Reine Mathematik

Invariant distributions on p-adic analytic groups

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Introduction

Let p be a prime number, L a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, K a spherically complete extension of L and G a locally L-analytic group of finite dimension with center Z and Lie algebra $\mathfrak{g}$.

The present paper is devoted to the study of the center $D(G,K)^G$ of the K-algebra $D(G,K)$ of locally analytic distributions on G whose definition is recalled in section 1.1. At the heart of our approach lies the simple observation that for locally analytic distributions on G there is a well-defined notion of support and that the support $supp(\delta)$ is a compact subset of G for any distribution $\delta \in D(G,K)$. It follows from the definition of the convolution product in $D(G,K)$ that any invariant distribution, i.e. any element of $D(G,K)^G$, is supported on a union of relatively compact conjugacy classes in G. If G is the group of L-rational points of a connected, reductive, linear algebraic group $\mathbb{G}$ all of whose simple factors are L-isotropic (e.g. an L-split group) then the only conjugacy classes of G which are relatively compact are the trivial ones, i.e. those belonging to the elements of Z. This is due to K.-Y. Sit (cf. [32], Theorem 2.4) generalizing work of J. Tits'. Therefore, we are led to the investigation of the K-algebra $D(G,K)_Z$ of centrally supported distributions on G.

If $\mathfrak{z}$ denotes the Lie algebra of Z then we let $U(\mathfrak{z},K)$ (resp. $U(\mathfrak{g},K)$) be the subalgebra of $D(Z,K)$ (resp. $D(G,K)$) consisting of distributions supported in the unit element. There is a natural continuous K-linear map

$$D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K) \longrightarrow D(G,K)_Z \tag{0.1}$$

of locally convex $D(Z,K)$-$U(\mathfrak{g},K)^{op}$-bimodules (here ι indicates the inductive tensor product topology). It is the main technical result of our work that under the assumption that K is discretely valued this map is a topological isomorphism (cf. Proposition 1.2.12). Its proof takes up most of section 1 and relies for one thing on certain compatibility conditions for global charts of small open subgroups of G and Z, respectively (cf. Proposition 1.3.5 and Corollary 1.3.6). On the other hand, we make extensive use of the fact that $D(G,K)$ is a K-Fréchet-Stein algebra (a notion introduced by P. Schneider and J. Teitelbaum) and a structure theorem of $D(G,K)$ as a module over $U(\mathfrak{g},K)$ after a certain completion process. The latter is due to H. Frommer who proved it for $\mathbb{Q}_p$ as a ground field. We generalize it to any finite extension $L|\mathbb{Q}_p$ (cf. Theorem 1.4.2).

G acts on $U(\mathfrak{g},K)$ and $D(G,K)_Z$. If G is an open subgroup of the group of L-rational points of a connected, algebraic L-group $\mathbb{G}$ then we derive from (0.1) a topological isomorphism

$$D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G \longrightarrow D(G,K)_Z^G \tag{0.2}$$

of K-algebras (cf. Theorem 2.2.1). If moreover G satisfies the hypotheses of Sit's theorem then $D(G,K)^G = D(G,K)_Z^G$ and it remains to examine the "*infinitesimal center*" $U(\mathfrak{g},K)^G$.

Consider $\mathfrak{g}$ as an abelian locally L-analytic group and let $S(\mathfrak{g},K)$ be the subalgebra of $D(\mathfrak{g},K)$ consisting of distributions supported in $0 \in \mathfrak{g}$. $S(\mathfrak{g},K)$ and $U(\mathfrak{g},K)$ carry actions of G and $\mathfrak{g}$. We show that Duflo's famous isomorphism $S(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{g})^{\mathfrak{g}}$ extends to a topological isomorphism $S(\mathfrak{g},K)^{\mathfrak{g}} \to U(\mathfrak{g},K)^{\mathfrak{g}}$ of K-Fréchet algebras (cf. Proposition 2.1.5; $S(\mathfrak{g})$ and $U(\mathfrak{g})$ denote the symmetric and the universal enveloping algebra of $\mathfrak{g}$, respectively). If $\mathfrak{g}$ is split semisimple with split maximal toral subalgebra $\mathfrak{t}$ and corresponding Weyl group $\mathfrak{W}$ then $\mathfrak{W}$ naturally acts on the algebra $S(\mathfrak{t},K)$ of locally analytic distributions on $\mathfrak{t}$ supported in $0 \in \mathfrak{t}$. We show that the classical isomorphism $S(\mathfrak{g})^{\mathfrak{g}} \to S(\mathfrak{t})^{\mathfrak{W}}$ extends to a topological isomorphism $S(\mathfrak{g},K)^{\mathfrak{g}} \simeq S(\mathfrak{t},K)^{\mathfrak{W}}$ of K-algebras (cf. Theorem 2.1.6). It follows that

$$U(\mathfrak{g},K)^{\mathfrak{g}} \simeq S(\mathfrak{t},K)^{\mathfrak{W}}. \tag{0.3}$$

Even more is true: Just as $S(\mathfrak{t})^{\mathfrak{W}}$ is a polynomial ring in $n := \dim_L(\mathfrak{t})$ variables, $S(\mathfrak{t}, K)^{\mathfrak{W}}$ is the algebra of holomorphic functions on the rigid analytic affine space $(\mathbb{A}^n_K)^{an}$ of dimension n over K (loc.cit.). Thus,

$$U(\mathfrak{g}, K)^{\mathfrak{g}} \simeq \mathcal{O}((\mathbb{A}^n_K)^{an}). \tag{0.4}$$

This isomorphism is constructed by showing that on the category of reduced affine K-varieties the passage to quotients by finite groups commutes with the rigid analytification functor (cf. Proposition 2.1.7 and Remark 2.1.8).

If G is the group of L-rational points of a connected, split reductive L-group $\mathbb{G}$ then (0.2) – (0.4) enable us to give two different, explicit descriptions of $D(G, K)^G$. Using results on the Fourier transform of Z obtained by M. Emerton, P. Schneider and J. Teitelbaum we deduce the existence of an explicitly computable quasi-Stein rigid analytic K-variety X_K and a continuous injection

$$D(G, K)^G \longrightarrow \mathcal{O}(X_K)$$

with dense image (cf. Corollary 2.3.4 and Remark 2.3.5). If $\mathbb{T}$ is a maximal L-split torus of $\mathbb{G}$, $T := \mathbb{T}(L)$ and $W := N_G(T)/T$ the corresponding Weyl group then we also construct a topological isomorphism

$$D(G, K)^G \simeq D(T, K)^W_Z$$

of separately continuous K-algebras extending Harish-Chandra's isomorphism $U(\mathfrak{g})^{\mathfrak{g}} \simeq S(\mathfrak{t})^W$ (cf. Theorem 2.4.2).

In the final section 2.5 we study the relation between $D(G, K)^G$ and the center $\hat{Z}$ of the category of smooth representations of G (known as the Bernstein center). We show that if G satisfies the hypotheses of Sit's theorem then the natural map $D(G, K)^G \to \hat{Z}$ has dense image with respect to the projective limit topology on $\hat{Z}$ if and only if G is abelian (cf. Proposition 2.5.3). Thus, in most cases of interest there is no way of directly studying the Bernstein center through the center of $D(G, K)$. Yet, since Harish-Chandra's isomorphism plays such a fundamental role in the representation theory of the Lie algebra $\mathfrak{g}$ it is to be hoped that our extension will prove important for the theory of locally analytic representations as studied by P. Schneider and J. Teitelbaum.

The present work comprises the author's thesis. He is deeply indebted to Prof. Dr. P. Schneider without whose guidance it would not have come into existence. He is also grateful to Prof. Dr. S. Bosch and Dr. M. Strauch for

many helpful discussions.

Conventions and notation. Throughout this paper p denotes a prime number and L a finite extension of $\mathbb{Q}_p$. Let $\mathfrak{o}_L$ be the ring of integers of L with maximal ideal $\mathfrak{m}_L$ and uniformizer π_L. We assume the valuation ω on L to be normalized such that $\omega(\pi) = 1$. Let further $e := \omega(p)$ be the ramification index of the extension $L|\mathbb{Q}_p$ and m its degree. The absolute value $|\cdot|$ of L corresponding to ω is assumed to be normalized through $|p| = p^{-1}$. We let K be a fixed spherically complete extension of L which for many results will have to be assumed to be discretely valued (cf. subsection 1.4, in particular). Let $\mathfrak{o}_K$ denote its ring of integers. We assume the absolute value $|\cdot|$ on K to extend the one on L. If V is a locally convex vector space over K then we let $V' := \mathrm{Hom}_K^{\mathrm{cont}}(V, K)$ denote the space of continuous functionals on V. We write V'_b (resp. V'_s) for the locally convex K-vector space V' endowed with the topology of strong (resp. weak) convergence. G will always be a locally L-analytic group of finite dimension d with center Z. The center of the Lie algebra $\mathfrak{g}$ of G will be denoted by $\mathfrak{z}$. We also fix an exponential map $exp: \mathfrak{g} \dashrightarrow G$ defined locally around zero in $\mathfrak{g}$.

1 Locally analytic distributions

1.1 Functoriality

Let M be a paracompact, locally L-analytic manifold of finite dimension d. Since M is automatically strictly paracompact (cf. [26], p. 35) the locally convex K-vector space $C^{an}(M, K)$ of locally analytic functions on M with values in K can be defined as in [18], Definition 2.1.10 (see also [29], section 2). It is the locally convex inductive limit

$$C^{an}(M, K) = \varinjlim_I \mathcal{F}_I(K),$$

where I runs through the inductive system of all "indices". Here an index I is a family of pairs $\{(D_i, \varphi_i)\}_{i\in I}$ such that $(D_i)_{i\in I}$ is a covering of M by disjoint open subsets and such that every D_i is analytically isomorphic to an affinoid ball in L^d via the chart φ_i of M. Further,

$$\mathcal{F}_I(K) := \prod_{i\in I} \mathcal{F}_{\varphi_i}(K)$$

is the locally convex direct product of the K-Banach spaces $\mathcal{F}_{\varphi_i}(K)$ of functions $f : D_i \to K$ such that $f \circ \varphi_i^{-1}$ is a K-valued rigid analytic function on the affinoid ball $\varphi_i(D_i)$. The space of locally analytic distributions on M is the locally convex K-vector space

$$D(M,K) := C^{an}(M,K)'_b.$$

If $(M_i)_{i\in I}$ is a covering of M by disjoint open subsets M_i then there is a topological isomorphism

$$C^{an}(M,K) \simeq \prod_{i\in I} C^{an}(M_i,K)$$

dualizing to a topological isomorphism

$$D(M,K) \simeq \bigoplus_{i\in I} D(M_i,K) \tag{1.1}$$

(cf. [18], Korollar 2.2.4). If M is compact, then $C^{an}(M,K)$ is a K-vector space of compact type and, in particular, is reflexive (cf. [25], Proposition 16.10). In this case $D(M,K)$ is a nuclear Fréchet space (cf. [29], Lemma 2.1 and Theorem 1.3).

There is an embedding $M \hookrightarrow D(M,K)$, sending $m \in M$ to the Dirac distribution $\delta_m := (f \mapsto f(m))$.

Lemma 1.1.1. *The subspace $K[M]$ of $D(M,K)$ generated by all Dirac distributions δ_m, $m \in M$, is dense.*

In [29] this statement is proved for a locally L-analytic group G (loc.cit. Lemma 3.1). The group structure on G is used in order to reduce to a situation where G is compact. However, this reduction step is not necessary because $C^{an}(M,K)$ is reflexive without any assumptions on M: choosing a covering of M by disjoint compact open subsets, the isomorphism preceding (1.1) shows that $C^{an}(M,K)$ is the locally convex direct product of reflexive K-vector spaces and hence is reflexive itself (cf. [25], Proposition 9.10 and Proposition 9.11). Therefore, all arguments remain unchanged. For the sake of completeness we repeat the proof:

Proof of Lemma 1.1.1: Let Δ be the closure of $K[M]$ in $D(M,K)$ and ℓ a continuous linear functional on $D(M,K)$ vanishing on Δ. By reflexivity of $C^{an}(M,K)$, ℓ corresponds to a locally analytic function f on M. To say that ℓ vanishes on Δ is to say that f is identically zero on M, i.e. $\ell = 0$. But then $\Delta = D(M,K)$ according to the Hahn-Banach theorem (cf. [25], Corollary 9.3 applied to $D(M,K)/\Delta$). $\square$

Now let N, M be paracompact, locally L-analytic manifolds of finite dimension and $\varphi : N \to M$ be a morphism. φ defines a K-linear map $\varphi^* : C^{an}(M,K) \to C^{an}(N,K)$ via $\varphi^*(f) := f \circ \varphi$ for $f \in C^{an}(M,K)$ (cf. [10], 5.4.5).

Proposition 1.1.2. *φ^* is continuous with respect to the locally convex topologies defined above.*

Proof: Let $(N_i)_{i\in I}$ be a covering of N by disjoint compact open subsets. Since φ^* is continuous if and only if for all $i \in I$ the compositions

$$C^{an}(M,K) \xrightarrow{\varphi^*} C^{an}(N,K) \simeq \prod_{i\in I} C^{an}(N_i,K) \longrightarrow C^{an}(N_i,K)$$

are continuous, we may assume N to be compact.

Consider the K-linear map $\tilde{\varphi} : K[N] \to D(M,K)$ defined by $\tilde{\varphi}(\delta_n) := \delta_{\varphi(n)}$, $n \in N$. We show that $\tilde{\varphi}$ is continuous if we endow $K[N]$ with the subspace topology of $D(N,K)$. Let $(\tau_k)_{k\in\mathbb{N}}$ be a zero sequence in $D(N,K)$ with $\tau_k \in K[N]$ for all $k \in \mathbb{N}$. Since strong convergence implies convergence in the weak topology, we have $\tau_k(f \circ \varphi) \to 0$ in K for all $f \in C^{an}(M,K)$. Hence we have $\lim_{k\to\infty} \tilde{\varphi}(\tau_k) = 0$ in $C^{an}(M,K)'_s$ given its weak topology. Following the arguments given at the end of section 2 in [29], we show that this limit formula even holds in $C^{an}(M,K)'_b = D(M,K)$.

As the strong dual of a reflexive K-vector space, $D(M,K)$ is reflexive itself (cf. [25], Lemma 15.4). Choose a closed, bounded $\mathfrak{o}_K$-submodule A of $C^{an}(M,K)'_s$ containing all $\tilde{\varphi}(\tau_k)$, $k \in \mathbb{N}$. Since $D(M,K)$ is reflexive, the strong topology on $C^{an}(M,K)'$ is admissible (cf. [25], Proposition 15.5 and Proposition 14.4). Hence A is a closed bounded $\mathfrak{o}_K$-submodule in $D(M,K)$ (loc.cit. Proposition 14.2) and therefore is compactoid (loc.cit. Proposition 15.3). Now on any such module the weak and the given topology coincide (loc.cit. Proposition 14.5). Thus, $(\tilde{\varphi}(\tau_k))_{k\in\mathbb{N}}$ converges to zero in $D(M,K)$. Since $D(N,K)$ (and hence $K[N]$) is metrizable and $\tilde{\varphi}$ is K-linear this suffices to show that $\tilde{\varphi}$ is continuous.

According to Lemma 1.1.1 $\tilde{\varphi}$ extends uniquely to a continuous K-linear map $D(N,K) \to D(M,K)$, again denoted by $\tilde{\varphi}$. The dual map $\tilde{\varphi}' : D(M,K)'_b \to D(N,K)'_b$ is continuous with respect to the strong topologies on both sides (loc.cit. Remark 16.1).

Let $\Delta_M : C^{an}(M,K) \to D(M,K)'_b$ and $\Delta_N : C^{an}(N,K) \to D(N,K)'_b$ be the canonical duality maps. They are topological isomorphisms because both

$C^{an}(M,K)$ and $C^{an}(N,K)$ are reflexive. The continuous K-linear map $\psi := \Delta_N^{-1} \circ \tilde{\varphi}' \circ \Delta_M : C^{an}(M,K) \to C^{an}(N,K)$ has the property that for all $f \in C^{an}(M,K)$ and all $n \in N$

$$\begin{aligned}\psi(f)(n) &= \delta_n(\psi(f)) = \Delta_N(\psi(f))(\delta_n) = \tilde{\varphi}'(\Delta_M(f))(\delta_n) = \Delta_M(f)(\tilde{\varphi}(\delta_n)) \\ &= \Delta_M(f)(\delta_{\varphi(n)}) = \delta_{\varphi(n)}(f) = f(\varphi(n)) = \varphi^*(f)(n),\end{aligned}$$

i.e. $\psi = \varphi^*$. Hence φ^* is continuous. □

Proposition 1.1.2 can also be proved more directly using the definition of $C^{an}(M,K)$ and $C^{an}(N,K)$ via indices (cf. [26], p. 65 or [18], Bemerkung 2.1.11). In any case, φ^* dualizes to a continuous K-linear map $\varphi_* : D(N,K) \to D(M,K)$ which, of course, coincides with the map $\tilde{\varphi}$ constructed in the above proof ($\tilde{\varphi}$ and φ_* coincide on the dense subspace $K[N]$ of $D(N,K)$).

Proposition 1.1.3. *Let $\varphi : N \to M$ be a closed embedding of paracompact, locally L-analytic manifolds of finite dimension, i.e. an immersion and a homeomorphism onto its closed image. Then $\varphi^* : C^{an}(M,K) \to C^{an}(N,K)$ is a strict surjection and $\varphi_* : D(N,K) \to D(M,K)$ is a topological embedding.*

Proof: Let $f \in C^{an}(N,K)$ and $a \in N$. Since φ is an immersion there is an open neighborhood U_a of a in N, an open neighborhood V_a of $\varphi(a)$ in M and a locally analytic manifold Z_a with the following properties: φ restricts to a morphism $\varphi_a : U_a \to V_a$ and there is an isomorphism $g : V_a \to U_a \times Z_a$ such that $pr_{U_a} \circ g \circ \varphi_a = id_{U_a}$ (cf. [10], 5.7.1; here pr_{U_a} is the projection onto U_a). It follows that $f|U_a = \varphi_a^*((pr_{U_a} \circ g)^*(f|U_a)) \in im(\varphi_a^*)$.

Let C be a closed and open subset of M with $\varphi(N) \subseteq C \subseteq \cup_{a\in N} V_a$. Such a set C exists because $\varphi(N)$ is closed, $\cup_{a\in N} V_a$ is open in M and M is an ultrametric topological space (cf. [26], p. 37). Now choose a refinement $(V_i)_{i\in I}$ of the open covering $(C \cap V_a)_{a\in N}$ of C consisting of disjoint open subsets V_i of C. Then $(M \setminus C, (V_i)_{i\in I})$ is a covering of M by disjoint open subsets. For each $i \in I$ choose a point $a \in N$ such that $V_i \subseteq V_a$. According to our above construction there is a function $g_a \in C^{an}(V_a,K)$ such that $\varphi_a^*(g_a) = f|U_a$. Set $g_i := g_a|V_i \in C^{an}(V_i,K)$ (note that V_i is open in V_a) and $g_{M\setminus C} := 0 \in C^{an}(M \setminus C, K)$. Then the family $g := (g_{M\setminus C}, (g_i)_{i\in I}) \in C^{an}(M,K)$ satisfies $\varphi^*(g) = f$ proving the surjectivity of φ^*.

In order to show that φ^* is open we may assume N and M to be compact: if $(M_i)_{i\in I}$ is a covering of M by disjoint compact open subsets then $(N_i :=$

$\varphi^{-1}(M_i))_{i\in I}$ is a covering of N of the same type. Under the topological isomorphisms

$$C^{an}(M,K) \simeq \prod_{i\in I} C^{an}(M_i,K) \text{ and } C^{an}(N,K) \simeq \prod_{i\in I} C^{an}(N_i,K)$$

the map φ^* is the direct product of the maps $\varphi_i^* : C^{an}(M_i,K) \to C^{an}(N_i,K)$ with $\varphi_i := \varphi|N_i : N_i \to M_i$. By definition of the product topology φ^* is open if and only if all φ_i^* are.
If M and N are compact then both $C^{an}(M,K)$ and $C^{an}(N,K)$ are locally convex K-vector spaces of compact type. In particular, they carry the locally convex final topology with respect to a countable family of BH-spaces. Therefore, the claim follows from [25], Proposition 8.8, and the surjectivity of φ^*.

If M and N are arbitrary again with coverings $(M_i)_{i\in I}$ and $(N_i)_{i\in I}$ as above then φ_* is the direct sum of the maps $(\varphi_i)_* : D(N_i,K) \to D(M_i,K)$. Since φ_i^* is strict surjective and $(\varphi_i)_*$ is the corresponding dual map, $(\varphi_i)_*$ is a topological embedding according to [29], Proposition 1.2 (i). The same is then true for φ_* by [25], Lemma 5.3 (i). □

Remark 1.1.4. In the situation of Proposition 1.1.3 we will henceforth write $D(N,K) \subseteq D(M,K)$ for the topological embedding $\varphi_* : D(N,K) \to D(M,K)$ of locally convex K-vector spaces. We shall see later that if M is a compact locally L-analytic group and N is a closed locally L-analytic subgroup then φ_* is even compatible with the structures of K-Fréchet-Stein algebras on $D(M,K)$ and $D(N,K)$, respectively (cf. Corollary 1.4.3).

If we assume $M = G$ to be a finite dimensional, locally L-analytic group then $D(G,K)$ carries the structure of a unital, associative K-algebra with separately continuous multiplication such that the natural inclusion $K[G] \hookrightarrow D(G,K)$ becomes a homomorphism of rings (cf. [29], section 2). It is explicitly given by

$$(\delta\cdot\delta')(f) = \delta'(g' \mapsto \delta(g \mapsto f(gg'))) \tag{1.2}$$

with $\delta, \delta' \in D(G,K)$ and $f \in C^{an}(G,K)$. If G_0 is an open subgroup of G then according to (1.1)

$$D(G,K) \simeq \bigoplus_{g\in G/G_0} D(g\cdot G_0,K) \simeq \bigoplus_{g\in G/G_0} \delta_g \cdot D(G_0,K).$$

If further H is a closed locally L-analytic subgroup of G then the topological embedding $D(H,K) \subseteq D(G,K)$ is a homomorphism of algebras. This is due to the fact that its restriction $K[H] \subseteq K[G]$ is a homomorphism of rings whence the claim follows from Lemma 1.1.1 and the separate continuity of the convolution product.

1.2 The notion of support

Definition 1.2.1. The support $supp(\delta)$ of a distribution $\delta \in D(M,K)$ is the complement of the largest open subset U of M such that $\delta(f) = 0$ for all $f \in C^{an}(M,K)$ with $supp(f) \subseteq U$. If C is a subset of M and $V \subseteq D(M,K)$ a subspace then we denote by V_C the subspace of all distributions $\delta \in V$ whose support is contained in C. Similarly, if W is a subspace of $C^{an}(M,K)$ then W_C denotes the subspace of all locally analytic functions $f \in W$ with $supp(f) \subseteq C$.

Remark 1.2.2. The existence of $supp(\delta)$ for a locally analytic distribution $\delta \in D(M,K)$ follows from the strict paracompactness of M: We need to show that if U_1, U_2 are open subsets of M such that $\delta(f) = 0$ for all $f \in C^{an}(M,K)$ with $supp(f) \subseteq U_1$ or $supp(f) \subseteq U_2$ then $\delta(f) = 0$ for all $f \in C^{an}(M,K)$ with $supp(f) \subseteq U_1 \cup U_2$. So let $f \in C^{an}(M,K)$ be supported on $U_1 \cup U_2$. Since $supp(f)$ is closed and $U_1 \cup U_2$ is open in M and since M is an ultrametric topological space, there is a closed and open subset A of M with $supp(f) \subseteq A \subseteq U_1 \cup U_2$ (cf. [26], p. 37). Since A is strictly paracompact, we can choose a refinement $(V_i)_{i\in I}$ of the covering $(U_1 \cap A, U_2 \cap A)$ of A consisting of disjoint open subsets V_i of A. Then $f|A \in C^{an}(A,K) = \prod_{i\in I} C^{an}(V_i,K)$ and we can write $f|A = (f_i)_{i\in I}$ with functions $f_i \in C^{an}(V_i,K)$ for all $i \in I$. Set $f^j := (f_i^j)_{i\in I}$, $j = 1,2$, with $f_i^1 := 0$ if $V_i \not\subseteq U_1 \cap A$ (i.e. $V_i \cap U_1 = \emptyset$), $f_i^1 := f_i$ if $V_i \subseteq U_1 \cap A$, $f_i^2 := 0$ if $V_i \subseteq U_1 \cap A$ and $f_i^2 := f_i$ if $V_i \not\subseteq U_1 \cap A$. Then $f^1, f^2 \in C^{an}(A,K)$ with $f^1 + f^2 = f|A$. Extending f^1, f^2 by zero outside of A we obtain functions $f^1, f^2 \in C^{an}(M,K)$ with $f^1 + f^2 = f$ and $supp(f^j) \subseteq U_j$, $j = 1,2$. By assumption $\delta(f) = \delta(f^1) + \delta(f^2) = 0$.

Remark 1.2.3. It follows from (1.1) that all locally analytic distributions on M are compactly supported, i.e. $supp(\delta)$ is a compact subset of M for all $\delta \in D(M,K)$.

If $M = G$ is again a locally L-analytic group, $g \in G$ and $\delta \in D(G,K)$ then according to (1.2)

$$supp(\delta_g \cdot \delta) = g \cdot supp(\delta) \text{ and } supp(\delta \cdot \delta_g) = supp(\delta) \cdot g. \tag{1.3}$$

More generally:

Lemma 1.2.4. *If $\delta_1, \delta_2 \in D(G,K)$ then $supp(\delta_1 \cdot \delta_2) \subseteq supp(\delta_1) \cdot supp(\delta_2)$.*

Proof: Assume that $g \in supp(\delta_1 \cdot \delta_2)$. Then for all open subgroups $H \subseteq G$ there is a locally analytic function $f \in C^{an}(G,K)$ supported on gH with $(\delta_1\delta_2)(f) = \delta_2(h \mapsto \delta_1(R_h f)) \neq 0$ (here R_h is the right translation operator associated with h). This implies (for fixed H and f) that $supp(\delta_2) \cap \{\gamma \in G : \delta_1(R_\gamma f) \neq 0\} \neq \emptyset$, i.e. there are elements $\gamma_2 \in supp(\delta_2)$ and $h \in H$ such that $supp(\delta_1) \cap (supp(f) \cdot h^{-1} \cdot \gamma_2^{-1}) \neq \emptyset$ (note that H is open in G). Since $supp(f) \subseteq gH$ there is $h' \in H$ and $\gamma_1 \in supp(\delta_1)$ such that $\gamma_1 = gh'h^{-1}\gamma_2^{-1}$, i.e. $g = \gamma_1\gamma_2 h(h')^{-1}$. It follows that $g \in supp(\delta_1) \cdot supp(\delta_2)$ because H is arbitrary and $supp(\delta_1) \cdot supp(\delta_2)$ is closed: note that according to Remark 1.2.3 $supp(\delta_1) \cdot supp(\delta_2)$ is even compact because both $supp(\delta_1)$ and $supp(\delta_2)$ are. □

For a closed subset C of G the locally convex K-vector space $C^\omega_C(G,K)$ of generalized germs in C was first introduced by C.T. Féaux de Lacroix (cf. [18], Definition 2.3.3). It is the quotient space

$$C^\omega_C(G,K) := C^{an}(G,K)/C^{an}(G,K)_{G\setminus C}. \tag{1.4}$$

If C is compact then there is a topological isomorphism

$$C^\omega_C(G,K) = \varinjlim_U C^{an}(U,K)$$

with U running through the inductive system of open subsets of G containing C and transition maps defined by restriction of functions (cf. the remarks preceding Definition 2.3.3 of [loc.cit.]). In this case the inductive limit topology on $C^\omega_C(G,K)$ is Hausdorff. If $C = \{g\}$ is a singleton we prefer to write $C^\omega_g(G,K)$ instead of $C^\omega_{\{g\}}(G,K)$.

Lemma 1.2.5. *$C^{an}(G,K)_C$ is a closed subspace of $C^{an}(G,K)$ for any subset C of G. If C is closed then $D(G,K)_C$ is a closed subspace of $D(G,K)$ and there is a topological isomorphism*

$$D(G,K)_C \simeq C^\omega_C(G,K)'_b. \tag{1.5}$$

If C is compact then this is an isomorphism of nuclear Fréchet spaces.

