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AF-embeddings of residually finite-dimensional C^*-algebras

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Abstract. It is shown that a separable exact residually finite-dimensional C^*-algebra with locally finitely generated (rational) K^0-homology embeds in an uniformly hyperfinite C^*-algebra.

1. Introduction

Kirchberg proved that any separable exact C^*-algebra embeds in the Cuntz algebra $\mathcal{O}_2$, see [6]. A related major open problem asks if any separable exact and quasi-diagonal C^*-algebra embeds in an almost finite-dimensional algebra (AF-algebra), see [1, Ch. 8]. Most positive results on AF-embeddability depend on the universal coefficient theorem in KK-theory (abbreviated UCT) [10], see for example [3, 8, 11]. A general result of Ozawa [7] shows that the cone over an exact separable C^*-algebra is AF-embeddable. Such cones are automatically quasi-diagonal by a theorem of Voiculescu [12]. While Ozawa's proof does not use the UCT explicitly, cones are contractible and in particular they do satisfy the UCT. Cones also play a key role in Rørdam's paper on purely infinite AH-algebras and AF-embeddings [9]. Indeed, Rørdam's C^*-algebra $\mathcal{A}[0,1]$, which he showed that contains the cone over $\mathcal{O}_2$ as a subalgebra, is itself an inductive limit of cones over matrix algebras and in particular it is KK-contractible. In a very recent paper [5], Gabe proves that a separable exact C^*-algebra for which its primitive spectrum has no nonempty compact open subsets embeds in $\mathcal{A}[0,1]$ and hence it is AF-embeddable. As far as I am aware, all previously known AF-embeddings results which do not assume the UCT factor through inductive limits of cones. Consequently, these results are not applicable to C^*-algebras that contain nonzero projections.

A C^*-algebra A is called *residually finite-dimensional* (abbreviated RFD) if the finite-dimensional representations of A separate the points of A. We have

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shown in [3] that a separable, exact, RFD C^*-algebra which satisfies the UCT is AF-embeddable and in fact it even embeds in a UHF-algebra $\bigotimes_{n=1}^{\infty} M_{k(n)}(\mathcal{C})$. In the present note we point out that the arguments of [3] can be adapted to obtain a UHF-embeddability result for RFD C^*-algebras which does not assume the UCT but requires (local) finite generation of the even K-homology, see Definition 2.1.

Theorem 1.1. *Let A be a separable exact residually finite-dimensional C^*-algebra. If the rational K^0-homology of A is locally finitely generated, then A embeds in a UHF-algebra.*

2. Preliminaries

Let A be a C^*-algebra. A family $\mathcal{D}$ of C^*-subalgebras of A is called exhaustive if for any finite subset $\mathcal{F} \subset A$ and any $\varepsilon > 0$, there exists $D \in \mathcal{D}$ such that $\mathcal{F} \subset_\varepsilon D$, i.e., for each $x \in \mathcal{F}$, there is $d \in D$ such that $\|x - d\| < \varepsilon$.

Definition 2.1. We say that the K^0-homology of A is *locally finitely generated* if there is an exhaustive family $\mathcal{D}$ of C^*-subalgebras of A such that for every $D \in \mathcal{D}$, the abelian group $K^0(D) = KK(D, \mathbb{C})$ is finitely generated. If instead we require the weaker condition that each $\mathbb{Q}$-vector space $K^0(D) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite-dimensional, then we say that the *rational K^0-homology* of A is locally finitely generated. The case when the vector space $K^0(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is itself finite-dimensional is an obvious first example.

Let $\mathcal{L}(\mathcal{H})$ denote the linear operators acting on a separable Hilbert space $\mathcal{H}$ and let $\mathcal{K}(\mathcal{H})$ denote the compact operators. We make the identifications $\mathcal{L}(\mathbb{C}^k) = \mathcal{K}(\mathbb{C}^k) \cong M_k(\mathbb{C})$. The unitary group of $M_k(\mathbb{C})$ is denoted by $U(k)$. Let A be a C^*-algebra, let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. If $\varphi\colon A \to \mathcal{L}(\mathcal{H}_\varphi)$ and $\psi\colon A \to \mathcal{L}(\mathcal{H}_\psi)$ are two maps, we write $\varphi \sim_{\mathcal{F},\varepsilon} \psi$ if there is a unitary $v\colon \mathcal{H}_\varphi \to \mathcal{H}_\psi$ such that $\|v\varphi(a)v^* - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.

