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Overconvergent de Rham–Witt cohomology for semi-stable varieties

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To Christopher Deninger, on the occasion of his sixtieth birthday

Abstract. We define an overconvergent version of the Hyodo–Kato complex for semi-stable varieties Y over perfect fields of positive characteristic, and prove that its hypercohomology tensored with $\mathbb{Q}$ recovers the log-rigid cohomology when Y is quasi-projective. We then describe the monodromy operator using the overconvergent Hyodo–Kato complex. Finally, we show that overconvergent Hyodo–Kato cohomology agrees with log-crystalline cohomology in the projective semi-stable case.

1. Introduction

For a proper and smooth variety Y over a perfect field k of characteristic $p > 0$, the hypercohomology of the Deligne–Illusie de Rham–Witt complex $W\Omega^\bullet_{Y/k}$ tensored with $\mathbb{Q}$ computes – using the comparison isomorphism with crystalline cohomology [1] – the rigid cohomology

$$H^*_{\mathrm{rig}}(Y/W(k)[1/p]) \cong \mathbb{H}^*(Y, W\Omega^\bullet_{Y/k}) \otimes \mathbb{Q}.$$

However, rigid cohomology is well-defined without any properness assumption on Y. In [4], Davis, Langer and Zink define an overconvergent de Rham–Witt complex $W^\dagger\Omega^\bullet_{Y/k}$, which is a subcomplex of $W\Omega^\bullet_{Y/k}$, and show that

$$H^*_{\mathrm{rig}}(Y/W(k)[1/p]) \cong \mathbb{H}^*(Y, W^\dagger\Omega^\bullet_{Y/k}) \otimes \mathbb{Q}$$

for Y quasi-projective and smooth over k.

On the other hand, one could instead relax the smoothness condition on Y. Let $S_0 = (\operatorname{Spec} k, \mathbb{N})$ be the standard log point, and let Y be a fine S_0-log scheme. Let $\mathfrak{S}_0$ be the (weak) formal log scheme $(\operatorname{Spf} W, \mathbb{N} \to W,\ 1 \mapsto 0)$. Then Grosse-Klönne [8] defines the log-rigid cohomology $H^*_{\text{log-rig}}(Y/\mathfrak{S}_0)$ of Y

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(we recall the definition in Section 2). Grosse-Klönne shows that the log-rigid cohomology of Y agrees with Shiho's log-convergent cohomology of Y whenever Y is a semi-stable variety whose irreducible components are proper. In particular, by the comparison between log-convergent and log-crystalline cohomology [19], there is an isomorphism

$$H^*_{\text{log-rig}}(Y/\mathfrak{S}_0) \cong \mathbb{H}^*(Y, W\omega^\bullet_{Y/k}) \otimes \mathbb{Q}$$

for proper semi-stable varieties Y over k, where $W\omega^\bullet_{Y/k}$ is the Hyodo–Kato complex [10].

There is, however, currently no Hyodo–Kato style theory available for non-proper semi-stable varieties. We shall define a complex $W^\dagger\omega^\bullet_{Y/k}$, functorial in Y, which computes log-rigid cohomology in non-proper situations. More precisely, we prove the following theorem.

Theorem 1.1. *Let Y be a quasi-projective semi-stable variety over S_0. Then there is a quasi-isomorphism*

$$R\Gamma_{\text{log-rig}}(Y/\mathfrak{S}_0) \cong \mathbb{R}\Gamma(Y, W^\dagger\omega^\bullet_{Y/k} \otimes \mathbb{Q}).$$

We then describe the monodromy operator on log-rigid cohomology in terms of the overconvergent Hyodo–Kato complex, using the method of [18].

Using the overconvergent Hyodo–Kato complex, we can formulate Grosse-Klönne's Hyodo–Kato isomorphism [8] in the non-proper case as follows: For a strictly semi-stable weak formal scheme $\mathfrak{X}$ over Spwf $W(k)$ with generic fibre X, a $K_0 = W(k)[1/p]$-dagger space, and a closed fibre Y which we assume to be quasi-projective, there is a canonical isomorphism between the de Rham cohomology of X and the (rational) overconvergent Hyodo–Kato cohomology of Y [8, Cor. 3.5].

In [3], log-rigid cohomology and overconvergent syntomic cohomology are used to study the p-adic cohomology of semi-stable p-adic Stein spaces, notably Drinfeld half-spaces, within the context of a hoped for p-adic local Langlands correspondence. We expect that the overconvergent Hyodo–Kato complex will have interesting applications in this area.

Note that even in the smooth case X/k, [4] only construct a map $R\Gamma_{\text{rig}}(X) \to \mathbb{R}\Gamma(X, W^\dagger\Omega^\bullet_{X/k} \otimes \mathbb{Q})$ when X is quasi-projective. We will see in the proof of Theorem 5.3 (= Theorem 1.1) in Section 5 why we need to assume quasi-projectivity. In a recent preprint [16], Lawless has been able to remove the quasi-projectivity hypothesis at least in the smooth case. For details, we refer to [16]. In a work in progress, Lawless also intends to remove this hypothesis in the semi-stable case and hence we expect Theorem 1.1 to hold unconditionally as well.

In the final part of the paper, we compare the usual and overconvergent Hyodo–Kato cohomology in the proper case:

Theorem 1.2. *Let Y be a projective semi-stable variety over S_0. Then the canonical map*

$$\mathbb{H}^*(Y, W^\dagger\omega^\bullet_{Y/k}) \to \mathbb{H}^*(Y, W\omega^\bullet_{Y/k}) = H^*_{\text{log-cris}}((Y, M)/(W(k), W(L)))$$

induced by the inclusion $W^\dagger\omega^\bullet_{Y/k} \subset W\omega^\bullet_{Y/k}$ *is an isomorphism of finite type* $W(k)$*-modules. Here* M *is the log structure on* Y *given by* $\mathcal{O}_Y \cap u_*\mathcal{O}_U^\times$*, where* $u\colon U \hookrightarrow Y$ *is a smooth dense open, and* $W(L)$ *is the canonical lifting of the log structure* L *on* $\operatorname{Spec} k$ *given by* $1 \mapsto 0$ (*previously denoted by* S_0).

It should be noted that in fact we give two definitions of the overconvergent Hyodo–Kato complex in this paper. The first is constructed in the style of [10], and the second in the more modern approach of [17]. We prove that the two complexes are the same. Along the way, this has the serendipitous consequence that we show that Matsuue's log de Rham–Witt complex $W\Lambda^\bullet_{Y/(R,\mathbb{N})}$ (see Section 3.1 for the notation) gives the Hyodo–Kato complex $W\omega^\bullet_{Y/k}$ in the special case that $R = k$, thus filling a gap in the literature.

We assume that the reader is familiar with the de Rham–Witt complex of Deligne–Illusie and its basic properties, including its explicit description for (Laurent)-polynomial algebras [14], étale base change results, and the overconvergent version in the smooth case proven in [4].

2. Log-rigid cohomology

Let k be a field of characteristic $p > 0$, let $W = W(k)$ be the Witt vectors of k and $K := \operatorname{Frac} W$. Set $W_n = W/p^n$ for $n \in \mathbb{N}$. We will write $\mathfrak{S}_0$ for the (weak) formal log scheme $(\operatorname{Spf} W, \mathbb{N} \to W,\ 1 \mapsto 0)$. The special fibre of $\mathfrak{S}_0$ is the standard log point $S_0 = (\operatorname{Spec} k, \mathbb{N} \to k,\ 1 \mapsto 0)$.

We briefly recall Grosse-Klönne's definition of the log-rigid cohomology of a fine S_0-log scheme Y. For the details, one should consult [8]. Let $Y = \bigcup_{i\in I} V_i$ be an open covering, and suppose there are exact closed immersions $V_i \hookrightarrow \mathfrak{P}_i$ into log smooth weak formal $\mathfrak{S}_0$-log schemes for each i. For each $H \subset I$, choose an exactification of the diagonal embedding

$$V_H := \bigcap_{i\in H} V_i \overset{\iota}{\hookrightarrow} \mathfrak{P}_H \overset{f}{\to} \prod_{i\in H} \mathfrak{P}_i$$

(this means that ι is an exact closed immersion and f is log-étale). Then the log de Rham complex $\omega^\bullet_{\mathfrak{P}_H/\mathfrak{S}_0}$ tensored with $\mathbb{Q}$ induces a complex of sheaves $\omega^\bullet_{\mathfrak{P}_{H,K}}$ on the K-dagger space $\mathfrak{P}_{H,K}$ associated to the generic fibre of $\mathfrak{P}_H$ (see [7]). Let $\operatorname{sp}\colon \mathfrak{P}_{H,K} \to \mathfrak{P}_H$ be the specialization map, and write $]V_H[^\dagger_{\mathfrak{P}_H} := \operatorname{sp}^{-1}(V_H)$ for the tubular neighborhood of V_H in $\mathfrak{P}_{H,K}$. Then

$$]V_H[^\dagger_{\mathfrak{P}_H} \quad \text{and} \quad \omega^\bullet_{]V_H[^\dagger_{\mathfrak{P}_H}} := \omega^\bullet_{\mathfrak{P}_{H,K}}|_{]V_H[^\dagger_{\mathfrak{P}_H}}$$

are independent of the choice of exactification above [8, Lem. 1.2]. Now, for $H_1 \subset H_2$, the projection $p_{12}\colon]V_{H_2}[^\dagger_{\mathfrak{P}_{H_2}} \to]V_{H_1}[^\dagger_{\mathfrak{P}_{H_1}}$ induces

$$p_{12}^{-1}\omega^\bullet_{]V_{H_1}[^\dagger_{\mathfrak{P}_{H_1}}} \to \omega^\bullet_{]V_{H_2}[^\dagger_{\mathfrak{P}_{H_2}}}.$$

This defines a complex of simplicial sheaves $\omega^{\bullet}_{]V_{H_\bullet}[^{\dagger}_{\mathfrak{P}_{H_\bullet}}}$ on a simplicial dagger space $]V_{H_\bullet}[^{\dagger}_{\mathfrak{P}_{H_\bullet}}$. The log-rigid cohomology of Y is given by

$$R\Gamma_{\text{log-rig}}(Y/\mathfrak{S}_0) := \mathbb{R}\Gamma(]V_{H_\bullet}[^{\dagger}_{\mathfrak{P}_{H_\bullet}}, \omega^{\bullet}_{]V_{H_\bullet}[^{\dagger}_{\mathfrak{P}_{H_\bullet}}}).$$

Then $R\Gamma_{\text{log-rig}}(Y/\mathfrak{S}_0)$ is independent on the choice of open cover $Y = \bigcup_{i\in I} V_i$ and the choice of embeddings $V_i \hookrightarrow \mathfrak{P}_i$ [8, Lem. 1.4].

3. The overconvergent Hyodo–Kato complex

Now suppose that k is a perfect field of characteristic $p > 0$. We follow closely the approach of [10]. Let X be a regular flat W-scheme and write

$$\begin{array}{ccccc} Y & \overset{i}{\hookrightarrow} & X & \overset{j}{\hookleftarrow} & X_K \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} W & \hookleftarrow & \operatorname{Spec} K \end{array}$$

for the special and generic fibres of X. We suppose that X has semi-stable reduction, that is to say we suppose that étale locally on X, there is a smooth morphism $X \to \operatorname{Spec} W[T_1, \dots, T_n]/(T_1 \cdots T_d - p)$ for some $n \geq d$. In particular, X_K is smooth and Y is a reduced normal crossings divisor on X. If we endow X with the log-structure induced by the special fibre and consider Y with the pullback log structure, then Y is a fine log-smooth S_0-log scheme. Étale locally on Y, the structure morphism factors as

$$Y \xrightarrow{f} (\operatorname{Spec} k[T_1, \dots, T_n]/(T_1 \cdots T_d), \mathbb{N}^d, e_i \mapsto T_i) \xrightarrow{\delta} S_0,$$

where f is exact and étale and δ is induced by the diagonal. We say that Y is semi-stable over S_0.

Since k is a perfect field, we can find a dense open subscheme $u\colon U \hookrightarrow Y$, which is smooth over k. We may therefore consider the pushforward of the overconvergent de Rham–Witt complex $W^{\dagger}\omega^{\bullet}_{U/k}$ of [4]. Let

$$d\log\colon i^{-1}j_*(\mathcal{O}^{\times}_{X_K}) \to u_* W\Omega^1_{U/k}$$

be the homomorphism considered in [10, Section 1]. Note that the image lies in $u_* W^{\dagger}\Omega^1_{U/k}$ (see [17, Prop. 10.1]). Then the Hyodo–Kato complex $W\omega^{\bullet}_{Y/k}$ is the p-adic completion of the $W(\mathcal{O}_Y)$-subalgebra of $u_* W\Omega^{\bullet}_{U/k}$ generated by $dW(\mathcal{O}_Y)$ and the image of $d\log$. Let $W^{\dagger}(\mathcal{O}_Y)$ denote the Zariski sheaf of overconvergent Witt vectors (see [5, Prop. 3.2]) on Y.

Definition. $W^{\dagger}\omega^{\bullet}_{Y/k}$ is the differential graded $W^{\dagger}(\mathcal{O}_Y)$-algebra

$$W\omega^{\bullet}_{Y/k} \cap u_* W^{\dagger}\Omega^{\bullet}_{U/k}.$$

Then $W^{\dagger}\omega^{\bullet}_{Y/k}$ is a subcomplex of the $W^{\dagger}(\mathcal{O}_Y)$-algebra $u_* W^{\dagger}\Omega^{\bullet}_{U/k}$ and inherits the operators F and V satisfying the usual de Rham–Witt relations, as in [10].

Now let $u_*W^\dagger\Omega^\bullet_{U/k}[\theta]/(\theta^2)$ be the complex given by adjoining an indeterminate θ in degree one, subject to $\theta a = (-1)^q a\theta$ for all $a \in u_*W^\dagger\Omega^q_{U/k}$ and $d\theta = 0$. Let

$$d\log\colon i^{-1}j_*(\mathcal{O}^\times_{X_K}) \to u_*W^\dagger\Omega^1_{U/k}[\theta]/(\theta^2)$$

be the unique homomorphism which induces on $u^{-1}i^{-1}(\mathcal{O}^\times_X)$ the composite map

$$u^{-1}i^{-1}(\mathcal{O}^\times_X) \to \mathcal{O}^\times_U \xrightarrow{d\log} W\Omega^1_{U/k}$$

and induces on $K^\times$ the map $a \mapsto \mathrm{ord}_K(a)\theta$ (again, see [10, Section 1]). The image of $d\log$ lies by definition inside $W\tilde{\omega}^\bullet_{Y/k}$, defined in [10, Section 1.4].

Definition. $W^\dagger\tilde{\omega}^\bullet_{Y/k}$ is the $W^\dagger(\mathcal{O}_Y)$-algebra $W\tilde{\omega}^\bullet_{Y/k} \cap u_*W^\dagger\Omega^\bullet_{U/k}[\theta]/(\theta^2)$.

Then we have a short exact sequence of complexes, induced by (see [10, Prop. 1.5])

$$\begin{cases} 0 \to W^\dagger\omega^\bullet_{Y/k}[-1] \to W^\dagger\tilde{\omega}^\bullet_{Y/k} \to W^\dagger\omega^\bullet_{Y/k} \to 0, \\ a \mapsto a \wedge \theta, \quad \theta \mapsto 0. \end{cases} \tag{1}$$

3.1. **An equivalent approach.** In this section we shall outline another definition of the overconvergent Hyodo–Kato complex, this time in the style of [17], and we will show that the two definitions are the same. This will become particularly useful in Section 7.