Proof: Let C be an arbitrary subset of G. As mentioned in [loc.cit.], section 2.3.1, $C^{an}(G,K)_C$ is the intersection of the kernels of all continuous surjections $C^{an}(G,K) \twoheadrightarrow C^\omega_g(G,K)$, $g \in G \setminus C$, hence is closed in $C^{an}(G,K)$. If C is closed in G it follows directly from the definition of support that $D(G,K)_C$ is the orthogonal space of $C^{an}(G,K)_{G\setminus C}$ with respect to the natural pairing

$$D(G,K) \times C^{an}(G,K) \to K.$$

Therefore, $D(G,K)_C$ is closed, as well. Further, the reflexivity of $D(G,K)$ implies by means of [9], IV.2.2 Corollary, that

$$(D(G,K)_C)'_b \simeq D(G,K)'_b/D(G,K)^{\circ}_C$$

where $D(G,K)^{\circ}_C$ denotes the orthogonal subspace of $D(G,K)_C$ with respect to the pairing $D(G,K)'_b \times D(G,K) \to K$. Since $C^{an}(G,K)$ is reflexive this pairing can be identified with the one above so that $D(G,K)^{\circ}_C \simeq C^{an}(G,K)^{\circ\circ}_{G\backslash C} = C^{an}(G,K)_{G\backslash C}$ because $C^{an}(G,K)_{G\backslash C}$ is closed (cf. [25], Corollary 13.5). It follows that

$$(D(G,K)_C)'_b \simeq C^{an}(G,K)/C^{an}(G,K)_{G\backslash C}.$$

But if G_0 is a compact open subgroup of G then by (1.1) and [25], Lemma 5.3 (i), there is a topological isomorphism

$$D(G,K)_C = \oplus_{g\in G/G_0} D(gG_0,K)_{gG_0\cap C}$$

showing that $D(G,K)_C$ is reflexive: by our above reasoning $D(gG_0,K)_{gG_0\cap C}$ is a closed subspace of the nuclear Fréchet space $D(gG_0,K)$ and hence is reflexive (loc.cit. Corollary 19.3 (ii) and Proposition 19.4 (i)). Thus, (1.5) follows. The last claim is a consequence of [29], Theorem 1.1 and Theorem 1.3 because if C is compact then $C^{\omega}_C(G,K)$ is a locally convex K-vector space of compact type (cf. [18], Satz 2.3.2). □

Corollary 1.2.6. *If C is a closed subset of G such that $1 \in C$ and $C\cdot C \subseteq C$ then $D(G,K)_C$ is a closed subalgebra of $D(G,K)$. If in addition C is compact then $D(G,K)_C$ is a nuclear K-Fréchet algebra.* □

Remark 1.2.7. Let G_0 be a compact open subgroup of G. If H is a locally L-analytic subgroup of G and $H_0 := H \cap G_0$ then as seen above

$$D(G,K)_H = \bigoplus_{g\in G/G_0} D(gG_0,K)_{gG_0\cap H}$$

as locally convex K-vector spaces. Note that $D(gG_0,K)_{gG_0\cap H} \neq 0$ if and only if $gG_0\cap H \neq \emptyset$ (for $0 \neq \delta \in D(gG_0,K)_{gG_0\cap H}$ we have $\emptyset \neq supp(\delta) \subseteq gG_0\cap H$). For any such $h \in gG_0 \cap H$

$$D(gG_0,K)_{gG_0\cap H} = D(hG_0,K)_{hG_0\cap H} \overset{(1.3)}{=} \delta_h \cdot D(G_0,K)_{H_0},$$

so that we get

$$D(G,K)_H = \bigoplus_{h\in H/H_0} \delta_h \cdot D(G_0,K)_{H_0}, \tag{1.6}$$

with h running through a set of right coset representatives for H/H_0.

According to [18], Bemerkung 3.1.2 and Satz 3.3.4, the Lie algebra $\mathfrak{g}$ of G acts on $C^{an}(G,K)$ via continuous endomorphisms defined by

$$\mathfrak{x}(f)(g) := \frac{d}{dt} f(exp(-t\mathfrak{x})g)|_{t=0} \text{ for } \mathfrak{x} \in \mathfrak{g} \text{ and } f \in C^{an}(G,K).$$

This action extends to an action of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ on $C^{an}(G,K)$.

According to Lemma 1.2.5 the space $C_1^\omega(G,K)'_b$ dual to the space of germs of locally analytic functions in $1 \in G$ is topologically isomorphic to $D(G,K)_{\{1\}}$ which is a K-Fréchet subalgebra of $D(G,K)$ by Corollary 1.2.6. Fixing an ordered L-basis $\mathfrak{X} = (\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ of the Lie algebra $\mathfrak{g}$ of G, the action of $U(\mathfrak{g})$ on $C^{an}(G,K)$ leads to the following explicit description of $C_1^\omega(G,K)'_b$ (cf. [29], Lemma 2.4):

$$C_1^\omega(G,K)'_b = \left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha |\ d_\alpha \in K, \forall r > 0 : \sup |d_\alpha \cdot \alpha!| r^{-|\alpha|} < \infty \right\},$$

where $|\alpha| := \alpha_1 + \ldots + \alpha_d$ and $\alpha! := \alpha_1! \cdot \ldots \cdot \alpha_d!$. Further, $\mathfrak{X}^\alpha := \mathfrak{x}_1^{\alpha_1} \cdots \mathfrak{x}_d^{\alpha_d}$ for a multi-index $\alpha \in \mathbb{N}^d$, and any such monomial is viewed as a distribution via

$$(1.7) \qquad \mathfrak{X}^\alpha(f) = ((-\mathfrak{x}_1)^{\alpha_1} \circ \ldots \circ (-x_d)^{\alpha_d}(f))(1) \text{ for } f \in C^{an}(G,K).$$

Finally, the Fréchet topology of $C_1^\omega(G,K)'_b$ is defined by the family of norms $(\nu'_r)_{r>0}$ with $\nu'_r(\sum_\alpha d_\alpha \mathfrak{X}^\alpha) := \sup |d_\alpha \cdot \alpha!| r^{-|\alpha|}$.

The explicit description above shows that letting $(\mathfrak{z} \mapsto \dot{\mathfrak{z}})$ denote the unique anti-automorphism of $U(\mathfrak{g}) \otimes_L K$ extending multiplication by -1 on $\mathfrak{g}$, the natural homomorphism $(\mathfrak{z} \mapsto (f \mapsto \dot{\mathfrak{z}}(f)(1))) : U(\mathfrak{g}) \otimes_L K \to C_1^\omega(G,K)'_b$ of K-algebras is injective. In fact, we can prove:

Proposition 1.2.8. *$U(\mathfrak{g}) \otimes_L K$ is dense in $C_1^\omega(G,K)'_b$. We have*

$$(1.8) \qquad C_1^\omega(G,K)'_b = \left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha |\ d_\alpha \in K, \forall r > 0 : \sup |d_\alpha| r^{-|\alpha|} < \infty \right\}$$

and the Fréchet topology of $C_1^\omega(G,K)'_b$ can be defined by the family of norms $(\nu_r)_{r>0}$ with $\nu_r(\sum_\alpha d_\alpha \mathfrak{X}^\alpha) := \sup_\alpha |d_\alpha| r^{-|\alpha|}$.

Proof: Since $|\alpha!| \leq 1$ the right hand side of (1.8) is contained in $C_1^\omega(G,K)'_b$. Conversely,

$$|\alpha!|^{-1} \leq p^{|\alpha|/(p-1)}$$

(cf. [24], Lemma 5.3.1), so that if $\sup_\alpha |d_\alpha| r^{-|\alpha|} < \infty$ for all $r > 0$ then also $\sup_\alpha |d_\alpha/\alpha!| r^{-|\alpha|} < \infty$ for all $r > 0$. This proves the reverse inclusion as well as the fact that the two families of norms $(\nu'_r)_{r>0}$ and $(\nu_r)_{r>0}$ are equivalent. It is clear that $U(\mathfrak{g}) \otimes_L K$ is dense in the completion

$$\left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid \lim_{|\alpha|\to\infty} |d_\alpha| r^{-|\alpha|} = 0 \right\}$$

of $C_1^\omega(G,K)'_b$ with respect to any norm ν_r. But then it is also dense in the topological projective limit (cf. [7], I.4.4 Corollaire). $\square$

Remark 1.2.9. When working with $C_1^\omega(G,K)'_b$ we will henceforth use the description given by (1.8) and assume its topology to be defined by the family of norms $(\nu_r)_{r>0}$. To simplify notation we write $U(\mathfrak{g},K) := C_1^\omega(G,K)'_b$ for the K-Fréchet algebra of all locally analytic distributions on G supported in $1 \in G$.

If H is a closed locally L-analytic subgroup of G there are two closed subalgebras of $D(G,K)$ which are canonically attached to H. On the one hand, there is $D(H,K) := C^{an}(H,K)'_b$ with inclusion morphism $D(H,K) \subseteq D(G,K)$ coming from the canonical restriction map

$$C^{an}(G,K) \twoheadrightarrow C^{an}(H,K)$$

(cf. Proposition 1.1.3). According to Lemma 1.1.1 the group ring $K[H] \subseteq D(H,K)$ is dense. On the other hand, there is the algebra $D(G,K)_H$ which contains $D(H,K)$: note that for any subset C of H we have $D(H,K)_C \subseteq D(G,K)_C$.

The only difference between $D(H,K)$ and $D(G,K)_H$ is the fact that according to Lemma 1.2.4 $D(G,K)_H$ is a (two-sided) $U(\mathfrak{g},K)$-module whereas $D(H,K)$ is not:

Lemma 1.2.10. *If C is a closed subset of G then the $U(\mathfrak{g},K)$-submodule of $D(G,K)_C$ generated by all Dirac distributions δ_c, $c \in C$, is dense.*

Proof: Let Δ be the closure of $\sum_{c\in C} \delta_c \cdot U(\mathfrak{g},K)$ in $D(G,K)$. It follows from Lemma 1.2.4 and Lemma 1.2.5 that $\Delta \subseteq D(G,K)_C$. During the proof of Lemma 1.2.5 we showed that the space $C^{an}(G,K)/C^{an}(G,K)_{G\setminus C}$ is the strong dual of the reflexive space $D(G,K)_C$. Hence it is reflexive itself (cf. [25], Lemma 15.4). Let ℓ be a continuous functional on $D(G,K)_C$ vanishing on Δ. By (1.5) and reflexivity, ℓ corresponds to an element $\overline{f}$ of $C^{an}(G,K)/C^{an}(G,K)_{G\setminus C}$. To say ℓ vanishes on Δ is to say that any representative f of $\overline{f}$ in $C^{an}(G,K)$ vanishes in an open neighborhood of C. Hence $f \in C^{an}(G,K)_{G\setminus C}$, i.e. $\overline{f} = 0$, and $\Delta = D(G,K)_C$ by the Hahn-Banach theorem. $\square$

Remark 1.2.11. Let B and C be locally convex K-vector spaces carrying separately continuous K-algebra structures. Assume that B and C possess a common K-subalgebra A. If $B\hat{\otimes}_{K,\iota}C$ denotes the Hausdorff completion of the algebraic tensor product $B \otimes_{K,\iota} C$ endowed with its inductive tensor product topology (cf. [25], section 17) then we let $B\hat{\otimes}_{A,\iota}C$ be the quotient of $B\hat{\otimes}_{K,\iota}C$ by the closure of the subspace generated by all elements of the form

$$ba \otimes c - b \otimes ac, \qquad a \in A, b \in B \text{ and } c \in C.$$

We endow $B\hat{\otimes}_{A,\iota}C$ with the corresponding quotient topology. If B and C are K-Fréchet spaces then the inductive and the projective tensor product topologies on $B \otimes_K C$ coincide. Therefore, we omit the ι from the notation and simply write $B\hat{\otimes}_K C$ and $B\hat{\otimes}_A C$.
Note that $B\hat{\otimes}_{A,\iota}C$ is naturally a B-C^{op}-bimodule where C^{op} is the K-algebra whose underlying vector space is C and whose multiplication is defined by $(c, c') \mapsto c' \cdot c$: any element $b \in B$ (resp. $c \in C$) defines a continuous K-linear endomorphism ℓ_b of B (resp. r_c of C) by left (resp. right) multiplication. The continuous K-linear endomorphism $\ell_b \otimes r_c$ of $B \otimes_{K,\iota} C$ extends to $B\hat{\otimes}_{K,\iota}C$ and leaves invariant the subspace constructed above. If A is contained in the center of B and C then, similarly, $B\hat{\otimes}_{A,\iota}C$ is naturally a module over $B \otimes_K C$ (via $\ell_b \otimes \ell_c$) and even over $B \otimes_A C$.

Let $\mathfrak{h}$ denote the Lie algebra of H. Since the multiplication map

$$D(H,K) \times U(\mathfrak{g},K) \longrightarrow D(G,K)_H \tag{1.9}$$

is K-bilinear and separately continuous it induces a continuous K-linear map

$$D(H,K) \otimes_{K,\iota} U(\mathfrak{g},K) \longrightarrow D(G,K)_H,$$

extending to a continuous K-linear map

$$D(H,K)\hat{\otimes}_{K,\iota}U(\mathfrak{g},K) \longrightarrow D(G,K)_H. \tag{1.10}$$

Let U be the closure of the subspace of $D(H,K)\hat{\otimes}_{K,\iota}U(\mathfrak{g},K)$ generated by all elements of the form

$$\lambda\mathfrak{y} \otimes \mathfrak{x} - \lambda \otimes \mathfrak{y}\mathfrak{x} \text{ with } \lambda \in D(H,K), \mathfrak{y} \in U(\mathfrak{h},K) \text{ and } \mathfrak{x} \in U(\mathfrak{g},K).$$

Since this space is contained in the kernel of the multiplication map (1.10), the latter induces a continuous K-linear map

$$\mu : D(H,K)\hat{\otimes}_{U(\mathfrak{h},K),\iota}U(\mathfrak{g},K) \longrightarrow D(G,K)_H.$$

Proposition 1.2.12. *If K is discretely valued then μ is a topological isomorphism of $D(H,K)$-$U(\mathfrak{g},K)^{op}$-bimodules.*

To prove the proposition we will first reduce to a local situation and then use results on the K-Fréchet-Stein structures of $D(H,K)$ and $D(G,K)$. These will be proved in subsection 1.4 under the assumption that K is discretely valued.

Lemma 1.2.13. *Let $(V_i)_{i\in I}$ and W be locally convex K-vector spaces. Then the canonical K-linear bijection*

$$(\bigoplus_{i\in I} V_i)\otimes_{K,\iota} W \simeq \bigoplus_{i\in I}(V_i\otimes_{K,\iota} W)$$

is a topological isomorphism.

Proof: For $i\in I$ we denote by $\varphi_i : V_i \to V := \oplus_{i\in I}V_i$ the natural inclusion. The composite map

$$V_i\times W \xrightarrow{\varphi_i\times id} V\times W \longrightarrow V\otimes_{K,\iota} W$$

is separately continuous so that by definition of the inductive tensor product topology the induced map $V_i\otimes_{K,\iota} W \to V\otimes_{K,\iota} W$ is continuous. This being true for all $i\in I$, the canonical K-linear bijection $\oplus_{i\in I}(V_i\otimes_{K,\iota} W)\to (\oplus_{i\in I}V_i)\otimes_{K,\iota} W$ is continuous.
Conversely, if $w\in W$ is fixed then the K-linear map

$$\cdot\otimes w : V_i \longrightarrow V_i\otimes_{K,\iota} W \longrightarrow \bigoplus_{i\in I}(V_i\otimes_{K,\iota} W)$$

is continuous for any $i\in I$. Therefore, so is the induced map $\cdot\otimes w : V\to \oplus_{i\in I}(V_i\otimes_{K,\iota} W)$. If on the other hand $v=(v_i)_{i\in I}\in V$ is fixed then for each $i\in I$ the K-linear map

$$v_i\otimes\cdot : W \longrightarrow V_i\otimes_{K,\iota} W \longrightarrow \bigoplus_{i\in I}(V_i\otimes_{K,\iota} W)$$

is continuous. Hence so is the finite sum $(\sum_{i\in I} v_i\otimes\cdot)\in \mathrm{Hom}_K(W, \oplus_{i\in I}(V_i\otimes_{K,\iota} W))$. Therefore, the natural map $V\times W\to \oplus_{i\in I}(V_i\otimes_{K,\iota} W)$ induces a continuous K-linear map $V\otimes_{K,\iota} W\to \oplus_{i\in I}(V_i\otimes_{K,\iota} W)$ inverse to the one above. □

Corollary 1.2.14. *If $(V_i)_{i\in I}$ and W are Hausdorff locally convex K-vector spaces then there is a topological isomorphism*

$$(\bigoplus_{i\in I} V_i)\hat{\otimes}_{K,\iota} W \simeq \bigoplus_{i\in I}(V_i\hat{\otimes}_{K,\iota} W).$$

Proof: This follows from the above lemma together with [25], Lemma 7.8.$\square$

Proof of Proposition 1.2.12: We will prove in Corollary 1.3.6 and Corollary 1.4.3 that there is a compact open subgroup G_0 of G with the following properties: $D(G_0, K)$ is a K-Fréchet-Stein algebra with respect to a family of norms $||\cdot||_{\overline{r}}$, $r \in p^{\mathbb{Q}}$, $1/p < r < 1$, such that the completion $D_r(G_0, K)$ of $D(G_0, K)$ with respect to the norm $||\cdot||_{\overline{r}}$ is finitely generated and free as a module over the closure $U_r(\mathfrak{g}, K)$ of $U(\mathfrak{g}, K)$ in $D_r(G_0, K)$; if $H_0 := H \cap G_0$ then $D(H_0, K)$ is a K-Fréchet-Stein algebra with respect to the family of norms $||\cdot||_{\overline{r}}$ restricted to $D(H_0, K)$; for each r the closure $D_r(H_0, K)$ of $D(H_0, K)$ in $D_r(G_0, K)$ ($\simeq$ the Hausdorff completion of $D(H_0, K)$ with respect to the norm $||\cdot||_{\overline{r}}$) is finitely generated and free as a module over the closure $U_r(\mathfrak{h}, K)$ of $U(\mathfrak{h}, K)$ in $D_r(H_0, K)$; $U_r(\mathfrak{g}, K)$ and $U_r(\mathfrak{h}, K)$ are noetherian K-Banach algebras.

By (1.6) and Corollary 1.2.14 the map (1.10) can be viewed as a continuous K-linear map

$$\bigoplus_{h \in H/H_0} (D(hH_0, K)\hat{\otimes}_K U(\mathfrak{g}, K)) \longrightarrow \bigoplus_{h \in H/H_0} \delta_h \cdot D(G_0, K)_{H_0}. \tag{1.11}$$

Considering supports of distributions on both sides, we see that it is the direct sum of the continuous K-linear maps

$$D(hH_0, K)\hat{\otimes}_K U(\mathfrak{g}, K) \longrightarrow \delta_h \cdot D(G_0, K)_{H_0}$$

induced by multiplication, with h running through a set of right coset representatives for H/H_0 containing $1 \in H_0$. Under the topological isomorphism

$$D(H, K)\hat{\otimes}_{K,\iota} U(\mathfrak{g}, K) \simeq \oplus_{h \in H/H_0} (D(hH_0, K)\hat{\otimes}_K U(\mathfrak{g}, K))$$

the closed subspace U constructed above corresponds to the locally convex direct sum $\oplus_{h \in H/H_0} U_h$, where U_h is the closure in $D(hH_0, K)\hat{\otimes}_K U(\mathfrak{g}, K)$ of the subspace generated by all elements of the form

$$\delta_h \lambda \mathfrak{y} \otimes \mathfrak{x} - \delta_h \lambda \otimes \mathfrak{y}\mathfrak{x} \text{ with } \lambda \in D(H_0, K), \mathfrak{y} \in U(\mathfrak{h}, K) \text{ and } \mathfrak{x} \in U(\mathfrak{g}, K).$$

By [25], Lemma 5.3, (1.11) induces a continuous K-linear map

$$\bigoplus_{h \in H/H_0} (D(hH_0, K)\hat{\otimes}_K U(\mathfrak{g}, K))/U_h \longrightarrow \bigoplus_{h \in H/H_0} \delta_h \cdot D(G_0, K)_{H_0}.$$

Hence, restricting to the case $h = 1$, we only need to show that the induced continuous K-linear map

$$D(H_0, K)\hat{\otimes}_{U(\mathfrak{h}, K)} U(\mathfrak{g}, K) \longrightarrow D(G_0, K)_{H_0}$$

is a topological isomorphism. We again denote this map by μ.
Let $r \in p^{\mathbb{Q}}$ with $1/p < r < 1$. The multiplication in $D_r(G_0, K)$ induces a continuous K-linear map

$$\mu_r : D_r(H_0, K) \otimes_K U_r(\mathfrak{g}, K) \longrightarrow D_r(G_0, K)_{H_0};$$

here $D_r(G_0, K)_{H_0}$ denotes the closure of $D(G_0, K)_{H_0}$ in $D_r(G_0, K)$. In the proof of Corollary 1.4.3 we will show that $D_r(G_0, K)_{H_0}$ is free and finitely generated as a module over $U_r(\mathfrak{g}, K)$ and has a basis $(\mathbf{b}^\alpha)_{\alpha \in A'}$ in $K[H_0]$ which is simultaneously a basis for the free $U_r(\mathfrak{h}, K)$-module $D_r(H_0, K)$. Hence μ_r induces a continuous K-linear bijection

$$D_r(H_0, K) \otimes_{U_r(\mathfrak{h},K)} U_r(\mathfrak{g}, K) \longrightarrow D_r(G_0, K)_{H_0}. \tag{1.12}$$

Note that both $D_r(H_0, K)$ and $U_r(\mathfrak{g}, K)$ are complete normed, continuous modules over the noetherian K-Banach algebra $U_r(\mathfrak{h}, K)$. Further, $D_r(H_0, K)$ is a finitely generated, free $U_r(\mathfrak{h}, K)$-module and therefore topologically isomorphic to a direct sum of copies of $U_r(\mathfrak{h}, K)$ (cf. [30], Proposition 2.1 (iii)). A straightforward generalization to the non-commutative setting of [4], 2.1.7 Proposition 6, shows that $D_r(H_0, K) \otimes_{U_r(\mathfrak{h},K)} U_r(\mathfrak{g}, K)$ is a complete normed space with respect to the tensor product norm. By the open mapping theorem (1.12) then is a topological isomorphism. In addition, 3.1 Corollaire 1 of [8] shows that

$$\begin{aligned} D_r(H_0, K) \otimes_{U_r(\mathfrak{h},K)} U_r(\mathfrak{g}, K) &= (D_r(H_0, K) \otimes_K U_r(\mathfrak{g}, K))/\ker\mu_r \\ &\simeq (D_r(H_0, K) \hat{\otimes}_K U_r(\mathfrak{g}, K))/\overline{\ker\mu_r}. \end{aligned}$$

Here $\overline{\ker\mu_r}$ is the closure of $\ker\mu_r$ in $D_r(H_0, K)\hat{\otimes}_K U_r(\mathfrak{g}, K)$. Thus, for each r as above, we obtain a short exact sequence of strict continuous K-linear maps between Banach spaces

$$0 \longrightarrow \overline{\ker\mu_r} \longrightarrow D_r(H_0, K)\hat{\otimes}_K U_r(\mathfrak{g}, K) \longrightarrow D_r(G_0, K)_{H_0} \longrightarrow 0.$$

Since for $r' < r$ the inclusion maps $D_r(H_0, K) \subseteq D_{r'}(H_0, K)$, $D_r(G_0, K) \subseteq D_{r'}(G_0, K)$, $U_r(\mathfrak{h}, K) \subseteq U_{r'}(\mathfrak{h}, K)$ and $U_r(\mathfrak{g}, K) \subseteq U_{r'}(\mathfrak{g}, K)$ are continuous homomorphisms of K-Banach algebras, the family of all these exact sequences forms a projective system. Recall that U_1 was defined to be the closure of the subspace generated in $D(H_0, K)\hat{\otimes}_K U(\mathfrak{g}, K)$ by all elements of the form

$$\lambda\mathfrak{y} \otimes \mathfrak{x} - \lambda \otimes \mathfrak{y}\mathfrak{x} \text{ with } \lambda \in D(H_0, K), \mathfrak{y} \in U(\mathfrak{h}, K) \text{ and } \mathfrak{x} \in U(\mathfrak{g}, K).$$

Since by (1.12) the kernel of μ_r is the vector space generated by all elements of the form

$$\lambda\mathfrak{y} \otimes \mathfrak{x} - \lambda \otimes \mathfrak{y}\mathfrak{x} \text{ with } \lambda \in D_r(H_0, K), \mathfrak{y} \in U_r(\mathfrak{h}, K) \text{ and } \mathfrak{x} \in U_r(\mathfrak{g}, K)$$

we obtain from the definition of the tensor product norm that $U_1 \subseteq \overline{\ker \mu_r}$ is dense for all r. Therefore, the system $(\overline{\ker \mu_r})$ with $r \in p^{\mathbb{Q}}$ and $1/p < r < 1$ satisfies the Mittag-Leffler property as formulated in [21], 13.2.4. By [loc.cit], 13.2.2, we obtain an exact sequence

$$0 \longrightarrow U_1 = \varprojlim_r \overline{\ker \mu_r} \longrightarrow D(H_0, K) \hat{\otimes}_K U(\mathfrak{g}, K) \longrightarrow D(G_0, K)_{H_0} \longrightarrow 0,$$

because

$$\varprojlim_r (D_r(H_0, K) \hat{\otimes}_K U_r(\mathfrak{g}, K)) \simeq (\varprojlim_r D_r(H_0, K)) \hat{\otimes}_K (\varprojlim_r U_r(\mathfrak{g}, K))$$

(cf. [17], Proposition 1.1.29). It induces a continuous K-linear bijection

$$D(H_0, K) \hat{\otimes}_{U(\mathfrak{h}, K)} U(\mathfrak{g}, K) \longrightarrow D(G_0, K)_{H_0}$$

which is a topological isomorphism by the open mapping theorem. That it coincides with μ is clear from the fact that for each r the restriction of μ_r to $D(H_0, K) \otimes_K U(\mathfrak{g}, K)$ is induced by the multiplication in $D(G_0, K)$. $\square$

Remark 1.2.15. Assume there is a compact open subgroup G_0 of G and a closed locally L-analytic subgroup C_0 of G_0 such that $G_0 = H_0 \times C_0$ as locally L-analytic groups with $H_0 := H \cap G_0$. Then the above proposition can be proved without any allusion to Fréchet-Stein structures and simplifies in the following manner: According to Proposition A.3 and Remark A.4 of [31] there is a topological isomorphism

$$D(H_0, K) \hat{\otimes}_K D(C_0, K) \longrightarrow D(G_0, K)$$

induced by multiplication. It follows from Lemma 1.2.10 and [25], Corollary 17.5 (ii) and Proposition 19.10 (i), that the preimage of $D(G_0, K)_{H_0}$ under this map is $D(H_0, K) \hat{\otimes}_K U(\mathfrak{c}, K)$ where $\mathfrak{c}$ is the Lie algebra of C_0. Hence we obtain from Corollary 1.2.14 that

$$D(G, K)_H \simeq D(H, K) \hat{\otimes}_{K,\iota} U(\mathfrak{c}, K).$$

1.3 Restriction of the base field

Let $L_0|\mathbb{Q}_p$ be a finite extension of fields with $L_0 \subseteq L$ and let $R^{L|L_0}$ be the functor "*restriction of the base field from L to L_0*" from the category of paracompact locally L-analytic manifolds to the category of locally analytic manifolds of the same type over L_0 (cf. [10], 5.14; note that if M is a locally L-analytic manifold then the underlying topological spaces of M and $R^{L|L_0}M$

are identical).

In the special case of a locally L-analytic group G there is a natural embedding

$$\tau : C^{an}(G, K) \longrightarrow C^{an}(R^{L|L_0}G, K)$$

mapping $C^{an}(G, K)$ homeomorphically onto its closed image (cf. [28], Lemma 1.2; note that its proof does not make use of the commutativity assumption on G).

Lemma 1.3.1. *The dual map $\tau' : D(R^{L|L_0}G, K) \to D(G, K)$ is a strict surjection and a homomorphism of K-algebras.*

Proof: Since τ' restricts distributions on $R^{L|L_0}G$ to the subspace $C^{an}(G, K)$ of $C^{an}(R^{L|L_0}G, K)$ it is clear that τ' is a homomorphism of K-algebras. Choose a compact open subgroup G_0 of G. Then τ is the direct product of the maps $\tau_g : C^{an}(gG_0, K) \to C^{an}(gR^{L|L_0}G_0, K)$, $g \in G/G_0$, and dually τ' is the direct sum of the maps $\tau'_g : D(gR^{L|L_0}G_0, K) \to D(gG_0, K)$. Since $D(gR^{L|L_0}G_0, K) = \delta_g \cdot D(R^{L|L_0}G_0, K)$, likewise for gG_0, and since τ' is a homomorphism of algebras we only need to show that τ'_1 is a strict surjection. But τ_1 is a topological embedding of spaces of compact type so that the claim follows from [29], Proposition 1.2 (i). $\square$

Consider the ideal $I := ker(\tau')$ of $D(R^{L|L_0}G, K)$. It is the orthogonal subspace of $C^{an}(G, K)$ with respect to the natural pairing

$$D(R^{L|L_0}G, K) \times C^{an}(R^{L|L_0}G, K) \longrightarrow K.$$

Since $D(R^{L|L_0}G, K)$ is reflexive we obtain by means of [9], IV.2.2 Corollary, that I'_b is topologically isomorphic to $C^{an}(R^{L|L_0}G, K)/C^{an}(G, K)$. The topological isomorphism $I \simeq \oplus_{g\in G/G_0} ker(\tau'_g)$ for a compact open subgroup G_0 of G shows that I itself is reflexive: For any $g \in G/G_0$ the kernel $ker(\tau'_g)$ of τ'_g is a closed subspace of a nuclear Fréchet space, hence is itself a nuclear Fréchet space and therefore reflexive (cf. [25], Corollary 19.3 (ii) and Proposition 19.4 (i)). Thus, there is a topological isomorphism

$$I \simeq (C^{an}(R^{L|L_0}G, K)/C^{an}(G, K))'_b. \tag{1.13}$$

In order to give an explicit description of the locally L-analytic functions inside $C^{an}(R^{L|L_0}G, K)$ we follow the arguments given in section 1 of [28]. Let $\mathfrak{g}_L$ be the Lie algebra of G. If we write $\mathfrak{g}_{L_0}$ for $\mathfrak{g}_L$ viewed as a Lie algebra over L_0 then $\mathfrak{g}_{L_0}$ can be identified with the Lie algebra of $R^{L|L_0}G$ and exp is also an exponential map for $R^{L|L_0}G$. $\mathfrak{g}_{L_0}$ (resp. $\mathfrak{g}_L$) acts on $C^{an}(R^{L|L_0}G, K)$ (resp. $C^{an}(G, K)$) via continuous endomorphisms defined by

$$\mathfrak{x}(f)(g) := \frac{d}{dt} f(exp(-t\mathfrak{x})g)|_{t=0}$$

for $\mathfrak{x} \in \mathfrak{g}_{L_0}$ (viewed also as an element of $\mathfrak{g}_L$) and $f \in C^{an}(R^{L|L_0}G, K)$ (resp. $f \in C^{an}(G, K)$). The orbit maps of these actions are L_0- and L-linear, respectively, and the action of $\mathfrak{g}_L$ on $C^{an}(G, K)$ is compatible with the inclusion $C^{an}(G, K) \subseteq C^{an}(R^{L|L_0}G, K)$.

