Münster J. of Math. **13** (2020), 597–619
DOI 10.17879/90169638010
urn:nbn:de:hbz:6-90169638355

**Münster Journal of Mathematics**


# Some specialization theorems for families of abelian varieties

Umberto Zannier

(Communicated by Peter Schneider)

*Dedicated to Christopher Deninger on the occasion of his 60th birthday*

**Abstract.** Consider an algebraic family $\pi\colon \mathcal{A} \to B$ of abelian varieties, defined over $\overline{\mathbb{Q}}$. We shall be concerned with properties of the generic fiber of $\mathcal{A}$ which are preserved on restricting to some (or 'many') suitable special fibers. We shall focus on instances like *torsion* for values of a section, endomorphism rings, existence of *generic* and *special* isogenies, illustrating some known results and some applications. Another, more recent, issue which we shall briefly discuss concerns the existence of abelian varieties over $\overline{\mathbb{Q}}$ not isogenous to a Jacobian. We shall conclude with a few comments on other specialization issues.

## 1. Introduction

The present short article arises from the lecture I had the honor and pleasure to deliver at the conference *Arithmetic and Analysis*, Münster, April 2018, in celebration of Christopher Deninger's 60th birthday. I had the fortune of meeting him many years ago and learn from him things far from my panorama.

For my talk, I chose a theme embracing a relevant part of my interests and work of the last few years. For a number of reasons, I preferred to adopt a survey style, illustrating the results with simple and special examples, rather than general detailed statements. In this note, though expanding the exposition and the topics a bit, I shall maintain the same principles. In particular, I shall be far from complete in any of the touched descriptions. I shall mention a few basic results, some of them going back to long ago, pausing with little more detail on what I am more familiar with.

1.1. **Specialization theorems.** As in the title and abstract, I shall consider some theorems concerning preservation of (arithmetical or geometrical) structures by 'specialization'. Such results on the one hand limit a certain type of 'degeneracy', and on the other hand allow to construct objects defined over residue fields, having 'generic' properties (see, e.g., Serre's first letter to Ribet in [30] for instances of this viewpoint). Well-known typical examples are the irreducibility theorem of Bertini and its arithmetical counterpart by Hilbert.

**Our context.** In this article I shall stick to families of abelian varieties. That is, our ambient is a *parameterized family* (i.e., a scheme) $\mathcal{A}/B$ of abelian varieties over an algebraic 'base' variety $B$. So, roughly, there is a morphism $\pi\colon \mathcal{A} \to B$ such that the fibers $A_b := \pi^{-1}(b)$, $b \in B$, are abelian varieties, there are morphisms $\mathrm{sum}\colon \mathcal{A} \times_B \mathcal{A} \to \mathcal{A}$ and $\mathrm{inv}\colon \mathcal{A} \to \mathcal{A}$ over $B$, representing the group laws on the various $A_b$, and there is a 0-section $o\colon B \to \mathcal{A}$ such that $o(b)$ is the origin of $A_b$. The parameter(s) run over $B(k)$, for a field $k$ over which the above data are defined. The field $k$ may be 'large' (e.g., algebraically closed) or quite restricted (e.g., a number field). For us, usually, it shall be $\overline{\mathbb{Q}}$ (in other issues it may be also $\mathbb{C}$, which is a more 'geometric' case).

**Generic and special properties.** We shall consider properties which hold 'generically' for the family, i.e., they hold for the generic fiber $A_t$ (where $t$ is a $k$-generic point of $B$); then our task is to study the special values $b \in B(k)$ of the parameters which destroy the property.

**Bad set:** This is the (hopefully) *exceptional* set of values of the relevant parameter(s) such that the property in question is destroyed. It will be denoted $\mathcal{E}$ or $\mathcal{E}(k)$.

**Main purpose:** To show that the bad-set is really 'sparse', for instance,

(i) at least not the whole $B(k)$ (a 'minimal' ambition),
(ii) or hopefully empty, or finite (a 'maximal' ambition),
(iii) or some intermediate conclusions.

Sometimes the issue (i) is already nontrivial or even false, and indeed we plan to describe both positive cases and failures.

**Remark 1.2.** Of course, any specialization principle may be formulated somewhat in a 'reverse' order, i.e., as a contrapositive statement, whose pattern would be: *If property (P) holds for sufficiently many special members, then it holds generically.* It is a matter of taste which phrasing to adopt. (For instance, the latter is more common to 'local-global principles' in arithmetic, see §6.2.) We also note that in all specialization principles that we shall meet a converse implication will hold trivially, i.e., in the last phrasing: *If property (P) holds generically, then it holds for (almost) all special members.*

**Important analogies:** reduction modulo $p$ is of course a kind of specialization; the corresponding results are commonly referred to as *local-global principles.* We shall leave such interesting context essentially out of our discussion, just mentioning a celebrated example in the final section.

## 2. A failure

Let us start with an example illustrating how the 'bad' set may be sometimes everything, contrary to (naive) expectation. Let us consider the Legendre elliptic curve $L_t$, defined (affinely) by the equation

$$L_t: \quad y^2 = x(x-1)(x-t) + \text{point } 0 \text{ at } \infty, \tag{1}$$

by which we mean that, in fact, we consider the projective closure of the curves $L_t \subset \mathbb{P}_2$.

Here $t$ is a (variable) parameter, confined to $\mathbb{P}_1 - \{0, 1, \infty\}$ if we want to avoid bad reduction, i.e., if we require that $L_t$ is an elliptic curve; this is the present base $B$. We choose 0 as the origin of $L_t$. This elliptic scheme, denoted $\mathcal{L}$, is a surface.

We may also view $L_t$ as an elliptic curve over $\mathbb{Q}(t)$; for many issues $t$ may be taken as a variable, or also a transcendental number. By specialization $t \to b \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, we obtain an elliptic curve $L_b$ defined over $\overline{\mathbb{Q}}$.

Let us now note that the coordinate map $x\colon L_t \to \mathbb{P}_1$ is branched precisely above the four points $0, 1, t, \infty$, which are its *critical values* (or branch points). Now, for some purposes it is desirable to have maps with few critical values, so it makes sense to ask the following question.

**Question.** Are there non-constant rational maps on $L_t$ with at most three critical values?

We assert that the answer is NO, i.e., *(at most) three critical values cannot be achieved for any rational map* $f = f_t\colon L_t \to \mathbb{P}_1$ (and this holds no matter the field of definition of $f$).

*Sketch of proof.* Otherwise, by composing $f$ on the left with a suitable automorphism of $\mathbb{P}_1$ (which is 3-transitive), we could carry the branch points inside $\{0, 1, \infty\}$, so the set of branch points could be assumed to be *independent of* $t$.

Now, by specializing $t$ to numbers $u$ in a small disk $U \subset \mathbb{C} - \{0, 1\}$, we see that all curves $L_u$, $u \in U$, would admit a rational map $f_u$ to $\mathbb{P}_1$, unbranched outside $0, 1, \infty$, and the degree of such maps would be independent of $u$. But it is well known that there are only finitely many compact Riemann surfaces (up to isomorphism) with a holomorphic map onto $\mathbb{P}_1(\mathbb{C})$ of given degree $d$ and given branch set (they are associated to the permutation representation in $S_d$ of the fundamental group of the complement in $\mathbb{P}_1(\mathbb{C})$ of the branch set). However, the $j$-invariant of $L_u$ is a non-constant rational function of $u$, hence the curves $L_u$, $u \in U$, represent infinitely many nonisomorphic classes of elliptic curves.

This conclusion, i.e., the *non-existence* of any map as above, or equivalently the fact that *the minimum number of branch points of a non-constant rational map is* 4 represents our 'generic' property for the present example.

Now, we may *specialize* $t$ to a complex number $u$ and ask the same question for the special curve $L_u$ in place of $L_t$.

If $u$ is a transcendental number, the question amounts to the same for the generic curve $L_t$, so the answer will be NO, and then it is sensible to ask this only for $u \in \overline{\mathbb{Q}}$. Now, for such $u$, the curve $L_u$ is defined over $\overline{\mathbb{Q}}$, and the following surprising fact holds.

**Theorem of Belyi.** *Each algebraic curve defined over $\overline{\mathbb{Q}}$ admits a non-constant rational map with at most three critical values.*

See, e.g., the book [4] by Bombieri and Gubler or Serre's [29] for a proof of Belyi's theorem, which is as short as ingenious.[1]

From Belyi's theorem, in particular, we deduce that *for each $b \in \overline{\mathbb{Q}}$, there exists a non-constant map $\phi_b \colon L_b \to \mathbb{P}_1$ branched over $\leq 3$ points*, so a specialization (or local-global) principle badly fails in our context: $\mathcal{E}$ is the whole $B(\overline{\mathbb{Q}})$.

