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Almost one-to-one extensions of Cantor minimal systems and order embeddings of simple dimension groups

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Abstract. Suppose π is an almost one-to-one factor map of Cantor minimal systems from (X,ϕ) to (Y,ψ). Then it is known that (1) the induced map π^* is an order embedding of $K^0(Y,\psi)$ to $K^0(X,\psi)$, (2) the cokernel of π^*, $K^0(X,\phi)/\pi^*(K^0(Y,\psi))$, is torsion free and (3) the induced affine map $\hat{\pi}$ from the state space $\mathcal{S}(K^0(X,\phi))$ to the state space $\mathcal{S}(K^0(Y,\psi))$ is surjective and sends $\operatorname{ex}\mathcal{S}(K^0(X,\phi))$, the set of all extreme points in $\mathcal{S}(K^0(X,\phi))$, onto $\operatorname{ex}\mathcal{S}(K^0(Y,\psi))$. In this paper we will show the dynamical realization problem of dimension groups by the following: Suppose G and H are simple dimension groups satisfying that (i) an injective unital order homomorphism $\iota : H \to G$ is an order embedding, (ii) the cokernel of ι, $G/\iota(H)$, is torsion free and (iii) an induced affine map ι^* of state spaces from $\mathcal{S}(G)$ to $\mathcal{S}(H)$ is surjective and $\iota^*(\operatorname{ex}\mathcal{S}(G)) = \operatorname{ex}\mathcal{S}(H)$. Then there exist Cantor minimal systems (X,ϕ) and (Y,ψ), an almost one-to-one factor map $\pi : X \to Y$ and a unital order isomorphism $\alpha : K^0(X,\phi) \to G$ and $\beta : K^0(Y,\psi) \to H$ such that (1), (2) and (3) above and $\alpha\circ\pi^* = \iota\circ\beta$ hold. This is a generalization of the results in [4].

1. Introduction

A topological dynamical system (Y,ψ) is called a *Cantor minimal system* if Y is the Cantor set (i.e., a compact totally disconnected metric space with no isolated points) and ψ is a homeomorphism on Y acting minimally (i.e., every ψ-orbit is dense in Y, or equivalently, the only closed ψ-invariant sets are Y and the empty set.).

Let (Y,ψ) be a Cantor minimal system, $C(Y,\mathbb{Z})$ be the set of integer-valued continuous functions on Y and $B_\psi := \{f - f\circ\psi \mid f \in C(Y,\mathbb{Z})\}$. We regard $C(Y,\mathbb{Z})$ as an abelian group with point-wise addition. Then B_ψ is a subgroup of $C(Y,\mathbb{Z})$. We define an ordered abelian group with a positive cone

$$K^0(Y,\psi) := C(Y,\mathbb{Z})/B_\psi,$$
$$K^0(Y,\psi)^+ := \{[f] \in K^0(Y,\psi) \mid f \geq 0,\ f \in C(Y,\mathbb{Z})\},$$

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where $[f]$ is the coset of $f \in C(Y, \mathbb{Z})$ in $K^0(Y, \psi)$. In the case where f is a constant function 1, we call $[1]$ *the distinguished order unit of* $K^0(Y, \psi)$. In [6], Herman, Putnam and Skau showed that $K^0(Y, \psi)$ is an acyclic (i.e., $K^0(Y, \psi) \ncong \mathbb{Z}$) simple dimension group and every acyclic simple dimension group arises from Cantor minimal systems. A triple (G, G^+, u), where G is a dimension group, G^+ its positive cone and u its distinguished order unit, is called a *dimension group triple*. Two dimension group triple (G, G^+, u) and (H, H^+, v) are *unital order isomorphic* if there is a group isomorphism $\iota : G \to H$ such that $\iota(G^+) = H^+$ and $\iota(u) = v$. In [3], Giordano, Putnam and Skau showed that the unital order isomorphic class of a dimension group triple $(K^0(Y, \psi), K^0(Y, \psi)^+, [1])$ is a complete invariant of the strong orbit equivalence class of (Y, ψ).

Given a dimension group triple (G, G^+, u), a state ω on G is a group homomorphism $\omega : G \to \mathbb{R}$ such that $\omega(G^+) \subset \mathbb{R}_+$ and $\omega(u) = 1$. Let $\mathcal{S}(G)$ denote the set of all states of G. Then $\mathcal{S}(G)$ is a metrizable Choquet simplex, that is, a compact convex metrizable space with the property that for any $\mu \in \mathcal{S}(G)$, there exists a unique probability measure τ on $\mathcal{S}(G)$ with $\tau(\operatorname{ex} \mathcal{S}(G)) = 1$, where $\operatorname{ex} \mathcal{S}(G)$ is the set of all extreme points in $\mathcal{S}(G)$, such that for any linear functional f on $\mathcal{S}(G)$,

$$f(\mu) = \int_{\nu \in \operatorname{ex} \mathcal{S}(G)} f(\nu)\ d\tau(\nu).$$

We write $\mu = \int_{\nu \in \operatorname{ex} \mathcal{S}(G)} \nu\ d\tau(\nu)$. (See [8] and [2].) If $G = K^0(Y, \psi)$, there is a bijection between $\mathcal{S}(K^0(Y, \psi))$ and the set of all ψ-invariant probability measures on Y, $\mathcal{M}_\psi(Y)$. In fact, define $\Phi : \mathcal{M}_\psi(Y) \to \mathcal{S}(K^0(Y, \psi))$ as

$$\Phi(\mu)[f] := \int f\, d\mu, \qquad f \in C(Y, \mathbb{Z}).$$

Then Φ is an affine isomorphism ([6, Thm. 5.4]).

Suppose (X, ϕ) and (Y, ψ) are topological dynamical systems. A continuous map $\pi : X \to Y$ is called a *factor map* if π is surjective and $\pi \circ \phi = \psi \circ \pi$. We say a factor map π is *almost one-to-one* if the set $\{x \in X \mid \#\pi^{-1}\pi(x) = 1\}$ is a residual set (i.e., the complement of a set of first category.). Then we call (Y, ψ) *an almost one-to-one factor of* (X, ϕ), or (X, ϕ) *an almost one-to-one extension of* (Y, ψ). In the case where (X, ϕ) is minimal, it suffices to verify the existence of x satisfying $\#\pi^{-1}\pi(x) = 1$.

Suppose that (X, ϕ) and (Y, ψ) are Cantor minimal systems and $\pi : X \to Y$ is a factor map. Define $\pi^* : K^0(Y, \psi) \to K^0(X, \phi)$ as

$$\pi^*[f] := [f \circ \pi].$$

By [5, Prop. 3.1], π^* is an *order embedding*, that is, an injective unital order homomorphism with the property that $[f] \in K^0(Y, \psi)^+$ if and only if $\pi^*[f] \in K^0(X, \phi)^+$.

Suppose that (X, ϕ) and (Y, ψ) are topological dynamical systems assuming that both transformation are homeomorphisms and $\pi : X \to Y$ is a factor map. Define $\tilde{\pi} : \mathcal{M}_\phi(X) \to \mathcal{M}_\psi(Y)$ as

$$\tilde{\pi}(\mu) := \mu \circ \pi^{-1}, \quad \mu \in \mathcal{M}_\phi(X).$$

Then $\tilde{\pi}$ is a surjective affine homomorphism ([1, Prop. 3.2 and 3.11]) and $\tilde{\pi}$ sends ϕ-invariant ergodic measures, $\operatorname{ex}\mathcal{M}_\phi(X)$ (extreme points of $\mathcal{M}_\phi(X)$), onto ψ-invariant ergodic measures, $\operatorname{ex}\mathcal{M}_\psi(Y)$. Indeed, suppose that E is a ψ-invariant Borel subset of Y and $\mu \in \operatorname{ex}\mathcal{M}_\phi(X)$. Since ϕ and ψ are homeomorphisms and

$$\pi \circ \phi \circ \pi^{-1}(E) = \psi \circ \pi \circ \pi^{-1}(E) = \psi(E) = E,$$
$$\pi \circ \phi^{-1} \circ \pi^{-1}(E) = \psi^{-1} \circ \pi \circ \pi^{-1}(E) = \psi^{-1}(E) = E,$$

we have $\phi \circ \pi^{-1}(E) \subset \pi^{-1}(E)$, $\phi^{-1} \circ \pi^{-1}(E) \subset \pi^{-1}(E)$ and hence $\phi \circ \pi^{-1}(E) = \pi^{-1}(E)$. Therefore $\pi^{-1}(E)$ is ϕ-invariant set and μ is ergodic, we have $\mu(\pi^{-1}(E)) = 0$ or 1. So $\tilde{\pi}(\mu)$ is also ergodic.

Suppose (Y, ψ) is a Cantor minimal system. It is not hard to show (Theorem 3.1) that if there are a Cantor minimal system (X, ϕ) and an almost one-to-one factor map $\pi : (X, \phi) \to (Y, \psi)$, then

(1) $\pi^* : K^0(Y, \psi) \to K^0(X, \phi)$ is an order embedding,
(2) the cokernel of π^*, $K^0(X, \phi)/\pi^*(K^0(Y, \psi))$, is torsion free,
(3) $\tilde{\pi} : \mathcal{M}_\phi(X) \to \mathcal{M}_\psi(Y)$ is a surjective affine homomorphism and $\tilde{\pi}(\operatorname{ex}\mathcal{M}_\phi(X)) = \operatorname{ex}\mathcal{M}_\psi(Y)$.

The condition (3) is equivalent to

(3') $\hat{\pi} : \mathcal{S}(K^0(X, \phi)) \to \mathcal{S}(K^0(Y, \psi))$ defined by $\hat{\pi}(\mu) := \mu \circ \pi^*$ is a surjective affine homomorphism and $\hat{\pi}(\operatorname{ex}\mathcal{S}(K^0(X, \phi))) = \operatorname{ex}\mathcal{S}(K^0(Y, \psi))$.

Then we have a problem of its converse, which is called *the dynamical realization problem of dimension groups*, as follows. Given a Cantor minimal system (Y, ψ) and a simple dimension group G satisfying that

(i) there is an order embedding $\iota : K^0(Y, \psi) \to G$,
(ii) $G/\iota(K^0(Y, \psi))$ is torsion free,
(iii) $\iota^* : \mathcal{S}(G) \to \mathcal{S}(K^0(Y, \psi))$ defined by $\iota^*(\mu) := \mu \circ \iota$ is a surjective affine homomorphism and $\iota^*(\operatorname{ex}\mathcal{S}(G)) = \operatorname{ex}\mathcal{S}(K^0(Y, \psi))$.

Then does there exist a Cantor minimal system (X, ϕ) such that the following statements hold?

(a) There is an almost one-to-one factor map $\pi : (X, \phi) \to (Y, \psi)$,
(b) there is a unital order isomorphism $\alpha : K^0(X, \phi) \to G$ such that $\alpha \circ \pi^* = \iota$ holds.

In [4], Giordano, Putnam and Skau showed the dynamical realization problem by assuming (i), (ii) above and the order dense condition. For an order embedding $\iota : H \to G$ of dimension groups, we say $\iota(H)$ is *order dense* in G if for any $g, g' \in G$ with $g < g'$, there is $h \in H$ such that $g < \iota(h) < g'$. It is known that $\iota(H)$ is order dense in G if and only if ι^* is injective ([4, Prop. 1.1]). So the order dense condition satisfies (iii) above.

We remark that by Proposition 2.3, ι is an order embedding if and only if ι^* is surjective. Moreover, if ι^* is affine and sends $\operatorname{ex}\mathcal{S}(G)$ onto $\operatorname{ex}\mathcal{S}(K^0(Y,\psi))$, then ι^* is surjective. So the conditions (i) and (iii) are equivalent to:

(iv) $\iota : K^0(Y,\psi) \to G$ is an injective unital order homomorphism and $\iota^* : \mathcal{S}(G) \to \mathcal{S}(K^0(Y,\psi))$ is an affine homomorphism so that $\iota^*(\operatorname{ex}\mathcal{S}(G)) = \operatorname{ex}\mathcal{S}(K^0(Y,\psi))$.

In this paper we will show the following.

Theorem 1.1. *In the situations* (ii) *and* (iv) *above, there exists a Cantor minimal system* (X,ϕ) *such that the conditions* (a) *and* (b) *above hold.*

We remark that the assumption (iv) (or (iii)) is important. In fact we construct $K^0(Y,\psi)$ and G satisfying the conditions (i) and (ii) but not (iii) (Example 5.1) and hence we cannot do the dynamical realization in this case.

By Theorem 1.1 and Theorem 3.1 it is easy to check the following corollaries:

Corollary 1.2. *Suppose that* (Y,ψ) *is a uniquely ergodic Cantor minimal system and* G *is a simple dimension group satisfying the assumptions* (i) *and* (ii) *above. Then there exists a Cantor minimal system* (X,ϕ) *such that the conditions* (a) *and* (b) *above hold.*

Corollary 1.3. *Suppose that* G *and* H *are acyclic simple dimension groups and* $\iota : H \to G$ *is an injective unital order homomorphism. Then the following statements are equivalent:*

(1) *There exist Cantor minimal systems* (X,ϕ) *and* (Y,ψ) *such that*
- *there is an almost one-to-one factor map* $\pi : (X,\phi) \to (Y,\psi)$,
- *there are unital order isomorphisms* $\alpha : K^0(X,\phi) \to G$ *and* $\beta : K^0(Y,\psi) \to H$ *such that* $\alpha \circ \pi^* = \iota \circ \beta$ *holds.*

$$\begin{array}{ccc} K^0(Y,\psi) & \overset{\pi^*}{\hookrightarrow} & K^0(X,\phi) \\ \beta \downarrow \cong & \circlearrowleft & \cong \downarrow \alpha \\ H & \underset{\iota}{\hookrightarrow} & G \end{array}$$

(2) $G/\iota(H)$ *is torsion free and* $\iota^* : \mathcal{S}(G) \to \mathcal{S}(H)$ *defined by* $\iota^*(\mu) := \mu \circ \iota$ *is an affine homomorphism and* $\iota^*(\operatorname{ex}\mathcal{S}(G)) = \operatorname{ex}\mathcal{S}(H)$.

Basically, we use notations and definitions in [6] and [3]. Here we will introduce some notations and definitions in this paper. Suppose $\mathcal{B} = (V,E,\geq)$ is a properly ordered (also called simply ordered) Bratteli diagram.

- Let $r : E \to V$ denote the range map and $s : E \to V$ denote the source map. Namely, $e \in E_n$ connects between $s(e) \in V_{n-1}$ and $r(e) \in V_n$.
- Let $M^{[n]} = [M^{[n]}_{uv}]$ denote the n-th incidence matrix of $\mathcal{B}$ (i.e., $M^{[n]}_{uv}$ is the number of edges connecting between $u \in V_n$ and $v \in V_{n-1}$). We also write $\mathcal{B} = (V,E,\{M^{[n]}\},\geq)$. Let $M^{[n]}_{*v} = (M^{[n]}_{uv})_{u\in V_n}$ denote the v's column vector of $M^{[n]}$ and $M^{[n]}_{u*} = (M^{[n]}_{uv})_{v\in V_{n-1}}$ denote the u's row

vector of $M^{[n]}$. For $n > k$, let $M^{[n,k)}$ (or $M^{[n,k+1]}$) denote the product of incidence matrices $M^{[n]}M^{[n-1]}\cdots M^{[k+1]}$.

- For a sequence $c_0 = 0 < c_1 < c_2 < c_3 < \cdots$ in $\mathbb{Z}_+$, we say that a Bratteli diagram $\mathcal{B}' = (V', E', \{M'^{[n]}\})$ is a *telescoping* (or *contraction*) of $\mathcal{B}$ to depths $\{c_n\}_{n=0}^{\infty}$, which we write $\mathcal{B}' = (\mathcal{B}, \{c_n\})$, if $V'_n := V_{c_n}$ and $M'^{[n]} := M^{[c_n, c_{n-1})}$. Especially, we define $\mathcal{B}_{\mathrm{odd}}$ as telescoping $\mathcal{B}$ to odd depths $(0, 1, 3, \cdots)$ and define $\mathcal{B}_{\mathrm{even}}$ as telescoping $\mathcal{B}$ to even depths $(0, 2, 4, \cdots)$.
- Let $(X_{\mathcal{B}}, S_{\mathcal{B}})$ denote the Bratteli-Vershik system of $\mathcal{B}$. Namely, $X_{\mathcal{B}}$ is the infinite path space of $\mathcal{B}$ and $S_{\mathcal{B}} : X_{\mathcal{B}} \to X_{\mathcal{B}}$ is the Vershik (lexicographic) map defined by the order $\geq$ on E. (See [6].)
- Define an equivalence relation $\sim$ on Bratteli diagrams as follows. $\mathcal{B} = (V, E) \sim (\tilde{V}, \tilde{E}) = \tilde{\mathcal{B}}$ if there exists a Bratteli diagram $\hat{\mathcal{B}} = (\hat{V}, \hat{E})$ such that $\hat{\mathcal{B}}_{\mathrm{odd}}$ yields a telescoping either $\mathcal{B}$ or $\tilde{\mathcal{B}}$, and $\hat{\mathcal{B}}_{\mathrm{even}}$ yields a telescoping of the other.