Lemma 1.3.2. *$C^{an}(G, K)$ is the closed subspace of all those functions $f \in C^{an}(R^{L|L_0}G, K)$ for which $(t\mathfrak{x})(f) = t \cdot \mathfrak{x}(f)$ for all $t \in L$ and all $\mathfrak{x} \in \mathfrak{g}_{L_0}$.*

Proof: Let W be the subspace of $C^{an}(R^{L|L_0}G, K)$ consisting of all functions with the above property. Then $C^{an}(G, K) \subseteq W$ and we need to show the reverse inclusion. Let $f \in W$. If $\mathfrak{x}, \mathfrak{y} \in \mathfrak{g}$ and $t \in L$ then

$$\begin{aligned}(t\mathfrak{x})(\mathfrak{y}(f)) &= \mathfrak{y}((t\mathfrak{x})(f)) + [t\mathfrak{x}, \mathfrak{y}](f) \\ &= \mathfrak{y}(t \cdot \mathfrak{x}(f)) + (t \cdot [\mathfrak{x}, \mathfrak{y}])(f) \\ &= t \cdot \mathfrak{y}(\mathfrak{x}(f)) + t \cdot [\mathfrak{x}, \mathfrak{y}](f) = t \cdot \mathfrak{x}(\mathfrak{y}(f))\end{aligned}$$

shows that W is $\mathfrak{g}_{L_0}$-invariant. Therefore, the proof of [loc.cit.], Lemma 1.1, generalizes to the non-commutative setting in the following manner: Fix an L-basis $\mathfrak{X} = (\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ of $\mathfrak{g}_L$. Choose an orthonormal basis $(v_1, \ldots, v_n)$ of L as a vector space over L_0 and put $\mathfrak{Y} := (v_1\mathfrak{x}_1, v_2\mathfrak{x}_1, \ldots, v_n\mathfrak{x}_d)$ which is an L_0-basis of $\mathfrak{g}_{L_0}$. The corresponding system θ_{L_0} of canonical coordinates of the second kind is defined by

$$\theta_{L_0}(\sum_{i,j} t_{ij} v_i \mathfrak{x}_j) := exp(t_{11}v_1\mathfrak{x}_1)exp(t_{21}v_2\mathfrak{x}_1) \cdots exp(t_{nd}v_n\mathfrak{x}_d)$$

for t_{ij} sufficiently close to zero in L_0 (cf. [5], III.4.3 Proposition 3). Given $g \in R^{L|L_0}G$ we have the expansion

$$(R_g f \circ \theta_{L_0})(\sum_{i,j} t_{ij} v_i \mathfrak{x}_j) = \sum_{\beta \in \mathbb{N}^n \times \mathbb{N}^d} c_\beta \mathbf{t}^\beta$$

converging for all t_{ij} near zero in L_0; here $c_\beta \in K$, $\mathbf{t}^\beta := \prod_{i,j} t_{ij}^{\beta_{ij}}$ and R_g is the right translation operator associated with g. Letting $\mathfrak{Y}^\beta(R_g f) := (v_1\mathfrak{x}_1)^{\beta_{11}} \circ (v_2\mathfrak{x}_1)^{\beta_{21}} \circ \cdots \circ (v_n\mathfrak{x}_d)^{\beta_{nd}}(R_g f)$ it follows from the remarks after Lemma 4.7.2 of [18] that

$$c_\beta = \frac{(-1)^{|\beta|}}{\beta!} \mathfrak{Y}^\beta(R_g f)(1) = \frac{(-1)^{|\beta|}}{\beta!} \mathfrak{Y}^\beta(f)(g)$$

for all $\beta \in \mathbb{N}^n \times \mathbb{N}^d$ where $|\beta|$ and $\beta!$ are as in subsection 1.2. Letting $\varphi(\beta) := (\alpha_1, \ldots, \alpha_d)$ with $\alpha_j := \beta_{1j} + \ldots + \beta_{nj}$, $b_{\varphi(\beta)} := c_{(\alpha_1,0,\ldots,\alpha_2,0,\ldots,\alpha_d,0,\ldots)}$ and $\mathfrak{X}^{\varphi(\beta)}(R_g f) := \mathfrak{x}_1^{\alpha_1} \circ \cdots \circ \mathfrak{x}_d^{\alpha_d}(R_g f)$ we deduce

$$\mathfrak{Y}^\beta(f)(g) = \prod_{i=1}^n v_i^{\beta_{i1}+\ldots+\beta_{id}} \cdot \mathfrak{X}^{\varphi(\beta)}(f)(g)$$

from the assumption on f and the $\mathfrak{g}_{L_0}$-invariance of W. Thus

$$c_\beta = b_{\varphi(\beta)} \frac{\varphi(\beta)!}{\beta!} \prod_{i=1}^n v_i^{\beta_{i1}+\ldots+\beta_{id}}$$

for all β. Since this is precisely the relation given in the proof of [28], Lemma 1.1, we may conclude that f is locally L-analytic at g: Setting $t_j := t_{1j}v_1 + \ldots + t_{nj}v_n$ we have

$$(R_g f \circ \theta_{L_0})(\sum_{i,j} t_{ij} v_i \mathfrak{x}_j) = (R_g f \circ \theta_L)(\sum t_j \mathfrak{x}_j) = \sum_{\alpha \in \mathbb{N}^d} b_\alpha t_1^{\alpha_1} \cdots t_d^{\alpha_d}$$

where θ_L denotes the system of canonical coordinates of the second kind corresponding to the L-basis $\mathfrak{X}$ of $\mathfrak{g}_L$. Note that for all t_{ij} sufficiently close to zero in L_0 we have

$$\begin{aligned} \theta_{L_0}(\sum_{i,j} t_{ij} v_i \mathfrak{x}_j) &= exp(t_{11}v_1\mathfrak{x}_1)exp(t_{21}v_2\mathfrak{x}_1) \cdots exp(t_{nd}v_n\mathfrak{x}_d) \\ &= exp(t_1\mathfrak{x}_1) \cdots exp(t_d\mathfrak{x}_d) = \theta_L(\sum_j t_j \mathfrak{x}_j) \end{aligned}$$

because $[v_i\mathfrak{x}_j, v_k\mathfrak{x}_j] = 0$ in $\mathfrak{g}_L$. □

Lemma 1.3.3. *If $J := I \cap (U(\mathfrak{g}_{L_0}) \otimes_{L_0} K)$ then the vector space $\sum_{g \in G} \delta_g \cdot J$ is dense in I.*

Proof: Let Δ be the closure of $\sum_{g \in G} \delta_g \cdot J$ in $D(R^{L|L_0}G, K)$ and ℓ a continuous functional on I vanishing on Δ. By (1.13) and the reflexivity of the space $C^{an}(R^{L|L_0}G, K)/C^{an}(G, K)$, ℓ can be identified with an element $\overline{f} \in C^{an}(R^{L|L_0}G, K)/C^{an}(G, K)$ represented by $f \in C^{an}(R^{L|L_0}G, K)$. Let $t \in L$ and $\mathfrak{x} \in \mathfrak{g}_{L_0}$. Since $(t\mathfrak{x}') - t \cdot \mathfrak{x}' \in J$ for all $\mathfrak{x}' \in \mathfrak{g}_{L_0}$, ℓ vanishing on Δ implies that for all $g \in G$

$$((t\mathfrak{x})(f) - t \cdot \mathfrak{x}(f))(g) = (\delta_g \cdot (t\mathfrak{x}') - \delta_g \cdot t \cdot \mathfrak{x}')(f) = 0,$$

where $\mathfrak{x}' := Ad(g^{-1})(-\mathfrak{x})$. Here we use that $g \cdot exp(s\mathfrak{x}) \cdot g^{-1} = exp(s \cdot Ad(g)(\mathfrak{x}))$ for all s sufficiently close to zero in L_0 (cf. [5], III.4.4 Corollaire 3). According to Lemma 1.3.2 we have $f \in C^{an}(G, K)$ whence $\ell = 0$ and $\Delta = I$ by the Hahn-Banach theorem. □

Lemma 1.3.4. *Let $C \subseteq G$ be a closed subset, considered also as a subset of $R^{L|L_0}G$. Then the image of $D(R^{L|L_0}G, K)_C$ under τ' is dense in $D(G,K)_C$.*

Proof: That $\tau'(D(R^{L|L_0}G, K)_C)$ is contained in $D(G,K)_C$ follows from

$$C^{an}(G,K)_{G\backslash C} = C^{an}(R^{L|L_0}G_0, K)_{G\backslash C} \cap C^{an}(G,K).$$

The same equation shows that τ induces a continuous injection

$$C^{an}(G,K)/C^{an}(G,K)_{G\backslash C} \hookrightarrow C^{an}(R^{L|L_0}G, K)/C^{an}(R^{L|L_0}G, K)_{R^{L|L_0}G\backslash C}.$$

We know from the proof of Lemma 1.2.10 that the locally convex K-vector spaces on both sides are reflexive so that as a consequence of the Hahn-Banach Theorem the dual map $\tau' : D(R^{L|L_0}G, K)_C \to D(G,K)_C$ has to have dense image. □

Now assume $L_0 = \mathbb{Q}_p$. For further applications we need the following technical results:

Proposition 1.3.5. *Let G be a locally L-analytic group. Then there is an open subgroup G_0 of G and a $\mathbb{Z}_p$-lattice $\Lambda \subset \mathfrak{g}_{\mathbb{Q}_p}$ with the following properties:*

- *i) there is an L-basis $(\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ of $\mathfrak{g}_L$ and a $\mathbb{Z}_p$-basis $(v_1, \ldots, v_m)$ of $\mathfrak{o}_L$ such that $(v_1\mathfrak{x}_1, \ldots, v_m\mathfrak{x}_d)$ is a $\mathbb{Z}_p$ basis of Λ;*
- *ii) the corresponding canonical coordinates of the second kind give a well defined isomorphism $\theta_{\mathbb{Q}_p} : \Lambda \to R^{L|\mathbb{Q}_p}G_0$ of locally $\mathbb{Q}_p$-analytic manifolds;*
- *iii) $R^{L|\mathbb{Q}_p}G_0$ is a uniform pro-p group (cf. [14], Definition 4.1).*

Proof: Let $(\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ be an L-basis of $\mathfrak{g}_L$ and θ_L the corresponding system of canonical coordinates of the second kind defined locally around zero in $\mathfrak{g}$. Since θ_L is étale in $0 \in \mathfrak{g}_L$ we may choose an open subgroup G' of G and an open neighborhood U of zero in $\mathfrak{g}_L$ such that $\theta_L : U \to G'$ is an isomorphism of locally L-analytic manifolds. Let Φ_L be its inverse. According to [5], III.7.3 Théorème 4 and its proof there is $\lambda \in L^*$ such that $\oplus_i \mathfrak{m}_L\mathfrak{x}_i \subseteq \lambda{\cdot}\Phi_L(G') = \lambda{\cdot}U$ and the group structure on $\oplus_i \lambda^{-1}\mathfrak{m}_L\mathfrak{x}_i$ obtained by transport of structure from G' is given by formal power series with coefficients in $\mathfrak{o}_L$: if $g, h \in G'$ and $\lambda \cdot \Phi_L(g) = \sum_i \lambda_i\mathfrak{x}_i$, $\lambda \cdot \Phi_L(h) = \sum_i \mu_i\mathfrak{x}_i$ with $\lambda_i, \mu_i \in \mathfrak{m}_L$ then

$$\lambda \cdot \Phi_L(gh^{-1}) = \sum_i F_i(\lambda_1, \ldots, \lambda_d, \mu_1, \ldots, \mu_d)\mathfrak{x}_i \qquad (1.14)$$

where $F_i(X_1, \ldots, X_d, Y_1, \ldots, Y_d) \in \mathfrak{o}_L[[(X_i), (Y_i)]]$ without constant term.

If p is odd set $\Lambda := \oplus_i \lambda^{-1}\mathfrak{m}_L^e \mathfrak{x}_i$ and $\Lambda := \oplus_i \lambda^{-1}\mathfrak{m}_L^{2e}\mathfrak{x}_i$ otherwise. By [loc.cit.], III.7.4 Proposition 5, $G_0 := \theta_L(\Lambda)$ is an open subgroup of G. Choosing a $\mathbb{Z}_p$-basis $(v_1, \ldots, v_m)$ of $\mathfrak{o}_L$ the canonical coordinates of the second kind

$$\theta_{\mathbb{Q}_p} : \mathfrak{g}_{\mathbb{Q}_p} \dashrightarrow R^{L|\mathbb{Q}_p} G$$

corresponding to the decomposition $\mathfrak{g}_{\mathbb{Q}_p} = \oplus_{i,j} \mathbb{Q}_p \lambda^{-1} v_j \mathfrak{x}_i$ coincide with θ_L (cf. the proof of Lemma 1.3.2).
Since $\mathfrak{m}_L^e = p\mathfrak{o}_L$ (resp. $4\mathfrak{o}_L$ if $p = 2$) (i) and (ii) are proved if for (i) we choose $(\lambda^{-1}p\mathfrak{x}_i)$ as an L-basis of $\mathfrak{g}_L$ (resp. $(\lambda^{-1}4\mathfrak{x}_i)$ if $p = 2$).

It remains to show that $\theta_{\mathbb{Q}_p}(\Lambda) = R^{L|\mathbb{Q}_p} G_0$ is a uniform pro-p group. According to [14], Theorem 8.31, we only need to show that $R^{L|\mathbb{Q}_p} G_0$ is a standard group in the sense of [loc.cit.], Definition 8.22. Let $g, h \in R^{L|\mathbb{Q}_p} G_0$ and write $\lambda \cdot \theta_{\mathbb{Q}_p}^{-1}(g) = \sum_{i,j} \lambda_{ij} v_j \mathfrak{x}_i$, $\lambda \cdot \theta_{\mathbb{Q}_p}^{-1}(h) = \sum_{i,j} \mu_{ij} v_j \mathfrak{x}_i$ with $\lambda_{ij}, \mu_{ij} \in p\mathbb{Z}_p$ (resp. $4\mathbb{Z}_2$ if $p = 2$). Then according to (1.14)

$$\begin{aligned} \lambda \cdot \theta_{\mathbb{Q}_p}^{-1}(gh^{-1}) = \lambda \cdot \Phi_L(gh^{-1}) &= \sum_i F_i((\sum_r \lambda_{kr} v_r)_k, (\sum_r \mu_{kr} v_r)_k)\mathfrak{x}_i \\ &= \sum_i \sum_j G_{ij}((\lambda_{kr}), (\mu_{kr})) v_j \mathfrak{x}_i. \end{aligned}$$

Since $v_j \in \mathfrak{o}_L$ for all $j = 1, \ldots, \ell$ we have $v_i v_j = \sum_k c_{ijk} v_k$ with $c_{ijk} \in \mathbb{Z}_p$. Using this, one obtains that the functions G_{ij} are given by formal power series with coefficients in $\mathbb{Z}_p$. Since $\lambda \cdot \theta_{\mathbb{Q}_p}^{-1} : R^{L|\mathbb{Q}_p} G_0 \to \oplus_{i,j} p\mathbb{Z}_p v_j \mathfrak{x}_i$ (resp. $\oplus_{i,j} 4\mathbb{Z}_2 v_j \mathfrak{x}_i$ if $p = 2$) is a global chart, the claim follows. $\square$

Recall [14], Theorem 4.9, that if $(a_1, \ldots, a_d)$ is a basis of topological generators of a uniform pro-p group G_0, $d = \dim G_0$, then every element has a unique expression of the form $a_1^{\lambda_1} \cdots a_d^{\lambda_d}$ with $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}_p$. If H_0 is a closed, uniform subgroup of G_0 then we say that H_0 is compatible with G_0 if there is a basis of topological generators of H_0 that can be extended to a basis of topological generators of G_0.

Corollary 1.3.6. *Let G be a locally L-analytic group and H a closed locally L-analytic subgroup. Then there is an open subgroup G_0 of G as in Proposition 1.3.5 such that $H_0 := H \cap G_0$, as an open subgroup of H, satisfies conditions (i) – (iii) of Proposition 1.3.5 and $R^{L|\mathbb{Q}_p} H_0$ is compatible with $R^{L|\mathbb{Q}_p} G_0$.*

Proof: Extend an L-basis $(\mathfrak{x}_1, \ldots, \mathfrak{x}_j)$ of the Lie algebra $\mathfrak{h}_L$ of H to an L-basis $(\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ of the Lie algebra $\mathfrak{g}_L$ of G, $j \leq d$. When restricted to $\mathfrak{h}_L$ the corresponding system $\theta_L : \mathfrak{g}_L \dashrightarrow G$ of canonical coordinates of the second kind is a system of canonical coordinates of the second kind for H because *exp* restricts to an exponential map for H. Since this system is étale in $0 \in \mathfrak{h}_L$ we may assume U and G' from the proof of Proposition 1.3.5 to satisfy $\Phi_L(H \cap G') \subseteq \mathfrak{h}_L$. Starting with G' define $\Lambda \subseteq U$ and $G_0 \subseteq G'$ as before. Then $\Lambda' := \Lambda \cap \mathfrak{h}_L$ is an open neighborhood of 0 in $\mathfrak{h}_L$ and a direct summand of Λ. Indeed, $\Lambda' = \oplus_{i=1}^{j} \lambda^{-1} \mathfrak{m}_L^e \mathfrak{x}_i$ (resp. $\oplus_{i=1}^{j} \lambda^{-1} \mathfrak{m}_L^{2e} \mathfrak{x}_i$ if $p = 2$). If $\mathfrak{x} \in \Lambda$ is such that $\theta_L(\mathfrak{x}) \in G_0 \cap H$ then $\mathfrak{x} \in \Lambda \cap \Phi_L(G_0 \cap H) \subseteq \Lambda \cap \Phi_L(G' \cap H) \subseteq \Lambda \cap \mathfrak{h}_L = \Lambda'$. Therefore, the restriction of θ_L from Λ to Λ' is an isomorphism $\Lambda' \to H_0 := G_0 \cap H$ of locally L-analytic manifolds. It follows as above that H_0 satisfies conditions (i) – (iii) of Proposition 1.3.5 with respect to Λ'. By definition of the canonical coordinates of the second kind Λ (resp. Λ') gives rise to the basis of topological generators $(exp(v_k \mathfrak{x}_i))$, $1 \leq k \leq m$, $1 \leq i \leq d$, (resp. $1 \leq k \leq m$, $1 \leq i \leq j$) of $R^{L|\mathbb{Q}_p} G_0$ (resp. $R^{L|\mathbb{Q}_p} H_0$): Note that $exp(n \cdot \mathfrak{x}) = exp(\mathfrak{x})^n$ for all $\mathfrak{x} \in \Lambda$ and all $n \in \mathbb{Z}$. Thus, $R^{L|\mathbb{Q}_p} G_0$ and $R^{L|\mathbb{Q}_p} H_0$ are compatible. $\square$

1.4 Explicit Fréchet-Stein structures

The notion of K-Fréchet-Stein algebra was first introduced by P. Schneider and J. Teitelbaum (cf. [30], section 3): A K-Fréchet algebra A is called a K-Fréchet-Stein algebra if there is a sequence $q_1 \leq q_2 \leq \ldots$ of continuous algebra seminorms on A defining its Fréchet topology such that for all $n \in \mathbb{N}$ the Hausdorff completion A_{q_n} of A with respect to q_n is a (left) noetherian K-Banach algebra and a flat $A_{q_{n+1}}$-module via the natural map $A_{q_{n+1}} \to A_{q_n}$. In this subsection we will assume K to be discretely valued.

Let G_0 be a uniform pro-p group with a basis $(a_1, \ldots, a_d)$ of topological generators. Putting $b_i := a_i - 1$ and $\mathbf{b}^\alpha := b_1^{\alpha_1} \ldots b_d^{\alpha_d}$ in $K[G_0]$ for a multi-index $\alpha \in \mathbb{N}^d$, it is shown in section 4 of [loc.cit.] that $D(G_0, K)$ admits the explicit description

$$D(G_0, K) = \left\{ \sum_\alpha d_\alpha \mathbf{b}^\alpha \mid d_\alpha \in K, \, \forall\, 0 < r < 1 : \, \sup_\alpha |d_\alpha| r^{\tau\alpha} < \infty \right\}.$$

Here $\tau\alpha = \sum \tau_i \alpha_i$ with rational numbers τ_i depending on the structure of G_0 as a p-valued group. The Fréchet topology of $D(G_0, K)$ can be defined by the family of norms $(||\cdot||_r)_{0<r<1}$ given by

$$\left|\left|\sum_\alpha d_\alpha \mathbf{b}^\alpha\right|\right|_r := \sup_\alpha |d_\alpha| r^{\tau\alpha}.$$

The norms $||\cdot||_r$ are independent of the choice of a basis $(a_1,\ldots,a_d)$ of topological generators. If we let $D_r(G_0,K) = \{\sum_\alpha d_\alpha \mathbf{b}^\alpha \mid \lim_{|\alpha|\to\infty} |d_\alpha| r^{\tau\alpha} = 0\}$ be the completion of $D(G_0,K)$ with respect to the norm $||\cdot||_r$ then

$$D(G_0,K) = \varprojlim_r D_r(G_0,K)$$

as K-Fréchet spaces. We summarize some of the main results of [30] in the following theorem (loc.cit. Theorem 4.5 and Theorem 4.9):

Theorem (Schneider-Teitelbaum). *If K is discretely valued, $r \in p^{\mathbb{Q}}$ and $1/p < r < 1$ then the algebra structure of $D(G_0,K)$ extends to $D_r(G_0,K)$ making it a K-Banach algebra with multiplicative norm $||\cdot||_r$. Moreover, for any two real numbers $r,r' \in p^{\mathbb{Q}}$ with $1/p < r' < r < 1$ the natural inclusion $D_r(G_0,K) \hookrightarrow D_{r'}(G_0,K)$ is a flat map of noetherian rings. In other words: $D(G_0,K)$ is a K-Fréchet-Stein algebra with respect to the family of norms $||\cdot||_r$, $r \in p^{\mathbb{Q}}$, $1/p < r < 1$.*

For $0 < r < 1$ we let $U_r(\mathfrak{g},K)$ be the closure of $U(\mathfrak{g},K)$ in $D_r(G_0,K)$ with respect to the norm $||\cdot||_r$. A careful analysis of orthogonal bases (cf. [20], section 1) leads to the following result (loc.cit. 1.4 Lemma 3, Corollaries 1, 2 and 3):

Theorem (Frommer). *If $r \in p^{\mathbb{Q}}$ and $1/p < r < 1$ then $U_r(\mathfrak{g},K)$ is a noetherian subalgebra of $D_r(G_0,K)$. In fact, there are integers $\ell_i > 0$ depending on r such that $D_r(G_0,K)$ is free as a (right) module over $U_r(\mathfrak{g},K)$ with basis consisting precisely of those $\mathbf{b}^\alpha \in K[G_0]$ for which $0 \le \alpha_i < \ell_i$ for all $i = 1,\ldots,d$. Further, $U_r(\mathfrak{g},K)$ is equal to the algebra*

$$U_r(\mathfrak{g},K) = \left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid d_\alpha \in K,\ \lim_{|\alpha|\to\infty} |d_\alpha| ||\mathfrak{X}^\alpha||_r = 0 \right\},$$

where $\mathfrak{X}$ is the $\mathbb{Q}_p$-basis $(\mathfrak{x}_i := \log(1+b_i))_{1\le i\le d}$ of $\mathfrak{g}$. The norm $||\cdot||_r$ can be computed via $||\sum_\alpha d_\alpha \mathfrak{X}^\alpha||_r = \sup_\alpha |d_\alpha| ||\mathfrak{X}^\alpha||_r$.

Using compatible uniform pro-p groups we can slightly extend this result:

Corollary 1.4.1. *Let G_0 be a uniform pro-p group with closed, compatible uniform subgroup H_0. Then $D(H_0,K)$ is a K-Fréchet-Stein algebra with* *respect to the family of norms* $||\cdot||_r$*,* $r \in p^{\mathbb{Q}}$*,* $1/p < r < 1$*, restricted to* $D(H_0, K)$*. The conclusions of Frommer's theorem hold for* $D(H_0, K)$*. If* $r \in p^{\mathbb{Q}}$ *is a real number with* $1/p < r < 1$ *then the closure* $D_r(G_0, K)_{H_0}$ *of* $D(G_0, K)_{H_0}$ *in* $D_r(G_0, K)$ *is a finitely generated, free* $U_r(\mathfrak{g}, K)$*-module possessing a basis contained in* $K[H_0]$*.*

Proof: Choose a basis $(a_1, \ldots, a_d)$ of topological generators of G_0 such that $(a_1, \ldots, a_j)$ is a basis of topological generators of H_0, $j := \dim H_0 \le d$. It follows directly from the definition of the norms $||\cdot||_r$ and the theorem of Schneider-Teitelbaum that $D(H_0, K)$ is a K-Fréchet-Stein algebra with respect to the restricted norms $||\cdot||_r$, $r \in p^{\mathbb{Q}}$, $1/p < r < 1$. Of course, H_0 has to be viewed as a p-valued group with respect to the valuation coming from G_0. It is also clear that Frommer's theorem applies to $D(H_0, K)$. Fix $r \in p^{\mathbb{Q}}$ with $1/p < r < 1$. Let $A \subset \mathbb{N}^d$ be the set of all multi-indices satisfying $0 \le \alpha_i < \ell_i$ for all i and $A' \subseteq A$ be the subset of all α such that $\alpha_{j+1} = \ldots = \alpha_d = 0$. If $\mathfrak{h}$ denotes the Lie algebra of H then $(\mathbf{b}^\alpha)_{\alpha \in A'}$ is a basis of the free $U_r(\mathfrak{h}, K)$-module $D_r(H_0, K)$: The proof of [20], 1.4 Lemma 3, shows that writing $\mathfrak{x}_i = \log(1 + b_i) = \sum_{n \ge 1} (-1)^{n+1} b_i^n / n$ one can choose

$$\ell_i = \max\{m \ge 1 \mid \sup_{n \ge 1} |1/n| r^{n\tau_i} = |1/m| r^{m\tau_i}\}.$$

Hence for $1 \le i \le j$ the integers ℓ_i do not depend on whether we consider b_i as an element of $K[G_0]$ or $K[H_0]$ as long as H_0 is viewed as a p-valued group with respect to the valuation coming from G_0.

Now consider the free, finitely generated $U_r(\mathfrak{g}, K)$-submodule D of $D_r(G_0, K)$ with basis $(\mathbf{b}^\alpha)_{\alpha \in A'}$. Since $\oplus_{\alpha \in A'} \mathbf{b}^\alpha U(\mathfrak{g}, K) \subseteq D(G_0, K)_{H_0}$, D is contained in $D_r(G_0, K)_{H_0}$. Conversely, D contains $D_r(H_0, K)$ and $U_r(\mathfrak{g}, K)$ and thereby a dense subspace of $D_r(G_0, K)_{H_0}$ (cf. Lemma 1.2.10). According to [30], Proposition 2.1 (ii), D is closed. Hence $D = D_r(G_0, K)_{H_0}$. $\square$

We are now going to extend Frommer's theorem and Corollary 1.4.1 to the case of a finite extension $L|\mathbb{Q}_p$. To do this we need to recall that if A is a K-Fréchet-Stein algebra with respect to a sequence $(q_n)_{n \ge 1}$ of continuous algebra seminorms and if I is a closed ideal of A then according to [30], Proposition 3.7 and its proof, A/I is a K-Fréchet-Stein algebra with respect to the sequence $(\overline{q_n})_{n \ge 1}$ of residue norms $\overline{q_n}$. It follows that if G_0 is a locally L-analytic group such that $R^{L|\mathbb{Q}_p} G_0$ is uniform pro-p then $D(G_0, K)$ is a K-Fréchet-Stein algebra (loc.cit. Theorem 5.1). Namely, $D(G_0, K)$ is topologically isomorphic as an algebra to the quotient of $D(R^{L|\mathbb{Q}_p} G_0, K)$ by $I := ker(\tau')$ (cf. subsection 1.3).