We remark that the maps $\phi_b$, $b \in \overline{\mathbb{Q}}$, provided by the known proof of Belyi's theorem, behave in a very puzzling irregular way, in particular, their degree grows wildly depending on $b$; of course the above says that these maps cannot be parameterized algebraically.

## 3. Torsion in abelian families

3.1. **Torsion in elliptic surfaces.** Let us consider again the Legendre elliptic scheme defined by (1), and let us consider a(n algebraic) section, i.e., a(n algebraic) map $\sigma \colon B \to \mathcal{L}$ such that $\pi \circ \sigma$ is the identity of $B$; such a section corresponds in practice to a point on the generic fiber $L_t$. To be specific (but the results below hold generally), let us choose, for instance,

$$\sigma(t) = (t+1, \sqrt{t(t+1)}).$$

Note that in fact $\sigma(t) \in L_t$. To be precise, we should specify the sign of the square root, by performing a base change from $\mathbb{P}_1 - \{0, 1, \infty\}$ to its double cover defined by $u^2 = t(t+1)$, but here we shall avoid such details, immaterial for the present discussion.

Now, an elliptic curve is endowed with a group law (here with origin 0), and an important issue for a point, or section, is whether it is *torsion* or not. When it is not torsion, we can produce infinitely many points by taking multiples. For instance, when we set $t = b := (a-1)^2/4a$ for a rational number $a \in \mathbb{Q}$, the value $\sigma(b)$ of the section is defined over $\mathbb{Q}$ and we obtain a rational point in $L_b(\mathbb{Q})$; if $\sigma(b)$ is not torsion, we obtain infinitely many rational points in $L_b$. This already illustrates a possible interest in studying when a section becomes torsion.

It is not difficult to see that $\sigma$ is not torsion identically in $t$, i.e., as a section. This fact itself could be established by specialization (e.g., $t \to b \in \mathbb{Q}$, and then using, for instance, the *Lutz–Nagell criterion* or similar ones, see [33]), but probably the simplest way is to notice that the minimal extension of $\mathbb{C}(t)$ over which $\sigma$ is defined is ramified above $t = -1$, so outside the locus $\{0, 1, \infty\}$ of bad reduction of $\mathcal{L}$, and it is known that this fact prevents torsion. Then the natural specialization issue now is:

[1] The history of Belyi's theorem, which was foreseen by Grothendieck but proved by Belyi without knowing this, is also interesting. One could expect an analog holding for curves defined over a field of transcendence degree $r$, admitting rational maps with $\leq 3 + r$ branch points; but this is unknown for any $r > 0$.

*What can be said of the complex numbers $b$ such that $\sigma(b)$ is torsion on $L_b$?* So, the 'bad' set for this example is $\mathcal{E} = \{b : n\sigma(b) = 0 \text{ for some } n > 0\}$.

Through the usual 'chord-tangent process' we can compute the multiples of $\sigma$ on $L_t$, obtaining $n\sigma(t) = (R_n(t), S_n(t)\sqrt{t(t+1)})$, for suitable rational functions $R_n, S_n \in \mathbb{Q}(t)$ (always defined since $\sigma$ is not torsion) whose degrees grow roughly proportionally to $n^2$, and similarly for the Weil-heights of their coefficients.[2]

So any torsion equation $n\sigma(b) = 0$ corresponds to $b$ being a pole of $R_n$, and in particular, $\mathcal{E}$ is automatically inside $\overline{\mathbb{Q}}$. This also follows *a priori* on noting that any equation $n\sigma(t) = 0$ defines a Zariski-closed subvariety of $\mathbb{P}_1$, defined over $\mathbb{Q}$, which is proper since $\sigma$ is non-torsion.

It would not be sensible to expect in this case that $\mathcal{E}$ is finite, since the above computation involves divisions which are unlikely to confine the poles of the various $R_n$ to a finite set. In fact, one can prove that $\mathcal{E}$ is infinite (and more) by appealing to Siegel's theorem on integral points (see [33]) over function fields: any infinite sequence of $n$ with $R_n$ having poles in the finite set $S \subset \mathbb{C}$ would yield an infinite sequence of $S$-integral points on $L_t$, i.e., for the ring of regular functions on $\mathbb{P}_1 - S$, contradicting the said result.

A much more precise conclusion can in fact be proved.

**Theorem 3.2.** *$\mathcal{E}$ is dense in $\mathbb{P}_1(\mathbb{C})$ for the complex topology.*

One can see this by considering the so-called *Betti map* of the section. This is obtained locally as follows. The elliptic curve $L_t$ is analytically isomorphic to $\mathbb{C}/\Lambda_t$ for a lattice $\Lambda_t$. Locally in simply connected domains $U_\alpha$ covering $\mathbb{C} - \{0, 1\}$, we can pick a basis of $\Lambda_t$ made up of (two) analytic functions of $t$; for instance in the domain $\{t \in \mathbb{C} : \max(|t|, |1-t|) < 1)\}$ this can be done by hypergeometric functions (see [16] and also §6.1 below). Further, we can express (locally in each $U_\alpha$) an elliptic logarithm of $\sigma(t)$ as a linear combinations with *real* coefficients of such basis. The pair of coefficients defines a map from $U_\alpha$ to $\mathbb{R}^2$. It may be proved that this map is (real analytic and) locally submersive on an open dense set, whence it assumes rational values on a dense set. On this set the section $\sigma$ assumes torsion values, by construction.

Actually, this also proves that the torsion orders can be chosen arbitrarily (if large enough). See the writer's book [36, Ch. III] for details of these arguments. The *Betti map*, which goes back implicitly to Manin, has been recently studied also in the case of higher dimensional base, which is rather more delicate, and admits applications of several kinds (as in past work by Krichever on the discrete Schrödinger operator, and in recent work by Voisin on Chow groups). See, for instance, the paper [3], joint with André and Corvaja (with an Appendix by Gao). In particular, the results allow to extend Theorem 3.2 to more general cases. See further the recent *Séminaire Bourbaki* by Serre [31] (relevant also for Application I below).

[2]See [19] for a study of such type of rational functions. Their degree is related to the canonical height of the section with respect to the function field.

**Galois equidistribution of $\mathcal{E}$.** In the case of elliptic surfaces with a section, even more comes from the recent paper [11] by DeMarco and Mavraki. They prove that there is a probability measure on $B$ (depending on the section) such that the elements of $\mathcal{E}$ tend to be *Galois equidistributed*, by which we roughly mean that as the degree of $b \in \mathcal{E}(\overline{\mathbb{Q}})$ tends to infinity, the percentage of conjugates of $b$ falling in a prescribed (sufficiently 'regular') region of $B$ tends to the measure of the region.[3]

So far we have seen that the set $\mathcal{E}$ is *large* in various meanings, hence going somewhat contrary to a 'positive' specialization conclusion. However this does not hold to the extent of the example illustrated in §2, and in fact now there are also rather sharp positive results.

**Theorem 3.3** (Silverman and Tate, 1980)**.** *$\mathcal{E}$ is a set of bounded Weil height.*

We skip here any definition of the (logarithmic) Weil height, except on recalling that for a rational $x = p/q$ in lowest terms, it is $h(x) = \log \max(|p|, |q|)$. (See [4] or [29] for the theory.)

It is a celebrated useful theorem of Northcott (admitting a short proof) that any set of algebraic numbers with bounded height and degree is finite. Therefore. we deduce the following corollary.

**Corollary 3.4.** *For each $D$, the set $\{b \in \mathcal{E} : [\mathbb{Q}(b) : \mathbb{Q}] \leq D\}$ is finite.*

In particular, we have finiteness of $\mathcal{E}(k)$ for any number field $k$, but the corollary is much stronger. We have not attempted to compute $\mathcal{E}(\mathbb{Q})$, and we only observe it contains $-1$. However, we remark that the result is effective and would allow the computation of the relevant finite set, for any given $D$ (also for any given section defined over $\overline{\mathbb{Q}}$).

**Remark 3.5.** (i) The above Theorem 3.3 is only an extremely special case of what had been proved by Silverman and Tate, and especially Silverman. For instance, one can consider several sections $\sigma_1, \ldots, \sigma_r$, linearly independent over $\mathbb{Z}$, and ask for the points $b \in B$ such that $\sigma_1(b), \ldots, \sigma_r(b)$ become dependent. We refer to Silverman's paper [32] for general statements. See also the book [36] for simple arguments valid for our examples (see [36, Appendix C], by Masser, for the case of several sections). These results inspired a wealth of research.