2. The state space $\mathcal{S}(K_0(\mathcal{B}))$

Recall notations of inductive limit of ordered groups. For a simple Bratteli diagram $\mathcal{B} = (V, E, \{M^{(n)}\})$, $K_0(\mathcal{B})$ is the inductive limit of a sequence

$$(2.1)\qquad \varinjlim(\mathbb{Z}^{V_{n-1}}, M^{[n]}) = \mathbb{Z}^{V_0} \xrightarrow{M^{[1]}} \mathbb{Z}^{V_1} \xrightarrow{M^{[2]}} \mathbb{Z}^{V_2} \xrightarrow{M^{[3]}} \cdots\cdots$$

with the distinguished order unit of $K_0(\mathcal{B})$ corresponding to $1 \in \mathbb{Z}^{V_0}$. For $\boldsymbol{g} \in \mathbb{Z}^{V_t}$, we write $[\boldsymbol{g}, t]_V \in K_0(\mathcal{B})$. We also write $[v, t]_V \in K_0(\mathcal{B})$ where we identify $v \in V_t$ with $v = (0, \ldots, 0, \overset{v}{1}, 0, \ldots, 0) \in \mathbb{Z}^{V_t}$. Suppose $\boldsymbol{g} \in \mathbb{Z}^{V_t}$ and $\boldsymbol{g}' \in \mathbb{Z}^{V_{t'}}$ with $[\boldsymbol{g}, t]_V = [\boldsymbol{g}', t']_V$. Then there is an $s > t, t'$ such that $M^{[s,t)}\boldsymbol{g} = M^{[s,t')}\boldsymbol{g}' =: \tilde{\boldsymbol{g}} \in \mathbb{Z}^{V_s}$ and hence $[\boldsymbol{g}, t]_V = [\boldsymbol{g}', t']_V = [\tilde{\boldsymbol{g}}, s]_V$. If G is a dimension group, then there exists $\mathcal{B} = (V, E, \{M^{(n)}\})$ such that $G \cong \varinjlim(\mathbb{Z}^{V_{n-1}}, M^{[n]})$.

For a simple Bratteli diagram $\mathcal{B}$, let us recall the definition of the state space $\mathcal{S}(K_0(\mathcal{B}))$. Let G be an ordered group with fixed order unit $u \in G^+$. We say that a homomorphism $\mu : G \to \mathbb{R}$ is a *state* if μ is positive (i.e., $\mu(g) \geq 0$ for $g \in G^+$) and $\mu(u) = 1$. Let $\mathcal{S}(G)$ denote the set of all states of G. In the case of $G = \mathbb{Z}^n$ and $u = (u_1, \ldots, u_n) \in (\mathbb{Z}^n)^+$, $\mu \in \mathcal{S}(\mathbb{Z}^n)$ is written by

$$\mu((z_1, \ldots, z_n)) = \sum_{i=1}^{n} \frac{s_i z_i}{u_i}, \quad \text{where } \sum_{i=1}^{n} s_i = 1 \text{ and } s_i \geq 0.$$

Therefore we may identify $\mathcal{S}(\mathbb{Z}^n)$ with the standard n-simplex Δ_n. I.e.,

$$\Delta_n = \left\{ (s_1, \ldots, s_n) \;\middle|\; \sum_{i=1}^{n} s_i = 1,\ s_i \geq 0 \right\}.$$

For $v \in V_n$, let p_v denote the number of paths between $v_0 (\in V_0)$ and v. Then $p_v = M^{[n,1]}_{v v_0}$ holds and $(p_v)_{v \in V_n} = M^{[n,1]}_{* v_0}$ is the distinguished order unit of $\mathbb{Z}^{V_n}$.

Let D_{V_n} denote the $V_n \times V_n$ diagonal matrix defined by $(D_{V_n})_{vv} := p_v$. Then by (2.1), the state space $\mathcal{S}(K_0(\mathcal{B}))$ is the inverse limit of a sequence

$$\varprojlim(\Delta_{V_{n-1}}, M^{[n]*}) = \Delta_{V_0} \xleftarrow{M^{[1]*}} \Delta_{V_1} \xleftarrow{M^{[2]*}} \Delta_{V_2} \xleftarrow{M^{[3]*}} \cdots\cdots,$$

where $M^{[n]*} := D_{V_n}^{-1} M^{[n]} D_{V_{n-1}}$ and we also identify $\mathcal{S}(K_0(\mathcal{B}))$ with

$$\Delta_{\mathcal{B}} := \left\{ (\boldsymbol{s}^{(i)}) \in \prod_{i=0}^{\infty} \Delta_{V_i} \;\middle|\; \boldsymbol{s}^{(i)} = \boldsymbol{s}^{(i+1)} M^{[i+1]*}, \quad i \geq 0 \right\}.$$

Let $\gamma^{[k]} : \Delta_{\mathcal{B}} \to \Delta_{V_k}$ denote a projection defined by $\gamma^{[k]}((\boldsymbol{s}^{(i)})) := \boldsymbol{s}^{(k)}$. Then $\gamma^{[k]}$ is an affine homomorphism. Define $\Delta_{V_k}(t) := \{\boldsymbol{s} M^{[t,k)*} \mid \boldsymbol{s} \in \Delta_{V_t}\}$. We have

- $\Delta_{V_k}(t) \supset \Delta_{V_k}(t+1)$ for any $t > k$,
- $\gamma^{[k]}(\Delta_{\mathcal{B}}) = \bigcap_{t>k} \Delta_{V_k}(t)$. (See Appendix, Proposition A.1.)

Let $\|\cdot\|_{V_n}$ denote the l_1-norm on $\mathbb{R}^{V_n}$, that is, for $\boldsymbol{s} = (s_v) \in \mathbb{R}^{V_n}$, $\|\boldsymbol{s}\|_{V_n} := \sum_{v \in V_n} |s_v|$. For a sequence $\{\boldsymbol{s}_n \in \Delta_{V_n}\}_{n=1}^{\infty}$, we write

$$\lim_{n\to\infty} \boldsymbol{s}_n = (\mu_k) =: \mu \quad \text{if } \lim_{n\to\infty} \|\boldsymbol{s}_n M^{[n,k)*} - \mu_k\|_{V_k} = 0 \text{ for any } k \in \mathbb{N}.$$

It is easy to see that $\mu \in \Delta_{\mathcal{B}}$. Now we identify $v \in V_n$ with an extreme point $v = (0, \ldots, 0, \overset{v}{1}, 0, \ldots, 0) \in \Delta_{V_n}$. For a sequence $\{v_n \in V_n\}_{n=1}^{\infty}$, we write

$$\lim_{n\to\infty} v_n = (\mu_k) \quad \text{if } \lim_{n\to\infty} \|M^{[n,k)*}_{v_n *} - \mu_k\|_{V_k} = 0 \text{ for any } k \in \mathbb{N}.$$

Proposition 2.1. *For any sequence* $\{\boldsymbol{x}_n \in \Delta_{V_n}\}_{n=1}^{\infty}$ ($\{v_n \in V_n\}_{n=1}^{\infty}$)*, there exist* $\{n_i\} \subset \mathbb{N}$ *and* $\mu \in \Delta_{\mathcal{B}}$ *such that* $\mu = \lim_{i\to\infty} \boldsymbol{x}_{n_i}$ ($\mu = \lim_{i\to\infty} v_{n_i}$) *holds.*

Proof. First we consider $\boldsymbol{x}_n M^{[n,1)*}$ for $n \in \mathbb{N}$. By the compactness of Δ_{V_1}, there exist $\mu_1 \in \Delta_{V_1}$ and $S_1 \subset \mathbb{N}$ with $\#S_1 = \infty$ such that

$$\lim_{S_1 \ni n \to\infty} \|\boldsymbol{x}_n M^{[n,1)*} - \mu_1\|_{V_1} = 0.$$

Take any $s_1 \in S_1$ and fix it. Next we consider $\boldsymbol{x}_n M^{[n,2)*}$ for $n \in S_1$. By the compactness of Δ_{V_2}, there exist $\mu_2 \in \Delta_{V_2}$ and $S_2 \subset S_1$ with $\#S_2 = \infty$ such that $s_1 \in S_2$ and

$$\lim_{S_2 \ni n \to\infty} \|\boldsymbol{x}_n M^{[n,2)*} - \mu_2\|_{V_2} = 0.$$

Take any $s_2 \in S_2$ with $s_2 > s_1$ and fix it. Repeating this argument we get $\{\mu_n \in \Delta_{V_n}\}_{n=1}^{\infty}$ and $S := \bigcap_{n \in \mathbb{N}} S_n$ with $\#S = \infty$ ($\because$ $S_n \supset \{s_i \mid 1 \leq i \leq n\}$, $S \supset \{s_n \mid n \in \mathbb{N}\}$) so that

$$\lim_{S \ni n \to\infty} \|\boldsymbol{x}_n M^{[n,k)*} - \mu_k\|_{V_k} = 0 \quad \text{for any } k \in \mathbb{N}.$$

Now we will show $(\mu_k) \in \Delta_{\mathcal{B}}$. It suffices to show that $\mu_k = \mu_{k+1} M^{[k]*}$ for all $k \in \mathbb{N}$.

$$\|\mu_k - \mu_{k+1} M^{[k]*}\|_{V_k} = \|(\mu_k - \boldsymbol{x}_n M^{[n,k)*}) + (\boldsymbol{x}_n M^{[n,k)*} - \mu_{k+1} M^{[k]*})\|_{V_k}$$

$$\leq \|\mu_k - \boldsymbol{x}_n M^{[n,k)*}\|_{V_k} + \|(\boldsymbol{x}_n M^{[n,k+1)*} - \mu_{k+1}) M^{[k]*}\|_{V_k}$$
$$\to 0 \quad \text{as } S \ni n \to \infty.$$

In the case of $\{v_n \in V_n\}_{n=1}^{\infty}$, we let $\boldsymbol{x}_n = v_n$ as an extreme point of Δ_{V_n}. □

Proposition 2.2. *Suppose G and H are dimension groups and $\iota : H \to G$ is a unital order group homomorphism. Then there exist Bratteli diagrams $\mathcal{B} = (V, E, \{M^{[n]}\})$, $\mathcal{C} = (W, F, \{N^{[n]}\})$ and $V_n \times W_n$ matrix $I^{[n]}$ such that $G \cong K_0(\mathcal{B})$ and $H \cong K_0(\mathcal{C})$ as unital order isomorphisms and the following diagrams commute:*

(2.2)
$$\begin{array}{ccccccccc}
K_0(\mathcal{C}) & = & \mathbb{Z}^{W_0} & \xrightarrow{N^{[1]}} & \mathbb{Z}^{W_1} & \xrightarrow{N^{[2]}} & \mathbb{Z}^{W_2} & \xrightarrow{N^{[3]}} & \cdots\cdots \\
\downarrow \iota & & \downarrow I^{[0]}=id. & & \downarrow I^{[1]} & & \downarrow I^{[2]} & & \\
K_0(\mathcal{B}) & = & \mathbb{Z}^{V_0} & \xrightarrow{M^{[1]}} & \mathbb{Z}^{V_1} & \xrightarrow{M^{[2]}} & \mathbb{Z}^{V_2} & \xrightarrow{M^{[3]}} & \cdots\cdots
\end{array}$$

Proof. Let $\mathcal{B} = (V, E, \{M^{[n]}\})$ and $\mathcal{C} = (W, F, \{N^{[n]}\})$ be Bratteli diagrams satisfying $G \cong K_0(\mathcal{B})$ and $H \cong K_0(\mathcal{C})$. In order to define $I^{[n]}$, we will modify $\mathcal{B}$ by telescoping to some suitable depths $\{t_n\}$. Let $I^{[0]} := [1]$. There exists $t_1 \geq 1$ such that for any $w \in W_1$, there is $\boldsymbol{g}_w \in \mathbb{Z}^{V_{t_1}}$ with $\iota([w, 1]_W) = [\boldsymbol{g}_w, t_1]_V$. We fix such t_1 and $\boldsymbol{g}_w$. Define $I^{[1]}_{*w} := \boldsymbol{g}_w$. Next there exists $t_2 > t_1$ such that for any $w' \in W_2$, there is $\boldsymbol{g}_{w'} \in \mathbb{Z}^{V_{t_2}}$ satisfying that

- $\iota([w', 2]_W) = [\boldsymbol{g}_{w'}, t_2]_V$,
- $M^{[t_2,t_1)} I^{[1]}_{*w} = \sum_{w' \in W_2} N^{[2]}_{w'w} \boldsymbol{g}_{w'}$.

Indeed, suppose $\iota([w', 2]_W) = [\boldsymbol{g}_{w'}, t_2]_V$ but $M^{[t_2,t_1)} I^{[1]}_{*w} \neq \sum_{w' \in W_2} N^{[2]}_{w'w} \boldsymbol{g}_{w'}$. Since

$$[M^{[t_2,t_1)} I^{[1]}_{*w}, t_2]_V = \iota([w, 1]_W) = \iota([N^{[2]}_{*w}, 2]_W) = \left[\sum_{w' \in W_2} N^{[2]}_{w'w} \boldsymbol{g}_{w'}, t_2 \right]_V,$$

there exists $t > t_2$ such that for any $w \in W_1$,

$$M^{[t,t_1)} I^{[1]}_{*w} = M^{[t,t_2)} \sum_{w' \in W_2} N^{[2]}_{w'w} \boldsymbol{g}_{w'}.$$

So resetting t_2 and $\boldsymbol{g}_{w'}$, we set $t_2 := t$ and $\boldsymbol{g}_{w'} := M^{[t,t_2)} \boldsymbol{g}_{w'}$. We fix such t_2 and $\boldsymbol{g}_{w'}$. Define $I^{[2]}_{*w'} := \boldsymbol{g}_{w'}$. Then for any $w \in W_1$,

$$M^{[t_2,t_1)} I^{[1]}_{*w} = I^{[2]} N^{[2]}_{*w} \quad \text{and hence} \quad M^{[t_2,t_1)} I^{[1]} = I^{[2]} N^{[2]}.$$

Repeating this process, we can define $I^{[n]}$ for all n. Telescoping $\mathcal{B}$ to $\{t_n\}$, we get the conclusion. □

In the situation above, ι induces an affine homomorphism $\iota^* : \Delta_{\mathcal{B}} \to \Delta_{\mathcal{C}}$ defined by $\iota^*(\mu) := \mu \circ \iota$. If $\mu = (\boldsymbol{s}^{(k)}) \in \Delta_{\mathcal{B}}$, we also write

$$\iota^*(\mu) = (\boldsymbol{s}^{(k)} I^{[k]*}) \quad \text{where} \quad I^{[k]*} := D^{-1}_{V_k} I^{[k]} D_{W_k}.$$

Then the following diagrams commute.

$$\begin{array}{ccccccccc}
\Delta_{\mathcal{C}} & = & \{1\} & \xleftarrow{N^{[1]*}} & \Delta_{W_1} & \xleftarrow{N^{[2]*}} & \Delta_{W_2} & \xleftarrow{N^{[3]*}} & \cdots\cdots \\
\uparrow \iota^* & & \uparrow I^{[0]*}=id. & & \uparrow I^{[1]*} & & \uparrow I^{[2]*} & & \\
\Delta_{\mathcal{B}} & = & \{1\} & \xleftarrow[M^{[1]*}]{} & \Delta_{V_1} & \xleftarrow[M^{[2]*}]{} & \Delta_{V_2} & \xleftarrow[M^{[3]*}]{} & \cdots\cdots
\end{array}$$

(Note that $\Delta_{V_0} = \Delta_{W_0} = \{1\}$.)

Proposition 2.3. *Suppose G and H are dimension groups and $\iota : H \to G$ is an injective unital order group homomorphism. Then following statements are equivalent:*

(1) *ι is an order embedding,*
(2) *ι^* is a surjection.*

Proof. **(1)⇒(2).** Let $\nu \in \mathcal{S}(H)$. For $x \in G^+$, define $\mu(x)$ as

$$\mu(x) := \begin{cases} \nu(y), & \text{if } x = \iota(y) \text{ for some } y \in H^+, \\ 0, & \text{if } x \in G^+ \setminus \iota(H^+). \end{cases}$$

Since ι is injective, μ is well-defined on G^+. Since for any $x \in G$, we can write $x = y - z$ for some $y, z \in G^+$ and hence we define $\mu(x) := \mu(y) - \mu(z)$. It is easy to check that μ is well-defined on G and $\mu \in \mathcal{S}(G)$. So $\nu = \mu \circ \iota = \iota^*(\mu)$ holds.