For $1/p < r < 1$ we denote by $||\cdot||_{\overline{r}}$ the residue norm on $D(G_0, K)$ induced by $||\cdot||_r$. The completion of $D(G_0, K)$ with respect to $||\cdot||_{\overline{r}}$ is denoted by $D_r(G_0, K)$. Let I_r be the closure of I in $D_r(R^{L|\mathbb{Q}_p}G_0, K)$ and consider the projection

$$\tau_r : D_r(R^{L|\mathbb{Q}_p}G_0, K) \longrightarrow D_r(R^{L|\mathbb{Q}_p}G_0, K)/I_r. \tag{1.15}$$

According to the proof of [30], Proposition 3.7, we have

$$D_r(G_0, K) = D_r(R^{L|\mathbb{Q}_p}G_0, K)/I_r. \tag{1.16}$$

As before we let $U_r(\mathfrak{g}_L, K)$ (resp. $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$) denote the closure of $U(\mathfrak{g}_L, K)$ (resp. $U(\mathfrak{g}_{\mathbb{Q}_p}, K)$) in $D_r(G_0, K)$ (resp. $D_r(R^{L|\mathbb{Q}_p}G_0, K)$). Set further $J_r := I_r \cap U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$.

Theorem 1.4.2. *Let G be a locally L-analytic group and G_0 as in Proposition 1.3.5. If $r \in p^{\mathbb{Q}}$ with $1/p < r < 1$ then $D_r(G_0, K)$ is a free, finitely generated module over the noetherian subalgebra $U_r(\mathfrak{g}_L, K)$ with the same basis in $K[G_0]$ as in Frommer's theorem applied to $R^{L|\mathbb{Q}_p}G_0$. Further, there is an L-basis $\mathfrak{X}$ of $\mathfrak{g}_L$ and a norm $\nu_{\overline{r}}$ on $U_r(\mathfrak{g}_L, K)$ equivalent to $||\cdot||_{\overline{r}}$ such that $U_r(\mathfrak{g}_L, K)$ is equal to the algebra*

$$U_r(\mathfrak{g}_L, K) = \left\{ \sum_{\alpha} d_\alpha \mathfrak{X}^\alpha \mid d_\alpha \in K, \lim_{|\alpha|\to\infty} |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha) = 0 \right\}.$$

The norm $\nu_{\overline{r}}$ can be computed via $\nu_{\overline{r}}(\sum_\alpha d_\alpha \mathfrak{X}^\alpha) = \sup_\alpha |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha)$.

Proof: Let $(\mathbf{b}^\alpha)_{\alpha\in A}$ be the $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-basis of $D_r(R^{L|\mathbb{Q}_p}G_0, K)$ considered before and D the (right) $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-submodule $D := \oplus_{\alpha\in A}\mathbf{b}^\alpha J_r$. We claim that D equals I_r. Since I_r is an ideal of $D_r(R^{L|\mathbb{Q}_p}G_0, K)$ containing J_r, we naturally have $D \subseteq I_r$. On the other hand, D contains a dense subspace of I_r according to Lemma 1.3.3 since $J := I \cap (U(\mathfrak{g}_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} K) \subseteq J_r$ and J_r is an ideal of $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$:

$$\sum_{g\in R^{L|\mathbb{Q}_p}G_0} \delta_g J \subseteq D_r(R^{L|\mathbb{Q}_p}G_0, K)\cdot J_r = \oplus_{\alpha\in A}\mathbf{b}^\alpha U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)\cdot J_r = D.$$

Since D is closed according to [30], Proposition 2.1 (ii), we also have $I_r \subseteq D$. It follows from (1.16) and Frommer's theorem that there is an isomorphism

$$D_r(G_0, K) \simeq \oplus_{\alpha\in A}\mathbf{b}^\alpha (U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)/J_r)$$

of (right) $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-modules. It becomes topological if $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)/J_r$ is equipped with its (Banach) quotient topology (cf. [30], Proposition 2.1). In particular, the image of $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$ under τ_r is closed. According to Lemma 1.3.4 it contains a dense subspace of $U_r(\mathfrak{g}_L, K)$ whence there is a topological isomorphism

$$U_r(\mathfrak{g}_L, K) \simeq U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)/J_r. \tag{1.17}$$

This proves the first statement of the theorem. We claim that the assertions concerning the explicit description of $U_r(\mathfrak{g}_L, K)$ hold if we equip $U_r(\mathfrak{g}_L, K)$ with the residue norm $\nu_{\overline{r}}$ coming from (1.17). We make use of the notations of Proposition 1.3.5.

According to Proposition 1.3.5 (i) there is an L-basis $\mathfrak{X} = (\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ of $\mathfrak{g}_L$ and a $\mathbb{Z}_p$-basis $(v_1, \ldots, v_m)$ of $\mathfrak{o}_L$ such that $\mathfrak{Y} := (v_i\mathfrak{x}_j)_{i,j}$ is a $\mathbb{Z}_p$-basis of Λ. According to Proposition 1.3.5 (ii) $\mathfrak{Y}$ gives rise to the set of topological generators $(exp(v_i\mathfrak{x}_j))_{i,j}$ of $R^{L|\mathbb{Q}_p}G_0$ so that by Frommer's theorem

$$U_r(\mathfrak{g}_{\mathbb{Q}_p}, K) = \left\{ \sum_\beta c_\beta \mathfrak{Y}^\beta \mid \lim_{|\beta|\to\infty} |c_\beta| ||\mathfrak{Y}^\beta||_r = 0 \right\}$$

with multiplicative norm $||\sum_\beta c_\beta \mathfrak{Y}^\beta||_r = \sup_\beta |c_\beta| ||\mathfrak{Y}^\beta||_r$. If $\beta = (\beta_{ij}) \in \mathbb{N}^m \times \mathbb{N}^d$ let $\varphi(\beta) := (\sum_{i=1}^m \beta_{ij})_{1\leq j\leq d} \in \mathbb{N}^d$. For any β with $\varphi(\beta) = \alpha$ we have $\tau'(\mathfrak{Y}^\beta) = \prod_{i,j} v_j^{\beta_{ij}} \mathfrak{X}^\alpha$ and $|\alpha| = |\beta|$. Since τ_r continuously extends τ' we have

$$\tau_r(\sum_\beta c_\beta \mathfrak{Y}^\beta) = \sum_\beta \tau_r(c_\beta \mathfrak{Y}^\beta) = \sum_{\alpha\in\mathbb{N}^d} \left(\sum_{\varphi(\beta)=\alpha} c_\beta \prod_{i,j} v_i^{\beta_{ij}} \right) \mathfrak{X}^\alpha$$

and also

$$\left| \sum_{\varphi(\beta)=\alpha} c_\beta \prod_{i,j} v_i^{\beta_{ij}} \right| \nu_{\overline{r}}(\mathfrak{X}^\alpha) \leq \max_{\varphi(\beta)=\alpha} |c_\beta| ||\mathfrak{Y}^\beta||_r \to 0 \text{ as } |\alpha| \to \infty.$$

Therefore, $U_r(\mathfrak{g}_L, K) \subseteq \{\sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid \lim_{|\alpha|\to\infty} |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha) = 0\}$. Conversely, any series $\sum_\alpha d_\alpha \mathfrak{X}^\alpha$ with $\lim_{|\alpha|\to\infty} |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha) = 0$ converges in $U_r(\mathfrak{g}_L, K)$ so that we get equality.

We claim that J is dense in J_r. Note first that J is dense in $I \cap U(\mathfrak{g}_{\mathbb{Q}_p}, K)$: If $\delta = \sum_\beta c_\beta \mathfrak{Y}^\beta \in U(\mathfrak{g}_{\mathbb{Q}_p}, K)$ then by (1.8) $\lim_{|\beta|\to\infty} |c_\beta| \rho^{-|\beta|} = 0$ for all $\rho > 0$. Hence $\tau'(\delta) = \sum_\alpha (\sum_{\varphi(\beta)=\alpha} c_\beta \prod_{i,j} v_i^{\beta_{ij}}) \mathfrak{X}^\alpha$ converges in $U(\mathfrak{g}_L, K)$ because for all $\rho > 0$

$$\left| \sum_{\varphi(\beta)=\alpha} c_\beta \prod_{i,j} v_i^{\beta_{ij}} \right| \rho^{-|\alpha|} \leq \max_{\varphi(\beta)=\alpha} |c_\beta| \rho^{-|\beta|} \to 0 \text{ as } |\alpha| \to \infty.$$

If now $\delta \in I \cap U(\mathfrak{g}_{\mathbb{Q}_p}, K)$ then due to uniqueness in $U(\mathfrak{g}_L, K)$ we have $\sum_{\varphi(\beta)=\alpha} c_\beta \prod_{i,j} v_i^{\beta_{ij}} = 0$ and hence $\sum_{\varphi(\beta)=\alpha} c_\beta \mathfrak{Y}^\beta \in J$ for all α. But the sequence $(\sum_{|\alpha|\leq N} \sum_{\varphi(\beta)=\alpha} c_\beta \mathfrak{Y}^\beta)_{N\geq 0}$ converges to δ as $N \to \infty$.
To see that $I \cap U(\mathfrak{g}_{\mathbb{Q}_p}, K)$ is dense in J_r we note that as a direct consequence of Frommer's theorem $U(\mathfrak{g}_{\mathbb{Q}_p}, K)$ is a K-Fréchet-Stein algebra with respect to the norms $||\cdot||_r$. As a closed ideal $I \cap U(\mathfrak{g}_{\mathbb{Q}_p}, K)$ is a coadmissible module over $U(\mathfrak{g}_{\mathbb{Q}_p}, K)$. Since J is dense in $I \cap U(\mathfrak{g}_{\mathbb{Q}_p}, K)$ we know from Theorem A (cf. [30], section 3) that the corresponding coherent sheaf is given by the $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-ideals J_r' where J_r' is the closure of J in $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$. The same reasoning as above shows that $I_r = \oplus_{\alpha\in A} \mathbf{b}^\alpha J_r'$. Since also $I_r = \oplus_{\alpha\in A} \mathbf{b}^\alpha J_r$ and $J_r' \subseteq J_r$ we obtain $J_r' = J_r$.

Let $\delta = \sum_\alpha d_\alpha \mathfrak{X}^\alpha \in U_r(\mathfrak{g}_L, K)$, i.e. $\lim_{|\alpha|\to\infty} |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha) = 0$. Let $\varepsilon > 0$ be given and choose $N \in \mathbb{N}$ so large that

$$\sup_{|\alpha|\leq N} |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha) = \sup_\alpha |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha) \text{ and } \nu_{\overline{r}}(\sum_{|\alpha|>N} d_\alpha \mathfrak{X}^\alpha) \leq \varepsilon.$$

Note that the preimage of $\sum_{|\alpha|\leq N} d_\alpha \mathfrak{X}^\alpha$ under τ_r contains elements in $U(\mathfrak{g}_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} K$. By our above claim there is then an element $\sum_\beta c_\beta \mathfrak{Y}^\beta \in U(\mathfrak{g}_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} K$ mapping to $\sum_{|\alpha|\leq N} d_\alpha \mathfrak{X}^\alpha$ under τ_r such that

$$\nu_{\overline{r}}(\sum_{|\alpha|\leq N} d_\alpha \mathfrak{X}^\alpha) \geq ||\sum_\beta c_\beta \mathfrak{Y}^\beta||_r - \varepsilon.$$

Uniqueness in $U(\mathfrak{g}_L) \otimes_L K$ implies that $\tau_r(\sum_{\varphi(\beta)=\alpha} c_\beta \mathfrak{Y}^\beta) = d_\alpha \mathfrak{X}^\alpha$ for all α with $|\alpha| \leq N$. Therefore,

$$\begin{aligned} ||\sum_\beta c_\beta \mathfrak{Y}^\beta||_r = \sup_\beta |c_\beta| ||\mathfrak{Y}^\beta||_r &\geq \sup_{|\alpha|\leq N} \left\{ \sup_{\varphi(\beta)=\alpha} |c_\beta| ||\mathfrak{Y}^\beta||_r \right\} \\ &\geq \sup_{|\alpha|\leq N} |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha) = \sup_\alpha |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha). \end{aligned}$$

Hence for all $\varepsilon > 0$

$$\max\{\varepsilon, \nu_{\overline{r}}(\delta)\} \geq \nu_{\overline{r}}(\sum_{|\alpha|\leq N} d_\alpha \mathfrak{X}^\alpha) \geq \sup_\alpha |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha) - \varepsilon,$$

i.e. $\nu_{\overline{r}}(\delta) \geq \sup_\alpha |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha)$. Since the opposite inequality holds trivially we have $\nu_{\overline{r}}(\sum_\alpha d_\alpha \mathfrak{X}^\alpha) = \sup_\alpha |d_\alpha| \nu_{\overline{r}}(\mathfrak{X}^\alpha)$. In particular, the expansion $\delta = \sum_\alpha d_\alpha \mathfrak{X}^\alpha$ of an element $\delta \in U_r(\mathfrak{g}_L, K)$ is unique. □

Corollary 1.4.3. *Let G be a locally L-analytic group, H a closed locally L-analytic subgroup and G_0 as in Corollary 1.3.6. If $H_0 := H \cap G_0$ then $D(H_0, K)$ is a K-Fréchet-Stein algebra with respect to the family of norms $|| \cdot ||_{\overline{r}}$, $r \in p^{\mathbb{Q}}$, $1/p < r < 1$, restricted from $D(G_0, K)$ to $D(H_0, K)$. The conclusions of Theorem 1.4.2 hold for $D(H_0, K)$. If $r \in p^{\mathbb{Q}}$ is a real number with $1/p < r < 1$ then the closure $D_r(G_0, K)_{H_0}$ of $D(G_0, K)_{H_0}$ in $D_r(G_0, K)$ is a finitely generated, free $U_r(\mathfrak{g}_L, K)$-module with the same basis in $K[H_0]$ as in Corollary 1.4.1 applied to the pair $(R^{L|\mathbb{Q}_p} G_0, R^{L|\mathbb{Q}_p} H_0)$.*

Proof: Since $R^{L|\mathbb{Q}_p} H_0$ is compatible with $R^{L|\mathbb{Q}_p} G_0$ we know from Corollary 1.4.1 that $D(R^{L|\mathbb{Q}_p} H_0, K)$ is a K-Fréchet-Stein algebra with respect to the family of norms $|| \cdot ||_r$, $r \in p^{\mathbb{Q}}$, $1/p < r < 1$, obtained by restriction from $D(R^{L|\mathbb{Q}_p} G_0, K)$. The commutativity of the diagram

$$\begin{array}{ccc} D(R^{L|\mathbb{Q}_p} H_0, K) & \hookrightarrow & D(R^{L|\mathbb{Q}_p} G_0, K) \\ \downarrow & & \downarrow {\scriptstyle \tau'} \\ D(H_0, K) & \hookrightarrow & D(G_0, K) \end{array}$$

shows that the kernel of the left vertical arrow is $I' := I \cap D(R^{L|\mathbb{Q}_p} H_0, K)$. Applying Theorem 1.4.2 to H_0 shows that if we let I'_r be the closure of I' in $D_r(R^{L|\mathbb{Q}_p} H_0, K)$ then $D(H_0, K)$ is a K-Fréchet-Stein algebra with respect to the corresponding quotient norms and

$$D_r(H_0, K) = D_r(R^{L|\mathbb{Q}_p} H_0, K)/I'_r$$

(cf. (1.16) applied to H_0). Recall that we have

$$D_r(R^{L|\mathbb{Q}_p} G_0, K) = \bigoplus_{\alpha \in A} \mathbf{b}^{\alpha} U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$$

as K-Banach spaces and similarly

$$D_r(R^{L|\mathbb{Q}_p} H_0, K) = \bigoplus_{\alpha \in A'} \mathbf{b}^{\alpha} U_r(\mathfrak{h}_{\mathbb{Q}_p}, K)$$

with $A' \subseteq A$ (cf. Corollary 1.4.1 and its proof). Moreover, we know from the proof of Theorem 1.4.2 that $I_r = \oplus_{\alpha \in A} \mathbf{b}^{\alpha} J_r$ with $J_r := I_r \cap U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$ and similarly $I'_r = \oplus_{\alpha \in A'} \mathbf{b}^{\alpha} (I'_r \cap U_r(\mathfrak{h}_{\mathbb{Q}_p}, K))$. It follows that $I'_r = I_r \cap D_r(R^{L|\mathbb{Q}_p} H_0, K)$ and hence that

$$D_r(H_0, K) = D_r(R^{L|\mathbb{Q}_p} H_0, K)/(I_r \cap D_r(R^{L|\mathbb{Q}_p} H_0, K)). \tag{1.18}$$

Once we can show that the image of $D_r(R^{L|\mathbb{Q}_p}H_0, K)$ under the quotient map (1.15) is closed it will follow from (1.16) that the right hand side of (1.18) is topologically isomorphic to the closure of $D(H_0, K)$ in $D_r(G_0, K)$ with respect to the residue norm $||\cdot||_{\overline{r}}$, thereby proving all statements on $D(H_0, K)$. Making use of the above direct sum decompositions it suffices to show that the image of $U_r(\mathfrak{h}_{\mathbb{Q}_p}, K)$ under τ_r is closed. We make use of the notation introduced earlier: By construction we may assume $\mathfrak{X}' := (\mathfrak{x}_1, \dots, \mathfrak{x}_j)$, $1 \leq j := \dim H_0 \leq d$, to be an L-basis of $\mathfrak{h}_L$. Recall that $U_r(\mathfrak{g}_L, K) = \{\sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid \lim_{|\alpha|\to\infty} |d_\alpha|\nu_{\overline{r}}(\mathfrak{X}^\alpha) = 0\}$ with $\nu_{\overline{r}}(\sum_\alpha d_\alpha \mathfrak{X}^\alpha) = \sup_\alpha |d_\alpha|\nu_{\overline{r}}(\mathfrak{X}^\alpha)$. We claim that

$$\tau_r(U_r(\mathfrak{h}_{\mathbb{Q}_p}, K)) = W := \{\sum_\alpha d_\alpha (\mathfrak{X}')^\alpha \mid \lim_{|\alpha|\to\infty} |d_\alpha|\nu_{\overline{r}}((\mathfrak{X}')^\alpha) = 0\}$$

which is a closed subspace of $U_r(\mathfrak{g}_L, K)$. Clearly, $\tau_r(U_r(\mathfrak{h}_{\mathbb{Q}_p}, K)) \subseteq W$. Conversely, let $\delta = \sum_\alpha d_\alpha (\mathfrak{X}')^\alpha \in W$ and $\tilde{\delta} = \sum_{\beta\in\mathbb{N}^m\times\mathbb{N}^d} c_\beta \mathfrak{Y}^\beta \in U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$ be a preimage of δ under τ_r (cf. (1.17)). Let $\tilde{\gamma}$ be the element of $U_r(\mathfrak{h}_{\mathbb{Q}_p}, K)$ obtained by summing up all those monomials $c_\beta \mathfrak{Y}^\beta$ for which $\beta_{ik} = 0$ whenever $k > j$. We claim that $\tau_r(\tilde{\gamma}) = \delta$ in which case we are done. Let $\tilde{\delta} - \tilde{\gamma} = \sum_\beta e_\beta \mathfrak{Y}^\beta$ so that

$$\tau_r(\tilde{\delta} - \tilde{\gamma}) = \sum_{\alpha\in\mathbb{N}^d} (\sum_{\varphi(\beta)=\alpha} e_\beta \prod_{i,j} v_i^{\beta_{ij}}) \mathfrak{X}^\alpha.$$

If $\alpha \in \mathbb{N}^d$ is such that $\alpha_{j+1} = \ldots = \alpha_d = 0$ then by construction $e_\beta = 0$ for all $\beta \in \mathbb{N}^n \times \mathbb{N}^d$ for which $\varphi(\beta) = \alpha$. If $\alpha_k \neq 0$ for some $k > j$ then $e_\beta = c_\beta$ for all β with $\varphi(\beta) = \alpha$. In this case the coefficient of $\mathfrak{X}^\alpha$ in the expansion of $\tau_r(\tilde{\delta} - \tilde{\gamma})$ equals the one of δ (here we make use of the uniqueness of the expansion). Since the latter coefficient is zero it follows that $\tau_r(\tilde{\delta} - \tilde{\gamma}) = 0$.

According to the proof of Corollary 1.4.1 there is a finite basis $(\mathbf{b}^\alpha)_{\alpha\in A}$ of the free $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-module $D_r(R^{L|\mathbb{Q}_p}G_0, K)$ and a subset $A' \subseteq A$ such that $(\mathbf{b}^\alpha)_{\alpha\in A'}$ is a basis of the free, finitely generated $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$-module $D_r(R^{L|\mathbb{Q}_p}G_0, K)_{H_0}$. It follows from the decomposition $I_r = \oplus_{\alpha\in A} \mathbf{b}^\alpha J_r$ that $I_r \cap D_r(R^{L|\mathbb{Q}_p}G_0, K)_{H_0} = \oplus_{\alpha\in A'} \mathbf{b}^\alpha J_r$. Thus, by (1.17)

$$(1.19)\quad D_r(R^{L|\mathbb{Q}_p}G_0, K)_{H_0}/(I_r \cap D_r(R^{L|\mathbb{Q}_p}G_0, K)_{H_0}) \simeq \oplus_{\alpha\in A'} \mathbf{b}^\alpha U_r(\mathfrak{g}_L, K).$$

In particular, the image of $D_r(R^{L|\mathbb{Q}_p}G_0, K)_{H_0}$ under τ_r is closed. It follows by means of Lemma 1.3.4 and (1.16) that the left hand side of (1.19) is topologically isomorphic to $D_r(G_0, K)_{H_0}$.
Note that by Theorem 1.4.2 $(\mathbf{b}^\alpha)_{\alpha\in A'}$ is also a basis for the free $U_r(\mathfrak{h}_L, K)$-module $D_r(H_0, K)$ and the free $U_r(\mathfrak{h}_{\mathbb{Q}_p}, K)$-module $D_r(R^{L|\mathbb{Q}_p}H_0, K)$. $\square$

Corollary 1.4.4. *If $L|L_0$ is an extension of local fields containing $\mathbb{Q}_p$ and G is a locally L-analytic group then the natural homomorphism*

$$D(R^{L|L_0}G, K)\hat{\otimes}_{U(\mathfrak{g}_{L_0},K),\iota}U(\mathfrak{g}_L, K) \longrightarrow D(G, K)$$

of $D(R^{L|L_0}G, K)$-$U(\mathfrak{g}_L, K)^{op}$-bimodules is a topological isomorphism.

Proof: Let G_0 be as in Proposition 1.3.5. According to Corollary 1.2.14 and (1.1) it suffices to show that the map

$$D(R^{L|L_0}G_0, K)\hat{\otimes}_{U(\mathfrak{g}_{L_0},K)}U(\mathfrak{g}_L, K) \longrightarrow D(G_0, K) \tag{1.20}$$

is a topological isomorphism. Let again I be the kernel of the surjection $\tau' := D(R^{L|L_0}G_0, K) \to D(G_0, K)$, $1/p < r < 1$ and I_r be the closure of I in $D_r(R^{L|L_0}G_0, K)$. According to Theorem 1.4.2 the modules $D_r(R^{L|L_0}G_0, K)$, resp. $D_r(G_0, K)$, are finitely generated and free over the noetherian Banach algebras $U_r(\mathfrak{g}_{L_0}, K)$, resp. $U_r(\mathfrak{g}_L, K)$, with a common basis $(\mathbf{b}^\alpha)_{\alpha\in A}$. Therefore, the same arguments as in the proof of Theorem 1.4.2 show that $I_r = \oplus_{\alpha\in A}\mathbf{b}^\alpha J_r$ with $J_r := I_r \cap U_r(\mathfrak{g}_{L_0}, K)$. It follows that the map

$$D_r(\mathfrak{g}_{L_0}, K) \otimes_{U_r(\mathfrak{g}_{L_0},K)} U_r(\mathfrak{g}_L, K) \longrightarrow D_r(G_0, K)$$

is an isomorphism of $D_r(\mathfrak{g}_{L_0}, K)$-$U_r(\mathfrak{g}_L, K)^{\mathrm{op}}$-bimodules. The arguments given in the proof of Proposition 1.2.12 show that it is bi-continuous and that we may pass to the projective limit in order to obtain that (1.20) is a topological isomorphism. $\square$

Corollary 1.4.5. *Let $L|L_0$ be an extension of local fields containing $\mathbb{Q}_p$ and G be a locally L-analytic group. If H is a closed, locally L-analytic subgroup of G then the map $\tau' : D(R^{L|L_0}G, K)_H \to D(G, K)_H$ is surjective.*

Proof: Let G_0 and H_0 be as in Corollary 1.3.6. According to (1.6) it suffices to show that the map $\tau' : D(R^{L|L_0}G_0, K)_{H_0} \to D(G_0, K)_{H_0}$ is surjective. Because of the commutativity of the diagram

$$\begin{array}{ccc} D(R^{L|L_0}G_0, K)_{H_0} & \longrightarrow & D(G_0, K)_{H_0} \\ \uparrow & & \uparrow \\ D(R^{L_0|\mathbb{Q}_p}R^{L|L_0}G_0, K)_{H_0} & = & D(R^{L|\mathbb{Q}_p}G_0, K)_{H_0} \end{array}$$

we may assume $L_0 = \mathbb{Q}_p$.
Let $1/p < r < 1$. According to Corollary 1.4.3 the modules $D_r(R^{L|\mathbb{Q}_p}G_0, K)_{H_0}$ and $D_r(G_0, K)_{H_0}$ are finitely generated and free over the noetherian Banach algebras $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$ and $U_r(\mathfrak{g}_L, K)$, respectively. Further, they have

a common basis $(\mathbf{b}^\alpha)_{\alpha\in A'}$ in $K[H_0]$. The map $U_r(\mathfrak{g}_{\mathbb{Q}_p}, K) \to U_r(\mathfrak{g}_L, K)$ is surjective (cf. (1.17)). It follows that the map $D_r(R^{L|\mathbb{Q}_p}G_0, K)_{H_0} \to D_r(G_0, K)_{H_0}$ is surjective for any r. According to the proofs of Theorem 1.4.2 and Corollary 1.4.3 its kernel is $\oplus_{\alpha\in A'}\mathbf{b}^\alpha J_r$ and contains the dense subspace $\sum_{g\in R^{L|\mathbb{Q}_p}H_0} \delta_g \cdot J \subseteq D(R^{L|\mathbb{Q}_p}G_0, K)_{H_0}$. Referring once more to the Mittag-Leffler arguments given in the proof of Proposition 1.2.12 we may conclude that in the projective limit the map $D(R^{L|\mathbb{Q}_p}G_0, K)_{H_0} \to D(G_0, K)_{H_0}$ is still surjective. $\square$

2 Invariant distributions

G acts on itself via conjugation inducing an action by continuous automorphisms on the space $C^{an}(G, K)$ of locally analytic functions on G (cf. Proposition 1.1.2). The contragredient action on $D(G, K)$ is explicitly given by $(g * \delta)(f) = \delta(h \mapsto f(ghg^{-1})) = (\delta_g\delta\delta_{g^{-1}})(f)$ for $g \in G$, $\delta \in D(G, K)$ and $f \in C^{an}(G, K)$, i.e.

$$g * \delta = \delta_g\delta\delta_{g^{-1}}. \tag{2.1}$$

We call a distribution $\delta \in D(G, K)$ invariant if $g * \delta = \delta$ for all $g \in G$. If U is a G-invariant subspace of $D(G, K)$ we denote by U^G the subspace of all invariant distributions contained in U.

The separate continuity of the multiplication together with the density of $K[G]$ in $D(G, K)$ imply by means of (2.1) that the subspace $D(G, K)^G$ of all invariant distributions on G coincides with the center of the ring $D(G, K)$.

For later use we introduce the subspace

$$D^{pt}(G, K) \;:=\; \sum_{g\in G} \delta_g \cdot (U(\mathfrak{g}) \otimes_L K)$$

of $D(G, K)$. It is the space of all point distributions in the sense of [10], 13.2.1.

2.1 The infinitesimal center

The exponential map *exp* restricts to an analytic isomorphism of locally L-analytic manifolds between a neighborhood of 0 in $\mathfrak{g}$ and a neighborhood of 1 in G such that $exp(0) = 1$. By Proposition 1.1.2 it induces a topological isomorphism

$$exp^* : C^\omega_1(G, K) \xrightarrow{\sim} C^\omega_0(\mathfrak{g}, K)$$

which does not depend on the choice of *exp* (cf. the remark following III.4.3 Définition 1 of [5]). Dualizing, we obtain a topological isomorphism

$$exp_* : C_0^\omega(\mathfrak{g}, K)'_b \xrightarrow{\sim} U(\mathfrak{g}, K) = C_1^\omega(G, K)'_b$$

of vector spaces which for $\delta \in C_0^\omega(\mathfrak{g}, K)'_b$ and $[f] \in C_1^\omega(G, K)$ is explicitly given by

$$(exp_*\delta)([f]) = \delta(exp^*[f]) = \delta([\mathfrak{x} \mapsto f(exp(\mathfrak{x}))]).$$

Here $[f]$ denotes the germ in 1 of a locally analytic function f defined in an open neighborhood of $1 \in G$.