(ii) Prior to Silverman and Tate's work, somewhat similar methods had been applied by Manin and Demianenko, who worked however over fixed number fields; see [29] for an account. Still before, Néron obtained somewhat weaker conclusions using the Hilbert irreducibility theorem; he wanted to achieve large rank over $\mathbb{Q}$ for an elliptic curve, by specializing from a curve of large rank over a function field. See again [29] for an account of these arguments. See also [5] and [37] for the case of multiplicative tori in place of abelian varieties.

[3] Recent work with Corvaja, Demeio, Masser, still in progress, shows that the said measure naturally comes from the Betti map.

(iii) In general, to achieve bounded height for the values of torsion (under the appropriate conditions) is much more difficult when $\dim B > 1$ (here one has at least to impose that the relative dimension is $\geq \dim B$), essentially due to the more varied possibilities for the Néron–Severi group; see the paper [15] by Habegger for cases when a bound for the height can be proved.

3.6. **Torsion in higher relative dimensions.** When the relative dimension is $g > 1$ (the base $B$ still being a curve) it is more stringent, a constraint that a non-torsion section becomes torsion; so to say, if $B$ is a curve and $g > 1$, we have 'roughly' one degree of freedom against $g$ constraints. In fact, in such cases, we can often go beyond the previous conclusions and prove finiteness for $\mathcal{E}$ (disregarding a bound on the degree as in Corollary 3.4). For instance, we have the following result, obtained with Masser in a series of papers culminating with [20] and [22].

**Theorem 3.7.** *Let $\mathcal{A} \to B$ be a complex abelian family of relative dimension $g \geq 2$ over a curve $B$, and let $\sigma\colon B \to \mathcal{A}$ be a section such that $\mathbb{Z}\sigma(B)$ is Zariski-dense. Then the set of $b \in B$ such that $\sigma(b)$ is torsion in $A_b$ is finite.*

This is a special case of the so-called *Zilber–Pink conjecture*, which in particular extended to a *relative* context the celebrated conjecture raised by Manin and Mumford (on torsion points on subvarieties of abelian varieties, solved by Raynaud in the 80s).

This result had been actually conjectured by Shou-Wu Zhang [39] already in 1998, prior to both Pink and Zilber. (Zhang was mainly interested in certain stronger height statements, but explicitly put forward this corollary as a conjecture, to our knowledge the first explicit statement in this direction.) It may be read as a (sharp) local-global principle.

We do not say anything here on our methods of proof, and refer to [36] for a rather extended discussion, at any rate of the basic ingredients. These usually work for varieties defined over $\overline{\mathbb{Q}}$, and something more is needed to extend to $\mathbb{C}$. In the paper [8], joint with Corvaja and Masser, this is achieved by specialization on a higher dimensional base, to reduce to the case of algebraic numbers (after viewing a finitely generated field of definition as a function field of a variety). So, in a sense specialization plays a double role in this result. Somewhat surprisingly, this last specialization does not run as smoothly as one would hope and expect, due mainly to the non-compactness of the base, so that the exceptional points coming from specializing $\mathcal{E}$ could 'escape' toward the boundary, so to say.

Here is an example coming from a previous question of Masser; it amounts to a special case of Theorem 3.7 and represented a first step towards it.

**Example 3.8.** Consider another section to $L_t$, for instance $\tau(t) := (t-1, \sqrt{(t-1)(2-t)})$, in addition to $\sigma$. We can now ask about the set $\mathcal{E}$ of $b \in \mathbb{C}$ such that *both* $\sigma(b), \tau(b)$ are torsion on $L_b$. Whereas, as we have seen, there are infinitely many $b$ which make torsion one of the sections, it follows from Theorem 3.7 that this $\mathcal{E}$ (the intersection of the 'bad' sets for the two sections) is finite: just consider the pair of sections as a single section to the fiber-square scheme $L_t \times_B L_t$, and apply Theorem 3.7. The assumption in this case amounts to the fact, not difficult to check, that $\sigma, \tau$ are linearly independent over $\mathbb{Z}$ on $L_t$ (one may use conjugation over $\mathbb{Q}(t)$).

The case of a base $B$ of dimension $> 1$ (with appropriate assumptions) presents new difficulties and is essentially open. However several authors (as Barroero, Bertrand, Capuano, Gao, Habegger, Pila, H. Schmidt, Tsimerman, among others) have obtained results of the same flavour in extended contexts, on which we cannot pause here. (See [36] and [38] for some references, which are however not updated with recent results.) Instead, we shall present a couple of further applications of Theorem 3.7.

**Application I: Families of Pell's equations in polynomials.** The celebrated *Pell's equation*, proposed in fact by Fermat and actually going back to several centuries ago, is $X^2 - DY^2 = 1$, with $D$ a non-square positive integer, to be solved in integers. But here we assume $D = D(z) \in \mathbb{C}[z] - \mathbb{C}$ and we want solutions in polynomials $X = p(z)$, $Y = q(z) \neq 0$.

This polynomial case is probably less known than the former, however shows as well a respectable history. For instance, already in 1826 Abel investigated this equation in connection with integration of differentials, some years later Chebyshev too dealt with it, and the equation subsequently appeared in a number of different mathematical issues, apparently distant. (See, for example, the survey [38] and the more recent *Séminaire Bourbaki* by Serre [31].)

Contrary to the classical case, in a sense this equation 'seldom' has solutions; when this happens, $D(z) \in k[z]$ is sometimes called *Pellian* (over $k$). Note that the equation defines a pencil of affine conics and to be Pellian amounts to the existence of a nontrivial section.

The distribution of complex Pellian polynomials leads to some intriguing issues, which belong to the 'specialization' context, since the general polynomial of any given degree $> 2$ is not Pellian. (See the quoted sources, and also [3] for mention of other issues related with this.)

To give specific examples within pencils of polynomials, we note that it is not difficult to prove that $z^4 + z + t$ is not Pellian over $\overline{\mathbb{C}(t)}$ (see [36]) but $z^4 + z + b$ becomes Pellian for an infinite set of $b \in \overline{\mathbb{Q}}$, of bounded height. This is related to the elliptic scheme defined by $w^2 = z^4 + z + t$ (to which one can apply the above arguments for torsion values after the *Criterion* recalled below). If we go to higher genus, Pellianity becomes a more stringent condition, and we can prove more. For example, as before one can check that $z^6 + z + t$ is not Pellian identically in $t$, but concerning specializations now we have a stronger assertion.

**Theorem 3.9.** *There are only finitely many $b \in \mathbb{C}$ such that $z^6 + z + b$ is Pellian.*

Here, $z^6 + z$ is Pellian: $(2z^5 + 1)^2 - (z^6 + z)(2z^2)^2 = 1$; and Stoll found an algebraic number $b_0$ of degree 10 for which $z^6 + z + b_0$ is Pellian. The continued fraction for $\sqrt{z^6+z+t} = [z^3, 2(z^2-tz+t^2), -(2t^3)^{-1}z-(2t^2)^{-1}, \ldots]$ helps to find these identities, but I do not know how to compute effectively the whole finite set in question.

The link with the above 'torsion' context is provided by the following simple known result.

**Criterion**: A squarefree $D(z)$ is Pellian if and only if the difference $\infty_+ - \infty_-$ of the points at infinity on the curve $w^2 = D(z)$ is torsion on its Jacobian.

Using this and the simplicity of the Jacobian of $w^2 = z^6 + z + t$ (checked by Stoll), it is an easy matter to deduce Theorem 3.9 from Theorem 3.7. See the cited sources for further information.

**Application II: Integration in finite terms.** The problem of expressing indefinite integrals in terms of simple functions goes back to long ago and appeared among the first examples of differential algebra. By 'simple' it was meant that the integral could be obtained by a finite tower of operations either of algebraic type, or taking exponentials or taking logarithms (starting from rational functions). We call IFT (*Integrable in Finite Terms*) a differential whose integral can be likewise expressed. We have already mentioned Abel in connection with Pell's equations in polynomials, and indeed that research of his involved also elementary integration. Subsequently the matter was studied by Liouville, Ritt and Kolchin, among others, giving rise, for instance, to differential Galois theory.

More recently, it was J. Davenport who investigated *pencils* of algebraic differentials, to be integrated in finite terms; he sought to prove that if the general member of the family cannot be likewise integrated, then the same happens for the special members up to finitely many exceptions.

Masser and I took up the topic, especially since it is related to torsion in (generalized) Jacobians, and thus to the results mentioned above. This link comes from the *Criterion* above and a result of Liouville, which gives a fairly simple necessary and sufficient condition for a differential to be IFT; in essence, this says that if an elementary integral exists at all, one has only to perform towers of length 1, seeking among sums of an exact differential plus a linear combination of logarithmic-exact differentials (all from the same function field). For a very good self-contained account, we refer to Rosenlichts's article [25].