(2)⇒(1). Since ι is an order homomorphism, $\iota(H^+) \subset G^+$ holds. So we will show $\iota^{-1}(G^+) \subset H^+$. Let $y \in \iota^{-1}(G^+) \setminus \{0\}$. Then there is a $x \in G^+$ such that $\iota(y) = x \neq 0$. Let $\nu \in \mathcal{S}(H)$. Then there is a $\mu \in \mathcal{S}(G)$ such that $\nu = \iota^*(\mu)$. In [2, Cor. 4.2], we see that

$$G^+ = \{z \in G \mid \mu(z) > 0 \text{ for any } \mu \in \mathcal{S}(G)\} \cup \{0\}.$$

Then $\mu(x) > 0$ and we have

$$0 < \mu(x) = \mu \circ \iota(y) = \iota^*(\mu)(y) = \nu(y).$$

Since ι^* is surjective, it is shown that $\nu(y) > 0$ for any $\nu \in \mathcal{S}(H)$. This implies $y \in H^+$. □

For $k \in \mathbb{N}$, $\boldsymbol{n} \in \mathbb{R}_+^{W_k} := \{(n_w) \in \mathbb{R}^{W_k} \mid n_w \geq 0\}$ and $\boldsymbol{n}^* \in \Delta_{W_k}$, we consider the linear equations for $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\boldsymbol{x} I^{[k]} = \boldsymbol{n}, \qquad \boldsymbol{y} I^{[k]*} = \boldsymbol{n}^*.$$

We set $S(\boldsymbol{n}) := \{\boldsymbol{x} \in \mathbb{R}^{V_k} \mid \boldsymbol{x} I^{[k]} = \boldsymbol{n}\}$, the set of solutions of $\boldsymbol{x} I^{[k]} = \boldsymbol{n}$, and $S^*(\boldsymbol{n}^*) := \{\boldsymbol{y} \in \mathbb{R}^{V_k} \mid \boldsymbol{y} I^{[k]*} = \boldsymbol{n}^*\}$, the set of solutions of $\boldsymbol{x}^* I^{[k]*} = \boldsymbol{n}^*$. Define maps $^* : \mathbb{R}_+^{V_k} \to \mathbb{R}_+^{V_k}$ and $^* : \mathbb{R}_+^{W_k} \to \mathbb{R}_+^{W_k}$ as

$$\mathbb{R}_+^{V_k} \ni \boldsymbol{x} \longmapsto \boldsymbol{x}^* := \frac{\boldsymbol{x} D_{V_k}}{\boldsymbol{x} M^{[k,1]}}, \qquad \mathbb{R}_+^{W_k} \ni \boldsymbol{y} \longmapsto \boldsymbol{y}^* := \frac{\boldsymbol{y} D_{W_k}}{\boldsymbol{y} N^{[k,1]}}.$$

Remark 2.4. We have the following properties with respect to *:
(1) $(\mathbb{R}_+^{V_k})^* = \Delta_{V_k}$ and $(\mathbb{R}_+^{W_k})^* = \Delta_{W_k}$,

(2) for $\boldsymbol{x} \in S(\boldsymbol{n})$, $\boldsymbol{x}^* = \boldsymbol{x}D_{V_k}/\boldsymbol{n}N^{[k,1]} \in S^*(\boldsymbol{n}^*)$,
(3) for a fixed $\boldsymbol{n}$, the restriction * to $S(\boldsymbol{n})$, $*|_{S(\boldsymbol{n})} : S(\boldsymbol{n}) \to S^*(\boldsymbol{n}^*)$ is bijective,
(4) the preimage of * to $S^*(\boldsymbol{n}^*)$, $(S^*(\boldsymbol{n}^*))^{*^{-1}}$, is

$$(S^*(\boldsymbol{n}^*))^{*^{-1}} = \bigcup_{r \in \mathbb{R}_+} S(r\boldsymbol{n}).$$

Proof. (1) We will show $(\mathbb{R}_+^{V_k})^* = \Delta_{V_k}$. Clearly $\boldsymbol{x}^* = (x_v^*) \in \mathbb{R}_+^{V_k}$ and we see that

$$\sum_{v \in V_k} x_v^* = \sum_{v \in V_k} \frac{(\boldsymbol{x}D_{V_k})_v}{\boldsymbol{x}M^{[k,1]}} = \sum_{v \in V_k} \frac{x_v p_v}{\boldsymbol{x}M^{[k,1]}} = \frac{\boldsymbol{x}M^{[k,1]}}{\boldsymbol{x}M^{[k,1]}} = 1.$$

So $\boldsymbol{x}^* \in \Delta_{V_k}$. If $\boldsymbol{x}' \in \Delta_{V_k}$, then $\boldsymbol{x}'D_{V_k}^{-1} \in \mathbb{R}_+^{V_k}$ and

$$(\boldsymbol{x}'D_{V_k}^{-1})^* = \frac{\boldsymbol{x}'}{\boldsymbol{x}'D_{V_k}^{-1}M^{[k,1]}} = \frac{\boldsymbol{x}'}{\sum_{v \in V_k} {x'}_v} = \boldsymbol{x}'.$$

So the map * is surjective.

(2) Since $M^{[k,1]} = I^{[k]}N^{[k,1]}$ and $\boldsymbol{x}I^{[k]} = \boldsymbol{n}$, we have

$$\boldsymbol{x}^* = \frac{\boldsymbol{x}D_{V_k}}{\boldsymbol{x}M^{[k,1]}} = \frac{\boldsymbol{x}D_{V_k}}{\boldsymbol{x}I^{[k]}N^{[k,1]}} = \frac{\boldsymbol{x}D_{V_k}}{\boldsymbol{n}N^{[k,1]}}.$$

Moreover,

$$\begin{aligned} \boldsymbol{x} \in S(\boldsymbol{n}) \;&\Leftrightarrow\; \boldsymbol{x}I^{[k]} = \boldsymbol{n} \\ &\Leftrightarrow\; (\boldsymbol{x}D_{V_k})D_{V_k}^{-1}I^{[k]}D_{W_k} = \boldsymbol{n}D_{W_k} \;\Leftrightarrow\; (\boldsymbol{x}D_{V_k})I^{[k]*} = \boldsymbol{n}D_{W_k} \\ &\Leftrightarrow\; \frac{\boldsymbol{x}D_{V_k}}{\boldsymbol{n}N^{[k,1]}}I^{[k]*} = \frac{\boldsymbol{n}D_{W_k}}{\boldsymbol{n}N^{[k,1]}} \;\Rightarrow\; \boldsymbol{x}^*I^{[k]*} = \boldsymbol{n}^* \;\Leftrightarrow\; \boldsymbol{x}^* \in S^*(\boldsymbol{n}^*). \end{aligned}$$

(3) Clearly $*|_{S(\boldsymbol{n})}$ is injective. For $\boldsymbol{x}' \in S^*(\boldsymbol{n}^*)$, it is easy to see that $(\boldsymbol{n}N^{[k,1]})\boldsymbol{x}'D_{V_k}^{-1} \in S(\boldsymbol{n})$ and $((\boldsymbol{n}N^{[k,1]})\boldsymbol{x}'D_{V_k}^{-1})^* = \boldsymbol{x}'$. So $*|_{S(\boldsymbol{n})}$ is surjective.

(4) Trivial. □

Let $\Delta_{V_k}^+ := \{\boldsymbol{x} = (x_v) \in \Delta_{V_k} \mid x_v > 0\ \forall v \in V_k\}$, $S_{\mathbb{R}_+}(\boldsymbol{n}) := S(\boldsymbol{n}) \cap \mathbb{R}_+^{V_k}$, $S_{\mathbb{Z}}(\boldsymbol{n}) := S(\boldsymbol{n}) \cap \mathbb{Z}^{V_k}$, $S_{\mathbb{N}}(\boldsymbol{n}) := S(\boldsymbol{n}) \cap \mathbb{N}^{V_k}$, $S_\Delta^*(\boldsymbol{n}^*) := S^*(\boldsymbol{n}^*) \cap \Delta_{V_k}$ and $S_{\Delta+}^*(\boldsymbol{n}^*) := S^*(\boldsymbol{n}^*) \cap \Delta_{V_k}^+$. For $\varepsilon > 0$ and $\boldsymbol{x} \in \mathbb{R}^{V_k}$, let $B_\varepsilon(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}^{V_k} \mid \|\boldsymbol{x} - \boldsymbol{y}\|_{V_k} < \varepsilon\}$ denote the open ball with center at $\boldsymbol{x}$ and radius ε.

Proposition 2.5. *Suppose $S_{\mathbb{Z}}(\boldsymbol{n}) \neq \varnothing$. Then there exists $\tau > 0$ such that for any $\boldsymbol{m}^* \in S^*(\boldsymbol{n}^*)$, there is $\boldsymbol{x}^* \in S^*(\boldsymbol{n}^*) \cap B_{p\tau/\boldsymbol{n}N^{[k,1]}}(\boldsymbol{m}^*)$ such that*

$$\boldsymbol{x} := (\boldsymbol{n}N^{[k,1]})\boldsymbol{x}^*D_{V_k}^{-1} \in S_{\mathbb{Z}}(\boldsymbol{n}) \cap B_\tau(\boldsymbol{m}),$$

where $p = \max_{v \in V_k} p_v$. Moreover if $S^*(\boldsymbol{n}^*) \cap B_{p\tau/\boldsymbol{n}N^{[k,1]}}(\boldsymbol{m}^*) \subset S_{\Delta+}^*(\boldsymbol{n}^*)$, then

$$\boldsymbol{x} \in S_{\mathbb{N}}(\boldsymbol{n}) \cap B_\tau(\boldsymbol{m}).$$

Proof. First we will show that there exists $\tau > 0$ such that for any $\boldsymbol{m} \in S(\boldsymbol{n})$, $S_{\mathbb{Z}}(\boldsymbol{n}) \cap B_\tau(\boldsymbol{m}) \neq \varnothing$. Let $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_r\}$ be a basis of $\mathrm{Ker}(I^{[k]})$. Since all entries of $I^{[k]}$ consist of positive integers, we may assume that $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_r \in \mathbb{Z}^{V_k}$. Let $\tau := \sum_{i=1}^r \|\boldsymbol{\xi}_i\|_{V_k}$. For any $\boldsymbol{a} \in S_{\mathbb{Z}}(\boldsymbol{n})$, there exist $a_i \in \mathbb{R}$ and $n_i \in \mathbb{Z}$ such that $\boldsymbol{m} = \boldsymbol{a} + \sum_{i=1}^r a_i \boldsymbol{\xi}_i$ and $n_i \leq a_i < n_i + 1$. Let $\boldsymbol{a}' := \boldsymbol{a} + \sum_{i=1}^r n_i \boldsymbol{\xi}_i$. Then $\boldsymbol{a}' \in S_{\mathbb{Z}}(\boldsymbol{n})$ and

$$\|\boldsymbol{m} - \boldsymbol{a}'\|_{V_k} = \left\| \sum_{i=1}^r (a_i - n_i) \boldsymbol{\xi}_i \right\|_{V_k} \leq \sum_{i=1}^r (a_i - n_i) \|\boldsymbol{\xi}_i\|_{V_k} < \tau.$$

Therefore $\boldsymbol{a}' \in S_{\mathbb{Z}}(\boldsymbol{n}) \cap B_\tau(\boldsymbol{m}) \neq \varnothing$.

We take $\boldsymbol{x} \in S_{\mathbb{Z}}(\boldsymbol{n}) \cap B_\tau(\boldsymbol{m})$ and fix it. Since $\|\boldsymbol{m}^* - \boldsymbol{x}^*\|_{V_k} \leq \frac{p}{\boldsymbol{n} N^{[k,1]}} \|\boldsymbol{m} - \boldsymbol{x}\|_{V_k}$, $\boldsymbol{x}^* = (\boldsymbol{n} N^{[k,1]})^{-1} \boldsymbol{x} D_{V_k} \in S^*(\boldsymbol{n}^*) \cap B_{p\tau/\boldsymbol{n} N^{[k,1]}}(\boldsymbol{m}^*)$. Moreover if $\boldsymbol{x}^* \in S^*_{\Delta^+}(\boldsymbol{n}^*)$, then $\boldsymbol{x} \in S_{\mathbb{R}_{++}}(\boldsymbol{n}) := \{(z_v) \in S(\boldsymbol{n}) \mid z_v > 0, \forall v \in V_k\}$. So we are done. □

3. Extensions of Cantor minimal systems and Bratteli diagrams

Theorem 3.1. *Suppose that (X, ϕ) and (Y, ψ) are Cantor minimal systems and $\pi : X \to Y$ is an almost one-to-one factor map. Then*

1. *$\pi^* : K^0(Y, \psi) \to K^0(X, \phi)$ defined by $\pi^*[f] := [f \circ \pi]$ is an order embedding,*
2. *$K^0(X, \phi)/\pi^*(K^0(Y, \psi))$ is torsion free,*
3. *$\tilde{\pi} : \mathcal{M}_\phi(X) \to \mathcal{M}_\psi(Y)$ defined by $\tilde{\pi}(\mu) := \mu \circ \pi^{-1}$ is the surjective affine homomorphism and $\tilde{\pi}(\mathrm{ex}\, \mathcal{M}_\phi(X)) = \mathrm{ex}\, \mathcal{M}_\psi(Y)$.*

Proof. We only show the statement (2). First we will construct a properly ordered Bratteli diagram for (Y, ψ) (See [6, Thm. 4.2]). Let $x_0 \in X$ satisfy $\# \pi^{-1} \circ \pi(x_0) = 1$ and $y_0 := \pi(x_0)$. Let $\mathcal{Q}_n := \{Y_n(i, h) \mid 1 \leq i \leq I_n,\ 1 \leq h \leq H_n(i)\}$ denote the n-th Kakutani-Rohlin (KR) clopen partition for (Y, ψ) and $\{\mathcal{Q}_n\}$ denote a sequence of KR partitions for (Y, ψ) satisfying the following conditions:

- $\mathcal{Q}_0 = \{Y = Y_0(1, 1)\}$,
- for $1 \leq h < H_n(i)$, $\psi Y_n(i, h) = Y_n(i, h+1)$,
- $\{y_0\} = \displaystyle\bigcap_{n \in \mathbb{Z}_+} \bigcup_{i=1}^{I_n} Y_n(i, 1)$,
- $\mathcal{Q}_n$ is finer than $\mathcal{Q}_{n-1}$ for all n (i.e., for any $Q \in \mathcal{Q}_n$, $Q \subset Q'$ for some $Q' \in \mathcal{Q}_{n-1}$),
- $\{\mathcal{Q}_n\}$ generates the topology of Y.

Let $\mathcal{C} = (W, F, \geq)$ denote the properly ordered Bratteli diagram arising from $\{\mathcal{Q}_n\}$. That is,

- $W_n := \{i_n \mid 1 \leq i \leq I_n\}$,
- $F_n := \{(i, i', h)_n \mid Y_n(i', h) \subset Y_{n-1}(i, 1)\}$,
- for $(i, i', h)_n \in F_n$, $s(i, i', h)_n := i_{n-1}$ and $r(i, i', h)_n := i'_n$,
- $(i, i', g)_n < (k, k', h)_n$ if and only if $i' = k'$ and $g < h$.

Then y_0 ($\psi^{-1}(y_0)$, resp.) corresponds to the unique minimal (maximal, resp.) path in $X_{\mathcal{C}}$.

Next we will construct a properly ordered Bratteli diagram for (X, ϕ). Let $\{\mathcal{P}'_n\}$ denote a sequence of clopen partitions for X which generates the topology of X. Let $\mathcal{P}_n = \{X_n(i^{(j)}, h) \mid 1 \le i \le I_n,\ 1 \le j \le J_n(i),\ 1 \le h \le H_n(i)\}$ denote the n-th KR clopen partition for (X, ϕ) associated with $\mathcal{Q}_n$ and $\{\mathcal{P}_n\}$ denote a sequence of KR partitions for (X, ϕ) such that

- $\mathcal{P}_0 = \{X = X_0(1^{(1)}, 1)\}$,
- $\bigcup_{j=1}^{J_n(i)} X_n(i^{(j)}, h) = \pi^{-1} Y_n(i, h)$,
- for $1 \le h < H_n(i)$, $\phi X_n(i^{(j)}, h) = X_n(i^{(j)}, h + 1)$,
- $\mathcal{P}_n$ is finer than $\mathcal{P}'_n$ and $\mathcal{P}_{n-1}$ for all $n \in \mathbb{N}$.

Then we see that

- $\{x_0\} = \bigcap_{n \in \mathbb{Z}_+} \bigcup_{i=1}^{I_n} \bigcup_{j=1}^{J_n(i)} X_n(i^{(j)}, 1)$,
- $\{\mathcal{P}_n\}$ generates the topology of X.

Let $\mathcal{B} = (V, E, \ge)$ denote the properly ordered Bratteli diagram arising from $\{\mathcal{P}_n\}$. That is,

- $V_n := \{i_n^{(j)} \mid 1 \le i \le I_n,\ 1 \le j \le J_n(i)\}$,
- $E_n := \{(i^{(j)}, i'^{(j')}, h)_n \mid X_n(i'^{(j')}, h) \subset X_{n-1}(i^{(j)}, 1)\}$,
- for $(i^{(j)}, i'^{(j')}, h)_n \in E_n$, $s(i^{(j)}, i'^{(j')}, h)_n := i_{n-1}^{(j)}$ and $r(i^{(j)}, i'^{(j')}, h)_n := i'^{(j')}_n$,
- $(i^{(j)}, i'^{(j')}, g)_n < (k^{(l)}, k'^{(l')}, h)_n$ if and only if $i'^{(j')} = k'^{(l')}$ and $g < h$.

Then x_0 ($\phi^{-1}(x_0)$, resp.) corresponds to the unique minimal (maximal, resp.) path in $X_{\mathcal{B}}$.

