by CBP there is some $\alpha_0 < \omega_2$, a canonical function f_{α_0} for α_0 and a club D_0 such that $f_{\alpha_0}(\xi) > f_0(\xi)$ for $\xi \in D_0$. In the induction step set $f_{i+1}(\xi) = \min(C \setminus (f_{\alpha_i}(\xi) + 1)$ for $\xi < \omega_1$. Note that by the choice of f_0 every α_i is $> \omega_1$. Set $D = \cap_{i<\omega} D_i$. Then for all $\xi \in D$

$$\sup_{i<\omega} f_{\alpha_i}(\xi) = \sup_{i<\omega} f_i(\xi) \in C,$$

since $\operatorname{ran}(f_i) \subset C$ by construction. By Lemma 1.6.5 $f : \omega_1 \to \omega_1$; $f(\xi) := \sup_{i<\omega} f_{\alpha_i}(\xi)$ is a canonical function for $\alpha := \sup_{i<\omega} \alpha_i$. Hence $f(\xi) \in C$ for all $\xi \in D$.

It remains to show the converse. Assume 2. holds. Let $f : \omega_1 \to \omega_1$ be a function and let $C := \{\beta < \omega_1 ; f``\beta \subset \beta\}$. By the hypothesis there is a club D, an $\alpha > \omega_1$ and a canonical function $f_\alpha : \omega_1 \to \omega_1$ for α such that $f_\alpha(\beta) \in C$ for all $\beta \in D$. Since $\alpha > \omega_1$ the set $\{\beta ; \beta \geq f_\alpha(\beta)\}$ is nonstationary. Hence we can assume without loss of generality that $f_\alpha(\beta) > \beta$ for $\beta < \omega_1$. If $\beta \in D$, then $f_\alpha(\beta) \in C$. Hence $f``f_\alpha(\beta) \subset f_\alpha(\beta) > \beta$ for $\beta \in D$. So especially $f(\beta) < f_\alpha(\beta)$ for all $\beta \in D$. □

Note that CBP implies that $\omega_2^V = \omega_1^N$ for every well-founded generic ultrapower $j : V \to N$, see Lemma 3.1.17

Lemma 3.1.6 ([Woo99, (proof of) 3.19]) acg $\implies$ CBP.

In the proof of the lemma and later on it is convenient to use the following function.

Definition 3.1.7 If x is a real we define a function looking for the next x-admissible above some ordinal.

$$\pi_x : \omega_1 + 1 \to \omega_2; \quad \pi_x(\alpha) := \text{the least } x\text{-admissible} > \alpha.$$

Note that we sometimes write π_x for $\pi_x \restriction \omega_1$, especially when equivalence classes of functions with domain ω_1 are discussed. Note the following: if $f = f_{\pi_x(\omega_1)}$ is a canonical function for $\pi_x(\omega_1)$, then f and π_x agree on a club.

We now show Lemma 3.1.6.

Proof. Let $f : \omega_1 \to \omega_1$. Let C be the club of ordinals α such that $f``\alpha \subset \alpha$. Let $x \in \omega^\omega$ such that $A_x \subset C$. If $\alpha < \omega_1$, then $\pi_x(\alpha) \in A_x$ and so $f``\pi_x(\alpha) \subset \pi_x(\alpha)$. Hence for all $\alpha \in \omega_1$

$$f(\alpha) < \pi_x(\alpha).$$

So clearly $f < \pi_x$ on a club. Let $\eta = \pi_x(\omega_1)$, so $\omega_1 < \eta < \omega_2$. We claim that f_η (i.e. an ηth canonical function) is a function that dominates π_x on a club (this clearly suffices to show). The set $D \subset \omega_1$ of ξ such that there is an $X \prec H_{\omega_2}$ such that the following conditions hold is club:

1. $\xi = X \cap \omega_1$,
2. $f_\eta(\xi) = otp(X \cap \eta)$.

Such X exist, by the following argument: pick some $X \prec \langle H_{\omega_2}, \in, \eta, g_\eta \rangle$, where $g_\eta : \omega_1 \to \eta$ bijective such that $f_\eta(\alpha) = otp(g_\eta``\alpha)$ (such a g_η exists by the definition of canonical functions). If $\xi = X \cap \omega_1$ then, by elementarity,

$$\rho \in X \cap \eta \iff X \models \exists \alpha < \omega_1 : g_\eta(\alpha) = \rho \iff \rho \in g_\eta``\xi,$$

Hence $X \cap \eta = g_\eta``\xi$. But then $f_\eta(\xi) = otp(g_\eta``\xi) = otp(X \cap \eta)$.
Without loss of generality we can assume that each witness X as above contains x as an element. We now show: if $\xi \in D$, then $\pi_x(\xi) \leq f_\eta(\xi)$. Pick X, a witness for $\xi \in D$. Let $\sigma : H \to H_{\omega_2}$ be the inverse of the transitive collapse of X. Let $\bar{\eta} = \sigma^{-1}(\eta)$. So $\bar{\eta} = f_\eta(\xi)$. Since $H_{\omega_2} \models$ "$\eta = \pi_x(\omega_1)$ is an x-admissible", the same holds for $\bar{\eta}$ in H by elementarity. Since H is transitive, $\bar{\eta}$ is really an x-admissible. So $\pi_x(\xi) \leq \bar{\eta} = f_\eta(\xi)$. □

Recall the following concepts:

Definition and Remark 3.1.8 By $\underset{\sim}{\delta}^1_2$ we denote the supremum of the lengths of all $\underset{\sim}{\Delta}^1_2$ well-orderings of the reals and by u_2 we denote the second uniform indiscernible, i.e. the least ordinal above ω_1 that is an x-indiscernible for all $x \in \mathbb{R}$. It is well known that if $x^\sharp$ exists for all reals x then $\underset{\sim}{\delta}^1_2 = u_2$. It is easy to see that always $\omega_1 < \underset{\sim}{\delta}^1_2 \leq \omega_2$ and $\omega_1 < u_2 \leq \omega_2$.

Lemma 3.1.9 ([Woo99, (proof of) 3.19]) $\mathsf{acg} \implies \underset{\sim}{\delta}^1_2 = \omega_2$.

We have not yet shown that acg implies that sharps for all reals exist but want to make use of this fact in the proof. We promise to prove the existence of sharps for reals in 3.1.14

Proof. By 3.1.14 acg implies that sharps for all reals exist. So $u_2 = \underset{\sim}{\delta}^1_2$. Hence it suffices to show that $u_2 = \omega_2$. If sharps for all reals exists, the second uniform indiscernible u_2 can be characterized as follows:

$$u_2 = \sup\{(\omega_1^V)^{+L[x]} \, ; \, x \in \omega^\omega\}.$$

Since $\pi_x(\omega_1) < (\omega_1^V)^{+L[x]}$, it suffices to show that the ordinals of the form $\pi_x(\omega_1)$ are cofinal in ω_2. Fix some $\eta < \omega_2$ and a canonical function $f_\eta : \omega_1 \to \omega_1$. We now apply acg as in the proof of the previous lemma to find a real x such that π_x dominates f_η on a club. So clearly $\pi_x(\omega_1) > \eta$. □

The following lemma is (3) $\implies$ (4) in [Woo99, 3.19]. Since there is no proof of this implication given, we give one here.

Lemma 3.1.10 *If $x^\sharp$ exists for every real x and additionally for every club $C \subset \omega_1$ there exists a club $C' \subset C$ that is constructible from a real, then* acg *holds.*

Proof. Fix a club $C \subset \omega_1$. By our hypothesis, there is a real x and a club $C' \in L[x]$ such that $C' \subset C$. Since $x^\sharp$ exists, we find x-indiscernibles $\nu_1 < ... < \nu_n < \omega_1$ and some formula ϕ, such that C' is the unique set (in $L[x]$) defined by ϕ using $\vec{\nu}$ and x as parameters. Pick some real y such that $x \in L[y]$ and $L[y] \models$ "$\nu_1, ..., \nu_n < \omega_1$". Since there is a club of countable x-indiscernibles there is clearly an x-indiscernible in C'. Hence all x-indiscernibles ξ such that $\nu_r < \xi < \omega_1$ are in C'. Note that every y-indiscernible $\xi < \omega_1$ is an x-indiscernible. Now look at $A := A_{y^\sharp} = \{\alpha < \omega_1 \, ; \, \alpha \text{ is } y^\sharp\text{-admissible}\}$. We claim that $A \subset C$. Pick some $\alpha \in A$. Hence $L_\alpha[y^\sharp] \models \mathsf{KP}$. Using only KP one can check that unboundedly many y-indiscernibles exist in $L_\alpha[y^\sharp]$. Hence α is a limit of y-indiscernibles and so also a y-indiscernible. Clearly $\alpha > \nu_n$, since y was chosen so that there is a surjection from ω to ν_r in $L[y]$. So $\alpha \in C'$. This suffices to show. □

By 3.1.14 acg implies that sharps exist for all reals. For any x the set A_x is clearly constructible from x. So we have the following corollary:

Corollary 3.1.11 *The following are equivalent:*

1. *Sharps for all reals exist and for every club $C \subset \omega_1$ there exists a club $C' \subset C$ that is constructible from a real.*

2. acg. □

3.1.2 The consistency strength of acg

To derive strength from acg it is crucial to know that sharps for reals exist if acg holds. We will now show that acg implies that $x^\sharp$ exists for every real x. There are two ways to show this; we are going to present both, since the more cumbersome way yields additional information about the generic iterability of substructures of H_{ω_2}.

Lemma 3.1.12 (Woodin) *Let $\langle X; \in, \mathsf{NS}_{\omega_1} \cap X\rangle \prec \langle H_{\omega_2}; \in, \mathsf{NS}_{\omega_1}\rangle$ be a countable substructure. Let N denote the transitive collapse of X. Then acg implies that N is (fully) generically iterable, i.e. if there is a generic iterate N_α of N in some transitive ZFC model $W \supseteq V$ such that N_α is countable in W then we can continue the generic iteration in W to an iteration of length $\omega_1^W + 1$.*

Proof. Fix a countable $\langle X; \mathsf{NS}_{\omega_1} \cap X\rangle \prec \langle H_{\omega_2}; \mathsf{NS}_{\omega_1}\rangle$ and let $\langle N; \in, \mathsf{NS}_{\omega_1}^N\rangle$ denote its transitive collapse. The proof of 3.1.9 shows that $\{\pi_x(\omega_1)\,;\, x \in \omega^\omega\}$ is cofinal in ω_2. If x, y are reals such that $\pi_x(\omega_1) < \pi_y(\omega_1)$ then there is a club C of $\alpha < \omega_1$ such that $\pi_x(\alpha) < \pi_y(\alpha)$; this can easily be seen by looking at collapses of elementary substructures of H_{ω_2}. Now acg implies that for each such C there exists a real z such that $A_z \subset C$. We now construct a sequence $\langle x_i\,;\, i \in \omega\rangle$ of reals such that

1. $\forall i \in \omega : x_i \in X$,
2. $\{\pi_{x_i}(\omega_1)\,;\, i \in \omega\}$ is cofinal in $X \cap \omega_2$,
3. for each $\alpha \in A_{x_{i+2}}$
$$\pi_{x_i}(\alpha) < \pi_{x_{i+1}}(\alpha).$$

Pick some countable sequence $\langle \alpha_i\,;\, i \in \omega\rangle$ of ordinals cofinal in $X \cap \omega_2$. Pick $x_0, x_1 \in X \cap \omega^\omega$ such that $\alpha_0 \le \pi_{x_0}(\omega_1) < \pi_{x_1}(\omega_1)$ and $\alpha_1 \le \pi_{x_1}(\omega_1)$. We now apply acg to find a real z_2 such that for all $\alpha \in A_{z_2}$ $\pi_{x_0}(\alpha) < \pi_{x_1}(\alpha)$. By elementarity z_2 can be picked in X. Pick some real z_2' such that $\alpha_2 \le \pi_{z_2'}(\omega_1)$. Again such a real exists in X. Let $x_2 \in X$ code z_2 and z_2', then clearly for all $\alpha \in A_{x_2}$ $\alpha_0 \le \pi_{x_0}(\alpha) < \pi_{x_1}(\alpha)$. We can now continue in this fashion to pick x_i for $2 < i$.
Since N is the transitive collapse of X it follows for $i \in \omega$ that $\pi_{x_i}(\omega_1^N)$ is the image of $\pi_{x_i}(\omega_1)$ under the collapsing map. Thus if $j : \langle N; \in, \mathsf{NS}_{\omega_1}^N\rangle \to \langle M; E, \mathsf{NS}_{\omega_1}^M\rangle$ is a generic iterate, $\{j(\pi_{x_i}(\omega_1^N))\,;\, i \in \omega\}$ is cofinal in the ordinals of M, simply because $j``\mathsf{OR} \cap N$ is cofinal in the ordinals of M and $\{\pi_{x_i}(\omega_1^N)\,;\, i \in \omega\}$ is cofinal in $\mathsf{OR} \cap N$.
We prove two claims and derive the desired result from them.
Claim 1. Suppose that $\langle N^*; E^*, I^*\rangle$ is a generic iterate of $\langle N; \in, \mathsf{NS}_{\omega_1}^N\rangle$ such that $\omega_1^{N^*}$ is well-founded. Then N^* is well-founded.
Proof of Claim 1. Let γ be the well-founded part of OR^{N^*}. Thus for each $x \in N^* \cap \omega^\omega$, $L_\gamma[x]$ is admissible, see [Bar75, Corollary II 8.5], i.e. the so called Truncation Lemma (the quoted result works for $x = \emptyset$; for $x \neq \emptyset$ it is easily checked that an analog holds). Since $\omega_1^{N^*}$ is well-founded $\omega_1^{N^*} \in \gamma$. Hence for all $x \in N^* \cap \omega^\omega$ we have $\pi_x(\omega_1^{N^*}) \le \gamma$. So $\sup\{\pi_x(\omega_1^{N^*})\,;\, x \in N^* \cap \omega^\omega\} \le \gamma$; this implies that $\gamma = \mathsf{OR}^{N^*}$. Hence N^* is well-founded. □(Claim 1)

Claim 2. Let $\langle N^*; \in, I^*\rangle$ be a well-founded iterate of N and let $\langle N^{**}; E^{**}, I^{**}\rangle$ be a generic ultrapower of $\langle N^*; \in, I^*\rangle$. Then

$$\omega_1^{N^{**}} = N^* \cap \mathsf{OR}.$$

Proof of Claim 2. The proof of 3.1.6 shows that for all $f : \omega_1 \to \omega_1$ a real x exists such that for all $\alpha \in A_x$ we have $f(\alpha) < \pi_x(\alpha)$. Hence this also holds in N and its iterates. Let $j^* : \langle N^*; \in, I^*\rangle \to \langle N^{**}; E^{**}, I^{**}\rangle$ be a generic ultrapower. All elements of $\omega_1^{N^{**}}$ are represented by functions $f : \omega_1^{N^*} \to \omega_1^{N^*}$. Fix such an f. It suffices to show that

$$j^*(f)(\omega_1^{N^*}) < \mathsf{OR} \cap N^*,$$

since the ordinals of N^* are clearly an initial segment of the ordinals of N^{**}. Now pick some real $x \in N^*$ such that for all $\alpha \in A_x^{N^*}$,

$$f(\alpha) < \pi_x(\alpha).$$

Again by elementarity we know that $\pi_x(\omega_1^{N^*}) \in N^*$. By absoluteness we have

$$\pi_x(\omega_1^{N^*}) = (\pi_x(\omega_1^{N^*}))^{N^{**}}.$$

By the elementarity of j^*

$$\forall \alpha \in A_x^{N^{**}} : j^*(f)(\alpha) < j^*(\pi_x)(\alpha).$$

Since $j(\pi_x^{N^*}) = \pi_x \restriction \omega_1^{N^{**}}$ we conclude

$$j^*(f)(\omega_1^{N^*}) < j^*(\pi_x)(\omega_1^{N^*}) < \mathsf{OR} \cap N^*.$$

Hence $j^*(\omega_1^{N^*}) = \mathsf{OR} \cap N^*$. □(Claim 2)

From the two claims it follows inductively that all generic iterates of N are well-founded: the successor case is a direct consequence of the two claims. For the limit case it suffices to check that for any putative generic iteration

$$\langle \langle N_i, \pi_{i,j}, \mathsf{NS}_{\omega_1}^{N_i}, \kappa_i; i \leq j \leq \gamma \rangle, \langle G_i; i < \gamma \rangle \rangle$$

of $\langle N; \in, \mathsf{NS}_{\omega_1}^N \rangle$, where γ is a countable limit ordinal, the ordinal $\omega_1^{N_\lambda}$ is well-founded. If not there is a decreasing chain $(a_i)_{i\in\omega}$ of countable ordinals in N_λ. Since all a_i are countable in N_λ there is some $\alpha < \lambda$ such that all a_i have a preimage $\bar{a}_i$ in N_α. But this contradicts the well-foundedness of N_α. Hence generic iterations of N can be continued in either case. □

Lemma 3.1.13 (Woodin) *If there is an* $\langle X; \in, \mathsf{NS}_{\omega_1} \cap X \prec \langle H_{\omega_2}; \in, \mathsf{NS}_{\omega_1} \rangle$ *such that the transitive collapse of* X *is (fully) generically iterable, then* $x^\sharp$ *exists for all reals* x*.*

Proof. Let $\langle M; \mathsf{NS}_{\omega_1}^M \rangle$ be the transitive collapse of some $X \prec H_{\omega_2}$ such that the model $\langle M; \mathsf{NS}_{\omega_1}^M \rangle$ is generically iterable. Note that sharps for all reals exists if and only if $H_{\omega_2} \models \forall x \in \omega^\omega : x^\sharp$ exists. We assume that sharps did not exist for all reals and prove that this contradicts Jensen's Covering Lemma. It clearly suffices to discuss only reals in M. So let z be a real in M and assume $z^\sharp$ did not exist.
Claim 1. If N is a countable, transitive, generically iterable model of ZFC^*+"ω_1 exists" and if t is a real in N then ω_1^N is a regular cardinal in $L[t]$.
Proof of Claim 1. Let $S = \{\kappa < \omega_1^N ; \kappa \text{ is singular in } L_{\omega_1^N}[t]\}$. We prove that S is nonstationary. Otherwise it was stationary in N. Let $j : N \to N^*$ be an iteration of length ω_1. So $j(\omega_1^N) = \omega_1$. Work in some generic extension $V[G]$ where ω_1 is countable. Continue the iteration by one step, say $k : N^* \to N^{**}$, such that $\omega_1 \in k(j(S))$. So ω_1^V is singular in $L_{\omega_1^{N^{**}}}[t]$, which is clearly a contradiction.
Hence there is a club $C \subset \omega_1^N$ in N such that all $\kappa \in C$ are regular in $L_{\omega_1^N}[t]$. Suppose ω_1^N was not regular in $L[t]$. Let $\beta < \omega_1$ such that ω_1^N is singular in $L_\beta[t]$. Let $j : N \to N^*$ be some iteration of N of length β. So $\beta \leq \omega_1^{N^*}$ and hence ω_1^N is singular in $L_{\omega_1^{N^*}}[t]$. But $\omega_1^N \in j(C)$, hence ω_1^N must be regular in $L_{\omega_1^{N^*}}[t]$. Contradiction. □(Claim 1)

We now use the claim to show that all uncountable cardinals in V are regular cardinals in $L[z]$. Let λ be an uncountable cardinal in V. Let $V[G]$ be a generic extension of V in which λ is countable. Let

$$\langle\langle M_i, \pi_{i,j}, \mathsf{NS}_{\omega_1}^{M_i}, \kappa_i; i \leq j \leq \lambda\rangle, \langle G_i; i < \lambda\rangle\rangle$$

be a generic iteration of $\langle M; \in, \mathsf{NS}_{\omega_1}^{M}\rangle$. By Lemma 3.15 of [Woo99] we have $j(\omega_1^M) = \lambda$. Also M_λ is generically iterable in $V[G]$. Hence an application of the claim implies that λ is regular in $L[z]$. But this implies that $\aleph_\omega^V$ is regular in $L[z]$ which clearly contradicts Jensen's Covering Lemma. □

Combining the lemmata above we have:

Lemma 3.1.14 ([Woo99, (proof of) 3.19]) acg *implies that sharps exist for all reals.*

Remark 3.1.15 Since acg implies that sharps for all reals exist and $\underset{\sim}{\delta}^1_2 = \omega_2$, it follows by [SW98], that ZFC+acg has at least consistency strength of ZFC$+\exists\lambda\exists\kappa > \lambda : \lambda$ is $<\kappa$-strong and κ is inaccessible. In fact it can be shown that n strong cardinals exist in some inner model for all $n \in \omega$. Unpublished work of Steel and Welch indicates that even κ-many strong cardinals exist in some inner model, for every cardinal κ. It is frequently conjectured that $\underset{\sim}{\delta}^1_2 = \omega_2$ and the existence of sharps for all reals has the consistency strength of one Woodin cardinal.

We now present a simplified proof of the fact that acg implies that sharps for all reals exists. Because the previous proof also gives information on the generic iterability of countable substructures of H_{ω_2}, we decided to present both.
First we show two lemmata about the Club Bounding Principle.

Lemma 3.1.16 (Folklore) *If* CBP *and $j : V \to N$ is any generic ultrapower (not necessarily well-founded), then each ordinal $< j(\omega_1)$ is represented by a canonical function.*

Proof. Let $f : \omega_1 \to \omega_1$. By CBP there is some $\eta < \omega_2$ such that $f_\eta > f$ on a club. The relation $<_S$ is well-founded for all stationary S and $||f_\eta||_S = \eta$. Hence

$$\mathbf{1}_{\mathcal{P}(\omega_1)\setminus\mathsf{NS}_{\omega_1}} \Vdash [f_\eta] = \eta \wedge [f] < [f_\eta].$$

But then for some $S \in G$ and some $\eta' < \eta$

$$S \Vdash [f] = \eta'.$$

Since $||f_{\eta'}||_S = \eta'$, we have

$$S \Vdash [f] = [f_{\eta'}].$$

□

The following is an immediate consequence of the previous lemma.

Lemma 3.1.17 (Folklore) *If* CBP, *then $\omega_2^V + 1 \subset$ wfp(N), where $N =$ Ult(V, G) and $G \subset \mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}$ is generic for V. Also $j(\omega_1) = \omega_2^V$, where j is the canonical generic ultrapower map $j : V \to N$.*

Lemma 3.1.18 *Let $a \subset \omega$. Then* CBP *implies that ω_1 is inaccessible in $L[a]$.*

Proof. Assume ω_1 was not inaccessible in some $L[a]$. So there is a countable ordinal α such that the following is true in V

$$\forall \beta < \omega_1 \exists \gamma < \omega_1 : L_\gamma[a] \models \text{ there is a surjection from } \alpha \text{ to } \beta.$$

Let $j : V \to N$ be any generic ultrapower. Then by elementarity the following holds in N

$$\forall \beta < j(\omega_1) \exists \gamma < j(\omega_1) : L_\gamma[a] \models \text{ there is a surjection from } \alpha \text{ to } \beta,$$

since $j(a) = a$. By the previous Lemma $\omega_2^V + 1 \subset \mathrm{wfp}(N)$, so setting $\beta = \omega_1^V$ we get some γ in the well-founded part of N such that $L_\gamma[a]$ contains a surjection from α to ω_1^V. But this is absurd, since $L_\gamma[a]$ is absolute between V and N. □

The following theorem is due to Silver and a (simplified) proof due to Paris was published by Harrington in [Har78].

Theorem 3.1.19 *Let $a \subset \omega$. Assume that all a-admissibles are L-cardinals. Then $0^\sharp$ exists in $L[a]$.*

It is straightforward to generalize the previous theorem to arbitrary sharps, so we do not give a proof of the following theorem.

Theorem 3.1.20 *Let $a, b \subset \omega$. Assume that all a-admissibles are $L[b]$-cardinals and that $b \in L[a]$. Then $b^\sharp$ exists in $L[a]$.*

Corollary 3.1.21 acg *implies that sharps exist for all reals.*

Proof. Fix an arbitrary real b. Since acg $\implies$ CBP we have that $C := \mathrm{Card}^{L[b]} \cap \omega_1$ is club in ω_1 by Lemma 3.1.18. By acg there is a real a that guesses C. Furthermore without loss of generality we can assume that $b \in L[a]$. Hence all a-admissibles are $L[b]$-cardinals. So $b^\sharp$ exists in $L[a]$ by the previous theorem. Since $b^\sharp$ is absolute, we are done. □

Remark 3.1.22 If acg then there are at least ω_2 many reals, see the proof 3.1.9. Nevertheless acg does not decide the size of the continuum, by the following argument: add many Cohen reals, say ω_3 many to a model of acg. Any club of the forcing extension has a subset which is a club in the ground model. Hence acg still holds in the forcing extension! This shows that relative to large cardinal assumptions, the continuum can be arbitrarily large and CBP can hold at the same time. Moreover this also implies that CBP does not bound the size of the continuum. In fact by [LS03], CBP + CH is consistent relative to some large cardinal.

An upper bound for the consistency strength of acg is the following.

Remark 3.1.23 If

$$\mathsf{ZFC} + \exists \delta \exists \kappa : \delta < \kappa \text{ and } \delta \text{ is Woodin and } \kappa \text{ is measurable}$$

is consistent then so is

$$\mathsf{ZFC} + \mathsf{acg}.$$

Proof. Let κ and δ be as above. Shelah showed that there is a forcing extension $V^{\mathbb{P}}$ of V such that NS_{ω_1} is saturated in $V^{\mathbb{P}}$. Since $\mathbb{P} \in V_\kappa$, κ is still measurable in $V^{\mathbb{P}}$. Hence $\mathcal{P}(\omega_1)^\sharp$ exists in $V^{\mathbb{P}}$. By [Woo99, 3.17] and [Woo99, 3.16] it follows that acg holds in $V^{\mathbb{P}}$. □

Clearly the upper bound given by the remark is suboptimal. We did not really need the measurable, an adequate sharp would have done. Also note the following interesting fact. If NS_{ω_1} is saturated, then it need not be the case that $\underset{\sim}{\delta}^1_2 = \omega_2$; Woodin, starting with one Woodin cardinal, has constructed a model of NS_{ω_1} is saturated and $\underset{\sim}{\delta}^1_2 \neq \omega_2$, see [Woo99, 3.28].

3.1.3 The consistency of acg

Claverie and Schindler have shown:

Theorem 3.1.24 ([CS09]) $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ *is precipitous* $\implies$ $\underset{\sim}{\delta}^1_2 = \omega_2$.

We will now refine the above theorem by showing the following.

Lemma 3.1.25 $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ *is precipitous* $\implies$ acg.

Proof. Fix some club C. We have to show that C is guessed in the sense of acg. Since NS_{ω_1} is precipitous we can construct $\mathbb{P}'(\omega_2, \mathsf{NS}_{\omega_1})$, see Theorem 2.4.9 or see [CS09] for the original construction. $\mathbb{P}'(\omega_2, \mathsf{NS}_{\omega_1})$ adds a countable generically iterable M_0 generically iterating in ω_1^V many steps to $\langle (H^V_{\omega_2})^\sharp; \in, \mathsf{NS}_{\omega_1}\rangle$, i.e. a generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle,$$

such that $\langle M_{\omega_1}; I_{\omega_1}\rangle = \langle (H^V_{\omega_2})^\sharp; \mathsf{NS}_{\omega_1}\rangle$. For brevity we write π_α instead of π_{α,ω_1}. So there is some $\alpha_0 < \omega_1$ such that $C \cap \omega_1^{M_{\alpha_0}} \in M_{\alpha_0}$ and $\pi_{\alpha_0}(C \cap \omega_1^{M_{\alpha_0}}) = C$. We can assume w.l.o.g. by changing some indices that $0 = \alpha_0$. We now show that in the extension by $\mathbb{P}'(\omega_2, \mathsf{NS}_{\omega_1})$ there is a real y such that $A_y \subset C$. Let x be a real that codes M_0 and let y code $x^\sharp$.
Writing $C_\alpha = C \cap \omega_1^{M_\alpha}$ we have $C_\alpha \in M_\alpha$ and $\pi_\alpha(C_\alpha) = C$ for all $\alpha < \omega_1$. By elementarity, C_α is unbounded in $\omega_1^{M_\alpha}$. So by the closedness of C we have $\omega_1^{M_\alpha} \in C$.

Claim 1. If α is an x-indiscernible and

$$\langle\langle M'_i, \pi'_{i,j}, I'_i, \kappa'_i; i \leq j \leq \alpha\rangle, \langle G'_i; i < \alpha\rangle\rangle$$

is an *arbitrary* generic iteration of $M = {M_0}'$ then $\alpha = \omega_1^{M'_\alpha}$.

Proof of Claim 1. First note that M is generically $\omega_1 + 1$ iterable by 2.4.9. Fix an x-indiscernible α and an iteration as above. Every x-indiscernible is inaccessible in $L[x]$, so for all $\beta < \alpha$

$$L[x]^{\mathrm{Col}(\omega,\beta)} \models \alpha \text{ is inaccessible.}$$

Let $g \subset \mathrm{Col}(\omega, \beta)$ be $L[x]$-generic. Assume w.l.o.g. that g is a real. Then, by [Woo99, 3.15] (compare Lemma 19 in [CS09]), $M'_\beta \cap OR < \omega_1^{L[x,g]}$. Hence $\omega_1^{M'_\beta} < \alpha$. This implies $\omega_1^{M'_\alpha} \leq \alpha$. So it follows easily that $\omega_1^{M'_\alpha} = \alpha$. □(Claim 1)

If α is $x^\sharp$-admissible, then α is x-indiscernible. Hence by the above claim it follows that each y-admissible $< \omega_1$ is in C. Hence $A_{x^\sharp} \subset C$. Since the existence of a real y such that $A_y \subset C$ can be recast as a Σ_1-statement over H_{ω_2} with C as a parameter, BMM implies that it is already true in V. □

3.2 The Sentences ϕ_{AC} and ψ_{AC}

We will now discuss the combinatorial principles ϕ_{AC} and ψ_{AC}. Both are Π_2 statements over H_{ω_2} and both will be shown to follow from $\mathsf{BMM} + \mathsf{NS}_{\omega_1}$ is precipitous. This was known for ψ_{AC}, but not for ϕ_{AC}.

3.2.1 Definition of ϕ_{AC}

We recall the Tilde Operation: Let $S \subset \omega_1$. Then we set

$$\tilde{S} := \{\alpha < \omega_2 \,;\, \omega_1 \leq \alpha \wedge \mathbf{1}_{\mathbb{B}} \Vdash \check{\alpha} \in j(\check{S})\},$$

where $\mathbb{B} = \mathrm{ro}(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1})$ and j is a name for the corresponding generic elementary embedding $\langle V; \in\rangle \to \langle M, E\rangle \subset V^{\mathbb{B}}$. Note that $\alpha \in \tilde{S}$ if and only if for all (equivalently one) canonical function f_α for α, there is a club C such that if $\beta \in C$ then $f_\alpha(\beta) \in S$.

Definition 3.2.1 (Woodin) Let $\vec{S} = \langle S_i \,;\, i \in \omega\rangle$, $\vec{T} = \langle T_i \,;\, i \in \omega\rangle$ be sequences of pairwise disjoint subsets of ω_1, such that all S_i are stationary and

$$\omega_1 = \bigcup\{T_i \,;\, i \in \omega\}.$$

$\varphi_{AC}(\vec{S}, \vec{T})$ is the conjunction of the following two statements:

1. There is an ω_1 sequence of distinct reals.[2]
2. There is $\gamma < \omega_2$ and a continuous increasing function $F : \omega_1 \to \gamma$ with range cofinal in γ such that for all $i \in \omega$

$$F``T_i \subset \tilde{S}_i.$$

$\varphi_{AC}(\vec{S}, \vec{T})$ is clearly $\Sigma_1(\{\vec{S}, \vec{T}\})$ in $\langle H_{\omega_2}; \in\rangle$. We set

$$\phi_{AC} :\equiv \forall\vec{S}\forall\vec{T}\varphi_{AC}(\vec{S}, \vec{T}).$$

Note that ϕ_{AC} is equivalent to a Π_2 statement in $\langle H_{\omega_2}; \in\rangle$.

By [Woo99, 5.9] MM implies ϕ_{AC}. Note that by an observation of Larson $\mathsf{MM}(c)$ already suffices, see [Woo99, p.200].
Our plan is as follows: we modify the forcing $\mathbb{P}'(\omega_2, \mathsf{NS}_{\omega_1})$ from [CS09] to show an arbitrary instance of ϕ_{AC} in the generic extension. An application of BMM will then give us the desired result. We need to prepare the proof a little:

3.2.2 Hitting many cardinals

The following lemma states that for a generically iterable $\langle M; I\rangle$ there is a generic iteration that realizes many cardinals.

Lemma 3.2.2 (Hitting many cardinals lemma) *Let $\langle M; I\rangle$ be a countable model of ZFC^*+ "ω_1 exists" and let I be a precipitous ideal on ω_1^M. Assume that $\mathcal{P}(\mathcal{P}(\omega_1))$ exists in M. Let $\alpha \in M$ be such that*

$$M \models 2^{2^{\omega_1}} = \aleph_\alpha,$$

[2] We are working in models of ZFC so this will trivially hold. It is more interesting if working in models of $\mathsf{ZF} + \mathsf{DC}$.

furthermore assume that

$$M \models (\aleph_{\alpha+\omega_1})^M \text{ exists.}$$

Let $\theta := (\aleph_{\alpha+\omega_1})^M$. *Then a generic iteration*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \theta\rangle, \langle G_i; i < \theta\rangle\rangle$$

of $\langle M_0; I_0\rangle = \langle M, I\rangle$ *exists such that for all* $0 < \beta < \omega_1^M$

$$\pi_{0,\aleph^M_{\alpha+\beta}}(\omega_1^M) = \aleph^M_{\alpha+\beta}.$$

Proof. First note the following fact: it suffices to construct a generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \theta\rangle, \langle G_i; i < \theta\rangle\rangle$$

such that for all $\beta < \omega_1^M$

$$\pi_{0,\aleph^M_{\alpha+\beta+1}}(\omega_1^M) = \aleph^M_{\alpha+\beta+1}.$$

To see this let $0 < \lambda < \omega_1^M$ a limit ordinal: then

$$\pi_{0,\aleph^M_{\alpha+\lambda}}(\omega_1^M) = \sup\{\pi_{0,\aleph^M_{\alpha+\beta+1}}(\omega_1^M)\,;\, \beta < \lambda\}.$$

So we will focus on the successor cardinals $< (\aleph_{\alpha+\omega_1})^M$ of M. Let $g \subset \mathrm{Col}(\omega, < \theta)$ be generic over M. Since M is countable in V the generic g can be chosen in V. Let $\mathbb{P} := \mathcal{P}(\omega_1^M)^M \setminus I$. For $\beta < \omega_1^M$ we set

$$g_{\alpha+\beta+1} := g \cap \mathrm{Col}(\omega, < \aleph^M_{\alpha+\beta+1}).$$

Clearly all the g_i defined in this fashion are generic over M. Recursively we construct a generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \theta\rangle, \langle G_i; i < \theta\rangle\rangle$$

such that for $\beta < \omega_1^M$ the sequence $\langle G_i; i < \aleph^M_{\alpha+\beta+1}\rangle$ is in $M[g_{\alpha+\beta+1}]$. We inductively maintain the following:

- For $\beta < \omega_1^M$ and $i < \aleph^M_{\alpha+\beta+1}$ the set

  $$D_i = \{d \in M_i\,;\, d \subset \pi_{0,i}(\mathbb{P}) \wedge M_i \models d \text{ is dense in } \pi_{0,i}(\mathbb{P})\}$$

  is countable in $M[g_{\alpha+\beta+1}]$.

Set $M_0 = M$, $I_0 = I$ and $\kappa_0 = \omega_1^M$. Assume we are at stage $i < \theta$ of the construction. Let $\beta < \omega_1^M$ be least such that $i < \aleph^M_{\alpha+\beta+1}$. Inductively we have that D_i is countable in $M[g_{\alpha+\beta+1}]$. Choose a D_i-generic G_i in $M[g_{\alpha+\beta+1}]$. At limit stages form direct limits.

Let us check our inductive hypotheses in the successor case, the limit case being an easy consequence of the fact that the sequence $\langle G_i; i < \aleph^M_{\alpha+\beta+1}\rangle$ is in $M[g_{\alpha+\beta+1}]$. For the successor case note that an appropriate hull of

$$\pi_{0,i+1}``(H_\theta)^{M_0} \cup \{\kappa_j\,;\, j < i+1\}$$

is $(H_{\theta_{i+1}})^{M_{i+1}}$ where $\theta_{i+1} = \pi_{0,i+1}(\theta)$. This hull can be calculated in $M[g_{\alpha+\beta+1}]$. Hence $D_{i+1} \subset (H_{\theta_{i+1}})^{M_{i+1}}$ is also countable in $M[g_{\alpha+\beta+1}]$. It is trivial to maintain that the sequence $\langle G_j; j < i+1\rangle$ is in $M[g_{\alpha+\beta+1}]$.

Now we need that $\aleph^M_{\alpha+\beta+1}$ is regular in M. Hence

$$\omega_1^{M[g_{\alpha+\beta+1}]} = \aleph^M_{\alpha+\beta+1}.$$

So an easy calculation shows that for all $\beta < \omega_1^M$

$$\pi_{0,\aleph^M_{\alpha+\beta+1}}(\omega_1^M) = \aleph^M_{\alpha+\beta+1}.$$

□

Clearly the previous lemma can be generalized further. Since we only need the case above, we refrained to state it in a more general fashion. Note that we have a lot of freedom when choosing the generics of the iteration; the only true restriction is that they come from small generic extensions. We will make use of this later.
We define a set of ordinals relative to a generic iteration. This set will come in handy in the proof of the main result of this section.