In the case of algebraic differentials, this criterion can be worked out, as done by Ritsch [24], to obtain explicit integrability conditions in terms of torsion conditions on certain divisor classes. This allows the applications of results like Theorem 3.7, and just a very special instance of the output is the following.

**Theorem 3.10.** *There are only finitely many $b \in \mathbb{C}$ such the integral*

$$\int \frac{(2z+b)\,\mathrm{d}z}{\sqrt{z^4+z+b}}$$

*is elementary.*

**Example 3.11.** The special value $b = 1/2$ is in the said finite set:

$$\int \frac{(2z+1/2)\,\mathrm{d}z}{\sqrt{z^4+z+1/2}} = \frac{1}{2}\log\big(4z^4-4z^3+2z^2+2z-1+(4z^2-4z+2)\sqrt{z^4+z+1/2}\big).$$

The analysis for such results now in fact involves torsion in *generalized Jacobians*, which are algebraic groups obtained as extensions of usual Jacobians by linear commutative groups. This requires additional results beyond Theorem 3.7, however, of the same nature (in cases like Theorem 3.10 this was worked out by H. Schmidt). For the present statement, the relevant generalized Jacobian (family) is a *non-split extension of the elliptic family* $w^2 = z^4 + z + t$ *by the additive group* $\mathbb{G}_a$. (The formula of the example corresponds to a point of order 4.)

In general, the 'obvious' expectation here would be Davenport's, i.e., that *a pencil of algebraic differentials not identically integrable in finite terms has only finitely many special members which are IFT.* In the above phrasing this would amount to the fact that *the bad set* $\mathcal{E}$ *for this problem is finite* (like in the case of Theorem 3.10).

However, this is not generally true, which was for us quite surprising.

**Example 3.12.** The differential $\frac{z\,\mathrm{d}z}{(z^2-t^2)\sqrt{z^3-z}}$ (over $\mathbb{C}(t)$) is not identically IFT but it becomes IFT for infinitely many specializations $t \to b$. Two proofs are given in [22].

In this example, note the underlying elliptic curve $w^2 = z^3 - z$ with CM: this is no coincidence, since it can be shown that if the (usual) Jacobian of the underlying curve (corresponding to the differential) does not contain CM elliptic curves, then $\mathcal{E}$ is indeed finite. More generally, there is a rather complicated analysis to decide if for a given pencil (defined over $\overline{\mathbb{Q}}$) it may happen that $\mathcal{E}$ is infinite, and here for brevity I do not pause further. See the paper [22] for complete detail.

## 4. ISOGENIES AND ENDOMORPHISM RINGS

4.1. **Specialization of endomorphism rings.** Given an abelian family $\mathcal{A}/B$ as above, say defined over $\overline{\mathbb{Q}}$, an important feature of it is the structure of the 'generic' endomorphism ring, namely, the ring $\mathrm{End}(A_t)$ (or the $\mathbb{Q}$-algebra $\mathbb{Q} \otimes \mathrm{End}(A_t)$), where $t$ is a generic point of $B$.

Then of course one may look, e.g., at algebraic points $b \in B(\overline{\mathbb{Q}})$ and ask about the structure of $\mathrm{End}(A_b)$: *what is the distribution of the bad set* $\mathcal{E}$ *for this situation?*

**Example 4.2.** (i) For instance, the property that $A_t$ (resp. $A_b$) is 'simple', meaning it does not contain nontrivial abelian subvarieties, falls into this realm, because simplicity amounts to the endomorphism ring to contain no zero-divisor.

(ii) Another illustrative example comes with a 'general' elliptic curve of $j$-invariant $t$ (e.g., $y^2 = 4x^3 - cx - c$, where $c = 27t/(t-1728)$): the generic endomorphism ring is $\mathbb{Z}$, and when $t = b \in \overline{\mathbb{Q}}$ is a *singular modulus*, i.e., equals the value $j(\tau_0)$ of the modular function $j(\tau)$ at an imaginary quadratic point $\tau_0 \in H$, the curve acquires CM, and the degree $[\mathbb{Q}(b) : \mathbb{Q}]$ equals the class-number of the corresponding order. Hence, to understand the distribution of these values in particular includes the class-number problem for imaginary quadratic fields.

In general, a first consideration is as follows: by a simple specialization argument it is not too difficult to check that for 'most' $b \in B$, namely, those in a suitable Zariski-open-dense set, $\mathrm{End}(A_t)$ injects into $\mathrm{End}(A_b)$ (one may look, e.g., at the action on regular 1-forms). And then we are essentially asking when $\mathrm{End}(A_b)$ can be strictly larger than the generic ring $\mathrm{End}(A_t)$.

That this may be a highly nontrivial issue is suggested, e.g., by the fact that, even in basic cases, already the *computation* of $\mathrm{End}(A)$, for a *given* abelian variety $A/\overline{\mathbb{Q}}$, is known to be a deep problem.[4]

Now, in 1996, Masser [18] proved a very general theorem about specializations of endomorphisms of such families of abelian varieties; namely, he proved that $\mathcal{E}$ is a *sparse* set, in a certain well-defined sense involving heights and degrees of the relevant algebraic numbers. Moreover, the bounds by Masser are completely effective. (The proofs rely among other things on his endomorphism estimates obtained with Wüstholz.)

It would be probably not ideal to state Masser's full results in this kind of exposition, so we limit to illustrate them through a simple (very) particular situation, as in the special case object of the following subsection.

4.3. **Specializations of non-isogenous elliptic curves.** Let us consider once more the Legendre curve defined by (1). For $t$ a variable (or a transcendental number), we can also consider simultaneously the elliptic curves $L_t, L_{2t}$ (defined over $\mathbb{Q}(t)$). We assert that they are not isogenous: their $j$-invariants are given, respectively, by $J(t), J(2t)$, where $J(t) = 2^8(t^2-t+1)^3t^{-2}(1-t)^{-2}$. These rational functions of $t$ have poles, respectively, at $t = 0, 1, \infty$ and $t = 0, 1/2, \infty$, hence none of the two is an integral over the ring generated by the other over $\mathbb{C}$. This excludes any isogeny, in view of the (monic) structure of modular equations.[5]

Then the following specialization question looks spontaneous.

**Question.** What can be said of the 'bad' set

$$\mathcal{E} = \{b \in \mathbb{C} : L_b, L_{2b} \text{ are isogenous}\}?$$

[4] In the elliptic case, this may be done with more elementary methods, see, for instance, [36, Rem. 4.2.1, p. 104]. As remarked below, already the case of dimension 2 to date requires more sophisticated tools.

[5] Other arguments are available; for instance, one can observe that $[\mathbb{C}(J(t), J(2t)) : \mathbb{C}(J(t)] \leq 6$, hence there are only a few modular equations to check.

**Remark 4.4.** We again stress that it appears to be a difficult issue even to decide effectively if a *given* $b$ is in $\mathcal{E}$. In fact, I know of only two algorithms to check whether two given elliptic curves, e.g., over $\mathbb{Q}$, are isogenous or not: one algorithm comes from isogeny-degree estimates due to Masser and Wüstholz, and the other one comes from an effective refinement by Serre of a theorem of Faltings on Galois representations.

As before, it is clear that $\mathcal{E} \subset \overline{\mathbb{Q}}$. And $\mathcal{E}$ is nonempty: for instance, we have $1/\sqrt{2} \in \mathcal{E}$.

Actually, it may be seen that *$\mathcal{E}$ is an infinite set in $\overline{\mathbb{Q}}$*. For a proof, one can look at the equation $\lambda(g\tau) = 2\lambda(\tau)$, where $\lambda$ is the fundamental modular function of level 2 and $g \in \mathrm{PGL}_2(\mathbb{Q})$ (a bit patient analysis yields also that $\mathcal{E}$ is dense in $\mathbb{C}$).

**Remark 4.5.** The work of Habegger [14] shows that there is no analog of Theorem 3.3 here, namely, $\mathcal{E}$ is a set of *unbounded* height (contrary to the case of torsion values of a section). Of course, we recover the easy fact that $\mathcal{E}$ is infinite, but Habegger's result is much subtler.

To go in the opposite direction, i.e., 'bounding $\mathcal{E}$ from above' so to say, is more delicate. The situation may be put into the framework of Masser's above mentioned results as follows. First, we note that, since the curves $L_t, L_{2t}$ are not (identically) isogenous, and neither has CM, the ring $\mathrm{End}(L_t \times L_{2t})$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. On the other hand, if $L_b, L_{2b}$ are isogenous, then $\mathrm{End}(L_b \times L_{2b})$ is certainly larger than $\mathbb{Z} \times \mathbb{Z}$, and hence is not isomorphic to the 'generic' ring.