We identify $\pi^* : K^0(Y, \psi) \to K^0(X, \phi)$ with $\pi^* : K_0(\mathcal{C}) \to K_0(\mathcal{B})$. That is,

$$\begin{aligned} \pi^* : \mathbb{Z}^{W_n} \ni i_n = (0, \ldots, 0, \overset{i_n}{1}, 0, \ldots, 0) \\ \mapsto \sum_{j=1}^{J_n(i)} i_n^{(j)} = \sum_{j=1}^{J_n(i)} (0, \ldots, 0, \overset{i_n^{(j)}}{1}, 0, \ldots, 0) \in \mathbb{Z}^{V_n}. \end{aligned}$$

$$\left(\pi^*([\boldsymbol{z}, n]_W) := \left[\sum_{i=1}^{I_n} \sum_{j=1}^{J_n(i)} z_{i_n} i_n^{(j)}, n \right]_V, \quad \text{where } \boldsymbol{z} = (z_{i_n}) \in \mathbb{Z}^{W_n}. \right)$$

In order to show that $K_0(\mathcal{B})/\pi^*(K_0(\mathcal{C}))$ is torsion free, it suffices to show that $\pi^*(\mathbb{Z}^{W_n})$ is a pure subgroup of $\mathbb{Z}^{V_n}$. Suppose that $\boldsymbol{z} = (z_{i_n^{(j)}}) \in \mathbb{Z}^{V_n}$ and $k\boldsymbol{z} \in \pi^*(\mathbb{Z}^{W_n})$ for some $k \in \mathbb{N}$. $k\boldsymbol{z} \in \pi^*(\mathbb{Z}^{W_n})$ implies that $kz_{i_n^{(j)}} = kz_{i'^{(j')}_n}$ if $i = i'$. So $\boldsymbol{z} \in \pi^*(\mathbb{Z}^{W_n})$ holds and hence $\pi^*(\mathbb{Z}^{W_n})$ is pure in $\mathbb{Z}^{V_n}$. So we finish the proof. □

Remark 3.2. Let $M^{[n]}$ ($N^{[n]}$, resp.) denote the n-th incidence matrix of $\mathcal{B}$ ($\mathcal{C}$, resp.) in Theorem 3.1. Then we see that

$$\sum_{l=1}^{J_{n-1}(k)} M^{[n]}_{i_n^{(j)} k_{n-1}^{(l)}} = N^{[n]}_{i_n k_{n-1}}. \tag{3.1}$$

π induces the surjection $\varphi : V \to W$ defined by $\varphi(i_n^{(j)}) := i_n$ for all n. Then we have

- $\varphi(V_n) = W_n$ for all n,
- for $v \in V_n$ and $w \in W_{n-1}$,

$$\sum_{u \in \varphi^{-1}(w)} M^{[n]}_{vu} = N^{[n]}_{\varphi(v)w}. \tag{3.2}$$

Note that (3.1) and (3.2) are equivalent.

Proposition 3.3. *Suppose that $\mathcal{C} = (W, F, \{N^{[n]}\}, \geq)$ is a properly ordered Bratteli diagram and $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \{\tilde{M}^{[n]}\})$ is a simple Bratteli diagram. Suppose $\varphi : \tilde{V} \to W$ is a surjection satisfying that*

(1) *$\varphi(\tilde{V}_n) = W_n$ for all n,*
(2) *for $v \in \tilde{V}_n$ and $w \in W_{n-1}$, $\sum_{u \in \varphi^{-1}(w)} \tilde{M}^{[n]}_{vu} = N^{[n]}_{\varphi(v)w}$.*

Then there exists a proper order $\geq$ on $\tilde{E}$ such that $(X_{\tilde{\mathcal{B}}}, S_{\tilde{\mathcal{B}}})$ is an almost one-to-one extension of $(X_{\mathcal{C}}, S_{\mathcal{C}})$, where $(X_{\tilde{\mathcal{B}}}, S_{\tilde{\mathcal{B}}})$ and $(X_{\mathcal{C}}, S_{\mathcal{C}})$ are the Bratteli-Vershik systems associated with $\tilde{\mathcal{B}}$ and $\mathcal{C}$ respectively.

Proof. We construct an almost one-to-one factor map $\pi : X_{\tilde{\mathcal{B}}} \to X_{\mathcal{C}}$. Since $\tilde{\mathcal{B}}$ and $\mathcal{C}$ are simple diagrams, we may assume that $\tilde{M}^{[n]}$ and $N^{[n]}$ are positive matrices (i.e., all entries of them are positive) for all n. Let $F_n^{\min}, F_n^{\max} \subset F_n$ denote the set of minimal, maximal edges in F_n respectively. For $f \in F_n^{\min}$ and $f' \in F_n^{\max}$, let $v_f^{\min}, v_{f'}^{\max} \in \tilde{V}_{n-1}$ satisfy $\varphi(v_f^{\min}) = s(f)$ and $\varphi(v_{f'}^{\max}) = s(f')$ and fix them. The condition (2) implies that for $v \in \tilde{V}_n$ there is a bijection between $r^{-1}(v) = \{e \in \tilde{E}_n \mid r(e) = v\}$ and $r^{-1}(\varphi(v))$. So we can define surjections $\pi_n : \tilde{E}_n \to F_n$ for all n and a partial order $\geq$ on $\tilde{E}_n$ as

(i) for $v \in \tilde{V}_n$, $\pi_n \circ r^{-1}(v) = r^{-1}(\varphi(v))$,
(ii) for $e \in \tilde{E}_n$, $s \circ \pi_n(e) = \varphi \circ s(e)$,
(iii) for any $f \in F_n^{\min}$ and $e \in \pi_n^{-1}(f)$, $s(e) = v_f^{\min}$,
(iv) for any $f \in F_n^{\max}$ and $e \in \pi_n^{-1}(f)$, $s(e) = v_f^{\max}$,
(v) $e, e' \in \tilde{E}_n$ and $e < e'$ if and only if $r(e) = r(e')$ and $\pi_n(e) < \pi_n(e')$.

Define $\pi : X_{\tilde{\mathcal{B}}} \to X_{\mathcal{C}}$ as

$$\pi(p) := (\pi_n(p_n)), \qquad \text{where } p = (p_n).$$

By the conditions (i) and (ii), π is well-defined. Now we will show that $\#\pi^{-1}(q^{\min}) = \#\pi^{-1}(q^{\max}) = 1$, where $q^{\min}, q^{\max} \in X_{\mathcal{C}}$ is a unique minimal, maximal path respectively. Let $p = (p_n), p' = (p'_n) \in X_{\tilde{\mathcal{B}}}$ satisfy $\pi(p) = \pi(p') = q^{\min}$. By the condition (v), p and p' are minimal paths in $X_{\tilde{\mathcal{B}}}$. Let $n \in \mathbb{N}$ be fixed. By $\pi_{n+1}(p_{n+1}) = \pi_{n+1}(p'_{n+1}) = q^{\min}_{n+1}$ and the conditions (ii) and (iii), $s(p_{n+1}) = s(p'_{n+1}) = v^{\min}_{q^{\min}_{n+1}}$. This implies that $r(p_n) = r(p'_n) = v^{\min}_{q^{\min}_{n+1}}$ and hence $p_n = p'_n$. So $p = p'$ holds. Therefore $\#\pi^{-1}(q^{\min}) = 1$. Similarly we can show $\#\pi^{-1}(q^{\max}) = 1$. This proof also means that $\tilde{\mathcal{B}}$ has a proper-order (i.e., there exist unique minimal, maximal path in $X_{\tilde{\mathcal{B}}}$).

Finally, it is easy to see that π is a continuous surjection and $\pi \circ S_{\tilde{\mathcal{B}}} = S_{\mathcal{C}} \circ \pi$. By $\#\pi^{-1}(q^{\min}) = 1$, π is an almost one-to-one factor map. $\square$

Remark 3.4. φ induces group homomorphisms (matrices) $\tilde{I}^{[n]} : \mathbb{Z}^{W_n} \to \mathbb{Z}^{\tilde{V}_n}$ by

$$\tilde{I}^{[n]}_{vw} := \begin{cases} 1, & \text{if } \varphi(\tilde{v}) = w, \\ 0, & \text{if } \varphi(\tilde{v}) \neq w. \end{cases}$$

Then $\tilde{I}^{[n]} N^{[n]} = \tilde{M}^{[n]} \tilde{I}^{[n-1]}$ holds.

4. Proof of Theorem 1.1

4.1. **Requirements of a simple Bratteli diagram for (X, ϕ).** Suppose that $\mathcal{C} = (W, F, \{N^{[n]}\}, \geq)$ is a properly ordered Bratteli diagram arising from (Y, ψ), $\mathcal{B} = (V, E, \{M^{[n]}\})$ is a simple Bratteli diagram arising from G, $\iota : K_0(\mathcal{C}) \to G$ is an injective unital order group homomorphism. The assumption (iv) of Theorem 1.1 implies that $\iota^* : \Delta_{\mathcal{B}} \to \Delta_{\mathcal{C}}$ defined by $\iota^*(\mu) := \mu \circ \iota$ is surjective. So by Proposition 2.3, ι is an order embedding. By Proposition 2.2, the following diagrams commute:

$$\begin{array}{ccccccccc}
K_0(\mathcal{C}) & = & \mathbb{Z}^{W_0} & \xrightarrow{N^{[1]}} & \mathbb{Z}^{W_1} & \xrightarrow{N^{[2]}} & \mathbb{Z}^{W_2} & \xrightarrow{N^{[3]}} & \cdots\cdots \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle I^{[0]}=id.} & & \downarrow{\scriptstyle I^{[1]}} & & \downarrow{\scriptstyle I^{[2]}} & & \\
G = K_0(\mathcal{B}) & = & \mathbb{Z}^{V_0} & \xrightarrow{M^{[1]}} & \mathbb{Z}^{V_1} & \xrightarrow{M^{[2]}} & \mathbb{Z}^{V_2} & \xrightarrow{M^{[3]}} & \cdots\cdots
\end{array}$$

Telescoping diagrams, we may assume that for all n, $M^{[n]}$, $N^{[n]}$ and $I^{[n]}$ are positive matrices. Also, the following diagrams commute:

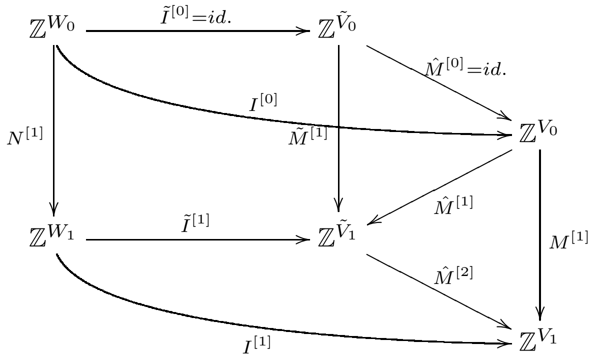
$$\begin{array}{ccccccccc}
\Delta_{\mathcal{C}} & = & \{1\} & \xleftarrow{N^{[1]*}} & \Delta_{W_1} & \xleftarrow{N^{[2]*}} & \Delta_{W_2} & \xleftarrow{N^{[3]*}} & \cdots\cdots \\
\uparrow{\scriptstyle \iota^*} & & \uparrow{\scriptstyle I^{[0]*}=id.} & & \uparrow{\scriptstyle I^{[1]*}} & & \uparrow{\scriptstyle I^{[2]*}} & & \\
\Delta_{\mathcal{B}} & = & \{1\} & \xleftarrow[M^{[1]*}]{} & \Delta_{V_1} & \xleftarrow[M^{[2]*}]{} & \Delta_{V_2} & \xleftarrow[M^{[3]*}]{} & \cdots\cdots
\end{array}$$

where

$$M^{[n]*} := D_{V_n}^{-1} M^{[n]} D_{V_{n-1}}, \quad N^{[n]*} := D_{W_n}^{-1} N^{[n]} D_{W_{n-1}}, \quad I^{[n]*} := D_{V_n}^{-1} I^{[n]} D_{W_n}.$$

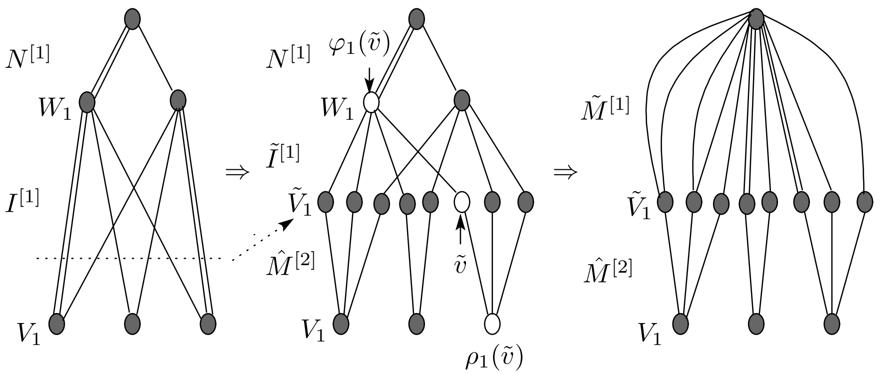
Also $M^{[n]*}$, $N^{[n]*}$ and $I^{[n]*}$ are positive matrices. Now we will construct a simple Bratteli diagram $\tilde{\mathcal{B}} = (\tilde{V}, \tilde{E}, \{\tilde{M}^{[n]}\})$ by an induction so that $\tilde{\mathcal{B}}$ satisfies the assumptions of Proposition 3.3. In order to construct $\tilde{\mathcal{B}}$, we need to telescope $\mathcal{B}$ and $\mathcal{C}$ to some suitable depths $\{t_n\}$.

The first step. Let $V_0 := \{v_0\}$, $W_0 := \{w_0\}$, $\tilde{V}_0 := \{\tilde{v}_0\}$ and $\varphi_0(\tilde{v}_0) := w_0$. We will construct the following commutative diagrams:



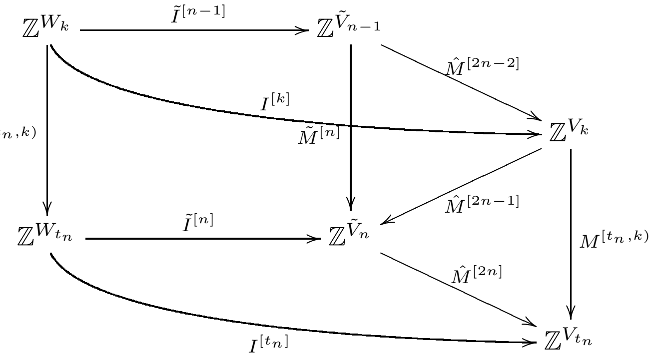
Define t_1, $\tilde{V}_1$, projections $\varphi_1 : \tilde{V}_1 \to W_1$ and $\rho_1 : \tilde{V}_1 \to V_1$, $\tilde{I}^{[1]}$, $\hat{M}^{[2]}$, $\hat{M}^{[1]}$ and $\tilde{M}^{[1]}$ as follows:

- $t_1 := 1$,
- $\tilde{V}_1 := \{(v, w, i) \mid v \in V_1,\ w \in W_1,\ 1 \le i \le I^{[1]}_{vw}\}$,
- $\varphi_1(v, w, i) := w$, $\rho_1(v, w, i) := v$
- $\tilde{I}^{[1]}_{\tilde{v}w} := \begin{cases} 1, & \text{if } \varphi_1(\tilde{v}) = w \\ 0, & \text{if } \varphi_1(\tilde{v}) \neq w \end{cases}$, $\quad \hat{M}^{[2]}_{v\tilde{v}} := \begin{cases} 1, & \text{if } \rho_1(\tilde{v}) = v \\ 0, & \text{if } \rho_1(\tilde{v}) \neq v \end{cases}$,
- $\hat{M}^{[1]}_{\tilde{v}v_0} := M^{[1]}_{\varphi_1(\tilde{v})v_0}$ for all $\tilde{v} \in \tilde{V}_1$,
- $\tilde{M}^{[1]}_{\tilde{v}\tilde{v}_0} := \hat{M}^{[1]}$.



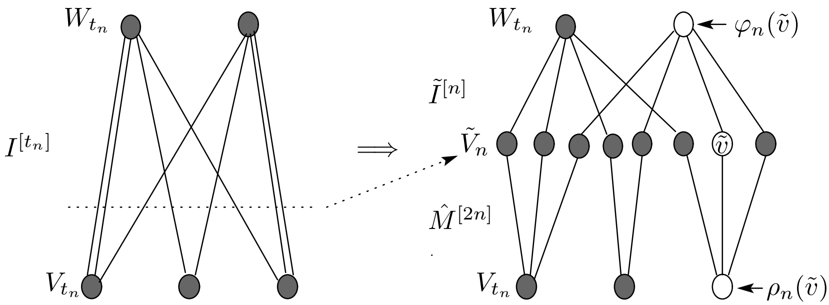
The *n*-th step. For $n \geq 2$, suppose $t_{n-1} \in \mathbb{N}$, $\tilde{V}_{n-1}$, $\varphi_{n-1} : \tilde{V}_{n-1} \to W_{t_{n-1}}$, $\tilde{M}^{[n-1]}$, $\tilde{I}^{[n-1]}$ and $\hat{M}^{[2n-2]}$ are defined. Now we simply write $k = t_{n-1}$. We will construct the following commutative diagrams:

(4.1)



When $t_n > k$ is decided, $\tilde{V}_n$, projections $\varphi_n : \tilde{V}_n \to W_{t_n}$ and $\rho_n : \tilde{V}_n \to V_{t_n}$, $\tilde{I}^{[n]}$ and $\hat{M}^{[2n]}$ are necessarily determined by the following:

- $\tilde{V}_n := \{(v, w, i) \mid v \in V_{t_n},\ w \in W_{t_n},\ 1 \le i \le I^{[t_n]}_{vw}\}$,
- $\varphi_n(v, w, i) := w$, $\rho_n(v, w, i) := v$,
- $\tilde{I}^{[n]}_{\tilde{v}w} := \begin{cases} 1, & \text{if } \varphi_n(\tilde{v}) = w \\ 0, & \text{if } \varphi_n(\tilde{v}) \neq w \end{cases}, \quad \hat{M}^{[2n]}_{v\tilde{v}} := \begin{cases} 1, & \text{if } \rho_n(\tilde{v}) = v \\ 0, & \text{if } \rho_n(\tilde{v}) \neq v \end{cases}.$



Moreover when $\hat{M}^{[2n-1]}$ is also decided, $\tilde{M}^{[n]}$ is necessarily determined by

- $\tilde{M}^{[n]} := \hat{M}^{[2n-1]}\hat{M}^{[2n-2]}$.