In [17, Section 3.4] Matsuue defines the log de Rham–Witt complex $W\Lambda^\bullet_{(S,Q)/(R,P)}$ for any morphism of pre-log rings $(R,P) \to (S,Q)$, where R is a $\mathbb{Z}_{(p)}$-algebra, as the initial object in the category of log F-V-procomplexes. The construction is a logarithmic generalization of the construction given in [14, Section 1.3].

Fix integers $n \geq d$ and let $(B := k[T_1,\dots,T_n], \mathbb{N}^d, e_i \mapsto T_i)$ be considered as a pre-log ring over $(k,\{*\})$, where $\{*\}$ denotes the trivial monoid. Then one has, in particular, the log de Rham–Witt complex $W\Lambda^\bullet_{(B,\mathbb{N}^d)/(k,\{*\})}$ as a special case. Any element of $W\Lambda^\bullet_{(B,\mathbb{N}^d)/(k,\{*\})}$ can be written as a convergent sum of basic log Witt differentials [17, Prop. 4.3]. Matsuue then defines a subcomplex $W^\dagger\Lambda^\bullet_{(B,\mathbb{N}^d)/(k,\{*\})}$ as those elements of $W\Lambda^\bullet_{(B,\mathbb{N}^d)/(k,\{*\})}$ which are overconvergent (see [17, Section 10.1]).

Now consider a pre-log ring (A,M,α) over $(k,\{*\})$ such that A is a finitely generated k-algebra. Then we may choose a surjective morphism of pre-log rings over $(k,\{*\})$:

$$\begin{array}{ccc} \mathbb{N}^d & \longrightarrow & B = k[T_1,\dots,T_n] \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha} & A, \end{array}$$

where the top morphism is $e_i \mapsto T_i$. This morphism of pre-log rings induces a morphism of log de Rham–Witt complexes

$$\lambda\colon W\Lambda^\bullet_{(B,\mathbb{N}^d)/(k,\{*\})} \to W\Lambda^\bullet_{(A,M)/(k,\{*\})}.$$

Matsuue defines $W^\dagger\Lambda^\bullet_{(A,M)/(k,\{*\})} := \lambda(W^\dagger\Lambda^\bullet_{(B,\mathbb{N}^d)/(k,\{*\})})$. Notice that one could have taken any log polynomial algebra over k which surjects onto (A,M), but Matsuue shows that $W^\dagger\Lambda^\bullet_{(A,M)/(k,\{*\})}$ is independent of this choice (see [17, Def. 10.2] and the subsequent discussion). By construction, $W^\dagger\Lambda^\bullet_{(A,\{*\})/(k,\{*\})}$ is the overconvergent de Rham–Witt complex $W^\dagger\omega^\bullet_{A/k}$ of [4]. In [4, Cor. 1.7], it is shown that this construction glues to give a complex of Zariski sheaves $W^\dagger\omega^\bullet_{X/k}$ on any variety X. In [17, Section 10.3] this is generalized to show that $W^\dagger\Lambda^\bullet_{(A,M)/(k,\{*\})}$ glues to give a complex of Zariski sheaves $W^\dagger\Lambda^\bullet_{(X,M)/(k,\{*\})}$, where (X,M) denotes the log scheme associated to the complement of a strict normal crossing divisor. We give a similar argument in the semi-stable case.

Let $\operatorname{Spec} A$ be a semi-stable affine k-scheme and let (A,M,α) be the associated pre-log ring. Define $P_jW\Lambda^r_{(A,M)/(k,\{*\})}$ to be the image of the map

$$W\Lambda^j_{(A,M)/(k,\{*\})} \otimes W\Omega^{r-j}_{A/k} \to W\Lambda^r_{(A,M)/(k,\{*\})}.$$

This gives a filtration $P_\bullet W\Lambda^\bullet_{(A,M)/(k,\{*\})}$ of the complex $W\Lambda^\bullet_{(A,M)/(k,\{*\})}$. Let $\{\operatorname{Spec} A_i\}_{i\in I}$ be the irreducible components of $\operatorname{Spec} A$. For subsets $J\subset I$, let $\bigcap_{i\in J}\operatorname{Spec} A_i = \operatorname{Spec} A_J$. Then the Poincaré residue maps give

$$\operatorname{Gr}_j W\Lambda^\bullet_{(A,M)/(k,\{*\})} \simeq \bigoplus_{\substack{J\subset I\\ |J|=j}} W\Omega^\bullet_{A_J/k}[-j].$$

Similarly, define $P_jW^\dagger\Lambda^r_{(A,M)/(k,\{*\})}$ to be the image of the map

$$W^\dagger\Lambda^j_{(A,M)/(k,\{*\})} \otimes W^\dagger\Omega^{r-j}_{A/k} \to W^\dagger\Lambda^r_{(A,M)/(k,\{*\})}$$

to get a filtration $P_\bullet W^\dagger\Lambda^\bullet_{(A,M)/(k,\{*\})}$ of the complex $W^\dagger\Lambda^\bullet_{(A,M)/(k,\{*\})}$. The argument in [17, Lem. 10.9] shows that the above residue isomorphism induces

$$\operatorname{Gr}_j W^\dagger\Lambda^\bullet_{(A,M)/(k,\{*\})} \simeq \bigoplus_{\substack{J\subset I\\ |J|=j}} W^\dagger\Omega^\bullet_{A_J/k}[-j].$$

Proposition 3.2. *Let* $\operatorname{Spec} A$ *be a semi-stable affine* k*-scheme and let* (A,M,α) *be the associated pre-log ring. Then the presheaf canonically determined by*

$$D(f) \mapsto W^\dagger\Lambda^r_{(A_f,M)/(k,\{*\})}$$

on the basis of distinguished opens is a Zariski sheaf on $\operatorname{Spec} A$. (*Given* $f\in A$, *we consider the localization* A_f *as a pre-log ring via the composition* $M \xrightarrow{\alpha} A \to A_f$.)

Proof. The proof is similar to the proof of [4, Prop. 1.2] and [17, Prop. 10.12]. Let $f_1,\dots,f_l\in A$ be a generating set for A. Write $A_{i_1\cdots i_s}$ for $A_{f_{i_1}\cdots f_{i_s}}$. Consider the Čech complex $C^\bullet$ given by

$$C^s = \bigoplus_{1\le i_1<\cdots<i_s\le l} W^\dagger\Lambda^r_{(A_{i_1\cdots i_s},M)/(k,\{*\})}$$

(so $C^0 = W^\dagger\Lambda^r_{(A,M)/(k,\{*\})}$). Then it suffices to show that $C^\bullet$ is exact. Define

$$P_jC^s = \bigoplus_{1\le i_1<\cdots<i_s\le l} P_jW^\dagger\Lambda^r_{(A_{i_1\cdots i_s},M)/(k,\{*\})}.$$

This gives a filtration $P_\bullet C^\bullet$ of the complex $C^\bullet$. The graded piece $\mathrm{Gr}_jC^\bullet$ is

$$\begin{aligned}\mathrm{Gr}_jC^s &\cong \bigoplus_{1\le i_1<\cdots<i_s\le l}\ \bigoplus_{\substack{J\subset I\\ |J|=j}} W^\dagger\Omega^{r-j}_{(A_{i_1\cdots i_s})_J/k}\\ &\cong \bigoplus_{1\le i_1<\cdots<i_s\le l}\ \bigoplus_{\substack{J\subset I\\ |J|=j}} W^\dagger\Omega^{r-j}_{(A_J)_{i_1\cdots i_s}/k}\end{aligned}$$

in degree s. Here the second direct sum in the first line runs over all j-fold intersections of irreducible components of $\operatorname{Spec} A_{i_1\cdots i_s}$. Therefore,

$$\mathrm{Gr}_jC^\bullet \cong \bigoplus_{\substack{J\subset I\\ |J|=j}} \tilde{C}^\bullet,$$

where

$$\tilde{C}^s = \bigoplus_{1\le i_1<\cdots<i_s\le l} W^\dagger\Omega^{r-j}_{(A_J)_{i_1\cdots i_s}/k}$$

is the Čech complex for $W^\dagger\omega^{r-j}_{A_J/k}$. This is exact by [4, Prop. 1.6]. Induction, using the short exact sequence of complexes

$$0\to P_{j-1}C^\bullet \to P_jC^\bullet \to \mathrm{Gr}_jC^\bullet \to 0,$$

shows that $P_jC^\bullet$ is exact for all j, and hence $C^\bullet$ is exact. □

One may therefore glue to define a complex of Zariski sheaves $W^\dagger\Lambda^\bullet_{Y/(k,\{*\})}$ for semi-stable schemes Y over S_0 (recall from Section 2 that S_0 is the standard log point $(\operatorname{Spec} k, \mathbb{N}, 1\mapsto 0)$). Finally, we define the overconvergent Hyodo–Kato complex $W^\dagger\Lambda^\bullet_{Y/S_0}$ to be the image of $W^\dagger\Lambda^\bullet_{Y/(k,\{*\})}$ under the projection

$$W\Lambda^\bullet_{Y/(k,\{*\})} \to W\Lambda^\bullet_{Y/(k,\mathbb{N})} = W\Lambda^\bullet_{Y/S_0}$$

of log de Rham–Witt complexes. This is again a complex of Zariski sheaves.

Proposition 3.3. *The overconvergent Hyodo–Kato complex $W^\dagger\omega^\bullet_{Y/k}$ of the previous section is the same as $W^\dagger\Lambda^\bullet_{Y/S_0}$.*

Proof. We first show that Matsuue's log de Rham–Witt complex $W\Lambda^\bullet_{Y/S_0}$ agrees with the Hyodo–Kato complex $W\omega^\bullet_{Y/k}$ of [10]. This must be well known to the experts, but the authors do not know of a proof recorded in the literature; it seems important to reconcile the two approaches, so we give a proof here. Since both complexes are complexes of Zariski sheaves, it suffices to construct a canonical isomomorphism, functorial in Y, if Y is affine.

Given a log F-V procomplex $\{E^\bullet_m\}_{m\in\mathbb{N}}$, define differential graded ideals

$$\mathrm{Fil}^sE^i_m := V^sE^i_{m-s} + dV^sE^{i-1}_{m-s} \subset E^i_m.$$

Then this gives a filtration of log F-V-procomplexes which is compatible with F, V, d and the projections, see [17, Section 3.5].

Now, $W_\bullet\Lambda^\bullet_{Y/S_0} = \{W_m\Lambda^\bullet_{Y/S_0}\}_{m\in\mathbb{N}}$ and $W_\bullet\omega^\bullet_{Y/k} = \{W_m\omega^\bullet_{Y/k}\}_{m\in\mathbb{N}}$ are log F-V-procomplexes, so we have a map of log F-V-procomplexes, evidently functorial in Y,

$$W_\bullet\Lambda^\bullet_{Y/S_0} \to W_\bullet\omega^\bullet_{Y/k},$$

by the universal property of $W\Lambda^\bullet_{Y/S_0}$. This map induces diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Fil}^m W_{m+1}\Lambda^\bullet_{Y/S_0} & \longrightarrow & W_{m+1}\Lambda^\bullet_{Y/S_0} & \longrightarrow & W_m\Lambda^\bullet_{Y/S_0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Fil}^m W_{m+1}\omega^\bullet_{Y/k} & \longrightarrow & W_{m+1}\omega^\bullet_{Y/k} & \longrightarrow & W_m\omega^\bullet_{Y/k} & \longrightarrow & 0 \end{array}$$

of short exact sequences (see [17, Prop. 3.6] for the top row, and [10, Thm. 4.4] for the bottom row) for each $m \in \mathbb{N}$. Now one notices that

$$W_1\Lambda^\bullet_{Y/S_0} = W_1\omega^\bullet_{Y/k} = \omega^\bullet_{Y/k}$$

is the usual logarithmic de Rham complex, by definition. This gives

$$\mathrm{Fil}^m W_{m+1}\Lambda^\bullet_{Y/S_0} = \mathrm{Fil}^m W_{m+1}\omega^\bullet_{Y/k}$$

and then the diagrams give $W_m\Lambda^\bullet_{Y/S_0} = W_m\omega^\bullet_{Y/k}$ for all $m \in \mathbb{N}$.

Since $W^\dagger\omega^\bullet_{Y/k}$ and $W^\dagger\Lambda^\bullet_{Y/S_0}$ are subcomplexes of Zariski sheaves of the completed versions, it suffices to show the claim for Y affine. By a result of Kedlaya [13, Thm. 2], Y can be covered by finitely many affines $\operatorname{Spec} B_i$ such that B_i is finite étale and free over $A_i = k[T_1,\dots,T_d]/(T_1\cdots T_r)$ for some r. Using again a sheaf argument, it suffices to prove the claim for B finite étale and free over $A = k[T_1,\dots,T_d]/(T_1\cdots T_r)$. By étale base change [17, Prop. 3.7] and the fact that $W(B)$ is again finite étale and free over $W(A)$, we have $W\Lambda^\bullet_{B/S_0} = W(B)\otimes_{W(A)} W\Lambda^\bullet_{A/S_0}$ and likewise $W\omega^\bullet_{B/k} = W(B)\otimes_{W(A)} W\omega^\bullet_{A/k}$. Note that the isomorphism $W\Lambda^\bullet_{A/S_0} \cong W\omega^\bullet_{A/k}$ is explicitly given by formula (14) in Section 7.

Now, [5, Cor. 2.46] implies that $W^\dagger(B)$ is finite étale and free as a $W^\dagger(A)$-module. The proofs of [4, Prop. 1.9] or, alternatively, [4, Prop. 3.19] transfer to the Hyodo–Kato complexes and provide étale base change in the overconvergent setting:

$$W^\dagger\Lambda^\ell_{B/S_0} \cong W^\dagger\Lambda^\ell_{A/S_0} \otimes_{W^\dagger(A)} W^\dagger(B),$$
$$W^\dagger\omega^\ell_{B/k} \cong W^\dagger\omega^\ell_{A/k} \otimes_{W^\dagger(A)} W^\dagger(B).$$

This finishes the proof, since evidently by definition $W^\dagger\Lambda^\ell_{A/S_0} \cong W^\dagger\omega^\ell_{A/k}$. □

4. Comparison with log-Monsky–Washnitzer cohomology

Let $Y = \operatorname{Spec} A$ be a semi-stable affine scheme over S_0. Let $Y \hookrightarrow Z = \operatorname{Spec} B$ be a closed embedding into a smooth affine k-scheme such that Y is a normal crossings divisor on Z, in other words, $A = B/(f_1\cdots f_r)$ and each

$B/(f_i)$ is smooth. Let $\tilde{B}$ be a smooth W-algebra lifting B (there always exists such a $\tilde{B}$ by [6]) and set $\tilde{A} := \tilde{B}/(\tilde{f}_1 \cdots \tilde{f}_r)$ for some liftings $\tilde{f}_i \in \tilde{B}$ of the f_i, such that $\tilde{Y} := \operatorname{Spec} \tilde{A}$ is a normal crossings divisor in $\tilde{Z} = \operatorname{Spec} \tilde{B}$. That is, we have a diagram

$$\begin{array}{ccc} \tilde{Y} = \operatorname{Spec} \tilde{A} & \hookrightarrow & \tilde{Z} = \operatorname{Spec} \tilde{B} \\ \downarrow & & \downarrow \\ Y = \operatorname{Spec} A & \hookrightarrow & Z = \operatorname{Spec} B. \end{array}$$

We define the complexes $W^\dagger\omega^\bullet_{Y/k}$ and $W^\dagger\tilde{\omega}^\bullet_{Y/k}$ as in Section 3. Indeed, $X = \operatorname{Spec} \tilde{B}/(\tilde{f}_1 \cdots \tilde{f}_r - p)$ is a regular scheme whose special fibre is Y, so the definition applies.