Definition 3.2.3 Let $\langle M; I\rangle$ be a countable model of ZFC^* + "ω_1 exists," such that $M \models I$ is precipitous. Let θ be a cardinal in M. Let

$$\mathcal{J} := \langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \rho\rangle, \langle G_i; i < \rho\rangle\rangle$$

be a generic iteration of $\langle M_0; I_0\rangle = \langle M; I\rangle$. We inductively define the *important ordinals of $\mathcal{J}$ relative to θ.*

1. 0 is an important ordinal.
2. If α is an important ordinal, then the least ordinal γ such that $\gamma = \kappa_\gamma$ and $\pi_{0,\alpha}(\theta) \leq \gamma$ is the next important ordinal.
3. Limits of important ordinals are important.

Remark 3.2.4 Let $\langle M; I\rangle$ be countable and as in the previous definition and let $\mathcal{J}$ be as in the previous definition and $\rho = \omega_1$. Then clearly the set of important ordinals of $\mathcal{J}$ relative to θ is a club in ω_1. Also, if $\alpha > 0$ is important, then $\kappa_\alpha = \alpha$.

3.2.3 Obtaining ϕ_{AC}

We will show the following theorem:

Theorem 3.2.5 *Suppose $2^{\omega_1} = \aleph_2$. Let $\aleph_\alpha = (2^{2^{\omega_1}})^+$. Let $\theta := \aleph_{\alpha+\omega_1}$. Let NS_{ω_1} be precipitous and suppose $H_\theta^\sharp$ exists. Let $F : \omega_1 \to \theta$ defined by*

$$F(\beta) = \aleph_{\alpha+\beta}.$$

Let $\vec{S} = \langle S_k\,;\, k \in \omega\rangle$, $\vec{T} = \langle T_k\,;\, k \in \omega\rangle$ be sequences of pairwise disjoint subsets of ω_1, such that all S_k are stationary and $\omega_1 = \bigcup\{T_k\,;\, k \in \omega\}$. There exists a forcing construction $\mathbb{P} = \mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, \vec{S}, \vec{T})$ that preserves stationary subsets of ω_1 such that if G is $\mathbb{P}$-generic over V, then in $V[G]$ there is generic iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that if $i < \omega_1$, then M_i is countable and $M_{\omega_1} = \langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}\rangle$. Moreover M_0 is generically ω_1-iterable. Additionally the following holds in $V[G]$ for all $k \in \omega$:

$$F``T_k \subset \tilde{S}_k.$$

We use a similar setup as in Theorem 2.4.8, i.e. we assume:

$$\theta = 2^{<\theta} < 2^\theta < \rho = 2^{<\rho},$$

for some cardinal ρ. For reasons of convenience we like to think of $\aleph_\alpha = (2^{2^{\omega_1}})^+$ as $\aleph_4$. This eases notation considerably. Note that we can force $\aleph_4 = (2^{2^{\omega_1}})^+$ with stationary set preserving forcing. Since $2^{\omega_1} = \aleph_2$, the precipitousness of NS_{ω_1} is preserved by forcing with $\mathrm{Col}(\omega_3, (2^{2^{\omega_1}}))$, since no new subsets of 2^{ω_1} are added, see [Jec03, 22.19]. Nevertheless the reader can verify that all of the following arguments

go through for an arbitrary $\aleph_\alpha$ instead of $\aleph_4$. If $\aleph_\alpha = \aleph_4$, then clearly $\theta = \aleph_{\omega_1}$.
At this point a remark is in order. In Theorem 2.4.8 θ is supposed to be regular. Nevertheless it is straightforward to check that if one can add generic iterations with last model H_η for arbitrarily large regular η you can also add generic iterations with last model H_θ by restricting the generic iteration with larger last model. We can hence work with a singular θ and use the theory of $\mathcal{L}$-forcings we have developed.
Fix a well-order $<$ of H_ρ such that $<\restriction H_\theta^\sharp$ is a well-ordering of $H_\theta^\sharp$ of ordertype $\mathsf{OR} \cap H_\theta^\sharp$. We now fix $\vec{S} = \langle S_k\,;\, k \in \omega\rangle$, $\vec{T} = \langle T_k\,;\, k \in \omega\rangle$ sequences of pairwise disjoint subsets of ω_1, such that all S_k are stationary and $\omega_1 = \bigcup\{T_k\,;\, k \in \omega\}$. We use

$$\mathcal{H} = \langle H_\rho; \in, H_\theta^\sharp, \mathsf{NS}_{\omega_1}, <\rangle$$

and

$$\mathcal{M} = \langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}, <\rangle$$

since we are defining a variant of $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1})$. We will now define our modified forcing construction $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, \vec{S}, \vec{T})$. For this we need a collection of statements in the language of set theory augmented by two constants $\dot\pi$, $\dot M$. It is convenient, but not necessary, to add further constants to the language we are working with: we add $\dot{\mathcal{J}}$, $\dot{\vec{G}}$ and $\dot D$.

Definition 3.2.6 By Φ we denote the collection of statements in the language of set theory augmented by the constants $\dot\pi$, $\dot M$, $\dot{\mathcal{J}}$, $\dot{\vec{G}}$ and $\dot D$ that contains:

1. "S is stationary in ω_1" for every $S \in \mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}$,
2. "$\dot{\vec{G}} = \langle G_i; i < \omega_1\rangle$ is a sequence of $(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1})^{\dot M_i}$-generics over $\dot M_i$",
3. "$\dot{\mathcal{J}}$ is a generic iteration

$$\langle\langle \dot M_i, \dot\pi_{i,j}, \mathsf{NS}_{\omega_1}^{\dot M_i}, \omega_1^{\dot M_i}; i \le j \le \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle,$$

and $\dot\pi_{i(i+1)} : \dot M_i \to \dot M_{i+1} \simeq \mathrm{Ult}(\dot M_i, G_i)$",

4. "$\dot D$ is the set of important ordinals of $\dot{\mathcal{J}}$ relative to $(\dot\pi_{0,\omega_1})^{-1}(\theta)$"
5. "If $\gamma \in \dot D$, then for all $\beta < \gamma = \omega_1^{\dot M_\gamma}$ and all $k \in \omega$.

$$\aleph_{4+\beta}^{\dot M_\gamma} \in S_k \iff \beta \in T_k."$$

We set $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, \vec{S}, \vec{T}) = \mathbb{P}_\Phi$.

Note that the only difference between the above forcing and $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1})$ is the requirement that a witness for a single instance of ϕ_{AC} is coded into the generic iteration.
We now show Theorem 3.2.5. First we show that $\mathbb{P}_\Phi \neq \emptyset$; i.e. the consistency of Φ.

Lemma 3.2.7 $\mathbb{P}_\Phi \neq \emptyset$.

Proof. We need to verify, that in $V^{\mathrm{Col}(\omega, 2^\theta)}$ there is a model which certifies the trivial condition with respect to $\mathcal{M}$. Let g be $\mathrm{Col}(\omega, <\rho)$-generic over V. We work in $V[g]$ until further notice. So $\langle V; \in, \mathsf{NS}_{\omega_1}\rangle$ is $\rho+1$ iterable, by [Woo99, 3.10, 3.11]. Hence $\langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}\rangle$ is also $\rho+1$ iterable. We prepare a book-keeping device: pick a bijection $g : [\rho]^{<\rho} \to \rho$ and a family $\langle U_\nu, \nu < \rho\rangle$ of pairwise disjoint stationary subsets of ρ. Now define $f : \rho \to [\rho]^{<\rho}$ by

$$f(i) = u \iff i \in U_{g(u)}.$$

Note that each u is enumerated stationarily often. We recursively construct a generic iteration

$$\mathcal{J} := \langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \rho\rangle, \langle G_i; i < \rho\rangle\rangle$$

of $M_0 = \langle V; \in, \mathsf{NS}_{\omega_1}\rangle$ together with a set of local generics g_i. Later the restriction of this iteration to $\langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}\rangle$ will be of interest. For each important ordinal i of the iteration a local generic g_i will be picked. Suppose we have already constructed $\mathcal{J}$ to some $i < \rho$. Note that we can calculate the important ordinals of $\mathcal{J}$ relative to θ while we construct $\mathcal{J}$. The following three clauses define the iteration.

1. If i is an important ordinal of $\mathcal{J}$ relative to θ, we continue the iteration as follows: $g_i := g \cap \mathrm{Col}(\omega, < \pi_{0,i}(\theta)) \in V[g]$ is generic over M_i. Then pick G_i in $M_i[g_i]$ with the following property: if for a (unique) j the set $\pi_{j,i}(f(i))$ is a stationary subset of $\omega_1^{M_i}$ in M_i, then $\pi_{j,i}(f(i)) \in G_i$. Note that j is unique because $f(i)$ can only be stationary in M_j if $\sup f(i) = \omega_1^{M_j}$.

2. If i is not important, γ is the largest important ordinal below i and $i = \omega_{4+\beta}^{M_\gamma}$ for some $\beta < \kappa_\gamma = \gamma$, we continue the iteration as follows: we already have fixed $g_\gamma \subset \mathrm{Col}(\omega, < \pi_{0,\gamma}(\theta))$ in $V[g]$ that is generic over M_γ. We pick some G_i in $M_\gamma[g_\gamma \cap \mathrm{Col}(\omega, < \omega_{4+\beta+1}^{M_\gamma})]$ such that

$$\beta \in \pi_{0,\gamma}(T_k) \iff \pi_{0,i}(S_k) \in G_i.$$

   Note that since $\vec{T}$ is a partition of ω_1, there is a unique k such that $\beta \in \pi_{0,i}(T_k)$.

3. If the first and second clause do not hold and γ is the largest important ordinal below i, we continue the iteration as follows: we already have fixed $g_\gamma \subset \mathrm{Col}(\omega, < \pi_{0,\gamma}(\theta))$ in $V[g]$ that is generic over M_γ. In the case that $i < \pi_{0,\gamma}(\theta)$ in M_γ, i is not a cardinal in M_γ, hence there is a least $\beta < \kappa_\gamma$ such that $i < \omega_{3+\beta+1}^{M_\gamma}$. We pick some arbitrary G_i in $M_\gamma[g_\gamma \cap \mathrm{Col}(\omega, < \omega_{3+\beta+1}^{M_\gamma})]$. Else we pick a completely arbitrary generic.

Fix some important $\gamma > 0$. So $\mathcal{J}$ restricted to $[\gamma, \pi_{0,\gamma}(\theta)[$ is an iteration like in the "Hitting many cardinals lemma" 3.2.2. Hence we know that the iteration is well-defined and additionally we have for $\beta < \kappa_\gamma = \gamma$ and $i := \aleph_{4+\beta}^{M_\gamma}$

$$i = \pi_{\gamma,i}(\kappa_\gamma) = \kappa_i.$$

By the second clause of the iteration we hence have for i as above and $k \in \omega$:

$$\beta \in \pi_{0,\gamma}(T_k) \iff \pi_{0,i}(S_k) \in G_i \iff \kappa_i \in \pi_{0,i+1}(S_k) \iff i \in \pi_{0,\rho}(S_k).$$

Let D denote the club of important ordinals and let U be a stationary subset of $\omega_1^{M_\rho} = \rho$. Let $j < \rho$ and u be such that $\pi_{j,\rho}(u) = U$. If $i \in D \setminus j$ and $f(i) = u$, then $\pi_{j,i}(u) \in G_i$. This shows that

$$D \cap U_{g(u)} \setminus j \subset \{i < \rho\, ;\, \kappa_i \in U\},$$

so that in fact U is stationary in $V[g]$.
Hence in $M_\rho^{\mathrm{Col}(\omega, \pi_{0,\rho(\theta)})}$ there is a model that certifies the empty condition with respect to $\pi_{0,\rho}(\langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}\rangle)$. Now we can literally complete our proof by following the last paragraph of the proof of 2.4.2. □

Clearly $\mathbb{P}_\Phi$ is resectionable. Since the Φ from Definition 2.4.1 is contained in our current Φ we have:

Lemma 3.2.8 *Let $G \subset \mathbb{P}$ be V-generic. Then in $V[G]$ there is a generic iteration*

$$\mathcal{J}_G := \langle\langle M_i, \pi_{i,j}, \mathsf{NS}^{M_i}_{\omega_1}, \kappa_i; i \leq j \leq \omega_1^V\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

of M_0 such that if $i < \omega_1$, then M_i is countable, and $M_{\omega_1} = \langle H^\sharp_\theta; \in, \mathsf{NS}_{\omega_1}\rangle$.

Let D_G denote the important ordinals of $\mathcal{J}_G$ relative to $\pi^{-1}_{0,\omega_1}(\theta)$. We can assume without loss of generality that there are $\vec{s}$, $\vec{t} \in M_0$ such that $\pi_{0,\omega_1}(\langle\vec{s}, \vec{t}\rangle) = \langle\vec{S}, \vec{T}\rangle$.

Lemma 3.2.9 *D_G is club and for all $\gamma \in D_G$ the following holds: if $\beta < \omega_1^{M_\gamma} = \gamma$ then for all $k \in \omega$*

$$\beta \in \pi_{0,\gamma}(t_k) \iff \aleph^{M_\gamma}_{4+\beta} \in \pi_{0,\omega_1}(s_k),$$

which by the choice of $\vec{s}$ and $\vec{t}$ means

$$\beta \in T_k \iff \aleph^{M_\gamma}_{4+\beta} \in S_k.$$

Proof. That D_G is club is obvious.
Claim 1. Let $p \in \mathbb{P}$. Then $p \Vdash \check{\gamma} \in D_{\dot{G}}$ if and only if for all $\mathfrak{A}$ which certify p, $\gamma \in \dot{D}^{\mathfrak{A}}$.

Proof of Claim 1. Fix p such that $p \Vdash \check{\gamma} \in D_{\dot{G}}$ and some structure $\mathfrak{A}$ which certifies p. Towards a contradiction suppose $\gamma \notin D^{\mathfrak{A}}$. Then there is some $\gamma' < \gamma$, $\gamma' \in D^{\mathfrak{A}}$ with

$$(\dot{\pi}^{\mathfrak{A}}_{\gamma',\omega_1})^{-1}(\theta) > \gamma.$$

By Lemma 2.2.1 we can extend p to p' also certified by $\mathfrak{A}$ such that $\gamma' \in \mathrm{dom}(p')$, $\pi^{p'}_{\gamma'}(\gamma') = \omega_1$ and $(\pi^{p'}_{\gamma'})^{-1}(\theta) > \gamma$. Then

$$p' \Vdash \check{\gamma} \notin D_{\dot{G}}.$$

Contradiction! The other direction is easy. $\square$(Claim 1)

Now if $\beta \in \pi_{0,\gamma}(t_k)$ and $\gamma \in D_G$ there is some $p \in G$ with $p \Vdash \check{\gamma} \in D_{\dot{G}}$ and $\beta \in (\pi^p_\gamma)^{-1} \circ \pi^p_0(t_k)$ (Note the following subtlety: π^p_0 is only defined on the ordinals, but using the well ordering $<$ on $H^\sharp_\theta$ we can assume that $\mathrm{dom}(\pi^p_0)$ contains t_k). Let $p' \leq p$ be arbitrary and let $\mathfrak{A}$ certify p'. Then $\aleph^{M^{\mathfrak{A}}_\gamma}_{4+\beta} \in S_k$ by the above claim and the fact that $\mathfrak{A}$ certifies p'. So we may extend p' to p'' making sure

$$p'' \Vdash \aleph^{M_\gamma}_{4+\beta} \in \pi_{0,\omega_1}(s_k).$$

Hence the set of p'' forcing the desired result is dense below p. The other direction is similar. $\square$

By Theorem 2.3.2 it is clear that $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, \vec{S}, \vec{T})$ is stationary set preserving. To finish the proof of 3.2.5 we have to show that in $V[G]$ for all $k \in \omega$

$$F``T_k \subset \tilde{S}_k.$$

For this fix $k \in \omega$ and some $\beta \in T_k$. By 3.2.9 we have for all $\gamma \in D_G \setminus (\beta + 1)$

$$\beta \in T_k \iff \aleph^{M_\gamma}_{4+\beta} \in S_k.$$

Recall that G adds a generic iteration, and hence a continuous chain, of the form

$$\mathcal{J}_G := \langle\langle M_i, \pi_{i,j}, \mathsf{NS}^{M_i}_{\omega_1}, \kappa_i; i \le j \le \omega_1^V\rangle, \langle G_i; i < \omega_1\rangle\rangle;$$

the following lemma is a consequence of this and Lemma 1.6.7.

Lemma 3.2.10 *The function $f : D_G \setminus (\beta + 1) \to \omega_1$*

$$\gamma \mapsto \aleph^{M_\gamma}_{4+\beta}$$

is a canonical function for $\aleph^V_{4+\beta} < \omega_2^{V[G]}$ in $V[G]$.

So the club $D_G \setminus (\beta + 1)$ and f from the previous lemma witness that in $V[G]$

$$\mathbf{1}_{\mathbb{B}} \Vdash \aleph^V_{4+\beta} \in j(S_i),$$

where $\mathbb{B}$ is $\mathrm{ro}(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1})^{V[G]}$ and j is a name for the generic embedding added by forcing with $\mathbb{B}$. Hence $\aleph^V_{4+\beta} \in \tilde{S}_i$. This finishes the proof of 3.2.5.

Observe that the single instance of ϕ_{AC} that holds in $V^{\mathbb{P}'(\theta,\mathsf{NS}_{\omega_1},\vec{S},\vec{T})}$ is a Σ_1 statement in H_{ω_2} in the parameters $\vec{S}$ and $\vec{T}$. Since $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, \vec{S}, \vec{T})$ preserves stationary subsets an application of BMM, noting Theorem 1.3.4 and Theorem 1.3.3, yields the following corollary.

Corollary 3.2.11 *If NS_{ω_1} is precipitous + BMM then ϕ_{AC}.* □

3.2.4 Obtaining ψ_{AC}

Definition 3.2.12 (Woodin) ψ_{AC} : Let $S \subset \omega_1$ and $T \subset \omega_1$ be stationary, costationary sets. Then there exists a canonical function f for some $\eta < \omega_2$ such that for some club $C \subset \omega_1$

$$\{\alpha < \omega_1\,;\, f(\alpha) \in T\} \cap C = S \cap C.$$

Note the following reformulation of the above definition in terms of generic ultrapowers: let j be a name for the ultrapower embedding induced by some generic $G \subset \mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}$: with S, T as above we have

$$\mathbf{1}_{\mathbb{P}(\omega_1)\setminus\mathsf{NS}_{\omega_1}} \Vdash \check{S} \in \dot{G} \iff \eta \in j(T).$$

Woodin has shown:

Theorem 3.2.13 ([Woo99, 10.95]) *If BMM + NS_{ω_1} is precipitous, then ψ_{AC}.*

With the technology from the previous section we developed for ϕ_{AC} it is possible to give a different proof of 3.2.13. Since this is very similar to the section on ϕ_{AC}, we shall only state the required results. The proofs are very similar to the ϕ_{AC} case.

Lemma 3.2.14 (Hitting regular cardinals lemma) *Let $\langle M; I\rangle$ be a countable model of ZFC^* and let I be a precipitous ideal on ω_1^M. Assume that $\mathcal{P}(\mathcal{P}(\omega_1))$ exists in M. Let $\theta \in M$ be such that*

$$M \models \mathrm{Card}(\mathcal{P}(\mathcal{P}(\omega_1)))^+ = \theta,$$

and let $\theta' \ge \theta$ such that θ' is a regular cardinal in M. Then a genericity iteration

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \le j \le \theta'\rangle, \langle G_i; i < \theta'\rangle\rangle$$

of $\langle M_0; I_0\rangle = \langle M, I\rangle$ exists in V such that $\pi_{0,\theta'}(\omega_1^M) = \theta'$. □

We again modify the forcing $\mathbb{P}'(\omega_2, \mathsf{NS}_{\omega_1})$ to show a weak form of ψ_{AC} in the generic extension. An application of BMM will then give us the desired result.

Theorem 3.2.15 *Suppose $2^{\omega_1} = \aleph_2$. Let NS_{ω_1} be precipitous and suppose $H_\theta^\sharp$ exists, where $\theta = (2^{2^{\aleph_1}})^+$. For all S, T stationary and costationary there exists a forcing construction $\mathbb{P} = \mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, S, T)$ that preserves stationary subsets of ω_1, such that if G is $\mathbb{P}$-generic over V, then in $V[G]$ there is generic iteration*

$$\langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that if $i < \omega_1$, then M_i is countable and $M_{\omega_1} = \langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}\rangle$. In particular, M_0 is generically ω_1-iterable. Additionally the following holds in $V[G]$: there is a club $C \subset \omega_1$, such that for all $\alpha \in C$

$$\omega_1^{M_\alpha} \in S \iff \theta_\alpha \in T,$$

where $\theta_\alpha = \pi_{\alpha,\omega_1}^{-1}(\theta)$.

We will now define our modified forcing construction $\mathbb{P} := \mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, S, T)$. For this we need a collection of statements in the language of set theory augmented by two constants $\dot{\pi}$, $\dot{M}$. Again it is convenient, but not necessary, to add further constants to the language we are working with: we add $\dot{\mathcal{J}}$, $\dot{\vec{G}}$ and $\dot{D}$.

Definition 3.2.16 By Φ we denote the collection of statements in the language of set theory augmented by the constants $\dot{\pi}$, $\dot{M}$, $\dot{\mathcal{J}}$, $\dot{\vec{G}}$ and $\dot{D}$ that contains:

1. "S is stationary in ω_1" for every $S \in \mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}$,
2. "$\dot{\vec{G}} = \langle G_i; i < \omega_1\rangle$ is a sequence of $(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1})^{\dot{M}_i}$-generics over $\dot{M}_i$",
3. "$\dot{\mathcal{J}}$ is a generic iteration
$$\langle\langle \dot{M}_i, \dot{\pi}_{i,j}, \mathsf{NS}_{\omega_1}^{\dot{M}_i}, \omega_1^{\dot{M}_i}; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle,$$
and $\dot{\pi}_{i(i+1)} : \dot{M}_i \to \dot{M}_{i+1} \simeq \mathrm{Ult}(\dot{M}_i, G_i)$",
4. "$\dot{D}$ is the set of important ordinals of $\dot{\mathcal{J}}$ relative to $(\dot{\pi}_{0,\omega_1})^{-1}(\theta)$"
5. "If $\gamma \in \dot{D}$
$$\omega_1^{\dot{M}_\alpha} \in S \iff \dot{\pi}_{\alpha,\omega_1}^{-1}(\theta) \in T."$$

We set $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, S, T) = \mathbb{P}_\Phi$.

Applying the "Hitting regular cardinals lemma" 3.2.14 one can show that certifying structures for Φ exist. Hence one has:

Lemma 3.2.17 $\mathbb{P}_\Phi \neq \emptyset$. □

Clearly $\mathbb{P}_\Phi$ is resectionable. Since the Φ from Definition 2.4.1 is contained in our current Φ we have:

Lemma 3.2.18 *Let $G \subset \mathbb{P}_\Phi$ is V-generic. Then in $V[G]$ there is a generic iteration*

$$\mathcal{J}_G := \langle\langle M_i, \pi_{i,j}, \mathsf{NS}_{\omega_1}^{M_i}, \kappa_i; i \leq j \leq \omega_1^V\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

of M_0 such that if $i < \omega_1$, then M_i is countable, and $M_{\omega_1} = \langle H_\theta^\sharp; \in, \mathsf{NS}_{\omega_1}\rangle$. □

We set

$$\theta_i := \pi^{-1}_{i,\omega_1}(\theta),$$

an we let D_G denote the club of limits of important ordinal of $\mathcal{J}$ relative to θ_0. A density argument shows:

Lemma 3.2.19 *D_G is club and for all $i \in D_G$*

$$\omega_1^{M_i} \in S \iff \theta_i \in T.$$

$\square$

By Lemma 1.6.7 the sequence $\langle \theta_i\,;\, i \in D_G \rangle$ is a canonical function for θ in the forcing extension, so we have

$$\mathbf{1}_{\mathbb{P}(\omega_1)\setminus \mathsf{NS}_{\omega_1}} \Vdash \check{S} \in \dot{G} \iff \theta \in j(T).$$

By Theorem 2.3.2 it is clear that $\mathbb{P}'(\theta, \mathsf{NS}_{\omega_1}, S, T)$ is stationary set preserving. Hence Theorem 3.2.15 follows. Again we have an obvious corollary:

Corollary 3.2.20 *If NS_{ω_1} is precipitous + BMM then ψ_{AC}.* $\square$

4 The axiom (†)

4.1 Introduction to (†)

Foreman, Magidor and Shelah isolated the following interesting axiom.

Definition 4.1.1 ([FMS88])

$$(\dagger) :\equiv \text{ Every stationary set preserving forcing is semiproper.}$$

We review some results on the consistency strength of (†):

Remark 4.1.2
1. If κ is supercompact, then $V^{\mathrm{Col}(\omega_1,<\kappa)} \models (\dagger)$. Also SPFA implies (†) and hence SPFA implies MM.
2. (†) implies that $\square_\kappa$ does not hold for any κ.

The failure of $\square_\kappa$ for a singular strong limit κ is known to imply $\mathsf{AD}^{L(\mathbb{R})}$, see [Ste05].

Proof. For the first item, see [FMS88]. For the second we need a short chain of reasoning. Shelah showed, that (†) is equivalent to *Semistationary Reflection* (SSR), see Definition 4.2.1 for a definition of SSR and see [She98, XIII.1.7(5)] for the result. SSR in turn was shown by Sakai to imply the following reflection property of stationary sets, see [Sak08]:

> If $\lambda \geq \omega_2$ is a regular cardinal, then for every stationary $S \subset E^\lambda_\omega = \{\alpha \in \lambda\,;\, \mathrm{cof}(\alpha) = \omega\}$ there is an ordinal $\beta < \lambda$ of cofinality ω_1 such that $S \cap \beta$ is stationary.

The above reflection principle contradicts the existence of a $\square_\kappa$-sequence for $\kappa^+ = \lambda$: assume towards a contradiction that a $\square_\kappa$-sequence $\langle C_\alpha; \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ exists. Let $f : E^{\kappa^+}_\omega \to \kappa^+$ be defined by $f(\alpha) = \mathrm{otp}(C_\alpha)$; so $f(\alpha) < \alpha$ if $\alpha > \kappa$. Hence there is an ordinal $\beta < \kappa$ of countable cofinality such that $S_\beta := \{\alpha \in E^{\kappa^+}_\omega\,;\, f(\alpha) = \beta\}$ is stationary. If S_β reflects to some $\gamma < \kappa^+$ of cofinality $\geq \omega_1$, then $C_\gamma \cap S_\beta$ is unbounded in γ, a contradiction to the fact that S_β has at most one point in common with each C_α. □

4.1.1 Other forms of (†)

Definition 4.1.3 Let Γ be a class of stationary set preserving forcings.

$$(\dagger)_\Gamma :\equiv \text{ every } \mathbb{P} \in \Gamma \text{ is semiproper.}$$

We study three examples of this dagger axiom. The first two examples will be weaker than (†) and the third will be shown to be equivalent to (†). The first example is taken from [She98].

Example 4.1.4 (When Namba is semiproper) Let $\mathbb{P}$ be the set of perfect subtrees of $[\omega_2]^{<\omega}$ ordered by reverse inclusion, i.e. Namba Forcing. Note that perfect here

means that every node of the tree has $\aleph_2$ many extensions. Forcing with $\mathbb{P}$ gives ω_2^V cofinality ω. If CH holds, then $\mathbb{P}$ adds no reals and hence preserves ω_1. Let $\Gamma = \{\mathbb{P}\}$. If κ is measurable, then $V^{\mathrm{Col}(\omega_1,<\kappa)} \models \mathsf{CH} \wedge (\dagger)_\Gamma$, see [She98, XII. 2.8] and [GJM78] respectively. Hence $(\dagger)_\Gamma$ is clearly weaker than (†), also in terms of consistency strength, since by Remark 4.1.2 (†) implies $\mathsf{AD}^{L(\mathbb{R})}$.
Note that by [She98, XII. 2.5] $(\dagger)_\Gamma$ implies $\mathrm{CC}^{**}(\omega_2)$, a principle defined in Definition 4.2.4. In fact they are equivalent, which we will show in 4.3.11.

Example 4.1.5 (The semiproperness of all sealing forcings) Let Γ denote the class of all sealing forcings, i.e. the forcings to seal antichains of $\mathcal{P}(\omega_1)$, see Defintion 2.4.13. We will show that if NS_{ω_1} is saturated, then $(\dagger)_\Gamma$ holds; like in the previous example this shows that $(\dagger)_\Gamma$ is clearly weaker than (†), also in terms consistency strength: to force that NS_{ω_1} is saturated one needs a Woodin cardinal, but by Remark 4.1.2, (†) implies $\mathsf{AD}^{L(\mathbb{R})}$. We need the following concept: A set $\mathcal{A} \subset \mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1}$ is *semiproper* if for any transitive M closed under sequences of length $2^{2^{\omega_1}}$ and for any countable $X \prec M$, $\mathcal{A} \in X$, there is $Y \prec M$ and some $S \in Y \cap \mathcal{A}$ such that $\omega_1 \cap X = \omega_1 \cap Y \in S$. In [FMS88] it is implicitly shown that some antichain $\mathcal{A}$ is semiproper if and only if the sealing forcing $\mathbb{P}_\mathcal{A}$ is semiproper, see the (proof of) [FMS88, Theorem 26]. By (the proof of) [Woo99, 3.12], every maximal antichain is semiproper if NS_{ω_1} is saturated. Hence $(\dagger)_\Gamma$ holds.
On the other hand if $(\dagger)_\Gamma$ holds, then by [FMS88, Theorem 26] NS_{ω_1} is precipitous. Hence $(\dagger)_\Gamma$ has at least the consistency strength of a measurable cardinal, see [JMMP80].

4.2 The Semiproperness of $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$

The third example of a (seemingly) weaker form of (†) requires more work. We will discuss the case that Γ is the class of all $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ for $\theta \geq \omega_2$ and show that $(\dagger)_\Gamma$ is equivalent to (†).
We have seen that $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ preserves stationary subsets of ω_1 provided that NS_{ω_1} is precipitous. The forcing $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ can clearly be semiproper if large cardinals are present, for example if (†) holds. We show that the semiproperness of the forcings $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ implies a generalization of Chang's Conjecture, CC^{**}, which in turn implies the semiproperness of all stationary set preserving forcings.

Definition 4.2.1 ([She98, XIII. 1.5])

- Let x, y be countable. We write $x \sqsubset y$ if $x \cap \omega_1 = y \cap \omega_1$ and $x \subset y$.
- A set $S \subset [W]^\omega$ is *semistationary in* $[W]^\omega$ if $\{y \in [W]^\omega\,;\, \exists x \in S : x \sqsubset y\}$ is stationary in $[W]^\omega$.
- Let $\lambda \geq \omega_2$. We denote by $\mathrm{SSR}([\lambda]^\omega)$ the following principle: For every S semistationary in $[\lambda]^\omega$ there is $W \subset \lambda$, $\mathrm{Card}(W) = \omega_1 \subset W$ and $S \cap [W]^\omega$ is semistationary in $[W]^\omega$.
- If $\mathrm{SSR}([\lambda]^\omega)$ holds for all cardinals $\lambda \geq \omega_2$ then we will say that *Semistationary Reflection* (SSR) holds.

Note that [She98] has a more general notation for the above reflection principles. In [She98] the principle $\mathrm{SSR}([\lambda]^\omega)$ is called $\mathrm{Rss}(\aleph_2, \lambda)$ and SSR is called $\mathrm{Rss}(\aleph_2)$.

Lemma 4.2.2 ([She98, XIII.1.7(3)]) *Semistationary Reflection implies that all stationary set preserving forcings are semiproper.*

Foreman, Magidor and Shelah have shown:

Lemma 4.2.3 ([FMS88, Theorem 26]) *If (†) holds, then NS_{ω_1} is precipitous.*

We will consider a generalization of Chang's Conjecture that we call CC**.

Definition 4.2.4 Let $\lambda \geq \omega_2$. CC$^*(\lambda)$ is the following axiom: There are arbitrarily large regular cardinals $\theta > \lambda$ such that for all well-orderings $<$ of H_θ and for all $a \in [\lambda]^{\omega_1}$ and for all countable $X \prec \langle H_\theta; \in, < \rangle$ there is a countable $Y \prec \langle H_\theta; \in, < \rangle$ such that $X \sqsubset Y$ and there is some $b \in Y \cap [\lambda]^{\omega_1}$ such that $a \subset b$.
CC** holds if CC$^*(\lambda)$ holds for all cardinals $\lambda \geq \omega_2$.

Note that CC$^*(\omega_2)$ implies Todorčević's CC*, i.e. the axiom

CC* :≡ There are arbitrarily large regular cardinals θ such that for all well-orderings $<$ of H_θ and for all countable $X \prec \langle H_\theta; \in, < \rangle$ there is a countable $Y \prec \langle H_\theta; \in, < \rangle$ such that $X \sqsubset Y$ and $X \cap \omega_2 \neq Y \cap \omega_2$.

To see that CC$^*(\omega_2)$ implies CC* pick an X as in CC* and set $a = \omega_1 \cup X \cap \omega_2$. Then CC$^*(\omega_2)$ applied to X, a yields a Y as desired. The axiom CC* was first studied in [Tod93].

Remark 4.2.5 CC* implies Chang's Conjecture.

Proof. We have to show that every model of type (ω_2, ω_1) has an elementary submodel of type (ω_1, ω). For this let $\langle M; A \rangle$ be a model of type (ω_2, ω_1), i.e. $\text{Card}(M) = \omega_2$ and $\text{Card}(A) = \omega_1$. Let θ be large enough such that the implication of CC* holds for H_θ. Inductively we define a sequence $\langle X_i; i \leq \omega_1 \rangle$ of elementary submodels of $\langle H_\theta; \in, < \rangle$ as follows: pick $X_0 \prec \langle H_\theta; \in, < \rangle$ such that $\langle M; A \rangle \in X_0$, at limit stages λ let $X_\lambda = \bigcup \{X_\alpha \,;\, \alpha < \lambda\}$ and at successor stages $\alpha + 1$ apply CC* to obtain some $X_{\alpha+1} \prec \langle H_\theta; \in, < \rangle$ such that $X_\alpha \sqsubset X_{\alpha+1}$ and $X_\alpha \cap \omega_2 \neq X_{\alpha+1} \cap \omega_2$. So $\omega_2 \cap (X_{\alpha+1} \setminus X_\alpha) \neq \emptyset$. Hence X_{ω_1} is a model of cardinality ω_1 such that $X_0 \cap \omega_1 = X_{\omega_1} \cap \omega_1$. Since $X_0 \models \text{Card}(A) = \omega_1$, we have that $A \cap X_0 = A \cap X_{\omega_1}$ is countable. By construction $\text{Card}(X_{\omega_1} \cap \omega_2) = \omega_1$, so $\text{Card}(M \cap X_{\omega_1}) = \omega_1$, hence $\langle M \cap X_{\omega_1}; A \cap X_{\omega_1} \rangle \prec \langle M; A \rangle$ is of type (ω_1, ω). □

The next theorem answers a question of Todorčević who asked Ralf Schindler under what circumstances $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ is semiproper.
We would like to thank Daiske Ikegami for telling us about Lemma 4.2.2 and for explaining that CC* implies a weak version of SSR.

Theorem 4.2.6 *The following are equivalent:*

1. *NS_{ω_1} is precipitous and for all regular $\theta \geq \omega_2$ the partial ordering $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ is semiproper.*

2. *For arbitrarily large $\theta \geq \omega_2$ there is a semiproper partial order $\mathbb{P}$ that adds a generic iteration*

$$\langle \langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1 \rangle, \langle G_i; i < \omega_1 \rangle \rangle.$$

   such that $H_\theta \subset M_{\omega_1}$ and all M_i are countable.

3. *CC***

4. *SSR*

5. (†)

Before we prove the above theorem note that the Namba-like forcing in [KLZ07] is stationary set preserving (cf. [Zap]) and hence $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ is not the only example witnessing the consistency of *2.*

Proof. 1. $\implies$ 2. is trivial and 4. $\implies$ 5. is Lemma 4.2.2.
5. $\implies$ 1. is clear since by 4.2.3, NS_{ω_1} is precipitous in this case and so by [CS09] the forcing $\mathbb{P}(\mathsf{NS}_{\omega_1}, \theta)$ exists for all regular $\theta \leq \omega_2$ and preserves stationary subsets of ω_1.
It remains to show 2. $\implies$ 3. and 3. $\implies$ 4. For the first implication we assume that CC^{**} does not hold and work toward a contradiction. So there is a least cardinal $\lambda_0 \geq \aleph_2$ for which $\mathrm{CC}^*(\lambda_0)$ fails. Since 2. holds there is a least $\theta_0 > \lambda_0$ such that a semiproper $\mathbb{P}$ exists that adds an iteration

$$\mathcal{J} = \langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle$$

such that $H_{\theta_0} \subset M_{\omega_1}$ and all M_i are countable. Let $\theta > \theta_0$ large enough so that a name for an iteration as above and $\mathcal{P}(\mathbb{P})$ are both in H_θ. Let $<$ be some well-ordering of H_θ. Now fix some arbitrary $X \prec \langle H_\theta; \in, <\rangle$ and some $a \in [\lambda_0]^{\omega_1}$. Our aim is now to construct a $Y \prec \langle H_\theta; \in, <\rangle$ like in CC^{**}. For this we first show that it suffices to do so in a generic extension:

Claim 1. If there is some generic extension W of V with $\omega_1^W = \omega_1^V$ that contains some $Y \prec \langle H_\theta; \in, <\rangle$ such that $X \sqsubset Y$ and there is some $b \in Y \cap [\lambda_0]^{\omega_1} \cap V$ such that $a \subset b$, then there is already some $Z \in V$ with $Z \prec \langle H_\theta; \in, <\rangle$, $X \sqsubset Z$ and $b \in Z$.