So we will show that there exist $t_n > k$ and $\hat{M}^{[2n-1]}$ such that

(n-1) for $\tilde{v} \in \tilde{V}_n$, $\hat{M}^{[2n-1]}_{\tilde{v}*} \in S_{\mathbb{N}}(N^{[t_n,k)}_{\varphi_n(\tilde{v})*}) := \{\boldsymbol{x} \in \mathbb{N}^{V_k} \mid \boldsymbol{x}I^{[k]} = N^{[t_n,k)}_{\varphi_n(\tilde{v})*}\}$,

(n-2) $\hat{M}^{[2n]}\hat{M}^{[2n-1]} = M^{[t_n,k)}$.

Suppose the recursive construction above is finished. First, we will show that $\tilde{\mathcal{B}}$ and $\mathcal{C}$ (more precisely, $\tilde{\mathcal{B}}$ and $(\mathcal{C}, \{t_n\})$) satisfy the assumptions of Proposition 3.3. Define $\varphi : \tilde{V} \to W$ as $\varphi|_{\tilde{V}_n} := \varphi_n$ for all n. Note that

$$\hat{M}^{[2n-1]}_{\tilde{v}*} \in \{\boldsymbol{m} \in \mathbb{N}^{V_k} \mid \boldsymbol{m}\hat{M}^{[2n-2]}\tilde{I}^{[n-1]} = N^{[t_n,k)}_{\varphi_n(\tilde{v})*}\}$$

and

$$\sum_{\tilde{u} \in \varphi^{-1}(w)} \hat{M}^{[2n-2]}_{*\tilde{u}} = \hat{M}^{[2n-2]}\tilde{I}^{[n-1]}_{*w}.$$

For $\tilde{v} \in \tilde{V}_n$ and $w \in W_k$,

$$\sum_{\tilde{u} \in \varphi^{-1}(w)} \tilde{M}^{[n]}_{\tilde{v}\tilde{u}} = \sum_{\tilde{u} \in \varphi^{-1}(w)} \hat{M}^{[2n-1]}_{\tilde{v}*} \hat{M}^{[2n-2]}_{*\tilde{u}} = \hat{M}^{[2n-1]}_{\tilde{v}*} \hat{M}^{[2n-2]} \tilde{I}^{[n-1]}_{*w} = N^{[t_n,k)}_{\varphi(v)w}.$$

So they satisfy the assumptions of Proposition 3.3. Then there exist a proper order $\geq$ on $\tilde{E}$ and an almost one-to-one factor map $\pi : X_{\tilde{\mathcal{B}}} \to X_{\mathcal{C}}$. We let $(X, \phi) := (X_{\tilde{\mathcal{B}}}, S_{\tilde{\mathcal{B}}})$.

Second, let $\hat{\mathcal{B}} = (\hat{V}, \hat{E})$ be the Bratteli diagram defined by $\{\hat{M}^{[n]}\}$. Then $\hat{\mathcal{B}}_{\text{odd}} = \tilde{\mathcal{B}}$ and $\hat{\mathcal{B}}_{\text{even}} = \mathcal{B}$. So $\mathcal{B} \sim \tilde{\mathcal{B}}$ and hence there is a unital order isomorphism $\alpha : K_0(\tilde{\mathcal{B}}) \to G$ defined by $\alpha([x,n]_{\tilde{V}}) = [\hat{M}^{[2n]}x, t_n]_V$ if $x \in \mathbb{Z}^{\tilde{V}_n}$. By Proposition 3.3, π^* is defined by $\pi^*([x,t_n]_W) = [\tilde{I}^{[n]}x, n]_{\tilde{V}}$ if $x \in \mathbb{Z}^{W_{t_n}}$. By the commutative diagrams (4.1) we have

$$\alpha \circ \pi^*([x,t_n]_W) = [\hat{M}^{[2n]}\tilde{I}^{[n]}x, t_n]_V = [I^{[t_n]}x, t_n]_V = \iota([x,t_n]_W).$$

4.2. **The condition $(\boldsymbol{n-1})$.** Now we simply write $t = t_n$. For $w \in W_t$, we set

$$S(w) := S(N^{[t,k)}_{w*}) = \{\boldsymbol{x} \in \mathbb{R}^{V_k} \mid \boldsymbol{x}I^{[k]} = N^{[t,k)}_{w*}\},$$
$$S^*(w) := S^*(N^{[t,k)*}_{w*}) = \{\boldsymbol{y} \in \mathbb{R}^{V_k} \mid \boldsymbol{y}I^{[k]*} = N^{[t,k)*}_{w*}\}.$$

(Note that the map $* : S(w) \ni \boldsymbol{m} \mapsto \boldsymbol{m}^* \in S^*(w)$ is bijective. See Remark 2.4.) In this subsection, we will show that for any sufficiently large t and any $\tilde{v} \in \tilde{V}_t$, $S_{\mathbb{N}}(\varphi_n(\tilde{v})) \neq \varnothing$. More simply, we will show that for sufficiently large t and $w \in W_t$,

(1) $S(w) \neq \varnothing$,
(2) $S_{\mathbb{Z}}(w) := S(w) \cap \mathbb{Z}^{V_k} \neq \varnothing$,
(3) $S_{\mathbb{N}}(w) := S(w) \cap \mathbb{N}^{V_k} \neq \varnothing$.

The proof of (1). Let $\acute{I} \in \mathbb{Z}^{(|V_k|+1)\times|W_k|}$ be the matrix defined by

$$\acute{I} = \begin{bmatrix} I^{[k]} \\ N^{[t,k)}_{w*} \end{bmatrix}.$$

We will show $\operatorname{rank}(I^{[k]}) = \operatorname{rank}(\acute{I})$. For any $\boldsymbol{c} \in \mathbb{Z}^{W_k}$ with $I^{[k]}\boldsymbol{c} = \boldsymbol{0}$, we have $[I^{[k]}\boldsymbol{c}, k]_V = 0$. This means $\iota([N^{[t',k)}\boldsymbol{c}, t']_W) = 0$ for any $t' > k$. Since ι is injective, $[N^{[t',k)}\boldsymbol{c}, t']_W = 0$. So there is a $T > k$ such that $N^{[T,k)}\boldsymbol{c} = \boldsymbol{0}$. This implies that for any $t \geq T$, $\acute{I}\boldsymbol{c} = \boldsymbol{0}$. Using this fact, we get $\operatorname{rank}(I^{[k]}) = \operatorname{rank}(\acute{I})$ and hence the equation $\boldsymbol{x}I^{[k]} = N^{[t,k)}_{w*}$ is solvable. So $S(w) \neq \varnothing$. □

The proof of (2). For a vector $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$, let $\operatorname{GCD}(\boldsymbol{a})$ denote the greatest common divider of $|a_1|, \ldots, |a_n|$. In this proof, we write $V_k = \{1, 2, \ldots, V\}$, $W_k = \{1, 2, \ldots, W\}$ and $I = I^{[k]}$. For $w \in W_k$ and $\boldsymbol{x} \in \mathbb{R}^{V_k}$, let $I_w(\boldsymbol{x}) := \sum_{v=1}^{V} I_{vw}x_v$. We will change x_v into X_v by the Euclidean algorithm substitution (cp. [7, p. 30, Thm. 2]). We may suppose that $1 \in V_k$ satisfies

$0 \neq |I_{1w}| = \min\{|I_{vw}| \mid v \in V_k\}$ because I is a positive matrix. Let k_v be an integer satisfying $|I_{1w}k_v + I_{vw}| < |I_{1w}|$. We apply a substitution

$$x_1 = X_1 + k_2x_2 + \cdots + k_Vx_V$$

and then $I_w(\boldsymbol{x}) = I'_{1w}X_1 + \sum_{v=2}^{V} I'_{vw}x_v$, where $I'_{vw} = I_{1w}k_v + I_{vw}$ ($v = 2, 3, \ldots, V$) and $I'_{1w} = I_{1w}$. It is easy to check that $\mathrm{GCD}(I_{*w}) = \mathrm{GCD}(I'_{*w})$. Suppose now that in the new form, I'_{2w} plays the role of I_{1w}. A substitution

$$x_2 = X_2 + l_1X_1 + l_3x_3 + \cdots + l_Vx_V,$$

where l_v is an integer satisfying $|I'_{2w}l_v + I'_{vw}| < |I'_{2w}|$. Continuing this process, we arrive at a *unimodular integral substitution*, which changes $I_w(\boldsymbol{x})$ into

$$I_w(\boldsymbol{x}) = \sum_{v=1}^{V} \varepsilon_v X_v, \quad \text{where } |\varepsilon_1| = \mathrm{GCD}(I_{*w}) \text{ and } \varepsilon_v = 0 \ (v = 2, 3, \ldots, V).$$

Here, consider the equation $I_w(\boldsymbol{x}) = a$ with $\mathrm{GCD}(I_{*w}) \mid a$. Then $X_1 = \frac{a}{\varepsilon_1}$ and the general solution is given linearly in terms of the $n-1$ variables $X_2, \ldots, X_V$.

Now for the matrix I we apply a unimodular integral substitution and the $I_w(\boldsymbol{x})$'s will be changed into the forms

$$I_w(\boldsymbol{x}) = \sum_{v=1}^{u} c_{vw}X_v \quad (u = 1, 2, \ldots, K), \tag{4.2}$$

where $c_{ww} \in \mathbb{N}$, $c_{1w}, c_{2w}, \ldots, c_{w-1w} \in \mathbb{Z}$ and $K = \mathrm{rank}(I)$. At first, by a unimodular integral substitution $I_1(\boldsymbol{x}) = c_{11}X_1$ and $c_{11} = \mathrm{GCD}(I_{*1})$ hold. This substitution changes $I_2(\boldsymbol{x})$ into $c_{12}X_1 + I_2^{(2)}(X_2, X_3, \ldots, X_V)$, where $I_2^{(2)}$ is a linear form in $X_2, X_3, \ldots, X_V$. By this substitution $I_2^{(2)}(X_2, X_3, \ldots, X_V)$ can be changed into $c_{22}X_2$, where c_{22} is the greatest common divisor of the coefficients of $I_2^{(2)}$. (If $c_{22} < 0$, we change X_2 into $-X_2$ and may suppose that $c_{22} > 0$.) This process can be continued and (4.2) follows.

Next we can replace the equation by

$$c_{11}X_1 = N_{v1}^{[t,k)}, c_{12}X_1 + c_{22}X_2 = N_{v2}^{[t,k)}, c_{13}X_1 + c_{23}X_2 + c_{33}X_3 = N_{v3}^{[t,k)}, \ldots$$

As $c_{11} = \mathrm{GCD}(I_{*w})$, we see that $c_{11} \mid N_{w1}^{[t,k)}$ and $c_{11}X_1 = N_{w1}^{[t,k)}$ is solvable in $\mathbb{Z}$. (See Appendix, Proposition A.2.) Moreover we eliminate X_1 from the equation $c_{12}X_1 + c_{22}X_2 = N_{w2}^{[t,k)}$ and get

$$c_{11}c_{22}X_2 = c_{11}N_{w2}^{[t,k)} - c_{12}N_{w1}^{[t,k)}. \tag{4.3}$$

Since $c_{11}c_{22}X_2 = c_{11}I_2(\boldsymbol{x}) - c_{12}I_1(\boldsymbol{x})$, we apply Proposition A.2 and get

$$\mathrm{GCD}(c_{11}L_{*2} - c_{12}I_{*1}) \mid \mathrm{GCD}(c_{11}N_{*2}^{[t,k)} - c_{12}N_{*1}^{[t,k)}).$$

As $\mathrm{GCD}(c_{11}I_{*2} - c_{12}I_{*1}) = c_{11}c_{22}$, the equation (4.3) is solvable in $\mathbb{Z}$. This process continues until X_K and we see that for sufficiently large t the equation $\boldsymbol{x}I^{[k]} = N_{w*}^{[t,k)}$ is solvable in $\mathbb{Z}^{V_k}$ and hence $S_{\mathbb{Z}}(w) \neq \varnothing$. □

The proof of (3). Suppose $S_{\mathbb{Z}}(w) \neq \varnothing$. Let $p_w := N^{[t,1]}_{w*}$. By Proposition 2.5, there exists $\tau > 0$ such that for any $\boldsymbol{m}^* \in S^*(w)$, there is $\boldsymbol{x}^* \in S^*(w) \cap B_{p\tau/p_w}(\boldsymbol{m}^*)$ such that $\boldsymbol{x} \in S_{\mathbb{Z}}(w) \cap B_\tau(\boldsymbol{m})$, where $p = \max_{v \in V_k} M^{[k,1]}_{v*}$. If $\boldsymbol{m}^* \in S^*_{\Delta^+}(w)$ satisfies

$$S^*(w) \cap B_{p\tau/p_w}(\boldsymbol{m}^*) \subset S^*_{\Delta^+}(w), \tag{4.4}$$

then $\boldsymbol{x} \in S_{\mathbb{N}}(w) \cap B_\tau(\boldsymbol{m})$ holds. So $S_{\mathbb{N}}(w) \neq \varnothing$. □

Let $H_{V_k} := \{\boldsymbol{x} = (x_v) \in \mathbb{R}^{V_k} \mid \sum_{v \in V_k} x_v = 1\}$. For $A \subset \Delta_{V_k}$ and $\delta > 0$, let $A^{+\delta} := \{\boldsymbol{x} \in H_{V_k} \mid \|\boldsymbol{x} - \boldsymbol{z}\|_{V_k} < \delta$ for some $\boldsymbol{z} \in A\}$. If A is convex, so is $A^{+\delta}$. Define

$$\varepsilon := \min_{v \in V_{k+1}} \sup\{\varepsilon' \geq 0 \mid B_{\varepsilon'}(M^{[k+1]*}_{v*}) \cap H_{V_k} \subset \Delta^+_{V_k}\}.$$

Since $M^{[k+1]*}_{v*} \in \Delta^+_{V_k}$ for any v, $\varepsilon > 0$ holds. Moreover, $\Delta_{V_k}(k+1)$ and $\Delta^+_{V_k}$ are convex and $\Delta_{V_k}(k+1) \subset \Delta^+_{V_k}$, it follows that

$$\Delta_{V_k}(k+1)^{+\varepsilon} \subset \Delta^+_{V_k}. \tag{4.5}$$

Here we will show

$$\Delta_{W_k}(t) \subset \iota^*(\Delta_{V_k}(k+1)) \tag{4.6}$$

for sufficiently large t. Since $\iota^* : \Delta_{\mathcal{B}} \to \Delta_{\mathcal{C}}$ is surjective and by Proposition A.1 in the Appendix, we have

$$\gamma^{[k]}(\Delta_{\mathcal{C}}) = \iota^* \circ \gamma^{[k]}(\Delta_{\mathcal{B}}) \;\Leftrightarrow\; \bigcap_{t>k} \Delta_{W_k}(t) = \bigcap_{t>k} \iota^*(\Delta_{V_k}(t)) =: \Delta.$$

(Note that $\Delta_{W_k}(t) \searrow \Delta$ and $\iota^*(\Delta_{V_k}(t)) \searrow \Delta$ as $t \to \infty$.) If $\Delta_{W_k}(T) = \Delta_{W_k}(T+1)$ for some T, then $\Delta_{W_k}(T) = \Delta$ holds and Δ has only one point. (See Appendix, Proposition A.3.) So we consider the following cases:

[1] there exists $T > k$ such that $\Delta_{W_k}(T) = \Delta_{W_k}(T+1)$,
[2] for any $t > k$, $\Delta_{W_k}(t) \supsetneq \Delta_{W_k}(t+1)$.

In the case of [1], Δ has only one point. Since $\Delta_{V_k}(t) \subset \Delta_{V_k}(k+1)$ for all $t \geq k+1$, it follows that for any $t \geq T$,

$$\Delta_{W_k}(t) = \bigcap_{t>k} \iota^*(\Delta_{V_k}(t)) \subset \iota^*(\Delta_{V_k}(k+1)).$$

In the case of [2], similarly $\Delta_{V_k}(t) \supsetneq \Delta_{V_k}(t+1)$ holds. Moreover we see that $\Delta \subsetneq \Delta_{W_k}(t)$ and $\Delta \subsetneq \iota^*(\Delta_{V_k}(t))$. So there exists $T > k$ such that for any $t > T$,

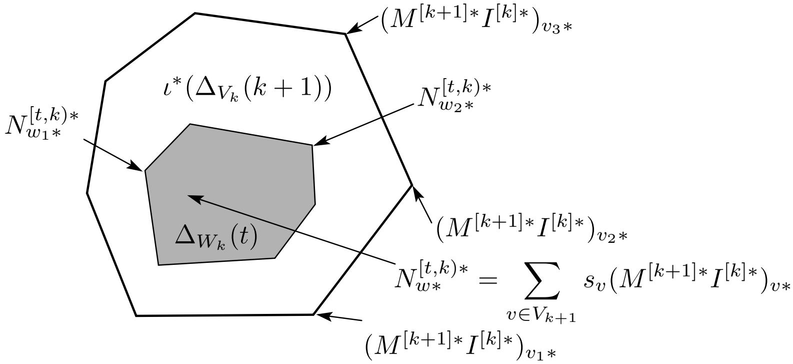
$$\Delta \subsetneq \Delta_{W_k}(t) \subset \iota^*(\Delta_{V_k}(k+1)).$$

Therefore (4.6) holds. Since $\min_{w \in W_t} p_w \to \infty$ as $t \to \infty$, we consider a sufficiently large t satisfying that

$$\frac{p\tau}{p_w} < \varepsilon \tag{4.7}$$

holds for any $w \in W_t$. We note that $\iota^*(\Delta_{V_k}(k+1)) = \{\boldsymbol{x}M^{[k+1]*}I^{[k]*} \mid \boldsymbol{x} \in \Delta_{V_{k+1}}\}$. (4.6) follows that for any $w \in W_t$, there exists $\boldsymbol{s} = (s_v) \in \Delta_{V_{k+1}}$ such that

$$N^{[t,k)*}_{w*} = \boldsymbol{s}M^{[k+1]*}I^{[k]*} = \sum_{v\in V_{k+1}} s_v (M^{[k+1]*}I^{[k]*})_{v*}.$$



So we see that

$$\boldsymbol{s}M^{[k+1]*} = \sum_{v\in V_{k+1}} s_v M^{[k+1]*}_{v*} \in S^*_{\Delta^+}(w). \tag{4.8}$$

Since $\boldsymbol{s}M^{[k+1]*} \in \Delta_{V_k}(k+1)$ and (4.5), we have

$$\sup\{\varepsilon' \geq 0 \mid B_{\varepsilon'}(\boldsymbol{s}M^{[k+1]*}) \cap H_{V_k} \subset \Delta^+_{V_k}\} \geq \varepsilon. \tag{4.9}$$

Therefore we set $\boldsymbol{m}^* := \boldsymbol{s}M^{[k+1]*}$. By (4.7), (4.8) and (4.9), (4.4) holds.