Viewing $\mathfrak{g}$ as an abelian locally L-analytic group which is its own Lie algebra, Proposition 1.2.8 shows that $C_0^\omega(\mathfrak{g}, K)'_b$ admits an explicit description

$$C_0^\omega(\mathfrak{g}, K)'_b = \left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha \mid d_\alpha \in K, \forall r > 0 : \sup |d_\alpha| r^{-|\alpha|} < \infty \right\}$$

in terms of power series with commutative multiplication. Here we put $\mathfrak{X}^\alpha := \mathfrak{x}_1^{\alpha_1} \cdots \mathfrak{x}_d^{\alpha_d}$, and the formula

$$\mathfrak{x}(\mathfrak{f}) = \frac{d}{dt}\mathfrak{f}(t\mathfrak{x})|_{t=0} = (\partial_\mathfrak{x}\mathfrak{f})(0) \text{ for } \mathfrak{x} \in \mathfrak{g} \text{ and } \mathfrak{f} \in C^{an}(\mathfrak{g}, K)$$

is the analog of (1.7). Again, the Fréchet topology of $C_0^\omega(\mathfrak{g}, K)'_b$ can be defined by the family of norms $(\nu_r)_{r>0}$ with

$$\nu_r(\sum_\alpha d_\alpha \mathfrak{X}^\alpha) := \sup |d_\alpha| r^{-|\alpha|}.$$

Since according to Proposition 1.2.8 the symmetric algebra $S(\mathfrak{g}) \otimes_L K$ of $\mathfrak{g}$ is dense in $C_0^\omega(\mathfrak{g}, K)'_b$ we prefer to write $S(\mathfrak{g}, K)$ instead of $C_0^\omega(\mathfrak{g}, K)'_b$.

G acts on itself via conjugation and on $\mathfrak{g}$ via the adjoint representation Ad. According to Proposition 1.1.2 this gives rise to actions of G by continuous automorphisms on $C^{an}(G, K)$ and $C^{an}(\mathfrak{g}, K)$, respectively: Note that multiplication by a fixed element and taking inverses are locally analytic automorphisms of G and that $Ad(g)$ for $g \in G$ is linear on $\mathfrak{g}$ and hence locally analytic. In fact, the above G-action on $C^{an}(G, K)$ descends to $C_1^\omega(G, K)$ which is a locally analytic G-representation in the sense of [29], section 3: Since the closed subspace $C^{an}(G, K)_{G\setminus\{1\}}$ is G-invariant it follows from (1.4) that G acts on $C_1^\omega(G, K)$. If G_0 is a compact open subgroup of G and the G-action on $C^{an}(G, K)$ is restricted to G_0 then the natural projection

$C^{an}(G,K) \to C_1^\omega(G,K)$ factors G_0-equivariantly through $C^{an}(G_0,K)$. By [18], Satz 3.3.4, the G_0-action on $C^{an}(G_0,K)$ is locally analytic whence so is the G_0-action on the barrelled quotient $C_1^\omega(G,K) = C_1^\omega(G_0,K)$ (cf. [17], Lemma 3.6.14). Since G_0 is open in G the claim follows.
Similarly, the action of G on $C^{an}(\mathfrak{g},K)$ descends to $C_0^\omega(\mathfrak{g},K)$ because the closed subspace $C^{an}(\mathfrak{g},K)_{\mathfrak{g}\setminus\{0\}}$ is G-invariant. We explicitly have

$$g * [f] = [h \mapsto f(g^{-1}hg)] \qquad \text{for } g \in G \text{ and } f \in C^{an}(G,K),$$
$$g * [\mathfrak{f}] = [\mathfrak{x} \mapsto \mathfrak{f}(Ad(g^{-1})(\mathfrak{x}))] \qquad \text{for } g \in G \text{ and } \mathfrak{f} \in C^{an}(\mathfrak{g},K).$$

Using the formula $g \cdot exp(\mathfrak{x}) \cdot g^{-1} = exp(Ad(g)(\mathfrak{x}))$ for $g \in G$ and all $\mathfrak{x}$ in a neighborhood of zero in $\mathfrak{g}$ depending on g (cf. [5], III.4.4 Corollaire 3) one deduces that the topological isomorphism exp^* is G-equivariant: For $g \in G$, $f \in C^{an}(G,K)$ and $\mathfrak{x}$ near zero in $\mathfrak{g}$ we have

$$\begin{aligned} exp^*(g * f)(\mathfrak{x}) &= (g * f)(exp(\mathfrak{x})) = f(g^{-1} \cdot exp(\mathfrak{x}) \cdot g) \\ &= f(exp(Ad(g^{-1})(\mathfrak{x}))) = (g * exp^* f)(\mathfrak{x}). \end{aligned}$$

Recall that if $n \in \mathbb{N}$, $\mathfrak{y}_1, \ldots, \mathfrak{y}_n \in \mathfrak{g}$ and $\mathfrak{y}_1 \cdots \mathfrak{y}_n$ is their product in $S(\mathfrak{g})$ then the symmetrization map $sym : S(\mathfrak{g}) \to U(\mathfrak{g})$ is defined by

$$sym(\mathfrak{y}_1 \cdots \mathfrak{y}_n) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathfrak{y}_{\sigma(1)} \cdots \mathfrak{y}_{\sigma(n)}$$

through L-linear continuation. Here $\mathfrak{S}_n$ denotes the symmetric group on n letters and the right hand side of the above equation is computed in $U(\mathfrak{g})$.

Proposition 2.1.1. *exp^*:$C_1^\omega(G,K) \to C_0^\omega(\mathfrak{g},K)$ is an isomorphism of locally analytic G-representations on locally convex K-vector spaces of compact type. The corresponding dual map $exp_* : S(\mathfrak{g},K) \to U(\mathfrak{g},K)$ is an isomorphism of separately continuous (left) $D(G,K)$-modules. Its restriction to $S(\mathfrak{g}) \otimes_L K$ coincides with $sym \otimes id$ and maps isomorphically onto $U(\mathfrak{g}) \otimes_L K$. Further, if the $D(G,K)$-actions on $S(\mathfrak{g},K)$ and $U(\mathfrak{g},K)$ are denoted by $*$ then the following formulae hold:*

i) $\mathfrak{x} * \mathfrak{y} = [\mathfrak{x}, \mathfrak{y}]$ *for all* $\mathfrak{x}, \mathfrak{y} \in \mathfrak{g}$ *where* $\mathfrak{x}$ *is considered as an element of* $D(G,K)$ *and* $\mathfrak{y}, [\mathfrak{x}, \mathfrak{y}]$ *as elements of* $S(\mathfrak{g},K)$ *(or* $U(\mathfrak{g},K)$*);*

ii) $\mathfrak{x} * \delta = \mathfrak{x} \cdot \delta - \delta \cdot \mathfrak{x}$ *in* $U(\mathfrak{g},K)$ *for all* $\mathfrak{x} \in \mathfrak{g}$ *and* $\delta \in U(\mathfrak{g},K)$*;*

iii) $\mathfrak{x} * (\delta_1 \cdots \delta_n) = (\mathfrak{x} * \delta_1)\delta_2 \cdots \delta_n + \ldots + \delta_1 \cdots \delta_{n-1}(\mathfrak{x} * \delta_n)$ *for all* $\mathfrak{x} \in \mathfrak{g}$ *and* $\delta_1, \ldots, \delta_n \in S(\mathfrak{g},K)$*.*

Proof: As seen above, conjugation of germs gives rise to a locally analytic G-representation on the space $C^\omega_1(G,K)$ which is a locally convex K-vector space of compact type (cf. [18], Satz 2.3.2). We also saw that $exp^* : C^\omega_1(G,K) \to C^\omega_0(\mathfrak{g},K)$ is a G-equivariant topological isomorphism. It is then automatic that also $C^\omega_0(\mathfrak{g},K)$ is a locally analytic G-representation, proving the first assertion of the proposition. It follows from general principles that the dual map $exp_* : S(\mathfrak{g},K) \to U(\mathfrak{g},K)$ is a topological isomorphism of nuclear Fréchet spaces carrying separately continuous $D(G,K)$-module structures for which exp_* is a homomorphism (cf. [29], Corollary 3.3). For the statement about the restriction of exp_* to $S(\mathfrak{g}) \otimes_L K$ confer [5], III.4.3 Théorème 4 and II.1.5 Proposition 9.

As for (i), note that the subspace $\mathfrak{g}\otimes_L K$ of $U(\mathfrak{g},K)$ is a $D(G,K)$-submodule: For $g \in G$, $\mathfrak{y} \in \mathfrak{g}$ and $f \in C^{an}(G,K)$ we have

$$\begin{aligned}(g * \mathfrak{y})(f) &= \mathfrak{y}(g^{-1} * f) = \frac{d}{dt} f(g \cdot exp(t\mathfrak{y}) \cdot g^{-1})|_{t=0} \\ &= \frac{d}{dt} f(exp(tAd(g)(\mathfrak{y})))|_{t=0} = Ad(g)(\mathfrak{y})(f).\end{aligned}$$

By linearity, $\mathfrak{g} \otimes_L K$ is $K[G]$-invariant and hence a $D(G,K)$-submodule because of the separate continuity of $*$ and because $\mathfrak{g} \otimes_L K$, as a finite dimensional subspace, is closed in $U(\mathfrak{g},K)$. We also see that the $D(G,K)$-module structure on $\mathfrak{g} \otimes_L K$ comes from the adjoint representation of G on $\mathfrak{g}$ which is an analytic representation in the sense of [5], III.1.2 Example 3: cf. [loc.cit.], III.3.11 Proposition 42. In this case, the action of $\mathfrak{g} \subseteq D(G,K)$ on $\mathfrak{g} \otimes_L K$ is obtained by differentiating the action of G (cf. the formula $\mathfrak{x} * v = d/dt(exp(t\mathfrak{x}) \cdot v)|_{t=0}$ in [29], section 3, for a locally analytic G-representation on a locally convex barrelled Hausdorff space V, as well as [loc.cit.], III.4.4 Corollaire 2). Thus, $\mathfrak{x} * \mathfrak{y} = ad(\mathfrak{x})(\mathfrak{y}) = [\mathfrak{x},\mathfrak{y}]$ (cf. [5], III.3.12 Proposition 44). The reasoning for $\mathfrak{g} \otimes_L K$ considered as a subspace of $S(\mathfrak{g},K)$ is analogous.

For (ii) let $n \geq 2$ and consider the continuous multilinear map

$$(\mathfrak{g} \otimes_L K)^n \to U(\mathfrak{g},K), \quad (\mathfrak{y}_i \otimes \lambda_i) \mapsto \prod_i \mathfrak{y}_i\lambda_i$$

whose image is contained in a G-invariant, finite dimensional and hence complete normed subspace of $U(\mathfrak{g},K)$. By (2.1) the G-action on $U(\mathfrak{g},K)$ has the property that

$$g * (\prod_i \mathfrak{y}_i\lambda_i) = \delta_g \cdot (\prod_i \mathfrak{y}_i\lambda_i) \cdot \delta_{g^{-1}} = \prod_i(\delta_g\mathfrak{y}_i\lambda_i\delta_{g^{-1}}) = \prod_i(g * (\mathfrak{y}_i\lambda_i)).$$

Let us put $\lambda_1 = \ldots = \lambda_n = 1$ for simplicity. Note that if V is a Banach space then the notion of a locally analytic G-representation as given in [29], section 3, coincides with the notion of an analytic Banach space representation in the sense of Bourbaki (cf. [18], Korollar 3.1.9). Using [5], III.3.11 Proposition 41 and (i) we may therefore conclude that

$$\begin{aligned}\mathfrak{x} * (\prod_i \mathfrak{y}_i) &= (\mathfrak{x} * \mathfrak{y}_1)\mathfrak{y}_2 \cdots \mathfrak{y}_n + \ldots + \mathfrak{y}_1 \cdots \mathfrak{y}_{n-1}(\mathfrak{x} * \mathfrak{y}_n) \\ &= [\mathfrak{x}, \mathfrak{y}_1]\mathfrak{y}_2 \cdots \mathfrak{y}_n + \ldots + \mathfrak{y}_1 \cdots \mathfrak{y}_{n-1}[\mathfrak{x}, \mathfrak{y}_n]\end{aligned}$$

for all $\mathfrak{x} \in \mathfrak{g}$. Since $[\mathfrak{x}, \mathfrak{y}_i] = \mathfrak{x}\mathfrak{y}_i - \mathfrak{y}_i\mathfrak{x}$ in $U(\mathfrak{g})$ we obtain statement (ii) for all $\mathfrak{x} \in \mathfrak{g}$ and $\delta \in U(\mathfrak{g}) \otimes_L K$. The general case follows by means of Proposition 1.2.8 and the separate continuity of $*$. Statement (iii) for $S(\mathfrak{g}, K)$ is proved analogously. □

According to the above proposition we have the following commutative diagram of continuous K-linear maps:

$$\begin{array}{ccccc} \mathfrak{g} \otimes_L K & \longrightarrow & U(\mathfrak{g}) \otimes_L K & \hookrightarrow & U(\mathfrak{g}, K) \\ \| & & \wr\uparrow {\scriptstyle sym \otimes id} & & \wr\uparrow exp_* \\ \mathfrak{g} \otimes_L K & \longrightarrow & S(\mathfrak{g}) \otimes_L K & \hookrightarrow & S(\mathfrak{g}, K), \end{array}$$

where we have identified $\mathfrak{g} \otimes_L K \subseteq S(\mathfrak{g}, K)$ with its image in $U(\mathfrak{g}, K)$ and the right horizontal arrows have dense image. Taking $\mathfrak{g}$-invariants one obtains the commutative diagram

$$\begin{array}{ccc} Z(\mathfrak{g}) \otimes_L K & \hookrightarrow & U(\mathfrak{g}, K)^{\mathfrak{g}} \\ \wr\uparrow & & \wr\uparrow \\ S(\mathfrak{g})^{\mathfrak{g}} \otimes_L K & \hookrightarrow & S(\mathfrak{g}, K)^{\mathfrak{g}} \end{array}$$

because exp_* is $\mathfrak{g}$-equivariant. Here $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$ is the center of $U(\mathfrak{g})$ and $U(\mathfrak{g}, K)^{\mathfrak{g}}$ is the center of the algebra $U(\mathfrak{g}, K)$ as follows from Proposition 2.1.1 (ii) and Proposition 1.2.8.

If $\delta = \sum_\alpha d_\alpha \mathfrak{X}^\alpha \in S(\mathfrak{g}, K)$ or $U(\mathfrak{g}, K)$ and $n \geq 0$ then we let $\delta^{\leq n} := \sum_{|\alpha| \leq n} d_\alpha \mathfrak{X}^\alpha$ and $\delta^{>n} := \sum_{|\alpha|>n} d_\alpha \mathfrak{X}^\alpha$ denote the sum of the homogeneous components of degree $\leq n$ and $> n$ of δ, respectively. Note that if $g \in G$ then $g * \delta^{\leq n}$ is of degree $\leq n$ for every $n \in \mathbb{N}$. This follows from writing $g * \mathfrak{x}_i = \sum_j a_j \mathfrak{x}_j$, $a_j \in L$, and noting that by (2.1)

$$g * (\lambda \cdot \prod_i \mathfrak{x}_i^{\alpha_i}) = \lambda \cdot \prod_i (g * \mathfrak{x}_i)^{\alpha_i}.$$

In particular, G acts on $S(\mathfrak{g}) \otimes_L K$ and $U(\mathfrak{g}) \otimes_L K$.

Proposition 2.1.2. *$Z(\mathfrak{g}) \otimes_L K$ and $U(\mathfrak{g})^G \otimes_L K$ are dense in $U(\mathfrak{g}, K)^{\mathfrak{g}}$ and $U(\mathfrak{g}, K)^G$, respectively.*

Proof: Since exp_* is equivariant for the actions of $\mathfrak{g}$ and G we may equally well show that $S(\mathfrak{g})^{\mathfrak{g}} \otimes_L K$ and $S(\mathfrak{g})^G \otimes_L K$ are dense in $S(\mathfrak{g}, K)^{\mathfrak{g}}$ and $S(\mathfrak{g}, K)^G$, respectively. If $\delta \in S(\mathfrak{g}, K)$ is homogeneous of degree n then it follows from Proposition 2.1.1 that for $\mathfrak{x} \in \mathfrak{g}$ either $\mathfrak{x} * \delta = 0$ or $\mathfrak{x} * \delta$ is again homogeneous of degree n (write $[\mathfrak{x}, \mathfrak{x}_i] = \sum_j a_j \mathfrak{x}_j$ for $\mathfrak{x} \in \mathfrak{g}$, $a_j \in L$). We have seen above that similarly $g * \delta$ will again be homogeneous of degree n. This shows that if $\delta \in S(\mathfrak{g}, K)$ is a general $\mathfrak{g}$-invariant (resp. G-invariant) element then both $\delta^{\leq n}$ and $\delta^{>n}$ are $\mathfrak{g}$-invariant (resp. G-invariant). Since $\delta^{\leq n} \in S(\mathfrak{g}) \otimes_L K$ and $\delta^{\leq n} \to \delta$ for $n \to \infty$, the assertion follows. $\square$

Remark 2.1.3. If G is an open subgroup of the group of L-rational points of a connected algebraic L-group $\mathbb{G}$ then [29], Proposition 3.7, shows that $Z(\mathfrak{g}) \otimes_L K$ consists of invariant distributions on G, i.e. $Z(\mathfrak{g}) \otimes_L K = U(\mathfrak{g})^G \otimes_L K$. According to Proposition 2.1.2 the same is then true for $U(\mathfrak{g}, K)^{\mathfrak{g}}$ and hence $U(\mathfrak{g}, K)^{\mathfrak{g}} = U(\mathfrak{g}, K)^G$. Similarly, $S(\mathfrak{g}, K)^{\mathfrak{g}} = S(\mathfrak{g}, K)^G$ in this case.

Remark 2.1.4. Let ν denote a norm on $S(\mathfrak{g}) \otimes_L K$ with respect to which the action of G (resp. $\mathfrak{g}$) is continuous. If the completion $S_\nu(\mathfrak{g}, K)$ of $S(\mathfrak{g}) \otimes_L K$ with respect to ν has the explicit description $\{\sum_\alpha d_\alpha \mathfrak{X}^\alpha | \lim_{|\alpha| \to \infty} |d_\alpha| \nu(\mathfrak{X}^\alpha) = 0\}$ with

$$\nu(\sum_\alpha d_\alpha \mathfrak{X}^\alpha) = \sup_\alpha |d_\alpha| \nu(\mathfrak{X}^\alpha),$$

then the above proof shows that $S(\mathfrak{g})^G \otimes_L K$ and $S(\mathfrak{g})^{\mathfrak{g}} \otimes_L K$ are even dense in $S_\nu(\mathfrak{g}, K)^G$ and $S_\nu(\mathfrak{g}, K)^{\mathfrak{g}}$, respectively.

In general, the restriction of exp_* to $S(\mathfrak{g}, K)^{\mathfrak{g}}$ is not an isomorphism of algebras although both $S(\mathfrak{g}, K)^{\mathfrak{g}}$ and $U(\mathfrak{g}, K)^{\mathfrak{g}}$ are commutative. Making use of a construction of M. Duflo's we will show, however, that one does obtain an isomorphism

$$\eta : S(\mathfrak{g}, K)^{\mathfrak{g}} \to U(\mathfrak{g}, K)^{\mathfrak{g}}$$

of K-Fréchet algebras if exp_* is suitably normalized. This result is similar to the conjecture of Kashiwara and Vergne for real Lie groups (cf. [1]) involving, however, distributions on germs of functions rather than germs of distributions.

For the following confer [15], p. 55. Let k be a field of characteristic zero and $\mathfrak{h}$ a Lie algebra of finite dimension over k. We identify $S(\mathfrak{h})$ with the algebra of polynomial functions on $\mathfrak{h}^*$ and $S(\mathfrak{h}^*)$ with the algebra of differential operators with constant coefficients on $\mathfrak{h}^*$: letting $\mathfrak{X} = (\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ and $\mathfrak{X}^* = (\mathfrak{x}_1^*, \ldots, \mathfrak{x}_d^*)$ be dual k-bases of $\mathfrak{h}$ and $\mathfrak{h}^*$, respectively, we identify $S(\mathfrak{h})$ and $S(\mathfrak{h}^*)$ with the polynomial algebras $k[\mathfrak{x}_1, \ldots, \mathfrak{x}_d]$ and $k[\mathfrak{x}_1^*, \ldots, \mathfrak{x}_d^*]$, respectively. If $f = \sum_\beta \mu_\beta \mathfrak{X}^\beta \in S(\mathfrak{h})$ is given then we identify f with the polynomial function

$$(\sum_{i=1}^{d} \lambda_i \mathfrak{x}_i^* \mapsto \sum_\beta \mu_\beta \lambda_1^{\beta_1} \cdots \lambda_d^{\beta_d}) : \mathfrak{h}^* \to k.$$

If $q = \sum_\alpha \lambda_\alpha (\mathfrak{X}^*)^\alpha \in S(\mathfrak{h}^*)$ is given then we let $D(q) := \sum_\alpha \lambda_\alpha (\partial\mathfrak{X})^\alpha \in \mathrm{End}_k(S(\mathfrak{h}))$ be the corresponding operator. Here $(\partial\mathfrak{X})^\alpha := (\partial\mathfrak{x}_1)^{\alpha_1} \circ \ldots \circ (\partial\mathfrak{x}_d)^{\alpha_d}$ and $\partial\mathfrak{x}_i$ formally differentiates a polynomial with respect to the variable $\mathfrak{x}_i$. The completion $\hat{S}(\mathfrak{h}^*)$ of $S(\mathfrak{h}^*)$ with respect to the topology defined by the maximal ideal $(\mathfrak{x}_1^*, \ldots, \mathfrak{x}_d^*)$ may be identified with the algebra of formal power series in the variables $\mathfrak{x}_i^*$ over k. If $f \in S(\mathfrak{h})$ is given and the order of $q \in S(\mathfrak{h}^*)$ is sufficiently large then $D(q)(f) = 0$. Hence for $q \in \hat{S}(\mathfrak{h}^*)$ one can define $D(q)(f)$ by continuity and set $\langle q, f\rangle := D(q)(f)(0)$. This identifies $S(\mathfrak{h})$ with the space $\hat{S}(\mathfrak{h}^*)'$ of continuous functionals on $\hat{S}(\mathfrak{h}^*)$ and $D(q)$ coincides with the transpose of multiplication by q in $\hat{S}(\mathfrak{h}^*)$ (loc.cit. Lemme II.1).

If $S(\mathfrak{h})$ is identified with the algebra of constant coefficient differential operators on $\mathfrak{h}$ and $f \in S(\mathfrak{h})$ then we let $D^*(f)$ be the corresponding operator. $D^*(f)$ is an endomorphism of $\hat{S}(\mathfrak{h}^*)$. If $q \in \hat{S}(\mathfrak{h}^*)$ is a power series we let $q(0)$ be its constant term. According to the remarks preceding Lemme II.2 of [loc.cit.] we have

$$D^*(f)(q)(0) = D(q)(f)(0) = \langle q, f\rangle \tag{2.2}$$

for all $q \in \hat{S}(\mathfrak{h}^*)$ and $f \in S(\mathfrak{h})$.

Let $ad(\mathfrak{X}) \in M_d(k[\mathfrak{x}_1^*, \ldots, \mathfrak{x}_d^*])$ be the matrix $ad(\mathfrak{X}) := \sum_i \mathfrak{x}_i^* A_i$ where $A_i \in M_d(k)$ represents $ad(\mathfrak{x}_i) \in \mathrm{End}_k(\mathfrak{h})$ with respect to the k-basis $\mathfrak{X}$ of $\mathfrak{h}$. If $B_{2n} \in \mathbb{Q}$ denote the Bernoulli numbers of even degree and $exp(t) \in \mathbb{Q}[[t]]$ is the usual exponential series then the formula

$$\begin{aligned} q = q(\mathfrak{x}_1^*, \ldots, \mathfrak{x}_d^*) &:= \det\left(\frac{exp(ad(\mathfrak{X})/2) - \ exp(-ad(\mathfrak{X})/2)}{ad(\mathfrak{X})}\right)^{1/2} \\ &= exp(\sum_{n=1}^{\infty} \frac{B_{2n}}{4n(2n)!} tr[ad(\mathfrak{X})^{2n}]) \end{aligned} \tag{2.3}$$

defines a formal power series in the indeterminates $\mathfrak{x}_i^*$ with coefficients in k, i.e. an element of $\hat{S}(\mathfrak{h}^*)$ (for the second formula cf. [1]). One of the main results of [16] is the following theorem (loc.cit. Théorème 2):

Theorem (Duflo). *If $\mathfrak{h}$ is a finite dimensional Lie algebra over a field k of characteristic zero then the normalized symmetrization map*

$$\eta := sym \circ D(q) : S(\mathfrak{h})^{\mathfrak{h}} \to Z(\mathfrak{h})$$

is an isomorphism of k-algebras.

It is known that in the case of Lie algebras $\mathfrak{h}$ over the fields $k = \mathbb{R}$ or $\mathbb{C}$, the formal power series q defines an analytic function around 0 in $\mathfrak{h}$. This is also true for the Lie algebra $\mathfrak{g}$ over the non-archimedean field L:

Proposition 2.1.5. *The formal power series q defines an analytic function in a neighborhood of 0 in $\mathfrak{g}$. If we let $[q] \in C_0^{\omega}(\mathfrak{g}, K)$ denote its germ in 0 then the normalized exponential map $\eta : S(\mathfrak{g}, K) \to U(\mathfrak{g}, K)$ defined by*

$$\eta(\delta)([f]) := \delta([q] \cdot exp^*[f]) \text{ for } \delta \in S(\mathfrak{g}, K) \text{ and } [f] \in C_1^{\omega}(G, K),$$

restricts to a topological isomorphism of K-Fréchet algebras

$$\eta : S(\mathfrak{g}, K)^{\mathfrak{g}} \xrightarrow{\sim} U(\mathfrak{g}, K)^{\mathfrak{g}}.$$

Proof: Let $\mathfrak{x} \in \mathfrak{g}$ and write $ad(\mathfrak{x}) = (\lambda_{ij}) \in M_d(L)$ with respect to the L-basis $\mathfrak{X}$ of $\mathfrak{g}$. Choose indices i_0 and j_0 such that $\lambda := \lambda_{i_0 j_0}$ is of maximal absolute value among all λ_{ij}. Since the entries of $ad(\mathfrak{x})^{2n}$ are homogeneous polynomials in the entries λ_{ij} of degree $2n$ we obtain

$$|tr[ad(\mathfrak{x})^{2n}]| \le |\lambda|^{2n}.$$

Using the estimates $|n!| \ge p^{-n/(p-1)}$ and $|B_{2n}| \le p$ (cf. [24], Lemma 5.3.1 and Corollary 5.5.5) we obtain

$$\left| \frac{B_{2n}}{4n(2n)!} tr[ad(\mathfrak{x})^{2n}] \right| \le 4np(|\lambda| p^{1/(p-1)})^{2n} \to 0$$

as $n \to \infty$ for $|\lambda|$ sufficiently small. Hence the formal power series

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{4n(2n)!} tr[ad(\mathfrak{X})^{2n}]$$

defines an analytic function near zero in $\mathfrak{g}$. Since its value at zero is $0 \in L$, restricting further (if necessary) it can be composed with the exponential map defined in a neighborhood of zero in L. This proves that q defines an analytic function in a neighborhood of zero in $\mathfrak{g}$.

The normalized exponential map $\eta : S(\mathfrak{g}, K) \to U(\mathfrak{g}, K)$ defined as above is still a topological isomorphism of K-Fréchet spaces: Note that $q(0) = 1$ so that $[q]$ is invertible in $C_0^\omega(\mathfrak{g}, K)$. If $\delta \in S(\mathfrak{g})$ and $[p] \in C_0^\omega(\mathfrak{g}, K)$ is represented by a formal power series $p \in \hat{S}(\mathfrak{g}^*)$ then by (2.2) and [15], Lemme II.1,

$$\begin{aligned}\delta([q] \cdot [p]) &= D^*(\delta)(qp)(0) = D(qp)(\delta)(0) = \langle qp, \delta \rangle \\ &= \langle p, D(q)(\delta) \rangle = D(q)(\delta)([p]).\end{aligned}$$

Since the restriction of exp_* to $S(\mathfrak{g}) \otimes_L K$ coincides with sym (cf. Proposition 2.1.1) it follows that $\eta | S(\mathfrak{g}, K)^{\mathfrak{g}}$ extends Duflo's isomorphism. Since by Proposition 2.1.2 $S(\mathfrak{g})^{\mathfrak{g}} \otimes_L K$ (resp. $U(\mathfrak{g})^{\mathfrak{g}} \otimes_L K$) is dense in $S(\mathfrak{g}, K)^{\mathfrak{g}}$ (resp. $U(\mathfrak{g}, K)^{\mathfrak{g}}$) it follows that η is an isomorphism of algebras onto $U(\mathfrak{g}, K)^{\mathfrak{g}}$. □

We are now going to explicitly compute $U(\mathfrak{g}, K)^{\mathfrak{g}}$ in the special case that $\mathfrak{g}$ is semisimple and contains a split maximal toral subalgebra $\mathfrak{t}$ (cf. [13], 1.9.10). The Weyl group $\mathfrak{W} = \mathfrak{W}(\mathfrak{g}, \mathfrak{t})$ acts on $\mathfrak{t}^*$ by L-linear endomorphisms and dually on $\mathfrak{t}$. It follows from Proposition 1.1.2 that $\mathfrak{W}$ acts continuously on $C^{an}(\mathfrak{t}, K)$. Since the closed subspace $C^{an}(\mathfrak{t}, K)_{\mathfrak{t} \setminus \{0\}}$ is $\mathfrak{W}$-invariant $\mathfrak{W}$ acts on the quotient $C_0^\omega(\mathfrak{t}, K)$ and hence on $S(\mathfrak{t}, K)$.