Proof of Claim 1. If Y is in some generic extension W of V with $\omega_1^W = \omega_1^V$, then by $b \in V$ there is a tree $T \in V$ searching for a countable $Z \prec \langle H_\theta; \in, <\rangle$ such that $b \in Z$ and $X \sqsubset Z$. So T has a branch in W, since this is clearly witnessed by Y. By the absoluteness of wellfoundedness we have a branch through T in V and hence there is some countable $Z \prec \langle H_\theta; \in, <\rangle$ with $X \sqsubset Z$ and $b \in Z$ in V. □(Claim 1)

By the minimality of λ_0 and θ_0 there is in X a semiproper forcing $\mathbb{P}'$ and a $\mathbb{P}'$-name $\dot{\mathcal{J}}$ for a generic iteration with the properties of $\mathcal{J}$. In an abuse of notation let us write $\mathbb{P}$ for $\mathbb{P}'$. Let $G \subset \mathbb{P}$ be generic over V such that G contains an $(X, \mathbb{P})$-semigeneric. By $X[G]$ we mean $\{\sigma^G\,;\, \sigma \in X \cap V^{\mathbb{P}}\}$.

Claim 2. $X[G] \prec H_\theta[G]$.
This claim is part of the folklore. For the readers convenience we give a short
Proof of Claim 2. An induction along the first order formulae will yield the desired result: let ϕ be a formula and let $\sigma \in X$ denote some name such that

$$H_\theta[G] \models \exists y \phi(y, \sigma^G).$$

Then by the fullness of the forcing names we have

$$H_\theta \models \exists\tau \forall p \in \mathbb{P}(p \Vdash \exists y\phi(y, \sigma) \implies p \Vdash \phi(\tau, \sigma)).$$

The above is a statement in the parameters $\mathbb{P}$ and σ, so by elementarity such a τ exists in X. By the inductive hypothesis we have

$$H_\theta[G] \models \phi(\tau^G, \sigma^G) \iff X[G] \models \phi(\tau^G, \sigma^G).$$

□(Claim 2)

By our hypothesis we can force the existence of a generic iteration

$$\dot{\mathcal{J}}^G = \langle\langle M_i, \pi_{i,j}, I_i, \kappa_i; i \leq j \leq \omega_1\rangle, \langle G_i; i < \omega_1\rangle\rangle.$$

with $M_{\omega_1} \supset H_\theta$. So by the regularity of θ we have $a \in M_{\omega_1}$. Note that $X[G]$ can calculate M_0.
We now apply the basic Lemma 1.5.2. In Lemma 1.5.2 we set $\beta = 0$ and $\alpha = \omega_1$, so we have that there is some $f \in M_0$, $f : \kappa_0^n \to M_0$ and $\vec{\xi} = \xi_1, ..., \xi_n < \omega_1$ such that

$$a = \pi_{0,\omega_1}(f)(\vec{\xi}).$$

This f is in $X[G]$. We set

$$b := \bigcup\{\pi_{0,\omega_1}(f)(\vec{\alpha})\,;\, \vec{\alpha} \in \omega_1^n \wedge \pi_{0,\omega_1}(f)(\vec{\alpha}) \in ([H_\theta]^{\omega_1})^V\}.$$

Clearly $a \subset b$ and $\mathrm{Card}(b) = \omega_1$. Since the parameters $\pi_{0,\omega_1}(f), [H_\theta]^{\omega_1}$ used in the definition of b are in V we have that $b \in V$. Also $b \in X[G]$. By the semiproperness of $\mathbb{P}$ and the choice of G we have $X \sqsubset X[G]$. So $X[G]$ witnesses that in some generic extension of V there is some Y as desired. This suffices to show by Claim 1.
We now show that 3. $\implies$ 4. This implication is a slight generalization of [Tod93, Lemma 6]. Let us assume that SSR does not hold and work toward a contradiction, say $\lambda \geq \omega_2$ and a semistationary $S \subset [\lambda]^\omega$ witness that. We set

$$\mathcal{W} := \{W \subset \lambda\,;\, \mathrm{Card}(W) = \omega_1 \subset W\}$$

and

$$T := \{y \in [\lambda]^\omega\,;\, \exists x \in S : x \sqsubset y\}.$$

By the very definition of semistationarity T is stationary. For all $W \in \mathcal{W}$

$$S_W := \{y \in [W]^\omega\,;\, \exists x \in S \cap [W]^\omega : x \sqsubset y\}$$

is nonstationary. For each $W \in \mathcal{W}$ we may hence pick a function

$$f_W : [W]^{<\omega} \to W$$

such that

$$S_W \cap \{x \in [W]^\omega\,;\, f_W\text{``}[x]^{<\omega} \subset x\} = \emptyset.$$

Let $\mathcal{F}$ denote the collection of these f_W. Let $\theta > \lambda$ be regular large enough such that $\mathcal{F}, \mathcal{W}, S, T \in H_\theta$ and such that the implications of $\mathrm{CC}^*(\lambda)$ hold for this θ. Let $<$ be a well-ordering of H_θ. Pick a countable $M \prec \langle H_\theta; \in, <\rangle$ such that $\mathcal{F}, \mathcal{W}, S, T, \lambda \in M$ and

$$M \cap \lambda \in T.$$

Let

$$a := (M \cap \lambda) \cup \omega_1.$$

Since $\mathrm{CC}^*(\lambda)$ holds for θ, there is a countable $M^* \prec H_\theta$ and some $W \in [\lambda]^{\omega_1}$ such that $M \sqsubset M^*$, $a \subset W$ and $W \in M^*$. So $f_W \in M^*$. Then by elementarity of M^*

$$f_W\text{``}[W \cap M^*]^{<\omega} \subset W \cap M^*.$$

By the choice of a and the properties of M^* we have

$$M \cap \lambda \sqsubset W \cap M^*.$$

Since we have $M \cap \lambda \in T$ there is some $x \in S$ such that $x \sqsubset M \cap \lambda$. Note that $x \in [W]^\omega$. By the transitivity of $\sqsubset$,

$$x \sqsubset W \cap M^*.$$

This implies $W \cap M^* \in S_W$. We thus have a contradiction to the choice of f_W. This finishes the proof. $\square$

4.3 RC implies (†)

We show that Rado's Conjecture, a combinatorial principle of large consistency strength, implies that the class of forcings preserving stationary subsets of ω_1 is equal to the class of semiproper forcings. First we will discuss Rado's Conjecture, then we will study a class of cut and choose games $G_\omega([\lambda]^{\omega_1}, \omega_1)$ that will be a key tool for showing that Rado's Conjecture implies (†).
We recall some basic concepts. A tree is *special* if it can be partitioned into ω-many antichains. An *interval* of a linear ordering $\langle A, < \rangle$ is a nonempty $a \subset A$ such that if $x < y < z$ and $x, z \in a$, then $y \in a$. A family of intervals is *σ-disjoint* if it is the union of countably many disjoint subfamilies. If $\langle T, < \rangle$ is a tree, then a *subtree* of $\langle T, < \rangle$ is a subset of T with the inherited tree structure.

Definition and Lemma 4.3.1 ([Tod83, Theorem 6]) *The following are equivalent*

1. *If T is a tree such that all subtrees of T of cardinality $\aleph_1$ are special, then T is special.*
2. *Every family of intervals of a linearly ordered set is σ-disjoint if and only if each of its subfamilies of size $\aleph_1$ is σ-disjoint.*

We refer to these equivalent statements as Rado's Conjecture, RC, *though we will always have the first statement in mind.*

Note that the second statement is the countable case of a conjecture of Rado, see [Tod93].
Let us collect some facts about (non)special trees. A tree $\langle T, < \rangle$ of countable height is always special: if T_α denotes the nodes of T at the αth level, then T_α is an antichain. Also if a tree $\langle T, < \rangle$ has height $> \omega_1$, then it is nonspecial: pick an element of height ω_1 then the subtree of T formed by all $s < t$ has size $\aleph_1$ and clearly can not be partitioned into ω-many antichains. Hence for the purpose of RC it will suffice to study trees of height ω_1. Also note that if we study a subtree U of size $\aleph_1$ of a tree of height $\leq \omega_1$, then we can always close U under initial segments without changing the cardinality of U.
When working with (non)special trees the following lemma is useful. A mapping $f : T \to T$ is called *regressive* if $t \in T \setminus \{\emptyset\}$ implies $f(t) < t$, here $\emptyset$ denotes the root of T which we assume to be always present.

Lemma 4.3.2 (Pressing Down Lemma for Trees, [Tod81]) *Every regressive mapping defined on a nonspecial tree must be constant on a nonspecial subtree.*

Since this result is not part of the standard literature, we prove it here.

Proof. Let $f : T \to T$ be regressive. Let $U_s = \{t \in T\,;\, f(t) = s\}$. It will suffice to show that if all U_s are special, then T is special. Let $g_s : U_s \to \omega$ be a specializing function. Since the tree order is well-founded, for every $t \in T \setminus \{\emptyset\}$ there is some $n \in \omega$ such that applying f n-times yields $\emptyset$. Hence for every $t \in T$ there is a unique sequence $\langle t_0, ..., t_n \rangle$ such that $t_0 = t$, $t_n = \emptyset$ and $t_{i+1} = f(t_i)$ for $i < n$. We define $g : T \setminus \{\emptyset\} \to [\omega]^{<\omega}$ by setting

$$g(t) = \langle g_{t_1}(t_0), ..., g_{t_n}(t_{n-1}) \rangle.$$

Hence if $s, t \in T \setminus \{\emptyset\}$, $s \neq t$ and $g(s) = g(t)$ then the unique sequences $\vec{s}$, $\vec{t}$ have the same length, say $n+1$. So $t_n = s_n = \emptyset$ and hence $g_{s_n}(s_{n-1}) = g_{t_n}(t_{n-1})$. Since $g_\emptyset$ is a specializing function either $s_{n-1} = t_{n-1}$ or s_{n-1} and t_{n-1} are incompatible. If the first case holds we continue in the same fashion until for some $i < n$ s_i

and t_i are incompatible. Hence, modulo a bijection $[\omega]^{<\omega} \to \omega$, g is a specializing function. $\square$

A tree $\langle T, <\rangle$ is *Baire*, if for all countable sequences $(D_n)_n$ of open dense sets in T the intersection $\bigcap_n D_n$ is dense; here a set $D \subset T$ is *dense* if for all $t \in T$ there is some $s \in D$ such that $t < s$ and $D \subset T$ is *open* if for all $t \in D$ and $s \in T$ if $t < s$, then $s \in D$. These notions clearly coincide with the notions for partial orders if one sees T as a partial order with $<$ reversed.

Remark 4.3.3 Every Baire tree is nonspecial.

Proof. We show that every special tree is not Baire. Fix a special tree T. By an application of Zorn's Lemma there is a specializing function $g : T \to \omega$ such that $g(s)$ is minimal for all $s \in T$ in the following sense: if $k < g(s)$, then there is some $t \in T$ such that $g(t) = k$ and $s < t$ or $t < s$. It easily follows that such a g is in fact a partition of T into countably many maximal antichains $(A_n)_{n\in\omega}$ of T. Set $D_n := \{t \in T\,;\, \exists s \in A_n : s \leq t\}$. Hence every D_n is a dense open set. Since $(A_n)_n$ is a partition $\bigcap_n D_n$ is empty. So T is not Baire. $\square$

In [Tod83] and [Tod93] Todorčević analysed RC in detail; we sum up the results of this analysis:

Remark 4.3.4

1. If κ is supercompact, then $V^{\mathrm{Col}(\omega_1,<\kappa)} \models \mathsf{RC} + \mathsf{CH}$.
2. If RC holds, then $\square_\kappa$ fails for all κ.
3. RC implies $2^{\aleph_0} \leq \aleph_2$.
4. If $V \models \mathsf{RC}$ and $W \subset V$ is a transitive inner model such that $\omega_2^W = \omega_2$, then $\mathbb{R}^V \subset W$.
5. MA_{ω_1} implies $\neg\mathsf{RC}$.
6. RC implies CC*.

$\square$

The implication RC $\implies$ CC* was shown in [Tod93]. We will generalize the argument for this from [Tod83] and show the following theorem.

Theorem 4.3.5 RC *implies CC**.*

The proof we are going to give for the above theorem is structured very much like Todorčević's original proof for RC $\implies$ CC* . Since CC** is equivalent to (†) by 4.2.6 the above theorem instantly yields the following corollary.

Corollary 4.3.6 RC *implies* (†). $\square$

For the proof of the above theorem we will need the following cut and choose game:

Definition 4.3.7 Let $\lambda \geq \omega_2$ denote an ordinal. We will call the following game $G_\omega([\lambda]^{\omega_1}, \omega_1)$:

$$\begin{array}{c|cccccc} I & f_0 & & f_1 & & & \\ II & & \delta_0 & & \delta_1 & \cdots & \end{array}$$

In the nth round player I splits $[\lambda]^{\omega_1}$ into ω_1 pieces, i.e. player I plays $f_n : [\lambda]^{\omega_1} \to \omega_1$. Player II responds by choosing some $\delta_n < \omega_1$. The game has ω many rounds. Player II wins a play $f_0, \delta_0, f_1, \delta_1, ...$ if and only if the set

$$\{a \in [\lambda]^{\omega_1} \, ; \, \forall n : f_n(a) < \sup\{\delta_i \, ; \, i \in \omega\}\}$$

is unbounded in $[\lambda]^{\omega_1}$, i.e. for all $b \in [\lambda]^{\omega_1}$ there is some a in the above set such that $b \subset a$.

The class of games above generalizes the game $G_\omega(\omega_2, \omega_1)$ studied in [Tod93] and also in [She98, XII. §2]; one obtains $G_\omega(\omega_2, \omega_1)$ if one replaces $[\lambda]^{\omega_1}$ by ω_2 in the above definition. The game $G_\omega(\omega_2, \omega_1)$ is due to Galvin. In [Tod93] it is shown that player II has a winning strategy in $G_\omega(\omega_2, \omega_1)$. We generalize this as follows:

Lemma 4.3.8 RC *implies that player II has a winning strategy in $G_\omega([\lambda]^{\omega_1}, \omega_1)$ for all $\lambda \geq \omega_2$.*

Proof. The proof of this lemma runs through several claims. Let us fix some $\lambda \geq \omega_2$. Set $F_\lambda := \omega_1^{([\lambda]^{\omega_1})} \cup \omega_1$, here $\omega_1^{([\lambda]^{\omega_1})}$ is the set of all functions $[\lambda]^{\omega_1} \to \omega_1$. For $X \in [F_\lambda]^\omega$, let

$$D_X := \{a \in [\lambda]^{\omega_1} \, ; \, f(a) \in X \text{ for all } f \in X\}.$$

Set

$$S := \{X \in F_\lambda \, ; \, X \cap \omega_1 \in \omega_1 \text{ and } D_X \text{ is bounded in } [\lambda]^{\omega_1}\}.$$

Claim 1. Player II has a winning strategy in $G_\omega([\lambda]^{\omega_1}, \omega_1)$ if and only if S is nonstationary in $[F_\lambda]^\omega$.

Proof of Claim 1. Let us first assume player II has a winning strategy σ in $G_\omega([\lambda]^{\omega_1}, \omega_1)$. Let $\theta > 2^\lambda$ and let $Y \prec H_\theta$ be countable such that $\sigma \in Y$. Set $X := Y \cap F_\lambda$. Let $(f_n)_{n\in\omega}$ enumerate all functions $[\lambda]^{\omega_1} \to \omega_1$ that are in Y. We play $(f_n)_n$ against σ. Since finite initial segments of this play are in Y and $Y \prec H_\lambda$, the responses $(\delta_n)_n$ of II according to σ are all countable in Y; hence for all n: $\delta_n < Y \cap \omega_1 = X \cap \omega_1$. Since σ is a winning strategy

$$\{a \in [\lambda]^{\omega_1} \, ; \, \forall n : f_n(a) < \sup\{\delta_i \, ; \, i \in \omega\}\}$$

is unbounded. Hence D_X, which contains the above set, is unbounded. Since there are club many $Y \prec H_\theta$ such that $\sigma \in Y$, there are also club many X such that $D_X \notin S$.

For the converse direction let C be a club witnessing that S is nonstationary. Player I will play functions f_n. We will choose player II's responses δ_n such that $\delta_n = X_n \cap \omega_1$ for some increasing sequence $(X_n)_n$ of elements of C, i.e. $X_n \subset X_{n+1}$. Suppose $f_0, \delta_0, ..., f_{n-1}, \delta_{n-1}, f_n$ are already played and $X_0, ..., X_{n-1} \in C$ are already picked. Since C is unbounded we find some $X_n \supset X_{n-1}$ such that $X_n \cap \omega_1 \geq \delta_{n-1}$ and $f_n \in X_n$. Let $\delta_n = X_n \cap \omega_1$. Since C is closed $X := \bigcup_n X_n \in C$. So

$$D_X := \{a \in [\lambda]^{\omega_1} \, ; \, f(a) \in X \text{ for all } f \in X\}$$

is unbounded. Hence

$$\{a \in [\lambda]^{\omega_1} \, ; \, \forall n : f_n(a) < \sup\{\delta_i \, ; \, i \in \omega\}\}$$

is also unbounded. □(Claim 1)

If $X, Y \in S$, then we will say that Y *strongly includes* X if $X \subset Y$ and $X \cap \omega_1 < Y \cap \omega_1$; this concept originates from [Tod93] and also the idea for the following tree construction is taken from [Tod93]. Let T be the tree of all countable continuous strong inclusion chains t of elements of S such that $\bigcup t$ is also an element of S. The ordering of T is end-extension. We want to apply RC to see that T is special. For this we have to check the following:

Claim 2. Every subtree of T of size $\aleph_1$ is special.

Proof of Claim 2. Fix some $U \subset T$ of size $\aleph_1$. Without loss of generality we can close U under initial segments. Since $D_{\bigcup t}$ is bounded for all $t \in U$, we can find a $d \in [\lambda]^{\omega_1}$ such that d is a bound for all $D_{\bigcup t}$, $t \in U$. We now define a regressive function H with the limit nodes of U as H's domain. Pick some $t \in U$ of limit length. Hence d is a bound for $D_{\bigcup t}$; so by the definition of $D_{\bigcup t}$ there is $f_t \in \bigcup t$ such that $f_t(d) \geq (\bigcup t) \cap \omega_1$. We can hence find some proper initial segment $H(t)$ of t such that $\bigcup H(t)$ contains f_t. Clearly H is regressive, since $H(t)$ is a proper initial segment of t. So by the Pressing Down Lemma for Trees 4.3.2 it suffices to show that every $H^{-1}``\{s\}$ is special for $s \in U$. For this fix some $f \in \bigcup s$ for some $s \in U$ and set

$$W_f := \{t\,;\, f_t = f \text{ and } H(t) = s\}.$$

Hence if $t \in W_f$, then $f(d) \geq (\bigcup t) \cap \omega_1$. Since t is a strong inclusion chain the length of t is bounded by $(\bigcup t) \cap \omega_1$, so the length of t is bounded by $f(d)$ which is a countable ordinal. It is straightforward to partition W_f: for $\alpha < f(d)$ set $W_f^\alpha = \{t \in W_f\,;\, \mathrm{lh}(t) = \alpha\}$. Since there are only countably many $f \in \bigcup s$, we clearly have that $H^{-1}``\{s\}$ is special. □(Claim 2)

So an application of RC yields that T is special. So it remains to show that player II has a winning strategy in $G_\omega([\lambda]^{\omega_1}, \omega_1)$. By Claim 1 it is enough to show: if S is stationary, then T is nonspecial. We even show that T is Baire. Pick a regular θ such that $\theta > 2^{2^\lambda}$. Fix a sequence $(D_n)_{n\in\omega}$ of dense open subsets of T and pick an arbitrary $t \in T$. Let $X \prec H_\theta$ be countable such that $t, (D_n)_n, S, T, \lambda \in X$ and $X \cap F_\lambda \in S$. Let $(x_n)_{n\in\omega}$ enumerate $X \cap F_\lambda$. We now construct a sequence $(t_n)_{n\in\omega}$ such that $t_n \in T \cap X$, $t_n \in D_n \cap X$, $t < t_0$, $t_n < t_{n+1}$ and $x_n \in t_n$. This is possible by the elementarity of X and the unboundedness of S. Set $t_\omega = \bigcup_n t_n$, then $\bigcup t_\omega = X \cap F_\lambda$, hence $t_\omega \in T$. Since t was arbitrary, we have that $\bigcap D_n$ is dense in T. So T is Baire. This finishes the proof. □

The proof of Theorem 4.3.5 will be completed once we show the next lemma, which generalizes the following implication from [Tod93]: if player II has a winning strategy in $G_\omega(\omega_2, \omega_1)$, then CC* holds.

Lemma 4.3.9 *If player II has a winning strategy in $G_\omega([\lambda]^{\omega_1}, \omega_1)$ for all $\lambda \geq \omega_2$, then CC** holds.*

Proof. Let us assume towards a contradiction that CC** fails; i.e. there is a least λ such that CC$^*(\lambda)$ does not hold. Fix some $\theta > 2^{2^\lambda}$ and some well-ordering $<$ of H_θ. Since λ is least such that CC$^*(\lambda)$ fails, it is definable in H_θ. Fix an arbitrary countable $X \prec H_\theta$ and an arbitrary $a \in [\lambda]^{\omega_1}$. Clearly $\lambda \in X$ and hence a winning strategy σ for player II in $G_\omega([\lambda]^{\omega_1}, \omega_1)$ is also in X. Let $(f_n)_{n\in\omega}$ enumerate $X \cap \{f\,;\, f : [\lambda]^{\omega_1} \to \omega_1\}$. We play $(f_n)_n$ against σ and obtain a sequence $(\delta_n)_{n\in\omega}$ of responses of player II. Each $\delta_n < X \cap \omega_1$, since initial segments of the play are in X. Hence $\delta = \sup_n \delta_n \leq X \cap \omega_1$ (in fact $\delta = X \cap \omega_1$, but we have no use for this fact). Since σ was winning we have that

$$\{b \in [\lambda]^{\omega_1}\,;\, \forall n : f_n(b) < \delta\}$$

is unbounded in $[\lambda]^{\omega_1}$. Pick some b in the set above such that $b \supset a$. Let Y denote the Skolem Hull of $X \cup \{b\}$ in $\langle H_\theta; \in, <\rangle$. It remains to show $Y \cap \omega_1 = X \cap \omega_1$. Since we assume all the Skolem functions to be defined relative to $<$, they are definable in X; furthermore we can assume they are closed under composition. Since b is the only new member of the Hull and we are only interested in $Y \cap \omega_1$, it suffices to look at functions of the form

$$f : H_\theta \times [\lambda]^{\omega_1} \to \omega_1.$$

Fix some such $f \in X$. For each $x \in X$, the function $f(x, \cdot) : [\lambda]^{\omega_1} \to \omega_1$ is a member of X, hence $f = f_n$ for some $n < \omega$. So for all x: $f(x, b) < \delta$. So $X \sqsubset Y$, a contradiction to the failure of $\mathrm{CC}^*(\lambda)$, since X and a were arbitrary. □

Clearly the previous two lemmata show Theorem 4.3.5.
For $\lambda = \omega_2$ the converse of the previous lemma holds.

Lemma 4.3.10 (Folklore) *$CC^*(\omega_2)$ holds if and only if player II has a winning strategy in $G_\omega([\omega_2]^{\omega_1}, \omega_1)$.*

Proof. In the previous lemma we have shown that a winning strategy for player II yields $\mathrm{CC}^*(\omega_2)$. For the converse we construct a strategy in a similar fashion as in the proof of 4.3.8. Let θ be large enough such that the consequences of $\mathrm{CC}^*(\omega_2)$ hold for $\langle H_\theta; \in, <\rangle$, where $<$ is some well-order of H_θ. Together with a run of $G_\omega([\omega_2]^{\omega_1}, \omega_1)$ we construct a sequence $(X_n)_{n\in\omega}$ of substructures of H_θ. Assume $f_0, ..., f_n$ and $\delta_0, ..., \delta_{n-1}$ are already played and $(X_i)_{i<n}$ are already picked such that $X_i \subset X_{i+1}$, $f_i \in X_i$ and $\delta_i = X_i \cap \omega_1$. Then pick a countable $X_n \prec H_\theta$ such that $X_{n-1} \subset X_n$ and $f_n \in X_n$; player II plays $\delta_n = X_n \cap \omega_1$. Now assume towards a contradiction that player II did not win this run; i.e. the set

$$D := \{a \in [\omega_2]^{\omega_1} \,;\, \forall n : f_n(a) < \sup_{i\in\omega} \delta_i\}$$

is bounded in $[\omega_2]^{\omega_1}$. So there is an ordinal $\alpha < \omega_2$ such that every $b \in D$ is contained in α. By $\mathrm{CC}^*(\omega_2)$ there is a $Y \prec H_\theta$ such that $X \sqsubset Y$ and $\alpha + 1 \subset b$ for some $b \in Y \cap [\omega_2]^{\omega_1}$. Hence for every $f \in Y$ such that $f : [\omega_2]^{\omega_1} \to \omega_1$ by elementarity $f(b) < \omega_1 \cap Y = \omega_1 \cap X = \sup_i \delta_i$. Hence the set

$$D_Y := \{c \in [\omega_2]^{\omega_1} \,;\, \forall f \in Y (f : [\omega_2]^{\omega_1} \to \omega_1 \implies f(c) < \sup\{\delta_i \,;\, i \in \omega\})\}$$

contains b. But clearly $D_Y \subset D$, a contradiction to the fact that D is bounded by $\alpha + 1 \subset b$. □

Corollary 4.3.11 (Folklore) *Namba forcing is semiproper if and only if $CC^*(\omega_2)$ holds.*

Proof. By [She98, XII. 2.2] Namba forcing is semiproper if and only if player II has a winning strategy in $G_\omega([\omega_2]^{\omega_1}, \omega_1)$. Note that the game used in [She98, XII. 2.2] is slightly different but is readily seen to be equivalent to ours. □

5 The Extender Algebra and Absoluteness

5.1 Introduction to absoluteness for $L(\mathbb{R})$

In this introductory section we will review some results about forcing absoluteness for $L(\mathbb{R})$. In what follows some concepts of inner model theory appear. All these concepts, especially the concepts of premice and (ω, ω_1+1)-iterability, can be found in [Ste]. We believe that stating the definitions from [Ste] without Steel's enlightening explanations would not help the reader understand what follows. A reader not too familiar with inner model theory is nevertheless encouraged to proceed: coarse iterations in the sense of [MS94] are sufficient to construct the main tool of this chapter, the extender algebra. We believe that many of proofs that follow, if not all, have natural versions in the coarse case.
Woodin has shown that $L(\mathbb{R})$ is Σ^2_1 absolute with respect to forcing extensions of V if large cardinals are present.

Theorem (Woodin) *Suppose $M^\sharp_{\mathsf{mw}}$ exists and is (ω, ω_1+1)-iterable in all set forcing extensions. Assume* CH *holds. Let $\mathbb{P}$ be a notion of forcing and let $G \subset \mathbb{P}$ be V-generic. Let z be a real in V. If in $V[G]$*

$$\exists A \subset \mathbb{R}^{V[G]} L(\mathbb{R}^{V[G]}, A) \models \phi(A, z),$$

then in V

$$\exists A \subset \mathbb{R}^{V} L(\mathbb{R}^{V}, A) \models \phi(A, z).$$

Furthermore if CH *holds in $V^{\mathbb{P}}$, then the converse is true.*

Here $M^\sharp_{\mathsf{mw}}$ is a fine-structural premouse that contains a measurable Woodin cardinal; we will give more details about $M^\sharp_{\mathsf{mw}}$ later. Note that the existence of $M^\sharp_{\mathsf{mw}}$ is not the original hypothesis of the Σ^2_1 absoluteness theorem; Woodin's first proof used class many measurable Woodins. We will give a detailed proof of the above theorem, see Theorem 5.4.1.
It is natural to ask if one can add ordinal parameters to the statement of the above theorem. Neeman and Zapletal showed, granted the large cardinal assumption A_κ, that the theory of $L(\mathbb{R})$ with ordinal parameters is stable under reasonable forcing. In the following theorem A_κ is a large cardinal assumption that follows from the existence and $(\omega, \kappa^+ + 1)$-iterability of $M^\sharp_\omega$.

Theorem (Embedding Theorem [NZ01]) *Assume A_κ holds. Let $\mathbb{P}$ be a reasonable forcing notion of size $\leq \kappa$, and let G be $\mathbb{P}$-generic over V. Then there exists an elementary embedding*

$$j : L(\mathbb{R}^V) \to L(\mathbb{R}^{V[G]})$$

which is the identity on all ordinals.

Note that a version of the Embedding Theorem exists which is shown using the stationary tower; this uses a weakly compact Woodin cardinal, see [NZ98].
Woodin studied a class of forcings larger than the reasonable forcings and obtained

the following result, using a related but different hypothesis. The conclusion of the following theorem is more general than the conclusion of the Embedding Theorem:

Theorem ([Woo99, 10.63]) *Let $\mathbb{P} \in V_\delta$ be a* weakly proper *notion of forcing; i.e. for all ordinals α $([\alpha]^\omega)^V$ is cofinal in $([\alpha]^\omega)^{V^{\mathbb{P}}}$. Assume $A \subset \mathbb{R}$, $L(A, \mathbb{R}) \models \mathsf{AD}$ and every set in $\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is δ-weakly homogeneously Suslin. Let $G \subset \mathbb{P}$ be generic and let*

$$j_G : L(A, \mathbb{R}) \to L(A_G, \mathbb{R}_G)$$

be the associated generic elementary embedding. Then $j_G(\alpha) = \alpha$ for all $\alpha \in \mathsf{OR}$.

Clearly A is $< \delta$-universally Baire in the previous theorem, so it makes sense to consider the natural reinterpretations A_G, $\mathbb{R}_G$ in the above theorem and also j_G is well-defined[1]. Note that Woodin has show relative to large cardinals that there is a semiproper forcing extension $V[G]$ of V such that

$$j_G : L(\mathbb{R}) \to L(\mathbb{R}_G)$$

is not the identity on the ordinals. So one cannot hope to generalize the above theorem to a larger class of forcings.
Another result in this direction due to Woodin is the following theorem published in [Woo05]. We state it with the reduced large cardinal assumption Larson obtained in [Lar04, Theorem 3.4.17].

Theorem (Woodin with stronger hypothesis, Larson) *Let Γ_{uB} denote the class of universally Baire sets. Suppose there is a proper class of Woodin cardinals. Suppose δ is supercompact and $V_{\delta+1}$ is countable in $V[G]$, G set generic over V. Let $V[G][g]$ be any set generic extension of $V[G]$. Then*

1. *$\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \Gamma_{uB})^{V[G]} = \Gamma_{uB}^{V[G]}$,*
2. *$\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \Gamma_{uB})^{V[G][g]} = \Gamma_{uB}^{V[G][g]}$,*
3. *$(\Gamma_{uB}^{V[G]})^\sharp \subset (\Gamma_{uB}^{V[G][g]})^\sharp$, where each set in $\Gamma_{uB}^{V[G]}$ is identified with its reinterpretation in $V[G][g]$.*

The above theorem says that the theory of $L(\mathbb{R}, \Gamma_{uB})$ is sealed with respect to set forcing and hence generalizes the Σ^2_1 absoluteness for $L(\mathbb{R})$.
All the above theorems are shown with modern set theoretic methods. Stationary tower forcing is one way to show Σ^2_1 absoluteness for $L(\mathbb{R})$ and is also used to show the above theorem. The second way to show Σ^2_1 absoluteness for $L(\mathbb{R})$ is the extender algebra; also the Embedding Theorem is shown using the extender algebra. Besides stationary tower forcing and the extender algebra there is yet another method to show Σ^2_1 absoluteness for $L(\mathbb{R})$: Todorčević imitated the stationary tower proof by Levy collapsing measurable Woodin cardinals to ω_2. In such a Levy collapse a ω_2-saturated ideal on ω_1 exists and one can force with this ideal. A detailed proof of Σ^2_1 absoluteness for $L(\mathbb{R})$ hay been published by Farah in [Far07].
In the literature there are other variants and extensions of Σ^2_1 absoluteness for $L(\mathbb{R})$. For example one can enrich the language and add predicates for universally Baire sets of reals; see [FL06] and [FKLM08] for such a result and other extensions of Σ^2_1 absoluteness.

[1] If T is a δ-weakly homogeneously Suslin tree such that $A = p[T]$, then the associated generic embedding j_G is uniquely determined by the following three clauses: (1) $j_G(A) = A_G = p[T]^{V[G]}$, (2) $\mathbb{R}_G = \mathbb{R}^{V[G]}$ and (3) $L(A_G, \mathbb{R}_G) = \{j_G(f)(a)\,;\, a \in \mathbb{R}_G, f : \mathbb{R} \to L(A, \mathbb{R}) \text{ and } f \in L(A, \mathbb{R})\}$

In this chapter the extender algebra will be our main tool. We construct the extender algebra and give a detailed proof of Σ^2_1 absoluteness for $L(\mathbb{R})$.
It is well known that given a real $x \subset \omega$ and a sufficiently iterable structure $\mathcal{M}$ (i.e. a (coarse) premice) that contains a Woodin cardinal one can make x generic over some countable iterate of $\mathcal{M}$. This technique, due to Woodin, is known as *genericity iteration*. The first result we present in this chapter is a generalization of the above technique, also due to Woodin: we explain how to make an arbitrary subset of ω_1 generic over an iterate of a fine structural model containing a measurable Woodin cardinal. We believe that all basic results not otherwise attributed to someone are due to Woodin.
We then look at $\dot{x} \subset \omega_1$ that lives in a c.c.c. generic extension and also construct a genericity iteration for $\dot{x}$. We apply this technique to show:

Theorem (Theorem 5.5.3) *Suppose $M^\sharp_{\mathsf{mw}}$ exists. Assume CH holds. Furthermore assume $\mathbb{P}$ is a c.c.c. forcing of size κ such that $V^{\mathbb{P}} \models \mathsf{CH}$. Let $G \subset \mathbb{P}$ be V-generic. Then*

$$V \models \exists A \subset \mathbb{R} : L(\mathbb{R}, A) \models \phi(A, z, \vec{\alpha})$$

if, and only if,

$$V[G] \models \exists A \subset \mathbb{R}^{V[G]} : L(\mathbb{R}^{V[G]}, A) \models \phi(A, z, \vec{\alpha}).$$

Here z is a real parameter and $\vec{\alpha}$ are finitely many ordinal parameters.

Note that it is not possible to substitute c.c.c. by ω-closed in the statement of the above theorem: let $G \subset \mathbb{P} = \mathrm{Col}(\omega_1, \omega_2)$ be V generic. Then the following statement in parameters ω_1^V and ω_2^V is true in $V[G]$ but absurd in V:

$$\exists A \subset \mathbb{R}^{V[G]} : L(\mathbb{R}^{V[G]}, A) \models A \text{ codes a surjection from } \omega_1^V \text{ onto } \omega_2^V.$$

So we turn to more restrictive subsets of ω_1: the sets $A \subset \omega_1$ that extend to a class with unique condensation. We develop a genericity iteration for $\dot{x} \subset \omega_1$ in (reasonable) forcing extensions that extend to a class with unique condensation and use this genericity iteration to show a weak absoluteness result.
These subsets of ω_1 can trivialize: granted a large cardinal hypothesis and an iterability hypothes, then every A that extends to a class with unique condensation is constructible from a real, see Theorem 5.6.14.

5.2 The Extender Algebra

We begin by recalling the Lindenbaum algebra and some basic facts. Then we will construct the extender algebra. We would like to mention the notes [Far] which were very helpful.

Definition 5.2.1 For a cardinal δ and an ordinal $\beta \leq \delta$ let $L_{\beta,\delta,0}$ be the propositional logic with β many propositional variables a_ξ, $\xi < \beta$, allowing conjunctions $\bigwedge_{\xi<\kappa} \phi_\xi$ for all $\kappa < \delta$. In addition to the axioms and rules for finitary propositional logic we have for all $\eta < \kappa < \delta$ and all $\langle \phi_\xi ; \xi < \kappa \rangle$ the abbreviation

$$\bigvee_{\xi<\kappa} \phi_\xi \equiv \neg \bigwedge_{\xi<\kappa} \neg \phi_\xi,$$

the axiom

$$\bigwedge_{\xi<\kappa} \phi_\xi \to \phi_\eta$$

and an infinitary rule of inference for each $\kappa < \delta$

$$\text{from } \vdash \phi_\xi \text{ for all } \xi < \kappa \text{ infer } \vdash \bigwedge_{\xi<\kappa} \phi_\xi.$$

Every $x \subset \beta$ naturally defines a valuation ν_x for $L_{\beta,\delta,0}$ via $\nu_x(a_\xi) = \mathsf{true}$ if, and only if, $\xi \in x$. For $\phi \in L_{\beta,\delta,0}$ let

$$A_\phi = \{x \subset \beta\,;\, x \models \phi\}.$$

If $T \subset L_{\beta,\delta,0}$ is a theory, we set $A_T = \{x \subset \beta\,;\, x \models T\}$.

Note that $x \models \phi$ is absolute between transitive models of ZFC containing x and ϕ; in particular collapsing δ to ω makes no difference.

Lemma 5.2.2 *For every $\phi \in L_{\beta,\delta,0}$ the following are equivalent.*

1. $\vdash \phi$.
2. $A_\phi = \mathcal{P}(\beta)$ *in all generic extensions.*
3. $A_\phi = \mathcal{P}(\beta)$ *in all generic extensions by* $\mathrm{Col}(\omega,\delta)$.

Furthermore: for every theory $T \subset L_{\beta,\delta,0}$ and every $\phi \in L_{\beta,\delta,0}$ the following are equivalent:

1. $T \vdash \phi$.
2. $A_{T\cup\{\phi\}} = A_T$ *in all generic extensions.*
3. $A_{T\cup\{\phi\}} = A_T$ *in all generic extensions by* $\mathrm{Col}(\omega,\delta)$.