4.3. **The condition $(n-2)$.** $M^{[t,k)} = \hat{M}^{[2n]}\hat{M}^{[2n-1]}$ is equivalent to $M^{[t,k)}_{v*} = \sum_{\tilde{v}\in\rho_n^{-1}(v)} \hat{M}^{[2n-1]}_{\tilde{v}*}$ for all $v \in V_t$. By the condition (n-1), we may assume $\hat{M}^{[2n-1]}_{\tilde{v}*} \in S_{\mathbb{N}}(\varphi_n(\tilde{v}))$. Now we will show, by three steps, that for any sufficiently large t and any $v \in V_t$,

$$M^{[t,k)}_{v*} \in \sum_{\tilde{v}\in\rho_n^{-1}(v)} S_{\mathbb{N}}(\varphi_n(\tilde{v})) = \sum_{w\in W_t} I^{[t]}_{vw} S_{\mathbb{N}}(w), \tag{4.10}$$

where

$$\sum_{w\in W_t} I^{[t]}_{vw} S_{\mathbb{N}}(w) := \left\{ \sum_{w\in W_t} \sum_{i=1}^{I^{[t]}_{vw}} \boldsymbol{m}(w,i) \;\middle|\; \boldsymbol{m}(w,i) \in S_{\mathbb{N}}(w) \right\}.$$

We explain (4.10) in terms of elements in Δ_{V_k} and Δ_{W_k}.

The first step. We consider a sequence $\{v_{n_i} \in V_{n_i}\}$ so that $\lim_{i\to\infty} v_{n_i} =: \mu \in \Delta_{\mathcal{B}}$ exists. Telescoping the diagrams, we may assume $\mu = \lim_{n\to\infty} v_n$. Let $\nu := \iota^*(\mu)$.

We recall the properties of linear algebra. We regard $I^{[k]*}$ as a linear map $I^{[k]*} : \mathbb{R}^{V_k} \to \mathbb{R}^{W_k}$. If $\boldsymbol{a} \in S^*(\boldsymbol{t})$, then we can write $S^*_{\Delta}(\boldsymbol{t}) = (\boldsymbol{a} + \mathrm{Ker}(I^{[k]*})) \cap$

Δ_{V_k}, where $S^*_\Delta(\boldsymbol{t}) := S^*(\boldsymbol{t}) \cap \Delta_{V_k}$ and $\boldsymbol{a}+\mathrm{Ker}(I^{[k]*}) := \{\boldsymbol{a}+\boldsymbol{x} \mid \boldsymbol{x} \in \mathrm{Ker}(I^{[k]*})\}$. Let H be the subspace of $\mathbb{R}^{V_k}$ so that $\mathrm{Ker}(I^{[k]*}) \oplus H = \mathbb{R}^{V_k}$. Then for $\boldsymbol{t} \in \Delta_{W_t}$, there exists a unique $\boldsymbol{a}(\boldsymbol{t}) \in H \cap \Delta_{V_k}$ such that

$$S^*_\Delta(\boldsymbol{t}) = (\boldsymbol{a}(\boldsymbol{t}) + \mathrm{Ker}(I^{[k]*})) \cap \Delta_{V_k}.$$

It is easy to see that the map $\boldsymbol{a} : \boldsymbol{t} \mapsto \boldsymbol{a}(\boldsymbol{t})$ is uniformly continuous and injective. For $w \in W_t$, we also define the map $\boldsymbol{a} : w \mapsto \boldsymbol{a}(w) := \boldsymbol{a}(N^{[t,k)*}_{w*})$.

Here we consider $\gamma^{[k]}(\Delta_\mathcal{B})$, $\Delta_{V_k}(t)$ and Δ_{V_k}. Since $\Delta_{V_k}(t)$ is convex and shrinks to $\gamma^{[k]}(\Delta_\mathcal{B})$ as $t \to \infty$, there exists $\varepsilon_0 > 0$ such that for any $t > k$,

$$\gamma^{[k]}(\Delta_\mathcal{B})^{+\varepsilon_0} \subset \Delta_{V_k}(t)^{+\varepsilon_0} \subsetneq \Delta^+_{V_k}. \tag{4.11}$$

In fact, we set $\varepsilon_0 := \min_{v \in V_{k+1}, v' \in V_k} M^{[k+1]*}_{vv'}$. Now we choose any ε with $0 < 8\varepsilon < \varepsilon_0$ and fix it. Then by the uniform continuity of $\boldsymbol{a} : \boldsymbol{t} \mapsto \boldsymbol{a}(\boldsymbol{t})$, there is a $\delta > 0$ such that for any $\boldsymbol{x}, \boldsymbol{y} \in \Delta_{W_k}$ with $\|\boldsymbol{x} - \boldsymbol{y}\|_{W_k} < \delta$,

$$\|\boldsymbol{a}(\boldsymbol{x}) - \boldsymbol{a}(\boldsymbol{y})\|_{V_k} < \varepsilon. \tag{4.12}$$

For such a $\delta > 0$, by the compactness of Δ_{W_k}, there exist $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_l \in \Delta_{W_k}$ such that $\Delta_{W_k} \subset \bigcup_{i=1}^{l} B_{\delta/2}(\boldsymbol{x}_i)$. We set $W_t(i) \subset W_t$ satisfying that

- $W_t = \bigcup_{i=1}^{l} W_t(i)$ as a disjoint union,
- for any $w \in W_t(i)$, $N^{[t,k)*}_{w*} \in B_{\delta/2}(\boldsymbol{x}_i)$.

Deleting and changing index i and telescoping the diagrams, we may assume $W_t(i) \neq \varnothing$ for any t and i. Let $b^{(i)}_t := \sum_{w \in W_t(i)} I^{[t]*}_{v_t w}$. $b^{(i)}_t \neq 0$ because $I^{[t]*}$ is a positive matrix. Define $\boldsymbol{s}^{(i)}_t \in \Delta_{W_t}$ as

$$\boldsymbol{s}^{(i)}_t := \frac{1}{b^{(i)}_t} \sum_{w \in W_t(i)} I^{[t]*}_{v_t w} w.$$

Clearly

$$I^{[t]*}_{v_t *} = \sum_{i=1}^{l} b^{(i)}_t \boldsymbol{s}^{(i)}_t \quad \text{and} \quad \boldsymbol{s}^{(i)}_t N^{[t,k)*} \in B_{\delta/2}(\boldsymbol{x}_i). \tag{4.13}$$

Renaming index i and telescoping diagrams again, we may assume that $\lim\limits_{t\to\infty} \boldsymbol{s}^{(i)}_t =: \nu^{(i)}$ and $\lim\limits_{t\to\infty} b^{(i)}_t =: b^{(i)}$ exist and there exists $1 \leq L \leq l$ such that $b^{(i)} > 0$ if $i \leq L$ and $b^{(i)} = 0$ if $i > L$. Then $\nu = \sum_{i \leq L} b^{(i)} \nu^{(i)}$. Now we will show that

$$\mu \in \sum_{i \leq L} b^{(i)} (\iota^*)^{-1}(\nu^{(i)}) := \left\{ \sum_{i \leq L} b^{(i)} \mu^{(i)} \;\middle|\; \mu^{(i)} \in (\iota^*)^{-1}(\nu^{(i)}) \right\}. \tag{4.14}$$

By the Choquet representation theorem, we have $\mu = \int_{\mu' \in \mathrm{ex}\,\Delta_\mathcal{B}} \mu' d\tau(\mu')$ and $\nu = \int_{\nu' \in \mathrm{ex}\,\Delta_\mathcal{C}} \nu' d\eta(\nu')$, where τ and η are unique invariant probability measures on $\Delta_\mathcal{B}$ and $\Delta_\mathcal{C}$ respectively with $\tau(\mathrm{ex}\,\Delta_\mathcal{B}) = 1$ and $\eta(\mathrm{ex}\,\Delta_\mathcal{C}) = 1$. Since

$\nu = \iota^*(\mu)$ and $\iota^*(\operatorname{ex}\Delta_{\mathcal{B}}) = \operatorname{ex}\Delta_{\mathcal{C}}$, $\tau \circ (\iota^*)^{-1} = \eta$ holds. For $\nu' \in \operatorname{ex}\Delta_{\mathcal{C}}$ with $\eta(\nu') \neq 0$, we define

$$\mu_{\nu'} := \frac{1}{\eta(\nu')} \int_{\mu' \in \operatorname{ex}\Delta_{\mathcal{B}} \cap (\iota^*)^{-1}(\nu')} \mu' d\tau(\mu').$$

Clearly $\mu_{\nu'} \in (\iota^*)^{-1}(\nu')$ and $\mu = \int_{\nu' \in \operatorname{ex}\Delta_{\mathcal{C}}} \eta(\nu')\mu_{\nu'} d\eta(\nu')$. So we write

$$\begin{aligned}(4.15)\quad \mu \in &\int_{\nu' \in \operatorname{ex}\Delta_{\mathcal{C}}} \eta(\nu')(\iota^*)^{-1}(\nu') d\eta(\nu') \\ &:= \left\{ \xi = \int_{\nu' \in \operatorname{ex}\Delta_{\mathcal{C}}} \eta(\nu')\mu' d\eta(\nu') \;\middle|\; \mu' \in (\iota^*)^{-1}(\nu') \right\}.\end{aligned}$$

Let $\eta^{(i)}$ denote the unique probability measure on $\Delta_{\mathcal{C}}$ with $\eta^{(i)}(\operatorname{ex}\Delta_{\mathcal{C}}) = 1$ satisfying $\nu^{(i)} = \int_{\nu' \in \operatorname{ex}\Delta_{\mathcal{C}}} \nu' d\eta^{(i)}(\nu')$. Then by the uniqueness of η,

$$(4.16)\qquad \eta = \sum_{i \le L} b^{(i)} \eta^{(i)}.$$

In general, for $\nu, \nu_1, \nu_2 \in \Delta_{\mathcal{C}}$ with $\nu = a\nu_1 + (1-a)\nu_2$, $(\iota^*)^{-1}(\nu) \supset a(\iota^*)^{-1}(\nu_1) + (1-a)(\iota^*)^{-1}(\nu_2)$ holds. So we have

$$(4.17)\qquad (\iota^*)^{-1}(\nu^{(i)}) \supset \int_{\nu' \in \operatorname{ex}\Delta_{\mathcal{C}}} (\iota^*)^{-1}(\nu') d\eta^{(i)}(\nu').$$

By (4.15), (4.16) and (4.17), we see that

$$\begin{aligned}\sum_{i \le L} b^{(i)} (\iota^*)^{-1}(\nu^{(i)}) &\supset \sum_{i \le L} b^{(i)} \int_{\nu' \in \operatorname{ex}\Delta_{\mathcal{C}}} (\iota^*)^{-1}(\nu') d\eta^{(i)}(\nu') \\ &= \int_{\nu' \in \operatorname{ex}\Delta_{\mathcal{C}}} \eta(\nu')(\iota^*)^{-1}(\nu') d\eta(\nu') \ni \mu.\end{aligned}$$

Therefore (4.14) holds.

The second step. Let $\boldsymbol{n}(w) := I^{[t]}_{v_t w} N^{[t,k)}_{w*}$ and $\boldsymbol{n}^{(i)} := \sum_{w \in W_t(i)} \boldsymbol{n}(w)$. In this step we will show that for any sufficiently large t, there exist $\boldsymbol{r}^{(i)} \in S_{\mathbb{N}}(\boldsymbol{n}^{(i)})$, $\boldsymbol{r}(w) \in S_{\mathbb{N}}(\boldsymbol{n}(w))$ and $\boldsymbol{r}(w,j) \in S_{\mathbb{N}}(w)$ such that

$$M^{[t,k)}_{v_t *} = \sum_{i=1}^{l} \boldsymbol{r}^{(i)}, \quad \boldsymbol{r}^{(i)} = \sum_{w \in W_t(i)} \boldsymbol{r}(w), \quad \boldsymbol{r}(w) = \sum_{j=1}^{I^{[t]}_{v_t w}} \boldsymbol{r}(w,j).$$

Then we have

$$M^{[t,k)}_{v_t *} = \sum_{i=1}^{l} \sum_{w \in W_t(i)} \sum_{j=1}^{I^{[t]}_{v_t w}} \boldsymbol{r}(w,j) \in \sum_{i=1}^{l} \sum_{w \in W_t(i)} I^{[t]}_{v_t w} S_{\mathbb{N}}(w) = \sum_{w \in W_t} I^{[t]}_{v_t w} S_{\mathbb{N}}(w).$$

The construction of $\boldsymbol{r}^{(i)}$. Let $p_{v_t} := M^{[t,1]}_{v_t *}$. Remark that

$$\boldsymbol{n}^{(i)*} = \boldsymbol{s}^{(i)}_t N^{[t,k)*}, \quad p_{v_t} b^{(i)}_t = \sum_{w \in W_t(i)} I^{[t]}_{v_t w} N^{[t,1]}_{w*} = \boldsymbol{n}^{(i)} N^{[k,1]}$$

and the local inverse map $*^{-1} : S^*_\Delta(\boldsymbol{n}^{(i)*}) \to S_{\mathbb{R}_+}(\boldsymbol{n}^{(i)})$ is given by

$$\boldsymbol{m} = (\boldsymbol{m}^*)^{*^{-1}} = p_{v_t} b_t^{(i)} \boldsymbol{m}^* D_{V_k}^{-1}.$$

$M^{[t,k)*} I^{[k]*} = I^{[t]*} N^{[t,k)*}$ implies $M^{[t,k)*}_{v_t*} \in S^*_\Delta(I^{[t]*}_{v_t*} N^{[t,k)*})$.
Let $B_t := \sum_{i \le L} b_t^{(i)}$. Since

$$\mu = \lim_{t\to\infty} v_t,\ \nu^{(i)} = \lim_{t\to\infty} \boldsymbol{s}_t^{(i)},\ \lim_{t\to\infty} b_t^{(i)} = b^{(i)},\ \lim_{t\to\infty} B_t = 1,\ \lim_{t\to\infty} \min_{w\in W_t} p_w = \infty,$$

we see that for any sufficiently large t,

$$\begin{gathered} \|\mu_k - M^{[t,k)*}_{v_t*}\|_{V_k} < \varepsilon, \quad \|\nu_k^{(i)} - \boldsymbol{n}^{(i)*}\|_{W_k} < \delta, \quad \max_{w\in W_t} \frac{2p\tau}{p_w} < \varepsilon, \\ \frac{(1-B_t)}{B_t} < \frac{\varepsilon}{8}, \quad \left|\frac{b^{(i)}}{b_t^{(i)}} - 1\right| < \frac{\varepsilon}{8} \quad \text{if } i \le L. \end{gathered} \tag{4.18}$$

We fix such a t. For $i > L$, choose any $\mu^{(i)} \in \Delta_{\mathcal{B}}$ with $\iota^*(\mu^{(i)}) = \nu^{(i)}$ and fix it. From (4.14), for $i \le L$, there exists $\mu^{(i)} \in (\iota^*)^{-1}(\nu^{(i)})$ such that $\mu = \sum_{i\le L} b^{(i)}\mu^{(i)}$. $\nu_k = \sum_{i\le L} b^{(i)}\nu_k^{(i)}$ follows that $\boldsymbol{a}(\nu_k) = \sum_{i\le L} b^{(i)}\boldsymbol{a}(\nu_k^{(i)})$. Since

$$I^{[t]*}_{v_t*} N^{[t,k)*} = \sum_{i=1}^{l} \sum_{w\in W_t(i)} I^{[t]*}_{v_t w} N^{[t,k)*}_{w*} = \sum_{i=1}^{l} b_t^{(i)} \boldsymbol{n}^{(i)*},$$

we have $\boldsymbol{a}(I^{[t]*}_{v_t*} N^{[t,k)*}) = \sum_{i=1}^{l} b_t^{(i)} \boldsymbol{a}(\boldsymbol{n}^{(i)*})$. Define $\boldsymbol{m}^{(i)*}$ as

if $i \le L$,

$$\begin{aligned} \boldsymbol{m}^{(i)*} :=& \boldsymbol{a}(\boldsymbol{n}^{(i)*}) + \frac{1}{B_t}(M^{[t,k)*}_{v_t*} - \boldsymbol{a}(I^{[t]*}_{v_t*} N^{[t,k)*})) \\ & + \frac{b^{(i)}}{b_t^{(i)}}\left(\mu_k^{(i)} - \mu_k + \boldsymbol{a}(\nu_k) - \boldsymbol{a}(\nu_k^{(i)})\right) - \frac{1}{B_t}\sum_{i'>L} b_t^{(i')}(\mu_k^{(i')} - \boldsymbol{a}(\nu_k^{(i')})), \end{aligned}$$

if $i > L$,

$$\boldsymbol{m}^{(i)*} := \boldsymbol{a}(\boldsymbol{n}^{(i)*}) + \mu_k^{(i)} - \boldsymbol{a}(\nu_k^{(i)}).$$