Then, on taking $\mathcal{A}$ as the abelian family with $\mathbb{Q}$-generic point $L_t \times L_{2t}$ (over $B = \mathbb{P}_1 \setminus \{0, 1/2, 1, \infty\}$) this reduces our issue to one in Masser's context. And now a very special corollary of his conclusions yields the following.

**Theorem 4.6** (A corollary of [18]). *There are computable numbers $c_1, c_2$ such that, for every $D, T > 0$, the set $\{b \in \mathcal{E} : [\mathbb{Q}(b) : \mathbb{Q}] \leq D, h(b) \leq T\}$ has at most $c_1(D+T)^{c_2}$ elements.*

Here $h(b)$ denotes Weil (logarithmic) height. Since the number of algebraic numbers of degree $\leq D$ and height $\leq T$ is $\gg \exp(T + D)$, this shows that the bad set is very sparse. For instance, the estimate immediately yields the following corollary.

**Corollary 4.7.** *There are at most $c_1(1 + \log T)^{c_2}$ integers $b \in \mathcal{E} \cap [0, T]$.*

As remarked, Masser's bounds are very general and maybe not very far from the truth in the most general case. However, for our special issue, we could now ask whether something stronger than Corollary 4.7 may be said. Indeed, the paper [13] shows, in particular, that the following analog of Corollary 3.4 holds.

**Theorem 4.8** ([13]). *For every $D$, the set $\{b \in \mathcal{E}, [\mathbb{Q}(b) : \mathbb{Q}] \leq D\}$ is finite.*

**Remark 4.9.** (i) Before Masser's paper, some interesting results in the same direction of this theorem were obtained by André, as a consequence of his theory of $G$-functions (see [1]). However, they required further assumptions, and we cannot pause on this here; see Masser's paper for comments, also on how the respective results could be joined in some cases.

André also used different methods, related to $\ell$-adic representation of endomorphisms, to obtain some very general results in [2] (see Thm. 5.2 (3)). In particular, he also proved the existence of 'good' algebraic specializations (i.e., not in the bad set $\mathcal{E}$).

(ii) Note that in view of Habegger's result quoted in Remark 4.5, this finiteness cannot be derived at once on invoking the theorem of Northcott (recalled in §3). In fact, the proof in [13] is much more delicate than that, and relying on deep tools.

Ellenberg, Hall and Kowalski actually prove in [13] several other results in the same vein; the above may be obtained just as a special case of their Theorem 6 (see also Example 15 in their paper).

Let us now very briefly comment on the differences of these last results compared to Masser's.

- *Masser's results* are effective, and work for arbitrary abelian schemes.
- *Ellenberg, Hall and Kowalski's results* are not effective, and, as they stand, work only for abelian schemes over *curves* (satisfying moreover certain additional hypotheses); the restriction to curves happens to be a severe one, since certain diophantine results at the basis of the methods are completely out of the present knowledge in dimension $> 1$. As an important counterpart, they give the finiteness of Theorem 4.8 (moreover, with effective bounds for cardinality).[6]

4.10. **A few words about the proofs.** Let us describe in a few words some of the principles which appear in the proofs of the two kinds of results, which are quite different. In this article of course we may offer at most a very vague description.

**Masser's results.** A basic ingredient of the arguments consists of estimates for degrees of a set of generators for the endomorphism ring, obtained by the author and Wüstholz. (More precisely, the 'Rosati quadratic form' is used to measure the 'length' of an endomorphsim.) This lies deep, and is employed to bound the length of a possible endomorphism of the specialized variety, not arising from specialization. Then still other tools are needed, as effective elimination theory (*Nullstellensatz* by Brownawell), which is used after parameterizing rationally the endomorphisms of bounded degree. Finally, also certain zero estimates from transcendence theory play a role, in the deduction that in fact the relevant endomorphism must come by specialization.

[6]Note also that Theorem 4.8 supersedes Corollary 4.7 but not the full Theorem 4.6, even ignoring effectivity.

**Ellenberg, Hall and Kowalski's results.** A fundamental principle is to look at the Galois representation associated to torsion points of large enough fixed order, say a prime $\ell$. On the one hand, since the generic elliptic curves $L_t, L_{2t}$ are non-isogenous, this representation has 'large' image; on the other hand, if the curves become isogenous by specialization $t \to b$ (with $b$ algebraic of bounded degree), the Galois image drastically decreases. On lifting $b$, this creates a rational point of bounded degree on a suitable curve (depending on $\ell$).[7] Now, due to celebrated results of Faltings, as applied by Frey, one may prove finiteness of points of degree $\leq D$ on a curve whose *gonality* is $> 2D$ (see, e.g., [12], especially the article by van der Geer); this transfers the issue in that of bounding below the gonality of the curves which appear. In turn, this is done by a combination of rather surprising (and deep) facts, which link the structure of Galois groups with *expansion theory* of graphs and the eigenvalues of suitable Laplacians. Lower bounds for the minimal such eigenvalue are applied, crucially for the argument, to bound from below the gonality, after a discovery of Li and Yau linking these quantities.

**Remark 4.11.** Part of these methods, applied to the situation of Example 4.2 (ii), lead to a lower bound for the class-number of imaginary quadratic fields, tending to $\infty$ with the discriminant. To my knowledge this kind of argument has not been written down; anyway most probably the bound will be very weak, and certainly ineffective (contrary to the lower bounds coming from Goldfeld, and Gross and Zagier). The corresponding ideas are at bottom not unrelated to those exploited by Heegner for the problem of class-number 1. See also the Appendix to [29].

**Remark 4.12.** A different 'sparseness' conclusion in this setting was obtained by Maulik and Poonen in [23], by entirely different, sophisticated, methods. Among other things they prove the very interesting fact that the 'bad set' is nowhere dense in the $p$-adic topology. The paper contains also reference to previous work by André and Serre, with different methods.

## 5. Moduli spaces of abelian varieties

To start with, we recall the definition of $\mathcal{A}_g$ as the *(coarse moduli) space of principally polarised abelian varieties (abbreviated p.p.a.v.) of dimension $g$* (where we think here of the complex points).

Namely, each *isomorphism class* (over $\mathbb{C}$) of p.p.a.v. of dimension $g$ corresponds to a complex point of $\mathcal{A}_g$. It turns out that $\mathcal{A}_g$ may be realized as a complex quasi-projective algebraic variety of dimension $g(g+1)/2$. (For this and some other facts implicit in the discussion below, see, for instance, Milne's and Rosen's articles in [7].)

Inside $\mathcal{A}_g$, we have the subvariety corresponding to Jacobians of curves of genus $g$. This is classically referred to as the *Torelli locus*, denoted here $\mathcal{T}_g$; it

[7] As mentioned in [18], this kind of idea was already foreseen, e.g., by Bertrand after a talk of Masser on his results.

is a quasi-projective variety as well, of dimension 1 for $g = 1$ and $3g - 3$ for $g > 1$ (as essentially proved by Riemann).

**Example 5.1.** For instance, for $g = 1$, we have $\mathcal{A}_1 = \mathcal{T}_1 = \mathbb{A}^1$, in the sense that $\mathcal{A}_1$ is the space of (isomorphism classes of) complex elliptic curves, parameterized by the value of their $j$-invariant, which can be any complex number, so in $\mathbb{A}_1(\mathbb{C}) = \mathbb{C}$; and moreover the Jacobian of any curve of genus 1 is an elliptic curve.[8]

For $g = 2, 3$, we have $\dim A_g = \dim T_g$, however, there is not exact equality, and $\mathcal{T}_g$ misses the points in certain closed sets in the respective $\mathcal{A}_g$.

For $g \geq 4$, we have $\dim T_g < \dim A_g$, so that definitely $\mathcal{T}_g$ is a subset of $\mathcal{A}_g$ 'much smaller' than it.

Now, while it is true that every abelian variety is isomorphic to a quotient of some Jacobian, in view of the results recalled in this example, we see that for any $g \geq 4$, there exist complex p.p.a.v. which are not isomorphic to any Jacobian.[9]

We can now replace *isomorphic* with *isogenous* (which is a weak version of the former notion), and ask about *p.p.a.v. isogenous to some Jacobian.* Now, isogenies depend on finitely many rational parameters, so to say, so the set of p.p.a.v. of dimension $g$ isogenous to some Jacobian forms a denumerable union of subvarieties of $\mathcal{A}_g$ of the same dimension as $\mathcal{T}_g$. Hence, for $g \geq 4$, again we deduce the existence of complex p.p.a.v. of dimension $g$ isogenous to no Jacobian: no complex algebraic variety can be the union of denumerably many subvarieties of smaller dimension.