Proof. We only show the first part since the characterization of $T \vdash \phi$ has almost the same proof. It suffices to show 1. $\Longrightarrow$ 2. and 3. $\Longrightarrow$ 1. Let us suppose that $\vdash \phi$. Since $\vdash \phi$ is upwards absolute, it holds in all generic extensions. So we need to verify the correctness of $L_{\beta,\delta,0}$, i.e. $\vdash \phi \implies x \models \phi$ for all $x \subset \beta$. We omit this argument since it is an easy induction on the rank of the proof for ϕ.
So let us suppose that $A_\phi = \mathcal{P}(\beta)$ holds in all generic extensions by $\mathrm{Col}(\omega,\delta)$. We assume that $\vdash \phi$ fails and construct a forcing of size δ that adds an $x \subset \beta$ such that $x \not\models \phi$. Since the forcing we are going to construct completely embeds into $\mathrm{ro}(\mathrm{Col}(\omega,\delta))$, this will suffice.
Let $\mathbb{P} = \{p \subset L_{\beta,\delta,0}\,;\, p \nvdash \phi \wedge \mathrm{Card}(p) < \delta\}$ ordered by reverse inclusion. For $p \in \mathbb{P}$ and $\psi \in L_{\beta,\delta,0}$ we claim that either $p \cup \{\psi\}$ or $p \cup \{\neg\psi\}$ belongs to $\mathbb{P}$. Otherwise we would have $p \vdash \psi \to \phi$ and $p \vdash \neg\psi \to \phi$. Hence, by elementary inference rules, we have $p \vdash \phi$, contradiction to $p \in \mathbb{P}$! So the set $D_\psi = \{p \in \mathbb{P}\,;\, \psi \in p \vee \neg\psi \in p\}$ is dense in $\mathbb{P}$, and hence every generic $\Gamma \subset \mathbb{P}$ is forced to be a complete theory such that $\Gamma \nvdash \phi$. In $V[\Gamma]$ define $x_\Gamma \subset \beta$ by $\xi \in x_\Gamma$ if, and only if, $a_\xi \in \Gamma$. Then $x_\Gamma \not\models \phi$. □

Let $B_{\beta,\delta,0}$ be the Lindenbaum algebra of $L_{\beta,\delta,0}$, i.e. we set

$$\phi \sim \psi \text{ iff } \vdash \phi \leftrightarrow \psi$$

and let $[\phi]$ denote the $\sim$-equivalence class of ϕ. Let

$$\phi \leq \psi \text{ iff } \vdash \phi \to \psi,$$

we then set $B_{\beta,\delta,0} = \langle\{[\phi]\,;\,\phi \in L_{\beta,\delta,0}\}, \leq / \sim\rangle$.
For a theory T we define the quotient Lindenbaum algebra $B_{\beta,\delta,0}/T$ as follows:

$$\phi \sim_T \psi \text{ iff } T \vdash \phi \leftrightarrow \psi$$

and let $[\phi]_T$ denote the $\sim_T$-equivalence class of ϕ. Let

$$\phi \leq_T \psi \text{ iff } T \vdash \phi \rightarrow \psi;$$

then $B_{\beta,\delta,0}/T = \langle\{[\phi]_T\,;\,\phi \in L_{\beta,\delta,0}\}, \leq_T / \sim_T\rangle$.

Lemma 5.2.3 *For every theory T if $B_{\beta,\delta,0}/T$ has the δ-chain condition, then $B_{\beta,\delta,0}/T$ is a complete Boolean algebra.*

Proof. $B_{\beta,\delta,0}$ is δ-complete, since for any $\kappa < \delta$

$$\Sigma_{\xi<\kappa}[\phi_\xi] = [\bigvee_{\xi<\kappa} \phi_\xi];$$

the same clearly holds for $B_{\beta,\delta,0}/T$. Let $X \subset B_{\beta,\delta,0}/T$. We have to show that ΣX exists. Fix an antichain Y that is maximal with respect to the following property: if $x \in X$, then there is some $y \in Y$ such that $y \leq x$. By the δ-chain condition, Y has cardinality $< \delta$, hence ΣY exists. It is easy to verify that $\Sigma Y = \Sigma X$. □

For $x \subset \beta$ such that $x \models T$ define an ultrafilter $\Gamma_x \subset B_{\beta,\delta,0}/T$ by

$$\Gamma_x = \{[\phi]_T\,;\,x \models \phi\}.$$

Note that Γ_x is well-defined on the $\sim_T$-equivalence classes since $x \models T$. For a generic $\Gamma \subset B_{\beta,\delta,0}/T$ we also set $x_\Gamma = \{\xi < \delta\,;\,[a_\xi]_T \in \Gamma\}$. Then $\Gamma_{x_\Gamma} = \Gamma$ and it is also not difficult to check that $x_{\Gamma_x} = x$ for any x such that $x \models T$.

Lemma 5.2.4 *Let δ be an ordinal. Assume M is a transitive model of* ZFC − Powerset + "$\mathcal{P}(\delta)$ *exists" such that for some $T \in M$ the Boolean algebra $B_{\beta,\delta,0}/T$ has the δ-chain condition. Then for every $x \subset \beta$ such that $x \models T$ the filter $\Gamma_x \subset B_{\beta,\delta,0}/T$ is generic over M. In particular, since Γ_x and x are interdefinable, x is generic over M.*

Proof. Fix $x \subset \beta$, $x \models T$. Assume $\{[\phi_\xi]\,;\,\xi < \kappa\}$ is a maximal antichain of $B_{\beta,\delta,0}/T$ that belongs to M. By Lemma 5.2.2 it suffices to verify $x \in A_{T\cup\{\phi_\xi\}}$ for some $\xi < \kappa$. Assume otherwise. Let $G \subset \mathrm{Col}(\omega, \delta)$ be V-generic. Note that G is also M-generic. Let $\{\psi_n\,;\,n \in \omega\}$ be an enumeration of $\{\phi_\xi\,;\,\xi < \kappa\}$ in order type ω in $M[G] \subset V[G]$. Since the statement "there is an $x \subset \beta$, $x \models T$ such that $x \not\models \psi_n$ for all $n < \omega$" is a $\underset{\sim}{\Sigma}^1_1$ statement true in $V[G]$, it is also true in $M[G]$. Therefore $\mathrm{Col}(\omega, \delta)$ forces over M that there is an $x \subset \beta$, $x \models T$ such that $x \not\models \bigvee_{\xi<\kappa} \phi_\xi$. Hence by Lemma 5.2.2 the sentence $\neg \bigvee_{\xi<\kappa} \phi_\xi$ is consistent with T. This statement is absolute and holds in M, contradicting the maximality of the antichain. □

We now define the extender algebra relative to a sequence of extenders $\vec{E}$. For details regarding extender sequences, premice and other concepts of inner model theory we refer the conscientious reader to [Ste]. Though we suppress many details we want to give the reader an intution of iteration strategies and iterability: given a premouse $\mathcal{M}$ one can define an iteration game for $\mathcal{M}$. Such a game exists for all ordinals α. In an iteration game of length α two players construct an iteration tree $\mathcal{T}$ of length α on $\mathcal{M}_0^{\mathcal{T}} = \mathcal{M}$. For each node β in the tree there is a model $\mathcal{M}_\beta^{\mathcal{T}}$, and the models at each direct $<_{\mathcal{T}}$-successor of a node β are obtained by forming

an ultrapower of (an initial segment of) $\mathcal{M}^{\mathcal{T}}_\beta$ with some extender. These extenders are chosen by player I at successor stages of the game. Player II only plays at limit stages. It is player II's responsibility to pick well-founded branches through the tree at limit stages; i.e. the direct limit of the models along the branch is well-founded. An (ω,α)-iteration strategy Σ for $\mathcal{M}$ is a winning strategy for player II in the iteration game of length α; here we suppress many details, especially why there is an ω in (ω,α)-iteration strategy. So Σ tells us what branches to pick. We will then choose player I's moves to obtain an iteration. A premouse is (ω,α)-iterable if there is an (ω,α)-iteration strategy.
Note that the extender algebra can also be defined for "coarse" iterable models with Woodin cardinals, see [Far]; in this case iterability refers to the concept in [MS94].

Definition 5.2.5 Let $\mathcal{M} = \langle J_\rho[\vec{E}]; \in, \vec{E}, E_\rho\rangle$ be a premouse such that $\mathcal{M} \models \delta$ is Woodin, let $\beta \leq \delta$ and let $\zeta < \rho$. Then $T(\vec{E}\restriction\zeta,\beta) \subset L_{\beta,\delta,0}$ is the theory containing the axioms

$$\bigvee_{\alpha<\kappa} \phi_\alpha \leftrightarrow \bigvee_{\alpha<\lambda} i_E(\langle \phi_\xi;\xi<\kappa\rangle)_\alpha$$

for E on the sequence $\vec{E}\restriction\zeta$ such that $\mathrm{crit}(E)=\kappa\leq\lambda$, and $\nu(E)$ is a $\mathcal{M}$-cardinal such that $i_E(\langle\phi_\xi;\xi<\kappa\rangle)\restriction\lambda \in \mathcal{J}^{\mathcal{M}}_{\nu(E)}$.
If $\delta=\zeta$, we will simply write $T(\vec{E},\beta)$ for $T(\vec{E}\restriction\delta,\beta)$. We will call $W_\delta(\vec{E},\beta) := B_{\beta,\delta,0}/T(\vec{E},\beta)$ the *extender algebra of* $\vec{E}$ *with* β*-many generators.* If $\beta=\delta=\zeta$, then we will write $W_\delta(\vec{E})$ and $T(\vec{E})$ respectively.

Note that the extender algebra of $\vec{E}$ with β many generators exists in $\mathcal{M}$. If β and $\vec{E}$ are clear from the context, we will omit them. Also note that the extender algebra only depends on $\vec{E}\restriction\delta$ and not on the whole sequence $\vec{E}$.

For us the most interesting case is $\beta=\delta$. The extender algebra with δ-many generators is used to make subsets of ω_1 generic. Sometimes it is convenient to use the extender algebra with less than δ many generators; we will especially need the case with ω-many generators to make reals generic over iterates.
Another well known trick is the following: one can restrict the extender sequence $\vec{E}$ such that only extenders with critical point $>\kappa$ for some $\kappa<\delta$ appear on $\vec{E}$. It is not difficult to see that it is possible to restrict in such a way, that the restricted sequence still witnesses that δ is Woodin. We cannot hope that the restriction of $\vec{E}$ is a fine extender sequence in the sense of [Ste].
Note that the extender algebra has atoms: for less than δ many generators this is easy to see. In the case of δ-many generators, look at the $L_{\delta,\delta,0}$ statement $\phi :\equiv \bigwedge_{\xi<\kappa} a_\xi$, where κ is a cardinal strong up to δ such that this strongness is witnessed by $\vec{E}$. For all $\kappa<\lambda<\delta$ we have, using the axioms induced by extenders with critical point κ,

$$T(\vec{E}) \vdash \bigvee_{\xi<\kappa} \phi \leftrightarrow \bigvee_{\xi<\lambda} (\bigwedge_{\xi<\lambda} a_\xi),$$

so

$$T(\vec{E}) \vdash \phi \leftrightarrow \bigwedge_{\xi<\lambda} a_\xi.$$

Hence the condition $[\bigwedge_{\xi<\kappa} a_\xi]_{T(\vec{E})}$ is an atom, since $\bigwedge_{\xi<\lambda} a_\xi \in [\bigwedge_{\xi<\kappa} a_\xi]_{T(\vec{E})}$ for $\kappa<\lambda<\delta$.
Let us recall some notation for iteration trees. Let $\mathcal{T}=\langle\alpha,<_{\mathcal{T}}\rangle$ be an iteration tree of length α in the sense of [Ste], then $\mathcal{M}^{\mathcal{T}}_\beta$ denotes the βth model of this tree. The set $[\beta,\gamma]_{\mathcal{T}}$ is the branch through $\mathcal{T}$ from β to γ. If γ is a $\mathcal{T}$-successor of β,

then there is an iteration map $j^{\mathcal{T}}_{\beta,\gamma}:\mathcal{M}^{\mathcal{T}}_{\beta}\to\mathcal{M}^{\mathcal{T}}_{\gamma}$. If an extender was picked to continue the iteration at stage β, then we denote this extender by $E^{\mathcal{T}}_{\beta}$. Any notions left undefined are to be found in [Ste].

5.3 The Genericity Iteration

Theorem 5.3.1 (Woodin and Steel independently) *Let $\mathfrak{M}=\langle J_\rho[\vec{E}];\in,\vec{E},E_\rho\rangle$ be a sound premouse that is active and has a (ω,ω_1+1)-iteration strategy Σ such that $\vec{E}$ witnesses the measurability and Woodiness of δ in $\mathfrak{M}$. Let $W_\delta:=W_\delta(\vec{E},\delta)$ denote the extender algebra of $\vec{E}$ with δ many generators. Let $x\subset\omega_1$. Then there is an iteration tree $\mathcal{T}$ on $\mathfrak{M}$ of height ω_1+1 such that $i^{\mathcal{T}}_{0,\omega_1}:\mathfrak{M}\to\mathcal{M}^{\mathcal{T}}_{\omega_1}$ such that x is $i^{\mathcal{T}}_{0,\omega_1}(W_\delta)$-generic over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$.*

Note that if $x\subset\omega$; i.e. the situation when the extender algebra is only constructed with ω-many generators, then the measurability of δ is not required, see [Ste, 7.14]. The proof we are about to give mainly follows the proof of [Ste, 7.14]; the notes [Far] were also very helpful.

Proof. The extender algebra W_δ is built using extenders witnessing that δ is Woodin; we will make use of this fact in the following claim:
Claim 1. W_δ is δ-c.c. in $\mathfrak{M}$.
Proof of Claim 1. Working in $\mathfrak{M}$ we pick a set $A=\{[\phi_\xi]_{T(\vec{E})};\xi<\delta\}$. We have to show that A is not an antichain. Let $\kappa<\delta$ be $\langle\phi_\xi;\xi<\delta\rangle$-reflecting and let this fact be witnessed by $\vec{E}$. Let ν be a cardinal such that $\langle\phi_\xi;\xi<\kappa+1\rangle\in\mathcal{J}^{\mathfrak{M}}_{\nu}$ and let F on $\vec{E}$ witness the reflection of κ at this ν. Let E be the trivial completion of $F\restriction\nu$. Then

$$i_E(\bigvee_{\xi<\kappa}\phi_\xi)\restriction(\kappa+1)=\bigvee_{\xi\le\kappa}\phi_\xi.$$

Hence

$$T(\vec{E})\vdash\bigvee_{\xi<\kappa}\phi_\xi\leftrightarrow\bigvee_{\xi\le\kappa}\phi_\xi$$

and hence also

$$T(\vec{E})\vdash\phi_\kappa\to\bigvee_{\xi<\kappa}\phi_\xi$$

Reformulating this fact gives $[\phi_\kappa]_{T(\vec{E})}\le[\bigvee_{\xi<\kappa}\phi_\xi]_{T(\vec{E})}$. So A is not an antichain.
□(Claim 1)

By Lemma 5.2.3 W_δ is a complete Boolean algebra. In general an arbitrary $y\subset\delta$ will not satisfy $T(\vec{E})$. We will produce a normal iteration tree $\mathcal{T}$ of height ω_1+1 such that for $i^{\mathcal{T}}_{0,\omega_1}:\mathfrak{M}\to\mathcal{M}^{\mathcal{T}}_{\omega_1}$

$$i^{\mathcal{T}}_{0,\omega_1}(\delta)=\omega_1 \text{ and } x\models i^{\mathcal{T}}_{0,\omega_1}(T(\vec{E})),$$

for a fixed $x\subset\omega_1$. If we achieve this, then by Lemma 5.2.4 the set x will be $j(W_\delta)$-generic over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$.
There is a normal measure U on δ such that U's trivial completion appears on $\vec{E}$. Let us assume that U's index is minimal; i.e. the trivial completion of U is E_{ζ_0}, where ζ_0 is minimal among all ordinals ζ' such $E_{\zeta'}$ is the trivial completion of a normal measure on δ.
Let $i_U:\mathfrak{M}\to\mathfrak{M}'\simeq\mathrm{Ult}(\mathfrak{M},U)$ and let $\vec{F}$ denote the extender sequence of $\mathfrak{M}'$. The

model $\mathfrak{M}$ can see a part of $i_U(T(\vec{E}))$: the coherence of $\mathfrak{M}$'s fine extender sequence implies

$$\vec{F}\restriction\zeta_0 = \vec{E}\restriction\zeta_0.$$

So if E_α is an extender on $\vec{E}$ with $\alpha < \zeta_0$ and $\mathrm{crit}(E) = \kappa$, then $E_\alpha = F_\alpha$; hence every axiom of the form

$$\bigvee_{\alpha<\kappa} \phi_\alpha \leftrightarrow \bigvee_{\alpha<\lambda} i_E(\langle\phi_\xi; \xi < \kappa\rangle)_\alpha,$$

with $i_{E_\alpha}(\langle\phi_\xi; \xi < \kappa\rangle)\restriction\lambda \in \mathcal{J}^{\mathfrak{M}'}_{\nu(E_\alpha)}$ is in $\mathfrak{M}$. We introduce a notation for this slightly "longer" theory: set $T(\vec{E})^+ = T(\vec{E}\restriction\zeta_0, \zeta_0)$.

We now recursively construct the iteration tree $\mathcal{T}$ for a fixed $x \subset \omega_1$. Before giving more details we outline our plan: we will show that for club many $\gamma < \omega_1$ we have

$$x \cap i^{\mathcal{T}}_{0,\gamma}(\delta) \models i^{\mathcal{T}}_{0,\gamma}(T(\vec{E})).$$

We call such a γ a *baby closure point*. At a baby closure point γ we would like to use the trivial completion of $i^{\mathcal{T}}_{0,\gamma}(U)$ to continue the iteration, but we need to ensure that the resulting iteration is normal. For this we define: γ is a *closure point*, if there is no extender with index $< i^{\mathcal{T}}_{0,\gamma}(\zeta_0)$ that induces an axiom not satisfied by $x \cap i^{\mathcal{T}}_{0,\gamma}(\zeta_0)$, or equivalently:

$$x \cap i^{\mathcal{T}}_{0,\gamma}(\zeta_0) \models i^{\mathcal{T}}_{0,\gamma}(T(\vec{E})^+).$$

Clearly every closure point is a baby closure point. Moreover we will show in the end that there are also club many closure points. Note that this is not trivial: using the agreement of models in an iteration tree, it is not difficult to see that limits of closure points are baby closure points, but in general such limits are not closure points.

We now give more details how to iterate away the least extender which induces an axiom not satisfied by x. Set $\mathcal{M}^{\mathcal{T}}_0 = \mathfrak{M}$ and suppose $\mathcal{T}$ on $\mathcal{M}$ has been constructed up to some countable stage β; furthermore suppose that $D^{\mathcal{T}} = \emptyset$, i.e. the tree has not dropped. If β is a limit ordinal we use the strategy Σ to continue the iteration. If β is a successor there are two cases: if β is a closure point, then we continue the construction of $\mathcal{T}^*$ by picking (the trivial completion of) the least normal measure witnessing that $i^{\mathcal{T}}_{0,\beta}(\delta)$ is measurable.

The second case is: β is not a closure point. Let E be on the $\mathcal{M}_\beta$-sequence such that E induces an axiom of $i_{0,\beta}(T(\vec{E})^+)$ not satisfied by x, and such that $\mathrm{lh}(E)$ is minimal among all extenders on the $\mathcal{M}_\beta$ sequence with this property. We set $E^{\mathcal{T}}_\beta = E$ and use E according to the rules for ω-maximal iteration trees to extend $\mathcal{T}$ one more step. Note that $\mathrm{lh}(E) < i^{\mathcal{T}}_{0,\beta}(\zeta_0)$ in this case.

The following is easily verified: if an extender E an axiom ϕ false of x, then $i_E(\phi)$ is true of x, where i_E is the ultrapower formed with E. In this sense we iterate away false axioms.

We check that all moves are valid in the iteration game. For this we must check that $\gamma < \beta \implies \mathrm{lh}(E^{\mathcal{T}}_\gamma) < \mathrm{lh}(E^{\mathcal{T}}_\beta)$ to see that E is a valid move of player I in the iteration game. There are four cases:

(1) If γ and β are closure points, then $\mathrm{lh}(E^{\mathcal{T}}_\gamma) = \mathrm{lh}(i^{\mathcal{T}}_{0,\gamma}(U)) = i^{\mathcal{T}}_{0,\gamma}(\zeta_0) < i^{\mathcal{T}}_{0,\beta}(\zeta_0) = \mathrm{lh}(i^{\mathcal{T}}_{0,\beta}(U)) = \mathrm{lh}(E^{\mathcal{T}}_\beta)$.

(2) If β is a closure point, then an easy induction, using the definition of closure point, yields that $\mathrm{lh}(E^{\mathcal{T}}_\beta) = i^{\mathcal{T}}_{0,\beta}(\zeta_0)$ is an upper bound for the length of all extenders used at stages $< \beta$ (note that (1) is a special case of (2)).

(3) Now suppose neither γ nor β is a closure point. Suppose the implication does not hold for $\gamma < \beta$. The agreement of models in an ω-maximal iteration tree implies that E_β is on the sequence of $\mathcal{M}_\gamma^\mathcal{T}$. We show that $\nu(E_\beta^\mathcal{T})$ is a cardinal of $\mathcal{M}_\gamma^\mathcal{T}$: $\nu(E_\gamma^\mathcal{T})$ is a cardinal of $\mathcal{M}_\gamma^\mathcal{T}$ and any cardinal $\leq \nu(E_\gamma^\mathcal{T})$ of $\mathcal{M}_\beta$ is a cardinal of $\mathcal{M}_\gamma^\mathcal{T}$. By our assumption $\nu(E_\beta^\mathcal{T}) < \mathrm{lh}(E_\beta^\mathcal{T}) \leq \mathrm{lh}(E_\gamma^\mathcal{T})$ and there are no cardinals in the open interval $]\nu(E_\gamma^\mathcal{T}), \mathrm{lh}(E_\gamma^\mathcal{T})[$, so $\nu(E_\beta^\mathcal{T}) \leq \nu(E_\gamma^\mathcal{T})$ is a cardinal in $\mathcal{M}_\gamma^\mathcal{T}$. So clearly the false axiom induced by $E_\beta^\mathcal{T}$ is also induced in $\mathcal{M}_\gamma^\mathcal{T}$. But this contradicts our choice of $E_\gamma^\mathcal{T}$, since $\mathrm{lh}(E_\gamma^\mathcal{T})$ was not minimal.

(4) Now suppose γ is a closure point and at stage $\beta > \gamma$ we used the extender $E_\beta^\mathcal{T}$ to iterate away a false axiom. Like in (3) we suppose towards a contradiction $\mathrm{lh}(E_\beta^\mathcal{T}) \leq \mathrm{lh}(E_\gamma^\mathcal{T})$. Then the argument for (3) yields that $E_\beta^\mathcal{T}$ is in $\mathcal{M}_\gamma^\mathcal{T}$, so in fact $\mathrm{lh}(E_\beta^\mathcal{T}) < \mathrm{lh}(E_\gamma^\mathcal{T})$. Moreover $E_\beta^\mathcal{T}$ also induces in $\mathcal{M}_\gamma^\mathcal{T}$ an axiom false of x, but then γ is not a closure point! Contradiction.

We must check that $[0, \beta+1]_\mathcal{T}$ does not drop; that is $E_\beta^\mathcal{T}$ measures all subsets of its critical point κ in the model $\mathcal{M}_\gamma^\mathcal{T}$ to which it is applied. In the closure point case this is clear. In the other case this is true because $\kappa < \nu(E_\gamma^\mathcal{T})$, $\nu(E_\gamma^\mathcal{T})$ is a cardinal of $\mathcal{M}_\gamma^\mathcal{T}$, and $\mathcal{M}_\beta^\mathcal{T}$ agrees with $\mathcal{M}_\gamma^\mathcal{T}$ below $\nu(E_\gamma^\mathcal{T})$. This finishes the successor step of the construction in both cases.
Set $\mathcal{M}^* = \mathcal{M}_{\omega_1}^\mathcal{T}$ and let $b = [0, \omega_1]_\mathcal{T}$ denote the branch that yields $\mathcal{M}^*$. We now show that b contains ω_1 many closure points.
So suppose not and aim for a contradiction, say the closure points are bounded by some ζ. Let H_η be large enough such that $x, \mathcal{T}, \mathfrak{M}, \Sigma, \zeta \in H_\eta$ and pick some countable, elementary

$$\pi : H \to H_\eta,$$

such that H is transitive and $\zeta < \gamma := \mathrm{crit}(\pi) = \omega_1^H$ and all the objects mentioned are in the range of π. Let $\pi(\bar{\mathcal{T}}) = \mathcal{T}$ and set $\gamma = \mathrm{crit}(\pi) = \omega_1^H$. Set $\delta^* = i_{0,\gamma}^\mathcal{T}(\delta)$ and $\zeta^* = i_{0,\gamma}^\mathcal{T}(\zeta_0)$. Like in the proof that the comparison process terminates we get the following claim.
Claim 2. We have

$$V_\gamma^{\mathcal{M}_\gamma^{\bar{\mathcal{T}}}} = V_\gamma^{\mathcal{M}_\gamma^\mathcal{T}}$$

and

$$\pi \restriction V_\gamma^{\mathcal{M}_\gamma^{\bar{\mathcal{T}}}} = i_{\gamma,\omega_1}^\mathcal{T} \restriction V_\gamma^{\mathcal{M}_\gamma^\mathcal{T}}.$$

□

Let $\beta+1 \in b$ be the $\mathcal{T}$-successor of γ. Because the critical points of the extenders used along b are increasing, we have $\mathrm{crit}(E_\beta^\mathcal{T}) = \mathrm{crit}(i_{\gamma,\omega_1}^\mathcal{T}) = \gamma$. Also we have an axiom

$$\bigvee_{\xi<\gamma} \phi_\xi \leftrightarrow i_{E_\beta^\mathcal{T}}(\bigvee_{\xi<\gamma} \phi_\xi) \restriction \lambda$$

of $i_{0,\beta}^\mathcal{T}(T(\vec{E})^+)$ induced by $E_\beta^\mathcal{T}$ that does not hold for $x \cap \zeta^*$. The falsity of this axiom means that the right hand side is true of $x \cap \zeta^*$, but the left hand side is not. But now $\bigvee_{\xi<\gamma} \phi_\xi$ is essentially a subset of γ, and therefore, by the agreement of the models of the iteration, contained as an element in $\mathcal{M}_\gamma^\mathcal{T}$. Recall that $\lambda < \nu(E_\beta^\mathcal{T})$; since generators are not moved on $\mathcal{T}$

$$i_{E_\beta^\mathcal{T}}(\bigvee_{\xi<\gamma} \phi_\xi) \restriction \lambda = i_{\gamma,\omega_1}^\mathcal{T}(\bigvee_{\xi<\gamma} \phi_\xi) \restriction \lambda = \pi(\bigvee_{\xi<\gamma} \phi_\xi) \restriction \lambda.$$

Now $\gamma \leq \zeta^*$ and $x \cap \zeta^* \not\models \bigvee_{\xi<\gamma} \phi_\xi$ implies that $x \cap \gamma \not\models \bigvee_{\xi<\gamma} \phi_\xi$. Since $x \cap \gamma \in H$ and $\pi(x \cap \gamma) = x$, we have $x \not\models \pi(\bigvee_{\xi<\gamma} \phi_\xi)$. This contradicts the fact that $x \cap \zeta^*$

satisfies the initial segment $i_{E^{\mathcal{T}}_\beta}(\bigvee_{\xi<\gamma}\phi_\xi)\restriction\lambda$ of this disjunction. In other words γ is a closure point: contradiction!
So b contains uncountably many closure points (in fact club many, but we have no use for this fact here), hence the least normal measure on δ witnessing the measurability of δ (resp. its image) was used ω_1-many times. For a closure point γ, note that $x\cap i^{\mathcal{T}}_{0,\gamma}(\delta)\models i^{\mathcal{T}}_{0,\gamma}(T(\vec{E}))$, so $x\models i^{\mathcal{T}}_{0,\gamma}(T(\vec{E}))$. Also note that the existence of unboundedly many closure points in b implies $i^{\mathcal{T}}_{0,\omega_1}(\delta)=\omega_1$. We need to show $x\models i^{\mathcal{T}}_{0,\omega_1}(T(\vec{E}))$, i.e. ω_1 is a baby closure point. For this fix some $\psi\in i^{\mathcal{T}}_{0,\omega_1}(T(\vec{E}))$. Clearly there is some $\bar\psi$ and some closure point $\gamma\in b$ such that $i^{\mathcal{T}}_{\gamma,\omega_1}(\bar\psi)=\psi$. But since $\bar\psi\in i^{\mathcal{T}}_{0,\gamma}(T(\vec{E}))$, it is basically a bounded subset of $i^{\mathcal{T}}_{0,\gamma}(\delta)=\mathrm{crit}(i^{\mathcal{T}}_{\gamma,\omega_1})$, hence $\bar\psi=\psi$. Since $x\cap i^{\mathcal{T}}_{0,\gamma}(\delta)\models\bar\psi$ clearly $x\models\psi$. □

We will call an iteration as above a *genericity iteration*. In the following we will refine the concept of genericity iteration. Note that the argument above for $x\models i^{\mathcal{T}}_{0,\omega_1}(T(\vec{E}))$ also yields that limits of closure points are baby closure points and moreover that the baby closure points are club in ω_1.

5.3.1 First applications of genericity iterations

We use genericity iterations to present Corollary 5.3.4, an absoluteness argument due to Steel and Woodin independently; this is not the most general result though, but the proof is quite easy to grasp. We will refine the argument later to add more parameters and to obtain Σ^2_1 absoluteness, see Theorem 5.4.1.

Definition 5.3.2 Let $x\subset\mathsf{OR}$ and let $\vec{E}$ be a fine extender sequence over x. We let $M^\sharp_{\mathsf{mw}}(x)=\langle J_\beta(x)^{\vec{E}};\in,x,\vec{E}\restriction\beta,E_\beta\rangle$ denote the minimal sound x-premouse that satisfies the following properties:

1. $M^\sharp_{\mathsf{mw}}(x)$ is active, i.e. $E_\beta\neq\emptyset$, and $\mathrm{crit}(E_\beta)>\delta$,
2. $M^\sharp_{\mathsf{mw}}(x)$ has a (ω,ω_1+1)-iteration strategy Σ,
3. $\vec{E}$ witnesses the measurability and Woodiness of δ.

Note that we demand that the witnesses for the measurability and Woodiness of δ are on $\vec{E}$. We can describe the top measure of $M^\sharp_{\mathsf{mw}}$; we do not prove the following fact, it follows from the minimality of $M^\sharp_{\mathsf{mw}}$. If $M^\sharp_{\mathsf{mw}}(x)=\langle J_\beta(x)^{\vec{E}};\in,x,\vec{E}\restriction\beta,E_\beta\rangle$, then on $\vec{E}\restriction\beta$ there is no extender witnessing the measurability of $M^\sharp_{\mathsf{mw}}(x)$'s measurable Woodin; E_β is in fact the only extender so that E_β is the trivial completion of a normal measure on δ. Furthermore, since we have the indexing of [Ste], $\beta=\delta^{++\mathrm{Ult}(M^\sharp_{\mathsf{mw}}(x),E_\beta)}$. Without a proof we state the following fact that we will make use of without further notice:

Remark 5.3.3 If $M^\sharp_{\mathsf{mw}}:=M^\sharp_{\mathsf{mw}}(\emptyset)$ exists, then $M^\sharp_{\mathsf{mw}}(x)$ exists for every $x\subset\omega$.

The key ingredients to show the above are the following: first one observes that even without the least Woodin cardinal η of $M^\sharp_{\mathsf{mw}}$ there are measure one many Woodins in $M^\sharp_{\mathsf{mw}}$, say the thinned out sequence of extenders with critical point $>\eta$ is called $\vec{F}$. Then one performs a genericity iteration to make x generic over some iterate of $M^\sharp_{\mathsf{mw}}$ for W_η, where W_η is constructed with ω-many generators. So in the generic extension containing x (the image of) $\vec{F}$ witnesses that there is still an iterable system of extenders.
We remark that the current inner model theory does not tell us under what circumstances $M^\sharp_{\mathsf{mw}}$ exists.

For the next theorem we introduce a quantifier Q. The interpretation of Q is as follows: if ϕ is a statement in the language of set theory with one free variable, then $\mathsf{Q}X\phi(X)$ if, and only if, there is some $X \subset \omega_1$ such that X is unbounded in ω_1 and $\phi(X)$ holds. This quantifier ensures that the next theorem is more than Shoenfield's Absoluteness Theorem.

Corollary 5.3.4 (Woodin and Steel independently) *Suppose $M^\sharp_{\mathsf{mw}}$ exists and is $(\omega, \omega_1 + 2)$-iterable. Let ϕ be a statement in the language of set theory with one free variable. There is a statement ϕ^* such that*

$$\mathsf{Q}X : L[X] \models \phi(X)$$

if, and only if,

$$M^\sharp_{\mathsf{mw}} \models \phi^*.$$

Proof. Let δ be the measurable Woodin cardinal of $M^\sharp_{\mathsf{mw}}$. We will define ϕ^* in a moment. Suppose $L[X] \models \phi(X)$ for some unbounded $X \subset \omega_1$. If necessary we modify ϕ and X a little so that $L[X] \models \omega_1 = \omega_1^V$. By Theorem 5.3.1 there is an elementary map $j : M^\sharp_{\mathsf{mw}} \to \mathfrak{M}^*$ such that $j(\delta) = \omega_1$ and X is generic over $\mathfrak{M}^*$. Then $\mathfrak{M}^*$ has a top measure U and the critical point of this top measure is $j(\delta) = \omega_1$. Let $h : \mathfrak{M}^{**} = \mathrm{Ult}(\mathfrak{M}^*, U)$ and note that the extender U must be applied to $\mathfrak{M}^*$ by the rules of the iteration game. Since $V_{\omega_1}^{\mathfrak{M}^{**}} = V_{\omega_1}^{\mathfrak{M}^*}$, we have that X is also generic over $\mathfrak{M}^{**}$. Because $h(\omega_1)$ is still measurable in $\mathfrak{M}^{**}[X]$, we have that $h(\omega_1)$ is an X-indiscernible. So

$$L_{h(\omega_1)}[X] \models \phi(X).$$

In $\mathfrak{M}^*$, the existence of an X, such that X is generic for the extender-algebra W_{ω_1} and $L_{h(\omega_1)}[X] \models \phi(X)$ in the ultrapower with U is a first order statement in the paramters ω_1 and U, call it $\phi^*(\omega_1, U)$. By elementarity $\phi^*(\delta, \bar{U})$ holds in $M^\sharp_{\mathsf{mw}}$, where $\bar{U}$ is $M^\sharp_{\mathsf{mw}}$'s top-measure.
For the other direction pick some $G \subset W_\delta$, $G \in V$ that is generic over $M^\sharp_{\mathsf{mw}}$ such that for some $Y \subset \omega_1^{M^\sharp_{\mathsf{mw}}[G]}$ unbounded in $\omega_1^{M^\sharp_{\mathsf{mw}}[G]}$, $Y \in M^\sharp_{\mathsf{mw}}[G]$ and Y is a witness for ϕ^*, say $p \in G$ is a condition that forces Y is a witness for ϕ^*. Then we iterate $M^\sharp_{\mathsf{mw}}$ linearly ω_1-many times using only its top measure on δ. We need to apply the technique we call "piecing together end-extending generics" from the proof of Theorem 5.4.1; since we give a very detailed and far more general version of this technique there, we omit the details of this construction and just sum up the result. Set $G_0 = G$. For each countable iterate $\mathfrak{M}_i$ of $M^\sharp_{\mathsf{mw}}$, $i < \omega_1$, obtained by linearly iterating, we have a generic $G_i \subset j_{0,i}(W_\delta)$, where $j_{0,i} : M^\sharp_{\mathsf{mw}} \to \mathfrak{M}_i$ is the iteration map. For $i < j$ the generics G_i, G_j end-extend each other, i.e.: $G_i \subset G_j$. Then $G_{\omega_1} = \bigcup\{G_i\,;\, i < \omega_1\}$ is generic over $\mathfrak{M}_{\omega_1}$. We have $p \in G_{\omega_1}$, so Y_{ω_1} has the desired properties, where Y_{ω_1} is calculated from G_{ω_1} in the same way as Y was calculated from G. □

We then have the following obvious corollary which looks like a bounded forcing axiom, except that it lacks interesting parameters.

Corollary 5.3.5 *Suppose $M^\sharp_{\mathsf{mw}}$ exists and furthermore suppose that the (ω, ω_1+1)-iterability of $M^\sharp_{\mathsf{mw}}$ is preserved in all generic extensions. Then for every forcing $\mathbb{P}$ and every Δ_0 statement ϕ with one free variable*

$$H_{\omega_2}^{V^{\mathbb{P}}} \models \mathsf{Q}X\phi(X) \implies H_{\omega_2} \models \mathsf{Q}X\phi(X).$$

□

The above corollary is suboptimal. With different methods one can show far more than the above corollary using a weaker large cardinal hypothesis: in [FL06, Theorem 5.2], assuming the existence of two Woodin cardinals but not the existence of $M^\sharp_{\mathsf{mw}}$, a similar absoluteness result is shown using a more expressive language as in the corollary above. The language in [FL06, Theorem 5.2] in addition contains a predicate for NS_{ω_1} and predicates for all universally Baire sets of reals, as well as constants for every member of H_{ω_1}.

5.3.2 Adding parameters

We now explore what parameters one can reasonably hope to add to the statement of the above corollaries. The arguments to follow are blueprints which can be applied for example to add parameters to the statement of Theorem 5.4.1. Let us first consider a real z: if we demand that $M^\sharp_{\mathsf{mw}}$ exists then we have remarked that $M^\sharp_{\mathsf{mw}}(z)$ exists.

Corollary 5.3.6 *Suppose $M^\sharp_{\mathsf{mw}}(z)$ exists. Let z be a real and suppose that in all generic extensions $M^\sharp_{\mathsf{mw}}(z)$ is (ω, ω_1+1)-iterable. Then for every forcing $\mathbb{P}$ and every Δ_0 statement ϕ with two free variables*

$$H^{V^{\mathbb{P}}}_{\omega_2} \models \mathsf{Q}X\phi(X,z) \implies H_{\omega_2} \models \mathsf{Q}X\phi(X,z).$$

□

We now study parameters for which forcing names exist in some generic extension of $M^\sharp_{\mathsf{mw}}(z)$. We need some notation first.