Now let $\Omega^\bullet_{\tilde{Z}/W}(\log \tilde{Y})$ denote the logarithmic de Rham complex of $\tilde{Z}$ with respect to the normal crossings divisor $\tilde{Y}$. We set

$$\tilde{\omega}^\bullet_{\tilde{Y}} := \mathcal{O}_{\tilde{Y}} \otimes_{\mathcal{O}_{\tilde{Z}}} \Omega^\bullet_{\tilde{Z}/W}(\log \tilde{Y})$$

and write

$$\tilde{\omega}^\bullet_{\tilde{Y}^\dagger} := \mathcal{O}_{\tilde{Y}^\dagger} \otimes_{\mathcal{O}_{\tilde{Z}^\dagger}} \Omega^\bullet_{\tilde{Z}^\dagger/W}(\log \tilde{Y}^\dagger)$$

for the weak completion.

Definition. The logarithmic Monsky–Washnitzer complex of Y is defined to be

$$\omega^\bullet_{\tilde{Y}^\dagger} := \tilde{\omega}^\bullet_{\tilde{Y}^\dagger}/(\tilde{\omega}^{\bullet-1}_{\tilde{Y}^\dagger} \wedge \theta),$$

where $\theta := d\log \tilde{f}_1 + \cdots + d\log \tilde{f}_r$. We define

$$H^*_{\text{log-MW}}(Y/K) := \mathbb{H}^*(Y, \omega^\bullet_{\tilde{Y}^\dagger} \otimes \mathbb{Q}).$$

This is the logarithmic Monsky–Washnitzer cohomology, as discussed in [8, Section 5]. It is clear that

$$H^*_{\text{log-MW}}(Y/K) \cong H^*_{\text{log-rig}}(Y/\mathfrak{S}_0),$$

and that there is a short exact sequence of complexes

$$\text{(2)} \qquad \begin{cases} 0 \to \omega^\bullet_{\tilde{Y}^\dagger}[-1] \to \tilde{\omega}^\bullet_{\tilde{Y}^\dagger} \to \omega^\bullet_{\tilde{Y}^\dagger} \to 0, \\ a \mapsto a \wedge \theta, \quad \theta \mapsto 0. \end{cases}$$

In this section we shall construct a morphism of short exact sequences from (2) to (1). We will then prove that the subsequent vertical arrows become quasi-isomorphisms after tensoring with $\mathbb{Q}$.

Let $\tilde{A}^\dagger$ and $\tilde{B}^\dagger$ denote the weak completion of $\tilde{A}$ and $\tilde{B}$, respectively. Then we have an induced diagram

$$\begin{array}{ccc} \tilde{B}^\dagger & \twoheadrightarrow & \tilde{A}^\dagger \\ {\scriptstyle t_F}\downarrow & & \downarrow{\scriptstyle t_F} \\ W(B) & \twoheadrightarrow & W(A), \end{array}$$

where the vertical arrows are the Lazard morphisms [11, Section 0, Eq. 1.3.6]. Since B is a smooth finitely generated k-algebra, $t_F\colon \tilde{B}^\dagger \to W(B)$ has image contained in the overconvergent Witt vectors $W^\dagger(B)$ [4, Prop. 3.2]. By functoriality of the Lazard morphisms and since $W(B) \twoheadrightarrow W(A)$ sends $W^\dagger(B)$ to $W^\dagger(A)$, we deduce that we in fact have a diagram

$$\begin{array}{ccc} \tilde{B}^\dagger & \twoheadrightarrow & \tilde{A}^\dagger \\ {\scriptstyle t_F}\downarrow & & \downarrow{\scriptstyle t_F} \\ W^\dagger(B) & \twoheadrightarrow & W^\dagger(A). \end{array}$$

Let u and $\tilde{u}$ denote the respective open immersions $U := Z \setminus Y \hookrightarrow Z$ and $\tilde{U} := \tilde{Z} \setminus \tilde{Y} \hookrightarrow \tilde{Z}$. Then the map

$$\tilde{u}_*\Omega^\bullet_{\tilde{U}/W} \to u_*W\Omega^\bullet_{U/k}, \quad d\log \tilde{f}_i \mapsto d\log[f_i],$$

sends logarithmic differentials along $\tilde{Y}$ to logarithmic differentials along Y. (Here $[f_i] \in W(\tilde{B})$ denotes the Teichmüller lift). In particular, it induces a map

$$\Omega^\bullet_{\tilde{Z}/W}(\log \tilde{Y}) \to W\Omega^\bullet_{Z/k}(\log Y),$$

and by the above discussion, this induces a map

$$\Omega^\bullet_{\tilde{Z}^\dagger/W}(\log \tilde{Y}^\dagger) \to W^\dagger\Omega^\bullet_{Z/k}(\log Y).$$

These maps were considered in [17, Section 10] and become quasi-isomorphisms after tensoring with $\mathbb{Q}$, by [17, Lem. 10.9]. In any case, this gives a map

$$\tilde{\omega}^\bullet_{\tilde{Y}^\dagger} \to W^\dagger(A) \otimes_{W^\dagger(B)} W^\dagger\Omega^\bullet_{Z/k}(\log Y). \tag{3}$$

Notice that the logarithmic differentials $d\log[f_i]$ along Y coincide with the logarithmic differentials as defined by Hyodo–Kato as the image of $d\log$ using the regular W-scheme $X = \operatorname{Spec} \tilde{B}/(\tilde{f}_1 \cdots \tilde{f}_r - p)$. Indeed, $\tilde{f}_i$ is mapped to $d\log[f_i]$ and p is mapped to $\theta = d\log[f_1] + \cdots + d\log[f_r]$.

Let $W_n\tilde{\omega}^i_{Y/k}$ be the sheaves introduced in [9, Section 1.6]. These are the same as $W_n\Lambda^i_{Y/(k,\{*\})}$. One way to see this is by mimicking the proof of Proposition 3.3. Recall that one can express the sheaves $W_n\tilde{\omega}^i_{Y/k}$ as quotients of the $W_n\Omega^i_{Z/k}(\log Y)$. Indeed, we may assume that $\tilde{Z}$ is an admissible lifting of Y (see [18, Section 2.4] for the definition). Set $\tilde{Z}_n := \tilde{Z} \times_W W_n$ and $\tilde{Y}_n := \tilde{Y} \times_W W_n$, so that $Z = \tilde{Z}_1$ and $Y = \tilde{Y}_1$. Then

$$W_n\tilde{\omega}^i_{Y/k} = W_n\Omega^i_{Z/k}(\log Y)/W_n\Omega^i_{Z/k}(-\log Y), \tag{4}$$

where we have identified

$$W_n\Omega^i_{Z/k}(\log Y) = \mathcal{H}^i\big(\Omega^\bullet_{\tilde{Z}_n/W_n}(\log \tilde{Y}_n)\big)$$

and

$$W_n\Omega^i_{Z/k}(-\log Y) = \mathcal{H}^i\big(\mathcal{J}_n \otimes_{\mathcal{O}_{\tilde{Z}_n}} \Omega^\bullet_{\tilde{Z}_n/W_n}(\log \tilde{Y}_n)\big),$$

where $\mathcal{J}_n = \ker(\mathcal{O}_{\tilde{Z}_n} \to \mathcal{O}_{\tilde{Y}_n})$. The fact that the right-hand side of (4) gives the same sheaf $W_n\tilde{\omega}^i_{Y/k}$ as defined in [10] is discussed in [18, Section 2.4]. Passing to the projective limit gives $W\tilde{\omega}^i_{Y/k}$ as a quotient of $W\Omega^i_{Z/k}(\log Y)$. We therefore deduce a map of complexes of sheaves on Y:

$$W(A) \otimes_{W(B)} W\Omega^\bullet_{Z/k}(\log Y) \to W\tilde{\omega}^\bullet_{Y/k}.$$

Let Σ be the singular locus of Y, $U = Y \setminus \Sigma$, $u\colon U \hookrightarrow Y$ and $i\colon Y \to Z$. Then we have canonical maps

$$W^\dagger\Omega^\bullet_{Z/k} \to i_*W^\dagger\Omega^\bullet_{Y/k} \to i_*u_*W^\dagger\Omega^\bullet_{U/k},$$

which extend to a map

$$W^\dagger\Omega^\bullet_{Z/k}(\log Y) \to i_*u_*W^\dagger\Omega^\bullet_{U/k}[\theta]/(\theta^2),$$

since the $d\log[f_i]$ are overconvergent [17, Prop. 10.1]. The image of this map lies – by the above interpretation of $W\tilde{\omega}^\bullet_{Y/k}$ – in

$$i_*W^\dagger\tilde{\omega}^\bullet_{Y/k} = i_*W\tilde{\omega}^\bullet_{Y/k} \cap i_*u_*W^\dagger\Omega^\bullet_{U/k}[\theta]/(\theta^2).$$

We get in each degree a $W^\dagger(B)$-module map

$$W^\dagger\Omega^r_{Z/k}(\log Y) \to i_*W^\dagger\tilde{\omega}^r_{Y/k}.$$

Hence, we get a canonical map

$$W^\dagger(A) \otimes_{W^\dagger(B)} W^\dagger\Omega^\bullet_{Z/k}(\log Y) \to W^\dagger\tilde{\omega}^\bullet_{Y/k}.$$

Composing with (3) defines a comparison morphism

$$\tilde{\omega}^\bullet_{\tilde{Y}^\dagger} \to W^\dagger\tilde{\omega}^\bullet_{Y/k}, \tag{5}$$

which sends $\theta = d\log\tilde{f}_1 + \cdots + d\log\tilde{f}_r$ to $\theta = d\log[f_1] + \cdots + d\log[f_r]$. Since the "divide by θ" projection $W^\dagger\tilde{\omega}^\bullet_{Y/k} \twoheadrightarrow W^\dagger\omega^\bullet_{Y/k}$ sends θ to 0, we get an induced comparison morphism

$$\omega^\bullet_{\tilde{Y}^\dagger} \to W^\dagger\omega^\bullet_{Y/k} \tag{6}$$

between the logarithmic Monsky–Washnitzer and overconvergent Hyodo–Kato complexes. Moreover, (5) and (6) give a diagram of exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega^\bullet_{\tilde{Y}^\dagger}[-1] & \longrightarrow & \tilde{\omega}^\bullet_{\tilde{Y}^\dagger} & \longrightarrow & \omega^\bullet_{\tilde{Y}^\dagger} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W^\dagger\omega^\bullet_{Y/k}[-1] & \longrightarrow & W^\dagger\tilde{\omega}^\bullet_{Y/k} & \longrightarrow & W^\dagger\omega^\bullet_{Y/k} & \longrightarrow & 0.
\end{array}$$

We will use the weight filtration of Steenbrink to show that the vertical arrows (5) and (6) become quasi-isomorphisms after tensoring with $\mathbb{Q}$.

Theorem 4.1. *The comparison morphisms* (5) *and* (6) *induce quasi-isomorphisms*

$$\tilde{\omega}^\bullet_{\tilde{Y}^\dagger} \otimes \mathbb{Q} \xrightarrow{\sim} W^\dagger\tilde{\omega}^\bullet_{Y/k} \otimes \mathbb{Q} \quad \textit{and} \quad \omega^\bullet_{\tilde{Y}^\dagger} \otimes \mathbb{Q} \xrightarrow{\sim} W^\dagger\omega^\bullet_{Y/k} \otimes \mathbb{Q}.$$

Proof. Recall that the weight filtration $P_\bullet\tilde{\omega}^\bullet_{\tilde{Y}^\dagger}$ of $\tilde{\omega}^\bullet_{\tilde{Y}^\dagger}$ (see [8, Section 5] for it in this context) is defined as

$$P_j\tilde{\omega}^i_{\tilde{Y}^\dagger} := \operatorname{image}\big(\Omega^j_{\tilde{Z}^\dagger/W}(\log\tilde{Y})\otimes\Omega^{i-j}_{\tilde{Z}^\dagger/W}\to\Omega^i_{\tilde{Z}^\dagger/W}(\log\tilde{Y})\big)\otimes_{\mathcal{O}_{\tilde{Z}^\dagger}}\mathcal{O}_{\tilde{Y}^\dagger}.$$

Via the Poincaré residue maps, the graded pieces of the filtration are identified as

$$\mathrm{Gr}_j(\tilde{\omega}^\bullet_{\tilde{Y}^\dagger}\otimes\mathbb{Q}) \xrightarrow{\sim} \bigoplus_{Y_I\in\mathcal{M}_j}\Omega^\bullet_{]Y_I[^\dagger_{\tilde{Z}^\dagger}}[-j],$$

where $\mathcal{M}_j$ denotes the collection of all (smooth) intersections of j different components of Y which lift to a smooth intersection of j different liftings in $\tilde{Y}$, with $\Omega^\bullet_{]Y_I[^\dagger_{\tilde{Z}^\dagger}}$ denoting the usual de Rham complex on the smooth affinoid dagger space $]Y_I[^\dagger_{\tilde{Z}^\dagger}$. By [8, Section 5.2], one has an isomorphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Gr}_0(\tilde{\omega}^\bullet_{\tilde{Y}^\dagger}\otimes\mathbb{Q}) & \xrightarrow{\wedge\theta} & \mathrm{Gr}_1(\tilde{\omega}^\bullet_{\tilde{Y}^\dagger}\otimes\mathbb{Q})[1] & \xrightarrow{\wedge\theta} & \mathrm{Gr}_2(\tilde{\omega}^\bullet_{\tilde{Y}^\dagger}\otimes\mathbb{Q})[2] & \xrightarrow{\wedge\theta} & \cdots \\
 & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\
0 & \longrightarrow & \Omega^\bullet_{\tilde{Y}^\dagger}\otimes\mathbb{Q} & \longrightarrow & \bigoplus_{Y_I\in\mathcal{M}_1}\Omega^\bullet_{]Y_I[^\dagger_{\tilde{Z}^\dagger}}\otimes\mathbb{Q} & \longrightarrow & \bigoplus_{Y_I\in\mathcal{M}_2}\Omega^\bullet_{]Y_I[^\dagger_{\tilde{Z}^\dagger}}\otimes\mathbb{Q} & \longrightarrow & \cdots
\end{array}$$

(the bottom row is exact because the Y_I are normal crossings intersections).

Similarly, consider the weight filtration of Mokrane [18] on $W\tilde{\omega}^\bullet_{Y/k}$:

$$P_jW\tilde{\omega}^i_{Y/k} := \operatorname{image}\big(W\tilde{\omega}^j_{Y/k}\otimes W\Omega^{i-j}_{Y/k}\to W\tilde{\omega}^i_{Y/k}\big)$$

and set

$$P_jW^\dagger\tilde{\omega}^i_{Y/k} := \operatorname{image}\big(W^\dagger\tilde{\omega}^j_{Y/k}\otimes W^\dagger\Omega^{i-j}_{Y/k}\to W^\dagger\tilde{\omega}^i_{Y/k}\big).$$

By construction, the comparison morphism (5) induces maps $P_j\tilde{\omega}^\bullet_{\tilde{Y}^\dagger}\to P_jW^\dagger\tilde{\omega}^\bullet_{Y/k}$ for each j, and therefore respects the weight filtrations. Moreover, we have $P_jW^\dagger\tilde{\omega}^\bullet_{Y/k} = W^\dagger\tilde{\omega}^\bullet_{Y/k}\cap P_jW\tilde{\omega}^\bullet_{Y/k}$ for each j. By [18, Section 3.7], the graded pieces of the weight filtration are identified, via the Poincaré residue maps, as

$$\mathrm{Gr}_jW\tilde{\omega}^\bullet_{Y/k}\xrightarrow{\sim}\bigoplus_{Y_I\in\mathcal{M}_j}W\Omega^\bullet_{Y_I/k}[-j],$$

and therefore

$$\mathrm{Gr}_jW^\dagger\tilde{\omega}^\bullet_{Y/k}\xrightarrow{\sim}\bigoplus_{Y_I\in\mathcal{M}_j}W^\dagger\Omega^\bullet_{Y_I/k}[-j].$$

Since each Y_I is smooth over k, we know by [4] that $W^\dagger\omega^\bullet_{Y_I/k}\otimes\mathbb{Q}$ is quasi-isomorphic to $\Omega^\bullet_{]Y_I[_{\tilde{Z}^\dagger}}$, and therefore conclude that the comparison morphism (5) is a quasi-isomorphism when tensored with $\mathbb{Q}$.