For $c \in K^*$ with $|c| > 1$ and $i \in \mathbb{N}$ let $T_{n,i}(K) := K\langle c^{-i}X_1, \ldots, c^{-i}X_n \rangle$ be the generalized Tate algebra of all power series $\sum_{\alpha \in \mathbb{N}^n} d_\alpha X^\alpha$ over K such that $\lim_{|\alpha| \to \infty} |d_\alpha||c^i|^{|\alpha|} = 0$. We endow $T_{n,i}(K)$ with the (complete) norm $|\cdot|_i$ defined by $|\sum_\alpha d_\alpha X^\alpha|_i := \sup_\alpha |d_\alpha||c^i|^{|\alpha|}$. Here $X = (X_1, \ldots, X_n)$ and $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. If $i \leq j$ then there is a continuous embedding $T_{n,j}(K) \subseteq T_{n,i}(K)$. The projective limit

$$\mathcal{O}((\mathbb{A}_K^n)^{an}) := \varprojlim_{i \in \mathbb{N}} T_{n,i}(K)$$

is the K-Fréchet algebra of holomorphic functions on the rigid analytic affine space $(\mathbb{A}_K^n)^{an}$ of dimension n over K.

Theorem 2.1.6. *If $\mathfrak{g}$ is split semisimple with $\mathfrak{t}$ and $\mathfrak{W}$ as above then there are isomorphisms*

$$U(\mathfrak{g}, K)^{\mathfrak{g}} \simeq S(\mathfrak{t}, K)^{\mathfrak{W}} \simeq \mathcal{O}((\mathbb{A}_K^n)^{an})$$

of K-Fréchet algebras with $n := \dim_L(\mathfrak{t})$.

In order to construct the above isomorphisms we need some preparation. Let k be a field which is complete with respect to a non-trivial, non-archimedean valuation and let X be an affine scheme of finite type over k. If Γ is a finite group acting on X by k-automorphisms then the quotient X/Γ exists and is again an affine scheme of finite type over k (cf. [3], Proposition 6.15). In fact, $X/\Gamma = \mathrm{Spec}(k[X]^\Gamma)$ if $k[X]$ denotes the ring of regular functions on X. The ring extension $k[X]|k[X]^\Gamma$ is finite and the quotient map $\pi : X \to X/\Gamma$ surjective. By X^{an} and $(X/\Gamma)^{an}$ we denote the rigid analytifications of X and X/Γ, respectively. Note that Γ acts on X^{an} by functoriality and that π induces a morphism $\pi^{an} : X^{an} \to (X/\Gamma)^{an}$ of rigid analytic k-varieties.

Proposition 2.1.7. *Let X be a reduced affine scheme of finite type over k and Γ a finite group of k-automorphisms of X whose order is prime to the characteristic of k. The presheaf $\mathcal{F}$ on $(X/\Gamma)^{an}$ defined by $\mathcal{F}(U) := \mathcal{O}_{X^{an}}((\pi^{an})^{-1}(U))^\Gamma$ is an $\mathcal{O}_{(X/\Gamma)^{an}}$-submodule of $\pi_*^{an}\mathcal{O}_{X^{an}}$ via the natural map $(\pi^{an})^\# : \mathcal{O}_{(X/\Gamma)^{an}} \to \pi_*^{an}\mathcal{O}_{X^{an}}$. In fact, $(\pi^{an})^\#$ is an isomorphism onto $\mathcal{F}$.*

Proof: Choose a representation $k[X] = k[\zeta_1, \ldots, \zeta_n]/\mathfrak{a}$ of $k[X]$ as a k-algebra of finite type with an ideal $\mathfrak{a}$ of $k[\zeta] := k[\zeta_1, \ldots, \zeta_n]$. If $c \in k^*$ with $|c| > 1$ let A_i be the k-affinoid algebra $A_i := k\langle c^{-i}\zeta\rangle/(\mathfrak{a})$. Letting $\overline{\zeta}_j$ be the class of ζ_j in $k[\zeta]/\mathfrak{a}$ we have $A_{i+1}\langle c^{-i}\overline{\zeta}\rangle = A_i$ so that $\mathrm{Sp}(A_i)$ is an admissible affinoid subdomain of $\mathrm{Sp}(A_{i+1})$. Pasting all $\mathrm{Sp}(A_i)$, one obtains X^{an} together with the admissible covering $(\mathrm{Sp}(A_i))_{i\in\mathbb{N}}$. The underlying set of X^{an} coincides with $\mathrm{Max}(k[X]) \subseteq \mathrm{Spec}(k[X]) = X$. Similarly, one obtains $(X/\Gamma)^{an} = \cup_{i\in\mathbb{N}}\mathrm{Sp}(B_i)$ with k-affinoid algebras $B_i := k\langle c^{-i}\overline{\xi}\rangle/(\mathfrak{b})$ once we choose a representation $k[X]^\Gamma = k[\xi]/\mathfrak{b}$ with an ideal $\mathfrak{b}$ of $k[\xi]$. The underlying set of $(X/\Gamma)^{an}$ coincides with $\mathrm{Max}(k[X]^\Gamma) \subseteq X/\Gamma$ (cf. [4], 9.3.4 Example 2).

With π also π^{an} is surjective and we have the following commutative diagram of locally G-ringed spaces:

(2.4)
$$\begin{array}{ccc} X^{an} & \xrightarrow{\pi^{an}} & (X/\Gamma)^{an} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & X/\Gamma. \end{array}$$

We see that if $U \subseteq (X/\Gamma)^{an}$ is admissible open then $V := (\pi^{an})^{-1}(U) \subseteq X^{an}$ is admissible open and Γ-invariant. Thus, Γ acts on $\mathcal{O}_{X^{an}}(V)$ so that the presheaf $\mathcal{F}$ is well-defined. If $(U_i)_{i\in I}$ is an admissible covering of U and $V_i := (\pi^{an})^{-1}(U_i)$ then we have the exact sequence

(2.5)
$$\mathcal{O}_{X^{an}}(V) \longrightarrow \prod_{i\in I} \mathcal{O}_{X^{an}}(V_i) \rightrightarrows \prod_{i,j\in I} \mathcal{O}_{X^{an}}(V_i \cap V_j).$$

It follows from the exactness on the left that an element $f \in \mathcal{O}_{X^{an}}(V)$ is Γ-invariant if and only if so are all restrictions $f|V_i \in \mathcal{O}_{X^{an}}(V_i)$ (consider the image of $f - \gamma^*(f)$ for $\gamma \in \Gamma$). Therefore, (2.5) restricts to a well-defined sequence

$$\mathcal{O}_{X^{an}}(V)^\Gamma \longrightarrow \prod_{i\in I} \mathcal{O}_{X^{an}}(V_i)^\Gamma \rightrightarrows \prod_{i,j\in I} \mathcal{O}_{X^{an}}(V_i \cap V_j)^\Gamma$$

which is still exact, and $\mathcal{F}$ is a sheaf on $(X/\Gamma)^{an}$.

Let U and V be as above. Since $\pi^{an} : V \to U$ is Γ-invariant it follows that for all $\gamma \in \Gamma$ and $g \in \mathcal{O}_{X^{an}}(U)$

$$\gamma^*((\pi^{an})^*(g)) = (\pi^{an} \circ \gamma)^*(g) = (\pi^{an})^*(g).$$

Hence $(\pi^{an})^\#$ is indeed a homomorphism $\mathcal{O}_{(X/\Gamma)^{an}} \to \mathcal{F}$ of $\mathcal{O}_{(X/\Gamma)^{an}}$-modules.

π is finite, hence proper, and according to [23], Satz 2.17, π^{an} is proper, too. The above diagram shows that π^{an} has finite fibres whence it is finite by [4], 9.6.3 Corollary 6. Thus, for each $i \in \mathbb{N}$, there is a k-affinoid algebra C_i such that $(\pi^{an})^{-1}(\mathrm{Sp}(B_i)) = \mathrm{Sp}(C_i)$ and the homomorphism $B_i \to C_i$ of k-algebras is finite. We already know that it factors as $B_i \to C_i^\Gamma \subseteq C_i$, and to prove that $(\pi^{an})^\# : \mathcal{O}_{(X/\Gamma)^{an}} \to \mathcal{F}$ is an isomorphism it suffices to show that the map $B_i \to C_i^\Gamma$ is an isomorphism for all $i \in \mathbb{N}$ (cf. [4], 9.4.1 Proposition 2). We denote this map by φ_i. Note that it is a finite homomorphism of k-affinoid algebras because C_i^Γ is a submodule of the finitely generated module C_i over the noetherian ring B_i (C_i^Γ is k-affinoid by [4], 6.3.3 Proposition 3).

We first show that the above map is injective. With X also X/Γ is reduced and hence so is $(X/\Gamma)^{an}$ (cf. [23], Folgerung 2.6). It follows that

$$\ker(\varphi_i) \subseteq \varphi_i^{-1}(\mathrm{rad}(C_i)) = \mathrm{rad}(B_i)$$

because $\pi^{an}|_{\mathrm{Sp}(C_i)} : \mathrm{Sp}(C_i) \to \mathrm{Sp}(B_i)$ is surjective. Thus, $ker(\pi^{an})^* = 0$.

There are indices $j, k \in \mathbb{N}$ depending on i such that $\mathrm{Sp}(C_i) \subseteq \mathrm{Sp}(A_j) \subseteq \mathrm{Sp}(C_k)$. π^{an} restricts to an affinoid map $\mathrm{Sp}(A_j) \to \mathrm{Sp}(B_k)$. Since $\mathrm{Sp}(B_i)$ is a Weierstrass domain in $\mathrm{Sp}(B_k)$ so is $\mathrm{Sp}(C_i)$ in $\mathrm{Sp}(A_j)$ (cf. [4], 7.2.3 Proposition 6). In particular, A_j is dense in C_i. Since $k[X]$ is dense in A_j it follows that $k[X]$ is dense in C_i. Therefore, the natural map $k[X] \otimes_{k[X]^\Gamma} B_i \to C_i$ is surjective: Since $B_i \to C_i$ is finite we may apply [4], 3.7.3 Proposition 1. Choose generators $\eta_1, \ldots, \eta_r$ of $k[X]$ over $k[X]^\Gamma$ and let $c \in C_i^\Gamma$. There are elements $b_1, \dots, b_r \in B_i$ such that $c = \sum_i \varphi_i(b_i)\eta_i$. But then

$$c = |\Gamma|^{-1} \sum_i (\sum_{\gamma \in \Gamma} \gamma^*(\eta_i))\varphi_i(b_i) = \varphi_i(|\Gamma|^{-1} \sum_i (\sum_{\gamma \in \Gamma} \gamma^*(\eta_i))b_i)$$

because $\sum_{\gamma \in \Gamma} \gamma^*(\eta_i) \in k[X]^\Gamma$ for all $i = 1, \dots, r$. Thus, $c \in im(\varphi_i)$. □

Remark 2.1.8. It follows from (2.4) that the underlying point space of $(X/\Gamma)^{an}$ is the set theoretical quotient of X^{an} modulo Γ. Since according to the above proposition the structure sheaf on $(X/\Gamma)^{an}$ is given by $\mathcal{O}_{(X/\Gamma)^{an}}(U) = \mathcal{O}_{X^{an}}((\pi^{an})^{-1}(U))^\Gamma$ it follows that $(X/\Gamma)^{an}$ can be identified with the rigid analytic quotient X^{an}/Γ whose existence is claimed (but not proved) in [19], 6.4.

Corollary 2.1.9. *Under the hypotheses of Proposition 2.1.7 there is an isomorphism*

$$\mathcal{O}_{(X/\Gamma)^{an}}((X/\Gamma)^{an}) \simeq \mathcal{O}_{X^{an}}(X^{an})^\Gamma$$

of k-algebras. □

Proof of Theorem 2.1.6: Let $\mathbf{t} = (t_1, \dots, t_n)$ be an L-basis of $\mathfrak{t}$ considered also as a K-basis of $\mathfrak{t} \otimes_L K$. Proposition 1.2.8 shows that there is a topological isomorphism $S(\mathfrak{t}, K) \to \mathcal{O}((\mathbb{A}^n_K)^{an})$ of K-Fréchet algebras identifying the subalgebra $S(\mathfrak{t}) \otimes_L K$ with the polynomial algebra $K[t_1, \dots, t_n]$ in the variables t_i, i.e. with the algebra of regular functions on the affine space $\mathbb{A}^n_K$ of dimension n over K. There is a family $\mathbf{s} = (s_1, \dots, s_n)$ of n algebraically independent, homogeneous elements in $(S(\mathfrak{t}) \otimes_L K)^{\mathfrak{W}}$ such that the natural map

$$\varphi : K[X_1, \dots, X_n] \longrightarrow (S(\mathfrak{t}) \otimes_L K)^{\mathfrak{W}}, \quad X_i \mapsto s_i \tag{2.6}$$

is an isomorphism (cf. [13], 11.1.14). According to Corollary 2.1.9 it extends to an isomorphism

$$\varphi : \mathcal{O}((\mathbb{A}^n_K)^{an}) \longrightarrow S(\mathfrak{t}, K)^{\mathfrak{W}} = \mathcal{O}((\mathbb{A}^n_K)^{an})^{\mathfrak{W}} \tag{2.7}$$

of K-algebras. If $c \in K^*$ with $|c| > 1$ and $i \in \mathbb{N}$ we denote by $|\cdot|_i$ the natural norm on the left hand side of (2.7) for which $(\mathbf{s}^\alpha)_{\alpha \in \mathbb{N}^n}$ is an orthogonal basis with $|s_j^{\alpha_j}|_i = |c^i|^{\alpha_j}$. Note that $|\cdot|_i$ is multiplicative according to [4], 6.1.5 Proposition 2. Similarly, $\nu_i := \nu_{|c^{-i}|}$ is the multiplicative norm on $S(\mathfrak{t}, K)$ for which $(\mathbf{t}^\alpha)_{\alpha \in \mathbb{N}^n}$ is an orthogonal basis with $\nu_i(t_j^{\alpha_j}) = |c^i|^{\alpha_j}$. Given $i \in \mathbb{N}$ choose $i_0 \in \mathbb{N}$ such that $\max_j \{\nu_i(\varphi(s_j))\} \leq |c^{i_o}|$. Then

$$\nu_i(\varphi(\sum_\alpha d_\alpha \mathbf{s}^\alpha)) \leq |\sum_\alpha d_\alpha \mathbf{s}^\alpha|_{i_0},$$

so that φ is continuous and in fact a topological isomorphism due to the open mapping theorem.

Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$ and choose an eigenvector X_α of α in $\mathfrak{g}$ for any $\alpha \in \Phi$. Extend $\mathbf{t}$ to the L-basis $\mathfrak{X} = (t_1, \ldots, t_n, (X_\alpha)_{\alpha\in\Phi})$ of $\mathfrak{g}$ and let $\overline{J}$ be the closed ideal of $S(\mathfrak{g}, K)$ generated by $\{X_\alpha\}_{\alpha\in\Phi}$. The explicit descriptions of $S(\mathfrak{g}, K)$ and $S(\mathfrak{t}, K)$ show that

$$S(\mathfrak{g}, K) = S(\mathfrak{t}, K) \oplus \overline{J}$$

first as abstract vector spaces but then also topologically due to the open mapping theorem. We claim that the induced continuous, surjective homomorphism $S(\mathfrak{g}, K) \to S(\mathfrak{t}, K)$ of K-algebras restricts to a topological isomorphism $\theta : S(\mathfrak{g}, K)^{\mathfrak{g}} \xrightarrow{\sim} S(\mathfrak{t}, K)^{\mathfrak{W}}$. By the open mapping theorem we only need to show that θ is bijective. To prove this we make use of the following fact from the theory of Lie algebras (cf. [13], Théorème 7.3.7): If $J := S(\mathfrak{g}) \cap \overline{J}$ then $S(\mathfrak{g}) = S(\mathfrak{t}) \oplus J$ and the corresponding projection $S(\mathfrak{g}) \to S(\mathfrak{t})$ restricts to an isomorphism

$$S(\mathfrak{g})^{\mathfrak{g}} \simeq S(\mathfrak{t})^{\mathfrak{W}} \tag{2.8}$$

of algebras.

As in the proof of Proposition 2.1.2 one sees that if δ is an element of $S(\mathfrak{g}, K)^{\mathfrak{g}}$ (resp. $\overline{J}$) then both $\delta^{\leq n}$ and $\delta^{>n}$ are elements of $S(\mathfrak{g}, K)^{\mathfrak{g}}$ (resp. $\overline{J}$). Since $(S(\mathfrak{g})^{\mathfrak{g}} \otimes_L K) \cap (J \otimes_L K) = 0$ it follows that $S(\mathfrak{g}, K)^{\mathfrak{g}} \cap \overline{J} = 0$ whence θ is injective.

Let $\tau \in S(\mathfrak{t}, K)^{\mathfrak{W}}$ be given. It follows from (1.2) that for $\mathfrak{x}_1, \mathfrak{x}_2 \in S(\mathfrak{t}) \otimes_L K$ and $w \in \mathfrak{W}$

$$w \cdot (\mathfrak{x}_1 \cdot \mathfrak{x}_2) = (w \cdot \mathfrak{x}_1) \cdot (w \cdot \mathfrak{x}_2).$$

Thus, the homogeneous components τ_k of τ of degree k with respect to the variables $\mathbf{t}$ are $\mathfrak{W}$-invariant for all $k \geq 0$. Write $\tau_k = \sum_\alpha d_\alpha(k)\mathbf{s}^\alpha$ and let $\xi_1, \ldots, \xi_n \in S(\mathfrak{g})^{\mathfrak{g}}$ be preimages of $s_1, \ldots, s_n$ under the map (2.8). Then $\gamma_k := \sum_\alpha d_\alpha(k)\xi^\alpha \in S(\mathfrak{g})^{\mathfrak{g}} \otimes_L K$ maps to τ_k and we need to show that the series $\sum_k \gamma_k$ converges in $S(\mathfrak{g}, K)$. Note that the Fréchet topology on $S(\mathfrak{g}, K)$ can be defined by a family of multiplicative norms $(\nu_i)_{i\in\mathbb{N}}$ extending the norms ν_i on $S(\mathfrak{t}, K)$ because $\mathfrak{X}$ extends the L-basis $\mathbf{t}$ of $\mathfrak{t}$ (cf. Proposition 1.2.8). Since φ^{-1} is continuous we have $\lim_{k\to\infty} |\varphi^{-1}(\tau_k)|_i = 0$ for all $i \in \mathbb{N}$.

Given $i \in \mathbb{N}$, choose $i_0 \in \mathbb{N}$ such that $\max_j\{\nu_i(\xi_j)\} \leq |c^{i_0}|$. Then

$$\begin{aligned}\nu_i(\gamma_k) &\leq \sup_\alpha |d_\alpha(k)|\nu_i(\xi^\alpha) \leq \sup_\alpha |d_\alpha(k)||c^{i_0}|^{|\alpha|} \\ &= |\sum_\alpha d_\alpha(k)\mathbf{s}^\alpha|_{i_0} = |\varphi^{-1}(\tau_k)|_{i_0} \to 0\end{aligned}$$

as $k \to \infty$.

Composing θ with the inverse of Duflo's isomorphism we obtain the topological isomorphism $\xi := \theta \circ \eta^{-1} : U(\mathfrak{g}, K)^{\mathfrak{g}} \to S(\mathfrak{t}, K)^{\mathfrak{W}}$ of K-Fréchet algebras. $\square$

Corollary 2.1.10. *Assume $\mathfrak{g}$ to be reductive with center $\mathfrak{z}$ and derived Lie algebra $\mathfrak{d}$. If $\mathfrak{d}$ contains a split maximal toral subalgebra $\mathfrak{t}'$ and $\mathfrak{W} = \mathfrak{W}(\mathfrak{d}, \mathfrak{t}')$ is the corresponding Weyl group then there are isomorphisms*

$$U(\mathfrak{g}, K)^{\mathfrak{g}} \simeq S(\mathfrak{z}, K)\hat{\otimes}_K S(\mathfrak{t}', K)^{\mathfrak{W}} \simeq \mathcal{O}((\mathbb{A}_K^n)^{an})$$

of K-Fréchet algebras with $n := \dim_L(\mathfrak{z}) + \dim_L(\mathfrak{t}')$.

Proof: Since $\mathfrak{g} = \mathfrak{z} \times \mathfrak{d}$ as locally L-analytic groups there is a topological isomorphism $D(\mathfrak{g}, K) \simeq D(\mathfrak{z}, K)\hat{\otimes}_{K,\iota} D(\mathfrak{d}, K)$ (cf. [31], Proposition A.3). It restricts to an isomorphism $S(\mathfrak{g}, K) \simeq S(\mathfrak{z}, K)\hat{\otimes}_K S(\mathfrak{d}, K)$ (cf. the arguments given in Remark 1.2.15). Applying the isomorphism exp_* to $S(\mathfrak{g}, K)$ and $S(\mathfrak{d}, K)$ one sees that the decomposition $U(\mathfrak{g}) \simeq S(\mathfrak{z}) \otimes_L U(\mathfrak{d})$ extends to a topological isomorphism $U(\mathfrak{g}, K) \simeq S(\mathfrak{z}, K)\hat{\otimes}_K U(\mathfrak{d}, K)$ of K-algebras (cf. Lemma 2.2.2 below). Since $(Z(\mathfrak{g})\otimes_L K) \simeq (S(\mathfrak{z})\otimes_L K)\otimes_K (Z(\mathfrak{d})\otimes_L K)$ under this isomorphism we obtain $U(\mathfrak{g}, K)^{\mathfrak{g}} \simeq S(\mathfrak{z}, K)\hat{\otimes}_K U(\mathfrak{d}, K)^{\mathfrak{d}}$ by taking completions (cf. [25], Proposition 17.5 (ii) and Proposition 19.10 (i)). By means of Theorem 2.1.6 we obtain the first isomorphism of the statement. Since by definition of the norms ν_r on $S(\mathfrak{z}, K)$ we have $S(\mathfrak{z}, K) = \mathcal{O}((\mathbb{A}_K^{\dim_L(\mathfrak{z})})^{an})$ the existence of the second isomorphism of the statement follows from Theorem 2.1.6 together with [17], Proposition 1.1.29, and [4], 6.1.1 Corollary 8. $\square$

Remark 2.1.11. Let us keep the notation of the above corollary. Even if $\mathfrak{g}$ does not split over L it is true that $Z(\mathfrak{g}) \otimes_L K = S(\mathfrak{z}) \otimes_L Z(\mathfrak{d}) \otimes_L K$ is a polynomial ring in $\dim_L(\mathfrak{z}) + \dim_L(\mathfrak{t})$ variables over K (cf. [13], Théorème 7.3.8 (ii)). According to Proposition 2.1.2 it is dense in $U(\mathfrak{g}, K)^{\mathfrak{g}}$.

2.2 Centrally supported invariant distributions

According to (1.3) and (2.1) the closed subalgebra $D(G, K)_Z$ of centrally supported distributions on G is invariant under the conjugation action of G.

We are going to investigate the subalgebra $D(G,K)^G_Z$ of all centrally supported invariant distributions on G.

G acts on $U(\mathfrak{g},K)$ by continuous automorphisms and (trivially) on $D(Z,K)$. This induces a G-action on $D(Z,K)\otimes_{K,\iota}U(\mathfrak{g},K)$ extending to the Hausdorff completion $D(Z,K)\hat{\otimes}_{K,\iota}U(\mathfrak{g},K)$. The closed subspace U of the latter, constructed in subsection 1.2, is G-invariant. Therefore, G acts on the corresponding quotient $D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)$.

Let similarly U' be the kernel of the quotient map

$$D(Z,K)\hat{\otimes}_{K,\iota}U(\mathfrak{g},K)^G \to D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G,$$

i.e. the closure of the subspace of $D(Z,K)\hat{\otimes}_{K,\iota}U(\mathfrak{g},K)^G$ generated by all elements of the form

$$\lambda\mathfrak{y}\otimes\mathfrak{x}-\lambda\otimes\mathfrak{y}\mathfrak{x} \text{ with } \lambda\in D(Z,K), \mathfrak{y}\in U(\mathfrak{z},K) \text{ and } \mathfrak{x}\in U(\mathfrak{g},K)^G.$$

Theorem 2.2.1. *If K is discretely valued and G is an open subgroup of the group of L-rational points of a connected, algebraic group defined over L then there are K-linear topological isomorphisms*

$$D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G \simeq (D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K))^G \simeq D(G,K)^G_Z$$

of separately continuous K-algebras induced by multiplication in $D(G,K)^G_Z$. In particular, the subspace $D^{pt}(G,K)^G_Z$ of centrally supported invariant point distributions is dense in $D(G,K)^G_Z$.

Proof: We endow $D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G$ and $(D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K))^G$ with the $D(Z,K)\otimes_{U(\mathfrak{z},K)}U(\mathfrak{g},K)^G$-module actions of Remark 1.2.11. Since $D(Z,K)$ and $U(\mathfrak{g},K)^G$ are contained in the center of $D(G,K)$ it is clear that the maps

$$\begin{aligned} D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G &\longrightarrow D(G,K)^G_Z \\ (D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K))^G &\longrightarrow D(G,K)^G_Z \end{aligned}$$

induced by multiplication are homomorphisms of $D(Z,K)\otimes_{U(\mathfrak{z},K)}U(\mathfrak{g},K)^G$-modules. If we can show them to be topological isomorphisms then it follows from the density of the image of $D(Z,K)\otimes_{U(\mathfrak{z},K)}U(\mathfrak{g},K)^G$ in the space $D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G$ that $D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K)^G$ and likewise $(D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K))^G$ carry unique K-algebra structures extending the action of $D(Z,K)\otimes_{U(\mathfrak{z},K)}U(\mathfrak{g},K)^G$ and for which the above maps are homomorphisms.

Now the $D(Z,K)$-$U(\mathfrak{g},K)^{op}$-bimodule isomorphism

$$\mu : D(Z,K)\hat{\otimes}_{U(\mathfrak{z},K),\iota}U(\mathfrak{g},K) \longrightarrow D(G,K)_Z$$

of Proposition 1.2.12 is G-equivariant by definition of the respective G-actions. This gives the second isomorphism of the theorem because the restriction to $U(\mathfrak{g},K)^G$ of the right $U(\mathfrak{g},K)^{op}$-action on either side coincides with the natural left $U(\mathfrak{g},K)^G$-action (note that $U(\mathfrak{g},K)^G$ is contained in the center of $U(\mathfrak{g},K)$).

Let Z_0 be a compact open subgroup of Z. According to Corollary 1.2.14

$$D(Z,K)\hat{\otimes}_{K,\iota}U(\mathfrak{g},K)^G \simeq \bigoplus_{z\in Z/Z_0} \delta_z \cdot D(Z_0,K)\hat{\otimes}_K U(\mathfrak{g},K)^G$$

and similarly for $D(Z,K)\hat{\otimes}_{K,\iota}U(\mathfrak{g},K)$ (recall Remark 1.2.11 for our convention on omitting the ι from the notation). It follows from [25], Lemma 5.3 and Corollary 17.5 (ii), that $D(Z,K)\hat{\otimes}_{K,\iota}U(\mathfrak{g},K)^G$ is a closed subspace of $D(Z,K)\hat{\otimes}_{K,\iota}U(\mathfrak{g},K)$ via the closed embedding of $U(\mathfrak{g},K)^G$ into $U(\mathfrak{g},K)$. We need to show that the corresponding restriction of (1.10) with $H=Z$ maps surjectively onto $D(G,K)_Z^G$ with kernel U'.

If G_0 is a compact open subgroup of G and $Z_0 := G_0 \cap Z$ then the subspace $D(G,K)_{Z_0} = D(G_0,K)_{Z_0}$ of $D(G,K)_Z$ is stable under the action of G (cf. (1.3) and (2.1)). It follows from (1.6) that

$$D(G,K)_Z^G = \bigoplus_{z\in Z/Z_0} \delta_z \cdot D(G_0,K)_{Z_0}^G$$

by taking G-invariants. Therefore, it is sufficient to show that the map

$$D(Z_0,K)\hat{\otimes}_{U(\mathfrak{z},K)}U(\mathfrak{g},K)^G \longrightarrow D(G_0,K)_{Z_0}^G$$

induced by multiplication is a topological isomorphism.