Definition 5.3.7 For $S \subset \omega_1$ let $\mathrm{code}(S) = \{x \in \mathsf{WO}\,;\, ||x|| \in S\}$. A set $A \subset \mathbb{R}$ is *closed under ordertypes* if $x \in A \cap \mathsf{WO}$ and $||x|| = ||y||$ for some y implies $y \in A$. Let $A \subset \mathbb{R}$, we then set $\mathrm{decode}(A) = \{\alpha < \omega_1\,;\, \exists x \in A \cap \mathsf{WO} : ||x|| = \alpha\}$. Let $\mathfrak{M}$ be a (ω, ω_1+1)-iterable premouse that contains a Woodin cardinal δ and let $A \subset \mathbb{R}$. We say a term for a set of reals $\tau \in \mathfrak{M}^{\mathrm{Col}(\omega,\delta)}$ *captures* A if for all iterations $\pi : \mathfrak{M} \to \mathfrak{M}^*$ and all $g \subset \pi(\mathrm{Col}(\omega,\delta))$ generic over $\mathfrak{M}^*$

$$\pi(\tau)^g = A \cap \mathfrak{M}^*.$$

We will say that $S \subset \omega_1$ is *captured by τ over $\mathfrak{M}$* if for all iterations $\pi : \mathfrak{M} \to \mathfrak{M}^*$ such that $\pi(\delta) = \omega_1$ and for all $g \subset \pi(\mathrm{Col}(\omega,\delta))$ generic over $\mathfrak{M}^*$

$$\pi(\tau)^g \cap \mathsf{WO} = \mathrm{code}(S) \cap \mathfrak{M}^*[g].$$

Note that equivalently we could say

$$\mathrm{decode}(\pi(\tau)^g) = S \cap \omega_1^{\mathfrak{M}^*[g]}$$

in the last part of the definitions above. Moreover note that in the presence of large cardinals lots of definable sets can be captured.

Lemma 5.3.8 *Let $\mathfrak{M} = \langle J_\beta[\vec{E}]; \in, \vec{E}, E_\beta\rangle$ be a sound premouse that is active and has a (ω, ω_1+1)-iteration strategy Σ such that $\vec{E}$ witnesses the measurability and Woodiness of δ. Furthermore assume that $S \subset \omega_1$ is captured by τ over $\mathfrak{M}$. Let ϕ be a statement in the language of set theory with two free variables, then*

$$\exists X \subset \omega_1 : L[X,S] \models \phi(X,S)$$

if, and only if,

$$\exists p \in W_\delta^{\mathfrak{M}} : p \Vdash^{\mathfrak{M}} \exists X \subset \check{\delta} : L_\kappa[X, \mathrm{decode}(\tau)] \models \phi(X, \mathrm{decode}(\tau)),$$

where κ *is the critical point of the top measure of* $\mathfrak{M}$.

Proof. The proof is similar to the proof of 5.3.4. First assume $L[X, S] \models \phi(X, S)$ for some $X \subset \omega_1$. Then produce a genericity iteration $\pi : \mathfrak{M} \to \mathfrak{M}^*$ such that $\pi(\delta) = \omega_1$ and X is $\pi(W_\delta)$-generic over $\mathfrak{M}^*$. So $\pi(\tau)^X \cap \mathsf{WO} = \mathrm{code}(S) \cap \mathfrak{M}^*[X]$ by our hypothesis. So $S \in \mathfrak{M}^*[X]$. Then by (X, S)-indiscernibility of $\pi(\kappa)$

$$L[X, S]_{\pi(\kappa)} \models \phi(X, S),$$

where κ is the critical point of the top measure of $\mathfrak{M}$. It remains to appeal to the elementarity of π.
The presence of τ does change the proof of the other direction. We need to piece together end-extending local generic objects for the other direction. Since a more complex argument of this type is given in the proof for 5.4.1, we omit it here. □

Unfortunately there are serious restrictions on the complexity of a parameters S such that $\mathrm{code}(S)$ is captured. Recall that a (κ, λ)-extender $E = \{E_a\,;\, a \in [\lambda]^{<\omega}\}$ with critical point κ on the sequence of a premouse $\mathfrak{M}$ is *complete*, if E_a measures $\mathcal{P}(\kappa^{|a|})^{\mathfrak{M}}$ for $a \in [\lambda]^{<\omega}$. If E is complete, then we can linearly iterate $\mathfrak{M}$ using only E and its images without dropping to an initial segment of $\mathfrak{M}$.

Lemma 5.3.9 *Let* $S \subset \omega_1$ *be such that* $\mathrm{code}(S)$ *is captured by some* τ *over some countable sound premouse* $\mathfrak{M}$ *that is active and* $(\omega, \omega_1 + 1)$*-iterable. Furthermore suppose that the* $\mathfrak{M}$*-extender sequence contains a complete measure on a regular* $\mathfrak{M}$*-cardinal. Then*

1. *there is a* $\mathbf{\Delta}^1_2$*-set* A *such that* $S = \mathrm{decode}(A)$*;*
2. *if furthermore sharps for all reals exist, then either* S *or* $\omega_1 \setminus S$ *contains a club.*

Proof. We show how to calculate $A \subset \mathsf{WO}$ with the desired properties. For this let us fix a a cardinal $\delta \in \mathfrak{M}$ such that there is a complete measure U on δ. Let $x \in \mathsf{WO}$, say $||x|| = \alpha$. Pick a countable linear iteration $\pi : \mathfrak{M} \to \mathfrak{M}^*$ that is obtained using only U and its images such that $\pi(\delta) > \alpha$. In V pick a $g \subset \mathrm{Col}(\omega, \pi(\delta))$ generic over $\mathfrak{M}^*$. If $x \in \mathfrak{M}^*[g]$ then by the choice of τ

$$x \in \pi(\tau)^g \iff \alpha \in S.$$

So we define A such that $x \in A$ if and only if

$$\forall \pi \forall g[\phi_0(\pi, U, \mathfrak{M}) \wedge \pi(\delta) > ||x|| \wedge \phi_1(\pi, g, \delta) \to \\ \exists y \in \mathsf{WO} \cap \mathfrak{M}^* : ||y|| = ||x|| \wedge y \in \pi(\tau)^g],$$

here $\phi_0(\pi, U, \mathfrak{M})$ expresses that π is a linear iteration of $\mathfrak{M}$ using only U and its images and $\phi_1(\pi, g, \delta)$ expresses that g is $\mathrm{Col}(\omega, \pi(\delta))$ generic over the last model of the iteration π and $\mathfrak{M}^*$ denotes π's last model.
We can also calculate A in the following fashion: $x \in A$ if and only if

$$\exists \pi \exists g[\phi_0(\pi, U, \mathfrak{M}) \wedge \pi(\delta) > ||x|| \wedge \phi_1(\pi, g, \delta) \wedge \\ (\exists y \in \mathsf{WO} \cap \mathfrak{M}^* : ||y|| = ||x|| \wedge y \in \pi(\tau)^g)].$$

By choosing a nice coding we see that the first formula defining A is $\Pi^1_2(z)$ where z is a real coding $(\mathfrak{M}, \delta, U)$ and the second is $\Sigma^1_2(z)$. Hence A is $\Delta^1_2(z)$.
This clearly implies that S is constructible from the real z. If $z^\sharp$ exists, then there is either a z-indiscernible in S or in $\omega_1 \setminus S$, hence there are either club many z-indiscernibles in S or in $\omega_1 \setminus S$. □

This shows that we cannot hope to capture (a code for) a stationary and costationary set if we have sharps for reals. Also we can not capture a ladder system for ω_1, since such a system would allow to partition ω_1 into ω_1-many stationary sets (a ladder system is in fact the amount of choice one needs to calculate such a partition).

5.4 Σ^2_1 absoluteness

We now work a little harder to obtain Σ^2_1 absoluteness which was first shown by Woodin. Our proof differs substantially from Woodin's original proof and uses genericity iterations instead of the stationary tower. The proof we are going to present is due to Steel and Woodin independently.

Theorem 5.4.1 (Woodin) *Suppose $M^\sharp_{\mathsf{mw}}$ exists and is (ω, ω_1+1)-iterable in all set forcing extensions. Assume* CH *holds. Let $\mathbb{P}$ be a notion of forcing and let $G \subset \mathbb{P}$ be V-generic. Let z be a real in V. If in $V[G]$*

$$\exists A \subset \mathbb{R}^{V[G]} L(\mathbb{R}^{V[G]}, A) \models \phi(A, z),$$

then in V

$$\exists A \subset \mathbb{R}^{V} L(\mathbb{R}^{V}, A) \models \phi(A, z).$$

Furthermore if CH *holds in $V^{\mathbb{P}}$, then the converse is true.*

Before we give proof, we want to state three Lemmata. The first one is part of the folklore; for a more general result see (for example) [Kan03, 10.10].

Lemma 5.4.2 *Let $\mathbb{P}$ and $\mathbb{Q}$ be notions of forcings in V such that in $V^{\mathbb{P}}$ for all $q \in \mathbb{Q}$ a $\mathbb{Q}$-generic containing q exists. Then a $\mathbb{Q}$ name $\dot{R}$ exists such that $V^{\mathbb{Q} * R} = V^{\mathbb{P}}$.* □

The above Lemma is shown using Boolean algebras. If $\mathbb{P}$ and $\mathbb{Q}$ are Boolean algebras, then the conclusion of the above Lemma reads: $\mathbb{Q}$ is a regular subalgebra of $\mathbb{P}$.
The second lemma is also part of the folklore; we do not explicitly state it for fine-structural models since it clearly also holds in the coarse case.

Lemma 5.4.3 *Let $\mathbb{P}$ be a complete Boolean algebra that satisfies the δ-c.c. and let $j : V \to M$ be an elementary embedding with critical point δ. Then j"$\mathbb{P}$ is a regular subalgebra of $j(\mathbb{P})$.*
Furthermore if δ is Woodin as witnessed by the extender-sequence $\vec{E}$, $\omega \leq \beta \leq \delta$ and $\mathbb{P} = W_\delta = W_\delta(\beta, \vec{E})$, then the embedding

$$[\phi]_{T(\vec{E})} \mapsto [\phi]_{j(T(\vec{E}))}$$

witnesses that W_δ is a regular subalgebra of $j(W_\delta)$.

Proof. Let A be a maximal antichain of $\mathbb{P}$. Then $\mathrm{Card}(A) < \delta$, so $j(A) = j''A$ is a maximal antichain of $j(\mathbb{P})$. Hence j"$\mathbb{P}$ is a regular subalgebra of $j(\mathbb{P})$.
For the second part let $A = \{[\phi_i]_{T(\vec{E})}\,; i < \kappa\}$, $\kappa < \delta$ be a maximal antichain of

W_δ. Notice $\bigvee_{i<\kappa} \phi_i \in V_\delta$. Hence by maximality $T(\vec{E}) \vdash \bigvee_{i<\kappa} \phi_i$. By elementarity $j(T(\vec{E})) \vdash \bigvee_{i<\kappa} \phi_i$. □

The third lemma discusses the relationship of the extender algebra with ω-many generators and small forcing. It is a slight generalization of the genericity iteration to make a real generic.

Lemma 5.4.4 (Woodin) *Let $\mathfrak{M} = \langle J_\rho[\vec{E}]; \in, \vec{E}, E_\rho\rangle$ be a sound premouse that is active and has a (ω, ω_1+1)-iteration strategy Σ such that $\vec{E}$ witnesses the Woodiness of δ in $\mathfrak{M}$. Let $\mathbb{P} \in V_\kappa^{\mathfrak{M}}$, $\kappa < \delta$, be a notion of forcing. Let $\vec{F}$ denote the complete extenders of $\vec{E}$ with critical point $> \kappa$ and index $< \delta$. Let $x \subset \omega$. Then the following hold:*

1. *If $g \subset \mathbb{P}$ is generic over $\mathfrak{M}$ and $\alpha < \delta$ such that $F_\alpha \neq \emptyset$, then there is a complete extender $\tilde{F}_\alpha \in \mathfrak{M}[g]$ such that $\tilde{F}_\alpha \cap \mathfrak{M} = F_\alpha$. We will say F_α* induces *$\tilde{F}_\alpha$.*
2. *For $g \subset \mathbb{P}$ is generic over $\mathfrak{M}$, let $W^g_\delta := W^g_\delta(\vec{F}, \omega)$ denote the extender algebra with ω many generators calculated from the set of induced extenders $\{\tilde{F}_\alpha\,;\, \alpha < \delta\}$ in $\mathfrak{M}[g]$ and let $W^{\dot{g}}_\delta$ denote a name for that forcing. If $g \subset \mathbb{P}$ is generic over $\mathfrak{M}$, then there is an iteration tree $\mathcal{T}$ on $\mathfrak{M}$ of some height $\alpha + 1 < \omega_1$ such that:*
   a) *if E is the extender we apply at stage β of the construction of $\mathcal{T}$, then E is on the sequence $i^{\mathcal{T}}_{0,\beta}(\vec{F})$;*
   b) $\mathrm{crit}(i^{\mathcal{T}}_{0,\alpha}) > \kappa$*;*
   c) *if $g \subset \mathbb{P}$ is generic over $\mathfrak{M}$, then g is generic over $\mathcal{M}^{\mathcal{T}}_\alpha$, and moreover x is generic for $i^{\mathcal{T}}_{0,\alpha}(W^{\dot{g}}_\delta)^g$ over $\mathcal{M}^{\mathcal{T}}_\alpha[g]$.*
3. *Moreover there is an iteration tree $\mathcal{T}$ on $\mathfrak{M}$ of some height $\alpha + 1 < \omega_1$ such that for all $g \subset \mathbb{P}$ generic over $\mathfrak{M}$ the real x is generic for $i^{\mathcal{T}}_{0,\alpha}(W^{\dot{g}}_\delta)^g$ over $\mathcal{M}^{\mathcal{T}}_\alpha[g]$.*

We will give the key ideas for this Lemma only. For *1.* one needs to run the argument that shows that the measurability of some cardinal is preserved under small forcing. Note that *2.* of the above lemma is identical to [Ste, 7.16] and *3.* has almost the same proof: one performs a genericity iteration for x using only the extenders from $\vec{F}$ and their images. We hint how to pick extenders to obtain a tree like in *3.* At stage β of the tree construction do the following: if there is a condition $p \in \mathbb{P}$ and an extender $E \in i^{\mathcal{T}}_{0,\beta}(\vec{F})$ such that p forces that $\tilde{E}$ induces an axiom false of x, then pick the minimal such E to continue the construction of $\mathcal{T}$. The rest runs similar to the proof of [Ste, 7.14]. It is routine to check that the extenders on $\vec{F}$ witness that δ is Woodin and that the extenders induced from $\vec{F}$ continue to do so in $\mathfrak{M}^{\mathbb{P}}$. So W^g_δ is well-defined and δ-c.c. We shall give no more details.
We now prove 5.4.1.

Proof. We fix $G \subset \mathbb{P}$ generic over V and some $A \in \mathcal{P}(\mathbb{R})^{V[G]}$ such that

$$\psi(A) :\equiv L(\mathbb{R}^{V[G]}, A) \models \phi(A, z),$$

where $z \in \mathbb{R}^V$. We force CH over $V[G]$ using $\mathrm{Col}(\omega_1, 2^\omega)^{V[G]}$ and call the resulting extension W. For a while we will work in W. We code A and $\mathbb{R}^{V[G]}$ by a set $B \subset \omega_1$. Clearly there is a formula ψ' such that $L[B] \models \psi'(B)$ if, and only if, $\psi(A)$ holds. By our hypothesis, we have that $\mathfrak{M} := M^\sharp_{\mathsf{mw}}(z)$ has a $(\omega, \omega_1 + 1)$-iteration strategy

Σ in W, so by Corollary 5.3.4 there is an iteration $j : \mathfrak{M} \to \mathfrak{M}^*$ such that B is generic over $\mathfrak{M}$ for the extender algebra. Let δ denote $\mathfrak{M}$'s measurable Woodin and let W_δ be the extender algebra calculated in $\mathfrak{M}$ relative to $\mathfrak{M}$'s extender sequence $\vec{E}$. Hence by elementarity of j there is a condition $p \in W_\delta$, say $p = [\phi]_{T(\vec{E})}$, such that

$$p \Vdash_{\mathfrak{M}} \check{\delta} = \omega_1 \wedge \exists \dot{B} : L_\kappa[\dot{B}] \models \psi'(\dot{B}),$$

where κ is the critical point of $\mathfrak{M}$'s top extender.
Our plan is as follows: we will construct in V an iteration tree $\mathcal{T}$ of length $\omega_1 + 1$ and Γ generic over the last model $\mathfrak{M}^*$ of $\mathcal{T}$ such that $p \in \Gamma$, $\mathbb{R}^V \subset \mathfrak{M}^*[\Gamma]$ and p is not moved by $j^{\mathcal{T}}_{0,\omega_1}$. The tree $\mathcal{T}$ will be constructed in ω_1 many rounds; for each round i there is an ordinal α_i, and in round i we will construct the map

$$j^{\mathcal{T}}_{\alpha_i,\alpha_{i+1}} : \mathcal{M}^{\mathcal{T}}_{\alpha_i} \to \mathcal{M}^{\mathcal{T}}_{\alpha_{i+1}}.$$

Before we can go into details we need to care for a minor technical thing. Recall that the members of W_δ are of the form $[\phi]_{T(\vec{E})}$; alternatively we could have constructed W_δ using the $<_{M^\sharp_{\mathsf{mw}}}$-least formula in an equivalence class. So for the rest of the proof we assume without loss of generality that W_δ contains formulae and so the maps of the form

$$[\phi]_{j^{\mathcal{T}}_{0,\alpha_i}(T(\vec{E}))} \mapsto [\phi]_{j^{\mathcal{T}}_{0,\alpha_{i+1}}(T(\vec{E}))}$$

are the identity on formulae. This identification eases the reasoning considerably, since $L_{\delta,\delta,0}$ formulae are not moved by maps with critical point δ. One consequence we will need later is that nice names for reals are not moved by such maps; another consequence of this and Lemma 5.4.3 is the following: if $j : \mathfrak{M} \to \mathfrak{M}'$ has critical point δ, then $W_\delta = j``W_\delta$ is a regular subalgebra of $j(W_\delta)$.
For book-keeping pick an enumeration $\{x_i\,;\, 0 < i < \omega_1 \text{ is not a limit ordinal}\}$ of the reals in V. We call what follows *piecing together end extending generics.* We now construct in V an iteration tree $\mathcal{T}$ of length $\omega_1 + 1$, a sequence of ordinals $\langle \alpha_i; i < \omega_1 + 1\rangle$ and a sequence of generics $\langle \Gamma_i; i < \omega_1 + 1\rangle$ such that

1. $\langle \alpha_i; i < \omega_1\rangle$ is a normal sequence, i.e. $\{\alpha_i\,;\, i < \omega_1\}$ is closed unbounded in ω_1 and $\alpha_{\omega_1} = \omega_1$,
2. $p \in \Gamma_0$,
3. p is not moved by $j^{\mathcal{T}}_{0,\omega_1}$,
4. $\mathrm{crit}(j^{\mathcal{T}}_{\alpha_i,\omega_1}) = j^{\mathcal{T}}_{0,\alpha_i}(\delta)$,
5. $\Gamma_i \subset j^{\mathcal{T}}_{0,\alpha_i}(W_\delta)$ is generic over $\mathcal{M}^{\mathcal{T}}_{\alpha_i}[\Gamma_j]$ for $j < i$,
6. if $i > 0$ is not a limit ordinal, then $x_i \in \mathcal{M}^{\mathcal{T}}_{\alpha_i}[\Gamma_i]$ and
7. if $i \leq j$, then $\Gamma_i \subset \Gamma_j$.

Let U denote (the trivial completion of) the least normal measure on δ that is on $\vec{E}$. Set $\mathcal{M}^{\mathcal{T}}_0 = \mathfrak{M}$ and set $\alpha_0 = 0$. In V we can pick Γ_0 such that $p \in \Gamma_0$. This finishes the construction of α_0 and Γ_0.
At all stages α_i of the iteration we use the trivial completion of $i^{\mathcal{T}}_{0,\alpha_i}(U)$ to continue the iteration. At limit stages $\lambda \leq \omega_1$ we set $\alpha_\lambda = \sup\{\alpha_i\,;\, i < \lambda\}$ and we use the iteration strategy Σ to continue the iteration. We set

$$\Gamma_\lambda := \bigcup\{\Gamma_i\,;\, i < \lambda\} \subset j^{\mathcal{T}}_{0,\alpha_\lambda}(W_\delta).$$

All antichains of the extender algebra are small and $\mathrm{crit}(j^{\mathcal{T}}_{\alpha_i,\omega_1}) = j^{\mathcal{T}}_{0,\alpha_i}(\delta)$ for $i < \lambda$, so we have that Γ_λ is generic over $\mathcal{M}^{\mathcal{T}}_{\alpha_\lambda}$.

We now discuss the successor case. Fix $i < \omega_1$ and let $\gamma = \alpha_i$. We continue the iteration by picking $j^{\mathcal{T}}_{0,\gamma}(U)$ as the next extender. At stage $\gamma+1$ let η_γ be the least Woodin cardinal in $\mathcal{M}^{\mathcal{T}}_{\gamma+1}$ in the open interval $]j^{\mathcal{T}}_{0,\gamma}(\delta), j^{\mathcal{T}}_{0,\gamma+1}(\delta)[$. Let $\vec{F}$ consist of the extenders on $\mathcal{M}^{\mathcal{T}}_{\gamma+1}$'s extender sequence with critical point $> j^{\mathcal{T}}_{0,\gamma}(\delta)$ and index $< \eta_\gamma$ that witness that η_γ is Woodin. As in Lemma 5.4.4 we define from $\vec{F}$ an extender algebra $W^{\Gamma_\gamma}_{\eta_\gamma} \in \mathcal{M}^{\mathcal{T}}_{\gamma+1}[\Gamma_\gamma]$ with ω-many generators. We now apply *2.* of Lemma 5.4.4: we continue the iteration tree $\mathcal{T}$ by performing a genericity iteration to make x_{i+1} generic for $i^{\mathcal{T}}_{\gamma+1,\beta}(W^{\Gamma_\gamma}_{\eta_\gamma})$ over $\mathcal{M}^{\mathcal{T}}_{\beta}[\Gamma_\gamma]$, where $\mathcal{M}^{\mathcal{T}}_{\beta}$ is some iterate of $\mathcal{M}^{\mathcal{T}}_{\gamma+1}$, such that $\mathrm{crit}(i^{\mathcal{T}}_{\gamma+1,\beta}) > j^{\mathcal{T}}_{0,\gamma}(\delta)$.
A model of the form $\mathcal{M}^{\mathcal{T}}_{\beta}[\Gamma_\gamma]$ is well-defined since Γ_γ is small forcing over $\mathcal{M}^{\mathcal{T}}_{\gamma+1}$ and $\mathrm{crit}(j^{\mathcal{T}}_{\gamma+1,\beta}) > j^{\mathcal{T}}_{0,\gamma}(\delta)$ by Lemma 5.4.4. Also the genericity iteration to make x_{i+1} generic over a small forcing extension of an iterate terminates after countably many steps. Note that we never apply extenders to models with index $< \gamma+1$ (every extender used in the construction of $W^{\Gamma_\gamma}_{\eta_\gamma}$, i.e. every extender on $\vec{F}$, has critical point $> j^{\mathcal{T}}_{0,\gamma}(\delta)$; since $\nu(E_\zeta) < j^{\mathcal{T}}_{0,\gamma}(\delta)$ for all $\zeta < \gamma$ we see that the extenders are never applied to models with index $< \gamma+1$). So we have that x_{i+1} is generic over $\mathcal{M}^{\mathcal{T}}_{\beta}[\Gamma_\gamma]$. We now want to apply Lemma 5.4.2 to find Γ_β. Let D denote the collection of all dense sets of $j^{\mathcal{T}}_{\gamma+1,\beta}(W^{\Gamma_\gamma}_{\eta_\gamma})$ computed in $\mathcal{M}^{\mathcal{T}}_{\beta}[\Gamma_\gamma]$. Recall that $p \Vdash \omega_1 = \check{\delta}$, hence we have for all $q \in j^{\mathcal{T}}_{\gamma+1,\beta}(W^{\Gamma_\gamma}_{\eta_\gamma})$

$$j^{\mathcal{T}}_{\alpha_0,\beta}(p) \Vdash^{\mathcal{M}^{\mathcal{T}}_{\beta}[\Gamma_\gamma]} \exists g \subset j^{\mathcal{T}}_{\gamma+1,\beta}(W^{\Gamma_\gamma}_{\eta_\gamma})[\check{q} \in g \wedge g \text{ meets every } d \in \check{D}].$$

So by Lemma 5.4.3 and Lemma 5.4.2 we find a generic filter Γ_β extending Γ_γ such that $x_{i+1} \in \mathcal{M}^{\mathcal{T}}_{\beta}[\Gamma_\beta]$. This finishes the construction of $\mathcal{T}$ and $\langle \Gamma_i; i \le \omega_1 \rangle$.
Let $b = [0, \omega_1]_{\mathcal{T}}$ be the uncountable branch through $\mathcal{T}$. By construction, b contains every α_i; hence $j^{\mathcal{T}}_{0,\omega_1}(\delta) = \omega_1^V$.
So $\Gamma := \Gamma_{\omega_1} \subset j^{\mathcal{T}}_{0,\omega_1}(W_\delta)$ is generic over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$, and $j^{\mathcal{T}}_{0,\omega_1}(p) \in \Gamma$. We have to check $\mathbb{R}^V \in \mathcal{M}^{\mathcal{T}}_{\omega_1}[\Gamma]$. Consider some $x_i \in \mathbb{R}^V$. By construction $x_i \in \mathcal{M}^{\mathcal{T}}_{\alpha_i}[\Gamma_i]$, so there is a nice name σ such that $x_i = \sigma^\Gamma_i$. By the δ-c.c. of W_δ, σ is not moved by $j^{\mathcal{T}}_{\alpha_i,\omega_1}$ and since $\Gamma_i = \Gamma \cap W_{\delta_i}$, we have $x_i = \sigma^\Gamma_i \in \mathcal{M}^{\mathcal{T}}_{\omega_1}$.
Recall that $p \in \Gamma$ and that p was not moved by $j^{\mathcal{T}}_{0,\omega_1}$. By elementarity it now suffices to iterate the top-extender of $\mathcal{M}^{\mathcal{T}}_{\omega_1}[\Gamma]$ out of the universe to obtain

$$V \models \exists A' \subset \mathbb{R}^V L(\mathbb{R}^V, A') \models \phi(A', z).$$

The same method yields a proof for the converse direction: basically one changes the roles of $V[G]$ and V; i.e. in $V[G]$ replace $\langle x_i; i < \omega_1 \rangle$ by an enumeration of the reals of $V[G]$, run the according tree construction in $V[G]$ □

It is possible to add parameters besides reals to the formulae above, using for example Lemma 5.3.8. Also one can add a subset of the reals captured by a term for example. Nevertheless the same restrictions to the complexity of such parameters as before apply, see Lemma 5.3.9.

5.5 Subsets of ω_1 in Forcing Extensions

The classic genericity iteration to make a fixed real generic has a generalization for reals living in forcing extensions. It is possible to produce a long iteration such that all interpretations of a name for a real are generic:

Theorem 5.5.1 (Woodin) *Let $\mathbb{P}$ be a forcing of size κ and suppose the sound premouse $\mathfrak{M} = \langle J_\beta[\vec{E}]; \in, \vec{E}, E_\beta \rangle$ is active and has a $(\omega, \kappa^+ + 1)$-iteration strategy*

Σ such that $\vec{E}$ witnesses the Woodiness of δ. Let W denote the extender algebra with ω many generators relative to $\vec{E}$. Let $\dot{x} \in V^{\mathbb{P}}$ be a name for a real. Then there exists an iteration $j : \mathfrak{M} \to \mathfrak{M}^*$ in V of length $< \kappa^+$ such that for all $G \subset \mathbb{P}$ generic over V the real $\dot{x}^G$ is $j(W)$-generic over $\mathfrak{M}$.

We do not give a proof of the above theorem but refer the reader to the appendix [NZ] of [NZ01]; we will give a proof of a more general result, Lemma 5.6.16, with a similar proof. We aim to generalize the above theorem to subsets of ω_1. The first generalization is the following theorem which allows us to make subsets of ω_1 in c.c.c. forcing extensions generic over an iterate living in V; clearly the following theorem also generalize Theorem 5.3.1. The second generalization is Lemma 5.6.16, which allows us to make certain subsets of ω_1 living in reasonable extensions generic over an iterate in V.

Theorem 5.5.2 *Let $\mathbb{P}$ be any c.c.c. forcing. Let $\dot{A}$ be a $\mathbb{P}$-name such that*

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{A} \subset \check{\omega}_1.$$

Let $\mathfrak{M} = \langle J_\beta[\vec{E}]; \in, \vec{E}, E_\beta \rangle$ be a sound premouse that is active and has a (ω, ω_1+1)-iteration strategy Σ such that $\vec{E}$ witnesses the Woodiness and measurability of δ. Then there exists an iteration $j : \mathfrak{M} \to \mathfrak{M}^*$ of length ω_1 in V such that for all $G \subset \mathbb{P}$ generic over V the set $\dot{A}^G$ is $j(W_\delta)$-generic over $\mathfrak{M}$.

The proof we are about to give is very similar to the one for Theorem 5.3.1; we will omit some details that we gave in the proof for Theorem 5.3.1.

Proof. Let U on $\vec{E}$ be the extender with the least index witnessing the measurability of δ, i.e. U is (the trivial completion of) a normal complete measure on δ. Let ζ_0 be the index of U. We construct an iteration tree $\mathcal{T}$ of length ω_1+1 on $\mathcal{M}_0^{\mathcal{T}} = \mathfrak{M}$. We will call an $\alpha \leq \omega_1$ a $\mathbb{P}$-*baby closure point for* $\dot{A}$ if for all $p \in \mathbb{P}$

$$p \Vdash \dot{A} \cap j_{0,\alpha}^{\mathcal{T}}(\delta) \models j_{0,\alpha}^{\mathcal{T}}(T(\vec{E})).$$

To ensure normality of the resulting iteration we need a more technical definition: $\alpha \leq \omega_1$ is a $\mathbb{P}$-*closure point for* $\dot{A}$ if for all $p \in \mathbb{P}$ and all $\zeta < j_{0,\alpha}^{\mathcal{T}}(\zeta_0)$ and $\vec{F}$ $\mathcal{M}_\alpha^{\mathcal{T}}$'s extender sequence

$$p \Vdash \dot{A} \cap j_{0,\alpha}^{\mathcal{T}}(\zeta_0) \text{ does not contradict any axiom induced by } \check{F}_\zeta.$$

Clearly any $\mathbb{P}$-closure point for $\dot{A}$ is a $\mathbb{P}$-baby closure point for $\dot{A}$ and limits of $\mathbb{P}$-closure points for $\dot{A}$ are $\mathbb{P}$-baby closure points for $\dot{A}$.
We define the iteration as follows: in the limit case we use Σ to continue the iteration. In the successor case there are two subcases: if $\alpha < \omega_1$ is a $\mathbb{P}$-closure point for $\dot{A}$, then we use $j_{0,\alpha}^{\mathcal{T}}(U)$ to continue the iteration. If α is not a closure point, then there is a least "bad" extender E on the extender sequence of $\mathcal{M}_\alpha^{\mathcal{T}}$ and some $p \in \mathbb{P}$ such that

$$p \Vdash \dot{A} \cap j_{0,\alpha}^{\mathcal{T}}(\zeta_0) \not\models \phi,$$

where ϕ is some axiom induced by E. We then use E to continue the iteration. This finishes the construction of $\mathcal{T}$. The arguments we have given before make sure $\mathcal{T}$ is a normal tree. Let $b = [0, \omega_1]_{\mathcal{T}}$ and let $j = j_{0,\omega_1}^{\mathcal{T}} : \mathfrak{M} \to \mathcal{M}_{\omega_1}^{\mathcal{T}}$. We set $\mathfrak{M}^* = \mathcal{M}_{\omega_1}^{\mathcal{T}}$. Let us now check that there are unboundedly many (in fact club many) $\mathbb{P}$-closure points for $\dot{A}$ in b; so suppose towards a contradiction that the set of closure points is bounded in ω_1 say by $\eta < \omega_1$. Pick a countable $X \prec V_\lambda$ for some large enough λ such that $\omega_1 \cap X > \eta$ and $\dot{A}, \mathcal{T}, \mathbb{P} \in X$. Let $\pi : H \to X$ denote the inverse of the transitive collapse of X and let $\alpha = \omega_1 \cap X$. Then $\pi \restriction \mathcal{M}_\alpha^{\mathcal{T}} = j_{\alpha,\omega_1}^{\mathcal{T}}$. Since $\alpha \in b$

there is a direct $\mathcal{T}$-successor of α, say $\gamma+1$. Then there is a $p \in \mathbb{P}$ that forces that the extender $E_\zeta^{\mathcal{M}_\gamma^\mathcal{T}}$ on the $\mathcal{M}_\gamma^\mathcal{T}$-sequence is the minimal extender that induces a bad axiom. Let $G \subset \mathbb{P}$ be V-generic such that $p \in G$. We show that G is generic over X: Let $D \in X$ be an antichain of $\mathbb{P}$; then $q \in G \cap D$ for a unique q. Since D can be enumerated in ordertype ω, we have $q \in X \cap G$. Moreover we show: $X[G] \cap V = X$, this follows from the following claim:

Claim 1. Let $\tau \in X$ be a $\mathbb{P}$-name, let $\mathbb{B} = \mathrm{ro}(\mathbb{P})$ and let $q := [[\tau \in \check{V}]]_\mathbb{B}$. Then $q \in X$ and there is a countable set $y \in X$ such that $q \Vdash \tau \in \check{y}$.
Proof of Claim 1. Clearly $q \in X$ by elementarity. Let

$$\mathcal{A} = \{q' \leq q\,;\, 0 \neq q' = [[\tau = \check{x}]] \text{ for some } x \in V\}.$$

Since $\mathbb{P}$ is c.c.c. $\mathcal{A}$ is countable. By elementarity $\mathcal{A} \in X$. Since $\mathcal{A}$ is countable we have $y \in X$. □(Claim 1)

Let $\hat{\pi} : \hat{H} \to X[G]$ denote the inverse of the transitive collapse of $X[G]$. Since $X[G] \cap V = X$, we have that $H \subset \hat{H}$ and $\hat{\pi} \restriction H = \pi$. Let $\bar{A}, \bar{G}$ be such that $\hat{\pi}(\bar{A}, \bar{G}) = \dot{A}, G$. So

$$\hat{H} \models \bar{A}^{\bar{G}} \cap j_{0,\alpha}^\mathcal{T}(\delta) \not\models j_{0,\alpha}^\mathcal{T}(T(\vec{E})).$$

As before we get the following claim:
Claim 2. We have

$$\hat{\pi} \restriction V_{\delta^*}^{\mathcal{M}_\alpha^{\bar{\mathcal{T}}}} = i_{\alpha,\omega_1}^\mathcal{T} \restriction V_{\delta^*}^{\mathcal{M}_\alpha^\mathcal{T}}.$$

□(Claim 2)

We have now reproduced the situation in the proof of Theorem 5.3.1 and can proceed like in that proof. Hence α is a $\mathbb{P}$-closure point for $\dot{A}$. By the argument at the end of the proof of Theorem 5.3.1 we have that ω_1 is a $\mathbb{P}$-baby closure point for $\dot{A}$. This suffices to show. □

We now refine the previous argument to show more Σ_1^2 absoluteness for the class of c.c.c. forcings; we allow not only real parameters but also ordinals.

5.5.1 Σ_1^2 Absoluteness and c.c.c. Forcing Extensions

Theorem 5.5.3 *Suppose $M_{\mathsf{mw}}^\sharp$ exists. Assume* CH *holds. Furthermore assume $\mathbb{P}$ is a c.c.c. forcing such that $V^\mathbb{P} \models \mathsf{CH}$. Let $G \subset \mathbb{P}$ be V-generic. Then*

$$V \models \exists A \subset \mathbb{R} : L(\mathbb{R}, A) \models \phi(A, z, \vec{\alpha})$$

if, and only if,

$$V[G] \models \exists A \subset \mathbb{R}^{V[G]} : L(\mathbb{R}^{V[G]}, A) \models \phi(A, z, \vec{\alpha}).$$

Here z is a real parameter and $\vec{\alpha}$ are finitely many ordinal parameters.

The proof will use ideas from the previous proof and from the proof of Theorem 5.4.1. It is convenient to introduce some notation: we will code two subsets of ω_1 into one. For this purpose we define the useful $\oplus$-operation and its reverse operations:

Definition 5.5.4 Given (maybe set-sized) classes $A, B \subset \mathsf{OR}$ we define the set $A \oplus B$ by $\gamma \in A \oplus B$ if and only if

$$(\exists \alpha \in A : \exists \alpha' \in \mathrm{Lim} : \exists n \in \omega : \alpha = \alpha' + n \wedge \gamma = \alpha' + 2n) \vee$$
$$(\exists \alpha \in B : \exists \alpha' \in \mathrm{Lim} : \exists n \in \omega : \alpha = \alpha' + n \wedge \gamma = \alpha' + 2n + 1).$$

Furthermore we implicitly define operations $(\cdot)_{\mathsf{even}}$ and $(\cdot)_{\mathsf{odd}}$ acting on classes of ordinals by demanding:

$$(A \oplus B)_{\mathsf{even}} = A$$

and

$$(A \oplus B)_{\mathsf{odd}} = B.$$

The intuition in the above definition is that A is mapped to the "even" ordinals and B to the "odd" ordinals. In the following we will make us of the following fact: if $\dot{A}$ and $\dot{B}$ are forcing names for sets of ordinals, then we can compute a forcing name $\dot{C}$ such that it is forced that $\dot{C} = \dot{A} \oplus \dot{B}$. In an abuse of notation, we will denote a name $\dot{C}$ as above by $\dot{A} \oplus \dot{B}$. We now show Theorem 5.5.3.