To show that the second comparison morphism induces a quasi-isomorphism after tensoring with $\mathbb{Q}$, define a double complex

$$\mathcal{A}^{\dagger i,j}_{\mathbb{Q}} := \frac{\tilde{\omega}^{i+j+1}_{\tilde{Y}^\dagger}\otimes\mathbb{Q}}{P_j\tilde{\omega}^{i+j+1}_{\tilde{Y}^\dagger}\otimes\mathbb{Q}}$$

with differential $\mathcal{A}_{\mathbb{Q}}^{\dagger i,j} \to \mathcal{A}_{\mathbb{Q}}^{\dagger i+1,j}$ induced by $(-1)^j d$, and the other differential $\mathcal{A}_{\mathbb{Q}}^{\dagger i,j} \to \mathcal{A}_{\mathbb{Q}}^{\dagger i,j+1}$ induced by $\omega \mapsto \omega \wedge \theta$. Let $\mathcal{A}_{\mathbb{Q}}^{\dagger \bullet}$ be the total complex of $\mathcal{A}_{\mathbb{Q}}^{\dagger \bullet,\bullet}$. Entirely similarly, define another double complex $\mathcal{B}_{\mathbb{Q}}^{\dagger \bullet,\bullet}$ by

$$\mathcal{B}_{\mathbb{Q}}^{\dagger i,j} := \frac{W^\dagger \tilde{\omega}_{Y/k}^{i+j+1} \otimes \mathbb{Q}}{P_j W^\dagger \tilde{\omega}_{Y/k}^{i+j+1} \otimes \mathbb{Q}},$$

with the differential $\mathcal{B}_{\mathbb{Q}}^{\dagger i,j} \to \mathcal{B}_{\mathbb{Q}}^{\dagger i+1,j}$ induced by $(-1)^j d$ and the differential $\mathcal{B}_{\mathbb{Q}}^{\dagger i,j} \to \mathcal{B}_{\mathbb{Q}}^{\dagger i,j+1}$ induced by $\omega \mapsto \omega \wedge \theta$, and let $\mathcal{B}_{\mathbb{Q}}^{\dagger \bullet}$ be the total complex of ${\mathcal{B}_{\mathbb{Q}}}^{\dagger \bullet,\bullet}$. Then $\mathcal{A}_{\mathbb{Q}}^{\dagger \bullet}$ is quasi-isomorphic to $\mathcal{B}_{\mathbb{Q}}^{\dagger \bullet}$, because the graded quotients $\Omega^\bullet_{]Y_I[^\dagger_{\tilde{Z}}}$ and $W^\dagger \omega^\bullet_{Y_I/k} \otimes \mathbb{Q}$ are quasi-isomorphic by the comparison theorem in the smooth case [4].

Now, the map

$$\tilde{\omega}^\bullet_{\tilde{Y}^\dagger} \otimes \mathbb{Q} \to \mathcal{A}_{\mathbb{Q}}^{\dagger \bullet,0}, \quad \omega \mapsto \omega \wedge \theta,$$

induces a quasi-isomorphism $\omega^\bullet_{\tilde{Y}^\dagger} \otimes \mathbb{Q} \xrightarrow{\sim} \mathcal{A}_{\mathbb{Q}}^{\dagger \bullet}$. This is [8, Section 5] in this context, but the argument goes back to [20]. The same argument shows that the map

$$W^\dagger \tilde{\omega}^\bullet_{Y/k} \otimes \mathbb{Q} \to \mathcal{B}_{\mathbb{Q}}^{\dagger \bullet,0}, \quad \omega \mapsto \omega \wedge \theta,$$

induces a quasi-isomorphism $W^\dagger \omega^\bullet_{Y/k} \otimes \mathbb{Q} \xrightarrow{\sim} \mathcal{B}_{\mathbb{Q}}^{\dagger \bullet}$. As we already noted that $\mathcal{A}_{\mathbb{Q}}^{\dagger \bullet} \cong \mathcal{B}_{\mathbb{Q}}^{\dagger \bullet}$, we conclude that the comparison morphism (6) is a quasi-isomorphism when tensored with $\mathbb{Q}$. □

Corollary 4.2. *Let Y be a semi-stable affine scheme over S_0. Then there is a canonical isomorphism*

$$H^*_{\text{log-rig}}(Y/\mathfrak{S}_0) \cong \mathbb{H}^*(Y, W^\dagger \omega^\bullet_{Y/k} \otimes \mathbb{Q}).$$

Proof. We showed that $\mathbb{H}^*(Y, W^\dagger \omega^\bullet_{Y/k} \otimes \mathbb{Q}) \cong H^*_{\text{log-MW}}(Y/K)$. The comparison between log-Monsky–Washnitzer cohomology and log-rigid cohomology is more or less by definition (see [8, Section 5.2]). □

5. Comparison with log-rigid cohomology

Our aim in this section is to globalize the comparison isomorphism between log-rigid and overconvergent Hyodo–Kato cohomology. We note here that given a W-scheme X, we shall always write $\hat{X}$ for the formal completion of X along the special fibre, and $\hat{X}_K$ for the associated rigid analytic generic fibre.

Definition. Let $Y = \operatorname{Spec} A$ be a semi-stable affine scheme over S_0. A semi-stable frame for Y is the data of a normal crossings divisor relative to W

$$G = \operatorname{Spec} C \hookrightarrow F = \operatorname{Spec} B$$

of affine W-schemes, where F is smooth over W and

$$Y \hookrightarrow G_k := G \times_W k$$

is an exact closed k-immersion.

Note that if (F,G) is a semi-stable frame for Y, then F is a special frame for Y in the sense of [4, Def. 4.1]. Recall from [4, p. 253] that the rigid tube $]Y[_{\hat{F}} \subset \hat{F}_K$ has a dagger space structure $]Y[^\dagger_{\hat{F}}$, which induces a dagger structure $]Y[^\dagger_{\hat{G}}$ on $]Y[_{\hat{G}}$. This is functorial in (Y,F,G).

Definition. An overconvergent semi-stable frame for $Y = \operatorname{Spec} A$ is the data of a semi-stable frame $(F = \operatorname{Spec} B, G = \operatorname{Spec} C)$ for Y along with a homomorphism $\varkappa\colon C \to W^\dagger(A)$ which lifts the comorphism $C \twoheadrightarrow A$ of the closed W-immersion $Y \hookrightarrow G$.

Let $Y = \operatorname{Spec} A$ be a semi-stable affine scheme over S_0 and suppose that $(F,G,\varkappa)$ is an overconvergent semi-stable frame for Y. Choose an embedding $F \hookrightarrow \mathbb{P}$ into a proper smooth W-scheme and write $\overline{G}$ and $\overline{F}$ for the respective closures of G and F inside $\mathbb{P}$. Let $\overline{G}_k$ and $\overline{F}_k$ be the special fibres of $\overline{G}$ and $\overline{F}$, and let $\overline{Y}$ be the closure of Y inside $\overline{G}_k$. Since $\hat{G}_K$ is defined in $\hat{F}_K$ by overconvergent functions, we can extend the normal crossings divisor $\hat{G}_K \hookrightarrow \hat{F}_K$ to a normal crossings divisor $V' \hookrightarrow V$, where V is a strict neighborhood of $]F_k[_{\hat{\mathbb{P}}}$ in $]\overline{F}_k[_{\hat{\mathbb{P}}}$ and V' is a strict neighborhood of $]G_k[_{\hat{\mathbb{P}}}$ in $]\overline{G}_k[_{\hat{\mathbb{P}}}$. We get the following diagram of strict neighborhoods:

$$\begin{array}{ccccc}
]F_k[_{\hat{\mathbb{P}}} & \subseteq & V & \subseteq &]\overline{F}_k[_{\hat{\mathbb{P}}} \\
\uparrow & & \uparrow & & \uparrow \\
]G_k[_{\hat{\mathbb{P}}} & \subseteq & V' & \subseteq &]\overline{G}_k[_{\hat{\mathbb{P}}} \\
\uparrow & & \uparrow & & \uparrow \\
]Y[_{\hat{G}} & \subseteq & \tilde{V} := V' \cap]\overline{Y}[_{\hat{\mathbb{P}}} & \subseteq &]\overline{Y}[_{\hat{\mathbb{P}}}.
\end{array}$$

In order to define the comparison morphism, we will find it useful to have a rigid analytic description of log-rigid cohomology in terms of sheaves on strict neighborhoods, in the style of Berthelot. Let $\tilde{\omega}^\bullet_{\tilde{V}}$ be the complex given by the restriction of $\Omega^\bullet_V(\log V') \otimes \mathcal{O}_{V'}$ to $\tilde{V}$, and $\omega^\bullet_{\tilde{V}} := \tilde{\omega}^\bullet_{\tilde{V}}/(\tilde{\omega}^{\bullet-1}_{\tilde{V}} \wedge \theta)$, where $\theta = d\log f_1 + \cdots + d\log f_s$ for the functions f_i cutting out the normal crossings divisor $\tilde{V}$ in V.

Recall that given an abelian sheaf $\mathcal{F}$ on a strict neighborhood W of $]Y[_{\hat{G}}$ in $]\overline{Y}[_{\hat{\overline{G}}}$, Berthelot's sheaf of overconvergent sections (see [1, Section 1.2]) is defined to be

$$j^\dagger \mathcal{F} := \varinjlim_{V} j_{W,V*} j^{-1}_{W,V} \mathcal{F},$$

where the limit is over strict neighborhoods of $]Y[_{\hat{G}}$ in W, and $j_{W,V}\colon V \hookrightarrow W$ is the inclusion.

We claim that we have the following Berthelot-style interpretation of log-rigid cohomology.

Lemma 5.1. *We have*

$$\mathbb{R}\Gamma(]Y[^\dagger_{\hat{G}}, \tilde{\omega}^\bullet_{]Y[^\dagger_{\hat{G}}}) = \mathbb{R}\Gamma(\tilde{V}, j^\dagger \tilde{\omega}^\bullet_{\tilde{V}})$$

and

$$R\Gamma_{\text{log-rig}}(Y/\mathfrak{S}_0) := \mathbb{R}\Gamma(]Y[^\dagger_{\hat{G}}, \omega^\bullet_{]Y[^\dagger_{\hat{G}}}) = \mathbb{R}\Gamma(\tilde{V}, j^\dagger \omega^\bullet_{\tilde{V}}).$$

Proof. We shall only prove the first statement, since the second is proved using exactly the same argument.

In order to prove the lemma, it suffices to prove that

$$\mathbb{R}\Gamma(]Y[^\dagger_{\hat{G}}, \tilde{\omega}^\bullet_{]Y[^\dagger_{\hat{G}}}) \cong \mathbb{R}\Gamma(]\overline{Y}[_{\hat{\overline{G}}}, j^\dagger \tilde{\omega}^\bullet_{]\overline{Y}[_{\hat{\overline{G}}}}).$$

Indeed, the right-hand side is the same as $\mathbb{R}\Gamma(\tilde{V}, j^\dagger \tilde{\omega}^\bullet_{\tilde{V}})$ by [1, Section 1.2 (iv)].

For a coherent sheaf $\mathcal{F}$ on $]\overline{Y}[_{\hat{\overline{G}}}$ considered as a dagger space with corresponding coherent sheaf $\mathcal{F}'$ on the rigid space $]\overline{Y}[_{\hat{\overline{G}}}$, let $\tilde{\mathcal{F}}$ be the restriction of $\mathcal{F}$ to the open subspace $]Y[^\dagger_{\hat{G}}$. Since the map $]Y[^\dagger_{\hat{G}} \xrightarrow{j}]\overline{Y}[_{\hat{\overline{G}}}$ is affinoid and any section of $\tilde{\mathcal{F}}$ is defined via a neighborhood of $]Y[^\dagger_{\hat{G}}$ in $]\overline{Y}[_{\hat{\overline{G}}}$, we have a canonical map

$$Rj_*\tilde{\mathcal{F}} = j_*\tilde{\mathcal{F}} \to j^\dagger \mathcal{F}'.$$

If $\mathcal{F}^\bullet$ is a complex of coherent sheaves on $]\overline{Y}[_{\hat{\overline{G}}}$, with corresponding complexes $\mathcal{F}'^\bullet$, $\tilde{\mathcal{F}}^\bullet$ defined as above, we get a canonical map

$$\mathbb{R}\Gamma(]Y[^\dagger_{\hat{G}}, \tilde{\mathcal{F}}^\bullet) \to \mathbb{R}\Gamma(]\overline{Y}[_{\hat{\overline{G}}}, j^\dagger \mathcal{F}'^\bullet).$$

Let $\tilde{\omega}^\bullet_{]\overline{Y}[^\dagger_{\hat{\overline{G}}}}$ be the log de Rham complex of the log morphism

$$]\overline{Y}[^\dagger_{\hat{\overline{G}}} \to (\mathrm{sp}^\dagger K, \{*\}).$$

Since $]\overline{Y}[_{\hat{\overline{G}}}$ is a partially proper rigid space and $\omega^\bullet_{]\overline{Y}[^\dagger_{\hat{\overline{G}}}}$ is a coherent $\mathcal{O}_{]\overline{Y}[^\dagger_{\hat{\overline{G}}}}$-module, we have canonical isomorphisms

$$H^j(]\overline{Y}[_{\hat{\overline{G}}}, j^\dagger_Y \tilde{\omega}^i_{]\overline{Y}[_{\hat{\overline{G}}}}) \cong H^j(]Y[^\dagger_{\hat{\overline{G}}}, \tilde{\omega}^i_{]\overline{Y}[^\dagger_{\hat{\overline{G}}}}|_{]Y[^\dagger_{\hat{\overline{G}}}}) = H^j(]Y[^\dagger_{\hat{\overline{G}}}, \tilde{\omega}^i_{]Y[^\dagger_{\hat{\overline{G}}}})$$

for all i, j, by [7, Thm. 5.1 (a)]. But $]Y[_{\hat{\overline{G}}} =]Y[_{\hat{G}}$, and therefore $]Y[^\dagger_{\hat{\overline{G}}} =]Y[^\dagger_{\hat{G}}$ too. Hence,

$$H^j(]\overline{Y}[_{\hat{\overline{G}}}, j^\dagger_Y \tilde{\omega}^i_{]\overline{Y}[_{\hat{\overline{G}}}}) \cong H^j(]Y[^\dagger_{\hat{G}}, \tilde{\omega}^i_{]Y[^\dagger_{\hat{G}}})$$

for all i, j. Since we have a canonical map

$$\mathbb{R}\Gamma(]Y[^\dagger_{\hat{G}}, \tilde{\omega}^\bullet_{]Y[^\dagger_{\hat{G}}}) \to \mathbb{R}\Gamma(]\overline{Y}[_{\hat{\overline{G}}}, j^\dagger \tilde{\omega}^\bullet_{]\overline{Y}[_{\hat{\overline{G}}}}),$$

we conclude from the first hypercohomology spectral sequence that

$$\mathbb{R}\Gamma(]\overline{Y}[_{\hat{\overline{G}}}, j^\dagger_Y \tilde{\omega}^\bullet_{]\overline{Y}[_{\hat{\overline{G}}}}) \cong \mathbb{R}\Gamma(]Y[^\dagger_{\hat{G}}, \tilde{\omega}^\bullet_{]Y[^\dagger_{\hat{G}}}),$$

as required.