Let $\boldsymbol{m}^{(i)} := p_{v_t} b_t^{(i)} \boldsymbol{m}^{(i)*} D_{V_k}^{-1}$. Then

$$M^{[t,k)*}_{v_t*} = \sum_{i=1}^{l} b_t^{(i)} \boldsymbol{m}^{(i)*}, \quad M^{[t,k)}_{v_t*} = \sum_{i=1}^{l} \boldsymbol{m}^{(i)} \quad \text{and } \boldsymbol{m}^{(i)} \in S(\boldsymbol{n}^{(i)})$$

hold. If $i \le L$, then

$$\begin{aligned} \|\boldsymbol{m}^{(i)*} - \mu_k^{(i)}\|_{V_k} \le& \|M^{[t,k)*}_{v_t*} - \mu_k\|_{V_k} + \|\boldsymbol{a}(\boldsymbol{n}^{(i)*}) - \boldsymbol{a}(\nu_k^{(i)})\|_{V_k} \\ & + \left\|\boldsymbol{a}(I^{[t]*}_{v_t*} N^{[t,k)*}) - \boldsymbol{a}(\nu_k)\right\|_{V_k} \\ & + \left|\frac{b^{(i)}}{b_t^{(i)}} - 1\right| (\|\mu_k\|_{V_k} + \|\boldsymbol{a}(\nu_k^{(i)})\|_{V_k} + \|\boldsymbol{a}(\nu_k)\|_{V_k} + \|\mu_k^{(i)}\|_{V_k}) \end{aligned}$$

$$+\frac{1-B_t}{B_t}(\|M^{[t,k)*}_{v_t*}\|_{V_k}+\|\boldsymbol{a}(I^{[t]*}_{v_t*}N^{[t,k)*})\|_{V_k}+2).$$

If $i>L$, then

$$\|\boldsymbol{m}^{(i)*}-\mu^{(i)}_k\|_{V_k}\le\|\boldsymbol{a}(\boldsymbol{n}^{(i)*})-\boldsymbol{a}(\nu^{(i)}_k)\|_{V_k}.$$

So by (4.12) and (4.18), we have

$$\|\boldsymbol{m}^{(i)*}-\mu^{(i)}_k\|_{V_k}<4\varepsilon<\varepsilon_0. \tag{4.19}$$

By (4.11) and $\mu^{(i)}_k\in\gamma^{[k]}(\Delta_{\mathcal{B}})$, $\boldsymbol{m}^{(i)*}\in S^*_{\Delta^+}(\boldsymbol{n}^{(i)*})$ holds.

Let $\tilde{\boldsymbol{m}}^{(i)}:=\sum_{\xi=1}^{i}\boldsymbol{m}^{(\xi)}$ and $\tilde{\boldsymbol{n}}^{(i)}:=\sum_{\xi=1}^{i}\boldsymbol{n}^{(\xi)}$. Then $\tilde{\boldsymbol{m}}^{(i)}\in S_{\mathbb{R}_+}(\tilde{\boldsymbol{n}}^{(i)})$. By Proposition 2.5, there is an $\boldsymbol{x}^{(i)*}\in S^*_{\Delta^+}(\tilde{\boldsymbol{n}}^{(i)*})\cap B_{p\tau/\tilde{\boldsymbol{n}}^{(i)}N^{[k,1]}}(\tilde{\boldsymbol{m}}^{(i)*})$ such that $\boldsymbol{x}^{(i)}=(\tilde{\boldsymbol{n}}^{(i)}N^{[k,1]})\boldsymbol{x}^{(i)*}D^{-1}_{V_k}\in S_{\mathbb{N}}(\tilde{\boldsymbol{n}}^{(i)})\cap B_\tau(\tilde{\boldsymbol{m}}^{(i)})$. Fix such an $\boldsymbol{x}^{(i)}$. Define

$$\boldsymbol{r}^{(i)}:=\boldsymbol{x}^{(i)}-\boldsymbol{x}^{(i-1)},\quad \boldsymbol{x}^{(0)}:=\mathbf{0}.$$

Then it follows $\boldsymbol{r}^{(i)}\in S_{\mathbb{Z}}(\boldsymbol{n}^{(i)})$ and $\sum_{i=1}^{l}\boldsymbol{r}^{(i)}=M^{[t,k)}_{v_t*}$. Here we will show $\boldsymbol{r}^{(i)}\in S_{\mathbb{N}}(\boldsymbol{n}^{(i)})$. Since

$$\boldsymbol{r}^{(i)*}=\frac{(\boldsymbol{x}^{(i)}-\boldsymbol{x}^{(i-1)})D_{V_k}}{\boldsymbol{n}^{(i)}N^{[k,1]}}=\frac{(\tilde{\boldsymbol{n}}^{(i)}N^{[k,1]})\boldsymbol{x}^{(i)*}-(\tilde{\boldsymbol{n}}^{(i-1)}N^{[k,1]})\boldsymbol{x}^{(i-1)*}}{\boldsymbol{n}^{(i)}N^{[k,1]}}$$

and

$$\boldsymbol{m}^{(i)*}=\frac{(\tilde{\boldsymbol{m}}^{(i)}-\tilde{\boldsymbol{m}}^{(i-1)})D_{V_k}}{\boldsymbol{n}^{(i)}N^{[k,1]}}=\frac{(\tilde{\boldsymbol{n}}^{(i)}N^{[k,1]})\tilde{\boldsymbol{m}}^{(i)*}-(\tilde{\boldsymbol{n}}^{(i-1)}N^{[k,1]})\tilde{\boldsymbol{m}}^{(i-1)*}}{\boldsymbol{n}^{(i)}N^{[k,1]}},$$

we have

$$\begin{aligned}\|\boldsymbol{r}^{(i)*}-\boldsymbol{m}^{(i)*}\|_{V_k}\le&\frac{\tilde{\boldsymbol{n}}^{(i)}N^{[k,1]}}{\boldsymbol{n}^{(i)}N^{[k,1]}}\|\boldsymbol{x}^{(i)*}-\tilde{\boldsymbol{m}}^{(i)*}\|_{V_k}\\&+\frac{\tilde{\boldsymbol{n}}^{(i-1)}N^{[k,1]}}{\boldsymbol{n}^{(i)}N^{[k,1]}}\|\boldsymbol{x}^{(i-1)*}-\tilde{\boldsymbol{m}}^{(i-1)*}\|_{V_k}\\\le&\frac{2p\tau}{\boldsymbol{n}^{(i)}N^{[k,1]}}=\frac{2p\tau}{\sum_{w\in W_t(i)}I^{[t]}_{v_tw}p_w}<\varepsilon.\end{aligned}\tag{4.20}$$

So by (4.19) and (4.20), we have

$$\|\mu^{(i)}_k-\boldsymbol{r}^{(i)*}\|_{V_k}<5\varepsilon<\varepsilon_0. \tag{4.21}$$

Therefore $\boldsymbol{r}^{(i)*}\in S^*_{\Delta^+}(\boldsymbol{n}^{(i)*})$ and hence $\boldsymbol{r}^{(i)}\in S_{\mathbb{N}}(\boldsymbol{n}^{(i)})$.

The construction of $\boldsymbol{r}(w)$. If $\#W_t(i)=1$, then let $W_t(i)=\{w\}$ and define $\boldsymbol{r}(w):=\boldsymbol{r}^{(i)}\in S_{\mathbb{N}}(\boldsymbol{n}^{(i)})$. So we consider the case where $\#W_t(i)\ge 2$. Remark that

$$\boldsymbol{n}^*(w)=N^{[t,k)*}_{w*},\quad b^{(i)}_t\boldsymbol{n}^{(i)*}=\sum_{w\in W_t(i)}I^{[t]*}_{v_tw}\boldsymbol{n}^*(w).$$

By the above, we see that $b^{(i)}_t\boldsymbol{a}(\boldsymbol{n}^{(i)*})=\sum_{w\in W_t(i)}I^{[t]*}_{v_tw}\boldsymbol{a}(\boldsymbol{n}^*(w))$. For any $w\in W_t(i)$, define

$$\boldsymbol{m}^*(w):=\boldsymbol{a}(\boldsymbol{n}^*(w))+\boldsymbol{r}^{(i)*}-\boldsymbol{a}(\boldsymbol{n}^{(i)*}),\quad \boldsymbol{m}(w):=I^{[t]}_{v_tw}p_w\boldsymbol{m}^*(w)D^{-1}_{V_k}.$$

Then

$$\boldsymbol{m}^*(w) \in S^*(\boldsymbol{n}^*(w)) = S^*(w), \quad \boldsymbol{m}(w) \in S(\boldsymbol{n}(w)),$$
$$b_t^{(i)}\boldsymbol{r}^{(i)*} = \sum_{w\in W_t(i)} I_{v_tw}^{[t]*}\boldsymbol{m}^*(w), \quad \boldsymbol{r}^{(i)} = \sum_{w\in W_t(i)} \boldsymbol{m}(w).$$

Let $s := \#W_t(i)$ and $\{w_1, w_2, \cdots, w_s\} = W_t(i)$. For $1 \le j \le s$, let $\tilde{\boldsymbol{m}}(j) := \sum_{\xi=1}^{j} \boldsymbol{m}(w_\xi)$ and $\tilde{\boldsymbol{n}}(j) := \sum_{\xi=1}^{j} \boldsymbol{n}(w_\xi)$. Clearly $\tilde{\boldsymbol{m}}(j) \in S(\tilde{\boldsymbol{n}}(j))$. By Proposition 2.5, there is $\boldsymbol{x}^*(j) \in S^*_{\Delta^+}(\tilde{\boldsymbol{n}}^*(j)) \cap B_{p\tau/\tilde{\boldsymbol{n}}(j)N^{[k,1]}}(\tilde{\boldsymbol{m}}^*(j))$ such that $\boldsymbol{x}(j) = (\tilde{\boldsymbol{n}}(j)N^{[k,1]})\boldsymbol{x}^*(j)D_{V_k}^{-1} \in S_{\mathbb{N}}(\tilde{\boldsymbol{n}}(j)) \cap B_\tau(\tilde{\boldsymbol{m}}(j))$. Fix such an $\boldsymbol{x}^*(j)$. Define

$$\boldsymbol{r}(w_j) := \boldsymbol{x}(j) - \boldsymbol{x}(j-1), \quad \boldsymbol{x}(0) := \boldsymbol{0}.$$

Then it follows $\boldsymbol{r}(w_j) \in S_{\mathbb{Z}}(\boldsymbol{n}(w_j))$ and $\sum_{j=1}^{s} \boldsymbol{r}(w_j) = \boldsymbol{r}^{(i)}$. Here we will show $\boldsymbol{r}(w_j) \in S_{\mathbb{N}}(\boldsymbol{n}(w_j))$. Since

$$\begin{aligned}\boldsymbol{r}^*(w_j) &= \frac{(\boldsymbol{x}(j) - \boldsymbol{x}(j-1))D_{V_k}}{\boldsymbol{n}(w_j)N^{[k,1]}}\\ &= \frac{(\tilde{\boldsymbol{n}}(j)N^{[k,1]})\boldsymbol{x}^*(j) - (\tilde{\boldsymbol{n}}(j-1)N^{[k,1]})\boldsymbol{x}^*(j-1)}{\boldsymbol{n}(w_j)N^{[k,1]}}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{m}^*(w_j) &= \frac{(\tilde{\boldsymbol{m}}(j) - \tilde{\boldsymbol{m}}(j-1))D_{V_k}}{\boldsymbol{n}(w_j)N^{[k,1]}}\\ &= \frac{(\tilde{\boldsymbol{n}}(j)N^{[k,1]})\tilde{\boldsymbol{m}}^*(j) - (\tilde{\boldsymbol{n}}(j-1)N^{[k,1]})\tilde{\boldsymbol{m}}^*(j-1)}{\boldsymbol{n}(w_j)N^{[k,1]}}\end{aligned}$$

we have

$$\begin{aligned}\|\boldsymbol{r}^*(w_j) - \boldsymbol{m}^*(w_j)\|_{V_k} \le& \frac{\tilde{\boldsymbol{n}}(j)N^{[k,1]}}{\boldsymbol{n}(w_j)N^{[k,1]}}\|\boldsymbol{x}^*(j) - \tilde{\boldsymbol{m}}^*(j)\|_{V_k}\\ &+ \frac{\tilde{\boldsymbol{n}}(j-1)N^{[k,1]}}{\boldsymbol{n}(w_j)N^{[k,1]}}\|\boldsymbol{x}^*(j-1) - \tilde{\boldsymbol{m}}^*(j-1)\|_{V_k}\\ \le& \frac{2p\tau}{\boldsymbol{n}(w_j)N^{[k,1]}} = \frac{2p\tau}{I_{v_tw_j}^{[t]}p_{w_j}} < \varepsilon.\end{aligned} \tag{4.22}$$

$\boldsymbol{n}^*(w_j), \boldsymbol{n}^{(i)*} \in B_{\delta/2}(\boldsymbol{x}_i)$ implies $\|\boldsymbol{n}^*(w_j) - \boldsymbol{n}^{(i)*}\|_{W_k} < \delta$. Then by (4.12) we have

$$\|\boldsymbol{m}^*(w_j) - \boldsymbol{r}^{(i)*}\|_{V_k} = \|\boldsymbol{a}(\boldsymbol{n}^*(w_j)) - \boldsymbol{a}(\boldsymbol{n}^{(i)*})\|_{V_k} < \varepsilon. \tag{4.23}$$

So by (4.21), (4.22) and (4.23), we have

$$\|\mu_k^{(i)} - \boldsymbol{r}^*(w_j)\|_{V_k} < 7\varepsilon < \varepsilon_0. \tag{4.24}$$

Therefore $\boldsymbol{r}(w) \in S_{\mathbb{N}}(\boldsymbol{n}(w))$ for all w.

The construction of $\boldsymbol{r}(w,j)$. Let $w \in W_t(i)$. If $I_{v_tw}^{[t]} = 1$, let $\boldsymbol{r}(w,1) := \boldsymbol{r}(w) \in S_{\mathbb{N}}(w)$ and we are done. So we consider the case where $I_{v_tw}^{[t]} \ge 2$.

Let $\boldsymbol{m}^* := \boldsymbol{r}^*(w)$ and $\boldsymbol{m} := p_w \boldsymbol{m}^* D_{V_k}^{-1}$. Then $\boldsymbol{r}(w) = I_{v_t w}^{[t]} \boldsymbol{m}$ holds. By Proposition 2.5, for any $1 \le j \le I_{v_t w}^{[t]}$, there exists $\boldsymbol{x}_j^* \in S^*(w) \cap B_{p\tau/jp_w}(\boldsymbol{m}^*)$ such that $\boldsymbol{x}_j = jp_w \boldsymbol{x}_j^* D_{V_k}^{-1} \in S_{\mathbb{N}}(jN_{w*}^{[t,k)}) \cap B_\tau(j\boldsymbol{m})$. Fix such an $\boldsymbol{x}_j$. Define

$$\boldsymbol{r}(w,j) := \boldsymbol{x}_j - \boldsymbol{x}_{j-1}, \quad \boldsymbol{x}_0 := \boldsymbol{0}.$$

Then it follows that $\boldsymbol{r}(w,j) \in S_{\mathbb{Z}}(w)$ and $\sum_{j=1}^{I_{v_t w}^{[t]}} \boldsymbol{r}(w,j) = \boldsymbol{r}(w)$. Now we will show $\boldsymbol{r}(w,j) \in S_{\mathbb{N}}(w)$. Since

$$\boldsymbol{r}(w,j)^* = \frac{(\boldsymbol{x}_j - \boldsymbol{x}_{j-1})D_{V_k}}{p_w} = \frac{\boldsymbol{x}_j D_{V_k}}{jp_w} j - \frac{\boldsymbol{x}_{j-1} D_{V_k}}{(j-1)p_w}(j-1) = j\boldsymbol{x}_j^* - (j-1)\boldsymbol{x}_{j-1}^*,$$

we have

$$\|\boldsymbol{r}(w,j)^* - \boldsymbol{r}^*(w)\|_{V_k} \le j\|\boldsymbol{x}_j^* - \boldsymbol{m}^*\|_{V_k} + (j-1)\|\boldsymbol{x}_{j-1}^* - \boldsymbol{m}^*\|_{V_k} \le \frac{2p\tau}{p_w} < \varepsilon.$$

So by (4.24), $\|\mu_k^{(i)} - \boldsymbol{r}(w,j)^*\|_{V_k} < 8\varepsilon < \varepsilon_0$ holds. This means that $\boldsymbol{r}(w,j)^* \in S_{\Delta+}^*(w)$ and hence $\boldsymbol{r}(w,j) \in S_{\mathbb{N}}(w)$.