According to [5], III.7.2 Proposition 3, there are compact open subgroups $\Lambda_\mathfrak{g}$ and G_0 of $\mathfrak{g}$ and G, respectively, such that $\Lambda_\mathfrak{g}$ lies in the domain of the exponential map and $exp : \Lambda_\mathfrak{g} \to G_0$ is an isomorphism of locally L-analytic manifolds. According to the proof of [loc.cit.], $\Lambda_\mathfrak{g}$ may be chosen to be contained in any open neighborhood of zero in $\mathfrak{g}$. If therefore $\Lambda_\mathfrak{z} := \Lambda_\mathfrak{g} \cap \mathfrak{z}$ and $Z_0 := G_0 \cap Z$ then we may assume exp to restrict to an isomorphism $\Lambda_\mathfrak{z} \to Z_0$ (note that exp is also an exponential map for Z_0). The K-linear topological isomorphism $exp_* : D(\Lambda_{\mathfrak{g}}, K) \to D(G_0, K)$ therefore restricts to isomorphisms

$$\begin{array}{rlll} exp_*: & D(\Lambda_{\mathfrak{z}}, K) & \longrightarrow D(Z_0, K) & \\ id: & S(\mathfrak{z}, K) & \longrightarrow U(\mathfrak{z}, K) & \text{and} \\ exp_*: & S(\mathfrak{g}, K) & \longrightarrow U(\mathfrak{g}, K). & \end{array}$$

Lemma 2.2.2. *If $\lambda \in D(\Lambda_{\mathfrak{z}}, K)$ and $\delta \in D(\Lambda_{\mathfrak{g}}, K)$ then $exp_*(\lambda \cdot \delta) = exp_*(\lambda) \cdot exp_*(\delta)$.*

Proof: Let $\mathfrak{y} \in \Lambda_{\mathfrak{z}}$ and $f \in C^{an}(G_0, K)$. Then

$$\begin{aligned} exp_*(\delta_{\mathfrak{y}} \cdot \delta)(f) &= (\delta_{\mathfrak{y}} \cdot \delta)(exp^* f) \\ &= \delta(\mathfrak{x} \mapsto f(exp(\mathfrak{y} + \mathfrak{x}))) \\ &= \delta(\mathfrak{x} \mapsto f(exp(\mathfrak{y}) \cdot exp(\mathfrak{x}))) \\ &= (exp_*(\delta_{\mathfrak{y}}) \cdot exp_*(\delta))(f), \end{aligned}$$

since $\mathfrak{y}$ commutes with all $\mathfrak{x} \in \mathfrak{g}$. Since $K[\Lambda_{\mathfrak{z}}]$ is dense in $D(\Lambda_{\mathfrak{z}}, K)$, the assertion follows from the linearity and continuity of exp_*. □

Together with Lemma 1.2.10 we obtain that exp_* restricts to an isomorphism $D(\Lambda_{\mathfrak{g}}, K)_{\Lambda_{\mathfrak{z}}} \to D(G_0, K)_{Z_0}$ and that the diagram

$$\begin{array}{ccc} D(\Lambda_{\mathfrak{z}}, K) \hat{\otimes}_{S(\mathfrak{z},K)} S(\mathfrak{g}, K) & \xrightarrow{\mu} & D(\Lambda_{\mathfrak{g}}, K)_{\Lambda_{\mathfrak{z}}} \\ \wr \downarrow exp_* \hat{\otimes} exp_* & & \wr \downarrow exp_* \\ D(Z_0, K) \hat{\otimes}_{U(\mathfrak{z},K)} U(\mathfrak{g}, K) & \xrightarrow{\mu} & D(G_0, K)_{Z_0} \end{array}$$

is commutative. G acts trivially on $D(\Lambda_{\mathfrak{z}}, K)$ and $D(Z_0, K)$. Moreover, G acts on $S(\mathfrak{g}, K)$ in such a way that $exp_* : S(\mathfrak{g}, K) \to U(\mathfrak{g}, K)$ is G-equivariant. Thus, there is an action of G on $D(\Lambda_{\mathfrak{g}}, K)_{\Lambda_{\mathfrak{z}}}$ such that $exp_* : D(\Lambda_{\mathfrak{g}}, K)_{\Lambda_{\mathfrak{z}}} \to D(G_0, K)_{Z_0}$ is G-equivariant. We obtain the commutative diagram

$$\begin{array}{ccc} D(\Lambda_{\mathfrak{z}}, K) \hat{\otimes}_{S(\mathfrak{z},K)} S(\mathfrak{g}, K)^G & \longrightarrow & D(\Lambda_{\mathfrak{g}}, K)^G_{\Lambda_{\mathfrak{z}}} \\ \wr \downarrow exp_* \hat{\otimes} exp_* & & \wr \downarrow exp_* \\ D(Z_0, K) \hat{\otimes}_{U(\mathfrak{z},K)} U(\mathfrak{g}, K)^G & \longrightarrow & D(G_0, K)^G_{Z_0} \end{array}$$

and may equally well show the above statements in the setting of $\Lambda_{\mathfrak{g}}$ and $\Lambda_{\mathfrak{z}}$.

Passing to an open subgroup of $\Lambda_{\mathfrak{g}}$, we may assume that $\Lambda_{\mathfrak{g}}$ and $\Lambda_{\mathfrak{z}}$ satisfy the compatibility conditions of Corollary 1.3.6. Hence for $r \in p^{\mathbb{Q}}$ with

$1/p < r < 1$ the K-Banach algebra $D_r(\Lambda_{\mathfrak{g}}, K)_{\Lambda_{\mathfrak{z}}}$ admits a finite direct sum decomposition

$$D_r(\Lambda_{\mathfrak{g}}, K)_{\Lambda_{\mathfrak{z}}} = \bigoplus_{\alpha \in A'} \mathbf{b}^\alpha S_r(\mathfrak{g}, K)$$

with $\mathbf{b}^\alpha \in K[\Lambda_{\mathfrak{z}}]$ for all $\alpha \in A'$ (cf. Corollary 1.4.3).

Lemma 2.2.3. *The action of $\mathfrak{g}$ on $D(\Lambda_{\mathfrak{g}}, K)_{\Lambda_{\mathfrak{z}}}$ induced by that of G extends to a $\mathfrak{g}$-action on $D_r(\Lambda_{\mathfrak{g}}, K)_{\Lambda_{\mathfrak{z}}}$.*

Proof: By the above direct sum decomposition it suffices to show that the action of $\mathfrak{g}$ on $S(\mathfrak{g}, K)$ extends to the closure $S_r(\mathfrak{g}, K)$ of $S(\mathfrak{g}, K)$ in $D_r(\Lambda_{\mathfrak{g}}, K)$. Note that by Corollary 1.4.5 there is a continuous K-linear surjection $\tau' : S(\mathfrak{g}_{\mathbb{Q}_p}, K) \to S(\mathfrak{g}, K)$. As a direct consequence of Frommer's theorem $S(\mathfrak{g}_{\mathbb{Q}_p}, K)$ is a K-Fréchet-Stein algebra. Therefore, the kernel J of τ' and $S(\mathfrak{g}, K)$ are coadmissible modules over $S(\mathfrak{g}_{\mathbb{Q}_p}, K)$. According to Theorem B (cf. [30], section 3) the coherent sheaf corresponding to J is given by the kernels J_r of the surjections $S_r(\mathfrak{g}_{\mathbb{Q}_p}, K) \to S_r(\mathfrak{g}, K)$ (cf. (1.17)). Since the Lie brackets on $\mathfrak{g}_{\mathbb{Q}_p}$ and $\mathfrak{g} = \mathfrak{g}_L$ coincide it can be seen directly from Proposition 2.1.1 (i) and (iii) that τ' is $\mathfrak{g}$-equivariant. In particular, J is $\mathfrak{g}$-invariant. If we can show that the action of $\mathfrak{g}$ on $S(\mathfrak{g}_{\mathbb{Q}_p}, K)$ extends continuously to $S_r(\mathfrak{g}_{\mathbb{Q}_p}, K)$ then, by the density of J in J_r (cf. Theorem A of [30], section 3), J_r will be $\mathfrak{g}$-invariant, too. The $\mathfrak{g}$-action will then descend to an action of $\mathfrak{g}$ on $S_r(\mathfrak{g}, K)$ extending the action of $\mathfrak{g}$ on $S(\mathfrak{g}, K)$. Thus, we may assume $L = \mathbb{Q}_p$ and hence $|| \cdot ||_{\overline{r}} = || \cdot ||_r$ to be multiplicative. We show that for each $\mathfrak{x} \in \mathfrak{g}$ the corresponding K-linear endomorphism of $S(\mathfrak{g}, K)$ is continuous with respect to the norm $|| \cdot ||_r$.

Recall from Frommer's theorem that there is a $\mathbb{Q}_p$-basis $\mathfrak{X} = (\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ of $\mathfrak{g}$ such that

$$S_r(\mathfrak{g}, K) = \left\{ \sum_\alpha d_\alpha \mathfrak{X}^\alpha \,|\, d_\alpha \in K,\ \lim_{|\alpha| \to \infty} |d_\alpha| ||\mathfrak{X}^\alpha||_r = 0 \right\}$$

with $|| \sum_\alpha d_\alpha \mathfrak{X}^\alpha ||_r = \sup_\alpha \{ |d_\alpha| \prod_{i=1}^d ||\mathfrak{x}_i||_r^{\alpha_i} \}$. For $\mathfrak{x} \in \mathfrak{g}$ choose $\lambda \in \mathbb{Q}_p^*$ such that $||ad(\lambda \mathfrak{x})(\mathfrak{x}_i)||_r \le ||\mathfrak{x}_i||_r$ for all i. Then $|\lambda| \cdot ||\mathfrak{x} * \mathfrak{X}^\alpha||_r \le ||\mathfrak{X}^\alpha||_r$ for all $\alpha \in \mathbb{N}^d$ (cf. Proposition 2.1.1 (i) and (iii)). It follows that $||\mathfrak{x} * \delta||_r \le |\lambda^{-1}| \cdot ||\delta||_r$ for all $\delta \in S(\mathfrak{g}, K)$. □

We obtain

$$D_r(\Lambda_{\mathfrak{g}}, K)^{\mathfrak{g}}_{\Lambda_{\mathfrak{z}}} = \bigoplus_{\alpha \in A'} \mathbf{b}^\alpha S_r(\mathfrak{g}, K)^{\mathfrak{g}}.$$

Since, as remarked in the proof of Corollary 1.4.3, $(\mathbf{b}^{\alpha})_{\alpha \in A'}$ is also a basis for the free $S_r(\mathfrak{z}, K)$-module $D_r(\Lambda_{\mathfrak{z}}, K)$ we obtain a topological isomorphism

$$D_r(\Lambda_{\mathfrak{z}}, K) \otimes_{S_r(\mathfrak{z},K)} S_r(\mathfrak{g}, K)^{\mathfrak{g}} \longrightarrow D_r(\Lambda_{\mathfrak{g}}, K)^{\mathfrak{g}}_{\Lambda_{\mathfrak{z}}}.$$

Passing to the projective limit we obtain a topological isomorphism

$$D(\Lambda_{\mathfrak{g}}, K) \hat{\otimes}_{S(\mathfrak{z},K)} S(\mathfrak{g}, K)^{\mathfrak{g}} \longrightarrow D(\Lambda_{\mathfrak{g}}, K)^{\mathfrak{g}}_{\Lambda_{\mathfrak{z}}}$$

as in the proof of Proposition 1.2.12: To satisfy the Mittag-Leffler condition we need to know that $S(\mathfrak{g}, K)^{\mathfrak{g}}$ is dense in $S_r(\mathfrak{g}, K)^{\mathfrak{g}}$ for all r. This is true according to Remark 2.1.4 and Theorem 1.4.2 and is in fact the reason for our working with $\Lambda_{\mathfrak{g}}$ and $\Lambda_{\mathfrak{z}}$ instead of with G_0 and Z_0. By our assumption on G and Remark 2.1.3 we have $S(\mathfrak{g}, K)^{\mathfrak{g}} = S(\mathfrak{g}, K)^G$. It follows that $D(\Lambda_{\mathfrak{g}}, K)^{\mathfrak{g}}_{\Lambda_{\mathfrak{z}}} \subseteq D(\Lambda_{\mathfrak{g}}, K)^G_{\Lambda_{\mathfrak{z}}}$. Since the reverse inclusion holds trivially it remains to prove the last assertion of the theorem.

Since by Lemma 1.1.1 and Proposition 2.1.2 $K[Z_0]$ and $U(\mathfrak{g})^G \otimes_L K$ are dense in $D(Z_0, K)$ and $U(\mathfrak{g}, K)^G$, respectively, it follows from [25], Lemma 19.10 (i), that the space $K[Z_0] \otimes_K (U(\mathfrak{g})^G \otimes_L K)$ is dense in $D(Z_0, K) \hat{\otimes}_K U(\mathfrak{g}, K)^G$. Therefore, so is its image in the quotient space $D(Z_0, K) \hat{\otimes}_{U(\mathfrak{z},K)} U(\mathfrak{g}, K)^G$. Since the image of $K[Z_0] \otimes_K (U(\mathfrak{g})^G \otimes_L K)$ under μ is precisely $D^{pt}(G_0, K)^G_{Z_0}$, the proof of the theorem is complete. □

Recall that a connected, reductive, linear algebraic L-group $\mathbb{G}$ is called L-isotropic if it contains a non-trivial torus $\mathbb{S}$ which is defined and split over L. The latter means that all rational characters of $\mathbb{S}$ are defined over L or, equivalently, that $\mathbb{S}$ is L-isomorphic to a closed subgroup of $\mathbb{G}^r_{m,L}$ for some r (cf. [3], Proposition 8.2 and its corollary). According to [loc.cit.], Proposition 14.2 and Theorem 22.10, $\mathbb{G}$ is the almost direct product of its center and the finitely many minimal, closed, connected, normal L-subgroups $\mathbb{G}_i$ of positive dimension of its derived subgroup $\mathbb{D}$. Let us call $\mathbb{G}$ sufficiently L-isotropic if all $\mathbb{G}_i$ are L-isotropic. This is the case, for example, if $\mathbb{G}$ is L-split, i.e. contains a maximal torus that is split over L (loc.cit. Proposition 22.9).

It is a consequence of results of K.-Y. Sit's that in many cases of interest the methods we have developed so far suffice to describe all of the center of $D(G, K)$. This is essentially due to the following theorem (cf. [32], Theorem 2.4):

Theorem (Sit). *Assume G to be the group of L-rational points of a connected, reductive, sufficiently L-isotropic L-group $\mathbb{G}$. If the conjugacy class* *of an element $g \in G$ is relatively compact in G then g is contained in the center of G.*

Corollary 2.2.4. *Assume G to be the group of L-rational points of a connected, reductive, sufficiently L-isotropic L-group $\mathbb{G}$. Then $D(G,K)^G = D(G,K)^G_Z$. Let $\mathbb{D}$ be the derived group of $\mathbb{G}$, D the group of L-rational points of $\mathbb{D}$ and $\mathfrak{d}$ the Lie algebra of D. If K is discretely valued then there is a topological isomorphism*

$$D(G,K)^G \simeq D(Z,K)\hat{\otimes}_{K,\iota}U(\mathfrak{d},K)^{\mathfrak{d}} \tag{2.9}$$

of separately continuous K-algebras.

Proof: According to (2.1) and Remark 1.2.3 any invariant distribution on G is supported on a union of relatively compact conjugacy classes. As a consequence of Sit's theorem we have $D(G,K)^G = D(G,K)^G_Z$.

Since $G = D \cdot Z$ with finite intersection $D \cap Z$ it follows from Remark 1.2.15 that there is a topological isomorphism

$$D(G,K)_Z \longrightarrow D(Z,K)\hat{\otimes}_{K,\iota}U(\mathfrak{d},K)$$

of $D(Z,K)$-$U(\mathfrak{d},K)^{op}$-bimodules. The image of $D^{pt}(G,K)^G$ under this isomorphism is $D^{pt}(Z,K) \otimes_K (Z(\mathfrak{d}) \otimes_L K)$: Note that $U(\mathfrak{d})^D = Z(\mathfrak{d})$ by Remark 2.1.3. Since $D^{pt}(G,K)^G$, $D^{pt}(Z,K)$ and $Z(\mathfrak{d}) \otimes_L K$ are dense in $D(G,K)^G$, $D(Z,K)$ and $U(\mathfrak{d},K)^{\mathfrak{d}}$, respectively, (cf. Theorem 2.2.1, Lemma 1.1.1 and Proposition 2.1.2) the above isomorphism restricts to an isomorphism $D(G,K)^G \simeq D(Z,K)\hat{\otimes}_{K,\iota}U(\mathfrak{d},K)^{\mathfrak{d}}$. The arguments given at the beginning of the proof of Theorem 2.2.1 show that it may naturally be viewed as a homomorphism of K-algebras. □

2.3 The Fourier transform

Let k be a field which is complete with respect to a non-trivial and non-archimedean absolute value. Recall that a rigid analytic k-variety X is called quasi-Stein if there is a countable, admissible affinoid covering $(X_i)_{i\in\mathbb{N}}$ of X such that $X_i \subseteq X_{i+1}$ and the image of the map $\mathcal{O}(X_{i+1}) \to \mathcal{O}(X_i)$ is dense for all $i \in \mathbb{N}$ (cf. [22], Definition 2.3).

Lemma 2.3.1. *Let X and Y be quasi-Stein, rigid analytic k-varieties and k' be a complete valued field extension of k.*

i) If X' is a rigid analytic k-variety admitting a finite morphism $f : X' \to X$ then X' is quasi-Stein.

ii) The fibre product $X \times_k Y$ of X and Y over k is quasi-Stein.

iii) X admits a base extension to k' and the resulting rigid analytic k'-variety $X_{k'}$ is quasi-Stein.

Proof: (i) Let $(X_i)_{i\in\mathbb{N}}$ be a covering of X as above and $X'_i := f^{-1}(X_i)$. Then $(X'_i)_{i\in\mathbb{N}}$ is an admissible covering of X' which, due to the finiteness of f, consists of affinoid subsets of X'. We clearly have $X'_i \subseteq X'_{i+1}$ for all $i \in \mathbb{N}$. Further, [4] 7.3.4 Proposition 2 and Proposition 6 show that X_i is a Weierstrass domain in X_{i+1}. According to [loc.cit.], 7.2.3 Proposition 6, X'_i is a Weierstrass domain in X'_{i+1} so that $\mathcal{O}(X'_{i+1})$ has dense image in $\mathcal{O}(X'_i)$ (loc.cit. 7.3.4 Proposition 2).

(ii) Let $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ be coverings of X and Y as quasi-Stein spaces where $X_i = \mathrm{Sp}(A_i)$ and $Y_i = \mathrm{Sp}(B_i)$ with affinoid k-algebras A_i and B_i. By construction $X \times_k Y$ has the admissible affinoid covering $(X_i \times_k Y_i)_{i\in\mathbb{N}}$ where $X_i \times_k Y_i := \mathrm{Sp}(A_i \hat{\otimes}_k B_i)$ and the natural map $X_i \times_k Y_i \to X_{i+1} \times_k Y_{i+1}$ is an open immersion. The maps $A_{i+1} \to A_i$ and $B_{i+1} \to B_i$ having dense image the same is true for the map $A_{i+1}\hat{\otimes}_k B_{i+1} \to A_i \hat{\otimes}_k B_i$.

(iii) Let $(X_i)_{i\in\mathbb{N}}$ be a covering of X as a quasi-Stein space and $U, U' \subseteq X$ be admissible open, affinoid subsets. Then there is an index $i_0 \in \mathbb{N}$ such that X_{i_0} contains both U and U'. The intersection $U \cap U' \subseteq X_{i_0}$ is then affinoid (loc.cit. 7.2.2 Corollary 5). According to [loc.cit.], 9.3.6, the base extension $X_{k'}$ exists and has $((X_i)_{k'})_{i\in\mathbb{N}}$ as an admissible affinoid covering where $(X_i)_{k'} := \mathrm{Sp}(A_i \hat{\otimes}_k k')$. As in (ii) one sees that $X_{k'}$ is quasi-Stein. □

Remark 2.3.2. If X is a quasi-Stein space over k and k' is a complete valued field extension of k then the algebra of global sections of $X_{k'}$ is a k'-Fréchet-Stein algebra: If $(X_i)_{i\in\mathbb{N}}$ is a covering of X as a quasi-Stein space then

$$\mathcal{O}_{X_{k'}}(X_{k'}) = \varprojlim_i \mathcal{O}_{X_{k'}}((X_i)_{k'}).$$

For each $i \in \mathbb{N}$ the algebra $\mathcal{O}_{X_{k'}}((X_i)_{k'})$ is a noetherian k'-Banach algebra for which the map $\mathcal{O}_{X_{k'}}((X_{i+1})_{k'}) \to \mathcal{O}_{X_{k'}}((X_i)_{k'})$ is flat (loc.cit. 7.3.2 Corollary 6). Moreover, the natural map $\mathcal{O}_{X_{k'}}(X_{k'}) \to \mathcal{O}_{X_{k'}}((X_i)_{k'})$ has dense image because this is true for all transition maps (cf. [7], I.4.4 Corollaire).

Recall that if Z is a commutative locally L-analytic group and X is a rigid analytic L-variety then the group $\hat{Z}(X)$ of locally analytic characters of Z with values in X consists of the homomorphisms $Z \to \mathcal{O}_X(X)^*$ of groups such that for any admissible open affinoid subset $X_0 = \mathrm{Sp}(A)$ of X the induced homomorphism $Z \to A^*$ is an element of $C^{an}(Z, A)$ (cf. [17], Definition

6.4.2). Here A carries its natural topology of an L-Banach algebra.
It is shown in [loc.cit.], Corollary 6.4.4, that $\hat{Z}$ is a functor on the category of all rigid analytic L-varieties. Generalizing work of P. Schneider and J. Teitelbaum, M. Emerton proves the following representability result (loc.cit. Proposition 6.4.5):

Theorem (Emerton-Schneider-Teitelbaum). *If Z is a commutative, locally L-analytic, topologically finitely generated group then the functor $\hat{Z}$ is representable by a strictly σ-affinoid rigid analytic space over L.*

Recall that according to [loc.cit.], Definition 2.1.17, a rigid analytic L-variety X is called strictly σ-affinoid if X has an admissible covering $(X_i)_{i\in\mathbb{N}}$ by affinoid subdomains X_i such that for every $i \in \mathbb{N}$ X_i is relatively compact in X_{i+1} in the sense of [4], 9.6.2. As a corollary to the construction of $\hat{Z}$ we obtain:

Corollary 2.3.3. *$\hat{Z}$ is quasi-Stein.*

Proof: By [17], Proposition 6.4.1, there is an isomorphism $Z \to \Lambda \times Z_0$ of locally L-analytic groups where Λ is a free abelian group of finite rank, say r, and Z_0 is a compact open subgroup of Z. Consequently, there is an isomorphism $\hat{Z} \to \hat{\Lambda} \times \hat{Z}_0$. $\hat{\Lambda}$ is represented by the r-fold direct product of the rigid analytification $\mathbb{G}_{m,L}^{an}$ of the multiplicative group $\mathbb{G}_{m,L}$ over L. As we saw in our review of rigid analytifications of affine schemes of finite type over a field at the beginning of the proof of Proposition 2.1.7, $\mathbb{G}_{m,L}^{an}$ is quasi-Stein. Further, $\hat{Z}_0$ admits a finite morphism to a finite direct product of copies of $\widehat{\mathfrak{o}_L}$ which is quasi-Stein by [28], p. 456. Thus, the assertion follows from Lemma 2.3.1 (i) and (ii). $\square$

It follows that $\hat{Z}$ admits a base extension to K which we denote by $\hat{Z}_K$. The ring of global sections of its structure sheaf is denoted by $\mathcal{O}(\hat{Z}_K)$. Since $\hat{Z}_K$ is quasi-Stein and strictly σ-affinoid it follows from Remark 2.3.2 and [17], Proposition 2.1.16, that $\mathcal{O}(\hat{Z}_K)$ is a nuclear K-Fréchet-Stein algebra. We make use of the following result:

Theorem (Emerton-Schneider-Teitelbaum). *If Z is a commutative, locally L-analytic, topologically finitely generated group then there is a natural continuous injection $D(Z,K) \to \mathcal{O}(\hat{Z}_K)$ of topological K-algebras with dense image.*

Since we will need it for the proof of the next corollary we briefly recall the construction of this map: As above we choose an isomorphism $Z \to \Lambda \times Z_0$ of locally L-analytic groups where Λ is a finitely generated free abelian group of rank r, say, and Z_0 is a compact open subgroup of Z. According to [31], Proposition A.3, there is a topological isomorphism

$$D(Z,K) \simeq D(\Lambda,K)\hat{\otimes}_{K,\iota} D(Z_0,K).$$

Λ being discrete, $D(\Lambda,K) = K[\Lambda]$ is the topological direct sum of one dimensional K-vector spaces. Hence $D(\Lambda,K) \otimes_{K,\iota} D(Z_0,K)$ is complete (cf. Corollary 1.2.14 and [25], Lemma 7.8) so that

$$D(Z,K) \simeq K[\Lambda] \otimes_{K,\iota} D(Z_0,K).$$

On the other hand, the Fourier transform of [28], Theorem 2.3, extends to an isomorphism $D(Z_0,K) \simeq \mathcal{O}((\widehat{Z_0})_K)$ of K-Fréchet algebras. Further, $D(\Lambda,K) = K[\Lambda]$ can be interpreted as the algebra of regular functions on the algebraic Cartier dual $D(\Lambda) = \mathbb{G}^r_{m,K}$ of Λ. It admits an embedding into $\mathcal{O}((\mathbb{G}^r_{m,K})^{an}) = \mathcal{O}(\hat{\Lambda}_K)$ with dense image. Since

$$\mathcal{O}(\hat{Z}_K) \simeq \mathcal{O}(\hat{\Lambda}_K)\hat{\otimes}_K \mathcal{O}((\hat{Z}_0)_K) \simeq \mathcal{O}(\hat{\Lambda}_K)\hat{\otimes}_{K,\iota} \mathcal{O}((\hat{Z}_0)_K)$$

the claim follows.

Corollary 2.3.4. *Let G be a locally L-analytic group and assume that either*

- *i) G is commutative and topologically finitely generated or*
- *ii) G is the group of L-rational points of a connected, split reductive L-group $\mathbb{G}$.*

If K is discretely valued then there is a quasi-Stein rigid analytic L-variety X and an injective, continuous homomorphism $D(G,K)^G \to \mathcal{O}(X_K)$ of K-algebras with dense image.

Proof: Case (i) is just the previous theorem because $D(G,K)^G = D(G,K)$. In case (ii) let Z be the center of G and n be the dimension of the derived group of $\mathbb{G}$. Being L-split, Z is isomorphic to the product of a finite number of copies of L^*. Thus, Z is topologically finitely generated and we may define $X := \hat{Z} \times_L (\mathbb{A}^n_L)^{an}$. Writing $Z = \Lambda \times Z_0$ as above we have $\mathcal{O}(X_K) \simeq \mathcal{O}(\hat{\Lambda}_K)\hat{\otimes}_K \mathcal{O}((\widehat{Z_0})_K)\hat{\otimes}_K \mathcal{O}((\mathbb{A}^n_K)^{an})$. Further, Corollary 2.2.4 yields

$$D(G,K)^G \simeq K[\Lambda] \otimes_{K,\iota} D(Z_0,K)\hat{\otimes}_{K,\iota} U(\mathfrak{d},K)^{\mathfrak{d}}, \tag{2.10}$$

where $\mathfrak{d}$ denotes the Lie algebra of the derived group of $\mathbb{G}$. It follows from our assumptions on G that $\mathfrak{d}$ is semisimple and split (cf. the beginning of subsection 2.4 below) whence by Theorem 2.1.6 there is a topological isomorphism $U(\mathfrak{d},K)^{\mathfrak{d}} \simeq \mathcal{O}((\mathbb{A}_K^n)^{an})$ of K-Fréchet algebras. Together with the isomorphism $D(Z_0,K) \simeq \mathcal{O}((\widehat{Z_0})_K)$ we obtain an isomorphism

$$D(Z_0,K)\hat{\otimes}_{K,\iota}U(\mathfrak{d},K)^{\mathfrak{d}} \simeq \mathcal{O}((\widehat{Z_0})_K)\hat{\otimes}_{K,\iota}\mathcal{O}((\mathbb{A}_K^n)^{an})$$

which, by tensoring the embedding $K[\Lambda] \subseteq \mathcal{O}(\hat{\Lambda}_K)$, gives a continuous K-linear injection

$$\begin{array}{rcl} D(G,K)^G & \hookrightarrow & \mathcal{O}(\hat{\Lambda}_K) \otimes_{K,\iota} \mathcal{O}((\widehat{Z_0})_K)\hat{\otimes}_{K,\iota}\mathcal{O}((\mathbb{A}_K^n)^{an}) \\ & \subseteq & \mathcal{O}(\hat{\Lambda}_K)\hat{\otimes}_{K,\iota}\mathcal{O}((\widehat{Z_0})_K)\hat{\otimes}_{K,\iota}\mathcal{O}((\mathbb{A}_K^n)^{an}) \\ & \simeq & \mathcal{O}(X_K) \end{array}$$

(for the last isomorphism confer our convention concerning ι in Remark 1.2.11). Since $K[\Lambda]$ is dense in $\mathcal{O}(\hat{\Lambda}_K)$ this injection has dense image (cf. [25], Lemma 19.10) and, by construction, is a homomorphism of K-algebras. □

Remark 2.3.5. The isomorphism (2.9) makes it possible to explicitly compute the center of $D(G,K)$ if $\mathbb{G}$ is L-split. As recalled above, the structure of $U(\mathfrak{d},K)^{\mathfrak{d}}$ has been determined in Theorem 2.1.6: if n is the rank of $\mathfrak{d}$, i.e. the dimension of a split maximal toral subalgebra, then $U(\mathfrak{d},K)^{\mathfrak{d}}$ is the K-algebra of all power series in n variables with infinite radius of convergence:

$$U(\mathfrak{d},K)^{\mathfrak{d}} \simeq \mathcal{O}((\mathbb{A}_K^n)^{an}).$$

There are also ways to make explicit the structure of the commutative algebra $D(Z,K)$: If r is the dimension of Z then Z contains an open subgroup isomorphic to $\mathfrak{o}_L^r$ (as follows, for example, from Proposition 1.3.5). Thus, $Z \simeq A \times \mathfrak{o}_L^r$ as locally L-analytic groups with a discrete, finitely generated abelian group A. Consequently,

$$D(Z,K) \simeq K[A] \otimes_{K,\iota} \underbrace{D(\mathfrak{o}_L,K)\hat{\otimes}_K \cdots \hat{\otimes}_K D(\mathfrak{o}_L,K)}_{r\text{-times}}$$

(cf. [31], Proposition A.3). The structure of $D(\mathfrak{o}_L,K)$ has been investigated in [28]. It is the K-algebra of holomorphic functions on a twisted form of the open unit disk.