The second statement is proved with exactly the same argument, but where one instead considers the log de Rham complex with respect to the base $(K, \mathbb{N})$. The proof is complete. □

In [4, Section 4], an explicit fundamental system of strict affinoid neighborhoods $V_{\lambda,\eta}$ of $]Y[_{\hat{G}}$ in $]\overline{Y}[_{\hat{\mathbb{P}}}$ (here $0 < \lambda, \eta < 1$) is constructed, as well as canonical morphisms (see [4, p. 251])

$$\Gamma(V_{\lambda,\eta}, j^\dagger \mathcal{O}_{V_{\lambda,\eta}}) \to W^\dagger(A) \otimes \mathbb{Q},$$

and therefore morphisms

$$\Gamma(\tilde{V}, j^\dagger \mathcal{O}_{\tilde{V}}) \to W^\dagger(A) \otimes \mathbb{Q}$$

for any strict neighborhood $\tilde{V}$ of $]Y[_{\hat{G}}$ in $]\overline{Y}[_{\hat{\mathbb{P}}}$. The universal property of the de Rham complex then gives a map

$$\Gamma(\tilde{V}, j^\dagger \tilde{\omega}^\bullet_{\tilde{V}}) \to u_* W^\dagger \Omega^\bullet_{U/k}[\theta]/(\theta^2) \otimes \mathbb{Q},$$

where $u\colon U \hookrightarrow Y$ is the smooth locus of $Y = \operatorname{Spec} A$, and this clearly factors through

$$\Gamma(\tilde{V}, j^\dagger \tilde{\omega}^\bullet_{\tilde{V}}) \to W^\dagger \tilde{\omega}^\bullet_{A/k} \otimes \mathbb{Q}. \tag{7}$$

The argument used after [4, Eq. (4.28)] can be used verbatim to show that this factors through a morphism

$$\mathbb{R}\Gamma(\tilde{V}, j^\dagger \tilde{\omega}^\bullet_{\tilde{V}}) \to W^\dagger \tilde{\omega}^\bullet_{A/k} \otimes \mathbb{Q}. \tag{8}$$

Indeed, $\tilde{V}$ contains some $V_{\lambda,\eta}$ and we can consider the restriction

$$\mathbb{R}\Gamma(\tilde{V}, j^\dagger \tilde{\omega}^\bullet_{\tilde{V}}) \to \mathbb{R}\Gamma(V_{\lambda,\eta}, j^\dagger \tilde{\omega}^\bullet_{V_{\lambda,\eta}}).$$

Given any strict neighborhood $\tilde{V}'$ of $]Y[_{\hat{G}}$ in $\tilde{V}$, let us write $\alpha_{\tilde{V}'}\colon \tilde{V}' \cap V_{\lambda,\eta} \hookrightarrow V_{\lambda,\eta}$ for the inclusion. Then, by the definition of $j^\dagger$, we have

$$j^\dagger \tilde{\omega}^\bullet_{V_{\lambda,\eta}} = \varinjlim_{\tilde{V}'} \alpha_{\tilde{V}'*} \tilde{\omega}^\bullet_{\tilde{V}' \cap V_{\lambda,\eta}},$$

where the direct limit runs over all strict neighborhoods $\tilde{V}'$ of $]Y[_{\hat{G}}$ in $\tilde{V}$. Therefore,

$$\mathbb{R}\Gamma(V_{\lambda,\eta}, j^\dagger \tilde{\omega}^\bullet_{V_{\lambda,\eta}}) = \mathbb{R}\Gamma(V_{\lambda,\eta}, \varinjlim_{\tilde{V}'} \alpha_{\tilde{V}'*} \tilde{\omega}^\bullet_{\tilde{V}' \cap V_{\lambda,\eta}}) \cong \varinjlim_{\tilde{V}'} \mathbb{R}\Gamma(V_{\lambda,\eta}, \alpha_{\tilde{V}'*} \tilde{\omega}^\bullet_{\tilde{V}' \cap V_{\lambda,\eta}}),$$

where the isomorphism is by the quasicompactness of $V_{\lambda,\eta}$. Now for each $\tilde{V}'$, one can find a λ' such that $V_{\lambda',\eta}$ is a strict affinoid neighborhood of $]Y[_{\hat{G}}$ in $\tilde{V}' \cap V_{\lambda,\eta}$. The restriction to the affinoids $V_{\lambda',\eta}$ gives a map

$$\mathbb{R}\Gamma(V_{\lambda,\eta}, j^\dagger \tilde{\omega}^\bullet_{V_{\lambda,\eta}}) \to \varinjlim_{\lambda'} \mathbb{R}\Gamma(V_{\lambda',\eta}, \tilde{\omega}^\bullet_{V_{\lambda',\eta}}) \cong \varinjlim_{\lambda'} \Gamma(V_{\lambda',\eta}, \tilde{\omega}^\bullet_{V_{\lambda',\eta}}) \to W^\dagger \tilde{\omega}^\bullet_{A/k} \otimes \mathbb{Q},$$

where the isomorphism is because each $V_{\lambda',\eta}$ is affinoid and the last map is induced by the morphisms $\Gamma(V_{\lambda',\eta}, \mathcal{O}_{V_{\lambda',\eta}}) \to W^\dagger(A) \otimes \mathbb{Q}$ constructed in [4, p. 251]. Precomposing with the restriction $\mathbb{R}\Gamma(\tilde{V}, j^\dagger \tilde{\omega}^\bullet_{\tilde{V}}) \to \mathbb{R}\Gamma(V_{\lambda,\eta}, j^\dagger \tilde{\omega}^\bullet_{V_{\lambda,\eta}})$ then gives the desired morphism.

If $f_1, \dots, f_s$ define the normal crossings divisor $\tilde{V}$ in V, then $f_1 \cdots f_s = 0$, and hence $\bar{f}_1 \cdots \bar{f}_s = 0$. Therefore, $d\log[\bar{f}_1] + \cdots + d\log[\bar{f}_s] = 0$ and the morphism (8) induces a morphism

$$R\Gamma_{\text{log-rig}}(Y/\mathfrak{S}_0) = \mathbb{R}\Gamma(\tilde{V}, j^\dagger \omega^\bullet_{\tilde{V}}) \to W^\dagger \omega^\bullet_{A/k} \otimes \mathbb{Q}. \tag{9}$$

Proposition 5.2. *The morphisms* (8) *and* (9) *for overconvergent semi-stable frames are isomorphisms in the derived category and do not depend on the choice of overconvergent semi-stable frame for Y.*

Proof. We first prove the independence assertion. Let $(F, G, \varkappa)$ and $(F', G', \varkappa')$ be two overconvergent semi-stable frames for Y. Let

$$F \xleftarrow{\text{pr}_1} F \times_W F' \xrightarrow{\text{pr}_2} F'$$

be the projections. Then the product

$$(F'', G'', \varkappa'') := (F \times_W F', \text{pr}_1^{-1}(G) + \text{pr}_2^{-1}(G'), \varkappa \otimes \varkappa')$$

is another overconvergent semi-stable frame for Y. Choose strict neighborhoods $\tilde{V}$, $\tilde{V}'$ and $\tilde{V}''$ such that $\tilde{V}''$ is sent to $\tilde{V}$ and $\tilde{V}'$ by the respective projections. By functoriality, the projections induce diagrams

$$\begin{array}{ccccc} \mathbb{R}\Gamma(\tilde{V}, j^\dagger \tilde{\omega}^\bullet_{\tilde{V}}) & \xrightarrow{\text{pr}_1^*} & \mathbb{R}\Gamma(\tilde{V}'', j^\dagger \tilde{\omega}^\bullet_{\tilde{V}''}) & \xleftarrow{\text{pr}_2^*} & \mathbb{R}\Gamma(\tilde{V}', j^\dagger \tilde{\omega}^\bullet_{\tilde{V}'}) \\ & \searrow & \downarrow & \swarrow & \\ & & W^\dagger \tilde{\omega}^\bullet_{A/k} \otimes \mathbb{Q} & & \end{array}$$

and

$$\begin{array}{ccccc} \mathbb{R}\Gamma(\tilde{V}, j^\dagger \omega^\bullet_{\tilde{V}}) & \xrightarrow{\text{pr}_1^*} & \mathbb{R}\Gamma(\tilde{V}'', j^\dagger \omega^\bullet_{\tilde{V}''}) & \xleftarrow{\text{pr}_2^*} & \mathbb{R}\Gamma(\tilde{V}', j^\dagger \omega^\bullet_{\tilde{V}'}) \\ & \searrow & \downarrow & \swarrow & \\ & & W^\dagger \omega^\bullet_{A/k} \otimes \mathbb{Q}, & & \end{array}$$

and we see therefore that the morphisms (8) and (9) do not depend on the choice of overconvergent semi-stable frame for Y.

To prove that the morphisms are isomorphisms, since we have already shown independence, we may as well work with the log-Monsky–Washnitzer frame $(\tilde{Z}, \tilde{Y}, \varkappa)$, as in Section 4 for the overconvergent semi-stable frame for Y (remember that Y is affine). We then conclude by Theorem 4.1. □

Theorem 5.3. *Let Y be a quasi-projective semi-stable scheme over S_0. Then the overconvergent Hyodo–Kato complex computes the log-rigid cohomology of Y:*

$$R\Gamma_{\text{log-rig}}(Y/\mathfrak{S}_0) \cong \mathbb{R}\Gamma(Y, W^\dagger \omega^\bullet_{Y/k} \otimes \mathbb{Q}).$$

Proof. Let $Y = \bigcup_{i\in I} Y_i$ be an open covering, and for $J = \{i_0, \dots, i_t\} \subset I$, let $Y_J := Y_{i_0} \cap \cdots \cap Y_{i_t}$. By choosing a possibly finer covering, we may assume that $Y_J = \operatorname{Spec} A_J$ is affine and that $A_J = (A_{i_0})_{\bar{g}}$ for some element $\bar{g} \in A_{i_0}$, where $Y_{i_0} = \operatorname{Spec} A_{i_0}$. It is here that we use the quasi-projectivity hypothesis (compare the argument in [4, Def. 4.33] and the subsequent discussion). For each $i \in J$, choose a smooth affine k-scheme $X_i = \operatorname{Spec} B_i$ such that Y_i is a normal crossings divisor in X_i. We may assume that each X_i is standard smooth in the sense of [4, Def. 4.33]. Let F_i be a smooth affine W-scheme lifting X_i, which is again standard smooth, and let Z_i be a lifting over W of Y_i which is a normal crossings divisor in F_i (compare with [12, Prop. 11.3]).

Now let $Z_{i_0} = \operatorname{Spec} \tilde{A}_{i_0}$ and $Z'_{i_0} = \operatorname{Spec}(\tilde{A}_{i_0})_g$ for some lifting g of $\bar{g}$, and let $F_{i_0} = \operatorname{Spec} \tilde{B}_{i_0}$ and $F'_{i_0} = \operatorname{Spec}(\tilde{B}_{i_0})_f$ for some lifting f of g. Set

$$E := \prod_{\substack{i\in J\\ i\neq i_0}} F_i.$$

Then, by the strong fibration theorem, the special frames $(Y_J, F_{i_0} \times E)$ and $(Y_J, F'_{i_0} \times E)$ have isomorphic dagger spaces. See the argument in the proof of [4, Prop. 4.35]. Since E is standard smooth, we can choose an étale map $E \to \mathbb{A}^n_W$ for some n. Again by the strong fibration theorem, the dagger spaces associated to $(Y_J, F'_{i_0} \times E)$ and $(Y_J, F'_{i_0} \times \mathbb{A}^n_W)$ are isomorphic. By the coordinate change argument in the proof of [4, Prop. 4.35], we may assume that the map $Y_J \to \mathbb{A}^n_W$ factors through the zero section $\operatorname{Spec} k \to \mathbb{A}^n_W$. Hence, the dagger space associated to $(Y_J, F'_{i_0} \times \mathbb{A}^n_W)$ is isomorphic to $Q \times \breve{D}^n$, where $\breve{D}$ is the open unit dagger disk and Q is the dagger space associated to the special frame (Y_J, F'_{i_0}). Using the notation of [4, p. 252], we write the dagger space associated to $(Y_J, F'_{i_0} \times \mathbb{A}^n_W)$ as $Q \times \breve{D}^n =:]Y_J[^{\dagger}_{\widehat{F'_{i_0} \times \mathbb{A}^n_W}}$, where $\widehat{F'_{i_0} \times \mathbb{A}^n_W}$ denotes the weak formal completion of $F'_{i_0} \times \mathbb{A}^n_W$ along p.

Now consider the embeddings

$$Y_J \hookrightarrow \prod_{i\in J} Z_i \hookrightarrow Z_{i_0} \times \prod_{\substack{j\in J\\ j\neq i_0}} F_j \hookrightarrow F_{i_0} \times E$$

and

$$Y_J \hookrightarrow Z'_{i_0} \times \prod_{\substack{i\in J\\ i\neq i_0}} Z_i \hookrightarrow \sum_{i\in J} \Big(Z'_i \times \prod_{\substack{j\in J\\ j\neq i}} F'_j \Big) \hookrightarrow F'_{i_0} \times E,$$

where

$$Z'_i := \begin{cases} Z_i & \text{if } i \neq i_0, \\ Z'_{i_0} & \text{if } i = i_0, \end{cases}$$

and likewise for F'_i. Note that

$$D_J := \sum_{i\in J} \Big(Z'_i \times \prod_{\substack{j\in J\\ j\neq i}} F'_j \Big) = \Big(Z'_{i_0} \times \prod_{\substack{i\in J\\ i\neq i_0}} F'_i \Big) + \sum_{\substack{i\in J\\ i\neq i_0}} \Big(F'_{i_0} \times Z_i \times \prod_{\substack{j\in J\\ j\neq i, i_0}} F_i \Big)$$

is a normal crossings divisor in $F'_{i_0} \times E$, and $]Y_J[^\dagger_{\hat{D}_J}$ is a normal crossings divisor in $]Y_J[^\dagger_{\widehat{F'_{i_0}\times E}}$. Applying the strong fibration theorem and coordinate change argument as above, we get a commutative diagram of dagger spaces

$$\begin{array}{ccccc}]Y_J[^\dagger_{\hat{D}_J} & \hookrightarrow &]Y_J[^\dagger_{\widehat{F'_{i_0}\times E}} & \hookrightarrow &]X_J[^\dagger_{\widehat{F'_{i_0}\times E}} \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ M_J & \hookrightarrow & Q \times \breve{D}^n & \hookrightarrow & \tilde{Q} \times \breve{D}^n, \end{array}$$

where the dagger space M_J, which is a normal crossings divisor in $Q \times \breve{D}^n$, is a sum of normal crossings divisors of the following form:

(a) $]Y_J[^\dagger_{\hat{Z}'_{i_0}} \times \breve{D}^n$, where $]Y_J[^\dagger_{\hat{Z}'_{i_0}}$ is a normal crossings divisor in Q,