If m is a positive integer and π is a representation, then $m\pi$ will denote the representation $\pi \oplus \cdots \oplus \pi$ (m-times). The infinite direct sum $\pi \oplus \pi \oplus \cdots$ is denoted by π_∞.

We need the following approximation result.

Proposition 2.2 ([2, Prop. 6.1])**.** *Let A be a unital separable exact C^*-algebra and let $(\chi_n)_{n\geq 1}$ be a sequence of unital representations of A that separates the elements of A and such that each representation in the sequence repeats itself infinitely many times. For any $\mathcal{F} \subset A$, a finite subset, and any $\varepsilon > 0$, there is an integer $r \geq 1$ such that if $\pi = \chi_1 \oplus \chi_2 \oplus \cdots \oplus \chi_r$, then for any unital faithful representation $\sigma\colon A \to \mathcal{L}(\mathcal{H})$, with $\sigma(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, one has $\sigma \sim_{\mathcal{F},\varepsilon} \pi_\infty$.*

Definition 2.3. Let A be a unital RFD C^*-algebra. Let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. A unital representation $\pi\colon A \to M_k(\mathbb{C})$ is called $(\mathcal{F}, \varepsilon)$-admissible if there is a unital faithful representation $\sigma\colon A \to \mathcal{L}(\mathcal{H})$, with $\sigma(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$ ($\mathcal{H} = \mathbb{C}^k \oplus \mathbb{C}^k \oplus \cdots$) such that

$$\|\sigma(a) - \pi_\infty(a)\| < \varepsilon \quad \text{for all } a \in \mathcal{F}. \tag{1}$$

Remark 2.4. Note that if π is $(\mathcal{F},\varepsilon)$-admissible, then so is $\pi\oplus\alpha$ for any unital finite-dimensional representation α. Moreover, $\|\pi(a)\| > \|a\| - \varepsilon$ for $a \in \mathcal{F}$. If a unital C^*-algebra A is separable exact and RFD, then Proposition 2.2 guaranties the existence of $(\mathcal{F},\varepsilon)$-admissible representations for any finite set $\mathcal{F} \subset A$ and any $\varepsilon > 0$.

The following proposition is crucial for our embedding result. It is based on a uniqueness theorem from [4].

Proposition 2.5. *Let A be a unital separable exact RFD C^*-algebra. Let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. Then for any $(\mathcal{F},\varepsilon)$-admissible representation $\pi\colon A \to M_k(\mathbb{C})$ and any two unital representations $\varphi,\psi\colon A \to M_r(\mathbb{C})$ such that $[\varphi] = [\psi] \in K^0(A)$, there exist a positive integer M and a unitary $u \in U(r + Mk)$ such that*

$$\|u(\varphi(a) \oplus M\pi(a))u^* - \psi(a) \oplus M\pi(a)\| < 3\varepsilon \quad \text{for all } a \in \mathcal{F}. \tag{2}$$

Proof. Fix $\mathcal{F}$, ε and π. Let σ be a unital faithful representation of A given by Definition 2.3. In particular, σ satisfies (1) and $\sigma(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Since $[\varphi] = [\psi] \in K^0(A)$, it follows that if we set $\Phi = \varphi \oplus \sigma$ and $\Psi = \psi \oplus \sigma$, then $\Phi(a) - \Psi(a)$ is a compact operator for all a, and the class of the Kasparov triple $(\Phi, \Psi, 1)$ in $KK(A,\mathbb{C}) = K^0(A)$ vanishes since

$$[\Phi, \Psi, 1] = [\varphi \oplus \sigma, \psi \oplus \sigma, 1_r \oplus 1_{\mathcal{H}}] = [\varphi] - [\psi] = 0.$$

Moreover, both Φ and Ψ are faithful representations whose images do not contain nonzero compact operators. This enables us to apply [4, Thm. 3.12] and obtain that Φ is asymptotically unitarily equivalent to Ψ via a continuous path of unitaries which are compact perturbations of the identity. In particular, there is a unitary $v \in U(\mathbb{C}^r \oplus \mathcal{H})$ of the form $v = 1 + x$, with $x \in \mathcal{K}(\mathbb{C}^r \oplus \mathcal{H})$, such that $\|v(\varphi(a) \oplus \sigma(a))v^* - \psi(a) \oplus \sigma(a)\| < \varepsilon$ for all $a \in \mathcal{F}$. Using (1), we obtain that

$$\|v(\varphi(a) \oplus \pi_\infty(a))v^* - \psi(a) \oplus \pi_\infty(a)\| < 3\varepsilon \quad \text{for all } a \in \mathcal{F}. \tag{3}$$