Proof. We will first give a detailed proof of the downwards direction of the absoluteness, i.e. we assume

$$L(\mathbb{R}^{V[G]}, \dot{B}^G) \models \phi(\dot{B}^G, z, \vec{\alpha})$$

for some $\dot{B}$, and we want to show

$$L(\mathbb{R}^V, B) \models \phi(B, z, \vec{\alpha})$$

for some $B \in V$. The converse direction of this absoluteness is a variant of what we are going to show now; we will mention some details for the upwards direction at the end of the proof.
We denote the measurable Woodin cardinal in $\mathfrak{M} = M^\sharp_{\mathsf{mw}}$ by δ and we let Σ denote $\mathfrak{M}$'s $(\omega, \omega_1 + 1)$-iteration strategy. Let us fix a $\mathbb{P}$-name $\dot{B}$ such that for all $G \subset \mathbb{P}$ generic over V

$$L(\mathbb{R}^{V[G]}, \dot{B}^G) \models \phi(\dot{B}^G, z, \vec{\alpha}) \wedge \dot{B}^G \subset \check{\omega}_1.$$

In the following we will suppress z and work with $\mathfrak{M} = M^\sharp_{\mathsf{mw}}$. We will construe $\mathfrak{M}$ and its iterates as class sized models if convenient (i.e. we will confuse $\mathfrak{M}$ with the class sized model one obtains when iterating $\mathfrak{M}$'s top measure out of the universe); we will need this fact to allow for arbitrarily large ordinal parameters at the end of this proof.
Set $\dot{A} = \dot{B} \oplus \dot{R}$ for some name $\dot{R}$ such that

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{R} \subset \check{\omega}_1 \wedge \dot{R} \text{ codes a well-ordering of } \mathbb{R}.$$

Our aim is to produce an iteration tree $\mathcal{T} \in V$, p and $\Gamma \in V$ such that

- $\mathcal{T}$ on $\mathcal{M}^{\mathcal{T}}_0 = \mathfrak{M}$ is of length $\omega_1 + 1$,
- for all $G \subset \mathbb{P}$ generic over V the set $\dot{A}^G$ is generic for $j^{\mathcal{T}}_{0,\omega_1}(W_\delta)$ over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$,
- $p \in j^{\mathcal{T}}_{0,\omega_1}(W_\delta)$ is such that

  $$p \Vdash L(\mathbb{R}, (\dot{\Gamma})_{\mathsf{even}}) \models \phi((\dot{\Gamma})_{\mathsf{even}}, \vec{\alpha}),$$

  where $\dot{\Gamma}$ is the canonical name for a $j^{\mathcal{T}}_{0,\omega_1}(W_\delta)$-generic,
- $\Gamma \subset j^{\mathcal{T}}_{0,\omega_1}(W_\delta)$ contains p and is generic over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$,
- $\mathbb{R}^V \subset \mathcal{M}^{\mathcal{T}}_{\omega_1}[\Gamma]$.

For this our stratgey is as follows: like in the proof for Theorem 5.4.1 we have to piece together end-extending generics. Again it is helpful to assume that the conditions of the extender algebra are not equivalence classes of formulae, but take the form of (minimal) formulae. In the proof of 5.4.1, we knew p from the beginning, in this proof we will have to consider all possible p; also we have ordinal parameters present which are moved in general by iterating, so we will have to arrange that in $V^{\mathbb{P}}$ the set $\dot{A}$ is generic over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$.
We will drop the superscript $\mathcal{T}$ in the rest of this proof; i.e. $\mathcal{T}$ has models $\mathcal{M}_\alpha$ and maps $j_{\alpha,\beta}$. We prepare a book-keeping device: let $\langle y_i; i < \omega_1\rangle$ be an enumeration of the reals of V and for $i < \omega_1$ let $x_i \in V$ be such that $\langle y_j; j \leq i\rangle \in L[x_i]$. Let U on $\vec{E}$ be the extender with the least index witnessing the measurability of δ, i.e. U is (the trivial completion of) a normal complete measure on δ, and let ζ_0 be the index of U, in fact ζ_0 is the height of $M^{\sharp}_{\mathsf{mw}}$ and U is $M^{\sharp}_{\mathsf{mw}}$'s top-measure. An ordinal α is a $\mathbb{P}$-*closure point for* $\dot{A}$ if for all $q \in \mathbb{P}$ and all $\zeta < j_{0,\alpha}(\zeta_0)$

$$q \Vdash \dot{A} \cap j_{0,\alpha}(\zeta_0) \text{ does not contradict any axiom induced by } \check{F}_\zeta,$$

where $\vec{F}$ denotes $\mathcal{M}_\alpha$'s extender sequence; in this case we clearly have: for all $q \in \mathbb{P}$

$$q \Vdash \dot{A} \cap j_{0,\alpha}(\delta) \models j_{0,\alpha}(T(\vec{E})),$$

we will call α a $\mathbb{P}$-*baby closure point for* $\dot{A}$ if it only satisfies this weaker property. As before we have that a limit of $\mathbb{P}$-baby closure points for $\dot{A}$ is also a $\mathbb{P}$-baby closure point for $\dot{A}$; in general a limit of $\mathbb{P}$-closure points for $\dot{A}$ is just a $\mathbb{P}$-baby closure point for $\dot{A}$.
We now formally define the iteration tree $\mathcal{T}$ in ω_1-many rounds; each round i starts at a stage α_i of $\mathcal{T}$. Set $\mathcal{M}_0 = \mathfrak{M}$. In each round α_i we have $\mathcal{T} \upharpoonright (\alpha_i + 1)$ defined and so $\mathcal{M}_{\alpha_i}$ exists. The tree $\mathcal{T}$ and the ordinals $\langle \alpha_i; i < \omega_1\rangle$ will have the following properties:

1. $\mathcal{T} \in V$ is an iteration tree on $\mathfrak{M}$ of length $\omega_1 + 1$,
2. the set $\{\alpha_i\,;\, i < \omega_1\}$ is a club of $\mathbb{P}$-baby closure points for $\dot{A}$.

Additionally, for $i < \omega_1$ and $p \in j_{0,\alpha_i}(W_\delta)$ such that

$$p \Vdash j_{0,\alpha_i}(\check{\delta}) = \dot{\omega}_1$$

we will pick a generic Γ^p_i, with $p \in \Gamma^p_i$. For this it is convenient to introduce some objects: we will define a partial regressive function j that maps α_i to the maximal $\alpha_j < \alpha_i$ such that the generic Γ^p_j can be extended to a generic Γ^p_i. We now define j formally: for $\gamma < \omega_1$ we inductively define

$$J(\gamma) := \{j\,;\, \alpha_j \in [0,\gamma[_{\mathcal{T}} \wedge \operatorname{crit}(j_{\alpha_j,\gamma}) = j_{0,\alpha_j}(\delta)\},$$

So if $j \in J(\gamma)$ we have $j_{\alpha_j,\gamma} \upharpoonright j_{0,\alpha_j}(\delta) = \mathrm{id}$. It is not difficult to check that $J(\gamma)$ is a closed set if γ is not a limit of α_i. If $\gamma = \alpha_\lambda$ for a limit λ, then $J(\gamma)$ might be unbounded in λ. We set

$$j(\gamma) = \max(J(\gamma)),$$

if $\max(J(\gamma)) < \gamma$ exists, and let $j(\gamma)$ be undefined else. Here we want the maximum of the empty set to be undefined. For $j \leq i$ look at the map $j_{\alpha_j,\alpha_i} : \mathcal{M}_{\alpha_j} \to \mathcal{M}_{\alpha_i}$, if it exists. If this map does not exist, then the following definition trivializes, i.e. $P_{j,i} = \emptyset$. Let

$$P_{j,i} := \{p \in j_{0,\alpha_j}(W_\delta)\,;\, j_{\alpha_j,\alpha_i} \upharpoonright j_{0,\alpha_j}(\delta) = \mathrm{id} \wedge p \Vdash j_{0,\alpha_j}(\check{\delta}) = \dot{\omega}_1\},$$

so that $P_{j,i}$ is empty if $j_{\alpha_j,\alpha_i} \restriction j_{0,\alpha_j}(\delta)$ is not the identity. Let

$$P_i := P_{j(\alpha_i),i}$$

if $j(\alpha_i)$ is defined and empty otherwise. Finally let

$$P^i := \{p \in j_{0,\alpha_i}(W_\delta)\,;\, p \Vdash j_{0,\alpha_i}(\check{\delta}) = \dot{\omega}_1 \wedge p \notin P_i\}.$$

The generic Γ_i^p will satisfy the following conditions:

3. if $p \in P_i \cup P^i$, then $p \in \Gamma_i^p \subset j_{0,\alpha_i}(W_\delta)$ and Γ_i^p is generic over $\mathcal{M}_{\alpha_i}$,
4. if $j(\alpha_i)$ is defined and if $p \in P_i$, then Γ_i^p end-extends $\Gamma_{j(\alpha_i)}^p$,
5. if λ is a limit ordinal and $J(\alpha_\lambda)$ is unbounded in λ, then

$$\Gamma_\lambda^p = \bigcup\{\Gamma_j^p\,;\, j \in J(\alpha_\lambda)\}$$

for all $p \in P^\lambda$,

6. if $p \in P_i$ and $j(\alpha_i)$ is defined, then the real $x_{j(\alpha_i)}$ is generic over $\mathcal{M}_{\alpha_i}[\Gamma_{j(\alpha_i)}^p]$ for a forcing of cardinality $< j_{0,\alpha_i}(\delta)$,
7. if $j(\alpha_i)$ is defined and $p \in P_i$, then $x_{j(\alpha_i)} \in \mathcal{M}_{\alpha_i}[\Gamma_i^p]$,
8. each Γ_i^p is generic over all models with index $\gamma \geq \alpha_i$.

Once we state how we construct the iteration in each round, the last item above will follow easily by the agreement of models of an iteration tree. At each limit ordinal $\leq \omega_1$ we use Σ to continue the iteration tree $\mathcal{T}$. Suppose we have already constructed the iteration with the above properties up to a stage α_i, i.e. we have produced $\mathcal{T} \restriction (\alpha_i + 1)$. We now describe a tree $\mathcal{U}$ of length $\omega_1 + 1$ that continues $\mathcal{T} \restriction (\alpha_i + 1)$. After we do so, we will decide which countable β is α_{i+1}, i.e. $\mathcal{T} \restriction (\alpha_{i+1} + 1) = \mathcal{U} \restriction \beta$ for some countable β.
Say the construction of $\mathcal{U}$ has reached a countable stage $\beta \geq \alpha_i$. There are three rules, (P1), (P2) and (P3) that define the iteration at a stage β. These rules tell us which extender we use; (P1) in fact gives rise to countably many rules. We use the minimal extender E with

(P1) $j_{0,\alpha_i}(\zeta_0) < \mathrm{crit}(E) < j_{0,\beta}(\delta)$ and there is some $j \leq i$, some $k \leq i$ and some $p \in P^j \cup P_j$ such that in $\mathcal{M}_\beta[\Gamma_j^p]$ the extender $\tilde{E}$ induces an axiom false of x_k, or

(P2) there is some $q \in \mathbb{P}$ such that

$$q \Vdash \dot{A} \cap j_{0,\beta}(\check{\zeta}_0) \text{ does contradict an axiom induced by } \check{E}.$$

Here $\tilde{E}$ is the induced extender in the sense of Lemma 5.4.4. We explicitly do not fix a system of extenders and a Woodin cardinal. One can define axioms induced by extenders independently of a Woodin cardinal. Of course later we will specify a system. Also note that $\mathcal{M}_\beta[\Gamma_j^p]$ in the definition of (P1) is well-defined by condition 8. For $\beta = \alpha_i$ the rule (P1) is trivial.
If none of the above rules imply that we have to use an extender, then rule (P3) tells us what to do:

(P3) If neither (P1) nor (P2) implies that we use an extender, use the top-measure $j_{0,\beta}(U)$ of $\mathcal{M}_\beta$, i.e. the measure witnessing that $j_{0,\beta}(\delta)$ is measurable.

So if we use $j_{0,\beta}(U)$, then especially β is a $\mathbb{P}$-closure point for $\dot{A}$. This fact and the fact that we always picked the minimal extenders at all stages yield that the extenders we used are of increasing length, i.e. the resulting iteration is normal.
We now show that in the construction of $\mathcal{U}$ we reach a stage where neither (P1) nor (P2) implies that we use an extender, so that the top-measure is actually used. Assume that this was not the case and work towards a contradiction. So in the ith round we produce a tree $\mathcal{U}$ of length $\omega_1 + 1$ such that rule (P3) was not used at a stage $\beta \geq \alpha_i$. This implies that (P1) was unboundedly often the reason why we had to apply an extender, otherwise by the argument from 5.5.2 we reach a $\mathbb{P}$-closure point for $\dot{A}$ after countably many steps.
Claim 1. Stationarily often (P1) was the reason why we had to continue the iteration.
Proof of Claim 1. If (P1) was at nonstationary many stages the reason why we had to continue the iteration, then there is a club $C \subset \omega_1$ of points such that (P1) was not the reason, and hence (P2) was. Now pick an elementary substructure $X \prec V_\lambda$ for a large enough λ such that $\gamma = X \cap \omega_1 \in C$ and $\mathcal{T} \in X$. Making use of $\mathbb{P}$'s c.c.c., like in the proof for Theorem 5.5.2, we can now study the embedding $X[G] \prec V_\lambda[G]$ for a $G \subset \mathbb{P}$ generic over V. But then an argument like in the proof for Theorem 5.5.2 shows that γ is a $\mathbb{P}$-closure point for $\dot{A}$. Contradiction! □(Claim 1)

By an application of Fodor's Theorem there is a stationary S, a k and a p such that for all $\beta \in S$: some extender $\tilde{E}$ induces an axiom false of x_k, in the sense of (P1). Pick an elementary substructure $X \prec V_\lambda$ for a large enough λ such that $\alpha_i < \gamma = X \cap \omega_1 \in S$ and $\mathcal{U} \in X$. But then an argument like in the proof for Theorem 5.3.1 shows that x_k is generic over $\mathcal{M}_\gamma[\Gamma_i^p]$ (here one has keep in mind that $\mathcal{M}_\gamma[\Gamma_i^p]$ is a small forcing extension of $\mathcal{M}_\gamma$, see Lemma 5.4.4). This contradicts $\gamma \in S$!
This shows that unboundedly often during the construction of $\mathcal{U}$ rule (P3) implied that we had to use the top-measure at a stage $\beta \geq \alpha_i$. An easier version of this argument shows that we reach unboundedly many stages β, such that β is a $\mathbb{P}$-baby closure point for $\dot{A}$ and (P1) is not the reason why we have to use an extender at stage β, call such a stage *extraordinary.* Then we let α_{i+1} be the least extraordinary stage such that we have used the top-measure at some stage γ, $\alpha_i \leq \gamma < \alpha_{i+1}$. We set $\mathcal{T} \restriction (\alpha_{i+1}+1) := \mathcal{U} \restriction (\alpha_{i+1}+1)$. Clearly α_{i+1} is a $\mathbb{P}$-baby closure point, since α_{i+1} is extraordinary. We have to show how to pick the generics of the form Γ_{i+1}^p such that conditions 3. through 8. are satisfied. We first show:
Claim 2. For $j \leq i$, $k \leq i$ and $p \in P^j \cup P_j$ the real x_k is generic over $\mathcal{M}_{\alpha_{i+1}}[\Gamma_j^p]$ for a forcing of cardinality $< j_{0,\alpha_{i+1}(\delta)}$.
Proof of Claim 2. Let $\eta < j_{0,\alpha_{i+1}(\delta)}$, $\eta > j_{0,\alpha_i}(\zeta_0)$ be a Woodin cardinal such that there are extenders $\vec{F}$ on $\mathcal{M}_{\alpha_{i+1}}$'s extender sequence that witness that η is Woodin. In the construction of $\mathcal{U}$ we picked α_{i+1} as an extraordinary stage, so none of the extenders from $\vec{F}$ induce an extender $\tilde{F}$ in $\mathcal{M}_{\alpha_{i+1}}[\Gamma_j^p]$ that induces an axiom false of x_k. So x_k is generic over $\mathcal{M}_{\alpha_{i+1}}[\Gamma_j^p]$ for the extender algebra with ω-many generators calculated from $\{\tilde{F} \in \mathcal{M}_{\alpha_{i+1}}[\Gamma_j^p]\,;\, F \text{ on } \vec{F}\}$. □(Claim 2)

This claim clearly shows more than what we demand in condition 6. Recall for any $j < \omega_1$ and any $p \in P^j \cup P_j$ we have $p \Vdash j_{0,\alpha_j}(\check{\delta}) = \dot{\omega}_1$, so if $j_{\alpha_j,\alpha_{i+1}}$ exists and p is not moved by $j_{\alpha_j,\alpha_{i+1}}$, we have by elementarity $p \Vdash j_{0,\alpha_{i+1}}(\check{\delta}) = \dot{\omega}_1$. So the powerset of any forcing of cardinality $< j_{0,\alpha_{i+1}}(\delta)$ is forced to be countable. We recall this fact because we are about to apply Lemma 5.4.2. Now pick a p in P_{i+1} and let $j = j(\alpha_{i+1})$. Inductively we already picked Γ_j^p. An argument like in the proof for Theorem 5.4.1, using $\mathrm{crit}(j_{\alpha_j,\alpha_{i+1}}) = j_{0,\alpha_j}(\delta)$, Lemmata 5.4.2 and 5.4.3, shows that we find Γ_{i+1}^p, an end-extension of Γ_j^p, such that $x_j \in \mathcal{M}_{\alpha_{i+1}}[\Gamma_{i+1}^p]$. This

choice satisfies condition 7. For $p \in P^{i+1}$ we pick $\Gamma^p_{i+1} \subset j_{0,\alpha_{i+1}}(W_\delta)$ generic over $\mathcal{M}_{\alpha+1}$, so 4. is satisfied. Note that since α_{i+1} is extraordinary, we will not pick an extender with length $< j_{0,\alpha_{i+1}}(\delta)$ when we continue the iteration at stage α_{i+1}, this shows that condition 8. holds.
For a limit λ of rounds we set $\alpha_\lambda = \sup\{\alpha_i\,;\, i < \lambda\}$. It is not difficult to see, using the agreement of models along the iteration, that α_λ is a $\mathbb{P}$-baby closure point for $\dot{A}$. We now have to pick the generics of the form Γ^p_λ. There are three cases. The first case is: $J(\alpha_\lambda)$ is unbounded in λ. Then we pick the generics according to condition 5.: for all $p \in P^\lambda$ set

$$\Gamma^p_\lambda = \bigcup\{\Gamma^p_j\,;\, j \in J(\alpha_\lambda)\}.$$

The second case is: $j(\alpha_\lambda)$ is undefined but the first case does not hold, then for $p \in P^{i+1}$ we pick $\Gamma^p_\lambda \subset j_{0,\alpha_\lambda}(W_\delta)$ generic over $\mathcal{M}_\lambda$. The third case is: $j := j(\alpha_\lambda)$ exists, then inductively we picked Γ^p_j. Recall that $j < \lambda$ and hence $\alpha_{j+1} < \alpha_\lambda$. The above claim shows: in the jth round we produced an iteration such that x_j was generic over $\mathcal{M}_{\alpha_{j+1}}$ for a forcing of cardinality $< j_{0,\alpha_{j+1}}(\delta)$. Since α_{j+1} is extraordinary, we used an extender with length $\geq j_{0,\alpha_{i+1}}(\delta)$ to continue the iteration, so by the agreement of models of $\mathcal{T}$

$$V^{\mathcal{M}_{\alpha_{j+1}}}_{j_{0,\alpha_{j+1}}(\delta)} = V^{\mathcal{M}_{\alpha_\lambda}}_{j_{0,\alpha_{j+1}}(\delta)}.$$

This implies that x_j is also generic over $\mathcal{M}_{\alpha_\lambda}$ for small forcing. This shows condition 6. By the argument from the successor case we find Γ^p_λ, an end-extension of Γ^p_j, such that $x_j \in \mathcal{M}_{\alpha_\lambda}[\Gamma^p_\lambda]$, making 7. true.
This finishes the construction of $\mathcal{T}$ and the family $(\Gamma^p_i)_i$.
Let $b = [0, \omega_1]_\mathcal{T}$ and let $j^* = j_{0,\omega_1} : \mathfrak{M} \to \mathcal{M}_{\omega_1}$. We set $\mathfrak{M}^* = \mathcal{M}_{\omega_1}$. We have shown that in each round we produce a countable iteration that terminates at an extraordinary stage. For this we showed that we actually use the top-measure. Another Skolem-hull argument of this type shows that there are unboundedly many $\beta \in b$ where we use the top-measure, so $j^*(\delta) = \omega_1$. It is easy to see that there are club many $\mathbb{P}$-baby closure points α_i for $\dot{A}$ in b such that $\mathrm{crit}(j_{\alpha_i,\omega_1}) = \alpha_i$. Since ω_1 is a limit of $\mathbb{P}$-baby closure points for $\dot{A}$, we have that ω_1 is also a $\mathbb{P}$-baby closure point for $\dot{A}$. Hence: if $G \subset \mathbb{P}$ is V-generic, then in $V[G]$ the set $\dot{A}^G$ is generic over $\mathfrak{M}^*$. Let $C \subset b$ denote a club of $\mathbb{P}$-baby closure points β for $\dot{A}$ such that $\mathrm{crit}(j_{\beta,\omega_1}) = \alpha_\beta = \beta = j_{0,\beta}(\delta)$.
We now iterate the top-measure of $\mathfrak{M}^*$ linearly and write $\mathfrak{M}^{**}$ for the resulting class sized model; we do this to make sure $\vec{\alpha} \in \mathfrak{M}^{**}$. Note that all generics of the for Γ^p_i are still generic over $\mathfrak{M}^{**}$ and $V^{\mathfrak{M}^*}_{\omega_1} = V^{\mathfrak{M}^{**}}_{\omega_1}$. Moreover if $G \subset \mathbb{P}$ is V-generic, then in $V[G]$ the set $\dot{A}^G$ is generic over $\mathfrak{M}^{**}$. So clearly there is a condition $p \in j(W_\delta)$ such that

$$p \Vdash^{j(W_\delta)}_{\mathfrak{M}^{**}} L(\dot{\mathbb{R}}, (\dot{\Gamma})_{\mathsf{even}}) \models \phi((\dot{\Gamma})_{\mathsf{even}}, \check{\vec{\alpha}}) \wedge j^*(\check{\delta}) = \dot{\omega}_1,$$

where $\dot{\Gamma}$ is the canonical name for the $j(W_\delta)$-generic. Then $p \in P_{j_0,\omega_1}$ for some countable $\alpha_{j_0} \in C$. Set

$$\Gamma = \bigcup\{\Gamma^p_i\,;\, j_0 < i \wedge i \in C\}.$$

We show that Γ is well defined: by the choice of C, we have that $C \cap \lambda \subset J(\alpha_\lambda)$ for every limit point $\lambda = \alpha_\lambda$ of C, so $J(\lambda)$ is unbounded in λ in this case. Then conditions 4. and 5. imply that the generics of the form Γ^p_i, $i \in C$, extend each other. Using that the antichains of $j^*(W_\delta)$ are of cardinality $< j^*(\delta)$, we see that $\Gamma \in V$ is generic over $\mathfrak{M}^*$ and hence over $\mathfrak{M}^{**}$.
We have to check $\mathbb{R}^V \subset \mathfrak{M}^*[\Gamma]$. If y is a real in V, then $y \in L[x_i]$ for all large enough i. Let $\alpha_i \in C$ and let α_j denote the least $\alpha_k > \alpha_i$ in b. We have $j(\alpha_j) = i$

and so $x_j \in \mathcal{M}_{\alpha_j}[\Gamma^p_j]$ by condition 7. Hence $x_i \in \mathfrak{M}^*[\Gamma]$, because a nice name for x_i is not moved by j_{α_i,ω_1}. Since i can be arbitrary large, we have $\mathbb{R}^V \in \mathfrak{M}^*[\Gamma]$. So $\mathbb{R}^V \in \mathfrak{M}^{**}[\Gamma]$.
By the choice of p

$$\mathfrak{M}^{**}[\Gamma] \models L(\mathbb{R}, (\Gamma)_{\mathsf{even}}) \models \phi((\Gamma)_{\mathsf{even}}, \vec{\alpha}).$$

This is what we needed to show for the downwards direction of the absoluteness.
For the upwards direction of the absoluteness, one runs a similar argument: we reverse the roles of $V[G]$ and V, still the construction takes place in V. Note for this we have to replace the sequence $\langle y_i; i < \omega_1\rangle$ by a sequence of names for reals. If $\dot{y} \in V^{\mathbb{P}}$ is a name for a real, then a genericity iteration for $\dot{y}$ still terminates after countably many steps, see [NZ01, Lemma 3]. This is the key fact one additionally needs in the converse direction. We replace $\dot{B}$ with an $B \in V$ such that $L(\mathbb{R}^V, B) \models \phi(B, z, \vec{\alpha})$ and so we replace $\mathbb{P}$-(baby) closure points for $\dot{A}$ with (baby)-closure points (for an appropriate A) in the sense of Theorem 5.3.1. We shall give no more details.
□

5.6 Sets that extend to a Class with unique Condensation

We mentioned that we cannot hope to generalize Theorem 5.5.3 to all proper forcings. One problem is that the witnesses for Σ^2_1 absoluteness are very general. If we restrict the choice of witnesses, then we can generalize Theorem 5.5.3. Sets that extend to a class with unique condensation, which we are about to define, are well-suited witnesses as we will see in Theorem 5.6.25.
We will systematically study the sets that extend to classes with unique condensation. Besides constructing examples, we will also show that a set that extends to a class with unique condensation, granted a large cardinal hypothesis, is constructible from a real.

Definition 5.6.1 Let $A \subset \omega_1$ and let $\kappa > \omega_1$ be a cardinal. We will say A *extends to* A^* *with unique condensation up to* κ if $A^* \subset \kappa$ is a set such that

1. $A^* \cap \omega_1 = A$,
2. if $\lambda > \kappa$ is a sufficiently large regular cardinal, then there is a club $C(A^*, \kappa, \lambda)$ of countable substructures $X \prec H_\lambda$ such that $A^* \in X$ and $A \cap \bar{\kappa} = \bar{A}^*$, where π is the inverse of the collapse of X and $\pi(\bar{\kappa}, \bar{A}^*) = \kappa, A^*$.

We will say A *extends to a class* A^* *with unique condensation* if $A^* \subset \mathsf{OR}$ is a class such that

1. $A^* \cap \omega_1 = A$,
2. for all cardinals $\kappa > \omega_1$ A extends to $A^* \cap \kappa$ with unique condensation.

In the above definition sufficiently large means that for every $\kappa > \omega_1$ as above there is some λ_0 such that for all $\lambda \geq \lambda_0$ the above property holds. If A extends to a class A^* with unique condensation, then $\omega_1 \setminus A$ also extends to $\mathsf{OR} \setminus A^*$ with unique condensation. In Lemma 5.6.7 we will show that all $A \subset \omega_1$ that live in a model of AD extend to a class with unique condensation. We now show that the term "unique" is justified in the above definition.

Lemma 5.6.2 *If $A \subset \omega_1$ extends to A^* with unique condensation, then A^* is unique with this property.*

Proof. Suppose A^* and A^{**} are both classes to which A extends with unique condensation. It suffices to show that all ordinals β are in A^* if and only if $\beta \in A^{**}$. Fix an ordinal β. Let $\kappa > \omega_1$ be a regular cardinal $> \beta$ and let $X \in C(A^*, \kappa, \lambda) \cap C(A^{**}, \kappa, \lambda)$ be a countable substructure such that $A^* \cap \kappa, A^{**} \cap \kappa, \beta \in X$; here λ is sufficiently large for A^* and A^{**}. Let $\pi : M \to X$ denote the inverse of the collapsing map of X and let $\pi(\bar{A}^*, \overline{A^{**}}, \bar{\beta}, \bar{\kappa}) = A^* \cap \kappa, A^{**} \cap \kappa, \beta, \kappa$. Then by our hypothesis

$$\bar{A}^* = A \cap \bar{\kappa} = \overline{A^{**}}.$$

So by elementarity of π

$$\beta \in A^* \iff \bar{\beta} \in \bar{A}^* \iff \bar{\beta} \in \overline{A^{**}} \iff \beta \in A^{**}.$$

□

We will call an A^* as above a *uniquely condensing extension of* A.

Lemma 5.6.3 *Let $A, B \subset \omega_1$.*

1. *If A extends to a class A^* with unique condensation, then A contains a club if $\omega_1 \in A^*$ and A is nonstationary if $\omega_1 \notin A^*$.*
2. *If A is bounded in ω_1, then A extends to a class with unique condensation.*
3. *If A and B both extend to a class with unique condensation, then $A \oplus B$ extends to a class with unique condensation, too.*
4. *If A extends to a class with unique condensation and A' is obtained from A by replacing a countable initial segment of A with another countable set, then A' extends to a class with unique condensation.*

Proof. For 1. we will show that if $\omega_1 \in A^*$, then A contains a club; the other implication then follows from the previous lemma and the fact that $\omega_1 \setminus A$ extends to $\mathsf{OR} \setminus A^*$ with unique condensation. The following set contains a club $C := \{\alpha < \omega_1 ; \exists X \in C(A^*, \omega_2, \theta) : \alpha = X \cap \omega_1\}$, for some sufficiently large θ. By the unique condensation of A^* and $\omega_1 \in A^*$, we have that $C \subset A$.
The rest of the lemma is straightforward to verify: we list the appropriate witnesses for each case: If A is bounded, then $A^* = A$ is a uniquely condensing extension of A. If A and B both extend to a class with unique condensation, then there are classes A^*, B^* that witness this fact; it is not difficult to see that $A^* \oplus B^*$ witnesses that $A \oplus B$ extends to a class with unique condensation. If A^* is a uniquely condensing extension of A and $A' = (A \setminus \alpha) \cup a$ for some $a \subset \alpha < \omega_1$, then $A'^* := (A^* \setminus \alpha) \cup a$ is a uniquely condensing extension of A'. □

If A^* is a uniquely condensing extension of some A, then A^* satisfies even better condensation properties, as the following lemma shows:

Lemma 5.6.4 *Let $\kappa > \omega_1$ be a cardinal and let A extend to a class A^* with unique condensation. Let $F : [H_\theta]^{<\omega} \to H_\theta$ be such that the club $C_F := \{X \in [H_\theta]^\omega ; X$ is closed under $F\} \subset C(A^*, \kappa, \theta)$. Let $X \subset H_\theta$ of cardinality $< \kappa$ such that X is closed under F and let $\pi : M \to X$ denote the inverse of the transitive collapse of X. Then*

$$A^* \cap \bar{\kappa} = \overline{A^* \cap \kappa},$$

where $\pi(\bar{\kappa}, \overline{A^* \cap \kappa}) = \kappa, A^* \cap \kappa$.

Proof. Let $\theta' > \theta$ be a regular cardinal such that $H_\theta \in H_{\theta'}$. Let Y be a countable substructure of $H_{\theta'}$ such that $X, F \in Y$. Then $Z := X \cap Y$ is a countable substructure of H_θ and by elementarity of Y, Z is closed under F. So $Z \in C(A^*, \kappa, \theta)$. Let $\sigma : N \to Y$ denote the inverse of the transitive collapse of Y, so

$$\rho := \sigma^{-1}(\pi) : \sigma^{-1}(M) \to Z$$

is the inverse of the transitive collapse of Z. Then

$$\rho^{-1}(A^* \cap \kappa) = A \cap \rho^{-1}(\kappa).$$

By elementarity of σ we have that π also has the above property, i.e.

$$\pi^{-1}(A^* \cap \kappa) = A \cap \pi^{-1}(\kappa).$$

□

Lemma 5.6.5 *Let* $\mathsf{OR} \subset M \subset N$ *denote two transitive models of set theory such that* $\omega_1^M = \omega_1^N$. *If* $A \subset \omega_1$ *extends to a class* A^* *with unique condensation in* M *and* A^* *is definable in* N, *then* A *also extends to a class with unique condensation in* N.

Proof. We suppose that A^* did not witness that A extends to class with unique condensation in N and work towards a contradiction. So given an uncountable N-cardinal $\lambda > \omega_1$, there are unboundedly many θ such that the second part of Definition 5.6.1 fails for λ and θ in N; i.e. the set of countable $X \prec H_\theta^N$ such that $A, A^* \cap \lambda \in X$ and $\pi^{-1}(A \cap \lambda) \neq A \cap \pi^{-1}(\lambda)$ is stationary. On the other hand, there is a club $C(A^*, \theta, \lambda)$ in M that witnesses that A^* has unique condensation. Say all countable structures closed under $F : [H_\theta]^{<\omega} \to H_\theta$ are in $C(A^*, \theta, \lambda)$ for some $F \in M$.
Pick $X \prec \langle H_\theta^N; \in, H_\theta^M \rangle$ countable with $A, A^* \cap \lambda, p \in X$ and $\pi^{-1}(A \cap \lambda) \neq A \cap \pi^{-1}(\lambda)$, where $\pi : \langle \bar{X}; \in, \bar{H} \rangle \to X$ denote the inverse of the transitive collapse of X. Since $\omega_1^M = \omega_1^N$, we can assume without loss of generality that $Y = X \cap H_\theta^M$ is closed under F. Then $\bar{H}$ is the transitive collapse of Y and $\pi \restriction \bar{H}$ is its uncollapsing map. So

$$(\pi \restriction \bar{H})^{-1}(A \cap \lambda) = \pi^{-1}(A \cap \lambda) \neq A \cap \pi^{-1}(\lambda) = A \cap (\pi \restriction \bar{H})^{-1}(\lambda).$$

We now study the tree T of height ω searching for a countable substructure Z of H_θ^M, Z closed under F such that $\sigma^{-1}(A \cap \lambda) \neq A \cap \sigma^{-1}(\lambda)$, where σ is the inverse of the transitive collapse of Z. Then $T \in M$ and Y witnesses that T is ill-founded in N. By absoluteness of well-foundedness, we have a branch Z through T, $Z \in M$. But since Z is closed under F, we have $Z \in C(A^*, \theta, \lambda)$, a contradiction! □

5.6.1 Constructing sets with uniquely condesing extensions

We now show that any $A \subset \omega_1$ coded by a universally Baire sets of reals extends to a class with unique condensation. For this let us fix a recursive function $\{(\cdot)^i \,;\, i < \omega\}$ that maps a real y to a countable set of reals $\{y^i \,;\, i < \omega\}$.

Lemma 5.6.6 *Let* $A \subset \omega_1$ *be unbounded in* ω_1 *and let* $B \subset \omega^\omega$ *be a set of reals with the following properties:*

1. *B is universally Baire;*
2. *if* $y \in B$ *and* $\{y^i \,;\, i < \omega\} \subset \mathsf{WO}$, *then* $\{||y^i|| \,;\, i < \omega\} = A \cap \alpha$ *for some* $\alpha < \omega_1$;

3. *for every* $\beta < \omega_1$ *there is some* $y \in B$ *such that* $\{y^i\,;\, i < \omega\} \subset \mathsf{WO}$ *and* $\{||y^i||\,;\, i < \omega\} = A \cap \alpha$ *for some* $\beta < \alpha < \omega_1$*;*

4. *if* $y \in B$ *and* $\{y^i\,;\, i < \omega\} \subset \mathsf{WO}$ *and* $z \in \omega^\omega$ *is such that* $\{z^i\,;\, i < \omega\} \subset \mathsf{WO}$ *and* $\{||y^i||\,;\, i < \omega\}$ *end-extends* $\{||z^i||\,;\, i < \omega\}$*, then* $z \in B$*.*

Then A *extends to a class with unique condensation.*

Proof. For every cardinal κ we fix trees T_κ, S_κ such that $B = p[T_\kappa]$ and $p[T_\kappa] = \omega^\omega \setminus p[S_\kappa]$ in all forcing extensions by forcings of cardinality $\leq \kappa$. We can now define the uniquely condensing extension A^* of A. We set $\alpha \in A^*$ if and only if

$$V^{\mathrm{Col}(\omega,\alpha)} \models \exists y \in p[\check{T}_\kappa] : \{y^i\,;\, i < \omega\} \subset \mathsf{WO} \wedge \check{\alpha} \in \{||y^i||\,;\, i < \omega\},$$

where κ is the least cardinal $> \alpha$. Note that by the homogeneity of $\mathrm{Col}(\omega,\alpha)$ the above statement is decided by $\mathbf{1}_{\mathrm{Col}(\omega,\alpha)}$.

Claim 1. A^* is definable from B and furthermore A^* does not depend on the choice of the family of trees $(T_\kappa, S_\kappa)_\kappa$.

Proof of Claim 1. It will suffice to show that the set A^* does not depend on the choice of the trees T_κ, S_κ. If we can show this, then A^* is definable from any class of trees witnessing the universal Baireness of B.