Corollary 2.3.6. *Under the assumptions of Corollary 2.3.4 any maximal ideal of $D(G,K)^G$ which is closed with respect to the topology induced by $\mathcal{O}(X_K)$ is of finite codimension.*

Proof: Let $\mathfrak{m}$ be a maximal ideal of $A := D(G,K)^G$ which is closed with respect to the metric topology induced by $\hat{A} := \mathcal{O}(X_K)$ and let $\hat{\mathfrak{m}}$ be the closure of $\mathfrak{m}$ in $\hat{A}$. Write $\hat{A} = \varprojlim_{i\in\mathbb{N}} A_i$ as a projective limit of K-affinoid algebras A_i exhibiting $\hat{A}$ as a K-Fréchet-Stein algebra (cf. Remark 2.3.2). Let $\mathfrak{m}_i$ be the the ideal of A_i generated by the image of $\hat{\mathfrak{m}}$ under the natural map $\hat{A} \to A_i$. Then $\hat{B} := \hat{A}/\hat{\mathfrak{m}} = \varprojlim_{i\in\mathbb{N}} A_i/\mathfrak{m}_i$ (cf. [30], Proposition 3.7 and its proof) where $B_i := A_i/\mathfrak{m}_i$ is a K-affinoid algebra. Since $\mathfrak{m}_{i+1}A_i = \mathfrak{m}_i$ there is a morphism $\varphi_i : \mathrm{Sp}(B_i) \longrightarrow \mathrm{Sp}(B_{i+1})$ making commutative the diagram

$$\begin{array}{ccc} \mathrm{Sp}(A_i) & \longrightarrow & \mathrm{Sp}(A_{i+1}) \\ \uparrow & & \uparrow \\ \mathrm{Sp}(B_i) & \xrightarrow{\varphi_i} & \mathrm{Sp}(B_{i+1}) \end{array}$$

Let $\mathfrak{n}$ be a maximal ideal of B_i denoting also its preimages in B_{i+1}, A_{i+1} and A_i. Since $\mathrm{Sp}(A_i) \longrightarrow \mathrm{Sp}(A_{i+1})$ is an open immersion there is an isomorphism $(A_i)_{\mathfrak{n}} \simeq (A_{i+1})_{\mathfrak{n}}$. But then

$$(B_i)_{\mathfrak{n}} \simeq (A_i)_{\mathfrak{n}}/\mathfrak{m}_i(A_i)_{\mathfrak{n}} = (A_i)_{\mathfrak{n}}/\mathfrak{m}_{i+1}(A_i)_{\mathfrak{n}} \simeq (B_{i+1})_{\mathfrak{n}}$$

showing that φ_i is an open immersion. According to Theorem A ([loc.cit.], section 3) the image of $\hat{B}$ in B_i is dense whence so is the image of B_{i+1} in B_i for every $i \in \mathbb{N}$. Thus, we may view $\hat{B}$ as the algebra of global sections of a quasi-Stein rigid analytic K-variety Y admitting the admissible open covering $Y = \bigcup_{i\in\mathbb{N}} \mathrm{Sp}(B_i)$.

Note that $\hat{A}/\hat{\mathfrak{m}}$ is topologically isomorphic to the Hausdorff completion of $A/\mathfrak{m}$ (cf. [8], 3.1 Corollaire 1) so that in particular $\hat{A}/\hat{m} \neq 0$. Therefore, the variety Y is nonempty and we may choose a point y lying in some $\mathrm{Sp}(B_i)$. Let $\mathfrak{m}_y$ be the corresponding maximal ideal of B_i. Consider the coherent $\mathcal{O}_Y$-ideal $I := \mathfrak{id}(\{y\})$ (cf. [4], 9.5.2 Corollary 6). Due to Theorem B (cf. [22], Satz 2.4) the exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y/I \longrightarrow 0$$

of coherent sheaves on Y gives rise to the exact sequence

$$0 \longrightarrow I(Y) \longrightarrow \hat{B} \longrightarrow \mathcal{O}_Y/I(Y) \longrightarrow 0$$

of $\hat{B}$-modules. The support of $\mathcal{O}_Y$ being $\{y\}$ we have

$$(\mathcal{O}_Y/I)(Y) = (\mathcal{O}_Y/I)_y \simeq \mathcal{O}_{Y,y}/\mathfrak{m}_y\mathcal{O}_{Y,y}$$

which is nonzero ($\mathfrak{m}_y\mathcal{O}_{Y,y}$ being the maximal ideal of the local ring $\mathcal{O}_{Y,y}$; cf. [4], 7.3.2 Proposition 1). But $\hat{B}$ is a field so that we must have $I(Y) = 0$ and (due to Theorem A again) $I(\mathrm{Sp}(B_i)) = \mathfrak{m}_y = 0$. Thus B_i is a field and, being K-affinoid, is of finite dimension over K (loc.cit. 6.1.2 Corollary 3). We obtain isomorphisms

$$A/\mathfrak{m} \simeq \hat{A}/\hat{\mathfrak{m}} \simeq A_i/\mathfrak{m}_i$$

because finite dimensional subspaces of a locally convex K-vector space are always closed. □

2.4 An extension of Harish-Chandra's isomorphism

Let $\mathbb{G}$ be a connected, split reductive, linear algebraic group defined over L. Let $\mathbb{D}$ and $\mathbb{Z}$ be the center and the derived group of $\mathbb{G}$, respectively. Both of them are defined over L (cf. [3], Corollary 2.3 and Theorem 18.2). Let further $\mathbb{T}$ be a maximal L-split torus of $\mathbb{G}$. Then $\mathbb{D}$ is L-split and $\mathbb{T}' := (\mathbb{D} \cap \mathbb{T})^\circ$ is a maximal L-split torus of $\mathbb{D}$ (loc.cit. 21.1). Let G, Z, D, T and T' be the group of L-rational points of $\mathbb{G}, \mathbb{Z}, \mathbb{D}, \mathbb{T}$ and $\mathbb{T}'$, respectively, and $\mathfrak{g}, \mathfrak{z}, \mathfrak{d}, \mathfrak{t}$ and $\mathfrak{t}'$ be the respective Lie algebras. Note that by [loc.cit.], Proposition 14.2, Proposition 7.8 and [12], II.6.2 Corollaire 2.2, $\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}]$ is a semisimple Lie algebra and that $\mathfrak{t}'$ is a maximal toral subalgebra of $\mathfrak{d}$. Since $\mathbb{T}'$ is L-split the pair $(\mathfrak{d}, \mathfrak{t}')$ is split in the sense of [13], 1.9.10. Let finally $W = W(G,T) := N_G(T)/T$ be the Weyl group of G with respect to T. W acts on T by conjugation and hence on $D(T,K)$. According to (1.3) and (2.1) the subalgebra $S(\mathfrak{t}, K)$ of $D(T,K)$ is stable under the action of W. W is also the Weyl group of D with respect to T', hence acts on T' and $D(T', K)$. The corresponding action on $S(\mathfrak{t}', K)$ is induced by the adjoint action of W on $\mathfrak{t}'$ (cf. the proof of Proposition 2.1.1). Since $\mathbb{T}'$ is connected and L of characteristic zero the homomorphism $Ad : W \to \mathrm{Aut}_L(\mathfrak{t}')$ is injective (cf. [12], II.6.2 Proposition 2.1). Recall that $S(\mathfrak{t}', K)$ is also acted on by the Weyl group $\mathfrak{W} = \mathfrak{W}(\mathfrak{d}, \mathfrak{t}')$ of the pair $(\mathfrak{d}, \mathfrak{t}')$ (cf. subsection 2.1). This action, too, is induced by viewing $\mathfrak{W}$ as a subgroup of $\mathrm{Aut}_L(\mathfrak{t}')$.

Lemma 2.4.1. *$Ad : W \to \mathfrak{W}$ is an isomorphism of groups. In particular, $S(\mathfrak{t}', K)^W = S(\mathfrak{t}', K)^{\mathfrak{W}}$.*

Proof: Since $Ad : W \to \mathrm{Aut}_L(\mathfrak{t}')$ is an injective homomorphism of groups it suffices to show that $Ad(W) = \mathfrak{W}$.

Let $\Phi = \Phi(\mathfrak{d}, \mathfrak{t}')$ be the root system of $(\mathfrak{d}, \mathfrak{t}')$ and let $\alpha \in \Phi$. If $\mathfrak{d}_\alpha$ denotes the corresponding eigenspace then $[\mathfrak{d}_\alpha, \mathfrak{d}_\alpha]$ contains a unique element H_α such that $\alpha(H_\alpha) = 2$. If $X_\alpha \in \mathfrak{d}_\alpha$ is different from zero then there is a unique element $X_{-\alpha} \in \mathfrak{d}_{-\alpha}$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$ (cf. [6], VIII.2.2 Théorème 1). Let $s_\alpha \in \mathfrak{W}$ be the simple reflection of $(\mathfrak{t}')^*$ corresponding to α. Its transpose acts on $\mathfrak{t}'$ via

$$\theta_\alpha := exp(ad(X_\alpha))exp(ad(X_{-\alpha}))exp(ad(X_\alpha))$$

(loc.cit. Théorème 2; here $exp \in \mathbb{Q}[[t]]$ is the usual exponential series). Since X_α and $X_{-\alpha}$ are nilpotent there are elements u_α and $u_{-\alpha}$ in D such that

$$exp(ad(X_\alpha)) = Ad(u_\alpha) \text{ and } exp(ad(X_{-\alpha})) = Ad(u_{-\alpha})$$

on $\mathfrak{d}$ (cf. [12], II.6.3 (3.7)), i.e. $\theta_\alpha = Ad(n_\alpha)$ with $n_\alpha := u_\alpha u_{-\alpha} u_\alpha$. Since $Ad(n_\alpha)(\mathfrak{t}') = \mathfrak{t}'$ we must have $n_\alpha \in N_D(T')$: Since L is of characteristic zero this is due to the connectedness of $\mathbb{T}'$ (loc.cit. II.6.2 Proposition 2.1). Thus, n_α represents an element of W and its action on $\mathfrak{t}'$ coincides with θ_α. It follows that $\mathfrak{W} \subseteq Ad(W)$ because the reflections s_α, $\alpha \in \Phi$, generate $\mathfrak{W}$.

Let $\mathfrak{B}^{\mathfrak{t}'}$ be the set of all Borel subalgebras of $\mathfrak{d}$ containing $\mathfrak{t}'$ and $\mathfrak{B}^{\mathbb{T}'}$ be the set of all L-Borel subgroups of $\mathbb{D}$ containing $\mathbb{T}'$. By [6], VIII.3.3 Remarque, (resp. [3], Proposition 11.19) $\mathfrak{W}$ (resp. W) acts simply transitively on $\mathfrak{B}^{\mathfrak{t}'}$ (resp. $\mathfrak{B}^{\mathbb{T}'}$). In particular, $|\mathfrak{W}| = |\mathfrak{B}^{\mathfrak{t}'}|$ and $|W| = |\mathfrak{B}^{\mathbb{T}'}|$. We claim that passage to the Lie algebra defines a bijection $\mathfrak{B}^{\mathbb{T}'} \to \mathfrak{B}^{\mathfrak{t}'}$. This will complete our proof.

Let $\mathbb{B} \in \mathfrak{B}^{\mathbb{T}'}$. Then $\mathbb{T}' \subseteq \mathbb{B}$ and hence $\mathfrak{t}' \subseteq \mathrm{Lie}(\mathbb{B})$. Further, $\mathbb{B}$ is a maximal closed, connected solvable subgroup of $\mathbb{D}$. Since for any two closed subgroups $\mathbb{D}_1$, $\mathbb{D}_2$ of $\mathbb{D}$ the Lie algebra of the commutator group $[\mathbb{D}_1, \mathbb{D}_2]$ contains $[\mathrm{Lie}(\mathbb{D}_1), \mathrm{Lie}(\mathbb{D}_2)]$ (cf. [3], Proposition 3.17) it follows that $\mathrm{Lie}(\mathbb{B})$ is solvable. For dimension reasons $\mathrm{Lie}(\mathbb{B})$ has to be a maximal solvable subalgebra of $\mathfrak{d}$. Indeed, according to the Cartan decomposition any such has dimension $\frac{1}{2}(\dim_L(\mathfrak{d}) + \dim_L(\mathfrak{t}'))$. But by [3], Theorem 13.18, this is precisely the dimension of $\mathrm{Lie}(\mathbb{B})$. Thus, the map $\mathfrak{B}^{\mathbb{T}'} \to \mathfrak{B}^{\mathfrak{t}'}$ is well-defined. Since by definition all elements of $\mathfrak{B}^{\mathbb{T}'}$ are connected subgroups of $\mathbb{D}$ it is injective (cf. [12], II.6.2 Proposition 2.1) and we obtain

$$|W| = |\mathfrak{B}^{\mathbb{T}'}| \leq |\mathfrak{B}^{\mathfrak{t}'}| = |\mathfrak{W}| \leq |W|. \qquad \square$$

Theorem 2.4.2. *Let G be the group of L-rational points of a connected, split reductive L-group $\mathbb{G}$ with T and W as above. If K is discretely valued then there is a topological isomorphism*

$$D(G,K)^G \simeq D(T,K)^W_Z$$

of separately continuous K-algebras.

Proof: According to Corollary 2.2.4 there is a topological isomorphism

$$\kappa : D(G,K)^G \longrightarrow D(Z,K)\hat{\otimes}_{K,\iota}U(\mathfrak{d},K)^{\mathfrak{d}}$$

of separately continuous K-algebras.

Since $T = Z \cdot T'$ with finite intersection $Z \cap T'$ one proves in an analogous manner that there is a topological isomorphism of separately continuous K-algebras

$$\psi : D(Z,K)\hat{\otimes}_{K,\iota}S(\mathfrak{t}',K)^W \longrightarrow D(T,K)^W_Z.$$

According to Theorem 2.1.6 and Lemma 2.4.1 there is a topological isomorphism $\xi : U(\mathfrak{d},K)^{\mathfrak{d}} \to S(\mathfrak{t}',K)^W$ of K-Fréchet algebras so that

$$\psi \circ (id\hat{\otimes}\xi) \circ \kappa : D(G,K)^G \to D(T,K)^W_Z$$

is as required. □

Remark 2.4.3. If $\mathbb{G}$ is semisimple then Z is finite and κ and ψ are the obvious isomorphisms

$$\begin{aligned} K[Z] \otimes_K U(\mathfrak{g},K)^G &\longrightarrow D(G,K)^G_Z = D(G,K)^G \text{and} \\ K[Z] \otimes_K S(\mathfrak{t},K)^W &\longrightarrow D(T,K)^W_Z. \end{aligned}$$

Since the construction of the isomorphism $\xi : U(\mathfrak{g},K)^G \to S(\mathfrak{t},K)^W$ was done without any restriction on K it follows that the isomorphism $D(G,K)^G \simeq D(T,K)^W_Z$ exists for any spherically complete coefficient field K.

2.5 Comparison with the Bernstein center

Let M be any paracompact, locally L-analytic manifold of finite dimension and $C^\infty_c(M,K)$ be the K-vector space of locally constant K-valued functions on M with compact support. The elements of the algebraic dual $C^\infty_c(M,K)^*$ of $C^\infty_c(M,K)$ are called distributions on M. As in Remark 1.2.2 one sees that for distributions on M there is a well-defined notion of support.

Let further $C^\infty(M,K) \subseteq C^{an}(M,K)$ be the K-vector space of all locally constant functions on M endowed with the subspace topology induced from $C^{an}(M,K)$. Set $D^\infty(M,K) := C^\infty(M,K)'_b$.

Lemma 2.5.1. *Via restriction of functionals from $C^\infty(M,K)$ to $C^\infty_c(M,K)$ the space $D^\infty(M,K)$ can be identified with the space of all compactly supported distributions on M.*

Proof: Let $(M_i)_{i\in I}$ be a covering of M by disjoint, compact open subsets M_i. As noted in [27], section 2, the space $C^\infty(M_i,K) \subseteq C^{an}(M_i,K)$ carries its finest locally convex topology for any $i \in I$. It follows from $C^{an}(M,K) = \prod_{i\in I} C^{an}(M_i,K)$ that $C^\infty(M,K) = \prod_{i\in I} C^\infty(M_i)$ and hence that

$$D^\infty(M,K) = \bigoplus_{i\in I} C^\infty(M_i)'_b = \bigoplus_{i\in I} C^\infty(M_i)^*.$$

This is precisely the subspace of

$$C_c^\infty(M,K)^* = (\bigoplus_{i\in I} C^\infty(M_i,K))^* = \prod_{i\in I} C^\infty(M_i,K)^*$$

consisting of compactly supported distributions. □

For the rest of this section we will again assume $M = G$ to be a finite dimensional, locally L-analytic group. G embeds in $D^\infty(G,K)$ via $(g \mapsto \delta_g)$. If δ and δ' are distributions on G at least one of which is compactly supported then one can define their convolution product $\delta \cdot \delta'$ as in (1.2). It is again a distribution on G. If both $supp(\delta)$ and $supp(\delta')$ are compact then so is $supp(\delta \cdot \delta')$; this follows from the analog of Lemma 1.2.4 for $D^\infty(G,K)$ (with the same proof). In particular, $D^\infty(G,K)$ is an associative K-algebra with unit element δ_1.

We let $\mathcal{H}_K(G)$ be the Hecke algebra of G over K, i.e. the subalgebra of $D^\infty(G,K)$ consisting of all compactly supported distributions δ for which there is a compact open subgroup G_0 of G such that $\delta_g \cdot \delta = \delta$ for all $g \in G_0$. Note that $\delta_1 \notin \mathcal{H}_K(G)$, i.e. $\mathcal{H}_K(G)$ is non-unital, unless G is discrete.

Remark 2.5.2. If μ denotes a (left invariant) Haar measure on G then it is well-known that the map

$$(f \mapsto f \cdot \mu) : C_c^\infty(G,K) \to \mathcal{H}_K(G)$$

is an isomorphism of K-algebras if $C_c^\infty(G,K)$ is endowed with the convolution product

$$(f * h)(x) := \int_G f(g)h(g^{-1}x)d\mu(g).$$

We will from now on tacitly make use of this identification and write $f \cdot h$ instead of $f * h$ if the functions $f, h \in C_c^\infty(G,K)$ are viewed as distributions.

If G_0 is a compact open subgroup of G let $\mathcal{H}_K(G,G_0)$ denote the subalgebra of $\mathcal{H}_K(G)$ consisting of G_0-biinvariant functions, i.e. of functions $f \in C_c^\infty(G,K)$

for which $f(g_1 g g_2) = f(g)$ for all $g \in G$ and $g_1, g_2 \in G_0$. $\mathcal{H}_K(G, G_0)$ is a ring with unit $e_{G_0} := \mu(G_0)^{-1}\chi_{G_0}$ where χ_{G_0} denotes the characteristic function of G_0. e_{G_0} is an idempotent in $\mathcal{H}_K(G)$, we have $\mathcal{H}_K(G, G_0) = e_{G_0}\mathcal{H}_K(G)e_{G_0}$ and $\mathcal{H}_K(G)$ is an idempotented algebra in the sense that

$$\mathcal{H}_K(G) = \varinjlim_{G_0} \mathcal{H}_K(G, G_0)$$

with G_0 running through the system of all compact open subgroups of G_0 (cf. [2], 1.1). If G_0 and G_0' are compact open subgroups of G with $G_0 \subseteq G_0'$ then $e_{G_0} \cdot e_{G_0'} = e_{G_0'} \cdot e_{G_0} = e_{G_0'}$ and there is a natural map

$$(f \cdot e_{G_0} \mapsto f \cdot e_{G_0'}) : \mathcal{H}_K(G)e_{G_0} \longrightarrow \mathcal{H}_K(G)e_{G_0'}.$$

The projective limit $\hat{\mathcal{H}}_K(G) := \varprojlim_{G_0} \mathcal{H}_K(G)e_{G_0}$ is the completion of $\mathcal{H}_K(G)$ with respect to the topology defined by the (left) annihilators of all e_{G_0} and can be identified with the space of all distributions δ on G such that $\delta \cdot e_{G_0}$ is compactly supported for all compact open subgroups G_0 of G (loc.cit. 1.2 and 1.4). The algebra structure on $\mathcal{H}_K(G)$ extends to $\hat{\mathcal{H}}_K(G)$ and we have the ascending chain

$$\mathcal{H}_K(G) \subseteq D^\infty(G, K) \subseteq \hat{\mathcal{H}}_K(G) \tag{2.11}$$

of K-algebras. The center $Z(\hat{\mathcal{H}}_K(G))$ of $\hat{\mathcal{H}}_K(G)$ is the projective limit of the centers $Z(\mathcal{H}_K(G, G_0))$ of the algebras $\mathcal{H}_K(G, G_0)$ (loc.cit. Lemme 1.5 (ii)). Since according to [loc.cit.], 1.3, the multiplication in $\hat{\mathcal{H}}_K(G)$ is continuous with respect to the above defined topology and since $D^\infty(G, K)$ is dense in $\hat{\mathcal{H}}_K(G)$ we have $Z(D^\infty(G, K)) \subseteq Z(\hat{\mathcal{H}}_K(G))$.

Let $I(\mathfrak{g})$ denote the closed ideal of $D(G, K)$ generated by the Lie algebra $\mathfrak{g}$ of G. Restricting continuous functionals from $C^{an}(G, K)$ to $C^\infty(G, K)$ induces a strict, continuous surjection

$$D(G, K) \twoheadrightarrow D^\infty(G, K) \tag{2.12}$$

of separately continuous K-algebras with kernel $I(\mathfrak{g})$ (cf. [27], section 2). It follows from Lemma 1.1.1 that $K[G]$ is dense in $D^\infty(G, K)$. Since G acts on $D^\infty(G, K)$ as in (2.1) we see that $D^\infty(G, K)^G$ is the center of the algebra $D^\infty(G, K)$. Further, the surjection (2.12) restricts to a map $D(G, K)^G \to D^\infty(G, K)^G$. Altogether we obtain the following sequence of homomorphisms of commutative K-algebras:

$$D(G, K)^G \longrightarrow D^\infty(G, K)^G \longrightarrow Z(\hat{\mathcal{H}}_K(G)). \tag{2.13}$$

If G is compact and abelian then both maps in (2.13) are surjective by (2.12) and since the right inclusion in (2.11) is an equality of commutative K-algebras. In the case that we have mostly been concerned with so far we find the following:

Proposition 2.5.3. *Assume G to be the group of L-rational points of a connected, reductive, sufficiently L-isotropic L-group $\mathbb{G}$. Then the first arrow in (2.13) is surjective. The second arrow has dense image with respect to the projective limit topology on $Z(\hat{\mathcal{H}}_K(G))$ if and only if $\mathbb{G}$ is abelian.*

Proof: Let us put $Z^\infty := D^\infty(G,K)^G$ and $\hat{Z} := Z(\hat{\mathcal{H}}_K(G))$ for simplicity. We know from Corollary 2.2.4 that $D(G,K)^G = D(G,K)_Z^G$ due to our assumption on G. The argument given for its proof shows that likewise $Z^\infty = D^\infty(G,K)_Z^G$ because all distributions in $D^\infty(G,K)$ are compactly supported (cf. Lemma 2.5.1). Thus, we obtain $Z^\infty = D^\infty(Z,K)$ from the fact that $D^\infty(G,K)_Z = D^\infty(Z,K)$. Since $D(Z,K) \subseteq D(G,K)_Z^G$ and since the map $D(Z,K) \to D^\infty(Z,K)$ is surjective, the first assertion is clear.

To prove the second assertion we need to find a compact open subgroup G_0 of G such that the restriction to Z^∞ of the projection $\hat{Z} \to \hat{Z}e_{G_0} = Z(\mathcal{H}_K(G,G_0))$ is not surjective unless $\mathbb{G}$ is abelian. Let $\mathbb{S}$ be a maximal L-split torus of $\mathbb{G}$, $\mathbb{M} := Z_{\mathbb{G}}(\mathbb{S})$, $S := \mathbb{S}(L)$ and $M := \mathbb{M}(L)$. Let A be the apartment of the Bruhat-Tits building $\mathcal{B}$ of G associated with S. We identify A with the real vector space $X_*(\mathbb{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ where $X_*(\mathbb{H})$ denotes the group of cocharacters defined over L of an algebraic L-group $\mathbb{H}$. Let $G_0 \subseteq G$ be the stabilizer of $0 \in \mathcal{B}$.

If Ω is a field containing both K and the field $\mathbb{C}$ of complex numbers (e.g. the completion of an algebraic closure of K) then

$$\mathcal{H}_\Omega(G,G_0) = \mathcal{H}_K(G,G_0) \otimes_K \Omega = \mathcal{H}_{\mathbb{C}}(G,G_0) \otimes_{\mathbb{C}} \Omega.$$

It follows that $\mathcal{H}_K(G,G_0)$ is commutative because $\mathcal{H}_{\mathbb{C}}(G,G_0)$ is. In fact, Satake's isomorphism (cf. [11], Theorem 4.1) gives a much more precise description of $\mathcal{H}_{\mathbb{C}}(G,G_0)$ as being isomorphic to the $\mathbb{C}$-algebra of Weyl invariants of the group ring $\mathbb{C}[\Lambda]$. Here Λ is a certain quotient of $X_*(\mathbb{M})$ and is a free abelian group of rank $\dim(\mathbb{S})$ parametrizing the unramified quasicharacters of $\mathbb{M}$. Thus, $Z(\mathcal{H}_K(G,G_0)) = \mathcal{H}_K(G,G_0)$ with the characteristic functions of the double cosets $G_0\backslash G/G_0$ as a K-basis.

Since $supp(e_{G_0}) = G_0$ we have $supp(\delta e_{G_0}) = supp(e_{G_0}\delta e_{G_0}) \subseteq G_0 Z G_0$ for any $\delta \in Z^\infty$. It follows that $Z^\infty e_{G_0} = \mathcal{H}_K(G,G_0)$ if and only if $G = G_0 Z G_0$.

Making use of the Iwasawa decomposition we will show that this forces $\mathbb{G}$ to be abelian.

Let $\Phi = \Phi(G, S)$ be the relative root system of the pair (G, S). Choose a set Φ^+ of positive roots in Φ and let $\mathbb{U}^+$ be the subgroup of $\mathbb{G}$ generated by all root groups $\mathbb{U}_\alpha$ with $\alpha \in \Phi^+$. If $U^+ := \mathbb{U}^+(L)$ then G is the disjoint union of the double cosets $G_0 m U^+$, $m \in M$ (cf. [33], 3.3.1). If $G = G_0 Z G_0 = G_0 Z = G_0 Z U^+$ then we must have $M = Z$ so that $\mathbb{M}$ is central (note that M is Zariski dense in $\mathbb{M}$ because L is infinite). Since $\mathbb{M}$ contains a maximal torus of $\mathbb{G}$ it follows from the conjugacy of all maximal tori that $\mathbb{G}$ has a unique maximal torus. But then $\mathbb{G}$ is abelian by [3], Theorem 12.3. Note that in this case $Z^\infty = D^\infty(G, K)$ is dense in $\hat{Z} = \hat{\mathcal{H}}_K(G)$ (cf. (2.11)). □

Remark 2.5.4. We do not know whether there are locally L-analytic groups G for which the map $D(G, K)^G \to D^\infty(G, K)^G$ is not surjective.

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