(b) $Q \times \breve{D}^n(m)$, where $\breve{D}^n(m)$ is the divisor in $\breve{D}^n$ corresponding to
$$\operatorname{Sp} K\langle T_1, \dots, T_n\rangle^\dagger/(T_1 \cdots T_m).$$

Let $\omega^\bullet_{M_J}$ denote the logarithmic de Rham complex on the normal crossings divisor M_J in $Q \times \breve{D}^n$, as defined in [8]. We rewrite the comparison morphism defined in (9) in terms of dagger spaces using Lemma 5.1. Then for the case (a), we have a map

$$\begin{aligned} \Gamma(]Y_J[^\dagger_{\hat{Z}'_{i_0}} \times \breve{D}^n, \omega^\bullet_{]Y_J[^\dagger_{\hat{Z}'_{i_0}} \times \breve{D}^n}) &= \Gamma\big(]Y_J[^\dagger_{\hat{Z}'_{i_0}} \times \breve{D}^n, \omega^\bullet_{]Y_J[^\dagger_{\hat{Z}'_{i_0}}} \otimes \Omega^\bullet_{\breve{D}^n}\big) \\ &\to \Gamma\big(]Y_J[^\dagger_{\hat{Z}'_{i_0}}, \omega^\bullet_{]Y_J[^\dagger_{\hat{Z}'_{i_0}}}\big) \to W^\dagger\omega^\bullet_{A_J/k} \otimes \mathbb{Q}, \end{aligned}$$

where $\Omega^\bullet_{\breve{D}^n}$ is the usual (non-logarithmic) de Rham complex on $\breve{D}^n$, and where the first map is the projection and the second comes from the comparison between the log-Monsky–Washnitzer complex and overconvergent Hyodo–Kato complex constructed in (6). For the case (b), we have a map

$$\begin{aligned} \Gamma(Q \times \breve{D}^n(m), \omega^\bullet_{Q\times \breve{D}^n(m)}) &= \Gamma(Q \times \breve{D}^n(m), \omega^\bullet_{]Y_J[^\dagger_{\hat{F}'_{i_0}} \times \breve{D}^n(m)}) \\ &= \Gamma(Q \times \breve{D}^n(m), \Omega^\bullet_{]Y_J[^\dagger_{\hat{F}'_{i_0}}} \otimes \omega^\bullet_{\breve{D}^n(m)}) \\ &\to \Gamma(]Y_J[^\dagger_{\hat{F}'_{i_0}}, \Omega^\bullet_{]Y_J[^\dagger_{\hat{F}'_{i_0}}}) \to W^\dagger\Omega^\bullet_{A_J/k} \otimes \mathbb{Q} \\ &\to W^\dagger\omega^\bullet_{A_J/k} \otimes \mathbb{Q}, \end{aligned}$$

where $\Omega^\bullet_{]Y_J[^\dagger_{\hat{F}'_{i_0}}}$ is the usual de Rham complex on $]Y_J[^\dagger_{\hat{F}'_{i_0}}$ and the first map is again the projection. Let $\operatorname{sp}\colon]Y_J[^\dagger_{\hat{D}_J} = M_J \to Y_J$ be the specialization map. Then, by the argument in [4, Eq. 4.32], we have a local version of the above morphisms and get morphisms of complexes of Zariski sheaves on Y_J

$$\operatorname{sp}_*\omega^\bullet_{]Y_J[^\dagger_{\hat{Z}'_{i_0}} \times \breve{D}^n} \to W^\dagger\omega^\bullet_{Y_J/k} \otimes \mathbb{Q}$$

and

$$\mathrm{sp}_*\omega^\bullet_{Q\times\breve{D}^n(m)} \to W^\dagger\omega^\bullet_{Y_J/k}\otimes\mathbb{Q},$$

which give rise to a morphism

$$\mathrm{sp}_*\omega^\bullet_{]Y_J[^\dagger_{\hat{D}_J}} = \mathrm{sp}_*\omega^\bullet_{M_J} \to W^\dagger\omega^\bullet_{Y_J/k}\otimes\mathbb{Q} \tag{10}$$

into the overconvergent Hyodo–Kato complex (tensored with $\mathbb{Q}$) of Y_J. For the convenience of the reader, we recall the argument in [4, pp. 253 and 254]. We use the notation as above. Let $Y_J = \operatorname{Spec} A_J$ with $A_J = (A_{i_0})_{\bar{g}}$, $Z'_{i_0} = \operatorname{Spec}(\tilde{A}_{i_0})_g$ with g a lifting of $\bar{g}$, and $F'_{i_0} = \operatorname{Spec}(\tilde{B}_{i_0})_f$ with f a lifting of g. Let $U = \operatorname{Spec}(A_J)_{\bar{h}}$, h a lifting of $\bar{h}$ in $(\tilde{A}_{i_0})_g$ and $\tilde{h}$ a lifting of h in $(\tilde{B}_{i_0})_f$. Let $Z''_{i_0} = \operatorname{Spec}((\tilde{A}_{i_0})_g)_h$ and $F''_{i_0} = \operatorname{Spec}((\tilde{B}_{i_0})_f)_{\tilde{h}}$. Then $]U[_{\hat{Z}'_{i_0}}$ is open in $]Y_J[_{\hat{Z}'_{i_0}}$ and $]U[_{\hat{F}'_{i_0}}$ is open in $]Y_J[_{\hat{F}'_{i_0}}$, hence $]U[_{\hat{Z}'_{i_0}}$ and $]U[_{\hat{F}'_{i_0}}$ inherit dagger space structures from $]Y_J[^\dagger_{\hat{Z}'_{i_0}}$ and $]Y_J[^\dagger_{\hat{F}'_{i_0}}$. By the strong fibration theorem applied in the same way as in the section following [4, Eq. 4.32], the dagger spaces $]U[^\dagger_{\hat{Z}'_{i_0}}$ and $]U[^\dagger_{\hat{Z}''_{i_0}}$ (resp. $]U[^\dagger_{\hat{F}'_{i_0}}$ and $]U[^\dagger_{\hat{F}''_{i_0}}$) coincide. This induces the two morphisms

$$\mathrm{sp}_*\omega^\bullet_{]Y_J[^\dagger_{\hat{Z}'_{i_0}}\times\breve{D}^n} \to W^\dagger\omega^\bullet_{Y_J/k}\otimes\mathbb{Q}$$

and

$$\mathrm{sp}_*\omega^\bullet_{Q\times\breve{D}^n(m)} \to W^\dagger\omega^\bullet_{Y_J/k}\otimes\mathbb{Q},$$

which combined give the morphism (10).

We claim, in analogy to [4, Cor. 4.38], that the canonical morphisms

$$\mathrm{sp}_*\omega^\bullet_{M_J} \to \mathbb{R}\mathrm{sp}_*\omega^\bullet_{M_J}$$

are quasi-isomorphisms. For now we will assume this claim; the proof is postponed until Proposition 5.4. Then (10), together with the claim, gives a morphism

$$\mathbb{R}\mathrm{sp}_*\omega^\bullet_{M_J} \to W^\dagger\omega^\bullet_{Y_J/k}\otimes\mathbb{Q}. \tag{11}$$

Let $U = \operatorname{Spec} C$ be affine open in Y_J. By applying $\mathbb{R}\Gamma(U,-)$ to (11), we get from Lemma 5.1 and Proposition 5.2 an isomorphism

$$\mathbb{R}\Gamma(U, \mathbb{R}\mathrm{sp}_*\omega^\bullet_{]\mathrm{sp}^{-1}(U)[^\dagger_{\hat{D}_J}}) \xrightarrow{\sim} \mathbb{R}\Gamma(U, W^\dagger\omega^\bullet_{U/k}\otimes\mathbb{Q}) \simeq W^\dagger\omega^\bullet_{C/k}\otimes\mathbb{Q},$$

where the latter isomorphism follows from the fact that $H^i(U, W^\dagger\omega^r_{U/k}) = 0$ for all $r \geq 0$, $i > 0$. This is the semi-stable analogue of [4, Prop. 1.2 (b)] and is derived from the smooth case by considering the graded quotients of the weight filtration on the Čech complex associated to $W^\dagger\omega^r_{C/k}$, as in the proof of Proposition 3.2. The above isomorphism shows that (11) is already an isomorphism.

Now, as we range through the subsets $J \subset I$, we get an augmented simplicial k-scheme $\theta\colon Y_\bullet := \{Y_J\}_{J\subset I} \to Y$. Let

$$\tilde{D}_J := \sum_{i\in J}\Big(Z_i \times \prod_{\substack{j\in J\\ j\neq i}} F_j\Big),$$

which is a normal crossings divisor in $F_{i_0} \times E$. Again by the strong fibration theorem, the dagger tubes $]Y_J[^\dagger_{\hat{\tilde{D}}_J}$ and $]Y_J[^\dagger_{\hat{D}_J}$ are isomorphic. We get a simplicial object of special frames $\{(Y_J, \tilde{D}_J)\}_{J\subset I}$, and this gives rise to a simplicial object of dagger spaces

$$M_\bullet := \{]Y_J[^\dagger_{\hat{\tilde{D}}_J}\}_{J\subset I} = \{M_J\}_{J\subset I}.$$

The quasi-isomorphisms (11) glue to give a quasi-isomorphism of simplicial complexes on $Y_\bullet$

$$\mathbb{R}\mathrm{sp}_*\omega^\bullet_{M_\bullet} \xrightarrow{\sim} W^\dagger\omega^\bullet_{Y_\bullet/k} \otimes \mathbb{Q}. \tag{12}$$

Therefore,

$$\mathbb{R}\theta_*\mathbb{R}\mathrm{sp}_*\omega^\bullet_{M_\bullet} \cong \mathbb{R}\theta_* W^\dagger\omega^\bullet_{Y_\bullet/k} \otimes \mathbb{Q} \cong W^\dagger\omega^\bullet_{Y/k} \otimes \mathbb{Q},$$

and we deduce that

$$R\Gamma_{\text{log-rig}}(Y/\mathfrak{S}_0) = \mathbb{R}\Gamma(Y, \mathbb{R}\theta_*\mathbb{R}\mathrm{sp}_*\omega^\bullet_{M_\bullet}) \cong \mathbb{R}\Gamma(Y, W^\dagger\omega^\bullet_{Y/k} \otimes \mathbb{Q}),$$

as desired. □

It therefore remains to prove the following proposition.

Proposition 5.4. *Let M_J be the dagger space considered in the proof of Theorem 5.3. Then the canonical morphism*

$$\mathrm{sp}_*\omega^\bullet_{M_J} \to \mathbb{R}\mathrm{sp}_*\omega^\bullet_{M_J}$$

is a quasi-isomorphism.

The proof will occupy us for the rest of the section. By using the Mayer–Vietoris exact sequence, it is easy to see that it suffices to prove the proposition separately for the two cases (a) and (b) above. That is, it suffices to prove that

$$\mathrm{sp}_*\omega^\bullet_{]Y_J[^\dagger_{\hat{Z}_{i_0}} \times \breve{D}^n} \to \mathbb{R}\mathrm{sp}_*\omega^\bullet_{]Y_J[^\dagger_{\hat{Z}_{i_0}} \times \breve{D}^n}$$

and

$$\mathrm{sp}_*\omega^\bullet_{Q\times\breve{D}^n(m)} \to \mathbb{R}\mathrm{sp}_*\omega^\bullet_{Q\times\breve{D}^n(m)}$$

are quasi-isomorphisms. We recall from the proof of Theorem 5.3 that we have

$$\omega^\bullet_{]Y_J[^\dagger_{\hat{Z}_{i_0}} \times \breve{D}^n} = \omega^\bullet_{]Y_J[^\dagger_{\hat{Z}'_{i_0}}} \otimes \Omega^\bullet_{\breve{D}^n}$$

and

$$\omega^\bullet_{Q\times\breve{D}^n(m)} = \Omega^\bullet_{]Y_J[^\dagger_{\hat{F}'_{i_0}}} \otimes \omega^\bullet_{\breve{D}^n(m)}.$$

The proof for case (a) is easy. Indeed, in [4, Prop. 4.37 and Cor. 4.38], it is not needed that Q is a smooth affinoid dagger space. What is needed is that Ω^p_Q is a locally free $\mathcal{O}_Q$-module and that $H^i(Q, \Omega^p_Q)$ vanishes for $i > 0$ (Tate-acyclicity). Both properties hold for the locally free $(\tilde{A}_{i_0})^\dagger_g \otimes \mathbb{Q}$-module $\omega^p_{]Y_J[^\dagger_{\hat{Z}_{i_0}}}$ as well; indeed,

$$H^i(]Y_J[^\dagger_{\hat{Z}_{i_0}}, \omega^p_{]Y_J[^\dagger_{\hat{Z}_{i_0}}}) = 0$$

for $i > 0$ because $]Y_J[^\dagger_{\hat{Z}_{i_0}}$ is affinoid. Hence, we can replace Q by $]Y_J[^\dagger_{\hat{Z}_{i_0}}$ and Ω^p_Q by $\omega^p_{]Y_J[^\dagger_{\hat{Z}_{i_0}}}$ in the proofs of [4, Prop. 4.37, Cor. 4.38 and Lem. 4.44–4.47] to obtain the desired quasi-isomorphism

$$\mathbb{R}\mathrm{sp}_* \omega^\bullet_{]Y_J[^\dagger_{\hat{Z}_{i_0}} \times \breve{D}^n} \cong \mathrm{sp}_* \omega^\bullet_{]Y_J[^\dagger_{\hat{Z}_{i_0}} \times \breve{D}^n}.$$

Now we will treat case (b), which is more subtle. Since Q is an open subspace in the smooth affinoid dagger space $\tilde{Q}$, it is enough to show that

$$\mathbb{R}\mathrm{sp}_* \omega^\bullet_{\tilde{Q} \times \breve{D}^n(m)} \cong \mathrm{sp}_* \omega^\bullet_{\tilde{Q} \times \breve{D}^n(m)}.$$

Note that we have

$$\omega^\bullet_{\tilde{Q} \times \breve{D}^n(m)} = \Omega^\bullet_{\tilde{Q}} \otimes \omega^\bullet_{\breve{D}^n(m)}.$$

We have the following analogues of [4, Lem. 4.45 and 4.47].