We fix another pair of trees T'_κ, S'_κ witnessing that B is κ-universally Baire. Assume for some $\alpha < \kappa$ there is some real $\dot{y} \in V^{\mathrm{Col}(\omega,\alpha)}$ such that $\dot{y} \in p[\check{T}_\kappa]$ and $\dot{y} \in p[\check{S}'_\kappa]$, then the tree U searching for a branch through T_κ and S'_κ is ill-founded in $V^{\mathrm{Col}(\omega,\alpha)}$; note that U is without loss of generality in V. By the absoluteness of well-foundedness, this tree is ill-founded in V. So there is some $z \in V$ such that $z \in p[T_\kappa] \cap p[S'_\kappa]$. This contradicts the fact that in V

$$p[T_\kappa] = \omega^\omega \setminus p[S_\kappa] = \omega^\omega \setminus p[S'_\kappa].$$

□(Claim 1)

We now have to show that $A^* \cap \omega_1 = A$. By the choice of B it is not difficult to see that $A \subset A^*$. So let $\alpha \in A^* \cap \omega_1$, we have to show $\alpha \in A$. Let $\dot{y} \in V^{\mathrm{Col}(\omega,\alpha)}$ be such that

$$V^{\mathrm{Col}(\omega,\alpha)} \models \dot{y} \in p[\check{T}_\kappa] \wedge \{\dot{y}^i\,;\, i < \omega\} \subset \mathsf{WO} \wedge \check{\alpha} \in \{||\dot{y}^i||\,;\, i < \omega\}.$$

Let λ be regular and large enough such that $\dot{y}, T_{\omega_1}, S_{\omega_1}, \mathbb{R} \in H_\lambda$ and let $X \prec H_\lambda$ be countable such that $\dot{y}, T_{\omega_1}, S_{\omega_1} \in X$ and $\alpha < X \cap \omega_1$. Let $\pi : \bar{H} \to X$ be the inverse of the transitive collapse of X and let $\pi(\bar{y}, \bar{T}, \bar{S}) = \dot{y}, T_{\omega_1}, S_{\omega_1}$. Let $g \in V$, $g \subset \mathrm{Col}(\omega,\alpha)$ be an arbitrary generic over $\bar{H}$. Then $\bar{y}^g \in p[\bar{T}]$, so for some f with domain ω we have $(\bar{y}^g, f) \in [\bar{T}]$. Hence back in V we have $(\bar{y}^g, \cup\{\pi(f \restriction n\,;\, n \in \omega\}) \in [T_\kappa]$, so $\bar{y}^g \in B$. This implies that $\alpha \in A$.

We have to show the second item in Definition 5.6.1; the argument for this will be similar to the argument we have just given for $A^* \cap \omega_1 = A$. So let us fix a cardinal $\kappa > \omega_1$ and let λ be regular and large enough such that $B, A^* \cap \kappa, T_\kappa, S_\kappa \in H_\lambda$. Pick $X \prec H_\lambda$ such that $A^* \cap \kappa, B \in X$ and let $\pi : \bar{H} \to X$ denote the inverse of the transitive collapse of X. Since $B \in X$, there are two trees $T, S \in X$ that witness that B is κ-universally Baire. Let $\pi(\bar{\kappa}, \bar{A}^*, \bar{T}, \bar{S}) = \kappa, A^* \cap \kappa, T, S$. We have to show $A \cap \bar{\kappa} = \bar{A}^*$. Fix $\alpha \in \bar{A}^*$. Then by elementarity there is some $\dot{y} \in \bar{H}^{\mathrm{Col}(\omega,\alpha)}$ such that

$$\bar{H}^{\mathrm{Col}(\omega,\alpha)} \models \dot{y} \in p[\check{\bar{T}}] \wedge \{\dot{y}^i\,;\, i < \omega\} \subset \mathsf{WO} \wedge \check{\alpha} \in \{||\dot{y}^i||\,;\, i < \omega\}.$$

Let $g \in V$, $g \subset \mathrm{Col}(\omega,\alpha)$ be an arbitrary $\bar{H}$ generic. Then $\alpha \in \{||\dot{y}^i||\,;\, i < \omega\}$ and $\dot{y}^g \in p[\bar{T}]$. By the same reasoning as before $\dot{y}^g \in p[T]$ in V. Hence $\dot{y}^g \in B$ and

$\alpha \in A \cap \bar{\kappa}$.
Let us assume the other inclusion fails and work towards a contradiction. Let $\alpha < \bar{\kappa}$ be minimal such that $\alpha \in A$ but $\alpha \notin \bar{A}^*$. Hence there is a condition $p \in \mathrm{Col}(\omega, \alpha)$ such that

$$\bar{H}^{\mathrm{Col}(\omega,\alpha)} \models p \Vdash \forall y : \{y^i \,;\, i < \omega\} \subset \mathsf{WO} \wedge \check{\alpha} \in \{||y^i|| \,;\, i < \omega\} \implies y \in p[\check{\bar{S}}].$$

Let $p \in g \in V$, $g \subset \mathrm{Col}(\omega, \alpha)$ generic over $\bar{H}$. Note that $\bar{H}^{\mathrm{Col}(\omega,\alpha)}$ can calculate $A \cap \alpha = \bar{A}^* \cap \alpha$. In $\bar{H}^{\mathrm{Col}(\omega,\alpha)}$ we find a real y such that $\{||y^i|| \,;\, i < \omega\} = A \cap (\alpha+1)$. Since $p \in g$, we have that $y \in p[\bar{S}]$. By the same argument as before $y \in p[S]$ and hence $y \notin B$. Hence by the properties of B there is no $z \in B$ such that $\{||y^i|| \,;\, i < \omega\}$ is end-extended by $\{||z^i|| \,;\, i < \omega\}$. Hence $\alpha \notin A$; a contradiction to our choice of α. This finishes the proof of the lemma. □

As a consequence to the previous lemma we can show that subsets of ω_1 living in determinacy models extend to uniquely condensing classes.

Lemma 5.6.7 *Let M be a transitive class sized model such that $\mathbb{R} \subset M \models \mathsf{ZF} + \mathsf{AD}$. Let $A \in M$ be a subset of $\omega_1^M = \omega_1^V$. Then A extends to a class with unique condensation.*

Proof. Let $A \in M \models \mathsf{AD}$, $A \subset \omega_1$. We aim to show that there is a universally Baire B that satisfies the properties in the statement of the previous lemma. For this we study the following well-known Solovay Game

$$G(A) : \begin{array}{c|cccccc} I & x_0 & & x_1 & & \\ II & & y_0 & & y_1 & \cdots \end{array}$$

Here player I is obliged to play some $x = \langle x_i; i < \omega\rangle \in \mathsf{WO}$, else II wins, and Player II has to respond by playing a real $y = \langle y_i; i < \omega\rangle$ such that y codes (in some fixed recursive way) a countable set $\{y^i \,;\, i \in \omega\} \subset \mathsf{WO}$, else I wins. Player II wins $G(A)$ if $\{||y^i|| \,;\, i < \omega\} = A \cap \alpha$ for some $\alpha > ||x||$.
We show that player I cannot have a winning strategy: let σ be a strategy (not necessarily winning) for I, then the set $\{\sigma * y \,;\, y \in \omega^\omega\}$ is a $\boldsymbol{\Sigma^1_1}$ subset of WO. Hence by boundedness there is a countable ordinal α such that $\alpha > ||\sigma * y||$ for all $y \in \omega^\omega$. So player II can play a y such that $\{||y^i||; i < \omega\} = A \cap \alpha$ and win against the strategy σ, hence σ is not winning.
By the determinacy hypothesis a winning strategy τ for player II exists. With the help of τ we will define B. Set $x \in B$ if and only if

$$\begin{aligned} \phi_0(x) \quad :\equiv \quad & \{x^i \,;\, i < \omega\} \subset \mathsf{WO} \wedge \\ & \exists y (y \in \mathsf{WO} \wedge \{||(y * \tau)^i|| \,;\, i < \omega\} \text{ end-extends } \{||x^i|| \,;\, i < \omega\}). \end{aligned}$$

A straightforward calculation shows that ϕ_0 is Σ^1_2 in a code for τ. We promise that the next claim shows that we can also define B as follows: $x \in B$ if and only if

$$\begin{aligned} \phi_1(x) :\equiv \{x^i \,;\, i < \omega\} \subset \mathsf{WO} \wedge \forall y [(y \in \mathsf{WO} \wedge ||y|| > \sup\{||x^i|| \,;\, i < \omega\}) \implies \\ \{||(y * \tau)^i|| \,;\, i < \omega\} \text{ end-extends } \{||x^i|| \,;\, i < \omega\}]. \end{aligned}$$

Another straightforward calculation shows that ϕ_1 is Π^1_2 in a code for τ. The following statement is true in V by the fact that τ is a winning strategy for player II in $G(A)$:

$$\begin{aligned} \forall y, z [y, z \in \mathsf{WO} \implies (&\{||(y * \tau)^i|| \,;\, i < \omega\} \text{ end-extends } \{||(z * \tau)^i|| \,;\, i < \omega\} \vee \\ &\{||(z * \tau)^i|| \,;\, i < \omega\} \text{ end-extends } \{||(y * \tau)^i|| \,;\, i < \omega\})]. \end{aligned}$$

It is not difficult to see that that ψ is a Π^1_2 statement in a code for τ. Hence by Shoenfield Absoluteness ψ holds in all forcing extensions of V.
Claim 1. If a transitive model of set theory containing τ satisfies ψ, then

$$\forall x : \phi_0(x) \iff \phi_1(x).$$

Proof of Claim 1. Clearly $\phi_1(x)$ implies $\phi_0(x)$. So suppose $\phi_0(x)$ and let $y \in \mathsf{WO}$ be such that Let $y \in \mathsf{WO}$ be such that $\{||(y*\tau)^i||\,;\, i<\omega\}$ end-extends $\{||x^i||\,;\, i<\omega\}$. Now let $z \in \mathsf{WO}$ be arbitrary such that $||z|| > \sup\{||x^i||\,;\, i<\omega\}$. Since ψ holds, we have that $\{||(z*\tau)^i||\,;\, i<\omega\}$ end-extends $\{||(y*\tau)^i||\,;\, i<\omega\}$ or $\{||(y*\tau)^i||\,;\, i<\omega\}$ end-extends $\{||(z*\tau)^i||\,;\, i<\omega\}$. In either case $\{||(z*\tau)^i||\,;\, i<\omega\}$ end-extends $\{||x^i||\,;\, i<\omega\}$. □(Claim 1)

So B is (in a weak sense) provably $\mathbf{\Delta^1_2}$, hence by Lemma 1.8.2 B is universally Baire.
We have to check that B satisfies the properties stated in the previous lemma; for all properties not obvious this is verified by using the fact that τ is a winning strategy for player I. Hence by the previous lemma A extends to a class with unique condensation. □

Note that if $A \subset \omega_1$ is in a model of AD, then A is constructible from a real; in fact $A \in L[\sigma]$ where σ is a winning strategy for player II in $G(A)$. In this sense, the set A trivializes. Nevertheless non-trivial examples of sets with uniquely condensing extensions exist if the universe has a uniform shape:

Example 5.6.8 Suppose sharps for all sets exist. Let $V = L^\sharp$, the smallest inner model that is closed under the $\sharp$ operation. Using the Gödel pairing function and the well order $<$ of $L^\sharp$, we can uniformly code initial segments of $L^\sharp$ in the following way: if $\alpha < \beta$ are limit ordinals, then the code A_α for $L^\sharp_\alpha$ is a subset of α and the code A_β for $L^\sharp_\beta$ end-extends A_α, i.e. $A_\beta \cap \alpha = A_\alpha$. By A^* we denote the class coding $L^\sharp$. Set $A = A_{\omega_1}$. We claim that A^* is a uniquely condensing extension of A. For $\theta > \kappa$ both regular uncountable cardinals consider a countable substructure $X \prec L^\sharp||\theta$ such that $A_\kappa = A^* \cap \kappa \in X$. Let $\pi : M \to X$ denote the transitive collapse of M and let $\pi(\bar\kappa, \bar A) = \kappa, A_\kappa$. By elementarity of π

$$M \models V = L^\sharp,$$

and for all $x \in M$ the set $(x^\sharp)^M$ is embedded into $(\pi(x))^\sharp$, hence $(x^\sharp)^M = x^\sharp$. Thus M is an initial segment of $L^\sharp$. So, since we defined the sets of the form A_α uniformly, we have $\bar A = A_{\bar\kappa}$.
We need to show that A is not constructible from a real. Suppose otherwise that $A \in L[z]$ for some real $z \in V = L^\sharp$. Then $z^\sharp$ exists and is clearly not in $L[z]$. But $z^\sharp \in L_{\omega_1}[A] \subset L[z]$, a contradiction!

Note that given any mouse operator J, the same construction works for L^J, the smallest inner model that is closed under J.

5.6.2 Sets with uniquely condensing extensions, precipitous ideals and CC*

We analyse how sets with uniquely condesing extensions behave in the presence of ideals and the combinatorial principle CC*.

Lemma 5.6.9 *Let I be a precipitous ideal on ω_1 with the following property: if $G \subset \mathbb{P} := \mathcal{P}(\omega_1) \setminus I$ is generic, then $j(\omega_1^V) = \omega_2^V$, where j is the generic ultrapower induced by G. Let A be a set that extends to a class A^* with unique condensation. If j is a generic ultrapower induced by some generic $G \subset \mathcal{P}(\omega_1) \setminus I$, then*

$$j(A) = A^* \cap \omega_2 = A \cup \tilde{A},$$

where $\tilde{A}$ is the set of $\omega_1 \leq \alpha < \omega_2$ such that there is a club C and a canonical function f_α such that $f_\alpha(\beta) \in A$ for every $\beta \in C$, i.e. the Tilde operation applied to A.

Proof. Fix some generic G and j as above. We first show $j(A) \subset A^* \cap \omega_2$. Let $\alpha \in j(A)$. So there is some I-positive $S \in G$ and some canonical function f_α such that $f_\alpha(\beta) \in A$ for all $\beta \in S$. The set

$$C := \{\beta < \omega_1\,;\, \beta = X \cap \omega_1 \text{ for some } X \in C(A^*, \omega_2, \theta) \text{ with } \alpha, f_\alpha \in X\}$$

is club, where θ is sufficiently large. Since S is stationary we find some $\beta \in S \cap C$, say $X \prec H_\theta$ witnesses $\beta \in C$. Let $\pi : M \to X$ be the inverse of the transitive collapse of X and let $\pi(\bar{\alpha}, \overline{A^* \cap \omega_2}) = \alpha, A^* \cap \omega_2$. Then $f_\alpha(\beta) = \text{otp}(X \cap \alpha) = \bar{\alpha} \in A$ and since A extends we have that $\bar{\alpha} \in \overline{A^* \cap \omega_2}$. Applying π yields: $\alpha \in A^* \cap \omega_2$.
We show $A^* \cap \omega_2 \subset A \cup \tilde{A}$. Let $\alpha \in A^* \cap \omega_2$, $\alpha \geq \omega_1$. Fix a surjection $g : \omega_1 \to \alpha$ and $f : \omega_1 \to \omega_1$ let be the canonical function induced by g. Consider the club

$$C := \{\beta < \omega_1\,;\, \beta = X \cap \omega_1 \text{ for some } X \in C(A^*, \omega_2, \theta) \text{ with } \alpha, g, f \in X\}$$

for some sufficiently large θ. Let $\beta \in C$ and $X \prec H_\theta$ be a witnesses for this, let $\pi : M \to X$ be the inverse of the transitive collapse of X and let $\pi(\bar{\alpha}) = \alpha$. Since $\alpha \in A^* \cap \omega_2$ we have $\bar{\alpha} = \text{otp}(X \cap \alpha) = \text{otp}(g``\beta) = f(\beta) \in A$. Hence C, f witness that $\alpha \in \tilde{A}$.
Trivially $A \cup \tilde{A} \subset j(A)$. This finishes the proof. □

We conjecture that the existence of a strong enough ideal on ω_1 implies that every set with a uniquely condensing extension is constructible from a real.
Using the combinatorial principle CC* we can show that only countably many reals can be constructed from a set with a uniquely condensing extension.

Lemma 5.6.10 *If CC* holds and A extends to a class A^* with unique condensation, then $L[A]$ only contains countably many reals.*

Proof. Suppose otherwise and work towards a contradiction. Then $L[A]$ contains uncountably many reals and any real of $L[A]$ is in some $L_\alpha[A]$ for a countable α. Let $f : \omega_1 \to \omega_1$ be such that $f(\alpha)$ is the least ordinal such that the $<_{L[A]}$-least real $a \notin L_\alpha[A]$ is in $L_{f(\alpha)}[A]$. Clearly $f \in L[A]$. Let $C = C(A^*, \omega_2, \theta)$ for some sufficiently large θ and suppose there is a function $F : [H_\theta]^{<\omega} \to H_\theta$ such that C contains exactly the $X \prec H_\theta$ closed under F. Let $X \prec H_{\theta^+}$ be countable such that $F, A, A^* \cap \omega_2 \in X$. Let $\alpha = X \cap \omega_1$ and let $\pi : M \to X$ be the inverse of the transitive collapse of X. In general we can not compute $f(\alpha)$ in M, we apply CC* $f(\alpha) + 1$-many times to find a countable $Y \supset X$, $Y \cap \omega_1 = \alpha$, $Y \prec H_{\theta^+}$ such that $\text{otp}(Y \cap \omega_2) > f(\alpha)$. Let $\sigma : N \to Y$ denote the inverse of the transitive collapse of Y and note that by elementarity $Y \cap H_\theta$ is closed under F. Let $\sigma(\bar{A}, \overline{A^* \cap \omega_2}, \beta) = A, A^* \cap \omega_2, \omega_2$. Then, since $Y \cap H_\theta \in C$, we have $\overline{A^* \cap \omega_2} = A \cap \beta$ and $\beta > f(\alpha)$. Hence $L_\beta[A] \in N$ and we can compute $f(\alpha)$ in N. By $F \in Y$ we have that $H_\theta \in Y$. In Y we find a countable $Y'' \prec H_\theta$ that contains $A, A^* \cap \omega_2, f(\alpha)$ and is closed under F, so $Y'' \in C$. Let $\pi(Y') = Y''$ and let $\rho : N' \to Y'$ be the inverse of the transitive collapse of Y'. Note that $N' \in N$ and $\mathrm{crit}(\rho) < \alpha$. Then in N the set $\overline{A^* \cap \omega_2}$ witnesses that $\bar{A}$ has a countable extension that condenses uniquely up to β. Let $\rho(\overline{\overline{A^* \cap \omega_2}}, \bar{\beta}) = \overline{A^* \cap \omega_2}, \beta$. Hence $\overline{\overline{A^* \cap \omega_2}} = \bar{A} \cap \bar{\beta}$ and so $\overline{\overline{A^* \cap \omega_2}} = A \cap \bar{\beta}$. In N' compute $L_{\bar{\beta}}[\overline{\overline{A^* \cap \omega_2}}] = L_{\bar{\beta}}[A]$. By elementarity $f(\alpha) \in L_{\bar{\beta}}[A]$, a contradiction to the fact that $\bar{\beta} < \alpha$. □

5.6.3 Sets with uniquely condensing extensions and term-capturing

We have seen: if an ω_1-dense ideal on ω_1 exists, then every set with a uniquely condensing extension is constructible from a real. If V contains ω-many Woodin cardinals and a measurable above and satisfies an iterability hypothesis, we can also show that sets with uniquely condensing extensions are constructible from a real. The key idea is the following: if $A \subset \omega_1$ is in a model of determinacy, then it is constructible from a real. So we aim to show $L(\mathbb{R}) = L(\mathbb{R}, A) \models \mathsf{AD}$, this is Theorem 5.6.14. This is of course similar to a proof of $\mathsf{AD}^{L(\mathbb{R})}$ from ω-many Woodin cardinals. There are various ways to show determinacy from large cardinals. We will use the technique of capturing sets of reals over sufficiently iterable premice. In contrast to the rest of this chapter, we will work with coarse premice in the sense of Martin and Steel [MS94], since we will apply [Nee95].

Definition 5.6.11 Let $B \subset \omega^\omega$. Let $\mathfrak{M}$ be a premouse with an iteration strategy Σ that contains ω-many Woodin cardinals $(\delta_i)_{i\in\omega}$. Let τ be a $\mathrm{Col}(\omega, \delta_0)$ term in $\mathfrak{M}$. We say *τ captures B with respect to* Σ if, and only if, for all countable iteration maps $i : \mathfrak{M} \to \mathfrak{M}^*$, $i \in V$, obtained by using Σ and for all $g \subset \mathrm{Col}(\omega, i(\delta_0))$, $g \in V$,

$$i(\tau)^g = B \cap \mathfrak{M}^*[g].$$

Definition 5.6.12 Let λ be an infinite ordinal. Let $G \subset \mathrm{Col}(\omega, < \lambda)$ be $\mathfrak{M}$-generic for some suitable $\mathfrak{M}$. Then we set

$$\mathbb{R}^*_G = \bigcup \{\mathbb{R} \cap \mathfrak{M}[G \cap \mathrm{Col}(\omega, < \alpha)] \,;\, \alpha < \lambda\}.$$

If $\mathfrak{M}$ is sufficiently iterable and contains ω-many Woodin cardinals, it is possible to use $\mathfrak{M}$ to verify parts of the theory of $L(\mathbb{R}^V)$; for this recall the following result:

Theorem 5.6.13 ([Ste, 7.15]) *Suppose that $\mathfrak{M} \models \lambda$ is a limit of Woodin cardinals, where λ is countable in V, and that Σ is an $\omega_1 + 1$-iteration strategy for $\mathfrak{M}$. Let H be* $\mathrm{Col}(\omega, \mathbb{R})$*-generic over V; then in $V[H]$ there is an iteration map $i : \mathfrak{M} \to \mathfrak{M}^*$ coming from an iteration tree all of whose proper initial segments are played by Σ, and a G which is* $\mathrm{Col}(\omega, < i(\lambda))$*-generic over $\mathfrak{M}^*$, such that*

$$\mathbb{R}^*_G = \mathbb{R}^V.$$

Moreover, given a $g \subset$ $\mathrm{Col}(\omega, \alpha)$ *for an $\alpha < \lambda$, we can construct i, G such that* $\mathrm{crit}(i) > \alpha^{++\mathfrak{M}^*}$ *for any given $\alpha < \lambda$ and G is generic over $\mathfrak{M}^*[g]$.* □

First note that [Ste, 7.15] deals with fine-structural premice, nevertheless the proof of [Ste, 7.15] works in the coarse case, too. As we stated earlier the extender algebra, which is the main tool in the proof of [Ste, 7.15], can also be used to construct genericity iterations with coarse premice. Also note that the moreover part of the above theorem is only implicit in [Ste]; it follows by a minor modification of the proof using [Ste, 7.16]. We apply the previous theorem to obtain:

Theorem 5.6.14 *Suppose*

$$V \models \lambda' \text{ is the limit of } \omega\text{-many Woodin cardinals and } \kappa' > \lambda' \text{ is measurable}$$

Suppose A extends to $A^* \subset \lambda'$ with unique condensation up to λ'. Let $\theta > \theta' > (2^{\kappa'})^+$ be large enough such that

1. *the club $C = C(A^*, \lambda', \theta')$ of countable substructures of $V_{\theta'}$ witnesses that A extends to A^* with unique condensation,*
2. *θ is large enough so that $\langle V_\theta; \in, \lambda' \rangle$ is a premouse in the sense of [MS94],*
3. *if $X \prec V_\theta$, $A^*, \lambda, C \in X$ is a countable elementary substructure with $\pi : \mathfrak{M} \to X$ the inverse of the transitive collapse, then*
   a) $\mathfrak{M}$ has a $\omega_1 + 1$-iteration strategy Σ, and
   b) (Re-embedding) if $i : \mathfrak{M} \to \mathfrak{M}^*$ is a countable iteration map obtained by using Σ, then there is an elementary $\pi_{\mathfrak{M}^*} : \mathfrak{M}^* \to V_\theta$ satisfying $\pi_{\mathfrak{M}^*} \circ i = \pi$.

Then

1. *$L(\mathbb{R}, A) \models \mathsf{AD}$,*
2. *A is constructible from a real, and*
3. *$L(\mathbb{R}, A) = L(\mathbb{R})$.*

Proof. We first discuss the conclusions: Clearly 2. implies 3. If A is contained in a model of AD, then it is a well-known fact that the determinacy of the Solovay-Game $G(A)$ implies that A is constructible from a real that codes a winning strategy for player II in $G(A)$. So it suffices to show 1.
For this assume $L(\mathbb{R}, A) \models \neg\mathsf{AD}$, hence there is a set of reals B that is not determined. By minimizing the ordinal parameters in the definition of B we can assume without loss of generality that B is definable without ordinal parameters in $L(\mathbb{R}, A)$. Let z be the only real parameter used in the definition of B. If we can show that B is captured by a term over some countable sufficiently iterable model then by [Nee95, Lemma 1.7] the set is determined, contradicting our assumption. So we aim to show that B is captured.
Assume $x \in B$ if and only if

$$L(\mathbb{R}, A) \models \phi(x, z, A).$$

Pick $X \prec V_\theta$ with $z, C, A^*, \lambda \in X$. Let $\pi : \mathfrak{M} \to X$ denote the inverse of the transitive collapse and let $\pi(\bar{A}, \lambda, \kappa) = A^*, \lambda', \kappa'$ and let $(\delta_i)_{i\in\omega}$ denote the countably many Woodin cardinals in $\mathfrak{M}$. In $\mathfrak{M}$ we define a $\mathrm{Col}(\omega, \delta_0)$-term τ as follows: if $g \subset \mathrm{Col}(\omega, \delta_0)$ is generic over $\mathfrak{M}$, then $x \in \tau^g$ if and only if

$$\psi(x, z, A) :\equiv \mathbf{1}_{\mathrm{Col}(\omega, <\lambda)} \Vdash J_{\check{\kappa}}(\mathbb{R}^*_{\dot{G}}, \check{\bar{A}}) \models \phi(\check{x}, \check{z}, \check{A}),$$

here $\dot{G}$ is a canonical name for a $\mathrm{Col}(\omega, < \lambda)$ generic and κ is the measurable $> \lambda$ in $\mathfrak{M}$. We need to verify that τ captures B. Assume $i : \mathfrak{M} \to \mathfrak{M}^*$ is a countable iteration according to Σ and let $g \subset \mathrm{Col}(\omega, i(\delta_0))$ be generic over $\mathfrak{M}^*$. Let $x \in \mathbb{R} \cap \mathfrak{M}^*[g]$. We have to show

$$(L(\mathbb{R}^V, A) \models \phi(x, z, A)) \iff (\mathfrak{M}^*[g] \models \psi(x, z, A)).$$

By the previous theorem, we find an iteration map $j : \mathfrak{M}^* \to \mathfrak{M}^{**}$, $j \in V^{\mathrm{Col}(\omega,\mathbb{R}^V)}$, with $\mathrm{crit}(j) > i(\delta_0)^{++\mathfrak{M}^*}$ coming from an iteration tree $\mathcal{T}$ of length ω_1+1 on $\mathfrak{M}^*$ all of whose proper initial segments are played by Σ, and a G which is $\mathrm{Col}(\omega, < j(i(\lambda)))$-generic over $\mathfrak{M}^{**}[g]$, such that

$$\mathbb{R}^*_G = \mathbb{R}^V.$$

For this note, that since $\mathrm{crit}(j)$ is large enough, g is a $\mathfrak{M}^{**}$-generic; moreover j lifts to

$$\hat{j} : \mathfrak{M}^*[g] \to \mathfrak{M}^{**}[g],$$

where $\hat{j}(\sigma^g) = j(\sigma)^g$. In an abuse of notation we shall write j for $\hat{j}$. By our re-embedding hypothesis, we have for $\alpha < \omega_1$ an elementary embedding

$$\pi_\alpha : \mathcal{M}^{\mathcal{T}}_\alpha \to V_\theta,$$

such that $\pi_\alpha(j^{\mathcal{T}}_{0,\alpha}(i(\bar{A}))) = A^*$. The model $\mathfrak{M}^{**}$ is the direct limit of the models $\mathcal{M}^{\mathcal{T}}_\alpha$, and hence there is a map

$$\pi^{**} : \mathfrak{M}^{**} \to V_\theta.$$

Since for every $\gamma < \omega_1^V$ a real $x_\gamma \in \mathsf{WO}$ with $||x|| = \gamma$ is in $\mathfrak{M}^{**}[g, G \cap \mathrm{Col}(\omega, < \alpha)]$ for some $\alpha < j(i(\lambda))$, we have that $j(i(\lambda)) \geq \omega_1^V$ and by a standard homogeneity argument and the symmetry of the name $\mathbb{R}^*_{\dot{G}}$ we have $\mathbb{R} \cap L(\mathbb{R}^*_G, A) = \mathbb{R}^*_G$ and hence $j(i(\lambda)) \leq \omega_1^V$. So $j(i(\lambda)) = \omega_1^V$. We now apply that A extends to A^* with unique condensation: for all $\alpha < \omega_1$ we have that $\mathrm{ran}(\pi_\alpha) \cap V_{\theta'} \in C$. Hence

$$\pi_\alpha^{-1}(A^*) = A \cap \pi_\alpha^{-1}(\lambda').$$

From $\sup\{\pi_\alpha^{-1}(\lambda')\,;\, \alpha < \omega_1\} = j(i(\lambda)) = \omega_1^V$ it follows $(\pi^{**})^{-1}(A^*) = A$. We can now calculate

$$J_{j(i(\kappa))}(\mathbb{R}^*_G, j(i(\bar{A})))^{\mathfrak{M}^{**}[G]} = J_{j(i(\kappa))}(\mathbb{R}^V, A).$$

Note that $\pi^{**} \circ j \circ i(\kappa) = \pi(\kappa) = \kappa'$. By elementarity of j and the fact that we can iterate the measure on $j(i(\kappa))$ out of the universe

$$\begin{aligned} & \mathfrak{M}^*[g] \models \psi(x,z,A) \\ \iff\ & J_{j(i(\kappa))}(\mathbb{R}^*_G, j(i(\bar{A}))) \models \phi(x,z,A) \\ \iff\ & J_{j(i(\kappa))}(\mathbb{R}^V, A) \models \phi(x,z,A) \\ \iff\ & L(\mathbb{R}^V, A) \models \phi(x,z,A). \end{aligned}$$

This shows that B is captured by $\mathfrak{M}$. This is what we needed to show. □

5.6.4 Sets with uniquely condensing extensions in forcing extensions

We work with fine-structural premice again. Given a forcing name $\dot{x} \in V^{\mathbb{P}}$ for a real and granted that $M^\sharp_{\mathsf{mw}}$ exists and is sufficiently iterable, one can construct an iteration tree $\mathcal{T}$ of length $< \mathrm{Card}(\mathbb{P})^+ + 1$ such that for every $G \subset \mathbb{P}$ generic over V the real $\dot{x}^G$ is generic over $\mathcal{T}$'s last model, see Theorem 5.5.1. In general such an iteration is uncountable. Neeman and Zapletal showed that, given one generic $G \subset \mathbb{P}$ for a reasonable forcing $\mathbb{P}$, one finds $\alpha < \omega_1$ such that $\dot{x}^G$ is generic over $\mathcal{M}^{\mathcal{T}}_\alpha$, see [NZ01, Lemma 3]. We generalize this to names for subset of ω_1 with uniquely condensing extensions; before we can state the lemma we need a definition:

Definition 5.6.15 Let κ be an ordinal and let $\mathfrak{M} = \langle J_\beta[\vec{E}]; \in, \vec{E}, E_\beta \rangle$ be a countable sound premouse that has a $(\omega, \kappa + 1)$-iteration strategy Σ. We will say Σ *condenses to fragments* if it satisfies the following property: if λ is a regular cardinal such that $\mathfrak{M}, \Sigma \in H_\lambda$, and if $X \prec H_\lambda$ is a countable with uncollapsing map $\pi : \bar{H} \to X$ and $\pi(\bar{\Sigma}) = \Sigma$, then $\bar{\Sigma} = \Sigma \restriction \mathrm{dom}(\bar{\Sigma})$.

Here we do not want to construct iteration strategies that condense to fragments; nevertheless let us note that there are (at least) two ways to see that they exist: in the large cardinal area below one Woodin cardinal one is always in the situation that there is at most one well-founded branch through an iteration tree, hence there is only at most one (highly absolute) iteration strategy, this is one of the main results of [MS94]; for a fine-structural version see [Ste, Theorem 6.10]. Beyond that one uses $\mathcal{Q}$-structures in the construction of iteration strategies. Under the assumption that the ultimate projectum drops below the least Woodin cardinal of a tame premouse $\mathfrak{M}$, there is a unique branch b through an iteration tree on $\mathfrak{M}$ such that b comes with a $\mathcal{Q}$-structure (if there is any). Again this gives rise to an absolute iteration strategy. For more details on $\mathcal{Q}$-structures and iteration trees see, for example, the introduction of [BS09].

Lemma 5.6.16 *Let $\mathfrak{M} = \langle J_\beta[\vec{E}]; \in, \vec{E}, E_\beta \rangle$ be a sound premouse that is active and has a $(\omega, \kappa^+ + 1)$-iteration strategy Σ such that $\vec{E}$ witnesses the Woodiness and measurability of $\delta < \beta$. Let $\mathbb{P}$ be a forcing of size $\leq \kappa$. Let $\dot{A} \in V^{\mathbb{P}}$ be a name such that*

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{A} \subset \check{\omega}_1 \textit{ extends to } \dot{A}^* \textit{ with unique condensation};$$

here we see $\dot{A}^*$ as a $\mathbb{P}$-name for a class definable from some set in $V^{\mathbb{P}}$.
It is possible to construct an iteration tree $\mathcal{T}$ of height $\kappa^+ + 1$ with the following properties:

1. *There are arbitrary large ordinals $\beta < \kappa^+$ such that for any $G \subset \mathbb{P}$ that is generic over V, the set $\dot{A}^{*G} \cap j^{\mathcal{T}}_{0,\beta}(\delta)$ is $j^{\mathcal{T}}_{0,\beta}(W_\delta)$-generic over $\mathcal{M}^{\mathcal{T}}_\beta$, where W_δ is the extender algebra with δ many generators calculated in $\mathfrak{M}$. We call such a β a* baby closure point.

2. *If $\mathbb{P}$ is a reasonable forcing and additionally Σ condenses to fragments, then in $V^{\mathbb{P}}$ there are club many* weak closure points *$\beta \in [0, \omega_1]_{\mathcal{T}}$; i.e. for any $G \subset \mathbb{P}$ that is generic over V there are club many β such that set $\dot{A}^G \cap j^{\mathcal{T}}_{0,\beta}(\delta)$ is $j^{\mathcal{T}}_{0,\beta}(W_\delta)$-generic over $\mathcal{M}^{\mathcal{T}}_\beta$.*

3. *Especially: if $\mathbb{P}$ is a reasonable forcing and additionally Σ condenses to fragments, then for any $G \subset \mathbb{P}$ that is generic over V, the set $\dot{A}^G \subset \omega_1$ is $j^{\mathcal{T}}_{0,\omega_1}(W_\delta)$-generic over $\mathcal{M}^{\mathcal{T}}_{\omega_1}$, where W_δ is the extender algebra with δ many generators calculated in $\mathfrak{M}$. So ω_1 is a baby closure point.*

Note that Theorem 5.5.1 is a special case of conclusion 1. above: if $\dot{A} \subset \omega$, then $\dot{A}$ extends to $\dot{A}$ with unique condensation.
Before we begin the proof of the above lemma, we need a suitable notation. Following Neeman and Zapletal, we extend our notation for the axioms that arise in the construction of the extender-algebra.

Definition 5.6.17 Let $\mathcal{M} = \langle J_\beta[\vec{E}]; \in, \vec{E}, E_\beta \rangle$ be a premouse such that $\mathcal{M} \models \delta$ is Woodin. Let $\vec{\phi} = \langle \phi_\xi; \xi < \kappa \rangle$ be a sequence of $L_{\delta,\delta,0}$-sentences and let $E = E_\rho$ be a extender on $\vec{E}$. Let λ such that $\mathrm{crit}(E) = \kappa \leq \lambda < \rho$, and suppose $\nu(E)$ is a

$\mathcal{M}$-cardinal such that $i_E(\langle\phi_\xi;\xi<\kappa\rangle)\restriction\lambda\in\mathcal{J}^{\mathcal{M}}_{\nu(E)}$. We set

$$a_{\kappa,\lambda,\rho,\vec{\phi}}:\equiv\bigvee_{\alpha<\kappa}\phi_\alpha\leftrightarrow\bigvee_{\alpha<\lambda}i_E(\langle\phi_\xi;\xi<\kappa\rangle)_\alpha.$$

Now we are ready to prove the lemma.

Proof. We construct $\mathcal{T}$ using the strategy we have used many times: iterate up to some closure point and then hit the measure (or its image respectively) witnessing the measurability of δ. This time the tree will be of height κ^++1. Let us fix $\mathbb{P}$ not necessarily reasonable. Let U be the least extender on the extender sequence of $\mathfrak{M}$ that witnesses that δ is measurable, and let ζ_0 be the ordinal where U is indexed. First we define what a *baby closure point* is in the context of this proof: We will call an $\alpha\leq\kappa^+$ a *baby closure point* if for all $p\in\mathbb{P}$

$$p\Vdash\dot{A}^*\cap j^{\mathcal{T}}_{0,\alpha}(\delta)\models j^{\mathcal{T}}_{0,\alpha}(T(\vec{E})).$$

Note the following subtlety: we are talking about $\dot{A}^*$ above, the uniquely condensing extension of $\dot{A}$; this allows us to discuss the case $\alpha>\omega_1$ in contrast to the previous proofs. Now $\alpha<\kappa^+$ is a *closure point for* if for all $p\in\mathbb{P}$ and all $\zeta<i^{\mathcal{T}}_{0,\alpha}(\zeta_0)$

$$p\Vdash\check{F}_\zeta\text{ does not induce an axiom false of }\dot{A}^*\cap j^{\mathcal{T}}_{0,\alpha}(\zeta_0),$$

where $\vec{F}$ is $\mathcal{M}^{\mathcal{T}}_\alpha$'s extender sequence. Clearly every closure point is a baby closure point. We will show something a little stronger than what we state in conclusion 1.; we actually show that there is a closure point.
We construct an iteration tree $\mathcal{T}$ of length κ^++1 on $\mathcal{M}^{\mathcal{T}}_0=\mathfrak{M}$. We will refer to this construction as a *genericity iteration for* $\dot{A}^*$. We define the iteration as follows: in the limit case we use Σ to continue the iteration. In the successor case there are subcases: if $\alpha<\omega_1$ is a closure point, then we use $j^{\mathcal{T}}_{0,\alpha}(U)$ to continue the iteration. If α is not a closure point, then there is a least "bad" extender $E^{\mathcal{M}^{\mathcal{T}}_\alpha}_\rho$ on the extender sequence of $\mathcal{M}^{\mathcal{T}}_\alpha$ and some $p_\alpha\in\mathbb{P}$ such that

$$p_\alpha\Vdash\dot{A}^*\cap j^{\mathcal{T}}_{0,\alpha}(\zeta_0)\not\models a,$$

where a is some axiom in $\mathcal{M}^{\mathcal{T}}_\alpha$ of the form $a_{\kappa_\alpha,\lambda_\alpha,\rho,\vec{\phi}}$ for some $(\lambda_\alpha,\vec{\phi})\in\mathcal{M}^{\mathcal{T}}_\alpha$. Furthermore we pick p_α so that it decides the value of a and minimizes λ_α, i.e. there is some $\vec{\phi}^\alpha\in\mathcal{M}^{\mathcal{T}}_\alpha$ such that

$$p_\alpha\Vdash\dot{A}^*\cap j^{\mathcal{T}}_{0,\alpha}(\zeta_0)\not\models a_{\kappa_\alpha,\lambda_\alpha,\rho,\vec{\phi}^\alpha},$$

and λ_α is minimal among all λ with

$$p_\alpha\Vdash\dot{A}^*\cap j^{\mathcal{T}}_{0,\alpha}(\zeta_0)\not\models a_{\kappa_\alpha,\lambda,\rho,\vec{\phi}'^\alpha}$$

for some ϕ'^α. We then use $E^{\mathcal{M}^{\mathcal{T}}_\alpha}_\rho$ to continue the iteration. This finishes the construction of $\mathcal{T}$. The arguments we have given before make sure $\mathcal{T}$ is a normal tree. Let $b=[0,\kappa^+]_{\mathcal{T}}$ and let $j=j^{\mathcal{T}}_{0,\kappa^+}:\mathfrak{M}\to\mathcal{M}^{\mathcal{T}}_\kappa$. Note that b is club in κ^+. We set $\mathfrak{M}^*=\mathcal{M}^{\mathcal{T}}_\kappa$. We aim to show the first part of the theorem, i.e. that there is a closure point $<\kappa^+$.
For every $\alpha\in b$ let α^+_b be the least ordinal such that $\alpha\mathcal{T}\alpha^+_b+1$. So there is an extender $E^{\mathcal{T}}_{\alpha^+_b}$ which we used to continue the iteration at stage α^+_b; let $\kappa_{\alpha^+_b}=\mathrm{crit}(E^{\mathcal{T}}_{\alpha^+_b})$. Since $\mathcal{M}^{\mathcal{T}}_\alpha$ and $\mathcal{M}^{\mathcal{T}}_{\alpha^+_b+1}$ agree on subsets of $\kappa_{\alpha^+_b}$, it follows that $\vec{\phi}^{\alpha^+_b}\in\mathcal{M}^{\mathcal{T}}_\alpha$. Let us denote $\vec{\phi}^{\alpha^+_b}$ by $\vec{\psi}^\alpha$.