This last argument involves denumerability and uses (implicitly) that $\mathbb{C}$ is not countable. But then the issue arises on what happens on replacing $\mathbb{C}$ with a denumerable field, e.g., $\overline{\mathbb{Q}}$. Note that the above remarks say that this may indeed be considered a specialization issue.

In spite of the above argument not working anymore, intuition seems to suggest that this change should not modify our conclusion; however, the example in §2 shows that one has to be careful before drawing quick conclusions in this kind of context.

And indeed, it seems not straight-forward to provide a definite answer, and N. Katz and Oort independently have raised[10] the following question.

**Question.** Are there a.v. over $\overline{\mathbb{Q}}$ not isogenous to any Jacobian?

By the way, restriction to a.v. which are p.p. is immaterial for the question and for its treatment, since any a.v. is isogenous to a p.p.a.v.

[8] In the case $g = 1$ a principal polarization always exists and is uniquely determined up to isomorphism, hence we can forget about it.

[9] There is here a subtlety in whether we impose the isomorphisms to respect or not the polarizations, and similarly for the considerations which follow. However, the assertion(s) remains true anyway.

[10] In fact, it seems that each author attributes somewhat the question to the other; but this quotation seems the standard reference, and we shall adopt it.

The question proved to be (surprisingly?) difficult and Chai and Oort [6] gave in 2012 an affirmative answer only assuming the so-called *André–Oort conjecture*, still unproven at that time.

Their construction was soon afterwards reconsidered by Tsimerman [34], who could prove a weak form of André–Oort conjecture, sufficient for an unconditional proof.[11]

It is to be remarked that these authors proved a more general result, valid on replacing the Torelli locus by any subvariety of $\mathcal{A}_g$ of smaller dimension (so the question and the answer become significant also for $g = 2, 3$).

The ingenious arguments of both papers of Chai and Oort, and Tsimerman were built on observing that if an a.v. has CM of a certain *type* (see any of the quoted papers for definitions) the same is true for any a.v. isogenous to it; so the proof heavily uses the CM type as an isogeny invariant.

Note also that these a.v. are not 'generic', so to say, because certainly the 'general' complex abelian variety has not CM (these notions may be put on more technical ground but we do not pause on this here). In fact, a further point is that their field of definition is necessarily inside $\overline{\mathbb{Q}}$, and there is no 'continuous' family with similar properties, and moreover the minimal degree of a field of definition is expected to grow to infinity (as happens for instance with elliptic curves with CM).

Hence, it looks natural to try to provide an answer to the question by means of 'generic' a.v. (in some sense). And since the very question considers the field of definition, it looks also natural to limit as far as possible this field for the sought a.v., i.e., those presented as evidence for an affirmative answer. (Indeed, similar requirements are explicitly present in [6, Question 2, p. 604].

In a recent (forthcoming) work, Masser and I have developed a completely different method to deal with this type of problem, obtaining conclusions which may be regarded as complementary to the previous ones, and answering the above issues.

We do not discuss this here in any detail, and we limit to state one of the results that can be obtained by such analysis:

**Theorem 5.2** (joint with Masser)**.** *Let $X$ be any proper closed subvariety of $\mathcal{A}_g$. There exist points $x \in \mathcal{A}_g(\overline{\mathbb{Q}})$ (corresponding to p.p.a.v.) that are Hodge-generic and not isogenous to any $y \in X$.*

*More precisely, for given $g$, such points may be taken to be defined over respective number-fields of bounded degree over $\mathbb{Q}$ (in terms of $g$), and so that they are pairwise not isogenous and complex-dense in $\mathcal{A}_g(\mathbb{C})$.*

[11]After steps by several authors, the André–Oort conjecture was proved for $g \leq 6$ in 2013 by Pila and Tsimerman and eventually in full by Tsimerman in 2015, who introduced a further ingredient related to a conjecture by Colmez on Faltings heights. In turn, a version of this, sufficient for the said purpose, was proved by other (groups of) authors (Yuan-Shouwu Zhang and Andreatta et al.). The André–Oort conjecture may be itself considered a specialization issue; however our presentation at the conference did not include it as a separate topic and we shall not further pause on this here as well.

By 'Hodge-generic', we mean that the so-called Mumford–Tate group is the whole of $\mathrm{GSp}_{2g}$; We skip here the latter definition and only say it is a property holding for all points in $\mathcal{A}_g$ outside a certain denumerable union of proper subvarieties. Actually, our proof of Theorem 5.2 allows to replace this property with *$\ell$-Galois-generic* (for any given prime $\ell$), which is known to imply the former. Roughly speaking, this last property says that the Galois group of the field generated by $\ell$-power torsion points over the ground number field is nearly maximal, i.e., open in $\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$.

In particular, all the a.v. with any of the said properties have a trivial ring of endomorphisms (and hence they are simple), so they are somewhat 'opposite' to being CM.

For the case $g = 4$, the relevant a.v. may be even chosen to be defined over $\mathbb{Q}$, and for $g = 5$ over a fixed number field. (This depends on the fact that $\mathcal{A}_4, \mathcal{A}_5$ are unirational. However, $\mathcal{A}_g$ is known to be of general type for $g \geq 7$, and well-known conjectures of Bombieri, Lang and Vojta would predict the rational points not to be Zariski-dense. Hence, for general $X$, the same conclusion is not expected to be true for $g \geq 7$.)

The method of proof depends on several ingredients, as Masser-Wüstholz isogeny estimates (actually in the more precise form involving the Rosati quadratic form), Pila-Wilkie estimates, Serre's method with Frattini groups for a Hilbert Irreducibility for infinite towers, and some theory of Shimura varieties.

All of this in fact yields more precise results, in the shape of counting theorems saying that in a well-defined sense the majority of p.p.a.v. over $\overline{\mathbb{Q}}$ are not isogenous to any point in $X$. We skip here any further details and refer to the paper [21].

**Remark 5.3.** (i) For the above alluded method devised by Serre, see [29] and especially [30], *Lettres a Ken Ribet*, this last reference containing explicit mention of specialization theorems. The method often allows to deal with specialization of the Mumford–Tate group, through Galois groups obtained by adding torsion points of $\ell$-power order. This principle was also applied by André in the paper [2], quoted in the previous section.

(ii) Of course, for $g = 1$, the above question is not sensible, but one can pose a *real* analog, namely, asking *whether, given a real algebraic curve $X \subset \mathcal{A}_1 \cong \mathbb{A}^1$, there are elliptic curves over $\overline{\mathbb{Q}}$ not isogenous to one with $j$-invariant in $X$*. For instance, are there elliptic curves over $\overline{\mathbb{Q}}$ not isogenous to anyone whose $j$-invariant is of the shape $a + ia^2$, $a \in \mathbb{R}$?

The methods lead to an affirmative answer also to this question (with an easier proof), actually providing infinitely many examples defined over any prescribed number field provided it is *non-real* (which is a necessary restriction for any elliptic curve defined over $\mathbb{R}$ has $j$-invariant on the real line, which is of the said type).

The restriction to real-*algebraic* curves looks natural, if not for the reason that is more likely that a curve contains 'many' algebraic points if it is an algebraic curve. Also, and more important, it may be seen by easy interpolation that the conclusion does not remain true if an arbitrary analytic curve is allowed. (On the other hand, difficult questions seem to arise if one allows analytic non-algebraic curves subject to other restrictions.)

## 6. Other contexts and open questions

In this section we discuss very briefly a couple of other contexts involving specialization problems. But again we shall leave aside issues related to the Bertini theorem or the Hilbert irreducibility theorem, which are probably better known (see, e.g., [4] or [29]).

6.1. **Non-commutative groups.** In place of an abelian variety, one may of course consider other algebraic groups, e.g., linear ones. The commutative linear groups are (up to isomorphism) of the shape $\mathbb{G}_{\mathrm{a}}^r \times \mathbb{G}_{\mathrm{m}}^s$ (so we have only iso-constant families) and here the situation is often similar to, and simpler than, the abelian case (see, for instance, [36, Ch. I]). On the other hand, for non-commutative linear groups, not much seems to be known, and already easily formulated problems, in basic cases, present open questions.

Borrowing from a similar example in [29], consider for instance the matrices

$$X_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad Y_t := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

where $t$ is a variable (or a transcendental number). They may be seen as sections to $\mathrm{SL}_2$, viewed as a 'constant' family over $\mathbb{A}^1$. It is known that they generate a free group (of sections), denoted here $\Gamma_t$.