Lemma 5.5. *Let A be a smooth dagger algebra, $Q = \mathrm{Sp}\, A$ the associated affinoid dagger space, and $D^n(m)$ the normal crossings divisor on the closed unit dagger n-ball D^n associated to*

$$\mathrm{Sp}\, K\langle T_1, \dots, T_n\rangle^\dagger / (T_1 \cdots T_m).$$

Let

$$\Lambda_n := \big(A \otimes_K \omega^0_{D^n(m)} \to A \otimes_K \omega^1_{D^n(m)} \to A \otimes_K \omega^2_{D^n(m)} \to \cdots\big)$$

be the complex with obvious differential. Let

$$d_t := \dim H^t(\omega^\bullet_{K\langle T_1, \dots, T_n\rangle^\dagger/(T_1 \cdots T_m)})$$

be the dimension of the log-Monsky–Washnitzer cohomology of $k[T_1, \dots, T_n]/(T_1 \cdots T_m)$. Then Λ_n is quasi-isomorphic to the complex (with zero differentials)

$$A \xrightarrow{0} A^{d_1} \xrightarrow{0} A^{d_2} \xrightarrow{0} \cdots .$$

Lemma 5.6. *With the same notation as above, let*

$$\breve{\Lambda}_n := \big(A \otimes_K \omega^0_{\breve{D}^n(m)} \to A \otimes_K \omega^1_{\breve{D}^n(m)} \to A \otimes_K \omega^2_{\breve{D}^n(m)} \to \cdots\big)$$

be the analogue complex for $\breve{D}^n$ and its closed normal crossings divisor $\breve{D}^n(m)$. Then $\breve{\Lambda}_n$ is quasi-isomorphic to the complex

$$A \xrightarrow{0} A^{d_1} \xrightarrow{0} A^{d_2} \xrightarrow{0} \cdots .$$

We can now follow the proof of [4, Prop. 4.37]. Let $\breve{D}^n = \bigcup_{i=1}^{\infty} U_i$ be a union of dagger balls of ascending radius, and let $\breve{D}^n(m) = \bigcup_{i=1}^{\infty} U_i(m)$ be the corresponding normal crossings divisors. For notational brevity, write $\omega^q := \omega^q_{\tilde{Q}\times U_i(m)}$. Since $\tilde{Q} \times U_i(m)$ is affinoid, $H^p(\tilde{Q} \times U_i(m), \omega^q)$ vanishes for $p \geq 1$ and $\mathbb{R}\Gamma(\tilde{Q} \times \breve{D}^n(m), \omega^q)$ is quasi-isomorphic to the global sections of the complex

$$\prod_{i=1}^{\infty} \omega^q(\tilde{Q} \times U_i(m)) \to \prod_{i=1}^{\infty} \omega^q(\tilde{Q} \times U_i(m)), \quad \prod s_i \mapsto \prod (s_i - s_{i+1}).$$

Note that

$$\omega^q_{\tilde{Q}\times U_i(m)} = \bigoplus_{\ell} \Omega^{\ell}_{\tilde{Q}} \otimes_K \omega^{q-\ell}_{U_i(m)}.$$

Then the complex $H^0(\tilde{Q} \times U_i(m), \omega^{\bullet}_{\tilde{Q}\times U_i(m)})$ is represented by the double complex with components

$$C^{p,q}(U_t(m)) = H^0(\tilde{Q}, \Omega^p_{\tilde{Q}}) \otimes_K H^0(U_i(m), \omega^q).$$

Therefore, the morphism of double complexes

$$\prod_{i=1}^{\infty} C^{\bullet,\bullet}(U_i(m)) \to \prod_{i=1}^{\infty} C^{\bullet,\bullet}(U_i(m))$$

given on the (p, q)-entry by

$$\prod_{i=1}^{\infty} C^{p,q}(U_i(m)) \to \prod_{i=1}^{\infty} C^{p,q}(U_i(m)), \quad \prod s_i \mapsto \prod (s_i - s_{i+1}),$$

induces a map of total complexes with kernel $H^0(\tilde{Q} \times \breve{D}^n(m), \omega^{\bullet}_{\tilde{Q}\times\breve{D}^n(m)})$ and cokernel $H^1(\tilde{Q} \times \breve{D}^n(m), \omega^{\bullet}_{\tilde{Q}\times\breve{D}^n(m)})$. It follows from Lemma 5.5 that the total complex associated to $C^{\bullet,\bullet}(U_i(m))$ is quasi-isomorphic to

$$\bigoplus_t (H^0(\tilde{Q}, \Omega^{\bullet}_{\tilde{Q}}))^{d_t}$$

with the correction $d_0 = 1$. Analogously, it follows from Lemma 5.6 that $H^0(\tilde{Q} \times \breve{D}^n(m), \omega^{\bullet}_{\tilde{Q}\times\breve{D}^n(m)})$ is quasi-isomorphic to

$$\bigoplus_t (H^0(\tilde{Q}, \Omega^{\bullet}_{\tilde{Q}}))^{d_t} = \Big(\bigoplus_t (H^0(\tilde{Q}, \Omega^0_{\tilde{Q}}))^{d_t} \to \bigoplus_t (H^0(\tilde{Q}, \Omega^1_{\tilde{Q}}))^{d_t} \to \cdots \Big).$$

Finally, $H^1(\tilde{Q} \times \breve{D}^n(m), \omega^{\bullet}_{\tilde{Q}\times\breve{D}^n(m)})$ is quasi-isomorphic to the total complex of the triple complex

$$H^0(\tilde{Q} \times \breve{D}^n(m), \Omega^{\bullet}_{\tilde{Q}} \otimes \omega^{\bullet}_{\breve{D}^n(m)}) \to \prod_{i=1}^{\infty} C^{\bullet,\bullet}(U_i(m)) \to \prod_{i=1}^{\infty} C^{\bullet,\bullet}(U_i(m)),$$

which is quasi-isomorphic to the total complex of the double complex

$$\bigoplus_t (H^0(\tilde{Q},\Omega^\bullet_{\tilde{Q}}))^{d_t} \to \prod_{i=1}^{\infty}\bigoplus_t H^0(\tilde{Q},\Omega^\bullet_{\tilde{Q}})^{d_t} \to \prod_{i=1}^{\infty}\bigoplus_t H^0(\tilde{Q},\Omega^\bullet_{\tilde{Q}})^{d_t},$$
$$\prod s_i \mapsto \prod (s_i - s_{i+1})$$

(we note that the direct sums are finite, since $d_t = 0$ for t greater than twice the dimension).

Since the double complex is acyclic with regard to the horizontal differential, the total complex is acyclic too, and hence $H^1(\tilde{Q}\times \check{D}^n(m), \omega^\bullet_{\tilde{Q}\times\check{D}^n(m)})$ is also acyclic. This proves [4, Prop. 4.37 and Cor. 4.38] for $\omega^\bullet_{\tilde{Q}\times\check{D}^n(m)}$, and hence we conclude that Proposition 5.4 holds.

6. The monodromy operator

We follow the argument in [18] but in the more general setting that Y need not be proper.

Let Y be a quasi-projective semi-stable scheme over S_0. Define a double complex

$$\mathcal{B}^{\dagger\bullet,\bullet} := \frac{W^\dagger\tilde{\omega}^{i+j+1}_{Y/k}}{P_j W^\dagger\tilde{\omega}^{i+j+1}_{Y/k}},$$

with the differential $\mathcal{B}^{\dagger i,j} \to \mathcal{B}^{\dagger i+1,j}$ given by $(-1)^j d$ and the differential $\mathcal{B}^{\dagger i,j} \to \mathcal{B}^{\dagger i,j+1}$ given by $\omega \mapsto \omega\wedge\theta$. Let $\mathcal{B}^{\dagger\bullet}$ be the total complex of $\mathcal{B}^{\dagger\bullet,\bullet}$. Then $\mathcal{B}^{\dagger\bullet}\otimes\mathbb{Q}$ is the complex $\mathcal{B}^{\dagger\bullet}_{\mathbb{Q}}$ considered in the proof of Theorem 4.1 in the log-Monsky–Washnitzer setting. Let Φ denote the map induced by $p^{i+1}F$ on $\mathcal{B}^{\dagger i,j}$. Define also a map ν by requiring that $(-1)^{i+j+1}\nu\colon \mathcal{B}^{\dagger i,j} \to \mathcal{B}^{\dagger i-1,j+1}$ is the projection. This induces a map on $\mathcal{B}^{\dagger\bullet}$, which we also call ν. The same argument as in the proof of Theorem 4.1 shows that the natural map $W^\dagger\tilde{\omega}^\bullet_{Y/k} \to \mathcal{B}^{\dagger\bullet}$ factors through $\Theta\colon W^\dagger\omega^\bullet_{Y/k} \to \mathcal{B}^{\dagger\bullet}$, and $\Theta\Phi = \Phi\Theta$. One also has that $\Theta\otimes\mathbb{Q}$ is a quasi-isomorphism. Indeed, this is a local question on Y, so we may reduce to the case that Y is a semi-stable affine scheme over S_0, and this case was already shown in the proof of Theorem 4.1.

Proposition 6.1. *The map $\nu\colon \mathcal{B}^{\dagger\bullet,\bullet} \to \mathcal{B}^{\dagger\bullet,\bullet}$ induces a nilpotent operator N on*

$$\mathbb{H}^*(Y, \mathcal{B}^{\dagger\bullet}_{\mathbb{Q}}) \cong \mathbb{H}^*(Y, W^\dagger\omega^\bullet_{Y/k}\otimes\mathbb{Q}) \cong H^*_{\text{log-rig}}(Y/\mathfrak{S}_0),$$

which coincides with the monodromy operator

$$N\colon H^*_{\text{log-rig}}(Y/\mathfrak{S}_0) \to H^*_{\text{log-rig}}(Y/\mathfrak{S}_0)$$

defined in [8, Section 5.4].

Proof. Let us define another double complex by

$$\mathcal{C}^{\dagger i,j} := \mathcal{B}^{\dagger, i-1,j} \oplus \mathcal{B}^{\dagger i,j}$$

for $i,j \geq 0$, with the differential $\mathcal{C}^{\dagger i,j} \to \mathcal{C}^{\dagger i+1,j}$ given by

$$(\omega_1,\omega_2) \mapsto ((-1)^j d\omega_1, (-1)^j d\omega_2)$$

and the differential $\mathcal{C}^{\dagger i,j} \to \mathcal{C}^{\dagger i,j+1}$ given by

$$(\omega_1, \omega_2) \mapsto (\omega_1 \wedge \theta + \nu\omega_2, \omega_2 \wedge \theta).$$

Let $\mathcal{C}^{\dagger\bullet}$ be the total complex of $\mathcal{C}^{\dagger\bullet,\bullet}$. Then we get a natural morphism

$$\Psi : W^\dagger\tilde{\omega}^\bullet_{Y/k} \to \mathcal{C}^{\dagger\bullet}$$

fitting into the following diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W^\dagger\omega^\bullet_{Y/k}[-1] & \xrightarrow{-\wedge\theta} & W^\dagger\tilde{\omega}^\bullet_{Y/k} & \longrightarrow & W^\dagger\omega^\bullet_{Y/k} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Theta[-1]} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Theta} & & \\ 0 & \longrightarrow & \mathcal{B}^{\dagger\bullet}[-1] & \xrightarrow{-\wedge\theta} & \mathcal{C}^{\dagger\bullet} & \longrightarrow & \mathcal{B}^{\dagger\bullet} & \longrightarrow & 0. \end{array}$$

Tensoring by $\mathbb{Q}$ gives the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W^\dagger\omega^\bullet_{Y/k}\otimes\mathbb{Q}[-1] & \xrightarrow{-\wedge\theta} & W^\dagger\tilde{\omega}^\bullet_{Y/k}\otimes\mathbb{Q} & \longrightarrow & W^\dagger\omega^\bullet_{Y/k}\otimes\mathbb{Q} & \longrightarrow & 0 \\ & & \wr\downarrow{\scriptstyle \Theta\otimes\mathbb{Q}[-1]} & & \downarrow{\scriptstyle \Psi\otimes\mathbb{Q}} & & \wr\downarrow{\scriptstyle \Theta\otimes\mathbb{Q}} & & \\ 0 & \longrightarrow & \mathcal{B}^{\dagger\bullet}_{\mathbb{Q}}[-1] & \xrightarrow{-\wedge\theta} & \mathcal{C}^{\dagger\bullet}\otimes\mathbb{Q} & \longrightarrow & \mathcal{B}^{\dagger\bullet}_{\mathbb{Q}} & \longrightarrow & 0, \end{array}$$

where the outermost vertical arrows are quasi-isomorphisms by the local argument given in the proof of Theorem 4.1, and hence we conclude that $\Psi\otimes\mathbb{Q}$ is also a quasi-isomorphism. By construction, this shows that the map

$$N\colon \mathbb{H}^*(Y, W^\dagger\omega^\bullet_{Y/k}\otimes\mathbb{Q}) \to \mathbb{H}^*(Y, W^\dagger\omega^\bullet_{Y/k}\otimes\mathbb{Q})$$

induced by $\nu\colon \mathcal{B}^{\dagger\bullet,\bullet} \to \mathcal{B}^{\dagger\bullet,\bullet}$ is exactly the connecting homomorphism on cohomology associated to the top short exact sequence.

It therefore suffices to prove that the connecting homomorphism gives the monodromy operator on $H^*_{\text{log-rig}}(Y/\mathfrak{S}_0)$. Let $Y_\bullet$ be the simplicial scheme and $M_\bullet :=]Y_\bullet[^\dagger_{\bar{D}_\bullet}$ the simplicial dagger space as constructed in the proof of Theorem 5.3. Then we have a diagram of short exact sequences of complexes of simplicial sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}\mathrm{sp}_*\omega^\bullet_{M_\bullet}[-1] & \xrightarrow{-\wedge\theta} & \mathbb{R}\mathrm{sp}_*\tilde{\omega}^\bullet_{M_\bullet} & \longrightarrow & \mathbb{R}\mathrm{sp}_*\omega^\bullet_{M_\bullet} & \longrightarrow & 0 \\ & & \wr\downarrow & & \downarrow & & \wr\downarrow & & \\ 0 & \longrightarrow & W^\dagger\omega^\bullet_{Y_\bullet/k}\otimes\mathbb{Q}[-1] & \xrightarrow{-\wedge\theta} & W^\dagger\tilde{\omega}^\bullet_{Y_\bullet/k}\otimes\mathbb{Q} & \longrightarrow & W^\dagger\omega^\bullet_{Y_\bullet/k}\otimes\mathbb{Q} & \longrightarrow & 0, \end{array}$$

where the outermost vertical arrows are the quasi-isomorphisms (12), and the middle arrow is constructed as follows.

We rewrite the morphism (7) in terms of dagger spaces for each J:

$$\Gamma(M_J, \tilde{\omega}^\bullet_{M_J}) \to W^\dagger\tilde{\omega}^\bullet_{Y_J/k}\otimes\mathbb{Q}.$$

Then applying the argument after [4, Eq. 4.32] gives a local version

$$\mathrm{sp}_*\tilde{\omega}^\bullet_{M_J} \to W^\dagger\tilde{\omega}^\bullet_{Y_J/k}\otimes\mathbb{Q}.$$

The proof of Proposition 5.4 holds verbatim with $\mathrm{sp}_*\omega^\bullet_{M_J}$ replaced by $\mathrm{sp}_*\tilde{\omega}^\bullet_{M_J}$ to show that the canonical morphism $\mathrm{sp}_*\tilde{\omega}^\bullet_{M_J} \to \mathbb{R}\mathrm{sp}_*\tilde{\omega}^\bullet_{M_J}$ is a quasi-isomorphism. This then defines the middle arrow in the diagram, which is therefore a quasi-isomorphism. The monodromy operator on the log-rigid cohomology of Y is, by definition, the connecting homomorphism on cohomology associated to the top short exact sequence.

7. Comparison with log-crystalline cohomology in the projective case

We prove a semi-stable analogue of a comparison, obtained for smooth projective varieties in [15] between overconvergent and usual de Rham–Witt cohomology, for Hyodo–Kato cohomology:

Theorem 7.1. *Let Y be a projective semi-stable scheme over S_0. Then the canonical map*

$$\mathbb{H}^*(Y, W^\dagger\omega^\bullet_{Y/k}) \to \mathbb{H}^*(Y, W\omega^\bullet_{Y/k})$$

is an isomorphism of $W(k)$-modules of finite type.

For the assumption of (quasi-)projectivity, see the remark below Theorem 1.1.

First we need the following lemma.