Let $S_1 = b \cap \mathrm{Lim}$. For $\alpha \in S_1$ the model $\mathcal{M}^{\mathcal{T}}_\alpha$ is a direct limit and contains $\vec{\psi}^\alpha$, so there is some $h(\alpha) < \alpha$ such that $\vec{\psi}^\alpha \in \mathrm{ran}(j^{\mathcal{T}}_{h(\alpha),\alpha})$. So Fodor's Theorem yields a stationary $S_2 \subset S_1$ such that $h(\alpha) = \beta$ for all $\alpha \in S_2$. Since $\mathcal{M}^{\mathcal{T}}_\beta$ has at most cardinality κ, further thinning of S_2 produces a stationary $S_3 \subset S_2$ and a fixed $\vec{\psi} \in \mathcal{M}^{\mathcal{T}}_\beta$ such that $\vec{\psi}^\alpha = j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi})$ for all $\alpha \in S_3$. Since $\mathbb{P}$ has cardinality $\leq \kappa$, we can also assume that there is a fixed $p \in \mathbb{P}$ such that $p_{\alpha_b^+} = p$ for all $\alpha \in S_3$.
Let α be any element of S_3 and set $\gamma = \alpha_b^+$ (hence $\gamma + 1 \in b$). So

$$p \Vdash \dot{A}^* \cap j^{\mathcal{T}}_{0,\gamma}(\zeta_0) \not\models a_{\kappa,\lambda_\gamma,\rho,j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi})},$$

where ρ satisfies our minimality assumption and $\kappa_\gamma = \mathrm{crit}(E^{\mathcal{T}}_\gamma)$ and $a_{\kappa,\lambda_\gamma,\rho,j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi})}$ is calculated in $\mathcal{M}^{\mathcal{T}}_\gamma$. Hence

$$p \Vdash \dot{A}^* \cap j^{\mathcal{T}}_{0,\gamma}(\zeta_0) \not\models \bigvee_{\xi<\kappa_\gamma} j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi})_\xi \text{ and } \dot{A}^* \cap j^{\mathcal{T}}_{0,\gamma}(\zeta_0) \models \bigvee_{\xi<\nu(E^{\mathcal{T}}_\gamma)} i^{\mathcal{M}^{\mathcal{T}}_\gamma}_{E^{\mathcal{T}}_\gamma}(j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi}))_\xi.$$

Note that $i^{\mathcal{M}^{\mathcal{T}}_\gamma}_{E^{\mathcal{T}}_\gamma}(j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi})) = i^{\mathcal{M}^{\mathcal{T}}_\alpha}_{E^{\mathcal{T}}_\gamma}(j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi}))$, so we will drop the superscript. Since $i_{E^{\mathcal{T}}_\gamma}(j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi}))$ is $j^{\mathcal{T}}_{\alpha,\gamma+1}(j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi}))$, we can rewrite the above statement as

$$(*)\ \ p \Vdash \dot{A}^* \cap j^{\mathcal{T}}_{0,\gamma}(\zeta_0) \not\models \textstyle\bigvee_{\xi<\kappa_\gamma} j^{\mathcal{T}}_{\beta,\alpha}(\vec{\psi})_\xi \text{ and } \dot{A}^* \cap j^{\mathcal{T}}_{0,\gamma}(\zeta_0) \models \bigvee_{\xi<\nu(E^{\mathcal{T}}_\gamma)} j^{\mathcal{T}}_{\beta,\gamma+1}(\vec{\psi})_\xi.$$

Let $\alpha' \in S_3$ such that $\alpha' > \gamma+1$. Then $\mathrm{crit}(j^{\mathcal{T}}_{\gamma+1,\alpha'}) \geq \nu(E^{\mathcal{T}}_\gamma)$ and so for $\xi < \nu(E^{\mathcal{T}}_\gamma)$, $j^{\mathcal{T}}_{\beta,\gamma+1}(\vec{\psi})_\xi$ is not moved by $j^{\mathcal{T}}_{\gamma+1,\alpha'}$. Thus

$$p \Vdash \bigvee_{\xi<\nu(E^{\mathcal{T}}_\gamma)} j^{\mathcal{T}}_{\beta,\alpha'}(\vec{\psi}).$$

But then clearly

$$p \Vdash \bigvee_{\xi<\kappa'} j^{\mathcal{T}}_{\beta,\alpha'}(\vec{\psi}),$$

where $\kappa' = \mathrm{crit}(E^{\mathcal{T}}_{\alpha'^+_b})$. This clearly contradicts $(*)$. Hence we have shown that there is a closure point $< \kappa^+$; in fact we did not need that $\dot{A}^*$ is (a name for) a uniquely condensing extension but our argument works for any subset of κ^+. Also it is obvious that there are arbitrarily large closure points $< \kappa^+$, since we could run the same argument starting with $S_1 \setminus \eta$ instead of S_1 for an arbitrary $\eta < \kappa^+$.
We now additionally assume that $\mathbb{P}$ is reasonable and that Σ condenses to fragments. We show the second part of the theorem; the third easily follows from the second. We fix a countable ordinal η and some $G \subset \mathbb{P}$ generic over V. We aim to find a *weak closure point*, i.e. some $\beta > \eta$, $\beta < \omega_1$ such that for some $q \in G$

$$q \Vdash \dot{A}^* \cap j^{\mathcal{T}}_{0,\beta}(\delta) \models j^{\mathcal{T}}_{0,\beta}(T(\vec{E})).$$

In V pick a countable $X \prec H_\lambda$ for some large enough regular λ such that $\omega_1 \cap X > \eta$ and $\dot{A}, \dot{A}^* \cap \check{\kappa}^+, \Sigma, \mathcal{T}, \mathbb{P} \in X$. Let $\pi : H \to X$ denote the inverse of the transitive collapse of X and let $\alpha = \omega_1 \cap X$. By the definition of reasonable forcings we can assume without loss of generality that $G \cap X$ is $\mathbb{P}$-generic over X and $X[G \cap X] \prec H_\lambda[G]$; this implies that π lifts, i.e.

$$\hat{\pi} : H[\bar{G}] \to H_\lambda[G],$$
$$\tau^{\bar{G}} \mapsto \pi(\tau)^G$$

is an elementary embedding, where $\bar{G} := \pi^{-1}``G$. We write π again for $\hat{\pi}$. Let $\pi(\bar{\kappa}, \bar{A}, \bar{A}^*, \bar{\Sigma}, \bar{\mathcal{T}}) = \kappa, \dot{A}^G, (\dot{A}^* \cap \check{\kappa}^+)^G, \Sigma, \mathcal{T}$. Since $(\dot{A}^*)^G$ is the uniquely condensing extension of $\dot{A}^G$ we have $\dot{A}^G \cap \bar{\kappa} = \bar{A}^*$. By the first part of the theorem there is an uncountable closure point $\beta \in \kappa^+ \cap X$. Let $\pi(\bar{\beta}) = \beta$, then by elementarity $\dot{A}^G \cap j^{\bar{\mathcal{T}}}_{0,\bar{\beta}}(\delta) = \bar{A}^* \cap j^{\bar{\mathcal{T}}}_{0,\bar{\beta}}(\delta)$ is $j^{\bar{\mathcal{T}}}_{0,\bar{\beta}}(W_\delta)$-generic over $\mathcal{M}^{\bar{\mathcal{T}}}_{\bar{\beta}}$. Also by elementarity the tree $\bar{\mathcal{T}}$ in H is built using the same rules as in the construction of $\mathcal{T}$, but at limit stages one uses the strategy $\bar{\Sigma}$ to pick branches. By our hypothesis we have that $\bar{\Sigma}$ is a fragment of Σ, hence we know that the branches picked by $\bar{\Sigma}$ in $\bar{H}$ are the same branches that Σ picked in V. This implies that the iteration tree $\bar{\mathcal{T}}$ is an initial segment of $\mathcal{T}$. So in $V[G]$ we have

$$V[G] \models \dot{A}^G \cap j^{\mathcal{T}}_{0,\bar{\beta}}(\delta) \models j^{\mathcal{T}}_{0,\bar{\beta}}(T(\vec{E})).$$

So there is a condition $q \in G$ that forces that $\bar{\beta}$ is a weak closue point. $\square$

Note that if $\mathbb{P}$ is reasonable but not c.c.c. we can not hope to show that in the previous tree construction there are countable stages α such that

$$\mathbf{1}_{\mathbb{P}} \Vdash j^{\mathcal{T}}_{0,\alpha}(\delta) \cap \dot{A} \models j^{\mathcal{T}}_{0,\alpha}(T(\vec{E})).$$

To see this pick a maximal antichain $\mathcal{A} \subset \mathbb{P}$ of cardinality ω_1 (we can do so if we without loss of generality suppose that $\mathbb{P}$ is a Boolean algebra). Assume CH and let $f : \mathcal{A} \to \mathcal{P}(\omega)$ be a surjection. Choose a name $\dot{A}$ such that $a \Vdash \dot{A} = f\check{(a)}$ for every $a \in \mathcal{A}$. Then clearly $\dot{A}$ is a name for a bounded subset of ω_1 and hence $\dot{A}$ extends to class with unique condensation. Let $\mathcal{T}$ be the iteration tree given by the previous lemma. Now assume towards a contradiction that for a countable α

$$\mathbf{1}_{\mathbb{P}} \Vdash j^{\mathcal{T}}_{0,\alpha}(\delta) \cap \dot{A} \models j^{\mathcal{T}}_{0,\alpha}(T(\vec{E})).$$

Let a be such that $f(a)$ codes an ordertype $> \mathcal{M}^{\mathcal{T}}_\alpha \cap \mathsf{OR}$. Then

$$a \Vdash f\check{(a)} \text{ is generic over } \mathcal{M}^{\mathcal{T}}_\alpha,$$

a contradiction.

Remark 5.6.18 Lemma 5.6.16 can be easily seen to generalize. Using the notation of Lemma 5.6.16 : if $\mathbb{P}$ is reasonable and Σ condenses to fragments and we have an arbitrary iteration tree $\mathcal{T}$ on $\mathfrak{M}$ of height $< \omega_2$, then one can continue $\mathcal{T}$ by performing a genericity iteration for A^* as in the proof of Lemma 5.6.16 of length $\kappa^+ + 1$. Note that in general one will have to apply extenders to models of $\mathcal{T}$ in this process. By the same reflection argument this genericity iteration terminates after $< \omega_2$-many steps. So we reach a closure point in the sense of Lemma 5.6.16 at some stage $< \omega_2$.

5.6.5 Applications of sets with uniquely condensing extensions

The following lemma is part of the folklore surrounding measurable cardinals and not too difficult to prove. A detailed proof can be found in [Lar04, 1.1.20, 1.1.19]

Lemma 5.6.19 (folklore) *Let κ be measurable and assume θ is a regular cardinal such that a normal measure μ on κ is in H_θ. Let $Z \prec M$ be a substructure such that $\mu \in Z$ and $Z \cap \mathcal{P}(\kappa)$ has cardinality $< \kappa$ and suppose* $\mathrm{Card}(Z) < \theta$. *Then for all $\gamma < \kappa$*

1. $Z[\gamma] = \{f(\gamma)\,;\, f \in Z, f : \kappa \to M\} \prec M$,

2. $\bigcap\{A\,;\,A \in Z \cap \mu\} \neq \emptyset$,
3. *if* $\gamma \in \cap\{A\,;\,A \in Z \cap \mu\} \neq \emptyset$, *then* $Z[\gamma] \cap \gamma = Z \cap \gamma$. □

Recall that Chang's Conjecture is equivalent to the statement: the set $\{X \subset \omega_2\,;\,\mathrm{otp}(X) = \omega_1\}$ intersects all strongly closed unbounded subsets of $[\omega_2]^{\omega_1}$, see Lemma 1.9.1. In the light of this, the following lemma can be seen as a generalization of Chang's Conjecture.

Lemma 5.6.20 (folklore) *Let κ be measurable. Let $M = H_\theta$ for some regular $\theta > 2^\kappa$. Then*

$$\{X \subset \kappa\,;\,\mathrm{otp}(X) = \omega_1 \wedge \exists Z \prec M : X = Z \cap \kappa\}$$

intersects all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$. Especially

$$\{X \subset \kappa\,;\,\mathrm{otp}(X) = \omega_1\}$$

intersects all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$.

Proof. Fix a function $F : [\kappa]^{<\omega} \to \kappa$. We have to show that

$$\{X \subset \kappa\,;\,\mathrm{otp}(X) = \omega_1 \wedge \exists Z \prec M : X = Z \cap \kappa\}$$

intersects C_F. We build a chain of length ω_1 of elementary substructures $\langle Z_\alpha; \alpha < \omega_1\rangle$ of M such that $Z_\alpha \subset Z_{\alpha+1}$ for $\alpha < \omega_1$ and $Z_\alpha \cap \kappa = Z_{\alpha+1} \cap \sup(Z_\alpha \cap \kappa)$. Let $Z_0 \prec M$ be a countable substructure with $\kappa, F \in Z_0$. At limit stages form unions. If Z_α is already constructed, then an application of Lemma 5.6.19 yields a $Z_{\alpha+1} \prec M$, $Z_\alpha \subset Z_{\alpha+1}$ with $Z_\alpha \cap \kappa = Z_{\alpha+1} \cap \sup(Z_\alpha \cap \kappa)$. Set $Z = \bigcup\{Z_\alpha\,;\,\alpha < \omega_1\}$. Then Z is closed under F and $\mathrm{otp}(Z \cap \kappa) = \omega_1$, the latter holds because the sets of the form $Z_\alpha \cap \kappa$, $\alpha < \kappa$, end-extend each other. This finishes the proof of the lemma. □

We show that the conclusion of the previous lemma is preserved under c.c.c. forcing.

Lemma 5.6.21 *Let $\mathbb{P}$ be notion of forcing that satisfies the c.c.c. Let $\kappa > \omega_1$ be a cardinal and let $S \subset [\kappa]^{\omega_1}$ be such that S intersect all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$. Then in $V^{\mathbb{P}}$ the set S also intersects all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$.*

Proof. Fix a name $\dot{F} \in V^{\mathbb{P}}$ such that

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{F} : [\check{\kappa}]^{<\check{\omega}} \to \kappa.$$

Let θ be large enough such that all the dense sets of $\mathbb{P}$, $\dot{F}$, $S \in V_\theta$. The set of C' of $Y \prec V_\theta$ of cardinality ω_1 such that $\mathbb{P}, \dot{F}, S \in Y$ is a strongly closed unbounded subsets of $[V_\theta]^{\omega_1}$, so the set $C = \{Y \cap \kappa\,;\,Y \in C\}$ is strongly closed unbounded in $[\kappa]^{\omega_1}$. Pick $Y \in C'$ such that $Y \cap \kappa \in C \cap S$. Then, by the argument for Claim 1 in the proof of Lemma 5.5.2, we have for all V-generics $G \subset \mathbb{P}$

$$Y[G] \prec V_\theta[G] \text{ and } Y[G] \cap V = Y.$$

By elementarity $Y \cap \kappa = Y[G] \cap \kappa$ is closed under $\dot{F}^G$. This suffices to show. □

Using the characterization of CC in Lemma 1.9.1 we obtain:

Corollary 5.6.22 (folklore) *Chang's Conjecture is preserved by c.c.c. forcing.* □

The following concept is not standard; we introduce it to state the following theorems.

Definition 5.6.23 Let $\kappa > \omega_1$ be a cardinal and let $S \subset [\kappa]^{\omega_1}$ such that S intersects all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$. A notion of forcing $\mathbb{P}$ *preserves* S, if in $V^{\mathbb{P}}$ the set S intersects all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$.

Theorem 5.6.24 *Let κ be measurable and let $\mathbb{P}$ be a forcing of cardinality $< \kappa$. Let $\dot{A}$ be a name such that*

$$\mathbf{1}_{\mathbb{P}} \Vdash L[\dot{A}] \models \phi(\dot{A}, \check{\vec{\alpha}}),$$

where $\vec{\alpha}$ are finitely many ordinal parameters, and $\dot{A}^*$ be a name for subset of κ such that

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{A} \text{ extends to } \dot{A}^* \text{ with unique condensation.}$$

Let $\theta \geq (2^\kappa)^+$ be large enough such that $\dot{A}^* \in H_\theta$ and set $M = \langle H_\theta; \in, \mathbb{P}, \dot{A}, \dot{A}^* \rangle$. Suppose $M_{\mathsf{mw}}^\sharp$ exists and has an $(\omega, \kappa+1)$-iteration strategy Σ that condenses to fragments. Suppose that

$$S := \{X \subset \kappa\,;\, \mathrm{otp}(X) = \omega_1 \wedge \exists Y \prec M : Y \cap \kappa = X\}^V$$

intersects all strongly closed unbounded subsets of $[\kappa]^{\omega_1}$ and $\mathbb{P}$ preserves S.
Then there is some $A \subset \omega_1$, $A \in V$ such that

$$L[A] \models \phi(A, \vec{\alpha}).$$

Proof. Set $\mathfrak{M} = M_{\mathsf{mw}}^\sharp$ and let δ denote $\mathfrak{M}$'s measurable Woodin. Let $U \in M_{\mathsf{mw}}^\sharp$ denote the (trivial completion of the) least normal measure on δ and let ζ_0 denote the index of U on $M_{\mathsf{mw}}^\sharp$'s extender sequence. Our strategy is as follows: using the ideas of Lemma 5.6.16 we build an iteration tree $\mathcal{T} \in V$ on $\mathfrak{M}$ of height $\kappa+1$ such that for all $G \subset \mathbb{P}$ the set $\dot{A}^{*G}$ is generic over $\mathcal{M}_\kappa^{\mathcal{T}}$. Once $\mathcal{T}$ is constructed it will follow from the hypothesis on S that we find some elementary substructure $X \prec M$, $X \in V$, $\mathrm{otp}(X \cap \kappa) = \omega_1$ such that $\dot{A}^G$ is generic over this substructure.
We construct $\mathcal{T}$; we will omit superscripts $\mathcal{T}$ where possible, so $\mathcal{T}$ has iteration maps $j_{\alpha,\beta}$ and model $\mathcal{M}_\alpha$. We will call $\alpha < \kappa$ a *closure point* if

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{A}^* \cap j_{0,\alpha}(\zeta_0) \models j_{0,\alpha}(W_\delta)$$

where W_δ is the extender algebra with δ-many generators calculated in $\mathfrak{M}$. If α is a closure point we use $j_{0,\alpha}(U)$ to continue the iteration. If α is not a closure point we perform a genericity iteration for $\dot{A}^*$ in the sense of Lemma 5.6.16. Since $\mathbb{P}$ has size $< \kappa$ we can apply conclusion 1. of Lemma 5.6.16; so we reach the next closure point after $< \kappa$-many stages. Literally Lemma 5.6.16 only tells us where the first baby closure point is; nevertheless from the proof we also obtain a closure point. Moreover it is easy to see the following: after we apply (the image of) U at a closure point α, we can perform another genericity iteration for $\dot{A}^*$ in the sense of Lemma 5.6.16. This completes the definition of $\mathcal{T}$.
Clearly $\mathcal{T} \in M$. Let $b = [0,\kappa]_{\mathcal{T}}$. By the argument we have given before b contains unboundedly many (and hence club many) closure points. So club often we have used (the images of) U to continue the iteration. Moreover $j_{0,\kappa}(\delta) = \kappa$ and κ is a baby closure point. Let $\mathfrak{M}^* = \mathcal{M}_\kappa$.
Now pick an arbitrary $G \subset \mathbb{P}$ generic over V. In $V[G]$ we consider the structure $M^+ := \langle H_\theta^V; \in, \mathbb{P}, \dot{A}, \dot{A}^*, \dot{A}^{*G} \rangle$. Trivially all substructures of M^+ contain $\mathcal{T}$. The set

$$C = \{X \subset \kappa\,;\, \mathrm{otp}(X) = \omega_1 \wedge \exists Y \prec M^+ : Y \cap \kappa = X\}^{V[G]}$$

is strongly closed unbounded. By our hypothesis on S, we find some $Y \prec M$, $Y \in V$ such that $\text{otp}(Y \cap \kappa) = \omega_1$ and there is some $Y^+ \in C$ such that $Y^+ \cap \kappa = Y \cap \kappa$. Since κ is a closure point $Y^+ \models \dot{A}^{*G} \cap \kappa$ is generic over $\mathfrak{M}^*$. Let $\pi^+ : N^+ \to Y^+$ denote the inverse of the transitive collapse of Y^+. By Lemma 5.6.4 and $\text{otp}(Y^+ \cap \kappa) = \omega_1$ we have $(\pi^+)^{-1}(\dot{A}^{*G} \cap \kappa) = \dot{A}^G$. Set $\bar{\mathcal{T}} = (\pi^+)^{-1}(\mathcal{T})$ and let $\pi : N \to Y$ denote the inverse of the transitive collapse of Y. Since $Y^+ \cap \kappa = Y \cap \kappa$ we have that $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T})$. Since Σ condenses to fragments, $\bar{\mathcal{T}}$ is build according to Σ. So by elementarity of π^+, $\bar{\mathcal{T}}$ contains ω_1-many closure points for $\dot{A}^G$ and hence $\dot{A}^G$ is generic over $\bar{\mathfrak{M}}^* = \mathcal{M}^{\bar{\mathcal{T}}}_{\omega_1}$. Summing up we have that $\dot{A}^G$ is generic over $\bar{\mathfrak{M}}^* \in V$. If necessary we iterate $\bar{\mathfrak{M}}^*$'s top-measure to make sure $\vec{\alpha} \in \bar{\mathfrak{M}}^*$. Pick a condition $q \in j^{\bar{\mathcal{T}}}_{0,\omega_1}(W_\delta)$ such that

$$q \Vdash \omega_1 = j^{\bar{\mathcal{T}}}_{0,\omega_1}(\delta)\check{} \wedge \exists A : L[A] \models \phi(A, \check{\vec{\alpha}}).$$

We want to find $\Gamma \in V$, $q \in \Gamma$, $\Gamma \subset j^{\bar{\mathcal{T}}}_{0,\omega_1}(W_\delta)$ generic over $\bar{\mathfrak{M}}^*$. For this we need to piece together end-extending generics; since we do not need to make sure that $\mathbb{R}^V \subset \bar{\mathfrak{M}}^*[\Gamma]$ the argument is much simpler than in the proof for Theorem 5.4.1. Especially we do not have to pick generics while iterating this time. Let $C \subset [0, \omega_1]_{\bar{\mathcal{T}}}$ denote the club of points α where we used $j_{0,\alpha}(U)$ to continue the iteration. Let $\alpha_0 \in C$ be such that $q \in \mathcal{M}^{\bar{\mathcal{T}}}_{\alpha_0}$ and q is not moved by $j^{\bar{\mathcal{T}}}_{\alpha_0,\omega_1}$. In V pick $\Gamma_0 \subset j^{\bar{\mathcal{T}}}_{0,\alpha_0}(W_\delta)$ generic over $\mathcal{M}^{\bar{\mathcal{T}}}_{\alpha_0}$. Let $\alpha_1 = \min(C \setminus (\alpha_0 + 1))$. Using Lemma 5.4.3, we can end-extend Γ_0 to some Γ_1 generic over $\mathcal{M}^{\bar{\mathcal{T}}}_{\alpha_1}$. In this fashion we continue all the way up through C: at successor stages repeat the argument we have just given, at limit stages $\lambda \in C$ form the union $\Gamma_\lambda = \bigcup\{\Gamma_i \,;\, i < \lambda\}$. Using the fact that anitchains are small, it is not difficult to see that Γ_λ is generic over $\mathcal{M}^{\bar{\mathcal{T}}}_\lambda$. Finally we set $\Gamma = \bigcup\{\Gamma_i \,;\, i < \omega_1\}$. Then Γ is as desired.
There is some $A \subset \omega_1$, $A \in \bar{\mathfrak{M}}^*[\Gamma]$ such that

$$L_\rho[A] \models \phi(A, \vec{\alpha}),$$

where ρ is the critical point of $\bar{\mathfrak{M}}^*$'s top measure. Iterating this top measure out of the universe we obtain

$$L[A] \models \phi(A, \vec{\alpha}).$$

This finishes the proof. □

The previous theorem has a variant:

Theorem 5.6.25 *Let $\mathbb{P}$ be a reasonable forcing of cardinality $< \kappa$, κ regular. Let $\dot{A}$ be a name such that*

$$\mathbf{1}_{\mathbb{P}} \Vdash L[\dot{A}] \models \phi(\dot{A}, \check{\vec{\alpha}}),$$

where $\vec{\alpha}$ are finitely many ordinal parameters, and $\dot{A}^*$ be a name for subset of κ such that

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{A} \textit{ extends to } \dot{A}^* \textit{ with unique condensation.}$$

Let $\theta \geq (2^\kappa)^+$ be large enough such that $\dot{A}^* \in H_\theta$ and set $M = \langle H_\theta; \in, \mathbb{P}, \dot{A}, \dot{A}^* \rangle$. Suppose $M^\sharp_{\mathsf{mw}}$ exists and has an $(\omega, \kappa + 1)$-iteration strategy Σ that condenses to fragments. Suppose that

$$S := \{X \subset \omega_2 \,;\, \text{otp}(X) = \omega_1 \wedge \exists Y \prec M : Y \cap \omega_2 = X\}^V$$

intersects all strongly closed unbounded subsets of $[\omega_2]^{\omega_1}$ and $\mathbb{P}$ preserves S.
Then there is some $A \subset \omega_1$, $A \in V$ such that

$$L[A] \models \phi(A, \vec{\alpha}).$$

Note that S intersects all strongly club sets in V is equivalent to Chang's Conjecture, compare Lemma 1.9.1. The proof of the above theorem is similar to the proof of the previous theorem. We nevertheless give some details, especially at the point where the reasonability of $\mathbb{P}$ is applied.

Proof. Set $\mathfrak{M} = M^\sharp_{\mathsf{mw}}$ and let δ denote $\mathfrak{M}$'s measurable Woodin. Let $U \in M^\sharp_{\mathsf{mw}}$ denote the (trivial completion of the) least normal measure on δ and let ζ_0 denote the index of U on $M^\sharp_{\mathsf{mw}}$'s extender sequence. Using Lemma 5.6.16 we are going to build an iteration tree $\mathcal{T} \in V$ on $\mathfrak{M}$ of height $\omega_2 + 1$ such that for all $G \subset \mathbb{P}$ the set $\dot{A}^{*G} \cap \omega_2$ is generic over $\mathcal{M}^{\mathcal{T}}_{\omega_2}$. Once $\mathcal{T}$ is constructed it will follow from the hypothesis on S that we find some elementary substructure $X \prec M$, $X \in V$, $\mathrm{otp}(X \cap \omega_2) = \omega_1$ such that $\dot{A}^G$ is generic over this substructure.
We construct $\mathcal{T}$; we will omit superscripts $\mathcal{T}$ where possible, so $\mathcal{T}$ has iteration maps $j_{\alpha,\beta}$ and model $\mathcal{M}_\alpha$. We will call $\alpha < \omega_2$ a *closure point* if

$$\mathbf{1}_{\mathbb{P}} \Vdash \dot{A}^* \cap j_{0,\alpha}(\zeta_0) \models j_{0,\alpha}(W_\delta)$$

where W_δ is the extender algebra with δ-many generators calculated in $\mathfrak{M}$. If α is a closure point we use $j_{0,\alpha}(U)$ to continue the iteration. If α is not a closure point we perform a genericity iteration for $\dot{A}^*$ in the sense of Lemma 5.6.16. Since $\mathbb{P}$ is reasonable and Σ condenses to fragments we can apply Lemma 5.6.16, so, taking note of Remark 5.6.18, we reach the next closure point after $< \omega_2$-many stages. This completes the definition of $\mathcal{T}$. Clearly $\mathcal{T} \in M$. Let $b = [0, \omega_2]_{\mathcal{T}}$. Clearly b contains unboundedly many (and hence club many) closure points. So club often we have used (the images of) U to continue the iteration. Moreover $j_{0,\omega_2}(\delta) = \omega_2$ and ω_2 is a baby closure point.
The rest follows like in the previous proof if one replaces κ with ω_2. □

Bibliography

[AW02] David Asperó and Philip D. Welch. Bounded Martin's maximum, weak Erdős cardinals, and ψ_{AC}. *J. Symbolic Logic*, 67(3):1141–1152, 2002.

[Bag00] Joan Bagaria. Bounded forcing axioms as principles of generic absoluteness. *Arch. Math. Logic*, 39(6):393–401, 2000.

[Bar75] Jon Barwise. *Admissible sets and structures.* Springer-Verlag, Berlin, 1975. An approach to definability theory, Perspectives in Mathematical Logic.

[Bau84] James E. Baumgartner. Applications of the proper forcing axiom. In *Handbook of set-theoretic topology*, pages 913–959. North-Holland, Amsterdam, 1984.

[BS09] Daniel Busche and Ralf Schindler. The strength of choiceless patterns of singular and weakly compact cardinals. *Ann. Pure Appl. Logic*, 159(1-2):198–248, 2009.

[CS09] Benjamin Claverie and Ralf Schindler. Increasing u_2 by a stationary set preserving forcing. *J. Symbolic Logic*, 74(1):187–200, 2009.

[Far] Ilijas Farah. The extender algebra. Circulated Notes.

[Far07] Ilijas Farah. A proof of the Σ^2_1-absoluteness theorem. In *Advances in logic*, volume 425 of *Contemp. Math.*, pages 9–22. Amer. Math. Soc., Providence, RI, 2007.

[FKLM08] Ilijas Farah, Richard Ketchersid, Paul Larson, and Menachem Magidor. Absoluteness for universally Baire sets and the uncountable. II. In *Computational prospects of infinity. Part II. Presented talks*, volume 15 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 163–191. World Sci. Publ., Hackensack, NJ, 2008.

[FL06] Ilijas Farah and Paul B. Larson. Absoluteness for universally Baire sets and the uncountable. I. In *Set theory: recent trends and applications*, volume 17 of *Quad. Mat.*, pages 47–92. Dept. Math., Seconda Univ. Napoli, Caserta, 2006.

[FM95] Matthew Foreman and Menachem Magidor. Large cardinals and definable counterexamples to the continuum hypothesis. *Ann. Pure Appl. Logic*, 76(1):47–97, 1995.

[FMS88] Matthew Foreman, Menachem Magidor, and Saharon Shelah. Martin's maximum, saturated ideals, and nonregular ultrafilters. I. *Ann. of Math. (2)*, 127(1):1–47, 1988.

[FMW92] Qi Feng, Menachem Magidor, and Hugh Woodin. Universally Baire sets of reals. In *Set theory of the continuum (Berkeley, CA, 1989)*, volume 26 of *Math. Sci. Res. Inst. Publ.*, pages 203–242. Springer, New York, 1992.

[GJM78] Fred Galvin, Thomas Jech, and Menachem Magidor. An ideal game. *J. Symbolic Logic*, 43(2):284–292, 1978.

[GS95] Martin Goldstern and Saharon Shelah. The bounded proper forcing axiom. *J. Symbolic Logic*, 60(1):58–73, 1995.

[Har78] Leo Harrington. Analytic determinacy and $0^\sharp$. *J. Symbolic Logic*, 43(4):685–693, 1978.

[Jec03] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[Jena] Ronald B. Jensen. $\mathcal{L}$-forcing. Handwritten notes.

[Jenb] Ronald B. Jensen. On some problems of Mitchell, Welch and Vickers. Handwritten notes.

[JMMP80] T. Jech, M. Magidor, W. Mitchell, and K. Prikry. Precipitous ideals. *J. Symbolic Logic*, 45(1):1–8, 1980.

[Kak81] Yuzuru Kakuda. On a condition for Cohen extensions which preserve precipitous ideals. *J. Symbolic Logic*, 46(2):296–300, 1981.

[Kan03] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[KLZ07] Richard Ketchersid, Paul Larson, and Jindřich Zapletal. Increasing δ^1_2 and Namba-style forcing. *J. Symbolic Logic*, 72(4):1372–1378, 2007.

[Lar04] Paul B. Larson. *The stationary tower*, volume 32 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2004. Notes on a course by W. Hugh Woodin.

[LS03] Paul Larson and Saharon Shelah. Bounding by canonical functions, with CH. *J. Math. Log.*, 3(2):193–215, 2003.

[Mag80] Menachem Magidor. Precipitous ideals and $\mathbf{\Sigma}^1_4$ sets. *Israel J. Math.*, 35(1-2):109–134, 1980.

[Moo05] Justin Tatch Moore. Set mapping reflection. *J. Math. Log.*, 5(1):87–97, 2005.

[MS70] Donald A. Martin and Robert M. Solovay. Internal Cohen extensions. *Ann. Math. Logic*, 2(2):143–178, 1970.

[MS94] Donald A. Martin and John R. Steel. Iteration trees. *J. Amer. Math. Soc.*, 7(1):1–73, 1994.

[Nee95] Itay Neeman. Optimal proofs of determinacy. *Bull. Symbolic Logic*, 1(3):327–339, 1995.

[NZ] Itay Neeman and Jindřich Zapletal. Proper forcing and $L(\mathbb{R})$. http://arxiv.org/abs/math/0003027.

[NZ98] Itay Neeman and Jindřich Zapletal. Proper forcings and absoluteness in $L(\mathbf{R})$. *Comment. Math. Univ. Carolin.*, 39(2):281–301, 1998.

[NZ01] Itay Neeman and Jindřich Zapletal. Proper forcing and $L(\mathbb{R})$. *J. Symbolic Logic*, 66(2):801–810, 2001.

[Sak08] Hiroshi Sakai. Semistationary and stationary reflection. *J. Symbolic Logic*, 73(1):181–192, 2008.

[Sch04] Ralf Schindler. Semi-proper forcing, remarkable cardinals, and bounded Martin's maximum. *MLQ Math. Log. Q.*, 50(6):527–532, 2004.

[Sch06] Ralf Schindler. Bounded Martin's maximum and strong cardinals. In *Set theory*, Trends Math., pages 401–406. Birkhäuser, Basel, 2006.

[She98] Saharon Shelah. *Proper and improper forcing.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

[SS09] Ralf Schindler and John Steel. The self-iterability of $L[E]$. *J. Symbolic Logic*, 74(3):751–779, 2009.

[Ste] John R. Steel. An outline of inner model theory. To appear in Handbook of Set Theory, see http://www.tau.ac.il/∼rinot/host.html.

[Ste05] John R. Steel. PFA implies $\mathrm{AD}^{L(\mathbb{R})}$. *J. Symbolic Logic*, 70(4):1255–1296, 2005.

[SW98] John R. Steel and Philip D. Welch. Σ^1_3 absoluteness and the second uniform indiscernible. *Israel J. Math.*, 104:157–190, 1998.

[Tod81] Stevo Todorčević. Stationary sets, trees and continuums. *Publ. Inst. Math. (Beograd) (N.S.)*, 29(43):249–262, 1981.

[Tod83] Stevo Todorčević. On a conjecture of R. Rado. *J. London Math. Soc. (2)*, 27(1):1–8, 1983.

[Tod93] Stevo Todorčević. Conjectures of Rado and Chang and cardinal arithmetic. In *Finite and infinite combinatorics in sets and logic (Banff, AB, 1991)*, volume 411 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 385–398. Kluwer Acad. Publ., Dordrecht, 1993.

[Woo99] W. Hugh Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, volume 1 of *de Gruyter Series in Logic and its Applications.* Walter de Gruyter & Co., Berlin, 1999.

[Woo05] W. Hugh Woodin. The continuum hypothesis. In *Logic Colloquium 2000*, volume 19 of *Lect. Notes Log.*, pages 143–197. Assoc. Symbol. Logic, Urbana, IL, 2005.

[Zap] Jindřich Zapletal. private communication.