This may be proved, e.g., by specialization $t \to 2$: we obtain a well-known group, denoted here $\Gamma_2$, which is 'almost' $\Gamma(2) := \{A \in \mathrm{SL}_2(\mathbb{Z}) : A \equiv I \pmod 2\}$, i.e., together with $-I$ it generates $\Gamma(2)$.

To discuss further $\Gamma_2$, since we met the Legendre family several times, it may be not out of place to recall that $\Gamma_2$ is naturally the image of the monodromy representation of $\mathbb{P}_1 - \{0, 1, \infty\}$ by action on the periods of the elliptic curve $L_\lambda$, given in the region $\max\{|\lambda|, |1-\lambda|\} < 1$ by the hypergeometric functions $\pi F(1/2, 1/2, 1, \lambda)$ and $i\pi F(1/2, 1/2, 1, 1-\lambda)$ (see [16, Ch. 9]). Here $\lambda$ may be seen as a variable in the said domain, but also as the well-known modular function for the group $\Gamma(2)$, which yields the universal covering map $\lambda\colon H \to \mathbb{P}_1 - \{0, 1, \infty\}$.[12]

Given the structure of $\pi_1(\mathbb{C} - \{0, 1\})$ as a free group on two generators, and observing that $\Gamma(2)/\pm I = \Gamma_2$ acts faithfully on $H$, we deduce that $\Gamma_2$ is also free, on the above generators, as asserted.

Another argument for this conclusion, really simpler, is by the so-called 'ping-pong' reasoning: if we define $A := \{(x, y) \in \mathbb{C}^2, |x| < |y|\}$ and $B = \{(x, y) \in \mathbb{C}^2 : |x| > |y|\}$, we see that $X_2^m A \subset B$ and $Y_2^m B \subset A$ for any nonzero $m \in \mathbb{Z}$, and it is then not difficult to check that this prevents any nontrivial relation among $X_2, Y_2$.

[12] Under a lifting to $H$, the ratio of the said periods becomes the natural variable $\tau$ on $H$.

It also follows that $X_t, Y_t$ generate a free group, i.e., $\Gamma_t$ is indeed free. (Note that this is the 'trivial' specialization implication '$\Gamma_b$ *free* $\implies$ $\Gamma_t$ *free*', whereas we are mainly interested in the converse one.)

If one specializes more generally $t \to b \in \overline{\mathbb{Q}}$, one obtains a group denoted here $\Gamma_b$ generated by the corresponding specializations $X_b, Y_b$. The natural analog of the problems of torsion considered in §3, and its extension to *dependence over* $\mathbb{Z}$ (see especially Remark 3.5 (i)) would be to ask for the possible *relations* among the specialized elements $X_b, Y_b$. The 'bad set' $\mathcal{E}$ for this problem consists of the $b$ such that $X_b, Y_b$ satisfy some nontrivial relation. As in many examples above, it is a set of algebraic numbers, hence countable.

The above shows that 2 does not belong to $\mathcal{E}$, and inspection through the second argument shows that $\mathcal{E}$ is in fact contained in the open disk $\{z \in \mathbb{C} : |z| < 2\}$.

In the opposite direction, it is not difficult to see that $\mathcal{E}$ is infinite: for instance, by the identites $X_t^m = X_{mt}$ and $Y_t^m = Y_{mt}$, we have $m^{-1}\mathcal{E} \subset \mathcal{E}$ for any nonzero integer $m$, so we only need to construct some nonzero element of $\mathcal{E}$. Here is just a possibility; consider the product $X_t Y_t^{\pm 1}$, for any choice of the sign. It is diagonalizable over a quadratic extension of $\mathbb{Q}(t)$, in fact over $\mathbb{Q}(t, \sqrt{t^2 \pm 4})$, with trace $2 \pm t^2$. Equating the trace to $2\cos(2\pi r)$ for a rational number $r$ with denominator $n > 2$ yields eigenvalues which are roots of unity of order $n$, with product 1, hence distinct eigenvalues producing a matrix of finite order $n$. This gives a nontrivial relation among the specialized matrices: $(X_b Y_b^{\pm 1})^n = I$. The solutions $t = b = \pm\sqrt{\pm 2(1 - \cos(2\pi r))}$ so obtained (on varying the sign) are dense in the segments $(-2, 2)$ and $i(-2, 2)$.

This kind of construction may be of course refined and extended. All of this even shows that there are infinitely many elements of $\mathcal{E}$ of bounded height, or also of bounded degree (hence unbounded height). So, many of the principles we have seen to be true for abelian families fail here.[13]

However, in spite of these first observations, an 'explicit' simple description of $\mathcal{E}$ (or of its closure) seems not known (see, e.g., the paper [17] by Lyndon and Ullmann for some results and methods).

6.2. **Families of quadratic forms.** We have mentioned in the introduction that specialization theorems may be (often) thought of as *local-global principles*, which is a terminology used especially in the arithmetical case, i.e., when specialization is replaced by reduction modulo a prime.

Among the best known such results is the *Hasse–Minkowski local-global principle* for quadratic forms, stating in its simplest nontrivial case that *if a ternary quadratic form* $aX^2 + bY^2 + cZ^2$ $(a, b, c \in \mathbb{Z})$ *admits a nontrivial zero* $(x_p : y_p : z_p) \in \mathbb{P}_2(\mathbb{Q}_p)$ *for each prime* $p$*, then it admits a nontrivial rational zero* $(x : y : z) \in \mathbb{P}_2(\mathbb{Q})$.

[13]Bounded height is also found to fail in the case of iso-constant abelian families, and in the case of families of tori, when the group of sections contains some nonidentical iso-constant element; see for instance [36], Ch. I. However here, though the family is constant, there are no constant sections $\neq I$ (since $\Gamma_t$ is free).

Actually, this ternary case is implicit in work by Legendre (see [28, p. 74] for a modern presentation of such proof). Hasse and Minkowski proved this for arbitrary quadratic forms over $\mathbb{Q}$. We also recall that the existence of a $p$-adic zero amounts essentially to a congruence modulo a suitable power of $p$, and is automatic for almost all primes.

One may consider a function-field analog, on replacing $\mathbb{Z}$ with a polynomial ring, e.g., $\mathbb{Q}[t]$. In place of the reduction modulo $p$ one then has a specialization $t \to s \in \mathbb{Q}$, whereas one would seek solutions this time in $\mathbb{P}_2(\mathbb{Q}(t))$. In this direction, Davenport, Lewis and Schinzel proved [10] the following elegant statement.

**Theorem 6.3** ([10]). *Let $a, b, c \in \mathbb{Q}[t]$ be nonzero polynomials. If every arithmetical progression contains an integer $n$ such that $a(n)X^2 + b(n)Y^2 + c(n)Z^2$ has a zero in $\mathbb{P}_2(\mathbb{Q})$, then $a(t)X^2 + b(t)Y^2 + c(t)Z^2$ has a zero in $\mathbb{P}_2(\mathbb{Q}(t))$.*

**Example 6.4.** We note that, unlike the classical case of integers, this result has no analog in an arbitrary number of variables, as the following example shows.

Consider the quadratic form in five variables over $\mathbb{Q}(t)$ given by $F_t := 2X_1^2 - X_2^2 + tX_3^2 + tX_4^2 + tX_5^2$. Note that for every $b \in \mathbb{Q}^*$, the resulting quadratic form $F_b$ is indefinite, hence by Mejer's theorem (see [28]) it represents 0 over $\mathbb{Q}$.

On the other hand, this form does not represent 0 over $\mathbb{Q}(t)$, for otherwise there would exist coprime polynomials $x_1(t), \dots, x_5(t) \in \mathbb{Q}[t]$ with $F_t(x_1(t), \dots, x_5(t)) = 0$. On setting $t = 0$, we would obtain $2x_1(0)^2 = x_2(0)^2$, hence $x_1, x_2$ would be both divisible by $t$ in $\mathbb{Q}[t]$: $x_i(t) = ty_i(t)$ for $i = 1, 2$. Therefore, we would have $x_3(t)^2 + x_4(t)^2 + x_5(t)^2 = ty_2(t)^2 - 2ty_1(t)^2$. But now setting $t = 0$ yields that all $x_i(t)$ would be divisible by $t$, a contradiction.

Therefore, this would violate a possible analog of Theorem 6.3 for this quadratic form. (For an example in four variables, see [26, p. 214].)

The proof of Theorem 6.3 given in [10] (see also [27, Ch. V]) worked on mimicking the proof by Legendre alluded to above. Another proof was possible on realizing a connection of this issue with the topic of *specialization of Brauer groups*, contributed to by Faddeev and Serre; for instance, we refer to our paper [35] for a self-contained (quantitative) argument, different from the one in [10], and for references to other works. Such an approach also allowed a function field analog of *Hasse's local-global principle for norms from cyclic extensions*, the case of ternary quadratic forms being equivalent to the quadratic case of such principle. This analog of Hasse's principle includes Theorem 6.3 as a special case.