Lemma 7.2. *Under the assumptions of Theorem 7.1, there is a commutative diagram*

$$\begin{array}{ccc}
\mathbb{H}^*(Y, W^\dagger\omega^\bullet_{Y/k}) & \longrightarrow & \mathbb{H}^*(Y, W\omega^\bullet_{Y/k}) \\
\downarrow & & \downarrow \\
\mathbb{H}^*(Y, W^\dagger\omega^\bullet_{Y/k}\otimes\mathbb{Q}) & \longrightarrow & \mathbb{H}^*(Y, W\omega^\bullet_{Y/k}\otimes\mathbb{Q}) \\
\wr\uparrow & & \wr\uparrow \\
H^*_{\text{log-rig}}(Y/\mathfrak{S}_0) & \xrightarrow{\sim} & H^*_{\text{log-cris}}((Y,M)/(W(k),W(L)))\otimes\mathbb{Q}.
\end{array}$$

Here M is the log structure on Y given by $\mathcal{O}_Y \cap u_*\mathcal{O}_U^\times$, where $u\colon U \hookrightarrow Y$ is a smooth dense open, and $W(L)$ is the canonical lifting of the log structure L on $\mathrm{Spec}\,k$ *given by* $1 \mapsto 0$ (*previously denoted by S_0*).

Proof. We need to show that the lower square commutes. The isomorphism on the left and right are the comparisons between log-rigid and overconvergent Hyodo–Kato (resp. between log-crystalline and Hyodo–Kato cohomology [10, Thm. 4.19]). These isomorphisms also hold if Y is only quasi-projective. The lower horizontal isomorphism is the logarithmic analogue of a comparison between rigid and crystalline cohomology defined in the proof of [1, Thm. 1.9]. If there exists a global semi-stable frame, the analogy with Berthelot's proof is clear, otherwise one has to proceed by simplicial methods. Using Grosse-Klönne's definition of log-rigid cohomology as the cohomology of simplicial dagger spaces [8, 1.5] one obtains a canonical map, by using p-adic formal schemes and rigid spaces instead of weak formal schemes and dagger spaces, to Shiho's analytic cohomology which is isomorphic to log-convergent cohomology by Shiho's log convergent Poincaré lemma [19, Cor. 2.3.9]. Using the isomorphism between log-convergent and log-crystalline cohomology [19, Thm. 3.1.1], one obtains the lower horizontal arrow for any semi-stable Y, not necessarily proper. If Y is proper, then the log-rigid cohomology is isomorphic to analytic, respectively, log-convergent cohomology by [8, Thm. 5.3], and hence the lower horizontal arrow is an isomorphism for Y proper semi-stable.

Hence, all maps in the lower square are defined for quasi-projective varieties as well. Using the Mayer–Vietoris sequence for cohomology, we may assume that Y is affine. Since the lower horizontal map in the diagram is independent from the choice of embeddings into log-smooth (weak) formal schemes, we may assume that $H^*_{\text{log-rig}}(Y/\mathfrak{S}_0)$ is given by the logarithmic Monsky–Washnitzer cohomology $H^*_{\text{log-MW}}(Y/K)$. In this case the map is given by a morphism of complexes

$$\omega^\bullet_{\tilde{Y}^\dagger} \to \omega^\bullet_{\hat{\tilde{Y}}},$$

i.e., by taking p-adic completion of the logarithmic Monsky–Washnitzer complex. The comparison maps to the overconvergent and usual Hyodo–Kato complexes evidently commute with taking p-adic completions. This proves the lemma. □

Next we show the analogue of [15, Prop. 2.2].

Proposition 7.3. *Under the assumptions of Theorem 7.1, we have quasi-isomorphisms*

$$W^\dagger\omega^\bullet_{Y/k}/p^n \cong W\omega^\bullet_{Y/k}/p^n \cong W_n\omega^\bullet_{Y/k}$$

for all $n \in \mathbb{N}$.

Only the first quasi-isomorphism requires a proof; the second quasi-isomorphism follows from [10, Cor. 4.5].

This is a Zariski-local question, so we may assume that Y is affine. Moreover, by a result of Kedlaya [13, Thm. 2], we may assume that $Y = \operatorname{Spec} B$ is finite étale and free over $\operatorname{Spec} A = \operatorname{Spec} k[T_1, \dots, T_d]/(T_1 \cdots T_r)$ for some r.

We note that [15, Prop. 2.3] is based on [5, Cor. 2.46] and does not need A being a smooth k-algebra, hence we conclude that $W^\dagger(B)$ is a finite étale $W^\dagger(A)$-algebra and free as a $W^\dagger(A)$-module.

The proof of [4, Prop. 1.9] transfers verbatim to the Hyodo–Kato complexes and extends the étale base change for the Hyodo–Kato complexes in [17, Prop. 3.7] to the overconvergent setting, hence we have

$$W^\dagger\omega^\ell_{A/k} \otimes_{W^\dagger(A)} W^\dagger(B) \xrightarrow{\sim} W^\dagger\omega^\ell_{B/k}.$$

Let $\kappa_A\colon \tilde{A}^\dagger = W(k)\langle T_1, \dots, T_d\rangle^\dagger/(T_1 \cdots T_r) \to W^\dagger(A)$ be the canonical map obtained by sending T_i to $[T_i]$ for $i = 1, \dots, d$. Note that $[T_1 \cdots T_r] = [T_1] \cdots [T_r]$ is zero in $W(A)$, hence κ_A is well defined. By reproducing the argument before [15, Prop. 2.5], we conclude that the above map extends uniquely to

$$\kappa_B \colon \tilde{B}^\dagger \to W^\dagger(B)$$

(note that this map is used to construct the comparison morphisms (5) and (6)). Then we have the following proposition.

Proposition 7.4. *Let B be finite étale and free over $A = k[T_1, \dots, T_d]/(T_1 \cdots T_r)$. Then there is a decomposition of $W^\dagger\omega^\bullet_{B/k}$ into subcomplexes*

$$W^\dagger\omega^\bullet_{B/k} = W^\dagger\omega^{\mathrm{int}\,\bullet}_{B/k} \oplus W^\dagger\omega^{\mathrm{frac}\,\bullet}_{B/k},$$

where $W^\dagger\omega^{\mathrm{frac}\,\bullet}_{B/k}$ is acyclic and $W^\dagger\omega^{\mathrm{int}\,\bullet}_{B/k}$ is isomorphic to $\omega^\bullet_{\tilde{B}^\dagger}$ via the morphism induced by

$$\kappa_B \colon \omega^\bullet_{\tilde{B}^\dagger} \to W^\dagger\omega^\bullet_{B/k}.$$

Proof. It is enough to treat the case $A = B$; the argument for this is the same as in the proof of [15, Prop. 2.5]. Indeed,

$$W^\dagger\omega^\ell_{B/k} \cong W^\dagger\omega^\ell_{A/k} \otimes_{W^\dagger(A)} W^\dagger(B)$$

by étale base change. Let $b_1, \dots, b_m$ be an A-module basis of B and lift these to an $\tilde{A}^\dagger$-module basis $u_1, \dots, u_m$ of $\tilde{B}^\dagger$. Then $\kappa_B(u_1), \dots, \kappa_B(u_m)$ is a $W^\dagger(A)$-module basis of $W^\dagger(B)$. Therefore,

$$W^\dagger\omega^\ell_{B/k} \cong W^\dagger\omega^\ell_{A/k} \otimes_{\tilde{A}^\dagger} \tilde{B}^\dagger.$$

If $W^\dagger\omega^\bullet_{A/k}$ decomposes as in the statement of the proposition, then we obtain a decomposition

$$W^\dagger\omega^\bullet_{B/k} = W^\dagger\omega^{\mathrm{int}\,\bullet}_{B/k} \oplus W^\dagger\omega^{\mathrm{frac}\,\bullet}_{B/k}$$

by tensoring the corresponding decomposition for A with $\tilde{B}^\dagger$. The same proof as that of [4, Thm. 3.19] shows that if $W^\dagger\omega^{\mathrm{frac}\,\bullet}_{A/k}$ is acyclic, then $W^\dagger\omega^{\mathrm{frac}\,\bullet}_{B/k}$ is acyclic. It therefore suffices to prove the proposition for the case $A = B$. For this we use the description of the de Rham–Witt complex of a (Laurent-)polynomial algebra given in [2, Section 10.4].

For a $\mathbb{Z}_{(p)}$-algebra R, any element ω in $W_n\Omega^\ell_{R[T_1^{\pm 1}, \dots, T_d^{\pm 1}]/R}$ can be uniquely written as a finite sum

$$\omega = \sum_{k, \mathcal{P}} e(\xi_{k,\mathcal{P}}, k, \mathcal{P}_{\le \rho}) \prod_{j=\rho+1}^{\ell} d\log\Big(\prod_{i \in I_j} [T_i]\Big), \tag{13}$$

where k ranges over the weight functions $k \colon [1,d] \to \mathbb{Z}[\frac{1}{p}] \cup \{\infty\}$ satisfying properties (i), (ii), (iii) in [2, Section 10.4], and $\mathcal{P} = \{I_0, I_1, \dots, I_\rho, I_{\rho+1}, \dots, I_\ell\}$ is a disjoint partition of $I = \operatorname{supp} k$, such that $\mathcal{P}_{\le \rho} = \{I_0, I_1, \dots, I_\rho\}$ and ρ is the integer denoted by ρ_2 in [2, Section 10.4], I_0 is possibly empty and $e(\xi_{k,\mathcal{P}}, k, \mathcal{P}_{\le \rho})$ is a basic Witt differential of type Case 1, Case 2, Case 3 given in [14, Eq. 2.15–2.17] (but where the exponents of the T_i for i occurring in I_j for $0 \le j \le \rho$ can be negative).

Consider now the log-scheme $\operatorname{Spec}(A, \mathbb{N}^r)$, where $\mathbb{N}^r \ni e_i \mapsto T_i$, $1 \le i \le r$, over the trivial base $\operatorname{Spec}(k, *)$. Then the complex $W\Lambda^\bullet_{(A,\mathbb{N}^r)/(k,*)}$, defined in [17], can be described as follows (our description differs from the description in [17] but is equivalent): any ω in $W\Lambda^{\ell}_{(A,\mathbb{N}^r)/(k,*)}$ has a unique expression as a convergent sum

$$\omega = \sum_{k,\mathcal{P}} e(\xi_{k,\mathcal{P}}, k, \mathcal{P}_{\leq\rho}) \prod_{j=\rho+1}^{\ell} d\log\Big(\prod_{i\in I_j}[T_i]\Big),$$

as in (13), where for any given m, the following hold:

- $\xi_{k,\mathcal{P}} \in V^m W(k) = p^m W(k)$ for all but finitely many k.
- All weight functions take nonnegative values, i.e., on $I_0\cup\cdots\cup I_\rho$, they take values in $\mathbb{Z}_{\geq 0}[\frac{1}{p}]$.
- $[1,r] \not\subset I_j$ for any $j = 0,\dots,\rho$, and for any i occurring in I_j for $j=\rho+1,\dots,\ell$, we have $i\in[1,\dots,r]$.

It is clear from this description that we get a decomposition

$$W\Lambda^{\bullet}_{(A,\mathbb{N}^r)/(k,*)} = W\Lambda^{\mathrm{int}\,\bullet}_{(A,\mathbb{N}^r)/(k,*)} \oplus W\Lambda^{\mathrm{frac}\,\bullet}_{(A,\mathbb{N}^r)/(k,*)},$$

given by integral and purely fractional weights, and that the fractional part is acyclic, as in the case of (Laurent-) polynomial algebras [2, Thm. 10.13].

Now we apply [17, Section 7.2]. Let $W_m\tilde{\Lambda}^{\bullet} := W_m\Lambda^{\bullet}_{(A,\mathbb{N}^r)/(k,*)}$ and $W_m\Lambda^{\bullet} := W_m\Lambda^{\bullet}_{(A,\mathbb{N}^r)/(k,\mathbb{N})} = W_m\Lambda^{\bullet}_{(A,\mathbb{N}^r)/S_0}$, which is isomorphic to the Hyodo–Kato complex $W_m\omega^{\bullet}_{Y/k}$ by the proof of Proposition 3.3. Then we have a short exact sequence (see [17, Lem. 7.4])

$$0 \longrightarrow W_m\Lambda^{\bullet-1} \xrightarrow{\wedge\theta_m} W_m\tilde{\Lambda}^{\bullet} \longrightarrow W_m\Lambda^{\bullet} \longrightarrow 0,$$

where $\theta_m := d\log[T_1] + \cdots + d\log[T_r]$. This implies that any element ω in $W_m\Lambda^{\ell}$ can be written uniquely as a sum

$$\omega = \sum_{k,\mathcal{P}} e(\xi_{k,\mathcal{P}}, k, \mathcal{P}_{\leq\rho}) \prod_{j=\rho+1}^{\ell} d\log\Big(\prod_{i\in I_j}[T_i]\Big) \tag{14}$$

with the following properties:

- $[1,r]\not\subset I_j$ for any $j=0,\dots,\ell$; ρ is equal to ρ_2 in [2, Section 10.4].
- For all $j=\rho+1,\dots,\ell$ we have $I_j\subset\{1,\dots,r\}$.
- $e(\xi_{k,\mathcal{P}}, k, \mathcal{P}_{\leq\rho})$, as before.

From the definitions it is clear that we again have a decomposition

$$W_m\Lambda^{\bullet} = W_m\Lambda^{\mathrm{int}\,\bullet} \oplus W_m\Lambda^{\mathrm{frac}\,\bullet},$$

and the fractional part is acyclic. Passing to the projective limit and overconvergent subcomplexes, we obtain decompositions

$$W\Lambda^{\bullet} = W\Lambda^{\mathrm{int}\,\bullet} \oplus W\Lambda^{\mathrm{frac}\,\bullet}$$

and

$$W^{\dagger}\Lambda^{\bullet} = W^{\dagger}\Lambda^{\mathrm{int}\,\bullet} \oplus W^{\dagger}\Lambda^{\mathrm{frac}\,\bullet},$$

and the fractional parts are acyclic subcomplexes (the acyclicity is inherited from the case of polynomial algebras). Hence, we have the desired decomposition

$$W^\dagger\omega^\bullet_{Y/k} = W^\dagger\omega^{\mathrm{int}\,\bullet}_{Y/k} \oplus W^\dagger\omega^{\mathrm{frac}\,\bullet}_{Y/k}$$

in the case that $Y = \operatorname{Spec} k[T_1, \dots, T_d]/(T_1 \cdots T_r)$. It is evident that $W^\dagger\omega^{\mathrm{int}\,\bullet}_{Y/k}$ is isomorphic to $\omega^\bullet_{\tilde{A}^\dagger}$. □

Since the $W^\dagger\omega^\ell_{Y/k}$ and $W\omega^\ell_{Y/k}$ are p-torsion-free [10, Cor. 4.5], we conclude that $W^\dagger\omega^{\mathrm{frac}\,\bullet}_{Y/k} \otimes \mathbb{Z}/p^n$ is acyclic too. It is clear that $\omega^\bullet_{\tilde{B}^\dagger} \otimes \mathbb{Z}/p^n$ is isomorphic to $\omega^\bullet_{\hat{\tilde{B}}} \otimes \mathbb{Z}/p^n$. This concludes the proof of Proposition 7.3.

Finally, since

$$\varprojlim_n \mathbb{H}^i(Y, W^\dagger\omega^\bullet_{Y/k}/p^n) = \varprojlim_n \mathbb{H}^i(Y, W_n\omega^\bullet_{Y/k}) = \mathbb{H}^i(Y, W\omega^\bullet_{Y/k}),$$

where the last equality holds because all $\mathbb{H}^i(Y, W_n\omega^\bullet_{Y/k})$ are of finite length if Y is proper [10, Section 3.2] and $\mathbb{H}^i(Y, W\omega^\bullet_{Y/k})$ are $W(k)$-modules of finite type, we can apply the arguments in [15, p. 1392] to conclude that

$$\mathbb{H}^*(Y, W^\dagger\omega^\bullet_{Y/k}) \cong \mathbb{H}^*(Y, W\omega^\bullet_{Y/k}).$$

This proves Theorem 7.1. □

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