The third step. Finally we will show (4.10). Suppose that for infinitely many t, there exists $v_t \in V_t$ such that

$$M_{v_t *}^{[t,k)} \notin \sum_{w \in W_t} I_{v_t w}^{[t]} S_{\mathbb{N}}(w).$$

Then we take a subsequence $\{v_{n_i}\} \subset \{v_t\}$ so that $\lim_{i\to\infty} v_{n_i}$ exists. By the second step there exists $I \in \mathbb{N}$ such that for any $i > I$,

$$M_{v_{n_i} *}^{[n_i,k)} \in \sum_{w \in W_{n_i}} I_{v_{n_i} w}^{[n_i]} S_{\mathbb{N}}(w).$$

This is a contradiction. So (4.10) holds.

5. Examples

Example 5.1. We will construct Bratteli diagrams $\mathcal{B}$, $\mathcal{C}$ and an injective, unital order group homomorphism $\iota : K_0(\mathcal{C}) \to K_0(\mathcal{B})$ so that (i) ι is an order embedding, (ii) $K_0/\iota(K_0(\mathcal{C}))$ is torsion free, and (iii) $\iota^* : \Delta_{\mathcal{B}} \to \Delta_{\mathcal{C}}$ is a surjective affine homomorphism but $\iota^*(\operatorname{ex}\Delta_{\mathcal{B}}) \neq \operatorname{ex}\Delta_{\mathcal{C}}$.

Let $\frac{1}{2} < \alpha < 1$. Define a sequence $\{a_n\}$ by the following: Let $a_1 := 0$. Let $a_2 \in \mathbb{N}$ satisfy $\frac{a_2-1}{a_2} \le \alpha < \frac{a_2}{a_2+1}$ (and hence $a_2 \ge 2$). Suppose for $n \ge 2$, a_n satisfies

$$0 < \frac{b_n}{p_n} - \alpha < \frac{1}{2}\left(\frac{b_{n-1}}{p_{n-1}} - \alpha\right),$$

where $p_n := \prod_{k=1}^{n}(a_k+1)$ and $b_n := (a_n - 1)b_{n-1} + p_{n-1}$, $b_1 := 1$. Define

$$f(x) := \frac{(x-1)b_n + p_n}{(x+1)p_n}.$$

Then it follows that $f(1) = 1/2$, $f(x)$ is monotonously increasing and

$$\lim_{x\to\infty} f(x) = \frac{b_n}{p_n}.$$

Define $a_{n+1} \in \mathbb{N}$ as $f(a_{n+1}-1) \le \alpha < f(a_{n+1})$. Then we see that

$$0 < f(a_{n+1}) - \alpha = \frac{b_{n+1}}{p_{n+1}} - \alpha,$$

$$\frac{1}{2}\left(\frac{b_n}{p_n} - \alpha\right) - \left(\frac{b_{n+1}}{p_{n+1}} - \alpha\right) > \frac{(2b_n - p_n)(a_{n+1}-1)}{2a_{n+1}p_n(a_{n+1}+1)} > 0.$$

Moreover we have $\lim_{n\to\infty} b_n/p_n = \alpha$. Remark that

$$f(a_{n+1}-1) \le \alpha < f(a_{n+1}) \quad \Longleftrightarrow \quad \frac{\alpha - 1 + b_n/p_n}{b_n/p_n - \alpha} < a_{n+1} \le \frac{2b_n/p_n - 1}{b_n/p_n - \alpha}.$$

Therefore $b_n/p_n \searrow \alpha$ means $a_{n+1} \nearrow \infty$. Let $\{c_n\}$ satisfy $c_n = (a_n - 2)c_{n-1} + p_{n-1}$ and $c_1 = 0$. Then $0 < c_n/p_n < b_n/p_n$ and

$$\left|\frac{c_n}{p_n} - \frac{c_{n+1}}{p_{n+1}}\right| < \frac{1}{a_{n+1}+1} \left|\frac{3c_n}{p_n} - 1\right| < \frac{2}{a_{n+1}+1} \to 0 \quad \text{as } n \to \infty.$$

So $\beta := \lim_{n\to\infty} c_n/p_n$ exists. We see that $\beta + \alpha \le 1$ because $b_n + c_n \le p_n$ holds for any n.

Suppose $\mathcal{B} = (V, E, \{M^{[n]}\})$ and $\mathcal{C} = (W, F, \{N^{[n]}\})$ are Bratteli diagrams and $I^{[n]}$ is a $V_n \times W_n$ matrix satisfying that

$$M^{[1]} := \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad N^{[1]} := \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad I^{[0]} := [1],$$

for $n \ge 2$,

$$M^{[n]} := \begin{bmatrix} a_n - 1 & 1 & 0 \\ 1 & a_n & 1 \\ 0 & 1 & a_n - 1 \end{bmatrix}, \quad N^{[n]} := \begin{bmatrix} a_n & 1 \\ 1 & a_n \end{bmatrix}, \quad I^{[n-1]} := \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since $\det M^{[n]} = (a_n+1)(a_n-1)(a_n-2)$ and $\det N^{[n]} = (a_n+1)(a_n-1)$, $M^{[n]}$ and $N^{[n]}$ are invertible ($n \ge 2$). (If $a_2 = 2$, then $M^{[2]}$ is not invertible. However, we telescope diagrams $\mathcal{B}$ and $\mathcal{C}$ to $\{0, 2, 3, 4, \cdots\}$, we may assume $a_n \ge 3$ for any $n \ge 2$.) Then $M^{[n]}I^{[n-1]} = I^{[n]}N^{[n]}$ for all $n \in \mathbb{N}$ and

$$M^{[n,1)} = \begin{bmatrix} b_n - c_n & c_n & p_n - b_n - c_n \\ c_n & p_n - c_n & c_n \\ p_n - b_n - c_n & c_n & b_n - c_n \end{bmatrix},$$

$$N^{[n,1)} = \begin{bmatrix} b_n & p_n - b_n \\ p_n - b_n & b_n \end{bmatrix},$$

$$M^{[n,1)*} = \frac{1}{p_n}\begin{bmatrix} b_n - c_n & 2c_n & p_n - b_n - c_n \\ c_n/2 & p_n - c_n & c_n/2 \\ p_n - b_n - c_n & 2c_n & b_n - c_n \end{bmatrix},$$

$$N^{[n,1)*} = \frac{1}{p_n} N^{[n,1)},$$

$$\lim_{n\to\infty} M^{[n,1)*} = \begin{bmatrix} \alpha-\beta & 2\beta & 1-\alpha-\beta \\ \beta/2 & 1-\beta & \beta/2 \\ 1-\alpha-\beta & 2\beta & \alpha-\beta \end{bmatrix},$$

$$\lim_{n\to\infty} N^{[n,1)*} = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix}.$$

Define $\iota : K_0(\mathcal{C}) \to K_0(\mathcal{B})$ as $\iota([\boldsymbol{z}, n]_W) := [I^{[n]}\boldsymbol{z}, n]_V$. Then ι is an injective, unital order group homomorphism. Since $\{I^{[n]}\boldsymbol{z} \mid \boldsymbol{z} \in \mathbb{Z}^{W_n}\}$ is a pure subgroup of $\mathbb{Z}^{V_n}$, $\iota(K_0(\mathcal{C}))$ is a pure subgroup of $K_0(\mathcal{B})$ and hence $K_0(\mathcal{B})/\iota(K_0(\mathcal{C}))$ is torsion free. Define $\mu^{(1)}, \mu^{(2)}, \mu^{(3)} \in \Delta_{\mathcal{B}}$ and $\nu^{(1)}, \nu^{(2)} \in \Delta_{\mathcal{C}}$ as

$$\mu_k^{(i)} := \lim_{n\to\infty} M_{i*}^{[n,k)*} = \begin{cases} [\alpha-\beta, 2\beta, 1-\alpha-\beta](M^{[k,1)*})^{-1}, & \text{if } i=1, \\ [\beta/2, 1-\beta, \beta/2](M^{[k,1)*})^{-1}, & \text{if } i=2, \\ [1-\alpha-\beta, 2\beta, \alpha-\beta](M^{[k,1)*})^{-1}, & \text{if } i=3, \end{cases}$$

$$\nu_k^{(i)} := \lim_{n\to\infty} N_{i*}^{[n,k)*} = \begin{cases} [\alpha, 1-\alpha](N^{[k,1)*})^{-1}, & \text{if } i=1, \\ [1-\alpha, \alpha](N^{[k,1)*})^{-1}, & \text{if } i=2. \end{cases}$$

It is easy to check that $\operatorname{ex}\Delta_{\mathcal{B}} = \{\mu^{(1)}, \mu^{(2)}, \mu^{(3)}\}$, $\operatorname{ex}\Delta_{\mathcal{C}} = \{\nu^{(1)}, \nu^{(2)}\}$ and

$$\Delta_{\mathcal{B}} = \left\{ \sum_{i=1}^{3} s_i \mu^{(i)} \;\middle|\; (s_1, s_2, s_3) \in \Delta_3 \right\}, \quad \Delta_{\mathcal{C}} = \left\{ \sum_{i=1}^{2} t_i \mu^{(i)} \;\middle|\; (t_1, t_2) \in \Delta_2 \right\}.$$

Define $\iota^* : \Delta_{\mathcal{B}} \to \Delta_{\mathcal{C}}$ as $\iota^*((\mu_k)) := (\mu_k I^{[k]*})$ where

$$I^{[k]*} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}.$$

Then $\iota^*(\mu^{(1)}) = \nu^{(1)}$, $\iota^*(\mu^{(2)}) = (\nu^{(1)} + \nu^{(2)})/2$ and $\iota^*(\mu^{(3)}) = \nu^{(2)}$. This means ι^* is surjective affine homomorphism. By Proposition 2.3, ι is order embedding. Moreover, we see that $\iota^*(\operatorname{ex}\Delta_{\mathcal{B}}) \neq \operatorname{ex}\Delta_{\mathcal{C}}$.

Appendix A

Proposition A.1. *For any $n \in \mathbb{N}$, $\gamma^{[n]}(\Delta_{\mathcal{B}}) = \bigcap_{k>n} \Delta_{V_n}(k)$.*

Proof. Clearly, $\Delta_{V_n}(k) \supset \Delta_{V_n}(k+1) \supset \gamma^{[n]}(\Delta_{\mathcal{B}})$ for any $k > n$. We have $\gamma^{[n]}(\Delta_{\mathcal{B}}) \subset \bigcap_{k>n} \Delta_{V_n}(k)$. Therefore we will show $\gamma^{[n]}(\Delta_{\mathcal{B}}) \supset \bigcap_{k>n} \Delta_{V_n}(k)$. Let $\boldsymbol{s}_n \in \bigcap_{k>n} \Delta_{V_n}(k)$. For any $k > n$, there exists $\boldsymbol{x}_k \in \Delta_{V_k}$ such that $\boldsymbol{s}_n = \boldsymbol{x}_k M^{[k,n)}$. We fix such $\{\boldsymbol{x}_k\}_{k>n}$. By Proposition 2.1, there exists subsequence $\{k_i\}$ such that $\lim_{i\to\infty} \boldsymbol{x}_{k_i} =: \mu \in \Delta_{\mathcal{B}}$. Then $\boldsymbol{s}_n = \gamma^{[n]}(\mu)$. Therefore $\boldsymbol{s}_n \in \gamma^{[n]}(\Delta_{\mathcal{B}})$. □

Proposition A.2. *Suppose that $\iota : K_0(\mathcal{C}) \to K_0(\mathcal{B})$ is an order embedding and $K_0(\mathcal{B})/\iota(K_0(\mathcal{C}))$ is torsion free. Suppose that $\boldsymbol{c} \in \mathbb{Z}^{W_k} \setminus \{\boldsymbol{0}\}$ satisfy that for any $t > k$, $N^{[t,k)}\boldsymbol{c} \neq \boldsymbol{0}$ and $I^{[k]}\boldsymbol{c} \neq \boldsymbol{0}$. Then there exists $T > k$ such that for all $t \geq T$,*

$$\mathrm{GCD}(I^{[k]}\boldsymbol{c}) \mid \mathrm{GCD}(N^{[t,k)}\boldsymbol{c}).$$

Proof. Set $c := \mathrm{GCD}(I^{[k]}\boldsymbol{c})$. Then

$$[I^{[k]}\boldsymbol{c}, k]_V = \iota[\boldsymbol{c}, k]_W \in \iota(K_0(\mathcal{C})) \quad \text{and} \quad \frac{I^{[k]}\boldsymbol{c}}{c} \in \mathbb{Z}^{V_k}.$$

As $K_0(\mathcal{B})/\iota(K_0(\mathcal{C}))$ is torsion free, $\iota(K_0(\mathcal{C}))$ is a pure subgroup of $K_0(\mathcal{B})$. So there exists $g \in K_0(\mathcal{C})$ such that $\iota(g) = [c^{-1}M^{[t,k)}I^{[k]}\boldsymbol{c}, t]_V$ for $t \geq k$ (if $t = k$, $M^{[k,t)} = id. \in \mathbb{Z}^{V_k \times V_k}$). Moreover there exist $T_1 > k$ and $\boldsymbol{g}_{T_1} \in \mathbb{Z}^{W_{T_1}}$ such that $g = [\boldsymbol{g}_{T_1}, T_1]_W$. Since $[I^{[T_1]}\boldsymbol{g}_{T_1}, T_1]_V = \iota(g)$, there is a $T_2 > T_1$ such that

$$I^{[T_2]}N^{[T_2,T_1)}\boldsymbol{g}_{T_1} = M^{[T_2,T_1)}I^{[T_1]}\boldsymbol{g}_{T_1} = \frac{M^{[T_2,k)}I^{[k]}\boldsymbol{c}}{c}.$$

The equality above and $M^{[T_2,k)}I^{[k]} = I^{[T_2]}N^{[T_2,k)}$ mean that $I^{[T_2]}(cN^{[T_2,T_1)}\boldsymbol{g}_{T_1}) = I^{[T_2]}N^{[T_2,k)}\boldsymbol{c}$. As ι is injective, we get $[cN^{[T_2,T_1)}\boldsymbol{g}_{T_1}, T_2]_W = [N^{[T_2,k)}\boldsymbol{c}, T_2]_W$. This implies that there is $T > T_2$ such that for $t \geq T$, $cN^{[t,T_1)}\boldsymbol{g}_{T_1} = N^{[t,k)}\boldsymbol{c}$. So we finish the proof. □

Proposition A.3. *Let $\Delta := \bigcap_{t>k} \Delta_{W_k}(t)$. Suppose $\Delta_{W_k}(T) = \Delta_{W_k}(T+1)$ for some T. Then $\Delta_{W_k}(T) = \Delta$ and Δ has only one point.*

Proof. Recall that $\Delta_{W_k}(T) = \{\boldsymbol{s}N^{[T,k)*} \mid \boldsymbol{s} \in \Delta_{W_T}\}$. We will show that for any $w, w' \in W_T$, $N^{[T,k)*}_{w*} = N^{[T,k)*}_{w*}$ holds. This means that $\Delta_{W_k}(T)$ has only one point. $\Delta_{W_k}(T) = \Delta_{W_k}(T+1)$ implies that for any $w \in W_T$, there exists $\boldsymbol{s} \in \Delta_{W_{T+1}}$ such that $N^{[T,k)*}_{w*} = \boldsymbol{s}N^{[T+1,k)*}$. We write $(t_v)_{v \in W_T} = \boldsymbol{s}N^{[T+1]*}$. Suppose $w \in W_T$ is an extreme point in $\Delta_{W_k}(T)$. Then

$$N^{[T,k)*}_{w*} = \boldsymbol{s}N^{[T+1,k)*} = \sum_{v \in W_T} t_v N^{[T,k)*}_{v*} \tag{A.1}$$

Since $0 < N^{[T+1]*}_{xy} < 1$ for all $x \in W_{T+1}$ and $y \in W_T$, $0 < t_v < 1$ holds for any v. Therefore by (A.1), $N^{[T,k)*}_{v*} = N^{[T,k)*}_{w*}$ for all v because $N^{[T,k)*}_{w*}$ is extreme in $\Delta_{W_k}(T)$. □

REFERENCES

[1] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Mathematics, Vol. **527**, Springer, Berlin, 1976. MR0457675 (56 #15879)

[2] E. G. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conference Series in Mathematics, **46**, Conf. Board Math. Sci., Washington, DC, 1981. MR0623762 (84k:46042)

[3] T. Giordano, I. F. Putnam and C. F. Skau, Topological orbit equivalence and C^*-crossed products, J. Reine Angew. Math. **469** (1995), 51–111. MR1363826 (97g:46085)

[4] T. Giordano, I. F. Putnam and C. F. Skau, K-theory and asymptotic index for certain almost one-to-one factors, Math. Scand. **89** (2001), no. 2, 297–319. MR1868179 (2002i:46071)

[5] E. Glasner and B. Weiss, Weak orbit equivalence of Cantor minimal systems, Internat. J. Math. **6** (1995), no. 4, 559–579. MR1339645 (96g:46054)

[6] R. H. Herman, I. F. Putnam and C. F. Skau, Ordered Bratteli diagrams, dimension groups and topological dynamics, Internat. J. Math. **3** (1992), no. 6, 827–864. MR1194074 (94f:46096)

[7] L. J. Mordell, *Diophantine equations*, Pure and Applied Mathematics, Vol. **30** Academic Press, London, 1969. MR0249355 (40 #2600)

[8] R. R. Phelps, *Lectures on Choquet's theorem*, second edition. Lecture Notes in Mathematics, 1757. Springer-Verlag, Berlin, 2001. MR1835574 (2002k:46001)

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