Theorem 6.3 may be phrased in terms of a pencil of conics over the affine line, the assumption becoming that the rational points in a certain set on the base can be lifted to rational points of the corresponding conics, the conclusion predicting the existence of a rational section defined over $\mathbb{Q}$.[14] One may ask

[14]We note that a theorem of Tsen asserts that a (rational) section defined over some number field always exists; see, for instance, [35] for a simple proof and an application.

what happens on replacing the base with another curve. Of course the issue is sensible only on assuming at least that such a curve has infinitely many rational points; but then by Faltings' theorem it must have genus 0 or 1. The case of genus 1 already had a negative answer concerning the existence of a section (see [9] for a simple example).

Other similar questions arise on considering pencils of subvarieties of abelian varieties, which brings us back nearer to the main context of this survey. The simplest case is when the ambient abelian family is *constant*, for instance, arising from a non-constant rational map $\pi\colon A \to \mathbb{P}_1$, the fibers being the relevant subvarieties. In [9, Thm. 3.47], it is shown, using deep theorems of Faltings (or Faltings–Vojta) that *there always exist infinitely many points in* $\mathbb{P}_1(\mathbb{Q})$ *which cannot be lifted to any rational point in the fiber* (equivalently, $\pi(A(\mathbb{Q}))$ has infinite complement in $\mathbb{P}_1(\mathbb{Q})$). In this direction, it would be not free of interest to have here some statements (even conjectural) for more general cases.

**Acknowledgements.** I wish to thank the organizers of the Conference for Deninger, namely, G. Kings, P. Schneider, A. Werner, for inviting me to deliver a talk, and the University of Münster and the staff, especially Ms. G. Dierkes, for kind hospitality and support. I also thank C. Voisin for pointing out relevant literature.

## References

[1] Y. André, *G-functions and geometry*, Aspects of Mathematics, E13, Friedr. Vieweg & Sohn, Braunschweig, 1989. MR0990016

[2] Y. André, Pour une théorie inconditionnelle des motifs, Inst. Hautes Études Sci. Publ. Math. No. 83 (1996), 5–49. MR1423019

[3] Y. André, P. Corvaja, and U. Zannier, The Betti map associated to a section of an abelian scheme. Preprint (2018).

[4] E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs, 4, Cambridge University Press, Cambridge, 2006. MR2216774

[5] E. Bombieri, D. Masser, and U. Zannier, Intersecting a curve with algebraic subgroups of multiplicative groups, Internat. Math. Res. Notices **1999** (1999), no. 20, 1119–1140. MR1728021

[6] C.-L. Chai and F. Oort, Aabelian varieties isogenous to a Jacobian, Ann. of Math. (2) **176** (2012), no. 1, 589–635. MR2925391

[7] G. Cornell and J. H. Silverman, *Arithmetic geometry*, Springer-Verlag, New York, 1986. MR0861969

[8] P. Corvaja, D. Masser, and U. Zannier, Torsion hypersurfaces on abelian schemes and Betti coordinates, Math. Ann. **371** (2018), no. 3-4, 1013–1045. MR3831262

[9] P. Corvaja and U. Zannier, *Applications of Diophantine approximation to integral points and transcendence*, Cambridge Tracts in Mathematics, 212, Cambridge University Press, Cambridge, 2018. MR3793125

[10] H. Davenport, D. J. Lewis, and A. Schinzel, Quadratic Diophantine equations with a parameter, Acta Arith. **11** (1965/1966), 353–358. MR0184902

[11] L. DeMarco and N. Mavraki, Variation of Canonical Height and Equidistribution, preprint (2017).

[12] B. Edixhoven and J.-H. Evertse, *Diophantine approximation and abelian varieties*, Lecture Notes in Mathematics, 1566, Springer-Verlag, Berlin, 1993. MR1288998

[13] J. S. Ellenberg, C. Hall, and E. Kowalski, Expander graphs, gonality, and variation of Galois representations, Duke Math. J. **161** (2012), no. 7, 1233–1275. MR2922374

[14] P. Habegger, Weakly bounded height on modular curves, Acta Math. Vietnam. **35** (2010), no. 1, 43–69. MR2642162

[15] P. Habegger, Special points on fibered powers of elliptic surfaces, J. Reine Angew. Math. **685** (2013), 143–179. MR3181568

[16] D. Husemoller, *Elliptic curves*, Graduate Texts in Mathematics, 111, Springer-Verlag, New York, 1987. MR0868861

[17] R. C. Lyndon and J. L. Ullman, Pairs of real 2-by-2 matrices that generate free products, Michigan Math. J. **15** (1968), 161–166. MR0228593

[18] D. W. Masser, Specializations of endomorphism rings of abelian varieties, Bull. Soc. Math. France **124** (1996), no. 3, 457–476. MR1415735

[19] D. Masser and U. Zannier, Bicyclotomic polynomials and impossible intersections, J. Théor. Nombres Bordeaux **25** (2013), no. 3, 635–659. MR3179679

[20] D. Masser and U. Zannier, Torsion points on families of simple abelian surfaces and Pell's equation over polynomial rings, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 9, 2379–2416. MR3420511

[21] D. Masser and U. Zannier, Abelian varieties isogenous to no Jacobian, preprint in progress (2018).

[22] D. Masser and U. Zannier, Torsion points, Pell's equations and integration in finite terms, preprint (2018).

[23] D. Maulik and B. Poonen, Néron–Severi groups under specialization, Duke Math. J. **161** (2012), no. 11, 2167–2206. MR2957700

[24] R. H. Risch, The solution of the problem of integration in finite terms, Bull. Amer. Math. Soc. **76** (1970), 605–608. MR0269635

[25] M. Rosenlicht, Liouville's theorem on functions with elementary integrals, Pacific J. Math. **24** (1968), 153–161. MR0223346

[26] A. Schinzel, *Selected topics on polynomials*, University of Michigan Press, Ann Arbor, MI, 1982. MR0649775

[27] A. Schinzel, *Polynomials with special regard to reducibility*, Encyclopedia of Mathematics and its Applications, 77, Cambridge University Press, Cambridge, 2000. MR1770638

[28] J.-P. Serre, *Cours d'arithmétique*, Collection SUP: "Le Mathématicien", 2, Presses Universitaires de France, Paris, 1970. MR0255476

[29] J.-P. Serre, *Lectures on the Mordell-Weil theorem*, translated from the French and edited by Martin Brown from notes by Michel Waldschmidt, Aspects of Mathematics, E15, Friedr. Vieweg & Sohn, Braunschweig, 1989. MR1002324

[30] J.-P. Serre, *Œuvres. Vol. I*, Springer-Verlag, Berlin, 1986. MR0926689

[31] J.-P. Serre, Distribution asymptotique des valeurs propres des endomorphismes de Frobenius [d'après Abel, Chebyshev, Robinson,...], Séminaire Bourbaki 1146 (2017–2018).

[32] J. H. Silverman, Heights and the specialization map for families of abelian varieties, J. Reine Angew. Math. **342** (1983), 197–211. MR0703488

[33] J. H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, 106, Springer-Verlag, New York, 1986. MR0817210

[34] J. Tsimerman, The existence of an abelian variety over $\overline{\mathbb{Q}}$ isogenous to no Jacobian, Ann. of Math. (2) **176** (2012), no. 1, 637–650. MR2925392

[35] U. Zannier, A local-global principle for norms from cyclic extensions of $\mathbf{Q}(t)$ (a direct, constructive and quantitative approach), Enseign. Math. (2) **45** (1999), no. 3-4, 357–377. MR1742338

[36] U. Zannier, *Some problems of unlikely intersections in arithmetic and geometry*, Annals of Mathematics Studies, 181, Princeton University Press, Princeton, NJ, 2012. MR2918151

[37] U. Zannier, *Lecture notes on Diophantine analysis*, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie), 8, Edizioni della Normale, Pisa, 2009. MR2517762
[38] U. Zannier, Unlikely intersections and Pell's equations in polynomials, in *Trends in contemporary mathematics*, 151–169, Springer INdAM Ser., 8, Springer, Cham, 2014. MR3586397
[39] S.-W. Zhang, Small points and Arakelov theory, Doc. Math. **1998**, Extra Vol. II, 217–225. MR1648072

Received August 27, 2018; accepted October 10, 2018

Umberto Zannier
Scuola Normale Superiore,
Piazza dei Cavalieri, 7, 56126 Pisa, Italy
E-mail: `umberto.zannier@sns.it`