Since π is a unital representation, it follows that the sequence of projections $p_n = 1_r \oplus n\pi(1)$ forms an approximate unit of $\mathcal{K}(\mathbb{C}^r \oplus \mathcal{H})$ and hence $[p_n, v] = [p_n, x] \to 0$ as $n \to \infty$. Since each p_n commutes with both $\varphi(a) \oplus \pi_\infty(a)$ and $\psi(a) \oplus \pi_\infty(a)$, we obtain from (3) that

$$\|(p_n v p_n)(\varphi(a) \oplus n\pi(a))(p_n v p_n)^* - \psi(a) \oplus n\pi(a)\| < 3\varepsilon$$

for all $a \in \mathcal{F}$ and all sufficiently large n. Moreover, one can perturb the almost unitary operator $p_n v p_n$ to a unitary u satisfying (2) for a sufficiently large value of n denoted M. ☐

3. Proof of Theorem 1.1

Without any loss of generality, we may assume that A is unital. We denote by $\mathrm{Rep}_{\mathrm{fd}}(A)$ the set of unital finite-dimensional representations of A. Since A is separable and RFD, there is a sequence $(\chi_n)_{n\geq 1}$ in $\mathrm{Rep}_{\mathrm{fd}}(A)$ which separates the points of A and such that each representation in the sequence repeats itself infinitely many times. Let $(x_n)_{n=1}^\infty$ be a dense sequence of elements of A and let $\varepsilon_n = 2^{-n}$. Since $K^0(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ is locally finitely generated, for each $n\geq 1$, there is a unital C^*-subalgebra A_n of A such that the $\mathbb{Q}$-vector space $K^0(A_n)\otimes_{\mathbb{Z}}\mathbb{Q}$ is finite-dimensional and $X_n := \{x_1,\dots,x_n\}\subset_{\varepsilon_n} A_n$. Fix a finite set $\mathcal{F}_n=\{a_{n,1},\dots,a_{n,n}\}\subset A_n$ such that $\|x_i - a_{n,i}\|<\varepsilon_n$ for all $1\leq i\leq n$. Since $K^0(A_n)\otimes_{\mathbb{Z}}\mathbb{Q}$ is finite-dimensional, its subspace V_n generated by all the classes $\{[\chi_i|_{A_n}]\otimes 1 \mid i\geq 1\}$ must also be finite-dimensional. Thus there is an integer r_n such that V_n is generated by just $\{[\chi_i|_{A_n}]\otimes 1 \mid 1\leq i\leq r_n\}$.

Define $\pi_n\in \mathrm{Rep}_{\mathrm{fd}}(A)$ by $\pi_n=\chi_1\oplus\chi_2\oplus\cdots\oplus\chi_{r_n}$. By Proposition 2.2, after increasing r_n, if necessary, we can moreover arrange that $\pi_n|_{A_n}$ is $(\mathcal{F}_n,\varepsilon_n)$-admissible.

With these choices, we are going to construct a sequence of unital representations $\gamma_n\colon A\to M_{k_n}(\mathbb{C})$ such that for all $n\geq 1$,

(i) $k_{n+1}=k_n m_n$ for some positive integer m_n,
(ii) γ_n is unitarily equivalent to $\pi_n\oplus\alpha_n$ for some $\alpha_n\in\mathrm{Rep}_{\mathrm{fd}}(A)$,
(iii) $\|\gamma_{n+1}(x)-m_n\gamma_n(x)\|<5\varepsilon_n$ for all $x\in X_n$.

We will see that in fact each γ_n is unitarily equivalent to a representation of the form $q_1\chi_1\oplus q_2\chi_2\oplus\cdots\oplus q_{r_n}\chi_{r_n}$, for integers $q_i\geq 0$.

Set $\gamma_1=\pi_1\oplus\chi_1$. Suppose now that α_i and γ_i were constructed for all $i\leq n$ such that the properties (i), (ii) and (iii) are satisfied. We construct α_{n+1} and γ_{n+1} as follows.

We need the following elementary observation. Suppose that G is an abelian group such that the vector space $G\otimes_{\mathbb{Z}}\mathbb{Q}$ is finite-dimensional and it is spanned by $g_1\otimes 1,\dots,g_r\otimes 1$, with $g_i\in G$. Then for any $g\in G$, there are strictly positive integers $p,m,q_1,\dots,q_r$, such that

$$pg+q_1g_1+\cdots+q_rg_r=m(g_1+\cdots+g_r).$$

By applying this observation to the abelian subgroup of $K^0(A_n)$ generated by $\{[\chi_i|_{A_n}]\mid i\geq 1\}$, with $g_i=[\chi_i|_{A_n}]$, $i=1,\dots,r_n$, one obtains strictly positive integers $p,m,q_1,\dots,q_{r_n}$ such that

$$p[\pi_{n+1}|_{A_n}]+\sum_{i=1}^{r_n} q_i[\chi_i|_{A_n}] = m\sum_{i=1}^{r_n}[\chi_i|_{A_n}] \quad \text{in } K^0(A_n).$$

Set $\alpha'_{n+1}=(p-1)\pi_{n+1}\oplus\big(\bigoplus_{i=1}^{r_n} q_i\chi_i\big)\oplus m\alpha_n$. Then

$$[(\pi_{n+1}\oplus\alpha'_{n+1})|_{A_n}]=\sum_{i=1}^{r_n} m[\chi_i|_{A_n}]+m[\alpha_n|_{A_n}]=m[\pi_n|_{A_n}]\oplus m[\alpha_n|_{A_n}],$$

and hence $[(\pi_{n+1}\oplus\alpha'_{n+1})|_{A_n}]=m[\gamma_n|_{A_n}]$, using (ii). In particular, the representations $\pi_{n+1}\oplus\alpha'_{n+1}$ and $m\gamma_n$ have the same dimension. Since $\pi_n|_{A_n}$ is $(\mathcal{F}_n,\varepsilon_n)$-admissible, so is $\gamma_n|_{A_n}$, as noted in Remark 2.4. By Proposition 2.5 applied to A_n, there is an integer $M\geq 1$ such that

$$(\pi_{n+1}\oplus\alpha'_{n+1})|_{A_n}\oplus M\gamma_n|_{A_n}\sim_{\mathcal{F}_n,3\varepsilon_n} m\gamma_n|_{A_n}\oplus M\gamma_n|_{A_n}.$$

Set $\alpha_{n+1} = \alpha'_{n+1} \oplus M\gamma_n$, $m_n = m + M$ and $\gamma_{n+1} = \pi_{n+1} \oplus \alpha_{n+1}$. Then

$$\gamma_{n+1}|_{A_n} \sim_{\mathcal{F}_n, 3\varepsilon_n} m_n \gamma_n|_{A_n}. \tag{4}$$

Since $\|x_i - a_{n,i}\| < \varepsilon_n$ for all $1 \le i \le n$, we deduce immediately from (4) that $\gamma_{n+1} \sim_{X_n, 5\varepsilon_n} m_n\gamma_n$. By conjugating γ_{n+1} by a suitable unitary, we can arrange that $\|\gamma_{n+1}(x) - m_n\gamma_n(x)\| < 5\varepsilon_n$, for all $x \in X_n$.

Consider the UHF algebra $B = \varinjlim M_{k(n)}(\mathbb{C})$ and let $\iota_n\colon M_{k(n)}(\mathbb{C}) \to B$ be the canonical inclusion. Having the sequence γ_n available, we construct a unital embedding $\gamma\colon A \to B$ by defining $\gamma(x)$, $x \in \{x_1, x_2, \dots\}$, to be the limit of the Cauchy sequence $(\iota_n\gamma_n(x))_{n\ge1}$ and then extend to A by continuity. Note that γ is a $*$-homomorphism, since all γ_n are $*$-homomorphisms. Moreover, $\|\gamma(x)\| = \|x\|$ for all $x \in A$, since $\|\gamma_n(a)\| > \|a\| - \varepsilon_n$ for $a \in \mathcal{F}_n$ (by Remark 2.4), hence $\|\gamma_n(x_i)\| \ge \|x_i\| - 3\varepsilon_n$, as $\|a_{n,i} - x_i\| < \varepsilon_n$ for $1 \le i \le n$.

Remark 3.1. It is clear from the proof that the conclusion of Theorem 1.1 holds under the weaker assumption that A admits a separating sequence of finite-dimensional representations $(\chi_n)_{n=1}^\infty$ such that for some exhaustive family $\mathcal{D}$ of C^*-subalgebras of A, the vector subspace of $K^0(D) \otimes_{\mathbb{Z}} \mathbb{Q}$ spanned by $\{[\chi_n|_D] \otimes 1 \mid n \ge 1\}$ is finite-dimensional for all $D \in \mathcal{D}$. For example, this condition is satisfied if A is the suspension of a separable exact RFD C^*-algebra. Indeed, in that case one can choose a separating sequence $(\chi_n)_{n=1}^\infty$ with the property that $[\chi_n] = 0$ in $K^0(A)$ for all $n \ge 1$.

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